Contents

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Part I

Foundations

Chapter 1

The Language of ∞-Categories

A principal goal of algebraic topology is to understand topological spaces by means of algebraic and combinatorial invariants. Let us consider some elementary examples. 0001

- To any topological space X, one can associate the set $\pi_0(X)$ of *path components* of X. This is the quotient of X by an equivalence relation $\simeq$, where $x \simeq y$ if there exists a continuous path $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$.
- To any topological space X equipped with a base point $x \in X$, one can associate the *fundamental group* $\pi_1(X, x)$. This is a group whose elements are homotopy classes of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x = p(1)$.

For many purposes, it is useful to combine the set $\pi_0(X)$ and the fundamental groups $\{\pi_1(X,x)\}_{x \in X}$ into a single mathematical object. To any topological space X, one can associate an invariant $\pi_{\leq 1}(X)$ called the *fundamental groupoid of* X. The fundamental groupoid $\pi_{\leq 1}(X)$ is a category whose objects are the points of X, where a morphism from a point $x \in X$ to a point $y \in X$ is given by a homotopy class of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$. The set of path components $\pi_0(X)$ can then be recovered as the set of isomorphism classes of objects of the category $\pi_{\leq 1}(X)$, and each fundamental group $\pi_1(X,x)$ can be identified with the automorphism group of the point x as an object of the category $\pi_{\leq 1}(X)$. The formalism of category theory allows us to assemble information about path components and fundamental groups into a single convenient package.

The fundamental groupoid $\pi_{\leq 1}(X)$ is a very important invariant of a topological space X, but is far from being a complete invariant. In particular, it does not contain any information about the *higher* homotopy groups $\{\pi_n(X,x)\}_{n \geq 2}$. We therefore ask the following:

Question 1.0.0.1. Let X be a topological space. Can one devise a "category-theoretic" invariant of X, in the spirit of the fundamental groupoid $\pi_{\leq 1}(X)$, which contains information about *all* the homotopy groups of X? 0002

We begin to address Question 1.0.0.1 in §1.1 by introducing the theory of *simplicial sets*. A simplicial set $S = S_\bullet$ is a collection of sets $\{S_n\}_{n \geq 0}$, which are related by *face operators* $\{d_i^n : S_n \to S_{n-1}\}_{0 \leq i \leq n}$ and *degeneracy operators* $\{s_i^n : S_n \to S_{n+1}\}_{0 \leq i \leq n}$ satisfying suitable identities (see Definition 1.1.0.6 and Proposition 1.1.2.14). Every topological space X determines a simplicial set $\mathrm{Sing}_\bullet(X)$, called the *singular simplicial set* of X, with the property that each $\mathrm{Sing}_n(X)$ is the collection of continuous maps from the topological n-simplex into X (Construction 1.2.2.2). Moreover, the homotopy groups of X can be reconstructed from the simplicial set $\mathrm{Sing}_\bullet(X)$ by a simple combinatorial procedure (see §3.2). Kan observed that this procedure can be applied more generally to any simplicial set S satisfying the following *Kan extension condition*:

$(*)$ For $0 \leq i \leq n$, every map $\sigma_0 : \Lambda_i^n \to S$ admits an extension $\sigma : \Delta^n \to S$.

Here Δ^n denotes a certain simplicial set called the *standard n-simplex* (Example 1.1.0.9), and Λ_i^n denotes a certain simplicial subset of Δ^n called the *ith horn* (Construction 1.2.4.1). Simplicial sets satisfying condition $(*)$ are called *Kan complexes*. Every simplicial set of the form $\mathrm{Sing}_\bullet(X)$ is a Kan complex (Proposition 1.2.5.8), and the converse is true up to homotopy. More precisely, Milnor proved in [44] that the construction $X \mapsto \mathrm{Sing}_\bullet(X)$ induces an equivalence from the (geometrically defined) homotopy theory of CW complexes to the (combinatorially defined) homotopy theory of Kan complexes; we will discuss this point in Chapter 3 (see Theorem 3.6.0.1).

The singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a natural candidate for the sort of invariant requested in Question 1.0.0.1: it is a mathematical object of a purely combinatorial nature which contains complete information about the homotopy groups of X and their interrelationship (from which we can even reconstruct X up to homotopy equivalence, provided that X has the homotopy type of a CW complex). But in order to see that it qualifies as a complete answer, we must address the following:

0003 **Question 1.0.0.2.** Let X be a topological space. To what extent does the simplicial set $\mathrm{Sing}_\bullet(X)$ behave like a category? What is the relationship between $\mathrm{Sing}_\bullet(X)$ with the fundamental groupoid of X?

Our answer to Question 1.0.0.2 begins with the observation that the theory of simplicial sets is closely related to category theory. To every category $\mathcal{C}$, one can associate a simplicial set $\mathrm{N}_\bullet(\mathcal{C})$, called the *nerve of* $\mathcal{C}$ (we will review the construction of $\mathrm{N}_\bullet(\mathcal{C})$ in §1.3; see Construction 1.3.1.1). The construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ is fully faithful (Proposition 1.3.3.1): in particular, a category $\mathcal{C}$ is determined (up to canonical isomorphism) by the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. Throughout much of this book, we will abuse notation by not distinguishing between a category $\mathcal{C}$ and its nerve $\mathrm{N}_\bullet(\mathcal{C})$: that is, we will view a category as a special kind of simplicial set. These special simplicial sets admit a simple characterization: according to

Proposition 1.3.4.1, a simplicial set S has the form $\mathrm{N}_\bullet(\mathcal{C})$ (for some category $\mathcal{C}$) if and only if it satisfies the following variant of the Kan extension condition (Proposition 1.3.4.1):

$(*')$ For $0 < i < n$, every morphism $\sigma_0 : \Lambda^n_i \to S$ admits a unique extension $\sigma : \Delta^n \to S$.

The extension conditions $(*)$ and $(*')$ are closely related, but differ in two important respects. The Kan extension condition requires that *every* map of simplicial sets $\sigma_0 : \Lambda^n_i \to S$ admits an extension $\sigma : \Delta^n \to S$. Condition $(*')$ requires the existence of an extension only in the case $0 < i < n$, but demands that the extension is unique. Neither of these conditions implies the other: a simplicial set of the form $\mathrm{N}_\bullet(\mathcal{C})$ satisfies condition $(*)$ if and only if the category $\mathcal{C}$ is a groupoid (Proposition 1.3.5.2), and a simplicial set of the form $\mathrm{Sing}_\bullet(X)$ satisfies condition $(*')$ if and only if every continuous path $[0, 1] \to X$ is constant. However, conditions $(*)$ and $(*')$ admit a common generalization. We will say that a simplicial set S is an ∞-*category* if it satisfies the following variant of $(*)$ and $(*')$, known as the *weak Kan extension condition*:

$(*'')$ For $0 < i < n$, every map $\sigma_0 : \Lambda^n_i \to S_\bullet$ admits an extension $\sigma : \Delta^n \to S_\bullet$.

The theory of ∞-categories can be viewed as a simultaneous generalization of homotopy theory and category theory. Every Kan complex is an ∞-category, and every category $\mathcal{C}$ determines an ∞-category (given by the nerve $\mathrm{N}_\bullet(\mathcal{C})$). In particular, the notion of ∞-category answers the first part of Question 1.0.0.2: simplicial sets of the form $\mathrm{Sing}_\bullet(X)$ are almost never (the nerves of) categories, but are always ∞-categories. At this point, the reader might reasonably object that this is terminological legerdemain: to address the spirit of Question 1.0.0.2, we must demonstrate that simplicial sets of the form $\mathrm{Sing}_\bullet(X)$ (or, more generally, all simplicial sets satisfying condition $(*'')$) really *behave* like categories. We begin in §1.4 by explaining how to extend various elementary category-theoretic ideas to the setting of ∞-categories. For example we can associate to each ∞-category $S = S_\bullet$ a collection of *objects* (these are the elements of the set S_0), a collection of *morphisms* (these are the elements of the set S_1), and a composition law on morphisms. In particular, we show that any ∞-category S determines an ordinary category $\mathrm{h}S$, called the *homotopy category of* S (Proposition 1.4.5.2). The construction of the homotopy category allows us to answer the second part of Question 1.0.0.2: for every topological space X, the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is an ∞-category, whose homotopy category $\mathrm{hSing}_\bullet(X)$ is the fundamental groupoid $\pi_{\leq 1}(X)$ (see Example 1.4.5.5).

Roughly speaking, the difference between an ∞-category S and its homotopy category $\mathrm{h}S$ is that the former can contain nontrivial homotopy-theoretic information (encoded by simplices of dimension $n \geq 2$, which can be loosely understood as "n-morphisms") which is lost upon passage to the homotopy category $\mathrm{h}S$. We can summarize the situation informally with the heuristic equation

$$\{\text{Categories}\} + \{\text{Homotopy Theory}\} = \{\infty\text{-Categories}\},$$

or more precisely with the diagram

$$\begin{tikzcd}
\{\text{Categories}\} \arrow[r, hook, "\mathrm{N}_\bullet"] & \{\infty\text{-Categories}\} \arrow[r, phantom, "\supset"] & \{\text{Kan Complexes}\} \\
& \cap & \\
& \{\text{Simplicial Sets}\} & \{\text{Topological Spaces}\} \arrow[uu, "\mathrm{Sing}_\bullet"']
\end{tikzcd}$$

1.1 Simplicial Sets

0004 In this section we provide an introduction to the theory of simplicial sets, which will play an essential role throughout this book. We begin with some preliminaries.

0009 **Notation 1.1.0.1.** For every nonnegative integer n, we let $[n]$ denote the linearly ordered set $\{0 < 1 < 2 < \cdots < n-1 < n\}$.

000A **Definition 1.1.0.2** (The Simplex Category)**.** We define a category $\mathbf{\Delta}$ as follows:

- The objects of $\mathbf{\Delta}$ are linearly ordered sets of the form $[n]$ for $n \geq 0$.
- A morphism from $[m]$ to $[n]$ in the category $\mathbf{\Delta}$ is a function $\alpha : [m] \to [n]$ which is *nondecreasing*: that is, for each $0 \leq i \leq j \leq m$, we have $0 \leq \alpha(i) \leq \alpha(j) \leq n$.

We will refer to $\mathbf{\Delta}$ as the *simplex category*.

000B **Remark 1.1.0.3.** The category $\mathbf{\Delta}$ is equivalent to the category of *all* nonempty finite linearly ordered sets, with morphisms given by nondecreasing maps. In fact, we can say something better: for every nonempty finite linearly ordered set I, there is a *unique* nondecreasing bijection $I \simeq [n]$, for some $n \geq 0$.

000C **Definition 1.1.0.4.** Let $\mathcal{C}$ be a category. A *simplicial object of* $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. A *cosimplicial object of* $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$.

000D **Notation 1.1.0.5.** We will often use an expression like $C_\bullet$ to denote a simplicial object of a category $\mathcal{C}$. In this case, we write C_n for the value of the functor $C_\bullet$ on the object $[n] \in \mathbf{\Delta}$. Similarly, we often use an expression like $C^\bullet$ to indicate a cosimplicial object of $\mathcal{C}$, and C^n for its value on $[n] \in \mathbf{\Delta}$.

We will be primarily interested in the following special case of Definition 1.1.0.4:

000H **Definition 1.1.0.6.** Let Set denote the category of sets. A *simplicial set* is a simplicial object of Set: that is, a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$.

Notation 1.1.0.7. We let $\mathrm{Set}_{\Delta} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ denote the category of functors from $\mathbf{\Delta}^{\mathrm{op}}$ to Set. We refer to Set_{Δ} as *the category of simplicial sets.* 04Z5

Remark 1.1.0.8. Since the category of sets has all (small) limits and colimits, the category of simplicial sets also has all (small) limits and colimits. Moreover, these limits and colimits are computed levelwise: for any functor 000J

$$S_{\bullet} : \mathcal{C} \to \mathrm{Set}_{\Delta} \qquad (C \in \mathcal{C}) \mapsto S_{\bullet}(C),$$

and any nonnegative integer n, we have canonical bijections

$$(\varinjlim_{C \in \mathcal{C}} S(C))_n \simeq \varinjlim_{C \in \mathcal{C}} (S_n(C)) \qquad (\varprojlim_{C \in \mathcal{C}} S(C))_n \simeq \varprojlim_{C \in \mathcal{C}} (S_n(C)).$$

Example 1.1.0.9 (The Standard Simplex)**.** Let $n \geq 0$ be an integer. We let Δ^n denote the functor 04Z6

$$\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set} \qquad [m] \mapsto \mathrm{Hom}_{\mathbf{\Delta}}([m], [n]).$$

Then Δ^n is a simplicial set, which we will refer to as the *standard n-simplex.* By convention, we extend this construction to the case $n = -1$ by setting $\Delta^{-1} = \emptyset$.

Example 1.1.0.10. The standard 0-simplex Δ^0 is a final object of the category of simplicial sets: that is, it carries each $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ to a set having a single element. 000M

Definition 1.1.0.11. Let $S_{\bullet}$ be a simplicial set and let n be a nonnegative integer. An *n-simplex of $S_{\bullet}$* is an element of the set S_n. We will also refer to elements of S_0 as *vertices of $S_{\bullet}$*, and to elements of S_1 as *edges* of $S_{\bullet}$. We often write $v \in S_{\bullet}$ to indicate that v is a vertex of $S_{\bullet}$. 04Z7

Proposition 1.1.0.12. *Let n be a nonnegative integer and regard the identity map $\mathrm{id}_{[n]} : [n] \to [n]$ as an n-simplex of Δ^n. For every simplicial set $S_{\bullet}$, evaluation on $\mathrm{id}_{[n]}$ induces a bijection* 04Z8

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, S_{\bullet}) \to S_n \qquad f \mapsto f(\mathrm{id}_{[n]}).$$

Proof. This is a special case of Yoneda's lemma. □

Notation 1.1.0.13. Let $S_{\bullet}$ be a simplicial set and let $\sigma \in S_n$ be an n-simplex of $\mathcal{C}$. By virtue of Proposition 1.1.0.12, there is a unique morphism $f_{\sigma} : \Delta^n \to S_{\bullet}$ in the category of simplicial sets which satisfies $f_{\sigma}(\mathrm{id}_{[n]}) = \sigma$. In practice, we will often abuse notation by identifying the n-simplex σ with the morphism f_{σ}. 04Z9

Remark 1.1.0.14 (Simplicial Subsets)**.** Let $S_{\bullet}$ be a simplicial set. Suppose that: 000P

- For every integer $n \geq 0$, we are given a subset $T_n \subseteq S_n$,

- For every morphism $\alpha : [m] \to [n]$ in the simplex category $\mathbf{\Delta}$, the associated map $S_n \to S_m$ carries T_n into T_m.

Then we the construction $[n] \mapsto T_n$ determines another simplicial set $T_\bullet$. In this case, we will say that $T_\bullet$ is a *simplicial subset* of $S_\bullet$ and write $T_\bullet \subseteq S_\bullet$.

000Q **Example 1.1.0.15.** Let $S_\bullet$ be a simplicial set and let v be a vertex of $S_\bullet$. Then v can be identified with a map of simplicial sets $\Delta^0 \to S_\bullet$. This map is automatically a monomorphism (note that Δ^0 has only a single n-simplex for every $n \geq 0$), whose image is a simplicial subset of $S_\bullet$. It will often be convenient to denote this simplicial subset by $\{v\}$. For example, we can identify vertices of the standard n-simplex Δ^n with integers i satisfying $0 \leq i \leq n$; every such integer i determines a simplicial subset $\{i\} \subseteq \Delta^n$ (whose k-simplices are the constant maps $[k] \to [n]$ taking the value i).

Our first goal in this section is to make Definition 1.1.0.6 more concrete. To a first degree of approximation, a simplicial set $S_\bullet$ can be viewed as a collection of sets $\{S_n\}_{n \geq 0}$. However, this collection is endowed with additional structure, arising from morphisms in the simplex category $\mathbf{\Delta}$. For example, let n be a positive integer. For each $0 \leq i \leq n$, there is a unique order-preserving bijection $[n-1] \simeq [n] \setminus \{i\} \subset [n]$. This induces a function $d^n_i : S_n \to S_{n-1}$ which we will refer to as a *face operator* for the simplicial set $S_\bullet$ (Construction 1.1.1.4). For $n \geq 2$ and $0 \leq i < j \leq n$, it is not difficult to show that these face operators satisfy the identity

04ZA
$$d^{n-1}_i(d^n_j(\sigma)) = d^{n-1}_{j-1}(d^n_i(\sigma)) \tag{1.1}$$

(see Remark 1.1.1.7). In §1.1.1, we prove a partial converse: a collection of sets $\{S_n\}$ and face operators $\{d^n_i : S_n \to S_{n-1}\}$ which satisfy (1.1), we can uniquely reconstruct the data of a *semisimplicial set*: that is, a (contravariant) set-valued functor on the subcategory $\mathbf{\Delta}_{\text{inj}} \subset \mathbf{\Delta}$ whose morphisms are strictly increasing functions (see Proposition 1.1.1.9).

To fully recover the structure of a simplicial set $S_\bullet$, it is not enough to remember the face operators alone: one also needs to encode the data supplied by non-injective maps in the simplex category $\mathbf{\Delta}$. For every pair of integers $0 \leq i \leq n$, there is a unique nondecreasing surjection $[n+1] \twoheadrightarrow [n]$ which is constant on the subset $\{i, i+1\}$. This induces a function $s^n_i : S_n \to S_{n+1}$, which we refer to as the *ith degeneracy operator* (Construction 1.1.2.1). In §1.1.2, we show that a simplicial set $S_\bullet$ can be reconstructed from its face and degeneracy operators, which are required only to satisfy a handful of compatibility conditions (Proposition 1.1.2.14).

Let $S_\bullet$ be a simplicial set. We say that an n-simplex $\sigma \in S_n$ is *degenerate* if it belongs to the image of some degeneracy operator $s^{n-1}_i : S_{n-1} \to S_n$ (Definition 1.1.2.3). We say that $S_\bullet$ *has dimension* $\leq k$ if every n-simplex of $S_\bullet$ is degenerate for $n > k$ (Definition 1.1.3.1). Simplicial sets of low dimension are easy to describe:

- A simplicial set of dimension ≤ 0 is essentially just an ordinary set. More precisely, in §1.1.5 we show that a simplicial set $S_\bullet$ has dimension ≤ 0 if and only if it is isomorphic to a constant functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ (Proposition 1.1.5.14); in this case, we will say that $S_\bullet$ is *discrete* (Definition 1.1.5.10).

- A simplicial set of dimension ≤ 1 is essentially a directed graph. More precisely, in §1.1.6 we construct a functor from the category of simplicial sets to the category of directed graphs, and show that it is an equivalence when restricted to simplicial sets of dimension ≤ 1 (Proposition 1.1.6.9).

Let S be an arbitrary simplicial set. For every integer k, there is a largest simplicial subset of S which has dimension $\leq k$. We will denote this simplicial subset by $\mathrm{sk}_k(S)$ and refer to it as the *k-skeleton* of S (Construction 1.1.4.1). Allowing k to vary, we can realize S as the union of an increasing sequence

$$\emptyset = \mathrm{sk}_{-1}(S) \subseteq \mathrm{sk}_0(S) \subseteq \mathrm{sk}_1(S) \subseteq \mathrm{sk}_2(S) \subseteq \cdots$$

which we refer to as the *skeletal filtration.* In §1.1.4, we analyze the transition maps which appear in the skeletal filtration. Our main result is that each of the inclusions $\mathrm{sk}_{k-1}(S) \hookrightarrow \mathrm{sk}_k(S)$ is a pushout of coproducts of the inclusion map $\partial\Delta^k \hookrightarrow \Delta^k$ (Proposition 1.1.4.12). Here $\partial\Delta^k = \mathrm{sk}_{k-1}(\Delta^k)$ denotes the *boundary* of the standard simplex Δ^k (Construction 1.1.4.10). Stated more informally, the k-skeleton $\mathrm{sk}_k(S)$ can be obtained from the $(k-1)$-skeleton $\mathrm{sk}_{k-1}(S)$ by attaching cells of dimension k.

1.1.1 Face Operators

For some applications, it is useful to work with variant of Definition 1.1.0.4. 04ZB

Notation 1.1.1.1. Let $\mathbf{\Delta}_{\mathrm{inj}}$ denote the category whose objects are linearly ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$ (where n is a nonnegative integer) and whose morphisms are *strictly increasing* functions $\alpha : [m] \hookrightarrow [n]$. 04ZC

Definition 1.1.1.2. Let $\mathcal{C}$ be a category. A *semisimplicial object of* $\mathcal{C}$ is a functor $\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$. We typically use the notation $C_\bullet$ to indicate a semisimplicial object of $\mathcal{C}$, whose value on an object $[n] \in \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$ we denote by C_n. A *semisimplicial set* is a semisimplicial object of the category of sets. 04ZD

Remark 1.1.1.3. The category $\mathbf{\Delta}_{\mathrm{inj}}$ of Notation 1.1.1.1 can be regarded as a (non-full) subcategory of the simplex category $\mathbf{\Delta}$ of Definition 1.1.0.2. Consequently, any simplicial object $C_\bullet$ of a category $\mathcal{C}$ has an underlying semisimplicial object, given by the composition 00BJ

$$\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{C_\bullet} \mathcal{C}.$$

We will often abuse notation by identifying a simplicial object of $\mathcal{C}$ with its underlying semisimplicial object.

The goal of this section is to make Definition 1.1.1.2 more concrete.

`04ZE` **Construction 1.1.1.4** (Face Operators)**.** Let n be a positive integer. For $0 \leq i \leq n$, we let $\delta^i_n : [n-1] \to [n]$ denote the unique strictly increasing function whose image does not contain the element i, given concretely by the formula

$$\delta^i_n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

If $C_\bullet$ is a (semi)simplicial object of a category $\mathcal{C}$, then we can evaluate $C_\bullet$ on the morphism δ^i_n to obtain a morphism from C_n to C_{n-1}. We will denote this morphism by $d^n_i : C_n \to C_{n-1}$ and refer to it as the *ith face operator*.

`04ZF` **Example 1.1.1.5.** Let n be a positive integer and let $S_\bullet$ be a simplicial set. For $0 \leq i \leq n$, the face operator d^n_i of Construction 1.1.1.4 carries each n-simplex σ of $S_\bullet$ to an $(n-1)$-simplex $d^n_i(\sigma)$, which we will refer to as the *ith face of* σ.

`04ZG` **Example 1.1.1.6.** Let $S_\bullet$ be a simplicial set and let $e \in S_\bullet$ be an edge of $S_\bullet$. Then $s = d^1_1(e)$ is a vertex of $S_\bullet$ which we refer to as the *source* of e, and $t = d^1_0(e)$ is a vertex of $S_\bullet$ which we refer to as the *target* of e. We will sometimes write $e : x \to y$ to indicate that e is an edge of $S_\bullet$ having source vertex x and target vertex y.

`04FP` **Remark 1.1.1.7** (Relations Among Face Operators)**.** Let $n \geq 2$ be an integer. For every pair of integers $0 \leq i < j \leq n$, the diagram of linearly ordered sets

$$\begin{tikzcd}
{[n-2]} \arrow[r, "\delta^i_{n-1}"] \arrow[d, "\delta^{j-1}_{n-1}"] & {[n-1]} \arrow[d, "\delta^j_n"] \\
{[n-1]} \arrow[r, "\delta^i_n"] & {[n]}
\end{tikzcd}$$

is commutative: both the clockwise and counterclockwise compositions can be identified with the unique order-preserving bijection $[n-2] \simeq [n] \setminus \{i < j\}$. It follows that, if $C_\bullet$ is a semisimplicial object of a category $\mathcal{C}$, then the face operators of $C_\bullet$ satisfy the following condition:

$(*)$ For $0 \leq i < j \leq n$, we have $d^{n-1}_i \circ d^n_j = d^{n-1}_{j-1} \circ d^n_i$ (as morphisms from C_n to C_{n-2}).

Example 1.1.1.8. Let $S_\bullet$ be a simplicial set and let σ be a 2-simplex of $S_\bullet$. Then σ has three faces: the edges $f = d^2_2(\sigma)$, $g = d^2_0(\sigma)$, and $h = d^2_1(\sigma)$. In this case, Remark 1.1.1.7 asserts the following: 04ZH

- The edges f and h have the same source vertex $x \in S_\bullet$.
- The edges g and h have the same target vertex $z \in S_\bullet$.
- The target of f and the source of g are the same vertex $y \in S_\bullet$.

These relationships can be encoded visually in the diagram

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z. \end{tikzcd}$$

Remark 1.1.1.7 admits the following converse:

Proposition 1.1.1.9. *Let $\mathcal{C}$ be a category and let $\{C_n\}_{n \geq 0}$ be a sequence of objects of $\mathcal{C}$. Then a system of morphisms $\{d^n_i : C_n \to C_{n-1}\}_{0 \leq i \leq n, n > 0}$ arise as the face operators of a semisimplicial object $C_\bullet$ of $\mathcal{C}$ if and only if they satisfy condition $(*)$ of Remark 1.1.1.7. Moreover, if this condition is satisfied, then $C_\bullet$ is uniquely determined.* 04FQ

Proof. Let $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$ denote the category which is freely generated by a collection of objects $\{[n]\}_{n \geq 0}$ and a collection of morphisms $\{\widetilde{\delta}^i_n : [n-1] \to [n]\}_{n > 0, 0 \leq i \leq n}$. Let $\overline{\mathbf{\Delta}}_{\mathrm{inj}}$ denote the quotient of $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$ obtained by imposing the relation

$$\widetilde{\delta}^j_n \circ \widetilde{\delta}^i_{n-1} = \widetilde{\delta}^i_n \circ \widetilde{\delta}^{j-1}_{n-1} \tag{1.2}$$ 04FR

for every integer $n \geq 2$ and every pair $0 \leq i < j \leq n$. Using Remark 1.1.1.7, we see that there is a unique functor $F_{\mathrm{inj}} : \overline{\mathbf{\Delta}}_{\mathrm{inj}} \to \mathbf{\Delta}_{\mathrm{inj}}$ which carries each object $[n] \in \overline{\mathbf{\Delta}}_{\mathrm{inj}}$ to itself, and each generating morphism $\widetilde{\delta}^i_n$ to the monomorphism $\delta^i_n : [n-1] \hookrightarrow [n]$ of Construction 1.1.1.4. To prove Proposition 1.1.1.9, it will suffice to show that the functor F_{inj} is an isomorphism of categories.

Fix integers $0 \leq m \leq n$, and set $b = n - m - 1$. In the category $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$, every morphism $\beta : [m] \to [n]$ admits a unique factorization $\beta = \widetilde{\delta}^{i_0}_n \circ \widetilde{\delta}^{i_1}_{n-1} \circ \cdots \circ \widetilde{\delta}^{i_b}_{n-b}$, where the superscripts are nonnegative integers satisfying $0 \leq i_a \leq n - a$ for $0 \leq a \leq b$. Let us say that β is in *standard form* if, in addition, the integers i_a satisfy the inequalities $i_0 > i_1 > i_2 > \cdots > i_b$. Note that, by repeatedly applying the relation (1.2), we can convert any morphism of $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$ to a morphism which is in standard form. More precisely, every morphism $\overline{\beta} : [m] \to [n]$ in $\overline{\mathbf{\Delta}}_{\mathrm{inj}}$ can be lifted to a morphism $\beta : [m] \to [n]$ which is in standard form.

By construction, the functor F_{inj} is bijective on objects. To complete the proof, it will suffice to show that for every morphism $\alpha : [m] \hookrightarrow [n]$, there is a unique morphism $\overline{\beta} : [m] \to [n]$ in $\overline{\mathbf{\Delta}}_{\mathrm{inj}}$ satisfying $F_{\mathrm{inj}}(\overline{\beta}) = \alpha$. By virtue of the preceding discussion, it will suffice to show that α can be lifted uniquely to a morphism $\beta : [m] \to [n]$ in the category $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$ which is in standard form. We now observe that $\beta = \widetilde{\delta}^{i_0}_{n} \circ \widetilde{\delta}^{i_1}_{n-1} \circ \cdots \circ \widetilde{\delta}^{i_b}_{n-b}$ is characterized by the requirement that $\{i_b < i_{b-1} < \cdots < i_0\} \subseteq [n]$ is the complement of the image of α. □

1.1.2 Degeneracy Operators

04ZJ Let $S_\bullet$ be a simplicial set. By virtue of Proposition 1.1.1.9, the underlying semisimplicial set is determined by the sequence of sets $\{S_n\}_{n \geq 0}$ together with the face operators $\{d^n_i : S_n \to S_{n-1}\}_{0 \leq i \leq n}$. To recover $S_\bullet$ as a simplicial set, we need more information.

04ZK **Construction 1.1.2.1** (Degeneracy Operators)**.** For every pair of integers $0 \leq i \leq n$ we let $\sigma^i_n : [n+1] \twoheadrightarrow [n]$ denote the nondecreasing function given by the formula

$$\sigma^i_n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

If $C_\bullet$ is a simplicial object of a category $\mathcal{C}$, then we can evaluate $C_\bullet$ on the morphism σ^i_n to obtain a morphism from C_n to C_{n+1}. We will denote this map by $s^n_i : C_n \to C_{n+1}$ and refer to it as the *ith degeneracy operator*.

04ZL **Notation 1.1.2.2.** Let $S_\bullet$ be a simplicial set. Then the degeneracy operator $s^0_0 : S_0 \to S_1$ carries each vertex x to an edge of $S_\bullet$ which we will denote by id_x. Note that the vertex x is both the source and target of the edge id_x (see Exercise 1.1.2.7).

0012 **Definition 1.1.2.3.** Let $S_\bullet$ be a simplicial set. We say that an n-simplex σ of $S_\bullet$ is *degenerate* if it belongs to the image of the degeneracy operator $s^{n-1}_i : S_{n-1} \to S_n$ for some integer $0 \leq i < n$. We say that σ is *nondegenerate* if it is not degenerate. In particular, every 0-simplex of $S_\bullet$ is nondegenerate.

04ZM **Example 1.1.2.4** (Degenerate Edges)**.** Let $S_\bullet$ be a simplicial set and let e be an edge of $S_\bullet$. Then e is degenerate if and only if it has the form id_x, for some vertex $x \in S_\bullet$. If this condition is satisfied, then the vertex x is uniquely determined (since it is both the source and target of the edge e).

0013 **Remark 1.1.2.5.** Let $f : S_\bullet \to T_\bullet$ be a map of simplicial sets. If σ is a degenerate n-simplex of $S_\bullet$, then $f(\sigma)$ is a degenerate n-simplex of $T_\bullet$. The converse holds if f is a monomorphism of simplicial sets (for example, if $S_\bullet$ is a simplicial subset of $T_\bullet$).

Remark 1.1.2.6. Let $f : S_\bullet \to T_\bullet$ be a morphism of simplicial sets. If every nondegenerate simplex of $T_\bullet$ belongs to the image of f, then f is an epimorphism: that is, it induces a surjection $S_n \twoheadrightarrow T_n$ for each $n \geq 0$. 04ZN

Exercise 1.1.2.7 (Relations Between Face and Degeneracy Operators)**.** Let $C_\bullet$ be a simplicial object of a category $\mathcal{C}$. Show that the face and degeneracy operators of $\mathcal{C}$ satisfy the following relations: 04FV

($*'$) For $0 \leq i \leq n$ and $0 \leq j \leq n+1$, we have an equality

$$d_j^{n+1} \circ s_i^n = \begin{cases} s_{i-1}^{n-1} \circ d_j^n & \text{if } j < i \\ \mathrm{id}_{C_n} & \text{if } j = i \text{ or } j = i+1 \\ s_i^{n-1} \circ d_{j-1}^n & \text{if } j > i+1 \end{cases}$$

(as morphisms from C_n to C_n).

Example 1.1.2.8 (Degenerate 2-Simplices)**.** Let $S_\bullet$ be a simplicial set and let σ be a 2-simplex of $S_\bullet$. We say that σ is *left-degenerate* if it has the form $s_0^1(e)$, for some edge $e : x \to y$ of $S_\bullet$. In this case, the faces of σ are depicted in the diagram 04ZP

$$\begin{tikzcd} & x \arrow[dr, "e"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "e"] & & y. \end{tikzcd}$$

We will say that σ is *right-degenerate* if it has the form $s_1^1(e)$, for some edge $e : x \to y$ of $S_\bullet$; in this case, the faces of σ are depicted in the diagram

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "e"] \arrow[rr, "e"] & & y. \end{tikzcd}$$

Note that σ is degenerate if and only if it is either left-degenerate or right-degenerate.

Exercise 1.1.2.9. Let $S_\bullet$ be a simplicial set and let σ be a 2-simplex of $S_\bullet$. Show that σ is both left-degenerate and right-degenerate if and only if it is constant: that is, it factors as a composition $\Delta^2 \twoheadrightarrow \Delta^0 \hookrightarrow S_\bullet$ (for a more general statement, see Proposition 1.1.3.8). 04ZQ

Proposition 1.1.2.10. *Let $S_\bullet$ be a simplicial set and let $\tau \in S_n$ be an n-simplex of $S_\bullet$ for some $n > 0$, which we will identify with a map of simplicial sets $\tau : \Delta^n \to S_\bullet$. The following conditions are equivalent:* 0011

(1) *The simplex τ belongs to the image of the degeneracy operator $s_i^{n-1} : S_{n-1} \to S_n$ for some $0 \leq i < n$ (see Construction 1.1.2.1).*

(2) *The map* τ *factors as a composition* $\Delta^n \xrightarrow{f} \Delta^{n-1} \to S_\bullet$*, where* f *corresponds to a surjective map of linearly ordered sets* $[n] \twoheadrightarrow [n-1]$*.*

(3) *The map* τ *factors as a composition* $\Delta^n \xrightarrow{f} \Delta^m \to S_\bullet$*, where* $m < n$ *and* f *corresponds to a surjective map of linearly ordered sets* $[n] \twoheadrightarrow [m]$*.*

(4) *The map* τ *factors as a composition* $\Delta^n \to \Delta^m \to S_\bullet$*, where* $m < n$*.*

(5) *The map* τ *factors as a composition* $\Delta^n \xrightarrow{\tau'} \Delta^m \to S_\bullet$*, where* τ' *is not injective on vertices.*

Proof. The implications (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate. We will complete the proof by showing that (5) implies (1). Assume that τ factors as a composition $\Delta^n \xrightarrow{\tau'} \Delta^m \xrightarrow{\sigma'} S_\bullet$, where τ' is not injective on vertices. Then there exists some integer $0 \leq i < n$ satisfying $\tau'(i) = \tau'(i+1)$. It follows that τ' factors through the map $\sigma^i_{n-1} : \Delta^n \to \Delta^{n-1}$ of Construction 1.1.2.1, so that τ belongs to the image of the degeneracy operator s^{n-1}_i. □

04FS **Remark 1.1.2.11** (Relations Among Degeneracy Operators)**.** For every triple of integers $0 \leq i \leq j \leq n$, the diagram of linearly ordered sets

$$\begin{tikzcd}
{[n+2]} \arrow[r, "\sigma^i_{n+1}"] \arrow[d, "\sigma^{j+1}_{n+1}"] & {[n+1]} \arrow[d, "\sigma^j_n"] \\
{[n+1]} \arrow[r, "\sigma^i_n"] & {[n]}
\end{tikzcd}$$

is commutative. It follows that, if $C_\bullet$ is a simplicial object of a category $\mathcal{C}$, then the degeneracy operators of $C_\bullet$ satisfy the following condition:

($*''$) For $0 \leq i \leq j \leq n$, we have an equality $s^{n+1}_i \circ s^n_j = s^{n+1}_{j+1} \circ s^n_i$ (as morphisms from C_n to C_{n+2}).

We close this section by showing that a simplicial object $C_\bullet$ of a category $\mathcal{C}$ can be recovered from the sequence of objects $\{C_n\}_{n \geq 0}$, together with the face and degeneracy operators given by Constructions 1.1.1.4 and 1.1.2.1 (Proposition 1.1.2.14). We begin by proving a simpler result, which involves only the degeneracy operators.

04ZR **Notation 1.1.2.12.** Let $\mathbf{\Delta}_{\mathrm{surj}}$ denote the category whose objects are the linearly ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and whose morphisms are nondecreasing surjective functions $[m] \twoheadrightarrow [n]$.

Proposition 1.1.2.13. *Let $\mathcal{C}$ be a category and let $\{C_n\}_{n \geq 0}$ be a sequence of objects of $\mathcal{C}$. Then a system of morphisms $\{s^n_i : C_n \to C_{n+1}\}_{0 \leq i \leq n}$ can be obtained from a functor $C_\bullet : \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{surj}} \to \mathcal{C}$ if and only if they satisfy condition $(*'')$ of Remark 1.1.2.11. In this case, the functor $C_\bullet$ is uniquely determined.* 04FT

Proof. We proceed as in the proof of Proposition 1.1.1.9. Let $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$ denote the category which is freely generated by a collection of objects $\{[n]\}_{n \geq 0}$ and a collection of morphisms $\{\widetilde{\sigma}^i_n : [n+1] \to [n]\}_{0 \leq i \leq n}$. Let $\overline{\mathbf{\Delta}}_{\mathrm{surj}}$ denote the quotient of $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$ obtained by imposing the relation

$$\widetilde{\sigma}^j_n \circ \widetilde{\sigma}^i_{n+1} = \widetilde{\sigma}^i_n \circ \widetilde{\sigma}^{j+1}_{n+1} \tag{1.3}$$ 04FU

for every triple of integers $0 \leq i \leq j \leq n$. Using Remark 1.1.2.11, we see that there is a unique functor $F_{\mathrm{surj}} : \overline{\mathbf{\Delta}}_{\mathrm{surj}} \to \mathbf{\Delta}_{\mathrm{surj}}$ which carries each object $[n] \in \overline{\mathbf{\Delta}}_{\mathrm{surj}}$ to itself, and each generating morphism $\widetilde{\sigma}^i_n$ to the epimorphism $\sigma^i_n : [n+1] \twoheadrightarrow [n]$ of Construction 1.1.2.1. To prove Proposition 1.1.2.13, it will suffice to show that the functor F_{surj} is an isomorphism of categories.

Fix integers $0 \leq m \leq n$, and set $b = n - m + 1$. In the category $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$, every morphism $\beta : [n] \to [m]$ admits a unique factorization $\beta = \widetilde{\sigma}^{i_0}_m \circ \widetilde{\sigma}^{i_1}_{m+1} \circ \cdots \circ \widetilde{\sigma}^{i_b}_{m+b}$, where the superscripts are nonnegative integers satisfying $0 \leq i_a \leq m + a$ for $0 \leq a \leq b$. Let us say that β is in *standard form* if, in addition, the integers i_a satisfy the inequalities $i_0 < i_1 < i_2 < \cdots < i_b$. Note that, by repeatedly applying the relation (1.3), we can convert any morphism of $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$ to a morphism which is in standard form. More precisely, every morphism $\overline{\beta} : [n] \to [m]$ in $\overline{\mathbf{\Delta}}_{\mathrm{surj}}$ can be lifted to a morphism $\beta : [m] \to [n]$ which is in standard form.

By construction, the functor F_{surj} is bijective on objects. To complete the proof, it will suffice to show that for every morphism $\alpha : [n] \twoheadrightarrow [m]$ in $\mathbf{\Delta}_{\mathrm{surj}}$, there is a unique morphism $\overline{\beta} : [n] \to [m]$ in $\overline{\mathbf{\Delta}}_{\mathrm{surj}}$ satisfying $F_{\mathrm{surj}}(\overline{\beta}) = \alpha$. By virtue of the preceding discussion, it will suffice to show that α can be lifted uniquely to a morphism $\beta : [n] \to [m]$ in the category $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$ which is in standard form. We now observe that $\beta = \widetilde{\sigma}^{i_0}_m \circ \widetilde{\sigma}^{i_1}_{m+1} \circ \cdots \circ \widetilde{\sigma}^{i_b}_{m+b}$ is characterized by the requirement that $\{i_0 < i_1 < \cdots < i_b\}$ is the collection of integers $0 \leq j < n$ satisfying $\alpha(j) = \alpha(j+1)$. □

Proposition 1.1.2.14. *Let $\mathcal{C}$ be a category containing a sequence of objects $\{C_n\}_{n \geq 0}$. Then morphisms* 04FW

$$\{d^n_i : C_n \to C_{n-1}\}_{0 \leq i \leq n, n > 0} \qquad \{s^n_i : C_n \to C_{n+1}\}_{0 \leq i \leq n}$$

are the face and degeneracy operators for a simplicial object $C_\bullet$ of $\mathcal{C}$ if and only if they satisfy condition $(*)$ of Remark 1.1.1.7, condition $(*')$ of Exercise 1.1.2.7, and condition $(*'')$ of Remark 1.1.2.11. In this case, the simplicial object $C_\bullet$ is uniquely determined.

Proof. We proceed as in the proofs of Propositions 1.1.1.9 and 1.1.2.13. Let $\widetilde{\mathbf{\Delta}}$ denote the category which is freely generated by a collection of objects $\{[n]\}_{n \geq 0}$ together with morphisms $\{\widetilde{\delta}^i_n : [n-1] \to [n]\}_{n>0, 0 \leq i \leq n}$ and $\{\widetilde{\sigma}^i_n : [n+1] \to [n]\}_{0 \leq i \leq n}$. Let $\overline{\mathbf{\Delta}}$ denote the quotient of $\widetilde{\mathbf{\Delta}}$ obtained by imposing the relations (1.2) and (1.3), together with the following:

04FX
$$\widetilde{\sigma}^j_n \circ \widetilde{\delta}^i_{n+1} = \begin{cases} \widetilde{\delta}^i_n \circ \widetilde{\sigma}^{j-1}_{n-1} & \text{if } i < j \\ \mathrm{id}_{[n]} & \text{if } i = j \text{ or } i = j+1 \\ \widetilde{\delta}^{i-1}_n \circ \widetilde{\sigma}^j_{n-1} & \text{if } i > j+1. \end{cases} \tag{1.4}$$

for every triple of integers $0 \leq i, j \leq n$. There is a unique functor $F : \overline{\mathbf{\Delta}} \to \mathbf{\Delta}$ which carries each object $[n] \in \overline{\mathbf{\Delta}}$ to itself and satisfies $F(\widetilde{\delta}^i_n) = \delta^i_n$ and $F(\widetilde{\sigma}^i_n) = \sigma^i_n$. To prove Proposition 1.1.2.14, it will suffice to show that the functor F is an isomorphism of categories.

Let $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$ and $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$ be the categories appearing in the proofs of Proposition 1.1.1.9 and Proposition 1.1.2.13, respectively. Let us identify $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$ and $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$ with (non-full) subcategories of $\widetilde{\mathbf{\Delta}}$. We will say that a morphism $\beta : [m] \to [n]$ of $\widetilde{\mathbf{\Delta}}$ is *weakly standard* if it factors as a composition $[m] \xrightarrow{\beta_{\mathrm{surj}}} [k] \xrightarrow{\beta_{\mathrm{inj}}} [n]$, where β_{inj} belongs to $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$ and β_{surj} belongs to $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$. In this case, the morphisms β_{inj} and β_{surj} are uniquely determined. We will say that β is in *standard form* if it is weakly standard and, in addition, the morphisms β_{inj} and β_{surj} are in standard form (as in the proofs of Propositions 1.1.1.9 and 1.1.2.13). Note that, by repeatedly applying the relation (1.4), we can convert any morphism of $\widetilde{\mathbf{\Delta}}$ into a morphism β which is weakly standard. Using the relations (1.2) and (1.3), we can further arrange that β is in standard form. It follows that every morphism $\overline{\beta} : [m] \to [n]$ in $\overline{\mathbf{\Delta}}$ can be lifted to a morphism $\beta : [m] \to [n]$ of $\widetilde{\mathbf{\Delta}}$ which is in standard form.

By construction, the functor F is bijective on objects. To complete the proof, it will suffice to show that for every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, there is a unique morphism $\overline{\beta} : [m] \to [n]$ in $\overline{\mathbf{\Delta}}$ satisfying $F(\overline{\beta}) = \alpha$. Let $\widetilde{F}$ denote the composite functor $\widetilde{\mathbf{\Delta}} \twoheadrightarrow \overline{\mathbf{\Delta}} \xrightarrow{F} \mathbf{\Delta}$. By virtue of the preceding discussion, it will suffice to show that there is a unique morphism $\beta : [m] \to [n]$ in $\widetilde{\mathbf{\Delta}}$ which is in standard form and satisfies $\widetilde{F}(\beta) = \alpha$. In the simplex category $\mathbf{\Delta}$, the morphism α factors uniquely as a composition $[m] \xrightarrow{\alpha_{\mathrm{surj}}} [k] \xrightarrow{\alpha_{\mathrm{inj}}} [n]$, where α_{inj} is an injection and α_{surj} is a surjection. If $\beta : [m] \to [n]$ is a weakly standard morphism of $\widetilde{\mathbf{\Delta}}$, then the identity $\widetilde{F}(\beta) = \alpha$ holds if and only if $\widetilde{F}(\beta_{\mathrm{inj}}) = \alpha_{\mathrm{inj}}$ and $\widetilde{F}(\beta_{\mathrm{surj}}) = \alpha_{\mathrm{surj}}$. We are therefore reduced to proving that α_{inj} and α_{surj} can be lifted uniquely to morphisms of $\widetilde{\mathbf{\Delta}}_{\mathrm{inj}}$ and $\widetilde{\mathbf{\Delta}}_{\mathrm{surj}}$ which are in standard form, which was established in the proofs of Proposition 1.1.1.9 and Proposition 1.1.2.13. □

1.1.3 Dimensions of Simplicial Sets

04ZS We now introduce an important complexity measure for simplicial sets.

0019 **Definition 1.1.3.1.** Let S be a simplicial set and let k be an integer. We will say that S *has dimension* $\leq k$ if every n-simplex of S is degenerate for $n > k$. If $k \geq 0$, we say that S

has dimension k if it has dimension $\leq k$ but does not have dimension $\leq k-1$. We say that S is *finite-dimensional* if it has dimension $\leq k$ for some $k \gg 0$.

Example 1.1.3.2. For each $n \geq 0$, the standard simplex Δ^n has dimension n. 04ZT

Remark 1.1.3.3. Let S be the coproduct of a collection of simplicial sets $\{S(a)\}_{a \in A}$. Then S has dimension $\leq k$ if and only if each $S(a)$ has dimension $\leq k$. 04ZU

Remark 1.1.3.4. Let $f : S \twoheadrightarrow T$ be an epimorphism of simplicial sets. If S has dimension $\leq n$, then T has dimension $\leq n$. 04ZV

Remark 1.1.3.5. Let k be an integer. If a simplicial set S has dimension $\leq k$, then every simplicial subset of S has dimension $\leq k$ (see Remark 1.1.2.5). 04ZW

Proposition 1.1.3.6. *Let S^- and S^+ be simplicial sets having dimensions $\leq k_-$ and $\leq k_+$, respectively. Then the product $S^- \times S^+$ has dimension $\leq k_- + k_+$.* 012R

Proof. Let $\sigma = (\sigma_-, \sigma_+)$ be a nondegenerate n-simplex of the product $S^- \times S^+$. Using Proposition 1.1.3.8, we see that σ_- and σ_+ admit factorizations

$$\Delta^n \xrightarrow{\alpha_-} \Delta^{n_-} \xrightarrow{\tau_-} S^- \qquad \Delta^n \xrightarrow{\alpha_+} \Delta^{n_+} \xrightarrow{\tau_+} S^+,$$

where τ_- and τ_+ are nondegenerate, so that $n_- \leq k_-$ and $n_+ \leq k_+$. It follows that σ factors as a composition

$$\Delta^n \xrightarrow{(\alpha_-, \alpha_+)} \Delta^{n_-} \times \Delta^{n_+} \xrightarrow{\tau_- \times \tau_+} S^- \times S^+.$$

The nondegeneracy of σ guarantees that the map of partially ordered sets $[n] \xrightarrow{(\alpha_-, \alpha_+)} [n_-] \times [n_+]$ is a monomorphism, so that $n \leq n_- + n_+ \leq k_- + k_+$. □

Exercise 1.1.3.7. Show that the inequality of Proposition 1.1.3.6 is sharp. That is, if S^- and S^+ are nonempty simplicial sets of dimensions k_- and k_+, respectively, then the product $S^- \times S^+$ has dimension $k_- + k_+$. 012S

We next show that, if S is a simplicial set of dimension $\leq k$, then it can be recovered from its n-simplices for $n \leq k$ (Proposition 1.1.3.11). Our proof will make use of the following:

Proposition 1.1.3.8. *Let $\sigma : \Delta^n \to S$ be a morphism of simplicial sets. Then σ can be factored as a composition* 0014

$$\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} S,$$

where α corresponds to a surjective map of linearly ordered sets $[n] \twoheadrightarrow [m]$ and τ is a nondegenerate m-simplex of S. Moreover, this factorization is unique.

Proof. Let m be the smallest nonnegative integer for which σ can be factored as a composition $\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\tau} S$. It follows from the minimality of m that α must induce a surjection of linearly ordered sets $[n] \twoheadrightarrow [m]$ (otherwise, we could replace $[m]$ by the image of α) and that the m-simplex τ is nondegenerate. This proves the existence of the desired factorization.

We now establish uniqueness. Suppose we are given another factorization of σ as a composition $\Delta^n \xrightarrow{\alpha'} \Delta^{m'} \xrightarrow{\tau'} S$, and assume that α' induces a surjection $[n] \twoheadrightarrow [m']$. We first claim that, for any pair of integers $0 \leq i < j \leq n$ satisfying $\alpha'(i) = \alpha'(j)$, we also have $\alpha(i) = \alpha(j)$. Assume otherwise. Then α admits a section $\beta : \Delta^m \hookrightarrow \Delta^n$ whose images include i and j. We then have

$$\tau = \tau \circ \alpha \circ \beta = \sigma \circ \beta = \tau' \circ \alpha' \circ \beta.$$

Our assumption that $\alpha'(i) = \alpha'(j)$ guarantees that the map $(\alpha' \circ \beta) : \Delta^m \to \Delta^{m'}$ is not injective on vertices, contradicting our assumption that τ is nondegenerate.

It follows from the preceding argument that α factors uniquely as a composition $\Delta^n \xrightarrow{\alpha'} \Delta^{m'} \xrightarrow{\alpha''} \Delta^m$, for some morphism $\alpha'' : \Delta^{m'} \to \Delta^m$ (which is also surjective on vertices). Let β' be a section of α', and note that we have

$$\tau' = \tau' \circ \alpha' \circ \beta' = \sigma \circ \beta' = \tau \circ \alpha \circ \beta' = \tau \circ \alpha'' \circ \alpha' \circ \beta' = \tau \circ \alpha''.$$

Consequently, if the simplex τ' is nondegenerate, then α'' must also be injective on vertices. It follows that $m' = m$ and α'' is the identity map, so that $\alpha = \alpha'$ and $\tau = \tau'$. □

00X4 **Construction 1.1.3.9** (The Category of Simplices)**.** Let $S_\bullet$ be a simplicial set. We define a category $\mathbf{\Delta}_S$ as follows:

- The objects of $\mathbf{\Delta}_S$ are pairs $([n], \sigma)$, where $[n]$ is an object of $\mathbf{\Delta}$ and σ is an n-simplex of S.
- A morphism from $([n], \sigma)$ to $([n'], \sigma')$ in the category $\mathbf{\Delta}_S$ is a nondecreasing function $f : [n] \to [n']$ with the property that the induced map $S_{n'} \to S_n$ carries σ' to σ.

We will refer to $\mathbf{\Delta}_S$ as the *category of simplices of* S. If k is an integer, we let $\mathbf{\Delta}_{S,\leq k}$ denote the full subcategory of $\mathbf{\Delta}_S$ spanned by those objects $([n], \sigma)$ satisfying $n \leq k$.

00X5 **Remark 1.1.3.10.** Passage from a simplicial set S to the category of simplices $\mathbf{\Delta}_S$ is a special case of the *category of elements* construction (see Variant 5.2.6.2), which we will return to in §5.2.6.

04ZX **Proposition 1.1.3.11.** *Let k be an integer and let S be a simplicial set. The following conditions are equivalent:*

(1) *The simplicial set S has dimension $\leq k$.*

(2) *The simplicial set S can be realized as the colimit of a diagram $\varinjlim_{J \in \mathcal{J}} S(J)$, where each $S(J)$ has dimension $\leq k$.*

(3) *The simplicial set S can be realized as the colimit of a diagram $\varinjlim_{J \in \mathcal{J}} S(J)$, where each $S(J)$ is a standard simplex of dimension $\leq k$.*

(4) *The tautological map*
$$\varinjlim_{([n],\sigma) \in \mathbf{\Delta}_{S, \leq k}} \Delta^n \to S$$
is an isomorphism of simplicial sets.

Proof. The implication (4) $\Rightarrow$ (3) is trivial, the implication (3) $\Rightarrow$ (2) follows from Example 1.1.3.2, and the implication (2) $\Rightarrow$ (1) follows from Remarks 1.1.3.3 and 1.1.3.4. It will therefore suffice to show that (1) implies (4). Assume that S has dimension $\leq k$, and let T denote the colimit $\varinjlim_{([n],\sigma) \in \mathbf{\Delta}_{S, \leq k}} \Delta^n$; we wish to show that the tautological map $f : T \to S$ is an isomorphism of simplicial sets. Since S has dimension $\leq k$, it follows immediately from the construction that the image of f contains every nondegenerate simplex of S. Applying Remark 1.1.2.6, we deduce that f is an epimorphism of simplicial sets. We will complete the proof by showing that f is injective. Let τ and τ' be ℓ-simplices of T satisfying $f(\tau) = f(\tau')$; we wish to show that $\tau = \tau'$. Choose an object $([n], \sigma) \in \mathbf{\Delta}_{S, \leq k}$ and a lift of τ to an ℓ-simplex $\widetilde{\tau}$ of Δ^n, which we can identify with a nondecreasing function from $[\ell]$ to $[n]$. Note that $\widetilde{\tau}$ factors uniquely as a composition $[\ell] \xrightarrow{\alpha} [m] \xrightarrow{\beta} [n]$, where α is surjective and β is injective. Replacing n by m and σ by the associated ℓ-simplex of S, we can reduce to the case where $\widetilde{\tau} : [\ell] \twoheadrightarrow [n]$ is a surjection. Using Proposition 1.1.3.8, we can factor σ as a composition
$$\Delta^n \xrightarrow{\gamma} \Delta^p \xrightarrow{\rho} S,$$
where γ is surjective and ρ is a nondegerate p-simplex of $S_{\bullet}$. Replacing $([n], \sigma)$ by $([p], \rho)$ and $\widetilde{\tau}$ by the composition $\gamma \circ \widetilde{\tau}$, we can further assume that σ is a nondegenerate n-simplex of $S_{\bullet}$. Similarly, we may assume that τ' lifts to an m-simplex $\widetilde{\tau}'$ of $\Delta^{n'}$, for some object $([n'], \sigma')$ of $\mathbf{\Delta}_{S, \leq k}$ where σ' is nondegenerate and $\widetilde{\tau}' : [m] \twoheadrightarrow [n']$ is surjective. We then have an equality
$$\sigma \circ \widetilde{\tau} = f(\tau) = f(\tau') = \sigma' \circ \widetilde{\tau}'.$$
The uniqueness assertion of Proposition 1.1.3.8 then implies that $([n], \sigma) = ([n'], \sigma')$ and $\widetilde{\tau} = \widetilde{\tau}'$, so that τ and τ' are the same m-simplex of T. □

Remark 1.1.3.12. Proposition 1.1.3.11 can be reformulated using the language of Kan extensions (see Definition 7.3.0.1): it asserts that a simplicial set $S : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ has dimension $\leq k$ if and only if it is left Kan extended from the full subcategory of $\mathbf{\Delta}^{\mathrm{op}}$ spanned by the objects $\{[n]\}_{n \leq k}$. 04ZY

002D **Remark 1.1.3.13.** It follows from the proof of Proposition 1.1.3.11 that every simplicial set S can be recovered as the colimit $\varinjlim_{([n],\sigma)\in\mathbf{\Delta}_S} \Delta^n$. In fact, this is a general feature of presheaf categories: see Theorem 8.4.2.1 for an ∞-categorical counterpart.

04ZZ **Corollary 1.1.3.14.** *Let k be an integer and let $f_\bullet : S_\bullet \to T_\bullet$ be a morphism between simplicial sets having dimension $\leq k$. Suppose that, for every nonnegative integer $n \leq k$, the map of sets $f_n : S_n \to T_n$ is a bijection. Then f is an isomorphism of simplicial sets.*

1.1.4 The Skeletal Filtration

0010 Roughly speaking, one can think of the simplicial sets Δ^n of Example 1.1.0.9 as elementary building blocks out of which more complicated simplicial sets can be constructed. In this section, we make this idea more precise by introducing the *skeletal filtration* of a simplicial set. This filtration allows us to write every simplicial set S as the union of an increasing sequence of simplicial subsets

$$\mathrm{sk}_0(S) \subseteq \mathrm{sk}_1(S) \subseteq \mathrm{sk}_2(S) \subseteq \mathrm{sk}_3(S) \subseteq \cdots,$$

where each $\mathrm{sk}_n(S)$ is obtained from $\mathrm{sk}_{n-1}(S)$ by attaching copies of Δ^n (see Proposition 1.1.4.12 below for a precise statement).

0015 **Construction 1.1.4.1.** Let $S = S_\bullet$ be a simplicial set and let k be an integer. For every integer n, we let $\mathrm{sk}_k(S)_n$ denote the subset of S_n consisting of those n-simplices $\sigma : \Delta^n \to S$ which satisfy the following condition:

$(*)$ In the category of simplicial sets, σ admits a factorizaton

$$\Delta^n \to \Delta^m \xrightarrow{\tau} S$$

where $m \leq k$.

It follows immediately from the definitions that the collection of subsets $\{\mathrm{sk}_k(S)_n \subseteq S_n\}_{n\geq 0}$ is stable under the face and degeneracy operators for the simplicial set $S_\bullet$, and therefore defines a simplicial subset $\mathrm{sk}_k(S) \subseteq S$. We will refer to $\mathrm{sk}_k(S)$ as the *k-skeleton* of S.

0018 **Example 1.1.4.2.** For every simplicial set S, the k-skeleton $\mathrm{sk}_k(S)$ is empty for $k < 0$.

0500 **Remark 1.1.4.3.** Let m and n be integers with $m \leq n$. Then, for every simplicial set S, the m-skeleton $\mathrm{sk}_m(S)$ is contained in the n-skeleton $\mathrm{sk}_n(S)$.

0016 **Remark 1.1.4.4.** Let S be a simplicial set and let k be an integer. If $n \leq k$, then $\mathrm{sk}_k(S)$ contains every n-simplex of S. In particular, the union $\bigcup_k \mathrm{sk}_k(S)$ is equal to S.

0017 **Remark 1.1.4.5.** Let S be a simplicial set and let σ be a *nondegenerate* n-simplex of S. Then σ is contained in the k-skeleton $\mathrm{sk}_k(S)$ if and only if $n \leq k$ (see Proposition 1.1.2.10).

Proposition 1.1.4.6. *Let S be a simplicial set and let k be an integer. Then:* 001A

(a) *The simplicial set $\mathrm{sk}_k(S)$ has dimension $\leq k$.*

(b) *For every simplicial set T of dimension $\leq k$, composition with the inclusion map $\mathrm{sk}_k(S) \hookrightarrow S$ induces a bijection*

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(T, \mathrm{sk}_k(S)) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(T, S).$$

In other words, the image of any map $T \to S$ is contained in $\mathrm{sk}_k(S)$.

Proof. Assertion (a) follows from Remark 1.1.4.5. To prove (b), suppose that $f : T \to S$ is a map of simplicial sets, where T has dimension $\leq k$. We wish to show that f carries every n-simplex σ of T to an n-simplex of $\mathrm{sk}_k(S)$. Using Proposition 1.1.3.8, we can reduce to the case where σ is a nondegenerate n-simplex of T. In this case, our assumption that T has dimension $\leq k$ guarantees that $n \leq k$, so that $f(\sigma)$ belongs to $\mathrm{sk}_k(S)$ by virtue of Remark 1.1.4.4. □

Corollary 1.1.4.7. *Let S be a simplicial set. For every integer k, the k-skeleton $\mathrm{sk}_k(S)$ is the largest simplicial subset of S of dimension $\leq k$.* 0501

Corollary 1.1.4.8. *Let k be an integer, let S be a simplicial set, and let $\mathbf{\Delta}_S^{\leq k}$ denote the category of simplices of S having dimension $\leq k$ (see Construction 1.1.3.9). Then the tautological map* 0502

$$\varinjlim_{([n],\sigma) \in \mathbf{\Delta}_S^{\leq k}} \Delta^n \to S$$

is a monomorphism, whose image is the k-skeleton $\mathrm{sk}_k(S) \subseteq S$.

Proof. By virtue of Remark 1.1.4.4, replacing S by the k-skeleton $\mathrm{sk}_k(S)$ does not change the category $\mathbf{\Delta}_S^{\leq k}$. We may therefore assume without loss of generality that S has dimension $\leq k$, in which case the desired result follows from Proposition 1.1.3.11. □

Corollary 1.1.4.9. *For every integer k, the skeleton functor $\mathrm{sk}_k : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ preserves small colimits.* 0503

Proof. Let $S : \mathcal{J} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets; we wish to show that the comparison map

$$\theta : \varinjlim_{J \in \mathcal{J}} \mathrm{sk}_k(S(J)) \to \mathrm{sk}_k(\varinjlim_{J \in \mathcal{J}} S(J))$$

is an isomorphism of simplicial sets. Using Propositions 1.1.4.6 and 1.1.3.11, we see that the source and target of θ are simplicial sets of dimension $\leq k$. It will therefore suffice to show that θ induces a bijection on n-simplices for $n \leq k$ (Corollary 1.1.3.14), which follows immediately from Remark 1.1.4.4 (and Remark 1.1.0.8). □

`000R` **Construction 1.1.4.10** (The Boundary of Δ^n)**.** Let $n \geq 0$ be an integer and let Δ^n denote the standard n-simplex (Example 1.1.0.9). We let $\partial \Delta^n$ denote the $(n-1)$-skeleton of Δ^n. We will refer to $\partial \Delta^n$ as the *boundary of* Δ^n. More explicitly, the simplicial set $(\partial \Delta^n) : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ is defined by the formula

$$(\partial \Delta^n)([m]) = \{\alpha \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n]) : \alpha \text{ is not surjective}\}.$$

`000S` **Example 1.1.4.11.** The simplicial set $\partial \Delta^0$ is empty.

Let S be a simplicial set. For each $k \geq 0$, we let S_k^{nd} denote the collection of all *nondegenerate* k-simplices of S. Every element $\sigma \in S_k^{\mathrm{nd}}$ determines a map of simplicial sets $\Delta^k \to \mathrm{sk}_k(S)$. Since the boundary $\partial \Delta^k \subseteq \Delta^k$ has dimension $\leq k-1$, this map carries $\partial \Delta^k$ into the $(k-1)$-skeleton $\mathrm{sk}_{k-1}(S)$ (Proposition 1.1.4.6).

`001B` **Proposition 1.1.4.12.** *Let S be a simplicial set and let $k \geq 0$. Then the construction outlined above determines a pushout square*

$$\begin{tikzcd}
\coprod_{\sigma \in S_k^{\mathrm{nd}}} \partial \Delta^k \arrow[r] \arrow[d] & \coprod_{\sigma \in S_k^{\mathrm{nd}}} \Delta^k \arrow[d] \\
\mathrm{sk}_{k-1}(S) \arrow[r] & \mathrm{sk}_k(S)
\end{tikzcd}$$

in the category Set_{Δ} *of simplicial sets.*

Proof. Unwinding the definitions, we must prove the following:

$(*)$ Let τ be an n-simplex of $\mathrm{sk}_k(S)$ which is not contained in $\mathrm{sk}_{k-1}(S)$. Then τ factors uniquely as a composition

$$\Delta^n \xrightarrow{\alpha} \Delta^k \xrightarrow{\sigma} S,$$

where σ is a nondegenerate simplex of S and α does not factor through the boundary $\partial \Delta^k$ (in other words, α is surjective on vertices).

Proposition 1.1.3.8 implies that *any* n-simplex of S admits a unique factorization $\Delta^n \xrightarrow{\alpha} \Delta^m \xrightarrow{\sigma} S$, where α is surjective on vertices and σ is nondegenerate. Our assumption that τ belongs to the $\mathrm{sk}_k(S)$ guarantees that $m \leq k$, and our assumption that τ does not belong to $\mathrm{sk}_{k-1}(S)$ guarantees that $m \geq k$. $\square$

We close this section by analyzing the simplicial sets $\partial \Delta^n$ of Construction 1.1.4.10 in a bit more detail. Note that, for every pair of integers $0 \leq k \leq n$, the morphism $\delta_n^k : \Delta^{n-1} \to \Delta^n$ of Construction 1.1.1.4 factors through the boundary $\partial \Delta^n$.

Proposition 1.1.4.13. *Let n be a positive integer. For every simplicial set $S_\bullet$, the map* 0504

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^n, S_\bullet) \to (S_{n-1})^{n+1} \qquad f \mapsto \{f \circ \delta^k_n\}_{0 \leq k \leq n}$$

is an injection, whose image consists of those tuples of $(\sigma_0, \sigma_1, \cdots, \sigma_n)$ of $(n-1)$-simplices of S which satisfy the identity $d^{n-1}_i(\sigma_j) = d^{n-1}_{j-1}(\sigma_i)$ for $0 \leq i < j \leq n$.

Example 1.1.4.14. When $n = 1$, Proposition 1.1.4.13 asserts that we can identify maps $\partial\Delta^1 \to S$ with ordered pairs (s, t) of vertices of S. Equivalently, the boundary $\partial\Delta^1$ can be identified with the coproduct of $\{0\}$ and $\{1\}$ (which we regard as simplicial subsets of Δ^1 as in Example 1.1.0.15). 0505

Example 1.1.4.15. When $n = 2$, Proposition 1.1.4.13 asserts that morphisms of simplicial sets $\partial\Delta^2 \to S$ can be identified with ordered triples (g, h, f) of edges of S having the property that f and h have the same source vertex $x \in S$, g and h have the same target vertex $z \in S$, and the target y of f coincides with the source of g; these relationships are summarized visually in the diagram 0506

$$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"] & & z.
\end{tikzcd}$$

Proof of Proposition 1.1.4.13. Let $w : \coprod_{0 \leq k \leq n} \Delta^{n-1} \to \partial\Delta^n$ be the map given on the kth summand by δ^k_n. To prove the first assertion of Proposition 1.1.4.13, we must show that w is an epimorphism of simplicial sets: that is, it is surjective on m-simplices for each $m \geq 0$. In fact, we can be a bit more precise. Let α be an m-simplex of Δ^n, which we identify with a nondecreasing function from $[m]$ to $[n]$. Then α belongs to the boundary $\partial\Delta^n$ if and only if it is not surjective: that is, if and only if there exists some integer $0 \leq i \leq n$ such that α factors through $[n] \setminus \{i\}$. In this case, there is a unique m-simplex β_i which belongs to the ith summand of $\coprod_{0 \leq k \leq n} \Delta^{n-1}$ and satisfies $w(\beta_i) = \alpha$.

For every integer $0 \leq k \leq n$, let $u_k : \coprod_{0 \leq i < k} \Delta^{n-2} \to \Delta^{n-1}$ be the map given on the ith summand by δ^i_{n-1}, and let $v_k : \coprod_{k < j \leq n} \Delta^{n-2} \to \Delta^{n-1}$ by the map given on the jth summand by δ^{j-1}_{n-1}. Passing to the coproduct over k and reindexing, we obtain a pair of maps

$$(u, v) : \coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{0 \leq k \leq n} \Delta^{n-1}.$$

Let $\mathrm{Coeq}(u, v)_\bullet$ denote the coequalizer of u and v in the category of simplicial sets. The morphism w satisfies $w \circ u = w \circ v$ (see Remark 1.1.1.7), and therefore factors uniquely through a map $\overline{w} : \mathrm{Coeq}(u, v)_\bullet \to \partial\Delta^n$. Proposition 1.1.4.13 asserts that $\overline{w}$ is an isomorphism of simplicial sets: that is, for every integer $m \geq 0$, it induces a bijection from $\mathrm{Coeq}(u, v)_m$ to the set of m-simplices of $\partial\Delta^n$. The surjectivity of this map was established above. To prove injectivity, it will suffice to observe that if $\alpha : [m] \to [n]$ is as above and we are given two elements $i, j \in [n]$ which do not belong to the image of α, then β_i and β_j have the same image in $\mathrm{Coeq}(u,v)_\bullet$. If $i = j$, this is automatic; we may therefore assume without loss of generality that $i < j$. In this case, the desired result follows from the observation that we can write $\beta_j = u(\gamma)$ and $\beta_i = v(\gamma)$, where γ is the m-simplex of the (i,j)th summand of $\coprod_{0 \leq i < j \leq n} \Delta^{n-2}$ corresponding to the nondecreasing function $[m] \xrightarrow{\alpha} [n] \setminus \{i < j\} \simeq [n-2]$. □

1.1.5 Discrete Simplicial Sets

00FQ Simplicial sets of dimension ≤ 0 admit a simple classification:

00FR **Proposition 1.1.5.1.** *The evaluation functor*

$$\mathrm{ev}_0 : \mathrm{Set}_\Delta \to \mathrm{Set} \qquad X_\bullet \mapsto X_0$$

restricts to an equivalence of categories

$$\{\text{Simplicial sets of dimension} \leq 0\} \simeq \mathrm{Set}\,.$$

We will give a proof of Proposition 1.1.5.1 at the end of this section. First, we make some general remarks which apply to simplicial objects of *any* category $\mathcal{C}$.

00FS **Construction 1.1.5.2.** Let $\mathcal{C}$ be a category. For each object $C \in \mathcal{C}$, we let $\underline{C}$ denote the constant functor $\mathbf{\Delta}^{\mathrm{op}} \to \{C\} \hookrightarrow \mathcal{C}$ taking the value C. We regard $\underline{C}$ as a simplicial object of $\mathcal{C}$, which we will refer to as the *constant simplicial object with value C*.

00FT **Remark 1.1.5.3.** Let C be an object of the category $\mathcal{C}$. The constant simplicial object $\underline{C}$ can be described concretely as follows:

- For each $n \geq 0$, we have $\underline{C}_n = C$.
- The face and degeneracy operators

  $$d^n_i : \underline{C}_n \to \underline{C}_{n-1} \qquad s^n_i : \underline{C}_n \to \underline{C}_{n+1}$$

  are the identity maps from C to itself.

00FU **Example 1.1.5.4.** Let $S = \{s\}$ be a set containing a single element. Then $\underline{S}$ is a final object of the category of simplicial sets: that is, it is isomorphic to the standard simplex Δ^0.

The constant simplicial object $\underline{C}$ of Construction 1.1.5.2 can be characterized by a universal mapping property:

00FV **Proposition 1.1.5.5.** *Let $\mathcal{C}$ be a category and let C be an object of $\mathcal{C}$. For any simplicial object $X_\bullet$ of $\mathcal{C}$, evaluation at the object $[0] \in \mathbf{\Delta}^{\mathrm{op}}$ induces a bijection*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathcal{C})}(\underline{C}, X_\bullet) \to \mathrm{Hom}_{\mathcal{C}}(C, X_0).$$

Proof. Let $f : C \to X_0$ be a morphism in $\mathcal{C}$; we wish to show that f can be promoted uniquely to a map of simplicial objects $f_\bullet : \underline{C} \to X_\bullet$. The uniqueness of $f_\bullet$ is clear. For existence, we define $f_\bullet$ to be the natural transformation whose value on an object $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ is given by the composite map

$$\underline{C}_n = C \xrightarrow{f} X_0 \xrightarrow{X_{\alpha(n)}} X_n,$$

where $\alpha(n)$ denotes the unique morphism in $\mathbf{\Delta}$ from $[n]$ to $[0]$. To prove the naturality of $f_\bullet$, we observe that for any nondecreasing map $\beta : [m] \to [n]$ we have a commutative diagram

$$\begin{tikzcd}
\underline{C}_n \arrow[r, equal] \arrow[d, "\underline{C}_\beta"] & C \arrow[r, "f"] \arrow[d, equal] & X_0 \arrow[r, "X_{\alpha(n)}"] \arrow[d, equal] & X_n \arrow[d, "X_\beta"] \\
\underline{C}_m \arrow[r, equal] & C \arrow[r, "f"] & X_0 \arrow[r, "X_{\alpha(m)}"] & X_m,
\end{tikzcd}$$

where the commutativity of the square on the right follows from the observation that $\alpha(m)$ is equal to the composition $[m] \xrightarrow{\beta} [n] \xrightarrow{\alpha(n)} [0]$. □

Example 1.1.5.6. Let $X_\bullet$ be a simplicial set and let $S = X_0$ be the set of vertices of $X_\bullet$. It follows from Proposition 1.1.5.5 that there is a unique morphism of simplicial sets $f : \underline{S} \to X_\bullet$ which is the identity map on 0-simplices. Using Proposition 1.1.4.12, we see that this map is an isomorphism from $\underline{S}$ to the 0-skeleton $\mathrm{sk}_0(X_\bullet)$. In particular, f is a monomorphism, which is an isomorphism if and only if $X_\bullet$ has dimension ≤ 0. 0507

Remark 1.1.5.7. Let $\mathcal{C}$ be a category. Proposition 1.1.5.5 can be rephrased as follows: 00FW

- For any simplicial object $X_\bullet$ of $\mathcal{C}$, the limit $\varprojlim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} X_n$ exists in the category $\mathcal{C}$.
- The canonical map $\varprojlim_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} X_n \to X_0$ is an isomorphism.

These assertions follow formally from the observation that $[0]$ is a *final object* of the category $\mathbf{\Delta}$ (and therefore an *initial object* of the category $\mathbf{\Delta}^{\mathrm{op}}$).

Corollary 1.1.5.8. *Let $\mathcal{C}$ be a category. Then the evaluation functor* 00FX

$$\mathrm{ev}_0 : \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C} \qquad X_\bullet \mapsto X_0$$

admits a left adjoint, given on objects by the formation of constant simplicial objects $C \mapsto \underline{C}$ described in Construction 1.1.5.2.

Corollary 1.1.5.9. *Let $\mathcal{C}$ be a category. Then the construction $C \mapsto \underline{C}$ determines a fully faithful embedding from $\mathcal{C}$ to the category* $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$ *of simplicial objects of $\mathcal{C}$.* 00FY

Proof. Let C and D be objects of $\mathcal{C}$; we wish to show that the canonical map

$$\theta : \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})}(\underline{C}, \underline{D})$$

is a bijection. This is clear, since θ is right inverse to the evaluation map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})}(\underline{C}, \underline{D}) \to \mathrm{Hom}_{\mathcal{C}}(C, D)$$

which is bijective by virtue of Proposition 1.1.5.5. □

We now specialize to the case where $\mathcal{C} = \mathrm{Set}$ is the category of sets.

`00FZ` **Definition 1.1.5.10.** Let $X_\bullet$ be a simplicial set. We will say that $X_\bullet$ is *discrete* if there exists a set S and an isomorphism of simplicial sets $X_\bullet \simeq \underline{S}$; here $\underline{S}$ denotes the constant simplicial set of Construction 1.1.5.2.

Specializing Corollary 1.1.5.9 to the case $\mathcal{C} = \mathrm{Set}$, we obtain the following:

`00G0` **Corollary 1.1.5.11.** *The construction $S \mapsto \underline{S}$ determines a fully faithful embedding* $\mathrm{Set} \hookrightarrow \mathrm{Set}_{\Delta}$*. The essential image of this embedding is the full subcategory of* Set_{Δ} *spanned by the discrete simplicial sets.*

`00G1` **Notation 1.1.5.12.** Let S be a set. We will often abuse notation by identifying S with the constant simplicial set $\underline{S}$ of Construction 1.1.5.2. (by virtue of Corollary 1.1.5.11, this is mostly harmless).

`00G2` **Remark 1.1.5.13.** The fully faithful embedding

$$\mathrm{Set} \hookrightarrow \mathrm{Set}_{\Delta} \qquad S \mapsto \underline{S}$$

preserves (small) limits and colimits (since limits and colimits of simplicial sets are computed levelwise; see Remark 1.1.0.8). It follows that the collection of discrete simplicial sets is closed under the formation of (small) limits and colimits in Set_{Δ}.

`00G3` **Proposition 1.1.5.14.** *Let $X_\bullet$ be a simplicial set. The following conditions are equivalent:*

(1) *The simplicial set $X_\bullet$ is discrete (Definition 1.1.5.10). That is, $X_\bullet$ is isomorphic to a constant simplicial set $\underline{S}$.*

(2) *For every morphism $\alpha : [m] \to [n]$ in the category $\mathbf{\Delta}$, the induced map $X_n \to X_m$ is a bijection.*

(3) *For every positive integer n, the 0th face operator $d^n_0 : X_n \to X_{n-1}$ is a bijection.*

(4) *The simplicial set $X_\bullet$ has dimension ≤ 0, in the sense of Definition 1.1.3.1. That is, $X_\bullet$ does not contain any nondegenerate n-simplices for $n > 0$.*

Proof. The implication (1) ⇒ (2) follows from Remark 1.1.5.3, and the implication (2) ⇒ (3) is immediate. To prove that (3) ⇒ (4), we observe that if the face operator $d_0^n : X_n \to X_{n-1}$ is bijective, then the degeneracy operator $s_0^{n-1} : X_{n-1} \to X_n$ is also bijective (since it is a right inverse of d_0^n). In particular, s_0^{n-1} is surjective, so every n-simplex of $X_\bullet$ is degenerate. The implication (4) ⇒ (1) follows from Example 1.1.5.6. □

Proof of Proposition 1.1.5.1. By virtue of Proposition 1.1.5.14, it will suffice to show that the construction $X_\bullet \mapsto X_0$ induces an equivalence of categories

$$\{\text{Discrete simplicial sets}\} \to \mathrm{Set}\,.$$

This follows immediately from Corollary 1.1.5.11. □

1.1.6 Directed Graphs as Simplicial Sets

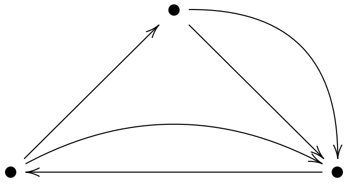
We now generalize Proposition 1.1.5.14 to obtain a concrete description of simplicial sets of dimension ≤ 1 (Proposition 1.1.6.9). `001D`

Definition 1.1.6.1. A *directed graph* G consists of the following data: `001E`

- A set $\mathrm{Vert}(G)$, whose elements we refer to as *vertices of* G.
- A set $\mathrm{Edge}(G)$, whose elements we refer to as *edges of* G.
- A pair of functions $s, t : \mathrm{Edge}(G) \to \mathrm{Vert}(G)$ which assign to each edge $e \in \mathrm{Edge}(G)$ a pair of vertices $s(e)$, $t(e) \in \mathrm{Vert}(G)$ that we refer to as the *source* and *target* of e, respectively.

Warning 1.1.6.2. The terminology of Definition 1.1.6.1 is not standard. Note that a directed graph G can have distinct edges $e \neq e'$ having the same source $s(e) = s(e')$ and target $t(e) = t(e')$ (for this reason, directed graphs in the sense of Definition 1.1.6.1 are sometimes called *multigraphs*). Definition 1.1.6.1 also allows graphs which contain loops: that is, edges e satisfying $s(e) = t(e)$. `001F`

Remark 1.1.6.3. It will sometimes be convenient to represent a directed graph G by a diagram, having a node for each vertex v of G and an arrow for each edge e of G, directed from the source of e to the target of e. For example, the diagram `001G`



represents a directed graph with three vertices and five edges.

001H **Example 1.1.6.4.** To every simplicial set X, we can associate a directed graph $\mathrm{Gr}(X)$ as follows:

- The vertex set $\mathrm{Vert}(\mathrm{Gr}(X))$ is the set of 0-simplices of the simplicial set X.
- The edge set $\mathrm{Edge}(\mathrm{Gr}(X))$ is the set of *nondegenerate* 1-simplices of the simplicial set X.
- For every edge $e \in \mathrm{Edge}(\mathrm{Gr}(X))$, the source $s(e)$ is the vertex $d^1_1(e)$, and the target $t(e)$ is the vertex $d^1_0(e)$ (here d^1_0 and d^1_1 denote the face operators of Construction 1.1.1.4).

It will be convenient to construe Example 1.1.6.4 as providing a functor from the category of simplicial sets to the category of directed graphs. First, we need an appropriate definition for the latter category.

001J **Definition 1.1.6.5.** Let G and G' be directed graphs (in the sense of Definition 1.1.6.1). A *morphism* from G to G' is a function $f : \mathrm{Vert}(G) \amalg \mathrm{Edge}(G) \to \mathrm{Vert}(G') \amalg \mathrm{Edge}(G')$ which satisfies the following conditions:

(a) For each vertex $v \in \mathrm{Vert}(G)$, the image $f(v)$ belongs to $\mathrm{Vert}(G')$.

(b) Let $e \in \mathrm{Edge}(G)$ be an edge of G with source $v = s(e)$ and target $w = t(e)$. Then exactly one of the following conditions holds:

  - The image $f(e)$ is an edge of G' having source $s(f(e)) = f(v)$ and target $t(f(e)) = f(w)$.
  - The image $f(e)$ is a vertex of G' satisfying $f(v) = f(e) = f(w)$.

We let Graph denote the category whose objects are directed graphs and whose morphisms are morphisms of directed graphs (with composition defined in the evident way).

001K **Warning 1.1.6.6.** Note that part (b) of Definition 1.1.6.5 allows the possibility that a morphism of directed graphs $G \to G'$ can "collapse" edges of G to vertices of G'. Many other notions of morphism between (directed) graphs appear in the literature; we single out Definition 1.1.6.5 because of its close connection with the theory of simplicial sets (see Proposition 1.1.6.7 below).

Let $X = X_\bullet$ be a simplicial set and let $\mathrm{Gr}(X)$ be the directed graph of Example 1.1.6.4. Then the disjoint union $\mathrm{Vert}(\mathrm{Gr}(X)) \amalg \mathrm{Edge}(\mathrm{Gr}(X))$ can be identified with the set X_1 of *all* 1-simplices of X (by identifying each vertex $x \in X$ with the degenerate edge id_x).

Proposition 1.1.6.7. *Let $X = X_\bullet$ and $Y = Y_\bullet$ be simplicial sets, and let $f : X \to Y$ be a morphism of simplicial sets. Then the induced map* 001L

$$\mathrm{Vert}(\mathrm{Gr}(X)) \amalg \mathrm{Edge}(\mathrm{Gr}(X)) \simeq X_1 \xrightarrow{f} Y_1 \simeq \mathrm{Vert}(\mathrm{Gr}(Y)) \amalg \mathrm{Edge}(\mathrm{Gr}(Y))$$

is a morphism of directed graphs from $\mathrm{Gr}(X)$ *to* $\mathrm{Gr}(Y)$*, in the sense of Definition 1.1.6.5.*

Proof. Since f commutes with the degeneracy operator s^0_0, it carries degenerate 1-simplices of X to degenerate 1-simplices of Y, and therefore satisfies requirement (a) of Definition 1.1.6.5. Requirement (b) follows from the fact that f commutes with the face operators d^1_0 and d^1_1. □

It follows from Proposition 1.1.6.7 that we can regard the construction $X \mapsto \mathrm{Gr}(X)$ as a functor from the category Set_Δ of simplicial sets to the category Graph of directed graphs.

Proposition 1.1.6.8. *Let X and Y be simplicial sets. If X has dimension ≤ 1, then the canonical map* 001M

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(X, Y) \to \mathrm{Hom}_{\mathrm{Graph}}(\mathrm{Gr}(X), \mathrm{Gr}(Y))$$

is bijective.

Proof. Set $G = \mathrm{Gr}(X)$. If X has dimension ≤ 1, then Proposition 1.1.4.12 supplies a pushout diagram

$$\begin{tikzcd}
\coprod_{e \in \mathrm{Edge}(G)} \partial \Delta^1 \arrow[r] \arrow[d] & \coprod_{e \in \mathrm{Edge}(G)} \Delta^1 \arrow[d] \\
\underline{\mathrm{Vert}(G)} \arrow[r] & X.
\end{tikzcd}$$

It follows that, for any simplicial set $Y = Y_\bullet$, we can identify $\mathrm{Hom}_{\mathrm{Set}_\Delta}(X, Y)$ with the fiber product

$$\Big(\prod_{e \in \mathrm{Edge}(G)} Y_1\Big) \times_{\prod_{e \in \mathrm{Edge}(G)} (Y_0 \times Y_0)} \Big(\prod_{v \in \mathrm{Vert}(G))} Y_0\Big),$$

which parametrizes morphisms of directed graphs from $\mathrm{Gr}(X)$ to $\mathrm{Gr}(Y)$. □

It follows from Proposition 1.1.6.8 that the theory of simplicial sets of dimension ≤ 1 is essentially equivalent to the theory of directed graphs.

Proposition 1.1.6.9. *Let* Set_Δ *denote the category of simplicial sets and let* $\mathrm{Set}_\Delta^{\leq 1} \subseteq \mathrm{Set}_\Delta$ *denote the full subcategory spanned by the simplicial sets of dimension ≤ 1. Then the construction $X \mapsto \mathrm{Gr}(X)$ induces an equivalence of categories* $\mathrm{Set}_\Delta^{\leq 1} \to \mathrm{Graph}$. 001N

Proof. It follows from Proposition 1.1.6.8 that the functor $X \mapsto \mathrm{Gr}(X)$ is fully faithful when restricted to simplicial sets of dimension ≤ 1. It will therefore suffice to show that it is essentially surjective. Let G be any directed graph, and form a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{e \in \mathrm{Edge}(G)} \partial \Delta^1 \arrow[r] \arrow[d, "(s,t)"] & \coprod_{e \in \mathrm{Edge}(G)} \Delta^1 \arrow[d] \\
\coprod_{v \in \mathrm{Vert}(G)} \Delta^0 \arrow[r] & X.
\end{tikzcd}$$

Then X is a simplicial set of dimension ≤ 1 (Proposition 1.1.3.11), and the directed graph $\mathrm{Gr}(X)$ is isomorphic to G. □

001P **Remark 1.1.6.10.** The proof of Proposition 1.1.6.9 gives an explicit description of the inverse equivalence $\mathrm{Graph} \simeq \mathcal{C} \hookrightarrow \mathrm{Set}_{\Delta}$: it carries a directed graph G to the 1-dimensional simplicial set $G_\bullet$ given by the colimit of the diagram

$$(\coprod_{v \in \mathrm{Vert}(G)} \Delta^0) \leftarrow (\coprod_{e \in \mathrm{Edge}(G)} \partial \Delta^1) \rightarrow (\coprod_{e \in \mathrm{Edge}(G)} \Delta^1).$$

00G4 **Example 1.1.6.11.** Let G be a directed graph and let $G_\bullet$ denote the associated simplicial set of dimension ≤ 1 (Remark 1.1.6.10). Then $G_\bullet$ has dimension ≤ 0 if and only if the edge set $\mathrm{Edge}(G)$ is empty. In this case, $G_\bullet$ can be identified with the constant simplicial set $\underline{\mathrm{Vert}(G)}$.

1.2 From Topological Spaces to Simplicial Sets

0508 Simplicial sets are connected to algebraic topology by two closely related constructions:

- To every topological space X, one can associate a simplicial set $\mathrm{Sing}_\bullet(X)$, whose n-simplices are given by continuous functions from the topological n-simplex

$$|\Delta^n| = \{(t_0, t_1, \ldots, t_n) \in [0,1]^{n+1} : t_0 + t_1 + \cdots + t_n = 1\}$$

  to X. We will refer to $\mathrm{Sing}_\bullet(X)$ as the *singular simplicial set of* X (Construction 1.2.2.2). These simplicial sets tend to be quite large: in any nontrivial example, the sets $\mathrm{Sing}_n(X)$ will be uncountable for every nonnegative integer n.

- Any simplicial set $S_\bullet$ can be regarded as a "blueprint" for constructing a topological space $|S_\bullet|$ called the *geometric realization* of $S_\bullet$, which can be obtained as a quotient of the disjoint union $\coprod_{n \geq 0} S_n \times |\Delta^n|$ by an equivalence relation determined by the face and degeneracy operators of $S_\bullet$. Many topological spaces of interest (for example, any space which admits a finite triangulation) can be realized as a geometric realization of a simplicial set $S_\bullet$ having only finitely many nondegenerate simplices.

These constructions determine adjoint functors

$$\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r, shift left, "| \; |"] & \mathrm{Top} \arrow[l, shift left, "\mathrm{Sing}_\bullet"] \end{tikzcd}$$

relating the category Set_Δ of simplicial sets to the category Top of topological spaces. We review the constructions of these functors in §1.2.2 and §1.2.3, viewing them as instances of a general paradigm (Variant 1.2.2.8 and Proposition 1.2.3.15) which will appear repeatedly in Chapter 2.

Under mild assumptions, the entire homotopy type of X can be recovered from the simplicial set $\mathrm{Sing}_\bullet(X)$. More precisely, there is a canonical map $|\mathrm{Sing}_\bullet(X)| \rightarrow X$ (given by the counit of the preceding adjunction), and Giever showed that it is always a weak homotopy equivalence (hence a homotopy equivalence when X has the homotopy type of a CW complex; see Proposition 3.6.3.8). Consequently, for the purpose of studying homotopy theory, nothing is lost by replacing X by $\mathrm{Sing}_\bullet(X)$ and working in the setting of simplicial sets, rather than topological spaces. In fact, it is possible to develop the theory of algebraic topology in entirely combinatorial terms, using simplicial sets as surrogates for topological spaces. In §1.2.1, we consider a simple example of this idea. We say that a simplicial set is *connected* if it is nonempty and cannot be decomposed as a disjoint union of nonempty simplicial subsets (Definition 1.2.1.6). Every simplicial set S decomposes uniquely as disjoint union of connected simplicial subsets (Proposition 1.2.1.13), indexed by a set which we denote by $\pi_0(S)$. In the special case where $S = \mathrm{Sing}_\bullet(X)$ is the singular simplicial set of a topological space X, this construction recovers the set $\pi_0(X)$ of path components of X (Remark 1.2.2.5).

The discussion of connectedness in §1.2.1 illustrates a general phenomenon: many useful concepts from topology have combinatorial counterparts in the setting of simplicial sets. However, one must take some care when applying those concepts to simplicial sets which are not of the form $\mathrm{Sing}_\bullet(X)$.

Warning 1.2.0.1. Let $f_0, f_1 : S \rightarrow T$ be morphisms of simplicial sets. We define a *homotopy from f_0 to f_1* to be a morphism of simplicial sets $h : \Delta^1 \times S \rightarrow T$ satisfying $h|_{\{0\} \times S} = f_0$ and $h|_{\{1\} \times S} = f_1$ (Definition 3.1.5.2). In the special case where $T = \mathrm{Sing}_\bullet(X)$ is the singular simplicial set of a topological space X, this recovers the usual definition of homotopy between the associated continuous functions $F_0, F_1 : |S| \rightarrow X$ (Example 3.1.5.5). Beware that, if T is a general simplicial set, then the definition of homotopy is not symmetric: the existence of 0509 a homotopy from f_0 to f_1 does not imply the existence of a homotopy from f_1 to f_0 (for example, take $T = \Delta^1$ to be the standard simplex, and $f_i : \{i\} \hookrightarrow \Delta^1$ to be the inclusion maps).

In §1.2.5, we introduce a class of simplicial sets called *Kan complexes*, for which the bad behavior described in Warning 1.2.0.1 cannot occur: if T is a Kan complex and S is any simplicial set, then homotopy determines an equivalence relation on the collection of morphisms $f : S \to T$ (see Proposition 3.1.5.4). By definition, T is a Kan complex if it satisfies an extension condition with respect to certain maps of simplicial sets $\Lambda^n_i \hookrightarrow \Delta^n$ called *horn inclusions*, which we introduce in §1.2.4. For every topological space X, the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a Kan complex (Proposition 1.2.5.8). Moreover, a classical theorem of Milnor ([44]) guarantees that the functor $X \mapsto \mathrm{Sing}_\bullet(X)$ induces an equivalence from the homotopy category of CW complexes to the homotopy category of Kan complexes. In particular, every Kan complex T is homotopy equivalent to a Kan complex of the form $\mathrm{Sing}_\bullet(X)$, where X is a topological space (in fact, we can take X to be the geometric realization $|T|$; see Theorem 3.6.4.1). Heuristically, one can think that Kan complexes are simplicial sets which “behave like” the singular simplicial sets of topological spaces. However, there are many other examples having a more combinatorial flavor: for example, any simplicial set which admits a group structure is automatically a Kan complex (Proposition 1.2.5.9).

1.2.1 Connected Components of Simplicial Sets

00G5 In this section, we introduce the notion of a *connected* simplicial set (Definition 1.2.1.6 and show that every simplicial set S decomposes uniquely as a disjoint union of connected subsets (Proposition 1.2.1.13), indexed by a set $\pi_0(S)$ which we call the *set of connected components of* S. Moreover, we characterize the construction $S \mapsto \pi_0(S)$ as a left adjoint to the functor $I \mapsto \underline{I}$ of Construction 1.1.5.2 (Corollary 1.2.1.21).

00G6 **Definition 1.2.1.1.** Let S be a simplicial set and let $S' \subseteq S$ be a simplicial subset of S (Remark 1.1.0.14). We will say that S' is a *summand* of S if the simplicial set S decomposes as a coproduct $S' \coprod S''$, for some other simplicial subset $S'' \subseteq S$.

00G7 **Remark 1.2.1.2.** In the situation of Definition 1.2.1.1, if $S'_\bullet \subseteq S_\bullet$ is a summand, then the complementary summand $S''_\bullet$ is uniquely determined: for each $n \geq 0$, we must have $S''_n = S_n \setminus S'_n$. Consequently, the condition that $S'_\bullet$ is a summand of $S_\bullet$ is equivalent to the condition that the construction

$$([n] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto S_n \setminus S'_n$$

is functorial: that is, that the face and degeneracy operators for the simplicial set $S_\bullet$ preserve the subsets $S_n \setminus S'_n$.

Remark 1.2.1.3. Let S be a simplicial set. Then the collection of all summands of S is closed under the formation of unions and intersections (this follows immediately from the criterion of Remark 1.2.1.2). 00G8

Remark 1.2.1.4 (Transitivity)**.** Let S be a simplicial set. If $S' \subseteq S$ is a summand of S and $S'' \subseteq S'$ is a summand of S', then S'' is a summand of S. 00G9

Remark 1.2.1.5. Let $f : S \to T$ be a map of simplicial sets and let $T' \subseteq T$ be a summand. Then the inverse image $f^{-1}(T') \simeq S \times_T T'$ is a summand of S. 00GA

Definition 1.2.1.6. Let S be a simplicial set. We will say that S is *connected* if it is nonempty and every summand $S' \subseteq S$ is either empty or coincides with S. 00GB

Example 1.2.1.7. For each $n \geq 0$, the standard n-simplex Δ^n is connected. 00GC

Definition 1.2.1.8 (Connected Components)**.** Let S be a simplicial set. We will say that a simplicial subset $S' \subseteq S$ is a *connected component* of S if S' is a summand of S (Definition 1.2.1.1) and S' is connected (Definition 1.2.1.6). We let $\pi_0(S)$ denote the set of all connected components of S. 00GD

Warning 1.2.1.9. Let S be a simplicial set. As we will soon see, the set $\pi_0(S)$ admits many different descriptions: 00GE

- We can identify $\pi_0(S)$ with the set of connected components of S (Definition 1.2.1.8).
- We can identify $\pi_0(S)$ with a colimit of the diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ given by the simplicial set S (Remark 1.2.1.20).
- We can identify $\pi_0(S)$ with the quotient of the set of vertices of S by an equivalence relation $\sim$ generated by the set of edges of S (Remark 1.2.1.23).
- We can identify $\pi_0(S)$ with the set of connected components of the directed graph $\mathrm{Gr}(S)$ introduced in §1.1.6 (Variant 1.2.1.24).
- If S is a Kan complex, we can identify $\pi_0(S)$ as the set of isomorphism classes of objects in the fundamental groupoid $\pi_{\leq 1}(S)$ (Remark 1.4.6.13).

Because of this abundance of perspectives, it often will be convenient to view $I = \pi_0(S)$ as an abstract index set which is equipped with a bijection

$$I \simeq \{\text{Connected components of } S\} \qquad (i \in I) \mapsto (S'_i \subseteq S),$$

rather than as the set of connected components itself.

00GF **Example 1.2.1.10.** Let I be a set and let $\underline{I}$ be the constant simplicial set associated to I (Construction 1.1.5.2). Then the connected components of $\underline{I}$ are exactly the simplicial subsets of the form $\underline{\{i\}}$ for $i \in I$. In particular, we have a canonical bijection $I \simeq \pi_0(\underline{I})$.

00GG **Proposition 1.2.1.11.** *Let $f : S \to T$ be a map of simplicial sets, and suppose that S is connected. Then there is a unique connected component $T' \subseteq T$ such that $f(S) \subseteq T'$.*

Proof. Let T' be the smallest summand of T which contains the image of f (the existence of T' follows from Remark 1.2.1.3: we can take T' to be the intersection of all those summands of T which contain the image of f). We will complete the proof by showing that T' is connected. Since S is nonempty, T' must be nonempty. Let $T'' \subseteq T'$ be a summand; we wish to show that $T'' = T'$ or $T'' = \emptyset$. Note that $f^{-1}(T'')$ is a summand of S (Remark 1.2.1.5). Since S is connected, we must have $f^{-1}(T'') = S$ or $f^{-1}(T'') = \emptyset$. Replacing T'' by its complement if necessary, we may assume that $f^{-1}(T'') = S$, so that f factors through T''. Since T'' is a summand of T (Remark 1.2.1.4), the minimality of T' guarantees that $T'' = T'$, as desired. □

00GH **Corollary 1.2.1.12.** *Let S be a simplicial set. The following conditions are equivalent:*

(*a*) *The simplicial set S is connected.*

(*b*) *For every set I, the canonical map*

$$I \simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^0, \underline{I}) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S, \underline{I})$$

is bijective.

Proof. The implication $(a) \Rightarrow (b)$ follows from Proposition 1.2.1.11 and Example 1.2.1.10. Conversely, suppose that (b) is satisfied. Applying (b) in the case $I = \emptyset$, we conclude that there are no maps from S to the empty simplicial set, so that S is nonempty. If S is a disjoint union of simplicial subsets $S', S'' \subseteq S$, then we obtain a map of simplicial sets

$$S \simeq S' \coprod S'' \to \Delta^0 \coprod \Delta^0$$

and assumption (b) guarantees that this map factors through one of the summands on the right hand side; it follows that either S' or S'' is empty. □

00GJ **Proposition 1.2.1.13.** *Let S be a simplicial set. Then S is the disjoint union of its connected components.*

Proof. Let σ be an n-simplex of S; we wish to show that there is a unique connected component of S which contains σ. This follows from Proposition 1.2.1.11, applied to the map $\Delta^n \to S$ classified by σ (since the standard n-simplex Δ^n is connected; see Example 1.2.1.7). □

Corollary 1.2.1.14. *Let S be a simplicial set. Then S is empty if and only if $\pi_0(S)$ is empty.* 00GK

Corollary 1.2.1.15. *Let S be a simplicial set. Then S is connected if and only if $\pi_0(S)$ has exactly one element.* 00GL

Exercise 1.2.1.16 (Classification of Summands)**.** Let S be a simplicial set. Show that a simplicial subset $S' \subseteq S$ is a summand if and only if it can be written as a union of connected components of S. Consequently, we have a canonical bijection 00GM

$$\{\text{Subsets of } \pi_0(S)\} \simeq \{\text{Summands of } S\}.$$

Remark 1.2.1.17 (Functoriality of π_0)**.** Let $f : S \to T$ be a map of simplicial sets. It follows from Proposition 1.2.1.11 that for each connected component $S' \subseteq S$, there is a unique connected component $T' \subseteq T$ such that $f(S') \subseteq T'$. The construction $S' \mapsto T'$ then determines a map of sets $\pi_0(f) : \pi_0(S) \to \pi_0(T)$. This construction is compatible with composition, and therefore allows us to view the construction $S \mapsto \pi_0(S)$ as a functor $\pi_0 : \mathrm{Set}_\Delta \to \mathrm{Set}$ from the category of simplicial sets to the category of sets. 00GN

We now show that the connected component functor $\pi_0 : \mathrm{Set}_\Delta \to \mathrm{Set}$ can be characterized by a universal property.

Construction 1.2.1.18 (The Component Map)**.** Let S be a simplicial set. For every n-simplex σ of S, Proposition 1.2.1.13 implies that there is a unique connected component $S' \subseteq S$ which contains σ. The construction $\sigma \mapsto S'$ then determines a map of simplicial sets 00GP

$$u : S \to \underline{\pi_0(S)},$$

where $\underline{\pi_0(S)}$ denotes the constant simplicial set associated to $\pi_0(S)$ (Construction 1.1.5.2). We will refer to u as the *component map.*

Proposition 1.2.1.19. *Let S be a simplicial set and let $u : S \to \underline{\pi_0(S)}$ be the component map of Construction 1.2.1.18. For every set J, composition with u induces a bijection* 00GQ

$$\mathrm{Hom}_{\mathrm{Set}}(\pi_0(S), J) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(S, \underline{J}).$$

Proof. Decomposing S as the union of its connected components, we can reduce to the case where S is connected, in which case the desired result is a reformulation of Corollary 1.2.1.12. □

Remark 1.2.1.20 (π_0 as a Colimit)**.** Let S be a simplicial set. It follows from Proposition 1.2.1.19 that the component map $u : S \to \underline{\pi_0(S)}$ exhibits $\pi_0(S)$ as the colimit of the diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ determined by S. 00GR

00GS **Corollary 1.2.1.21.** *The connected component functor*

$$\pi_0 : \mathrm{Set}_\Delta \to \mathrm{Set} \qquad S \mapsto \pi_0(S)$$

of Remark 1.2.1.17 is left adjoint to the constant simplicial set functor

$$\mathrm{Set} \to \mathrm{Set}_\Delta \qquad I \mapsto \underline{I}$$

of Construction 1.1.5.2. More precisely, the construction $S \mapsto (u : S \to \underline{\pi_0(S)})$ *is the unit of an adjunction.*

We now make Remark 1.2.1.20 more concrete.

00GT **Proposition 1.2.1.22.** *Let* $S_\bullet$ *be a simplicial set, and let* $u_0 : S_0 \to \pi_0(S_\bullet)$ *be the map of sets given by the component map of Construction 1.2.1.18. Then* u_0 *exhibits* $\pi_0(S_\bullet)$ *as the coequalizer of the face operators* $d^1_0, d^1_1 : S_1 \rightrightarrows S_0$*.*

00GU **Remark 1.2.1.23.** Let $S_\bullet$ be a simplicial set. Proposition 1.2.1.22 supplies a coequalizer diagram of sets

$$\begin{tikzcd} S_1 \arrow[r, shift left, "d^1_0"] \arrow[r, shift right, "d^1_1"'] & S_0 \arrow[r] & \pi_0(S_\bullet). \end{tikzcd}$$

In other words, it allows us to identify $\pi_0(S_\bullet)$ with the quotient of $S_0/\sim$, where $\sim$ is the equivalence relation generated by the set of edges of $S_\bullet$ (that is, the smallest equivalence relation with the property that $d^1_0(e) \sim d^1_1(e)$, for every edge $e \in S_1$). In particular, the set $\pi_0(S_\bullet)$ depends only on the 1-skeleton of $S_\bullet$.

00GV **Variant 1.2.1.24.** Let $S_\bullet$ be a simplicial set. Then the set of connected components $\pi_0(S_\bullet)$ can also be described as the coequalizer of the pair of maps $d^1_0, d^1_1 : S^{\mathrm{nd}}_1 \rightrightarrows S_0$, where $S^{\mathrm{nd}}_1 \subseteq S_1$ denotes the set of nondegenerate edges of $S_\bullet$ (since every degenerate edge $e \in S_1$ automatically satisfies $d^1_0(e) = d^1_1(e)$). We therefore have a coequalizer diagram of sets

$$\begin{tikzcd} \mathrm{Edge}(G) \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathrm{Vert}(G) \arrow[r] & \pi_0(S_\bullet), \end{tikzcd}$$

where $G = \mathrm{Gr}(S_\bullet)$ is the directed graph of Example 1.1.6.4. In other words, we can identify $\pi_0(S_\bullet)$ with the set of connected components of G, in the usual graph-theoretic sense.

050A **Corollary 1.2.1.25.** *For* $n \geq 2$*, the simplicial set* $\partial\Delta^n$ *is connected.*

Proof. Example 1.2.1.7 guarantees that the standard simplex Δ^n is connected. The desired result now follows from Proposition 1.2.1.22, since the inclusion map $\partial\Delta^n \hookrightarrow \Delta^n$ is bijective on simplices of dimension ≤ 1. □

Proof of Proposition 1.2.1.22. Let I be a set and let $f : S_0 \to I$ be a function satisfying $f \circ d^1_0 = f \circ d^1_1$ (as functions from S_1 to I). We wish to show that f factors uniquely as a composition

$$S_0 \xrightarrow{u_0} \pi_0(S_\bullet) \to I.$$

By virtue of Proposition 1.2.1.19, this is equivalent to the assertion that there is a unique map of simplicial sets $F : S_\bullet \to \underline{I}$ which coincides with f on simplices of degree zero. Let σ be an n-simplex of $S_\bullet$, which we identify with a map of simplicial sets $\sigma : \Delta^n \to S_\bullet$. For $0 \leq i \leq n$, we regard $\sigma(i)$ as a vertex of $S_\bullet$. Note that if $0 \leq i \leq j \leq n$, then we have $f(\sigma(i)) = f(\sigma(j))$: to prove this, we can assume without loss of generality that $i = 0$ and $j = n = 1$, in which case it follows from our hypothesis that $f \circ d^1_0 = f \circ d^1_1$. It follows that there is a unique element $F(\sigma) \in I$ such that $F(\sigma) = f(\sigma(i))$ for each $0 \leq i \leq n$. The construction $\sigma \mapsto F(\sigma)$ defines a map of simplicial sets $F : S_\bullet \to \underline{I}$ with the desired properties. □

Proposition 1.2.1.26. *The collection of connected simplicial sets is closed under finite products.* 00GW

Proof. Since the final object $\Delta^0 \in \mathrm{Set}_\Delta$ is connected (Example 1.2.1.7), it will suffice to show that the collection of connected simplicial sets is closed under pairwise products. Let $S_\bullet$ and $T_\bullet$ be connected simplicial sets; we wish to show that $S \times T$ is connected. Equivalently, we wish to show that $\pi_0(S_\bullet \times T_\bullet)$ consists of a single element (Corollary 1.2.1.15). By virtue of Proposition 1.2.1.22, the component map supplies a surjection

$$u_0 : S_0 \times T_0 \twoheadrightarrow \pi_0(S_\bullet \times T_\bullet).$$

It will therefore suffice to show that every pair of vertices $(s,t), (s',t') \in S_0 \times T_0$ belong to the same connected component of $S_\bullet \times T_\bullet$. Let $K_\bullet \subseteq S_\bullet \times T_\bullet$ be the connected component which contains the vertex (s',t). Since $S_\bullet$ is connected, the map

$$S_\bullet \simeq S_\bullet \times \{t\} \hookrightarrow S_\bullet \times T_\bullet$$

factors through a unique connected component of $S_\bullet \times T_\bullet$, which must be equal to $K_\bullet$. It follows that $K_\bullet$ contains the vertex (s,t). A similar argument (with the roles of $S_\bullet$ and $T_\bullet$ reversed) shows that $K_\bullet$ contains (s',t'). □

Corollary 1.2.1.27. *The functor* $\pi_0 : \mathrm{Set}_\Delta \to \mathrm{Set}$ *preserves finite products.* 00GX

Proof. Since $\pi_0(\Delta^0)$ is a singleton (Example 1.2.1.7), it will suffice to show that for every pair of simplicial sets $S_\bullet$ and $T_\bullet$, the canonical map

$$\pi_0(S_\bullet \times T_\bullet) \to \pi_0(S_\bullet) \times \pi_0(T_\bullet)$$

is bijective. Writing $S_\bullet$ and $T_\bullet$ as a disjoint union of connected components (Proposition 1.2.1.13), we can reduce to the case where $S_\bullet$ and $T_\bullet$ are connected, in which case the desired result follows from Proposition 1.2.1.26. □

00GY **Warning 1.2.1.28.** The collection of connected simplicial sets is not closed under infinite products (so the functor $\pi_0 : \mathrm{Set}_\Delta \to \mathrm{Set}$ does not commute with infinite products). For example, let G be the directed graph with vertex set $\mathrm{Vert}(G) = \mathbf{Z}_{\geq 0} = \mathrm{Edge}(G)$, with source and target maps

$$s, t : \mathrm{Edge}(G) \to \mathrm{Vert}(G) \qquad s(n) = n \qquad t(n) = n + 1.$$

More informally, G is the directed graph depicted in the diagram

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & \cdots \end{tikzcd}$$

The associated 1-dimensional simplicial set $G_\bullet$ is connected. However, the infinite product $S_\bullet = \prod_{n \in \mathbf{Z}_{\geq 0}} G_\bullet$ is not connected. By definition, the vertices of $S_\bullet$ can be identified with functions $f : \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$. It is not difficult to see that two such functions $f, g : \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ belong to the same connected component of $S_\bullet$ if and only if the function $n \mapsto |f(n) - g(n)|$ is bounded. In particular, the identity function $n \mapsto n$ and the zero function $n \mapsto 0$ do not belong to the same connected component of $S_\bullet$.

1.2.2 The Singular Simplicial Set of a Topological Space

001Q Topology provides an abundant supply of examples of simplicial sets.

050B **Notation 1.2.2.1** (The n-Simplex)**.** For each integer $n \geq 0$, we let $|\Delta^n|$ denote the set of $(n+1)$-tuples of nonnegative real numbers $(t_0, t_1, \cdots, t_n)$ which satisfy the equation $t_0 + t_1 + \cdots + t_n = 1$. We regard $|\Delta^n|$ as a topological space (with the topology inherited from standard topology on Euclidean space $\mathbf{R}^{n+1}$). If X is a topological space, we will refer to a continuous function $\sigma : |\Delta^n| \to X$ as a *singular n-simplex in X*.

001R **Construction 1.2.2.2.** Let X be a topological space. We define a simplicial set $\mathrm{Sing}_\bullet(X)$ as follows:

- To each object $[n] \in \mathbf{\Delta}$, we assign the set $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$ of singular n-simplices in X.
- To each non-decreasing map $\alpha : [m] \to [n]$, we assign the map $\mathrm{Sing}_n(X) \to \mathrm{Sing}_m(X)$ given by precomposition with the continuous map

$$|\Delta^m| \to |\Delta^n|$$

$$(t_0, t_1, \ldots, t_m) \mapsto \Big(\sum_{\alpha(i)=0} t_i, \sum_{\alpha(i)=1} t_i, \ldots, \sum_{\alpha(i)=n} t_i \Big).$$

We will refer to $\operatorname{Sing}_\bullet(X)$ as the *singular simplicial set of* X. We view the construction $X \mapsto \operatorname{Sing}_\bullet(X)$ as a functor from the category of topological spaces to the category of simplicial sets, which we will denote by $\operatorname{Sing}_\bullet : \operatorname{Top} \to \operatorname{Set}_\Delta$.

Example 1.2.2.3. Let X be a topological space and let $\operatorname{Sing}_\bullet(X)$ be its singular simplicial set. Then: `001S`

- Vertices of $\operatorname{Sing}_\bullet(X)$ can be identified with points of X.
- Edges of $\operatorname{Sing}_\bullet(X)$ can be identified with continuous paths $p : [0,1] \to X$. Here the source of p is the point $x = p(0)$, and the target of p is the point $y = p(1)$.

Remark 1.2.2.4. The functor $X \mapsto \operatorname{Sing}_\bullet(X)$ carries limits in the category of topological spaces to limits in the category of simplicial sets (in fact, the functor $\operatorname{Sing}_\bullet$ admits a left adjoint; see Corollary 1.2.3.5). It does not preserve colimits in general. However, it does carry coproducts of topological spaces to coproducts of simplicial sets: this follows from the observation that the topological n-simplex $|\Delta^n|$ is connected for every $n \geq 0$. `0217`

Remark 1.2.2.5 (Connected Components of $\operatorname{Sing}_\bullet(X)$)**.** Let X be a topological space. We let $\pi_0(X)$ denote the set of *path components* of X: that is, the quotient of X by the equivalence relation `00GZ`

$$(x \sim y) \Leftrightarrow (\exists p : [0,1] \to X)[p(0) = x \text{ and } p(1) = y].$$

It follows from Remark 1.2.1.23 that we have a canonical bijection $\pi_0(\operatorname{Sing}_\bullet(X)) \simeq \pi_0(X)$. That is, we can identify connected components of the simplicial set $\operatorname{Sing}_\bullet(X)$ (in the sense of Definition 1.2.1.8) with path components of the topological space X.

Remark 1.2.2.6 (Connectedness of $\operatorname{Sing}_\bullet(X)$)**.** Let X be a topological space. Then the simplicial set $\operatorname{Sing}_\bullet(X)$ is connected if and only if X is path connected (this follows from Remark 1.2.2.5). `00H0`

Warning 1.2.2.7. Let X be a topological space. If the simplicial set $\operatorname{Sing}_\bullet(X)$ is connected, then the topological space X is path connected and therefore connected. Beware that the converse is not necessarily true: there exist topological spaces X which are connected but not path connected, in which case the singular simplicial set $\operatorname{Sing}_\bullet(X)$ will not be connected. `00H1`

It will be convenient to consider a generalization of Construction 1.2.2.2.

Variant 1.2.2.8. Let $\mathcal{C}$ be a category and let Q be a cosimplicial object of $\mathcal{C}$, which we view as a functor $\mathbf{\Delta}$ to $\mathcal{C}$. For every object $X \in \mathcal{C}$, the construction $([n] \in \mathbf{\Delta}) \mapsto \operatorname{Hom}_{\mathcal{C}}(Q([n]), X)$ determines a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to the category of sets, which we can view as a simplicial set. We will denote this simplicial set by $\operatorname{Sing}^Q_\bullet(X)$, so that we have canonical bijections $\operatorname{Sing}^Q_n(X) \simeq \operatorname{Hom}_{\mathcal{C}}(Q^n, X)$. We view the construction $X \mapsto \operatorname{Sing}^Q_\bullet(X)$ as a functor from $\mathcal{C}$ to the category of simplicial sets, which we denote by $\operatorname{Sing}^Q_\bullet : \mathcal{C} \to \operatorname{Set}_\Delta$. `001T`

001U **Example 1.2.2.9.** The construction $[n] \mapsto |\Delta^n|$ determines a functor from the simplex category $\mathbf{\Delta}$ to the category Top of topological spaces, which assigns to each morphism $\alpha : [m] \to [n]$ the continuous map

$$|\Delta^m| \to |\Delta^n| \qquad (t_0, \ldots, t_m) \mapsto (\sum_{\alpha(i)=0} t_i, \ldots, \sum_{\alpha(i)=n} t_i).$$

We regard this functor as a cosimplicial topological space, which we denote by $|\Delta^{\bullet}|$. Applying Variant 1.2.2.8 to this cosimplicial space yields a functor $\mathrm{Sing}_{\bullet}^{|\Delta|} : \mathrm{Top} \to \mathrm{Set}_{\Delta}$, which coincides with the singular simplicial set functor $\mathrm{Sing}_{\bullet}$ of Construction 1.2.2.2.

001V **Example 1.2.2.10.** The construction $[n] \mapsto \Delta^n$ determines a functor from the simplex category $\mathbf{\Delta}$ to the category $\mathrm{Set}_{\Delta} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ of simplicial sets (this is the *Yoneda embedding* for the simplex category $\mathbf{\Delta}$). We regard this functor as a cosimplicial object of Set_{Δ}, which we denote by $\Delta^{\bullet}$. Applying Variant 1.2.2.8 to this cosimplicial object, we obtain a functor from the category of simplicial sets to itself, which is canonically isomorphic to the identity functor $\mathrm{id}_{\mathrm{Set}_{\Delta}} : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ (see Proposition 1.1.0.12).

001W **Remark 1.2.2.11.** The cosimplicial space $|\Delta^{\bullet}|$ of Example 1.2.2.9 can be described more informally as follows:

- To each nonempty finite linearly ordered set I, it assigns a topological simplex $|\Delta^I|$ whose vertices are the elements of I: that is, the convex hull of the set I inside the real vector space $\mathbf{R}[I]$ generated by I.
- To every nondecreasing map $\alpha : I \to J$, the induced map $|\Delta^I| \to |\Delta^J|$ is given by the restriction of the $\mathbf{R}$-linear map $\mathbf{R}[I] \to \mathbf{R}[J]$ determined by α. Equivalently, it is the unique affine map which coincides with α on the vertices of the simplex $|\Delta^I|$.

1.2.3 The Geometric Realization of a Simplicial Set

001X Let X be a topological space. By definition, n-simplices of the simplicial set $\mathrm{Sing}_{\bullet}(X)$ are continuous functions $|\Delta^n| \to X$. Using Proposition 1.1.0.12, we obtain a bijection

$$\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X) \simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathrm{Sing}_{\bullet}(X)).$$

We now consider a generalization of this observation, where we replace Δ^n by an arbitrary simplicial set.

001Y **Definition 1.2.3.1.** Let S be a simplicial set and let Y be a topological space. We will say that a map of simplicial sets $u : S \to \mathrm{Sing}_{\bullet}(Y)$ *exhibits Y as a geometric realization of S* if, for every topological space X, the composite map

$$\mathrm{Hom}_{\mathrm{Top}}(Y, X) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sing}_{\bullet}(Y), \mathrm{Sing}_{\bullet}(X)) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S, \mathrm{Sing}_{\bullet}(X))$$

is a bijection.

Example 1.2.3.2. For each $n \geq 0$, the identity map $\mathrm{id} : |\Delta^n| \simeq |\Delta^n|$ determines an n-simplex of the simplicial set $\mathrm{Sing}_{\bullet}(|\Delta^n|)$, which we can identify with a morphism of simplicial sets $u : \Delta^n \to \mathrm{Sing}_{\bullet}(|\Delta^n|)$. It follows from Proposition 1.1.0.12 that u exhibits the topological space $|\Delta^n|$ as a geometric realization of the simplicial set Δ^n. 001Z

Notation 1.2.3.3. Let S be a simplicial set. It follows immediately from the definitions that if there exists a map $u : S \to \mathrm{Sing}_{\bullet}(Y)$ which exhibits Y as a geometric realization of S, then the topological space Y is determined up to homeomorphism and depends functorially on S. We will emphasize this dependence by writing $|S|$ to denote a geometric realization of S. By virtue of Example 1.2.3.2, this is compatible with the convention of Notation 1.2.2.1 in the special case where $S = \Delta^n$ is a standard simplex. 0020

Every simplicial set admits a geometric realization:

Proposition 1.2.3.4. *For every simplicial set S, there exists a topological space Y and a map $u : S \to \mathrm{Sing}_{\bullet}(Y)$ which exhibits Y as a geometric realization of S.* 0021

Corollary 1.2.3.5. *The singular simplicial set functor* $\mathrm{Sing}_{\bullet} : \mathrm{Top} \to \mathrm{Set}_{\Delta}$ *admits a left adjoint, given by the geometric realization construction $S \mapsto |S|$.* 0022

Our proof of Proposition 1.2.3.4 will make use of the following formal observation:

Lemma 1.2.3.6. *Let $\mathcal{J}$ be a small category equipped with a functor $S : \mathcal{J} \to \mathrm{Set}_{\Delta}$. Suppose that, for each $J \in \mathcal{J}$, the simplicial set $S(J)$ admits a geometric realization $|S(J)|$. Then the colimit $T = \varinjlim_{J \in \mathcal{J}} S(J)$ also admits a geometric realization, given by the colimit $Y = \varinjlim_{J \in \mathcal{J}} |S(J)|$ in the category of topological spaces.* 0023

Proof. For each $J \in \mathcal{J}$, choose a topological space $|S(J)|$ and a map $u_J : S(J) \to \mathrm{Sing}_{\bullet}(|S(J)|)$ which exhibits $|S(J)|$ as a geometric realization of $S(J)$. We can then amalgamate the composite maps

$$S(J) \xrightarrow{u_J} \mathrm{Sing}_{\bullet}(|S(J)|) \to \mathrm{Sing}_{\bullet}(Y)$$

to a single map of simplicial sets $u : T \to \mathrm{Sing}_{\bullet}(Y)$. We claim that u exhibits Y as a geometric realization of the simplicial set T. Let X be any topological space; we wish to show that the composite map

$$\mathrm{Hom}_{\mathrm{Top}}(Y, X) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sing}_{\bullet}(Y), \mathrm{Sing}_{\bullet}(X)) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(T, \mathrm{Sing}_{\bullet}(X))$$

is a bijection. This is clear, since this composite map can be written as an inverse limit of the bijections $\mathrm{Hom}_{\mathrm{Top}}(|S(J)|, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S(J), \mathrm{Sing}_{\bullet}(X))$ determined by the maps u_J. $\square$

It is possible to prove Proposition 1.2.3.4 in a completely formal way from Lemma 1.2.3.6, since every simplicial set can be presented as a colimit of simplices (see Proposition 1.2.3.15 below). However, we will instead give a less formal argument which yields some additional information about the structure of the geometric realization $|S|$. We begin by studying simplicial subsets of the standard simplex Δ^n.

0024 **Notation 1.2.3.7.** Let $n \geq 0$ be an integer and let $\mathcal{U}$ be a collection of nonempty subsets of $[n] = \{0, 1, \ldots, n\}$. We will say that $\mathcal{U}$ is *downward closed* if $\emptyset \neq I \subseteq J \in \mathcal{U}$ implies that $I \in \mathcal{U}$. If this condition is satisfied, we let $\Delta^n_{\mathcal{U}}$ denote the simplicial subset of Δ^n whose m-simplices are nondecreasing maps $\alpha : [m] \to [n]$ for which the image of α is an element of $\mathcal{U}$. Similarly, we set

$$|\Delta^n|_{\mathcal{U}} = \{(t_0, \ldots, t_n) \in |\Delta^n| : \{i \in [n] : t_i \neq 0\} \in \mathcal{U}\}.$$

0025 **Example 1.2.3.8.** For each $n \geq 0$, the boundary $\partial \Delta^n$ of Construction 1.1.4.10 is given by $\Delta^n_{\mathcal{U}}$, where $\mathcal{U}$ is the collection of all nonempty proper subsets of $[n]$.

0027 **Exercise 1.2.3.9.** Show that every simplicial subset of the standard n-simplex Δ^n has the form $\Delta^n_{\mathcal{U}}$, where $\mathcal{U}$ is some (uniquely determined) downward closed collection of nonempty subsets of $[n]$.

0028 **Proposition 1.2.3.10.** *Let n be a nonnegative integer and let $\mathcal{U}$ be a downward closed collection of nonempty subsets of $[n]$. Then the canonical map $\Delta^n \to \operatorname{Sing}_\bullet(|\Delta^n|)$ restricts to a map of simplicial sets $f_{\mathcal{U}} : \Delta^n_{\mathcal{U}} \to \operatorname{Sing}_\bullet(|\Delta^n|_{\mathcal{U}})$, which exhibits the topological space $|\Delta^n|_{\mathcal{U}}$ as a geometric realization of $\Delta^n_{\mathcal{U}}$.*

Proof. We proceed by induction on the cardinality of $\mathcal{U}$. If $\mathcal{U}$ is empty, then the simplicial set $\Delta^n_{\mathcal{U}}$ and the topological space $|\Delta^n|_{\mathcal{U}}$ are both empty, in which case there is nothing to prove. We may therefore assume that $\mathcal{U}$ is nonempty. Choose some $S \in \mathcal{U}$ whose cardinality is as large as possible. Set

$$\mathcal{U}_0 = \mathcal{U} \setminus \{S\} \qquad \mathcal{U}_1 = \{T \subseteq S : T \neq \emptyset\} \qquad \mathcal{U}_{01} = \mathcal{U}_0 \cap \mathcal{U}_1 .$$

Our inductive hypothesis implies that the maps $f_{\mathcal{U}_0}$ and $f_{\mathcal{U}_{01}}$ exhibit $|\Delta^n|_{\mathcal{U}_0}$ and $|\Delta^n|_{\mathcal{U}_{01}}$ as geometric realizations of $\Delta^n_{\mathcal{U}_0}$ and $\Delta^n_{\mathcal{U}_{01}}$, respectively. Moreover, if $S = \{i_0 < i_1 < \cdots < i_m\} \subseteq [n]$, then we can identify $f_{\mathcal{U}_1}$ with the tautological map $\Delta^m \to \operatorname{Sing}_\bullet(|\Delta^m|)$, so that $f_{\mathcal{U}_1}$ exhibits $|\Delta^n|_{\mathcal{U}_1}$ as a geometric realization of $\Delta^n_{\mathcal{U}_1}$ by virtue of Example 1.2.3.2. It follows immediately from the definitions that the diagram of simplicial sets

$$\begin{tikzcd}
\Delta^n_{\mathcal{U}_{01}} \arrow[r] \arrow[d] & \Delta^n_{\mathcal{U}_0} \arrow[d] \\
\Delta^n_{\mathcal{U}_1} \arrow[r] & \Delta^n_{\mathcal{U}}
\end{tikzcd}$$

is a pushout square. By virtue of Lemma 1.2.3.6, we are reduced to proving that the diagram of topological spaces

$$\begin{tikzcd}
{|\Delta^n|_{\mathcal{U}_{01}}} \arrow[r] \arrow[d] & {|\Delta^n|_{\mathcal{U}_0}} \arrow[d] \\
{|\Delta^n|_{\mathcal{U}_1}} \arrow[r] & {|\Delta^n|_{\mathcal{U}}}
\end{tikzcd}$$

is also a pushout square. This is clear, since $|\Delta^n|_{\mathcal{U}_0}$ and $|\Delta^n|_{\mathcal{U}_1}$ are closed subsets of $|\Delta^n|$ whose union is $|\Delta^n|_{\mathcal{U}}$ and whose intersection is $|\Delta^n|_{\mathcal{U}_{01}}$. □

Example 1.2.3.11. Let n be a nonnegative integer. Combining Example 1.2.3.8 with Proposition 1.2.3.10, we see that the inclusion map $\partial \Delta^n \hookrightarrow \Delta^n$ induces a homeomorphism from $| \partial \Delta^n |$ to the *boundary* of the topological n-simplex $|\Delta^n|$, given by 0029

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j\}.$$

Proof of Proposition 1.2.3.4. Let $S = S_\bullet$ be a simplicial set; we wish to show that S admits a geometric realization $|S|$. We first show that for each $n \geq -1$, the n-skeleton $\mathrm{sk}_n(S)$ admits a geometric realization. The proof proceeds by induction on n, the case $n = -1$ being trivial (since $\mathrm{sk}_{-1}(S)$ is empty). Let C denote the collection of nondegenerate n-simplices of C. we note that Proposition 1.1.4.12 provides a pushout diagram

$$\begin{tikzcd}
\coprod_{\sigma \in C} \partial \Delta^n \arrow[r] \arrow[d] & \coprod_{\sigma \in C} \Delta^n \arrow[d] \\
\mathrm{sk}_{n-1}(S) \arrow[r] & \mathrm{sk}_n(S).
\end{tikzcd}$$

Combining our inductive hypothesis, Example 1.2.3.2, Example 1.2.3.11, and Lemma 1.2.3.6, we deduce that $\mathrm{sk}_n(S)$ admits a geometric realization $| \mathrm{sk}_n(S)|$ which fits into a pushout diagram of topological spaces

$$\begin{tikzcd}
\coprod_{\sigma \in C} | \partial \Delta^n | \arrow[r] \arrow[d] & \coprod_{\sigma \in C} |\Delta^n| \arrow[d] \\
| \mathrm{sk}_{n-1}(S)| \arrow[r] & | \mathrm{sk}_n(S)|.
\end{tikzcd}$$

Combining the equality $S = \bigcup_n \mathrm{sk}_n(S)$ of Remark 1.1.4.4 with Lemma 1.2.3.6, we deduce that the simplicial set S also admits a geometric realization, given by the direct limit $\varinjlim_n |\mathrm{sk}_n(S)|$. □

002B **Remark 1.2.3.12.** The proof of Proposition 1.2.3.4 shows that the geometric realization $|S|$ of a simplicial set S has a canonical realization as a CW complex, having one cell of dimension n for each nondegenerate n-simplex σ of S; this cell can be described explicitly as the image of the map

$$|\Delta^n| \setminus |\partial\Delta^n| \hookrightarrow |\Delta^n| \xrightarrow{\sigma} |S|.$$

The proof of Proposition 1.2.3.4 also yields the following fact, which we will use often throughout this book:

00H2 **Lemma 1.2.3.13.** *Let $\mathcal{U}$ be a full subcategory of the category Set_Δ of simplicial sets. Suppose that $\mathcal{U}$ satisfies the following three conditions:*

(1) *Suppose we are given a pushout diagram of simplicial sets*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y', \end{tikzcd}$$

where f is a monomorphism. If X, Y, and X' belong to $\mathcal{U}$, then Y' belongs to $\mathcal{U}$.

(2) *Suppose we are given a sequence of monomorphisms of simplicial sets*

$$X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow X(3) \hookrightarrow \cdots$$

If each $X(m)$ belongs to $\mathcal{U}$, then the sequential colimit $\varinjlim_m X(m)$ belongs to $\mathcal{U}$.

(3) *For each $n \geq 0$ and every set I, the coproduct $\coprod_{i \in I} \Delta^n$ belongs to $\mathcal{U}$.*

Then every simplicial set belongs to $\mathcal{U}$.

Proof. Let S be a simplicial set; we wish to show that S belongs to $\mathcal{U}$. By virtue of Remark 1.1.4.4, we can identify S with the colimit $\varinjlim_n \mathrm{sk}_n(S)$. By virtue of (2), it will suffice to show that each skeleton $\mathrm{sk}_n(S)$ belongs to $\mathcal{U}$. We may therefore assume without loss of generality that S has dimension $\leq n$, for some integer n. We proceed by induction on n. In the case $n = -1$, the simplicial set S is empty and the desired result is a special case of (3). To carry out the inductive step, we invoke Proposition 1.1.4.12 to choose a pushout diagram

$$\begin{tikzcd}
\coprod_{\sigma \in C} \partial \Delta^n \arrow[r] \arrow[d] & \coprod_{\sigma \in C} \Delta^n \arrow[d] \\
\mathrm{sk}_{n-1}(S) \arrow[r] & S,
\end{tikzcd}$$

where C is the collection of nondegenerate n-simplices of S. By virtue of assumption (1), it will suffice to show that the simplicial sets $\mathrm{sk}_{n-1}(S)$, $\coprod_{\sigma \in C} \partial \Delta^n$, and $\coprod_{\sigma \in C} \Delta^n$ belong to $\mathcal{U}$. In the first two cases, this follows from our inductive hypothesis. In the third, it follows from assumption (3). □

Remark 1.2.3.14. In the statement of Lemma 1.2.3.13, we can replace (3) by the following pair of conditions: 00H3

($3'$) For each $n \geq 0$, the standard n-simplex Δ^n belongs to $\mathcal{U}$.

($3''$) The subcategory $\mathcal{U} \subseteq \mathrm{Set}_{\Delta}$ is closed under the formation of coproducts.

In Chapter 2, we will encounter a number of variants of the geometric realization construction $S \mapsto |S|$, which can be obtained from the following generalization of Corollary 1.2.3.5:

Proposition 1.2.3.15. *Let $\mathcal{C}$ be a category, let $Q^{\bullet}$ be a cosimplicial object of $\mathcal{C}$, and let $\mathrm{Sing}^{Q}_{\bullet} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be the functor of Variant 1.2.2.8. If the category $\mathcal{C}$ admits small colimits, then the functor $\mathrm{Sing}^{Q}_{\bullet}$ admits a left adjoint $\mathrm{Set}_{\Delta} \to \mathcal{C}$, which we will denote by $S \mapsto |S|^{Q}$.* 002F

Proof. Let S be a simplicial set; we wish to show that the functor

$$\lambda : \mathcal{C} \to \mathrm{Set} \qquad C \mapsto \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S, \mathrm{Sing}^{Q}_{\bullet}(C))$$

is corepresentable by an object $|S|^{Q} \in \mathcal{C}$. Since $\mathcal{C}$ admits small colimits, the collection of corepresentable functors from $\mathcal{C}$ to Set is closed under the formation of small limits. Using Remark 1.1.3.13 (or Lemma 1.2.3.13), we can reduce to the case where $S = \Delta^n$ is a standard simplex. In this case, the functor λ is corepresented by the object $Q^n \in \mathcal{C}$ (see Proposition 1.1.0.12). □

Remark 1.2.3.16. From the proof of Proposition 1.2.3.15, we can extract an explicit description of the realization $|S|^{Q}$: it can be realized as the colimit of the composite functor 050C

$$\mathbf{\Delta}_S \to \mathbf{\Delta} \xrightarrow{Q} \mathcal{C},$$

where $\mathbf{\Delta}_S$ denotes the category of simplices of S (Construction 1.1.3.9).

00H4 **Remark 1.2.3.17.** The functor $\pi_0 : \mathrm{Set}_\Delta \to \mathrm{Set}$ of Corollary 1.2.1.21 can be regarded as special case of Proposition 1.2.3.15: it agrees with the functor $| \bullet |^Q$, where $Q^\bullet : \mathbf{\Delta} \to \mathrm{Set}$ is a constant functor whose value is a singleton set $* \in \mathrm{Set}_\Delta$ (see Proposition 1.2.1.19).

00H5 **Proposition 1.2.3.18.** *Let S be a simplicial set. The following conditions are equivalent:*

(1) *The geometric realization $|S|$ is a path-connected topological space.*

(2) *The geometric realization $|S|$ is a connected topological space.*

(3) *The simplicial set S is connected, in the sense of Definition 1.2.1.6.*

Proof. The implication (1) ⇒ (2) holds for any topological space. To prove that (2) ⇒ (3), we observe that any decomposition $S \simeq S' \coprod S''$ into disjoint nonempty simplicial subsets determines a homeomorphism $|S| \simeq |S'| \coprod |S''|$. We will complete the proof by showing that (3) ⇒ (1). Let $\mathbf{\Delta}_S$ denote the category of simplices of S (Construction 1.1.3.9). We then have a commutative diagram of sets

$$\begin{tikzcd}
\varinjlim_{([n],\sigma) \in \mathbf{\Delta}_S} |\Delta^n| \arrow[r, "\sim"] \arrow[d] & {|S|} \arrow[d] \\
\varinjlim_{([n],\sigma) \in \mathbf{\Delta}_S} \pi_0(|\Delta^n|) \arrow[r] & \pi_0(|S|),
\end{tikzcd}$$

where the upper horizontal map is bijective (Remark 1.2.3.16) and the right vertical map is surjective. It follows that the lower horizontal map is also surjective. Since each of the topological spaces $|\Delta^n|$ is path connected, the colimit in the lower left can be identified with the set $\pi_0(S)$ (Remark 1.2.3.17). If S is connected, the set $\pi_0(S)$ consists of a single element, so that $\pi_0(|S|)$ is also a singleton. □

00H6 **Corollary 1.2.3.19.** *For every simplicial set S, we have a canonical bijection*

$$\pi_0(S) \simeq \pi_0(|S|).$$

Proof. Writing S as a disjoint union of connected components (Proposition 1.2.1.11, we can reduce to the case where S is connected, in which case both sets have a single element (Proposition 1.2.3.18). □

1.2.4 Horns

000K We now consider some elementary examples of simplicial sets which will play an important role throughout this book.

Construction 1.2.4.1 (The Horn Λ^n_i)**.** Suppose we are given a pair of integers $0 \leq i \leq n$ with $n > 0$. We define a simplicial set $\Lambda^n_i : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ by the formula 000U

$$(\Lambda^n_i)([m]) = \{\alpha \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n]) : [n] \not\subseteq \alpha([m]) \cup \{i\}\}.$$

We regard Λ^n_i as a simplicial subset of the boundary $\partial \Delta^n \subseteq \Delta^n$. We will refer to Λ^n_i as the *ith horn in* Δ^n. We will say that Λ^n_i is an *inner horn* if $0 < i < n$, and an *outer horn* if $i = 0$ or $i = n$.

Remark 1.2.4.2. Roughly speaking, one can think of the horn Λ^n_i as obtained from the n-simplex Δ^n by removing its interior together with the face opposite its ith vertex (see Remark 1.2.4.6). 000V

Example 1.2.4.3. The horn $\Lambda^1_0 \subset \Delta^1$ is the vertex $\{0\}$, and the horn $\Lambda^1_1 \subset \Delta^1$ is the vertex $\{1\}$ (see Example 1.1.0.15). In particular, Λ^1_0 and Λ^1_1 are abstractly isomorphic to the standard 0-simplex Δ^0. Moreover, the boundary $\partial \Delta^1$ is the disjoint union of Λ^1_0 and Λ^1_1. 000X

Example 1.2.4.4. The horns contained in Δ^2 are depicted in the following diagram: 000W

$$\begin{tikzcd}
& \{1\} \arrow[dr, dashed] & & & \{1\} \arrow[dr] & & & \{1\} \arrow[dr] & \\
\{0\} \arrow[ur] \arrow[rr] & \Lambda^2_0 & \{2\} & \{0\} \arrow[ur] \arrow[rr, dashed] & \Lambda^2_1 & \{2\} & \{0\} \arrow[ur, dashed] \arrow[rr] & \Lambda^2_2 & \{2\}.
\end{tikzcd}$$

Here the dotted arrows indicate edges of Δ^2 which are not contained in the corresponding horn.

Remark 1.2.4.5. Let $0 \leq i \leq n$ be integers with $n > 0$. Then the horn Λ^n_i is connected. If $n = 1$ or $n = 2$, this follows by inspection (see Examples 1.2.4.3 and 1.2.4.4). For $n \geq 3$, the inclusion map $\Lambda^n_i \hookrightarrow \Delta^n$ is bijective on simplices of dimension ≤ 1, so the desired result follows from Proposition 1.2.1.22 (together with the connectedness of the standard simplex Δ^n; see Example 1.2.1.7). 050D

Remark 1.2.4.6. Let $0 \leq i \leq n$ be integers with $n > 0$. It follows from Proposition 1.2.3.10 that the inclusion map $\Lambda^n_i \hookrightarrow \Delta^n$ induces a homeomorphism from the geometric realization $|\Lambda^n_i|$ to the closed subset of $|\Delta^n|$ given by 050E

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j \neq i\}.$$

Let n be a positive integer. For every pair of *distinct* integers $i, j \in [n]$, the inclusion map δ^j_n of Construction 1.1.1.4 can be regarded as a morphism of simplicial sets from Δ^{n-1} to the horn Λ^n_i. We have the following counterpart of Proposition 1.1.4.13:

050F **Proposition 1.2.4.7.** *Let $0 \leq i \leq n$ be integers with $n > 0$. For any simplicial set $S_{\bullet}$, the map*

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Lambda^n_i, S_{\bullet}) \rightarrow (S_{n-1})^n \qquad f \mapsto \{f \circ \delta^j_n\}_{0 \leq j \leq n, j \neq i}$$

is an injection, whose image is the collection of "incomplete" sequences

$$(\sigma_0, \ldots, \sigma_{i-1}, \bullet, \sigma_{i+1}, \ldots, \sigma_n)$$

which satisfy the identity $d^{n-1}_j(\sigma_k) = d^{n-1}_{k-1}(\sigma_j)$ for $j, k \in [n] \setminus \{i\}$ with $j < k$.

Proof. We proceed as in the proof of Proposition 1.1.4.13, with minor modifications. Set $Q = [n] \setminus \{i\}$ and let $w : \coprod_{\ell \in Q} \Delta^{n-1} \rightarrow \Lambda^n_i$ be the map given on the ℓth summand by δ^{ℓ}_n. To prove the first assertion of Proposition 1.2.4.7, we must show that w is an epimorphism of simplicial sets: that is, it is surjective on m-simplices for each $m \geq 0$. In fact, we can be a bit more precise. Let α be an m-simplex of Δ^n, which we identify with a nondecreasing function from $[m]$ to $[n]$. Then α belongs to the boundary Λ^n_i if and only its image does not contain Q: that is, if and only if there exists some integer $j \in Q$ such that α factors through $[n] \setminus \{j\}$. In this case, there is a unique m-simplex β_j which belongs to the jth summand of $\coprod_{\ell \in Q} \Delta^{n-1}$ and satisfies $w(\beta_j) = \alpha$.

For every integer $\ell \in Q$, let $u_{\ell} : \coprod_{j \in Q, j < \ell} \Delta^{n-2} \rightarrow \Delta^{n-1}$ be the map given on the jth summand by δ^j_{n-1}, and let $v_{\ell} : \coprod_{k \in Q, k > \ell} \Delta^{n-2} \rightarrow \Delta^{n-1}$ be the map given on the kth summand by δ^{k-1}_{n-1}. Passing to the coproduct over ℓ and reindexing, we obtain a pair of maps

$$(u, v) : \coprod_{j,k \in Q, j<k} \Delta^{n-2} \rightrightarrows \coprod_{\ell \in Q} \Delta^{n-1}.$$

Let $\mathrm{Coeq}(u, v)_{\bullet}$ denote the coequalizer of u and v in the category of simplicial sets. The morphism w satisfies $w \circ u = w \circ v$ (see Remark 1.1.1.7), and therefore factors uniquely through a map $\overline{w} : \mathrm{Coeq}(u, v) \rightarrow \Lambda^n_i$. Proposition 1.2.4.7 asserts that $\overline{w}$ is an isomorphism of simplicial sets: that is, for every integer $m \geq 0$, it induces a bijection from $\mathrm{Coeq}(u, v)_m$ to the set of m-simplices of Λ^n_i. The surjectivity of this map was established above. To prove injectivity, it will suffice to observe that if $\alpha : [m] \rightarrow [n]$ is as above and we are given two elements $j, k \in Q$ which do not belong to the image of α, then β_j and β_k have the same image in $\mathrm{Coeq}(u, v)_{\bullet}$. If $j = k$, this is automatic; we may therefore assume without loss of generality that $j < k$. In this case, the desired result follows from the observation that we can write $\beta_k = u(\gamma)$ and $\beta_j = v(\gamma)$, where γ is the m-simplex of the (j, k)th summand of $\coprod_{j,k \in Q, j<k} \Delta^{n-2}$ corresponding to the nondecreasing function $[m] \xrightarrow{\alpha} [n] \setminus \{j < k\} \simeq [n - 2]$. □

1.2.5 Kan Complexes

002G We now articulate an important property enjoyed by simplicial sets of the form $\mathrm{Sing}_{\bullet}(X)$.

Definition 1.2.5.1. Let S be a simplicial set. We will say that S is a *Kan complex* if it satisfies the following condition: 002H

$(*)$ For every pair of integers $0 \leq i \leq n$ with $n > 0$, every morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to S$ can be extended to a map $\sigma : \Delta^n \to S$. Here $\Lambda^n_i \subseteq \Delta^n$ denotes the ith horn (see Construction 1.2.4.1).

Exercise 1.2.5.2. Show that for $n > 0$, the standard simplex Δ^n is not a Kan complex (for a more general statement, see Proposition 1.3.5.2). 002J

Example 1.2.5.3 (Products of Kan Complexes)**.** Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of simplicial sets parametrized by a set A, and let $S = \prod_{\alpha \in A} S_\alpha$ be their product. If each S_α is a Kan complex, then S is a Kan complex. The converse holds provided that each S_α is nonempty. 00H8

Example 1.2.5.4 (Coproducts of Kan Complexes)**.** Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of simplicial sets parametrized by a set A, and let $S = \coprod_{\alpha \in A} S_\alpha$ be their coproduct. For every pair of integers $0 \leq i \leq n$ with $n > 0$, the restriction map 00H9

$$\theta : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, S)$$

can be identified with the coproduct (formed in the arrow category $\mathrm{Fun}([1], \mathrm{Set})$) of restriction maps $\theta_\alpha : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S_\alpha) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, S_\alpha)$; this follows from the connectedness of the simplicial sets Δ^n and Λ^n_i (see Example 1.2.1.7 and Remark 1.2.4.5). It follows that θ is surjective if and only if each θ_α is surjective. Allowing n and i to vary, we conclude that S is a Kan complex if and only if each summand S_α is a Kan complex.

Remark 1.2.5.5. Let S be a simplicial set. Combining Example 1.2.5.4 with Proposition 1.2.1.13, we deduce that $S_\bullet$ is a Kan complex if and only if each connected component of S is a Kan complex. 00HA

Example 1.2.5.6. Let S be a discrete simplicial set (Definition 1.1.5.10). Then every connected component of S is isomorphic to the standard simplex Δ^0, which is a Kan complex. Applying Remark 1.2.5.5, we see that S is a Kan complex. 00HB

Example 1.2.5.7. Let S be a simplicial set of dimension exactly 1 (that is, a simplicial set S which arises from a directed graph with at least one edge). Then S is not a Kan complex. 00H7

Proposition 1.2.5.8. *Let X be a topological space. Then the singular simplicial set* $\mathrm{Sing}_\bullet(X)$ *is a Kan complex.* 002K

Proof. Let $\sigma_0 : \Lambda^n_i \to \mathrm{Sing}_\bullet(X)$ be a map of simplicial sets for $n > 0$; we wish to show that σ_0 can be extended to an n-simplex of X. Using the geometric realization functor, we can identify σ_0 with a continuous map of topological spaces $f_0 : |\Lambda^n_i| \to X$; we wish to show that f_0 factors as a composition

$$|\Lambda^n_i| \to |\Delta^n| \xrightarrow{f} X.$$

Using Remark 1.2.4.6, we can identify $|\Lambda^n_i|$ with the subset

$$\{(t_0, \ldots, t_n) \in |\Delta^n| : t_j = 0 \text{ for some } j \neq i\} \subseteq |\Delta^n|.$$

In this case, we can take f to be the composition $f_0 \circ r$, where r is any continuous retraction of $|\Delta^n|$ onto the subset $|\Lambda^n_i|$. For example, we can take r to be the map given by the formula

$$r(t_0, \ldots, t_n) = (t_0 - c, \ldots, t_{i-1} - c, t_i + nc, t_{i+1} - c, \ldots, t_n - c)$$

$$c = \min\{t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}.$$

□

Algebra furnishes another rich supply of examples of Kan complexes:

`00MG` **Proposition 1.2.5.9.** *Let $G_\bullet$ be a simplicial group (that is, a simplicial object of the category of groups). Then (the underlying simplicial set of) $G_\bullet$ is a Kan complex.*

Proof. Let n be a positive integer and $\vec{\sigma} : \Lambda^n_i \to G_\bullet$ be a map of simplicial sets for some $0 \leq i \leq n$, which we will identify with a tuple $(\sigma_0, \sigma_1, \ldots, \sigma_{i-1}, \bullet, \sigma_{i+1}, \ldots, \sigma_n)$ of elements of the group G_{n-1} (Proposition 1.2.4.7). We wish to prove that there exists an element $\tau \in G_n$ satisfying $d^n_j \tau = \sigma_j$ for $j \neq i$. Let e denote the identity element of G_{n-1}. We first treat the special case where $\sigma_{i+1} = \cdots = \sigma_n = e$. If, in addition, we have $\sigma_0 = \sigma_1 = \cdots = \sigma_{i-1} = e$, then we can take τ to be the identity element of G_n. Otherwise, there exists some smallest integer $j < i$ such that $\sigma_j \neq e$. We proceed by descending induction on j. Set $\tau'' = s^{n-1}_j \sigma_j \in G_n$, and consider the map $\vec{\sigma}' : \Lambda^n_i \to G_\bullet$ given by the tuple $(\sigma'_0, \sigma'_1, \ldots, \sigma'_{i-1}, \bullet, \sigma'_{i+1}, \ldots, \sigma'_n)$ with $\sigma'_k = \sigma_k (d^n_k \tau'')^{-1}$. We then have $\sigma'_0 = \sigma'_1 = \cdots = \sigma'_j = e$ and $\sigma'_{i+1} = \cdots = \sigma'_n = e$. Invoking our inductive hypothesis we conclude that there exists an element $\tau' \in G_n$ satisfying $d^n_k \tau' = \sigma'_k$ for $k \neq i$. We can then complete the proof by taking τ to be the product $\tau' \tau''$.

If not all of the equalities $\sigma_{i+1} = \cdots = \sigma_n = e$ hold, then there exists some largest integer $j > i$ such that $\sigma_j \neq e$. We now proceed by ascending induction on j. Set $\tau'' = s^{n-1}_{j-1} \sigma_j$ and let $\vec{\sigma}' : \Lambda^n_i \to G_\bullet$ be the map given by the tuple $(\sigma'_0, \sigma'_1, \ldots, \sigma'_{i-1}, \bullet, \sigma'_{i+1}, \ldots, \sigma'_n)$ with $\sigma'_k = \sigma_k (d^n_k \tau'')^{-1}$, as above. We then have $\sigma'_j = \sigma'_{j+1} = \cdots = \sigma'_n = e$, so the inductive hypothesis guarantees the existence of an element $\tau' \in G_n$ satisfying $d^n_k \tau' = \sigma'_k$ for $k \neq i$. As before, we complete the proof by setting $\tau = \tau' \tau''$. □

Let $S = S_\bullet$ be a simplicial set. According to Remark 1.2.1.23, we can identify the set of connected components $\pi_0(S)$ with the quotient $S_0/\sim$, where $\sim$ is the equivalence relation generated by the image of the map $(d^1_0, d^1_1) : S_1 \to S_0 \times S_0$. In the special case where $S = \mathrm{Sing}_\bullet(X)$ is the singular simplicial set of a topological space X, this description simplifies: the image of the map $(d^1_0, d^1_1) : \mathrm{Sing}_1(X) \to \mathrm{Sing}_0(X) \times \mathrm{Sing}_0(X) = X \times X$ is already an equivalence relation, and $\pi_0(S_\bullet)$ can be identified with the set of path components $\pi_0(X)$ (Remark 1.2.2.5). A similar phenomenon occurs for any Kan complex:

Proposition 1.2.5.10. *Let S be a Kan complex and let x and y be vertices of S. Then x and y belong to the same connected component of S if and only if there exists an edge e of S having source x and target y.* 00HC

Proof. Let S_0 denote the set of vertices of S. Let R be the collection of pairs $(x, y) \in S_0$ for which there exists an edge e of S having source x and target y. Using Remark 1.2.1.23, we can identify $\pi_0(S)$ with the quotient of S_0 by the equivalence relation generated by R. It will therefore suffice to show that R is already an equivalence relation on S_0. To prove this, we must verify three things:

- The relation R is reflexive. This follows from the observation that for every vertex $x \in S_0$, the degenerate edge id_x has source x and target x.

- The relation R is symmetric. Suppose that $(x, y) \in R$: that is, there exists an edge e of S having source x and target y. Then the tuple $(\bullet, \mathrm{id}_x, e)$ determines a map of simplicial sets $\sigma_0 : \Lambda^2_0 \to S$ (see Proposition 1.2.4.7), which we depict as a diagram

$$\begin{tikzcd}
 & y \arrow[dr, dashed] & \\
x \arrow[ur, "e"] \arrow[rr, "\mathrm{id}_x"] & & x.
\end{tikzcd}$$

  Since S is a Kan complex, we can complete this diagram to a 2-simplex $\sigma : \Delta^2 \to S$. Then $e' = d^2_0(\sigma)$ is an edge of S having source y and target x, so the pair (y, x) also belongs to R.

- The relation R is transitive. Suppose that we are given vertices $x, y, z \in S_0$ with $(x, y) \in R$ and $(y, z) \in R$; we wish to show that $(x, z) \in R$. Let e be an edge of S having source x and target y, and let e' be an edge of S having source y and target z. Then the tuple $(e', \bullet, e)$ determines a map of simplicial sets $\tau_0 : \Lambda^2_1 \to S$ (see

Proposition 1.2.4.7), which we depict as a diagram

$$\begin{tikzcd}
 & y \arrow[dr, "e'"] & \\
x \arrow[ur, "e"] \arrow[rr, dashed] & & z.
\end{tikzcd}$$

Our assumption that S is a Kan complex guarantees that we can extend τ_0 to a 2-simplex $\tau : \Delta^2 \to S$. Then $e'' = d^2_1(\tau)$ is an edge of S having source x and target z, so that (x, z) belongs to R.

□

00HD **Corollary 1.2.5.11.** *Let* $\{S_\alpha\}_{\alpha \in A}$ *be a collection of Kan complexes parametrized by a set* A*, and let* $S = \prod_{\alpha \in A} S_{\alpha \bullet}$ *denote their product. Then the canonical map*

$$\pi_0(S) \to \prod_{\alpha \in A} \pi_0(S_\alpha)$$

is bijective. In particular, S *is connected if and only if each factor* S_α *is connected.*

1.3 From Categories to Simplicial Sets

002L In §1.1, we introduced the theory of simplicial sets and discussed its relationship to the theory of topological spaces. Every topological space X determines a simplicial set $\mathrm{Sing}_\bullet(X)$ (Construction 1.2.2.2), and simplicial sets of the form $\mathrm{Sing}_\bullet(X)$ have a special property: they are Kan complexes (Proposition 1.2.5.8). In this section, we will study a different class of simplicial sets, which arise instead from the theory of categories. In §1.3.1, we associate to every category $\mathcal{C}$ a simplicial set $\mathrm{N}_\bullet(\mathcal{C})$, called the *nerve of* $\mathcal{C}$. We show in §1.3.3 that the construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ is fully faithful (Proposition 1.3.3.1). In §1.3.4, we show that a simplicial set S belongs to the essential image of the functor $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ if and only if it satisfies a certain lifting condition (Proposition 1.3.4.1). This lifting condition is similar to the Kan extension condition (Definition 1.2.5.1), but not identical to it: in §1.3.5, we show that a simplicial set of the form $\mathrm{N}_\bullet(\mathcal{C})$ is a Kan complex if and only if every morphism in $\mathcal{C}$ is invertible (Proposition 1.3.5.2).

In §1.3.6, we show that the construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ has a left adjoint, which associates to each simplicial set S a category $\mathrm{h}S$ which we call the *homotopy category of* S (Definition 1.3.6.1). This category admits a particularly simple description in the case where the simplicial set S has dimension ≤ 1: in §1.3.7, we show that it can be identified with the *path category* of the directed graph G corresponding to S (under the equivalence of Proposition 1.1.6.9).

1.3.1 The Nerve of a Category

We begin with a few definitions. 002M

Construction 1.3.1.1. For every integer $n \geq 0$, let us view the linearly ordered set $[n] = \{0 < 1 < \cdots < n-1 < n\}$ as a category (where there is a unique morphism from i to j when $i \leq j$). For any category $\mathcal{C}$, we let $\mathrm{N}_n(\mathcal{C})$ denote the set of all functors from $[n]$ to $\mathcal{C}$. Note that for any nondecreasing map $\alpha : [m] \to [n]$, precomposition with α determines a map of sets $\mathrm{N}_n(\mathcal{C}) \to \mathrm{N}_m(\mathcal{C})$. We can therefore view the construction $[n] \mapsto \mathrm{N}_n(\mathcal{C})$ as a simplicial set. We will denote this simplicial set by $\mathrm{N}_\bullet(\mathcal{C})$ and refer to it as the *nerve of* $\mathcal{C}$. 002N

Remark 1.3.1.2 (The Classifying Space of a Category)**.** Let $\mathcal{C}$ be a category. Then the topological space $|\mathrm{N}_\bullet(\mathcal{C})|$ is called the *classifying space* of the category $\mathcal{C}$. 002P

Remark 1.3.1.3. Let $\mathcal{C}$ be a category and let $n \geq 1$. Elements of $\mathrm{N}_n(\mathcal{C})$ can be identified with diagrams 002Q

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n$$

in the category $\mathcal{C}$ (see Remark 1.5.7.8). In other words, we can identify elements of $\mathrm{N}_n(\mathcal{C})$ with n-tuples $(f_1, \ldots, f_n)$ of morphisms of $\mathcal{C}$ having the property that, for $0 < i < n$, the source of f_{i+1} coincides with the target of f_i.

Example 1.3.1.4. Let $\mathcal{C}$ be a category. Then: 002R

- Vertices of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ can be identified with objects of the category $\mathcal{C}$.
- Edges of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ can be identified with morphisms in the category $\mathcal{C}$.
- Let $f : X \to Y$ be a morphism in $\mathcal{C}$, regarded as an edge of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. Then the faces of f are given by the target $d^1_0(f) = Y$ and the source $d^1_1(f) = X$, respectively.
- Let X be an object of $\mathcal{C}$, which we regard as a vertex of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. Then the degenerate edge $s^0_0(X)$ is the identity morphism $\mathrm{id}_X : X \to X$.

Exercise 1.3.1.5. Let $\mathcal{C}$ be a category. Show that the restriction map 050G

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{N}_\bullet(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^n, \mathrm{N}_\bullet(\mathcal{C}))$$

is an injection for $n = 2$ and a bijection for $n > 2$.

Variant 1.3.1.6. Let $\mathcal{C}$ be a category. For every integer $n \geq 0$, we let $\mathrm{N}_{\leq n}(\mathcal{C})$ denote the n-skeleton of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. In the special case $n = 0$, this recovers the discrete simplicial set associated to the set of objects $\mathrm{Ob}(\mathcal{C})$ (Example 1.3.1.4). 03X2

002S **Remark 1.3.1.7** (Face Operators on $\mathrm{N}_\bullet(\mathcal{C})$)**.** Let $\mathcal{C}$ be a category and suppose we are given an n-simplex σ of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ for some $n > 0$, which we identify with a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \rightarrow \cdots \xrightarrow{f_n} C_n.$$

Then:

- The 0th face $d_0^n(\sigma) \in \mathrm{N}_{n-1}(\mathcal{C})$ can be identified with the diagram

  $$C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3 \rightarrow \cdots \xrightarrow{f_n} C_n$$

  obtained from σ by "deleting" the object C_0 (and the morphism f_1 with source C_0).

- The nth face $d_n^n(\sigma) \in \mathrm{N}_{n-1}(\mathcal{C})$ can be identified with the diagram

  $$C_0 \xrightarrow{f_1} C_1 \rightarrow \cdots \rightarrow C_{n-2} \xrightarrow{f_{n-1}} C_{n-1}$$

  obtained from σ by "deleting" the object C_n (and the morphism f_n with target C_n).

- For $0 < i < n$, the ith face $d_i^n(\sigma) \in \mathrm{N}_{n-1}(\mathcal{C})$ can be identified with the diagram

  $$C_0 \xrightarrow{f_1} C_1 \rightarrow \cdots \rightarrow C_{i-1} \xrightarrow{f_{i+1} \circ f_i} C_{i+1} \rightarrow \cdots \xrightarrow{f_n} C_n$$

  obtained by "deleting" the object C_i (and composing the morphisms f_i and f_{i+1}).

002T **Remark 1.3.1.8** (Degeneracy Operators on $\mathrm{N}_\bullet(\mathcal{C})$)**.** Let $\mathcal{C}$ be a category and suppose we are given an n-simplex σ of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ which we identify with a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \rightarrow \cdots \xrightarrow{f_n} C_n.$$

Then, for $0 \leq i \leq n$, we can identify the degenerate simplex $s_i^n(\sigma) \in \mathrm{N}_{n+1}(\mathcal{C})$ with the diagram

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{i-1}} C_{i-1} \xrightarrow{f_i} C_i \xrightarrow{\mathrm{id}_{C_i}} C_i \xrightarrow{f_{i+1}} C_{i+1} \rightarrow \cdots \xrightarrow{f_n} C_n$$

obtained from σ by "inserting" the identity morphism id_{C_i}.

002U **Remark 1.3.1.9.** Let $\mathcal{C}$ be a category and let σ be an n-simplex of $\mathrm{N}_\bullet(\mathcal{C})$, corresponding to a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \rightarrow \cdots \xrightarrow{f_n} C_n.$$

Then σ is degenerate (Definition 1.1.2.3) if and only if some f_i is an identity morphism of $\mathcal{C}$ (in which case we must have $C_{i-1} = C_i$).

Remark 1.3.1.10. Let I be a set equipped with a partial ordering $\leq_I$. Then we can regard I as a category whose objects are the elements of I, with morphisms given by 002V

$$\mathrm{Hom}_I(i,j) = \begin{cases} * & \text{if } i \leq_I j \\ \emptyset & \text{otherwise.} \end{cases}$$

We will denote the nerve of this category by $\mathrm{N}_\bullet(I)$, and refer to it as the *nerve of the partially ordered set I*. For each $n \geq 0$, we can identify n-simplices of $\mathrm{N}_\bullet(I)$ with monotone functions $[n] \to I$: that is, with nondecreasing sequences $(i_0 \leq_I i_1 \leq_I \cdots \leq_I i_n)$ of elements of I.

Example 1.3.1.11. For each $n \geq 0$, the nerve $\mathrm{N}_\bullet([n])$ can be identified with the standard n-simplex Δ^n of Example 1.1.0.9. 002W

Remark 1.3.1.12. The construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ determines a functor $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$ from the category Cat of (small) categories to the category Set_Δ of simplicial sets. This is a special case of the construction described in Variant 1.2.2.8. More precisely, we can identify $\mathrm{N}_\bullet$ with the functor $\mathrm{Sing}^Q_\bullet$, where $Q : \mathbf{\Delta} \to \mathrm{Cat}$ is the functor which carries each object $[n] \in \mathbf{\Delta}$ to itself, regarded as a category. It follows from Proposition 1.2.3.15 that this functor admits a left adjoint, which we will study in §1.3.6. 002X

1.3.2 Example: Monoids as Simplicial Sets

We now specialize Construction 1.3.1.1 to categories having a single object. 04FY

Definition 1.3.2.1. A *monoid* is a set M equipped with a multiplication map 04FZ

$$m : M \times M \to M \qquad (x, y) \mapsto xy$$

which satisfies the following conditions:

(a) The multiplication m is associative. That is, we have $x(yz) = (xy)z$ for each triple of elements $x, y, z \in M$.

(b) There exists an element $e \in M$ such that $ex = x = xe$ for each $x \in M$ (in this case, the element e is uniquely determined; we refer to it as the *unit element* of M).

Monoids are ubiquitous in mathematics:

Example 1.3.2.2. Let $\mathcal{C}$ be a category and let X be an object of $\mathcal{C}$. An *endomorphism of* X is a morphism from X to itself in the category $\mathcal{C}$. We let $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ denote the set of all endomorphisms of X. The composition law on $\mathcal{C}$ determines a map 00BM

$$\mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(X) \qquad (f, g) \mapsto f \circ g,$$

which exhibits $\mathrm{End}_{\mathcal{C}}(X)$ as a monoid; the unit element of $\mathrm{End}_{\mathcal{C}}(X)$ is the identity morphism $\mathrm{id}_X : X \to X$. We refer to $\mathrm{End}_{\mathcal{C}}(X)$ as the *endomorphism monoid of* X.

The collection of monoids can be organized into a category:

`00BR` **Definition 1.3.2.3.** Let M and M' be monoids having unit elements e and e', respectively. A function $f : M \to M'$ is a *monoid homomorphism* if it satisfies the identities

$$f(e) = e' \qquad f(xy) = f(x)f(y)$$

for every pair of elements $x, y \in M$. We let Mon denote the category whose objects are monoids and whose morphisms are monoid homomorphisms.

`00BS` **Remark 1.3.2.4.** The construction $\mathcal{C} \mapsto \mathrm{End}_{\mathcal{C}}(X)$ of Example 1.3.2.2 induces an equivalence

$$\{\text{Categories } \mathcal{C} \text{ with } \mathrm{Ob}(\mathcal{C}) = \{X\}\} \xrightarrow{\sim} \{\text{Monoids}\}.$$

More precisely, there is a pullback diagram of categories

$$\begin{tikzcd}
\mathrm{Mon} \arrow[r, "M \mapsto BM"] \arrow[d] & \mathrm{Cat} \arrow[d, "\mathrm{Ob}"] \\
\{*\} \arrow[r] & \mathrm{Set},
\end{tikzcd}$$

where $* = \{X\}$ is the set having a single element X. Here the upper horizontal functor assigns to each monoid M the category BM of Construction 1.3.2.5, given concretely by

$$\mathrm{Ob}(BM) = \{X\} \qquad \mathrm{Hom}_{BM}(X, X) = M.$$

`04G0` **Construction 1.3.2.5.** Let M be a monoid. We let $B_\bullet M$ denote the nerve of the category BM described in Remark 1.3.2.4. We will refer to $B_\bullet M$ as the *classifying simplicial set* of the monoid M.

`04G1` **Remark 1.3.2.6.** Let M be a monoid with unit element e and let $B_\bullet M$ denote its classifying simplicial set. By definition, n-simplices of the simplicial set $B_\bullet M$ are functors from the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$ to the category BM. Such a functor can be identified with a collection of elements $\{\alpha_{j,i} \in M\}_{0 \leq i \leq j \leq n}$ (where $\alpha_{j,i}$ denotes the image in BM of the unique element of $\mathrm{Hom}_{[n]}(i, j)$) which are required to satisfy the identities

$$\alpha_{i,i} = e \qquad \alpha_{k,i} = \alpha_{k,j}\alpha_{j,i} \text{ for } 0 \leq i \leq j \leq k \leq n.$$

For each $n \geq 0$, the construction

$$\{\alpha_{j,i}\}_{0 \leq i \leq j \leq n} \mapsto (\alpha_{n,n-1}, \alpha_{n-1,n-2}, \cdots, \alpha_{1,0})$$

induces a bijection $B_n M \simeq M^n$. Under the resulting identification, the face and degeneracy operators of $B_\bullet M$ are given concretely by the formulae

$$d^n_i(x_n, x_{n-1}, \ldots, x_1) = \begin{cases} (x_n, x_{n-1}, \ldots, x_2) & \text{if } i = 0 \\ (x_n, \ldots, x_{i+2}, x_{i+1}x_i, x_{i-1}, \ldots, x_1) & \text{if } 0 < i < n \\ (x_{n-1}, x_{n-2}, \ldots, x_1) & \text{if } i = n \end{cases}$$

$$s^n_i(x_n, x_{n-1}, \ldots, x_1) = (x_n, \ldots, x_{i+1}, e, x_i, \ldots, x_1)$$

(see Remarks 1.3.1.7 and 1.3.1.8).

Proposition 1.3.2.7. *The construction $M \mapsto B_\bullet M$ determines a fully faithful embedding* $\mathrm{Mon} \hookrightarrow \mathrm{Set}_\Delta$. *The essential image of this functor consists of those simplicial sets $S_\bullet$ which satisfy the following condition for each $n \geq 0$:* 04G2

($*_n$) *For $1 \leq i \leq n$, let $\rho_i : S_n \to S_1$ denote the map associated to the inclusion of linearly ordered sets* $[1] \simeq \{i-1, i\} \hookrightarrow [n]$. *Then the maps $\{\rho_i\}_{1 \leq i \leq n}$ determine a bijection* $S_n \to \prod_{1 \leq i \leq n} S_1$.

We will give the proof of Proposition 1.3.2.7 at the end of this section. As a first step, we establish a simpler result in the setting of semisimplicial sets.

Variant 1.3.2.8. A *nonunital monoid* is a set M equipped with a map 04G3

$$m : M \times M \to M \qquad (x, y) \mapsto xy$$

which satisfies the associative law $x(yz) = (xy)z$ for $x, y, z \in M$. If M and M' are nonunital monoids, a function $f : M \to M'$ is a *nonunital monoid homomorphism* if it satisfies the equation $f(xy) = f(x)f(y)$ for every pair of elements $x, y \in M$. We let $\mathrm{Mon}^{\mathrm{nu}}$ denote the category whose objects are nonunital monoids and whose morphisms are nonunital monoid homomorphisms.

Warning 1.3.2.9. The terminology of Variant 1.3.2.8 is not standard. Many authors use the term *semigroup* for what we call a *nonunital monoid.* 00BQ

Remark 1.3.2.10. The category Mon of monoids (Definition 1.3.2.1) can be regarded as a subcategory of the category $\mathrm{Mon}^{\mathrm{nu}}$ of nonunital monoids (Variant 1.3.2.8). Beware that this subcategory is not full. If M and M' are monoids containing unit elements e and e', respectively, then a nonunital monoid homomorphism $f : M \to M'$ need not satisfy the identity $f(e) = e'$. 04G4

Remark 1.3.2.11. Let M be a nonunital monoid, and let $M^+ = M \cup \{e\}$ be the enlargement of M obtained by formally adjoining a new element e. Then the multiplication on M extends 04G5 uniquely to a monoid structure on M^+ having unit element e. Moreover, if M' is any other monoid, then the restriction map $f \mapsto f|_M$ induces a bijection

$$\begin{tikzcd}
\{\text{Monoid homomorphisms } f : M^+ \to M'\} \arrow[d] \\
\{\text{Nonunital monoid homomorphisms } f_0 : M \to M'\}.
\end{tikzcd}$$

Consequently, the inclusion functor $\mathrm{Mon} \hookrightarrow \mathrm{Mon}^{\mathrm{nu}}$ has a left adjoint, given on objects by the construction $M \mapsto M^+$.

04G6 **Variant 1.3.2.12.** Let M be a nonunital monoid. We let $B_\bullet M$ denote the semisimplicial set which assigns to each object $[n] \in \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}$ the collection of tuples $\{\alpha_{j,i} \in M\}_{0 \leq i < j \leq n}$ which satisfy the identity $\alpha_{k,i} = \alpha_{k,j}\alpha_{j,i}$ for $0 \leq i < j < k \leq n$. As in Remark 1.3.2.6, the construction

$$\{\alpha_{j,i}\}_{0 \leq i < j \leq n} \mapsto (\alpha_{n,n-1}, \alpha_{n-1,n-2}, \cdots, \alpha_{1,0})$$

induces an identification $B_n M \simeq M^n$. Under this identification, the face operators of $B_\bullet M$ are given by the formula

$$d^n_i(x_n, x_{n-1}, \ldots, x_1) = \begin{cases} (x_n, x_{n-1}, \ldots, x_2) & \text{if } i = 0 \\ (x_n, \ldots, x_{i+2}, x_{i+1}x_i, x_{i-1}, \ldots, x_1) & \text{if } 0 < i < n \\ (x_{n-1}, x_{n-2}, \ldots, x_1) & \text{if } i = n. \end{cases}$$

04G7 **Remark 1.3.2.13.** Construction 1.3.2.5 and Variant 1.3.2.12 are compatible: if M is a monoid and $B_\bullet M$ is the classifying simplicial set of Construction 1.3.2.5, then the underlying semisimplicial set of $B_\bullet M$ is given by Variant 1.3.2.12.

Proposition 1.3.2.7 has the following nonunital counterpart:

04G8 **Proposition 1.3.2.14.** *The construction $M \mapsto B_\bullet M$ determines a fully faithful functor from the category* $\mathrm{Mon}^{\mathrm{nu}}$ *of nonunital monoids to the category of semisimplicial sets. The essential image of this functor consists of those semisimplicial sets which satisfy condition $(*_n)$ of Proposition 1.3.2.7, for each $n \geq 0$.*

Proof. We first show that the functor $M \mapsto B_\bullet M$ is fully faithful. Fix a pair of nonunital monoids M and M', and let $f_\bullet : B_\bullet M \to B_\bullet M'$ be a morphism of semisimplicial sets. We wish to show that there is a unique nonunital monoid homomorphism $g : M \to M'$ such that $f_\bullet$ can be recovered by applying the functor $B_\bullet$ to g. Let us abuse notation by identifying M and M' with the sets $B_1 M$ and $B_1 M'$, respectively, so that $f_\bullet$ determines a function $f_1 : M \to M'$. The uniqueness of g is now clear: if $f_\bullet = B_\bullet g$, then g must coincide with f_1 (as a function). To prove existence, we must establish the following:

(1) The function $f_1 : M \to M'$ is a nonunital monoid homomorphism.

(2) The morphism of semisimplicial sets $f_\bullet$ is obtained by applying the functor $B_\bullet$ to the homomorphism f_1.

We first prove (1). Fix a pair of elements $x, y \in M$ and regard the pair (x, y) as a 2-simplex σ of the semisimplicial set $B_\bullet M$. Since $f_\bullet$ is a morphism of semisimplicial sets, we have

$$f_1(xy) = f_1(d^2_1(\sigma)) = d^2_1(f_2(\sigma)) = f_1(x)f_1(y).$$

Assertion (1) now follows by allowing x and y to vary. To prove (2), let $f'_\bullet : B_\bullet M \to B_\bullet M'$ be the morphism of semisimplicial sets determined by the homomorphism f_1, and let τ be an n-simplex of $B_\bullet M$; we wish to show that $f_n(\tau) = f'_n(\tau)$. Since τ is determined by its 1-dimensional faces, we can assume without loss of generality that $n = 1$, in which case the result is clear. This completes the proof that the functor $M \mapsto B_\bullet M$ is fully faithful.

Now suppose that $S_\bullet$ is a semisimplicial set which satisfies condition $(*_n)$ of Proposition 1.3.2.7 for every integer $n \geq 0$, and set $M = S_1$. For every n-tuple of elements $(x_n, x_{n-1}, \cdots, x_1)$ of M, condition $(*_n)$ guarantees that there is a unique n-simplex $\sigma_{x_n,\cdots,x_1}$ of $S_\bullet$ satisfying $\rho_i(\sigma) = x_i$, where $\rho_i : S_n \to S_1 = M$ is the function induced by the inclusion map $[1] \simeq \{i-1 < i\} \hookrightarrow [n]$. We can then define a multiplication $m : M \times M \to M$ by the formula $m(x, y) = d^2_1(\sigma_{x,y})$. This multiplication is associative: for every triple of elements $x, y, z \in M$, we compute

$$\begin{aligned}
m(m(x,y),z) &= m(d^2_1(\sigma_{x,y}), z) \\
&= d^2_1(\sigma_{d^2_1(\sigma_{x,y}),z}) \\
&= d^2_1(d^3_2(\sigma_{x,y,z})) \\
&= d^2_1(d^3_1(\sigma_{x,y,z})) \\
&= d^2_1(\sigma_{x,d^2_1(\sigma_{y,z})}) \\
&= m(x, d^2_1(\sigma_{y,z})) \\
&= m(x, m(y,z)).
\end{aligned}$$

It follows that we can regard M as a nonunital commutative monoid. Moreover, for every integer $n \geq 0$, the construction $(x_n, \cdots, x_1) \mapsto \sigma_{x_n,\cdots,x_1}$ determines a bijection $f_n : B_n M \to S_n$. We will complete the proof by showing that the collection $\{f_n\}_{n \geq 0}$ is an isomorphism of semisimplicial sets: that is, that it commutes with the face operators. Fix an integer $n > 0$ and an n-simplex τ of $B_\bullet M$; we wish to show that $d^n_i(f_n(\tau)) = f_{n-1}(d^n_i(\tau))$ for $0 \leq i \leq n$. Let us identify τ with a tuple of elements $(x_n, x_n, \cdots, x_1)$ of M; we wish to verify the identity

$$d_i^n(\sigma_{x_n,x_{n-1},\cdots,x_1}) = \begin{cases} \sigma_{x_n,x_{n-1},\cdots,x_2} & \text{if } i = 0 \\ \sigma_{x_n,\cdots,x_{i+2},m(x_{i+1},x_i),x_{i-1},\cdots,x_1} & \text{if } 0 < i < n \\ \sigma_{x_{n-1},\cdots,x_1} & \text{if } i = n. \end{cases}$$

For $1 \leq j \leq n-1$, let $\rho_j : S_{n-1} \to S_1 = M$ be defined as above; we can then rewrite the preceding identity as

$$\rho_j(d_i^n(\sigma_{x_n,x_{n-1},\cdots,x_1})) = \begin{cases} x_j & \text{if } j < i \\ m(x_{j+1}, x_j) & \text{if } j = i \\ x_{j+1} & \text{if } j > i. \end{cases}$$

This follows immediately from the definition of the simplex $\sigma_{x_n,x_{n-1},\cdots,x_1}$ in the case $j \neq i$, and from the construction of the multiplication m in the case $j = i$. □

Proof of Proposition 1.3.2.7. We first show that Construction 1.3.2.5 is fully faithful. Fix monoids M and M' and let $f_\bullet : B_\bullet M \to B_\bullet M'$ be a morphism of simplicial sets. Applying Proposition 1.3.2.14 (together with Remark 1.3.2.13), we deduce that there is a unique nonunital monoid homomorphism $g : M \to M'$ such that $f_\bullet$ coincides with $B_\bullet g$ (as a morphism of semisimplicial sets). Since $f_\bullet$ is a morphism of simplicial sets, it carries the (unique) degenerate edge of $B_\bullet M$ to the (unique) degenerate edge of $B_\bullet M'$. It follows that g carries the unit element of M to the unit element of M': that is, it is a monoid homomorphism.

Now suppose that $S_\bullet$ is a simplicial set satisfying condition $(*_n)$ for each $n \geq 0$. Applying Proposition 1.3.2.14, we deduce that there is a nonunital monoid M and an isomorphism of semisimplicial sets $f_\bullet : B_\bullet M \to S_\bullet$, which carries each n-tuple $(x_n, \cdots, x_1) \in M$ to the n-simplex $\sigma_{x_n,\cdots,x_1}$ of $S_\bullet$ appearing in the proof of Proposition 1.3.2.14. Let $e \in M$ be the element corresponding to the unique degenerate 1-simplex of $S_\bullet$. For $0 \leq i \leq n$, the degeneracy operator $s_i^n : S_n \to S_{n+1}$ satisfies the identity

04G9
$$s_i^n(\sigma_{x_n,\cdots,x_1}) = \sigma_{x_n,\cdots,x_{i+1},e,x_i,\cdots,x_1}. \tag{1.5}$$

Specializing to the case $i = n = 1$ and applying the face operator d_1^1, we obtain an equality

$$\begin{aligned} \sigma_x &= d_1^2(s_1^1(\sigma_x)) \\ &= d_1^2(\sigma_{e,x}) \\ &= \sigma_{ex}; \end{aligned}$$

that is, e is a left unit with respect to the multiplication on M. A similar argument shows that e is a right unit with respect to the multiplication on M: that is, M is a monoid with unit element e. To complete the proof, it will suffice to show that $f_\bullet : B_\bullet M \to S_\bullet$ is an isomorphism of simplicial sets: that is, it commutes with degeneracy operators as well as face operators. This is a restatement of the identity (1.5). □

1.3.3 Recovering a Category from its Nerve

Passage from a category $\mathcal{C}$ to the nerve $\mathrm{N}_\bullet(\mathcal{C})$ does not lose any information: 002Y

Proposition 1.3.3.1. *The nerve functor* $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$ *is fully faithful.* 002Z

Throughout this book, we will often abuse terminology by identifying a category $\mathcal{C}$ with its nerve $\mathrm{N}_\bullet(\mathcal{C})$. By virtue of Proposition 1.3.3.1, this is essentially harmless: the nerve construction allows us to identify categories with certain kinds of simplicial sets.

Remark 1.3.3.2. If we restrict our attention to categories having a single object, Proposition 1.3.3.1 follows from Proposition 1.3.2.7 (see Remark 1.3.2.4). 04GA

Proof of Proposition 1.3.3.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. We wish to show that the nerve functor $\mathrm{N}_\bullet$ induces a bijection

$$\theta : \mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{C}') \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{C}')).$$

Here the source of θ is the *set* of all functors from $\mathcal{C}$ to $\mathcal{C}'$. We first note that θ is injective: a functor $F : \mathcal{C} \to \mathcal{C}'$ is determined by its behavior on the objects and morphisms of $\mathcal{C}$, and therefore by the behavior of $\theta(F)$ on the vertices and edges of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ (see Example 1.3.1.4). Let us prove the surjectivity of θ. Let $f : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C}')$ be a morphism of simplicial sets; we wish to show that there exists a functor $F : \mathcal{C} \to \mathcal{C}'$ such that $f = \theta(F)$. For each $n \geq 0$, the morphism f determines a map of sets $\mathrm{N}_n(\mathcal{C}) \to \mathrm{N}_n(\mathcal{C}')$, which we will also denote by f. In the case $n = 0$, this map carries each object $C \in \mathcal{C}$ to an object of $\mathcal{C}'$, which we will denote by $F(C)$. For every pair of objects $C, D \in \mathcal{C}$, the map f carries each morphism $u : C \to D$ to a morphism $f(u)$ in the category $\mathcal{C}'$. Since f commutes with face operators, the morphism $f(u)$ has source $F(C)$ and target $F(D)$ (see Example 1.3.1.4), and can therefore be regarded as an element of $\mathrm{Hom}_{\mathcal{C}'}(F(C), F(D))$; we denote this element by $F(u)$. We will complete the proof by verifying the following:

(a) The preceding construction determines a functor $F : \mathcal{C} \to \mathcal{C}'$.

(b) We have an equality $f = \theta(F)$ of maps from $\mathrm{N}_\bullet(\mathcal{C})$ to $\mathrm{N}_\bullet(\mathcal{C}')$.

To prove (a), we first note that the compatibility of f with degeneracy operators implies that we have $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$ for each $C \in \mathcal{C}$ (see Example 1.3.1.4). It will therefore suffice to show that for every pair of composable morphisms $u : C \to D$ and $v : D \to E$ in the category $\mathcal{C}$, we have $F(v) \circ F(u) = F(v \circ u)$ as elements of the set $\mathrm{Hom}_{\mathcal{C}'}(F(C), F(E))$. For this, we observe that the diagram $C \xrightarrow{u} D \xrightarrow{v} E$ can be identified with a 2-simplex σ of $\mathrm{N}_{\bullet}(\mathcal{C})$. Using the equality $d^2_i(f(\sigma)) = f(d^2_i(\sigma))$ for $i = 0, 2$, we see that $f(\sigma)$ corresponds to the diagram $F(C) \xrightarrow{F(u)} F(D) \xrightarrow{F(v)} F(E)$ in $\mathcal{C}'$. We now compute

$$F(v) \circ F(u) = d^2_1(f(\sigma)) = f(d^2_1(\sigma)) = F(v \circ u).$$

This completes the proof of (a). To prove (b), we must show that $f(\tau) = \theta(F)(\tau)$ for each n-simplex τ of $\mathrm{N}_{\bullet}(\mathcal{C})$. This follows by construction in the case $n \leq 1$, and follows in general since an n-simplex of $\mathrm{N}_{\bullet}(\mathcal{C}')$ is determined by its 1-dimensional faces (see Remark 1.3.1.3). □

1.3.4 Characterization of Nerves

0030 We now describe the essential image of the functor $\mathrm{N}_{\bullet} : \mathrm{Cat} \to \mathrm{Set}_{\Delta}$.

0031 **Proposition 1.3.4.1.** *Let S be a simplicial set. Then S is isomorphic to the nerve of a category if and only if it satisfies the following condition:*

$(*')$ *For every pair of integers $0 < i < n$ and every map of simplicial sets $\sigma_0 : \Lambda^n_i \to S$, there exists a unique map $\sigma : \Delta^n \to S$ such that $\sigma_0 = \sigma|_{\Lambda^n_i}$.*

The proof of Proposition 1.3.4.1 will require some preliminaries. We begin by establishing the necessity of condition $(*')$.

0032 **Lemma 1.3.4.2.** *Let $\mathcal{C}$ be a category. Then the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C})$ satisfies condition $(*')$ of Proposition 1.3.4.1.*

Proof. Choose integers $0 < i < n$ together with a map of simplicial sets $\sigma_0 : \Lambda^n_i \to \mathrm{N}_{\bullet}(\mathcal{C})$; we wish to show that σ_0 can be extended uniquely to a n-simplex of $\mathrm{N}_{\bullet}(\mathcal{C})$. For $0 \leq j \leq n$, let $C_j \in \mathcal{C}$ denote the image under σ_0 of the jth vertex of Δ^n (which belongs to the horn Λ^n_i). We first consider the case where $n \geq 3$. In this case, Λ^n_i contains every edge of Δ^n. For $0 \leq j \leq k \leq n$, let $f_{k,j} : C_j \to C_k$ denote the 1-simplex of $\mathrm{N}_{\bullet}(\mathcal{C})$ obtained by evaluating σ_0 on the edge of Δ^n corresponding to the pair (j, k). We claim that the construction

$$j \mapsto C_j \qquad (j \leq k) \mapsto f_{k,j}$$

determines a functor $[n] \to \mathcal{C}$, which we can then identify with an n-simplex of $\mathrm{N}_{\bullet}(\mathcal{C})$ having the desired properties. It is easy to see that $f_{j,j} = \mathrm{id}_{C_j}$ for each $0 \leq j \leq n$, so it will suffice to show that $f_{\ell,k} \circ f_{k,j} = f_{\ell,j}$ for every triple $0 \leq j \leq k \leq \ell \leq n$. The triple (j, k, ℓ) determines a 2-simplex τ of Δ^n. If τ is contained in Λ^n_i, then $\tau' = \sigma_0(\tau)$ is a 2-simplex of $\mathrm{N}_{\bullet}(\mathcal{C})$ satisfying $d^2_0(\tau') = f_{\ell,k}$, $d^2_1(\tau') = f_{\ell,j}$, and $d^2_2(\tau') = f_{k,j}$, so that τ' "witnesses" the identity $f_{\ell,k} \circ f_{k,j} = f_{\ell,j}$. It will therefore suffice to treat the case where the simplex τ does *not* belong to the Λ^n_i. In this case, our assumption that $n \geq 3$ guarantees that we must have $\{j, k, \ell\} = [n] \setminus \{i\}$. It follows that $n = 3$, so that either $i = 1$ or $i = 2$. We will treat the case $i = 1$ (the case $i = 2$ follows by a similar argument). Note that Λ^3_1 contains all of the nondegenerate 2-simplices of Δ^3 other than τ; applying the map σ_0, we obtain 2-simplices of $\mathrm{N}_\bullet(\mathcal{C})$ which witness the identities

$$f_{3,0} = f_{3,1} \circ f_{1,0} \qquad f_{3,1} = f_{3,2} \circ f_{2,1} \qquad f_{2,0} = f_{2,1} \circ f_{1,0}.$$

We now compute

$$f_{3,0} = f_{3,1} \circ f_{1,0} = (f_{3,2} \circ f_{2,1}) \circ f_{1,0} = f_{3,2} \circ (f_{2,1} \circ f_{1,0}) = f_{3,2} \circ f_{2,0}$$

so that $f_{\ell,j} = f_{\ell,k} \circ f_{k,j}$, as desired.

It remains to treat the case $n = 2$. In this case, the inequality $0 < i < n$ guarantees that $i = 1$. The morphism $\sigma_0 : \Lambda^n_i \to \mathrm{N}_\bullet(\mathcal{C})$ can then be identified with a pair of composable morphisms $f_{1,0} : C_0 \to C_1$ and $f_{2,1} : C_1 \to C_2$ in the category $\mathcal{C}$. This data extends uniquely to a 2-simplex σ of $\mathcal{C}$ satisfying $d^2_1(\sigma) = f_{2,1} \circ f_{1,0}$ (see Remark 1.3.1.3). □

Lemma 1.3.4.3. *Let $f : S \to T$ be a morphism of simplicial sets which is bijective on both vertices and edges. If both S and T satisfy condition $(*')$ of Proposition 1.3.4.1, then f is an isomorphism.* 0033

Proof. We claim that, for every simplicial set K, composition with f induces a bijection

$$\theta_K : \mathrm{Hom}_{\mathrm{Set}_\Delta}(K, S) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(K, T).$$

Writing K as a union of its skeleta $\mathrm{sk}_n(K)$, we can reduce to the case where K has dimension $\leq n$, for some integer $n \geq -1$ (see Definition 1.1.3.1). We now proceed by induction on n. The case $n = -1$ is trivial (since a simplicial set of dimension ≤ -1 is empty). Let us therefore assume that $n \geq 0$, so that Proposition 1.1.4.12 supplies a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod \partial\Delta^n \arrow[r] \arrow[d] & \coprod \Delta^n \arrow[d] \\
\mathrm{sk}_{n-1}(K) \arrow[r] & K.
\end{tikzcd}$$

It follows from our inductive hypothesis that the maps $\theta_{\partial\Delta^n}$ and $\theta_{\mathrm{sk}_{n-1}(K)}$ are bijective. Consequently, to show that θ_K is bijective, it will suffice to show that θ_{Δ^n} is bijective: that is, that f induces a bijection on n-simplices. For $n \leq 1$, this follows from our hypothesis. To handle the case $n \geq 2$, we observe that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S) \arrow[r] \arrow[d, "\theta_{\Delta^n}"] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_1, S) \arrow[d, "\theta_{\Lambda^n_1}"] \\
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, T) \arrow[r] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_1, T).
\end{tikzcd}$$

Here the right vertical map is bijective by virtue of our inductive hypothesis, and the horizontal maps are bijective by virtue of our assumption that both S and T satisfy condition $(*')$. It follows that the left vertical map is also bijective, as desired. ☐

Proof of Proposition 1.3.4.1. Let S be a simplicial set satisfying condition $(*')$ of Proposition 1.3.4.1; we will show that there is a category $\mathcal{C}$ and an isomorphism of simplicial sets $u : S \to \mathrm{N}_\bullet(\mathcal{C})$ (the converse follows from Lemma 1.3.4.2). It follows from Proposition 1.3.3.1 that the category $\mathcal{C}$ is uniquely determined (up to isomorphism), and from the proof of Proposition 1.3.3.1 we can extract an explicit construction of $\mathcal{C}$:

- The objects of $\mathcal{C}$ are the vertices of S.
- Given a pair of objects $C, D \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathcal{C}}(C, D)$ denote the collection of edges e of S having source C and target D.
- For each object $C \in \mathcal{C}$, we define the identity morphism $\mathrm{id}_C \in \mathrm{Hom}_{\mathcal{C}}(C, C)$ to be the degenerate edge $s^0_0(C)$.
- Given a triple of objects $C, D, E \in \mathcal{C}$ and a pair of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(D, E)$, we can apply hypothesis $(*')$ (in the special case $n = 2$ and $i = 1$) to conclude that there is a unique 2-simplex σ of $S_\bullet$ satisfying $d^2_2(\sigma) = f$ and $d^2_0(\sigma) = g$. We define the composition $g \circ f \in \mathrm{Hom}_{\mathcal{C}}(C, E)$ to be the edge $d^2_1(\sigma)$.

We claim that $\mathcal{C}$ is a category. For this, we must check the following:

- The composition law on $\mathcal{C}$ is unital: for every morphism $f : C \to D$ in $\mathcal{C}$, we have equalities
$$\mathrm{id}_D \circ f = f = f \circ \mathrm{id}_C .$$
  Let us verify the identity on the left; the proof in the other case is similar. For this, we must construct a 2-simplex σ of S such that $d^2_0(\sigma) = \mathrm{id}_D$ and $d^2_1(\sigma) = d^2_2(\sigma) = f$. The degenerate 2-simplex $s^1_1(f)$ has these properties.

- The composition law on $\mathcal{C}$ is associative. That is, for every triple of composable morphisms
$$f : W \to X \qquad g : X \to Y \qquad h : Y \to Z$$
in $\mathcal{C}$, we have an identity $h \circ (g \circ f) = (h \circ g) \circ f$ in $\mathcal{C}$. Applying condition $(*')$ repeatedly, we deduce the following:
  - There is a unique 2-simplex σ_0 of S satisfying $d^2_0(\sigma_0) = h$ and $d^2_2(\sigma_0) = g$ (it follows that $d^2_1(\sigma_0) = h \circ g$).
  - There is a unique 2-simplex σ_3 of S satisfying $d^2_0(\sigma_3) = g$ and $d^2_2(\sigma_3) = f$ (it follows that $d^2_1(\sigma_3) = g \circ f$).
  - There is a unique 2-simplex σ_2 of S satisfying $d^2_0(\sigma_2) = h \circ g$ and $d^2_2(\sigma_2) = f$ (it follows that $d^2_1(\sigma_2) = (h \circ g) \circ f$).
  - There is a unique 3-simplex τ of S satisfying $d^3_0(\tau) = \sigma_0$, $d^3_2(\tau) = \sigma_2$, and $d^3_3(\tau) = \sigma_3$ (this follows by applying $(*')$ to the horn inclusion $\Lambda^3_1 \hookrightarrow \Delta^3$).

  The 3-simplex τ can be depicted in the following diagram
$$\begin{tikzcd}
 & X \arrow[r, "g"] \arrow[drr, "h\circ g"] & Y \arrow[dr, "h"] & \\
W \arrow[ur, "f"] \arrow[urr, "g\circ f"] \arrow[rrr, "(h\circ g)\circ f"] & & & Z.
\end{tikzcd}$$
  Set $\sigma_1 = d^3_1(\tau)$. Then σ_1 is a 2-simplex of S satisfying $d^2_0(\sigma_1) = h$, $d^2_1(\sigma_1) = (h \circ g) \circ f$, and $d^2_2(\sigma_1) = g \circ f$. It follows that σ_1 "witnesses" the identity $h \circ (g \circ f) = (h \circ g) \circ f$.

Note that every n-simplex $\sigma : \Delta^n \to S$ determines a functor $[n] \to \mathcal{C}$, given on objects by the values of σ on the vertices of Δ^n and on morphisms by the values of σ on the edges of Δ^n. This construction determines a map of simplicial sets $u : S \to \mathrm{N}_\bullet(\mathcal{C})$ which is bijective on simplices of dimension ≤ 1. Since the simplicial sets S and $\mathrm{N}_\bullet(\mathcal{C})$ both satisfy condition $(*')$ (Lemma 1.3.4.2), it follows from Lemma 1.3.4.3 that u is an isomorphism. □

Remark 1.3.4.4. The characterization of Proposition 1.3.4.1 has many variants. For example, one can replace condition $(*')$ by the following *a priori* weaker condition: 0034

$(*'_0)$ For every $n \geq 2$ and every morphism of simplicial sets $\sigma_0 : \Lambda^n_1 \to S$, there is a unique n-simplex $\sigma : \Delta^n \to S_\bullet$ satisfying $\sigma_0 = \sigma|_{\Lambda^n_1}$.

See Corollary 1.5.7.9 for a closely related result.

1.3.5 The Nerve of a Groupoid

0035 According to Proposition 1.3.3.1, every category $\mathcal{C}$ can be recovered, up to canonical isomorphism, from the nerve $\mathrm{N}_\bullet(\mathcal{C})$. In particular, any isomorphism-invariant condition on a category $\mathcal{C}$ can be reformulated as a condition on the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. We now illustrate this principle with a simple example.

0036 **Definition 1.3.5.1.** Let $\mathcal{C}$ be a category. Recall that a morphism $f : C \to D$ in $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g : D \to C$ satisfying the identities

$$f \circ g = \mathrm{id}_D \qquad g \circ f = \mathrm{id}_C .$$

In this case, the morphism g is uniquely determined and we write $g = f^{-1}$. We say that $\mathcal{C}$ is a *groupoid* if every morphism in $\mathcal{C}$ is an isomorphism.

0037 **Proposition 1.3.5.2.** *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is a groupoid (Definition 1.3.5.1) if and only if the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ is a Kan complex (Definition 1.2.5.1).*

0038 **Example 1.3.5.3.** Let G be a group. Then the category BG of Remark 1.3.2.4 is a groupoid. It follows from Proposition 1.3.5.2 that the simplicial set $B_\bullet G$ of Construction 1.3.2.5 is a Kan complex. The geometric realization $|B_\bullet G|$ is a topological space called the *classifying space* of G. It can be characterized (up to homotopy equivalence) by the fact that it is a CW complex with either of the following properties:

- The space $|B_\bullet G|$ is connected, and its homotopy groups (with respect to any choice of base point) are given by the formula

$$\pi_*(|B_\bullet G|) \simeq \begin{cases} G & \text{if } * = 1 \\ 0 & \text{if } * > 1. \end{cases}$$

- For any paracompact topological space X, there is a canonical bijection

$$\{\text{Continuous maps } f : X \to |B_\bullet G|\}/\text{homotopy} \simeq \{G\text{-torsors } P \to X\}/\text{isomorphism}.$$

We refer the reader to [43] for a more detailed discussion (including an extension to the setting of topological groups).

Proof of Proposition 1.3.5.2. Suppose first that $\mathrm{N}_\bullet(\mathcal{C})$ is a Kan complex; we wish to show that $\mathcal{C}$ is a groupoid. Let $f : C \to D$ be a morphism in $\mathcal{C}$. Using the surjectivity of the map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^2, \mathrm{N}_\bullet(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^2_2, \mathrm{N}_\bullet(\mathcal{C}))$, we see that there exists a 2-simplex σ of $\mathrm{N}_\bullet(\mathcal{C})$ satisfying $d^2_0(\sigma) = f$ and $d^2_1(\sigma) = \mathrm{id}_D$. Setting $g = d^2_2(\sigma)$, we conclude that $f \circ g = \mathrm{id}_D$: that is, g is a left inverse to f. Similarly, the surjectivity of the map

$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^2, \mathrm{N}_{\bullet}(\mathcal{C})) \rightarrow \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Lambda^2_0, \mathrm{N}_{\bullet}(\mathcal{C}))$ allows us to construct a map $h : D \rightarrow C$ satisfying $h \circ f = \mathrm{id}_C$. The calculation

$$g = \mathrm{id}_C \circ g = (h \circ f) \circ g = h \circ (f \circ g) = h \circ \mathrm{id}_D = h$$

then shows that $g = h$ is an inverse of f, so that f is invertible as desired.

Now suppose that $\mathcal{C}$ is a groupoid. We wish to show that, for $0 \leq i \leq n$, every map $\sigma_0 : \Lambda^n_i \rightarrow \mathrm{N}_{\bullet}(\mathcal{C})$ can be extended to an n-simplex $\sigma : \Delta^n \rightarrow \mathrm{N}_{\bullet}(\mathcal{C})$. For $0 < i < n$, this follows from Lemma 1.3.4.2 (and does not require the assumption that $\mathcal{C}$ is a groupoid). We will treat the case where $i = 0$; the case $i = n$ follows by similar reasoning. We consider several cases:

- In the case $n = 1$, the map $\sigma_0 : \Lambda^n_0 \rightarrow \mathrm{N}_{\bullet}(\mathcal{C})$ can be identified with an object $C \in \mathcal{C}$. In this case, we can take σ to be an edge of $\mathrm{N}_{\bullet}(\mathcal{C})$ corresponding to any morphism with target C (for example, we can take σ to be the identity morphism id_C).
- In the case $n = 2$, we can identify σ_0 with a pair of morphisms in $\mathcal{C}$ having the same source, which we can depict as a diagram

  $$\begin{tikzcd} & D \arrow[dr, dashed] & \\ C \arrow[ur, "f"] \arrow[rr, "g"] & & E. \end{tikzcd}$$

  Our assumption that $\mathcal{C}$ is a groupoid guarantees that we can extend this diagram to a 2-simplex of $\mathcal{C}$, whose 0th face is given by the morphism $g \circ f^{-1} : D \rightarrow E$.
- In the case $n \geq 3$, the map σ_0 determines a collection of objects $\{C_i\}_{0 \leq i \leq n}$ and morphisms $f_{j,i} : C_i \rightarrow C_j$ for $i \leq j$ (as in the proof of Lemma 1.3.4.2). We wish to show that these morphisms determine a functor $[n] \rightarrow \mathcal{C}$ (which we can then identify with an n-simplex σ of $\mathrm{N}_{\bullet}(\mathcal{C})$ satisfying $\sigma|_{\Lambda^n_0} = \sigma_0$). For this, we must verify the identity $f_{k,j} \circ f_{j,i} = f_{k,i}$ for $0 \leq i \leq j \leq k \leq n$. Note that this identity is satisfied whenever the triple $(i \leq j \leq k)$ determines a 2-simplex of Δ^n belonging to the horn Λ^n_0. This is automatic unless $n = 3$ and $(i, j, k) = (1, 2, 3)$. To handle this exceptional case, we compute

  $$\begin{aligned} (f_{3,2} \circ f_{2,1}) \circ f_{1,0} &= f_{3,2} \circ (f_{2,1} \circ f_{1,0}) \\ &= f_{3,2} \circ f_{2,0} \\ &= f_{3,0} \\ &= f_{3,1} \circ f_{1,0}. \end{aligned}$$

Since $\mathcal{C}$ is a groupoid, composing with $f_{1,0}^{-1}$ on the right yields the desired identity $f_{3,2} \circ f_{2,1} = f_{3,1}$.

□

We close this section by introducing some notation which will be useful later.

007G **Construction 1.3.5.4.** Let $\mathcal{C}$ be a category. We define a subcategory $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ as follows:

- Every object of $\mathcal{C}$ belongs to $\mathcal{C}^{\simeq}$.
- A morphism $f : X \to Y$ of $\mathcal{C}$ belongs to $\mathcal{C}^{\simeq}$ if and only if f is an isomorphism.

We will refer to $\mathcal{C}^{\simeq}$ as the *core* of $\mathcal{C}$.

007H **Remark 1.3.5.5.** Let $\mathcal{C}$ be a category. The core $\mathcal{C}^{\simeq}$ is determined (up to isomorphism) by the following properties:

- The category $\mathcal{C}^{\simeq}$ is a groupoid.
- If $\mathcal{D}$ is a groupoid, then every functor $F : \mathcal{D} \to \mathcal{C}$ factors (uniquely) through $\mathcal{C}^{\simeq}$.

1.3.6 The Homotopy Category of a Simplicial Set

00HE We now show that the functor $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ of Construction 1.3.1.1 admits a left adjoint (Corollary 1.3.6.5).

004J **Definition 1.3.6.1.** Let $\mathcal{C}$ be a category. We will say that a map of simplicial sets $u : S \to \mathrm{N}_\bullet(\mathcal{C})$ *exhibits $\mathcal{C}$ as the homotopy category of S* if, for every category $\mathcal{D}$, the composite map

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_\Delta}(S, \mathrm{N}_\bullet(\mathcal{D}))$$

is bijective (note that the map on the left is always bijective, by virtue of Proposition 1.3.3.1).

00HF **Exercise 1.3.6.2.** Let X be a topological space and let $\pi_{\leq 1}(X)$ denote its fundamental groupoid. Show that there is a unique map of simplicial sets $u : \mathrm{Sing}_\bullet(X) \to \mathrm{N}_\bullet(\pi_{\leq 1}(X))$ with the following properties:

- On 0-simplices, u carries each point $x \in X$ (regarded as a vertex of $\mathrm{Sing}_\bullet(X)$) to itself (regarded as an object of $\pi_{\leq 1}(X)$).
- On 1-simplices, u carries each path $p : [0,1] \to X$ (regarded as an edge of $\mathrm{Sing}_\bullet(X)$) to its homotopy class $[p]$ (regarded as a morphism of the category $\pi_{\leq 1}(X)$).

Moreover, u exhibits the fundamental groupoid $\pi_{\leq 1}(X)$ as a homotopy category of the singular simplicial set $\mathrm{Sing}_\bullet(X)$. For a generalization, see Proposition 1.4.5.7.

Notation 1.3.6.3. Let S be a simplicial set. It follows immediately from the definition that if there exists a category $\mathcal{C}$ and a morphism $u : S \to \mathrm{N}_\bullet(\mathcal{C})$ which exhibits $\mathcal{C}$ as a homotopy category of S, then the category $\mathcal{C}$ is unique up to isomorphism and depends functorially on S. To emphasize this dependence, we will refer to $\mathcal{C}$ as *the* homotopy category of S and denote it by $\mathrm{h}S$. 004K

Proposition 1.3.6.4. *Let $S = S_\bullet$ be a simplicial set. Then there exists a category $\mathcal{C}$ and a map of simplicial sets $u : S \to \mathrm{N}_\bullet(\mathcal{C})$ which exhibits $\mathcal{C}$ as a homotopy category of S.* 004M

Proof. Let $Q^\bullet$ denote the cosimplicial object of Cat given by the inclusion $\mathbf{\Delta} \hookrightarrow \mathrm{Cat}$. Unwinding the definitions, we see that a homotopy category of S can be identified with a realization $|S|^Q$, whose existence is a special case of Proposition 1.2.3.15. Alternatively, we can give a direct construction of the homotopy category $\mathrm{h}S$:

- The objects of $\mathrm{h}S$ are the vertices of S.
- Every edge e of S determines a morphism $[e]$ in $\mathrm{h}S$, whose source is the vertex $d^1_1(e)$ and whose target is the vertex $d^1_0(e)$.
- The collection of morphisms in $\mathrm{h}S$ is generated under composition by morphisms of the form $[e]$, subject only to the relations

$$[s^0_0(x)] = \mathrm{id}_x \text{ for } x \in S_0 \qquad [d^2_1(\sigma)] = [d^2_0(\sigma)] \circ [d^2_2(\sigma)] \text{ for } \sigma \in S_2.$$

□

Corollary 1.3.6.5. *The nerve functor $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$ admits a left adjoint, given on objects by the construction $S \mapsto \mathrm{h}S$.* 004N

Remark 1.3.6.6. Let $\mathcal{C}$ be a category. Then the counit of the adjunction described in Corollary 1.3.6.5 induces an isomorphism of categories $\mathrm{hN}_\bullet(\mathcal{C}) \xrightarrow{\sim} \mathcal{C}$ (this is a restatement of Proposition 1.3.3.1). In other words, every category $\mathcal{C}$ can be recovered as the homotopy category of its nerve $\mathrm{N}_\bullet(\mathcal{C})$. 00HG

Warning 1.3.6.7. Let S be a simplicial set. The proof of Proposition 1.3.6.4 gives a construction of the homotopy category $\mathrm{h}S$ by generators and relations. The result of this construction is not always easy to describe. If x and y are vertices of S, then every morphism from x to y in $\mathrm{h}S$ can be represented by a composition 00HH

$$[e_n] \circ [e_{n-1}] \circ \cdots \circ [e_1],$$

where $\{e_i\}_{0 \leq i \leq n}$ is a sequence of edges satisfying

$$d^1_1(e_1) = x \qquad d^1_0(e_i) = d^1_1(e_{i+1}) \qquad d^1_0(e_n) = y.$$

In general, it can be difficult to determine whether or not two such compositions represent the same morphism of $\mathrm{h}S$ (even for finite simplicial sets, this question is algorithmically undecidable). However, there are two situations in which the homotopy category $\mathrm{h}S$ admits a simpler description:

- Let S be a simplicial set of dimension ≤ 1, which we can identify with a directed graph G (Proposition 1.1.6.9). In this case, the homotopy category $\mathrm{h}S$ is generated *freely* by the vertices and edges of the graph G: that is, it can be identified with the *path category* of G (Proposition 1.3.7.5) which we study in §1.3.7.

- Let S be an ∞-category. In this case, every morphism in the homotopy category $\mathcal{C} = \mathrm{h}S$ can be represented by a single edge of S, rather than a composition of edges (in other words, the canonical map $u : S \to \mathrm{N}_\bullet(\mathcal{C})$ is surjective on edges), and two edges of S represent the same morphism in $\mathrm{h}S$ if and only if they are *homotopic* (Definition 1.4.3.1). This leads to a more explicit description of the homotopy category $\mathcal{C}$ (generalizing Exercise 1.3.6.2) which we will discuss in §1.4.5 (see Proposition 1.4.5.7).

1.3.7 Example: The Path Category of a Directed Graph

`00HJ` Let S be a simplicial set of dimension ≤ 1. In this section, we will show that the homotopy category $\mathrm{h}S$ of Notation 1.3.6.3 admits a concrete description, which can be conveniently described using the language of directed graphs.

`00HK` **Construction 1.3.7.1** (The Path Category)**.** Let G be a directed graph (Definition 1.1.6.1). For each edge $e \in \mathrm{Edge}(G)$, we let $s(e), t(e) \in \mathrm{Vert}(G)$ denote the source and target of e, respectively. If x and y are vertices of $\mathrm{Vert}(G)$, then a *path* from x to y is a sequence of edges $(e_n, e_{n-1}, \ldots, e_1)$ satisfying

$$s(e_1) = x \qquad t(e_i) = s(e_{i+1}) \qquad t(e_m) = y,$$

By convention, we regard the empty sequence of edges as a path from each vertex $x \in \mathrm{Vert}(G)$ to itself.

We define a category $\mathrm{Path}[G]$ as follows:

- The objects of $\mathrm{Path}[G]$ are the vertices of G.

- For every pair of vertices $x, y \in \mathrm{Vert}(G)$, we let $\mathrm{Hom}_{\mathrm{Path}[G]}(x, y)$ denote the set of all paths $(e_m, \ldots, e_1)$ from x to y.

- For every vertex $x \in \mathrm{Vert}(G)$, the identity morphism id_x in the category $\mathrm{Path}[G]$ is the empty path from x to itself.
- Let $x, y, z \in \mathrm{Vert}(G)$. Then the composition law

  $$\circ : \mathrm{Hom}_{\mathrm{Path}[G]}(y, z) \times \mathrm{Hom}_{\mathrm{Path}[G]}(x, y) \to \mathrm{Hom}_{\mathrm{Path}[G]}(x, z)$$

  is described by the formula

  $$(e_n, \cdots, e_1) \circ (e'_m, \ldots, e'_1) = (e_n, \cdots, e_1, e'_m, \ldots, e'_1).$$

  In other words, composition in $\mathrm{Path}[G]$ is given by concatentation of paths.

We will refer to $\mathrm{Path}[G]$ as the *path category* of the directed graph G.

Example 1.3.7.2. Fix an integer $n \geq 0$. Let G be the directed graph with vertex set $\mathrm{Vert}(G) = \{v_0, v_1, \ldots, v_n\}$, and edge set $\mathrm{Edge}(G) = \{e_1, \ldots, e_n\}$, where each edge e_i has source $s(e_i) = v_{i-1}$ and target $t(e_i) = v_i$; we can represent G graphically by the diagram 00HL

$$\begin{tikzcd} v_0 \arrow[r, "e_1"] & v_1 \arrow[r, "e_2"] & \cdots \arrow[r, "e_{n-1}"] & v_{n-1} \arrow[r, "e_n"] & v_n. \end{tikzcd}$$

Let v_i and v_j be a pair of vertices of G. Then:

- If $i \leq j$, there is a unique path from v_i to v_j, given by the sequence of edges $(e_j, e_{j-1}, \ldots, e_{i+1})$.
- If $i > j$, then there are no paths from v_i to v_j.

It follows that the path category $\mathrm{Path}[G]$ is isomorphic to the linearly ordered set $[n] = \{0 < 1 < 2 < \cdots < n\}$ (regarded as a category).

Example 1.3.7.3. Let G be a directed graph having a single vertex $\mathrm{Vert}(G) = \{x\}$. Then the path category $\mathrm{Path}[G]$ has a single object x, and can therefore be identified with the category BM associated to the monoid $M = \mathrm{End}_{\mathrm{Path}[G]}(x) = \mathrm{Hom}_{\mathrm{Path}[G]}(x, x)$ (see Construction 1.3.2.5). Note that the elements of M can be identified with (possibly empty) sequences of elements of the set $\mathrm{Edge}(G)$, and that the multiplication on M is given by concatenation of sequences. In other words, M can be identified with the *free* monoid generated by the set $\mathrm{Edge}(M)$ (this identification is not completely tautological: it can be regarded as a special case of Proposition 1.3.7.5 below). 00HM

Example 1.3.7.4. Let G be a directed graph having a single vertex $\mathrm{Vert}(G) = \{x\}$ and a single edge $\mathrm{Edge}(G) = \{e\}$ (necessarily satisfying $s(e) = x = t(e)$). Then the path category $\mathrm{Path}[G]$ has a single object x whose endomorphism monoid $\mathrm{End}_{\mathrm{Path}[G]}(x) = \mathrm{Hom}_{\mathrm{Path}[G]}(x, x)$ can be identified with the set $\mathbf{Z}_{\geq 0}$ of nonnegative integers (with monoid structure given by addition). 00HN

Let G be a directed graph, and let $G_\bullet$ denote the associated 1-dimensional simplicial set (see Proposition 1.1.6.9). Then there is an evident map of simplicial sets $u : G_\bullet \to \mathrm{N}_\bullet(\mathrm{Path}[G])$, which carries each vertex $v \in \mathrm{Vert}(G)$ to itself and each edge $e \in \mathrm{Edge}(G)$ to the path consisting of the single edge e.

00HP **Proposition 1.3.7.5.** *Let G be a directed graph. Then the map of simplicial sets $u : G_\bullet \to \mathrm{N}_\bullet(\mathrm{Path}[G])$ exhibits $\mathrm{Path}[G]$ as the homotopy category of the simplicial set $G_\bullet$, in the sense of Definition 1.3.6.1. In other words, for every category $\mathcal{C}$, the composite map*

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Path}[G], \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet(\mathrm{Path}[G]), \mathrm{N}_\bullet(\mathcal{C})) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_\Delta}(G_\bullet, \mathrm{N}_\bullet(\mathcal{C}))$$

is a bijection.

Proof. Let $f : G_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$ be a morphism of simplicial sets. We wish to show that there is a unique functor $F : \mathrm{Path}[G] \to \mathcal{C}$ for which the composite map

$$G_\bullet \xrightarrow{u} \mathrm{N}_\bullet(\mathrm{Path}[G]) \xrightarrow{\mathrm{N}_\bullet(F)} \mathrm{N}_\bullet(\mathcal{C})$$

coincides with f. Unwinding the definitions, we see that this agreement imposes the following requirements on F:

(a) For each vertex $v \in \mathrm{Vert}(G)$, we have $F(x) = f(x)$ (as objects of $\mathcal{C}$).

(b) For each edge $e \in \mathrm{Edge}(G)$ having $x = s(e)$ and target $y = t(e)$, the functor F carries the path (e) to the morphism $f(e) : f(x) \to f(y)$ in $\mathcal{C}$.

The existence and uniqueness of the functor F is now clear: it is determined on objects by property (a), and on morphisms by the formula

$$F(e_n, e_{n-1}, \cdots, e_1) = f(e_n) \circ f(e_{n-1}) \circ \cdots \circ f(e_1).$$

□

00HQ **Remark 1.3.7.6.** In the proof of Proposition 1.3.7.5, we have implicitly invoked the fact that every category $\mathcal{C}$ satisfies the *generalized associative law*: every sequence of composable morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n$$

has a well-defined composition $f_n \circ f_{n-1} \circ \cdots \circ f_1$, which can be computed in terms of the binary composition law by inserting parentheses arbitrarily. One might object that this logic is circular: the generalized associative law is essentially equivalent to Proposition 1.3.7.5 (applied to the directed graph G described in Example 1.3.7.2). In §1.5.7, we will establish an ∞-categorical generalization of Proposition 1.3.7.5 (Theorem 1.5.7.1), whose proof will avoid this sort of circular reasoning (see Remark 1.5.7.4).

Definition 1.3.7.7. A category $\mathcal{C}$ is *free* if it is isomorphic to $\operatorname{Path}[G]$, for some directed graph G. 00HR

We close this section with a characterization of those categories which are free in the sense of Definition 1.3.7.7.

Definition 1.3.7.8. Let $\mathcal{C}$ be a category. We will say that a morphism $f : X \to Y$ in $\mathcal{C}$ is *indecomposable* if f is not an identity morphism, and for every factorization $f = g \circ h$ have either $g = \mathrm{id}_Y$ (so $h = f$) or $h = \mathrm{id}_X$ (so $g = f$). 00HS

Example 1.3.7.9. Let G be a directed graph and let $\vec{e}$ be a morphism in the path category $\operatorname{Path}[G]$, given by a sequence of edges $(e_n, e_{n-1}, \dots, e_1)$ satisfying $t(e_i) = s(e_{i+1})$. Then $\vec{e}$ is indecomposable if and only if $n = 1$. 00HT

Warning 1.3.7.10. Definitions 1.3.7.7 and 1.3.7.8 are not invariant under equivalence of categories. If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories and $\mathcal{C}$ is free, then $\mathcal{D}$ need not be free; if f is an indecomposable morphism in $\mathcal{C}$, then $F(f)$ need not be an indecomposable morphism of $\mathcal{D}$. 00HU

Let $\mathcal{C}$ be any category. We define a directed graph $\mathrm{Gr}_0(\mathcal{C})$ as follows:

- The vertices of $\mathrm{Gr}_0(\mathcal{C})$ are the objects of $\mathcal{C}$.
- The edges of $\mathrm{Gr}_0(\mathcal{C})$ are the indecomposable morphisms of $\mathcal{C}$ (where an indecomposable morphism $f : X \to Y$ is regarded as an edge with source $s(f) = X$ and target $t(f) = Y$).

By construction, the graph $\mathrm{Gr}_0(\mathcal{C})$ comes equipped with a canonical map $\mathrm{Gr}_0(\mathcal{C})_\bullet \to \mathrm{N}_\bullet(\mathcal{C})$, which we can identify (by means of Proposition 1.3.7.5) with a functor $F : \operatorname{Path}[\mathrm{Gr}_0(\mathcal{C})] \to \mathcal{C}$.

Proposition 1.3.7.11. *Let $\mathcal{C}$ be a category. The following conditions on $\mathcal{C}$ are equivalent:* 00HV

(a) *The category $\mathcal{C}$ is free. That is, there exists a directed graph G and an isomorphism of categories $\mathcal{C} \simeq \operatorname{Path}[G]$.*

(b) *The functor $F : \operatorname{Path}[\mathrm{Gr}_0(\mathcal{C})] \to \mathcal{C}$ is an isomorphism of categories.*

(c) *The functor $F : \operatorname{Path}[\mathrm{Gr}_0(\mathcal{C})] \to \mathcal{C}$ is an equivalence of categories.*

(d) *The functor $F : \operatorname{Path}[\mathrm{Gr}_0(\mathcal{C})] \to \mathcal{C}$ is fully faithful.*

(e) *Every morphism f in $\mathcal{C}$ admits a unique factorization $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$, where each f_i is an indecomposable morphism of $\mathcal{C}$.*

Proof. The functor F is bijective on objects, which shows that (b) ⇔ (c) ⇔ (d). The equivalence of (d) and (e) follows from the definition of morphisms in the path category $\operatorname{Path}[\mathrm{Gr}_0(\mathcal{C})]$. The implication ($b$) ⇒ ($a$) is immediate, and the converse follows from Example 1.3.7.9. □

1.4 ∞-Categories

0039 In §1.1 and §1.3, we considered two closely related conditions on a simplicial set S:

($*$) For $n > 0$ and $0 \leq i \leq n$, every morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to S$ can be extended to an n-simplex $\sigma : \Delta^n \to S$.

($*'$) For $0 < i < n$, every morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to S_\bullet$ can be extended uniquely to an n-simplex $\sigma : \Delta^n \to S_\bullet$.

Simplicial sets satisfying ($*$) are called Kan complexes and form the basis for a combinatorial approach to homotopy theory, while simplicial sets satisfying ($*'$) can be identified with categories (Propositions 1.3.3.1 and 1.3.4.1). These notions admit a common generalization:

003A **Definition 1.4.0.1.** An ∞-*category* is a simplicial set S which satisfies the following condition:

($*''$) For $0 < i < n$, every morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to S$ can be extended to an n-simplex $\sigma : \Delta^n \to S_\bullet$.

003B **Remark 1.4.0.2.** Condition ($*''$) is commonly known as the *weak Kan extension condition*. It was introduced by Boardman and Vogt in [5], who refer to ∞-categories as *weak Kan complexes*. The theory was developed further by Joyal ([32] and [31]), who refers to ∞-categories as *quasicategories*.

003C **Example 1.4.0.3.** Every Kan complex is an ∞-category. In particular, if X is a topological space, then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is an ∞-category.

003D **Example 1.4.0.4.** For every category $\mathcal{C}$, the nerve $\mathrm{N}_\bullet(\mathcal{C})$ is an ∞-category.

003E **Remark 1.4.0.5.** We will often abuse terminology by identifying a category $\mathcal{C}$ with its nerve $\mathrm{N}_\bullet(\mathcal{C})$ (this abuse is essentially harmless by virtue of Proposition 1.3.3.1). Adopting this convention, we can state Example 1.4.0.4 more simply: every category is an ∞-category. To minimize the possibility of confusion, we will sometimes refer to categories as *ordinary categories*.

00HW **Example 1.4.0.6** (Products of ∞-Categories)**.** Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of simplicial sets parametrized by a set A, and let $S = \prod_{\alpha \in A} S_\alpha$ denote their product. If each S_α is an ∞-category, then S is an ∞-category. The converse holds provided that each factor S_α is nonempty.

00HX **Example 1.4.0.7** (Coproducts of ∞-Categories)**.** Let $\{S_\alpha\}_{\alpha \in A}$ be a collection of simplicial sets parametrized by a set A, and let $S = \coprod_{\alpha \in A} S_\alpha$ denote their coproduct. For each $0 < i < n$, the restriction map

$$\theta : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, S)$$

can be identified with the coproduct (formed in the arrow category $\mathrm{Fun}([1], \mathrm{Set})$) of restriction maps $\theta_\alpha : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S_\alpha) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, S_\alpha)$; this follows from the connectedness of the simplicial sets Δ^n and Λ^n_i (see Example 1.2.1.7 and Remark 1.2.4.5). It follows that θ is a surjection if and only if each θ_α is a surjection. Allowing n and i to vary, we conclude that S is an ∞-category if and only if each summand S_α is an ∞-category.

Remark 1.4.0.8. Let S be a simplicial set. Combining Example 1.4.0.7 with Proposition 1.2.1.13, we deduce that S is an ∞-category if and only if each connected component of S is an ∞-category. 00HY

Remark 1.4.0.9. Suppose we are given a filtered diagram of simplicial sets $\{S(\alpha)\}$ having colimit $S = \varinjlim S(\alpha)$. If each $S(\alpha)$ is an ∞-category, then S is an ∞-category. 01G8

Throughout this book, we will generally use calligraphic letters (like $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$) to denote ∞-categories, and we will generally describe them using terminology borrowed from category theory. For example, if $\mathcal{C} = S$ is an ∞-category, then we will refer to *vertices* of the simplicial set S as *objects* of the ∞-category $\mathcal{C}$, and to *edges* of the simplicial set S as *morphisms* of the ∞-category $\mathcal{C}$ (see §1.4.1). One of the central themes of this book is that ∞-categories behave much like ordinary categories. In particular, for any ∞-category $\mathcal{C}$, there is a notion of composition for morphisms of $\mathcal{C}$, which we study in §1.4.4. Given a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$ (corresponding to edges $f, g \in S_1$ satisfying $d^1_0(f) = d^1_1(g)$), the pair (f, g) defines a map of simplicial sets $\sigma_0 : \Lambda^2_1 \to \mathcal{C}$. Applying condition $(*'')$, we can extend σ_0 to a 2-simplex σ of $\mathcal{C}$, which we can think of heuristically as a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, dashed, "h"] & & Z.
\end{tikzcd}$$

In this case, we will refer to the morphism $h = d^2_1(\sigma)$ as a *composition of f and g*. However, this comes with a caveat: the extension σ is usually not unique, so the morphism h is not completely determined by f and g. However, we will show that it is unique up to a certain notion of *homotopy* which we study in §1.4.3. We apply this observation in §1.4.5 to give a concrete description of the homotopy category $\mathrm{h}\mathcal{C}$ (in the sense of Definition 1.3.6.1) when $\mathcal{C}$ is an ∞-category (see Definition 1.4.5.3 and Proposition 1.4.5.7).

1.4.1 Objects and Morphisms

We begin by introducing some terminology. 003F

003G **Definition 1.4.1.1.** Let $\mathcal{C} = S_\bullet$ be an ∞-category. An *object of* $\mathcal{C}$ is a vertex of the simplicial set $S_\bullet$ (that is, an element of the set S_0). A *morphism of* $\mathcal{C}$ is an edge of the simplicial set $S_\bullet$ (that is, an element of S_1). If $f \in S_1$ is a morphism of $\mathcal{C}$, we will refer to the object $X = d^1_1(f)$ as the *source* of f and to the object $Y = d^1_0(f)$ as the *target* of f. In this case, we will say that f is a *morphism from* X *to* Y. For any object X of $\mathcal{C}$, we can regard the degenerate edge $s^0_0(X)$ as a morphism from X to itself; we will denote this morphism by id_X and refer to it as the *identity morphism* of X.

003H **Notation 1.4.1.2.** Let $\mathcal{C}$ be an ∞-category. We will often write $X \in \mathcal{C}$ to indicate that X is an object of $\mathcal{C}$. We use the phrase "$f : X \to Y$ is a morphism of $\mathcal{C}$" to indicate that f is a morphism of $\mathcal{C}$ having source X and target Y.

003J **Example 1.4.1.3.** Let $\mathcal{C}$ be an ordinary category, and regard the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ as an ∞-category. Then:

- The objects of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ are the objects of $\mathcal{C}$.
- The morphisms of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ are the morphisms of $\mathcal{C}$. Moreover, the source and target of a morphism of $\mathcal{C}$ coincide with the source and target of the corresponding morphism in $\mathrm{N}_\bullet(\mathcal{C})$.
- For every object $X \in \mathcal{C}$, the identity morphism id_X does not depend on whether we view X as an object of the category $\mathcal{C}$ or the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$.

003K **Example 1.4.1.4.** Let X be a topological space, and regard the simplicial set $\mathrm{Sing}_\bullet(X)$ as an ∞-category. Then:

- The objects of $\mathrm{Sing}_\bullet(X)$ are the points of X.
- The morphisms of $\mathrm{Sing}_\bullet(X)$ are continuous paths $f : [0,1] \to X$. The source of a morphism f is the point $f(0)$, and the target is the point $f(1)$.
- For every point $x \in X$, the identity morphism id_x is the constant path $[0,1] \to X$ taking the value x.

03X3 **Definition 1.4.1.5** (Endomorphisms)**.** Let $\mathcal{C}$ be an ∞-category. An *endomorphism in* $\mathcal{C}$ is a morphism $f : X \to X$ of $\mathcal{C}$ for which the source and target of f are the same. In this case, we will say that f is an *endomorphism of* X.

1.4.2 The Opposite of an ∞-Category

003L Let $\mathcal{C}$ be an ordinary category. Then we can construct a new category $\mathcal{C}^{\mathrm{op}}$, called the *opposite category of* $\mathcal{C}$, as follows:

- The objects of the opposite category $\mathcal{C}^{\mathrm{op}}$ are the objects of $\mathcal{C}$.
- For every pair of objects $C, D \in \mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C, D) = \mathrm{Hom}_{\mathcal{C}}(D, C)$.
- Composition of morphisms in $\mathcal{C}^{\mathrm{op}}$ is given by the composition of morphisms in $\mathcal{C}$, with the order reversed.

The construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ admits a straightforward generalization to the setting of ∞-categories. In fact, it can be extended to arbitrary simplicial sets.

Notation 1.4.2.1. Let Lin denote the category whose objects are finite linearly ordered sets and whose morphisms are nondecreasing functions. Let I be an object of Lin, regarded as a set with a linear ordering $\leq_I$. We let I^{op} denote the same set with the opposite ordering, so that 003M

$$(i \leq_{I^{\mathrm{op}}} j) \Leftrightarrow (j \leq_I i).$$

The construction $I \mapsto I^{\mathrm{op}}$ determines an equivalence from the category Lin to itself.

Recall that the simplex category $\mathbf{\Delta}$ of Definition 1.1.0.2 is the full subcategory of Lin spanned by objects of the form $[n] = \{0 < 1 < \cdots < n\}$, and is equivalent to the full subcategory of Lin spanned by those linearly ordered sets which are finite and nonempty (Remark 1.1.0.3). There is a unique functor $\mathrm{Op} : \mathbf{\Delta} \to \mathbf{\Delta}$ for which the diagram

$$\begin{tikzcd}
\mathbf{\Delta} \arrow[r] \arrow[d, "\mathrm{Op}"] & \mathrm{Lin} \arrow[d, "I \mapsto I^{\mathrm{op}}"] \\
\mathbf{\Delta} \arrow[r] & \mathrm{Lin}
\end{tikzcd}$$

commutes up to isomorphism, where the horizontal maps are given by the inclusion. The functor Op can be described more concretely as follows:

- For each object $[n] \in \mathbf{\Delta}$, we have $\mathrm{Op}([n]) = [n]$ (note that the construction $i \mapsto n - i$ determines an isomorphism of $[n]$ with the opposite linear ordering $[n]^{\mathrm{op}}$).
- For each morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, the morphism $\mathrm{Op}(\alpha) : [m] \to [n]$ is given by the formula $\mathrm{Op}(\alpha)(i) = n - \alpha(m - i)$.

Construction 1.4.2.2. Let S be a simplicial set, which we regard as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. We let S^{op} denote the simplicial set given by the composition 003N

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{Op}} \mathbf{\Delta}^{\mathrm{op}} \xrightarrow{S} \mathrm{Set},$$

where Op is the functor described in Notation 1.4.2.1. We will refer to S^{op} as the *opposite* of the simplicial set S.

003P **Remark 1.4.2.3.** Let $S_\bullet$ be a simplicial set. Then the opposite simplicial set $S_\bullet^{\mathrm{op}}$ can be described more concretely as follows:

- For each $n \geq 0$, we have $S_n^{\mathrm{op}} = S_n$.
- The face and degeneracy operators of $S_\bullet^{\mathrm{op}}$ are given by

$$(d_i^n : S_n^{\mathrm{op}} \to S_{n-1}^{\mathrm{op}}) = (d_{n-i}^n : S_n \to S_{n-1})$$
$$(s_i^n : S_n^{\mathrm{op}} \to S_{n+1}^{\mathrm{op}}) = (s_{n-i}^n : S_n \to S_{n+1}).$$

003Q **Example 1.4.2.4.** Let $\mathcal{C}$ be a category. For each $n \geq 0$, we can identify n-simplices σ of $\mathrm{N}_\bullet(\mathcal{C})$ with diagrams

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n$$

in the category $\mathcal{C}$. Then σ determines an n-simplex σ' of $\mathrm{N}_\bullet(\mathcal{C}^{\mathrm{op}})$, given by the diagram

$$C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} C_1 \xrightarrow{f_1} C_0$$

in the opposite category $\mathcal{C}^{\mathrm{op}}$. The construction $\sigma \mapsto \sigma'$ determines an isomorphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{N}_\bullet(\mathcal{C}^{\mathrm{op}})$.

003R **Example 1.4.2.5.** Let X be a topological space. Then there is a canonical isomorphism of simplicial sets $\mathrm{Sing}_\bullet(X) \simeq \mathrm{Sing}_\bullet(X)^{\mathrm{op}}$, which carries each singular n-simplex $\sigma : |\Delta^n| \to X$ to the composite map

$$|\Delta^n| \xrightarrow{r} |\Delta^n| \xrightarrow{\sigma} X$$

where r is denotes the homeomorphism of $|\Delta^n|$ with itself given by $r(t_0, t_1, \ldots, t_{n-1}, t_n) = (t_n, t_{n-1}, \ldots, t_1, t_0)$.

003S **Proposition 1.4.2.6.** *Let $\mathcal{C}$ be an ∞-category. Then the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is also an ∞-category.*

Proof. Let $\sigma_0 : \Lambda_i^n \to \mathcal{C}^{\mathrm{op}}$ be a map of simplicial sets for $0 < i < n$; we wish to show that σ_0 can be extended to an n-simplex of $\mathcal{C}^{\mathrm{op}}$. Passing to opposite simplicial sets, we are reduced to showing that the map $\sigma_0^{\mathrm{op}} : (\Lambda_i^n)^{\mathrm{op}} \to \mathcal{C}$ can be extended to a map $(\Delta^n)^{\mathrm{op}} \to \mathcal{C}$. This follows from our assumption that $\mathcal{C}$ is an ∞-category, since there is a unique isomorphism $(\Delta^n)^{\mathrm{op}} \simeq \Delta^n$ which carries the simplicial subset $(\Lambda_i^n)^{\mathrm{op}}$ to Λ_{n-i}^n. □

003T **Remark 1.4.2.7.** Let $\mathcal{C}$ be an ∞-category. We will refer to the ∞-category $\mathcal{C}^{\mathrm{op}}$ of Proposition 1.4.2.6 as the *opposite* of the ∞-category $\mathcal{C}$. Note that:

- The objects of $\mathcal{C}^{\mathrm{op}}$ are the objects of $\mathcal{C}$.
- Given a pair of objects $X, Y \in \mathcal{C}$, the datum of a morphism from X to Y in $\mathcal{C}^{\mathrm{op}}$ is equivalent to the datum of a morphism from Y to X in $\mathcal{C}$.

050H **Variant 1.4.2.8.** If X is a Kan complex, then the opposite simplicial set X^{op} is also a Kan complex.

1.4.3 Homotopies of Morphisms

For any topological space X, we can view the singular simplicial set $\mathrm{Sing}_{\bullet}(X)$ as an ∞-category, where a morphism from a point $x \in X$ to a point $y \in X$ is given by a continuous path $f : [0,1] \to X$ satisfying $f(0) = x$ and $f(1) = y$. For many purposes (for example, in the study of the fundamental group $\pi_1(X, x)$), it is useful to work not with paths but with *homotopy classes* of paths (having fixed endpoints). This notion can be generalized to an arbitrary ∞-category: 003U

Definition 1.4.3.1. Let $\mathcal{C}$ be an ∞-category and let $f, g : C \to D$ be a pair of morphisms in $\mathcal{C}$ having the same source and target. A *homotopy from f to g* is a 2-simplex σ of $\mathcal{C}$ satisfying $d^2_0(\sigma) = \mathrm{id}_D$, $d^2_1(\sigma) = g$, and $d^2_2(\sigma) = f$, as depicted in the diagram 003V

$$\begin{tikzcd}
& D \arrow[dr, "\mathrm{id}_D"] & \\
C \arrow[ur, "f"] \arrow[rr, "g"] & & D.
\end{tikzcd}$$

We will say that f and g are *homotopic* if there exists a homotopy from f to g.

Example 1.4.3.2. Let $\mathcal{C}$ be an ordinary category. Then a pair of morphisms $f, g : C \to D$ in $\mathcal{C}$ (having the same source and target) are homotopic as morphisms of the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$ if and only if $f = g$. 003W

Example 1.4.3.3. Let X be a topological space. Suppose we are given points $x, y \in X$ and a pair of continuous paths $f, g : [0,1] \to X$ satisfying $f(0) = x = g(0)$ and $f(1) = y = g(1)$. Then f and g are homotopic as morphisms of the ∞-category $\mathrm{Sing}_{\bullet}(X)$ (in the sense of Definition 1.4.3.1) if and only if the paths f and g are homotopic relative to their endpoints: that is, if and only if there exists a continuous function $H : [0,1] \times [0,1] \to X$ satisfying 003X

$$H(s,0) = f(s) \quad H(s,1) = g(s) \quad H(0,t) = x \quad H(1,t) = y$$

(see Exercise 1.4.3.4 for a more precise statement).

Exercise 1.4.3.4. Let $\pi : [0,1] \times [0,1] \to |\Delta^2|$ denote the continuous function given by the formula $\pi(s,t) = (1 - s, (1-t)s, ts)$. For any topological space X, the construction $\sigma \mapsto \sigma \circ \pi$ determines a map from the set $\mathrm{Sing}_2(X)$ of singular 2-simplices of X to the set of all continuous functions $H : [0,1] \times [0,1] \to X$. Show that, if $f, g : [0,1] \to X$ are continuous paths satisfying $f(0) = g(0)$ and $f(1) = g(1)$, then the construction $\sigma \mapsto \sigma \circ \pi$ induces a bijection from the set of homotopies from f to g (in the sense of Definition 1.4.3.1) to the set of continuous functions H satisfying the requirements of Example 1.4.3.3. 003Y

003Z **Proposition 1.4.3.5.** *Let $\mathcal{C}$ be an ∞-category containing objects $X, Y \in \mathcal{C}$, and let E denote the collection of all morphisms from X to Y in $\mathcal{C}$. Then homotopy is an equivalence relation on E.*

Proof. We first observe that for any morphism $f : X \to Y$ in $\mathcal{C}$, the degenerate 2-simplex $s^1_1(f)$ is a homotopy from f to itself. It follows that homotopy is a reflexive relation on E. We will complete the proof by establishing the following:

$(*)$ Let $f, g, h : X \to Y$ be three morphisms from X to Y. If f is homotopic to g and f is homotopic to h, then g is homotopic to h.

Let us first observe that assertion $(*)$ implies Proposition 1.4.3.5. Note that in the special case $f = h$, $(*)$ asserts that if f is homotopic to g, then g is homotopic to f (since f is always homotopic to itself). That is, the relation of homotopy is symmetric. We can therefore replace the hypothesis that f is homotopic to g in assertion $(*)$ by the hypothesis that g is homotopic to f, so that $(*)$ is equivalent to the transitivity of the relation of homotopy.

It remains to prove $(*)$. Let σ_2 and σ_3 be 2-simplices of $\mathcal{C}$ which are homotopies from f to h and f to g, respectively, and let σ_0 be the 2-simplex given by the constant map $\Delta^2 \to \Delta^0 \xrightarrow{Y} \mathcal{C}$. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (see Proposition 1.2.4.7), depicted informally by the diagram

$$\begin{tikzcd}
& Y \arrow[rr, "\mathrm{id}_Y"] \arrow[drrr, "\mathrm{id}_Y"] & & Y \arrow[dr, dashed, "\mathrm{id}_Y"] & \\
X \arrow[ur, "f"] \arrow[urrr, dashed, "g"] \arrow[rrrr, dashed, "h"] & & & & Y;
\end{tikzcd}$$

here the dotted arrows represent the boundary of the "missing" face of the horn Λ^3_1. Our hypothesis that $\mathcal{C}$ is an ∞-category guarantees that τ_0 can be extended to a 3-simplex τ of $\mathcal{C}$. We can then regard the face $d^2_1(\tau)$ as a homotopy from g to h. □

Note that there is a potential asymmetry in Definition 1.4.3.1: if $f, g : X \to Y$ are two morphisms in an ∞-category $\mathcal{C}$, then the datum of a homotopy from f to g in the ∞-category $\mathcal{C}$ is not identical to the datum of a homotopy from f to g in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. Nevertheless, we have the following:

0040 **Proposition 1.4.3.6.** *Let $\mathcal{C}$ be an ∞-category, and let $f, g : X \to Y$ be morphisms of $\mathcal{C}$ having the same source and target. Then f and g are homotopic if and only if they are homotopic when regarded as morphisms of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. In other words, the following conditions are equivalent:*

(1) *There exists a 2-simplex σ of $\mathcal{C}$ satisfying $d_0^2(\sigma) = \mathrm{id}_Y$, $d_1^2(\sigma) = g$, and $d_2^2(\sigma) = f$, as depicted in the diagram*

$$\begin{tikzcd}
 & Y \arrow[dr, "\mathrm{id}_Y"] & \\
X \arrow[ur, "f"] \arrow[rr, "g"] & & Y.
\end{tikzcd}$$

(2) *There exists a 2-simplex τ of $\mathcal{C}$ satisfying $d_0^2(\tau) = f$, $d_1^2(\tau) = g$, and $d_2^2(\tau) = \mathrm{id}_X$, as depicted in the diagram*

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "g"] & & Y.
\end{tikzcd}$$

Proof. We will show that (1) implies (2); the proof of the reverse implication is similar. Assume that f is homotopic to g. Since the relation of homotopy is symmetric (Proposition 1.4.3.5), it follows that g is also homotopic to f. Let σ be a homotopy from g to f. Then we can regard the tuple of 2-simplices $(\sigma, s_1^1(g), \bullet, s_0^1(g))$ as a map of simplicial sets $\rho_0 : \Lambda_2^3 \to \mathcal{C}$ (see Proposition 1.2.4.7), depicted informally in the diagram

$$\begin{tikzcd}
 & X \arrow[r, "g"] \arrow[drr, dashed, "f"] & Y \arrow[dr, "\mathrm{id}_Y"] & \\
X \arrow[ur, dashed, "\mathrm{id}_X"] \arrow[urr, "g"] \arrow[rrr, dashed, "g"] & & & Y,
\end{tikzcd}$$

where the dotted arrows indicate the boundary of the "missing" face of the horn Λ_2^3. Using our assumption that $\mathcal{C}$ is an ∞-category, we can extend ρ_0 to a 3-simplex ρ of $\mathcal{C}$. Then the face $\tau = d_2^3(\rho)$ has the properties required by (2). □

Using Proposition 1.4.3.6, we can formulate the notion of homotopy in a more symmetric form:

Corollary 1.4.3.7. *Let $\mathcal{C}$ be an ∞-category, and let $f, g : X \to Y$ be morphisms of $\mathcal{C}$ having the same source and target. Then f and g are homotopic (in the sense of Definition 1.4.3.1)* `00V0` *if and only if there exists a map of simplicial sets* $H : \Delta^1 \times \Delta^1 \to \mathcal{C}$ *satisfying* $H|_{\{0\}\times\Delta^1} = f$, $H|_{\{1\}\times\Delta^1} = g$, $H|_{\Delta^1\times\{0\}} = \mathrm{id}_X$, *and* $H|_{\Delta^1\times\{1\}} = \mathrm{id}_Y$, *as indicated in the diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"] \arrow[dr, "h"] & Y \arrow[d, "\mathrm{id}_Y"] \arrow[dl, phantom, "\sigma", near end] \\
X \arrow[r, "g"] \arrow[ur, phantom, "\tau", near end] & Y.
\end{tikzcd}$$

Proof. The "only if" direction is clear: if σ is a homotopy from f to g (in the sense of Definition 1.4.3.1), then we can extend σ to a map $H : \Delta^1 \times \Delta^1 \to \mathcal{C}$ by taking τ to be the degenerate simplex $s^1_0(g)$. Conversely, suppose that there exists a map $\Delta^1 \times \Delta^1 \to \mathcal{C}$, as indicated in the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] \arrow[dr, "h"] & Y \arrow[d, "\mathrm{id}_Y"] \arrow[dl, phantom, "\sigma", near start] \\
X \arrow[r, "g"] \arrow[ur, phantom, "\tau", near start] & Y.
\end{tikzcd}$$

Then the 2-simplex σ is a homotopy from f to h, and the 2-simplex τ guarantees that g is homotopic to h (by virtue of Proposition 1.4.3.6). Since homotopy is an equivalence relation (Proposition 1.4.3.5), it follows that f is homotopic to g. $\square$

1.4.4 Composition of Morphisms

We now introduce a notion of composition for morphisms in an ∞-category. 0041

Definition 1.4.4.1. Let $\mathcal{C}$ be an ∞-category. Suppose we are given objects $X, Y, Z \in \mathcal{C}$ and morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$. We will say that *h is a composition of f and g* if there exists a 2-simplex σ of $\mathcal{C}$ satisfying $d_0^2(\sigma) = g$, $d_1^2(\sigma) = h$, and $d_2^2(\sigma) = f$. In this case, we will also say that the 2-simplex σ *witnesses h as a composition of f and g*. 0042

Beware that, in the situation of Definition 1.4.4.1, the morphism h is not determined by f and g. However, it is determined up to homotopy:

Proposition 1.4.4.2. *Let $\mathcal{C}$ be an ∞-category containing morphisms $f : X \to Y$ and $g : Y \to Z$. Then:* 0043

(1) *There exists a morphism $h : X \to Z$ which is a composition of f and g.*

(2) *Let $h : X \to Z$ be a composition of f and g, and let $h' : X \to Z$ be another morphism in $\mathcal{C}$ having the same source and target. Then h' is a composition of f and g if and only if h' is homotopic to h.*

Proof. The tuple $(g, \bullet, f)$ determines a map of simplicial sets $\sigma_0 : \Lambda^2_1 \to \mathcal{C}$ (Proposition 1.2.4.7). Since $\mathcal{C}$ is an ∞-category, we can extend σ_0 to a 2-simplex σ of $\mathcal{C}$. Then σ witnesses the morphism $h = d^2_1(\sigma)$ as a composition of f and g. This proves (1). To prove (2), let us first suppose that $h' : X \to Z$ is some other morphism in $\mathcal{C}$ which is a composition of f and g. We will show that h is homotopic to h'. Choose a 2-simplex σ' which witnesses h' as a composition of f and g. Then the tuple $(s^1_1(g), \bullet, \sigma', \sigma)$ determines a morphism of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (Proposition 1.2.4.7), which we depict informally as a diagram

$$\begin{tikzcd}
& Y \arrow[r, "g"] \arrow[drr, "g"] & Z \arrow[dr, dashed, "\mathrm{id}_Z"] & \\
X \arrow[ur, "f"] \arrow[urr, dashed, "h"] \arrow[rrr, dashed, "h'"] & & & Z
\end{tikzcd}$$

where the dotted arrows indicate the boundary of the "missing" face of the horn Λ^3_1. Using our assumption that $\mathcal{C}$ is an ∞-category, we can extend τ_0 to a 3-simplex τ of $\mathcal{C}$. Then the face $d^2_1(\tau)$ is a homotopy from h to h'.

We now prove the converse. Let σ be a 2-simplex of $\mathcal{C}$ which witnesses h as a composition of f and g, and let $h' : X \to Z$ be a morphism of $\mathcal{C}$ which is homotopic to h. Let σ'' be a 2-simplex of $\mathcal{C}$ which is a homotopy from h to h'. Then the tuple $(s^1_1(g), \sigma'', \bullet, \sigma)$ determines a map of simplicial sets $\rho_0 : \Lambda^3_2 \to \mathcal{C}$ (Proposition 1.2.4.7), which we depict informally as a diagram

$$\begin{tikzcd}
& Y \arrow[r, "g"] \arrow[drr, dashed, "g"] & Z \arrow[dr, "\mathrm{id}_Z"] & \\
X \arrow[ur, dashed, "f"] \arrow[urr, "h"] \arrow[rrr, dashed, "h'"] & & & Z.
\end{tikzcd}$$

Our assumption that $\mathcal{C}$ is an ∞-category guarantees that we can extend ρ_0 to a 3-simplex ρ of $\mathcal{C}$. Then the face $d^2_2(\rho)$ witnesses h' as a composition of f and g. □

0044 **Notation 1.4.4.3.** Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ and $g : Y \to Z$ be a pair of morphisms in $\mathcal{C}$. We will write $h = g \circ f$ to indicate that h is a composition of f and g (in the sense of Definition 1.4.4.1). In this case, it should be implicitly understood that we have chosen a 2-simplex that witnesses h as a composition of f and g. We will sometimes abuse terminology by referring to h as *the* composition of f and g. However, the reader should beware that only the homotopy class of h is well-defined (Proposition 1.4.4.2).

Example 1.4.4.4. Let $\mathcal{C}$ be an ordinary category containing a pair of morphisms $f : X \to Y$ and $g : Y \to Z$. Then there is a unique morphism $h : X \to Z$ in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ which is a composition of f and g, given by the usual composition $g \circ f$ in the category $\mathcal{C}$. 0045

Example 1.4.4.5. Let X be a topological space and suppose we are given continuous paths $f, g : [0, 1] \to X$ which are composable in the sense that $f(1) = g(0)$, and let $g \star f : [0, 1] \to X$ denote the path obtained by concatenating f and g, given concretely by the formula 0046

$$(g \star f)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq 1/2 \\ g(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Then $g \star f$ is a composition of f and g in the ∞-category $\mathrm{Sing}_\bullet(X)$. More precisely, the continuous map

$$\sigma : |\Delta^2| \to X \qquad \sigma(t_0, t_1, t_2) = \begin{cases} f(t_1 + 2t_2) & \text{if } t_0 \geq t_2 \\ g(t_2 - t_0) & \text{if } t_0 \leq t_2. \end{cases}$$

can be regarded as a 2-simplex of $\mathrm{Sing}_\bullet(X)$ which witnesses $g \star f$ as a composition of f and g.

Warning 1.4.4.6. In the situation of Example 1.4.4.5, the concatenation $g \star f$ is not the only path which is a composition of f and g in the ∞-category $\mathrm{Sing}_\bullet(X)$. Any path in X which is homotopic to $g \star f$ (with endpoints fixed) has the same property, by virtue of Proposition 1.4.4.2 (and Example 1.4.3.3). For example, we can replace $g \star f$ by a reparametrization, such as the path 0047

$$(s \in [0, 1]) \mapsto \begin{cases} f(3s) & \text{if } 0 \leq s \leq 1/3 \\ g(\frac{3}{2}s - \frac{1}{2}) & \text{if } 1/3 \leq s \leq 1. \end{cases}$$

When viewing $\mathrm{Sing}_\bullet(X)$ as an ∞-category, all of these paths have an equal claim to be regarded as "the" composition of f and g.

We now show that composition respects the relation of homotopy:

Proposition 1.4.4.7. *Let $\mathcal{C}$ be an ∞-category. Suppose we are given a pair of homotopic morphisms $f, f' : X \to Y$ in $\mathcal{C}$ and a pair of homotopic morphisms $g, g' : Y \to Z$ in $\mathcal{C}$. Let h be a composition of f and g, and let h' be a composition of f' and g'. Then h is homotopic to h'.* 0048

Proof. Let h'' be a composition of f and g'. Since homotopy is an equivalence relation (Proposition 1.4.3.5), it will suffice to show that both h and h' are homotopic to h''. We will show that h is homotopic to h''; the proof that h' is homotopic to h'' is similar. Let σ_3 be a 2-simplex of $\mathcal{C}$ which witnesses h as a composition of f and g, let σ_2 be a 2-simplex of $\mathcal{C}$ which witnesses h'' as a composition of f and g', and let σ_0 be a 2-simplex of $\mathcal{C}$ which is a homotopy from g to g'. Then the tuple $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (Proposition 1.2.4.7), which we depict informally as a diagram

$$\begin{tikzcd}
& Y \arrow[r, "g"] \arrow[drr, "g'"] & Z \arrow[dr, dashed, "\mathrm{id}_Z"] & \\
X \arrow[ur, "f"] \arrow[urr, dashed, "h"] \arrow[rrr, dashed, "h''"] & & & Z
\end{tikzcd}$$

where the dotted arrows indicate the boundary of the "missing" face of the horn Λ^3_1. Using our assumption that $\mathcal{C}$ is an ∞-category, we can extend τ_0 to a 3-simplex τ of $\mathcal{C}$. Then the face $d^2_1(\tau)$ is a homotopy from h to h''. □

1.4.5 The Homotopy Category of an ∞-Category

0049 To any topological space X, one can associate a category $\pi_{\leq 1}(X)$, called the *fundamental groupoid of X*. This category can be described informally as follows:

- The objects of $\pi_{\leq 1}(X)$ are the points of X.
- Given a pair of points $x, y \in X$, we can identify $\mathrm{Hom}_{\pi_{\leq 1}(X)}(x, y)$ with the set of *homotopy classes* of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$.
- Composition in $\pi_{\leq 1}(X)$ is given by concatenation of paths (see Example 1.4.4.5).

All of the concepts needed to define the fundamental groupoid $\pi_{\leq 1}(X)$ (such as points, paths, homotopies, and concatenation) can be formulated in terms of singular n-simplices of X (for $n \leq 2$). Consequently, one can view the fundamental groupoid $\pi_{\leq 1}(X)$ as an invariant of the simplicial set $\mathrm{Sing}_\bullet(X)$, rather than the topological space X. In this section, we describe an extension of this invariant, where the simplicial set $\mathrm{Sing}_\bullet(X)$ is replaced by an arbitrary ∞-category $\mathcal{C}$. In this case, the fundamental groupoid $\pi_{\leq 1}(X)$ is replaced by a category $\mathrm{h}\mathcal{C}$ which we call the *homotopy category* of $\mathcal{C}$ (beware that the homotopy category $\mathrm{h}\mathcal{C}$ is generally not a groupoid: in fact, we will later see that it is a groupoid if and only if $\mathcal{C}$ is a Kan complex (Proposition 4.4.2.1).

Construction 1.4.5.1. Let $\mathcal{C}$ be an ∞-category. For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ denote the set of homotopy classes of morphisms from X to Y in $\mathcal{C}$. For every morphism $f : X \to Y$, we let $[f]$ denote its equivalence class in $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$. 004A

It follows from Propositions 1.4.4.2 and 1.4.4.7 that, for every triple of objects $X, Y, Z \in \mathcal{C}$, there is a unique composition law

$$\circ : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Z)$$

satisfying the identity $[g] \circ [f] = [h]$ whenever $h : X \to Z$ is a composition of f and g in the ∞-category $\mathcal{C}$.

Proposition 1.4.5.2. *Let $\mathcal{C}$ be an ∞-category. Then:* 004B

(1) *The composition law of Construction 1.4.5.1 is associative. That is, for every triple of composable morphisms $f : W \to X$, $g : X \to Y$, and $h : Y \to Z$ in $\mathcal{C}$, we have an equality $([h] \circ [g]) \circ [f] = [h] \circ ([g] \circ [f])$ in $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(W, Z)$.*

(2) *For every object $X \in \mathcal{C}$, the homotopy class $[\mathrm{id}_X] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, X)$ is a two-sided identity with respect to the composition law of Construction 1.4.5.1. That is, for every morphism $f : W \to X$ in $\mathcal{C}$ and every morphism $g : X \to Y$ in $\mathcal{C}$, we have $[\mathrm{id}_X] \circ [f] = [f]$ and $[g] \circ [\mathrm{id}_X] = [g]$.*

Proof. We first prove (1). Let $u : W \to Y$ be a composition of f and g, let $v : X \to Z$ be a composition of g and h, and let $w : W \to Z$ be a composition of f and v. Then $([h] \circ [g]) \circ [f] = [w]$ and $[h] \circ ([g] \circ [f]) = [h] \circ [u]$. It will therefore suffice to show that w is a composition of u and h. Choose a 2-simplex σ_0 of $\mathcal{C}$ which witnesses v as a composition of g and h, a 2-simplex σ_2 of $\mathcal{C}$ which witnesses w as a composition of f and v, and a 2-simplex σ_3 of $\mathcal{C}$ which witnesses u as a composition of f and g. Then the sequence $(\sigma_0, \bullet, \sigma_2, \sigma_3)$ determines a map of simplicial sets $\tau_0 : \Lambda^3_1 \to \mathcal{C}$ (Proposition 1.2.4.7), which we depict informally as a diagram

$$\begin{tikzcd}
 & X \arrow[r, "g"] \arrow[drr, "v"] & Y \arrow[dr, dashed, "h"] & \\
W \arrow[ur, "f"] \arrow[urr, dashed, "u"] \arrow[rrr, dashed, "w"] & & & Z.
\end{tikzcd}$$

Using our assumption that $\mathcal{C}$ is an ∞-category, we can extend τ_0 to a 3-simplex τ of $\mathcal{C}$. Then the 2-simplex $d^3_1(\tau)$ witnesses w as a composition of u and h.

We now prove (2). Fix an object $X \in \mathcal{C}$ and a morphism $g : X \to Y$ in $\mathcal{C}$; we will show that $[g] \circ [\mathrm{id}_X] = [g]$ (the analogous identity $[\mathrm{id}_X] \circ [f] = [f]$ follows by a similar argument). For this, it suffices to observe that the degenerate 2-simplex $s_0^1(g)$ witnesses g as a composition of id_X and g. □

004C **Definition 1.4.5.3** (The Homotopy Category)**.** Let $\mathcal{C}$ be an ∞-category. We define a category $\mathrm{h}\mathcal{C}$ as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ denote the collection of homotopy classes of morphisms from X to Y in the ∞-category $\mathcal{C}$ (as in Construction 1.4.5.1).
- For every object $X \in \mathcal{C}$, the identity morphism from X to itself in $\mathrm{h}\mathcal{C}$ is given by the homotopy class $[\mathrm{id}_X]$.
- Composition of morphisms is defined as in Construction 1.4.5.1.

We will refer to $\mathrm{h}\mathcal{C}$ as *the homotopy category* of the ∞-category $\mathcal{C}$.

004D **Example 1.4.5.4.** Let $\mathcal{C}$ be an ordinary category. Then the homotopy category of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ can be identified with $\mathcal{C}$. In particular, for each $n \geq 0$, the homotopy category $\mathrm{h}\Delta^n$ can be identified with $[n] = \{0 < 1 < \cdots < n\}$.

004E **Example 1.4.5.5.** Let X be a topological space, and regard the singular simplicial set $\mathrm{Sing}_\bullet(X)$ as an ∞-category. Then the homotopy category $\mathrm{hSing}_\bullet(X)$ can be identified with the fundamental groupoid $\pi_{\leq 1}(X)$. More precisely, we can regard the contents of §1.4, when specialized to ∞-categories of the form $\mathrm{Sing}_\bullet(X)$, as providing a *construction* of the fundamental groupoid of X. By virtue of Exercise 1.4.3.4 and Example 1.4.4.5, the resulting category $\mathrm{hSing}_\bullet(X)$ matches the informal description of $\pi_{\leq 1}(X)$ given in the introduction to §1.4.5.

Let $\mathcal{C}$ be an ∞-category. Beware that we have now introduced two different definitions of the homotopy category $\mathrm{h}\mathcal{C}$:

- The homotopy category $\mathrm{h}\mathcal{C}$ of Definition 1.4.5.3, defined by an explicit construction using the assumption that $\mathcal{C}$ is an ∞-category.
- The homotopy category $\mathrm{h}\mathcal{C}$ of Notation 1.3.6.3, defined for any simplicial set $\mathcal{C}$ by a universal mapping property.

We conclude this section by showing that these definitions are equivalent (Proposition 1.4.5.7).

Construction 1.4.5.6. Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^n \to \mathcal{C}$ be an n-simplex of $\mathcal{C}$. For $0 \leq i \leq n$, let C_i denote the object of $\mathcal{C}$ given by the image of the ith vertex of Δ^n. For $0 \leq i \leq j \leq n$, let $f_{ij} : C_i \to C_j$ denote the image under σ of the edge of Δ^n joining the ith vertex to the jth vertex, and let $[f_{ij}] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C_i, C_j)$ denote the homotopy class of f_{ij}. Then we can regard $(\{C_i\}_{0 \leq i \leq n}, \{[f_{ij}]\}_{0 \leq i \leq j \leq n})$ as a functor from the linearly ordered set $[n]$ to the homotopy category $\mathrm{h}\mathcal{C}$. Let $u(\sigma)$ denote the corresponding n-simplex of $\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$. Then the construction $\sigma \mapsto u(\sigma)$ determines a map of simplicial sets 004G

$$u : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C}).$$

The comparison map of Construction 1.4.5.6 has the following universal property:

Proposition 1.4.5.7. *Let $\mathcal{C}$ be an ∞-category and let $u : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ be as in Construction 1.4.5.6. Then u exhibits $\mathrm{h}\mathcal{C}$ as a homotopy category of the simplicial set $\mathcal{C}$, in the sense of Definition 1.3.6.1. In other words, for every category $\mathcal{D}$, the composite map* 004H

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet(\mathrm{h}\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathrm{N}_\bullet(\mathcal{D}))$$

is a bijection.

Proof. Let $F : \mathcal{C} \to \mathrm{N}_\bullet(\mathcal{D})$ be a morphism of simplicial sets. Then F induces a functor of homotopy categories $G : \mathrm{h}\mathcal{C} \to \mathrm{hN}_\bullet(\mathcal{D}) \simeq \mathcal{D}$ (where the second identification comes from Example 1.4.5.4). By construction, the morphism of simplicial sets

$$\mathcal{C} \xrightarrow{u} \mathrm{N}_\bullet(\mathrm{h}\mathcal{C}) \xrightarrow{\mathrm{N}_\bullet(G)} \mathrm{N}_\bullet(\mathcal{D})$$

coincides with F on the vertices and edges of $\mathcal{C}$, and therefore coincides with F (since a simplex of $\mathrm{N}_\bullet(\mathcal{D})$ is determined by its 1-dimensional facets; see Remark 1.3.1.3). We leave it to the reader to verify that G is the unique functor with this property. □

1.4.6 Isomorphisms

Recall that a morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ satisfying $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. This notion has an ∞-categorical analogue: 004Q

Definition 1.4.6.1. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. We will say that f is an *isomorphism* if the homotopy class $[f]$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$. We will say that two objects $X, Y \in \mathcal{C}$ are *isomorphic* if there exists an isomorphism from X to Y (that is, if X and Y are isomorphic as objects of the homotopy category $\mathrm{h}\mathcal{C}$). 004R

004S **Example 1.4.6.2.** Let $\mathcal{C}$ be an ordinary category. Then a morphism $f : X \to Y$ of $\mathcal{C}$ is an isomorphism if and only if it is an isomorphism when regarded as a morphism of the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$.

004U **Remark 1.4.6.3** (Two-out-of-three)**.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in an ∞-category $\mathcal{C}$ and let h be a composition of f and g. If any two of the morphisms f, g, and h is an isomorphism, then so is the third.

004V **Definition 1.4.6.4.** Let $\mathcal{C}$ be an ∞-category and suppose we are given a pair of morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$. We say that g is a *left homotopy inverse* of f if the identity morphism id_X is a composition of f and g: that is, if we have an equality $[\mathrm{id}_X] = [g] \circ [f]$ in the homotopy category $\mathrm{h}\mathcal{C}$. We say that g is a *right homotopy inverse* of f if the identity morphism id_Y is a composition of g and f: that is, if we have an equality $[\mathrm{id}_Y] = [f] \circ [g]$ in the homotopy category $\mathrm{h}\mathcal{C}$. We will say that g is a *homotopy inverse* of f if it is both a left and a right homotopy inverse of f.

004W **Remark 1.4.6.5.** Let $f : X \to Y$ and $g : Y \to X$ be morphisms in an ∞-category $\mathcal{C}$. Then the condition that g is a left homotopy inverse (right homotopy inverse, homotopy inverse) to f depends only on the homotopy classes $[f]$ and $[g]$.

004X **Remark 1.4.6.6.** Let $f : X \to Y$ and $g : Y \to X$ be morphisms in an ∞-category $\mathcal{C}$. Then g is left homotopy inverse to f if and only if f is right homotopy inverse to g. Both of these conditions are equivalent to the existence of a 2-simplex σ of $\mathcal{C}$ satisfying $d_0^2(\sigma) = g$, $d_1^2(\sigma) = \mathrm{id}_X$, and $d_2^2(\sigma) = f$, as depicted in the diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_X"] & & X.
\end{tikzcd}$$

004Y **Remark 1.4.6.7.** Let $f : X \to Y$ be a morphism in an ∞-category $\mathcal{C}$. Suppose that f admits a left homotopy inverse g and a right homotopy inverse h. Then g and h are homotopic: this follows from the calculation

$$[g] = [g] \circ [\mathrm{id}_Y] = [g] \circ ([f] \circ [h]) = ([g] \circ [f]) \circ [h] = [\mathrm{id}_Y] \circ [h] = [h].$$

It follows that both g and h are homotopy inverse to f.

004Z **Remark 1.4.6.8.** Let $f : X \to Y$ be a morphism in the ∞-category $\mathcal{C}$. It follows from Remark 1.4.6.7 that the following conditions are equivalent:

(1) The morphism f is an isomorphism.

(2) The morphism f admits a homotopy inverse g.

(3) The morphism f admits both left and right homotopy inverses.

In this case, the morphism g is uniquely determined up to homotopy; moreover, any left or right homotopy inverse of f is homotopic to g. We will sometimes abuse notation by writing f^{-1} to denote a homotopy inverse to f.

Warning 1.4.6.9. Let $f : X \to Y$ be a morphism in an ∞-category $\mathcal{C}$, and suppose that $g, h : Y \to X$ are left homotopy inverses to f. If f does not admit a right homotopy inverse, then g and h need not be homotopic. 0050

Proposition 1.4.6.10. *Let $\mathcal{C}$ be a Kan complex. Then every morphism in $\mathcal{C}$ is an isomorphism.* 0052

Remark 1.4.6.11. We will see later that the converse to Proposition 1.4.6.10 is also true: if $\mathcal{C}$ is an ∞-category in which every morphism is an isomorphism, then $\mathcal{C}$ is a Kan complex (Proposition 4.4.2.1). 0053

Proof of Proposition 1.4.6.10. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then the tuple $(\bullet, \mathrm{id}_X, f)$ determines a map of simplicial sets $\sigma_0 : \Lambda^2_0 \to \mathcal{C}$ (Proposition 1.2.4.7), which we depict as

$$\begin{tikzcd}
& Y \arrow[dr, dashed] & \\
X \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_X"] & & X.
\end{tikzcd}$$

If $\mathcal{C}$ is a Kan complex, then we can extend σ_0 to a 2-simplex σ of $\mathcal{C}$. Then σ exhibits the morphism $g = d^2_0(\sigma)$ as a left homotopy inverse to f. A similar argument shows that f admits a right homotopy inverse, so that f is an isomorphism by virtue of Remark 1.4.6.8. □

Definition 1.4.6.12 (The Fundamental Groupoid of a Kan Complex)**.** Let X be a Kan complex. It follows from Proposition 1.4.6.10 that the homotopy category $\mathrm{h}X$ of Definition 1.4.5.3 is a groupoid. We will denote this groupoid by $\pi_{\leq 1}(X)$ and refer to it as the *fundamental groupoid* of X. 00HZ

Remark 1.4.6.13. Let X be a Kan complex. By construction, the objects of the fundamental groupoid $\pi_{\leq 1}(X)$ are the vertices of X, and a pair of vertices $x, y \in X$ are isomorphic in $\pi_{\leq 1}(X)$ if and only if there exists an edge $e : x \to y$ in X. Applying Proposition 1.2.5.10, we deduce that $x, y \in X$ are isomorphic if and only if they belong to the same connected component of X. In other words, we have a canonical bijection 00J0

$$\pi_0(X) \simeq \{\text{Objects of } \pi_{\leq 1}(X)\}/\text{Isomorphism}.$$

0054 **Example 1.4.6.14.** Let X be a topological space. Then the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a Kan complex (Proposition 1.2.5.8), and its fundamental groupoid $\pi_{\leq 1}(\mathrm{Sing}_\bullet(X))$ can be identified with the usual fundamental groupoid $\pi_{\leq 1}(X)$ of the topological space X (where objects are the points of X and morphisms are given by homotopy classes of paths in X).

1.5 Functors of ∞-Categories

0055 Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathrm{N}_\bullet(\mathcal{C})$ and $\mathrm{N}_\bullet(\mathcal{D})$ denote the corresponding ∞-categories. According to Proposition 1.3.3.1, the nerve functor $\mathrm{N}_\bullet$ induces a bijection

$$\{\text{Functors } F : \mathcal{C} \to \mathcal{D}\} \simeq \{\text{Morphisms of simplicial sets } \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})\}.$$

Consequently, the notion of functor admits an obvious generalization to the setting of ∞-categories:

0056 **Definition 1.5.0.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. A *functor from $\mathcal{C}$ to $\mathcal{D}$* is a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$.

This section is devoted to the study of functors between ∞-categories, in the sense of Definition 1.5.0.1. We begin in §1.5.1 with some simple examples, which illustrate the meaning of Definition 1.5.0.1 in the case of ∞-categories which arise from ordinary categories (via the construction $\mathcal{E} \mapsto \mathrm{N}_\bullet(\mathcal{E})$) or topological spaces (via the construction $X \mapsto \mathrm{Sing}_\bullet(X)$).

In ordinary category theory, one can think of a functor $F : \mathcal{C} \to \mathcal{D}$ as a kind of *commutative diagram* in $\mathcal{D}$, having vertices indexed by the objects of $\mathcal{C}$ and arrows indexed by the morphisms of $\mathcal{C}$. This perspective is quite useful: if the category $\mathcal{C}$ is sufficiently small, one can communicate the datum of a functor by drawing a graphical representation of the corresponding diagram. In §1.5.2, we discuss the notion of commutative diagram in an ∞-category (Convention 1.5.2.12) and describe some dangers associated with diagrammatic reasoning in the higher-categorical setting (Remark 1.5.2.13).

If $\mathcal{C}$ and $\mathcal{D}$ are ordinary categories, then the collection of all functors from $\mathcal{C}$ to $\mathcal{D}$ can itself be organized into a category, which we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. In §1.5.3, we describe a counterpart of this construction in the setting of ∞-categories. For every pair of simplicial sets S and T, one can form a new simplicial set $\mathrm{Fun}(S, T)$ whose vertices are maps from S to T (Construction 1.5.3.1). The main result of this section asserts that if T is an ∞-category, then $\mathrm{Fun}(S, T)$ is also an ∞-category (Theorem 1.5.3.7). Moreover, our notation is consistent: in the case where S and T are isomorphic to the nerves of categories $\mathcal{C}$ and $\mathcal{D}$, the ∞-category $\mathrm{Fun}(S, T)$ is isomorphic to the nerve of the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (Proposition 1.5.3.3).

In order to prove Theorem 1.5.3.7, we will need to introduce some auxiliary ideas. Recall that if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms in an ∞-category $\mathcal{C}$, then we can form a composition of f and g by choosing a 2-simplex σ of $\mathcal{C}$ which satisfies $d^2_0(\sigma) = g$ and $d^2_2(\sigma) = f$, as indicated in the diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & Z.
\end{tikzcd}$$

We proved in §1.4.4 that the resulting morphism $g \circ f$ is well-defined up to homotopy (Proposition 1.4.4.2). In §1.5.6, we prove a variant of this assertion which asserts that the 2-simplex σ is "unique up to a contractible space of choices" (see Corollary 1.5.6.2 for a precise statement, and §1.5.7 for an extension to more general path categories). Moreover, we show that a strong version of this uniqueness result is *equivalent* to the assumption that $\mathcal{C}$ is an ∞-category (Theorem 1.5.6.1), and deduce the existence of functor ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as a consequence (Theorem 1.5.3.7). The precise formulation and proof of Theorem 1.5.6.1 will require some general ideas about categorical lifting properties and the homotopy theory of simplicial sets, which we develop in §1.5.4 and §1.5.5, respectively.

1.5.1 Examples of Functors

Let us begin by illustrating Definition 1.5.0.1 in some special cases. 0057

Example 1.5.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories. It follows from Proposition 1.3.3.1 that the formation of nerves induces a bijection 0058

$$\begin{tikzcd}
\{\text{Functors of ordinary categories from } \mathcal{C} \text{ to } \mathcal{D}\} \arrow[d, "\sim"] \\
\{\text{Functors of } \infty\text{-categories from } \mathrm{N}_\bullet(\mathcal{C}) \text{ to } \mathrm{N}_\bullet(\mathcal{D})\}.
\end{tikzcd}$$

In other words, Definition 1.5.0.1 can be regarded as a generalization of the usual notion of functor to the setting of ∞-categories.

Example 1.5.1.2. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}$ be an ordinary category. Using Proposition 1.4.5.7, we obtain a bijection 0059

$$\begin{tikzcd}
\{\text{Functors of } \infty\text{-categories from } \mathcal{C} \text{ to } \mathrm{N}_\bullet(\mathcal{D})\} \arrow[d, "\sim"] \\
\{\text{Functors of ordinary categories from } \mathrm{h}\mathcal{C} \text{ to } \mathcal{D}\}.
\end{tikzcd}$$

005A **Remark 1.5.1.3.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:

(a) To each object $X \in \mathcal{C}$ the functor F assigns an object of $\mathcal{D}$, which we will denote by $F(X)$ (or sometimes more simply by FX).

(b) To each morphism $f : X \to Y$ in the ∞-category $\mathcal{C}$, the functor F assigns a morphism $F(f) : F(X) \to F(Y)$ in the ∞-category $\mathcal{D}$.

(c) For every object $X \in \mathcal{C}$, the functor F carries the identity morphism $\mathrm{id}_X : X \to X$ in $\mathcal{C}$ to the identity morphism $\mathrm{id}_{F(X)} : F(X) \to F(X)$ in $\mathcal{D}$.

(d) If $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{C}$ and h is a composition of f and g (in the sense of Definition 1.4.4.1), then the morphism $F(h) : F(X) \to F(Z)$ is a composition of $F(f)$ and $F(g)$.

005B **Warning 1.5.1.4.** To define a functor F from an ordinary category $\mathcal{C}$ to an ordinary category $\mathcal{D}$, it suffices to specify the values of F on objects and morphisms (as described in (a) and (b) of Remark 1.5.1.3) and to verify that F is compatible with the formation of composition and identity morphisms (as described in (c) and (d) of Remark 1.5.1.3). In the ∞-categorical setting, this is not enough: to give a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$, one must specify its values on simplices of *all* dimensions. Roughly speaking, these values encode the requirement that F is compatible with composition "up to coherent homotopy." For example, suppose that we are given objects $X, Y, Z \in \mathcal{C}$ and morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$. Part (d) of Remark 1.5.1.3 asserts that if h is a composition of f and g, then $F(h)$ is a composition of $F(f)$ and $F(g)$. However, we can say more: if σ is a 2-simplex of $\mathcal{C}$ which *witnesses* h as a composition of f and g, then $F(\sigma)$ is a 2-simplex of $\mathcal{D}$ which witnesses $F(h)$ as a composition of $F(f)$ and $F(g)$.

005C **Remark 1.5.1.5.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. If $f, g : X \to Y$ are homotopic morphisms of $\mathcal{C}$, then $F(f), F(g) : F(X) \to F(Y)$ are homotopic morphisms of $\mathcal{D}$. More precisely, the functor F carries homotopies from f to g (viewed as 2-simplices of $\mathcal{C}$) to homotopies from $F(f)$ to $F(g)$ (viewed as 2-simplices of $\mathcal{D}$).

005D **Remark 1.5.1.6.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. If $f : X \to Y$ is a morphism in $\mathcal{C}$ and $g : Y \to X$ is a homotopy inverse to f, then $F(g)$ is a homotopy inverse to $F(f)$. In particular, if f is an isomorphism in $\mathcal{C}$, then $F(f)$ is also an isomorphism in $\mathcal{D}$.

005E **Example 1.5.1.7.** Let X be a topological space and let $\mathcal{C}$ be an ordinary category. To specify a functor of ∞-categories $F : \mathrm{Sing}_\bullet(X) \to \mathrm{N}_\bullet(\mathcal{C})$, one must give a rule which assigns to each continuous map $\sigma : |\Delta^n| \to X$ (viewed as an n-simplex of $\mathrm{Sing}_\bullet(X)$) a diagram $F(\sigma) = (C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n)$. In particular:

(a) To each point $x \in X$, the functor F assigns an object $F(x) \in \mathcal{C}$.

(*b*) To each continuous path $f : [0,1] \to X$ starting at the point $x = f(0)$ and ending at the point $y = f(1)$, the functor F assigns a morphism $F(f) : F(x) \to F(y)$ in the category $\mathcal{C}$. The morphism $F(f)$ is automatically an isomorphism (by virtue of Proposition 1.4.6.10 and Remark 1.5.1.6).

(*c*) For each continuous map $\sigma : |\Delta^2| \to X$ with boundary behavior as depicted in the diagram

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z, \end{tikzcd}$$

we have an identity $F(h) = F(g) \circ F(f)$ in $\mathrm{Hom}_{\mathcal{C}}(F(x), F(z))$.

The data of a collection of objects $\{F(x)\}_{x \in X}$ and isomorphisms $\{F(f)\}_{f:[0,1]\to X}$ satisfying (*c*) is called a *$\mathcal{C}$-valued local system on X*. The preceding discussion determines a bijection

$$\begin{tikzcd} \{\text{Functors of } \infty\text{-categories from } \mathrm{Sing}_\bullet(X) \text{ to } \mathrm{N}_\bullet(\mathcal{C})\} \arrow[d, "\sim"] \\ \{\, \mathcal{C}\text{-valued local systems on } X\}. \end{tikzcd}$$

By virtue of Example 1.5.1.2, we can also identify local systems with functors from the fundamental groupoid $\pi_{\leq 1}(X)$ into $\mathcal{C}$.

Remark 1.5.1.8. Let X be a topological space and let $\mathcal{C}$ be an arbitrary ∞-category. `005F` Motivated by Example 1.5.1.7, one can define a *$\mathcal{C}$-valued local system on X* to be a functor of ∞-categories $\mathrm{Sing}_\bullet(X) \to \mathcal{C}$. Beware that this notion generally cannot be reformulated in terms of the fundamental groupoid $\pi_{\leq 1}(X)$.

Example 1.5.1.9. Let $\mathcal{C}$ be an ∞-category and let X be a topological space. Then we have `005G` a canonical bijection

$$\begin{tikzcd} \{\text{Functors of } \infty\text{-categories from } \mathcal{C} \text{ to } \mathrm{Sing}_\bullet(X)\} \arrow[d, "\sim"] \\ \{\text{Continuous functions from } |\,\mathcal{C}\,| \text{ to } X\}. \end{tikzcd}$$

Here $|\,\mathcal{C}\,|$ denotes the geometric realization of the simplicial set $\mathcal{C}$ (see Definition 1.2.3.1). Beware that neither side has an obvious interpretation in terms of functors between ordinary categories (even in the special case where $\mathcal{C}$ is the nerve of a category).

1.5.2 Commutative Diagrams

005H We now consider a variant of the terminology introduced in §1.5.1.

005J **Definition 1.5.2.1.** Let $\mathcal{C}$ be an ∞-category. A *diagram in $\mathcal{C}$* is a map of simplicial sets $f : K \to \mathcal{C}$. We will also refer to a map $f : K \to \mathcal{C}$ as a *diagram in $\mathcal{C}$ indexed by K*, or a *K-indexed diagram in $\mathcal{C}$*.

If $\mathcal{C}$ is an ordinary category, then a *(K-indexed) diagram in $\mathcal{C}$* is a (K-indexed) diagram in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$.

In the special case where K is the nerve $\mathrm{N}_\bullet(I)$ of a partially ordered set I (Remark 1.3.1.10), we will refer to a map $f : K \to \mathcal{C}$ as a *diagram in $\mathcal{C}$ indexed by I*, or an *I-indexed diagram in $\mathcal{C}$*.

005K **Remark 1.5.2.2.** In the case where K is an ∞-category, Definition 1.5.2.1 is superfluous: a K-indexed diagram in $\mathcal{C}$ (in the sense of Definition 1.5.2.1) is just a functor from K to $\mathcal{C}$ (in the sense of Definition 1.5.0.1). However, the redundant terminology will be useful to signal a shift in emphasis. We will generally refer to a map $f : \mathcal{C} \to \mathcal{D}$ as a *functor* when we wish to regard the ∞-categories $\mathcal{C}$ and $\mathcal{D}$ on an equal footing. By contrast, we will refer to a morphism $f : K \to \mathcal{C}$ as a *diagram* if we are primarily interested in the ∞-category $\mathcal{C}$ (in many cases, K will be a very simple simplicial set).

005L **Remark 1.5.2.3** (Diagrams of Dimension ≤ 1)**.** Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set of dimension ≤ 1, corresponding to a directed graph G (Proposition 1.1.6.9). In this case, a diagram $K \to \mathcal{C}$ can be identified with a pair $(\{C_v\}_{v \in \mathrm{Vert}(G)}, \{f_e\}_{e \in \mathrm{Edge}(G)})$, where each C_v is an object of the ∞-category $\mathcal{C}$ and each $f_e : C_{s(e)} \to C_{t(e)}$ is a morphism of $\mathcal{C}$ (here $s(e)$ and $t(e)$ denote the source and target of the edge e). It is often convenient to specify diagrams $K \to \mathcal{C}$ by drawing a graphical representation of G (as in Remark 1.1.6.3), where each node is labelled by an object of $\mathcal{C}$ and each arrow is labelled by a morphism in $\mathcal{C}$ (having the indicated source and target).

005M **Example 1.5.2.4** (Non-Commuting Squares)**.** Let K denote the boundary of the product $\Delta^1 \times \Delta^1$: that is, the simplicial subset of $\Delta^1 \times \Delta^1$ given by the union of the simplicial subsets $\partial \Delta^1 \times \Delta^1$ and $\Delta^1 \times \partial \Delta^1$. Then $K_\bullet$ is a 1-dimensional simplicial set, corresponding to a directed graph which we can depict as

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet .
\end{tikzcd}$$

We can then display a K-indexed diagram in an ∞-category $\mathcal{C}$ pictorially

$$\begin{tikzcd}
C_{00} \arrow[r, "f"] \arrow[d, "g"] & C_{01} \arrow[d, "g'"] \\
C_{10} \arrow[r, "f'"] & C_{11},
\end{tikzcd}$$

where each C_{ij} is an object of $\mathcal{C}$, f is a morphism in $\mathcal{C}$ from C_{00} to C_{01}, g is a morphism in $\mathcal{C}$ from C_{00} to C_{10}, f' is a morphism in $\mathcal{C}$ from C_{10} to C_{11}, and g' is a morphism in $\mathcal{C}$ from C_{01} to C_{11}.

In classical category theory, it is useful to extend the notational conventions of Remark 1.5.2.3 to more general situations by introducing the notion of a *commutative diagram*.

Definition 1.5.2.5. Let K be a simplicial set of dimension ≤ 1, which we will identify with a directed graph G (see Proposition 1.1.6.9). Assume that G satisfies the following additional conditions: `005N`

(a) For every pair of vertices $v, w \in \mathrm{Vert}(G)$, there is at most one edge of G with source v and target w. We will denote this edge (if it exists) by $(v, w) \in \mathrm{Edge}(G)$.

(b) The graph G has no directed cycles. That is, if there exists a sequence of vertices $v_0, v_1, \ldots, v_n \in \mathrm{Vert}(G)$ with the property that the edges (v_{i-1}, v_i) exist for $1 \leq i \leq n$, then either $n = 0$ or $v_0 \neq v_n$.

Let $\mathcal{C}$ be an ordinary category and suppose we are given a diagram $\sigma : K \to \mathrm{N}_\bullet(\mathcal{C})$, which we identify with a pair $(\{C_v\}_{v \in \mathrm{Vert}(G)}, \{f_{w,v} : C_v \to C_w\}_{(v,w) \in \mathrm{Edge}(G)})$. We will say that the diagram σ *commutes* (or that σ *is a commutative diagram*) if the following additional condition is satisfied:

(c) Let v and w be vertices of G which are joined by directed paths $(v = v_0, v_1, \ldots, v_m = w)$ and $(v = v'_0, v'_1, \ldots, v'_n = w)$ (so that the edges $(v_{i-1}, v_i), (v'_{j-1}, v'_j) \in \mathrm{Edge}(G)$ exist for $1 \leq i \leq m$ and $1 \leq j \leq n$). Then we have an identity

$$f_{v_m, v_{m-1}} \circ f_{v_{m-1}, v_{m-2}} \circ \cdots \circ f_{v_1, v_0} = f_{v'_n, v'_{n-1}} \circ f_{v'_{n-1}, v'_{n-2}} \circ \cdots \circ f_{v'_1, v'_0}$$

in the set $\mathrm{Hom}_{\mathcal{C}}(C_v, C_w)$.

Proposition 1.5.2.6. *Let K be a simplicial set of dimension ≤ 1, corresponding to a directed graph G which satisfies conditions (a) and (b) of Definition 1.5.2.5. Let $\mathcal{C}$ be an ordinary category, and let $\sigma : K \to \mathrm{N}_\bullet(\mathcal{C})$ be a diagram. Then:* `005P`

(1) *There is a partial ordering $\leq$ on the vertex set* $\mathrm{Vert}(G)$*, where we have $v \leq w$ if and only if there exists a sequence of vertices $(v = v_0, v_1, \ldots, v_n = w)$ with the property that the edges $(v_{i-1}, v_i) \in \mathrm{Edge}(G)$ exist for $1 \leq i \leq n$.*

(2) *There is a unique monomorphism of simplicial sets $K \hookrightarrow \mathrm{N}_{\bullet}(\mathrm{Vert}(G))$ which carries each vertex to itself.*

(3) *The diagram σ extends to a map $\overline{\sigma} : \mathrm{N}_{\bullet}(\mathrm{Vert}(G)) \to \mathrm{N}_{\bullet}(\mathcal{C})$ (that is, to a functor $\mathrm{Vert}(G) \to \mathcal{C}$) if and only if it is commutative, in the sense of Definition 1.5.2.5. Moreover, if the extension $\overline{\sigma}$ exists, then it is unique.*

Proof. It follows immediately from the definitions that the relation $\leq$ defined in (1) is reflexive and transitive. Antisymmetry follows from our assumption that the graph G has no directed loops (condition (b) of Definition 1.5.2.5). By construction, we have $v \leq w$ whenever v and w are connected by an edge $(v, w) \in \mathrm{Edge}(G)$. From the description of the simplicial set K given in Remark 1.1.6.10, we immediately see that there is a unique map of simplicial sets $i : K \to \mathrm{N}_{\bullet}(\mathrm{Vert}(G))$ which is the identity on vertices. It follows from assumption (a) of Definition 1.5.2.5 that the map i is a monomorphism. Let us henceforth identify K with a simplicial subset of $\mathrm{N}_{\bullet}(\mathrm{Vert}(G))$ given by the image of i. Let us identify σ with a pair $(\{C_v\}_{v \in \mathrm{Vert}(G)}, \{f_{w,v} : C_v \to C_w\}_{(v,w) \in \mathrm{Edge}(G)})$. Suppose that the diagram σ extends to a functor $\overline{\sigma} : \mathrm{N}_{\bullet}(\mathrm{Vert}(G)) \to \mathcal{C}$. If v and w are a pair of vertices of G with $v \leq w$, then we can choose a directed path $(v = v_0, v_1, \ldots, v_n = w)$ from v to w. The compatibility of $\overline{\sigma}$ with composition then guarantees that $\overline{\sigma}$ must carry the edge (v, w) of $\mathrm{N}_{\bullet}(\mathrm{Vert}(G))$ to the iterated composition $f_{v_n, v_{n-1}} \circ f_{v_{n-1}, v_{n-2}} \circ \cdots \circ f_{v_1, v_0} \in \mathrm{Hom}_{\mathcal{C}}(C_v, C_w)$. Since the morphism $\overline{\sigma}(v, w)$ is independent of the choice of directed path, it follows that the diagram σ is commutative. Conversely, if σ is commutative, then we can define $\overline{\sigma}$ on morphisms by the formula $\overline{\sigma}(v, w) = f_{v_n, v_{n-1}} \circ f_{v_{n-1}, v_{n-2}} \circ \cdots \circ f_{v_1, v_0}$ to obtain the desired extension of σ. □

00J1 **Remark 1.5.2.7.** In the situation of Proposition 1.5.2.6, an arbitrary morphism of simplicial sets $\sigma : K \to \mathrm{N}_{\bullet}(\mathcal{C})$ can be identified with a functor $F : \mathrm{Path}[G] \to \mathcal{C}$, where $\mathrm{Path}[G]$ denotes the path category of the graph G (Proposition 1.3.7.5). The commutativity of the diagram σ is equivalent to the requirement that F factors through the quotient functor $\mathrm{Path}[G] \twoheadrightarrow \mathrm{Vert}(G)$: that is, the value of F on a path p depends only the endpoints of p.

005Q **Example 1.5.2.8** (Commutative Squares in a Category)**.** Let $K = \partial(\Delta^1 \times \Delta^1)$ be as in Example 1.5.2.4. For any ordinary category $\mathcal{C}$, we can display a diagram $\sigma : K \to \mathrm{N}_{\bullet}(\mathcal{C})$

pictorially as

$$\begin{tikzcd}
C_{00} \arrow[r, "f"] \arrow[d, "g"] & C_{01} \arrow[d, "g'"] \\
C_{10} \arrow[r, "f'"] & C_{11}.
\end{tikzcd}$$

The diagram σ is commutative if and only if we have $g' \circ f = f' \circ g$ in $\mathrm{Hom}_{\mathcal{C}}(C_{00}, C_{11})$. In this case, Proposition 1.5.2.6 ensures that σ extends uniquely to a diagram $\overline{\sigma} : \Delta^1 \times \Delta^1 \rightarrow \mathrm{N}_{\bullet}(\mathcal{C})$, or equivalently to a functor of ordinary categories $[1] \times [1] \rightarrow \mathcal{C}$.

In the setting of ∞-categories, assertion (3) of Proposition 1.5.2.6 is false in general.

Example 1.5.2.9 (Square Diagrams in an ∞-Category)**.** Let I denote the partially ordered set $[1] \times [1]$. The simplicial set $\mathrm{N}_{\bullet}(I) \simeq \Delta^1 \times \Delta^1$ has four vertices (given by the elements of I), five nondegenerate edges, and two nondegenerate 2-simplices. Unwinding the definitions, we see that an I-indexed diagram in an ∞-category $\mathcal{C}$ is equivalent to the following data: `005R`

- A collection of objects $\{C_{ij}\}_{0 \leq i,j \leq 1}$ in $\mathcal{C}$.

- A collection of morphisms $f : C_{00} \rightarrow C_{01}$, $g : C_{00} \rightarrow C_{10}$, $f' : C_{10} \rightarrow C_{11}$, $g' : C_{01} \rightarrow C_{11}$, and $h : C_{00} \rightarrow C_{11}$.

- A 2-simplex σ of $\mathcal{C}$ which witnesses h as a composition of f with g', and a 2-simplex τ of $\mathcal{C}$ which witnesses h as a composition of g with f'.

This data can be depicted graphically as follows:

$$\begin{tikzcd}
C_{00} \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, "h"] & C_{01} \arrow[d, "g'"] \\
C_{10} \arrow[r, "f'"] & C_{11}.
\end{tikzcd}$$

σ

τ

Beware that such a diagram is usually not determined by its restriction to the simplicial subset $K \subseteq \mathrm{N}_\bullet(I)$ of Example 1.5.2.8.

005S **Exercise 1.5.2.10.** Let $\mathcal{C}$ be an ∞-category and let $K \subseteq \Delta^1 \times \Delta^1$ be the simplicial subset appearing in Example 1.5.2.8. Suppose we are given a diagram $\sigma : K \to \mathcal{C}$, which we depict graphically as

$$\begin{tikzcd}
C_{00} \arrow[r, "f"] \arrow[d, "g"] & C_{01} \arrow[d, "g'"] \\
C_{10} \arrow[r, "f'"] & C_{11}.
\end{tikzcd}$$

Composing with the unit map $\mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$, we obtain a diagram σ' in the homotopy category $\mathrm{h}\mathcal{C}$, which we can depict as

$$\begin{tikzcd}
C_{00} \arrow[r, "{[f]}"] \arrow[d, "{[g]}"] & C_{01} \arrow[d, "{[g']}"] \\
C_{10} \arrow[r, "{[f']}"] & C_{11}.
\end{tikzcd}$$

Show that the diagram σ' is commutative if and only if σ can be extended to a map $\overline{\sigma} : \Delta^1 \times \Delta^1 \to \mathcal{C}$. Beware that this extension is generally not unique.

Warning 1.5.2.11. Let I be a partially ordered set and let $\mathcal{C}$ be an ∞-category. In the case $I = [1] \times [1]$, Exercise 1.5.2.10 implies that every functor of ordinary categories $I \to \mathrm{h}\mathcal{C}$ can be lifted to a functor of ∞-categories $\mathrm{N}_{\bullet}(I) \to \mathcal{C}$. Beware that this conclusion is generally false for more complicated partially ordered sets. For example, it fails for the partially ordered set $I = [1] \times [1] \times [1]$. `005T`

Example 1.5.2.9 illustrates that the notion of "commutative diagram" becomes considerably more subtle in the setting of ∞-categories. To specify an I-indexed diagram $F : \mathrm{N}_{\bullet}(I) \to \mathcal{C}$ of an ∞-category $\mathcal{C}$, one generally needs to specify the values of F on *all* the simplices of the simplicial set $\mathrm{N}_{\bullet}(I)$. In general, it is not feasible to graphically encode *all* of this data in a comprehensible way. On the other hand, the formalism of commutative diagrams is too useful to completely abandon. We will therefore sacrifice some degree of mathematical precision in favor of clarity of exposition.

Convention 1.5.2.12. Let $\mathcal{C}$ be an ∞-category and let G be a directed graph satisfying conditions (a) and (b) of Definition 1.5.2.5, so that the vertex set $\mathrm{Vert}(G)$ inherits a partial ordering (Proposition 1.5.2.6). We will sometimes refer to the notion of a *commutative* diagram σ in $\mathcal{C}$, which we indicate graphically by a collection of objects $\{C_v\}_{v \in \mathrm{Vert}(G)}$ of $\mathcal{C}$, connected by arrows which are labelled by morphisms $\{f_e\}_{e \in \mathrm{Edge}(G)}$. In this case, it should be understood that σ is a diagram $\mathrm{N}_{\bullet}(\mathrm{Vert}(G)) \to \mathcal{C}$, which carries each vertex v of $\mathrm{N}_{\bullet}(\mathrm{Vert}(G))$ to the object $C_v \in \mathcal{C}$ and each edge $e = (v, w)$ of G to the morphism f_e in $\mathcal{C}$. Beware that in this case, the map σ need not be completely determined by the pair $(\{C_v\}_{v \in \mathrm{Vert}(G)}, \{f_e\}_{e \in \mathrm{Edge}(G)})$ (this pair can instead be identified with the restriction $\sigma|_K$, where K is the 1-dimensional simplicial subset of $\mathrm{N}_{\bullet}(\mathrm{Vert}(G))$ corresponding to G). `005U`

Remark 1.5.2.13. In the situation of Convention 1.5.2.12, suppose that $\mathcal{C} = \mathrm{N}_{\bullet}(\mathcal{C}_0)$, where $\mathcal{C}_0$ is an ordinary category. Then giving a commutative diagram in the ∞-category $\mathcal{C}$ (in the sense of Convention 1.5.2.12) is equivalent to giving a commutative diagram in the ordinary category $\mathcal{C}_0$ (in the sense of Definition 1.5.2.5). In this case, commutativity is a *property* that the underlying diagram (indexed by a 1-dimensional simplicial set) does or does not possess. For a general ∞-category $\mathcal{C}$, commutativity of a diagram in $\mathcal{C}$ is not a property but a *structure*; to promote a diagram to a commutative diagram, one must specify additional data to *witness* the requisite commutativity. `005V`

005W **Example 1.5.2.14.** Let $\mathcal{C}$ be an ∞-category. If we refer to a commutative diagram σ :

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z, \end{tikzcd}$$

then we mean that σ is a 2-simplex of $\mathcal{C}$ satisfying $d_0^2(\sigma) = g$, $d_1^2(\sigma) = h$, and $d_2^2(\sigma) = f$. In other words, we mean that σ is a 2-simplex which witnesses h as a composition of f and g, in the sense of Definition 1.4.4.1.

005X **Example 1.5.2.15.** Let $\mathcal{C}$ be an ∞-category. If we refer to a commutative diagram σ :

$$\begin{tikzcd} C_{00} \arrow[r, "f"] \arrow[d, "g"] & C_{01} \arrow[d, "g'"] \\ C_{10} \arrow[r, "f'"] & C_{11}, \end{tikzcd}$$

we implicitly assume that σ is a map from the entire simplicial set $\Delta^1 \times \Delta^1$ to $\mathcal{C}$. In other words, we assume that we have specified another morphism $h : C_{00} \to C_{11}$, which is not indicated in the picture, together with a 2-simplex σ witnessing h as the composition of f and g' and a 2-simplex τ witnessing h as the composition of g and f'.

005Y **Warning 1.5.2.16.** In ordinary category theory, it is sometimes useful to refer to the commutativity of diagrams in situations which do not fit the paradigm of Definition 1.5.2.5. For example, the commutativity of a diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & Z \end{tikzcd}$$

is often understood as the requirement that $u \circ f = v \circ f$. Beware that this usage is potentially ambiguous (from the shape of the diagram alone, it is not clear that commutativity should enforce the identity $u \circ f = v \circ f$, but not the identity $u = v$), so we will take special care when applying similar terminology in the ∞-categorical setting.

1.5.3 The ∞-Category of Functors

005Z Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then we can form a new category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations. In this section, we describe an analogous construction in the setting of ∞-categories.

Construction 1.5.3.1. Let S and T be simplicial sets. Then the construction 0060

$$([n] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n \times S, T)$$

determines a functor from the category $\mathbf{\Delta}^{\mathrm{op}}$ to the category of sets. We regard this functor as a simplicial set which we will denote by $\mathrm{Fun}(S,T)$.

Note that, given an n-simplex f of $\mathrm{Fun}(S,T)$ and an n-simplex σ of S, we can construct an n-simplex $\mathrm{ev}(f,\sigma)$ of T, given by the composition

$$\Delta^n \xrightarrow{\delta} \Delta^n \times \Delta^n \xrightarrow{\mathrm{id} \times \sigma} \Delta^n \times S \xrightarrow{f} T.$$

This construction determines a map of simplicial sets $\mathrm{ev} : \mathrm{Fun}(S,T) \times S \to T$, which we will refer to as *the evaluation map.*

Proposition 1.5.3.2. *Let S, T, and U be simplicial sets. Then the composite map* 0061

$$\begin{aligned} \theta : \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(U, \mathrm{Fun}(S,T)) &\rightarrow \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(U \times S, \mathrm{Fun}(S,T) \times S) \\ &\xrightarrow{\mathrm{ev} \circ} \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(U \times S, T) \end{aligned}$$

is bijective.

Proof. Let $f : U \times S \to T$ be a map of simplicial sets. For each n-simplex σ of U, the composite map

$$\Delta^n \times S \xrightarrow{\sigma \times \mathrm{id}} U \times S \xrightarrow{f} T$$

can be regarded as an n-simplex of $\mathrm{Fun}(S,T)$, which we will denote by $g(\sigma)$. The construction $\sigma \mapsto g(\sigma)$ determines a map of simplicial sets $g : U \to \mathrm{Fun}(S,T)$. We leave as an exercise for the reader to verify that g is the unique map satisfying $\theta(g) = f$. □

Beware that the notation of Construction 1.5.3.1 is potentially confusing, because it conflicts with our use of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ to denote the category of functors from a category $\mathcal{C}$ to a category $\mathcal{D}$. However, these usages are compatible:

Proposition 1.5.3.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $e : \mathrm{Fun}(\mathcal{C},\mathcal{D}) \times \mathcal{C} \to \mathcal{D}$ denote the evaluation functor, given on objects by the formula $e(F,C) = F(C)$. Then the composite map* 0062

$$\mathrm{N}_{\bullet}(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \times \mathrm{N}_{\bullet}(\mathcal{C}) \simeq \mathrm{N}_{\bullet}(\mathrm{Fun}(\mathcal{C},\mathcal{D}) \times \mathcal{C}) \xrightarrow{\mathrm{N}_{\bullet}(e)} \mathrm{N}_{\bullet}(\mathcal{D})$$

corresponds, under the bijection of Proposition 1.5.3.2, to an isomorphism of simplicial sets $\rho : \mathrm{N}_{\bullet}(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \to \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}), \mathrm{N}_{\bullet}(\mathcal{D}))$.

Proof. For each $n \geq 0$, the map ρ is given on n-simplices by the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{N}_\bullet(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))) &\simeq \mathrm{Hom}_{\mathrm{Cat}}([n], \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Cat}}([n] \times \mathcal{C}, \mathcal{D}) \\
&\xrightarrow{v} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet([n] \times \mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet([n]) \times \mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n \times \mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{Fun}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D}))).
\end{aligned}$$

It will therefore suffice to show that v is bijective, which is a special case of Proposition 1.3.3.1. □

Passing to homotopy categories, we obtain the following weaker result:

0063 **Corollary 1.5.3.4.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then there is a canonical isomorphism of categories*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{hFun}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet(\mathcal{D})).$$

We can also generalize Proposition 1.5.3.3 as follows:

0064 **Corollary 1.5.3.5.** *Let S be a simplicial set having homotopy category $\mathrm{h}S$. Then, for any category $\mathcal{D}$, the composite map*

$$\mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}S, \mathcal{D})) \times S \to \mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}S, \mathcal{D})) \times \mathrm{N}_\bullet(\mathrm{h}S) \simeq \mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}S, \mathcal{D}) \times \mathrm{h}S) \to \mathrm{N}_\bullet(\mathcal{D})$$

induces an isomorphism of simplicial sets $\rho_S : \mathrm{N}_\bullet(\mathrm{Fun}(\mathrm{h}S, \mathcal{D})) \simeq \mathrm{Fun}(S, \mathrm{N}_\bullet(\mathcal{D}))$.

Proof. The construction $S \mapsto \rho_S$ carries colimits (in the category Set_Δ of simplicial sets) to limits (in the category $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$ of morphisms between simplicial sets). Since every simplicial set can be realized as a colimit of standard simplices (Remark 1.1.3.13), it will suffice to prove Corollary 1.5.3.5 in the special case where $S = \Delta^n$ for some $n \geq 0$. In this case, the desired result follows from Proposition 1.5.3.3, since S is isomorphic to the nerve of the category $\mathcal{C} = [n]$. □

0065 **Corollary 1.5.3.6.** *The formation of homotopy categories determines a functor $\mathrm{Set}_\Delta \to \mathrm{Cat}$ which commutes with finite products.*

Proof. Since the construction $S \mapsto \mathrm{h}S$ preserves final objects, it will suffice to show that for any pair of simplicial sets S and T, the canonical map $u : \mathrm{h}(S \times T) \to \mathrm{h}S \times \mathrm{h}T$. s an isomorphism of categories. In other words, we wish to show that for any category $\mathcal{C}$, composition with u induces a bijection

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}S \times \mathrm{h}T, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}(S \times T), \mathcal{C}).$$

Unwinding the definitions, we see that this map is given by the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}S \times \mathrm{h}T, \mathcal{C}) & \simeq & \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}S, \mathrm{Fun}(\mathrm{h}T, \mathcal{C})) \\
& \simeq & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S, \mathrm{N}_{\bullet}(\mathrm{Fun}(\mathrm{h}T, \mathcal{C}))) \\
& \xrightarrow{\rho_T \circ} & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S, \mathrm{Fun}(T, \mathrm{N}_{\bullet}(\mathcal{C}))) \\
& \simeq & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S \times T, \mathrm{N}_{\bullet}(\mathcal{C})) \\
& \simeq & \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{h}(S \times T), \mathcal{C}),
\end{aligned}$$

where ρ_T is the isomorphism appearing in the statement of Corollary 1.5.3.5. □

We will be primarily interested in the special case of Construction 1.5.3.1 where the target simplicial set T is an ∞-category. In this case, we have the following result:

Theorem 1.5.3.7. *Let S be a simplicial set and let $\mathcal{D}$ be an ∞-category. Then the simplicial set* $\mathrm{Fun}(S, \mathcal{D})$ *is an ∞-category.* 0066

The proof of Theorem 1.5.3.7 will require some combinatorial preliminaries; we defer the proof to §1.5.6.

Definition 1.5.3.8. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. It follows from Theorem 1.5.3.7 that the simplicial set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also an ∞-category. We will refer to $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as *the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$*. 0067

Remark 1.5.3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. By definition, the objects of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ can be identified with functors from $\mathcal{C}$ to $\mathcal{D}$, in the sense of Definition 1.5.0.1 (that is, with maps of simplicial sets from $\mathcal{C}$ to $\mathcal{D}$). 0068

Remark 1.5.3.10. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and suppose we are given a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$. We define a *natural transformation from F to G* to be a map of simplicial sets $u : \Delta^1 \times \mathcal{C} \to \mathcal{D}$ satisfying $u|_{\{0\} \times \mathcal{C}} = F$ and $u|_{\{1\} \times \mathcal{C}} = G$. In other words, a natural transformation from F to G is a morphism from F to G in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. 0069

Remark 1.5.3.11. Let us abuse notation by identifying each ordinary category $\mathcal{E}$ with the ∞-category $\mathrm{N}_{\bullet}(\mathcal{E})$. In this case, Corollary 1.5.3.5 implies that when $\mathcal{C}$ is an ∞-category and $\mathcal{D}$ is an ordinary category, then we have a canonical isomorphism $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathcal{D})$. In particular, the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is also an ordinary category. 006A

1.5.4 Digression: Lifting Properties

We now review some categorical terminology which will be useful in the proof of Theorem 1.5.3.7, and in several other parts of this book. 006B

006C **Definition 1.5.4.1.** Let $\mathcal{C}$ be a category. A *lifting problem* in $\mathcal{C}$ is a commutative diagram σ :

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & X \arrow[d, "g"] \\
B \arrow[r, "v"] & Y
\end{tikzcd}$$

in $\mathcal{C}$. A *solution to the lifting problem* σ is a morphism $h : B \to X$ in $\mathcal{C}$ satisfying $g \circ h = v$ and $h \circ f = u$, as indicated in the diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & X \arrow[d, "g"] \\
B \arrow[r, "v"] \arrow[ur, "h"] & Y.
\end{tikzcd}$$

006D **Remark 1.5.4.2.** In the situation of Definition 1.5.4.1, we will often indicate a lifting problem by a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & X \arrow[d, "g"] \\
B \arrow[r, "v"] \arrow[ur, dashed, "h"] & Y,
\end{tikzcd}$$

which includes a dotted arrow representing a hypothetical solution.

006E **Definition 1.5.4.3.** Let $\mathcal{C}$ be a category and suppose we are given a morphism $f : A \to B$ and $g : X \to Y$ in $\mathcal{C}$. We will say that f *is weakly left orthogonal to* g if, for every pair of morphisms $u : A \to X$ and $v : B \to Y$ satisfying $g \circ u = v \circ f$, the associated lifting problem

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & X \arrow[d, "g"] \\
B \arrow[r, "v"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution (that is, there exists a map $h : B \to X$ satisfying $g \circ h = v$ and $h \circ f = u$). In this case, we will also say that g *is weakly right orthogonal to* f.

If S and T are collections of morphisms of $\mathcal{C}$, we say that S *is weakly left orthogonal to* T if every morphism $f \in S$ is weakly left orthogonal to every morphism $g \in T$. In this case, we also say that T *is weakly right orthogonal to* S. In the special case where $S = \{f\}$ is a singleton, we abbreviate this condition by saying that f *is weakly left orthogonal to* T, or T

is weakly right orthogonal to f. In the special case $T = \{g\}$ is a singleton, we abbreviate this condition by saying that g *is weakly right orthogonal to* S, or S *is weakly left orthogonal to* g.

Let T be a collection of morphisms in a category $\mathcal{C}$. We now summarize some closure properties enjoyed by the collection of morphisms which are weakly left orthogonal to T.

Definition 1.5.4.4. Let $\mathcal{C}$ be a category which admits pushouts and let S be a collection of morphisms of $\mathcal{C}$. We will say that S is *closed under pushouts* if, for every pushout diagram 006F

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

in the category $\mathcal{C}$ where the morphism f belongs to S, the morphism f' also belongs to S.

Proposition 1.5.4.5. *Let $\mathcal{C}$ be a category which admits pushouts, let T be a collection of morphisms of $\mathcal{C}$, and let S be the collection of all morphisms of $\mathcal{C}$ which are weakly left orthogonal to T. Then S is closed under pushouts.* 006G

Proof. Suppose we are given a pushout diagram σ :

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r, "t"] & B' \end{tikzcd}$$

where f belongs to S. We wish to show that f' also belongs to S. For this, we must show that every lifting problem

$$\begin{tikzcd} A' \arrow[r, "u"] \arrow[d, "f'"] & X \arrow[d, "g"] \\ B' \arrow[r, "v"] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a solution, provided that the morphism g belongs to T. Using our assumption that σ is a pushout square, we are reduced to solving the associated lifting problem

$$\begin{tikzcd} A \arrow[r, "u \circ s"] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r, "v \circ t"] \arrow[ur, dashed] & Y, \end{tikzcd}$$

which is possible by virtue of our assumption that f is weakly left orthogonal to g. □

006H **Definition 1.5.4.6.** Let $\mathcal{C}$ be a category containing a pair of objects C and C'. We will say that C *is a retract of* C' if there exist maps $i : C \to C'$ and $r : C' \to C$ such that $r \circ i = \mathrm{id}_C$.

006J **Variant 1.5.4.7.** Let $\mathcal{C}$ be a category. We will say that a morphism $f : C \to D$ of $\mathcal{C}$ is a *retract* of another morphism $f' : C' \to D'$ if it is a retract of f' when viewed as an object of the functor category $\mathrm{Fun}([1], \mathcal{C})$. In other words, we say that f is a retract of f' if there exists a commutative diagram

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "f"] & C' \arrow[r, "r"] \arrow[d, "f'"] & C \arrow[d, "f"] \\
D \arrow[r, "\bar{i}"] & D' \arrow[r, "\overline{r}"] & D
\end{tikzcd}$$

in the category $\mathcal{C}$, where $r \circ i = \mathrm{id}_C$ and $\overline{r} \circ \bar{i} = \mathrm{id}_D$.

We say that a collection of morphisms T of $\mathcal{C}$ is *closed under retracts* if, for every pair of morphisms f, f' in $\mathcal{C}$, if f is a retract of f' and f' belongs to T, then f also belongs to T.

006K **Exercise 1.5.4.8.** Let $\mathcal{C}$ be a category and let S be the collection of all monomorphisms in $\mathcal{C}$. Show that S is closed under retracts.

006L **Proposition 1.5.4.9.** *Let $\mathcal{C}$ be a category, let T be a collection of morphisms of $\mathcal{C}$, and let S be the collection of all morphisms of $\mathcal{C}$ which are weakly left orthogonal to S. Then S is closed under retracts.*

Proof. Let f' be a morphism of $\mathcal{C}$ which belongs to S and let f be a retract of f', so that there exists a commutative diagram

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "f"] & C' \arrow[r, "r"] \arrow[d, "f'"] & C \arrow[d, "f"] \\
D \arrow[r, "\bar{i}"] & D' \arrow[r, "\overline{r}"] & D
\end{tikzcd}$$

with $r \circ i = \mathrm{id}_C$ and $\overline{r} \circ \bar{i} = \mathrm{id}_D$. We wish to show that f also belongs to S. Consider a lifting problem σ :

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "f"] & X \arrow[d, "g"] \\
D \arrow[r, "v"] \arrow[ur, dashed, "h"] & Y,
\end{tikzcd}$$

where g belongs to T. Our assumption $f' \in S$ ensures that the associated lifting problem

$$\begin{tikzcd}
C' \arrow[r, "u \circ r"] \arrow[d, "f'"] & X \arrow[d, "g"] \\
D' \arrow[r, "v \circ \overline{r}"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution: that is, we can choose a morphism $h' : D' \to X$ satisfying $g \circ h' = v \circ \overline{r}$ and $h' \circ f' = u \circ r$. Then the morphism $h = h' \circ \overline{i}$ is a solution to the lifting problem σ, by virtue of the calculations

$$g \circ h = g \circ h' \circ \overline{i} = v \circ \overline{r} \circ \overline{i} = v$$

$$h \circ f = h' \circ \overline{i} \circ f = h' \circ f' \circ i = u \circ r \circ i = u.$$

□

In what follows, we assume that the reader is familiar with the theory of ordinals (see §4.7.1 for a quick review).

Definition 1.5.4.10. For every ordinal α, let $\mathrm{Ord}_{\leq \alpha} = \{\beta : \beta \leq \alpha\}$ denote the collection of all ordinal numbers which are less than or equal to α, regarded as a linearly ordered set. 006M

Let $\mathcal{C}$ be a category and let S be a collection of morphisms of $\mathcal{C}$. We will say that a morphism f of $\mathcal{C}$ is a *transfinite composition of morphisms of S* if there exists an ordinal α and a functor $F : \mathrm{Ord}_{\leq \alpha} \to \mathcal{C}$, given by a collection of objects $\{C_\beta\}_{\beta \leq \alpha}$ and morphisms $\{f_{\gamma,\beta} : C_\beta \to C_\gamma\}_{\beta \leq \gamma}$ with the following properties:

(a) For every nonzero limit ordinal $\lambda \leq \alpha$, the functor F exhibits C_λ as a colimit of the diagram $(\{C_\beta\}_{\beta < \lambda}, \{f_{\gamma,\beta}\}_{\beta \leq \gamma < \lambda})$.

(b) For every ordinal $\beta < \alpha$, the morphism $f_{\beta+1,\beta}$ belongs to S.

(c) The morphism f is equal to $f_{\alpha,0} : C_0 \to C_\alpha$.

We will say that S is *closed under transfinite composition* if, for every morphism f which is a transfinite composition of morphisms of S, we have $f \in S$.

Proposition 1.5.4.11. *Let $\mathcal{C}$ be a category, let T be a collection of morphisms in $\mathcal{C}$, and let S be the collection of all morphisms of $\mathcal{C}$ which are weakly left orthogonal to T. Then S is closed under transfinite composition.* 006R

Proof. Let α be an ordinal and suppose we are given a functor $\mathrm{Ord}_{\leq \alpha} \to \mathcal{C}$, given by a pair

$$(\{C_\beta\}_{\beta \leq \alpha}, \{f_{\gamma,\beta}\}_{\beta \leq \gamma \leq \alpha})$$

which satisfies condition (a) of Definition 1.5.4.10. Assume that each of the morphisms $f_{\beta+1,\beta}$ belongs to S. We wish to show that the morphism $f_{\alpha,0}$ also belongs to S. For this, we must show that every lifting problem σ :

$$\begin{tikzcd}
C_0 \arrow[r, "u"] \arrow[d, "f_{\alpha,0}"'] & X \arrow[d, "g"] \\
C_\alpha \arrow[r, "v"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution, provided that g belongs to T. We construct a collection of morphisms $\{u_\beta : C_\beta \to X\}_{\beta \leq \alpha}$, satisfying the requirements $g \circ u_\beta = v \circ f_{\alpha,\beta}$ and $u_\beta = u_\gamma \circ f_{\gamma,\beta}$ for $\beta \leq \gamma$, using transfinite recursion. Fix an ordinal $\gamma \leq \alpha$, and assume that the morphisms $\{u_\beta\}_{\beta<\gamma}$ have been constructed. We consider three cases:

- If $\gamma = 0$, we set $u_\gamma = u$.
- If γ is a nonzero limit ordinal, then our hypothesis that C_γ is the colimit of the diagram $\{C_\beta\}_{\beta<\gamma}$ guarantees that there is a unique morphism $u_\gamma : C_\gamma \to X$ satisfying $u_\beta = u_\gamma \circ f_{\gamma,\beta}$ for $\beta < \gamma$. Moreover, our assumption that the equality $g \circ u_\beta = v \circ f_{\alpha,\beta}$ holds for $\beta < \gamma$ guarantees that it also holds for $\beta = \gamma$.
- Suppose that $\gamma = \beta + 1$ is a successor ordinal. In this case, we take u_γ to be any solution to the lifting problem

$$\begin{tikzcd}
C_\beta \arrow[r, "u_\beta"] \arrow[d, "f_{\beta+1,\beta}"'] & X \arrow[d, "g"] \\
C_{\beta+1} \arrow[r, "v \circ f_{\alpha,\beta+1}"] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

  which exists by virtue of our assumption that $f_{\beta+1,\beta}$ belongs to S.

We now complete the proof by observing that u_α is a solution to the lifting problem σ. □

Motivated by the preceding discussion, we introduce the following:

006S **Definition 1.5.4.12.** Let $\mathcal{C}$ be a category which admits small colimits and let S be a collection of morphisms of $\mathcal{C}$. We will say that S is *weakly saturated* if it is closed under pushouts (Definition 1.5.4.4), retracts (Variant 1.5.4.7), and transfinite composition (Definition 1.5.4.10).

Proposition 1.5.4.13. *Let $\mathcal{C}$ be a category which admits small colimits, let T be a collection of morphisms of $\mathcal{C}$, and let S be the collection of all morphisms of $\mathcal{C}$ which are weakly left orthogonal to T. Then S is weakly saturated.* 006T

Proof. Combine Propositions 1.5.4.5, 1.5.4.9, and 1.5.4.11. □

Remark 1.5.4.14. Let $\mathcal{C}$ be a category and let S_0 be a collection of morphisms of $\mathcal{C}$. Then there exists a smallest collection of morphisms S of $\mathcal{C}$ such that $S_0 \subseteq S$ and S is weakly saturated (for example, we can take S to be the intersection of all the weakly saturated collections of morphisms containing S_0). We will refer to S as the *weakly saturated collection of morphisms generated by* S_0. It follows from Proposition 1.5.4.13 that if S_0 is weakly left orthogonal to some collection of morphisms T, then S has the same property. 006U

1.5.5 Trivial Kan Fibrations

We now specialize the ideas of §1.5.4 to the category of simplicial sets. 006V

Definition 1.5.5.1. Let $q : X \to Y$ be a morphism of simplicial sets. We say that p is a *trivial Kan fibration* if, for each $n \geq 0$, every lifting problem 006W

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d, "i"] & X \arrow[d, "q"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution; here $i : \partial\Delta^n \hookrightarrow \Delta^n$ denotes the inclusion map.

Remark 1.5.5.2. Suppose we are given a pullback diagram of simplicial sets 006X

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\
Y' \arrow[r] & Y.
\end{tikzcd}$$

If q is a trivial Kan fibration, then so is q' (this follows from Proposition 1.5.4.5, applied to the opposite of the category Set_Δ).

Remark 1.5.5.3. The collection of trivial Kan fibrations is closed under filtered colimits (when regarded as a full subcategory of the arrow category $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$). 02L0

Proposition 1.5.5.4. *Let $p : X \to Y$ be a map of simplicial sets. The following conditions are equivalent:* 006Y

(1) *The map p is a trivial Kan fibration (in the sense of Definition 1.5.5.1).*

(2) *The map p is weakly right orthogonal to every monomorphism of simplicial sets $i : A \hookrightarrow B$. In other words, every lifting problem*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution, provided that i is a monomorphism.

We will give the proof of Proposition 1.5.5.4 at the end of this section.

006Z **Corollary 1.5.5.5.** *Let $p : X \to Y$ be a trivial Kan fibration of simplicial sets. Then:*

(a) *The map p admits a section: that is, there is a map of simplicial sets $s : Y \to X$ such that the composition $p \circ s$ is the identity map $\mathrm{id}_Y : Y \to Y$.*

(b) *Let s be any section of p. Then the composition $s \circ p : X \to X$ is fiberwise homotopic to the identity. That is, there exists a map of simplicial sets $h : \Delta^1 \times X \to X$, compatible with the projection to Y, such that $h|_{\{0\}\times X} = s \circ p$ and $h|_{\{1\}\times X} = \mathrm{id}_X$.*

Proof. To prove (a), we observe that a section of p can be described as a solution to the lifting problem

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
Y \arrow[r, "\mathrm{id}"] \arrow[ur, dashed, "s"] & Y,
\end{tikzcd}$$

which exists by virtue of Proposition 1.5.5.4. Given any section s, a fiberwise homotopy from $s \circ p$ to the identity can be identified with a solution to the lifting problem

$$\begin{tikzcd}
\partial \Delta^1 \times X \arrow[r, "(s\circ p, \mathrm{id})"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^1 \times X \arrow[r] \arrow[ur, dashed, "h"] & Y,
\end{tikzcd}$$

which again exists by virtue of Proposition 1.5.5.4. □

Corollary 1.5.5.6. *Let $p : X \to Y$ be a trivial Kan fibration of simplicial sets and let $i : A \to B$ be a monomorphism of simplicial sets. Then the canonical map* 0070

$$\theta : \operatorname{Fun}(B, X) \to \operatorname{Fun}(B, Y) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(A, X)$$

is also a trivial Kan fibration.

Proof. Fix an integer $n \geq 0$; we wish to show that every lifting problem

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \operatorname{Fun}(B, X) \arrow[d, "\theta"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(B, Y) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(A, X)
\end{tikzcd}$$

admits a solution. Unwinding the definitions, we see that this is equivalent to solving an associated lifting problem

$$\begin{tikzcd}
(\partial \Delta^n \times B) \coprod_{\partial \Delta^n \times A} (\Delta^n \times A) \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\
\Delta^n \times B \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

This is possible by virtue of Proposition 1.5.5.4, since p is a trivial Kan fibration and i is a monomorphism. □

Corollary 1.5.5.7. *Let $p : X \to Y$ be a trivial Kan fibration of simplicial sets. Then, for every simplicial set B, the induced map* $\operatorname{Fun}(B, X) \to \operatorname{Fun}(B, Y)$ *is a trivial Kan fibration.* 0071

Proof. Apply Corollary 1.5.5.6 in the special case $A = \emptyset$. □

Definition 1.5.5.8. Let X be a simplicial set. We say that X is a *contractible Kan complex* if the projection map $X \to \Delta^0$ is a trivial Kan fibration (Definition 1.5.5.1). In other words, X is a contractible Kan complex if every map $\sigma_0 : \partial \Delta^n \to X$ can be extended to an n-simplex of X. 0072

Example 1.5.5.9. Let X be a topological space. Then the singular simplicial set $\operatorname{Sing}_\bullet(X)$ is a contractible Kan complex if and only if the space X is *weakly contractible*: that is, if and 0073 only if every continuous map $\sigma_0 : S^{n-1} \to X$ is nullhomotopic (here $S^{n-1} \simeq |\,\partial\Delta^n\,|$ denotes the sphere of dimension $n-1$, so that σ_0 is nullhomotopic if and only if it extends to a continuous map defined on the disk $D^n \simeq |\Delta^n|$). In particular, if the topological space X is contractible, then the simplicial set $\mathrm{Sing}_\bullet(X)$ is a contractible Kan complex.

0074 **Remark 1.5.5.10.** Let $p : X \to Y$ be a trivial Kan fibration. Then, for every vertex y of Y, the fiber $X \times_Y \{y\}$ is a contractible Kan complex (this is a special case of Remark 1.5.5.2). For a partial converse, see Proposition 3.3.7.6.

050J **Proposition 1.5.5.11.** *Let $p : X \to Y$ be a trivial Kan fibration of simplicial sets. Then:*

(1) *If X is a Kan complex, then Y is a Kan complex.*

(2) *If X is a contractible Kan complex, then Y is a contractible Kan complex.*

(3) *If X is an ∞-category, then Y is an ∞-category.*

Proof. We will prove (1); the proofs of (2) and (3) are similar. Suppose we are given a pair of integers $0 \leq i \leq n$ with $n > 0$; we wish to show that every morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to Y$ can be extended to an n-simplex of Y. Since p is a trivial Kan fibration, we can write $\sigma_0 = p \circ \tau_0$ for some morphism $\tau_0 : \Lambda^n_i \to X$ (Proposition 1.5.5.4). If X is a Kan complex, we can extend τ_0 to an n-simplex τ of X. Then $\sigma = p \circ \tau$ is an n-simplex of Y satisfying $\sigma_0 = \sigma|_{\Lambda^n_i}$. □

Applying Proposition 1.5.5.4 in the case $Y = \Delta^0$, we obtain the following:

0075 **Corollary 1.5.5.12.** *Let X be a simplicial set. The following conditions are equivalent:*

(1) *The simplicial set X is a contractible Kan complex.*

(2) *For every monomorphism of simplicial sets $i : A \hookrightarrow B$ and every map of simplicial sets $f_0 : A \to X$, there exists a map $f : B \to X$ such that $f_0 = f \circ i$.*

0076 **Corollary 1.5.5.13.** *Let X be a contractible Kan complex. Then X is a Kan complex. In particular, X is an ∞-category.*

We will deduce Proposition 1.5.5.4 from the following:

0077 **Proposition 1.5.5.14.** *Let T be the collection of all monomorphisms in the category* Set_Δ *of simplicial sets. Then:*

(*a*) *The collection T is weakly saturated, in the sense of Definition 1.5.4.12.*

(*b*) *As a weakly saturated collection of morphisms, T is generated by the collection of inclusion maps $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n \geq 0}$ (see Remark 1.5.4.14).*

Proof. To prove (a), we must establish the following:

- The collection T is closed under pushouts. That is, if we are given a pushout diagram of simplicial sets

  $$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

  where f is a monomorphism, then f' is also a monomorphism. This is clear, since we have a pushout diagram

  $$\begin{tikzcd} A_n \arrow[r] \arrow[d] & A'_n \arrow[d] \\ B_n \arrow[r] & B'_n \end{tikzcd}$$

  in the category of sets for each $n \geq 0$ (where the left vertical map is injective, so the right vertical map is injective as well).

- The collection T is closed under retracts. This is a special case of Exercise 1.5.4.8.

- The collection T is closed under transfinite composition. Suppose we are given an ordinal α and a functor $S : \mathrm{Ord}_{\leq \alpha} \to \mathrm{Set}_{\Delta}$, given by a collection of simplicial sets $\{S(\beta)\}_{\beta \leq \alpha}$ and transition maps $f_{\gamma,\beta} : S(\beta) \to S(\gamma)$. Assume that the maps $f_{\beta+1,\beta}$ are monomorphisms for $\beta < \alpha$ and that, for every nonzero limit ordinal $\lambda \leq \alpha$, the induced map $\varinjlim_{\beta < \lambda} S(\beta) \to S(\lambda)$ is an isomorphism. We must show that the map $f_{\alpha,0} : S(0) \to S(\alpha)$ is a monomorphism of simplicial sets. In fact, we claim that for each $\gamma \leq \alpha$, the map $f_{\gamma,0} : S(0) \to S(\gamma)$ is a monomorphism. The proof proceeds by transfinite induction on γ. In the case $\gamma = 0$, the map $f_{\gamma,0} = \mathrm{id}_{S(0)}$ is an isomorphism. If γ is a nonzero limit ordinal, then the desired result follows from our inductive hypothesis, since the collection of monomorphisms in Set_{Δ} is closed under filtered colimits. If $\gamma = \beta + 1$ is a successor ordinal, then we can identify $f_{\gamma,0}$ with the composition

  $$S(0) \xrightarrow{f_{\beta,0}} S(\beta) \xrightarrow{f_{\gamma,\beta}} S(\gamma),$$

  where $f_{\gamma,\beta}$ is a monomorphism by assumption and $f_{\beta,0}$ is a monomorphism by virtue of our inductive hypothesis.

We now prove (b). Let T' be a collection of morphisms in Set_{Δ} which is weakly saturated and contains each of the inclusions $\partial \Delta^n \hookrightarrow \Delta^n$; we wish to show that every monomorphism

$i : A \to B$ belongs to T'. For each $k \geq -1$, let $B(k) \subseteq B$ denote the simplicial subset given by the union of the skeleton $\mathrm{sk}_k(B)$ (Construction 1.1.4.1) with the image of i. Then the inclusion i can be written as a transfinite composition

$$A \simeq B(-1) \hookrightarrow B(0) \hookrightarrow B(1) \hookrightarrow B(2) \hookrightarrow \cdots$$

Since T' is closed under transfinite composition, it will suffice to show that each of the inclusion maps $B(k-1) \hookrightarrow B(k)$ belongs to T'. Applying Proposition 1.1.4.12 to both A and B, we obtain a pushout diagram

$$\begin{tikzcd}
\coprod_{\sigma \in Q} \partial \Delta^k \arrow[r] \arrow[d] & \coprod_{\sigma \in Q} \Delta^k \arrow[d] \\
B(k-1) \arrow[r] & B(k)
\end{tikzcd}$$

where Q denotes the collection of all nondegenerate k-simplices of B which do not belong to the image of i. Since T' is closed under pushouts, we are reduced to showing that the inclusion map

$$j : \coprod_{\sigma \in Q} \partial \Delta^k \hookrightarrow \coprod_{\sigma \in Q} \Delta^k$$

belongs to T'. By virtue of Theorem 4.7.1.34, the set Q admits a well-ordering. Then j can be written as a transfinite composition of morphisms

$$j_\sigma : (\coprod_{\tau \geq \sigma} \partial \Delta^k) \amalg (\coprod_{\tau < \sigma} \Delta^k) \hookrightarrow (\coprod_{\tau > \sigma} \partial \Delta^k) \amalg (\coprod_{\tau \leq \sigma} \Delta^k),$$

each of which is a pushout of the inclusion $\partial \Delta^k \hookrightarrow \Delta^k$. □

Proof of Proposition 1.5.5.4. Let $p : X \to Y$ be a trivial Kan fibration of simplicial sets and let S be the collection of all morphisms in Set_Δ which are weakly left orthogonal to p. Then S contains each of the inclusions $\partial \Delta^n \hookrightarrow \Delta^n$ (by virtue of our assumption that p is a trivial Kan fibration) and is weakly saturated (Proposition 1.5.4.13). It follows from Proposition 1.5.5.14 that every monomorphism of simplicial sets $i : A \hookrightarrow B$ belongs to S (and is therefore weakly left orthogonal to p). □

1.5.6 Uniqueness of Composition

Let $\mathcal{C}$ be an ∞-category. Given a composable pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, one can form a composition $g \circ f$ by choosing a 2-simplex σ with $d^2_0(\sigma) = g$ and $d^2_2(\sigma) = f$, as indicated in the diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, dashed, "g\circ f"] & & Z.
\end{tikzcd}$$

In general, neither the 2-simplex σ nor the resulting morphism $g \circ f = d^2_1(\sigma)$ is uniquely determined. However, we saw in §1.4.4 that the composition $g \circ f$ is unique up to homotopy (Proposition 1.4.4.2). We now prove a stronger result, which asserts that the 2-simplex σ (hence also the composite morphism $g \circ f = d^2_1(\sigma)$) is unique up to a contractible space of choices.

Theorem 1.5.6.1 (Joyal)**.** *Let S be a simplicial set. The following conditions are equivalent:* 0079

(1) *The simplicial set S is an ∞-category.*

(2) *The inclusion of simplicial sets $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a trivial Kan fibration*

$$\operatorname{Fun}(\Delta^2, S) \to \operatorname{Fun}(\Lambda^2_1, S).$$

Corollary 1.5.6.2. *Let $f : X \to Y$ and $g : Y \to Z$ be a composable pair of morphisms in an ∞-category $\mathcal{C}$, so that the tuple $(g, \bullet, f)$ determines a map of simplicial sets $\Lambda^2_1 \to \mathcal{C}$ (see Proposition 1.2.4.7). Then the fiber product* 007A

$$\operatorname{Fun}(\Delta^2, \mathcal{C}) \times_{\operatorname{Fun}(\Lambda^2_1, \mathcal{C})} \{(g, \bullet, f)\}$$

is a contractible Kan complex.

Proof. Combine Theorem 1.5.6.1 with Remark 1.5.5.10. □

Remark 1.5.6.3. In the situation of Corollary 1.5.6.2, one can think of the simplicial set 007B

$$Z = \operatorname{Fun}(\Delta^2, \mathcal{C}) \underset{\operatorname{Fun}(\Lambda^2_1, \mathcal{C})}{\times} \{(g, \bullet, f)\}$$

as a “parameter space” for all choices of 2-simplex σ satisfying $d^2_0(\sigma) = g$ and $d^2_2(\sigma) = f$ (note that such 2-simplices can be identified with the vertices of Z).

We will give the proof of Theorem 1.5.6.1 at the end of this section. First, let us note one of its consequences.

Proof of Theorem 1.5.3.7. Let S be a simplicial set and let $\mathcal{D}$ be an ∞-category. We wish to show that the simplicial set $\operatorname{Fun}(S, \mathcal{D})$ is an ∞-category. By virtue of Theorem 1.5.6.1, it will suffice to show that the restriction map

$$r : \operatorname{Fun}(\Delta^2, \operatorname{Fun}(S, \mathcal{D})) \to \operatorname{Fun}(\Lambda^2_1, \operatorname{Fun}(S, \mathcal{D}))$$

is a trivial Kan fibration. Note that we can identify r with the canonical map

$$\operatorname{Fun}(S, \operatorname{Fun}(\Delta^2, \mathcal{D})) \to \operatorname{Fun}(S, \operatorname{Fun}(\Lambda^2_1, \mathcal{D})),$$

which is a trivial Kan fibration by virtue of Corollary 1.5.5.7 and Theorem 1.5.6.1. □

We now introduce some terminology which will be useful for the proof of Theorem 1.5.6.1.

007C **Definition 1.5.6.4.** Let $f : A \to B$ be a morphism of simplicial sets. We will say that f is *inner anodyne* if it belongs to the weakly saturated class of morphisms generated by the collection of all inner horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ (so that $0 < i < n$).

007D **Remark 1.5.6.5.** Let $f : A \to B$ be an inner anodyne map of simplicial sets. Then f is a monomorphism. This follows from the observation that the collection of monomorphisms is weakly saturated (Proposition 1.5.5.14), since every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is a monomorphism.

01C3 **Exercise 1.5.6.6.** Let $f : A \hookrightarrow B$ be an inner anodyne morphism of simplicial sets. Show that the underlying map on vertices $A_0 \to B_0$ is a bijection.

007E **Proposition 1.5.6.7.** *Let S be a simplicial set. The following conditions are equivalent:*

(1) *The simplicial set S is an ∞-category.*

(2) *For every inner anodyne map of simplicial sets $i : A \hookrightarrow B$ and every map $f_0 : A \to S$, there exists a map $f : B \to S$ such that $f_0 = f \circ i$.*

Proof. The implication (2) ⇒ (1) is immediate (since every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is inner anodyne). Conversely, if (1) is satisfied, then every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is weakly left orthogonal to the projection map $p : S \to \Delta^0$. It then follows from Remark 1.5.4.14 that every inner anodyne map is weakly left orthogonal to p. □

00J2 **Variant 1.5.6.8.** Let S be a simplicial set. The following conditions are equivalent:

(1) The simplicial set S is isomorphic to the nerve of a category.

(2) For every inner anodyne map of simplicial sets $i : A \hookrightarrow B$ and every map $f_0 : A \to S$, there exists a unique map $f : B \to S$ such that $f_0 = f \circ i$.

Proof. Let us regard the simplicial set S as fixed, and let T be the collection of all morphisms of simplicial sets $i : A \to B$ for which the induced map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(B, S) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(A, S)$ is bijective. Then T is weakly saturated (in the sense of Definition 1.5.4.12). It follows that (2) is equivalent to the following *a priori* weaker assertion:

($2'$) For every pair of integers $0 < i < n$, the map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_i, S)$ is bijective.

The equivalence of (1) and ($2'$) is the content of Proposition 1.3.4.1. □

We will deduce Theorem 1.5.6.1 from the following technical result:

Lemma 1.5.6.9 (Joyal)**.** 007F

(a) *For every monomorphism of simplicial sets $i : A \hookrightarrow B$, the induced map*

$$(B \times \Lambda^2_1) \coprod_{A \times \Lambda^2_1} (A \times \Delta^2) \subseteq B \times \Delta^2$$

is inner anodyne.

(b) *The collection of inner anodyne morphisms is generated (as a weakly saturated class) by the inclusion maps*

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$$

for $m \geq 0$.

Proof. Let T be the weakly saturated class of morphisms generated by all inclusions of the form

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2,$$

and let S be the collection of all morphisms of simplicial sets $A \to B$ for which the map

$$(B \times \Lambda^2_1) \coprod_{A \times \Lambda^2_1} (A \times \Delta^2) \subseteq B \times \Delta^2$$

belongs to T. By construction, S contains all inclusions of the form $\partial \Delta^m \hookrightarrow \Delta^m$. Moreover, since T is weakly saturated, the class S is also weakly saturated. It follows that every monomorphism of simplicial sets belongs to S (Proposition 1.5.5.14). Consequently, to prove Lemma 1.5.6.9, it will suffice to show that T coincides with the class of inner anodyne morphisms of Set_Δ. We first show that every inner anodyne morphism belongs to T. Since T

is weakly saturated, we are reduced to showing that every inner horn inclusion $f : \Lambda^n_i \hookrightarrow \Delta^n$ belongs to T. Since f belongs to S, the monomorphism

$$\overline{f} : (\Delta^n \times \Lambda^2_1) \coprod_{\Lambda^n_i \times \Lambda^2_1} (\Lambda^n_i \times \Delta^2) \subseteq \Delta^n \times \Delta^2.$$

belongs to T. We conclude by observing that the morphism f is a retract of $\overline{f}$. More precisely, we have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, "f"] & (\Delta^n \times \Lambda^2_1) \coprod_{\Lambda^n_i \times \Lambda^2_1} (\Lambda^n_i \times \Delta^2) \arrow[r] \arrow[d, "\overline{f}"] & \Lambda^n_i \arrow[d, "f"] \\
\Delta^n \arrow[r, "s"] & \Delta^n \times \Delta^2 \arrow[r, "r"] & \Delta^n,
\end{tikzcd}$$

where the maps s and r are given on vertices by the formulae

$$s(j) = \begin{cases} (j, 0) & \text{if } j < i \\ (j, 1) & \text{if } j = i \\ (j, 2) & \text{if } j > i \end{cases}$$

$$r(j, k) = \begin{cases} j & \text{if } j < i, k = 0 \\ j & \text{if } j > i, k = 2 \\ i & \text{otherwise.} \end{cases}$$

We now show that every morphism of T is inner anodyne. Since the collection of inner anodyne morphisms is weakly saturated, it will suffice to show that the inclusion map

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$$

is inner anodyne for each $m \geq 0$. For each $0 \leq i \leq j < m$, we let σ_{ij} denote the $(m+1)$-simplex of $\Delta^m \times \Delta^2$ given by the map of partially ordered sets

$$f_{ij} : [m+1] \to [m] \times [2]$$

$$f_{ij}(k) = \begin{cases} (k, 0) & \text{if } 0 \leq k \leq i \\ (k-1, 1) & \text{if } i+1 \leq k \leq j+1 \\ (k-1, 2) & \text{if } j+2 \leq k \leq m+1. \end{cases}$$

For each $0 \leq i \leq j \leq m$, we let τ_{ij} denote the $(m+2)$-simplex of $\Delta^m \times \Delta^2$ given by the map of partially ordered sets

$$g_{ij} : [m+2] \to [m] \times [2]$$

$$g_{ij}(k) = \begin{cases} (k,0) & \text{if } 0 \leq k \leq i \\ (k-1,1) & \text{if } i+1 \leq k \leq j+1 \\ (k-2,2) & \text{if } j+2 \leq k \leq m+2. \end{cases}$$

We will regard each σ_{ij} and τ_{ij} as a simplicial subset of $\Delta^m \times \Delta^2$.

Set $X(0) = (\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2)$. For $0 \leq j < m$, we let

$$X(j+1) = X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{jj}.$$

We have a chain of inclusions

$$X(j) \subseteq X(j) \cup \sigma_{0j} \subseteq \cdots \subseteq X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{jj} = X(j+1).$$

Each of these inclusions fits into a pushout diagram

$$\begin{tikzcd}
\Lambda^{m+1}_{i+1} \arrow[r] \arrow[d] & X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{(i-1)j} \arrow[d] \\
\sigma_{ij} \arrow[r] & X(j) \cup \sigma_{0j} \cup \cdots \cup \sigma_{ij},
\end{tikzcd}$$

and is therefore inner anodyne. Set $Y(0) = X(m)$, so that the inclusion $X(0) \subseteq Y(0)$ is inner anodyne. We now set $Y(j+1) = Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{jj}$ for $0 \leq j \leq m$. As before, we have a chain of inclusions

$$Y(j) \subseteq Y(j) \cup \tau_{0j} \subseteq \cdots \subseteq Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{jj} = Y(j+1),$$

each of which fits into a pushout diagram

$$\begin{tikzcd}
\Lambda^{m+2}_{i+1} \arrow[r] \arrow[d] & Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{(i-1)j} \arrow[d] \\
\tau_{ij} \arrow[r] & Y(j) \cup \tau_{0j} \cup \cdots \cup \tau_{ij},
\end{tikzcd}$$

and is therefore inner anodyne. It follows that each inclusion $Y(j) \subseteq Y(j+1)$ is inner anodyne. Since the collection of inner anodyne morphisms is closed under composition, we conclude that the inclusion map $X(0) \hookrightarrow Y(0) \hookrightarrow Y(1) \hookrightarrow \cdots Y(m+1) = \Delta^m \times \Delta^2$ is inner anodyne, as desired. □

Proof of Theorem 1.5.6.1. Let S be a simplicial set and let $p : \mathrm{Fun}(\Delta^2, S) \to \mathrm{Fun}(\Lambda^2_1, S)$ denote the restriction map. Then p is a trivial Kan fibration if and only if every lifting problem

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, S) \arrow[d, "p"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\Lambda^2_1, S)
\end{tikzcd}$$

admits a solution. Unwinding the definitions, we see that this is equivalent to the requirement that every lifting problem of the form

$$\begin{tikzcd}
(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial\, \Delta^m \times \Delta^2) \arrow[r] \arrow[d, "i"] & S \arrow[d] \\
\Delta^m \times \Delta^2 \arrow[r] \arrow[ur, dashed] & \Delta^0
\end{tikzcd}$$

admits a solution. Let T be the collection of all morphisms of simplicial sets which are weakly left orthogonal to the projection $S \to \Delta^0$. Then p is a trivial Kan fibration if and only if T contains each of the inclusion maps

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial\, \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2.$$

Since T is weakly saturated (Proposition 1.5.4.13), this is equivalent to the requirement that T contains all inner anodyne morphisms (Lemma 1.5.6.9), which is in turn equivalent to the requirement that S is an ∞-category (Proposition 1.5.6.7). □

1.5.7 Universality of Path Categories

00J3 Let G be a directed graph, let $G_\bullet$ denote the associated 1-dimensional simplicial set (see Proposition 1.1.6.9), and let $\mathrm{Path}[G]$ denote the path category of G (Construction 1.3.7.1). There is an evident map of simplicial sets $u : G_\bullet \to \mathrm{N}_\bullet(\mathrm{Path}[G])$. By virtue of Proposition 1.3.7.5, this map exhibits $\mathrm{Path}[G]$ as the homotopy category of the simplicial set $G_\bullet$. In other words, the path category $\mathrm{Path}[G]$ is universal among categories $\mathcal{C}$ which are equipped with a $G_\bullet$-indexed diagram (see Definition 1.5.2.1). Our goal in this section is to establish a variant of this statement in the setting of ∞-categories:

00J4 **Theorem 1.5.7.1.** *Let G be a directed graph and let $\mathcal{C}$ be an ∞-category. Then composition with the map of simplicial sets $u : G_\bullet \to \mathrm{N}_\bullet(\mathrm{Path}[G])$ induces a trivial Kan fibration of simplicial sets $\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Path}[G]), \mathcal{C}) \to \mathrm{Fun}(G_\bullet, \mathcal{C})$.*

More informally, Theorem 1.5.7.1 asserts that any G-indexed diagram in an ∞-category $\mathcal{C}$ admits an essentially unique extension to a functor of ∞-categories $\mathrm{N}_\bullet(\mathrm{Path}[G]) \to \mathcal{C}$.

Example 1.5.7.2. Let G be the directed graph depicted in the diagram 00J5

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet. \end{tikzcd}$$

Then the map $u : G_\bullet \to \mathrm{N}_\bullet(\mathrm{Path}[G])$ can be identified with the inclusion of simplicial sets $\Lambda^2_1 \hookrightarrow \Delta^2$. In this case, Theorem 1.5.7.1 reduces to the statement that the map

$$\mathrm{Fun}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Lambda^2_1, \mathcal{C})$$

is a trivial Kan fibration, which is equivalent to the assumption that $\mathcal{C}$ is an ∞-category by virtue of Theorem 1.5.6.1.

We will deduce Theorem 1.5.7.1 from the following more precise assertion.

Proposition 1.5.7.3. *Let G be a directed graph. Then the map of simplicial sets $u : G_\bullet \hookrightarrow \mathrm{N}_\bullet(\mathrm{Path}[G])$ is inner anodyne (Definition 1.5.6.4).* 00J6

Remark 1.5.7.4. Let G be a directed graph and let $\mathcal{C}$ be an ordinary category. Combining Proposition 1.5.7.3 with Variant 1.5.6.8, we deduce that the canonical map 00J7

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{N}_\bullet(\mathrm{Path}[G]), \mathrm{N}_\bullet(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(G_\bullet, \mathrm{N}_\bullet(\mathcal{C}))$$

is bijective. Combining this observation with Proposition 1.3.3.1, we obtain a bijection

$$\mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Path}[G], \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(G_\bullet, \mathrm{N}_\bullet(\mathcal{C})).$$

Allowing $\mathcal{C}$ to vary, we recover the assertion that $u : G_\bullet \to \mathrm{N}_\bullet(\mathrm{Path}[G])$ exhibits $\mathrm{Path}[G]$ as the homotopy category of $G_\bullet$ (Proposition 1.3.7.5).

Let us first show that Proposition 1.5.7.3 implies Theorem 1.5.7.1.

Lemma 1.5.7.5. *Let $f : X \hookrightarrow Y$ and $f' : X' \hookrightarrow Y'$ be monomorphisms of simplicial sets. If f is inner anodyne, then the induced map* 00J8

$$u_{f,f'} : (Y \times X') \coprod_{(X \times X')} (X \times Y') \hookrightarrow Y \times Y'$$

is inner anodyne.

Proof. Let us regard the morphism $f' : X' \hookrightarrow Y'$ as fixed. Let T be the collection of all morphisms $f : X \to Y$ for which the map $u_{f,f'}$ is inner anodyne. Then T is weakly saturated. To prove Lemma 1.5.7.5, we must show that T contains all inner anodyne morphisms of simplicial sets. By virtue of Lemma 1.5.6.9, it will suffice to show that T contains every morphism of the form

$$u_{i,j} : (B \times \Lambda^2_1) \coprod_{A \times \Lambda^2_1} (A \times \Delta^2) \subseteq B \times \Delta^2,$$

where $i : A \hookrightarrow B$ is a monomorphism of simplicial sets and $j : \Lambda^2_1 \hookrightarrow \Delta^2$ is the inclusion. Setting

$$A' = (B \times X') \coprod_{(A \times X')} (A \times Y') \qquad B' = B \times Y',$$

we are reduced to the problem of showing that the map

$$u_{i',j} : (B' \times \Lambda^2_1) \coprod_{A' \times \Lambda^2_1} (A' \times \Delta^2) \subseteq B' \times \Delta^2,$$

is inner anodyne, which follows from Lemma 1.5.6.9. □

00J9 **Proposition 1.5.7.6.** *Let $\mathcal{C}$ be an ∞-category and let $f : X \hookrightarrow Y$ be an inner anodyne morphism of simplicial sets. Then the induced map $p : \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ is a trivial Kan fibration.*

Proof. To show that p is a trivial Kan fibration, it will suffice to show that it is weakly right orthogonal to every monomorphism of simplicial sets $f' : X' \hookrightarrow Y'$. This is equivalent to the assertion that every map of simplicial sets

$$g_0 : (Y \times X') \coprod_{(X \times X')} (X \times Y') \to \mathcal{C}$$

can be extended to a map $g : Y \times Y' \to \mathcal{C}$. This follows from Proposition 1.5.6.7, since $\mathcal{C}$ is an ∞-category and the map

$$u_{f,f'} : (Y \times X') \coprod_{(X \times X')} (X \times Y') \hookrightarrow Y \times Y'$$

is inner anodyne (Lemma 1.5.7.5). □

Proof of Theorem 1.5.7.1. Let G be a graph and let $\mathcal{C}$ be an ∞-category; we wish to show that the canonical map

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Path}[G]), \mathcal{C}) \to \mathrm{Fun}(G_\bullet, \mathcal{C})$$

is a trivial Kan fibration. This follows from Proposition 1.5.7.6, since the inclusion $G_\bullet \hookrightarrow \mathrm{N}_\bullet(\mathrm{Path}[G])$ is inner anodyne (Proposition 1.5.7.3). □

Before giving the proof of Proposition 1.5.7.3, let us illustrate its contents with some examples.

Example 1.5.7.7 (The Spine of a Simplex)**.** Let $n \geq 0$ and let Δ^n be the standard n-simplex (Example 1.1.0.9). We let $\mathrm{Spine}[n]$ denote the simplicial subset of Δ^n whose k-simplices are monotone maps $\sigma : [k] \to [n]$ satisfying $\sigma(k) \leq \sigma(0) + 1$. We will refer to $\mathrm{Spine}[n]$ as the *spine* of the simplex Δ^n. More informally, it is comprised of all vertices of Δ^n, together with those edges which join adjacent vertices. The spine $\mathrm{Spine}[n]$ is a simplicial set of dimension ≤ 1, which we can identify with the directed graph G depicted in the diagram 00JA

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n. \end{tikzcd}$$

Under this identification, the map $u : G_\bullet \to \mathrm{N}_\bullet(\mathrm{Path}[G])$ corresponds to the inclusion $\mathrm{Spine}[n] \hookrightarrow \Delta^n$ (see Example 1.3.7.2). Invoking Proposition 1.5.7.3 and Theorem 1.5.7.1, we obtain the following:

(a) The inclusion $\mathrm{Spine}[n] \hookrightarrow \Delta^n$ is inner anodyne.

(b) For any ∞-category $\mathcal{C}$, the restriction map $\mathrm{Fun}(\Delta^n, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Spine}[n], \mathcal{C})$ is a trivial Kan fibration.

Remark 1.5.7.8 (The Generalized Associative Law)**.** Let $\mathcal{C}$ be an ordinary category and let $n \geq 0$ be an integer. Applying Remark 1.5.7.4 to the inner anodyne inclusion $\mathrm{Spine}[n] \hookrightarrow \Delta^n$ of Example 1.5.7.7, we deduce that every diagram 00JB

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n$$

can be extended uniquely to a functor $[n] \to \mathcal{C}$. In particular, it shows that $\mathcal{C}$ satisfies the "generalized associative law": the iterated composition $f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ is well-defined (that is, it does not depend on a choice of parenthesization). In essence, Proposition 1.5.7.3 can be regarded as an extension of this generalized associative law to the setting of ∞-categories.

Remark 1.5.7.8 admits a converse:

Corollary 1.5.7.9. *Let S be a simplicial set. Then S is isomorphic to the nerve of a category if and only if it satisfies the following condition for every integer $n \geq 0$:* 05HU

($*_n$) *Every morphism of simplicial sets* $\mathrm{Spine}[n] \to S$ *extends uniquely to an n-simplex of S.*

Proof. Assume that S satisfies condition ($*_n$) for each $n \geq 0$; we will show that S is isomorphic to the nerve of a category (the reverse implication follows from Remark 1.5.7.8). By virtue of Proposition 1.3.4.1, it will suffice to show that for $0 < i < n$, every inner horn $\sigma : \Lambda^n_i \to S$ can be extended to an n-simplex of S. Note that Λ^n_i contains the spine $\mathrm{Spine}[n]$. Applying condition ($*_n$), we deduce that there is a unique n-simplex σ' of S satisfying $\sigma'|_{\mathrm{Spine}[n]} = \sigma|_{\mathrm{Spine}[n]}$. We will complete the proof by showing that $\sigma = \sigma'|_{\Lambda^n_i}$. If $n = 2$, then $\Lambda^n_i = \operatorname{Spine}[n]$ and there is nothing to prove. We may therefore assume that $n > 2$, so that Λ^n_i contains the 1-skeleton of Δ^n. For every pair of integers $0 \leq j \leq k \leq n$, let $e_{j,k}$ denote the corresponding edge of Δ^n. Using condition $(*_{n-1})$, we are reduced to showing that $\sigma(e_{j,k}) = \sigma'(e_{j,k})$ for every pair of integers $0 \leq j \leq k \leq n$. We proceed by induction on the difference $k - j$. If $k - j \leq 1$, then $e_{j,k}$ is contained in the spine $\operatorname{Spine}[n]$ and there is nothing to prove. Otherwise, we can choose an integer ℓ satisfying $j < \ell < k$. Let τ denote the 2-simplex of Δ^n given by the triple $(j < \ell < k)$. Replacing ℓ by i if necessary, we can arrange that τ is contained in the horn Λ^n_i. Our inductive hypothesis guarantees that σ and σ' coincide on the edges $e_{j,\ell}$ and $e_{\ell,k}$. Invoking $(*_2)$, we conclude that $\sigma \circ \tau = \sigma' \circ \tau$, so that σ and σ' also agree on the edge $e_{j,k}$. □

01C4 **Remark 1.5.7.10.** Let $\mathcal{C}$ be an ∞-category and let $\mathrm{h}\mathcal{C}$ denote its homotopy category (Definition 1.4.5.3). Then the canonical map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ is an epimorphism of simplicial sets: that is, it induces a surjection on n-simplices for each $n \geq 0$. To prove this, we note that there is a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathcal{C}) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})) \arrow[d, "\sim"] \\
\operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\operatorname{Spine}[n], \mathcal{C}) \arrow[r] & \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\operatorname{Spine}[n], \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})),
\end{tikzcd}$$

where the left vertical map is surjective (Example 1.5.7.7) and the right vertical map is bijective (Remark 1.5.7.8). It therefore suffices to show that the bottom horizontal map is surjective: that is, every sequence of composable morphisms

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} X_3 \to \cdots \xrightarrow{f_n} X_n$$

in the homotopy category $\mathrm{h}\mathcal{C}$ can be lifted to a sequence of composable morphisms in $\mathcal{C}$, which is immediate from the definition of $\mathrm{h}\mathcal{C}$.

00JC **Example 1.5.7.11** (The Simplicial Circle)**.** Let $\Delta^1 / \partial \Delta^1$ denote the simplicial set obtained from Δ^1 by collapsing the boundary $\partial \Delta^1$ to a point, so that we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
\partial \Delta^1 \arrow[r] \arrow[d] & \Delta^1 \arrow[d] \\
\Delta^0 \arrow[r] & \Delta^1 / \partial \Delta^1 .
\end{tikzcd}$$

We will refer to $\Delta^1 / \partial \Delta^1$ as the *simplicial circle*; note that the geometric realization $|\Delta^1 / \partial \Delta^1|$ is isomorphic to the standard circle S^1 as a topological space. The simplicial set $\Delta^1 / \partial \Delta^1$ has dimension ≤ 1, and can therefore be identified with the directed graph G depicted in the diagram

Note that the path category $\mathrm{Path}[G]$ can be identified with the category $B\,\mathbf{Z}_{\geq 0}$ associated to the monoid $\mathbf{Z}_{\geq 0}$ of nonnegative numbers under addition (Example 1.3.7.4) whose nerve is the simplicial set $B_\bullet\,\mathbf{Z}_{\geq 0}$ of Construction 1.3.2.5. Invoking Proposition 1.5.7.3 and Theorem 1.5.7.1, we obtain the following:

(a) The inclusion of simplicial sets $\Delta^1 / \partial \Delta^1 \hookrightarrow B_\bullet\,\mathbf{Z}_{\geq 0}$ is inner anodyne.

(b) For any ∞-category $\mathcal{C}$, the restriction map $\mathrm{Fun}(B_\bullet\,\mathbf{Z}_{\geq 0}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C})$ is a trivial Kan fibration.

If $\mathcal{C}$ is an ∞-category, then a morphism of simplicial sets $\Delta^1 / \partial \Delta^1 \to \mathcal{C}$ can be identified with a pair (X, f), where X is an object of $\mathcal{C}$ and $f : X \to X$ is an endomorphism of X (Definition 1.4.1.5). Theorem 1.5.7.1 then guarantees that the pair (X, f) can be extended to a functor of ∞-categories $B_\bullet\,\mathbf{Z}_{\geq 0} \to \mathcal{C}$.

Example 1.5.7.12 (Free Monoids)**.** Let M be the free monoid generated by a set E. Then 00JD
we can identify BM with the path category $\mathrm{Path}[G]$ of a directed graph G satisfying

$$\mathrm{Vert}(G) = \{x\} \qquad \mathrm{Edge}(G) = E;$$

see Example 1.3.7.3. Invoking Proposition 1.5.7.3 and Theorem 1.5.7.1, we obtain the following:

(a) The inclusion of simplicial sets $G_\bullet \hookrightarrow B_\bullet M$ is inner anodyne.

(b) For any ∞-category $\mathcal{C}$, the restriction map $\mathrm{Fun}(B_\bullet M, \mathcal{C}) \to \mathrm{Fun}(G_\bullet, \mathcal{C})$ is a trivial Kan fibration.

Note that if $\mathcal{C}$ is an ∞-category, then a map of simplicial sets $\sigma_0 : G_\bullet \to \mathcal{C}$ can be identified with a choice of object $X \in \mathcal{C}$ together with a collection of morphisms $\{f_e : X \to X\}_{e \in E}$ indexed by E. It follows from (b) that any such map admits an (essentially unique) extension to a functor $\sigma : B_\bullet M \to \mathcal{C}$, which we can interpret as an *action* of the monoid M on the object $X \in \mathcal{C}$.

Proof of Proposition 1.5.7.3. Let G be a directed graph and let $\mathrm{Path}[G]$ denote its path category. By definition, a morphism from $x \in \mathrm{Vert}(G)$ to $y \in \mathrm{Vert}(G)$ in the category $\mathrm{Path}[G]$ is given by a sequence of edges $\vec{e} = (e_m, e_{m-1}, \ldots, e_1)$ satisfying

$$s(e_1) = x \qquad t(e_i) = s(e_{i+1}) \qquad t(e_m) = y.$$

In this case, we will refer to m as the *length* of the morphism $\vec{e}$ and write $m = \ell(\vec{e})$. If $\sigma : \Delta^n \to \mathrm{N}_\bullet(\mathrm{Path}[G])$ is an n-simplex given by a diagram

$$x_0 \xrightarrow{\vec{e}_1} x_1 \xrightarrow{\vec{e}_2} \cdots \xrightarrow{\vec{e}_n} x_n$$

in $\mathrm{Path}[G]$, we define the *length* $\ell(\sigma)$ to be the sum $\ell(\vec{e}_1) + \cdots + \ell(\vec{e}_n) = \ell(\vec{e}_n \circ \cdots \circ \vec{e}_1)$. For each positive integer k, let $\mathrm{N}_\bullet^{\leq k}(\mathrm{Path}[G])$ denote the simplicial subset of $\mathrm{N}_\bullet(\mathrm{Path}[G])$ consisting of those simplices having length $\leq k$. We then have inclusions

$$\mathrm{N}_\bullet^{\leq 1}(\mathrm{Path}[G]) \subseteq \mathrm{N}_\bullet^{\leq 2}(\mathrm{Path}[G]) \subseteq \mathrm{N}_\bullet^{\leq 3}(\mathrm{Path}[G]) \subseteq \mathrm{N}_\bullet^{\leq 4}(\mathrm{Path}[G]) \subseteq \cdots,$$

where $\mathrm{N}_\bullet^{\leq 1}(\mathrm{Path}[G]) = G_\bullet$ and $\mathrm{N}_\bullet(\mathrm{Path}[G]) = \bigcup \mathrm{N}_\bullet^{\leq k}(\mathrm{Path}[G])$. Consequently, to show that the inclusion $G_\bullet \hookrightarrow \mathrm{N}_\bullet(\mathrm{Path}[G])$ is inner anodyne, it will suffice to show that each of the inclusion maps $\mathrm{N}_\bullet^{\leq k}(\mathrm{Path}[G]) \hookrightarrow \mathrm{N}_\bullet^{\leq k+1}(\mathrm{Path}[G])$ is inner anodyne.

We henceforth regard the integer $k \geq 1$ as fixed. Let $\sigma : \Delta^n \to \mathrm{N}_\bullet(\mathrm{Path}[G])$ be an n-simplex of $\mathrm{N}_\bullet(\mathrm{Path}[G])$ having length $k+1$, corresponding to a diagram

$$x_0 \xrightarrow{\vec{e}_1} x_1 \xrightarrow{\vec{e}_2} \cdots \xrightarrow{\vec{e}_n} x_n$$

as above. Note that σ is nondegenerate if and only if each $\vec{e}_i$ has positive length. We will say that σ is *normalized* if it is nondegenerate and $\ell(\vec{e}_1) = 1$. Let $S(n)$ be the collection of all normalized n-simplices of $\mathrm{N}_\bullet^{\leq k+1}(\mathrm{Path}[G])$ having length $k+1$. We make the following observations:

(*i*) If σ belongs to $S(n)$, then the faces $d_0^n(\sigma)$ and $d_n^n(\sigma)$ have length $\leq k$, and are therefore contained in $\mathrm{N}_\bullet^{\leq k}(\mathrm{Path}[G])$.

(*ii*) If σ belongs to $S(n)$ and $1 < i < n$, then the face $d_i^n(\sigma)$ is a normalized $(n-1)$-simplex of $\mathrm{N}_\bullet^{\leq k+1}(\mathrm{Path}[G])$ of length $k+1$, and therefore belongs to $S(n-1)$.

(*iii*) If σ belongs to $S(n)$, then the face $d_1^n(\sigma)$ is *not* normalized. Moreover, the construction $\sigma \mapsto d_1^n(\sigma)$ induces a bijection from $S(n)$ to the collection of $(n-1)$-simplices of $\mathrm{N}_\bullet^{\leq k+1}(\mathrm{Path}[G])$ which are nondegenerate, of length $k+1$, and not normalized.

For each $n \geq 1$, let $X(n)$ denote the simplicial subset of $\mathrm{N}_\bullet^{\leq k+1}(\mathrm{Path}[G])$ given by the union of the $(n-1)$-skeleton $\mathrm{sk}_{n-1}(\mathrm{N}_\bullet^{\leq k+1}(\mathrm{Path}[G]))$, the simplicial set $\mathrm{N}_\bullet^{\leq k}(\mathrm{Path}[G])$, and the collection of normalized n-simplices of $\mathrm{N}_\bullet^{\leq k+1}(\mathrm{Path}[G])$. We have inclusions

$$X(1) \subseteq X(2) \subseteq X(3) \subseteq X(4) \subseteq \cdots,$$

where $\mathrm{N}_\bullet^{\leq k}(\mathrm{Path}[G]) = X(1)$ and $\mathrm{N}_\bullet^{\leq k+1}(\mathrm{Path}[G]) = \bigcup_n X(n)$. It will therefore suffice to show that the inclusion maps $X(n-1) \hookrightarrow X(n)$ are inner anodyne for $n \geq 2$. We conclude by observing that (*i*), (*ii*), and (*iii*) guarantee the existence of a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{\sigma \in S(n)} \Lambda^n_1 \arrow[r] \arrow[d] & \coprod_{\sigma \in S(n)} \Delta^n \arrow[d] \\
X(n-1) \arrow[r] & X(n).
\end{tikzcd}$$

□

Chapter 2

Examples of ∞-Categories

007J In Chapter 1, we introduced the notion of an ∞-*category*: that is, a simplicial set which satisfies the weak Kan extension condition (Definition 1.4.0.1). The theory of ∞-categories can be understood as a synthesis of classical category theory and algebraic topology. This perspective is supported by the two main examples of ∞-categories that we have encountered so far:

- Every ordinary category $\mathcal{C}$ can be regarded as an ∞-category, by identifying $\mathcal{C}$ with the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C})$ of Construction 1.3.1.1.
- Every Kan complex is an ∞-category. In particular, for every topological space X, the singular simplicial set $\mathrm{Sing}_{\bullet}(X)$ is an ∞-category.

Beware that, individually, both of these examples are rather special. An ∞-category $\mathcal{C}$ can be regarded as a mathematical structure which encodes information not only about objects and morphisms (given by the vertices and edges of $\mathcal{C}$, respectively), but also about *homotopies* between morphisms (Definition 1.4.3.1). When $\mathcal{C}$ is (the nerve of) an ordinary category, the notion of homotopy is trivial: two morphisms in $\mathcal{C}$ (having the same source and target) are homotopic if and only if they are identical. On the other hand, if $\mathcal{C}$ is a Kan complex, then every morphism in $\mathcal{C}$ is invertible up to homotopy (Proposition 1.4.6.10); from a category-theoretic perspective, this is a very restrictive condition.

Our goal in this chapter is to supply a larger class of examples of ∞-categories, which are more representative of the subject as a whole. To this end, we introduce three variants of the nerve construction $\mathcal{C} \mapsto \mathrm{N}_{\bullet}(\mathcal{C})$ which can be used to produce ∞-categories out of other (possibly more familiar) mathematical structures. To describe these constructions in a uniform way, it will be convenient to employ the language of *enriched* category theory, which we review in §2.1. Let $\mathcal{A}$ be a monoidal category: that is, a category equipped with a tensor product operation $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which is unital and associative up to (specified) isomorphisms (see Definition 2.1.2.10). An $\mathcal{A}$-*enriched category* is a mathematical structure $\mathcal{C}$ consisting of the following data (see Definition 2.1.7.1):

- A collection $\mathrm{Ob}(\mathcal{C})$ whose elements we refer to as *objects* of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a *mapping object* $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \in \mathcal{A}$.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition law*

  $$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z),$$

  which we require to be unital and associative.

Taking our cues from Examples [?], [?], and [?], we consider three examples of this paradigm:

- Let $\mathcal{A} = \mathrm{Set}_{\Delta}$ be the category of simplicial sets, equipped with the monoidal structure given by cartesian product. In this case, we refer to an $\mathcal{A}$-enriched category as a *simplicial category* (Definition 2.4.1.1). In §2.4, we associate to each simplicial category $\mathcal{C}$ a simplicial set $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$, which we refer to as the *homotopy coherent nerve* of $\mathcal{C}$ (Definition 2.4.3.5). Moreover, we show that if each of the simplicial sets $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ is a Kan complex, then the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.4.5.1).
- Let $\mathcal{A} = \mathrm{Ch}(\mathbf{Z})$ be the category of chain complexes of abelian groups, equipped with the monoidal structure given by tensor product of chain complexes. In this case, we refer to an $\mathcal{A}$-enriched category as a *differential graded category* (Definition 2.5.2.1). In §2.5, we associate to each differential graded category $\mathcal{C}$ a simplicial set $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$, which we refer to as the *differential graded nerve of* $\mathcal{C}$ (Definition 2.5.3.7), and show that $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ is always an ∞-category (Theorem 2.5.3.10).
- Let $\mathcal{A} = \mathrm{Cat}$ be the category of (small) categories, equipped with the monoidal structure given by the cartesian product. In this case, we refer to an $\mathcal{A}$-enriched category as a *strict* 2-*category* (Definition 2.2.0.1). This is a special case of the more general notion of 2-*category* (or *bicategory*, in the terminology of Bénabou), which we review in §2.2. In §2.3, we will associate to each 2-category $\mathcal{C}$ a simplicial set $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$, which we refer to as the *Duskin nerve* of $\mathcal{C}$ (Construction 2.3.1.1). Moreover, we show that if each of the categories $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ is a groupoid, then $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.3.2.1).

Simplicial categories, differential graded categories, and 2-categories are ubiquitous in algebraic topology, homological algebra, and category theory, respectively. Consequently, the constructions of this section furnish a rich supply of examples of ∞-categories.

2.1 Monoidal Categories

00BL Recall that a *monoid* is a set M equipped with a multiplication map

$$M \times M \to M \qquad (x, y) \mapsto xy$$

which is unital and associative (Definition 1.3.2.1). In the setting of category theory, one often encounters analogous structures which satisfy a more subtle form of associativity.

00BN **Example 2.1.0.1.** Let k be a field and let U, V, and W be vector spaces over k. Recall that a function $b : U \times V \to W$ is said to be *k-bilinear* if it satisfies the identities

$$b(u + u', v) = b(u, v) + b(u', v) \qquad b(u, v + v') = b(u, v) + b(u, v')$$

$$b(\lambda u, v) = \lambda b(u, v) = b(u, \lambda v) \text{ for } \lambda \in k.$$

We say that a k-bilinear map $b : U \times V \to W$ is *universal* if, for any k-vector space W', composition with b induces a bijection

$$\{k\text{-linear maps } W \to W'\} \simeq \{k\text{-bilinear maps } U \times V \to W'\}.$$

If this condition is satisfied, then W is determined (up to unique isomorphism) by U and V; we refer to W as the *tensor product of U and V* and denote it by $U \otimes_k V$. The construction $(U, V) \mapsto U \otimes_k V$ then determines a functor

$$\otimes_k : \mathrm{Vect}_k \times \mathrm{Vect}_k \to \mathrm{Vect}_k,$$

which we will refer to as the *tensor product functor*. It is associative in the following sense: for every triple of vector spaces $U, V, W \in \mathrm{Vect}_k$, there exists a canonical isomorphism

$$U \otimes_k (V \otimes_k W) \xrightarrow{\sim} (U \otimes_k V) \otimes_k W \qquad u \otimes (v \otimes w) \mapsto (u \otimes v) \otimes w.$$

Our goal in this section is to review the theory of *monoidal categories*, which axiomatizes the essential features of Example 2.1.0.1. To simplify the discussion, we begin by developing the nonunital version of this theory. In §2.1.1, we introduce the notion of a *nonunital monoidal structure* on a category $\mathcal{C}$ (Definition 2.1.1.5). Roughly speaking, a nonunital monoidal structure on $\mathcal{C}$ is a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative up to isomorphism. More precisely, it consists of the functor $\otimes$ together with a *choice* of isomorphism $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow{\sim} (X \otimes Y) \otimes Z$ for every triple of objects $X, Y, Z \in \mathcal{C}$ (these isomorphisms are called the *associativity constraints* of $\mathcal{C}$). The isomorphisms $\alpha_{X,Y,Z}$ are required to depend functorially on X, Y, and Z, and to satisfy a further coherence condition called the *pentagon identity* (this condition was introduced by MacLane in [40], and is sometimes known as *MacLane's pentagon identity*).

By definition, a nonunital monoid M is a monoid if and only if there exists an element $e \in M$ satisfying $ex = x = xe$ for each $x \in M$. If this condition is satisfied, then the element e is uniquely determined. The categorical analogue of this statement is a bit more subtle. Let X be an object of a nonunital monoidal category $\mathcal{C}$, and let $\ell_X, r_X : \mathcal{C} \to \mathcal{C}$ denote the functors given by $\ell_X(Y) = X \otimes Y$ and $r_X(Y) = Y \otimes X$. In §2.1.2, we define a *unit* in $\mathcal{C}$ to be an object $\mathbf{1}$ with the property that the functors $\ell_{\mathbf{1}}$ and $r_{\mathbf{1}}$ are fully faithful, together with a choice of isomorphism $\upsilon : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$. In this case, the pair $(\mathbf{1}, \upsilon)$ is not unique; however, it is unique up to (unique) isomorphism (Proposition 2.1.2.9). One can use υ to construct natural isomorphisms

$$\lambda_Y : \mathbf{1} \otimes Y \xrightarrow{\sim} Y \qquad \rho_Y : Y \otimes \mathbf{1} \xrightarrow{\sim} Y,$$

so that $\mathbf{1}$ really behaves like a unit for the tensor product $\otimes$ (Construction 2.1.2.17). We define a *monoidal category* to be a nonunital monoidal category $\mathcal{C}$ together with a choice of unit $(\mathbf{1}, \upsilon)$ (Definition 2.1.2.10). A basic prototype is the category Vect_k of vector spaces over a field k (equipped with the tensor product and associativity constraints given in Example 2.1.0.1, and the unit given by the object $k \in \mathrm{Vect}_k$). We give a more detailed description of this and other examples in §2.1.3.

Most of the rest of this section is devoted to studying functors between monoidal categories. We start in §2.1.4 with the nonunital case. If $\mathcal{C}$ and $\mathcal{C}'$ are nonunital monoidal categories, we define a *nonunital monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ to be a functor $F : \mathcal{C} \to \mathcal{C}'$ together with a collection of isomorphisms

$$\mu_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y),$$

which depend functorially on $X, Y \in \mathcal{C}$ and are compatible with the associativity constraints on $\mathcal{C}$ and $\mathcal{C}'$ (Definition 2.1.4.4). We also introduce the more general notion of *nonunital lax monoidal functor*, where we do not require the morphisms $\mu_{X,Y}$ to be isomorphisms (Definition 2.1.4.3). Both of these definitions have unital analogues, which we study in §2.1.6 and §2.1.5, respectively.

We conclude this section in §2.1.7 with a brief review of *enriched* category theory. If $\mathcal{A}$ is a monoidal category, then an $\mathcal{A}$-*enriched category* $\mathcal{C}$ consists of a collection $\mathrm{Ob}(\mathcal{C})$ of *objects of* $\mathcal{C}$, a collection of mapping objects $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \in \mathcal{A}$ for each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, and a composition law

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$$

which is required to be unital and associative (see Definition 2.1.7.1). Enriched category theory will play an important role throughout this chapter: we will be particularly interested in the special case where $\mathcal{A} = \mathrm{Cat}$ is the category of small categories (in which case we recover the notion of *strict* 2-*category*, which we study in §2.2), where $\mathcal{A} = \mathrm{Set}_\Delta$ is the category of simplicial sets (in which case we recover the notion of *simplicial category*, which we study in §2.4), and where $\mathcal{A} = \mathrm{Ch}(\mathbf{Z})(\mathrm{Ab})$ is the category of chain complexes of abelian groups (In which case we recover the notion of *differential graded category*, which we study in §2.5.

2.1.1 Nonunital Monoidal Categories

00BT Let Cat denote the category whose objects are (small) categories and whose morphisms are functors. Then Cat admits finite products. One can therefore consider (nonunital) monoids in Cat: that is, small categories $\mathcal{C}$ equipped with a strictly associative multiplication $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. For the convenience of the reader, we spell out this definition in detail (and abandon the smallness assumption on $\mathcal{C}$):

00BU **Definition 2.1.1.1.** Let $\mathcal{C}$ be a category. A *nonunital strict monoidal structure* on $\mathcal{C}$ is a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad (X, Y) \mapsto X \otimes Y$$

which is strictly associative in the following sense:

- For every triple of objects $X, Y, Z \in \mathcal{C}$, we have an equality $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ (as objects of $\mathcal{C}$).
- For every triple of morphisms $f : X \to X'$, $g : Y \to Y'$, $h : Z \to Z'$, we have an equality
$$f \otimes (g \otimes h) = (f \otimes g) \otimes h$$
of morphisms in $\mathcal{C}$ from the object $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ to the object $X' \otimes (Y' \otimes Z') = (X' \otimes Y') \otimes Z'$.

A *nonunital strict monoidal category* is a pair $(\mathcal{C}, \otimes)$, where $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a nonunital strict monoidal structure on $\mathcal{C}$.

00BV **Remark 2.1.1.2.** We will often abuse terminology by identifying a nonunital strict monoidal category $(\mathcal{C}, \otimes)$ with the underlying category $\mathcal{C}$. If we refer to a category $\mathcal{C}$ as a nonunital strict monoidal category, we implicitly assume that $\mathcal{C}$ has been endowed with a tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is strictly associative in the sense of Definition 2.1.1.1.

00BW **Example 2.1.1.3.** Let M be a set, which we regard as a category having only identity morphisms. Then nonunital strict monoidal structures on M (in the sense of Definition 2.1.1.1) can be identified with nonunital monoid structures on M (in the sense of Variant 1.3.2.8). In particular, any nonunital monoid can be regarded as a nonunital strict monoidal category (having only identity morphisms).

Example 2.1.1.4 (Endomorphism Categories)**.** Let $\mathcal{C}$ be a category, and let $\mathrm{End}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ denote the category of functors from $\mathcal{C}$ to itself. Then the composition functor 00BX

$$\circ : \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{C}) \qquad (F, G) \mapsto F \circ G;$$

is a nonunital strict monoidal structure on $\mathrm{End}(\mathcal{C})$.

For many purposes, Definition 2.1.1.1 is too restrictive. Note that if k is a field, then the tensor product functor $\otimes_k : \mathrm{Vect}_k \times \mathrm{Vect}_k \to \mathrm{Vect}_k$ of Example 2.1.0.1 does not quite fit the framework described in Definition 2.1.1.1. Given vector spaces X, Y, and Z over k, there is no reason to expect the iterated tensor products $X \otimes_k (Y \otimes_k Z)$ and $(X \otimes_k Y) \otimes_k Z$ to be identical. In fact, this is impossible to determine based from the definition sketched in Example 2.1.0.1. To construct the functor $\otimes_k$ explicitly, we need to make certain choices: namely, a choice of universal bilinear map $b : U \times V \to U \otimes_k V$ for every pair of vector spaces $U, V \in \mathrm{Vect}_k$. Without an explicit convention for how these choices are to be made, we cannot answer the question of whether the vector spaces $X \otimes_k (Y \otimes_k Z)$ and $(X \otimes_k Y) \otimes_k Z$ are equal. However, this is arguably the wrong question to consider: in the setting of vector spaces, the appropriate notion of "sameness" is not equality, but isomorphism. The iterated tensor products $X \otimes_k (Y \otimes_k Z)$ and $(X \otimes_k Y) \otimes_k Z$ are isomorphic, because they can be characterized by the same universal property: both are universal among vector spaces W equipped with a k-trilinear map $t : X \times Y \times Z \to W$. Even better, there is a canonical isomorphism

$$\alpha_{X,Y,Z} : X \otimes_k (Y \otimes_k Z) \to (X \otimes_k Y) \otimes_k Z,$$

which depends functorially on X, Y, and Z. Motivated by this example, we introduce the following generalization of Definition 2.1.1.1:

Definition 2.1.1.5. Let $\mathcal{C}$ be a category. A *nonunital monoidal structure* on $\mathcal{C}$ consists of the following data: 00BY

- A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which we will refer to as the *tensor product functor*.
- A collection of isomorphisms $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$, for $X, Y, Z \in \mathcal{C}$, called the *associativity constraints of* $\mathcal{C}$. We demand that the associativity constraints $\alpha_{X,Y,Z}$ depend functorially on X, Y, Z in the following sense: for every triple of morphisms $f : X \to X'$, $g : Y \to Y'$, and $h : Z \to Z'$, the diagram

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha_{X,Y,Z}", "\sim"'] \arrow[d, "f \otimes (g \otimes h)"] & (X \otimes Y) \otimes Z \arrow[d, "(f \otimes g) \otimes h"] \\
X' \otimes (Y' \otimes Z') \arrow[r, "\alpha_{X',Y',Z'}", "\sim"'] & (X' \otimes Y') \otimes Z'
\end{tikzcd}$$

is commutative. In other words, we require that $\alpha = \{\alpha_{X,Y,Z}\}_{X,Y,Z \in \mathcal{C}}$ can be regarded as a natural isomorphism from the functor

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \xrightarrow{(X,Y,Z) \mapsto X \otimes (Y \otimes Z)} \mathcal{C}$$

to the functor

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \xrightarrow{(X,Y,Z) \mapsto (X \otimes Y) \otimes Z} \mathcal{C}.$$

The associativity constraints of $\mathcal{C}$ are required to satisfy the following additional condition:

(P) For every quadruple of objects $W, X, Y, Z \in \mathcal{C}$, the diagram of isomorphisms

$$\begin{tikzcd}
& W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\alpha_{W,X \otimes Y,Z}", "\sim"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[dr, "\alpha_{W,X,Y} \otimes \mathrm{id}_Z", "\sim"'] & \\
W \otimes (X \otimes (Y \otimes Z)) \arrow[ur, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}", "\sim"'] \arrow[drr, "\sim", "\alpha_{W,X,Y \otimes Z}"'] & & & & ((W \otimes X) \otimes Y) \otimes Z \\
& & (W \otimes X) \otimes (Y \otimes Z) \arrow[urr, "\sim", "\alpha_{W \otimes X,Y,Z}"'] & &
\end{tikzcd}$$

commutes.

A *nonunital monoidal category* is a triple $(\mathcal{C}, \otimes, \alpha)$, where $\mathcal{C}$ is a category and $(\otimes, \alpha)$ is a nonunital monoidal structure on $\mathcal{C}$.

`00BZ` **Remark 2.1.1.6.** In the setting of Definition 2.1.1.5, we will refer to (P) as the *pentagon identity*. It is a prototypical example of a *coherence* condition: the associativity constraints $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$ "witness" the requirement that the tensor product is associative up to isomorphism, and the pentagon identity is a sort of "higher order" associative law required of the witnesses themselves.

`00C0` **Example 2.1.1.7.** Let $\mathcal{C}$ be a category equipped with a nonunital strict monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (in the sense of Definition 2.1.1.1). Then $\otimes$ determines a nonunital monoidal structure on $\mathcal{C}$ (in the sense of Definition 2.1.1.5) by taking the associativity constraints $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$ to be identity morphisms. Conversely, if $\mathcal{C}$ is equipped with a nonunital monoidal structure $(\otimes, \alpha)$ where each of the associativity constraints $\alpha_{X,Y,Z}$ is an identity morphism, then $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a nonunital strict monoidal structure on $\mathcal{C}$.

`00C1` **Remark 2.1.1.8.** Let $\mathcal{C}$ be a category equipped with a nonunital monoidal structure $(\otimes, \alpha)$. We will often abuse terminology by identifying the nonunital monoidal structure $(\otimes, \alpha)$ with the underlying tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. If we refer to a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

as a *nonunital monoidal structure on* $\mathcal{C}$, we implicitly assume that $\mathcal{C}$ has been equipped with associativity constraints $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$ satisfying the pentagon identity of Definition 2.1.1.5. Beware that, in the non-strict case, the associativity constraints are an essential part of the data: it is possible to have inequivalent nonunital monoidal categories $(\mathcal{C}, \otimes, \alpha)$ and $(\mathcal{C}', \otimes', \alpha')$ with $\mathcal{C} = \mathcal{C}'$ and $\otimes = \otimes'$ (see Example 2.1.3.3).

Remark 2.1.1.9 (Full Subcategories of Nonunital Monoidal Categories)**.** Let $\mathcal{C}$ be a category equipped with a nonunital monoidal structure $(\otimes, \alpha)$, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that, for every pair of objects $X, Y \in \mathcal{C}_0$, the tensor product $X \otimes Y$ also belongs to $\mathcal{C}_0$. Then $\mathcal{C}_0$ inherits a nonunital monoidal structure, with tensor product functor given by the composition 00MH

$$\mathcal{C}_0 \times \mathcal{C}_0 \subseteq \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

(which factors through $\mathcal{C}_0$ by hypothesis), and associativity constraints given by those of $\mathcal{C}$.

Remark 2.1.1.10 (Nonunital Monoidal Structures on Functor Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then every nonunital monoidal structure $(\otimes, \alpha)$ on $\mathcal{D}$ determines a nonunital monoidal structure on the functor category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$, whose underlying tensor product is given by the composition 00MJ

$$\operatorname{Fun}(\mathcal{C}, \mathcal{D}) \times \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \simeq \operatorname{Fun}(\mathcal{C}, \mathcal{D} \times \mathcal{D}) \xrightarrow{\otimes \circ} \operatorname{Fun}(\mathcal{C}, \mathcal{D})$$

and whose associativity constraint assigns to each triple of functors $F, G, H : \mathcal{C} \to \mathcal{D}$ the natural isomorphism

$$F \otimes (G \otimes H) \xrightarrow{\sim} (F \otimes G) \otimes H \qquad C \mapsto \alpha_{F(C), G(C), H(C)}.$$

2.1.2 Monoidal Categories

We now introduce unital versions of Definitions 2.1.1.1 and 2.1.1.5. 00C2

Definition 2.1.2.1. Let $\mathcal{C}$ be a category. A *strict monoidal structure* on $\mathcal{C}$ is a nonunital strict monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for which there exists an object $\mathbf{1} \in \mathcal{C}$ satisfying the following condition: 00C3

$(*)$ For every object $X \in \mathcal{C}$, we have $X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$ (as objects of $\mathcal{C}$). Moreover, for every morphism $f : X \to X'$ in $\mathcal{C}$, we have $f \otimes \mathrm{id}_{\mathbf{1}} = f = \mathrm{id}_{\mathbf{1}} \otimes f$ (as morphisms from X to X').

A *strict monoidal category* is a pair $(\mathcal{C}, \otimes)$, where $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a strict monoidal structure on $\mathcal{C}$.

00C4 **Remark 2.1.2.2.** Let $\mathcal{C}$ be a nonunital strict monoidal category. We will say that an object $\mathbf{1} \in \mathcal{C}$ is a *strict unit* if it satisfies condition $(*)$ of Definition 2.1.2.1. Note that if such an object exists, then it is uniquely determined: it can be characterized as the unit element of the monoid $\mathrm{Ob}(\mathcal{C})$.

It follows from Remark 2.1.2.2 that the notion of strict unit is not invariant under isomorphism. To address this, it will be convenient to consider a more general notion of unit object, which makes sense in the non-strict setting as well. We will use an efficient formulation due to Saavedra ([49]); see also [38]. To motivate the definition, we begin with a simple observation about units in a more elementary setting.

00C5 **Proposition 2.1.2.3.** *Let M be a nonunital monoid, let e be an element of M, and let $\ell_e : M \to M$ denote the function given by the formula $\ell_e(x) = ex$. The following conditions are equivalent:*

(a) The element e is a left unit of M: that is, ℓ_e is the identity function from M to itself.

(b) The element e is idempotent (that is, it satisfies $ee = e$) and the function $\ell_e : M \to M$ is a bijection.

(c) The element e is idempotent and the function $\ell_e : M \to M$ is a monomorphism.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are immediate. To complete the proof, assume that e satisfies condition (c) and let x be an element of M. Using the assumption that e is idempotent (and the associativity of the multiplication on M), we obtain an identity $\ell_e(x) = ex = (ee)x = e(ex) = \ell_e(ex)$. Since ℓ_e is a monomorphism, it follows that $x = ex$. □

00C6 **Corollary 2.1.2.4.** *Let M be a nonunital monoid. Then an element $e \in M$ is a unit if and only if the following conditions are satisfied:*

(i) The element e is idempotent: that is, we have $ee = e$.

(ii) The element e is left cancellative: that is, the function $x \mapsto ex$ is a monomorphism from M to itself.

(iii) The element e is right cancellative: that is, the function $x \mapsto xe$ is a monomorphism from M to itself.

We now adapt the characterization of Corollary 2.1.2.4 to the setting of nonunital monoidal categories.

00C7 **Definition 2.1.2.5.** Let $\mathcal{C}$ be a nonunital monoidal category. A *unit* of $\mathcal{C}$ is a pair $(\mathbf{1}, \upsilon)$, where $\mathbf{1}$ is an object of $\mathcal{C}$ and $\upsilon : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ is an isomorphism, which satisfies the following additional condition:

$(*)$ The functors

$$\mathcal{C} \to \mathcal{C} \qquad C \mapsto \mathbf{1} \otimes C$$
$$\mathcal{C} \to \mathcal{C} \qquad C \mapsto C \otimes \mathbf{1}$$

are fully faithful.

Remark 2.1.2.6. Condition $(*)$ of Definition 2.1.2.5 depends only on the object $\mathbf{1} \in \mathcal{C}$, and not on the choice of isomorphism $\upsilon : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$. 00C8

Example 2.1.2.7. Let $\mathcal{C}$ be a strict monoidal category, and let $\mathbf{1} \in \mathcal{C}$ be the strict unit (Remark 2.1.2.2). Then $(\mathbf{1}, \mathrm{id}_{\mathbf{1}})$ is a unit of $\mathcal{C}$. 00C9

Example 2.1.2.8. Let M be a nonunital monoid, regarded as a (strict) nonunital monoidal category having only identity morphisms (Example 2.1.1.3). Then the converse of Example 2.1.2.7 holds: a pair $(\mathbf{1}, \upsilon)$ is a unit structure on M (in the sense of Definition 2.1.2.5) if and only if $\mathbf{1}$ is a unit element of M and $\upsilon = \mathrm{id}_{\mathbf{1}}$. This is a restatement of Corollary 2.1.2.4. 00CA

If M is a nonunital monoid, then a unit element $e \in M$ is unique if it exists. For nonunital monoidal categories, the analogous statement is more subtle. If a nonunital monoidal category $\mathcal{C}$ admits a unit $(\mathbf{1}, \upsilon)$, then it has many others: we can replace $\mathbf{1}$ by any object $\mathbf{1}'$ which is isomorphic to it, and υ by any choice of isomorphism $\upsilon' : \mathbf{1}' \otimes \mathbf{1}' \simeq \mathbf{1}'$. Nevertheless, we have the following strong uniqueness result:

Proposition 2.1.2.9 (Uniqueness of Units)**.** *Let $\mathcal{C}$ be a nonunital monoidal category equipped with units $(\mathbf{1}, \upsilon)$ and $(\mathbf{1}', \upsilon')$ (in the sense of Definition 2.1.2.5). Then there is a unique isomorphism $u : \mathbf{1} \xrightarrow{\sim} \mathbf{1}'$ for which the diagram* 00CB

$$\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "\upsilon"] \arrow[d, "u \otimes u"] & \mathbf{1} \arrow[d, "u"] \\
\mathbf{1}' \otimes \mathbf{1}' \arrow[r, "\upsilon'"] & \mathbf{1}'
\end{tikzcd}$$

commutes.

We will give the proof of Proposition 2.1.2.9 at the end of this section.

Definition 2.1.2.10. Let $\mathcal{C}$ be a category. A *monoidal structure on* $\mathcal{C}$ is a nonunital monoidal structure $(\otimes, \alpha)$ on $\mathcal{C}$ (Definition 2.1.1.5) together with a choice of unit $(\mathbf{1}, \upsilon)$ (in the sense of Definition 2.1.2.5). A *monoidal category* is a category $\mathcal{C}$ together with a monoidal structure $(\otimes, \alpha, \mathbf{1}, \upsilon)$ on $\mathcal{C}$. In this case, we refer to $\mathbf{1}$ as the *unit object* of $\mathcal{C}$ and the isomorphism $\upsilon : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ as the *unit constraint* of $\mathcal{C}$. 00CC

00CD **Remark 2.1.2.11.** It is possible to adopt the following variant of Definition 2.1.2.10:

- A *monoidal category* is a nonunital monoidal category $\mathcal{C}$ which admits a unit, in the sense of Definition 2.1.2.5.

This is essentially equivalent to Definition 2.1.2.10, since a unit $(\mathbf{1}, \upsilon)$ of $\mathcal{C}$ is uniquely determined up to unique isomorphism (Proposition 2.1.2.9). However, for our purposes it will be more convenient to adopt the convention that a monoidal structure on a category $\mathcal{C}$ includes a *choice* of unit object $\mathbf{1} \in \mathcal{C}$ and unit constraint $\upsilon : \mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1}$.

00CE **Remark 2.1.2.12.** Let $\mathcal{C}$ be a category. We will sometimes abuse terminology by identifying a monoidal structure $(\otimes, \alpha, \mathbf{1}, \upsilon)$ with the underlying nonunital monoidal structure $(\otimes, \alpha)$ on $\mathcal{C}$ (or with the underlying tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$). This is essentially harmless, by virtue of Remark 2.1.2.11. We will also abuse terminology (in a less harmless way) by identifying a monoidal category $(\mathcal{C}, \otimes, \alpha, \mathbf{1}, \upsilon)$ with the underlying category $\mathcal{C}$.

00CF **Notation 2.1.2.13.** Let $\mathcal{C}$ be a monoidal category. We will generally use the symbol $\mathbf{1}$ to denote the unit object of $\mathcal{C}$. In situations where this notation is potentially confusing (for example, if we are comparing $\mathcal{C}$ with another monoidal category), we will often disambiguate by instead writing $\mathbf{1}_{\mathcal{C}}$ for the unit object of $\mathcal{C}$.

00CG **Example 2.1.2.14.** Let $\mathcal{C}$ be a category. Then every strict monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (in the sense of Definition 2.1.2.1) can be promoted to a monoidal structure $(\otimes, \alpha, \mathbf{1}, \upsilon)$ on $\mathcal{C}$, by taking $\mathbf{1}$ to be the strict unit of $\mathcal{C}$ and the associativity and unit constraints to be identity morphisms of $\mathcal{C}$. Conversely, if $\mathcal{C}$ is equipped with a monoidal structure $(\otimes, \alpha, \mathbf{1}, \upsilon)$ for which the associativity and unit constraints are identity morphisms, then $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a strict monoidal structure on $\mathcal{C}$ and $\mathbf{1}$ is the strict unit.

00MK **Example 2.1.2.15.** Let $\mathcal{C}$ be a monoidal category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Assume that $\mathcal{C}_0$ contains the unit object $\mathbf{1}$ and is closed under the formation of tensor products in $\mathcal{C}$. Then $\mathcal{C}_0$ inherits the structure of a monoidal category: the underlying nonunital monoidal structure on $\mathcal{C}_0$ is given by the construction of Remark 2.1.1.9, and the unit $(\mathbf{1}, \upsilon)$ of $\mathcal{C}_0$ coincides with the unit of $\mathcal{C}$.

00ML **Example 2.1.2.16.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then every monoidal structure on $\mathcal{D}$ determines a monoidal structure on the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, whose underlying nonunital monoidal structure is given by the construction of Remark 2.1.1.10 and whose unit object is the constant functor $\mathcal{C} \to \{\mathbf{1}\} \hookrightarrow \mathcal{D}$ (and whose unit constraint $\upsilon : \mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1}$ is the constant natural transformation induced by the unit constraint of $\mathcal{D}$).

Let $\mathcal{C}$ be a monoidal category. In general, the unit object $\mathbf{1}$ of $\mathcal{C}$ need not be strict, in the sense that the functors

$$\mathcal{C} \to \mathcal{C} \qquad X \mapsto \mathbf{1} \otimes X$$

$$\mathcal{C} \to \mathcal{C} \qquad X \mapsto X \otimes \mathbf{1}$$

need not be *equal* to the identity functor $\mathrm{id}_{\mathcal{C}}$. However, they are always (canonically) isomorphic to $\mathrm{id}_{\mathcal{C}}$.

Construction 2.1.2.17 (Left and Right Unit Constraints)**.** Let $\mathcal{C} = (\mathcal{C}, \otimes, \alpha, \mathbf{1}, \upsilon)$ be a monoidal category. For each object $X \in \mathcal{C}$, we have canonical isomorphisms 00CH

$$\mathbf{1} \otimes (\mathbf{1} \otimes X) \xrightarrow{\alpha_{\mathbf{1},\mathbf{1},X}} (\mathbf{1} \otimes \mathbf{1}) \otimes X \xrightarrow{\upsilon \otimes \mathrm{id}_X} \mathbf{1} \otimes X.$$

Since the functor $Y \mapsto \mathbf{1} \otimes Y$ is fully faithful, it follows that there is a unique isomorphism $\lambda_X : \mathbf{1} \otimes X \xrightarrow{\sim} X$ for which the diagram

$$\begin{tikzcd}
\mathbf{1} \otimes (\mathbf{1} \otimes X) \arrow[rr, "\alpha_{\mathbf{1},\mathbf{1},X}", "\sim"'] \arrow[dr, "\sim", "\mathrm{id}_{\mathbf{1}} \otimes \lambda_X"'] & & (\mathbf{1} \otimes \mathbf{1}) \otimes X \arrow[dl, "\sim"', "\upsilon \otimes \mathrm{id}_X"] \\
& \mathbf{1} \otimes X &
\end{tikzcd}$$

commutes. We will refer to λ_X as the *left unit constraint*. Similarly, there is a unique isomorphism $\rho_X : X \otimes \mathbf{1} \simeq X$ for which the diagram

$$\begin{tikzcd}
X \otimes (\mathbf{1} \otimes \mathbf{1}) \arrow[rr, "\alpha_{X,\mathbf{1},\mathbf{1}}", "\sim"'] \arrow[dr, "\sim", "\mathrm{id}_X \otimes \upsilon"'] & & (X \otimes \mathbf{1}) \otimes \mathbf{1} \arrow[dl, "\sim"', "\rho_X \otimes \mathrm{id}_{\mathbf{1}}"] \\
& X \otimes \mathbf{1} &
\end{tikzcd}$$

commutes; we refer to ρ_X as the *right unit constraint*.

Remark 2.1.2.18. Let $\mathcal{C}$ be a monoidal category. Then the left and right unit constraints $\lambda_X : \mathbf{1} \otimes X \xrightarrow{\sim} X$ and $\rho_X : X \otimes \mathbf{1} \xrightarrow{\sim} X$ depend functorially on X. In other words, for every morphism $f : X \to Y$, the diagram 00CJ

$$\begin{tikzcd}
\mathbf{1} \otimes X \arrow[r, "\lambda_X", "\sim"'] \arrow[d, "\mathrm{id}_{\mathbf{1}} \otimes f"] & X \arrow[d, "f"] & X \otimes \mathbf{1} \arrow[l, "\rho_X"', "\sim"] \arrow[d, "f \otimes \mathrm{id}_{\mathbf{1}}"] \\
\mathbf{1} \otimes Y \arrow[r, "\lambda_Y", "\sim"'] & Y & Y \otimes \mathbf{1} \arrow[l, "\rho_Y"', "\sim"]
\end{tikzcd}$$

is commutative.

00CK **Proposition 2.1.2.19** (The Triangle Identity)**.** *Let $\mathcal{C}$ be a monoidal category with unit object $\mathbf{1}$. Let X and Y be objects of $\mathcal{C}$, and let $\rho_X : X \otimes \mathbf{1} \simeq X$ and $\lambda_Y : \mathbf{1} \otimes Y \to Y$ be the right and left unit constraints of Construction 2.1.2.17. Then the diagram of isomorphisms*

$$\begin{tikzcd}
X \otimes (\mathbf{1} \otimes Y) \arrow[rr, "\alpha_{X,\mathbf{1},Y}", "\sim"'] \arrow[dr, "\sim", "\mathrm{id}_X \otimes \lambda_Y"'] & & (X \otimes \mathbf{1}) \otimes Y \arrow[dl, "\sim"', "\rho_X \otimes \mathrm{id}_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

is commutative.

Proof. We have a diagram of isomorphisms

$$\begin{tikzcd}
& X \otimes ((\mathbf{1} \otimes \mathbf{1}) \otimes Y) \arrow[rr, "\alpha"] \arrow[d, "\upsilon_Y"] & & (X \otimes (\mathbf{1} \otimes \mathbf{1})) \otimes Y \arrow[d, "\upsilon_Y"'] \arrow[dr, "\alpha"] & \\
& X \otimes (\mathbf{1} \otimes Y) \arrow[rr, "\alpha"] \arrow[d, "\alpha"] & & (X \otimes \mathbf{1}) \otimes Y \arrow[dll, "\mathrm{id}"] & \\
X \otimes (\mathbf{1} \otimes (\mathbf{1} \otimes Y)) \arrow[uur, "\alpha"] \arrow[ur, "\lambda_Y"'] \arrow[drr, "\alpha"'] & (X \otimes \mathbf{1}) \otimes Y \arrow[r, "\rho_X"] & X \otimes Y & X \otimes (\mathbf{1} \otimes Y) \arrow[l, "\lambda_Y"'] \arrow[u, "\alpha"'] & ((X \otimes \mathbf{1}) \otimes \mathbf{1}) \otimes Y \arrow[ul, "\rho_X"] \\
& & (X \otimes \mathbf{1}) \otimes (\mathbf{1} \otimes Y). \arrow[ul, "\lambda_Y"] \arrow[ur, "\rho_X"'] \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

Here the outer cycle commutes by the pentagon identity (P) of Definition 2.1.1.5, the upper rectangle and outer quadrilaterals by the functoriality of the associativity constraint, the side triangles by the definition of the left and right unit constraints, and the lower quadrilateral by the functoriality of the tensor product $\otimes$. It follows that the middle square is also commutative, which is equivalent to the statement of Proposition 2.1.2.19. □

00CL **Exercise 2.1.2.20.** Let $\mathcal{C}$ be a monoidal category with unit object $\mathbf{1}$. Show that, for every pair of objects $X, Y \in \mathcal{C}$, the diagrams

$$\begin{tikzcd}
X \otimes (Y \otimes \mathbf{1}) \arrow[rr, "\alpha_{X,Y,\mathbf{1}}"] \arrow[dr, "\mathrm{id}_X \otimes \rho_Y"'] & & (X \otimes Y) \otimes \mathbf{1} \arrow[dl, "\rho_{X \otimes Y}"] \\
& X \otimes Y &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1} \otimes (X \otimes Y) \arrow[rr, "\alpha_{\mathbf{1},X,Y}"] \arrow[dr, "\lambda_{X \otimes Y}"'] & & (\mathbf{1} \otimes X) \otimes Y \arrow[dl, "\lambda_X \otimes \mathrm{id}_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

are commutative (for a more general statement, see Proposition 2.2.1.16).

Corollary 2.1.2.21. *Let $\mathcal{C}$ be a monoidal category with unit object $\mathbf{1}$. Then the left and right unit constraints $\lambda_{\mathbf{1}}, \rho_{\mathbf{1}} : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ are equal to the unit constraint $\upsilon : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$.* 00CM

Proof. Let X be any object of $\mathcal{C}$. Then the left unit contraint λ_X is characterized by the commutativity of the diagram

$$\begin{tikzcd}
\mathbf{1} \otimes (\mathbf{1} \otimes X) \arrow[rr, "\alpha_{\mathbf{1},\mathbf{1},X}", "\sim"'] \arrow[dr, "\sim", "\mathrm{id}_{\mathbf{1}} \otimes \lambda_X"'] & & (\mathbf{1} \otimes \mathbf{1}) \otimes X \arrow[dl, "\sim"', "\upsilon \otimes \mathrm{id}_X"] \\
& \mathbf{1} \otimes X. &
\end{tikzcd}$$

Using Proposition 2.1.2.19, we deduce that $\upsilon \otimes \mathrm{id}_X = \rho_{\mathbf{1}} \otimes \mathrm{id}_X$ as morphisms from $(\mathbf{1} \otimes \mathbf{1}) \otimes X$ to $\mathbf{1} \otimes X$. In other words, the morphisms $\upsilon, \rho_{\mathbf{1}} : \mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$ have the same image under the functor

$$\mathcal{C} \to \mathcal{C} \qquad Y \mapsto Y \otimes X.$$

In the case $X = \mathbf{1}$, this functor is fully faithful; it follows that $\upsilon = \rho_{\mathbf{1}}$. The equality $\upsilon = \lambda_{\mathbf{1}}$ follows by a similar argument. □

Proof of Proposition 2.1.2.9. Let $\mathcal{C}$ be a nonunital monoidal category equipped with units $(\mathbf{1}, \upsilon)$ and $(\mathbf{1}', \upsilon')$. We can then regard $\mathcal{C}$ as a monoidal category with unit object $\mathbf{1}$ and unit constraint υ. For each object $X \in \mathcal{C}$, let $\lambda_X : \mathbf{1} \otimes X \xrightarrow{\sim} X$ be the left unit constraint of

Construction 2.1.2.17. We wish to show that there is a unique isomorphism $u : \mathbf{1} \simeq \mathbf{1}'$ for which the outer rectangle in the diagram of isomorphisms

$$\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "\lambda_{\mathbf{1}}"] \arrow[d, "\mathrm{id}_{\mathbf{1}} \otimes u"] & \mathbf{1} \arrow[d, "u"] \\
\mathbf{1} \otimes \mathbf{1}' \arrow[r, "\lambda_{\mathbf{1}'}"] \arrow[d, "u \otimes \mathrm{id}_{\mathbf{1}'}"] & \mathbf{1}' \arrow[d, "\mathrm{id}_{\mathbf{1}'}"] \\
\mathbf{1}' \otimes \mathbf{1}' \arrow[r, "\upsilon'"] & \mathbf{1}'
\end{tikzcd}$$

is commutative. Since the upper square commutes (Remark 2.1.2.18), this is equivalent to the commutativity of the lower square. The existence and uniqueness of u now follows from the assumption that the functor $X \mapsto X \otimes \mathbf{1}'$ is fully faithful. □

00CN **Remark 2.1.2.22.** Let $\mathcal{C}$ be a nonunital monoidal category. Suppose we are given objects $\mathbf{1}, \mathbf{1}' \in \mathcal{C}$ together with isomorphisms

$$\upsilon : \mathbf{1} \otimes \mathbf{1} \simeq \mathbf{1} \qquad \upsilon' : \mathbf{1}' \otimes \mathbf{1}' \simeq \mathbf{1}'.$$

To carry out the proof of Proposition 2.1.2.9, it is sufficient to assume that the functors

$$\mathcal{C} \to \mathcal{C} \qquad X \mapsto \mathbf{1} \otimes X$$

$$\mathcal{C} \to \mathcal{C} \qquad X \mapsto X \otimes \mathbf{1}'$$

are fully faithful: the first assumption is sufficient to construct the left unit constraints of Construction 2.1.2.17, and the second is used at the end of the proof. This can be regarded as a categorical analogue of the observation that if a nonunital monoid admits a left unit e and a right unit e', then we must have $e = e'$.

2.1.3 Examples of Monoidal Categories

00CP We now illustrate Definition 2.1.2.10 with some examples.

00CQ **Example 2.1.3.1.** Let k be a field and let Vect_k denote the category of vector spaces over k (where morphisms are k-linear maps). For every pair of vector spaces $V, W \in \mathrm{Vect}_k$, let us *choose* a vector space $V \otimes_k W$ and a bilinear map

$$V \times W \to V \otimes_k W \qquad (v, w) \mapsto v \otimes w$$

which exhibits $V \otimes_k W$ as a tensor product of V and W (see Example 2.1.0.1). The construction $(V, W) \mapsto V \otimes_k W$ determines a functor

$$\otimes_k : \mathrm{Vect}_k \times \mathrm{Vect}_k \to \mathrm{Vect}_k,$$

whose value on a pair of k-linear maps $\varphi : V \to V'$, $\psi : W \to W'$ is characterized by the identity

$$(\varphi \otimes_k \psi)(v \otimes w) = \varphi(v) \otimes \psi(w).$$

For every triple of vector spaces $U, V, W \in \mathrm{Vect}_k$, there is a canonical isomorphism

$$\alpha_{U,V,W} : U \otimes_k (V \otimes_k W) \xrightarrow{\sim} (U \otimes_k V) \otimes_k W,$$

characterized by the identity $\alpha_{U,V,W}(u \otimes (v \otimes w)) = (u \otimes v) \otimes w$ for $u \in U$, $v \in V$, and $w \in W$. The pair $(\otimes_k, \alpha) = (\otimes_k, \{\alpha_{U,V,W}\}_{U,V,W \in \mathrm{Vect}_k})$ is then a nonunital monoidal structure on the category Vect_k, in the sense of Definition 2.1.1.5. We can upgrade this to a monoidal structure by taking the unit object $\mathbf{1}$ to be the field k (regarded as a vector space over itself), and the unit constraint $v : \mathbf{1} \otimes_k \mathbf{1} \simeq \mathbf{1}$ to be the linear map corresponding to the multiplication on k (so that $v(a \otimes b) = ab$).

Example 2.1.3.2 (Cartesian Products)**.** Let $\mathcal{C}$ be a category. Assume that every pair of objects $X, Y \in \mathcal{C}$ admits a product in $\mathcal{C}$. This product is not unique: it is only unique up to (canonical) isomorphism. However, let us *choose* an object $X \times Y$ together with a pair of morphisms 00CR

$$X \xleftarrow{\pi_{X,Y}} X \times Y \xrightarrow{\pi'_{X,Y}} Y$$

which exhibit $X \times Y$ as a product of X and Y in the category $\mathcal{C}$. Then the construction $(X, Y) \mapsto X \times Y$ determines a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, given on morphisms by the construction

$$((f : X \to X'), (g : Y \to Y')) \mapsto ((f \times g) : (X \times Y) \to (X' \times Y')),$$

where $f \times g$ is the unique morphism for which the diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & X \times Y \arrow[l, "\pi_{X,Y}"'] \arrow[r, "\pi'_{X,Y}"] \arrow[d, "f \times g"] & Y \arrow[d, "g"] \\
X' & X' \times Y' \arrow[l, "\pi_{X',Y'}"'] \arrow[r, "\pi'_{X',Y'}"] & Y'
\end{tikzcd}$$

is commutative.

For every triple of objects $X, Y, Z \in \mathcal{C}$, there is a canonical isomorphism $\alpha_{X,Y,Z} : X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$, which is characterized by the commutativity of the diagram

$$\begin{tikzcd}
 & X \times (Y \times Z) \arrow[rr, "\alpha_{X,Y,Z}", "\sim"'] \arrow[ddl, "\pi_{X,Y \times Z}"'] \arrow[dr] & & (X \times Y) \times Z \arrow[dll] \arrow[ddr, "\pi'_{X \times Y,Z}"] & \\
 & X \times Y \arrow[dl, "\pi_{X,Y}"] \arrow[dr, "\pi'_{X,Y}"'] & & Y \times Z \arrow[dl, "\pi_{Y,Z}"] \arrow[dr, "\pi'_{Y,Z}"'] & \\
X & & Y & & Z.
\end{tikzcd}$$

The category $\mathcal{C}$ admits a nonunital monoidal structure, with tensor product given by the functor $(X, Y) \mapsto X \times Y$, and associativity constraints given by $(X, Y, Z) \mapsto \alpha_{X,Y,Z}$.

If we assume also that the category $\mathcal{C}$ has a final object $\mathbf{1}$ (so that $\mathcal{C}$ admits all finite products), then we can upgrade the nonunital monoidal structure above to a monoidal structure, where the unit object of $\mathcal{C}$ is $\mathbf{1}$ and the unit constraint υ is the unique morphism from $\mathbf{1} \times \mathbf{1}$ to $\mathbf{1}$ in $\mathcal{C}$. We refer to this monoidal structure as the *cartesian monoidal structure* on $\mathcal{C}$.

00CS **Example 2.1.3.3** (Group Cocycles)**.** Let G be a group with identity element $1 \in G$, and let Γ be an abelian group on which G acts by automorphisms; we denote the action of an element $g \in G$ by $(\gamma \in \Gamma) \mapsto g(\gamma) \in \Gamma$. A 3-*cocycle on* G *with values in* Γ is a map of sets

$$\alpha : G \times G \times G \to \Gamma \qquad (x, y, z) \mapsto \alpha_{x,y,z}.$$

which satisfies the equations

0087 $$w(\alpha_{x,y,z}) - \alpha_{wx,y,z} + \alpha_{w,xy,z} - \alpha_{w,x,yz} + \alpha_{w,x,y} = 0 \tag{2.1}$$

for every quadruple of elements $w, x, y, z \in G$.

Let $\mathcal{C}$ denote the category whose objects are the elements of G, and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}}(g, h) = \begin{cases} \Gamma & \text{if } g = h \\ \emptyset & \text{otherwise.} \end{cases}$$

Using the action of G on Γ, we can construct a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

given on objects by $(g, h) \mapsto gh$ and on morphisms by

$$((\gamma : g \to g), (\delta : h \to h)) \mapsto (\gamma + g(\delta) : gh \to gh).$$

Unwinding the definitions, one sees that upgrading the functor $\otimes$ to a nonunital monoidal structure on the category $(\otimes, \alpha)$ on $\mathcal{C}$ is equivalent to choosing a 3-cocycle $\alpha : G \times G \times G \to \Gamma$. More precisely, *any* map $\alpha : G \times G \times G \to \Gamma$ can be regarded as a natural transformation of functors

$$\bullet \otimes (\bullet \otimes \bullet) \to (\bullet \otimes \bullet) \otimes \bullet,$$

and pentagon identity (P) of Definition 2.1.1.5 translates to the cocycle condition (2.1) above.

For any choice of cocycle $\alpha : G \times G \times G \to \Gamma$, we can upgrade the associated nonunital monoidal structure $(\otimes, \alpha)$ to a monoidal structure on the category $\mathcal{C}$, by taking the unit object of $\mathcal{C}$ to be the identity element $1 \in G$ and the unit constraint $\upsilon : 1 \otimes 1 \simeq 1$ to be the element $0 \in \Gamma$.

Example 2.1.3.4 (The Opposite of a Monoidal Category)**.** Let $\mathcal{C}$ be a category equipped with a nonunital monoidal structure $(\otimes, \{\alpha_{X,Y,Z}\}_{X,Y,Z \in \mathcal{C}})$. Then the opposite category $\mathcal{C}^{\mathrm{op}}$ inherits a nonunital monoidal structure, which can be described concretely as follows: `00CT`

- The tensor product on $\mathcal{C}^{\mathrm{op}}$ is obtained from the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ by passing to opposite categories.

- Let X, Y, and Z be objects of $\mathcal{C}$, and let us write X^{op}, Y^{op}, and Z^{op} for the corresponding objects of $\mathcal{C}^{\mathrm{op}}$. Then the associativity constraint $\alpha_{X^{\mathrm{op}},Y^{\mathrm{op}},Z^{\mathrm{op}}}$ for $\mathcal{C}^{\mathrm{op}}$ is the *inverse* of the associativity constraint $\alpha_{X,Y,Z}$ for $\mathcal{C}$.

If the nonunital monoidal category $\mathcal{C}$ is equipped with a unit structure $(\mathbf{1}, \upsilon)$, then we can regard $(\mathbf{1}^{\mathrm{op}}, \upsilon^{-1})$ as a unit structure for the nonunital monoidal category $\mathcal{C}^{\mathrm{op}}$. In particular, every monoidal structure on a category $\mathcal{C}$ determines a monoidal structure on the opposite category $\mathcal{C}^{\mathrm{op}}$.

Example 2.1.3.5 (The Reverse of a Monoidal Structure)**.** Let $\mathcal{C}$ be a category equipped with a nonunital monoidal structure $(\otimes, \{\alpha_{X,Y,Z}\}_{X,Y,Z \in \mathcal{C}})$. Then we can equip $\mathcal{C}$ with another nonunital monoidal structure $(\otimes^{\mathrm{rev}}, \{\alpha^{\mathrm{rev}}_{X,Y,Z}\}_{X,Y,Z \in \mathcal{C}})$, defined as follows: `00CU`

- The tensor product functor $\otimes^{\mathrm{rev}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is given on objects by the formula $X \otimes^{\mathrm{rev}} Y = Y \otimes X$ (and similarly on morphisms).

- The associativity constraint on $\otimes^{\mathrm{rev}}$ is given by the formula $\alpha^{\mathrm{rev}}_{X,Y,Z} = \alpha^{-1}_{Z,Y,X}$.

We will refer to the nonunital monoidal structure $(\otimes^{\mathrm{rev}}, \{\alpha^{\mathrm{rev}}_{X,Y,Z}\}_{X,Y,Z \in \mathcal{C}})$ as the *reverse* of the nonunital monoidal structure $(\otimes, \{\alpha_{X,Y,Z}\}_{X,Y,Z \in \mathcal{C}})$. In this case, we will write $\mathcal{C}^{\mathrm{rev}}$ to denote the nonunital monoidal category whose underlying category is $\mathcal{C}$, equipped with the nonunital monoidal structure $(\otimes^{\mathrm{rev}}, \{\alpha^{\mathrm{rev}}_{X,Y,Z}\}_{X,Y,Z \in \mathcal{C}})$.

If the nonunital monoidal category $\mathcal{C}$ is equipped with a unit structure $(\mathbf{1}, \upsilon)$, then we can also regard $(\mathbf{1}, \upsilon)$ as a unit structure for the nonunital monoidal category $\mathcal{C}^{\mathrm{rev}}$. In other words, if $\mathcal{C}$ is a monoidal category, then we can regard $\mathcal{C}^{\mathrm{rev}}$ as a monoidal category (having the same underlying category and unit object, but "reversed" tensor product).

2.1.4 Nonunital Monoidal Functors

00CV We now study functors between (nonunital) monoidal categories.

00CW **Definition 2.1.4.1** (Nonunital Strict Monoidal Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories (Definition 2.1.1.5). A *nonunital strict monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following properties:

- The diagram of functors

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "F \times F"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \times \mathcal{D} \arrow[r, "\otimes"] & \mathcal{D}
\end{tikzcd}$$

  is strictly commutative. In particular, for every pair of objects $X, Y \in \mathcal{C}$, we have an equality $F(X) \otimes F(Y) = F(X \otimes Y)$ of objects of $\mathcal{D}$.

- For every triple of objects $X, Y, Z \in \mathcal{C}$, the functor F carries the associativity constraint $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$ (for the monoidal structure on $\mathcal{C}$) to the associativity constraint $\alpha_{F(X),F(Y),F(Z)} : F(X) \otimes (F(Y) \otimes F(Z)) \simeq (F(X) \otimes F(Y)) \otimes F(Z)$ (for the monoidal structure on $\mathcal{D}$).

00CX **Example 2.1.4.2.** Let $\mathcal{C}$ be a nonunital monoidal category. Then the identity functor $\mathrm{id}_{\mathcal{C}}$ is a nonunital strict monoidal functor from $\mathcal{C}$ to itself.

For many applications, Definition 2.1.4.1 is too restrictive. In practice, the definition of a (nonunital) monoidal structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$ often involves constructions which are only well-defined up to isomorphism (see Examples 2.1.3.1 and 2.1.3.2). In such cases, it is unreasonable to require that a functor $F : \mathcal{C} \to \mathcal{D}$ has the property that $F(X) \otimes F(Y)$ and $F(X \otimes Y)$ are the same object of $\mathcal{D}$. Instead, we should ask for any isomorphism $\mu_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$. To get a well-behaved theory, we should further demand that the isomorphisms $\mu_{X,Y}$ depend functorially on X and Y, and are suitably compatible with the associativity constraints on $\mathcal{C}$ and $\mathcal{D}$. We begin by considering a slightly more general situation, where the morphisms $\mu_{X,Y}$ are not required to be invertible.

Definition 2.1.4.3 (Nonunital Lax Monoidal Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor from $\mathcal{C}$ to $\mathcal{D}$. A *nonunital lax monoidal structure* on F is a collection of morphisms $\mu = \{\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)\}_{X,Y \in \mathcal{C}}$ which satisfy the following pair of conditions: `00CY`

(a) The morphisms $\mu_{X,Y}$ depend functorially on X and Y: that is, for every pair of morphisms $f : X \to X'$, $g : Y \to Y'$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\mu_{X,Y}"] \arrow[d, "F(f)\otimes F(g)"] & F(X \otimes Y) \arrow[d, "F(f\otimes g)"] \\
F(X') \otimes F(Y') \arrow[r, "\mu_{X',Y'}"] & F(X' \otimes Y')
\end{tikzcd}$$

commutes (in the category $\mathcal{D}$). In other words, we can regard μ as a natural transformation of functors as indicated in the diagram

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{C} \arrow[r, "\otimes"] \arrow[d, "F\times F"'] & \mathcal{C} \arrow[d, "F"] \arrow[dl, phantom, ""{name=U}] \\
\mathcal{D} \times \mathcal{D} \arrow[r, "\otimes"] \arrow[ur, Rightarrow, "\mu"'] & \mathcal{D}.
\end{tikzcd}$$

(b) The morphisms $\mu_{X,Y}$ are compatible with the associativity constraints on $\mathcal{C}$ and $\mathcal{D}$ in the following sense: for every triple of objects $X, Y, Z \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "\alpha_{F(X),F(Y),F(Z)}"] \arrow[d, "\mathrm{id}_{F(X)} \otimes \mu_{Y,Z}"] & (F(X) \otimes F(Y)) \otimes F(Z) \arrow[d, "\mu_{X,Y} \otimes \mathrm{id}_{F(Z)}"] \\
F(X) \otimes F(Y \otimes Z) \arrow[d, "\mu_{X,Y\otimes Z}"] & F(X \otimes Y) \otimes F(Z) \arrow[d, "\mu_{X\otimes Y,Z}"] \\
F(X \otimes (Y \otimes Z)) \arrow[r, "F(\alpha_{X,Y,Z})"] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

commutes (in the category $\mathcal{D}$).

A *nonunital lax monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a pair (F, μ), where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ is a nonunital lax monoidal structure on F. In this case, we will refer to the morphisms $\{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ as the *tensor constraints* of F.

00CZ **Definition 2.1.4.4.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor from $\mathcal{C}$ to $\mathcal{D}$. A *nonunital monoidal structure* on F is a lax nonunital monoidal structure $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ on F with the property that each of the tensor constraints $\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ is an isomorphism.

A *nonunital monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a pair (F, μ), where $F : \mathcal{C} \to \mathcal{D}$ is a functor and μ is a nonunital monoidal structure on F.

00D0 **Example 2.1.4.5.** Let k be a field and let Vect_k denote the category of vector spaces over k, endowed with the monoidal structure of Example 2.1.3.1. The construction of this monoidal structure involved certain choices: for every pair of vector spaces $U, V \in \mathrm{Vect}_k$, we selected a universal k-bilinear map $b_{U,V} : U \times V \to U \otimes_k V$. The collection of functions $b = \{b_{U,V}\}_{U,V \in \mathrm{Vect}_k}$ is then a nonunital lax monoidal structure on the forgetful functor $\mathrm{Vect}_k \to \mathrm{Set}$ (where we equip Set with the monoidal structure given by cartesian products; see Example 2.1.3.2). Note that the tensor product functor $\otimes_k : \mathrm{Vect}_k \times \mathrm{Vect}_k \to \mathrm{Vect}_k$ is characterized by the requirement that it is given on objects by $(U, V) \mapsto U \otimes_k V$ and satisfies condition (a) of Definition 2.1.4.3, and the associativity constraint on Vect_k is characterized by the requirement that it satisfies condition (b) of Definition 2.1.4.3. Note that b is *not* a nonunital monoidal structure: the bilinear maps $b_{U,V} : U \times V \to U \otimes_k V$ are never bijective, except in the trivial case where $U \simeq 0 \simeq V$.

00D1 **Example 2.1.4.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital strict monoidal functor. Then F admits a nonunital monoidal structure $\{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$, where we take each $\mu_{X,Y}$ to be the identity morphism from $F(X) \otimes F(Y) = F(X \otimes Y)$ to itself.

Conversely, if (F, μ) is a nonunital monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ with the property that the tensor constraints $\mu_{X,Y}$ is an identity morphism in $\mathcal{D}$, then F is a nonunital strict monoidal functor.

00D2 **Example 2.1.4.7.** Let M and M' be nonunital monoids, regarded as nonunital monoidal categories having only identity morphisms (Example 2.1.1.3). Then nonunital lax monoidal functors from M to M' (in the sense of Definition 2.1.4.3) can be identified with nonunital monoid homomorphisms from M to M' (in the sense of Variant 1.3.2.8). Moreover, every nonunital lax monoidal functor from M to M' is automatically strict.

00D3 **Example 2.1.4.8** (The Left Regular Representation)**.** Let $\mathcal{C}$ be a nonunital monoidal category and let $\mathrm{End}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ be the category of functors from $\mathcal{C}$ to itself, endowed with the strict monoidal structure of Example 2.1.1.4. For each object $X \in \mathcal{C}$, let $\ell_X : \mathcal{C} \to \mathcal{C}$ denote the functor given on objects by the formula $\ell_X(Y) = X \otimes Y$. The construction $X \mapsto \ell_X$ then determines a functor $\ell : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$. For every pair of objects $X, Y \in \mathcal{C}$, there is a natural isomorphism $\mu_{X,Y} : \ell_X \circ \ell_Y \xrightarrow{\sim} \ell_{X \otimes Y}$, whose value on an object $Z \in \mathcal{C}$ is given by the associativity constraint

$$(\ell_X \circ \ell_Y)(Z) = X \otimes (Y \otimes Z) \xrightarrow{\alpha_{X,Y,Z}} (X \otimes Y) \otimes Z = \ell_{X \otimes Y}(Z).$$

Then $\mu = \{\mu_{X,Y}\}_{X,Y}$ is a nonunital monoidal structure on the functor $X \mapsto \ell_X$: property (a) of Definition 2.1.4.3 follows from the naturality of the associativity constraint on $\mathcal{C}$, and property (b) is a reformulation of the pentagon identity.

Warning 2.1.4.9. Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories. A nonunital strict monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ possessing certain *properties.* However, a nonunital (lax) monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ together with additional *structure*, given by the tensor constraints $\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$. We will often abuse terminology by identifying a nonunital (lax) monoidal functor (F, μ) with the underlying functor F; in this case, we implicitly assume that the tensor constraints $\mu_{X,Y}$ have been specified. 00D4

Definition 2.1.4.10. Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories. Let $F, F' : \mathcal{C} \to \mathcal{D}$ be functors equipped with nonunital lax monoidal structures μ and μ', respectively. We say that a natural transformation of functors $\gamma : F \to F'$ is *nonunital monoidal* if, for every pair of objects $X, Y \in \mathcal{C}$, the diagram 00D5

$$\begin{tikzcd}
F(X) \otimes F(Y) \arrow[r, "\mu_{X,Y}"] \arrow[d, "\gamma(X) \otimes \gamma(Y)"] & F(X \otimes Y) \arrow[d, "\gamma(X \otimes Y)"] \\
F'(X) \otimes F'(Y) \arrow[r, "\mu'_{X,Y}"] & F'(X \otimes Y)
\end{tikzcd}$$

is commutative.

We let $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D})$ denote the category whose objects are nonunital lax monoidal functors (F, μ) from $\mathcal{C}$ to $\mathcal{D}$, and whose morphisms are nonunital monoidal natural transformations, and we let $\mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D})$ spanned by the nonunital monoidal functors (F, μ) from $\mathcal{C}$ to $\mathcal{D}$.

Example 2.1.4.11 (Nonunital Algebras)**.** Let $\mathcal{C}$ be a nonunital monoidal category and let A be an object of $\mathcal{C}$. A *nonunital algebra structure on A* is a map $m : A \otimes A \to A$ for which 00D6 the diagram

$$\begin{tikzcd}
 & A \otimes (A \otimes A) \arrow[r, "\alpha_{A,A,A}"'] \arrow[dl, "\mathrm{id} \otimes m"'] & (A \otimes A) \otimes A \arrow[dr, "m \otimes \mathrm{id}"] & \\
A \otimes A \arrow[drr, "m"] & & & A \otimes A \arrow[dl, "m"'] \\
 & & A &
\end{tikzcd}$$

is commutative. A *nonunital algebra object of* $\mathcal{C}$ is a pair (A, m), where A is an object of $\mathcal{C}$ and m is a nonunital algebra structure on A. If (A, m) and (A', m') are nonunital algebra objects of $\mathcal{C}$, then we say that a morphism $f : A \to A'$ is a *nonunital algebra homomorphism* if the diagram

$$\begin{tikzcd}
A \otimes A \arrow[r, "m"] \arrow[d, "f \otimes f"] & A \arrow[d, "f"] \\
A' \otimes A' \arrow[r, "m'"] & A'
\end{tikzcd}$$

is commutative. We let $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ denote the category whose objects are nonunital algebra objects of $\mathcal{C}$ and whose morphisms are nonunital algebra homomorphisms.

Let $\{e\}$ denote the trivial monoid, regarded as a (strict) monoidal category having only identity morphisms (Example 2.1.1.3). Then we can identify objects $A \in \mathcal{C}$ with functors $F : \{e\} \to \mathcal{C}$ (by means of the formula $A = F(e)$). Unwinding the definitions, we see that nonunital lax monoidal structures on the functor F (in the sense of Definition 2.1.4.3) can be identified with nonunital algebra structures on the object $A = F(e)$. Under this identification, nonunital monoidal natural transformations correspond to homomorphisms of nonunital algebras. We therefore have an *isomorphism* of categories $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\{e\}, \mathcal{C}) \simeq \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$.

00D7 **Example 2.1.4.12.** Let Set denote the category of sets, endowed with the monoidal structure given by cartesian product of sets (Example 2.1.3.2). For each set S, we can identify nonunital algebra structures on S (in the sense of Example 2.1.4.11) with nonunital monoid structures on S (in the sense of Variant 1.3.2.8). This observation supplies an isomorphism of categories $\mathrm{Alg}^{\mathrm{nu}}(\mathrm{Set}) \simeq \mathrm{Mon}^{\mathrm{nu}}$, where $\mathrm{Mon}^{\mathrm{nu}}$ is the category of nounital monoids.

00D8 **Example 2.1.4.13.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and let $\mathcal{C}^{\mathrm{rev}}$ and $\mathcal{D}^{\mathrm{rev}}$ denote the same categories with the reversed nonunital monoidal structure (Example 2.1.3.5).

Then every functor $F : \mathcal{C} \to \mathcal{D}$ can be also regarded as a functor from $\mathcal{C}^{\mathrm{rev}}$ to $\mathcal{D}^{\mathrm{rev}}$, which we will denote by F^{rev}. There is a canonical bijection

$$\begin{tikzcd}
\{\text{Nonunital lax monoidal structures on } F\} \arrow[d, "\sim"] \\
\{\text{Nonunital lax monoidal structures on } F^{\mathrm{rev}}\},
\end{tikzcd}$$

which carries a nonunital lax monoidal structure μ to the nonunital lax monoidal structure μ^{rev} given by the formula $\mu^{\mathrm{rev}}_{X,Y} = \mu_{Y,X}$. Using these bijections, we obtain a canonical isomorphism of categories $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})$, which restricts to an isomorphism $\mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})$.

Example 2.1.4.14. Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and regard the opposite categories $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$ as equipped with the nonunital monoidal structures of Example 2.1.3.4. Then every functor $F : \mathcal{C} \to \mathcal{D}$ determines a functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. There is a canonical bijection 00D9

$$\{\text{Nonunital monoidal structures on } F\} \simeq \{\text{Nonunital monoidal structures on } F^{\mathrm{op}}\},$$

which carries a nonunital monoidal structure μ on F to a nonunital monoidal structure μ' on F^{op}, given concretely by $\mu'_{X,Y} = \mu^{-1}_{X,Y}$. Using these bijections, we obtain a canonical isomorphism of categories $\mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$.

Warning 2.1.4.15. The analogue of Example 2.1.4.14 for nonunital lax monoidal functors is false. The notion of nonunital lax monoidal functor is not self-opposite: in general, there is no simple relationship between the categories $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$. 00DA

Motivated by Warning 2.1.4.15, we introduce the following:

Variant 2.1.4.16. Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A *nonunital colax monoidal structure* on F is a nonunital lax monoidal structure on the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ (Definition 2.1.4.3). In other words, a colax monoidal structure on F is a collection of morphisms $\mu = \{\mu_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)\}_{X,Y \in \mathcal{C}}$ which satisfy the following pair of conditions: 00MM

(a) The morphisms $\mu_{X,Y}$ depend functorially on X and Y: that is, for every pair of morphisms $f : X \to X'$, $g : Y \to Y'$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd}
F(X \otimes Y) \arrow[r, "\mu_{X,Y}"] \arrow[d, "F(f \otimes g)"] & F(X) \otimes F(Y) \arrow[d, "F(f) \otimes F(g)"] \\
F(X' \otimes Y') \arrow[r, "\mu_{X',Y'}"] & F(X') \otimes F(Y')
\end{tikzcd}$$

commutes (in the category $\mathcal{D}$).

(b) For every triple of objects $X, Y, Z \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
F(X \otimes (Y \otimes Z)) \arrow[r, "F(\alpha_{X,Y,Z})"] \arrow[d, "\mu_{X,Y \otimes Z}"] & F((X \otimes Y) \otimes Z) \arrow[d, "\mu_{X \otimes Y,Z}"] \\
F(X) \otimes F(Y \otimes Z) \arrow[d, "\mathrm{id} \otimes \mu_{Y,Z}"] & F(X \otimes Y) \otimes F(Z) \arrow[d, "\mu_{X,Y} \otimes \mathrm{id}"] \\
F(X) \otimes (F(Y) \otimes F(Z)) \arrow[r, "\alpha_{F(X),F(Y),F(Z)}"] & (F(X) \otimes F(Y)) \otimes F(Z)
\end{tikzcd}$$

commutes.

`00DB` **Construction 2.1.4.17** (Composition of Nonunital Monoidal Functors)**.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be nonunital monoidal categories, and suppose we are given a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$. If $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ is a nonunital lax monoidal structure on the functor F and $\nu = \{\nu_{U,V}\}_{U,V \in \mathcal{D}}$ is a nonunital lax monoidal structure on G, then the composite functor $G \circ F$ inherits a nonunital lax monoidal structure, which associates to each pair of objects $X, Y \in \mathcal{C}$ the composite map

$$(G \circ F)(X) \otimes (G \circ F)(Y) \xrightarrow{\nu_{F(X),F(Y)}} G(F(X) \otimes F(Y)) \xrightarrow{G(\mu_{X,Y})} (G \circ F)(X \otimes Y).$$

This construction determines a composition law

$$\circ : \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}, \mathcal{E}).$$

`00DC` **Remark 2.1.4.18.** In the situation of Construction 2.1.4.17, suppose that μ and ν are nonunital monoidal structures on F and G, respectively: that is, assume that all of the tensor constraints

$$\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y) \qquad \nu_{U,V} : G(U) \otimes G(V) \to G(U \otimes V)$$

are isomorphisms. Then Construction 2.1.4.17 supplies a nonunital monoidal structure on the composite functor $G \circ F$. We therefore obtain a composition law

$$\circ : \mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{C}, \mathcal{E}).$$

We close this section by describing an alternative perspective on nonunital lax monoidal functors. First, we need to review a bit of terminology.

Notation 2.1.4.19 (Oriented Fiber Products)**.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories, and suppose we are given a pair of functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$. We let $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ denote the iterated pullback $\mathcal{C} \times_{\mathrm{Fun}(\{0\}, \mathcal{E})} \mathrm{Fun}([1], \mathcal{E}) \times_{\mathrm{Fun}(\{1\}, \mathcal{E})} \mathcal{D}$. We will refer to $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ as the *oriented fiber product of* $\mathcal{C}$ *with* $\mathcal{D}$ *over* $\mathcal{E}$. More concretely: 00MN

- An object of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ is a triple (C, D, η) where C is an object of the category $\mathcal{C}$, D is an object of the category $\mathcal{D}$, and $\eta : F(C) \to G(D)$ is a morphism in the category $\mathcal{E}$.

- If (C, D, η) and (C', D', η') are objects of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$, then a morphism from (C, D, η) to (C', D', η') is a pair (u, v), where $u : C \to C'$ is a morphism in the category $\mathcal{C}$, $v : D \to D'$ is a morphism in the category $\mathcal{D}$, and the diagram

$$\begin{tikzcd}
F(C) \arrow[r, "\eta"] \arrow[d, "F(u)"] & G(D) \arrow[d, "G(v)"] \\
F(C') \arrow[r, "\eta'"] & G(D')
\end{tikzcd}$$

commutes in the category $\mathcal{E}$.

Remark 2.1.4.20. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. The oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ is often referred to in the literature as the *comma construction* on the functors F and G, and is commonly denoted by $F \downarrow G$. 02AP

Proposition 2.1.4.21. *Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories, let $G : \mathcal{D} \to \mathcal{C}$ be a functor, and let $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ denote the oriented fiber product of Notation 2.1.4.19. Then:* 00MP

- *Let $\mu = \{\mu_{D,D'}\}_{D,D' \in \mathcal{D}}$ be a nonunital lax monoidal structure on the functor G. Then there is a unique nonunital monoidal structure $\otimes_{\mu}$ on the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ with the following properties:*

  (1) *The forgetful functor*

  $$U : \mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \to \mathcal{C} \times \mathcal{D} \qquad (C, D, \eta) \mapsto (C, D)$$

  is a strict nonunital monoidal functor.

(2) *On objects, the tensor product* $\otimes_\mu$ *is given by the formula*

$$(C, D, \eta) \otimes_\mu (C', D', \eta') = (C \otimes C', D \otimes D', t(\eta, \eta')),$$

where $t(\eta, \eta')$ *is the composition* $C \otimes C' \xrightarrow{\eta \otimes \eta'} G(D) \otimes G(D') \xrightarrow{\mu_{D,D'}} G(D \otimes D')$.

- *The construction* $\mu \mapsto \otimes_\mu$ *induces a bijection*

$$\begin{tikzcd}
\{\text{Nonunital lax monoidal structures on } G\} \arrow[d] \\
\{\text{Nonunital monoidal structures on } \mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \text{ satisfying (1)}\}.
\end{tikzcd}$$

00MQ **Remark 2.1.4.22.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories. We can summarize Proposition 2.1.4.21 more informally as follows: for any functor $G : \mathcal{D} \to \mathcal{C}$, choosing a nonunital lax monoidal structure on G is equivalent to choosing a nonunital monoidal structure on the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ which is compatible with the existing nonunital monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, respectively.

Proof of Proposition 2.1.4.21. Unwinding the definitions, we see that to describe nonunital monoidal structure on the category $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ satisfying condition (1), one must give the following data:

- For every pair of objects (C, D, η) and (C', D', η') of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$, we must supply a tensor product $(C, D, \eta) \otimes (C', D', \eta')$. By virtue of the assumption that U is nonunital strict monoidal, this tensor product must be given as a triple $(C \otimes C', D \otimes D', t(\eta, \eta'))$, for some morphism $t(\eta, \eta') : C \otimes C' \to G(D \otimes D')$ in the category $\mathcal{D}$.

- For every pair of morphisms $(u, v) : (C, D, \eta) \to (\overline{C}, \overline{D}, \overline{\eta})$ and $(u', v') : (C', D', \eta') \to (\overline{C}', \overline{D}', \overline{\eta}')$ in the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$, we must supply a tensor product morphism $(C \otimes C', D \otimes D', t(\eta, \eta')) \to (\overline{C} \otimes \overline{C}', \overline{D} \otimes \overline{D}', t(\overline{\eta}, \overline{\eta}'))$. Note that this morphism is uniquely determined: for U to be a nonunital strict monoidal functor, it must be the pair $(u \otimes u', v \otimes v')$. However, the *existence* of this morphism imposes the following condition:

(i) If the diagrams

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[d, "u"] & G(D) \arrow[d, "G(v)"] \\ \overline{C} \arrow[r, "\overline{\eta}"] & G(\overline{D}) \end{tikzcd} \qquad \begin{tikzcd} C' \arrow[r, "\eta'"] \arrow[d, "u'"] & G(D') \arrow[d, "G(v')"] \\ \overline{C}' \arrow[r, "\overline{\eta}'"] & G(\overline{D}') \end{tikzcd}$$

commute (in the category $\mathcal{C}$), then the diagram

$$\begin{tikzcd} C \otimes C' \arrow[r, "t(\eta,\eta')"] \arrow[d, "u\otimes u'"] & G(D \otimes D') \arrow[d, "G(v\otimes v')"] \\ \overline{C} \otimes \overline{C}' \arrow[r, "t(\overline{\eta},\overline{\eta}')"] & G(\overline{D} \otimes \overline{D}') \end{tikzcd}$$

also commutes.

- For every triple of objects (C, D, η), (C', D', η'), and (C'', D'', η'') of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$, we must supply an associativity constraint

$$(C, D, \eta) \otimes ((C', D', \eta') \otimes (C'', D'', \eta'')) \simeq ((C, D, \eta) \otimes (C', D', \eta')) \otimes (C'', D'', \eta'')$$

in $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$. By virtue of our assumption that U is nonunital strict monoidal, this associativity constraint is uniquely determined: it must be the pair $(\alpha_{C,C',C''}, \alpha_{D,D',D''})$ given by the associativity constraints for the nonunital monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, respectively. However, the *existence* of this morphism imposes the following condition:

(ii) For every triple of morphisms $\eta : C \to G(D)$, $\eta' : C' \to G(D')$, and $\eta'' : C'' \to G(D'')$, the diagram

$$\begin{tikzcd} C \otimes (C' \otimes C'') \arrow[r, "\alpha_{C,C',C''}"] \arrow[d, "t(\eta,t(\eta',\eta''))"] & (C \otimes C') \otimes C'' \arrow[d, "t(t(\eta,\eta'),\eta'')"] \\ G(D \otimes (D' \otimes D'')) \arrow[r, "G(\alpha_{D,D',D''})"] & G((D \otimes D') \otimes D'') \end{tikzcd}$$

commutes (in the category $\mathcal{C}$).

> If this condition is satisfied, then the associativity constraints are automatically functorial and satisfy the pentagon identity (since the analogous conditions hold in the categories $\mathcal{C}$ and $\mathcal{D}$, respectively).

Given a collection of morphisms $t(\eta, \eta')$ satisfying these conditions, we define $\mu = \{\mu_{D,D'}\}_{D,D' \in \mathcal{D}}$ by the formula $\mu_{D,D'} = t(\mathrm{id}_{G(D)}, \mathrm{id}_{G(D')})$. Note that, if (C, D, η) and (C', D', η') are arbitrary objects of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$, then we have canonical maps

$$(\eta, \mathrm{id}_D) : (C, D, \eta) \to (G(D), D, \mathrm{id}_{G(D)}) \qquad (\eta', \mathrm{id}_{D'}) : (C', D', \eta') \to (G(D'), D', \mathrm{id}_{G(D')}).$$

Applying condition (i), we see that the morphism $t(\eta, \eta')$ can then be recovered as the composition

$$C \otimes C' \xrightarrow{\eta \otimes \eta'} G(D) \otimes G(D') \xrightarrow{\mu_{D,D'}} G(D \otimes D').$$

To complete the proof, it will suffice to show that if we are given *any* system of morphisms $\mu = \{\mu_{D,D'} : G(D) \otimes G(D') \to G(D \otimes D')\}_{D,D' \in \mathcal{D}}$ and we define $t(\eta, \eta')$ as above, then μ is a nonunital lax monoidal structure on G if and only if conditions (i) and (ii) are satisfied.

Using the formula for $t(\eta, \eta')$ in terms of μ, we can rewrite condition (i) as follows:

(i') If the diagrams

$$\begin{tikzcd}
C \arrow[r, "\eta"] \arrow[d, "u"] & G(D) \arrow[d, "G(v)"] \\
\overline{C} \arrow[r, "\overline{\eta}"] & G(\overline{D})
\end{tikzcd} \qquad \begin{tikzcd}
C' \arrow[r, "\eta'"] \arrow[d, "u'"] & G(D') \arrow[d, "G(v')"] \\
\overline{C}' \arrow[r, "\overline{\eta}'"] & G(\overline{D}')
\end{tikzcd}$$

commute (in the category $\mathcal{C}$), then the outer rectangle in the diagram

$$\begin{tikzcd}
C \otimes C' \arrow[r, "\eta \otimes \eta'"] \arrow[d, "u \otimes u'"] & G(D) \otimes G(D') \arrow[r, "\mu_{D,D'}"] \arrow[d, "G(v) \otimes G(v')"] & G(D \otimes D') \arrow[d, "G(v \otimes v')"] \\
\overline{C} \otimes \overline{C}' \arrow[r, "\overline{\eta} \otimes \overline{\eta}'"] & G(\overline{D}) \otimes G(\overline{D}') \arrow[r, "\mu_{\overline{D}, \overline{D}'}"] & G(\overline{D} \otimes \overline{D}')
\end{tikzcd}$$

commutes.

Note that the left square appearing in this diagram is automatically commutative. Assertion (i') is therefore a consequence of the following:

(a) For every pair of morphisms $v : D \to \overline{D}$ and $v' : D' \to \overline{D}'$ in the category $\mathcal{D}$, the diagram

$$\begin{tikzcd}
G(D) \otimes G(D') \arrow[r, "\mu_{D,D'}"] \arrow[d, "G(v) \otimes G(v')"] & G(D \otimes D') \arrow[d, "G(v \otimes v')"] \\
G(\overline{D}) \otimes G(\overline{D}') \arrow[r, "\mu_{\overline{D},\overline{D}'}"] & G(\overline{D} \otimes \overline{D}')
\end{tikzcd}$$

commutes (in the category $\mathcal{C}$).

Conversely, if (i') is satisfied, then (a) can be deduced by specializing to the case $\eta = \mathrm{id}_{G(D)}$, $\eta' = \mathrm{id}_{G(D')}$, $\overline{\eta} = \mathrm{id}_{G(\overline{D})}$, and $\overline{\eta}' = \mathrm{id}_{G(\overline{D}')}$. It follows that (i) is satisfied if and only if (a) is satisfied: that is, if and only if $\mu = \{\mu_{D,D'}\}_{D,D' \in \mathcal{D}}$ is a natural transformation.

We can reformulate condition (ii) as follows:

(ii') For every triple of morphisms $\eta : C \to G(D)$, $\eta' : C' \to G(D')$, and $\eta'' : C'' \to G(D'')$, the outer rectangle in the diagram

$$\begin{tikzcd}
C \otimes (C' \otimes C'') \arrow[r, "\alpha_{C,C',C''}"] \arrow[d, "\eta \otimes (\eta' \otimes \eta'')"] & (C \otimes C') \otimes C'' \arrow[d, "(\eta \otimes \eta') \otimes \eta''"] \\
G(D) \otimes (G(D') \otimes G(D'')) \arrow[r, "\alpha_{G(D),G(D'),G(D'')}"] \arrow[d, "\mathrm{id}_{G(D)} \otimes \mu_{D,D'}"] & (G(D) \otimes G(D')) \otimes G(D'') \arrow[d, "\mu_{D,D'} \otimes \mathrm{id}_{G(D'')}"] \\
G(D) \otimes G(D' \otimes D'') \arrow[d, "\mu_{D,D' \otimes D''}"] & (G(D) \otimes G(D')) \otimes G(D'') \arrow[d, "\mu_{D \otimes D', D''}"] \\
G(D \otimes (D' \otimes D'')) \arrow[r, "G(\alpha_{D,D',D''})"] & G((D \otimes D') \otimes D'')
\end{tikzcd}$$

commutes (in the category $\mathcal{C}$).

Since the upper square in this diagram automatically commutes (by the naturality of the associativity constraints on $\mathcal{C}$), assertion (ii') is a consequence of the following simpler assertion:

(*b*) For every triple of objects $D, D', D'' \in \mathcal{D}$, the diagram

$$\begin{tikzcd}
G(D) \otimes (G(D') \otimes G(D'')) \arrow[r, "\alpha_{G(D),G(D'),G(D'')}"] \arrow[d, "\mathrm{id}_{G(D)} \otimes \mu_{D,D'}"] & (G(D) \otimes G(D')) \otimes G(D'') \arrow[d, "\mu_{D,D'} \otimes \mathrm{id}_{G(D'')}"] \\
G(D) \otimes G(D' \otimes D'') \arrow[d, "\mu_{D,D' \otimes D''}"] & (G(D) \otimes G(D')) \otimes G(D'') \arrow[d, "\mu_{D \otimes D', D''}"] \\
G(D \otimes (D' \otimes D'')) \arrow[r, "G(\alpha_{D,D',D''})"] & G((D \otimes D') \otimes D'')
\end{tikzcd}$$

commutes (in the category $\mathcal{C}$).

Conversely, if (ii') is satisfied, then (*b*) can be deduced by specializing to the case $\eta = \mathrm{id}_{G(D)}$, $\eta' = \mathrm{id}_{G(D')}$, and $\eta'' = \mathrm{id}_{G(D'')}$. We conclude by observing that conditions (*a*) and (*b*) assert precisely that μ is a nonunital lax monoidal structure (Definition 2.1.4.3). □

00MR **Remark 2.1.4.23** (Adjoint Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be nonunital monoidal categories and suppose we are given a pair of adjoint functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, so that we have an isomorphism of oriented fiber products $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \simeq \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ (see Notation 2.1.4.19). Applying Proposition 2.1.4.21 (and the dual characterization of nonunital colax monoidal functors), we see that the following are equivalent:

- The datum of a nonunital lax monoidal structure on the functor $G : \mathcal{D} \to \mathcal{C}$.
- The datum of a nonunital colax monoidal structure on the functor $F : \mathcal{C} \to \mathcal{D}$.
- The datum of a nonunital monoidal structure on the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \simeq \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ which is compatible with the nonunital monoidal structures on $\mathcal{C}$ and $\mathcal{D}$ (meaning that the projection map $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ is a nonunital strict monoidal functor).

2.1.5 Lax Monoidal Functors

00DD We now introduce a unital version of Definition 2.1.4.3. To motivate the discussion, we begin with a special case.

00DE **Definition 2.1.5.1.** Let $\mathcal{C}$ be a monoidal category with unit object $\mathbf{1}$, and let A be a nonunital algebra object of $\mathcal{C}$ (Example 2.1.4.11) with multiplication $m : A \otimes A \to A$. We say that a morphism $\epsilon : \mathbf{1} \to A$ is a *left unit* for A if the composite map

$$A \xrightarrow{\lambda_A^{-1}} \mathbf{1} \otimes A \xrightarrow{\epsilon \otimes \mathrm{id}_A} A \otimes A \xrightarrow{m} A$$

is the identity map from A to itself; here $\lambda_A : \mathbf{1} \otimes A \xrightarrow{\sim} A$ denotes the left unit constraint of Construction 2.1.2.17. We say that ϵ is a *right unit of A* if the composite map

$$A \xrightarrow{\rho_A^{-1}} A \otimes \mathbf{1} \xrightarrow{\mathrm{id}_A \otimes \epsilon} A \otimes A \xrightarrow{m} A$$

is equal to the identity. We say that ϵ is a *unit of A* if it is both a left and a right unit of A.

By virtue of Example 2.1.4.11, we can view the theory of nonunital algebras as a special case of the theory of nonunital lax monoidal functors $F : \mathcal{C} \to \mathcal{D}$, where we take $\mathcal{C}$ to be the trivial monoid $\{e\}$ (regarded as a category having only identity morphisms). Definition 2.1.5.1 has an analogue for nonunital lax monoidal functors in general.

Definition 2.1.5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories with unit objects $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$, respectively. Let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital lax monoidal functor with tensor constraints $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$. Let $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ be a morphism in $\mathcal{D}$. We say that ϵ is a *left unit for F* if, for every object $X \in \mathcal{C}$, the left unit constraint $\lambda_{F(X)} : \mathbf{1}_{\mathcal{D}} \otimes F(X) \xrightarrow{\sim} F(X)$ in the category $\mathcal{D}$ is equal to the composition `00DG`

$$\mathbf{1}_{\mathcal{D}} \otimes F(X) \xrightarrow{\epsilon \otimes \mathrm{id}_{F(X)}} F(\mathbf{1}_{\mathcal{C}}) \otimes F(X) \xrightarrow{\mu_{\mathbf{1}_{\mathcal{C}},X}} F(\mathbf{1}_{\mathcal{C}} \otimes X) \xrightarrow{F(\lambda_X)} F(X),$$

where $\lambda_X : \mathbf{1}_{\mathcal{C}} \otimes X \xrightarrow{\sim} X$ is the left unit constraint in the monoidal category $\mathcal{C}$. We say that ϵ is a *right unit for F* if, for every object $X \in \mathcal{C}$, the right unit constraint $\rho_{F(X)} : F(X) \otimes \mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} F(X)$ is equal to the composition

$$F(X) \otimes \mathbf{1}_{\mathcal{D}} \xrightarrow{\mathrm{id}_{F(X)} \otimes \epsilon} F(X) \otimes F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\mu_{X,\mathbf{1}_{\mathcal{C}}}} F(X \otimes \mathbf{1}_{\mathcal{C}}) \xrightarrow{F(\rho_X)} F(X).$$

We say that ϵ is a *unit for F* if it is both a left and a right unit for F.

Example 2.1.5.3. Let $\mathcal{C}$ be a monoidal category and let A be a nonunital algebra object of $\mathcal{C}$, which we identify with a nonunital lax monoidal functor $F : \{e\} \to \mathcal{C}$ as in Example 2.1.4.11. Then a map $\epsilon : \mathbf{1} \to A = F(e)$ is a unit (left unit, right unit) for A (in the sense of Definition 2.1.5.1) if and only if it is a unit (left unit, right unit) for F (in the sense of Definition 2.1.5.2). `00DH`

We now show that if a nonunital lax monoidal functor F admits a unit ϵ, then ϵ is uniquely determined. This is a consequence of the following:

Proposition 2.1.5.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories with unit objects $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$, respectively, and let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital lax monoidal functor. Suppose that F admits a left unit $\epsilon_L : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ and a right unit $\epsilon_R : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$. Then $\epsilon_L = \epsilon_R$.* `00DJ`

Proof. We first observe that there is a commutative diagram

$$\begin{tikzcd}
\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} \arrow[r, "\mathrm{id} \otimes \epsilon_R"] \arrow[dd, "\lambda_{\mathbf{1}_{\mathcal{D}}}"] & \mathbf{1}_{\mathcal{D}} \otimes F(\mathbf{1}_{\mathcal{C}}) \arrow[r, "\epsilon_L \otimes \mathrm{id}"] \arrow[ddr, "\lambda_{F(\mathbf{1}_{\mathcal{C}})}"'] & F(\mathbf{1}_{\mathcal{C}}) \otimes F(\mathbf{1}_{\mathcal{C}}) \arrow[d, "\mu_{\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}}"] \\
 & & F(\mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}}) \arrow[d, "F(\lambda_{\mathbf{1}_{\mathcal{C}}})"] \\
\mathbf{1}_{\mathcal{D}} \arrow[rr, "\epsilon_R"] & & F(\mathbf{1}_{\mathcal{C}});
\end{tikzcd}$$

the left square commutes by the naturality of the left unit constraints for $\mathcal{C}$ (Remark 2.1.2.18), and the right square commutes by virtue of our assumption that ϵ_L is a left unit for $\mathcal{C}$. Using Corollary 2.1.2.21, we see that the unit constraints

$$\upsilon_{\mathcal{C}} : \mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}} \xrightarrow{\sim} \mathbf{1}_{\mathcal{C}} \qquad \upsilon_{\mathcal{D}} : \mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} \mathbf{1}_{\mathcal{D}}$$

are equal to the left unit constraints $\lambda_{\mathbf{1}_{\mathcal{C}}}$ and $\lambda_{\mathbf{1}_{\mathcal{D}}}$, respectively. It follows that the composition $\epsilon_R \circ \upsilon_{\mathcal{D}}$ coincides with the composition

$$\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} \xrightarrow{\epsilon_L \otimes \epsilon_R} F(\mathbf{1}_{\mathcal{C}}) \otimes F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\mu_{\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}}} F(\mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}}) \xrightarrow{F(\upsilon_{\mathcal{C}})} F(\mathbf{1}_{\mathcal{C}}).$$

A similar argument shows that this composition coincides with $\epsilon_L \circ \upsilon_{\mathcal{D}}$. Since $\upsilon_{\mathcal{D}}$ is an isomorphism, it follows that $\epsilon_R = \epsilon_L$. □

00DK **Corollary 2.1.5.5.** *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital lax monoidal functor. Then F admits a unit $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ if and only if it has both a left unit and a right unit. In this case, the unit ϵ is unique.*

00MS **Proposition 2.1.5.6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories with unit objects $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$, respectively. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor equipped with a nonunital lax monoidal structure, which we will identify with the corresponding nonunital monoidal structure on the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ (see Proposition 2.1.4.21). Let $\epsilon : \mathbf{1}_{\mathcal{C}} \to G(\mathbf{1}_{\mathcal{D}})$ be a morphism in $\mathcal{C}$, and regard the triple $\mathbf{1} = (\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{D}}, \epsilon)$ as an object of $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$. Then:*

(1) *The morphism ϵ is a left unit for G if and only if, for every object (C, D, η) of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$, the left unit constraints $\lambda_C : \mathbf{1}_{\mathcal{C}} \otimes C \simeq C$ and $\lambda_D : \mathbf{1}_{\mathcal{D}} \otimes D \simeq D$ determine an isomorphism $(\lambda_C, \lambda_D) : \mathbf{1} \otimes (C, D, \eta) \simeq (C, D, \eta)$ in the category $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$.*

(2) *The morphism ϵ is a right unit for G if and only if, for every object (C, D, η) of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$, the right unit constraints $\rho_C : C \otimes \mathbf{1}_{\mathcal{C}} \simeq C$ and $\rho_D : D \otimes \mathbf{1}_{\mathcal{D}} \simeq D$ determine an isomorphism $(\rho_C, \rho_D) : (C, D, \eta) \otimes \mathbf{1} \simeq (C, D, \eta)$ in the category $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$.*

Proof. We will prove (1); the proof of (2) is similar. Fix an object (C, D, η) of the oriented fiber product $\mathcal{C} \otimes_{\mathcal{C}} \mathcal{D}$. Unwinding the definitions, we see that the pair (λ_C, λ_D) determines a morphism from $\mathbf{1} \otimes (C, D, \eta)$ to (C, D, η) in $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ if and only if the outer rectangle of the diagram

$$\begin{tikzcd}
\mathbf{1}_{\mathcal{C}} \otimes C \arrow[r, "\lambda_C"] \arrow[d, "\mathrm{id} \otimes \eta"] & C \arrow[d, "\eta"] \\
\mathbf{1}_{\mathcal{C}} \otimes G(D) \arrow[r, "\lambda_{G(D)}"] \arrow[d, "\epsilon \otimes \mathrm{id}"] & G(D) \arrow[dd, "\mathrm{id}"] \\
G(\mathbf{1}_{\mathcal{D}}) \otimes G(D) \arrow[d, "\mu"] & \\
G(\mathbf{1}_{\mathcal{D}} \otimes D) \arrow[r, "G(\lambda_D)"] & G(D)
\end{tikzcd}$$

is commutative. Here the upper square commutes by the functoriality of the left unit constraints in $\mathcal{C}$ (Remark 2.1.2.18), and the commutativity of the lower rectangle follows from the assumption that ϵ is a left unit. This proves the "only if" direction of (1). The converse follows by specializing to the case where $C = G(D)$ and η is the identity map. ☐

Corollary 2.1.5.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories with units $(\mathbf{1}_{\mathcal{C}}, \upsilon_{\mathcal{C}})$ and $(\mathbf{1}_{\mathcal{D}}, \upsilon_{\mathcal{D}})$, respectively. Let $G : \mathcal{D} \to \mathcal{C}$ be a nonunital lax monoidal functor. Let $\epsilon : \mathbf{1}_{\mathcal{C}} \to G(\mathbf{1}_{\mathcal{D}})$ be a morphism in $\mathcal{C}$ and regard the triple $\mathbf{1} = (\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{D}}, \epsilon)$ as an object of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$. The following conditions are equivalent:* `00MT`

(1) *The morphism ϵ is a unit for G (in the sense of Definition 2.1.5.2).*

(2) *The pair $\upsilon = (\upsilon_{\mathcal{C}}, \upsilon_{\mathcal{D}})$ is a morphism from $\mathbf{1} \otimes \mathbf{1}$ to $\mathbf{1}$ in the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$, and the pair $(\mathbf{1}, \upsilon)$ is a unit with respect to the tensor product $\otimes_{\mu}$ of Proposition 2.1.4.21.*

Proof. Assume first that (1) is satisfied. Then Proposition 2.1.5.6 implies that the functors

$$\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \to \mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \qquad X \mapsto \mathbf{1} \otimes X, X \mapsto X \otimes \mathbf{1}$$

are naturally isomorphic to the identity, and are therefore fully faithful. To complete the proof of (2), it will suffice to show that the pair $(\upsilon_{\mathcal{C}}, \upsilon_{\mathcal{D}})$ is a morphism from $\mathbf{1} \otimes \mathbf{1}$ to $\mathbf{1}$ in

$\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$. This also follows from Proposition 2.1.5.6, by virtue of the identities $\upsilon_{\mathcal{C}} = \lambda_{\mathbf{1}_{\mathcal{C}}}$ and $\upsilon_{\mathcal{D}} = \lambda_{\mathbf{1}_{\mathcal{D}}}$ (Corollary 2.1.2.21).

Now suppose that (2) is satisfied, so that we can regard $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ as a monoidal category with unit $(\mathbf{1}, \upsilon)$. It follows that the forgetful functor $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ carries the left and right unit constraints of $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ to the left and right unit constraints of $\mathcal{C}$ and $\mathcal{D}$. Applying Proposition 2.1.5.6, we conclude that ϵ is both a left and right unit for the nonunital lax monoidal functor G. □

00DL **Definition 2.1.5.8.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A *lax monoidal structure* on F is a nonunital lax monoidal structure $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ (Definition 2.1.4.3) for which there exists a unit $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$.

A *lax monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a pair (F, μ), where $F : \mathcal{C} \to \mathcal{D}$ is functor and μ is a lax monoidal structure on F. In this case, we will refer to the morphism $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ as the *unit of F*.

00DM **Remark 2.1.5.9.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital lax monoidal functor. The condition that F is a lax monoidal functor depends only on the underlying nonunital monoidal structures on $\mathcal{C}$ and $\mathcal{D}$, and not on the particular choice of units $(\mathbf{1}_{\mathcal{C}}, \upsilon_{\mathcal{C}})$ and $(\mathbf{1}_{\mathcal{D}}, \upsilon_{\mathcal{D}})$ for $\mathcal{C}$ and $\mathcal{D}$, respectively (see Remark 2.1.2.11).

Combining Proposition 2.1.4.21 with Corollary 2.1.5.7, we obtain the following:

00MU **Corollary 2.1.5.10.** *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, let $G : \mathcal{D} \to \mathcal{C}$ be a functor, let $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D}$ be the oriented fiber product of Notation 2.1.4.19, and let $U : \mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ denote the forgetful functor $(C, D, \eta) \mapsto (C, D)$. Then the construction $\mu \mapsto \otimes_{\mu}$ of Proposition 2.1.4.21 restricts to a bijection*

$$\begin{tikzcd}
\{\text{Lax monoidal structures on } G\} \arrow[d] \\
\left\{\begin{matrix}\text{Monoidal structures on } \mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \\ \text{with } U \text{ strict monoidal}\end{matrix}\right\}
\end{tikzcd}$$

(see Example 2.1.6.5).

00MV **Variant 2.1.5.11.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A *colax monoidal structure* on F is a lax monoidal structure on the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$: that is, a collection of maps $\mu = \{\mu_{X,Y} : F(X \otimes Y) \to F(X) \otimes F(Y)\}_{X,Y \in \mathcal{C}}$ satisfying the requirements of Variant 2.1.4.16, together with the additional condition that there exists a *counit* $\epsilon : F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ having the property that, for every object $X \in \mathcal{C}$, the left and right unit constraints of $F(X)$ the inverses of the composite maps

$$F(X) \xrightarrow{F(\lambda_X)} F(\mathbf{1}_{\mathcal{C}} \otimes X) \xrightarrow{\mu_{\mathbf{1}_{\mathcal{C}},X}} F(\mathbf{1}_{\mathcal{C}}) \otimes F(X) \xrightarrow{\epsilon \otimes \mathrm{id}} \mathbf{1}_{\mathcal{D}} \otimes F(X)$$

$$F(X) \xrightarrow{F(\rho_X)} F(X \otimes \mathbf{1}_{\mathcal{C}}) \xrightarrow{\mu_{X,\mathbf{1}_{\mathcal{C}}}} F(X) \otimes F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\mathrm{id} \otimes \epsilon} F(X) \otimes \mathbf{1}_{\mathcal{C}}.$$

Remark 2.1.5.12 (Adjoint Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and suppose we are given a pair of adjoint functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, given by an isomorphism of oriented fiber products $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \simeq \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ (see Notation 2.1.4.19). Applying Corollary 2.1.5.10 (and the dual characterization of colax monoidal functors), we see that the following are equivalent: `00MW`

- The datum of a lax monoidal structure on the functor $G : \mathcal{D} \to \mathcal{C}$.
- The datum of a colax monoidal structure on the functor $F : \mathcal{C} \to \mathcal{D}$.
- The datum of a monoidal structure on the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{D} \simeq \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ which is compatible with the monoidal structures on $\mathcal{C}$ and $\mathcal{D}$.

The compatibility conditions appearing in Definition 2.1.5.2 can be formulated more directly in terms of the unit constraints of $\mathcal{C}$ and $\mathcal{D}$ (without referring the left and right unit constraints of Construction 2.1.2.17).

Proposition 2.1.5.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories with unit objects $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$, respectively, let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital lax monoidal functor, and let $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ be a morphism in $\mathcal{C}$. Then ϵ is a left unit for F if and only if it satisfies the following pair of conditions:* `00DN`

(1) *The diagram*

$$\begin{tikzcd}
\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} \arrow[r, "\epsilon \otimes \epsilon"] \arrow[dd, "\upsilon_{\mathcal{D}}"] & F(\mathbf{1}_{\mathcal{C}}) \otimes F(\mathbf{1}_{\mathcal{C}}) \arrow[d, "\mu_{\mathbf{1}_{\mathcal{C}},\mathbf{1}_{\mathcal{C}}}"] \\
& F(\mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}}) \arrow[d, "F(\upsilon_{\mathcal{C}})"] \\
\mathbf{1}_{\mathcal{D}} \arrow[r, "\epsilon"] & F(\mathbf{1}_{\mathcal{C}})
\end{tikzcd}$$

commutes (in the category $\mathcal{D}$). Here $\upsilon_{\mathcal{C}}$ and $\upsilon_{\mathcal{D}}$ denote the unit constraints of $\mathcal{C}$ and $\mathcal{D}$, respectively.

(2) *For every object $X \in \mathcal{C}$, the composite map*

$$\mathbf{1}_{\mathcal{D}} \otimes F(X) \xrightarrow{\epsilon \otimes \mathrm{id}_{F(X)}} F(\mathbf{1}_{\mathcal{C}}) \otimes F(X) \xrightarrow{\mu_{\mathbf{1}_{\mathcal{C}},X}} F(\mathbf{1}_{\mathcal{C}} \otimes X)$$

is a monomorphism in the category $\mathcal{C}$.

Moreover, if these conditions are satisfied, then the map

$$\mathbf{1}_{\mathcal{D}} \otimes F(X) \xrightarrow{\epsilon \otimes \mathrm{id}_{F(X)}} F(\mathbf{1}_{\mathcal{C}}) \otimes F(X) \xrightarrow{\mu_{\mathbf{1}_{\mathcal{C}},X}} F(\mathbf{1}_{\mathcal{C}} \otimes X)$$

is an isomorphism for each $X \in \mathcal{C}$.

00DP **Example 2.1.5.14.** In the special case where $\mathcal{C} = \{e\}$, we can identify a nonunital lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ with a nonunital algebra object A of $\mathcal{D}$. In this case, Proposition 2.1.5.13 asserts that a morphism $\epsilon : \mathbf{1}_{\mathcal{D}} \to A$ is a left unit (in the sense of Definition 2.1.5.1) if and only if the diagram

$$\begin{tikzcd}
\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} \arrow[r, "\epsilon \otimes \epsilon"] \arrow[d, "\upsilon"] & A \otimes A \arrow[d, "m"] \\
\mathbf{1}_{\mathcal{D}} \arrow[r, "\epsilon"] & A
\end{tikzcd}$$

is commutative (that is, ϵ is idempotent) and the map

$$\mathbf{1}_{\mathcal{D}} \otimes A \xrightarrow{\epsilon \otimes \mathrm{id}_A} A \otimes A \xrightarrow{m} A$$

is a monomorphism in $\mathcal{D}$ (that is, ϵ is left cancellative). When $\mathcal{D}$ is the category of sets (equipped with the cartesian monoidal structure of Example 2.1.3.2), this reduces to the statement of Proposition 2.1.2.3.

Proof of Proposition 2.1.5.13. To simplify the notation, let us use the symbol $\mathbf{1}$ to denote the unit objects of both $\mathcal{C}$ and $\mathcal{D}$, $\upsilon : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ for the unit constraints of both $\mathcal{C}$ and $\mathcal{D}$, and λ for the unit constraints of both $\mathcal{C}$ and $\mathcal{D}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor equipped with a nonunital lax monoidal structure $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$. Suppose first that $\epsilon : \mathbf{1} \to F(\mathbf{1})$ is a left unit for F. Then the diagram

$$\begin{tikzcd}
\mathbf{1} \otimes \mathbf{1} \arrow[r, "\mathrm{id}_{\mathbf{1}} \otimes \epsilon"] \arrow[dd, "\lambda_{\mathbf{1}}"'] & \mathbf{1} \otimes F(\mathbf{1}) \arrow[r, "\epsilon \otimes \mathrm{id}_{F(\mathbf{1})}"] \arrow[ddr, "\lambda_{F(\mathbf{1})}"'] & F(\mathbf{1}) \otimes F(\mathbf{1}) \arrow[d, "\mu_{\mathbf{1},\mathbf{1}}"] \\
 & & F(\mathbf{1} \otimes \mathbf{1}) \arrow[d, "F(\lambda_{\mathbf{1}})"] \\
\mathbf{1} \arrow[rr, "\epsilon"] & & F(\mathbf{1})
\end{tikzcd}$$

commutes: the region on the left commutes by the naturality of the left unit constraints for $\mathcal{D}$ (Remark 2.1.2.18), and the region on the right commutes by virtue of our assumption that ϵ is a left unit. The commutativity of the outer square shows that ϵ satisfies condition (1) of Proposition 2.1.5.13 (by virtue of the fact that the unit constraints of $\mathcal{C}$ and $\mathcal{D}$ are given by $\upsilon = \lambda_{\mathbf{1}}$; see Corollary 2.1.2.21). For every object $X \in \mathcal{C}$, the composition

$$\mathbf{1} \otimes F(X) \xrightarrow{\epsilon \otimes \mathrm{id}_{F(X)}} F(\mathbf{1}) \otimes F(X) \xrightarrow{\mu_{\mathbf{1},X}} F(\mathbf{1} \otimes X) \xrightarrow{F(\lambda_X)} F(X)$$

is the left unit constraint $\lambda_{F(X)}$, which is an isomorphism. Since $F(\lambda_X)$ is also an isomorphism, it follows that the composition $\mu_{\mathbf{1},X} \circ (\epsilon \otimes \mathrm{id}_{F(X)})$ is an isomorphism.

Now suppose that ϵ satisfies conditions (1) and (2); we wish to show that it is a left unit for F. Fix an object $X \in \mathcal{C}$, and let $f : \mathbf{1} \otimes F(X) \to F(X)$ denote the composition

$$\mathbf{1} \otimes F(X) \xrightarrow{\epsilon \otimes \mathrm{id}_{F(X)}} F(\mathbf{1}) \otimes F(X) \xrightarrow{\mu_{\mathbf{1},X}} F(\mathbf{1} \otimes X) \xrightarrow{F(\lambda_X)} F(X).$$

We wish to show that f is equal to the left unit constraint $\lambda_{F(X)}$ for the monoidal category $\mathcal{D}$. Unwinding the definitions, this is equivalent to the assertion that $\mathrm{id}_{\mathbf{1}} \otimes f$ is equal to the composition

$$\mathbf{1} \otimes (\mathbf{1} \otimes F(X)) \xrightarrow{\alpha_{\mathbf{1},\mathbf{1},X}} (\mathbf{1} \otimes \mathbf{1}) \otimes F(X) \xrightarrow{\upsilon \otimes \mathrm{id}_{F(X)}} \mathbf{1} \otimes F(X).$$

By virtue of assumption (2), it will suffice to prove that these morphisms agree after postcomposition with the monomorphism

$$\mathbf{1} \otimes F(X) \xrightarrow{\epsilon \otimes \mathrm{id}_{F(X)}} F(\mathbf{1}) \otimes F(X) \xrightarrow{\mu_{\mathbf{1},X}} F(\mathbf{1} \otimes X).$$

This is equivalent to the commutativity of the outer rectangle in the diagram

$$\begin{tikzcd}
\mathbf{1} \otimes (\mathbf{1} \otimes F(X)) \arrow[r, "\epsilon"] \arrow[dd, "\alpha"] \arrow[dr, "\epsilon \otimes \epsilon"] & \mathbf{1} \otimes (F(\mathbf{1}) \otimes F(X)) \arrow[r, "\mu"] \arrow[d, "\epsilon"] & \mathbf{1} \otimes F(\mathbf{1} \otimes X) \arrow[r, "F(\lambda_X)"] \arrow[d, "\epsilon"] & \mathbf{1} \otimes F(X) \arrow[d, "\epsilon"] \\
 & F(\mathbf{1}) \otimes (F(\mathbf{1}) \otimes F(X)) \arrow[r, "\mu"] \arrow[d, "\alpha"] & F(\mathbf{1}) \otimes F(\mathbf{1} \otimes X) \arrow[r, "F(\lambda_X)"] \arrow[d, "\mu"] & F(\mathbf{1}) \otimes F(X) \arrow[d, "\mu"] \\
(\mathbf{1} \otimes \mathbf{1}) \otimes F(X) \arrow[r, "\epsilon \otimes \epsilon"] \arrow[dd, "\upsilon"] & (F(\mathbf{1}) \otimes F(\mathbf{1})) \otimes F(X) \arrow[d, "\mu"] & F(\mathbf{1} \otimes (\mathbf{1} \otimes X)) \arrow[r, "F(\mathrm{id} \otimes \lambda_X)"] \arrow[d, "F(\alpha)"] & F(\mathbf{1} \otimes X) \arrow[dd, "\mathrm{id}"] \\
 & F(\mathbf{1} \otimes \mathbf{1}) \otimes F(X) \arrow[r, "\mu"] \arrow[d, "F(\upsilon)"] & F((\mathbf{1} \otimes \mathbf{1}) \otimes X) \arrow[dr, "F(\upsilon \otimes \mathrm{id})"] & \\
\mathbf{1} \otimes F(X) \arrow[r, "\epsilon"] & F(\mathbf{1}) \otimes F(X) \arrow[rr, "\mu"] & & F(\mathbf{1} \otimes X).
\end{tikzcd}$$

In fact, the whole diagram commutes: the rectangle on the lower left commutes by virtue of our assumption that ϵ satisfies (1), the rectangle in the middle commutes by virtue of the compatibility of the μ with the associativity constraints of $\mathcal{C}$ and $\mathcal{D}$, the square on the lower right commutes by the construction of the left unit constraint λ_X, and the remaining regions commute by naturality. □

00DQ **Example 2.1.5.15.** Let k be a field, let Vect_k denote the category of vector spaces over k, and let $F : \mathrm{Vect}_k \to \mathrm{Set}$ be the forgetful functor, endowed with the nonunital lax monoidal structure described in Example 2.1.4.5. Then F is a lax monoidal functor: the function

$$\epsilon : \{*\} \to F(k) \qquad \epsilon(*) = 1 \in k$$

is a left and right unit for F.

Example 2.1.5.15 illustrates a special case of a general phenomenon:

00DR **Example 2.1.5.16.** Let $\mathcal{C}$ be a monoidal category, and let $F : \mathcal{C} \to \mathrm{Set}$ denote the functor corepresented by the unit object $\mathbf{1} \in \mathcal{C}$, given concretely by the formula $F(X) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, X)$. For every pair of objects $X, Y \in \mathcal{C}$, we have a canonical map

$$\mu_{X,Y} : F(X) \times F(Y) \to F(X \otimes Y),$$

which carries a pair of elements $x \in F(X)$, $y \in F(Y)$ to the composite map

$$\mathbf{1} \xrightarrow{v^{-1}} \mathbf{1} \otimes \mathbf{1} \xrightarrow{x \otimes y} X \otimes Y.$$

The collection of maps $\{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ determines a lax monoidal structure on the functor F, with unit given by the map

$$\epsilon : \{*\} \to F(\mathbf{1}) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1}) \qquad \epsilon(*) = \mathrm{id}_{\mathbf{1}}.$$

Example 2.1.5.17. Let $\mathcal{C}$ and $\mathcal{D}$ be categories which admit finite products, and regard $\mathcal{C}$ and $\mathcal{D}$ as endowed with the cartesian monoidal structures described in Example 2.1.3.2. Let $F : \mathcal{C} \to \mathcal{D}$ be any functor, and let $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ be the induced functor of opposite categories. Then the functor F^{op} admits a lax monoidal structure, which associates to each pair of objects $X, Y \in \mathcal{C}$ the canonical map $\mu_{X,Y} : F(X \times Y) \to F(X) \times F(Y)$ in the category $\mathcal{D}$ (which we can view as a morphism from $F^{\mathrm{op}}(X) \otimes F^{\mathrm{op}}(Y) \to F^{\mathrm{op}}(X \otimes Y)$ in the category $\mathcal{D}^{\mathrm{op}}$). The unit for F is given by the unique morphism $\epsilon : F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ in the category $\mathcal{D}$ (where $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$ are final objects of $\mathcal{C}$ and $\mathcal{D}$, respectively). 00DS

Definition 2.1.5.18. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $F, F' : \mathcal{C} \to \mathcal{D}$ be lax monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. We will say that a natural transformation $\gamma : F \to F'$ is *monoidal* if it satisfies the following pair of conditions: 00DT

- The natural transformation γ is nonunital monoidal, in the sense of Definition 2.1.4.10. That is, for every pair of objects $X, Y \in \mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "\mu_{X,Y}"] \arrow[d, "\gamma(X) \otimes \gamma(Y)"] & F(X \otimes Y) \arrow[d, "\gamma(X \otimes Y)"] \\ F'(X) \otimes F'(Y) \arrow[r, "\mu'_{X,Y}"] & F'(X \otimes Y) \end{tikzcd}$$

  commutes, where μ and μ' are the tensor constraints of F and F', respectively.

- The unit of F' is equal to the composition $\mathbf{1}_{\mathcal{D}} \xrightarrow{\epsilon} F(\mathbf{1}_{\mathcal{C}}) \xrightarrow{\gamma(\mathbf{1}_{\mathcal{C}})} F'(\mathbf{1}_{\mathcal{C}})$, where ϵ is the unit of F.

We let $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ denote the category whose objects are lax monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are monoidal natural transformations, which we regard as a (non-full) subcategory of the category $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D})$ introduced in Definition 2.1.4.10.

Remark 2.1.5.19 (Compatibility with Reversal)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital lax monoidal functor, and let $F^{\mathrm{rev}} : \mathcal{C}^{\mathrm{rev}} \to \mathcal{D}^{\mathrm{rev}}$ be as in Example 00DU

2.1.4.13. Then F is a lax monoidal functor if and only if F^{rev} is a lax monoidal functor. This observation (and its counterpart for monoidal natural transformations) supplies an isomorphism of categories $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})$.

`00DV` **Remark 2.1.5.20** (Closure under Composition)**.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors equipped with nonunital lax monoidal structures μ and ν, respectively, so that the composite functor $G \circ F$ inherits a nonunital lax monoidal structure (Construction 2.1.4.17). If F and G admit units

$$\delta : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}}) \qquad \epsilon : \mathbf{1}_{\mathcal{E}} \to G(\mathbf{1}_{\mathcal{D}}),$$

then the composite map

$$\mathbf{1}_{\mathcal{E}} \xrightarrow{\epsilon} G(\mathbf{1}_{\mathcal{D}}) \xrightarrow{G(\delta)} (G \circ F)(\mathbf{1}_{\mathcal{C}})$$

is a unit for the composite functor $G \circ F$. This observation (and its counterpart for monoidal natural transformations) imply that the composition law of Construction 2.1.4.17 restricts to a functor

$$\circ : \mathrm{Fun}^{\mathrm{lax}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{E}).$$

`00DW` **Example 2.1.5.21** (Algebra Objects)**.** Let $\mathcal{C}$ be a monoidal category. An *algebra object of* $\mathcal{C}$ is a pair (A, m), where A is an object of $\mathcal{C}$ and $m : A \otimes A \to A$ is a nonunital algebra structure on A (Example 2.1.4.11) for which there exists a unit $\epsilon : \mathbf{1} \to A$ (in the sense of Definition 2.1.5.1). If (A, m) and (A', m') are algebra objects of $\mathcal{C}$ with units $\epsilon : \mathbf{1} \to A$ and $\epsilon' : \mathbf{1} \to A'$, then we say that a morphism $f : A \to A'$ is an *algebra homomorphism* if it is a nonunital algebra homomorphism (Example 2.1.4.11) which satisfies $\epsilon' = f \circ \epsilon$. We let $\mathrm{Alg}(\mathcal{C})$ denote the category whose objects are algebra objects of $\mathcal{C}$ and whose morphisms are algebra homomorphisms. We regard $\mathrm{Alg}(\mathcal{C})$ as a (non-full) subcategory of the category $\mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ of nonunital algebra objects of $\mathcal{C}$ defined in Example 2.1.4.11.

Let $\{e\}$ denote the trivial monoid, regarded as a (strict) monoidal category having only identity morphisms (Example 2.1.1.3). Then algebra objects of $\mathcal{C}$ can be identified with lax monoidal functors $\{e\} \to \mathcal{C}$. More precisely, the isomorphism $\mathrm{Fun}^{\mathrm{lax}}_{\mathrm{nu}}(\{e\}, \mathcal{C}) \simeq \mathrm{Alg}^{\mathrm{nu}}(\mathcal{C})$ of Example 2.1.4.11 specializes to an isomorphism of (non-full) subcategories $\mathrm{Fun}^{\mathrm{lax}}(\{e\}, \mathcal{C}) \simeq \mathrm{Alg}(\mathcal{C})$.

`00DF` **Example 2.1.5.22.** Let Set denote the category of sets, equipped with the cartesian monoidal structure of Example 2.1.3.2. Then we can identify algebra objects of Set with monoids. More precisely, there is a canonical isomorphism of categories $\mathrm{Alg}(\mathrm{Set}) \simeq \mathrm{Mon}$, where Mon denotes the category of monoids (Definition 1.3.2.3).

For later use, we record the following elementary fact about algebra objects of a monoidal category $\mathcal{C}$:

Proposition 2.1.5.23. *Let $\mathcal{C}$ be a monoidal category and let (A, m) be an algebra object of $\mathcal{C}$. The following conditions are equivalent:* `02AQ`

(1) *The unit map $\epsilon : \mathbf{1} \to A$ is an isomorphism in $\mathcal{C}$.*

(2) *The object A is invertible: that is, there exists an object $B \in \mathcal{C}$ for which the tensor products $A \otimes B$ and $B \otimes A$ are isomorphic to $\mathbf{1}$.*

(3) *The construction $X \mapsto A \otimes X$ determines a fully faithful functor from $\mathcal{C}$ to itself.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are immediate. We will prove that (3) implies (1). It follows from assumption that (3) that there is a unique morphism $f : A \to \mathbf{1}$ for which the lower right triangle in the diagram

$$\begin{tikzcd}
A \otimes \mathbf{1} \arrow[r, "\mathrm{id}_A \otimes \epsilon"] \arrow[d, "\rho_A"'] & A \otimes A \arrow[dl, "m"'] \arrow[d, "\mathrm{id}_A \otimes f"] \\
A \arrow[r, "\rho_A^{-1}"] & A \otimes \mathbf{1}
\end{tikzcd}$$

commutes. The upper left triangle also commutes, since ϵ is a right unit with respect to the multiplication m. It follows that the square commutes: that is, the composition

$$A \otimes \mathbf{1} \xrightarrow{\mathrm{id}_A \otimes \epsilon} A \otimes A \xrightarrow{\mathrm{id}_A \otimes f} A \otimes \mathbf{1}$$

is equal to the identity. Invoking assumption (3), we conclude that f is a left inverse to ϵ: that is, the composition $f \circ \epsilon$ is equal to the identity on the unit object $\mathbf{1}$.

We now show that f is also a right inverse to ϵ: that is, the composition $\epsilon \circ f$ is equal to the identity morphism id_A. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\lambda_A^{-1}"] \arrow[dd, "f"] & \mathbf{1} \otimes A \arrow[r, "\epsilon \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_{\mathbf{1}} \otimes f"] & A \otimes A \arrow[d, "\mathrm{id}_A \otimes f"] \\
 & \mathbf{1} \otimes \mathbf{1} \arrow[r, "\epsilon \otimes \mathrm{id}_{\mathbf{1}}"] \arrow[dl, "\upsilon"'] & A \otimes \mathbf{1} \arrow[d, "\rho_A"] \\
\mathbf{1} \arrow[rr, "\epsilon"] & & A.
\end{tikzcd}$$

The defining property of f guarantees that the vertical composition on the right coincides with the multiplication map $m : A \otimes A \to A$. The assumption that ϵ is a left unit with respect to the multiplication m shows that clockwise composition around the diagram gives the identity map $\mathrm{id}_A : A \to A$. To complete the proof, it will suffice to show that the diagram commutes. The commutativity of the upper right square follows from the functoriality of the tensor product, the commutativity of the trapezoidal region on the left follows from the functoriality of the left unit constraints of $\mathcal{C}$, and the commutativity of the trapezoidal region on the bottom from the functoriality of the right unit constraints of $\mathcal{C}$ (here we invoke the fact that the map $\upsilon : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ coincides with both $\lambda_{\mathbf{1}}$ and $\rho_{\mathbf{1}}$; see Corollary 2.1.2.21). □

2.1.6 Monoidal Functors

00DX We now introduce the unital analogue of Definition 2.1.4.4.

00DY **Definition 2.1.6.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A *monoidal structure* on F is a nonunital lax monoidal structure $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$ (Definition 2.1.4.3) which satisfies the following additional conditions:

- For every pair of objects $X, Y \in \mathcal{C}$, the tensor constraint $\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ is an isomorphism in $\mathcal{D}$ (that is, μ is a nonunital monoidal structure on F).
- There exists an isomorphism $\epsilon : \mathbf{1}_{\mathcal{D}} \xrightarrow{\sim} F(\mathbf{1}_{\mathcal{C}})$ which is a unit for F (in the sense of Definition 2.1.5.2).

A *monoidal functor from $\mathcal{C}$ to $\mathcal{D}$* is a pair (F, μ), where F is a functor from $\mathcal{C}$ to $\mathcal{D}$ and μ is a monoidal structure on F.

00DZ **Remark 2.1.6.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. We will generally abuse terminology by identifying a monoidal functor (F, μ) from $\mathcal{C}$ to $\mathcal{D}$ with the underlying functor $F : \mathcal{C} \to \mathcal{D}$. If we refer to F as a monoidal functor, we implicitly assume that it has been equipped with a monoidal structure $\mu = \{\mu_{X,Y}\}_{X,Y \in \mathcal{C}}$.

00E0 **Warning 2.1.6.3.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital lax monoidal functor. If F is a monoidal functor from $\mathcal{C}$ to $\mathcal{D}$, then it is both a nonunital monoidal functor (that is, the tensor constraints $\mu_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ are isomorphisms) and a lax monoidal functor (that is, it admits a unit $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$). However, the converse is false: to qualify as a monoidal functor, F must satisfy the additional condition that ϵ is an isomorphism.

00E1 **Remark 2.1.6.4.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital monoidal functor. Let $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ be an isomorphism in the category $\mathcal{C}$. Then ϵ automatically satisfies condition (2) of Proposition 2.1.5.13: for each $X \in \mathcal{C}$, both of the maps

$$\mathbf{1}_{\mathcal{D}} \otimes F(X) \xrightarrow{\epsilon \otimes \mathrm{id}_{F(X)}} F(\mathbf{1}_{\mathcal{C}}) \otimes F(X) \xrightarrow{\mu_{\mathbf{1}_{\mathcal{C}},X}} F(\mathbf{1}_{\mathcal{C}} \otimes X)$$

are isomorphisms. It follows that ϵ is a unit for F if and only if it satisfies condition (1) of Proposition 2.1.5.13: that is, if and only if the diagram

$$\begin{tikzcd}
\mathbf{1}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} \arrow[r, "\epsilon \otimes \epsilon"] \arrow[dd, "\upsilon_{\mathcal{D}}"] & F(\mathbf{1}_{\mathcal{C}}) \otimes F(\mathbf{1}_{\mathcal{C}}) \arrow[d, "\mu_{\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}}"] \\
 & F(\mathbf{1}_{\mathcal{C}} \otimes \mathbf{1}_{\mathcal{C}}) \arrow[d, "F(\upsilon_{\mathcal{C}})"] \\
\mathbf{1}_{\mathcal{D}} \arrow[r, "\epsilon"] & F(\mathbf{1}_{\mathcal{C}})
\end{tikzcd}$$

is commutative. By virtue of Proposition 2.1.2.9, there exists an isomorphism ϵ satisfying this condition if and only if the pair $(F(\mathbf{1}_{\mathcal{C}}), F(\upsilon_{\mathcal{C}}) \circ \mu_{\mathbf{1}_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}}})$ is a unit of $\mathcal{C}$ (in the sense of Definition 2.1.2.5).

In other words, a nonunital monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is monoidal if and only if the functors

$$\mathcal{D} \to \mathcal{D} \qquad X \mapsto F(\mathbf{1}_{\mathcal{C}}) \otimes X$$

$$\mathcal{D} \to \mathcal{D} \qquad X \mapsto X \otimes F(\mathbf{1}_{\mathcal{C}})$$

are fully faithful (in which case they are both canonically isomorphic to the identity functor $\mathrm{id}_{\mathcal{D}} : \mathcal{D} \simeq \mathcal{D}$).

Example 2.1.6.5 (Strict Monoidal Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be strict monoidal categories (Definition 2.1.2.1). We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *strict monoidal* if it is a nonunital strict monoidal functor (Definition 2.1.4.1) which carries the strict unit object $\mathbf{1}_{\mathcal{C}}$ to the strict unit object $\mathbf{1}_{\mathcal{D}}$. 00E2

Every strict monoidal functor $F : \mathcal{C} \to \mathcal{D}$ can be regarded as a monoidal functor from $\mathcal{C}$ to $\mathcal{D}$, by taking each tensor constraint $\mu_{X,Y}$ to be the identity morphisms from $F(X) \otimes F(Y) = F(X \otimes Y)$ to itself. Conversely, if (F, μ) is a monoidal functor for which the tensor constraints $\mu_{X,Y}$ and the unit morphism $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ are identity morphisms in $\mathcal{D}$, then F is a strict monoidal functor from $\mathcal{C}$ to $\mathcal{D}$.

Example 2.1.6.6. Let M and M' be monoids, regarded as monoidal categories having only identity morphisms (Example 2.1.2.8). Then lax monoidal functors from M to M' (in the sense of Definition 2.1.5.8) can be identified with monoid homomorphisms from M to M' (in the sense of Definition 1.3.2.3). Moreover, every lax monoidal functor from M to M' is automatically strict monoidal (and therefore monoidal). 00E3

00E4 **Example 2.1.6.7.** Let $\mathcal{C}$ be a monoidal category, and let $\ell : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ be the nonunital monoidal functor of Example 2.1.4.8 (carrying each object $X \in \mathcal{C}$ to the functor $\ell_X : \mathcal{C} \to \mathcal{C}$ given by $\ell_X(Y) = X \otimes Y$). Then ℓ is a monoidal functor: it admits a unit $\epsilon : \mathrm{id}_{\mathcal{C}} \to \ell_{\mathbf{1}}$ given by the inverse of the left unit constraint of Construction 2.1.2.17. To prove this, it suffices to verify that ϵ satisfies property (1) of Proposition 2.1.5.13 (Remark 2.1.6.4). Unwinding the definitions, this is equivalent to the assertion that for every object $X \in \mathcal{C}$, the outer cycle of the diagram

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}_X"] & \mathbf{1} \otimes X \arrow[l, "\lambda_X"'] \arrow[ddr, "\mathrm{id}_{\mathbf{1} \otimes X}"'] & \mathbf{1} \otimes (\mathbf{1} \otimes X) \arrow[l, "\mathrm{id}_{\mathbf{1}} \otimes \lambda_X"'] \arrow[d, "\alpha_{\mathbf{1},\mathbf{1},X}"] \\
 & & (\mathbf{1} \otimes \mathbf{1}) \otimes X \arrow[d, "\upsilon \otimes \mathrm{id}_X"] \\
X & & \mathbf{1} \otimes X \arrow[ll, "\lambda_X"']
\end{tikzcd}$$

is commutative. In fact, the whole diagram commutes: for the inner cycle on the left this is immediate, and for the inner cycle on the right it follows from the definition of the left unit constraing λ_X (Construction 2.2.1.11).

00E5 **Example 2.1.6.8** (2-Cochains as Monoidal Structures)**.** Let G be a group and let Γ be an abelian group equipped with an action of G. Let $\mathcal{C}$ be the category introduced in Example 2.1.3.3, whose objects are the elements of G and morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}}(g, h) = \begin{cases} \Gamma & \text{if } g = h \\ \emptyset & \text{otherwise.} \end{cases}$$

Then every 3-cocycle $\alpha : G \times G \times G \to \Gamma$ can be regarded as the associativity constraint for a monoidal structure $(\otimes, \alpha)$ on $\mathcal{C}$. Let us write $\mathcal{C}(\alpha)$ to indicate the category $\mathcal{C}$, endowed with the monoidal structure $(\otimes, \alpha)$.

Suppose that we are given a pair of cocycles $\alpha, \alpha' : G \times G \times G \to \Gamma$. Unwinding the definitions, we see that monoidal structures on the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C}(\alpha) \to \mathcal{C}(\alpha')$ are given by functions

$$\mu : G \times G \to \Gamma \qquad (x, y) \mapsto \mu_{x,y}$$

which satisfy the identity

$$\alpha_{x,y,z} + \mu_{x,yz} + x(\mu_{y,z}) = \mu_{xy,z} + \mu_{x,y} + \alpha'_{x,y,z}$$

for $x, y, z \in G$. We can rewrite this identity more compactly as an equation $\alpha + d\mu = \alpha'$, where

$$d : \{\text{2-Cochains } G \times G \to \Gamma\} \to \{\text{3-Cochains } G \times G \times G \to \Gamma\}$$

is defined by the formula $(d\mu)_{x,y,z} = x(\mu_{y,z}) - \mu_{xy,z} + \mu_{x,yz} - \mu_{x,y}$.

In particular, the identity functor $\mathrm{id}_{\mathcal{C}}$ can be promoted to a monoidal functor from $\mathcal{C}(\alpha)$ to $\mathcal{C}(\alpha')$ if and only if the cocycles α and α' are *cohomologous*: that is, they represent the same element of the cohomology group $\mathrm{H}^3(G;\Gamma)$.

Notation 2.1.6.9. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, and let $F, F' : \mathcal{C} \to \mathcal{D}$ be monoidal functors. We say that a natural transformation $\gamma : F \to F'$ is *monoidal* if it is monoidal when viewed as a natural transformation of lax monoidal functors (Definition 2.1.5.18). We let $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ denote the category whose objects are monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are monoidal natural transformations. We regard $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ as a full subcategory of the category $\mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, \mathcal{D})$ of Definition 2.1.5.18 (or as a non-full subcategory of the category $\mathrm{Fun}^{\otimes}_{\mathrm{nu}}(\mathcal{C}, \mathcal{D})$ of nonunital monoidal functors from $\mathcal{C}$ to $\mathcal{D}$). 00E6

Warning 2.1.6.10. We will not be consistent in our usage of Notation 2.1.6.9. For example, if $\mathcal{C}$ and $\mathcal{D}$ are *symmetric monoidal categories* ([?]), then we will sometimes write $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ to denote the category of *symmetric* monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ (which is a full subcategory of the category of monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ defined in Notation 2.1.6.9). 00E7

Remark 2.1.6.11 (Compatibility with Reversal)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital lax monoidal functor, and let $F^{\mathrm{rev}} : \mathcal{C}^{\mathrm{rev}} \to \mathcal{D}^{\mathrm{rev}}$ be as in Example 2.1.4.13. Then F is a monoidal functor if and only if F^{rev} is a monoidal functor. This observation (and its counterpart for monoidal natural transformations) supplies an isomorphism of categories $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\otimes}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})$. 00E8

Remark 2.1.6.12 (Opposite Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, let $F : \mathcal{C} \to \mathcal{D}$ be a nonunital monoidal functor, and let $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ be the induced nonunital monoidal functor on opposite categories (Example 2.1.4.14). Then F is a monoidal functor if and only if F^{op} is a monoidal functor. This observation (and its counterpart for monoidal natural transformations) supplies an isomorphism of categories $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{\otimes}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$. 00E9

Remark 2.1.6.13 (Composition of Monoidal Functors)**.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors equipped with nonunital lax monoidal structures μ and ν, respectively, so that the composite functor $G \circ F$ inherits a nonunital lax monoidal structure (Construction 2.1.4.17). If μ and ν are monoidal structures on F and G, then $G \circ F$ inherits a monoidal structure. This observation (and its counterpart for monoidal natural transformations) imply that the composition law of Construction 2.1.4.17 restricts to a functor 00EA

$$\circ : \mathrm{Fun}^{\otimes}(\mathcal{D}, \mathcal{E}) \times \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{E}).$$

Example 2.1.6.14. Let $\mathcal{C}$ and $\mathcal{D}$ be categories which admit finite products, endowed with the cartesian monoidal structure described in Example 2.1.3.2. For any functor $F : \mathcal{C} \to \mathcal{D}$, 00EB we can regard the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ as endowed with the lax monoidal structure described in Example 2.1.5.17. This lax monoidal structure is a monoidal structure if and only if the functor F preserves finite products. If this condition is satisfied, then the original functor F inherits a monoidal structure (Remark 2.1.6.12).

`00EC` **Example 2.1.6.15** (1-Cochains as Natural Transformations)**.** Let G be a group, let Γ be an abelian group equipped with an action of G, and choose a pair of 3-cocycles

$$\alpha, \alpha' : G \times G \times G \to \Gamma,$$

which we can regard as associativity constraints for monoidal categories $\mathcal{C}(\alpha)$ and $\mathcal{C}(\alpha')$ having the same underlying category $\mathcal{C}$ (Example 2.1.6.8). Suppose we are given a pair of monoidal structures μ and μ' on the identity functor $\mathrm{id}_{\mathcal{C}}$, which we can identify with 2-cochains $\mu, \mu' : G \times G \to \Gamma$ satisfying

$$\alpha + d\mu = \alpha' \qquad \alpha + d\mu' = \alpha'.$$

Then the difference $\nu = \mu - \mu'$ is a 2-*cocycle*: that is, it satisfies the identity

$$x\nu_{y,z} - \nu_{xy,z} + \nu_{x,yz} - \nu_{x,y} = 0$$

for every triple of elements $x, y, z \in G$.

Note that a natural transformation from the identity functor $\mathrm{id}_{\mathcal{C}}$ to itself can be identified with a function

$$\gamma : G \to \Gamma \qquad x \mapsto \gamma_x;$$

that is, with a 1-*cochain on* G taking values in the group Γ. Unwinding the definitions, we see that the natural transformation γ is monoidal (with respect to the monoidal structures supplied by μ and μ', respectively) if and only if it satisfies the identity

$$\mu'_{x,y} + x\gamma_y + \gamma_x = \mu_{x,y} + \gamma_{xy}$$

for every pair of elements $x, y \in G$. We can rewrite this identity more conceptually as $\mu' + d\gamma = \mu$, where

$$d : \{\text{1-Cochains } G \to \Gamma\} \to \{\text{2-Cochains } G \times G \to \Gamma\}$$

is defined by the formula $(d\gamma)_{x,y} = x(\gamma_y) - \gamma_{xy} + \gamma_x$. In particular, the monoidal functors $(\mathrm{id}_{\mathcal{C}}, \mu)$ to $(\mathrm{id}_{\mathcal{C}}, \mu')$ are isomorphic if and only if the 2-cocycle $\nu = \mu - \mu'$ is a coboundary: that is, it has vanishing image in the cohomology group $\mathrm{H}^2(G; \Gamma)$.

2.1.7 Enriched Category Theory

Let $\mathcal{C}$ be a category. For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ denote the set of morphisms from X to Y in $\mathcal{C}$. In many cases of interest, the sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ can be endowed with additional structure, which are respected by the composition law on $\mathcal{C}$. To give a systematic discussion of this phenomenon, it is convenient to use the formalism of *enriched* category theory. 00ED

Definition 2.1.7.1. Let $\mathcal{A}$ be a monoidal category with unit object $\mathbf{1}$. An $\mathcal{A}$-*enriched category* $\mathcal{C}$ consists of the following data: 00EE

(1) A collection $\mathrm{Ob}(\mathcal{C})$, whose elements we refer to as *objects of* $\mathcal{C}$. We will often abuse notation by writing $X \in \mathcal{C}$ to indicate that X is an element of $\mathrm{Ob}(\mathcal{C})$.

(2) For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, an object $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ of the monoidal category $\mathcal{A}$.

(3) For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a morphism

$$c_{Z,Y,X} : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$$

in the category $\mathcal{A}$, which we will refer to as the *composition law*.

(4) For every object $X \in \mathrm{Ob}(\mathcal{C})$, a morphism $e_X : \mathbf{1} \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ in the category $\mathcal{A}$, which we refer to as the *identity of* X.

These data are required to satisfy the following conditions:

(A) For every quadruple of objects $W, X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
 & \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(W,Y) \arrow[dd, "c_{Z,Y,W}"] \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \otimes (\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(W,X)) \arrow[ur, "\mathrm{id} \otimes c_{Y,X,W}"] \arrow[dd, "\alpha"] & \\
 & \underline{\mathrm{Hom}}_{\mathcal{C}}(W,Z) \\
(\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(W,X) \arrow[dr, "c_{Z,Y,X} \otimes \mathrm{id}"'] & \\
 & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(W,X) \arrow[uu, "c_{Z,X,W}"]
\end{tikzcd}$$

commutes. Here α denotes the associativity constraint on the monoidal category $\mathcal{A}$.

(U) For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the diagrams

$$\begin{tikzcd}
\mathbf{1} \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[rr, "e_Y \otimes \mathrm{id}"] \arrow[dr, "\lambda"'] & & \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[dl, "c_{Y,Y,X}"] \\
 & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) &
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \otimes \mathbf{1} \arrow[rr, "\mathrm{id} \otimes e_X"] \arrow[dr, "\rho"'] & & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X) \arrow[dl, "c_{Y,X,X}"] \\
 & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) &
\end{tikzcd}$$

commute, where λ and ρ denote the left and right unit constraints on $\mathcal{A}$ (see Construction 2.1.2.17).

`00EF` **Example 2.1.7.2** (Categories Enriched Over Sets)**.** Let $\mathcal{A} = \mathrm{Set}$ be the category of sets, endowed with the monoidal structure given by the cartesian product (see Example 2.1.3.2). Then an $\mathcal{A}$-enriched category (in the sense of Definition 2.1.7.1) can be identified with a category in the usual sense.

Example 2.1.7.3. Let $\mathcal{A}$ be a monoidal category. If $\mathcal{C}$ is a category enriched over $\mathcal{A}$ and X is an object of $\mathcal{C}$, then the composition law 00EG

$$c_{X,X,X} : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X)$$

exhibits $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,X)$ as an algebra object of $\mathcal{A}$, in the sense of Example 2.1.5.21. Moreover, this construction induces a bijection

$$\{\mathcal{A}\text{-Enriched Categories } \mathcal{C} \text{ with } \mathrm{Ob}(\mathcal{C}) = \{X\}\} \simeq \{\text{Algebra objects of } \mathcal{A}\}.$$

Consequently, the theory of enriched categories can be regarded as a generalization of the theory of associative algebras (See Example 2.1.7.14 for a more precise statement).

Remark 2.1.7.4 (Functoriality)**.** Let $\mathcal{A}$ and $\mathcal{A}'$ be monoidal categories, and let $F : \mathcal{A} \to \mathcal{A}'$ be a lax monoidal functor (with tensor constraints $\mu_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$ and unit $\epsilon : \mathbf{1}_{\mathcal{A}'} \to F(\mathbf{1}_{\mathcal{A}})$). Then every $\mathcal{A}$-enriched category $\mathcal{C}$ determines an $\mathcal{A}'$-enriched category $\mathcal{C}'$, which can be described concretely as follows: 00EH

- The objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$: that is, we have $\mathrm{Ob}(\mathcal{C}') = \mathrm{Ob}(\mathcal{C})$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}')$, we set $\underline{\mathrm{Hom}}_{\mathcal{C}'}(X,Y) = F(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y))$.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C}')$, the composition law $c'_{Z,Y,X}$ for $\mathcal{C}'$ is given by the composition

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathcal{C}'}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}'}(X,Y) &= F(\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z)) \otimes F(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)) \\ &\xrightarrow{\mu} F(\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)) \\ &\xrightarrow{F(c_{Z,Y,X})} F(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z)) \\ &= \underline{\mathrm{Hom}}_{\mathcal{C}'}(X,Z). \end{aligned}$$

- For every object $X \in \mathrm{Ob}(\mathcal{C}')$, the identity morphism e'_X for X in $\mathcal{C}'$ is given by the composition

$$\mathbf{1}_{\mathcal{A}'} \xrightarrow{\epsilon} F(\mathbf{1}_{\mathcal{A}}) \xrightarrow{F(e_X)} F(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,X)) = \underline{\mathrm{Hom}}_{\mathcal{C}'}(X,X).$$

Example 2.1.7.5 (The Underlying Category of an Enriched Category)**.** Let $\mathcal{A}$ be a monoidal category and let $F : \mathcal{A} \to \mathrm{Set}$ be the functor given by $F(A) = \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, A)$, endowed with the lax monoidal structure of Example 2.1.5.16. If $\mathcal{C}$ is a category enriched over $\mathcal{A}$, then we can apply the construction of Remark 2.1.7.4 to obtain a Set-enriched category, which we can identify with an ordinary category (Example 2.1.7.2). We will refer to this category as the *underlying category* of the $\mathcal{A}$-enriched category $\mathcal{C}$, and we will generally abuse notation by denoting it also by $\mathcal{C}$. Concretely, this underlying category has the same objects as the enriched category $\mathcal{C}$, with morphism sets given by the formula $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y))$. 00EJ

00MX **Remark 2.1.7.6.** Let $\mathcal{A}$ be a monoidal category and let $\mathcal{C}$ be an ordinary category. We define an *$\mathcal{A}$-enrichment* of $\mathcal{C}$ to be an $\mathcal{A}$-enriched category $\widetilde{\mathcal{C}}$ together with an identification of $\mathcal{C}$ with the underlying category of $\widetilde{\mathcal{C}}$, in the sense of Example 2.1.7.5.

00EK **Example 2.1.7.7** (Enrichment in Vector Spaces)**.** Let k be a field and let Vect_k denote the category of vector spaces over k, endowed with the monoidal structure given by tensor product over k (Example 2.1.3.1). Then choosing an Vect_k-enrichment of $\mathcal{C}$ is equivalent to endowing each of the sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ with the structure of a k-vector space, for which the composition maps

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

are k-bilinear.

00EL **Example 2.1.7.8** (Topologically Enriched Categories)**.** Let Top denote the category of topological spaces, endowed with the monoidal structure given by the cartesian product (Example 2.1.3.2). We will refer to a Top-enriched category as a *topologically enriched category*. Note that the functor F of Example 2.1.7.5 is (canonically isomorphic to) the forgetful functor Top → Set. Consequently, if $\mathcal{C}$ is a topologically enriched category, then the underlying ordinary category $\mathcal{C}_0$ can be described concretely as follows:

- The objects of the ordinary category $\mathcal{C}_0$ are the objects of the Top-enriched category $\mathcal{C}$.
- Given a pair of objects $X, Y \in \mathcal{C}_0$, a morphism f from X to Y (in the ordinary category $\mathcal{C}_0$) is a point of the topological space $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$.
- Given a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}_0$, the composition $g \circ f$ is given by the image of (g, f) under the continuous map

$$c_{Z,Y,X} : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z).$$

It follows that, for any ordinary category $\mathcal{C}_0$, promoting $\mathcal{C}_0$ to a topologically enriched category $\mathcal{C}$ is equivalent to endowing each of the morphism sets $\mathrm{Hom}_{\mathcal{C}_0}(X, Y)$ with a topology for which the composition maps $\circ : \mathrm{Hom}_{\mathcal{C}_0}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}_0}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_0}(X, Z)$ are continuous.

00EM **Exercise 2.1.7.9** (Uniqueness of Identities)**.** Let $\mathcal{A}$ be a monoidal category. A *nonunital $\mathcal{A}$-enriched category* $\mathcal{C}$ consists of a collection $\mathrm{Ob}(\mathcal{C})$ of *objects of* $\mathcal{C}$, together with objects $\{\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$ of the category $\mathcal{A}$ and composition laws

$$c_{Z,Y,X} : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$$

which satisfy the associative law (A) appearing in Definition 2.1.7.1. Show that, if a nonunital $\mathcal{A}$-enriched category $\mathcal{C}$ can be promoted to an $\mathcal{A}$-enriched category $\overline{\mathcal{C}}$, then $\overline{\mathcal{C}}$ is unique: that is, the identity maps $e_X : \mathbf{1} \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ are determined by axiom (U) of Definition 2.1.7.1.

Definition 2.1.7.10. Let $\mathcal{A}$ be a monoidal category, and let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{A}$-enriched categories. An *$\mathcal{A}$-enriched functor* $F : \mathcal{C} \to \mathcal{D}$ consists of the following data: 00JE

(1) For every object $X \in \mathrm{Ob}(\mathcal{C})$, and object $F(X) \in \mathrm{Ob}(\mathcal{D})$.

(2) For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a morphism

$$F_{X,Y} : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Y))$$

in the category $\mathcal{A}$.

These data are required to satisfy the following conditions:

- For every object $X \in \mathrm{Ob}(\mathcal{C})$, the morphism $e_{F(X)} : \mathbf{1} \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(X))$ factors as a composition

$$\mathbf{1} \xrightarrow{e_X} \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X) \xrightarrow{F_{X,X}} \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(X)).$$

- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d, "F_{Y,Z} \otimes F_{X,Y}"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(F(Y),F(Z)) \otimes \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Y)) \arrow[r] & \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Z))
\end{tikzcd}$$

commutes (in the category $\mathcal{A}$); here the horizontal maps are given by the composition laws on $\mathcal{C}$ and $\mathcal{D}$.

Notation 2.1.7.11 (The Category of Enriched Categories)**.** Let $\mathcal{A}$ be a monoidal category. We say that an $\mathcal{A}$-enriched category $\mathcal{C}$ is *small* if the collection of objects $\mathrm{Ob}(\mathcal{C})$ is small. The collection of small $\mathcal{A}$-enriched categories can itself be organized into a category $\mathrm{Cat}(\mathcal{A})$, whose morphisms are given by $\mathcal{A}$-enriched functors (in the sense of Definition 2.1.7.10). 00JF

Example 2.1.7.12. Let $\mathcal{C}$ and $\mathcal{D}$ be small categories, which we regard as Set-enriched categories by means of Example 2.1.7.2. Then Set-enriched functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.1.7.10) can be identified with functors from $\mathcal{C}$ to $\mathcal{D}$ in the usual sense. This identification determines an isomorphism of categories $\mathrm{Cat} \simeq \mathrm{Cat}(\mathrm{Set})$. 00JG

Remark 2.1.7.13. Let $F : \mathcal{A} \to \mathcal{A}'$ be a lax monoidal functor between monoidal categories. Then the construction of Remark 2.1.7.4 determines a functor $\mathrm{Cat}(\mathcal{A}) \to \mathrm{Cat}(\mathcal{A}')$. In the 00JH special case where $\mathcal{A}' = \mathrm{Set}$ and F is the functor $A \mapsto \underline{\mathrm{Hom}}_{\mathcal{A}}(\mathbf{1}, \mathcal{A})$ corepresented by the unit object $\mathbf{1} \in \mathcal{A}$, we obtain a forgetful functor

$$\mathrm{Cat}(\mathcal{A}) \to \mathrm{Cat}(\mathrm{Set}) \simeq \mathrm{Cat},$$

which assigns to each (small) $\mathcal{A}$-enriched category $\mathcal{C}$ its underlying ordinary category (Example 2.1.7.5).

00JJ **Example 2.1.7.14.** Let $\mathcal{A}$ be a monoidal category, let A be an algebra object of $\mathcal{A}$, which we can identify with an $\mathcal{A}$-enriched category $\mathcal{C}_A$ having a single object X (Example 2.1.7.3). For any $\mathcal{A}$-enriched category $\mathcal{D}$ containing an object Y, we have a canonical bijection

$$\begin{tikzcd}
\{\mathcal{A}\text{-Enriched Functors } F : \mathcal{C}_A \to \mathcal{D} \text{ with } F(X) = Y\} \arrow[d, "\sim"] \\
\{\text{Algebra homomorphisms } A \to \underline{\mathrm{Hom}}_{\mathcal{D}}(Y, Y)\}.
\end{tikzcd}$$

In particular, if $\mathcal{D} = \mathcal{C}_B$ for some other algebra object $B \in \mathrm{Alg}(\mathcal{A})$, we obtain a bijection

$$\mathrm{Hom}_{\mathrm{Cat}(\mathcal{A})}(\mathcal{C}_A, \mathcal{C}_B) \simeq \mathrm{Hom}_{\mathrm{Alg}(\mathcal{A})}(A, B).$$

In other words, the construction $A \mapsto \mathcal{C}_A$ induces a fully faithful embedding $\mathrm{Alg}(\mathcal{A}) \to \mathrm{Cat}(\mathcal{A})$, whose essential image is spanned by those $\mathcal{A}$-enriched categories having a single object.

2.2 The Theory of 2-Categories

007K The collection of (small) categories can itself be organized into a (large) category Cat, whose objects are small categories and whose morphisms are functors. However, the structure of Cat as an abstract category fails to capture many of the essential features of category theory:

(*i*) Given a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$ with the same source and target, we are usually not interested in the question of whether or not F and G are *equal*. Instead, we should regard F and G as interchangeable if there exists a natural isomorphism $\alpha : F \simeq G$. This sort of information is not encoded in the structure of the category Cat.

(*ii*) Given a pair of categories $\mathcal{C}$ and $\mathcal{D}$, we are usually not interested in the question of whether or not $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*. Instead, we should regard $\mathcal{C}$ and $\mathcal{D}$ as interchangeable if there exists an *equivalence of categories* from $F : \mathcal{C} \to \mathcal{D}$. In this case, the functor F need not be invertible when regarded as a morphism in Cat.

To remedy the situation, it is useful to contemplate a more elaborate mathematical structure.

Definition 2.2.0.1. A *strict* 2*-category* $\mathcal{C}$ consists of the following data: 007L

- A collection $\mathrm{Ob}(\mathcal{C})$, whose elements we refer to as *objects of* $\mathcal{C}$. We will often abuse notation by writing $X \in \mathcal{C}$ to indicate that X is an element of $\mathrm{Ob}(\mathcal{C})$.

- For every pair of objects $X, Y \in \mathcal{C}$, a category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$. We refer to objects f of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ as 1*-morphisms from* X *to* Y and write $f : X \to Y$ to indicate that f is a 1-morphism from X to Y. Given a pair of 1-morphisms $f, g \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$, we refer to morphisms from f to g in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ as 2*-morphisms from* f *to* g.

- For every triple of objects $X, Y, Z \in \mathcal{C}$, a *composition functor*
$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z).$$

- For every object $X \in \mathcal{C}$, an *identity* 1*-morphism* $\mathrm{id}_X \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$.

These data are required to satisfy the following conditions:

(1) For each object $X \in \mathcal{C}$, the identity 1-morphism id_X is a unit for both right and left composition. That is, for every object $Y \in \mathcal{C}$, the functors
$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \qquad f \mapsto f \circ \mathrm{id}_X$$
$$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, X) \qquad g \mapsto \mathrm{id}_X \circ g$$
are both equal to the identity.

(2) The composition law of $\mathcal{C}$ is strictly associative. That is, for every quadruple of objects $W, X, Y, Z \in \mathcal{C}$, the diagram of categories
$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(W, X) \arrow[r, "\mathrm{id} \times \circ"] \arrow[d, "\circ \times \mathrm{id}"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(W, Y) \arrow[d, "\circ"] \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(W, X) \arrow[r, "\circ"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(W, Z)
\end{tikzcd}$$
commutes (in the ordinary category Cat).

`00EN` **Remark 2.2.0.2** (Strict 2-Categories as Enriched Categories)**.** Let Cat denote the category whose objects are (small) categories and whose morphisms are functors. Then Cat admits finite products, and therefore admits a monoidal structure given by the formation of cartesian products (Example 2.1.3.2). Neglecting set-theoretic technicalities, a strict 2-category (in the sense of Definition 2.2.0.1) can be identified with a Cat-enriched category (in the sense of Definition 2.1.7.1).

`00EP` **Remark 2.2.0.3.** To every strict 2-category $\mathcal{C}$, we can associate an ordinary category $\mathcal{C}_0$, whose objects and morphisms are given by

$$\mathrm{Ob}(\mathcal{C}_0) = \mathrm{Ob}(\mathcal{C}) \qquad \mathrm{Hom}_{\mathcal{C}_0}(X, Y) = \mathrm{Ob}(\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)).$$

We will refer to $\mathcal{C}_0$ as the *underlying ordinary category* of $\mathcal{C}$ (note that $\mathcal{C}_0$ can be obtained from $\mathcal{C}$ by the general procedure of Example 2.1.7.5). More informally, the underlying category $\mathcal{C}_0$ is obtained from $\mathcal{C}$ by "forgetting" its 2-morphisms.

`007M` **Example 2.2.0.4.** We define a strict 2-category **Cat** as follows:

- The objects of **Cat** are (small) categories.
- For every pair of small categories $\mathcal{C}, \mathcal{D} \in \mathbf{Cat}$, we take $\underline{\mathrm{Hom}}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ to be the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$.
- The composition law on **Cat** is given by the usual composition of functors.

We will refer to **Cat** as the *strict* 2*-category of (small) categories*. Note that the underlying ordinary category of **Cat** is the category Cat (whose objects are small categories and morphisms are functors).

We can obtain many more examples by studying categories equipped with additional structure.

`00EQ` **Example 2.2.0.5.** We define a strict 2-category **MonCat** as follows:

- The objects of **MonCat** are (small) monoidal categories.
- For every pair of small monoidal categories $\mathcal{C}$ and $\mathcal{D}$, we take $\underline{\mathrm{Hom}}_{\mathbf{MonCat}}(\mathcal{C}, \mathcal{D})$ to be the category $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ of monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ (Notation 2.1.6.9).
- The composition law on **MonCat** is given by the composition of monoidal functors described in Remark 2.1.6.13.

There are several obvious variants on this construction: for example, we can work with nonunital monoidal categories in place of monoidal categories, or lax monoidal functors in place of monoidal functors.

Example 2.2.0.6 (Ordinary Categories)**.** Every ordinary category can be regarded as a strict 2-category. More precisely, to each category $\mathcal{C}$ we can associate a strict 2-category $\mathcal{C}'$ as follows: 007W

- The objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, objects of the category $\underline{\mathrm{Hom}}_{\mathcal{C}'}(X, Y)$ are elements of the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, and every morphism in $\underline{\mathrm{Hom}}_{\mathcal{C}'}(X, Y)$ is an identity morphism.
- For every triple of objects $X, Y, Z \in \mathcal{C}$, the composition functor
$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}'}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}'}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}'}(X, Z)$$
is given on objects by the composition map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$.
- For every object $X \in \mathcal{C}$, the identity object $\mathrm{id}_X \in \underline{\mathrm{Hom}}_{\mathcal{C}'}(X, X)$ coincides with the identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$.

In this situation, we will generally abuse terminology by identifying the strict 2-category $\mathcal{C}'$ with the ordinary category $\mathcal{C}$ (see Example 2.2.5.7).

Remark 2.2.0.7 (Endomorphism Categories)**.** Let $\mathcal{C}$ be a strict 2-category and let X be an object of $\mathcal{C}$. We will write $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ for the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$. Then the composition law 00ER
$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$$
determines a strict monoidal structure on the category $\underline{\mathrm{End}}_{\mathcal{C}}(X)$.

Note that, if $\mathcal{C}$ is an ordinary category (regarded as a strict 2-category by means of Example 2.2.0.6), then the endomorphism category $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ can be identified with the endomorphism monoid $\mathrm{End}_{\mathcal{C}}(X)$ of Example 1.3.2.2, regarded as a (strict) monoidal category via Example 2.1.2.8.

Example 2.2.0.8 (Delooping)**.** Let $\mathcal{M}$ be a category equipped with a strict monoidal structure $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (Definition 2.1.2.1). We define a strict 2-category $B\,\mathcal{M}$ as follows: 00ES

- The set of objects $\mathrm{Ob}(B\,\mathcal{M})$ is the singleton set $\{X\}$.
- The category $\underline{\mathrm{Hom}}_{B\,\mathcal{M}}(X, X)$ is equal to $\mathcal{M}$.
- The composition functor $\circ : \underline{\mathrm{Hom}}_{B\,\mathcal{M}}(X, X) \times \underline{\mathrm{Hom}}_{B\,\mathcal{M}}(X, X) \to \underline{\mathrm{Hom}}_{B\,\mathcal{M}}(X, X)$ is equal to the tensor product $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$.
- The identity morphism id_X is the strict unit object of $\mathcal{M}$.

We will refer to $B\,\mathcal{M}$ as the *delooping of* $\mathcal{M}$.

Note that the constructions

$$\mathcal{M} \mapsto B\,\mathcal{M} \qquad \mathcal{C} \mapsto \underline{\mathrm{End}}_{\mathcal{C}}(X)$$

induce mutually inverse bijections

$$\{\text{Strict Monoidal Categories } \mathcal{M}\} \simeq \{\text{Strict 2-Categories } \mathcal{C} \text{ with } \mathrm{Ob}(\mathcal{C}) = \{X\}\},$$

generalizing the identification of Remark 1.3.2.4.

The reader might at this point object that the definition of strict 2-category violates a fundamental principle of category theory: axioms (1) and (2) of Definition 2.2.0.1 require that certain functors are *equal*. In practice, one often encounters mathematical structures $\mathcal{C}$ which do not quite fit in the framework of Definition 2.2.0.1, because the associative law for composition of 1-morphisms in $\mathcal{C}$ holds only up to isomorphism. To address this point, Bénabou introduced a more general type of structure which he called a *bicategory*, which we will refer to here as a *2-category*.

Our goal in this section is to give a brief introduction to the theory of 2-categories. We begin in §2.2.1 by reviewing the definition of a 2-category (Definition 2.2.1.1) and establishing some notational and terminological conventions. Every strict 2-category can be regarded as a 2-category (Example 2.2.1.4), but many of the 2-categories which arise "in nature" fail to be strict: we discuss several examples of this phenomenon in §2.2.2.

To articulate the relationship between 2-categories and strict 2-categories more precisely, it is convenient to view each as the objects of a suitable (ordinary) category. In §2.2.4, we introduce the notion of a *functor* between 2-categories (Definition 2.2.4.5). Roughly speaking, a functor $F : \mathcal{C} \to \mathcal{D}$ is an operation which carries objects, 1-morphisms, and 2-morphisms of $\mathcal{C}$ to objects, 1-morphisms, and 2-morphisms of $\mathcal{D}$, which is compatible with the composition laws on $\mathcal{C}$ and $\mathcal{D}$. Here again there are several possible definitions, depending on whether one demands that the compatibility holds strictly (in which case we say that F is a *strict functor*), up to isomorphism (in which case we say that F is a *functor*), or up to possible non-invertible 2-morphism (in which case we say that F is a *lax functor*). We use this notion in §2.2.5 to introduce an (ordinary) category 2Cat, whose objects are 2-categories and whose morphisms are functors between 2-categories (and consider several other variations on this theme).

The notion of 2-category is more general than the notion of strict 2-category defined above: in general, a 2-category $\mathcal{C}$ need not be strict or even isomorphic (as an object of 2Cat) to a strict 2-category $\mathcal{C}'$. However, we will prove in §2.2.7 that every 2-category $\mathcal{C}$ is isomorphic to a *strictly unitary* 2-category $\mathcal{C}'$: that is, a 2-category $\mathcal{C}'$ in which the composition law is strictly unital, but not necessarily strictly associative (Proposition 2.2.7.7). The proof will make use of a certain twisting procedure in the setting of 2-categories (Construction 2.2.6.8), which we will describe in 2.2.6.

Remark 2.2.0.9. Let $\mathcal{C}$ be a 2-category. It is generally not possible to find a strict 2-category $\mathcal{C}'$ which is *isomorphic* to $\mathcal{C}$ (as an object of the category 2Cat we will introduce in §2.2.5). However, it is always possibly to find a strict 2-category $\mathcal{C}'$ which is *equivalent* to $\mathcal{C}$; we will return to this point in §[?]. 007N

2.2.1 2-Categories

Let $\mathcal{C}$ be a strict 2-category (Definition 2.2.0.1). Then the composition of 1-morphisms in $\mathcal{C}$ is strictly associative: that is, given a triple of composable 1-morphisms 007P

$$f : W \to X \qquad g : X \to Y \qquad h : Y \to Z$$

of $\mathcal{C}$, we have an equality $h \circ (g \circ f) = (h \circ g) \circ f$. Our goal in this section is to introduce the more general notion of (non-strict) 2-*category*, where we weaken the associativity requirement: rather than demand that the 1-morphisms $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are identical, we instead ask for a specified isomorphism $\alpha_{h,g,f} : h \circ (g \circ f) \xRightarrow{\sim} (h \circ g) \circ f$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(W, Z)$. In order to obtain a sensible theory, we must require that these isomorphisms satisfy an analogue of the pentagon identity which appears in Definition 2.1.1.5.

Definition 2.2.1.1 (Bénabou)**.** A 2-*category* $\mathcal{C}$ consists of the following data: 007Q

- A collection $\mathrm{Ob}(\mathcal{C})$, whose elements we refer to as *objects of* $\mathcal{C}$. We will often abuse notation by writing $X \in \mathcal{C}$ to indicate that X is an element of $\mathrm{Ob}(\mathcal{C})$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$. We refer to objects f of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ as 1-*morphisms from* X *to* Y and write $f : X \to Y$ to indicate that f is a 1-morphism from X to Y. Given a pair of 1-morphisms $f, g \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$, we refer to morphisms from f to g in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ as 2-*morphisms from* f *to* g. We will sometimes write $\gamma : f \Rightarrow g$ or $f \xRightarrow{\gamma} g$ to indicate that γ is a 2-morphism from f to g.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a *composition functor*

$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z).$$

- For every object $X \in \mathrm{Ob}(\mathcal{C})$, a 1-morphism $\mathrm{id}_X \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$, which we call the *identity* 1-*morphism* from X to itself.
- For every object $X \in \mathrm{Ob}(\mathcal{C})$, an isomorphism $\upsilon_X : \mathrm{id}_X \circ \mathrm{id}_X \xRightarrow{\sim} \mathrm{id}_X$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$. We refer to the 2-morphisms $\{\upsilon_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ as the *unit constraints* of $\mathcal{C}$.

- For every quadruple of objects $W, X, Y, Z \in \mathcal{C}$, a natural isomorphism α from the functor

  $$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(W,X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(W,Z) \qquad (h,g,f) \mapsto h \circ (g \circ f)$$

  to the functor

  $$\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(W,X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(W,Z) \qquad (h,g,f) \mapsto (h \circ g) \circ f.$$

  We denote the value of α on a triple (h,g,f) by $\alpha_{h,g,f} : h \circ (g \circ f) \overset{\sim}{\Longrightarrow} (h \circ g) \circ f$. We refer to these isomorphisms as the *associativity constraints* of $\mathcal{C}$.

These data are required to satisfy the following pair of conditions:

(C) For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the functors

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \qquad f \mapsto f \circ \mathrm{id}_X$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \qquad f \mapsto \mathrm{id}_Y \circ f$$

are fully faithful.

(P) For every quadruple of composable 1-morphisms

$$V \xrightarrow{e} W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

in $\mathcal{C}$, the diagram of isomorphisms

$$\begin{tikzcd}
 & h \circ ((g \circ f) \circ e) \arrow[rr, Rightarrow, "\alpha_{h,g \circ f,e}", "\sim"'] & & (h \circ (g \circ f)) \circ e \arrow[dr, Rightarrow, "\alpha_{h,g,f} \circ \mathrm{id}_e", "\sim"'] & \\
h \circ (g \circ (f \circ e)) \arrow[ur, Rightarrow, "\mathrm{id}_h \circ \alpha_{g,f,e}", "\sim"'] \arrow[drr, Rightarrow, "\sim", "\alpha_{h,g,f \circ e}"'] & & & & ((h \circ g) \circ f) \circ e \\
 & & (h \circ g) \circ (f \circ e) \arrow[urr, Rightarrow, "\sim", "\alpha_{h \circ g,f,e}"'] & &
\end{tikzcd}$$

commutes in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(V,Z)$.

007R **Remark 2.2.1.2.** An equivalent formulation of Definition 2.2.1.1 was given by Bénabou in [4]. Beware that Bénabou uses the term *bicategory* for what we call a *2-category*.

007S **Remark 2.2.1.3.** In the situation of Definition 2.2.1.1, we will refer to axiom (P) as the *pentagon identity*.

Example 2.2.1.4 (Strict 2-Categories)**.** Let $\mathcal{C}$ be any strict 2-category (in the sense of Definition 2.2.0.1). Then $\mathcal{C}$ can be viewed as a 2-category (in the sense of Definition 2.2.1.1) by taking the unit and associativity constraints υ_X and $\alpha_{h,g,f}$ to be identity 2-morphisms in $\mathcal{C}$. 007T

Warning 2.2.1.5. Let $\mathcal{C}$ be a 2-category. If $\mathcal{C}$ is strict, then we can extract from $\mathcal{C}$ an underlying ordinary category having the same objects and 1-morphisms (Remark 2.2.0.3). However, this operation has no counterpart for a general 2-category $\mathcal{C}$: in general, composition of 1-morphisms in $\mathcal{C}$ is associative only up to isomorphism. 00EU

Remark 2.2.1.6. Let $\mathcal{C}$ be a 2-category. Then $\mathcal{C}$ can be obtained from an ordinary category (via the construction of Example 2.2.0.6) if and only if every 2-morphism in $\mathcal{C}$ is an identity 2-morphism (note that a 2-category with this property is automatically strict, by virtue of Example 2.2.1.4). 007X

Remark 2.2.1.7 (Endomorphism Categories)**.** Let $\mathcal{C}$ be a 2-category and let X be an object of $\mathcal{C}$. We will denote the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,X)$ by $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ and refer to it as the *endomorphism category* of X. The category $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ has a monoidal structure, with tensor product given by the composition law 00ET

$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,X),$$

unit object given by the identity 1-morphism id_X, and the unit and associativity constraints of $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ given by υ_X and the associativity constraints of $\mathcal{C}$, respectively.

Notation 2.2.1.8. Let $\mathcal{C}$ be a 2-category. We will generally follow the convention of denoting objects of $\mathcal{C}$ by capital Roman letters, 1-morphisms of $\mathcal{C}$ by lowercase Roman letters, and 2-morphisms of $\mathcal{C}$ by lowercase Greek letters. However, we will often violate this convention when discussing specific examples. For instance, when studying the (strict) 2-category **Cat** of small categories (Example 2.2.0.4), we denote objects using calligraphic letters (such as $\mathcal{C}$ and $\mathcal{D}$) and 1-morphisms using uppercase Roman letters (such as F and G). 007Y

Warning 2.2.1.9. Let $\mathcal{C}$ be a 2-category. Then there are two different notions of composition for the 2-morphisms of $\mathcal{C}$: 007Z

(V) Let X and Y be objects of $\mathcal{C}$. Suppose we are given 1-morphisms $f, g, h : X \to Y$ and a pair of 2-morphisms

$$\gamma : f \Rightarrow g \qquad \delta : g \Rightarrow h.$$

We can then apply the composition law in the ordinary category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$ to obtain a 2-morphism $f \Rightarrow h$, which we refer to as the *vertical composition of* γ *and* δ.

(H) Let X, Y, and Z be objects of $\mathcal{C}$. Suppose we are given 2-morphisms $\gamma : f \Rightarrow g$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$ and $\gamma' : f' \Rightarrow g'$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z)$. Then the image of (γ', γ) under the composition law

$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z),$$

is a 2-morphism from $f' \circ f$ to $g' \circ g$, which will refer to as the *horizontal composition* of γ and γ'.

The terminology is motivated by the following graphical representations of the data described in (V) and (H):

$$\begin{tikzcd}
X \arrow[r, bend left=70, "f"{name=F}] \arrow[r, "g"{name=G}] \arrow[r, bend right=70, "h"'{name=H}] & Y
\arrow[from=F, to=G, phantom, "\Downarrow \gamma"]
\arrow[from=G, to=H, phantom, "\Downarrow \delta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \gamma"] & Y \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow \gamma'"] & Z.
\end{tikzcd}$$

To avoid confusion, we will generally denote the vertical composition of 2-morphisms γ and δ by $\delta\gamma$ and the horizontal composition of 2-morphisms γ and γ' by $\gamma' \circ \gamma$.

`00EV` **Remark 2.2.1.10.** Let $\mathcal{C}$ be a 2-category. For each object $X \in \mathrm{Ob}(\mathcal{C})$, the identity 1-morphism id_X and the unit constraint υ_X are determined (up to unique isomorphism) by the composition law and associativity constraints. More precisely, given any other choice of identity morphism id'_X and unit constraint $\upsilon'_X : \mathrm{id}'_X \circ \mathrm{id}'_X \xRightarrow{\sim} \mathrm{id}'_X$, there exists a unique invertible 2-morphism $\gamma : \mathrm{id}_X \xRightarrow{\sim} \mathrm{id}'_X$ for which the diagram

$$\begin{tikzcd}
\mathrm{id}_X \circ \mathrm{id}_X \arrow[r, Rightarrow, "\upsilon_X"] \arrow[d, Rightarrow, "\gamma \circ \gamma"] & \mathrm{id}_X \arrow[d, Rightarrow, "\gamma"] \\
\mathrm{id}'_X \circ \mathrm{id}'_X \arrow[r, Rightarrow, "\upsilon'_X"] & \mathrm{id}'_X
\end{tikzcd}$$

commutes. This follows from Proposition 2.1.2.9, applied to the monoidal category $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ of Remark 2.2.1.7.

It is possible to adopt a variant of Definition 2.2.1.1 where we do not require the identity morphisms $\{\mathrm{id}_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ (or unit constraints $\{\upsilon_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$) to be explicitly specified. This variant is equivalent to Definition 2.2.1.1 for many purposes. However, it is not suitable for our applications: in §2.3, we associate to each 2-category $\mathcal{C}$ a simplicial set $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ called the *Duskin nerve* of $\mathcal{C}$, whose degeneracy operators depend on the choice of identity morphisms and unit constraints in $\mathcal{C}$ (though the face operators do not: see Warning 2.3.1.11).

Axiom (C) of Definition 2.2.1.1 requires that, for every pair of objects X and Y of a 2-category $\mathcal{C}$, the functors

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \qquad f \mapsto f \circ \mathrm{id}_X, \mathrm{id}_Y \circ f$$

are fully faithful. In fact, we can say more: they are canonically isomorphic to the identity functor from $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$ to itself.

Construction 2.2.1.11 (Left and Right Unit Constraints)**.** Let $\mathcal{C}$ be a 2-category. For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, we have canonical isomorphisms 00EW

$$\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \xRightarrow{\alpha_{\mathrm{id}_Y, \mathrm{id}_Y, f}} (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \xRightarrow{\upsilon_Y \circ \mathrm{id}_f} \mathrm{id}_Y \circ f.$$

Since composition on the left with id_Y is fully faithful, it follows that there is a unique isomorphism $\lambda_f : \mathrm{id}_Y \circ f \xRightarrow{\sim} f$ for which the diagram

$$\begin{tikzcd}
\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha_{\mathrm{id}_Y, \mathrm{id}_Y, f}", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\mathrm{id}_{\mathrm{id}_Y} \circ \lambda_f"'] & & (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\sim"', "\upsilon_Y \circ \mathrm{id}_f"] \\
& \mathrm{id}_Y \circ f &
\end{tikzcd}$$

commutes. We will refer to λ_f as the *left unit constraint*. Similarly, there is a unique isomorphism $\rho_f : f \circ \mathrm{id}_X \xRightarrow{\sim} f$ for which the diagram

$$\begin{tikzcd}
f \circ (\mathrm{id}_X \circ \mathrm{id}_X) \arrow[rr, Rightarrow, "\alpha_{f, \mathrm{id}_X, \mathrm{id}_X}", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\mathrm{id}_f \circ \upsilon_X"'] & & (f \circ \mathrm{id}_X) \circ \mathrm{id}_X \arrow[dl, Rightarrow, "\sim"', "\rho_f \circ \mathrm{id}_{\mathrm{id}_X}"] \\
& f \circ \mathrm{id}_X &
\end{tikzcd}$$

commutes; we refer to ρ_f as the *right unit constraint*.

Remark 2.2.1.12. Let $\mathcal{C}$ be a 2-category and let X be an object of $\mathcal{C}$. For every 1-morphism $f : X \to X$ in $\mathcal{C}$, the left and right unit constraints 00EX

$$\lambda_f : \mathrm{id}_X \circ f \xRightarrow{\sim} f \qquad \rho_f : f \circ \mathrm{id}_X \xRightarrow{\sim} f$$

of Construction 2.2.1.11 coincide with the left and right unit constraints of Construction 2.1.2.17, applied to the monoidal category $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ of Remark 2.2.1.7.

00EY **Remark 2.2.1.13** (Naturality of Unit Constraints)**.** Let $\mathcal{C}$ be a 2-category, let X and Y be objects of $\mathcal{C}$, and let $\gamma : f \Rightarrow g$ be a morphism in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$. Then the diagram of 2-morphisms

$$\begin{tikzcd}
\mathrm{id}_Y \circ f \arrow[r, Rightarrow, "\lambda_f"] \arrow[d, Rightarrow, "\mathrm{id}_{\mathrm{id}_Y} \circ \gamma"] & f \arrow[d, Rightarrow, "\gamma"] \\
\mathrm{id}_Y \circ g \arrow[r, Rightarrow, "\lambda_g"] & g
\end{tikzcd}$$

commutes. In other words, the construction $f \mapsto \lambda_f$ determines a natural isomorphism from the functor

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \qquad f \mapsto \mathrm{id}_Y \circ f$$

to the identity functor. Similarly, the construction $f \mapsto \rho_f$ determines a natural isomorphism from the functor

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \qquad f \mapsto f \circ \mathrm{id}_X$$

to the identity functor.

We have the following generalization of Proposition 2.1.2.19:

00EZ **Proposition 2.2.1.14** (The Triangle Identity)**.** *Let $\mathcal{C}$ be a 2-category containing a pair of 1-morphisms $f : X \to Y$ and $g : Y \to Z$. Then the diagram of 2-morphisms*

$$\begin{tikzcd}
g \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha_{g,\mathrm{id}_Y,f}", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\mathrm{id}_g \circ \lambda_f"'] & & (g \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\sim"', "\rho_g \circ \mathrm{id}_f"] \\
& g \circ f &
\end{tikzcd}$$

is commutative.

Proof. We have a diagram of isomorphisms

$$\begin{tikzcd}
 & g \circ ((\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f) \arrow[rr, "\alpha"] \arrow[d, "\upsilon_Y"] & & (g \circ (\mathrm{id}_Y \circ \mathrm{id}_Y)) \circ f \arrow[d, "\upsilon_Y"'] \arrow[ddr, "\alpha"] & \\
 & g \circ (\mathrm{id}_Y \circ f) \arrow[rr, "\alpha"] \arrow[d, "\alpha"] & & (g \circ \mathrm{id}_Y) \circ f \arrow[dll, "\mathrm{id}"] & \\
g \circ (\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f)) \arrow[uur, "\alpha"] \arrow[ur, "\lambda_f"'] \arrow[drr, "\alpha"'] & (g \circ \mathrm{id}_Y) \circ f \arrow[r, "\rho_g"] & g \circ f & g \circ (\mathrm{id}_Y \circ f) \arrow[l, "\lambda_f"'] \arrow[u, "\alpha"] & ((g \circ \mathrm{id}_Y) \circ \mathrm{id}_Y) \circ f \arrow[ul, "\rho_g"'] \\
 & & (g \circ \mathrm{id}_Y) \circ (\mathrm{id}_Y \circ f). \arrow[ul, "\lambda_f"'] \arrow[ur, "\rho_g"] \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

Here the outer cycle commutes by the pentagon identity (P) of Definition 2.2.1.1, the upper rectangle by the functoriality of the associativity constraint, the upper side triangles by the definition of the left and right unit constraints, the quadrilaterals on the lower sides by the functoriality of the associativity constraints, and the lower region by the functoriality of composition. It follows that the middle square is also commutative, which is equivalent to the statement of Proposition 2.2.1.14. □

It follows from Proposition 2.2.1.14 that we can recover the unit constraints $\{\upsilon_X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ of a 2-category $\mathcal{C}$ from the left and right unit constraints defined in Construction 2.2.1.11.

Corollary 2.2.1.15. *Let $\mathcal{C}$ be a 2-category and let X be an object of $\mathcal{C}$. Then the left and right unit constraints* 0082

$$\lambda_{\mathrm{id}_X} : \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X \qquad \rho_{\mathrm{id}_X} : \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$$

are both equal to the unit constraint $\upsilon_X : \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$.

Proof. For any 1-morphism $f : Y \to X$ in $\mathcal{C}$, the left unit constraint λ_f is characterized by the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{id}_X \circ (\mathrm{id}_X \circ f) \arrow[rr, Rightarrow, "\alpha_{\mathrm{id}_X, \mathrm{id}_X, f}", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\mathrm{id}_{\mathrm{id}_X} \circ \lambda_f"'] & & (\mathrm{id}_X \circ \mathrm{id}_X) \circ f \arrow[dl, Rightarrow, "\sim"', "\upsilon_X \circ \mathrm{id}_f"] \\
 & \mathrm{id}_X \circ f. &
\end{tikzcd}$$

Using Proposition 2.2.1.14, we deduce that $\upsilon_X \circ \mathrm{id}_f = \rho_{\mathrm{id}_X} \circ \mathrm{id}_f$ as 2-morphisms from $(\mathrm{id}_X \circ \mathrm{id}_X) \circ f$ to $\mathrm{id}_X \circ f$. In other words, the 2-morphisms $\upsilon_X, \rho_{\mathrm{id}_X} : \mathrm{id}_X \circ \mathrm{id}_X \Rightarrow \mathrm{id}_X$ have the same image under the functor

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(Y,X) \qquad g \mapsto g \circ f.$$

In the special case where $Y = X$ and $f = \mathrm{id}_X$, this functor is fully faithful. It follows that $\upsilon_X = \rho_{\mathrm{id}_X}$. The equality $\upsilon_X = \lambda_{\mathrm{id}_X}$ follows by a similar argument. □

We will also need some variants of Proposition 2.2.1.14 (generalizing Exercise 2.1.2.20):

0080 **Proposition 2.2.1.16.** *Let $\mathcal{C}$ be a 2-category containing a pair of composable 1-morphisms $f : X \to Y$ and $g : Y \to Z$. Then:*

(1) *The associativity constraint $\alpha_{\mathrm{id}_Z,g,f} : \mathrm{id}_Z \circ (g \circ f) \Rightarrow (\mathrm{id}_Z \circ g) \circ f$ is given by the (vertical) composition*

$$\mathrm{id}_Z \circ (g \circ f) \xRightarrow{\lambda_{g\circ f}} g \circ f \xRightarrow{\lambda_g^{-1} \circ \mathrm{id}_f} (\mathrm{id}_Z \circ g) \circ f.$$

(2) *The associativity constraint $\alpha_{g,f,\mathrm{id}_X} : g \circ (f \circ \mathrm{id}_X) \Rightarrow (g \circ f) \circ \mathrm{id}_X$ is given by the (vertical) composition*

$$g \circ (f \circ \mathrm{id}_X) \xRightarrow{\mathrm{id}_g \circ \rho_f} g \circ f \xRightarrow{\rho_{g\circ f}^{-1}} (g \circ f) \circ \mathrm{id}_X$$

Proof of Proposition 2.2.1.16. We will prove (2); the proof of (1) is similar. Set $e = \mathrm{id}_X$, and consider the diagram of isomorphisms

$$\begin{tikzcd}
 & g \circ ((f \circ e) \circ e) \arrow[rr, Rightarrow, "\alpha_{g,f\circ e,e}"] \arrow[d, Rightarrow, "\rho_f"] & & (g \circ (f \circ e)) \circ e \arrow[dl, Rightarrow, "\rho_f"'] \arrow[dr, Rightarrow, "\alpha_{g,f,e}"] & \\
g \circ (f \circ (e \circ e)) \arrow[ur, Rightarrow, "\alpha_{f,e,e}"] \arrow[r, Rightarrow, "\lambda_e"] \arrow[drr, Rightarrow, "\alpha_{g,f,e\circ e}"'] & g \circ (f \circ e) \arrow[r, Rightarrow, "\alpha_{g,f,e}"] & (g \circ f) \circ e & & ((g \circ f) \circ e) \circ e \arrow[ll, Rightarrow, "\rho_{g\circ f}"'] \\
 & & (g \circ f) \circ (e \circ e). \arrow[u, Rightarrow, "\lambda_e"'] \arrow[urr, Rightarrow, "\alpha_{g\circ f,e,e}"'] & &
\end{tikzcd}$$

Here the outer cycle of the diagram commutes by the pentagon identity for $\mathcal{C}$, the triangles on the upper left and lower right commute by virtue of Proposition 2.2.1.14, and the upper and lower square diagrams commute by the functoriality of the associativity constraints. It follows that the triangle on the upper right commutes: that is, the identity $\alpha_{g,f,\mathrm{id}_X} = \rho_{g\circ f}^{-1}(\mathrm{id}_g \circ \rho_f)$ holds after applying the functor $(\bullet \circ \mathrm{id}_X) : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z)$. Since this functor is fully faithful (in fact, it is isomorphic to the identity functor by means of the right unit constraint ρ), we conclude that the identity $\alpha_{g,f,\mathrm{id}_X} = \rho_{g\circ f}^{-1}(\mathrm{id}_g \circ \rho_f)$ holds in $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z)$ itself. □

2.2.2 Examples of 2-Categories

We now collect some examples of 2-categories which arise naturally. 0083

Example 2.2.2.1 (Cospans)**.** Let $\mathcal{C}$ be a category containing a pair of objects X and Y. A *cospan from X to Y* is an object $B \in \mathcal{C}$ together with a pair of morphisms $X \xrightarrow{f} B \xleftarrow{g} Y$ in $\mathcal{C}$. The cospans from X to Y can be regarded as the objects of a category $\mathcal{B}_{X,Y}$, where a morphism from (B, f, g) to (B', f', g') in $\mathcal{B}_{X,Y}$ is a morphism $u : B \to B'$ in the category $\mathcal{C}$ which satisfies $f' = u \circ f$ and $g' = u \circ g$, so that the diagram 0084

$$\begin{tikzcd} & B \arrow[dd, "u"] & \\ X \arrow[ur, "f"] \arrow[dr, "f'"'] & & Y \arrow[ul, "g"'] \arrow[dl, "g'"] \\ & B' & \end{tikzcd}$$

is commutative.

Assume now that the category $\mathcal{C}$ admits pushouts. We can then construct a 2-category $\mathrm{Cospan}(\mathcal{C})$ as follows:

- The objects of $\mathrm{Cospan}(\mathcal{C})$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, we define $\underline{\mathrm{Hom}}_{\mathrm{Cospan}(\mathcal{C})}(X, Y)$ to be the category $\mathcal{B}_{X,Y}$; in particular, 1-morphisms from X to Y in the 2-category $\mathrm{Cospan}(\mathcal{C})$ can be identified with cospans from X to Y.
- For every triple of objects $X, Y, Z \in \mathcal{C}$, the composition law

  $$\circ : \underline{\mathrm{Hom}}_{\mathrm{Cospan}(\mathcal{C})}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathrm{Cospan}(\mathcal{C})}(X, Y) \to \underline{\mathrm{Hom}}_{\mathrm{Cospan}(\mathcal{C})}(X, Z)$$

  is given on objects by the construction $(C, B) \mapsto C \amalg_Y B$.
- For every object $X \in \mathcal{C}$, the identity 1-morphism from X to itself in $\mathcal{C}$ is given by the cospan $X \xrightarrow{\mathrm{id}_X} X \xleftarrow{\mathrm{id}_X} X$, and the unit constraint υ_X is given by the canonical isomorphism $X \amalg_X X \xrightarrow{\sim} X$.
- For every triple of composable 1-morphisms

  $$W \xrightarrow{A} X \xrightarrow{B} Y \xrightarrow{C} Z$$

in Cospan($\mathcal{C}$), the associativity constraint $\alpha_{C,B,A}$ is the canonical isomorphism of iterated pushouts

$$C \amalg_Y (B \amalg_X A) \xrightarrow{\sim} (C \amalg_Y B) \amalg_X A.$$

We will refer to Cospan($\mathcal{C}$) as the 2-*category of cospans in* $\mathcal{C}$.

03J5 **Variant 2.2.2.2** (Spans)**.** Let $\mathcal{C}$ be a category. If X and Y are objects of $\mathcal{C}$, we define a *span from* X *to* Y to be a diagram $X \leftarrow M \rightarrow Y$ in the category $\mathcal{C}$. If $\mathcal{C}$ admits fiber products, then we can dualize Example 2.2.2.1 to produce a 2-category Span($\mathcal{C}$) having the same objects, where 1-morphisms from X to Y in Span($\mathcal{C}$) are given by spans from X to Y in $\mathcal{C}$. More precisely, we define Span($\mathcal{C}$) to be the *conjugate* of the 2-category Cospan($\mathcal{C}^{\mathrm{op}}$).

00F0 **Remark 2.2.2.3.** Let $\mathcal{C}$ be a category which admits finite limits, and let $\mathbf{1}$ denote a final object of $\mathcal{C}$. Then the endomorphism category $\underline{\mathrm{End}}_{\mathrm{Span}(\mathcal{C})}(\mathbf{1})$ can be identified with the category $\mathcal{C}$ itself, equipped with the Cartesian monoidal structure of Example 2.1.3.2.

0085 **Example 2.2.2.4** (Bimodules)**.** We define a 2-category Bimod as follows:

- The objects of Bimod are associative rings.
- For every pair of associative rings A and B, we take $\underline{\mathrm{Hom}}_{\mathrm{Bimod}}(B, A)$ to be the category whose objects are A-B bimodules: that is, abelian groups $M = {}_A M_B$ equipped with commuting actions of A on the left and B on the right.
- For every triple of associative rings A, B, and C, we take the composition law

  $$\underline{\mathrm{Hom}}_{\mathrm{Bimod}}(B, A) \times \underline{\mathrm{Hom}}_{\mathrm{Bimod}}(C, B) \rightarrow \underline{\mathrm{Hom}}_{\mathrm{Bimod}}(C, A)$$

  to be the relative tensor product functor

  $$(M, N) \mapsto M \otimes_B N$$

- For every associative ring A, we take the identity object of $\underline{\mathrm{Hom}}_{\mathrm{Bimod}}(A, A)$ to be the ring A (regarded as a bimodule over itself) and the unit constraint $\upsilon_A : A \otimes_A A \xrightarrow{\sim} A$ is the map given by $\upsilon_A(x \otimes y) = xy$.
- For every quadruple of associative rings A, B, C, and D equipped with bimodules $M = {}_A M_B$, $N = {}_B N_C$, and $P = {}_C P_D$, we define the associativity constraint

  $$\alpha_{M,N,P} : M \otimes_B (N \otimes_C P) \xRightarrow{\sim} (M \otimes_B N) \otimes_C P$$

  to be the isomorphism characterized by the identity $\alpha_{M,N,P}(x \otimes (y \otimes z)) = (x \otimes y) \otimes z$.

00F1 **Example 2.2.2.5** (Delooping a Monoidal Category)**.** Let $\mathcal{C}$ be a monoidal category. We define a 2-category $B\,\mathcal{C}$ as follows:

- The 2-category $B\mathcal{C}$ has a single object, which we will denote by X.
- The category $\underline{\mathrm{Hom}}_{B\mathcal{C}}(X,X)$ is the category $\mathcal{C}$.
- The composition functor

$$\circ : \underline{\mathrm{Hom}}_{B\mathcal{C}}(X,X) \times \underline{\mathrm{Hom}}_{B\mathcal{C}}(X,X) \to \underline{\mathrm{Hom}}_{B\mathcal{C}}(X,X)$$

  is the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

- The identity morphism $\mathrm{id}_X \in \underline{\mathrm{Hom}}_{B\mathcal{C}}(X,X)$ is the unit object $\mathbf{1} \in \mathcal{C}$.
- The associativity and unit constraints of $B\mathcal{C}$ are the associativity and unit constraints for the monoidal structure on $\mathcal{C}$.

We will refer to the 2-category $B\mathcal{C}$ as the *delooping of* $\mathcal{C}$. Note that $B\mathcal{C}$ is strict as a 2-category if and only if the monoidal structure on $\mathcal{C}$ is strict (in which case we recover the delooping construction of Example 2.2.0.8). The construction $\mathcal{C} \mapsto B\mathcal{C}$ induces a bijection

$$\{\text{Monoidal Categories } \mathcal{C}\} \xrightarrow{\sim} \{\text{2-Categories } \mathcal{E} \text{ with } \mathrm{Ob}(\mathcal{E}) = \{X\}\}$$

which can be viewed as an equivalence of categories (see Remark 2.2.5.8).

Remark 2.2.2.6. Let M be a monoid, which we view as a (strict) monoidal category having only identity morphisms. Then the 2-category BM of Example 2.2.2.5 can be identified with the ordinary category BM appearing in Remark 1.3.2.4. 00F2

2.2.3 Opposite and Conjugate 2-Categories

Recall that every ordinary category $\mathcal{C}$ has an *opposite category* $\mathcal{C}^{\mathrm{op}}$, in which the objects are the same but the order of composition is reversed. In the setting of 2-categories, this operation generalizes in two essentially different ways: we can independently reverse the order of either vertical or horizontal composition. To avoid confusion, we will use different terminology when discussing these two operations. 008A

Construction 2.2.3.1 (The Opposite of a 2-Category)**.** Let $\mathcal{C}$ be a 2-category. We define a new 2-category $\mathcal{C}^{\mathrm{op}}$ as follows: 008B

- The objects of $\mathcal{C}^{\mathrm{op}}$ are the objects of $\mathcal{C}$. To avoid confusion, for each object $X \in \mathcal{C}$ we will write X^{op} for the corresponding object of $\mathcal{C}^{\mathrm{op}}$.
- For every pair of objects $X, Y \in \mathcal{C}$, we have $\underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, Y^{\mathrm{op}}) = \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, X)$. In particular, every 1-morphism $f : Y \to X$ in the 2-category $\mathcal{C}$ can be regarded as a 1-morphism from X^{op} to Y^{op} in the 2-category $\mathcal{C}^{\mathrm{op}}$, which we will denote by

$f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$. Similarly, if we are given a pair of 1-morphisms $f, g : Y \to X$ in the 2-category $\mathcal{C}$ having the same source and target, then every 2-morphism $\gamma : f \Rightarrow g$ in $\mathcal{C}$ determines a 2-morphism from f^{op} to g^{op} in $\mathcal{C}^{\mathrm{op}}$, which we will denote by $\gamma^{\mathrm{op}} : f^{\mathrm{op}} \Rightarrow g^{\mathrm{op}}$.

- For every triple of objects $X, Y, Z \in \mathcal{C}$, the composition functor

$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}}}(Y^{\mathrm{op}}, Z^{\mathrm{op}}) \times \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, Y^{\mathrm{op}}) \to \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, Z^{\mathrm{op}})$$

  for the 2-category $\mathcal{C}^{\mathrm{op}}$ is given by the composition functor

$$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, X) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(Z, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(Z, X).$$

  on the 2-category $\mathcal{C}$; in particular, it is given on objects by the formula $f^{\mathrm{op}} \circ g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$.

- For every object $X \in \mathcal{C}$, the identity 1-morphism $\mathrm{id}_{X^{\mathrm{op}}} \in \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{op}}, X^{\mathrm{op}})$ is given by $\mathrm{id}_X^{\mathrm{op}}$, where $\mathrm{id}_X \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ is the identity 1-morphism associated to X in the 2-category $\mathcal{C}$, and the unit constraint $\upsilon_{X^{\mathrm{op}}}$ is the isomorphism $\upsilon_X^{\mathrm{op}} : \mathrm{id}_{X^{\mathrm{op}}} \circ \mathrm{id}_{X^{\mathrm{op}}} \xRightarrow{\sim} \mathrm{id}_{X^{\mathrm{op}}}$.

- For every triple of composable 1-morphisms

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

  in the 2-category $\mathcal{C}$, the associativity constraint

$$\alpha_{f^{\mathrm{op}}, g^{\mathrm{op}}, h^{\mathrm{op}}} : f^{\mathrm{op}} \circ (g^{\mathrm{op}} \circ h^{\mathrm{op}}) \xRightarrow{\sim} (f^{\mathrm{op}} \circ g^{\mathrm{op}}) \circ h^{\mathrm{op}}$$

  in the 2-category $\mathcal{C}^{\mathrm{op}}$ is given by the inverse $(\alpha_{h,g,f}^{\mathrm{op}})^{-1}$ of the associativity constraint $\alpha_{h,g,f} : h \circ (g \circ f) \xRightarrow{\sim} (h \circ g) \circ f$ in the 2-category $\mathcal{C}$.

We will refer to $\mathcal{C}^{\mathrm{op}}$ as the *opposite* of the 2-category $\mathcal{C}$.

008C **Example 2.2.3.2.** Let $\mathcal{C}$ be a category which admits pushouts, and let $\mathrm{Cospan}(\mathcal{C})$ be the 2-category of cospans in $\mathcal{C}$ (see Example 2.2.2.1). Then the opposite 2-category $\mathrm{Cospan}(\mathcal{C})^{\mathrm{op}}$ can be identified with $\mathrm{Cospan}(\mathcal{C})$ itself (every cospan from X to Y in $\mathcal{C}$ can also be viewed as a cospan from Y to X).

00F3 **Example 2.2.3.3.** Let $\mathcal{C}$ be a monoidal category, and let $B\,\mathcal{C}$ be the 2-category obtained by delooping $\mathcal{C}$ (Example 2.2.2.5). Then the opposite 2-category $(B\,\mathcal{C})^{\mathrm{op}}$ can be identified with $B(\mathcal{C}^{\mathrm{rev}})$, where $\mathcal{C}^{\mathrm{rev}}$ denotes the reverse of the monoidal category $\mathcal{C}$ (Example 2.1.3.5).

008D **Construction 2.2.3.4** (The Conjugate of a 2-Category)**.** Let $\mathcal{C}$ be a 2-category. We define a new 2-category $\mathcal{C}^{\mathrm{c}}$ as follows:

- The objects of $\mathcal{C}^{\mathrm{c}}$ are the objects of $\mathcal{C}$. To avoid confusion, for each object $X \in \mathcal{C}$ we will write X^{c} for the corresponding object of $\mathcal{C}^{\mathrm{c}}$.

- For every pair of objects $X, Y \in \mathcal{C}$, we have $\underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{c}}}(X^{\mathrm{c}}, Y^{\mathrm{c}}) = \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^{\mathrm{op}}$. In particular, every 1-morphism $f : X \to Y$ in the 2-category $\mathcal{C}$ can be regarded as a 1-morphism from X^{c} to Y^{c} in the 2-category $\mathcal{C}^{\mathrm{c}}$, which we will denote by $f^{\mathrm{c}} : X^{\mathrm{c}} \to Y^{\mathrm{c}}$. Similarly, if we are given a pair of 1-morphisms $f, g : X \to Y$ in the 2-category $\mathcal{C}$ having the same source and target, then every 2-morphism $\gamma : f \Rightarrow g$ in $\mathcal{C}$ determines a 2-morphism from g^{c} to f^{c} in $\mathcal{C}^{\mathrm{c}}$, which we will denote by $\gamma^{\mathrm{c}} : g^{\mathrm{c}} \Rightarrow f^{\mathrm{c}}$.

- For every triple of objects $X, Y, Z \in \mathcal{C}$, the composition functor

  $$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{c}}}(Y^{\mathrm{c}}, Z^{\mathrm{c}}) \times \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{c}}}(X^{\mathrm{c}}, Y^{\mathrm{c}}) \to \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}}}(X^{\mathrm{c}}, Z^{\mathrm{c}})$$

  for the 2-category $\mathcal{C}^{\mathrm{c}}$ is induced by the composition functor

  $$\circ : \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z).$$

  on $\mathcal{C}$ by passing to opposite categories. In particular, it is given on objects by the formula $g^{\mathrm{c}} \circ f^{\mathrm{c}} = (g \circ f)^{\mathrm{c}}$.

- For every object $X \in \mathcal{C}$, the identity 1-morphism $\mathrm{id}_{X^{\mathrm{c}}} \in \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{c}}}(X^{\mathrm{c}}, X^{\mathrm{c}})$ is given by $\mathrm{id}_X^{\mathrm{c}}$, where $\mathrm{id}_X \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ is the identity 1-morphism associated to X in the 2-category $\mathcal{C}$, and the unit constraint $\upsilon_{X^{\mathrm{c}}}$ is the isomorphism $(\upsilon_X^{\mathrm{c}})^{-1} : \mathrm{id}_{X^{\mathrm{c}}} \circ \mathrm{id}_{X^{\mathrm{c}}} \xRightarrow{\sim} \mathrm{id}_{X^{\mathrm{c}}}$.

- For every triple of composable 1-morphisms

  $$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

  in the 2-category $\mathcal{C}$, the associativity constraint

  $$\alpha_{h^{\mathrm{c}}, g^{\mathrm{c}}, f^{\mathrm{c}}} : h^{\mathrm{c}} \circ (g^{\mathrm{c}} \circ f^{\mathrm{c}}) \xRightarrow{\sim} (h^{\mathrm{c}} \circ g^{\mathrm{c}}) \circ f^{\mathrm{c}}$$

  in the 2-category $\mathcal{C}^{\mathrm{c}}$ is given by the inverse $(\alpha_{h,g,f}^{\mathrm{c}})^{-1}$ of the associativity constraint $\alpha_{h,g,f} : h \circ (g \circ f) \xRightarrow{\sim} (h \circ g) \circ f$ in the 2-category $\mathcal{C}$.

We will refer to $\mathcal{C}^{\mathrm{c}}$ as the *conjugate* of the 2-category $\mathcal{C}$.

Example 2.2.3.5. Let $\mathcal{C}$ be a monoidal category, and let $B\,\mathcal{C}$ be the 2-category obtained by delooping $\mathcal{C}$ (Example 2.2.2.5). Then the conjugate 2-category $(B\,\mathcal{C})^{\mathrm{c}}$ can be identified with $B(\mathcal{C}^{\mathrm{op}})$, where we endow the opposite category $\mathcal{C}^{\mathrm{op}}$ with the monoidal structure of Example 2.1.3.4. 00F4

008E **Remark 2.2.3.6.** Constructions 2.2.3.1 and 2.2.3.4 are analogous but not identical. At the level of 2-morphisms, passage from a 2-category $\mathcal{C}$ to its opposite $\mathcal{C}^{\mathrm{op}}$ reverses the order of horizontal composition, but preserves the order of vertical composition; passage from $\mathcal{C}$ to its conjugate $\mathcal{C}^{\mathrm{c}}$ preserves the order of horizontal composition and reverses the order of vertical composition. Following the notation of Warning 2.2.1.9, we have

$$\delta^{\mathrm{op}}\gamma^{\mathrm{op}} = (\delta\gamma)^{\mathrm{op}} \qquad \gamma^{\mathrm{op}} \circ \gamma'^{\mathrm{op}} = (\gamma' \circ \gamma)^{\mathrm{op}}$$

$$\gamma^{\mathrm{c}}\delta^{\mathrm{c}} = (\delta\gamma)^{\mathrm{c}} \qquad \gamma'^{\mathrm{c}} \circ \gamma^{\mathrm{c}} = (\gamma' \circ \gamma)^{\mathrm{c}}.$$

008F **Example 2.2.3.7.** Let $\mathcal{C}$ be an ordinary category, which we regard as a 2-category having only identity 2-morphisms (Example 2.2.0.6). Then the opposite 2-category $\mathcal{C}^{\mathrm{op}}$ of Construction 2.2.3.1 coincides with the opposite of $\mathcal{C}$ as an ordinary category (which we can again regard as a 2-category having only identity morphisms). The conjugate 2-category $\mathcal{C}^{\mathrm{c}}$ of Construction 2.2.3.4 can be identified with $\mathcal{C}$ itself.

2.2.4 Functors of 2-Categories

008G Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Roughly speaking, a *functor* $F : \mathcal{C} \to \mathcal{D}$ should be an operation which carries objects, 1-morphisms, and 2-morphisms of $\mathcal{C}$ to objects, 1-morphisms, and 2-morphisms of $\mathcal{D}$, which is suitably compatible with (horizontal and vertical) composition. Here it is useful to distinguish between different notions of functor, which are differentiated by the degree of compatibility which is assumed.

008H **Definition 2.2.4.1** (Strict Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A *strict functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

- For every object $X \in \mathcal{C}$, an object $F(X)$ in $\mathcal{D}$.
- For every pair of objects $X, Y \in \mathcal{C}$, a functor of ordinary categories

  $$F_{X,Y} : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Y)).$$

  We will generally abuse notation by writing $F(f)$ for the value of the functor $F_{X,Y}$ on an object f of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$, and $F(\gamma)$ for the value of F on a morphism γ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$.

This data is required to satisfy the following compatibility conditions:

(1) For every object $X \in \mathcal{C}$, we have $\mathrm{id}_{F(X)} = F(\mathrm{id}_X)$.

(2) For every triple of objects $X, Y, Z \in \mathcal{C}$, the diagram of categories

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, "\circ"] \arrow[d, "F_{Y,Z} \times F_{X,Y}"] & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(F(Y),F(Z)) \times \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Y)) \arrow[r, "\circ"] & \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Z))
\end{tikzcd}$$

is strictly commutative.

(3) For every object $X \in \mathcal{C}$, the functor $F_{X,X}$ carries the unit constraint $\upsilon_X : \mathrm{id}_X \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} \mathrm{id}_X$ to the unit constraint $\upsilon_{F(X)} : \mathrm{id}_{F(X)} \circ \mathrm{id}_{F(X)} \overset{\sim}{\Longrightarrow} \mathrm{id}_{F(X)}$.

(4) For every composable triple of 1-morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in $\mathcal{C}$, we have $F(\alpha_{h,g,f}) = \alpha_{F(h),F(g),F(f)}$. In other words, F carries the associativity constraints of $\mathcal{C}$ to the associativity constraints of $\mathcal{D}$.

Remark 2.2.4.2. In the situation of Definition 2.2.4.1, conditions (3) and (4) are automatically satisfied if the 2-categories $\mathcal{C}$ and $\mathcal{D}$ are strict. `008J`

Example 2.2.4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be strict 2-categories, which we regard as Cat-enriched categories (Remark 2.2.0.2). Then strict functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.2.4.1) can be identified with Cat-enriched functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.1.7.10). `00JK`

Exercise 2.2.4.4. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a strict functor. Show that, for each morphism $f : X \to Y$ in $\mathcal{C}$, the functor $F_{X,Y} : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Y))$ carries the left and right unit constraints $\lambda_f : \mathrm{id}_Y \circ f \overset{\sim}{\Longrightarrow} f$ and $\rho_f : f \circ \mathrm{id}_X \overset{\sim}{\Longrightarrow} f$ to $\lambda_{F(f)}$ and $\rho_{F(f)}$, respectively (see Construction 2.2.1.11). `00F5`

Note that axiom (2) of Definition 2.2.4.1 implies in particular that for every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $\mathcal{C}$, we have an *equality* $F(g) \circ F(f) = F(g \circ f)$ between objects of the category $\underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Z))$. In practice, this requirement is often too strong: it is often better to allow a more liberal notion of functor, which is only required to preserve composition *up to isomorphism*.

Definition 2.2.4.5 (Lax Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. A *lax functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data: `008K`

- For every object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$.

- For every pair of objects $X, Y \in \mathcal{C}$, a functor of ordinary categories
$$F_{X,Y} : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Y)).$$
We will generally abuse notation by writing $F(f)$ for the value of the functor $F_{X,Y}$ on an object f of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$, an $F(\gamma)$ for the value of F on a morphism γ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$.

- For every object $X \in \mathcal{C}$, a 2-morphism $\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X)$ in the 2-category $\mathcal{D}$, which we will refer to as the *identity constraint.*

- For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $\mathcal{C}$, a 2-morphism
$$\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f),$$
which we will refer to as the *composition constraint.* We require that, if the objects X, Y, and Z are fixed, then the construction $(g, f) \mapsto \mu_{g,f}$ is functorial: that is, we can regard μ as a natural transformation of functors as indicated in the diagram
$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, "\circ"] \arrow[d, "F_{Y,Z} \times F_{X,Y}"'] & \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z) \arrow[d, "F_{X,Z}"] \arrow[dl, phantom, ""{name=U}] \\
\underline{\mathrm{Hom}}_{\mathcal{D}}(F(Y),F(Z)) \times \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Y)) \arrow[r, "\circ"] \arrow[ur, Rightarrow, "\mu"'] & \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Z))
\end{tikzcd}$$

This data is required to be compatible with the unit and associativity constraints of $\mathcal{C}$ and $\mathcal{D}$ in the following sense:

(a) For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, the left unit constraint $\lambda_{F(f)}$ in $\mathcal{D}$ is given by the vertical composition
$$\mathrm{id}_{F(Y)} \circ F(f) \xRightarrow{\epsilon_Y \circ \mathrm{id}_{F(f)}} F(\mathrm{id}_Y) \circ F(f) \xRightarrow{\mu_{\mathrm{id}_Y, f}} F(\mathrm{id}_Y \circ f) \xRightarrow{F(\lambda_f)} F(f).$$

(b) For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, the right unit constraint $\rho_{F(f)}$ in $\mathcal{D}$ is given by the vertical composition
$$F(f) \circ \mathrm{id}_{F(X)} \xRightarrow{\mathrm{id}_{F(f)} \circ \epsilon_X} F(f) \circ F(\mathrm{id}_X) \xRightarrow{\mu_{f, \mathrm{id}_X}} F(f \circ \mathrm{id}_X) \xRightarrow{F(\rho_f)} F(f).$$

(c) For every triple of composable 1-morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in the 2-category $\mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd}
F(h) \circ (F(g) \circ F(f)) \arrow[r, Rightarrow, "\alpha_{F(h),F(g),F(f)}"] \arrow[d, Rightarrow, "\mathrm{id}_{F(h)} \circ \mu_{g,f}"] & (F(h) \circ F(g)) \circ F(f) \arrow[d, Rightarrow, "\mu_{h,g} \circ \mathrm{id}_{F(f)}"] \\
F(h) \circ F(g \circ f) \arrow[d, Rightarrow, "\mu_{h,g\circ f}"] & F(h \circ g) \circ F(f) \arrow[d, Rightarrow, "\mu_{h\circ g,f}"] \\
F(h \circ (g \circ f)) \arrow[r, Rightarrow, "F(\alpha_{h,g,f})"] & F((h \circ g) \circ f)
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{D}}(F(W), F(Z))$.

A *functor* from $\mathcal{C}$ to $\mathcal{D}$ is a lax functor $F : \mathcal{C} \to \mathcal{D}$ with the property that the identity and composition constraints

$$\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X) \qquad \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$$

are isomorphisms.

Warning 2.2.4.6. The terminology of Definition 2.2.4.5 is not standard. In [4], Bénabou uses the term *morphism* for what we call a lax functor of 2-categories, *homomorphism* for what we call a functor of 2-categories, and *strict homomorphism* for what we call a strict functor of 2-categories. Other authors refer to functors of 2-categories (in the sense of Definition 2.2.4.5) as *weak functors* or *pseudofunctors* (to avoid confusion with the notion of strict functor). 008L

Remark 2.2.4.7. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a lax functor from $\mathcal{C}$ to $\mathcal{D}$. Then, for each object $X \in \mathrm{Ob}(\mathcal{C})$, we can regard $F_{X,X} : \underline{\mathrm{End}}_{\mathcal{C}}(X) \to \underline{\mathrm{End}}_{\mathcal{D}}(F(X))$ as a lax monoidal functor from $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ (endowed with the monoidal structure of Remark 2.2.1.7) to $\underline{\mathrm{End}}_{\mathcal{D}}(F(X))$: the tensor and unit constraints on $F_{X,X}$ are given by the composition and identity constraints on F, respectively. If F is a functor, then $F_{X,X}$ is a monoidal functor. 00F6

Remark 2.2.4.8. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a lax functor from $\mathcal{C}$ to $\mathcal{D}$. Then the identity constraints $\{\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ are uniquely determined by the other data of Definition 2.2.4.5. This follows from Proposition 2.1.5.4, applied to the lax monoidal functor $F_{X,X} : \underline{\mathrm{End}}_{\mathcal{C}}(X) \to \underline{\mathrm{End}}_{\mathcal{D}}(F(X))$ of Remark 2.2.4.7. 00F7

Remark 2.2.4.9. Let $\mathcal{C}$ be a monoidal category, let $B\mathcal{C}$ be the 2-category obtained by delooping $\mathcal{C}$ (Example 2.2.2.5), and let X denote the unique object of $B\mathcal{C}$. Let $\mathcal{D}$ be any 00F8

2-category, and let Y be an object of $\mathcal{D}$. Then the construction of Remark 2.2.4.7 induces bijections

$$\{\text{Lax Functors } F : B\,\mathcal{C} \to \mathcal{D} \text{ with } F(X) = Y\} \simeq \{\text{Lax monoidal functors } \mathcal{C} \to \underline{\mathrm{End}}_{\mathcal{D}}(Y)\}$$

$$\{\text{Functors } F : B\,\mathcal{C} \to \mathcal{D} \text{ with } F(X) = Y\} \simeq \{\text{Monoidal functors } \mathcal{C} \to \underline{\mathrm{End}}_{\mathcal{D}}(Y)\ \}.$$

Applying this observation in the case where $\mathcal{D} = B\,\mathcal{C}'$ for some other monoidal category $\mathcal{C}'$, we deduce that (lax) monoidal functors from $\mathcal{C}$ to $\mathcal{C}'$ can be identified with (lax) functors of 2-categories from $B\,\mathcal{C}$ to $B\,\mathcal{C}'$.

01NL **Example 2.2.4.10** (Algebras as Lax Functors)**.** Let $[0]$ denote the category having a single object and a single morphism, which we regard as a (strict) 2-category, and let $\mathcal{D}$ be any 2-category. Combining Remark 2.2.4.9 and Example 2.1.5.21, we deduce that lax functors $[0] \to \mathcal{D}$ can be identified with pairs (Y, A), where $Y \in \mathcal{D}$ is an object and A is an algebra object of the monoidal category $\underline{\mathrm{End}}_{\mathcal{D}}(Y)$.

008M **Example 2.2.4.11.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a strict functor (in the sense of Definition 2.2.4.1). Then we can regard F as a functor from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.2.4.5) by taking the identity and composition constraints

$$\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X) \qquad \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$$

to be the identity maps (note that in this case, conditions (a), (b), and (c) of Definition 2.2.4.5 reduce to conditions (3) and (4) of Definition 2.2.4.1). Conversely, if $F : \mathcal{C} \to \mathcal{D}$ is a lax functor having the property that each of the identity and composition constraints ϵ_X and $\mu_{g,f}$ is an identity 2-morphism of $\mathcal{D}$, then we can regard F as a strict 2-functor from $\mathcal{C}$ to $\mathcal{D}$. We therefore have inclusions

$$\{\text{Strict functors } F : \mathcal{C} \to \mathcal{D}\} \subseteq \{\text{Functors } F : \mathcal{C} \to \mathcal{D}\} \subseteq \{\text{Lax functors } F : \mathcal{C} \to \mathcal{D}\}.$$

In general, neither of these inclusions is reversible.

00F9 **Example 2.2.4.12** (Enriched Categories as Lax Functors)**.** Let S be a set, and let $\mathcal{E}_S$ denote the *indiscrete* category with object set S: that is, the objects of $\mathcal{E}_S$ are the elements of S, and $\mathrm{Hom}_{\mathcal{E}_S}(X, Y)$ is a singleton for every pair of elements $X, Y \in S$. Regard $\mathcal{E}_S$ as a (strict) 2-category having only identity 2-morphisms (Example 2.2.0.6). Let $\mathcal{C}$ be a monoidal category, and let $B\,\mathcal{C}$ be its delooping (Example 2.2.2.5). Unwinding the definitions, we see that lax functors $F : \mathcal{E}_S \to B\,\mathcal{C}$ (in the sense of Definition 2.2.4.5) can be identified with $\mathcal{C}$-enriched categories having object set S (in the sense of Definition 2.1.7.1).

008N **Warning 2.2.4.13.** Let $\mathcal{C}$ and $\mathcal{D}$ be strict 2-categories, and let $\mathcal{C}_0$ and $\mathcal{D}_0$ denote their underlying ordinary categories (obtained by ignoring the 2-morphisms of $\mathcal{C}$ and $\mathcal{D}$, respectively).

Every strict functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor of ordinary categories $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$. However, if a functor $F : \mathcal{C} \to \mathcal{D}$ is not strict, then it need not give rise to a functor from $\mathcal{C}_0$ to $\mathcal{D}_0$. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a composable pair of 1-morphisms in $\mathcal{C}$, then Definition 2.2.4.5 guarantees that the 1-morphisms $F(g) \circ F(f)$ and $F(g \circ f)$ are isomorphic (via the composition constraint $\mu_{g,f}$), but not that they are identical.

Example 2.2.4.14. Let $\mathcal{C}$ be a 2-category and let $\mathcal{D}$ be an ordinary category, which we regard as a 2-category having only identity 2-morphisms. If $F : \mathcal{C} \to \mathcal{D}$ is lax functor of 2-categories, then its values on the 1-morphisms of $\mathcal{C}$ must satisfy the following conditions: `008P`

(1) If $u, v : X \to Y$ are 1-morphisms of $\mathcal{C}$ having the same source and target and $\gamma : u \Rightarrow v$ is a 2-morphism of $\mathcal{C}$, then $F(u) = F(v)$ (since $F(\gamma) : F(u) \Rightarrow F(v)$ must be an identity 2-morphism of $\mathcal{D}$).

(2) If $u : X \to Y$ and $v : Y \to Z$ are composable 1-morphisms of $\mathcal{C}$, then $F(v \circ u) = F(v) \circ F(u)$ (since the composition constraint $\mu_{v,u} : F(v) \circ F(u) \Rightarrow F(v \circ u)$ is an identity 2-morphism of $\mathcal{D}$).

(3) For every object $X \in \mathcal{C}$, $F(\mathrm{id}_X)$ is the identity morphism $\mathrm{id}_{F(X)}$ in $\mathcal{D}$ (since the identity constraint $\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X)$ is an identity 2-morphism of $\mathcal{D}$).

Conversely, any specification of the values of F on objects and 1-morphisms which satisfies conditions (1), (2), and (3) extends uniquely to a strict functor $F : \mathcal{C} \to \mathcal{D}$ (the coherence conditions appearing in Definition 2.2.4.5 are automatic, by virtue of the fact that every 2-morphism of $\mathcal{D}$ is an identity). In particular, every lax functor $F : \mathcal{C} \to \mathcal{D}$ is automatically strict. Beware that the analogous statement is generally false if the roles of $\mathcal{C}$ and $\mathcal{D}$ are reversed.

Notation 2.2.4.15. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. To supply a lax 2-functor $F : \mathcal{C} \to \mathcal{D}$, one must specify not only the values of F on objects, 1-morphisms, and 2-morphisms of $\mathcal{C}$, but also the identity and composition constraints `008Q`

$$\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X) \qquad \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f).$$

In situations where we need to consider more than one lax functor at a time, we will denote these 2-morphisms by ϵ_X^F and $\mu_{g,f}^F$ (to avoid ambiguity).

Exercise 2.2.4.16. In the situation of Definition 2.2.4.5, show that we can replace (a) and (b) by the following alternative conditions: `00FA`

- For every object $X \in \mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathrm{id}_{F(X)} \circ \mathrm{id}_{F(X)} \arrow[r, Rightarrow, "\upsilon_{F(X)}"] \arrow[d, Rightarrow, "\epsilon_X \circ \epsilon_X"] & \mathrm{id}_{F(X)} \arrow[dd, Rightarrow, "\epsilon_X"] \\
F(\mathrm{id}_X) \circ F(\mathrm{id}_X) \arrow[d, Rightarrow, "\mu_{\mathrm{id}_X, \mathrm{id}_X}"] & \\
F(\mathrm{id}_X \circ \mathrm{id}_X) \arrow[r, Rightarrow, "F(\upsilon_X)"] & F(\mathrm{id}_X)
\end{tikzcd}$$

  commutes (in the endomorphism category $\underline{\mathrm{End}}_{\mathcal{C}}(X)$).

- For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, the vertical compositions

$$\mathrm{id}_{F(Y)} \circ F(f) \xRightarrow{\epsilon_Y \circ \mathrm{id}_{F(f)}} F(\mathrm{id}_Y) \circ F(f) \xRightarrow{\mu_{\mathrm{id}_Y, f}} F(\mathrm{id}_Y \circ f)$$

$$F(f) \circ \mathrm{id}_{F(X)} \xRightarrow{\mathrm{id}_{F(f)} \circ \epsilon_X} F(f) \circ F(\mathrm{id}_X) \xRightarrow{\mu_{f, \mathrm{id}_X}} F(f \circ \mathrm{id}_X)$$

  are monomorphisms in the category $\underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Y))$.

See Proposition 2.1.5.13.

Let $F : \mathcal{C} \to \mathcal{D}$ be a (lax) functor between 2-categories. According to Example 2.2.4.11, F is strict if and only if the identity and composition constraints

$$\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X) \qquad \mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$$

are identity 2-morphisms in $\mathcal{D}$. In §2.3.1, it will be useful to consider a weaker version of this condition, where we require strict compatibility with the formation of identity morphisms but not with respect to composition in general.

008R **Definition 2.2.4.17.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a lax functor. We say that F is *unitary* if, for every object $X \in \mathcal{C}$, the identity constraint $\epsilon_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X)$ is an invertible 2-morphism of $\mathcal{D}$. We say that F is *strictly unitary* if, for every object $X \in \mathcal{C}$, we have an equality $\mathrm{id}_{F(X)} = F(\mathrm{id}_X)$ and the identity constraint ϵ_X is the identity 2-morphism from $\mathrm{id}_{F(X)}$ to itself.

008S **Remark 2.2.4.18.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Every functor $F : \mathcal{C} \to \mathcal{D}$ is unitary when viewed as a lax functor from $\mathcal{C}$ to $\mathcal{D}$. Every strict functor $F : \mathcal{C} \to \mathcal{D}$ is strictly unitary when viewed as a lax functor from $\mathcal{C}$ to $\mathcal{D}$.

Remark 2.2.4.19. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a unitary lax functor. Then one can modify F to produce a strictly unitary lax functor $F' : \mathcal{C} \to \mathcal{D}$ by the following explicit procedure: `008U`

- For every object $X \in \mathcal{C}$, we set $F'(X) = F(X)$.
- For every 1-morphism $f : X \to Y$ in $\mathcal{C}$ which is not an identity morphism, we set $F'(f) = F(f)$; if $X = Y$ and $f = \mathrm{id}_X$ we instead set $F'(f) = \mathrm{id}_{F(X)}$. In either case, we have an invertible 2-morphism $\varphi_f : F'(f) \xRightarrow{\sim} F(f)$, given by
$$\varphi_f = \begin{cases} \epsilon^F_X & \text{if } f = \mathrm{id}_X \\ \mathrm{id}_{F(f)} & \text{otherwise.} \end{cases}$$
- Let X and Y be objects of $\mathcal{C}$, and let $\gamma : f \Rightarrow g$ be a 2-morphism between 1-morphisms $f, g : X \to Y$. We define $F'(\gamma)$ to be the vertical composition $\varphi_g^{-1} F(\gamma) \varphi_f$.
- For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $\mathcal{D}$, we define the composition constraint $\mu^{F'}_{g,f} : F'(g) \circ F'(f) \Rightarrow F'(g \circ f)$ to be the vertical composition
$$F'(g) \circ F'(f) \xRightarrow{\varphi_g \circ \varphi_f} F(g) \circ F(f) \xRightarrow{\mu^F_{g,f}} F(g \circ f) \xRightarrow{\varphi^{-1}_{g \circ f}} F'(g \circ f).$$

Consequently, it is generally harmless to assume that a unitary lax functor of 2-categories $F : \mathcal{C} \to \mathcal{D}$ is strictly unitary.

2.2.5 The Category of 2-Categories

We now show that 2-categories can be regarded as the objects of a category 2Cat, in which the morphisms are functors between 2-categories (Definition 2.2.5.5). There are several variants of this construction, depending on what sort of functors we allow. `008V`

Construction 2.2.5.1 (Composition of Lax Functors)**.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be 2-categories, and suppose we are given a pair of lax functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$. We define a lax functor $GF : \mathcal{C} \to \mathcal{E}$ as follows: `008W`

- On objects, the lax functor GF is given by $(GF)(X) = G(F(X))$.
- For every pair of objects $X, Y \in \mathcal{C}$, the functor
$$(GF)_{X,Y} : \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{E}}((GF)(X), (GF)(Y))$$

is given by the composition of functors

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \xrightarrow{F_{X,Y}} \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X),F(Y)) \xrightarrow{G_{F(X),F(Y)}} \underline{\mathrm{Hom}}_{\mathcal{E}}((GF)(X),(GF)(Y)).$$

In other words, the lax functor GF is given on 1-morphisms and 2-morphisms by the formulae

$$(GF)(f) = G(F(f)) \qquad (GF)(\gamma) = G(F(\gamma)).$$

- For each object $X \in \mathcal{C}$, the identity constraint $\epsilon_X^{GF} : \mathrm{id}_{(GF)(X)} \Rightarrow (GF)(\mathrm{id}_X)$ is given by the composition

$$\mathrm{id}_{(GF)(X)} \xRightarrow{\epsilon^G_{F(X)}} G(\mathrm{id}_{F(X)}) \xRightarrow{G(\epsilon^F_X)} (GF)(\mathrm{id}_X).$$

- For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $\mathcal{C}$, the composition constraint $\mu^{GF}_{g,f} : (GF)(g) \circ (GF)(f) \to (GF)(g \circ f)$ is given by the composition

$$(GF)(g) \circ (GF)(f) \xRightarrow{\mu^G_{F(g),F(f)}} G(F(g) \circ F(f)) \xRightarrow{G(\mu^F_{g,f})} (GF)(g \circ f).$$

We will refer to GF as the *composition of F with G*, and will sometimes denote it by $G \circ F$.

008X **Exercise 2.2.5.2.** Check that the composition of lax functors is well-defined. That is, if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are lax functors between 2-categories, then the identity and composition constraints ϵ^{GF}_X and $\mu^{GF}_{g,f}$ of Construction 2.2.5.1 are compatible with the unit constraints and associativity constraints of $\mathcal{C}$ and $\mathcal{E}$, as required by Definition 2.2.4.5.

008Y **Remark 2.2.5.3.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be lax functors of 2-categories, and let $GF : \mathcal{C} \to \mathcal{E}$ be their composition. Then:

- If F and G are unitary, then the composition GF is unitary.
- If F and G are functors, then the composition GF is a functor.
- If F and G are strictly unitary, then the composition GF is strictly unitary.
- If F and G are strict functors, then the composition GF is a strict functor.

008Z **Example 2.2.5.4.** Let $\mathcal{C}$ be a 2-category. We let $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ be the strict functor which carries every object, 1-morphism, and 2-morphism of $\mathcal{C}$ to itself. We will refer to $\mathrm{id}_{\mathcal{C}}$ as the *identity functor* on $\mathcal{C}$. Note that it is both a left and right unit for the composition of lax functors given in Construction 2.2.5.1.

Definition 2.2.5.5. We let $2\mathrm{Cat}_{\mathrm{Lax}}$ denote the ordinary category whose objects are (small) 2-categories and whose morphisms are lax functors between 2-categories (Definition 2.2.4.5), with composition given by Construction 2.2.5.1 and identity morphisms given by Example 2.2.5.4. We define (non-full) subcategories 0090

$$2\mathrm{Cat}_{\mathrm{Str}} \subsetneq 2\mathrm{Cat} \subsetneq 2\mathrm{Cat}_{\mathrm{Lax}} \supsetneq 2\mathrm{Cat}_{\mathrm{ULax}}$$

- The objects of 2Cat are 2-categories, and the morphisms of 2Cat are functors.
- The objects of $2\mathrm{Cat}_{\mathrm{Str}}$ are strict 2-categories, and the morphisms of $2\mathrm{Cat}_{\mathrm{Str}}$ are strict functors.
- The objects of $2\mathrm{Cat}_{\mathrm{ULax}}$ are 2-categories, and the morphisms of $2\mathrm{Cat}_{\mathrm{ULax}}$ are strictly unitary lax functors.

We will refer to 2Cat as the *category of 2-categories*, and to $2\mathrm{Cat}_{\mathrm{Str}}$ as the *category of strict 2-categories*.

Remark 2.2.5.6. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then the collection $\mathrm{Hom}_{2\mathrm{Cat}}(\mathcal{C}, \mathcal{D})$ of functors from $\mathcal{C}$ to $\mathcal{D}$ can be identified with the set of objects of a certain 2-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, called the *2-category of functors from $\mathcal{C}$ to $\mathcal{D}$*. We will return to this point in more detail in §[?]. 0092

Example 2.2.5.7. Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories, which we regard as 2-categories having only identity 2-morphisms (see Example 2.2.0.6). Then every lax functor of 2-categories from $\mathcal{C}$ to $\mathcal{D}$ is automatically strict (Example 2.2.4.14), and can be identified with a functor from $\mathcal{C}$ to $\mathcal{D}$ in the usual sense. In other words, we can view Example 2.2.0.6 as supplying fully faithful embeddings (of ordinary categories) 0093

$$\mathrm{Cat} \hookrightarrow 2\mathrm{Cat}_{\mathrm{Str}} \qquad \mathrm{Cat} \hookrightarrow 2\mathrm{Cat} \qquad \mathrm{Cat} \hookrightarrow 2\mathrm{Cat}_{\mathrm{Lax}} \qquad \mathrm{Cat} \hookrightarrow 2\mathrm{Cat}_{\mathrm{ULax}}.$$

Remark 2.2.5.8. Let MonCat denote the ordinary category whose objects are monoidal categories and whose morphisms are monoidal functors (that is, the underlying category of the strict 2-category **MonCat** of Example 2.2.0.5). Then the construction $\mathcal{C} \mapsto B\mathcal{C}$ determines a fully faithful embedding from MonCat to the category 2Cat of Definition 2.2.5.5, which fits into a pullback diagram 00FB

$$\begin{tikzcd}
\mathrm{MonCat} \arrow[r, "\mathcal{C} \mapsto B\mathcal{C}"] \arrow[d] & 2\mathrm{Cat} \arrow[d, "\mathcal{C} \mapsto \mathrm{Ob}(\mathcal{C})"] \\
\{*\} \arrow[r] & \mathrm{Set};
\end{tikzcd}$$

here $* = \{X\}$ denotes a set containing a single fixed object X. Similarly, the ordinary category of monoidal categories and lax monoidal functors can be regarded as a full subcategory of $2\mathrm{Cat}_{\mathrm{Lax}}$.

0094 **Remark 2.2.5.9** (Functors on Opposite 2-Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$ denote their opposites (Construction 2.2.3.1). Then every lax functor $F : \mathcal{C} \to \mathcal{D}$ induces a lax functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, given explicitly by the formulae

$$F^{\mathrm{op}}(X^{\mathrm{op}}) = F(X)^{\mathrm{op}} \qquad F^{\mathrm{op}}(f^{\mathrm{op}}) = F(f)^{\mathrm{op}} \qquad F^{\mathrm{op}}(\gamma^{\mathrm{op}}) = F(\gamma)^{\mathrm{op}}$$

$$\epsilon_{X^{\mathrm{op}}} = (\epsilon_X)^{\mathrm{op}} \qquad \mu_{g^{\mathrm{op}}, f^{\mathrm{op}}} = (\mu_{f,g})^{\mathrm{op}}.$$

In this case, F is a functor if and only if F^{op} is a functor, and a strict functor if and only if F^{op} is a strict functor. This operation is compatible with composition, and therefore induces equivalences of categories

$$2\mathrm{Cat}_{\mathrm{Str}} \simeq 2\mathrm{Cat}_{\mathrm{Str}} \qquad 2\mathrm{Cat} \simeq 2\mathrm{Cat} \qquad 2\mathrm{Cat}_{\mathrm{Lax}} \simeq 2\mathrm{Cat}_{\mathrm{Lax}} \qquad 2\mathrm{Cat}_{\mathrm{ULax}} \simeq 2\mathrm{Cat}_{\mathrm{ULax}}.$$

0095 **Remark 2.2.5.10** (Functors on Conjugate 2-Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $\mathcal{C}^{\mathrm{c}}$ and $\mathcal{D}^{\mathrm{c}}$ denote their conjugates (Construction 2.2.3.4). Then every functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $F^{\mathrm{c}} : \mathcal{C}^{\mathrm{c}} \to \mathcal{D}^{\mathrm{c}}$, given explicitly by the formulae

$$F^{\mathrm{c}}(X^{\mathrm{c}}) = F(X)^{\mathrm{c}} \qquad F^{\mathrm{c}}(f^{\mathrm{c}}) = F(f)^{\mathrm{c}} \qquad F^{\mathrm{c}}(\gamma^{\mathrm{c}}) = F(\gamma)^{\mathrm{c}}$$

$$\epsilon_{X^{\mathrm{c}}} = (\epsilon_X^{-1})^{\mathrm{c}} \qquad \mu_{g^{\mathrm{c}}, f^{\mathrm{c}}} = (\mu_{g,f}^{-1})^{\mathrm{c}}.$$

In this case, the functor F is strict if and only if F^{c} is strict. This operation is compatible with composition, and therefore induces equivalences of categories

$$2\mathrm{Cat}_{\mathrm{Str}} \simeq 2\mathrm{Cat}_{\mathrm{Str}} \qquad 2\mathrm{Cat} \simeq 2\mathrm{Cat}$$

0096 **Warning 2.2.5.11.** The construction of Remark 2.2.5.10 requires that the identity and composition constraints of F are invertible, and therefore does not extend to lax functors between 2-categories. In general, one cannot identify lax functors from $\mathcal{C}$ to $\mathcal{D}$ with lax functors from $\mathcal{C}^{\mathrm{c}}$ to $\mathcal{D}^{\mathrm{c}}$: the definition of lax functor is asymmetrical with respect to vertical composition.

2.2.6 Isomorphisms of 2-Categories

0097 We now study isomorphisms between 2-categories.

0098 **Definition 2.2.6.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. We will say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism* if it is an isomorphism in the category 2Cat of Definition 2.2.5.5. That is, F is an isomorphism if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $GF = \mathrm{id}_{\mathcal{C}}$ and $FG = \mathrm{id}_{\mathcal{C}}$. We say that 2-categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic* if there exists an isomorphism from $\mathcal{C}$ to $\mathcal{D}$.

Remark 2.2.6.2. Let $F : \mathcal{C} \to \mathcal{D}$ be an isomorphism of 2-categories, and let $G : \mathcal{D} \to \mathcal{C}$ be the inverse isomorphism. Then: 0099

- The functor F is strictly unitary if and only if G is strictly unitary. In this case, we say that F is a *strictly unitary isomorphism.*

- The functor F is strict if and only if G is strict. In this case, we say that F is a *strict isomorphism.*

We say that 2-categories $\mathcal{C}$ and $\mathcal{D}$ are *strictly isomorphic* if there is a strict isomorphism from $\mathcal{C}$ to $\mathcal{D}$.

Warning 2.2.6.3. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories which are strictly isomorphic. Then $\mathcal{C}$ is strict if and only if $\mathcal{D}$ is strict. If we assume only that $\mathcal{C}$ and $\mathcal{D}$ are isomorphic (rather than strictly isomorphic), then we cannot draw the same conclusion. In other words, the condition that a 2-category $\mathcal{C}$ is strict is invariant under *strict* isomorphism, but not under isomorphism. 009A

Warning 2.2.6.4. The notions of isomorphism and strict isomorphism of 2-categories are somewhat artificial. As in classical category theory, there is notion of *equivalence of* 2-*categories* (Definition [?]) which is more general than isomorphism and more appropriate for describing what it means for 2-categories to be "the same." 009B

Remark 2.2.6.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of 2-categories. Then F is an isomorphism (in the sense of Definition 2.2.6.1) if and only if it satisfies the following conditions: 009C

- The functor F induces a bijection from the set of objects of $\mathcal{C}$ to the set of objects of $\mathcal{D}$.

- For every pair of objects $X, Y \in \mathcal{C}$, the functor F induces an isomorphism of categories $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Y))$.

One might be tempted to consider a more liberal version of Definition 2.2.6.1 working with lax functors rather than functors. However, the resulting notion of isomorphism turns out to be the same.

Proposition 2.2.6.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a lax functor which is an isomorphism in the category* $2\mathrm{Cat}_{\mathrm{Lax}}$. *Then F is a functor.* 009D

Proof. We will show that, for every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the 2-category $\mathcal{C}$, the composition constraint $\mu^F_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ is an isomorphism (in the ordinary category $\underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Z))$); the analogous statement for the identity constraints $\epsilon^F_X : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X)$ follows by a similar (but easier) argument.

Let $G : \mathcal{D} \to \mathcal{C}$ be a lax functor which is an inverse of F in the category $2\mathrm{Cat}_{\mathrm{Lax}}$. For any pair of composable 1-morphisms $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$ in the 2-category $\mathcal{D}$, the composition constraint $\mu^{F\circ G}_{g',f'}$ for the lax functor $F \circ G$ is given by the vertical composition

$$(F \circ G)(g') \circ (F \circ G)(f') \xRightarrow{\mu^F_{G(g'),G(f')}} F(G(g') \circ G(f')) \xRightarrow{F(\mu^G_{g',f'})} (F \circ G)(g' \circ f').$$

Since $F \circ G$ coincides with $\mathrm{id}_{\mathcal{D}}$ as a lax functor, this composition is the identity 2-morphism from $g' \circ f'$ to itself. In particular, we see that $F(\mu^G_{g',f'})$ has a right inverse in the category $\underline{\mathrm{Hom}}_{\mathcal{D}}(X', Z')$. It follows that $\mu^G_{g',f'} = G(F(\mu^G_{g',f'}))$ has a right inverse in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(G(X'), G(Z'))$.

Applying the same argument with the roles of F and G reversed, we see that the composition constraint $\mu^{G\circ F}_{g,f} = \mathrm{id}_{g\circ f}$ factors as a vertical composition

$$(G \circ F)(g) \circ (G \circ F)(f) \xRightarrow{\mu^G_{F(g),F(f)}} G(F(g) \circ F(f)) \xRightarrow{G(\mu^F_{g,f})} (G \circ F)(g \circ f).$$

In particular, this shows that $\mu^G_{F(g),F(f)}$ has a left inverse (in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$). Applying the preceding argument in the case $g' = F(g)$ and $f' = F(f)$, we see that $\mu^G_{F(g),F(f)}$ also has a right inverse. It follows that $\mu^G_{F(g),F(f)}$ is an isomorphism in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$. Since $G(\mu^F_{g,f})$ is a left inverse of $\mu^G_{F(g),F(f)}$, it must also be an isomorphism. It follows that $F(G(\mu^F_{g,f})) = \mu^F_{g,f}$ is an isomorphism in the category $\underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Z))$, as desired. □

We now construct some examples of non-strict isomorphisms of 2-categories.

009E **Notation 2.2.6.7.** Let $\mathcal{C}$ be a 2-category. A *twisting cochain for* $\mathcal{C}$ is a datum which assigns, to every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, a 1-morphism $(g \circ' f) : X \to Z$ and an invertible 2-morphism $\mu_{g,f} : g \circ' f \overset{\sim}{\Longrightarrow} g \circ f$. In this case, we will (slightly) abuse notation by identifying the twisting cochain with the collection of 2-morphisms $\{\mu_{g,f}\}$.

009F **Construction 2.2.6.8.** Let $\mathcal{C}$ be a 2-category equipped with a twisting cochain

$$\{\mu_{g,f}\} = \{\mu_{g,f} : (g \circ' f) \Rightarrow (g \circ f)\}.$$

We define a new 2-category $\mathcal{C}'$ as follows:

- The objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, we define $\underline{\mathrm{Hom}}_{\mathcal{C}'}(X, Y)$ to be the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$. In particular, we can identify 1-morphisms of $\mathcal{C}'$ with 1-morphisms of $\mathcal{C}$, 2-morphisms of $\mathcal{C}'$ with 2-morphisms of $\mathcal{C}$, and the vertical composition of 2-morphisms in $\mathcal{C}'$ with the vertical composition of 2-morphisms in $\mathcal{C}$.

- For every object $X \in \mathcal{C}$, the identity 1-morphism from X to itself in the 2-category $\mathcal{C}'$ is the same as the identity morphism from X to itself in the 2-category $\mathcal{C}$.

- For every triple of objects $X, Y, Z \in \mathcal{C}$, the composition functor
$$\underline{\mathrm{Hom}}_{\mathcal{C}'}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}'}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}'}(X,Z)$$
is given on objects by $(g,f) \mapsto g \circ' f$ and on morphisms by the construction
$$(\delta : g \Rightarrow g', \gamma : f \Rightarrow f') \mapsto \mu_{g',f'}^{-1}(\delta \circ \gamma)\mu_{g,f}.$$

- For every object $X \in \mathcal{C}$, the unit constraint $\upsilon'_X : \mathrm{id}_X \circ' \mathrm{id}_X \xRightarrow{\sim} \mathrm{id}_X$ for the 2-category $\mathcal{C}'$ is given by the composition
$$\mathrm{id}_X \circ' \mathrm{id}_X \xRightarrow{\mu_{\mathrm{id}_X,\mathrm{id}_X}} \mathrm{id}_X \circ \mathrm{id}_X \xRightarrow{\upsilon_X} \mathrm{id}_X \, .$$

- For every triple of composable 1-morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ of $\mathcal{C}$, the associativity constraint of $\mathcal{C}'$ is given by the composition
$$\begin{aligned} h \circ' (g \circ' f) &\xRightarrow{\mu_{h, g\circ' f}} h \circ (g \circ' f) \\ &\xRightarrow{\mathrm{id}_h \circ \mu_{g,f}} h \circ (g \circ f) \\ &\xRightarrow{\alpha_{h,g,f}} (h \circ g) \circ f \\ &\xRightarrow{\mu_{h,g}^{-1} \circ \mathrm{id}_f} (h \circ' g) \circ f \\ &\xRightarrow{\mu_{h \circ' g, f}^{-1}} (h \circ' g) \circ' f. \end{aligned}$$

We will refer to $\mathcal{C}'$ as the *twist of $\mathcal{C}$ with respect to* $\{\mu_{g,f}\}$.

Exercise 2.2.6.9. Let $\mathcal{C}$ be a 2-category equipped with a twisting cochain $\{\mu_{g,f}\}$. Show that the 2-category $\mathcal{C}'$ of Construction 2.2.6.8 is well-defined. Moreover, there is a strictly unitary isomorphism of 2-categories $\mathcal{C} \to \mathcal{C}'$ which carries each object, 1-morphism, and 2-morphism of $\mathcal{C}$ to itself, where the composition constraints are given by $\{\mu_{g,f}\}$. 009G

Exercise 2.2.6.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a strictly unitary isomorphism of 2-categories. Show that there is a unique twisting cochain $\{\mu_{g,f}\}$ on the 2-category $\mathcal{C}$ such that F factors as a composition $\mathcal{C} \xrightarrow{G} \mathcal{C}' \xrightarrow{H} \mathcal{D}$, where G is the strictly unitary isomorphism of Exercise 2.2.6.9 and H is a strict isomorphism of 2-categories. In other words, the notion of twisting cochain (in the sense of Notation 2.2.6.7) measures the difference between strictly unitary isomorphisms and strict isomorphisms in the setting of 2-categories. 009H

009J **Remark 2.2.6.11.** It is possible to consider a generalization of the twisting procedure of Construction 2.2.6.8 in which one modifies not only the composition law for 1-morphisms of $\mathcal{C}$, but also the choice of identity 1-morphisms of $\mathcal{C}$. Since we will not need this generalization, we leave the details to the reader.

009L **Example 2.2.6.12.** Let G be a group with identity element $1 \in G$, let Γ be an abelian group on which G acts by automorphisms, let $\alpha : G \times G \times G \to \Gamma$ be a 3-cocycle, let $\mathcal{C}$ be the monoidal category of Example 2.1.3.3, and let $B\mathcal{C}$ be the 2-category obtained by delooping $\mathcal{C}$ (Example 2.2.2.5). A twisting cochain for the 2-category $B\mathcal{C}$ (in the sense of Notation 2.2.6.7) can be identified with a map of sets

$$\mu : G \times G \to \Gamma \qquad (g, f) \mapsto \mu_{g,f}.$$

Let $(B\mathcal{C})'$ denote the twist of $B\mathcal{C}$ with respect to μ. Unwinding the definitions, we see that $(B\mathcal{C})'$ is obtained by delooping the same category $\mathcal{C}$ with respect to a different monoidal structure: namely, the monoidal structure supplied by the 3-cocycle $\alpha' : G \times G \times G \to \Gamma$ given by the formula

$$\alpha'_{h,g,f} = \alpha_{h,g,f} + h(\mu_{g,f}) - \mu_{hg,f} + \mu_{h,gf} - \mu_{h,g}.$$

We can summarize the situation as follows:

- To every 3-cocycle $\alpha : G \times G \times G \to \Gamma$, we can associate a 2-category $B\mathcal{C}$ in which the 1-morphisms are the elements of G, the 2-morphisms are the elements of Γ, and the associativity constraint is given by α.
- If $\alpha, \alpha' : G \times G \times G \to \Gamma$ are cohomologous 3-cocycles on G with values in Γ, then the associated 2-categories $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic (though not necessarily strictly isomorphic). More precisely, every choice of 2-cocycle $\mu : G \times G \to \Gamma$ satisfying $\alpha' = \alpha + \partial(\mu)$ determines a strictly unitary isomorphism from $\mathcal{C}$ to $\mathcal{C}'$. Here ∂ denotes the boundary operator from 2-cochains to 3-cocycles, given concretely by the formula

$$(\partial\mu)_{h,g,f} = h(\mu_{g,f}) - \mu_{hg,f} + \mu_{h,gf} - \mu_{h,g}.$$

009N **Example 2.2.6.13.** The 2-categories Bimod and $\mathrm{Cospan}(\mathcal{C})$ of Examples 2.2.2.4 and 2.2.2.1 both depend on certain auxiliary choices:

- Let A, B, and C be associative rings, and suppose we are given a pair of bimodules $M = {}_AM_B$ and $N = {}_BN_C$. Then we can regard M and N as 1-morphisms in the 2-category Bimod, whose composition is defined to be the relative tensor product $M \otimes_B N$. This tensor product is well-defined up to (unique) isomorphism: it is universal among abelian groups P which are equipped with a B-bilinear map $M \times N \to P$.

However, it is possible to give many different constructions of an abelian group with this universal property, each of which gives a (slightly) different composition law for the 1-morphisms in the 2-category Bimod.

- Let $\mathcal{C}$ be a category which admits pushouts, and suppose we are given a pair of cospans

  $$X \leftarrow B \rightarrow Y \qquad Y \leftarrow C \rightarrow Z$$

  in $\mathcal{C}$. Then B and C can be regarded as 1-morphisms in the 2-category $\operatorname{Cospan}(\mathcal{C})$, whose composition is given by the pushout $C \amalg_Y B$ (regarded as a cospan from X to Z). This pushout is well-defined up to (unique) isomorphism as an object of $\mathcal{C}$, but there is generally no preferred representative of its isomorphism class. Consequently, different choices of pushout lead to (slightly) different definitions for the composition of 1-morphisms in the 2-category $\operatorname{Cospan}(\mathcal{C})$.

By making a different choice of conventions in these examples, one can obtain 2-categories Bimod' and $\operatorname{Cospan}'(\mathcal{C})$ having the same objects, 1-morphisms, and 2-morphisms as the 2-categories Bimod and $\operatorname{Cospan}(\mathcal{C})$, but different composition laws for 1-morphisms. In this case, the 2-categories Bimod' and $\operatorname{Cospan}'(\mathcal{C})$ can be obtained from Bimod and $\operatorname{Cospan}(\mathcal{C})$ (respectively) by the twisting procedure of Construction 2.2.6.8. In particular, the resulting 2-categories Bimod' and $\operatorname{Cospan}'(\mathcal{C})$ are isomorphic (though not necessarily *strictly* isomorphic) to the 2-categories Bimod and $\operatorname{Cospan}(\mathcal{C})$, respectively.

2.2.7 Strictly Unitary 2-Categories

We now introduce a special class of 2-categories. `00FC`

Definition 2.2.7.1. Let $\mathcal{C}$ be a 2-category. We will say that $\mathcal{C}$ is *strictly unitary* if, for each 1-morphism $f : X \to Y$ in $\mathcal{C}$, the left and right unit constraints `00FD`

$$\lambda_f : \mathrm{id}_Y \circ f \xRightarrow{\sim} f \qquad \rho_f : f \circ \mathrm{id}_X \xRightarrow{\sim} f$$

are identity 2-morphisms of $\mathcal{C}$.

Proposition 2.2.7.2. *Let $\mathcal{C}$ be a 2-category. Then $\mathcal{C}$ is strictly unitary if and only if the following conditions are satisfied:* `00FE`

(a) *For each 1-morphism $f : X \to Y$ in $\mathcal{C}$, we have $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.*

(b) *For each object X of $\mathcal{C}$, the unit constraint $\upsilon_X : \mathrm{id}_X \circ \mathrm{id}_X \xRightarrow{\sim} \mathrm{id}_X$ is the identity morphism from $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$ to itself.*

(c) *For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, the associativity constraints $\alpha_{\mathrm{id}_Y, \mathrm{id}_Y, f}$ and $\alpha_{f, \mathrm{id}_X, \mathrm{id}_X}$ are equal to the identity (as 2-morphisms from f to itself).*

Proof. If $\mathcal{C}$ is strictly unitary, then (a) is clear and (b) follows from Corollary 2.2.1.15. Assume that (a) and (b) are satisfied. For any 1-morphism $f : X \to Y$ in $\mathcal{C}$, the left unit constraint λ_f is characterized by the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{id}_Y \circ (\mathrm{id}_Y \circ f) \arrow[rr, Rightarrow, "\alpha_{\mathrm{id}_Y,\mathrm{id}_Y,f}"] \arrow[dr, Rightarrow, swap, "\mathrm{id}_{\mathrm{id}_Y} \circ \lambda_f"] & & (\mathrm{id}_Y \circ \mathrm{id}_Y) \circ f \arrow[dl, Rightarrow, "\upsilon_Y \circ \mathrm{id}_f"] \\
& \mathrm{id}_Y \circ f, &
\end{tikzcd}$$

and is therefore the identity 2-morphism if and only if $\alpha_{\mathrm{id}_Y,\mathrm{id}_Y,f}$ is an identity 2-morphism (from f to itself). Similarly, the right unit constraint ρ_f is an identity 2-morphism if and only if $\alpha_{f,\mathrm{id}_X,\mathrm{id}_X}$ is an identity 2-morphism in $\mathcal{C}$. □

00FF **Remark 2.2.7.3.** Let $\mathcal{C}$ be a strictly unitary 2-category. Then $\mathcal{C}$ satisfies the following stronger versions of conditions (a) and (c) of Proposition 2.2.7.2:

(a') For every pair of objects $X, Y \in \mathcal{C}$, the functors

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \qquad f \mapsto \mathrm{id}_Y \circ f$$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \qquad f \mapsto f \circ \mathrm{id}_X$$

are equal to the identity.

(c') For every pair of 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, the associativity constraints $\alpha_{g,f,\mathrm{id}_X}$, $\alpha_{g,\mathrm{id}_Y,f}$, and $\alpha_{\mathrm{id}_Z,g,f}$ are equal to the identity (as 2-morphisms from $g \circ f$ to itself).

Here (a') follows from the naturality of the left and right unit constraints (Remark 2.2.1.13), and (c') follows from Propositions 2.2.1.14 and 2.2.1.16.

0089 **Example 2.2.7.4.** Let G be a group with identity element $1 \in G$, let Γ be an abelian group on which G acts by automorphisms, let $\alpha : G \times G \times G \to \Gamma$ be a 3-cocycle, let $\mathcal{C}$ be the monoidal category of Example 2.1.3.3, and let $B\mathcal{C}$ be the 2-category obtained by delooping $\mathcal{C}$ (Example 2.2.2.5). The following conditions are equivalent:

- The 3-cocycle α is *normalized*: that is, it satisfies the equations

$$\alpha_{x,y,1} = \alpha_{x,1,y} = \alpha_{1,x,y} = 0$$

  for every pair of elements $x, y \in G$.

- The 2-category $B\mathcal{C}$ is strictly unitary, in the sense of Definition 2.2.7.1.

Remark 2.2.7.5. Let $\mathcal{C}$ and $\mathcal{D}$ be strictly unitary 2-categories (Definition 2.2.7.1). Then a strictly unitary lax functor $F : \mathcal{C} \to \mathcal{D}$ is given by the following data: 008T

- For each object $X \in \mathcal{C}$, an object $F(X) \in \mathcal{D}$.
- For every pair of objects $X, Y \in \mathcal{C}$, a functor of ordinary categories
$$F_{X,Y} : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Y)).$$
- For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, a composition constraint $\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$, depending functorially on f and g.

This data must be required to satisfy axiom (c) of Definition 2.2.4.5, together with the identities $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for each object $X \in \mathcal{C}$ and $\mu_{\mathrm{id}_Y, f} = \mathrm{id}_{F(f)} = \mu_{f, \mathrm{id}_X}$ for each 1-morphism $f : X \to Y$ of $\mathcal{C}$.

Remark 2.2.7.6. Let $\mathcal{C}$ be a strictly unitary 2-category, let $\{\mu_{g,f}\}$ be a twisting cochain for $\mathcal{C}$ (see Notation 2.2.6.7), and let $\mathcal{C}'$ denote the twist of $\mathcal{C}'$ with respect to $\{\mu_{g,f}\}$ (Construction 2.2.6.8). The following conditions are equivalent: 009K

(1) The 2-category $\mathcal{C}'$ is strictly unitary.

(2) For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, both μ_{f,id_X} and $\mu_{\mathrm{id}_Y, f}$ are identity 2-morphisms (from $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$ to itself).

If these conditions are satisfied, we will say that the twisting cochain $\{\mu_{g,f}\}$ is *normalized.*

It is generally harmless to assume that a 2-category $\mathcal{C}$ is strictly unitary, by virtue of the following:

Proposition 2.2.7.7. *Let $\mathcal{C}$ be a 2-category. Then there exists a strictly unitary isomorphism $\mathcal{C} \simeq \mathcal{C}'$, where $\mathcal{C}'$ is a strictly unitary 2-category.* 00FG

Proof. Let $\mu = \{\mu_{g,f}\}$ be the twisting cochain on $\mathcal{C}$ given on composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ by the formula
$$\mu_{g,f} = \begin{cases} \lambda_f^{-1} : f \Rightarrow g \circ f & \text{if } g = \mathrm{id}_Y \\ \rho_g^{-1} : g \Rightarrow g \circ f & \text{if } f = \mathrm{id}_Y \\ \mathrm{id}_{g \circ f} : g \circ f \Rightarrow g \circ f & \text{otherwise.} \end{cases}$$

Note that this prescription is consistent, since $\lambda_f = \upsilon_Y = \rho_g$ in the special case where $f = \mathrm{id}_Y = g$ (Corollary 2.2.1.15). Let $\mathcal{C}'$ be the twist of $\mathcal{C}$ with respect to the cocycle $\{\mu_{g,f}\}$ (Construction 2.2.6.8). Then $\mathcal{C}'$ is a strictly unitary 2-category (in the sense of Definition 2.2.7.1), and Exercise 2.2.6.9 supplies a strictly unitary isomorphism of 2-categories $\mathcal{C} \simeq \mathcal{C}'$ □

0091 **Remark 2.2.7.8.** Let $2\mathrm{Cat}'_{\mathrm{ULax}}$ denote the subcategory of $2\mathrm{Cat}_{\mathrm{Lax}}$ (and full subcategory of $2\mathrm{Cat}_{\mathrm{ULax}}$) whose objects are strictly unitary 2-categories and whose morphisms are strictly unitary lax functors. It follows from Proposition 2.2.7.7 that the inclusion $2\mathrm{Cat}'_{\mathrm{ULax}} \hookrightarrow 2\mathrm{Cat}_{\mathrm{ULax}}$ is an equivalence of categories.

00FH **Remark 2.2.7.9.** Let G be a group and let Γ be an abelian group with an action of G. When applied to the 2-categories described in Example 2.2.7.4, Proposition 2.2.7.7 reduces to the assertion that every 3-cocycle $\alpha : G \times G \times G \to \Gamma$ is cohomologous to a normalized 3-cocycle $\alpha' : G \times G \times G \to \Gamma$.

2.2.8 The Homotopy Category of a 2-Category

02GD Every ordinary category can be regarded as a 2-category having only identity 2-morphisms (Remark 2.2.1.6). Conversely, to every 2-category $\mathcal{C}$ one can associate ordinary category $\mathrm{hPith}(\mathcal{C})$ having the same objects, in which morphisms are given by isomorphism classes of 1-morphisms in $\mathcal{C}$. We will refer to $\mathrm{hPith}(\mathcal{C})$ as the *homotopy category* of the 2-category $\mathcal{C}$ (Construction 2.2.8.12). It will be convenient to view this construction as a composition of two different operations:

- To every 2-category $\mathcal{C}$, one can associate a subcategory $\mathrm{Pith}(\mathcal{C}) \subseteq \mathcal{C}$ by removing the non-invertible 2-morphisms of $\mathcal{C}$; we will refer to $\mathrm{Pith}(\mathcal{C})$ as the *pith of* $\mathcal{C}$ (Construction 2.2.8.9).
- To every 2-category $\mathcal{C}$, one can associate an ordinary category $\mathrm{h}\mathcal{C}$ by "collapsing" all 2-morphisms of $\mathcal{C}$ to identity 2-morphisms (Construction 2.2.8.2). We will refer to $\mathrm{h}\mathcal{C}$ as the *coarse homotopy category* of the 2-category $\mathcal{C}$.

We begin by formulating the latter construction more precisely.

02AR **Definition 2.2.8.1.** Let $\mathcal{C}$ be a 2-category and let $\mathcal{H}$ be an ordinary category, viewed as a 2-category having only identity 2-morphisms. We say that a functor $F : \mathcal{C} \to \mathcal{H}$ *exhibits* $\mathcal{H}$ *as a coarse homotopy category of* $\mathcal{C}$ if, for every ordinary category $\mathcal{E}$, precomposition with F induces a bijection

$$\begin{tikzcd}
\{\text{Functors of ordinary categories from } \mathcal{H} \text{ to } \mathcal{E}\} \arrow[d] \\
\{\text{Functors of 2-categories from } \mathcal{C} \text{ to } \mathcal{E}\}.
\end{tikzcd}$$

It follows immediately from the definitions that if a 2-category $\mathcal{C}$ admits a coarse homotopy category $\mathcal{H}$, then $\mathcal{H}$ is uniquely determined up to isomorphism. We will prove existence by an explicit construction.

Construction 2.2.8.2 (The Coarse Homotopy Category of a 2-Category)**.** Let $\mathcal{C}$ be a 2-category. We define a category $\mathrm{h}\mathcal{C}$ as follows: 02AS

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- If X and Y are objects of $\mathcal{C}$, then $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)$ is the set of connected components of the simplicial set $\mathrm{N}_\bullet(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y))$.
- For objects X, Y, and Z of $\mathcal{C}$, the composition of morphisms in $\mathrm{h}\mathcal{C}$ is given by the map

$$\begin{aligned}\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y,Z)\times\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y) &= \pi_0(\mathrm{N}_\bullet\,\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z))\times\pi_0(\mathrm{N}_\bullet\,\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y))\\ &\simeq \pi_0(\mathrm{N}_\bullet(\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Z)\times\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)))\\ &\xrightarrow{\circ} \pi_0(\mathrm{N}_\bullet\,\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z))\\ &= \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Z).\end{aligned}$$

We will refer to $\mathrm{h}\mathcal{C}$ as the *coarse homotopy category* of $\mathcal{C}$.

The terminology of Construction 2.2.8.2 is consistent with that of Definition 2.2.8.1, by virtue of the following:

Proposition 2.2.8.3. *Let $\mathcal{C}$ be a 2-category and let $\mathrm{h}\mathcal{C}$ be the ordinary category of Construction 2.2.8.2, regarded as a 2-category having only identity 2-morphisms. Then there is a unique functor of 2-categories $F:\mathcal{C}\to\mathrm{h}\mathcal{C}$ with the following properties:* 02AT

- *The functor F carries each object of $\mathcal{C}$ to itself (regarded as an object of $\mathrm{h}\mathcal{C}$).*
- *The functor F carries each 1-morphism $u:X\to Y$ of $\mathcal{C}$ to the connected component of u, regarded as a vertex of the nerve $\mathrm{N}_\bullet(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y))$.*

Moreover, the functor F exhibits $\mathrm{h}\mathcal{C}$ as a coarse homotopy category of $\mathcal{C}$, in the sense of Definition 2.2.8.1.

Proof. The existence of F follows from Example 2.2.4.14. Let $\mathcal{E}$ be an ordinary category, and suppose we are given a functor of 2-categories $G:\mathcal{C}\to\mathcal{E}$. We wish to show that there is a unique functor of ordinary categories $\overline{G}:\mathrm{h}\mathcal{C}\to\mathcal{E}$ satisfying $G=\overline{G}\circ F$. The uniqueness is clear (since the functor F is surjective on objects and on 1-morphisms). To prove existence, we define $\overline{G}$ on objects by the formula $\overline{G}(X)=G(X)$ and on morphism by using the map of simplicial sets

$$\mathrm{N}_\bullet(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y))\to\mathrm{Hom}_{\mathcal{E}}(G(X),G(Y))$$

and passing to connected components. $\square$

02AU **Corollary 2.2.8.4.** *Let* Cat *denote the category of (small) categories and let* 2Cat *denote the category of (small)* 2*-categories (Definition 2.2.5.5. Then the inclusion* $\mathrm{Cat} \hookrightarrow 2\mathrm{Cat}$ *admits a left adjoint, given on objects by the construction* $\mathcal{C} \mapsto \mathrm{h}\mathcal{C}$*.*

In general, passage from a 2-category $\mathcal{C}$ to its coarse homotopy category $\mathrm{h}\mathcal{C}$ is a very destructive procedure: if $u, v : X \to Y$ are 1-morphisms of $\mathcal{C}$ having the same source and target, then the existence of *any* 2-morphism $\gamma : u \Rightarrow v$ in $\mathcal{C}$ guarantees that u and v have the same image in $\mathrm{h}\mathcal{C}$. For many purposes, it is more appropriate to work with a variant of $\mathrm{h}\mathcal{C}$ which identifies only *isomorphic* 1-morphisms of $\mathcal{C}$ (Construction 2.2.8.12). First, let us introduce some terminology.

009Q **Definition 2.2.8.5.** A $(2,1)$-*category* is a 2-category $\mathcal{C}$ with the property that every 2-morphism in $\mathcal{C}$ is invertible.

009R **Remark 2.2.8.6.** The terminology of Definition 2.2.8.5 fits into a general paradigm. Given $0 \leq m \leq n \leq \infty$, let us informally use the term (n,m)-*category* to refer to an n-category $\mathcal{C}$ having the property that every k-morphism of $\mathcal{C}$ is invertible for $k > m$. Following this convention, the ∞-categories of Definition 1.4.0.1 should really be called $(\infty, 1)$-categories.

02AV **Example 2.2.8.7.** Let $\mathcal{C}$ be an ordinary category, viewed as a 2-category having only identity 2-morphisms (Remark 2.2.1.6). Then $\mathcal{C}$ is a $(2,1)$-category.

009S **Remark 2.2.8.8.** Let $\mathcal{C}$ be a $(2,1)$-category. Then every lax functor of 2-categories $F : \mathcal{D} \to \mathcal{C}$ is automatically a functor. Consequently, there is no need to distinguish between functors and lax functors when working in the setting of $(2,1)$-categories.

00AL **Construction 2.2.8.9** (The Pith of a 2-Category)**.** Let $\mathcal{C}$ be a 2-category. We define a new 2-category $\mathrm{Pith}(\mathcal{C})$ as follows:

- The objects of $\mathrm{Pith}(\mathcal{C})$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, the category $\underline{\mathrm{Hom}}_{\mathrm{Pith}(\mathcal{C})}(X, Y)$ is the core $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^{\simeq}$ of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ (see Construction 1.3.5.4).
- The composition law, associativity constraints, and unit constraints of $\mathrm{Pith}(\mathcal{C})$ are given by restricting the composition law, associativity constraints, and unit constraints of $\mathcal{C}$.

Then $\mathrm{Pith}(\mathcal{C})$ is a $(2,1)$-category which we will refer to as the *pith* of $\mathcal{C}$.

More informally: for any 2-category $\mathcal{C}$, the $(2,1)$-category $\mathrm{Pith}(\mathcal{C})$ is obtained by discarding the non-invertible 2-morphisms of $\mathcal{C}$.

00AM **Remark 2.2.8.10** (The Universal Property of the Pith)**.** Let $\mathcal{C}$ be a 2-category. Then $\mathrm{Pith}(\mathcal{C})$ is characterized (up to isomorphism) by the following properties:

- The pith $\mathrm{Pith}(\mathcal{C})$ is a $(2,1)$-category.
- For every $(2,1)$-category $\mathcal{D}$, every functor $F : \mathcal{D} \to \mathcal{C}$ factors (uniquely) through $\mathrm{Pith}(\mathcal{C})$.

Warning 2.2.8.11. In the situation of Remark 2.2.8.10, it is not true that a *lax* functor $F : \mathcal{D} \to \mathcal{C}$ factors through the pith $\mathrm{Pith}(\mathcal{C})$ (even when $\mathcal{D}$ is a $(2,1)$-category): any lax functor which admits such a factorization is automatically a functor, by virtue of Remark 2.2.8.8. `00AN`

Construction 2.2.8.12 (The Homotopy Category of a 2-Category)**.** Let $\mathcal{C}$ be a 2-category. We define a category $\mathrm{hPith}(\mathcal{C})$ as follows: `02AW`

- The objects of $\mathrm{hPith}(\mathcal{C})$ are the objects of $\mathcal{C}$.
- If X and Y are objects of $\mathcal{C}$, then $\mathrm{Hom}_{\mathrm{hPith}(\mathcal{C})}(X,Y)$ is the set of isomorphism classes of objects in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$. If $f : X \to Y$ is a 1-morphism from X to Y, we typically denote its isomorphism class by $[f] \in \mathrm{Hom}_{\mathrm{hPith}(\mathcal{C})}(X,Y)$.
- The composition law on $\mathrm{hPith}(\mathcal{C})$ is determined by the requirement that $[g] \circ [f] = [g \circ f]$ for every pair of composable 1-morphisms $f : X \to Y$ and $g : Y \to Z$ (this composition law is associative by virtue of the *existence* of the associativity contraints of the 2-category $\mathcal{C}$).
- For every object $Y \in \mathcal{C}$, the identity morphism from Y to itself in $\mathrm{hPith}(\mathcal{C})$ is the isomorphism class of the identity morphism id_Y in $\mathcal{C}$. For 1-morphisms $f : X \to Y$ and $g : Y \to Z$, the identities

  $$[\mathrm{id}_Y] \circ [f] = [f] \qquad [g] \circ [\mathrm{id}_Y] = [g]$$

  follow from the existence of left and right unit constraints (see Construction 2.2.1.11).

We will refer to $\mathrm{hPith}(\mathcal{C})$ as the *homotopy category* of $\mathcal{C}$.

Remark 2.2.8.13. Let $\mathcal{C}$ be a 2-category. For every pair of objects $X, Y \in \mathcal{C}$, the category `02AX`

$$\underline{\mathrm{Hom}}_{\mathrm{Pith}(\mathcal{C})}(X,Y) = \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)^{\simeq}$$

is a groupoid, so that the nerve $\mathrm{N}_{\bullet}(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)^{\simeq})$ is a Kan complex. It follows that 1-morphisms $u, v : X \to Y$ belong to the same connected component of $\mathrm{N}_{\bullet}(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)^{\simeq})$ if and only if they are connected by an edge of $\mathrm{N}_{\bullet}(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)^{\simeq})$ (Remark 1.4.6.13): that is, if and only if u and v are isomorphic as objects of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$. It follows that the homotopy category $\mathrm{hPith}(\mathcal{C})$ of Construction 2.2.8.12 can be identified with the coarse homotopy category of the 2-category $\mathrm{Pith}(\mathcal{C})$ (as suggested by the notation).

`02AY` **Warning 2.2.8.14.** Let $\mathcal{C}$ be a 2-category and let $\mathrm{hPith}(\mathcal{C})$ be the homotopy category of $\mathcal{C}$, which we regard as a 2-category having only identity 2-morphisms. In general, there is no functor which directly relates $\mathcal{C}$ to the homotopy category $\mathrm{hPith}(\mathcal{C})$. Instead, there is a commutative diagram of 2-categories

$$\begin{tikzcd}
\mathrm{Pith}(\mathcal{C}) \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathrm{hPith}(\mathcal{C}) \arrow[r] & \mathrm{h}\mathcal{C}.
\end{tikzcd}$$

Here the functor $\mathrm{hPith}(\mathcal{C}) \to \mathrm{h}\mathcal{C}$ is bijective on objects and *full*: that is, for every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathrm{hPith}(\mathcal{C})}(X,Y) = \pi_0(\mathrm{N}_\bullet \, \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)^{\simeq}) \to \pi_0(\mathrm{N}_\bullet \, \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)$$

is surjective.

`02AZ` **Example 2.2.8.15.** Let $\mathcal{C}$ be a $(2,1)$-category, so that $\mathrm{Pith}(\mathcal{C}) = \mathcal{C}$. In particular, the inclusion $\mathrm{Pith}(\mathcal{C}) \hookrightarrow \mathcal{C}$ induces an isomorphism of categories $\mathrm{hPith}(\mathcal{C}) \simeq \mathrm{h}\mathcal{C}$. In this situation, we will generally abuse notation by identifying $\mathrm{h}\mathcal{C}$ with $\mathrm{hPith}(\mathcal{C})$ and referring to it as the *homotopy category* of $\mathcal{C}$.

`02B0` **Remark 2.2.8.16** (Functoriality)**.** Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of 2-categories. Then there is a unique functor of ordinary categories $\mathrm{hPith}(U) : \mathrm{hPith}(\mathcal{C}) \to \mathrm{hPith}(\mathcal{D})$ with the following properties:

- For each object $X \in \mathcal{C}$, the functor $\mathrm{hPith}(U)$ carries X to the object $U(X) \in \mathcal{D}$.
- For each 1-morphism $f : X \to Y$ of $\mathcal{C}$, the functor $\mathrm{hPith}(U)$ carries the isomorphism class $[f]$ to the isomorphism class of the 1-morphism $U(f) : U(X) \to U(Y)$.

Beware that the analogous assertion does not hold if U is only assumed to be a lax functor of 2-categories.

`02B1` **Definition 2.2.8.17.** Let $\mathcal{C}$ be a 2-category. We say that a 1-morphism $f : X \to Y$ in $\mathcal{C}$ is an *isomorphism* if the homotopy class $[f]$ is an isomorphism in the homotopy category $\mathrm{hPith}(\mathcal{C})$. Equivalently, f is an isomorphism if there exists another 1-morphism $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are isomorphic to id_X and id_Y as objects of the categories $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,X)$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(Y,Y)$, respectively. In this case, g is also an isomorphism in $\mathcal{C}$, which we will refer to as a *homotopy inverse* to f.

Example 2.2.8.18. Let $\mathcal{C}$ be an ordinary category, regarded as a 2-category having only identity 2-morphisms (Remark 2.2.1.6). Then a morphism $f : X \to Y$ in $\mathcal{C}$ is an isomorphism in the sense of Definition 2.2.8.17 if and only if it is an isomorphism in the usual sense: that is, if and only if there exists a morphism $g : Y \to X$ satisfying $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. 02B2

Warning 2.2.8.19. Let $\mathcal{C}$ be a strict 2-category. We can then consider two different notions of isomorphism in $\mathcal{C}$: 02B3

- We say that a morphism $f : X \to Y$ is a *strict isomorphism* if it is an isomorphism in the underlying category of $\mathcal{C}$: that is, if there exists a 1-morphism $g : Y \to X$ satisfying $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.
- We say that a morphism $f : X \to Y$ is an *isomorphism* if the homotopy class $[f]$ is an isomorphism in the homotopy category $\mathrm{hPith}(\mathcal{C})$: that is, if there exists a 1-morphism $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are isomorphic to id_X and id_Y as objects of the categories $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, X)$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Y)$, respectively.

Every strict isomorphism in $\mathcal{C}$ is an isomorphism. However, the converse is false in general (see Example 2.2.8.20).

Example 2.2.8.20. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between (small) categories. Then F is an equivalence of categories if and only if it is an isomorphism when regarded as a 1-morphism in the 2-category **Cat** of Example 2.2.0.4. 02B4

Remark 2.2.8.21. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between 2-categories. Then F carries isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$ (see Remark 2.2.8.16). Beware that the analogous assertion need not hold if we assume only that F is a lax functor of 2-categories. 02B5

Remark 2.2.8.22. Let $\mathcal{C}$ be a 2-category and let $f : X \to Y$ and $g : Y \to Z$ be 1-morphisms of $\mathcal{C}$. If any two of the 1-morphisms f, g, and $g \circ f$ is an isomorphism, then so is the third. In particular, the collection of isomorphisms is closed under composition. 02B6

Remark 2.2.8.23. Let $\mathcal{C}$ be a 2-category and let $f, g : X \to Y$ be 1-morphisms in $\mathcal{C}$ having the same source and target. If f and g are isomorphic (as objects of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$), then f is an isomorphism if and only if g is an isomorphism. 02B7

We close this section by discussing a strengthening of Definition 2.2.8.5.

Definition 2.2.8.24. Let $\mathcal{C}$ be a 2-category. We say that $\mathcal{C}$ is a 2-*groupoid* if every 1-morphism in $\mathcal{C}$ is an isomorphism and every 2-morphism of $\mathcal{C}$ is an isomorphism. 02B8

Remark 2.2.8.25. A 2-category $\mathcal{C}$ is a 2-groupoid if and only if it is a $(2,1)$-category and the homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid. 02B9

02BA **Example 2.2.8.26.** Let $\mathcal{C}$ be an ordinary category. Then $\mathcal{C}$ is a groupoid if and only if it is a 2-groupoid (when viewed as a 2-category having only identity 2-morphisms).

02BB **Construction 2.2.8.27** (The Core of a 2-Category)**.** Let $\mathcal{C}$ be a 2-category. We define a new 2-category $\mathcal{C}^{\simeq}$ as follows:

- The objects of $\mathcal{C}^{\simeq}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, the category $\underline{\mathrm{Hom}}_{\mathcal{C}^{\simeq}}(X, Y)$ is the full subcategory of $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)^{\simeq}$ spanned by the isomorphisms $f : X \to Y$.
- The composition law, associativity constraints, and unit constraints of $\mathcal{C}^{\simeq}$ are obtained by restricting the composition law, associativity constraints, and unit constraints of $\mathcal{C}$ (which is well-defined by virtue of Remark 2.2.8.22).

We will refer to $\mathcal{C}^{\simeq}$ as the *core* of the 2-category $\mathcal{C}$.

02BC **Example 2.2.8.28.** Let $\mathcal{C}$ be a category. Then the core $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ of Construction 1.3.5.4 coincides with the core $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ of Construction 2.2.8.27, where we regard $\mathcal{C}$ as a 2-category having only identity 2-morphisms.

02BD **Remark 2.2.8.29.** Let $\mathcal{C}$ be a 2-category. Then the inclusion functor $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ is a functor of 2-categories, which induces an isomorphism of categories from $\mathrm{h}(\mathcal{C}^{\simeq})$ to the core $\mathrm{hPith}(\mathcal{C})^{\simeq}$ of the homotopy category $\mathrm{hPith}(\mathcal{C})$.

02BE **Remark 2.2.8.30.** Let $\mathcal{C}$ be a 2-category. Then the core $\mathcal{C}^{\simeq}$ is a 2-groupoid. This follows from Remark 2.2.8.25: it is immediate from the construction that $\mathcal{C}^{\simeq}$ is a $(2,1)$-category, and the homotopy category $\mathrm{h}(\mathcal{C}^{\simeq})$ is a groupoid by virtue of the isomorphism $\mathrm{h}(\mathcal{C}^{\simeq}) \simeq \mathrm{hPith}(\mathcal{C})^{\simeq}$ of Remark 2.2.8.29.

02BF **Remark 2.2.8.31** (The Universal Property of the Core)**.** Let $\mathcal{C}$ be a 2-category. Then the core $\mathcal{C}^{\simeq}$ is characterized by the following properties:

- The 2-category $\mathcal{C}^{\simeq}$ is a 2-groupoid (Remark 2.2.8.30).
- For every 2-groupoid $\mathcal{D}$, every functor $F : \mathcal{D} \to \mathcal{C}$ factors (uniquely) through $\mathcal{C}^{\simeq}$.

2.3 The Duskin Nerve of a 2-Category

009P In §1.4, we defined an ∞-*category* to be a simplicial set $X_{\bullet}$ which satisfies the weak Kan extension condition. Beware that this terminology is potentially misleading. Roughly speaking, an ∞-category (in the sense of Definition 1.4.0.1) should be viewed as a higher category $\mathcal{C}$ with the property that every k-morphism in $\mathcal{C}$ is invertible for $k \geq 2$. The framework of weak Kan complexes does not capture the entirety of higher category theory, or even the entirety of the theory of 2-categories (as described in §2.2). Nevertheless, we will show in this section that the theory of ∞-categories *can* be viewed as a generalization of the theory of $(2,1)$-categories. Recall that, to every category $\mathcal{C}$, one can associate a simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ called the *nerve of* $\mathcal{C}$ (Construction 1.3.1.1). We proved in Chapter 1 that $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ determines a fully faithful embedding from the category Cat of small categories to the category Set_Δ of simplicial sets (Proposition 1.3.3.1), and that every simplicial set of the form $\mathrm{N}_\bullet(\mathcal{C})$ is an ∞-category (Example 1.4.0.4). The construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ has a generalization to the setting of 2-categories. In §2.3.1, we associate to each 2-category $\mathcal{C}$ a simplicial set $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ called the *Duskin nerve* of $\mathcal{C}$ (introduced by Duskin and Street; see [17] and [54]). This construction has the following features (both established by Duskin in [17]):

- If $\mathcal{C}$ is a $(2,1)$-category, then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ is an ∞-category (Theorem 2.3.2.1). We prove this in §2.3.2 as a consequence of a more general result which applies to the Duskin nerve of *any* 2-category (Theorem 2.3.2.5), whose proof we defer to §2.3.3.

- Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. In §2.3.4, we show that passage to the Duskin nerve induces a bijection

$$\begin{tikzcd}
\{\text{Strictly unitary lax functors } F : \mathcal{C} \to \mathcal{D}\} \arrow[d, "\sim"] \\
\{\text{Maps of simplicial sets } \mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{D})\};
\end{tikzcd}$$

  see Theorem 2.3.4.1. In other words, the formation of Duskin nerves induces a fully faithful embedding from the category $2\mathrm{Cat}_{\mathrm{ULax}}$ of Definition 2.2.5.5 to the category of simplicial sets.

By virtue of Theorem 2.3.4.1, it is mostly harmless to abuse terminology by *identifying* a 2-category $\mathcal{C}$ with the simplicial set $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ (each can be recovered from the other, up to canonical isomorphism). Theorem 2.3.2.1 then asserts that, under this identification, every $(2,1)$-category can be regarded as an ∞-category (see Remark 2.3.4.2 for a more precise statement).

In §2.3.5, we study the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ in the case where $\mathcal{C}$ is a *strict* 2-category. In this case, we show that n-simplices of $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ can be identified with *strict* functors $\mathrm{Path}_{(2)}[n] \to \mathcal{C}$ (Corollary 2.3.5.7). Here $\mathrm{Path}_{(2)}[n]$ denotes a certain 2-categorical variant of the path category introduced in §1.3.7, which will play an important role in our discussion of the homotopy coherent nerve of a simplicial category (see §2.4.3).

2.3.1 The Duskin Nerve

009T In §1.3, we associated to each category $\mathcal{C}$ a simplicial set $\mathrm{N}_\bullet(\mathcal{C})$, called the *nerve of* $\mathcal{C}$. This construction has a natural generalization to the setting of 2-categories.

009U **Construction 2.3.1.1** (The Duskin Nerve)**.** Let n be a nonnegative integer and let $[n]$ denote the linearly ordered set $\{0 < 1 < 2 < \cdots < n\}$. We will regard $[n]$ as a category, hence also as a 2-category having only identity 2-morphisms (Example 2.2.0.6). For any 2-category $\mathcal{C}$, we let $\mathrm{N}^{\mathrm{D}}_n(\mathcal{C})$ denote the set of all strictly unitary lax functors from $[n]$ to $\mathcal{C}$ (Definition 2.2.4.17). The construction $[n] \mapsto \mathrm{N}^{\mathrm{D}}_n(\mathcal{C})$ determines a simplicial set, given as a functor by the composition

$$\boldsymbol{\Delta}^{\mathrm{op}} \hookrightarrow \mathrm{Cat}^{\mathrm{op}} \hookrightarrow 2\mathrm{Cat}^{\mathrm{op}}_{\mathrm{ULax}} \xrightarrow{\mathrm{Hom}_{2\mathrm{Cat}_{\mathrm{ULax}}}(\bullet, \mathcal{C})} \mathrm{Set}.$$

We will denote this simplicial set by $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ and refer to it as the *Duskin nerve* of the 2-category $\mathcal{C}$.

009V **Remark 2.3.1.2.** In the setting of strict 2-categories, the Duskin nerve $\mathcal{C} \mapsto \mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ was introduced by Street in [54]. The generalization to arbitrary 2-categories was given by Duskin in [17].

009W **Example 2.3.1.3.** Let $\mathcal{C}$ be an ordinary category, viewed as a 2-category having only identity 2-morphisms (Example 2.2.0.6). Then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ can be identified with the nerve $\mathrm{N}_\bullet(\mathcal{C})$ of $\mathcal{C}$ as an ordinary category (Construction 1.3.1.1).

009X **Remark 2.3.1.4.** Let $\mathcal{C}$ be a 2-category and let $\mathcal{C}^{\mathrm{op}}$ denote the opposite 2-category (see Construction 2.2.3.1). Then we have a canonical isomorphism of simplicial sets $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})^{\mathrm{op}}$, where $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})^{\mathrm{op}}$ denotes the opposite of the simplicial set $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ (see Notation 1.4.2.1).

009Y **Warning 2.3.1.5.** Let $\mathcal{C}$ be a 2-category and let $\mathcal{C}^{\mathrm{c}}$ be the conjugate of $\mathcal{C}$, obtained by reversing vertical composition (Construction 2.2.3.4). There is no simple relationship between Duskin nerves of $\mathcal{C}$ and $\mathcal{C}^{\mathrm{c}}$ (since the operation $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{c}}$ is not functorial with respect to lax functors; see Warning 2.2.5.11).

009Z **Remark 2.3.1.6** (Functoriality)**.** The construction $\mathcal{C} \mapsto \mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ determines a functor from the category $2\mathrm{Cat}_{\mathrm{ULax}}$ of small 2-categories (with morphisms given by strictly unitary lax functors) to the category Set_Δ of simplicial sets. This functor fits into the general paradigm of Variant 1.2.2.8: it arises from a cosimplicial object of the category $2\mathrm{Cat}_{\mathrm{ULax}}$, given by the inclusion $\boldsymbol{\Delta} \hookrightarrow \mathrm{Cat} \hookrightarrow 2\mathrm{Cat}_{\mathrm{ULax}}$. Beware that, unlike the usual nerve functor $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$, the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet : 2\mathrm{Cat}_{\mathrm{ULax}} \to \mathrm{Set}_\Delta$ does not admit a left adjoint: Proposition 1.2.3.15 does not apply, because the category $2\mathrm{Cat}_{\mathrm{ULax}}$ does not admit small colimits (one can address this problem by restricting to *strict* 2-categories: we will return to this point in §2.3.5).

Remark 2.3.1.7. Let $\mathcal{C}$ be a 2-category, let $\{\mu_{g,f}\}$ be a twisting cochain for $\mathcal{C}$ (Notation 2.2.6.7), and let $\mathcal{C}'$ be the twist of $\mathcal{C}$ with respect to $\{\mu_{g,f}\}$ (Construction 2.2.6.8). Then the twisting cochain $\{\mu_{g,f}\}$ defines a strictly unitary isomorphism of 2-categories $\mathcal{C} \simeq \mathcal{C}'$, and therefore induces an isomorphism of simplicial sets $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \simeq \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}')$. In other words, the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ cannot detect the difference between $\mathcal{C}$ and $\mathcal{C}'$. This should be regarded as a feature, rather than a bug. Defining the composition law for 1-morphisms in a 2-category $\mathcal{C}$ often requires certain arbitrary (but ultimately inessential) choices (see Example 2.2.6.13). In such cases, one can often give a more direct description of the simplicial set $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ which avoids such choices. See Example 2.3.1.17 and Corollary 8.1.3.15. `00A0`

Remark 2.3.1.8. Let us make Construction 2.3.1.1 more explicit. Fix a 2-category $\mathcal{C}$. Unwinding the definitions, we see that an element of $\mathrm{N}^{\mathrm{D}}_{n}(\mathcal{C})$ consists of the following data: `00A1`

(0) A collection of objects $\{X_i\}_{0 \leq i \leq n}$ of the 2-category $\mathcal{C}$.

(1) A collection of 1-morphisms $\{f_{j,i} : X_i \to X_j\}_{0 \leq i \leq j \leq n}$ in the 2-category $\mathcal{C}$

(2) A collection of 2-morphisms $\{\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i}\}_{0 \leq i \leq j \leq k \leq n}$ in the 2-category $\mathcal{C}$.

These data are required to satisfy the following conditions:

(a) For $0 \leq i \leq n$, the 1-morphism $f_{i,i} : X_i \to X_i$ is the identity 1-morphism id_{X_i}.

(b) For $0 \leq i \leq j \leq n$, the 2-morphisms

$$\mu_{j,j,i} : f_{j,j} \circ f_{j,i} \Rightarrow f_{j,i} \qquad \mu_{j,i,i} : f_{j,i} \circ f_{i,i} \Rightarrow f_{j,i}$$

are the left unit constraints $\lambda_{f_{j,i}}$ and the right unit constraints $\rho_{f_{j,i}}$, respectively.

(c) For $0 \leq i \leq j \leq k \leq \ell \leq n$, we have a commutative diagram

$$\begin{tikzcd}
f_{\ell,k} \circ (f_{k,j} \circ f_{j,i}) \arrow[rr, Rightarrow, "\alpha_{f_{\ell,k}, f_{k,j}, f_{j,i}}"] \arrow[d, Rightarrow, "\mathrm{id}_{f_{\ell,k}} \circ \mu_{k,j,i}"'] & & (f_{\ell,k} \circ f_{k,j}) \circ f_{j,i} \arrow[d, Rightarrow, "\mu_{\ell,k,j} \circ \mathrm{id}_{f_{j,i}}"] \\
f_{\ell,k} \circ f_{k,i} \arrow[dr, Rightarrow, "\mu_{\ell,k,i}"'] & & f_{\ell,j} \circ f_{j,i} \arrow[dl, Rightarrow, "\mu_{\ell,j,i}"] \\
& f_{\ell,i} &
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X_i, X_\ell)$.

In the description of Remark 2.3.1.8, it is possible to be more efficient by eliminating some of the "redundant" information.

`00A2` **Proposition 2.3.1.9.** *Let $\mathcal{C}$ be a 2-category and let n be a nonnegative integer. Suppose we are given the following data:*

(0) *A collection of objects $\{X_i\}_{0 \leq i \leq n}$ of the 2-category $\mathcal{C}$.*

(1′) *A collection of 1-morphisms $\{f_{j,i} : X_i \to X_j\}_{0 \leq i < j \leq n}$ in the 2-category $\mathcal{C}$*

(2′) *A collection of 2-morphisms $\{\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i}\}_{0 \leq i < j < k \leq n}$ in the 2-category $\mathcal{C}$.*

This data can be extended uniquely to an n-simplex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ (as described in Remark 2.3.1.8) if and only if the following condition is satisfied:

(c') *For $0 \leq i < j < k < \ell \leq n$, we have a commutative diagram*

$$\begin{tikzcd}
f_{\ell,k} \circ (f_{k,j} \circ f_{j,i}) \arrow[rr, Rightarrow, "\alpha_{f_{\ell,k}, f_{k,j}, f_{j,i}}"] \arrow[d, Rightarrow, "\mathrm{id}_{f_{\ell,k}} \circ \mu_{k,j,i}"'] & & (f_{\ell,k} \circ f_{k,j}) \circ f_{j,i} \arrow[d, Rightarrow, "\mu_{\ell,k,j} \circ \mathrm{id}_{f_{j,i}}"] \\
f_{\ell,k} \circ f_{k,i} \arrow[dr, Rightarrow, "\mu_{\ell,k,i}"'] & & f_{\ell,j} \circ f_{j,i} \arrow[dl, Rightarrow, "\mu_{\ell,j,i}"] \\
& f_{\ell,i} &
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X_i, X_\ell)$.

Proof. We wish to show that there is a unique way to choose 1-morphisms $f_{j,i} : X_i \to X_j$ for $i = j$ and 2-morphisms $\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i}$ for $i = j \leq k$ and $i \leq j = k$ so that conditions (a), (b), and (c) of Remark 2.3.1.8 are satisfied. The uniqueness is clear: to satisfy condition (a), we must have $f_{i,i} = \mathrm{id}_{X_i}$ for $0 \leq i \leq n$, and to satisfy condition (b) we must have $\mu_{k,j,i} = \rho_{f_{j,i}}$ when $i = j$ and $\mu_{k,j,i} = \lambda_{f_{k,j}}$ when $j = k$. To complete the proof, it will suffice to verify the following:

(I) The prescription above is consistent. That is, when $i = j = k$, we have $\rho_{f_{j,i}} = \lambda_{f_{k,j}}$ (as morphisms of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X_i, X_k)$).

(II) The prescription above satisfies condition (c) of Remark 2.3.1.8. That is, the diagram

$$\begin{tikzcd}
f_{\ell,k} \circ (f_{k,j} \circ f_{j,i}) \arrow[rr, Rightarrow, "\alpha_{f_{\ell,k}, f_{k,j}, f_{j,i}}"] \arrow[d, Rightarrow, "\mathrm{id}_{f_{\ell,k}} \circ \mu_{k,j,i}"'] & & (f_{\ell,k} \circ f_{k,j}) \circ f_{j,i} \arrow[d, Rightarrow, "\mu_{\ell,k,j} \circ \mathrm{id}_{f_{j,i}}"] \\
f_{\ell,k} \circ f_{k,i} \arrow[dr, Rightarrow, "\mu_{\ell,k,i}"'] & & f_{\ell,j} \circ f_{j,i} \arrow[dl, Rightarrow, "\mu_{\ell,j,i}"] \\
& f_{\ell,i} &
\end{tikzcd}$$

commutes in the special cases $0 \leq i = j \leq k \leq \ell \leq n$, $0 \leq i \leq j = k \leq \ell \leq n$, and $0 \leq i \leq j \leq k = \ell \leq n$.

Assertion (I) follows from Corollary 2.2.1.15. Assertion (II) follows from the triangle identity in $\mathcal{C}$ in the case $j = k$, and from Proposition 2.2.1.16 in the cases $i = j$ and $k = \ell$. □

Corollary 2.3.1.10. *Let $\mathcal{C}$ be a 2-category. Then the restriction map* 00A3

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^n, \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}))$$

is bijective for $n \geq 4$ and injective when $n = 3$.

Warning 2.3.1.11. Let $\mathcal{C}$ be a 2-category. By virtue of Proposition 2.3.1.9, we can identify n-simplices of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ with triples 00A4

$$(\{X_i\}_{0 \leq i \leq n}, \{f_{j,i}\}_{0 \leq i < j \leq n}, \{\mu_{k,j,i}\}_{0 \leq i < j < k \leq n})$$

satisfying condition (c') of Proposition 2.3.1.9. This gives a description of $\mathrm{N}^{\mathrm{D}}_{n}(\mathcal{C})$ which makes no reference to the identity 1-morphisms of $\mathcal{C}$ or the left and right unit constraints of $\mathcal{C}$. The resulting identification is functorial with respect to *injective* maps of linearly ordered sets $[m] \to [n]$. In other words, we can construct the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ as a *semisimplicial* set (see Definition 1.1.1.2) without knowing the left and right unit constraints of $\mathcal{C}$. However, the left and right unit constraints of $\mathcal{C}$ are needed to define the degeneracy operators on the simplicial set $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$.

Remark 2.3.1.12. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a lax functor. If F is strictly unitary, then composition with F induces a map of simplicial sets $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$. However, even without the assumption that F is strictly unitary, one can use the description of Proposition 2.3.1.9 to obtain a collection of maps $\mathrm{N}^{\mathrm{D}}_{n}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{n}(\mathcal{D})$ which are compatible 00A5 with the face operators on the simplicial sets $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ and $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ (though not necessarily with the degeneracy operators). In other words, if we regard the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ as a *semisimplicial set*, then it is functorial with respect to all (lax) functors between 2-categories.

`00A6` **Example 2.3.1.13** (Vertices of the Duskin Nerve)**.** Let $\mathcal{C}$ be a 2-category. Using Proposition 2.3.1.9, we can identify vertices of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ with objects of the 2-category $\mathcal{C}$.

`00A7` **Example 2.3.1.14** (Edges of the Duskin Nerve)**.** Let $\mathcal{C}$ be a 2-category. Using Proposition 2.3.1.9, we can identify edges of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ with 1-morphisms $f : X \to Y$ of the 2-category $\mathcal{C}$. Under this identification, the face and degeneracy operators

$$d^1_0, d^1_1 : \mathrm{N}^{\mathrm{D}}_1(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_0(\mathcal{C}) \qquad s^0_0 : \mathrm{N}^{\mathrm{D}}_0(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_1(\mathcal{C})$$

are given by $d^1_0(f : X \to Y) = Y$, $d^1_1(f : X \to Y) = X$, and $s^0_0(X) = \mathrm{id}_X$.

`00A8` **Example 2.3.1.15** (2-Simplices of the Duskin Nerve)**.** Let $\mathcal{C}$ be a 2-category. Using Proposition 2.3.1.9, we see that a 2-simplex σ of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ can be identified with the following data:

- A triple of objects $X, Y, Z \in \mathcal{C}$.
- A triple of 1-morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ in the 2-category $\mathcal{C}$ (corresponding to the faces $d^2_2(\sigma)$, $d^2_0(\sigma)$, and $d^2_1(\sigma)$, respectively).
- A 2-morphism $\mu : g \circ f \Rightarrow h$, which we depict as a diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow \mu"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z.
\end{tikzcd}$$

`00A9` **Example 2.3.1.16** (3-Simplices of the Duskin Nerve)**.** Let $\mathcal{C}$ be a 2-category. Using Proposition 2.3.1.9, we see that a map of simplicial sets $\partial\Delta^3 \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ can be identified with the following data:

- A collection of objects $\{X_i\}_{0 \le i \le 3}$ of the 2-category $\mathcal{C}$.
- A collection of 1-morphisms $\{f_{j,i} : X_i \to X_j\}_{0 \le i < j \le 3}$.
- A quadruple of 2-morphisms

$$\mu_{2,1,0} : f_{2,1} \circ f_{1,0} \Rightarrow f_{2,0} \qquad \mu_{3,2,1} : f_{3,2} \circ f_{2,1} \Rightarrow f_{3,1}$$

$$\mu_{3,1,0} : f_{3,1} \circ f_{1,0} \Rightarrow f_{3,0} \qquad \mu_{3,2,0} : f_{3,2} \circ f_{2,0} \Rightarrow f_{3,0}.$$

This data can be conveniently visualized as a pair of diagrams

$$\begin{tikzcd}
 & X_1 \arrow[r, "f_{2,1}"] \arrow[d, phantom, "\Downarrow \mu_{2,1,0}"] & X_2 \arrow[dr, "f_{3,2}"] \arrow[d, phantom, "\Swarrow \mu_{3,2,0}"] & \\
X_0 \arrow[ur, "f_{1,0}"] \arrow[urr, "f_{2,0}"'] \arrow[rrr, "f_{3,0}"] & {} & {} & X_3
\end{tikzcd}$$

$$\begin{tikzcd}
 & X_1 \arrow[r, "f_{2,1}"] \arrow[drr, "f_{3,1}"'] \arrow[d, phantom, "\mu_{3,1,0} \Searrow"] & X_2 \arrow[dr, "f_{3,2}"] \arrow[d, phantom, "\Swarrow \mu_{3,2,1}"] & \\
X_0 \arrow[ur, "f_{1,0}"] \arrow[rrr, "f_{3,0}"] & {} & {} & X_3,
\end{tikzcd}$$

representing “front” and “back” perspectives of the boundary of a 3-simplex. A 3-simplex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ can be identified with a map $\partial\,\Delta^3 \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ as above which satisfies an additional compatibility condition: namely, the commutativity of the diagram

$$\begin{tikzcd}
 & f_{3,2} \circ (f_{2,1} \circ f_{1,0}) \arrow[rr, Rightarrow, "\alpha_{f_{3,2}, f_{2,1}, f_{1,0}}"] \arrow[dl, Rightarrow, "\mathrm{id}_{f_{3,2}} \circ \mu_{2,1,0}"'] & & (f_{3,2} \circ f_{2,1}) \circ f_{1,0} \arrow[dr, Rightarrow, "\mu_{3,2,1} \circ \mathrm{id}_{f_{1,0}}"] & \\
f_{3,2} \circ f_{2,0} \arrow[drr, Rightarrow, "\mu_{3,2,0}"] & & & & f_{3,1} \circ f_{1,0} \arrow[dll, Rightarrow, "\mu_{3,1,0}"'] \\
 & & f_{3,0} & &
\end{tikzcd}$$

in the ordinary category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X_0, X_3)$.

Example 2.3.1.17 (The Duskin Nerve of Bimod)**.** Let Bimod denote the 2-category of Example 2.2.2.4. Then an n-simplex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Bimod})$ can be identified with a collection of abelian groups $\{A_{j,i}\}_{0 \leq i \leq j \leq n}$ equipped with unit elements $e_i \in A_{i,i}$ and bilinear multiplication maps $\cdot : A_{k,j} \times A_{j,i} \to A_{k,i}$ satisfying the identities $e_j \cdot x = x = x \cdot e_i$ for $x \in A_{j,i}$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for $x \in A_{\ell,k}$, $y = A_{k,j}$, and $z \in A_{j,i}$ (where $0 \leq i \leq j \leq k \leq \ell \leq n$). In this case, the multiplication equips each $A_{i,i}$ with the structure of an associative ring (which 00AA is an object of the 2-category Bimod), each $A_{j,i}$ with the structure of an $A_{j,j}$-$A_{i,i}$ bimodule (which is a 1-morphism in the 2-category Bimod). For $0 \leq i \leq j \leq k \leq n$, the bilinear map $A_{k,j} \times A_{j,i} \rightarrow A_{k,i}$ can be identified with a map of bimodules $\mu_{k,j,i} : A_{k,j} \otimes_{A_{j,j}} A_{j,i} \rightarrow A_{k,i}$, which we can regard as a 2-morphism in the category Bimod.

00FJ **Example 2.3.1.18** (The Classifying Simplicial Set of a Monoidal Category)**.** Let $\mathcal{C}$ be a monoidal category (Definition 2.1.2.10) and let $B\mathcal{C}$ denote the 2-category obtained by delooping $\mathcal{C}$ (Example 2.2.2.5). We will denote the Duskin nerve of $B\mathcal{C}$ by $B_\bullet \mathcal{C}$ and refer to it as the *classifying simplicial set of* $\mathcal{C}$. By virtue of Proposition 2.3.1.9, we can identify n-simplices of the simplicial set $B_\bullet \mathcal{C}$ with pairs

$$(\{C_{j,i}\}_{0 \leq i < j \leq n}, \{\mu_{k,j,i}\}_{0 \leq i < j < k \leq n})$$

where each $C_{j,i}$ is an object of $\mathcal{C}$ and each $\mu_{k,j,i}$ is a morphism from $C_{k,j} \otimes C_{j,i}$ to $C_{k,i}$, satisfying the following coherence condition:

- For $0 \leq i < j < k < \ell \leq n$, the diagram

$$\begin{tikzcd}
C_{\ell,k} \otimes (C_{k,j} \otimes C_{j,i}) \arrow[rr, "\alpha_{C_{\ell,k}, C_{k,j}, C_{j,i}}"] \arrow[d, "\mathrm{id}_{C_{\ell,k}} \otimes \mu_{k,j,i}"'] & & (C_{\ell,k} \otimes C_{k,j}) \otimes C_{j,i} \arrow[d, "\mu_{\ell,k,j} \otimes \mathrm{id}_{C_{j,i}}"] \\
C_{\ell,k} \otimes C_{k,i} \arrow[dr, "\mu_{\ell,k,i}"'] & & C_{\ell,j} \otimes C_{j,i} \arrow[dl, "\mu_{\ell,j,i}"] \\
& C_{\ell,i} &
\end{tikzcd}$$

  is commutative.

00FK **Remark 2.3.1.19.** Let G be a monoid, regarded as a monoidal category having only identity morphisms. Then the classifying simplicial set $B_\bullet G$ of Example 2.3.1.18 agrees (up to canonical isomorphism) with the simplicial set $B_\bullet G$ given by Construction 1.3.2.5.

2.3.2 From 2-Categories to ∞-Categories

00AB We now use Construction 2.3.1.1 to connect the theory of 2-categories (in the sense of Definition 2.2.1.1) to the theory of ∞-categories (in the sense of Definition 1.4.0.1).

00AC **Theorem 2.3.2.1** (Duskin [17])**.** *Let $\mathcal{C}$ be a 2-category. Then $\mathcal{C}$ is a $(2,1)$-category if and only if the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ is an ∞-category.*

Example 2.3.2.2. Let $\mathcal{C}$ be a monoidal category and suppose that every morphism in $\mathcal{C}$ is an isomorphism. Then the classifying simplicial set $B_\bullet \mathcal{C}$ of Example 2.3.1.18 is an ∞-category. 00FL

We will deduce Theorem 2.3.2.1 from a more general statement (Theorem 2.3.2.5), which gives a filling criterion for inner horns in the Duskin nerve $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ for an arbitrary 2-category $\mathcal{C}$. First, we need a bit of terminology.

Definition 2.3.2.3. Let $X_\bullet$ be a simplicial set. We will say that a 2-simplex σ of $X_\bullet$ is *thin* if it satisfies the following condition: 00AD

$(*)$ Let $n \geq 3$, let $0 < i < n$, and let τ denote the 2-simplex of Λ^n_i given by the map

$$[2] \simeq \{i-1, i, i+1\} \subseteq [n].$$

Then any map of simplicial sets $f_0 : \Lambda^n_i \to X_\bullet$ satisfying $f_0(\tau) = \sigma$ can be extended to an n-simplex of $X_\bullet$.

Example 2.3.2.4. Let $X_\bullet$ be a simplicial set. If $X_\bullet$ is an ∞-category (in the sense of Definition 1.4.0.1), then every 2-simplex of $X_\bullet$ is thin. Conversely, if every 2-simplex of $X_\bullet$ is thin, then $X_\bullet$ is an ∞-category if and only if every map of simplicial sets $f_0 : \Lambda^2_1 \to X_\bullet$ can be extended to a 2-simplex of $X_\bullet$. 00AE

We will deduce Theorem 2.3.2.1 from the following result, whose proof will be given in §2.3.3:

Theorem 2.3.2.5. *Let $\mathcal{C}$ be a 2-category and let σ be a 2-simplex of the Duskin nerve $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$, corresponding to a diagram* 00AF

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow \gamma"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

(see Example 2.3.1.15). Then σ is thin if and only if $\gamma : g \circ f \Rightarrow h$ is an isomorphism in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$.

Proof of Theorem 2.3.2.1 from Theorem 2.3.2.5. Let $\mathcal{C}$ be a 2-category. If the Duskin nerve $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ is an ∞-category, then every 2-simplex of $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ is thin (Example 2.3.2.4), so that every 2-morphism in $\mathcal{C}$ is invertible by virtue of Theorem 2.3.2.5. Conversely, if $\mathcal{C}$ is a $(2,1)$-category, then every 2-simplex of $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ is thin (Theorem 2.3.2.5). Consequently, to show that $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ is an ∞-category, it will suffice to show that every map of simplicial sets $u_0 : \Lambda^2_1 \to \mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ can be extended to a 2-simplex of $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$. Note that we can identify u_0 with a composable pair of 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$. To extend this to a 2-simplex of $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$, it suffices to choose a 1-morphism $h : X \to Z$ and a 2-morphism $\gamma : g \circ f \Rightarrow h$. This is always possible: for example, we can take $h = g \circ f$ and γ to be the identity 2-morphism. □

`00AG` **Remark 2.3.2.6.** Let $\mathcal{C}$ be a $(2,1)$-category, so that the Duskin nerve $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ is an ∞-category. Then:

- Objects of the ∞-category $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ can be identified with objects of the 2-category $\mathcal{C}$.
- If X and Y are objects of $\mathcal{C}$, then morphisms from X to Y in the ∞-category $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ can be identified with 1-morphisms from X to Y in the 2-category $\mathcal{C}$.
- If $f, g : X \to Y$ are 1-morphisms in $\mathcal{C}$ having the same domain and codomain, then f and g are homotopic when regarded as morphisms of the ∞-category $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ (Definition 1.4.3.1) if and only if they are isomorphic when viewed as objects of the groupoid $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$. More precisely, vertical composition with the left unit constraint $\lambda_f : \mathrm{id}_Y \circ f \xRightarrow{\sim} f$ induces a bijection

$$\begin{tikzcd}
\{\text{Isomorphisms from } f \text{ to } g \text{ in the groupoid } \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)\} \arrow[d, "\sim"] \\
\{\text{Homotopies from } f \text{ to } g \text{ in the } \infty\text{-category } \mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})\}.
\end{tikzcd}$$

Let us now collect some other consequences of Theorem 2.3.2.5.

`00AH` **Corollary 2.3.2.7.** *Let $\mathcal{C}$ be a 2-category. Then every degenerate 2-simplex of the Duskin nerve $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ is thin.*

Proof. Combine Theorem 2.3.2.5 with the observation that, for every 1-morphism $f : X \to Y$ of $\mathcal{C}$, the left and right unit constraints

$$\lambda_f : \mathrm{id}_Y \circ f \Rightarrow f \qquad \rho_f : f \circ \mathrm{id}_X \Rightarrow f$$

are isomorphisms (in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$). □

`00AJ` **Corollary 2.3.2.8.** *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a strictly unitary lax functor. Then F is a functor if and only if the induced map of simplicial sets $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C}) \Rightarrow \mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{D})$ carries thin 2-simplices of $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{C})$ to thin 2-simplices of $\mathrm{N}_\bullet^{\mathrm{D}}(\mathcal{D})$.*

Proof. Let σ be a 2-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$, corresponding to a diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\gamma"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

in $\mathcal{C}$. Let σ' denote the image of σ in $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$, corresponding to the diagram

$$\begin{tikzcd}
& F(Y) \arrow[dr, "F(g)"] \arrow[d, phantom, "\Downarrow\gamma'"] & \\
F(X) \arrow[ur, "F(f)"] \arrow[rr, "F(h)"'] & & F(Z)
\end{tikzcd}$$

where γ' is given by the (vertical) composition

$$F(g) \circ F(f) \xRightarrow{\mu_{g,f}} F(g \circ f) \xRightarrow{F(\gamma)} F(h).$$

Since σ is thin, the 2-morphism γ is an isomorphism (Theorem 2.3.2.5). It follows that σ' is thin if and only if $\mu_{g,f}$ is an isomorphism. In particular, the strictly unitary lax functor F preserves thin 2-simplices if and only if $\mu_{g,f}$ is an isomorphism for *every* pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $\mathcal{C}$: that is, if and only if F is a functor. □

Warning 2.3.2.9. Let $\mathcal{C}$ be a 2-category. Let us say that a 2-simplex σ of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is *special* if it corresponds to a diagram `00AK`

$$\begin{tikzcd}
& Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\gamma"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z,
\end{tikzcd}$$

where $h = g \circ f$ and $\gamma = \mathrm{id}_{g \circ f}$. Arguing as in the proof of Corollary 2.3.2.8, we see that a strictly unitary lax functor $F : \mathcal{C} \to \mathcal{D}$ is *strict* if and only if it carries special 2-simplices of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ to special 2-simplices of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$. Beware, however, that the special 2-simplices of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ and $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ do not have an *intrinsic* description in terms of the simplicial sets $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ and $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ themselves. In particular, it is possible to have an isomorphism of simplicial sets $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \simeq \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ which does not preserve special 2-simplices (corresponding to an isomorphism of 2-categories which is strictly unitary but not strict).

In general, passage from a 2-category $\mathcal{C}$ to its Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ involves a slight loss of information. From the simplicial set $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$, we can recover the objects of $\mathcal{C}$ (these can be identified with vertices of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$) and the collection of 1-morphisms $f : X \to Y$ from an object X to an object Y (these can be identified with edges of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ having source X and target Y). However, the composition $g \circ f$ of a pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ cannot be recovered from the structure of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ as an abstract simplicial set. The best we can do is to ask for a thin 2-simplex σ of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ satisfying $d^2_0(\sigma) = g$ and $d^2_2(\sigma) = f$. Such a simplex can be viewed as "witnessing" the presence of an isomorphism of the edge $h = d^2_1(\sigma)$ with the composition $g \circ f$. Put another way, the abstract simplicial set $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ contains enough information to reconstruct the composition $g \circ f$ up to (unique) isomorphism, but not enough information to select a canonical representative of its isomorphism class. This can be viewed as a feature, rather than a bug: the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ often admits a more invariant description than the 2-category $\mathcal{C}$ itself (since the information lost by passing from $\mathcal{C}$ to $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ depends on choices that one would prefer not make in the first place; see Remark 2.3.1.7).

If $\mathcal{C}$ is a 2-category which contains non-invertible 2-morphisms, then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is not an ∞-category. However, we can extract an ∞-category by applying the Duskin nerve to the pith $\mathrm{Pith}(\mathcal{C})$ introduced in Construction 2.2.8.9.

00AP **Remark 2.3.2.10.** Let $\mathcal{C}$ be a 2-category. Then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathcal{C}))$ is an ∞-category (Theorem 2.3.2.1). Unwinding the definitions, we see that $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathcal{C}))$ can be identified with the largest simplicial subset $X_{\bullet}$ of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ having the property that each 2-simplex of $X_{\bullet}$ is thin when regarded as a 2-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ (so that an n-simplex $\sigma \in \mathrm{N}^{\mathrm{D}}_{n}(\mathcal{C})$ belongs to $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathcal{C}))$ if and only if, for every map $\Delta^2 \to \Delta^n$, the composition $\Delta^2 \to \Delta^n \xrightarrow{\sigma} \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is thin).

2.3.3 Thin 2-Simplices of a Duskin Nerve

00AQ Let $\mathcal{C}$ be a 2-category and let σ be a 2-simplex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$, corresponding to a diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\gamma"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z.
\end{tikzcd}$$

Our goal is to prove Theorem 2.3.2.5, which asserts that σ is thin (in the sense of Definition 2.3.2.3) if and only if the 2-morphism $\gamma : g \circ f \Rightarrow h$ is invertible. This follows from Propositions 2.3.3.1 and Proposition 2.3.3.2 below.

Proposition 2.3.3.1. *Let $\mathcal{C}$ be a 2-category, let $n \geq 3$, and let $u : \Lambda^n_\ell \to \mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ be a map of simplicial sets for some $0 < \ell < n$. Let σ denote the 2-simplex of $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ obtained by composing u with the map $\Delta^2 \to \Lambda^n_\ell$ given by the map of linearly ordered sets* 00AR

$$[2] \simeq \{\ell - 1, \ell, \ell + 1\} \subseteq [n],$$

corresponding to a diagram

$$\begin{tikzcd}
& X_\ell \arrow[dr] \arrow[d, phantom, "\Downarrow \gamma"] & \\
X_{\ell-1} \arrow[ur] \arrow[rr] & & X_{\ell+1}
\end{tikzcd}$$

in the 2-category $\mathcal{C}$. If γ is invertible, then u extends uniquely to an n-simplex of $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$.

Proof. Using Examples 2.3.1.13 and 2.3.1.14, we see that the restriction of u to the 1-skeleton of Λ^n_ℓ is given by a collection of objects $\{X_i\}_{0 \leq i \leq n}$ of $\mathcal{C}$, together with 1-morphisms $\{f_{ji} : X_i \to X_j\}_{0 \leq i < j \leq n}$. For $n \geq 5$, the horn Λ^n_ℓ contains the 3-skeleton of Δ^n, so the existence and uniqueness of the desired extension is automatic by virtue of Corollary 2.3.1.10 (in particular, we do not need to assume that $0 < \ell < n$ or that γ is invertible). We now treat the case $n = 3$. We will assume that $\ell = 1$ (the case $\ell = 2$ follows by symmetry), so that we can use Example 2.3.1.15 to identify u with a triple of 2-morphisms

$$\mu_{210} : f_{21} \circ f_{10} \Rightarrow f_{20} \quad \mu_{310} : f_{31} \circ f_{10} \Rightarrow f_{30} \quad \mu_{321} : f_{32} \circ f_{21} \Rightarrow f_{31}.$$

Using the description of 3-simplices of $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ supplied by Example 2.3.1.16, we see an extension of u to a 3-simplex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ can be identified with a 2-morphism $\mu_{320} : f_{32} \circ f_{20} \Rightarrow f_{30}$ satisfying the equation

$$\mu_{320}(\mathrm{id}_{f_{32}} \circ \mu_{210}) = \mu_{310}(\mu_{321} \circ \mathrm{id}_{f_{10}}) \alpha_{f_{32}, f_{21}, f_{10}}.$$

Our assumption guarantees that $\gamma = \mu_{210}$ is an isomorphism; it follows that the preceding equation has a unique solution, given by

$$\mu_{320} = \mu_{310}(\mu_{321} \circ \mathrm{id}_{f_{10}}) \alpha_{f_{32}, f_{21}, f_{10}} (\mathrm{id}_{f_{32}} \circ \mu_{210}^{-1}).$$

We now treat the case $n = 4$. For simplicity, we will assume that $\ell = 2$ (the cases $\ell = 1$ and $\ell = 3$ follow by a similar argument). To simplify the notation in what follows, we will denote the composition of a pair of 1-morphisms of $\mathcal{C}$ by hg, rather than $h \circ g$. Note that the horn Λ^n_ℓ contains the 2-skeleton of Δ^n, so the morphism u can be identified with a collection of 2-morphisms $\mu_{kji} : f_{kj} f_{ji} \Rightarrow f_{ki}$. Using Example 2.3.1.16, we note that the extension of u to a 4-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is automatically unique, and exists if and only if the outer cycle commutes in the diagram

$$\begin{tikzcd}
f_{43}(f_{31}f_{10}) \arrow[rrrr, Rightarrow, "\sim"] \arrow[ddddd, Rightarrow, "\mu_{310}"] & & & & (f_{43}f_{31})f_{10} \arrow[ddddd, Rightarrow, "\mu_{431}"] \\
& f_{43}((f_{32}f_{21})f_{10}) \arrow[ul, Rightarrow, "\mu_{321}", "\sim"'] \arrow[rr, Rightarrow, "\sim"] & & (f_{43}(f_{32}f_{21}))f_{10} \arrow[ur, Rightarrow, "\mu_{321}"', "\sim"] \arrow[d, Rightarrow, "\sim"] & \\
& f_{43}(f_{32}(f_{21}f_{10})) \arrow[u, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\mu_{210}"] \arrow[dr, Rightarrow, "\sim"] & & ((f_{43}f_{32})f_{21})f_{10} \arrow[d, Rightarrow, "\mu_{432}"] & \\
& f_{43}(f_{32}f_{20}) \arrow[d, Rightarrow, "\sim"] \arrow[ddl, Rightarrow, "\mu_{320}"'] & (f_{43}f_{32})(f_{21}f_{10}) \arrow[ur, Rightarrow, "\sim"] \arrow[dl, Rightarrow, "\mu_{210}"'] \arrow[dr, Rightarrow, "\mu_{432}"] & (f_{42}f_{21})f_{10} \arrow[d, Rightarrow, "\sim"'] \arrow[ddr, Rightarrow, "\mu_{421}"] & \\
& (f_{43}f_{32})f_{20} \arrow[r, Rightarrow, "\mu_{432}"'] & f_{42}f_{20} \arrow[d, Rightarrow, "\mu_{420}"] & f_{42}(f_{21}f_{10}) \arrow[l, Rightarrow, "\mu_{210}"] & \\
f_{43}f_{30} \arrow[rr, Rightarrow, "\mu_{430}"] & & f_{04} & & f_{41}f_{10}; \arrow[ll, Rightarrow, "\mu_{410}"']
\end{tikzcd}$$

here the unlabeled 2-morphisms are induced by the associativity constraints of $\mathcal{C}$. This follows from a diagram chase, since $\mu_{321} = \gamma$ is an isomorphism and each of the inner cycles of the diagram commutes (the 4-cycles commute by functoriality, the central 5-cycle commutes by the pentagon identity in $\mathcal{C}$, and the remaining 5-cycles commute by virtue of our assumption that u is defined on the 0th, 1st, 3rd, and 4th face of the simplex Δ^4). ☐

00AS **Proposition 2.3.3.2.** *Let $\mathcal{C}$ be a 2-category and let σ be a 2-simplex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$, corresponding to a diagram*

$$\begin{tikzcd}
& Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\gamma"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z.
\end{tikzcd}$$

in the 2-category $\mathcal{C}$. Assume that the following condition is satisfied:

(∗) *Let $n \in \{3, 4\}$ and let $u : \Lambda^n_1 \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ be a map of simplicial sets such that $u|_{\Delta^2} = \sigma$; here we identify Δ^2 with a simplicial subset of $\Lambda^n_1 \subseteq \Delta^n$ via the inclusion map $[2] \hookrightarrow [n]$. Then u extends to an n-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$.*

Then γ is invertible.

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is strictly unitary (Proposition 2.2.7.7). Applying $(*)$ in the case $n = 3$, we can extend σ to a 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ which is represented by the pair of diagrams

$$\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[d, phantom, "\Downarrow \gamma"] & Z \arrow[drr, "\mathrm{id}_Z"] \arrow[d, phantom, "\Downarrow \delta"] & & \\
X \arrow[ur, "f"] \arrow[urr, "h"'] \arrow[rrrr, "g \circ f"'] & {} & {} & & Z
\end{tikzcd}$$

$$\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[drrr, "g"'] \arrow[d, phantom, "\Downarrow \mathrm{id}_{g \circ f}"] & Z \arrow[drr, "\mathrm{id}_Z"] \arrow[d, phantom, "\Downarrow \mathrm{id}_g"] & & \\
X \arrow[ur, "f"] \arrow[rrrr, "g \circ f"'] & {} & {} & & Z.
\end{tikzcd}$$

It follows that γ admits a left inverse, given by the vertical composition $\delta : h \Rightarrow g \circ f$. To show that this composition is also a right inverse, we apply $(*)$ in the case $n = 4$ to construct a 4-simplex τ of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ whose two-dimensional faces correspond to the 2-morphisms

$$\mu_{2,1,0} = \mu_{4,1,0} = \gamma \qquad \mu_{3,1,0} = \mathrm{id}_{g\circ f} \quad \mu_{3,2,0} = \delta \quad \mu_{4,2,0} = \mathrm{id}_h$$

$$\mu_{4,3,0} = \gamma \qquad \mu_{3,2,1} = \mu_{4,2,1} = \mu_{4,3,1} = \mathrm{id}_g \qquad \mu_{4,3,2} = \mathrm{id}_{\mathrm{id}_Z}.$$

The 3-simplex $d^4_1(\tau)$ then witnesses the identity

$$\mu_{4,2,0}(\mu_{4,3,2} \circ \mathrm{id}_h) = \mu_{4,3,0}(\mathrm{id}_{\mathrm{id}_Z} \circ \mu_{3,2,0}),$$

which shows that δ is also a right inverse to γ. □

2.3.4 Recovering a 2-Category from its Duskin Nerve

In §1.3.3, we proved that the nerve functor 00AT

$$\mathrm{N}_{\bullet} : \mathrm{Cat} \to \mathrm{Set}_{\Delta}$$

is fully faithful. This result generalizes to the setting of 2-categories:

`00AU` **Theorem 2.3.4.1** (Duskin [17]). *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then passage to the Duskin nerve induces a bijection*

$$\{\text{Strictly unitary lax functors } \mathcal{C} \to \mathcal{D}\} \to \{\text{Morphisms of simplicial sets } \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})\}.$$

In other words, the Duskin nerve functor $\mathrm{N}^{\mathrm{D}}_{\bullet} : 2\mathrm{Cat}_{\mathrm{ULax}} \to \mathrm{Set}_{\Delta}$ is fully faithful.

`00AV` **Remark 2.3.4.2.** Combining Theorem 2.3.4.1, Theorem 2.3.2.1, and Remark 2.2.8.8, we see that the construction $\mathcal{C} \mapsto \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ determines a fully faithful embedding from the ordinary category of $(2,1)$-categories (where morphisms are strictly unitary functors in the sense of Definition 2.2.4.17) to the ordinary category of ∞-categories (where morphisms are functors in the sense of Definition 1.5.0.1).

`00AW` **Remark 2.3.4.3.** In [17], Duskin proves a stronger version of Theorem 2.3.4.1, which also identifies the essential image of the functor $\mathrm{N}^{\mathrm{D}}_{\bullet} : 2\mathrm{Cat}_{\mathrm{ULax}} \to \mathrm{Set}_{\Delta}$.

`00FM` **Example 2.3.4.4.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. We say that a lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is *strictly unitary* if the unit $\epsilon : \mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$ is an identity morphism of $\mathcal{D}$. It follows from Theorem 2.3.4.1 and Remark 2.2.4.9 that the formation of classifying simplicial sets induces a bijection

$$\begin{tikzcd}
\{\text{Strictly unitary lax monoidal functors } F : \mathcal{C} \to \mathcal{D}\} \arrow[d, "\sim"] \\
\{\text{Maps of simplicial sets } B_{\bullet}\mathcal{C} \to B_{\bullet}\mathcal{D}\}.
\end{tikzcd}$$

`00AX` **Corollary 2.3.4.5.** *Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then passage to the Duskin nerve induces a bijection*

$$\begin{tikzcd}
\{\text{Strictly unitary functors } \mathcal{C} \to \mathcal{D}\} \arrow[d] \\
\{\text{Maps } \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D}) \text{ preserving thin 2-simplices}\}.
\end{tikzcd}$$

Proof. Combine Theorem 2.3.4.1 with Corollary 2.3.2.8. □

`02BG` **Corollary 2.3.4.6.** *Let $\mathcal{C}$ be a 2-category, let $\mathrm{h}\mathcal{C}$ be its coarse homotopy category, and let $F : \mathcal{C} \to \mathrm{h}\mathcal{C}$ be the functor of Proposition 2.2.8.3. Then the induced map of simplicial sets*

$$\mathrm{N}^{\mathrm{D}}_{\bullet}(F) : \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{h}\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$$

exhibits $\mathrm{h}\mathcal{C}$ as the homotopy category of the simplicial set $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$, in the sense of Definition 1.3.6.1.

Proof. Let $\mathcal{D}$ be a category, which we regard as a 2-category having only identity morphisms. We wish to show that every morphism of simplicial sets $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{D})$ factors uniquely through the morphism $\mathrm{N}^{\mathrm{D}}_{\bullet}(F)$. By virtue of Theorem 2.3.4.1, this is equivalent to the assertion that every strictly unitary lax functor $G : \mathcal{C} \to \mathcal{D}$ factors uniquely through F, which follows from Proposition 2.2.8.3. □

Proof of Theorem 2.3.4.1. By virtue of Proposition 2.2.7.7, we may assume without loss of generality that the 2-categories $\mathcal{C}$ and $\mathcal{D}$ are strictly unitary (this assumption will simplify some of the notation in what follows). Let $U : \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ be a map of simplicial sets. Then:

- Each object X of $\mathcal{C}$ can be identified with a vertex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ (Example 2.3.1.13), whose image under U is a vertex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$. This vertex can be identified with an object of $\mathcal{D}$, which we denote by $U_0(X)$.

- Each 1-morphism $f : X \to Y$ of $\mathcal{C}$ can be identified with an edge of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ (Example 2.3.1.14), whose image under U is an edge of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$. This edge can be identified with a 1-morphism of $\mathcal{D}$, which we will denote by $U_1(f) : U_0(X) \to U_0(Y)$.

- Let $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ be 1-morphisms of $\mathcal{C}$, and let $\gamma : g \circ f \Rightarrow h$ be a 2-morphism of $\mathcal{C}$. The 2-morphism γ determines a 2-simplex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ (Example 2.3.1.15). The image of this 2-simplex under U is a 2-simplex of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$, which we can identify with a 2-morphism $U_2(\gamma) : U_1(g) \circ U_1(f) \Rightarrow U_1(h)$ in $\mathcal{D}$. Beware that this notation is slightly abusive: the 2-morphism $U_2(\gamma)$ is *a priori* dependent not only on γ, but also on the factorization of the source of γ as a composition $g \circ f$.

Let $F : \mathcal{C} \to \mathcal{D}$ be a strictly unitary lax functor. Unwinding the definitions, we see that the induced map of simplicial sets $\mathrm{N}^{\mathrm{D}}_{\bullet}(F) : \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ coincides with U if and only if the following conditions are satisfied:

(0) For every object $X \in \mathcal{C}$, we have $F(X) = U_0(X)$ (as objects of $\mathcal{D}$).

(1) For every 1-morphism $f : X \to Y$ in $\mathcal{C}$, we have $F(f) = U_1(f)$ (as 1-morphisms from $F(X) = U_0(X)$ to $F(Y) = U_0(Y)$ in $\mathcal{D}$).

(2) For every triple of 1-morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ in $\mathcal{C}$ and every 2-morphism $\gamma : g \circ f \Rightarrow h$, the 2-morphism $U_2(\gamma) : U_1(g) \circ U_1(f) \Rightarrow U_1(h)$ of $\mathcal{D}$ is given by the (vertical) composition

$$U_1(g) \circ U_1(f) = F(g) \circ F(f) \xRightarrow{\mu_{g,f}} F(g \circ f) \xRightarrow{F(\gamma)} F(h) = U_1(h),$$

Let us note two special cases of condition (2). Taking $h = g \circ f$ and $\gamma : g \circ f \Rightarrow h$ to be the identity 2-morphism, we obtain the following:

(2_0) For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of $\mathcal{C}$, the composition constraint $\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ coincides with the 2-morphism $U_2(\mathrm{id}_{g \circ f})$.

Taking g to be the identity morphism $\mathrm{id}_Y : Y \to Y$ and invoking our assumption that $\mathcal{C}$ and $\mathcal{D}$ are strictly unitary, we also obtain:

(2_1) For every pair of 1-morphisms $f, h : X \to Y$ in $\mathcal{C}$ and every 2-morphism $\gamma : f \Rightarrow h$, we have

$$U_2(\gamma) = F(\gamma)\mu_{\mathrm{id}_Y, f} = F(\gamma)$$

(here the second identity follows from Remark 2.2.7.5, since the 2-categories $\mathcal{C}$ and $\mathcal{D}$ are strictly unitary).

We wish to show that there is a unique strictly unitary lax functor $F : \mathcal{C} \to \mathcal{D}$ satisfying conditions (0), (1), and (2). The uniqueness is clear: by virtue of the analysis above, the functor F must be given on objects, 1-morphisms, and 2-morphisms of $\mathcal{C}$ by the formulae

$$F(X) = U_0(X) \qquad F(f) = U_1(f) \qquad F(\gamma) = U_2(\gamma)$$

(where, in the third formula, we identify the domain of each 2-morphism $\gamma : f \Rightarrow h$ in $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ with the composition $\mathrm{id}_Y \circ f$), and the composition constraint $\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ must be given by $\mu_{g,f} = U_2(\mathrm{id}_{g \circ f})$. To complete the proof, it will suffice to show that these formulae supply a well-defined lax functor $F : \mathcal{C} \Rightarrow \mathcal{D}$, and that F satisfies condition (2) above (note that F satisfies conditions (0) and (1) by construction).

We first show that F satisfies condition (2). Suppose we are given a triple of 1-morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$, together with a 2-morphism $\gamma : g \circ f \Rightarrow h$ in the 2-category $\mathcal{C}$. Consider the map $\partial \Delta^3 \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ represented by the pair of diagrams

$$\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[d, phantom, "\Downarrow \mathrm{id}_{g\circ f}"] & Z \arrow[dr, "\mathrm{id}_Z"] \arrow[d, phantom, "\Swarrow \gamma"] & \\
X \arrow[ur, "f"] \arrow[urr, "g\circ f"'] \arrow[rrr, "h"'] & {} & {} & Z
\end{tikzcd}$$

$$\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[drr, "g"'] \arrow[d, phantom, "\gamma \Searrow"] & Z \arrow[dr, "\mathrm{id}_Z"] \arrow[d, phantom, "\Swarrow \mathrm{id}_g"] & \\
X \arrow[ur, "f"] \arrow[rrr, "h"'] & {} & {} & Z
\end{tikzcd}$$

(see Example 2.3.1.16). Using the identity $\alpha_{\mathrm{id}_Z, g, f} = \mathrm{id}_{g \circ f}$ (Remark 2.2.7.3), we see that these diagrams satisfy the compatibility condition of Example 2.3.1.16, and can therefore be regarded as a 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$. Applying the map of simplicial sets U, we deduce that the diagrams

$$\begin{tikzcd}
 & F(Y) \arrow[r, "F(g)"] \arrow[d, phantom, "\Downarrow \mu_{g,f}"] & F(Z) \arrow[dr, "\mathrm{id}_{F(Z)}"] \arrow[d, phantom, "\Downarrow F(\gamma)"] & \\
F(X) \arrow[ur, "F(f)"] \arrow[urr, "F(g \circ f)"'] \arrow[rrr, "F(h)"'] & & & F(Z)
\end{tikzcd}$$

$$\begin{tikzcd}
 & F(Y) \arrow[r, "F(g)"] \arrow[drr, "F(g)"'] \arrow[d, phantom, "\Downarrow U_2(\gamma)"] & F(Z) \arrow[dr, "\mathrm{id}_{F(Z)}"] \arrow[d, phantom, "\Downarrow \mathrm{id}_{F(g)}"] & \\
F(X) \arrow[ur, "F(f)"] \arrow[rrr, "F(h)"'] & & & F(Z)
\end{tikzcd}$$

determine a 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$: that is, we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{id}_{F(Z)} \circ (F(g) \circ F(f)) \arrow[rr, Rightarrow, "\alpha_{\mathrm{id}_{F(Z)}, F(g), F(f)}"] \arrow[dl, Rightarrow, "\mu_{g,f}"'] & & (\mathrm{id}_{F(Z)} \circ F(g)) \circ F(f) \arrow[dr, Rightarrow, "\mathrm{id}"] & \\
\mathrm{id}_Z \circ F(g \circ f) \arrow[drr, Rightarrow, "F(\gamma)"] & & & & F(g) \circ F(f) \arrow[dll, Rightarrow, "U_2(\gamma)"'] \\
 & & F(h). & &
\end{tikzcd}$$

By virtue of Remark 2.2.7.3, we see that this is equivalent to the identity $U_2(\gamma) = F(\gamma)\mu_{g,f}$ asserted by (2).

Note that from condition (2), we can deduce that F satisfies the dual of condition (2_1): that is, for every 2-morphism $\gamma : g \Rightarrow h$ in $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$, we have $F(\gamma) = U_2(\gamma)$, where the right hand side is computed by regarding γ as a 2-morphism with domain $g \circ \mathrm{id}_X$. It follows that the construction of F from U is invariant under the operation of replacing $\mathcal{C}$ and $\mathcal{D}$ by the opposite 2-categories $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{D}^{\mathrm{op}}$ (this will be useful in what follows, since it reduces the number of identities that we need to check).

We now show that, for every pair of objects $X, Y \in \mathcal{C}$, the construction of F on 1-morphisms and 2-morphisms determines a functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Y))$. For this, we must establish the following:

- For each 1-morphism $f : X \to Y$ in $\mathcal{C}$, we have $F(\mathrm{id}_f) = \mathrm{id}_{F(f)}$ (as 2-morphisms from $F(f)$ to itself in $\mathcal{D}$). By definition, this is equivalent to the identity $U_2(\mathrm{id}_f) = \mathrm{id}_{F(f)}$, which follows from the compatibility of the map $U : \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ with the degeneracy operators
$$s^1_1 : \mathrm{N}^{\mathrm{D}}_1(\mathcal{C}) \Rightarrow \mathrm{N}^{\mathrm{D}}_2(\mathcal{C}) \qquad s^1_1 : \mathrm{N}^{\mathrm{D}}_1(\mathcal{D}) \Rightarrow \mathrm{N}^{\mathrm{D}}_2(\mathcal{D}).$$

- For every triple of 1-morphisms $f, g, h : X \Rightarrow Y$ in $\mathcal{C}$ and every pair of 2-morphisms $\gamma : f \Rightarrow g$, $\delta : g \Rightarrow h$, we have $F(\delta\gamma) = F(\delta)F(\gamma)$. To prove this, consider the map $\partial\Delta^3 \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ represented by the pair of diagrams
$$\begin{tikzcd}
& Y \arrow[r, "\mathrm{id}_Y"] \arrow[d, phantom, "\Downarrow \gamma"] & Y \arrow[dr, "\mathrm{id}_Y"] \arrow[d, phantom, "\Downarrow \delta"] & \\
X \arrow[ur, "f"] \arrow[urr, "g"'] \arrow[rrr, "h"] & & & Y
\end{tikzcd}$$
$$\begin{tikzcd}
& Y \arrow[r, "\mathrm{id}_Y"] \arrow[drr, "\mathrm{id}_Y"'] \arrow[d, phantom, "\Downarrow \delta\gamma"] & Y \arrow[dr, "\mathrm{id}_Y"] \arrow[d, phantom, "\Downarrow \mathrm{id}_{\mathrm{id}_Y}"] & \\
X \arrow[ur, "f"] \arrow[rrr, "h"] & & & Y
\end{tikzcd}$$
(see Example 2.3.1.16). It follows from Remark 2.2.7.3 that the associativity constraint $\alpha_{\mathrm{id}_Y, \mathrm{id}_Y, f}$ is the identity, so that the diagrams above satisfy the compatibility condition of Example 2.3.1.16 and therefore determine a 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$. Applying the map of simplicial sets U, we deduce that there exists a 3-simplex of the Duskin nerve $\mathrm{N}^{\mathcal{D}}_{\bullet}$ whose boundary is given by the diagrams
$$\begin{tikzcd}
& F(Y) \arrow[r, "\mathrm{id}_{F(Y)}"] \arrow[d, phantom, "\Downarrow F(\gamma)"] & F(Y) \arrow[dr, "\mathrm{id}_{F(Y)}"] \arrow[d, phantom, "\Downarrow F(\delta)"] & \\
F(X) \arrow[ur, "F(f)"] \arrow[urr, "F(G)"'] \arrow[rrr, "F(h)"] & & & F(Y)
\end{tikzcd}$$
$$\begin{tikzcd}
& F(Y) \arrow[r, "\mathrm{id}_{F(Y)}"] \arrow[drr, "\mathrm{id}_{F(Y)}"'] \arrow[d, phantom, "\Downarrow F(\delta\gamma)"] & F(Y) \arrow[dr, "\mathrm{id}_{F(Y)}"] \arrow[d, phantom, "\Downarrow \mathrm{id}_{\mathrm{id}_{F(Y)}}"] & \\
F(X) \arrow[ur, "F(f)"] \arrow[rrr, "F(h)"] & & & F(Y).
\end{tikzcd}$$

> Using the criterion of Example 2.3.1.16, we see that this is equivalent to the identity $F(\delta\gamma) = F(\delta)F(\gamma)$.

We now show that, for every triple of objects $X, Y, Z \in \mathcal{C}$, the composition constraints $\mu_{g,f} : F(g) \circ F(f) \Rightarrow F(g \circ f)$ depends functorially on $f \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ and $g \in \underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z)$. We will argue that for fixed f, the construction $g \mapsto \mu_{g,f}$ is functorial; functoriality in g will then follow by symmetry. Suppose we are given a 2-morphism $\gamma : g \Rightarrow h$ in $\mathcal{C}$; we wish to show that the diagram τ :

$$\begin{tikzcd}
F(g) \circ F(f) \arrow[r, Rightarrow, "F(\gamma)\circ \mathrm{id}_{F(f)}"] \arrow[d, Rightarrow, "\mu_{g,f}"] & F(h) \circ F(f) \arrow[d, Rightarrow, "\mu_{h,f}"] \\
F(g \circ f) \arrow[r, Rightarrow, "F(\gamma \circ \mathrm{id}_f)"] & F(h \circ f)
\end{tikzcd}$$

commutes in the category $\underline{\mathrm{Hom}}_{\mathcal{D}}(F(X), F(Z))$. To prove this, we consider the map $\partial\,\Delta^3 \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ represented by the pair of diagrams

$$\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[d, phantom, "\Downarrow \mathrm{id}_{g\circ f}"] & Z \arrow[dr, "\mathrm{id}_Z"] \arrow[d, phantom, "\Swarrow \gamma\circ\mathrm{id}_f"] & \\
X \arrow[ur, "f"] \arrow[urr, "g\circ f"'] \arrow[rrr, "h\circ f"] & {} & {} & Z
\end{tikzcd}$$

$$\begin{tikzcd}
 & Y \arrow[r, "g"] \arrow[drr, "h"'] \arrow[d, phantom, "\Searrow \mathrm{id}_{h\circ f}"] & Z \arrow[dr, "\mathrm{id}_Z"] \arrow[d, phantom, "\Swarrow \gamma"] & \\
X \arrow[ur, "f"] \arrow[rrr, "h\circ f"] & {} & {} & Z.
\end{tikzcd}$$

Using the identity $\alpha_{\mathrm{id}_Z, g, f} = \mathrm{id}_{g\circ f}$ supplied by Remark 2.2.7.3, we see that this diagram defines a 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$. Applying the map of simplicial sets U, we deduce that there is a 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ whose boundary is represented by the pair of diagrams

$$\begin{tikzcd}
 & F(Y) \arrow[r, "F(g)"] \arrow[d, phantom, "\Downarrow \mu_{g,f}"] & F(Z) \arrow[dr, "\mathrm{id}_{F(Z)}"] \arrow[d, phantom, "\Swarrow F(\gamma\circ\mathrm{id}_f)"] & \\
F(X) \arrow[ur, "F(f)"] \arrow[urr, "F(g\circ f)"'] \arrow[rrr, "F(h\circ f)"] & {} & {} & F(Z)
\end{tikzcd}$$

$$\begin{tikzcd}
 & F(Y) \arrow[r, "F(g)"] \arrow[rrd, "F(h)"'] \arrow[d, phantom, "\overset{\mu_{h,f}}{\Rightarrow}"] & F(Z) \arrow[rd, "\mathrm{id}_{F(Z)}"] \arrow[d, phantom, "\overset{F(\gamma)}{\Leftarrow}"] & \\
F(X) \arrow[ru, "F(f)"] \arrow[rrr, "F(h\circ f)"'] & {} & {} & F(Z).
\end{tikzcd}$$

This translates to the commutativity of the diagram

$$\begin{tikzcd}
 & \mathrm{id}_{F(Z)} \circ (F(g) \circ F(f)) \arrow[r, Rightarrow, "\alpha_{\mathrm{id}_{F(Z)}, F(g), F(f)}"] \arrow[ld, Rightarrow, "\mu_{g,f}"'] & (\mathrm{id}_{F(Z)} \circ F(g)) \circ F(f) \arrow[rd, Rightarrow, "F(\gamma)"] & \\
\mathrm{id}_{F(Z)} \circ F(g \circ f) \arrow[rrd, Rightarrow, "F(\gamma \circ \mathrm{id}_f)"] & & & F(h) \circ F(f) \arrow[lld, Rightarrow, "\mu_{h,f}"'] \\
 & & F(h \circ f), &
\end{tikzcd}$$

which (again by virtue of Remark 2.2.7.3) is equivalent to the commutativity of the diagram τ.

To complete the proof, it will suffice to show that F and μ satisfy conditions (a), (b), and (c) of Definition 2.2.4.5. Condition (a) is immediate from the construction, and (b) follows by symmetry. To verify (c), suppose we are given a triple of composable 1-morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in the 2-category $\mathcal{C}$. Consider the 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ represented by the pair of diagrams

$$\begin{tikzcd}
 & X \arrow[r, "g"] \arrow[d, phantom, "\overset{\mathrm{id}_{g\circ f}}{\Rightarrow}"] & Z \arrow[rd, "h"] \arrow[d, phantom, "\overset{\alpha_{h,g,f}}{\Leftarrow}"] & \\
W \arrow[ru, "f"] \arrow[rru, "g\circ f"'] \arrow[rrr, "(h\circ g)\circ f"'] & {} & {} & Z
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \arrow[r, "g"] \arrow[rrd, "h\circ g"'] \arrow[d, phantom, "\overset{\mathrm{id}_{(h\circ g)\circ f}}{\Rightarrow}"] & Y \arrow[rd, "h"] \arrow[d, phantom, "\overset{\mathrm{id}_{h\circ g}}{\Leftarrow}"] & \\
W \arrow[ru, "f"] \arrow[rrr, "(h\circ g)\circ f"'] & {} & {} & Z.
\end{tikzcd}$$

Applying U, we obtain a 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ represented by the pair of diagrams

$$\begin{tikzcd}
& F(X) \arrow[r, "F(g)"] \arrow[d, phantom, "\Downarrow \mu_{g,f}"] & F(Z) \arrow[dr, "F(h)"] \arrow[d, phantom, "\Downarrow U_2(\alpha_{h,g,f})"] & \\
F(W) \arrow[ur, "F(f)"] \arrow[urr, "F(g\circ f)"'] \arrow[rrr, "F((h\circ g)\circ f)"] & & & F(Z)
\end{tikzcd}$$

$$\begin{tikzcd}
& F(X) \arrow[r, "F(g)"] \arrow[drr, "F(h\circ g)"'] \arrow[d, phantom, "\Downarrow \mu_{h\circ g,f}"] & F(Y) \arrow[dr, "F(h)"] \arrow[d, phantom, "\Downarrow \mu_{h,g}"] & \\
F(W) \arrow[ur, "F(f)"] \arrow[rrr, "F((h\circ g)\circ f)"] & & & F(Z),
\end{tikzcd}$$

which is equivalent to the commutativity of the pentagon appearing in the diagram

$$\begin{tikzcd}
F(h)\circ(F(g)\circ F(f)) \arrow[r, Rightarrow, "\alpha_{F(h),F(g),F(f)}"] \arrow[d, Rightarrow, "\mathrm{id}_{F(h)}\circ\mu_{g,f}"] & (F(h)\circ F(g))\circ F(f) \arrow[d, Rightarrow, "\mu_{h,g}\circ\mathrm{id}_{F(f)}"] \\
F(h)\circ F(g\circ f) \arrow[d, Rightarrow, "\mu_{h,g\circ f}"] \arrow[dr, Rightarrow, "U_2(\alpha_{h,g,f})"] & F(h\circ g)\circ F(f) \arrow[d, Rightarrow, "\mu_{h\circ g,f}"] \\
F(h\circ(g\circ f)) \arrow[r, Rightarrow, "F(\alpha_{h,g,f})"] & F((h\circ g)\circ f)
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{D}}(F(W), F(Z))$. Since the triangle on the lower left commutes by virtue of (2), it follows that the outer cycle of the diagram commutes, as desired. □

2.3.5 The Duskin Nerve of a Strict 2-Category

Let $\mathcal{C}$ be a strict 2-category (Definition 2.2.0.1). Then we can regard $\mathcal{C}$ as a 2-category (in which the associativity and unit constraints are identity morphisms), and form the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ by applying Construction 2.3.1.1. However, the Duskin nerve of a strict 2-category admits a more direct description, which can be formulated entirely in terms of strict 2-categories (and strict functors between them). The proof is based on a construction which will play an important role in §2.4.3. 00B9

00BA **Construction 2.3.5.1** (The Path 2-Category of a Partially Ordered Set)**.** Let $(Q, \leq)$ be a partially ordered set. We define a strict 2-category $\mathrm{Path}_{(2)}[Q]$ as follows:

- The objects of $\mathrm{Path}_{(2)}[Q]$ are the elements of Q.
- Given elements $x, y \in Q$, we let $\underline{\mathrm{Hom}}_{\mathrm{Path}[Q]_{(2)}}(x, y)$ denote the partially ordered set of all finite linearly ordered subsets
$$S = \{x = x_0 < x_1 < \cdots < x_n = y\} \subseteq Q$$
having least element x and greatest element y, ordered by *reverse* inclusion. We regard the partially ordered set $\underline{\mathrm{Hom}}_{\mathrm{Path}[Q]_{(2)}}(x, y)$ as a category, having a unique morphism $S \Rightarrow T$ when T is contained in S.
- For every element $x \in Q$, the identity 1-morphism $\mathrm{id}_x \in \underline{\mathrm{Hom}}_{\mathrm{Path}[Q]}(x, x)$ is given by the singleton $\{x\}$ (regarded as a linearly ordered subset of Q, having greatest and least element x).
- For every triple of objects $x, y, z \in Q$, the composition functor
$$\circ : \underline{\mathrm{Hom}}_{\mathrm{Path}_{(2)}[Q]}(y, z) \times \underline{\mathrm{Hom}}_{\mathrm{Path}_{(2)}[Q]}(x, y) \to \underline{\mathrm{Hom}}_{\mathrm{Path}_{(2)}[Q]}(x, z)$$
is given on objects by the construction $(S, T) \mapsto S \cup T$.

‘ We will refer to $\mathrm{Path}_{(2)}[Q]$ as the *path 2-category of Q*.

00JL **Remark 2.3.5.2** (Comparison with the Path Category)**.** Let $(Q, \leq)$ be a partially ordered set. We let $\mathrm{Path}[Q]$ denote the underlying category of the strict 2-category $\mathrm{Path}_{(2)}[Q]$. The category $\mathrm{Path}[Q]$ can be described concretely as follows:

- The objects of $\mathrm{Path}[Q]$ are the elements of Q.
- If x and y are elements of Q, then a morphism from x to y in $\mathrm{Path}[Q]$ is given by a finite linearly ordered subset
$$S = \{x = x_0 < x_1 < x_2 < \cdots < x_n = y\} \subseteq Q$$
having least element x and largest element y.

Note that $\mathrm{Path}[Q]$ can also be realized as the path category of a directed graph $\mathrm{Gr}(Q)$ (as defined in Construction 1.3.7.1). Here $\mathrm{Gr}(Q)$ denotes the underlying directed graph of the category Q, given concretely by

$$\mathrm{Vert}(\mathrm{Gr}(Q)) = Q \qquad \mathrm{Edge}(\mathrm{Gr}(Q)) = \{(x, y) \in Q : x < y\}$$

where we regard each ordered pair $(x, y) \in \mathrm{Edge}(\mathrm{Gr}(Q))$ as an edge with source $s(x, y) = x$ and target $t(x, y) = y$.

Remark 2.3.5.3. Let $(Q, \leq)$ be a partially ordered set, which we regard as a category (having a unique morphism from x to y when $x \leq y$). Note that, for every pair of elements $x, y \in Q$, the category $\underline{\mathrm{Hom}}_{\mathrm{Path}[Q]_{(2)}}(x, y)$ is empty unless $x \leq y$. It follows that there is a unique (strict) functor $\mathrm{Path}[Q]_{(2)} \to Q$ which is the identity on objects. 00BB

Construction 2.3.5.4. Let $(Q, \leq)$ be a partially ordered set, which we regard as a category having a unique morphism $e_{y,x}$ for every pair of elements $x, y \in Q$ with $x \leq y$. We define a strictly unitary lax functor $T_Q : Q \to \mathrm{Path}_{(2)}[Q]$ as follows: 00BC

- On objects, the lax functor T_Q is given by $T_Q(x) = x$.
- On 1-morphisms, the lax functor T_Q is given by $T_Q(e_{y,x}) = \{y, x\} \in \underline{\mathrm{Hom}}_{\mathrm{Path}_{(2)}[Q]}(x, y)$ whenever $x \leq y$ in Q.
- For every triple of elements $x, y, z \in Q$ satisfying $x \leq y \leq z$, the composition constraint $\mu_{z,y,x} : T_Q(e_{z,y}) \circ T_Q(e_{y,z}) \Rightarrow T_Q(e_{z,x})$ is the 2-morphism of $\mathrm{Path}_{(2)}[Q]$ corresponding to the inclusion of linearly ordered sets

$$T_Q(e_{z,x}) = \{z, x\} \subseteq \{z, y, x\} = \{z, y\} \cup \{y, x\} = T_Q(e_{z,y}) \circ T_Q(e_{y,x}).$$

Remark 2.3.5.5. Let $(Q, \leq)$ be a partially ordered set, let $T_Q : Q \to \mathrm{Path}_{(2)}[Q]$ be the lax functor of Construction 2.3.5.4, and let $F : \mathrm{Path}_{(2)}[Q] \to Q$ be the functor of Remark 2.3.5.3 (so that F is the identity on objects). Then the composition 00BD

$$Q \xrightarrow{T_Q} \mathrm{Path}_{(2)}[Q] \xrightarrow{F} Q$$

is the identity functor from Q to itself. Beware that the composition

$$\mathrm{Path}_{(2)}[Q] \xrightarrow{F} Q \xrightarrow{T_Q} \mathrm{Path}_{(2)}[Q]$$

is not the identity (as a lax functor from $\mathrm{Path}_{(2)}[Q]$ to itself). This composition carries each object of $\mathrm{Path}_{(2)}[Q]$ to itself, but is given on 1-morphism by the construction $\{x_0 < x_1 < \cdots < x_n\} \mapsto \{x_0 < x_n\}$.

The 2-category $\mathrm{Path}_{(2)}[Q]$ of Construction 2.3.5.1 is characterized by the following universal property:

Theorem 2.3.5.6. *Let Q be a partially ordered set and let $T_Q : Q \to \mathrm{Path}_{(2)}[Q]$ be the lax functor of Construction 2.3.5.4. For every strict 2-category $\mathcal{C}$, composition with T_Q induces a bijection* 00BE

$$\{\text{Strict functors } F^+ : \mathrm{Path}_{(2)}[Q] \to \mathcal{C}\} \to \{\text{Strictly unitary lax functors } F : Q \to \mathcal{C}\}.$$

Before giving the proof of Theorem 2.3.5.6, let us note one of its consequences. The construction $[n] \mapsto \mathrm{Path}_{(2)}[n]$ determines a functor from the simplex category $\boldsymbol{\Delta}$ of Definition 1.1.0.2 to the (ordinary) category $2\mathrm{Cat}_{\mathrm{Str}}$ of strict 2-categories (Definition 2.2.5.5). We will view this functor as a cosimplicial object of $2\mathrm{Cat}_{\mathrm{Str}}$ which we denote by $\mathrm{Path}_{(2)}[\bullet]$. Applying the construction of Variant 1.2.2.8, we obtain a functor $\mathrm{Sing}_{\bullet}^{\mathrm{Path}_{(2)}[\bullet]} : 2\mathrm{Cat}_{\mathrm{Str}} \to \mathrm{Set}_{\Delta}$, which carries each strict 2-category $\mathcal{C}$ to the simplicial set $[n] \mapsto \mathrm{Hom}_{2\mathrm{Cat}_{\mathrm{Str}}}(\mathrm{Path}_{(2)}[\bullet], \mathcal{C})$. Using Theorem 2.3.5.6, we can identify this construction with the Duskin nerve functor

$$2\mathrm{Cat}_{\mathrm{Str}} \hookrightarrow 2\mathrm{Cat}_{\mathrm{ULax}} \xrightarrow{\mathrm{N}_{\bullet}^{\mathrm{D}}} \mathrm{Set}_{\Delta}\,.$$

In particular, we have the following:

00BF **Corollary 2.3.5.7.** *For every strict 2-category $\mathcal{C}$, there is a canonical isomorphism of simplicial sets*

$$\mathrm{Sing}_{\bullet}^{\mathrm{Path}_{(2)}[\bullet]}(\mathcal{C}) \simeq \mathrm{N}_{\bullet}^{\mathrm{D}}(\mathcal{C}),$$

given on n-simplices by composition with the lax functor $T_{[n]} : [n] \to \mathrm{Path}[n]$ of Construction 2.3.5.4. In other words, the Duskin nerve $\mathrm{N}_{\bullet}^{\mathrm{D}}(\mathcal{C})$ is given by

$$\mathrm{N}_{n}^{\mathrm{D}}(\mathcal{C}) \simeq \{\text{Strict functors } \mathrm{Path}_{(2)}[n] \to \mathcal{C}\}.$$

00BG **Remark 2.3.5.8.** It is not difficult to show that the category $2\mathrm{Cat}_{\mathrm{Str}}$ of strict 2-categories admits small colimits (beware that this is not true for the larger category 2Cat). Combining Corollary 2.3.5.7 with Proposition 1.2.3.15, we deduce that the Duskin nerve functor $\mathrm{N}_{\bullet}^{\mathrm{D}} : 2\mathrm{Cat}_{\mathrm{Str}} \to \mathrm{Set}_{\Delta}$ admits a left adjoint $\mathrm{Set}_{\Delta} \to 2\mathrm{Cat}_{\mathrm{Str}}$, which carries a simplicial set $S_{\bullet}$ to the generalized geometric realization $|S_{\bullet}|^{\mathrm{Path}_{(2)}[\bullet]}$. Composing this left adjoint with the fully faithful embedding $\mathrm{N}_{\bullet}^{\mathrm{D}} : 2\mathrm{Cat}_{\mathrm{ULax}} \to \mathrm{Set}_{\Delta}$ (Theorem 2.3.4.1), we deduce that the inclusion functor $2\mathrm{Cat}_{\mathrm{Str}} \hookrightarrow 2\mathrm{Cat}_{\mathrm{ULax}}$ has a left adjoint, given by the construction $\mathcal{C} \mapsto |\,\mathrm{N}_{\bullet}^{\mathrm{D}}(\mathcal{C})|^{\mathrm{Path}_{(2)}[\bullet]}$. We can regard Theorem 2.3.5.6 as providing an explicit description of this left adjoint in a special case: it carries each partially ordered set Q to the strict 2-category $\mathrm{Path}_{(2)}[Q]$ given by Construction 2.3.5.1.

Proof of Theorem 2.3.5.6. Let $\mathcal{C}$ be a strict 2-category, let Q be a partially ordered set, and let $F : Q \to \mathcal{C}$ be a strictly unitary lax functor. We wish to show that F factors uniquely as a composition

$$Q \xrightarrow{T_Q} \mathrm{Path}_{(2)}[Q] \xrightarrow{F^{+}} \mathcal{C},$$

where T_Q is the strictly unitary lax functor of Construction 2.3.5.4 and F^{+} is a strict functor from $\mathrm{Path}_{(2)}[Q]$ to $\mathcal{C}$.

For every pair of elements $x, y \in Q$ satisfying $x \leq y$, we let $e_{y,x} : x \to y$ denote the unique morphism from x to y in the category Q, and for every triple $x, y, z \in Q$ satisfying

$x \leq y \leq z$, we let $\mu_{z,y,x} : F(e_{z,y}) \circ F(e_{y,x}) \Rightarrow F(e_{z,x})$ denote the composition constraint for the lax monoidal functor F. Unwinding the definitions, we see that a strict functor $F^{+} : \mathrm{Path}_{(2)}[Q] \to \mathcal{C}$ satisfies $F^{+} \circ T_Q = F$ if and only if the following conditions are satisfied:

(0) For every element $x \in Q$, we have $F^{+}(x) = F(x)$ (as objects of the 2-category $\mathcal{C}$).

(1) For every pair of elements $x, y \in Q$ satisfying $x \leq y$, we have $F^{+}(\{y, x\}) = F(e_{y,x})$ (as 1-morphisms from $F(x)$ to $F(y)$ in the strict 2-category $\mathcal{C}$).

(2) For every triple of elements $x, y, z \in Q$ satisfying $x \leq y \leq z$, the functor F^{+} carries the inclusion $\{z, x\} \subseteq \{z, y, x\}$ (regarded as a 2-morphism from $\{z, y\} \circ \{y, x\}$ to $\{z, x\}$ in the strict 2-category $\mathrm{Path}_{(2)}[Q]$) to $\mu_{z,y,x}$ (regarded as a 2-morphism from $F(e_{z,y}) \circ F(e_{y,x})$ to $F(e_{z,x})$ in the strict 2-category $\mathcal{C}$).

Note that, since we are requiring F^{+} to be a strict functor, we can replace (1) by the following stronger condition:

(1′) For every nonempty finite linearly ordered subset $S = \{x_0 < x_1 < \cdots < x_n\} \subseteq Q$, the functor F^{+} carries S (regarded as a 1-morphism from x_0 to x_n in the strict 2-category $\mathrm{Path}_{(2)}[Q]$) to the composition $F(e_{x_n,x_{n-1}}) \circ \cdots \circ F(e_{x_1,x_0})$ (regarded as a 1-morphism from $F(x_0)$ to $F(x_n)$ in the strict 2-category $\mathcal{C}$). In what follows, we will denote this composition by $F(S)$.

Let $S = \{x_0 < x_1 < \cdots < x_n\}$ be a nonempty finite linearly ordered subset of Q. For each $0 \leq i \leq j \leq n$, set $f_{j,i} = F(e_{x_j,x_i})$, which we regard as a 1-morphism from $F(x_i)$ to $F(x_j)$ in the 2-category $\mathcal{C}$. Let x_i be an element of S which is neither the largest nor the smallest (so that $0 < i < n$). In this case, we let $\gamma_{S,x_i} : F(S) \Rightarrow F(S \setminus \{x_i\})$ denote the 2-morphism of $\mathcal{C}$ given by the horizontal composition

$$\gamma_{S,x_i} = \mathrm{id}_{f_{n,n-1}} \circ \cdots \circ \mathrm{id}_{f_{i+2,i+1}} \circ \mu_{x_{i+1},x_i,x_{i-1}} \circ \mathrm{id}_{f_{i-1,i-2}} \circ \cdots \circ \mathrm{id}_{f_{1,0}}.$$

More generally, given a sequence of distinct elements $s_1, s_2, \cdots, s_m \in S \setminus \{x_0, x_n\}$, we let $\gamma_{S,s_1,\ldots,s_m} : F(S) \Rightarrow F(S \setminus \{s_1, \ldots, s_m\})$ denote the 2-morphism of $\mathcal{C}$ given by the vertical composition

$$F(S) \xRightarrow{\gamma_{S,s_1}} F(S \setminus \{s_1\}) \xRightarrow{\gamma_{S \setminus \{s_1\},s_2}} F(S \setminus \{s_1, s_2\}) \Rightarrow \cdots \Rightarrow F(S \setminus \{s_1, \ldots, s_m\}).$$

Since the strict functor F^{+} is required to be compatible with vertical and horizontal composition, we can replace (2) by the following stronger condition:

(2′) Let $S = \{x_0 < x_1 < \cdots < x_n\}$ be a nonempty finite linearly ordered subset of Q. Then, for every sequence of distinct elements $s_1, \ldots, s_m \in S \setminus \{x_0, x_n\}$, the functor

F^+ carries the inclusion $S \setminus \{s_1, \dots, s_m\} \subseteq S$ (regarded as a 2-morphism from S to $S \setminus \{s_1, \dots, s_m\}$ in the strict 2-category $\mathrm{Path}_{(2)}[Q]$) to the 2-morphism $\gamma_{S, s_1, \dots, s_m}$ (regarded as a 2-morphism from $F(S)$ to $F(S \setminus \{s_1, \dots, s_m\})$ in the strict 2-category $\mathcal{C}$).

It is now clear that the functor F^+ is unique if it exists: its values on objects, 1-morphisms, and 2-morphisms of $\mathrm{Path}_{(2)}[Q]$ are determined by conditions (0), (1′), and (2′), respectively. To prove existence, it will suffice to show that this prescription is well-defined: namely, that the 2-morphism $\gamma_{S, s_1, \dots, s_m}$ defined above depends only on the sets S and $T = S \setminus \{s_1, \dots, s_m\}$, and not on the order of the sequence $(s_1, \dots, s_m)$ (it then follows easily from the construction that the definition of F^+ on 2-morphisms is compatible with vertical and horizontal composition). Since the group of all permutations of the set $\{s_1, \dots, s_m\}$ is generated by transpositions of adjacent elements, it will suffice to show that we have

$$\gamma_{S, s_1, \cdots, s_{i-1}, s_i, s_{i+1}, s_{i+2}, \cdots, s_m} = \gamma_{S, s_1, \cdots, s_{i-1}, s_{i+1}, s_i, s_{i+2}, \cdots, s_m}$$

for each $1 \leq i < m$. Replacing S by $S \setminus \{s_1, \dots, s_{i-1}\}$, we are reduced to proving that $\gamma_{S,s,t} = \gamma_{S,t,s}$ whenever $s < t$ are elements of $S - \{x_0, x_n\}$. We now distinguish two cases:

- Suppose that the elements s and t are *non-consecutive* elements of S: that is, we have $s = x_i$ and $t = x_j$ where $j > i + 1$. In this case, we can identify both $\gamma_{S,s,t}$ and $\gamma_{S,t,s}$ with the horizontal composition

  $$\mathrm{id}_{f_{n,n-1}} \circ \cdots \circ \mu_{x_{j+1}, x_j, x_{j-1}} \circ \cdots \circ \mu_{x_{i+1}, x_i, x_{i-1}} \circ \cdots \circ \mathrm{id}_{f_{1,0}}.$$

- Suppose that the elements s and t are consecutive: that is, we have

  $$S = \{x_0 < \cdots < r < s < t < u < \cdots < x_n\}.$$

  In this case, to verify the identity $\gamma_{S,s,t} = \gamma_{S,t,s}$, we can replace S by the subset $\{r < s < t < u\}$ and thereby reduce to checking the commutativity of the diagram

  $$\begin{tikzcd}
  F(e_{u,t}) \circ F(e_{t,s}) \circ F(e_{s,r}) \arrow[r, Rightarrow, "\mathrm{id}_{F(e_{u,t})} \circ \mu_{t,s,r}"] \arrow[d, Rightarrow, "\mu_{u,t,s} \circ \mathrm{id}_{F(e_{s,r})}"] & F(e_{u,t}) \circ F(e_{t,r}) \arrow[d, Rightarrow, "\mu_{u,t,r}"] \\
  F(e_{u,s}) \circ F(e_{s,r}) \arrow[r, Rightarrow, "\mu_{u,s,r}"] & F(e_{u,r})
  \end{tikzcd}$$

  in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(F(r), F(u))$, which is the coherence condition required by the composition contraints for the lax functor F (axiom (c) of Definition 2.2.4.5).

□

2.4 Simplicial Categories

Let Top denote the category of topological spaces. By definition, a morphism in the category Top is a continuous function $f : X \to Y$. In homotopy theory, one is fundamentally concerned not only with continuous functions themselves, but also with *homotopies* between them: that is, continuous functions $h : [0,1] \times X \to Y$. More generally, for each $n \geq 0$, one can consider the set `00JN`

$$\mathrm{Hom}_{\mathrm{Top}}(X, Y)_n = \{\text{Continuous functions } \sigma : |\Delta^n| \times X \to Y\};$$

here $|\Delta^n|$ denotes the topological simplex of dimension n. The sets $\{\mathrm{Hom}_{\mathrm{Top}}(X,Y)_n\}_{n \geq 0}$ can be assembled into a simplicial set $\mathrm{Hom}_{\mathrm{Top}}(X,Y)_\bullet$, and the construction $(X,Y) \mapsto \mathrm{Hom}_{\mathrm{Top}}(X,Y)_\bullet$ endows Top with the structure of a *simplicial category*: that is, a category which is enriched over simplicial sets, in the sense of Definition 2.1.7.1. Much as the singular simplicial set $\mathrm{Sing}_\bullet(X) = \mathrm{Hom}_{\mathrm{Top}}(*, X)_\bullet$ can be regarded as a combinatorial encoding of the *homotopy type* of an individual topological space X, the simplicial enrichment of Top can be regarded as a combinatorial encoding of the *homotopy theory* of topological spaces.

Our goal in this section is to provide an introduction to the theory of simplicial categories. We begin in §2.4.1 by defining the notion of simplicial category (Definition 2.4.1.1). The collection of (small) simplicial categories can itself be organized into a category Cat_Δ, in which the morphisms are given by simplicial functors (Definition 2.4.1.11). In §2.4.2 we provide many examples of how simplicial categories arise in nature: in particular, we explain that Cat_Δ can be regarded as an enlargement of the usual category Cat of small categories (Example 2.4.2.4), and also of the category $2\mathrm{Cat}_{\mathrm{Str}}$ of strict 2-categories (Example 2.4.2.8).

Recall that to every category $\mathcal{C}$ we can associate a simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ called the *nerve* of $\mathcal{C}$ (Construction 1.3.1.1). In §2.4.3, we describe a generalization of this construction (due to Cordier) which associates to each simplicial category $\mathcal{C}_\bullet$ a simplicial set $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ called the *homotopy coherent nerve* of $\mathcal{C}_\bullet$ (Definition 2.4.3.5). This construction specializes to the ordinary nerve in the case where $\mathcal{C}_\bullet$ is an ordinary category (and to the Duskin nerve in the case where $\mathcal{C}_\bullet$ arises from a strict 2-category: see Example 2.4.3.11). It is particularly well-behaved in the special case where $\mathcal{C}_\bullet$ is locally Kan (meaning that simplicial Hom-sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ are Kan complexes): in this case, a theorem of Cordier and Porter asserts that the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is an ∞-category (Theorem 2.4.5.1).

In §2.4.4, we show that the homotopy coherent nerve functor $\mathrm{N}^{\mathrm{hc}}_\bullet : \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta$ admits a left adjoint (Corollary 2.4.4.4). This left adjoint carries each simplicial set $S_\bullet$ to a simplicial category $\mathrm{Path}[S]_\bullet$ which we will refer to as the (simplicial) *path category of* $S_\bullet$. The construction $S_\bullet \mapsto \mathrm{Path}[S]_\bullet$ is a generalization of the classical path category studied in §1.3.7: when $S_\bullet$ is the 1-dimensional simplicial set associated to a directed graph G, the simplicial category $\mathrm{Path}[S]_\bullet$ can be identified with the ordinary category $\mathrm{Path}[G]$ of Construction 1.3.7.1 (see Proposition 2.4.4.7). For a general simplicial set $S_\bullet$, the path category $\mathrm{Path}[S]_\bullet$ is a complicated object. However, in each fixed simplicial degree m it is relatively simple: the ordinary category $\mathrm{Path}[S]_m$ can be identified with the classical path category of a certain directed graph G_m which can be described concretely in terms of the combinatorics of $S_\bullet$ (Theorem 2.4.4.10). We will exploit this description in §2.4.5 to carry out the proof of Theorem 2.4.5.1, and again in §2.4.6 to compare the homotopy category of a (locally Kan) simplicial category $\mathcal{C}_\bullet$ to the homotopy category of its associated ∞-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ (Proposition 2.4.6.9).

00JP **Warning 2.4.0.1.** The ordinary nerve functor $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ determines a fully faithful embedding from the category Cat of small categories to the category Set_Δ of simplicial sets (Proposition 1.3.3.1). However, the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet : \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta$ is not fully faithful when regarded as a functor of ordinary categories. Phrased differently, the adjoint functors

$$\begin{tikzcd}\mathrm{Set}_\Delta \arrow[r, shift left, "\mathrm{Path}[-]_\bullet"] & \mathrm{Cat}_\Delta \arrow[l, shift left, "\mathrm{N}^{\mathrm{hc}}_\bullet"]\end{tikzcd}$$

associate to each simplicial category $\mathcal{C}_\bullet$ a *counit map* $v : \mathrm{Path}[\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}_\bullet)]_\bullet \to \mathcal{C}_\bullet$, which is almost never an *isomorphism* of simplicial categories. However, we will see later that v is a *weak equivalence* of simplicial categories whenever $\mathcal{C}_\bullet$ is locally Kan ([?]). Moreover, the construction $\mathcal{C}_\bullet \mapsto \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ establishes an equivalence from the homotopy theory of (locally Kan) simplicial categories $\mathcal{C}_\bullet$ with the homotopy theory of ∞-categories ([?]).

2.4.1 Simplicial Enrichment

00JQ Let Set_Δ denote the category of simplicial sets (Definition 1.1.0.6). Then Set_Δ admits cartesian products (Remark 1.1.0.8), and can therefore be endowed with the cartesian monoidal structure described in Example 2.1.3.2. We will use the term *simplicial category* to refer to a category which is enriched over Set_Δ, in the sense of Definition 2.1.7.1. For the reader's convenience, we spell this definition out in detail (and establish some notation we will use when discussing simplicial categories, which differs somewhat from the general conventions of §2.1.7).

00JR **Definition 2.4.1.1** (Simplicial Categories). A *simplicial category* $\mathcal{C}_\bullet$ consists of the following data:

(1) A collection $\mathrm{Ob}(\mathcal{C}_\bullet)$, whose elements we refer to as *objects of* $\mathcal{C}_\bullet$. We will often abuse notation by writing $X \in \mathcal{C}_\bullet$ to indicate that X is an element of $\mathrm{Ob}(\mathcal{C}_\bullet)$.

(2) For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}_\bullet)$, a simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet$.

(3) For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C}_\bullet)$, a morphism of simplicial sets

$$c_{Z,Y,X} : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(X, Z)_\bullet,$$

which we will refer to as the *composition law*.

(4) For every object $X \in \mathrm{Ob}(\mathcal{C})$, a vertex $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)_0$, which we will refer to as the *identity morphism of X*.

These data are required to satisfy the following conditions:

(A) For every quadruple of objects $W,X,Y,Z \in \mathrm{Ob}(\mathcal{C}_\bullet)$, the diagram of simplicial sets

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(W,X)_\bullet \arrow[dl, "\mathrm{id} \times c_{Y,X,W}"'] \arrow[dr, "c_{Z,Y,X} \times \mathrm{id}"] & \\
\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(W,Y)_\bullet \arrow[dr, "c_{Z,Y,W}"'] & & \mathrm{Hom}_{\mathcal{C}}(X,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(W,X)_\bullet \arrow[dl, "c_{Z,X,W}"] \\
 & \mathrm{Hom}_{\mathcal{C}}(W,Z)_\bullet &
\end{tikzcd}$$

commutes (in other words, the composition law of (3) is associative).

(U) For every pair of objects $X,Y \in \mathrm{Ob}(\mathcal{C}_\bullet)$, the maps of simplicial sets

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \times \{\mathrm{id}_X\} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,X)_\bullet \xrightarrow{c_{Y,X,X}} \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$$

$$\{\mathrm{id}_Y\} \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y,Y)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \xrightarrow{c_{Y,Y,X}} \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$$

coincide with the projection maps onto the factor $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$.

Warning 2.4.1.2. The terminology of Definition 2.4.1.1 is not standard. Many authors use the term *simplicial category* to mean a simplicial object of the category Cat, and the term *simplicially enriched category* to mean a category enriched over simplicial sets. These notions are closely related: see Remark 2.4.1.12. 00JS

Construction 2.4.1.3. Let $\mathcal{C}_\bullet$ be a simplicial category. For every nonnegative integer $n \geq 0$, we define an ordinary category $\mathcal{C}_n$ as follows: 00JT

- The objects of $\mathcal{C}_n$ are the objects of $\mathcal{C}_\bullet$.
- Let $X,Y \in \mathrm{Ob}(\mathcal{C}_n) = \mathrm{Ob}(\mathcal{C}_\bullet)$ be objects of $\mathcal{C}_n$. A morphism from X to Y in the category $\mathcal{C}_n$ is an n-simplex of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$. In other words, we have an equality of sets $\mathrm{Hom}_{\mathcal{C}_n}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)_n$.
- For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}_n$, the composition $g \circ f : X \to Z$ is given by the image of the n-simplex (g,f) under the map of simplicial sets

$$c_{Z,Y,X} : \mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(X,Z)_\bullet.$$

- For every object $X \in \mathrm{Ob}(\mathcal{C}_n) = \mathrm{Ob}(\mathcal{C}_\bullet)$, the identity morphism from X to itself in the category $\mathcal{C}_n$ is the n-simplex of $\mathrm{Hom}_{\mathcal{C}}(X,X)_\bullet$ which corresponds to the composite map
$$\Delta^n \to \Delta^0 \xrightarrow{\mathrm{id}_X} \mathrm{Hom}_{\mathcal{C}}(X,X)_\bullet$$

00JU **Example 2.4.1.4** (The Underlying Category of a Simplicial Category)**.** Let $\mathcal{C}_\bullet$ be a simplicial category. We let $\mathcal{C} = \mathcal{C}_0$ denote the ordinary category obtained by applying Construction 2.4.1.3 in the case $n = 0$. We will refer to $\mathcal{C}$ as the *underlying category* of the simplicial category $\mathcal{C}_\bullet$. Note that $\mathcal{C}$ can also be obtained from $\mathcal{C}_\bullet$ by applying the general procedure described in Example 2.1.7.5.

We will sometimes abuse terminology by identifying a simplicial category $\mathcal{C}_\bullet$ with its underlying category $\mathcal{C}$. In particular, if X and Y are objects of $\mathcal{C}_\bullet$, we will write $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ to denote the set $\mathrm{Hom}_{\mathcal{C}_0}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)_0$ of morphisms from X to Y in the category $\mathcal{C}$.

00JV **Example 2.4.1.5** (Topological Spaces)**.** Let Top denote the category whose objects are topological spaces and whose morphisms are continuous functions. Then Top can be promoted to a simplicial category $\mathrm{Top}_\bullet$: given a pair of topological spaces X and Y, we define the simplicial set $\mathrm{Hom}_{\mathrm{Top}}(X,Y)_\bullet$ informally by the formula
$$\mathrm{Hom}_{\mathrm{Top}}(X,Y)_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n| \times X, Y)$$
In particular, a vertex of $\mathrm{Hom}_{\mathrm{Top}}(X,Y)_\bullet$ can be identified with a continuous function $f : X \to Y$. Moreover, for any topological space Y, we have a canonical isomorphism of simplicial sets $\mathrm{Hom}_{\mathrm{Top}}(*,Y)_\bullet \simeq \mathrm{Sing}_\bullet(Y)$, where $\mathrm{Sing}_\bullet(Y)$ is the singular simplicial set of Construction 1.2.2.2.

Let $\mathcal{C}$ be a category. Roughly speaking, a simplicial enrichment $\mathcal{C}_\bullet$ of $\mathcal{C}$ can be viewed as a datum which allows us to "do homotopy theory" in $\mathcal{C}$. For example, it allows us to define a notion of homotopy between morphisms of $\mathcal{C}$:

00JW **Definition 2.4.1.6.** Let $\mathcal{C}_\bullet$ be a simplicial category, and let $f, g : X \to Y$ be two morphisms in the underlying category $\mathcal{C} = \mathcal{C}_0$ having the same source and target. A *homotopy* from f to g is an edge $h \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_1$ satisfying $d^1_1(h) = f$ and $d^1_0(h) = g$.

00JX **Example 2.4.1.7.** Let X and Y be topological spaces and let $f, g : X \to Y$ be continuous functions, which we regard as morphisms in the simplicial category $\mathrm{Top}_\bullet$ of Example 2.4.1.5. Then a homotopy from f to g in the sense of Definition 2.4.1.6 is a homotopy in the usual sense: a continuous function $h : [0,1] \times X = |\Delta^1| \times X \to Y$ satisfying $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in X$.

In a general simplicial category $\mathcal{C}$, the notion of homotopy (in the sense of Definition 2.4.1.6) need not be well-behaved: for example, the existence of a homotopy from f to g need not imply the existence of a homotopy from g to f. To remedy the situation, it is convenient to restrict attention to a special class of simplicial categories:

Definition 2.4.1.8. Let $\mathcal{C}_\bullet$ be a simplicial category. We will say that $\mathcal{C}_\bullet$ is *locally Kan* if, for every pair of objects $X, Y \in \mathcal{C}_\bullet$, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ is a Kan complex (Definition 1.2.5.1). `00JY`

Remark 2.4.1.9. Let $\mathcal{C}_\bullet$ be a locally Kan simplicial category, and let $f, g : X \to Y$ be a pair of morphisms in the underlying category $\mathcal{C} = \mathcal{C}_0$ having the same source and target. Invoking Proposition 1.2.5.10, we see that the following conditions are equivalent: `00JZ`

(a) There exists a homotopy from f to g, in the sense of Definition 2.4.1.6.

(b) The morphisms f and g belong to the same connected component of the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$.

In particular, condition (a) defines an equivalence relation on the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Exercise 2.4.1.10. Let $\mathrm{Top}_\bullet$ be the simplicial category of Example 2.4.1.5. Show that $\mathrm{Top}_\bullet$ is locally Kan (hint: generalize the proof of Proposition 1.2.5.8). `00K0`

Specializing Definition 2.1.7.10 to the setting of simplicial enrichments, we obtain the following:

Definition 2.4.1.11 (Simplicial Functors)**.** Let $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$ be simplicial categories. A *simplicial functor* $F : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ consists of the following data: `00K1`

(1) For every object $X \in \mathrm{Ob}(\mathcal{C}_\bullet)$, an object $F(X) \in \mathrm{Ob}(\mathcal{D}_\bullet)$.

(2) For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}_\bullet)$, a map of simplicial sets $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))_\bullet$.

These data are required to satisfy the following conditions:

- For every object $X \in \mathrm{Ob}(\mathcal{C}_\bullet)$, the map of simplicial sets $F_{X,X} : \mathrm{Hom}_{\mathcal{C}}(X,X)_\bullet \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(X))_\bullet$ carries the vertex id_X to the vertex $\mathrm{id}_{F(X)}$.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C}_\bullet)$, the diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \arrow[r] \arrow[d, "F_{Y,Z} \times F_{X,Y}"] & \mathrm{Hom}_{\mathcal{C}}(X,Z)_\bullet \arrow[d, "F_{X,Z}"] \\
\mathrm{Hom}_{\mathcal{D}}(F(Y),F(Z))_\bullet \times \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))_\bullet \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Z))_\bullet
\end{tikzcd}$$

  is commutative.

We let Cat_{Δ} denote the category whose objects are (small) simplicial categories and whose morphisms are simplicial functors.

00K2 **Remark 2.4.1.12.** Let $\mathcal{C}_\bullet$ be a (small) simplicial category. Then the construction $[n] \mapsto \mathcal{C}_n$ determines a functor from the simplex category $\mathbf{\Delta}^{\mathrm{op}}$ (Definition 1.1.0.2) to the category Cat of (small) categories. Allowing $\mathcal{C}_\bullet$ to vary, we obtain a functor $\mathrm{Cat}_{\Delta} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat})$, which fits into a pullback diagram of categories

$$\begin{tikzcd}
\mathrm{Cat}_{\Delta} \arrow[r, "\mathcal{C}_\bullet \mapsto ([n] \mapsto \mathcal{C}_n)"] \arrow[d, "\mathrm{Ob}"] & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat}) \arrow[d, "\mathrm{Ob}"] \\
\mathrm{Set} \arrow[r] & \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}),
\end{tikzcd}$$

where the lower horizontal map carries each set S to the constant functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ taking the value S.

Phrased more informally: simplicial categories can be identified with simplicial objects $\mathcal{C}_\bullet$ of Cat for which the underlying simplicial set of objects $[n] \mapsto \mathrm{Ob}(\mathcal{C}_n)$ is constant. In particular, the functor $\mathrm{Cat}_{\Delta} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat})$ is a fully faithful embedding.

00K3 **Proposition 2.4.1.13.** *The category* Cat_{Δ} *admits small limits and colimits.*

Proof. The category Cat admits small limits and colimits, which are preserved by the forgetful functor $\mathrm{Ob} : \mathrm{Cat} \to \mathrm{Set}$. It follows that the category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat})$ of simplicial objects in Cat also admits small limits and colimits, which are computed pointwise. Remark 2.4.1.12 supplies a fully faithful embedding $\mathrm{Cat}_{\Delta} \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat})$ whose essential image is closed under small limits and colimits, so that Cat_{Δ} admits small limits and colimits as well. □

2.4.2 Examples of Simplicial Categories

00K4 We now supply some examples of simplicial categories.

00K5 **Example 2.4.2.1** (Simplicial Sets)**.** Let Set_{Δ} denote the category of simplicial sets. Then Set_{Δ} can be regarded as (the underlying ordinary category of) a simplicial category, which we will also denote by Set_{Δ}: given a pair of simplicial sets $X_\bullet$ and $Y_\bullet$, we define $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(X_\bullet, Y_\bullet)_\bullet$ to be the simplicial set $\mathrm{Fun}(X_\bullet, Y_\bullet)$ parametrizing morphisms from $X_\bullet$ to $Y_\bullet$ (see Construction 1.5.3.1).

0323 **Example 2.4.2.2** (Functor Categories)**.** Let $\mathcal{C}$ be a category and let $Y : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor. For every simplicial set K, we let $Y^K : \mathcal{C} \to \mathrm{Set}_{\Delta}$ denote the functor given on objects by the formula $Y^K(C) = \mathrm{Fun}(K, Y(C))$. If $X : \mathcal{C} \to \mathrm{Set}_\Delta$ is another functor, we let $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta)}(X,Y)_\bullet$ denote the simplicial set given by the functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set} \qquad [n] \mapsto \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta)}(X, Y^{\Delta^n}).$$

Together with the evident composition maps

$$\circ : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta)}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta)}(X,Y)_\bullet \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta)}(X,Z)_\bullet,$$

this construction endows $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)$ with the structure of a simplicial category.

Example 2.4.2.3 (Delooping)**.** Let $M_\bullet$ be a simplicial monoid. We let $BM_\bullet$ denote the simplicial category having a single object X, with $\mathrm{Hom}_{BM}(X,X)_\bullet = M_\bullet$ and the composition law is given by the monoid structure on $M_\bullet$. We will refer to $BM_\bullet$ as the *delooping* of the simplicial monoid $M_\bullet$. Note that the construction $M_\bullet \mapsto BM_\bullet$ induces an equivalence of categories 00MY

$$\{\text{Simplicial Monoids}\} \simeq \{\text{Simplicial Categories } \mathcal{C} \text{ with } \mathrm{Ob}(\mathcal{C}) = \{X\}\}.$$

We can produce many more examples using the construction of Remark 2.1.7.4. If $\mathcal{A}$ is a monoidal category equipped with a (lax) monoidal functor $F : \mathcal{A} \to \mathrm{Set}_\Delta$, then every $\mathcal{A}$-enriched category can also be regarded as a simplicial category. We now consider four instances of this construction:

- We can take $F : \mathrm{Set} \hookrightarrow \mathrm{Set}_\Delta$ to be the functor which carries each set S to the associated constant simplicial set (Construction 1.1.5.2).
- We can take $F : \mathrm{Cat} \hookrightarrow \mathrm{Set}_\Delta$ to be the functor which carries each category $\mathcal{C}$ to its nerve $\mathrm{N}_\bullet(\mathcal{C})$ (Construction 1.3.1.1).
- We can take $F : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ to be the functor which carries each simplicial set $S_\bullet$ to the opposite simplicial set $S_\bullet^{\mathrm{op}}$.
- We can take $F : \mathrm{Top} \to \mathrm{Set}_\Delta$ to be the functor which carries each topological space to the singular simplicial set $\mathrm{Sing}_\bullet(\mathcal{C})$ (Construction 1.2.2.2).

Example 2.4.2.4 (Ordinary Categories as Simplicial Categories)**.** Let $\mathcal{C}$ be an ordinary category. We define a simplicial category $\underline{\mathcal{C}}_\bullet$ as follows: 00K6

- The objects of $\underline{\mathcal{C}}_\bullet$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\underline{\mathcal{C}}_\bullet) = \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\underline{\mathcal{C}}}(X,Y)_\bullet$ is the constant simplicial set associated to the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (see Construction 1.1.5.2).

- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C}_\bullet) = \mathrm{Ob}(\mathcal{C})$, the composition law
$$c_{Z,Y,X} : \mathrm{Hom}_{\underline{\mathcal{C}}}(Y,Z)_\bullet \times \mathrm{Hom}_{\underline{\mathcal{C}}}(X,Y)_\bullet \to \mathrm{Hom}_{\underline{\mathcal{C}}}(X,Z)_\bullet$$
on $\underline{\mathcal{C}}_\bullet$ is determined by the composition law $\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$ on $\mathcal{C}$.

We will refer to $\underline{\mathcal{C}}_\bullet$ as the *constant simplicial category* associated to $\mathcal{C}$. Under the fully faithful embedding $\mathrm{Cat}_\Delta \hookrightarrow \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Cat})$ of Remark 2.4.1.12, it corresponds to the constant functor $\mathbf{\Delta}^{\mathrm{op}} \to \{\mathcal{C}\} \hookrightarrow \mathrm{Cat}$ (see Construction 1.1.5.2). In particular, the underlying category of $\underline{\mathcal{C}}_\bullet$ (in the sense of Example 2.4.1.4) is the ordinary category $\mathcal{C}$.

00K7 **Remark 2.4.2.5.** It follows from Corollary 1.1.5.9 and Remark 2.4.1.12 that the construction
$$\mathrm{Cat} \to \mathrm{Cat}_\Delta \qquad \mathcal{C} \mapsto \underline{\mathcal{C}}_\bullet$$
of Example 2.4.2.4 is fully faithful. Its essential image consists of those simplicial categories $\mathcal{E}_\bullet$ having the property that, for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{E}_\bullet)$, the simplicial set $\mathrm{Hom}_{\mathcal{E}}(X,Y)_\bullet$ is discrete (Definition 1.1.5.10). We will sometimes abuse notation by not distinguishing between the ordinary category $\mathcal{C}$ and the constant simplicial category $\underline{\mathcal{C}}_\bullet$.

00K8 **Remark 2.4.2.6.** Let $\mathcal{C}$ be an ordinary category and let $\mathcal{D}_\bullet$ be a simplicial category. Applying Proposition 1.1.5.5 (and Remark 2.4.1.12), we deduce that the restriction map
$$\{\text{Simplicial functors } \underline{\mathcal{C}}_\bullet \to \mathcal{D}_\bullet\} \simeq \{\text{Functors } \mathcal{C} \to \mathcal{D}_0\},$$
is bijective. In other words, the fully faithful embedding
$$\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\Delta \qquad \mathcal{C} \mapsto \underline{\mathcal{C}}_\bullet$$
of Remark 2.4.2.5 is left adjoint to the forgetful functor
$$\mathrm{Cat}_\Delta \to \mathrm{Cat} \qquad \mathcal{D}_\bullet \mapsto \mathcal{D}_0\,.$$
of Example 2.4.1.4.

00K9 **Remark 2.4.2.7.** Let $\mathcal{C}$ be an ordinary category. Then the simplicial category $\underline{\mathcal{C}}_\bullet$ of Example 2.4.2.4 is locally Kan (since constant simplicial sets are Kan complexes; see Example 1.2.5.6).

00KA **Example 2.4.2.8** (Strict 2-Categories as Simplicial Categories)**.** Let $\mathcal{C}$ be strict 2-category (Definition 2.2.0.1). Then we can associate to $\mathcal{C}$ a simplicial category $\mathcal{C}_\bullet$ as follows:

- The objects of $\mathcal{C}_\bullet$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}_\bullet) = \mathrm{Ob}(\mathcal{C})$, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ is the nerve of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$.

- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C}_\bullet) = \mathrm{Ob}(\mathcal{C})$, the composition law

  $$\mathrm{Hom}_{\mathcal{C}}(Y, Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(X, Z)_\bullet$$

  of $\mathcal{C}_\bullet$ is given by the nerve of the composition functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$.

Remark 2.4.2.9. In the situation of Example 2.4.2.8, we will generally abuse notation by identifying the strict 2-category $\mathcal{C}$ with the associated simplicial category $\mathcal{C}_\bullet$. Note that the underlying category of $\mathcal{C}_\bullet$ (in the sense of Example 2.4.1.4) agrees with the underlying category of $\mathcal{C}$ (in the sense of Remark 2.2.0.3). Moreover, since the nerve functor $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$ is fully faithful (Proposition 1.3.3.1), the construction of Example 2.4.2.8 supplies a fully faithful embedding 00KB

$$2\mathrm{Cat}_{\mathrm{Str}} \hookrightarrow \mathrm{Cat}_\Delta \qquad \mathcal{C} \mapsto \mathcal{C}_\bullet,$$

where $2\mathrm{Cat}_{\mathrm{Str}}$ denotes the category of strict 2-categories (see Definition 2.2.5.5).

Remark 2.4.2.10. Let $\mathcal{C}$ be an ordinary category, regarded as a strict 2-category having only identity 2-morphisms (Example 2.2.0.6). Then the simplicial category $\mathcal{C}_\bullet$ associated to $\mathcal{C}$ by Example 2.4.2.8 agrees with the simplicial category associated to $\mathcal{C}$ by Example 2.4.2.4. 00KC

Remark 2.4.2.11. Let $\mathcal{C}$ be a strict 2-category. Then the simplicial category $\mathcal{C}_\bullet$ of Example 2.4.2.8 is locally Kan if and only if every 2-morphism in $\mathcal{C}$ is invertible: that is, if and only if $\mathcal{C}$ is a $(2,1)$-category (in the sense of Definition 2.2.8.5). This follows from Proposition 1.3.5.2. 00KD

Example 2.4.2.12 (The Conjugate of a Simplicial Category)**.** Let $\mathcal{C}_\bullet$ be a simplicial category. We define a new simplicial category $\mathcal{C}^{\mathrm{c}}_\bullet$ as follows: 00KE

- The objects of $\mathcal{C}^{\mathrm{c}}_\bullet$ are the objects of $\mathcal{C}_\bullet$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}^{\mathrm{c}}_\bullet) = \mathrm{Ob}(\mathcal{C}_\bullet)$, we have an equality of simplicial sets

  $$\mathrm{Hom}_{\mathcal{C}^{\mathrm{c}}}(X, Y)_\bullet = \mathrm{Hom}_{\mathcal{C}}(X, Y)^{\mathrm{op}}_\bullet;$$

  here the right hand side denotes the opposite of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet$ (Construction 1.4.2.2).
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C}^{\mathrm{c}}_\bullet) = \mathrm{Ob}(\mathcal{C}_\bullet)$, the composition law

  $$\mathrm{Hom}_{\mathcal{C}^{\mathrm{c}}}(Y, Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}^{\mathrm{c}}}(X, Y)_\bullet \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{c}}}(X, Z)_\bullet$$

  on $\mathcal{C}^{\mathrm{c}}_\bullet$ is obtained from the composition law on $\mathcal{C}_\bullet$ by passing to opposite simplicial sets.

We will refer to $\mathcal{C}_\bullet^{\mathrm{c}}$ as the *conjugate* of the simplicial category $\mathcal{C}_\bullet$.

00KF **Remark 2.4.2.13.** Let $\mathcal{C}_\bullet$ be a simplicial category and let $\mathcal{C}_\bullet^{\mathrm{c}}$ denote the conjugate simplicial category (Example 2.4.2.12). Then, when regarded as a simplicial object of Cat, the conjugate simplicial category $\mathcal{C}_\bullet^{\mathrm{c}}$ is given by the functor

$$\boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{Op}} \boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{[n] \mapsto \mathcal{C}_n} \mathrm{Cat};$$

here Op denotes the involution of $\boldsymbol{\Delta}$ described in Notation 1.4.2.1. In particular, the underlying ordinary categories of $\mathcal{C}_\bullet$ and $\mathcal{C}_\bullet^{\mathrm{c}}$ are the same.

00KG **Remark 2.4.2.14.** Let $\mathcal{C}$ be a strict 2-category and let $\mathcal{C}_\bullet$ denote the associated simplicial category (Example 2.4.2.8). Then the conjugate simplicial category $(\mathcal{C}_\bullet)^{\mathrm{c}}$ can be identified with the simplicial category $(\mathcal{C}^{\mathrm{c}})_\bullet$ associated to the conjugate 2-category $\mathcal{C}^{\mathrm{c}}$ of Construction 2.2.3.4. In particular, if $\mathcal{C}$ is an ordinary category, then we have a canonical isomorphism $\mathcal{C}_\bullet^{\mathrm{c}} \simeq \mathcal{C}_\bullet$.

00KH **Remark 2.4.2.15.** Let $\mathcal{C}_\bullet$ be a simplicial category. Then $\mathcal{C}_\bullet$ is locally Kan if and only if the conjugate simplicial category $\mathcal{C}_\bullet^{\mathrm{c}}$ (Example 2.4.2.12) is locally Kan.

00KJ **Example 2.4.2.16** (Topologically Enriched Categories)**.** Let Top denote the category of topological spaces. The formation of singular simplicial sets (Construction 1.2.2.2) determines a functor

$$\mathrm{Sing}_\bullet : \mathrm{Top} \to \mathrm{Set}_\Delta \qquad X \mapsto \mathrm{Sing}_\bullet(X)$$

which preserves finite products (in fact, it preserves all small limits), and can therefore be regarded as a monoidal functor from Top to Set_Δ (where we endow both Top and Set_Δ with the cartesian monoidal structure). Applying Remark 2.1.7.4, we see that every topologically enriched category $\mathcal{C}$ can be regarded as a simplicial category $\mathcal{C}_\bullet$ having the same objects, with morphism simplicial sets given by

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet = \mathrm{Sing}_\bullet(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y));$$

here $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$ denotes the set of morphisms from X to Y, endowed with the topology determined by the topological enrichment of $\mathcal{C}$ (see Example 2.1.7.8).

00KK **Remark 2.4.2.17.** Let $\mathcal{C}$ be a topologically enriched category, and let $\mathcal{C}_\bullet$ denote the associated simplicial category (Example 2.4.2.16). Then $\mathcal{C}_\bullet$ is locally Kan (since the singular simplicial set $\mathrm{Sing}_\bullet(X)$ of any topological space X is a Kan complex; see Proposition 1.2.5.8).

00KL **Warning 2.4.2.18.** Let $\mathrm{Top}_{\mathrm{LCH}}$ denote the full subcategory of Top spanned by the locally compact Hausdorff spaces. Then we can view $\mathrm{Top}_{\mathrm{LCH}}$ as a topologically enriched category, where we endow each of the sets

$$\mathrm{Hom}_{\mathrm{Top}_{\mathrm{LCH}}}(X,Y) = \{\text{Continuous functions } f : X \to Y\}$$

with the *compact-open topology*, generated by open sets of the form $\{f \in \mathrm{Hom}_{\mathrm{Top}}(X,Y) : f(K) \subseteq U\}$ where $K \subseteq X$ is compact and $U \subseteq Y$ is open. On this subcategory, the simplicial enrichment of Example 2.4.2.16 coincides with the simplicial enrichment of Example 2.4.1.5. Beware that some technical issues arise if we allow spaces which are not locally compact:

- Given topological spaces X, Y, and Z, the composition map

  $$\mathrm{Hom}_{\mathrm{Top}}(Y,Z) \times \mathrm{Hom}_{\mathrm{Top}}(X,Y) \to \mathrm{Hom}_{\mathrm{Top}}(X,Z) \qquad (g,f) \mapsto g \circ f$$

  need not be continuous (with respect to the compact-open topologies on $\mathrm{Hom}_{\mathrm{Top}}(X,Y)$, $\mathrm{Hom}_{\mathrm{Top}}(Y,Z)$, and $\mathrm{Hom}_{\mathrm{Top}}(X,Z)$) when $Y \notin \mathrm{Top}_{\mathrm{LCH}}$. Consequently, the construction of compact-open topologies does not determine a topological enrichment of Top (in the sense of Example 2.1.7.8).

- Given topological spaces X and Y, a function $|\Delta^n| \to \mathrm{Hom}_{\mathrm{Top}}(X,Y)$ which is continuous (for the compact-open topology on $\mathrm{Hom}_{\mathrm{Top}}(X,Y)$) need not correspond to a continuous function $|\Delta^n| \times X \to Y$ when $X \notin \mathrm{Top}_{\mathrm{LCH}}$.

One can remedy these difficulties by replacing Top by the subcategory of *compactly generated weak Hausdorff spaces* introduced in [42].

2.4.3 The Homotopy Coherent Nerve

Let Top denote the category of topological spaces and let $\mathrm{N}_\bullet(\mathrm{Top})$ denote its nerve (Construction 1.3.1.1). Then $\mathrm{N}_\bullet(\mathrm{Top})$ is a simplicial set whose 2-simplices can be identified with diagrams of topological spaces σ : 00KM

$$\begin{tikzcd}
& X_1 \arrow[dr, "f_{21}"] & \\
X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & & X_2
\end{tikzcd}$$

which commute in the sense that $f_{21} \circ f_{10}$ is *equal* to f_{20}. In the study of algebraic topology, one often encounters diagrams which commute in the weaker sense that the composition $f_{21} \circ f_{10}$ *homotopic* to f_{20}. By definition, this means that there exists a continuous function $h : [0,1] \times X_0 \to X_2$ which satisfies the boundary conditions

$$h|_{\{0\} \times X_0} = f_{21} \circ f_{10} \qquad h|_{\{1\} \times X_0} = f_{20}.$$

In this case, we say that the function h is a *homotopy* from $f_{21} \circ f_{10}$ to f_{20}, and that h is a *witness* to the homotopy commutativity of the diagram σ. In this section, we will introduce an enlargement $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Top})$ of the simplicial set $\mathrm{N}_{\bullet}(\mathrm{Top})$, whose 2-simplices are given by pairs (σ, h) where σ is a (possibly noncommutative) diagram as above, and h is a witness to the homotopy commutativity of σ. This is a special case of a general construction (Definition 2.4.3.5) which can be applied to any simplicial category.

`00KN` **Notation 2.4.3.1** (Simplicial Path Categories)**.** Let $(Q, \leq)$ be a partially ordered set, and let $\mathrm{Path}_{(2)}[Q]$ denote the path 2-category of Q (Construction 2.3.5.1). We let $\mathrm{Path}[Q]_{\bullet}$ denote the simplicial category obtained from the strict 2-category $\mathrm{Path}_{(2)}[Q]$ by applying the construction of Example 2.4.2.8. More concretely, we can describe the simplicial category $\mathrm{Path}[Q]_{\bullet}$ as follows:

- The objects of $\mathrm{Path}[Q]_{\bullet}$ are the elements of the partially ordered set Q.
- If x and y are elements of $Q = \mathrm{Ob}(\mathrm{Path}[Q]_{\bullet})$, then $\mathrm{Hom}_{\mathrm{Path}[Q]}(x, y)_{\bullet}$ is the nerve of the partially ordered set of finite linearly ordered subsets $\{x = x_0 < x_1 < \cdots < x_m = y\} \subseteq Q$ with least element x and largest element y, ordered by *reverse* inclusion.
- For each element $x \in Q = \mathrm{Ob}(\mathrm{Path}[Q]_{\bullet})$, the identity morphism id_x is the singleton $\{x\} \in \mathrm{Hom}_{\mathrm{Path}[Q]}(x, x)_0$.
- For $x, y, z \in Q = \mathrm{Ob}(\mathrm{Path}[Q])$, the composition law
$$\mathrm{Hom}_{\mathrm{Path}[Q]}(y, z)_{\bullet} \times \mathrm{Hom}_{\mathrm{Path}[Q]}(x, y)_{\bullet} \to \mathrm{Hom}_{\mathrm{Path}[Q]}(x, z)_{\bullet}$$
is given on vertices by the construction $(S, T) \mapsto S \cup T$.

In the special case where $Q = [n] = \{0 < 1 < \cdots < n\}$, we denote the simplicial category $\mathrm{Path}[Q]_{\bullet}$ by $\mathrm{Path}[n]_{\bullet}$.

`00KP` **Remark 2.4.3.2.** Let Q be a partially ordered set. The simplicial category $\mathrm{Path}[Q]_{\bullet}$ can be regarded as a "thickened version" of Q. For every pair of elements $x, y \in Q$, the simplicial set $\mathrm{Hom}_{\mathrm{Path}[Q]}(x, y)_{\bullet}$ is empty if $x \not\leq y$, and weakly contractible (see Definition 3.2.4.15) if $x \leq y$ (since it is the nerve of a partially ordered set with a largest element $\{x, y\}$). In particular, there is a unique simplicial functor $\pi : \mathrm{Path}[Q]_{\bullet} \to Q$ which is the identity on objects (where we abuse notation by identifying Q with the associated constant simplicial category of Example 2.4.2.4). The simplicial functor π is a prototypical example of a *weak equivalence* in the setting of simplicial categories (see Definition 4.6.8.7).

`00KQ` **Remark 2.4.3.3.** A topologically enriched variant of $\mathrm{Path}[Q]_{\bullet}$ appears in the work of Leitch ([39]); see appendix B of [51] for a related construction.

`00KR` **Remark 2.4.3.4** (Relationship with Ordinary Path Categories)**.** Let Q be a partially ordered set and let $\mathrm{Gr}(Q)$ denote the associated directed graph, given concretely by

$$\mathrm{Vert}(\mathrm{Gr}(Q)) = Q \qquad \mathrm{Edge}(\mathrm{Gr}(Q)) = \{(x, y) \in Q : x < y\}.$$

Then the path category $\mathrm{Path}[\mathrm{Gr}(Q)]$ of Construction 1.3.7.1 is the underlying category of the simplicial category $\mathrm{Path}[Q]_\bullet$ of Notation 2.4.3.1 (see Remark 2.3.5.2). In other words, we can regard $\mathrm{Path}[Q]_\bullet$ as a simplicially enriched version of $\mathrm{Path}[\mathrm{Gr}(Q)]$. Beware that the simplicial enrichment is nontrivial in general: that is, the simplicial mapping sets $\mathrm{Hom}_{\mathrm{Path}[Q]}(x,y)_\bullet$ are usually not constant.

Definition 2.4.3.5 (The Homotopy Coherent Nerve)**.** Let $\mathcal{C}_\bullet$ be a simplicial category. We let $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ denote the simplicial set given by the construction 00KS

$$([n] \in \boldsymbol{\Delta}^{\mathrm{op}}) \mapsto \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathrm{Path}[n]_\bullet, \mathcal{C}_\bullet) = \{\text{Simplicial functors } \mathrm{Path}[n]_\bullet \to \mathcal{C}_\bullet\}.$$

We will refer to $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ as the *homotopy coherent nerve* of the simplicial category $\mathcal{C}_\bullet$.

Remark 2.4.3.6. The homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ was introduced by Cordier in [11] (motivated by earlier work of Vogt on the theory of homotopy coherence; see [59]). Beware that Cordier uses slightly different conventions: [11] defines the homotopy coherent nerve of a simplicial category $\mathcal{C}$ to be the simplicial set $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}^{\mathrm{c}})$, where $\mathcal{C}^{\mathrm{c}}$ denotes the *conjugate* of the simplicial category $\mathcal{C}$ (Example 2.4.2.12). 00KT

Remark 2.4.3.7. The homotopy coherent nerve of Definition 2.4.3.5 determines a functor $\mathrm{N}^{\mathrm{hc}}_\bullet(-)$ from the category Cat_Δ of simplicial categories (Definition 2.4.1.11) to the category Set_Δ of simplicial sets (Definition 1.1.0.6). This is a special case of the general construction described in Variant 1.2.2.8, associated to the cosimplicial object of Cat_Δ given by 00KU

$$\boldsymbol{\Delta} \to \mathrm{Cat}_\Delta \qquad [n] \mapsto \mathrm{Path}[n]_\bullet.$$

Remark 2.4.3.8 (Comparison with the Nerve)**.** Let $\mathcal{C}_\bullet$ be a simplicial category and let $\mathcal{C} = \mathcal{C}_0$ denote the underlying ordinary category. For every partially ordered set Q, composition with the simplicial functor $\mathrm{Path}[Q]_\bullet \to Q$ of Remark 2.4.3.2 induces a monomorphism 00KV

$$\{\text{Ordinary functors } Q \to \mathcal{C}\} \hookrightarrow \{\text{Simplicial functors } \mathrm{Path}[Q]_\bullet \to \mathcal{C}_\bullet\}.$$

Restricting this construction to partially ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$, we obtain a monomorphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C}) \hookrightarrow \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$, where $\mathrm{N}_\bullet(\mathcal{C})$ is the nerve of Construction 1.3.1.1 and $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is the homotopy coherent nerve of Definition 2.4.3.5.

Example 2.4.3.9 (Vertices and Edges of the Homotopy Coherent Nerve)**.** In the cases $Q = [0]$ and $Q = [1]$, the map $\pi : \mathrm{Path}[Q]_\bullet \to Q$ is an equivalence of simplicial categories (since a path in Q is uniquely determined by its endpoints). It follows that for every simplicial category $\mathcal{C}_\bullet$, the comparison map $\mathrm{N}_\bullet(\mathcal{C}) \hookrightarrow \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ of Remark 2.4.3.8 is bijective on vertices and edges. In particular: 00KW

- Vertices of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ can be identified with objects X of the underlying category $\mathcal{C}$.

- Edges of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ can be identified with morphisms $f : X \to Y$ of the underlying category $\mathcal{C}$.
- The face operators $d^1_0, d^1_1 : \mathrm{N}^{\mathrm{hc}}_1(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_0(\mathcal{C})$ carry a morphism $f : X \to Y$ to its target $Y = d^1_0(f)$ and source $f = d^1_1(f)$, respectively.
- The degeneracy operator $s^0_0 : \mathrm{N}^{\mathrm{hc}}_0(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_1(\mathcal{C})$ carries an object $X \in \mathcal{C}$ to the identity morphism $\mathrm{id}_X : X \to X$.

00KX **Example 2.4.3.10** (2-Simplices of the Homotopy Coherent Nerve)**.** Let $Q = \{x_0 < x_1 < x_2\}$ be a linearly ordered set with three elements. Then the map $\pi : \mathrm{Path}[Q]_{\bullet} \to Q$ is *not* an equivalence of simplicial categories. In the underlying category $\mathrm{Path}[Q]$, the diagram

$$\begin{tikzcd}
& x_1 \arrow[dr, "\{x_1<x_2\}"] & \\
x_0 \arrow[ur, "\{x_0<x_1\}"] \arrow[rr, "\{x_0<x_2\}"'] & & x_2
\end{tikzcd}$$

does not commute: the composition of the diagonal maps is the path $\{x_0 < x_1 < x_2\}$. However, it commutes in a weak sense: there is an edge of the simplicial set $\mathrm{Hom}_{\mathrm{Path}[Q]}(x_0, x_2)_{\bullet}$ having source $\{x_0 < x_1 < x_2\}$ and target $\{x_0 < x_2\}$. It follows that for any simplicial category $\mathcal{C}_{\bullet}$, a choice of 2-simplex

$$\sigma \in \mathrm{N}^{\mathrm{hc}}_2(\mathcal{C}) = \mathrm{Hom}_{\mathrm{Cat}_{\Delta}}(\mathrm{Path}[2]_{\bullet}, \mathcal{C}_{\bullet}) \simeq \mathrm{Hom}_{\mathrm{Cat}_{\Delta}}(\mathrm{Path}[Q]_{\bullet}, \mathcal{C}_{\bullet})$$

determines a (possibly non-commutative) diagram σ_0:

$$\begin{tikzcd}
& X_1 \arrow[dr, "f_{21}"] & \\
X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"] & & X_2,
\end{tikzcd}$$

in $\mathcal{C}$, together with a homotopy h from $f_{21} \circ f_{10}$ to f_{20} (in the sense of Definition 2.4.1.6). Conversely, every choice of homotopy from $f_{21} \circ f_{10}$ to f_{20} determines a unique 2-simplex of $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ (see Proposition 2.4.6.10).

00KY **Example 2.4.3.11** (Comparison with the Duskin Nerve)**.** Let $\mathcal{C}$ be a strict 2-category and let $\mathcal{C}_{\bullet}$ denote the associated simplicial category (Example 2.4.2.8). For any partially ordered set Q, Remark 2.4.2.9 and Theorem 2.3.5.6 supply bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Cat}_{\Delta}}(\mathrm{Path}[Q]_{\bullet}, \mathcal{C}_{\bullet}) &\simeq \mathrm{Hom}_{2\mathrm{Cat}_{\mathrm{Str}}}(\mathrm{Path}_{(2)}[Q], \mathcal{C}) \\
&\simeq \mathrm{Hom}_{2\mathrm{Cat}_{\mathrm{ULax}}}(Q, \mathcal{C}).
\end{aligned}$$

Restricting to partially ordered sets of the form $[n] = \{0 < 1 < \cdots < n\}$, we obtain an isomorphism of simplicial sets $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}) \simeq \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$, where $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is the homotopy coherent nerve of Definition 2.4.3.5 and $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is the Duskin nerve of Construction 2.3.1.1.

Example 2.4.3.12 (The Case of an Ordinary Category)**.** Let $\mathcal{C}$ be an ordinary category, regarded as a constant simplicial category $\underline{\mathcal{C}}_{\bullet}$ via the construction of Example 2.4.2.4. Combining Examples 2.3.1.3 and Examples 2.4.3.11, we obtain isomorphisms 00KZ

$$\mathrm{N}_{\bullet}(\mathcal{C}) \simeq \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \simeq \mathrm{N}^{\mathrm{hc}}_{\bullet}(\underline{\mathcal{C}}).$$

Unwinding the definitions, we see that the composite isomorphism $\mathrm{N}_{\bullet}(\mathcal{C}) \simeq \mathrm{N}^{\mathrm{hc}}_{\bullet}(\underline{\mathcal{C}})$ is the comparison map of Remark 2.4.3.8. In other words, when restricted to constant simplicial categories, the homotopy coherent nerve of Definition 2.4.3.5 reduces to the classical nerve of Construction 1.3.1.1.

2.4.4 The Path Category of a Simplicial Set

Let G be a directed graph, which we identify with a simplicial set $G_{\bullet}$ of dimension ≤ 1 (Proposition 1.1.6.9). In §1.3.7, we introduced a category $\mathrm{Path}[G]$ called the *path category* of G (Construction 1.3.7.1). The category $\mathrm{Path}[G]$ is characterized (up to isomorphism) by a universal property: for any category $\mathcal{C}$, Proposition 1.3.7.5 supplies a bijection 00L0

$$\{\text{Functors } F : \mathrm{Path}[G] \to \mathcal{C}\} \simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(G_{\bullet}, \mathrm{N}_{\bullet}(\mathcal{C})).$$

In this section, we introduce a generalization of the construction $G \mapsto \mathrm{Path}[G]$, where we replace directed graphs by arbitrary simplicial sets (not necessarily of dimension ≤ 1) and categories by simplicial categories.

Definition 2.4.4.1. Let $S_{\bullet}$ be a simplicial set and let $\mathcal{C}_{\bullet}$ be a simplicial category. We will say that a morphism of simplicial sets $u : S_{\bullet} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ *exhibits $\mathcal{C}_{\bullet}$ as a path category of $S_{\bullet}$* if, for every simplicial category $\mathcal{D}_{\bullet}$, composition with u induces a bijection 00L1

$$\{\text{Simplicial functors } F : \mathcal{C}_{\bullet} \to \mathcal{D}_{\bullet}\} \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S_{\bullet}, \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{D})).$$

Notation 2.4.4.2 (The Path Category of a Simplicial Set)**.** Let $S_{\bullet}$ be a simplicial set. It follows immediately from the definitions that if there exists a map of simplicial sets $u : S_{\bullet} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ which exhibits $\mathcal{C}_{\bullet}$ as the path category of $S_{\bullet}$, then the simplicial category $\mathcal{C}_{\bullet}$ (and the morphism u) are uniquely determined up to isomorphism and depend functorially on $S_{\bullet}$. We will emphasize this dependence by denoting $\mathcal{C}_{\bullet}$ by $\mathrm{Path}[S]_{\bullet}$ and referring to it as the *path category* of the simplicial set $S_{\bullet}$. 00L2

Proposition 2.4.4.3. *Let $S_{\bullet}$ be a simplicial set. Then there exists a simplicial category $\mathcal{C}_{\bullet}$ and a morphism of simplicial sets $u : S_{\bullet} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ which exhibits $\mathcal{C}_{\bullet}$ as a path category of $S_{\bullet}$.* 00L3

Proof. This is a special case of Proposition 1.2.3.15, since the category Cat_{Δ} admits small colimits (Proposition 2.4.1.13). Explicitly, the simplicial path category of a simplicial set $S_{\bullet}$ is given by the generalized geometric realization

$$|S_{\bullet}|^{\mathrm{Path}[-]_{\bullet}} = \varinjlim_{\Delta^n \to S_{\bullet}} \mathrm{Path}[n]_{\bullet},$$

where $\mathrm{Path}[-]_{\bullet}$ denotes the cosimplicial object of Cat_{Δ} defined in Notation 2.4.3.1. □

00L4 **Corollary 2.4.4.4.** *The homotopy coherent nerve functor* $\mathrm{N}^{\mathrm{hc}}_{\bullet} : \mathrm{Cat}_{\Delta} \to \mathrm{Set}_{\Delta}$ *admits a left adjoint*

$$\mathrm{Path}[-]_{\bullet} : \mathrm{Set}_{\Delta} \to \mathrm{Cat}_{\Delta},$$

which associates to each simplicial set $S_{\bullet}$ *the path category* $\mathrm{Path}[S]_{\bullet}$ *of Notation 2.4.4.2.*

00L5 **Warning 2.4.4.5.** We have now introduced several different notions of path category:

(a) To every directed graph G, Construction 1.3.7.1 associates an ordinary category $\mathrm{Path}[G]$.

(b) To every partially ordered set Q, Notation 2.4.3.1 associates a simplicial category $\mathrm{Path}[Q]_{\bullet}$.

(c) To every simplicial set $S_{\bullet}$, Proposition 2.4.4.3 associates a simplicial category $\mathrm{Path}[S]_{\bullet}$.

We will show below that these constructions are closely related:

(1) If G is a directed graph and $S_{\bullet}$ denotes the associated simplicial set of dimension ≤ 1 (Proposition 1.1.6.9), then the simplicial category $\mathrm{Path}[S]_{\bullet}$ of (c) is constant, associated to the ordinary category $\mathrm{Path}[G]$ of (a) (Proposition 2.4.4.7).

(2) If Q is a partially ordered set, then the simplicial category $\mathrm{Path}[Q]_{\bullet}$ of (b) can be identified with the simplicial category $\mathrm{Path}[\mathrm{N}(Q)]_{\bullet}$ of (c), where $\mathrm{N}_{\bullet}(Q)$ denotes the nerve of Q (Proposition 2.4.4.15).

(3) For any simplicial set $S_{\bullet}$, the simplicial category $\mathrm{Path}[S]_{\bullet}$ of (c) has an underlying ordinary category $\mathrm{Path}[S]_0$, which can be described as the category $\mathrm{Path}[G]$ associated by (a) to the underlying directed graph $G = \mathrm{Gr}(S_{\bullet})$ of $S_{\bullet}$ (Proposition 2.4.4.13).

Assertions (1) and (2) imply that the path category constructions of §1.3.7 and §2.4.3 can be regarded as special cases of the construction $S_{\bullet} \mapsto \mathrm{Path}[S]_{\bullet}$. Assertion (3) is a partial converse, which guarantees that the simplicial path category $\mathrm{Path}[S]_{\bullet}$ can be regarded as a simplicially enriched version of the classical path category studied in §1.3.7.

In the special case where Q is a linearly ordered set of the form $[n] = \{0 < 1 < \cdots < n\}$, assertion (2) of Warning 2.4.4.5 is immediate from the definitions:

Example 2.4.4.6 (The Path Category of a Simplex)**.** Let $n \geq 0$ be a nonnegative integer and let $\mathrm{Path}[n]_\bullet$ denote the simplicial category of Notation 2.4.3.1. For any simplicial category $\mathcal{C}_\bullet$, we have canonical bijections 00L6

$$\mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathrm{Path}[n]_\bullet, \mathcal{C}_\bullet) \simeq \mathrm{N}^{\mathrm{hc}}_n(\mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})).$$

It follows that $\mathrm{Path}[n]_\bullet$ is a path category for the standard simplex Δ^n, in the sense of Definition 2.4.4.1.

Proposition 2.4.4.7. *Let G be a directed graph, let $\mathrm{Path}[G]$ denote the path category of Construction 1.3.7.1, and let $\underline{\mathrm{Path}[G]}_\bullet$ denote the associated constant simplicial category (Example 2.4.2.4). Then the comparison map $u : G_\bullet \to \mathrm{N}_\bullet(\mathrm{Path}[G]) \simeq \mathrm{N}^{\mathrm{hc}}_\bullet(\underline{\mathrm{Path}[G]})$ exhibits $\underline{\mathrm{Path}[G]}_\bullet$ as a path category of the simplicial set $G_\bullet$.* 00L7

Proof. Unwinding the definitions, we must show that for every simplicial category $\mathcal{D}_\bullet$, the composite map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\underline{\mathrm{Path}[G]}_\bullet, \mathcal{D}_\bullet) &\to \mathrm{Hom}_{\mathrm{Cat}}(\mathrm{Path}[G], \mathcal{D}) \\ &\to \mathrm{Hom}_{\mathrm{Set}_\Delta}(G_\bullet, \mathrm{N}_\bullet(\mathcal{D})) \\ &\to \mathrm{Hom}_{\mathrm{Set}_\Delta}(G_\bullet, \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{D})) \end{aligned}$$

is a bijection. Here the first map is bijective because the simplicial category $\underline{\mathrm{Path}[G]}_\bullet$ is constant (Remark 2.4.2.6), the second by virtue of Proposition 1.3.7.5, and the third because $G_\bullet$ has dimension ≤ 1 and the comparison map $\mathrm{N}_\bullet(\mathcal{D}) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{D})$ is an isomorphism on simplices of dimension ≤ 1 (Example 2.4.3.9). □

Warning 2.4.4.8. It follows from Proposition 2.4.4.7 that if $S_\bullet$ is a simplicial set of dimension ≤ 1, then the simplicial category $\mathrm{Path}[S]_\bullet$ is constant. Beware that this is never true for simplicial sets of dimension > 1 (see Theorem 2.4.4.10 below). 00L8

The proof of Proposition 2.4.4.3 given above is somewhat unsatisfying: it constructs the path category of a simplicial set $S_\bullet$ abstractly, as the colimit of a certain diagram in Cat_Δ. In general, colimits in Cat_Δ (like colimits in Cat) can be difficult to describe. However, the (simplicial) path category $\mathrm{Path}[S]_\bullet$ actually has a relatively simple structure. For each nonnegative integer m, the category $\mathrm{Path}[S]_m$ is *free* in the sense of Definition 1.3.7.7: that is, it can be realized as the (ordinary) path category of a directed graph. To formulate a more precise statement, we need a bit of (temporary) notation.

Notation 2.4.4.9. Let $S_\bullet$ be a simplicial set. For each nonnegative integer m, we let $E(S, m)$ denote the collection of pairs $(\sigma, \overrightarrow{I})$, where $\sigma : \Delta^n \to S_\bullet$ is a nondegenerate simplex of $S_\bullet$ of dimension $n > 0$ and $\overrightarrow{I} = (I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{m-1} \supseteq I_m)$ is a chain of subsets of $[n]$ 00L9 satisfying $I_0 = [n]$ and $I_m = \{0, n\}$. Here we will view $\overrightarrow{I}$ as a m-simplex of the simplicial set $\mathrm{Hom}_{\mathrm{Path}[n]}(0, n)_\bullet$.

Let $\mathcal{C}_\bullet$ be a simplicial category and let $u : S_\bullet \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ be a morphism of simplicial sets. For each element $(\sigma, \overrightarrow{I}) \in E(S, m)$, the composite map

$$\Delta^n \xrightarrow{\sigma} S_\bullet \xrightarrow{u} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$$

can be identified with a simplicial functor $u(\sigma) : \mathrm{Path}[n] \to \mathcal{C}$. This functor carries $\overrightarrow{I}$ to a morphism in the ordinary category $\mathcal{C}_m$, which we will denote by $u(\sigma, \overrightarrow{I})$.

00LA **Theorem 2.4.4.10.** *Let $S_\bullet$ be a simplicial set and let $u : S_\bullet \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Path}[S])$ be a morphism of simplicial sets which exhibits $\mathrm{Path}[S]_\bullet$ as a path category of $S_\bullet$. Then:*

(1) *The map u induces a bijection from the set of vertices of $S_\bullet$ to the set of objects of $\mathrm{Path}[S]_\bullet$.*

(2) *For each nonnegative integer $m \geq 0$, the category $\mathrm{Path}[S]_m$ is free (in the sense of Definition 1.3.7.7).*

(3) *For each nonnegative integer $m \geq 0$, the construction $(\sigma, \overrightarrow{I}) \mapsto u(\sigma, \overrightarrow{I})$ of Notation 2.4.4.9 induces a bijection from $E(S, m)$ to the set of indecomposable morphisms of the category $\mathrm{Path}[S]_m$.*

00LB **Remark 2.4.4.11.** Let $S_\bullet$ be a simplicial set. Then the path category $\mathrm{Path}[S]_\bullet$ is characterized (up to isomorphism) by properties (1), (2), and (3) of Theorem 2.4.4.10. More precisely, suppose that $\mathcal{C}_\bullet$ is a simplicial category and that we are given a comparison map $u' : S_\bullet \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ satisfying the following three conditions:

(1′) The map u' induces a bijection from the set of vertices of $S_\bullet$ to the set of objects of $\mathcal{C}_\bullet$.

(2′) For each nonnegative integer $m \geq 0$, the category $\mathcal{C}_m$ is free.

(3′) For each nonnegative integer $m \geq 0$, the construction $(\sigma, \overrightarrow{I}) \mapsto u'(\sigma, \overrightarrow{I})$ induces a bijection from $E(S, m)$ to the set of indecomposable morphisms of the category $\mathcal{C}_m$.

Then u' exhibits $\mathcal{C}_\bullet$ as a path category of $S_\bullet$, in the sense of Definition 2.4.4.1. To prove this, we invoke the universal property of $\mathrm{Path}[S]_\bullet$ to deduce that there is a unique simplicial functor $F : \mathrm{Path}[S]_\bullet \to \mathcal{C}_\bullet$ for which the composite map

$$S_\bullet \xrightarrow{u} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Path}[S]) \xrightarrow{\mathrm{N}^{\mathrm{hc}}_\bullet(F)} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$$

is equal to u'. Combining Theorem 2.4.4.10 with assumptions (1′), (2′), and (3′), we deduce that for each $m \geq 0$, the induced functor $\mathrm{Path}[S]_m \to \mathcal{C}_m$ is a map between free categories which is bijective on objects and indecomposable morphisms, and is therefore an isomorphism of categories.

Remark 2.4.4.12. Let $u : S_\bullet \hookrightarrow S'_\bullet$ be a monomorphism of simplicial sets. Then, for each $m \geq 0$, u induces a monomorphism of sets $E(S, m) \hookrightarrow E(S', m)$ (see Notation 2.4.4.9). It follows from Theorem 2.4.4.10 that if x and y are vertices of $S_\bullet$, then the induced map of simplicial sets $\mathrm{Hom}_{\mathrm{Path}[S]}(x, y)_\bullet \rightarrow \mathrm{Hom}_{\mathrm{Path}[S']}(u(x), u(y))_\bullet$ is a monomorphism. 01GD

Before giving the proof of Theorem 2.4.4.10, let us use it to deduce assertions (2) and (3) of Warning 2.4.4.5.

Proposition 2.4.4.13. *Let $S_\bullet$ be a simplicial set and let G be its underlying directed graph (Example 1.1.6.4), so that $G_\bullet$ can be identified with the 1-skeleton of $S_\bullet$. Let $u : S_\bullet \rightarrow \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Path}[S])$ denote the unit map. Then:* 00LC

- *The restriction $u|_{G_\bullet}$ factors uniquely as a composition*
$$G_\bullet \xrightarrow{u_0} \mathrm{N}_\bullet(\mathrm{Path}[S]_0) \rightarrow \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Path}[S]).$$
- *The map u_0 induces an isomorphism of categories $\mathrm{Path}[G] \xrightarrow{\sim} \mathrm{Path}[S]_0$.*

Proof. The first assertion follows immediately from Example 2.4.3.9, since $G_\bullet$ is a simplicial set of dimension ≤ 1. To prove the second assertion, we note that Theorem 2.4.4.10 guarantees that $\mathrm{Path}[S]_0$ is a free category, whose objects can be identified with the vertices of $S_\bullet$ and whose indecomposable morphisms can be identified with elements of the set $E(S, 0)$ of Notation 2.4.4.9. By definition, $E(S, m)$ consists of pairs $(\sigma, \overrightarrow{I})$, where σ is a nondegenerate n-simplex of $S_\bullet$ for $n > 0$ and $\overrightarrow{I} = (I_0 \supseteq \cdots \supseteq I_m)$ is a chain of subsets of $[n]$ satisfying $I_0 = [n]$ and $I_m = \{0, n\}$. In the case $m = 0$, the equality $I_0 = I_m$ forces $n = 1$, so that $E(S, 0)$ can be identified (via the morphism u_0) with the collection of nondegenerate 1-simplices of $S_\bullet$: that is, with the collection of edges of the graph G. The freeness of $\mathrm{Path}[S]_0$ now guarantees that the induced map $\mathrm{Path}[G] \simeq \mathrm{Path}[S]_0$ is an isomorphism of categories (see Proposition 1.3.7.11). □

Exercise 2.4.4.14. Use Theorem 2.4.4.10 to give a different proof of Proposition 2.4.4.7 (show that if $S_\bullet$ is a simplicial set of dimension ≤ 1, then the sets $E(S, m)$ appearing in Notation 2.4.4.9 do not depend on m). 00LD

Let Q be a partially ordered set. Note that every n-simplex $\sigma \in \mathrm{N}_\bullet(Q)$ can be identified with a map of partially ordered sets $[n] \rightarrow Q$, and therefore induces a simplicial functor $\mathrm{Path}[n]_\bullet \rightarrow \mathrm{Path}[Q]_\bullet$ which we can view as an n-simplex of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Path}[Q])$. This construction determines a map of simplicial sets $u : \mathrm{N}_\bullet(Q) \rightarrow \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Path}[Q])$.

Proposition 2.4.4.15. *Let Q be a partially ordered set. Then the comparison map $u : \mathrm{N}_\bullet(Q) \rightarrow \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Path}[Q])$ described above exhibits $\mathrm{Path}[Q]_\bullet$ as a path category for the simplicial set $\mathrm{N}_\bullet(Q)$ (in the sense of Definition 2.4.4.1).* 00LE

Proposition 2.4.4.15 follows immediately from Remark 2.4.4.11 together with the following:

`00LF` **Lemma 2.4.4.16.** *Let Q be a partially ordered set. Then the comparison map $u : \mathrm{N}_\bullet(Q) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Path}[Q])$ satisfies conditions $(1')$, $(2')$, and $(3')$ of Remark 2.4.4.11.*

Proof. Assertion $(1')$ is immediate (the morphism u is bijective on vertices by construction). For each $m \geq 0$, the category $\mathrm{Path}[Q]_m$ can be described concretely as follows:

- The objects of $\mathrm{Path}[Q]_m$ are the elements of Q.
- If x and y are elements of Q, then a morphism from x to y in $\mathrm{Path}[Q]_m$ is a chain

$$\overrightarrow{J} = (J_0 \supseteq J_1 \supseteq \cdots \supseteq J_m)$$

  of finite linearly ordered subsets of Q, where each J_i has least element x and greatest element y.

Note that a morphism $\overrightarrow{J}$ from x to y is indecomposable (in the sense of Definition 1.3.7.8) if and only if $x < y$ and $J_m = \{x, y\}$. Moreover, an arbitrary morphism $\overrightarrow{J}$ from x to y with $J_m = \{x = x_0 < x_1 < \cdots < x_k = y\}$ decomposes uniquely as a composition of indecomposable morphisms

$$x_0 \xrightarrow{\overrightarrow{J(1)}} x_1 \xrightarrow{\overrightarrow{J(2)}} x_2 \to \cdots \xrightarrow{\overrightarrow{J(k)}} x_k$$

where $J(a)_b = \{z \in J_b : x_{a-1} \leq z \leq x_a\}$. Applying Proposition 1.3.7.11, we deduce that the category $\mathrm{Path}[Q]_m$ is free, which proves $(2')$. To prove $(3')$, we observe that every indecomposable morphism $\overrightarrow{J}$ can be written uniquely in the form $u(\sigma, \overrightarrow{I})$, where $(\sigma, \overrightarrow{I})$ is an element of the set $E(S, m)$ of Notation 2.4.4.9. Writing $J_0 = \{x = x_0 < \cdots < x_n = y\}$, we see that σ must be the nondegenerate n-simplex of $\mathrm{N}_\bullet(Q)$ given by the map

$$[n] \to Q \qquad i \mapsto x_i,$$

and $\overrightarrow{I}$ must be the chain $(\sigma^{-1}(J_0) \supseteq \sigma^{-1}(J_1) \supseteq \cdots \supseteq \sigma^{-1}(J_m))$ of subsets of $[n]$. □

Proof of Theorem 2.4.4.10. Let m be a nonnegative integer, which we regard as fixed throughout the proof. For each simplicial set S, let $G(S)$ denote the directed graph given by

$$\mathrm{Vert}(G(S)) = \{\text{Vertices of } S\} \qquad \mathrm{Edge}(G(S)) = E(S, m),$$

where we regard each element

$$(\sigma : \Delta^n \to S_\bullet, \overrightarrow{I} \in \mathrm{Hom}_{\mathrm{Path}[n]}(0, n)_m) \in \mathrm{Edge}(G(S))$$

as an edge of $G(S)$ having source $\sigma(0) \in \operatorname{Vert}(G(S))$ and target $\sigma(n) \in \operatorname{Vert}(G(S))$. Let $u_S : S \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\operatorname{Path}[S])$ exhibit the simplicial category $\operatorname{Path}_{\bullet}[S]$ as a path category of S. Then u_S induces a map of simplicial sets $G(S)_{\bullet} \to \mathrm{N}_{\bullet}(\operatorname{Path}[S]_m)$, which we can identify with a functor of ordinary categories $F_S : \operatorname{Path}[G(S)] \to \operatorname{Path}[S]_m$. Let us say that the simplicial set S is *good* if F_S is an isomorphism of categories. Theorem 2.4.4.10 is equivalent to the assertion that every simplicial set is good (for every choice of nonnegative integer m). We will prove this by verifying that the collection of good simplicial sets satisfies the hypotheses of Lemma 1.2.3.13:

- Suppose we are given a pushout diagram of simplicial sets σ :

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & T \arrow[d] \\
S' \arrow[r] & T',
\end{tikzcd}$$

  where the horizontal maps are monomorphisms. Suppose that S, T, and S' are good; we wish to show that $T'_{\bullet}$ is good. Note that the horizonal maps induce monomorphisms of directed graphs

$$G(S) \hookrightarrow G(T) \qquad G(S') \hookrightarrow G(T').$$

  Define subgraphs $G_0(S) \subseteq G(S)$ and $G_0(T) \subseteq G(T)$ by the formulae

$$\operatorname{Vert}(G_0(S)) = \operatorname{Vert}(G(S)) = S_0 \qquad \operatorname{Vert}(G_0(T)) = \operatorname{Vert}(G(T)) = T_0$$

$$\operatorname{Edge}(G_0(S)) = \emptyset \qquad \operatorname{Edge}(G_0(T)) = \operatorname{Edge}(G(T)) \setminus \operatorname{Edge}(G(S)).$$

  We then have a commutative diagram of categories

$$\begin{tikzcd}
\operatorname{Path}[G_0(S)] \arrow[r] \arrow[d] & \operatorname{Path}[G_0(T)] \arrow[d] \\
\operatorname{Path}[G(S')] \arrow[r] \arrow[d, "F_{S'}"] & \operatorname{Path}[G(T')] \arrow[d, "F_{T'}"] \\
\operatorname{Path}[S']_m \arrow[r] & \operatorname{Path}[T']_m.
\end{tikzcd}$$

  We wish to show that the functor $F_{T'}$ is an isomorphism of categories, and the map $F_{S'}$ is an isomorphism by assumption. It will therefore suffice to show that the lower square in this diagram is a pushout. Note that the upper square is a pushout (since it is obtained from a pushout diagram in the category of directed graphs by passing to path categories). We are therefore reduced to showing that the outer rectangle is a pushout. We can rewrite this as the outer rectangle in another commutative diagram of categories

$$\begin{tikzcd}
\mathrm{Path}[G_0(S)] \arrow[r] \arrow[d] & \mathrm{Path}[G_0(T)] \arrow[d] \\
\mathrm{Path}[G(S)] \arrow[r] \arrow[d, "F_S"] & \mathrm{Path}[G(T)] \arrow[d, "F_T"] \\
\mathrm{Path}[S]_m \arrow[r] \arrow[d] & \mathrm{Path}[T]_m \arrow[d] \\
\mathrm{Path}[S']_m \arrow[r] & \mathrm{Path}[T']_m .
\end{tikzcd}$$

We now conclude by observing that the upper square in this diagram is a pushout (because it is obtained from a pushout diagram of directed graphs by passing to path categories), the middle square is a pushout (since F_S and F_T are isomorphisms), and the lower square is a pushout (since the construction $X_\bullet \mapsto \mathrm{Path}[X]_m$ preserves colimits).

- Suppose we are given a sequence of monomorphisms of simplicial sets

  $$S(0) \hookrightarrow S(1) \hookrightarrow S(2) \hookrightarrow \cdots$$

  with colimit S. Then the functor $F_S : \mathrm{Path}[G(S)] \to \mathrm{Path}[S]_m$ can be written as a filtered colimit of functors $F_{S(i)} : \mathrm{Path}[G(S(i))] \to \mathrm{Path}[S(i)]_m$. Consequently, if each $S(i)$ is good, then S is good.

- Let S be a simplicial set which can be written as a coproduct $\coprod_{i \in I} \Delta^n$; we must show that S is good. Without loss of generality, we may assume that I is a singleton, so that $S = \Delta^n$. In this case, Example 2.4.4.6 supplies an equivalence of simplicial categories $\mathrm{Path}[S]_\bullet \simeq \mathrm{Path}[n]_\bullet$. The desired result now follows from Lemma 2.4.4.16.

□

Remark 2.4.4.17. Let $S_\bullet$ be a simplicial set. For each $m \geq 0$, Theorem 2.4.4.10 guarantees that $\mathrm{Path}[S]_m$ can be realized as the path category of a directed graph G_m (Construction 1.3.7.1), which can be described explicitly as follows: 00LG

- The vertices of G_m are the vertices of the simplicial set $S_\bullet$.
- The edges of G_m are the elements of the set $E(S, m)$ of Notation 2.4.4.9.

It follows that we can regard the construction $[m] \mapsto \mathrm{Path}[G_m]$ as a simplicial object of Cat. The face and degeneracy operators on this simplicial object can be described as follows:

- For $0 \leq i \leq m$, the degeneracy operator $s^m_i : \mathrm{Path}[G_m] \to \mathrm{Path}[G_{m+1}]$ is induced by a map of directed graphs from G_m to G_{m+1}, which is the identity on vertices and given on edges by the construction
$$(\sigma, I_0 \supseteq \cdots \supseteq I_m) \mapsto (\sigma, I_0 \supseteq \cdots \supseteq I_{i-1} \supseteq I_i \supseteq I_i \supseteq I_{i+1} \supseteq \cdots \supseteq I_m).$$
- For $0 < i < m$, the face operator $d^m_i : \mathrm{Path}[G_m] \to \mathrm{Path}[G_{m-1}]$ is induced by a map of directed graphs from G_m to G_{m-1}, which is the identity on vertices and given on edges by the construction
$$(\sigma, I_0 \supseteq \cdots \supseteq I_m) \mapsto (\sigma, I_0 \supseteq \cdots \supseteq I_{i-1} \supseteq I_{i+1} \supseteq \cdots \supseteq I_m).$$
- Each of the face operators $d^m_0 : \mathrm{Path}[G_m] \to \mathrm{Path}[G_{m-1}]$ is induced by a morphism directed graphs $f : G_m \to G_{m-1}$ which is the identity on vertices. Let $(\sigma, \overrightarrow{I})$ be an edge of G_m, given by a nondegenerate simplex $\sigma : \Delta^n \to S_\bullet$ and a chain of subsets $\overrightarrow{I} = (I_0 \supseteq \cdots \supseteq I_m)$ of $[n]$. Then the subset $I_1 \subseteq I_0 = [n]$ is the image of a unique monotone injection $\alpha : [n'] \hookrightarrow [n]$, and the composite map $\Delta^{n'} \xrightarrow{\alpha} \Delta^n \xrightarrow{\sigma} S_\bullet$ factors uniquely as a composition $\Delta^{n'} \twoheadrightarrow \Delta^{n''} \xrightarrow{\tau} S_\bullet$, where the first map is surjective on vertices and τ is a nondegenerate n''-simplex of $S_\bullet$. For $0 \leq i < m$, let $J_i \subseteq [n'']$ denote the image of the composite map $I_{i+1} \hookrightarrow I_1 \simeq [n'] \twoheadrightarrow [n'']$, and set $\overrightarrow{J} = (J_0 \supseteq J_1 \supseteq \cdots \supseteq J_{m-1})$. In the case $n'' = 0$, the morphism f carries $(\sigma, \overrightarrow{I})$ to the vertex $\tau \in \mathrm{Vert}(G_{m-1})$. In the case $n'' > 0$ the morphism f carries $(\sigma, \overrightarrow{I})$ to the edge $(\tau, \overrightarrow{J}) \in \mathrm{Edge}(G_{m-1})$.
- The face operators $d^m_m : \mathrm{Path}[G_m] \to \mathrm{Path}[G_{m-1}]$ are generally not induced by maps of directed graphs $G_m \to G_{m-1}$: that is, they do not carry indecomposable morphisms of $\mathrm{Path}[G_m]$ to indecomposable morphisms of $\mathrm{Path}[G_{m-1}]$. More precisely, if $(\sigma, \overrightarrow{I})$ is an edge of G_n with $I_{m-1} = \{0 = i_0 < i_1 < \cdots < i_k = m\}$, then d^m_0 carries $(\sigma, \overrightarrow{I})$ to a path of length k in the category $\mathrm{Path}[G_{m-1}]$.

Let us record a consequence of Remark 2.4.4.12 which will be useful later.

0324 **Corollary 2.4.4.18.** *Let Q be a partially ordered set, let $q \in Q$ be an element, and set $Q_- = \{q_- \in Q : q_- \leq q\}$ and $Q_+ = \{q_+ \in Q : q \leq q_+\}$. Let $\mathcal{C}$ be the smallest simplicial subcategory of* $\mathrm{Path}[Q]_\bullet$ *which contains* $\mathrm{Path}[Q_-]_\bullet$ *and* $\mathrm{Path}[Q_+]_\bullet$*. Then the diagram*

$$\begin{tikzcd}
\{q\} \arrow[r] \arrow[d] & \mathrm{Path}[Q_-]_\bullet \arrow[d] \\
\mathrm{Path}[Q_+]_\bullet \arrow[r] & \mathcal{C}
\end{tikzcd}$$

is a pushout square of simplicial categories.

Proof. Using Proposition 2.4.4.15, we can identify the pushout $\mathrm{Path}[Q_-]_\bullet \coprod_{\{q\}} \mathrm{Path}[Q_+]_\bullet$ with the simplicial path category of the simplicial set $S = \mathrm{N}_\bullet(Q_-) \coprod_{\{q\}} \mathrm{N}_\bullet(Q_+)$. The tautological map $S \to \mathrm{N}_\bullet(Q)$ is a monomorphism of simplicial sets, and therefore induces an equivalence from $\mathrm{Path}[S]_\bullet$ to a simplicial subcategory $\mathcal{C} \subseteq \mathrm{Path}[Q]_\bullet$ (Remark 2.4.4.12). It is clear that this subcategory contains both $\mathrm{Path}[Q_-]_\bullet$ and $\mathrm{Path}[Q_+]_\bullet$. To complete the proof, it will suffice to show that if $\mathcal{D}$ is any other simplicial subcategory of $\mathrm{Path}[Q]_\bullet$ which contains $\mathrm{Path}[Q_-]_\bullet$ and $\mathrm{Path}[Q_+]_\bullet$, then $\mathcal{D}$ contains $\mathcal{C}$. This is clear: the universal property of $\mathcal{C}$ guarantees that there is a unique simplicial functor $F : \mathcal{C} \to \mathcal{D}$ which is the identity on both $\mathrm{Path}[Q_-]_\bullet$ and $\mathrm{Path}[Q_+]_\bullet$. Invoking the universal property of $\mathcal{C}$ again, we deduce that the composite functor $\mathcal{C} \xrightarrow{F} \mathcal{D} \hookrightarrow \mathrm{Path}[Q]_\bullet$ coincides with the inclusion map, so that $\mathcal{C} \subseteq \mathcal{D}$. □

0325 **Remark 2.4.4.19.** In the situation of Corollary 2.4.4.18, the simplicial subcategory $\mathcal{C} \subseteq \mathrm{Path}[Q]_\bullet$ can be described more concretely:

- The objects of $\mathcal{C}$ are elements of the subset $Q_- \cup Q_+ \subseteq Q$.
- Let a and b be objects of $\mathcal{C}$, and write $\mathrm{Hom}_{\mathrm{Path}[Q]}(a,b)_\bullet = \mathrm{N}_\bullet(P_{a,b})$, where $P_{a,b}$ is the collection finite linearly ordered $J \subseteq Q$ having smallest element a and largest element b, ordered by reverse inclusion. Then $\mathrm{Hom}_{\mathcal{C}}(a,b)_\bullet$ can be identified with the nerve of the partially ordered subset $P'_{a,b} \subseteq P_{a,b}$ given by

$$P'_{a,b} = \begin{cases} \{J \in P_{a,b} : q \in J\} & \text{if } a \leq q \leq b \\ P_{a,b} & \text{otherwise.} \end{cases}$$

Stated more informally, $\mathcal{C}$ is a simplicial subcategory of $\mathrm{Path}[Q]_\bullet$ whose morphisms are paths which, when possible, contain the element q.

Corollary 2.4.4.20. *Let Q be a partially ordered set, let $q \in Q$ be an element, and suppose that $Q = Q_- \cup Q_+$ for $Q_- = \{q_- \in Q : q_- \leq q\}$ and $Q_+ = \{q_+ \in Q : q \leq q_+\}$ (this condition is automatically satisfied, for example, if Q is linearly ordered). Then the simplicial functor* 0326

$$\mathrm{Path}[Q_-]_\bullet \coprod_{\{q\}} \mathrm{Path}[Q_+]_\bullet \to \mathrm{Path}[Q]_\bullet$$

has a unique left inverse $R : \mathrm{Path}[Q]_\bullet \to \mathrm{Path}[Q_-]_\bullet \coprod_{\{q\}} \mathrm{Path}[Q_+]_\bullet$.

Proof. By virtue of Corollary 2.4.4.18, we can identify the pushout $\mathrm{Path}[Q_-]_\bullet \coprod_{\{q\}} \mathrm{Path}[Q_+]_\bullet$ with a simplicial subcategory $\mathcal{C} \subseteq \mathrm{Path}[Q]_\bullet$; we wish to show that there is a unique simplicial functor $R : \mathrm{Path}[Q]_\bullet \to \mathcal{C}$ satisfying $R|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$. Our assumption that $Q = Q_- \cup Q_+$ guarantees that $\mathcal{C}$ contains every object of $\mathrm{Path}[Q]_\bullet$. To prove existence, we take the simplicial functor R to be the identity on objects and given on morphisms by the maps

$$\mathrm{Hom}_{\mathrm{Path}[Q]}(a,b)_\bullet = \mathrm{N}_\bullet(P_{a,b}) \to \mathrm{N}_\bullet(P'_{a,b}) = \mathrm{Hom}_{\mathcal{C}}(a,b)_\bullet$$

$$(J \in P_{a,b}) \mapsto \begin{cases} J \cup \{q\} & \text{if } a \leq q \leq b \\ J & \text{otherwise,} \end{cases}$$

where $P_{a,b}$ and the subset $P'_{a,b} \subseteq P_{a,b}$ are defined as in Remark 2.4.4.19.

To prove uniqueness, let $R' : \mathrm{Path}[Q]_\bullet \to \mathcal{C}$ be another simplicial functor satisfying $R'|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$; we wish to show that $R' = R$. It is clear that R and R' agree at the level of objects. For every pair of elements $a, b \in Q$, the simplicial functors R and R' induce maps $\theta, \theta' : \mathrm{Hom}_{\mathrm{Path}[Q]}(a,b)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(a,b)_\bullet$; we wish to show that $\theta = \theta'$. Since $\mathrm{Hom}_{\mathcal{C}}(a,b)_\bullet$ can be identified with the nerve of the partially ordered set $P'_{a,b}$, it will suffice to show that θ and θ' agree on vertices. For every finite linearly ordered subset $J \subseteq Q$ having least element a and greatest element b, let $f_J : a \to b$ denote the corresponding morphism in the path category $\mathrm{Path}[Q]$; we wish to show that $\theta(f_J) = \theta'(f_J)$. Without loss of generality, we may assume that the morphism f_J is indecomposable: that is, that we have $a \neq b$ and that $J = \{a < b\}$. We may further assume that $a < q < b$ (otherwise, f_J is a morphism in the category $\mathcal{C}$ and we have $\theta(f_J) = f_J = \theta'(f_J)$). Set $J^+ = \{a < q < b\}$, so that $\theta(f_J) = f_{J^+}$. Write $\theta'(f_J) = f_K$ where $K \subseteq Q$ is a finite linearly ordered subset having least element a and greatest element b. Since f_{J^+} is a morphism of $\mathcal{C}$, we have $\theta'(f_{J^+}) = f_{J^+}$. The inclusion $J \subseteq J^+$ then implies that $K \subseteq J^+$. On the other hand, f_K is also a morphism of $\mathcal{C}$, so we must have $q \in K$. It follows that $K = J^+$, so that $\theta(f_J) = f_{J^+} = f_K = \theta'(f_J)$ as desired. □

2.4.5 From Simplicial Categories to ∞-Categories

Our goal in this section is to prove the following result (see [12]): 00LH

Theorem 2.4.5.1 (Cordier-Porter)**.** *Let $\mathcal{C}_\bullet$ be a simplicial category. If $\mathcal{C}_\bullet$ is locally Kan, then the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is an ∞-category.* 00LJ

The proof of Theorem 2.4.5.1 will require some preliminaries. We begin by analyzing the relationship of the simplicial path category $\mathrm{Path}[\Delta^n]_\bullet \simeq \mathrm{Path}[n]_\bullet$ with the subcategory $\mathrm{Path}[\Lambda^n_i]_\bullet$, where $\Lambda^n_i \subseteq \Delta^n$ is an inner horn.

00LK **Notation 2.4.5.2** (Cubes as Simplicial Sets)**.** Let I be a set. We let $\square^I$ denote the simplicial set given by the product $\prod_{i \in I} \Delta^1$. We will refer to $\square^I$ as the *I-cube*. Equivalently, we can describe $\square^I$ as the nerve of the power set $P(I) = \{I_0 \subseteq I\}$, where we regard $P(I)$ as partially ordered with respect to inclusion.

In the special case where I is the set $\{1, 2, \ldots, n\}$ for some nonnegative integer n, we will denote the simplicial set $\square^I$ by $\square^n$ and refer to it as the *standard n-cube*.

00LL **Remark 2.4.5.3.** Let I be a finite set and let $\square^I$ be the I-cube of Notation 2.4.5.2. Then the geometric realization $|\square^I|$ can be identified with the topological space $\prod_{i \in I}[0,1]$. In particular, the geometric realization $|\square^n|$ is homeomorphic to the standard cube

$$\{(t_1, t_2, \ldots, t_n) \in \mathbf{R}^n : 0 \leq t_i \leq 1\}.$$

This is a tautology in the case $n = 1$, and follows in general from the compatibility of geometric realizations with products of finite simplicial sets (see Corollary 3.6.2.2).

00LM **Remark 2.4.5.4.** Let $n \geq 0$ be a nonnegative integer. For every pair of integers $0 \leq i < j \leq n$, we can identify morphisms from i to j in the path category $\mathrm{Path}[n]$ with subsets $S \subseteq [n]$ having least element i and largest element j. The construction $S \mapsto (\{i, i+1, \ldots, j-1, j\} \setminus S)$ then induces a bijection $\mathrm{Hom}_{\mathrm{Path}[n]}(i,j) \simeq P(\{i+1, i+2, \ldots, j-2, j-1\})$, which extends to uniquely to an isomorphism of simplicial sets

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Path}[n]}(i,j)_\bullet &\simeq \mathrm{N}_\bullet(P(\{i+1, i+2, \ldots, j-2, j-1\})) \\
&\simeq \square^{\{i+1, i+2, \ldots, j-2, j-1\}} \\
&\simeq \square^{j-i-1}.
\end{aligned}$$

In particular, we have a canonical isomorphism of simplicial sets $\mathrm{Hom}_{\mathrm{Path}[n]}(0,n)_\bullet \simeq \square^{n-1}$.

Under these isomorphisms, the composition law on $\mathrm{Path}[n]_\bullet$ is given for $i < j < k$ by the construction

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Path}[n]}(j,k)_\bullet \times \mathrm{Hom}_{\mathrm{Path}[n]}(i,j)_\bullet &\simeq \square^{\{j+1, \ldots, k-1\}} \times \square^{\{i+1, \ldots, j-1\}} \\
&\simeq \square^{\{j+1, \ldots, k-1\}} \times \{0\} \times \square^{\{i+1, \ldots, j-1\}} \\
&\hookrightarrow \square^{\{j+1, \ldots, k-1\}} \times \Delta^1 \times \square^{\{i+1, \ldots, j-1\}} \\
&\simeq \square^{\{i+1, \ldots, k-1\}} \\
&\simeq \mathrm{Hom}_{\mathrm{Path}[n]}(i,k)_\bullet.
\end{aligned}$$

Notation 2.4.5.5 (Subsets of the I-Cube)**.** Let I be a finite set and let $\square^I$ denote the I-cube of Notation 2.4.5.2. For each element $i \in I$, we can identify $\square^I$ with the product $\Delta^1 \times \square^{I \setminus \{i\}}$. Using this identification, we obtain simplicial subsets 00LN

$$\{0\} \times \square^{I \setminus \{i\}} \subseteq \square^I \supseteq \{1\} \times \square^{I \setminus \{i\}}$$

which we will refer to as *faces* of $\square^I$. The (disjoint) union of these two faces is another simplicial subset of $\square^I$, which we can identify with the product $\partial \Delta^1 \times \square^{I \setminus \{i\}}$.

We let $\partial \square^I$ denote the simplicial subset of $\square^I$ given by the union

$$\bigcup_{i \in I} (\partial \Delta^1 \times \square^{I \setminus \{i\}})$$

of all its faces. We will refer to $\partial \square^I$ as the *boundary of the I-cube* $\square^I$.

For $i \in I$, we let $\sqcap^I_i \subseteq \square^I$ denote the simplicial subset of $\square^I$ given by the union of the face $(\{0\} \times \square^{I \setminus \{i\}})$ with $\bigcup_{j \in I \setminus \{i\}} (\partial \Delta^1 \times \square^{I \setminus \{j\}})$. Similarly, we let $\sqcup^I_i$ denote the simplicial subset of $\square^I$ given by the union of the face $(\{1\} \times \square^{I \setminus \{i\}})$ with $\bigcup_{j \in I \setminus \{i\}} (\partial \Delta^1 \times \square^{I \setminus \{j\}})$. We will refer to the simplicial subsets $\sqcap^I_i, \sqcup^I_i \subseteq \square^I$ as *hollow I-cubes.*

In the special case where $I = \{1, \ldots, n\}$ for some nonnegative integer n, we will denote the simplicial sets $\partial \square^I$, $\sqcap^I_i$, and $\sqcup^I_i$ by $\partial \square^n$, $\sqcap^n_i$, and $\sqcup^n_i$, respectively.

Remark 2.4.5.6. Roughly speaking, one can think of the simplicial set $\partial \square^n$ as obtained from the n-cube $\square^n$ by removing its interior, while the subsets $\sqcap^n_i, \sqcup^n_i$ are obtained from $\square^n$ by removing the interior together with a single face. 00LP

Example 2.4.5.7. The standard 2-cube $\square^2 \simeq \Delta^1 \times \Delta^1$ is depicted in the diagram 00LQ

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet .
\end{tikzcd}$$

It is a simplicial set of dimension 2, having exactly two nondegenerate 2-simplices (represented by the triangular regions in the preceding diagram) and five nondegenerate edges. The boundary $\partial \square^2$ is a 1-dimensional simplicial subset of $\square^2$, obtained by removing the nondegenerate 2-simplices along with the "internal" edge to obtain the directed graph depicted in the diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet .
\end{tikzcd}$$

Each of the hollow 2-cubes $\sqcap^2_1, \sqcap^2_2, \sqcup^2_1, \sqcup^2_2$ can be obtained from $\partial \square^2$ by further deletion of a single edge, represented in the diagrams

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[dd] & & \bullet & & \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] \\
& \sqcap^2_1 & & & & \sqcap^2_2 & \\
\bullet \arrow[rr] & & \bullet & & \bullet & & \bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\bullet \arrow[rr] & & \bullet \arrow[dd] & & \bullet \arrow[dd] & & \bullet \arrow[dd] \\
& \sqcup^2_1 & & & & \sqcup^2_2 & \\
\bullet \arrow[rr] & & \bullet & & \bullet \arrow[rr] & & \bullet.
\end{tikzcd}$$

00LR **Proposition 2.4.5.8.** *Let $0 < i < n$ be positive integers and let $F : \mathrm{Path}[\Lambda^n_i]_\bullet \to \mathrm{Path}[\Delta^n]_\bullet$ be the simplicial functor induced by the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$. Then:*

(a) *The functor F is bijective on objects; in particular, we can identify the objects of $\mathrm{Path}[\Lambda^n_i]_\bullet$ with elements of the set $[n] = \{0 < 1 < \cdots < n\}$.*

(b) *For $(j, k) \neq (0, n)$, the functor F induces an isomorphism of simplicial sets*

$$\mathrm{Hom}_{\mathrm{Path}[\Lambda^n_i]}(j, k)_\bullet \simeq \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(j, k)_\bullet.$$

(c) *The functor F induces a monomorphism of simplicial sets*

$$\mathrm{Hom}_{\mathrm{Path}[\Lambda^n_i]}(0, n)_\bullet \hookrightarrow \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0, n)_\bullet,$$

whose image can be identified with the hollow cube

$$\sqcap^{n-1}_i \subseteq \square^{n-1} \simeq \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0, n)_\bullet$$

introduced in Notation 2.4.5.5.

Proof. Assertion (a) is immediate from Theorem 2.4.4.10. To prove (b) and (c), fix an integer $m \geq 0$. Using Lemma 2.4.4.16, we see that $\mathrm{Path}[\Delta^n]_m$ can be identified with the path category $\mathrm{Path}[G]$ of a directed graph G which can be described concretely as follows:

- The vertices of G are the elements of the set $[n] = \{0 < 1 < \cdots < n\}$.

- For $0 \leq j < k \leq n$, an edge of G with source j and target k is a chain of subsets

$$\{j, j+1, \ldots, k-1, k\} \supseteq I_0 \supseteq \cdots \supseteq I_m = \{j, k\}$$

Using Theorem 2.4.4.10, we see that $\mathrm{Path}[\Lambda^n_i]_m$ can be identified with the path category of the directed subgraph $G' \subseteq G$ having the same vertices, where an edge $\overrightarrow{I} = (I_0 \supseteq \cdots \supseteq I_m)$ of G belongs to G' if and only if the subset $I_0 \subseteq [n]$ corresponds to a simplex of Δ^n which belongs to the horn Λ^n_i: that is, if and only if $[n] \setminus \{i\} \nsubseteq I_0$. In particular, we see that for $(j,k) \neq (0,n)$, every edge of G with source j and target k is contained in G'. It follows that the simplicial functor F induces a bijection $\mathrm{Hom}_{\mathrm{Path}[\Lambda^n_i]}(j,k)_m \to \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(j,k)_m$ for $(j,k) \neq (0,n)$, which proves (b). Moreover, the map $\mathrm{Hom}_{\mathrm{Path}[\Lambda^n_i]}(0,n)_m \to \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n)_m$ is a monomorphism, whose image consists of those chains

$$\overrightarrow{I} = (I_0 \supseteq I_1 \supseteq \cdots \supseteq I_m)$$

where either $I_m \neq \{0,n\}$ or $([n] \setminus \{i\}) \nsubseteq I_0$. Under the identification of $\mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n)_\bullet$ with the cube $\square^{n-1} \simeq \mathrm{N}_\bullet(P(\{1,\ldots,n-1\}))$ described in Remark 2.4.5.4, this corresponds to collection of m-simplices of $\square^{n-1}$ given by chains of subsets

$$J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m \subseteq \{1, \ldots, n-1\}$$

where either $J_0 \nsubseteq \{i\}$ or $J_m \subsetneq \{1,\ldots,n-1\}$, which is exactly the set of m-simplices which belong to the hollow cube $\sqcap^{n-1}_i$. □

To apply Proposition 2.4.5.8, we record the following elementary observation about simplicial categories:

Proposition 2.4.5.9. *Let $\mathcal{E}_\bullet$ be a simplicial category containing a pair of objects $x, y \in$ $\mathrm{Ob}(\mathcal{E}_\bullet)$. Assume that, for each object $z \in \mathrm{Ob}(\mathcal{E}_\bullet)$, we have* 00LS

$$\mathrm{Hom}_{\mathcal{E}}(z,x)_\bullet = \begin{cases} \{\mathrm{id}_x\} & \text{if } z = x \\ \emptyset & \text{otherwise.} \end{cases} \qquad \mathrm{Hom}_{\mathcal{E}}(y,z)_\bullet = \begin{cases} \{\mathrm{id}_y\} & \text{if } z = y \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathcal{D}_\bullet \subseteq \mathcal{E}_\bullet$ denote a simplicial subcategory having the same objects, which satisfies

$$\mathrm{Hom}_{\mathcal{D}}(a,b)_\bullet = \mathrm{Hom}_{\mathcal{E}}(a,b)_\bullet$$

unless $(a,b) = (x,y)$. Let $F : \mathcal{D}_\bullet \to \mathcal{C}_\bullet$ be a functor of simplicial categories carrying x to an object $X = F(x)$ and y to an object $Y \in F(y)$, so that F induces a map of simplicial sets $F_{x,y} : \mathrm{Hom}_{\mathcal{D}}(x,y)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$. Then the construction $\overline{F} \mapsto \overline{F}_{x,y}$ induces a bijection

$$\begin{tikzcd} \{\text{Simplicial functors } \overline{F} : \mathcal{E}_\bullet \to \mathcal{C}_\bullet \text{ extending } F\} \arrow[d, "\sim"] \\ \{\text{Maps } \lambda : \mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \text{ extending } F_{x,y}\}. \end{tikzcd}$$

Proof. Fix a map of simplicial sets $\lambda : \mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ which extends $F_{x,y}$. We wish to show that there is a unique simplicial functor $\overline{F} : \mathcal{E}_\bullet \to \mathcal{C}_\bullet$ such that $F = \overline{F}|_{\mathcal{D}_\bullet}$ and $\overline{F}_{x,y} = \lambda$. The uniqueness is clear: the simplicial functor $\overline{F}$ must coincide with F on objects and satisfy $\overline{F}_{x',y'} = F_{x',y'}$ for $(x',y') \neq (x,y)$. To prove existence, one must show that this prescription defines a simplicial functor: that is, that for every triple of objects $a, b, c \in \mathrm{Ob}(\mathcal{E}_\bullet)$, the resulting diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(b,c)_\bullet \times \mathrm{Hom}_{\mathcal{E}}(a,b)_\bullet \arrow[r] \arrow[d, "\overline{F}_{a,b} \otimes \overline{F}_{b,c}"] & \mathrm{Hom}_{\mathcal{E}}(a,c)_\bullet \arrow[d, "\overline{F}_{a,c}"] \\
\mathrm{Hom}_{\mathcal{C}}(F(b),F(c))_\bullet \times \mathrm{Hom}_{\mathcal{C}}(F(a),F(b))_\bullet \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F(a),F(c))_\bullet
\end{tikzcd}$$

is commutative. We consider several cases:

- Suppose that $(a,b) = (x,y)$. If $c \neq y$, then the simplicial set $\mathrm{Hom}_{\mathcal{E}}(b,c)_\bullet$ is empty and the commutativity of the diagram is automatic. If $c = y$, then both compositions can be identified with the map

$$\{\mathrm{id}_y\} \times \mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet \simeq \mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet \xrightarrow{\lambda} \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet.$$

- Suppose that $(b,c) = (x,y)$. If $a \neq x$, then the simplicial set $\mathrm{Hom}_{\mathcal{E}}(a,b)_\bullet$ is empty and the commutativity of the diagram is automatic. If $a = x$, then both compositions can be identified with the map

$$\mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet \times \{\mathrm{id}_x\} \simeq \mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet \xrightarrow{\lambda} \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet.$$

- If neither $(a,b) = (x,y)$ or $(b,c) = (x,y)$, then the desired result follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(b,c)_\bullet \times \mathrm{Hom}_{\mathcal{D}}(a,b)_\bullet \arrow[r] \arrow[d, "F_{a,b} \otimes F_{b,c}"] & \mathrm{Hom}_{\mathcal{D}}(a,c)_\bullet \arrow[d, "F_{a,c}"] \\
\mathrm{Hom}_{\mathcal{C}}(F(b),F(c))_\bullet \times \mathrm{Hom}_{\mathcal{C}}(F(a),F(b))_\bullet \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(F(a),F(c))_\bullet
\end{tikzcd}$$

(since F is assumed to be a simplicial functor).

□

It follows from Proposition 2.4.5.8 that for $0 < i < n$, the hypotheses of Proposition 2.4.5.9 are satisfied by the inclusion $\mathcal{D}_\bullet = \mathrm{Path}[\Lambda^n_i]_\bullet \hookrightarrow \mathrm{Path}[\Delta^n]_\bullet = \mathcal{E}_\bullet$ and the objects $x = 0$ and $y = n$. We therefore obtain the following:

Corollary 2.4.5.10. *Let $\mathcal{C}_\bullet$ be a simplicial category, let $0 < i < n$, and let $\sigma_0 : \Lambda^n_i \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ be a map of simplicial sets, which we can identify with a simplicial functor $F : \mathrm{Path}[\Lambda^n_i]_\bullet \to \mathcal{C}_\bullet$ inducing a map of simplicial sets* 00LT

$$\lambda_0 : \sqcap^{n-1}_i \simeq \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_i]}(0, n)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet.$$

Then we have a canonical bijection

$$\begin{tikzcd}
\{\text{Maps } \sigma : \Delta^n \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \text{ with } \sigma_0 = \sigma|_{\Lambda^n_i}\} \arrow[d] \\
\{\text{Maps } \lambda : \square^{n-1} \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet \text{ with } \lambda_0 = \lambda|_{\sqcap^{n-1}_i}\}.
\end{tikzcd}$$

To deduce Theorem 2.4.5.1 from Corollary 2.4.5.10, we will need the following standard characterization of Kan complexes (for a proof, see Proposition 4.4.2.1):

Theorem 2.4.5.11 (Homotopy Extension Lifting Property)**.** *Let $X_\bullet$ be a simplicial set. The following conditions are equivalent:* 00LU

(1) *The simplicial set $X_\bullet$ is a Kan complex.*

(2) *The inclusion of simplicial sets $\{0\} \hookrightarrow \Delta^1$ induces a trivial Kan fibration* $\mathrm{Fun}(\Delta^1, X_\bullet) \to \mathrm{Fun}(\{0\}, X_\bullet) \simeq X_\bullet$.

(3) *The inclusion of simplicial sets $\{1\} \hookrightarrow \Delta^1$ induces a trivial Kan fibration* $\mathrm{Fun}(\Delta^1, X_\bullet) \to \mathrm{Fun}(\{1\}, X_\bullet) \simeq X_\bullet$.

Corollary 2.4.5.12. *Let $X_\bullet$ be a Kan complex and let I be a finite set containing a distinguished element i. Then:* 00LV

(*a*) *Every map of simplicial sets $f : \sqcup^I_i \to X_\bullet$ can be extended to a map $\overline{f} : \square^I \to X_\bullet$.*

(*b*) *Every map of simplicial sets $g : \sqcap^I_i \to X_\bullet$ can be extended to a map $\overline{g} : \square^I \to X_\bullet$.*

Proof. Unwinding the definitions, we see that $\sqcup^I_i$ can be identified with the pushout

$$(\{1\} \times \square^{I \setminus \{i\}}) \coprod_{\{1\} \times \partial \square^{I \setminus \{i\}}} (\Delta^1 \times \partial \square^{I \setminus \{i\}}).$$

Consequently, a map of simplicial sets $f : \sqcup_i^I \to X_\bullet$ can be identified with a commutative diagram of solid arrows

$$\begin{tikzcd}
\partial \square^{I \setminus \{i\}} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, X_\bullet) \arrow[d] \\
\square^{I \setminus \{i\}} \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\{1\}, X_\bullet),
\end{tikzcd}$$

and an extension $\overline{f} : \square^I \to X_\bullet$ of f can be identified with a solution to the associated lifting problem. If $X_\bullet$ is a Kan complex, then the right vertical arrow is a trivial Kan fibration (Theorem 2.4.5.11), so the desired extension exists by virtue of Proposition 1.5.5.4. This proves (a); the proof of (b) is similar. □

Proof of Theorem 2.4.5.1. Let $\mathcal{C}_\bullet$ be a locally Kan simplicial category; we wish to show that the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ is an ∞-category. Fix positive integers $0 < i < n$; we wish to show that every map of simplicial sets $\sigma_0 : \Lambda_i^n \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ can be extended to an n-simplex $\sigma : \Delta^n \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$. Let us identify σ_0 with a simplicial functor $F : \mathrm{Path}[\Lambda_i^n]_\bullet \to \mathcal{C}_\bullet$ inducing a map of simplicial sets $\lambda_0 : \sqcap_i^{n-1} \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet$. By virtue of Corollary 2.4.5.10, it will suffice to show that λ_0 can be extended to a map of simplicial sets $\lambda : \square^{n-1} \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet$. The existence of this extension follows from Corollary 2.4.5.12, by virtue of our assumption that $\mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet$ is a Kan complex. □

2.4.6 The Homotopy Category of a Simplicial Category

00LW For every simplicial set $S_\bullet$, we let $\pi_0(S_\bullet)$ denote the set of connected components of $S_\bullet$ (Definition 1.2.1.8). Recall that the functor $\pi_0 : \mathrm{Set}_\Delta \to \mathrm{Set}$ preserves finite products (Corollary 1.2.1.27). Applying Remark 2.1.7.4, we obtain the following:

00LX **Construction 2.4.6.1** (The Homotopy Category of a Simplicial Category)**.** Let $\mathcal{C}_\bullet$ be a simplicial category. We define an ordinary category $\mathrm{h}\mathcal{C}$ as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of the simplicial category $\mathcal{C}_\bullet$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathrm{h}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$, we have

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet).$$

- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathrm{h}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$, the composition map

$$\circ : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Z)$$

is given by the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y) &= \pi_0(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet) \times \pi_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet) \\
&\xleftarrow{\sim} \pi_0(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet) \\
&\rightarrow \pi_0(\mathrm{Hom}_{\mathcal{C}}(X,Z)_\bullet) \\
&= \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Z).
\end{aligned}$$

We will refer to $\mathrm{h}\mathcal{C}$ as the *homotopy category of* $\mathcal{C}$.

Remark 2.4.6.2 (The Component Functor)**.** Let $\mathcal{C}_\bullet$ be a simplicial category and let $\mathrm{h}\mathcal{C}$ be its homotopy category (Construction 2.4.6.1). For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}_\bullet) = \mathrm{Ob}(\mathrm{h}\mathcal{C})$, Construction 1.2.1.18 supplies a map of simplicial sets 00LY

$$u_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \underline{\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)}_\bullet.$$

Here $\underline{\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)}_\bullet$ denotes the constant simplicial set associated to the set $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)$, and $u_{X,Y}$ carries each n-simplex of $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ to the connected component which contains it. Allowing X and Y to vary, we obtain a simplicial functor $u : \mathcal{C}_\bullet \to \underline{\mathrm{h}\mathcal{C}}_\bullet$ which is the identity on objects; we will refer to u as the *component functor*.

Remark 2.4.6.3. Let $\mathcal{C}_\bullet$ be a simplicial category with underlying category $\mathcal{C} = \mathcal{C}_0$. Then the simplicial functor $u : \mathcal{C}_\bullet \to \underline{\mathrm{h}\mathcal{C}}_\bullet$ induces a functor of ordinary categories $u_0 : \mathcal{C} \to \mathrm{h}\mathcal{C}$, which can be described as follows: 00LZ

- On objects, the functor u_0 is the identity map from $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathrm{h}\mathcal{C})$ to itself.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathrm{h}\mathcal{C})$, the induced map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)$ is a surjection, which we will denote by $f \mapsto [f]$.
- Given a pair of morphisms $f, g; X \to Y$ in $\mathcal{C}$ having the same source and target, we have $[f] = [g]$ if and only if f and g belong to the same connected component of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$.

Remark 2.4.6.4. Let $\mathcal{C}_\bullet$ be a simplicial category with underlying category $\mathcal{C} = \mathcal{C}_0$, and let $f, g : X \to Y$ be a pair of morphisms of $\mathcal{C}$ having the same source and target. Using Remark 1.2.1.23, we see that the following conditions are equivalent: 00M0

(*a*) The morphisms f and g represent the same morphism in the homotopy category $\mathrm{h}\mathcal{C}$: that is, we have $[f] = [g]$.

(*b*) There exists a sequence of morphisms $f = f_0, f_1, f_2, \ldots, f_n = g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that, for $1 \leq i \leq n$, either there exists a homotopy from f_{i-1} to f_i or a homotopy from f_i to f_{i-1} (in the sense of Definition 2.4.1.6).

If $\mathcal{C}_\bullet$ is locally Kan, then we can replace (b) by the following simpler condition:

(c) There exists a homotopy from f to g (in the sense of Definition 2.4.1.6).

See Remark 2.4.1.9.

02BH **Example 2.4.6.5.** Let $\mathcal{C}$ be a strict 2-category (Definition 2.2.0.1) and let $\mathcal{C}_\bullet$ denote the associated simplicial category (Example 2.4.2.8). Then the homotopy category $\mathrm{h}\mathcal{C}_\bullet$ of the simplicial category $\mathcal{C}_\bullet$ (in the sense of Construction 2.4.6.1) can be identified with the coarse homotopy category $\mathrm{h}\mathcal{C}$ of $\mathcal{C}$ (in the sense of Construction 2.2.8.2).

00M1 **Example 2.4.6.6** (The Homotopy Category of Top)**.** Let Top denote the category of topological spaces and continuous functions, endowed with the simplicial enrichment $\mathrm{Top}_\bullet$ described in Example 2.4.1.5. Then the homotopy category hTop is the homotopy category of *all* topological spaces: the objects of hTop are topological spaces, and the morphisms of hTop are homotopy classes of continuous maps.

The homotopy category of a simplicial category can be characterized by a universal mapping property:

00M2 **Proposition 2.4.6.7.** *Let $\mathcal{C}_\bullet$ be a simplicial category and let $u : \mathcal{C}_\bullet \to \underline{\mathrm{h}\mathcal{C}}_\bullet$ be the simplicial functor described in Remark 2.4.6.2. Then, for any category $\mathcal{D}$, composition with u induces a bijection*

$$\{\text{Ordinary Functors } f : \mathrm{h}\mathcal{C} \to \mathcal{D}\} \to \{\text{Simplicial Functors } F : \mathcal{C}_\bullet \to \underline{\mathcal{D}}_\bullet\}.$$

Proof. Use Proposition 1.2.1.19. □

00M3 **Corollary 2.4.6.8.** *The fully faithful embedding*

$$\mathrm{Cat} \hookrightarrow \mathrm{Cat}_\Delta \qquad \mathcal{D} \mapsto \underline{\mathcal{D}}_\bullet$$

of Example 2.4.2.4 admits a left adjoint, given on objects by the formation of homotopy categories $\mathcal{C}_\bullet \mapsto \mathrm{h}\mathcal{C}$.

We have now introduced two different notions of homotopy category:

- The homotopy category $\mathrm{h}\mathcal{C}$ of a simplicial category $\mathcal{C}_\bullet$, given by Construction 2.4.6.1.
- The homotopy category $\mathrm{h}S_\bullet$ of a simplicial set $S_\bullet$, defined in Definition 1.3.6.1 (and described more explicitly in §1.4.5 when $S_\bullet$ is an ∞-category).

These constructions are related. Let $\mathcal{C}_\bullet$ be a simplicial category. Applying the homotopy coherent nerve to the component functor u of Remark 2.4.6.2, we obtain a map of simplicial sets

$$\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \xrightarrow{\mathrm{N}^{\mathrm{hc}}_\bullet(u)} \mathrm{N}^{\mathrm{hc}}_\bullet(\underline{\mathrm{h}\mathcal{C}}) \simeq \mathrm{N}_\bullet(\mathrm{h}\mathcal{C}),$$

which we can identify with a functor of ordinary categories $U : \mathrm{h}\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \to \mathrm{h}\mathcal{C}$.

Proposition 2.4.6.9. *Let $\mathcal{C}_\bullet$ be a locally Kan simplicial category. Then the construction above induces an isomorphism of categories $U : \mathrm{hN}^{\mathrm{hc}}_\bullet(\mathcal{C}) \xrightarrow{\sim} \mathrm{h}\mathcal{C}$.* 00M4

To prove Proposition 2.4.6.9, we need to analyze the 2-simplices of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$. Recall that the vertices and edges of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ can be identified with objects and morphisms in the underlying category $\mathcal{C} = \mathcal{C}_0$ (Example 2.4.3.9). In particular, a map of simplicial sets $\sigma_0 : \partial\,\Delta^2 \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ can be identified with a (possibly noncommutative) diagram

$$\begin{tikzcd}
 & X_1 \arrow[dr, "f_{21}"] & \\
X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & & X_2
\end{tikzcd}$$

in the category $\mathcal{C}$. We will need the following:

Proposition 2.4.6.10. *Let $\mathcal{C}_\bullet$ be a simplicial category and let $\sigma_0 : \partial\,\Delta^2 \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ be a map of simplicial sets, which we identify with a diagram* 00M5

$$\begin{tikzcd}
 & X_1 \arrow[dr, "f_{21}"] & \\
X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & & X_2
\end{tikzcd}$$

as above. Then the construction of Example 2.4.3.10 induces a bijection

$$\begin{tikzcd}
\{\text{Maps } \sigma : \Delta^2 \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \text{ with } \sigma|_{\partial\,\Delta^2} = \sigma_0\} \arrow[d, "\sim"] \\
\{\text{Homotopies from } f_{21} \circ f_{10} \text{ to } f_{20}\}.
\end{tikzcd}$$

It is not difficult to deduce Proposition 2.4.6.10 directly from the definition of the homotopy coherent nerve. We will instead deduce it from a more general result (Corollary 2.4.6.13), which supplies an analogous description of the n-simplices of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ for all $n > 0$. First, let us note some consequences of Proposition 2.4.6.10.

Example 2.4.6.11. Let $\mathcal{C}_\bullet$ be a locally Kan simplicial category, so that the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is an ∞-category (Theorem 2.4.5.1). Suppose we are given a pair of 00M6 morphisms $f, g : X \to Y$ in the underlying category $\mathcal{C} = \mathcal{C}_0$ having the same source and target. Let $\sigma_0 : \partial\,\Delta^2 \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ be the map corresponding to the (possibly noncommutative) diagram

$$\begin{tikzcd} & Y \arrow[dr, "\mathrm{id}_Y"] & \\ X \arrow[ur, "f"] \arrow[rr, "g"] & & Y. \end{tikzcd}$$

Applying Proposition 2.4.6.10 we obtain a bijection from the set of homotopies from f to g in the ∞-category $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ (in the sense of Definition 1.4.3.1) to the set of homotopies from f to g in the simplicial category $\mathcal{C}_{\bullet}$ (in the sense of Definition 2.4.1.6). In particular, we see that f and g are homotopic in $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ if and only if they are homotopic in $\mathcal{C}_{\bullet}$.

Proof of Proposition 2.4.6.9. Let $\mathcal{C}_{\bullet}$ be a locally Kan simplicial category; we wish to show that the comparison map $U : \mathrm{hN}^{\mathrm{hc}}_{\bullet}(\mathcal{C}) \xrightarrow{\sim} \mathrm{h}\mathcal{C}$ is an isomorphism of categories. By construction, U is bijective on objects. It will therefore suffice to show that for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the induced map

$$U_{X,Y} : \mathrm{Hom}_{\mathrm{hN}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$$

is a bijection. This is precisely the content of Example 2.4.6.11. □

We will deduce Proposition 2.4.6.10 from the following variant of Proposition 2.4.5.8:

00M7 **Proposition 2.4.6.12.** *Let n be a positive integer and let $F : \mathrm{Path}[\partial\Delta^n]_{\bullet} \to \mathrm{Path}[\Delta^n]_{\bullet}$ be the simplicial functor induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$. Then:*

(*a*) *The functor F is bijective on objects; in particular, we can identify objects of $\mathrm{Path}[\partial\Delta^n]_{\bullet}$ with elements of the set $[n] = \{0 < 1 < \cdots < n\}$.*

(*b*) *For $(j,k) \neq (0,n)$, the functor F induces an isomorphism of simplicial sets*

$$\mathrm{Hom}_{\mathrm{Path}[\partial\Delta^n]}(j,k)_{\bullet} \simeq \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(j,k)_{\bullet}.$$

(*c*) *The functor F induces a monomorphism of simplicial sets $\mathrm{Hom}_{\mathrm{Path}[\partial\Delta^n]}(0,n)_{\bullet} \hookrightarrow \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n)_{\bullet}$, whose image can be identified with the boundary $\partial\square^{n-1} \subseteq \square^{n-1} \simeq \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n)_{\bullet}$ introduced in Notation 2.4.5.5.*

Proof. Assertion (*a*) is immediate from Theorem 2.4.4.10. To prove (*b*) and (*c*), fix an integer $m \geq 0$ and let us identify $\mathrm{Path}[\Delta^n]_m$ with the path category $\mathrm{Path}[G]$ of the directed graph G appearing in the proof of Proposition 2.4.5.8. Using Theorem 2.4.4.10, we see that $\mathrm{Path}[\partial\Delta^n]_m$ can be identified with the path category of the directed subgraph $G' \subseteq G$

having the same vertices, where an edge $\overrightarrow{I} = (I_0 \supseteq \cdots \supseteq I_m)$ of G belongs to G' unless $I_0 = [n]$. In particular, we see that for $(j,k) \neq (0,n)$, every edge of G with source j and target k is contained in G'. It follows that the simplicial functor F induces a bijection

$$\mathrm{Hom}_{\mathrm{Path}[\partial \Delta^n]}(j,k)_m \to \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(j,k)_m$$

for $(j,k) \neq (0,n)$, which proves (b). Moreover, the map

$$\mathrm{Hom}_{\mathrm{Path}[\partial \Delta^n]}(0,n)_m \to \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n)_m$$

is a monomorphism, whose image consists of those chains

$$\overrightarrow{I} = (I_0 \supseteq I_1 \supseteq \cdots \supseteq I_m)$$

where either $I_0 \neq [n]$ or $I_m \neq \{0,n\}$. Under the identification of $\mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n)_\bullet$ with the cube $\square^{n-1} \simeq \mathrm{N}_\bullet(P(\{1,\ldots,n-1\}))$ described in Remark 2.4.5.4, this is exactly the set of m-simplices which belong to the boundary $\partial \square^{n-1} \subseteq \square^{n-1}$. □

Combining Propositions 2.4.6.12 and 2.4.5.9, we obtain the following:

Corollary 2.4.6.13. *Let $\mathcal{C}_\bullet$ be a simplicial category, let $n > 0$, and let $\sigma_0 : \partial \Delta^n \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ be a map of simplicial sets, which we identify with a simplicial functor $F : \mathrm{Path}[\partial \Delta^n]_\bullet \to \mathcal{C}_\bullet$ inducing a map of simplicial sets* 00M8

$$\lambda_0 : \partial \square^{n-1} \simeq \mathrm{Hom}_{\mathrm{Path}[\partial \Delta^n]}(0,n)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(F(0),F(n))_\bullet.$$

Then we have a canonical bijection

$$\begin{tikzcd}
\{\text{Maps } \sigma : \Delta^n \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C}) \text{ with } \sigma_0 = \sigma|_{\partial \Delta^n}\} \arrow[d] \\
\{\text{Maps } \lambda : \square^{n-1} \to \mathrm{Hom}_{\mathcal{C}}(F(0),F(n))_\bullet \text{ with } \lambda_0 = \lambda|_{\partial \square^{n-1}}\}
\end{tikzcd}.$$

Example 2.4.6.14 (1-Simplices of the Homotopy Coherent Nerve)**.** Let $\mathcal{C}_\bullet$ be a simplicial category. By definition, giving a map of simplicial sets $\sigma_0 : \partial\, \Delta^1 \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ is equivalent to giving a pair of objects $X_0 = \sigma_0(0)$ and $X_1 = \sigma_0(1)$ of the underlying category $\mathcal{C} = \mathcal{C}_0$. Applying Corollary 2.4.6.13, we deduce that extending σ_0 to a 1-simplex of $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ is equivalent to supplying a morphism $f : X_0 \to X_1$ in the category $\mathcal{C}$ (see Example 2.4.3.9). 00M9

Proof of Proposition 2.4.6.10. Apply Corollary 2.4.6.13 in the case $n = 2$. □

00MA **Example 2.4.6.15** (3-Simplices of the Homotopy Coherent Nerve)**.** Let $\mathcal{C}_\bullet$ be a simplicial category. Using Proposition 2.4.6.10, we see that a map of simplicial sets $\sigma_0 : \partial\,\Delta^3 \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ can be identified with the following data:

- A collection of four objects $\{X_i \in \mathcal{C}\}_{0 \leq i \leq 3}$.
- A collection of six morphisms $\{f_{ji} \in \mathrm{Hom}_{\mathcal{C}}(X_i, X_j)\}_{0 \leq i < j \leq 3}$.
- A collection of four 1-simplices $\{h_{kji} \in \mathrm{Hom}_{\mathcal{C}}(X_i, X_k)_1\}_{0 \leq i < j < k \leq 3}$, where each h_{kji} is a homotopy from $f_{kj} \circ f_{ji}$ to f_{ki}.

From this data, we can assemble a map of simplicial sets $\lambda_0 : \partial \square^2 \to \mathrm{Hom}_{\mathcal{C}}(X_0, X_3)_\bullet$, which is represented by the diagram

$$\begin{tikzcd}
f_{32} \circ f_{21} \circ f_{10} \arrow[r, "h_{321} \circ \mathrm{id}_{f_{10}}"] \arrow[d, "\mathrm{id}_{f_{32}} \circ h_{210}"] & f_{31} \circ f_{10} \arrow[d, "h_{310}"] \\
f_{32} \circ f_{20} \arrow[r, "h_{320}"] & f_{30}.
\end{tikzcd}$$

Corollary 2.4.6.13 then asserts that extending σ_0 to a 3-simplex of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is equivalent to extending λ_0 to a map of simplicial sets $\lambda : \square^2 \to \mathrm{Hom}_{\mathcal{C}}(X_0, X_3)_\bullet$. Stated more informally, the map σ_0 supplies two potentially different paths from the composition $f_{32} \circ f_{21} \circ f_{10}$ to f_{30} in the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X_0, X_3)_\bullet$. To extend σ_0 to a 3-simplex of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$, one must supply additional data which "witnesses" that these paths are homotopic.

We close this section with a refinement of Construction 2.4.6.1:

02BJ **Construction 2.4.6.16** (The Homotopy 2-Category of a Simplicial Category)**.** Let $\mathcal{C}_\bullet$ be a simplicial category. We define a strict 2-category $\mathrm{h}_2\mathcal{C}$ as follows:

- The objects of $\mathrm{h}_2\mathcal{C}$ are the objects of the simplicial category $\mathcal{C}_\bullet$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathrm{h}_2\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$, the category $\underline{\mathrm{Hom}}_{\mathrm{h}_2\mathcal{C}}(X, Y)$ is the homotopy category of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet$.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathrm{h}_2\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$, the composition map

$$\circ : \underline{\mathrm{Hom}}_{\mathrm{h}_2\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathrm{h}_2\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathrm{h}_2\mathcal{C}}(X, Z)$$

is given by the composition

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathrm{h}_2\mathcal{C}}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathrm{h}_2\mathcal{C}}(X,Y) &= (\mathrm{hHom}_{\mathcal{C}}(Y,Z)_\bullet) \times (\mathrm{hHom}_{\mathcal{C}}(X,Y)_\bullet) \\
&\xleftarrow{\sim} \mathrm{h}(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet) \\
&\to \mathrm{hHom}_{\mathcal{C}}(X,Z)_\bullet \\
&= \mathrm{Hom}_{\mathrm{h}_2\mathcal{C}}(X,Z),
\end{aligned}$$

where the isomorphism is supplied by Corollary 1.5.3.6.

We will refer to $\mathrm{h}_2\mathcal{C}$ as the *homotopy 2-category of $\mathcal{C}$*.

Remark 2.4.6.17. Let $\mathcal{C}_\bullet$ be a simplicial category and let $\mathrm{h}_2\mathcal{C}$ denote the homotopy 2-category of $\mathcal{C}$. Then the underlying category $\mathcal{C}_0$ of $\mathcal{C}_\bullet$ (in the sense of Example 2.4.1.4) coincides with the underlying category of the strict 2-category $\mathrm{h}_2\mathcal{C}$ (in the sense of Remark 2.2.0.3). 02BK

Remark 2.4.6.18. Let $\mathcal{C}_\bullet$ be a simplicial category. Then the homotopy category of $\mathcal{C}_\bullet$ can be identified with the coarse homotopy category of the homotopy 2-category $\mathrm{h}_2\mathcal{C}$ of Construction 2.4.6.16, in the sense of Construction 2.2.8.2. That is, we have a canonical isomorphism $\mathrm{h}\mathcal{C} \simeq \mathrm{h}(\mathrm{h}_2\mathcal{C})$. 02BL

Remark 2.4.6.19. Let $\mathcal{C}_\bullet$ be a simplicial category, let $\mathrm{h}_2\mathcal{C}$ be the homotopy 2-category of $\mathcal{C}$, and let $(\mathrm{h}_2\mathcal{C})_\bullet$ denote the simplicial category obtained from $\mathrm{h}_2\mathcal{C}$ by applying the construction of Example 2.4.2.8. Then there is a simplicial functor $U : \mathcal{C}_\bullet \to (\mathrm{h}_2\mathcal{C})_\bullet$, given on objects by the identity map and on morphism spaces by the tautological maps 02BM

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{N}_\bullet(\mathrm{hHom}_{\mathcal{C}}(X,Y)_\bullet).$$

Passing to the homotopy coherent nerve (and invoking Example 2.4.3.11), we obtain a map of simplicial sets $V : \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{h}_2\mathcal{C})$, which restricts to the identity on the nerve $\mathrm{N}_\bullet(\mathcal{C})$ (which we can regard as a simplicial subset of both $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ and $\mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{h}_2\mathcal{C})$).

Remark 2.4.6.20. Let $\mathcal{C}_\bullet$ be a simplicial category. The comparison map $V : \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{h}_2\mathcal{C})$ of Remark 2.4.6.19 is always bijective at the level of vertices (which can be identified with the objects of the category $\mathcal{C}_0$ underlying $\mathcal{C}_\bullet$) and edges (which can be identified with morphisms of $\mathcal{C}_0$). Suppose that, for every pair of objects $C, D \in \mathcal{C}_0$, the simplicial set $\underline{\mathrm{Hom}}_{\mathcal{C}}(C,D)_\bullet$ is an ∞-category. In this case, the map V is also surjective (but not necessarily injective) at the level of 2-simplices. By virtue of Example 2.3.1.15, we can identify 2-simplices 02BN

$\overline{\sigma}$ of $\mathrm{N}_{\bullet}^{\mathrm{D}}(\mathrm{h}_2\mathcal{C})$ with diagrams

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow{[\mu]}"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z,
\end{tikzcd}$$

where $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ are morphisms in $\mathcal{C}_0$, and $[\mu] : g \circ f \to h$ is a morphism in the *homotopy category* of the ∞-category $\mathrm{Hom}_{\mathcal{C}}(X,Z)_{\bullet}$. To lift $\overline{\sigma}$ to a 2-simplex σ of the homotopy coherent nerve $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})$, one must choose a morphism $\mu : g \circ f \to h$ in the ∞-category $\mathrm{Hom}_{\mathcal{C}}(X,Z)_{\bullet}$ which represents the homotopy class $[\mu]$ (see Example 2.4.3.10). Such a representative always exists, but is not necessarily unique.

Using the universal property of the homotopy category, we immediately obtain the following variant of Proposition 2.4.6.7:

02BP **Proposition 2.4.6.21.** *Let $\mathcal{C}_{\bullet}$ be a simplicial category and let $U : \mathcal{C}_{\bullet} \to (\mathrm{h}_2\mathcal{C})_{\bullet}$ be the simplicial functor described in Remark 2.4.6.19. Then, for any strict 2-category $\mathcal{D}$, composition with U induces a bijection*

$$\{\text{Strict functors } f : \mathrm{h}_2\mathcal{C} \to \mathcal{D}\} \to \{\text{Simplicial Functors } F : \mathcal{C}_{\bullet} \to \mathcal{D}_{\bullet}\};$$

here $\mathcal{D}_{\bullet}$ denote the simplicial category associated to $\mathcal{D}$ by Example 2.4.2.8.

2.4.7 Example: Braid Monoids

00MZ In general, the path category $\mathrm{Path}[S]_{\bullet}$ associated to a simplicial set $S_{\bullet}$ is a fairly complicated object. In this section, we describe one situation in which it admits a particularly concrete description, which arises in the theory of Coxeter groups. Let us begin by reviewing some terminology.

00N0 **Definition 2.4.7.1.** A *Coxeter system* is a pair (W, S), where W is a group and $S \subseteq W$ is a subset with the following properties:

- Each element of S has order 2.
- For each $s, t \in S$, let $m_{s,t} \in \mathbf{Z}_{>0} \cup \{\infty\}$ denote the order of the product st in the group W. Then the inclusion $S \hookrightarrow W$ exhibits W as the quotient of the free group generated by S by the relations $(st)^{m_{s,t}} = 1$ (indexed by those pairs (s,t) with $m_{s,t} < \infty$).

Remark 2.4.7.2. We will use the term *Coxeter group* to refer to a group W together with a choice of subset $S \subseteq W$ for which the pair (W, S) is a Coxeter system. Beware that the subset S is not determined by the structure of W as an abstract group: for example, if (W, S) is a Coxeter system, then so is (W, wSw^{-1}) for each $w \in W$. In other words, a Coxeter group is not merely a group, but a group equipped with some additional structure (namely, the structure of a Coxeter system (W, S)). 00N1

Notation 2.4.7.3 (Lengths)**.** Let (W, S) be a Coxeter system. Then the group W is generated by S: that is, every element of W can be written as a product of elements of S. For each $w \in W$, we let $\ell(w)$ denote the smallest nonnegative integer n for which w factors as a product $s_1 s_2 \cdots s_n$, where each s_i belongs to S. We will refer to $\ell(w)$ as the *length* of w. 00N2

Remark 2.4.7.4. Let (W, S) be a Coxeter system. Then the length function $\ell : W \to \mathbf{Z}_{\geq 0}$ has the following properties: 00N3

- An element $w \in W$ satisfies $\ell(w) = 0$ if and only if $w = 1$ is the identity element of W.
- An element $w \in W$ satisfies $\ell(w) = 1$ if and only if w belongs to S.
- For every pair of elements $w, w' \in W$, we have $\ell(ww') \leq \ell(w) + \ell(w')$. Moreover, we also have $\ell(ww') \equiv \ell(w) + \ell(w') \pmod 2$.

Construction 2.4.7.5 (The Braid Group)**.** Let (W, S) be a Coxeter system. We let $\mathrm{Br}(W)$ denote the quotient of the free group generated by S by the relations 00N4

$$s \cdot t \cdot s \cdots = t \cdot s \cdot t \cdots ; \tag{2.2}$$ 05HV

here s and t range over *distinct* elements of S satisfying $m_{s,t} < \infty$ (where $m_{s,t}$ is the order of st in the group W), and the product appearing on each side of (2.2) has $m_{s,t}$-factors. We will refer to $\mathrm{Br}(W)$ as the *braid group* of the Coxeter system (W, S). By construction, the braid group $\mathrm{Br}(W)$ is equipped with a surjective group homomorphism $\mathrm{Br}(W) \twoheadrightarrow W$, which exhibits W as the quotient of $\mathrm{Br}(W)$ by the relations $s^2 = 1$ for $s \in S$.

Let $\mathrm{Br}^+(W)$ denote the submonoid of $\mathrm{Br}(W)$ generated by the elements of S. We will refer to $\mathrm{Br}^+(W)$ as the *braid monoid* of the Coxeter system (W, S).

In [13], Deligne gave a convenient simplicial presentation for the braid monoid $\mathrm{Br}^+(W)$ in the case where the Coxeter group W is finite. To formulate it, we need a bit more terminology.

Notation 2.4.7.6. Let W be a Coxeter group with identity element 1. We let $M_0(W)$ denote the free monoid generated by the set $W \setminus \{1\}$. We will identify the elements of $M_0(W)$ with finite sequences $\vec{w} = (w_1, w_2, \ldots, w_n)$, where each w_i is an element of $W \setminus \{1\}$. 00N5

We will say that $\vec{v}$ *is a refinement of* $\vec{w}$ if there exists a strictly increasing sequence of integers $0 = i_0 < i_1 < \cdots < i_n = m$ having the property that

$$w_j = v_{i_{j-1}+1} v_{i_{j-1}+2} \cdots v_{i_j}$$

$$\ell(w_j) = \ell(v_{i_{j-1}+1}) + \ell(v_{i_{j-1}+2}) + \cdots + \ell(v_{i_j})$$

for $1 \leq j \leq n$. We write $\vec{v} \preceq \vec{w}$ to indicate that $\vec{v}$ is a refinement of $\vec{w}$. Then $\preceq$ determines a partial ordering on the set $M_0(W)$. We denote the nerve of this partially ordered set by $M_\bullet(W)$. Note that the multiplication on $M_0(W)$ (given by concatenation) endows $M_\bullet(W)$ with the structure of a simplicial monoid.

00N6 **Exercise 2.4.7.7.** Let W be a Coxeter group, and let $\vec{v} = (v_1, v_2, \ldots, v_m)$ and $\vec{w} = (w_1, \ldots, w_n)$ be elements of $M(W)$. Show that, if $\vec{v}$ is a refinement of $\vec{w}$, then there is a *unique* sequence of integers $0 = j_0 < j_1 < \cdots < j_m = n$ satisfying the condition specified in Notation 2.4.7.6.

00N7 **Remark 2.4.7.8.** Let (W, S) be a Coxeter system. Then an element $\vec{w} = (w_1, w_2, \ldots, w_n)$ of $M_0(W)$ is minimal (with respect to the refinement ordering $\preceq$) if and only if each w_i belongs to S. Moreover, every element $\vec{w} \in M_0(W)$ admits a refinement $\vec{s} = (s_1, s_2, \ldots, s_m)$ which is minimal in $M_0(W)$ (given by choosing a decomposition of each w_i as a product of elements of S). In particular, every connected component of the simplicial set $M_\bullet(W)$ contains a vertex $\vec{s} = (s_1, \ldots, s_m)$, where each s_i belongs to S.

00N8 **Theorem 2.4.7.9** (Deligne)**.** *Let (W, S) be a Coxeter system for which the underlying Coxeter group W is finite, and let $\mathrm{Br}^+(W)$ denote the braid monoid of Construction 2.4.7.5. Then:*

- *(a) There is an isomorphism of monoids $f : \pi_0(M_\bullet(W)) \to \mathrm{Br}^+(W)$ which is uniquely determined by the following property: if $\vec{s} = (s_1, s_2, \ldots, s_m) \in M_0(W)$ is a sequence of elements of S, then f carries the connected component of $\vec{s}$ to the product $s_1 s_2 \cdots s_m \in \mathrm{Br}^+(W)$.*
- *(b) Each connected component of $M_\bullet(W)$ is weakly contractible (Definition 3.2.4.15).*

In other words, the isomorphism f determines a weak homotopy equivalence of simplicial monoids $M_\bullet(W) \to \mathrm{Br}^+(W)$.

Proof. This is a special case of Théorème 2.4 of [13]. □

We now reformulate the definition of the simplicial monoid $M_\bullet(W)$ using the theory of simplicial path categories.

Notation 2.4.7.10. Let (W, S) be a Coxeter system and let $B_\bullet W$ denote the classifying simplicial set of the group W (Construction 1.3.2.5). For each nonnegative integer n, let us identify $B_n W$ with the collection of all n-tuples $(w_n, w_{n-1}, \ldots, w_1)$ of elements of W. Let $B_n^\circ W$ denote the subset of $B_n W$ consisting of those sequences $(w_n, w_{n-1}, \ldots, w_1)$ satisfying the identity 00N9

$$\ell(w_1 w_2 \cdots w_n) = \ell(w_1) + \ell(w_2) + \cdots + \ell(w_n).$$

It is easy to see that the collection of subsets $B_n^\circ W \subseteq B_n W$ are stable under the face and degeneracy operators of $B_\bullet W$, and therefore determine a simplicial subset $B_\bullet^\circ W \subseteq B_\bullet W$.

Construction 2.4.7.11. Let (W, S) be a Coxeter system, let $M_\bullet(W)$ be the simplicial monoid of Notation 2.4.7.6, and let $BM_\bullet(W)$ denote the simplicial category obtained by delooping $M_\bullet(W)$ (Example 2.4.2.3), having a single object X with $\mathrm{Hom}_{BM(W)}(X, X)_\bullet = M_\bullet(W)$. 00NA

Let $\sigma = (w_n, \ldots, w_1)$ be a nondegenerate n-simplex of the simplicial set $B_\bullet^\circ(W)$ (Notation 2.4.7.10). Then σ determines a simplicial functor $u(\sigma) : \mathrm{Path}[n]_\bullet \to BM_\bullet(W)$, which carries each object of $\mathrm{Path}[n]_\bullet$ to the unique object X of $BM_\bullet(W)$, and each morphism $I = \{i_0 < \ldots < i_k\} \in \mathrm{Hom}_{\mathrm{Path}[n]}(i_0, i_k)$ to the sequence

$$(v_1, v_2, \ldots, v_k) \in M_0(W) \qquad v_j = w_{i_{j-1}+1} w_{i_{j-1}+2} \cdots w_{i_j}.$$

Regarding $u(\sigma)$ as an n-simplex of the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(BM(W))$, the construction $\sigma \mapsto u(\sigma)$ extends to a map of simplicial sets $u : B_\bullet^\circ(W) \to \mathrm{N}_\bullet^{\mathrm{hc}}(BM(W))$.

Proposition 2.4.7.12. *Let (W, S) be a Coxeter system. Then the map of simplicial sets $u : B_\bullet^\circ(W) \to \mathrm{N}_\bullet^{\mathrm{hc}}(BM(W))$ of Construction 2.4.7.11 exhibits $BM_\bullet(W)$ as a path category of the simplicial set $B_\bullet^\circ(W)$, in the sense of Definition 2.4.4.1.* 00NB

Proof. Fix an integer $m \geq 0$. Then $BM_m(W)$ is the delooping of the monoid $M_m(W)$ whose elements are tuples

$$\overrightarrow{w}_0 \preceq \overrightarrow{w}_1 \preceq \overrightarrow{w}_2 \preceq \cdots \preceq \overrightarrow{w}_m,$$

where each $\overrightarrow{w}_i \in M_0(W)$ is a sequence $(w_{i,1}, w_{i,2}, \ldots, w_{i,n_i})$ of elements of $W \setminus \{1\}$. Moreover, the monoid structure on $M_m(W)$ is given by concatenation. From this description, it is easy to see that the monoid $M_m(W)$ is freely generated by its indecomposable elements, which are precisely those sequences for which the sequence $\overrightarrow{w}_m$ has length 1. In this case, the relation $\overrightarrow{w}_0 \preceq \overrightarrow{w}_m$ guarantees that $\overrightarrow{w}_0$ is a nondegenerate n_0-simplex of the simplicial set $B_\bullet^\circ(W)$. It follows that the map u induces a bijection from the set $E(B^\circ(W), m)$ of Notation 2.4.4.9 to the set of indecomposable elements of the monoid $M_m(W)$. The desired result now follows from the criterion of Remark 2.4.4.11. □

Corollary 2.4.7.13. *Let W be a finite Coxeter group, and let $B_\bullet^\circ(W) \subseteq B_\bullet(W)$ be the simplicial subset of Notation 2.4.7.10. Then the simplicial path category* $\mathrm{Path}[B^\circ(W)]_\bullet$ *has* 00NC *a single object X, whose endomorphism monoid* $\mathrm{Hom}_{\mathrm{Path}[B^{\circ}(W)]}(X,X)_{\bullet}$ *is weakly homotopy equivalent to the braid monoid* $\mathrm{Br}^{+}(W)$ *of Construction 2.4.7.5.*

Proof. Combine Proposition 2.4.7.12 with Theorem 2.4.7.9. □

2.5 Differential Graded Categories

00ND Homological algebra provides a plentiful supply of examples of ∞-categories. Let us begin by reviewing some terminology.

00NE **Definition 2.5.0.1.** Let $\mathcal{A}$ be an additive category (Definition [?]). A *chain complex with values in* $\mathcal{A}$ is a pair (C_*, ∂), where $C_* = \{C_n\}_{n\in\mathbf{Z}}$ is a collection of objects of $\mathcal{A}$ and $\partial = \{\partial_n\}_{n\in\mathbf{Z}}$ is a collection of morphisms $\partial_n : C_n \to C_{n-1}$ in $\mathcal{A}$ with the property that each composition $\partial_n \circ \partial_{n+1}$ is the zero morphism from C_{n+1} to C_{n-1}.

00NF **Notation 2.5.0.2.** Let $\mathcal{A}$ be an additive category. Then a chain complex (C_*, ∂) with values in $\mathcal{A}$ can be graphically represented by a diagram

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \cdots$$

in which each successive composition is equal to zero. We will generally abuse terminology by identifying (C_*, ∂) with the underlying collection $C_* = \{C_n\}_{n\in\mathbf{Z}}$, which we will refer to as a *graded object of* $\mathcal{A}$. We view $\partial = \{\partial_n\}_{n\in\mathbf{Z}}$ as an endomorphism of C_* which is homogeneous of degree -1, which we refer to as the *differential* or the *boundary operator* of the chain complex C_*. We will generally abuse notation by omitting the subscript from the expression ∂_n; that is, we denote each of the boundary operators $C_n \to C_{n-1}$ by the same symbol ∂ (or ∂_C, when we need to emphasize its association with the particular chain complex C_*).

Chain complexes with values in an additive category $\mathcal{A}$ can themselves be organized into a category.

00NG **Definition 2.5.0.3.** Let (C_*, ∂_C) and (D_*, ∂_D) be chain complexes with values in an additive category $\mathcal{A}$. A *chain map from* (C_*, ∂_C) *and* (D_*, ∂_D) is a collection $f = \{f_n\}_{n\in\mathbf{Z}}$, where each f_n is a morphism from C_n to D_n in the category $\mathcal{A}$, for which each of the diagrams

$$\begin{tikzcd}
C_n \arrow[r, "\partial_C"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_n \arrow[r, "\partial_D"] & D_{n-1}
\end{tikzcd}$$

is commutative.

If $\mathcal{A}$ is an additive category, we let $\mathrm{Ch}(\mathcal{A})$ denote the category whose objects are chain complexes with values in $\mathcal{A}$ and whose morphisms are chain maps.

Notation 2.5.0.4. Let k be a commutative ring. We will write $\mathrm{Ch}(k)$ for the category $\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A}$ is the category of k-modules and k-module homomorphisms. In particular, we will write $\mathrm{Ch}(\mathbf{Z})$ for the category of chain complexes of abelian groups. `00NH`

Definition 2.5.0.5 (Chain Homotopy)**.** Let $\mathcal{A}$ be an additive category and let (C_*, ∂_C) and (D_*, ∂_D) be chain complexes with values in $\mathcal{A}$. Let $f = \{f_n\}_{n \in \mathbf{Z}}$ and $f' = \{f'_n\}_{n \in \mathbf{Z}}$ be chain maps from C_* to D_*. A *chain homotopy from f to f'* is a collection of maps $h = \{h_n : C_n \to D_{n+1}\}$ which satisfy the identity `00NJ`

$$f'_n - f_n = \partial_D \circ h_n + h_{n-1} \circ \partial_C$$

for every integer n.

We say that f and f' are *chain homotopic* if there exists a chain homotopy from f to f'. We will say that f is a *chain homotopy equivalence* if there exists a chain map $g : D_* \to C_*$ such that $g \circ f$ and $f \circ g$ are chain homotopic to the identity morphisms id_{C_*} and id_{D_*}, respectively.

Example 2.5.0.6. Let $\mathcal{A}$ be an additive category and suppose we are given a chain complex `04GB`

$$\cdots \to C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} C_{-2} \to \cdots$$

with values in $\mathcal{A}$. A *contracting homotopy* for (C_*, ∂) is a chain homotopy from the zero morphism $0 : C_* \to C_*$ to the identity morphism $\mathrm{id} : C_* \to C_*$ (in the sense of Definition 2.5.0.5). More concretely, a contracting homotopy is a system of morphisms $\{h_n : C_n \to C_{n+1}\}_{n \in \mathbf{Z}}$ which satisfy the identity $\mathrm{id}_{C_n} = \partial \circ h_n + h_{n-1} \circ \partial$ for every integer n. We will say that the complex (C_*, ∂) is *contractible* if it admits a contracting homotopy.

Remark 2.5.0.7. Let C_* and D_* be chain complexes with values in an additive category $\mathcal{A}$. Then chain homotopy determines an equivalence relation on the set of chain maps $f : C_* \to D_*$. More precisely: `00NK`

- Every chain map $f : C_* \to D_*$ is chain homotopic to itself, via the chain homotopy given by the collection of zero maps $\{0 : C_n \to D_{n+1}\}$.
- Let $f, f' : C_* \to D_*$ be chain maps. If f is chain homotopic to f', then f' is chain homotopic to f. More precisely, if h is a chain homotopy from f to f', then $-h$ is a chain homotopy from f' to f.
- Let $f, f', f'' : C_* \to D_*$ be chain maps. If f is chain homotopic to f' and f' is chain homotopic to f'', then f is chain homotopic to f''. More precisely, if h is a chain homotopy from f to f' and h' is a chain homotopy from f' to f'', then $h + h'$ is a chain homotopy from f to f''.

00NL **Remark 2.5.0.8.** Let C_* and D_* be chain complexes with values in an additive category $\mathcal{A}$, and let $f, f' : C_* \to D_*$ be chain maps which are chain homotopic. Then:

- For every chain map $g : D_* \to E_*$, the composite maps $g \circ f$ and $g \circ f'$ are chain homotopic. More precisely, if $h = \{h_n\}_{n \in \mathbf{Z}}$ is a chain homotopy from f to f', then the collection of composite maps $\{g_{n+1} \circ h_n\}$ is a chain homotopy from $g \circ f$ to $g \circ f'$.
- For every chain map $e : B_* \to C_*$, the composite maps $f \circ e$ and $f' \circ e$ are chain homotopic. More precisely, if $h = \{h_n\}_{n \in \mathbf{Z}}$ is a chain homotopy from f to f', then the collection of composite maps $\{h_n \circ e_n\}$ is a chain homotopy from $f \circ e$ to $f' \circ e$.

00NM **Construction 2.5.0.9** (The Homotopy Category of Chain Complexes)**.** Let $\mathcal{A}$ be an additive category. We define a category $\mathrm{hCh}(\mathcal{A})$ as follows:

- The objects of $\mathrm{hCh}(\mathcal{A})$ are chain complexes with values in $\mathcal{A}$.
- If C_* and D_* are chain complexes with values in $\mathcal{A}$, then $\mathrm{Hom}_{\mathrm{hCh}(\mathcal{A})}(C_*, D_*)$ is the quotient of $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, D_*)$ by the relation of chain homotopy equivalence. If $f : C_* \to D_*$ is a chain map, we denote its equivalence class by $[f] \in \mathrm{Hom}_{\mathrm{hCh}(\mathcal{A})}(C_*, D_*)$.
- If C_*, D_*, and E_* are chain complexes with values in $\mathcal{A}$, then the composition law

  $$\circ : \mathrm{Hom}_{\mathrm{hCh}(\mathcal{A})}(D_*, E_*) \times \mathrm{Hom}_{\mathrm{hCh}(\mathcal{A})}(C_*, D_*) \to \mathrm{Hom}_{\mathrm{hCh}(\mathcal{A})}(C_*, E_*)$$

  is uniquely determined by the requirement that $[g] \circ [f] = [g \circ f]$ for every pair of chain maps $f : C_* \to D_*$ and $g : D_* \to E_*$ (this operation is well-defined by virtue of Remark 2.5.0.8).

We will refer to $\mathrm{hCh}(\mathcal{A})$ as the *homotopy category* of $\mathrm{Ch}(\mathcal{A})$.

The definition of the homotopy category $\mathrm{hCh}(\mathcal{A})$ of chain complexes is analogous to the definition of the homotopy category hTop of topological spaces: the latter is obtained by working with continuous functions up to homotopy, and the former by working with chain maps up to chain homotopy. As with its topological counterpart, passage from $\mathrm{Ch}(\mathcal{A})$ to $\mathrm{hCh}(\mathcal{A})$ is a destructive procedure. By enforcing the equality $[f] = [f']$ whenever there *exists* a chain homotopy h from f to f', we sacrifice the ability to extract information which depends on a particular *choice* of chain homotopy. The situation can be remedied by contemplating a more elaborate structure.

00NN **Construction 2.5.0.10** (Mapping Complexes)**.** Let (C_*, ∂_C) and (D_*, ∂_D) be chain complexes with values in an additive category $\mathcal{A}$. For each integer d, we let $[C, D]_d$ denote the abelian group $\prod_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{A}}(C_n, D_{n+d})$ consisting of maps from C_* to D_* which are homogeneous of degree d. These abelian groups can be organized into a chain complex

$$\cdots \xrightarrow{\partial} [C,D]_2 \xrightarrow{\partial} [C,D]_1 \xrightarrow{\partial} [C,D]_0 \xrightarrow{\partial} [C,D]_{-1} \xrightarrow{\partial} [C,D]_{-2} \xrightarrow{\partial} \cdots,$$

whose boundary operator $\partial : [C,D]_d \to [C,D]_{d-1}$ is given by the formula $\partial\{f_n : C_n \to D_{n+d}\}_{n \in \mathbf{Z}} = \{\partial_D \circ f_n - (-1)^d f_{n-1} \circ \partial_C\}_{n \in \mathbf{Z}}$. We will refer to $[C,D]_*$ as the *mapping complex* associated to the chain complexes C_* and D_*.

Note that from the mapping complexes $[C,D]_*$, we can extract *both* the set of chain maps $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, D_*)$ and the set of homotopy equivalence classes $\mathrm{Hom}_{\mathrm{hCh}(\mathcal{A})}(C_*, D_*)$:

- Chain maps from C_* to D_* can be identified with 0-*cycles* of the chain complex $[C,D]_*$: that is, with elements $f = \{f_n\}_{n \in \mathbf{Z}} \in [C,D]_0$ satisfying $\partial(f) = 0$.
- Given a pair of chain maps $f, f' : C_* \to D_*$, a chain homotopy from f to f' is an element $h = \{h_n\}_{n \in \mathbf{Z}} \in [C,D]_1$ satisfying $\partial(h) = f' - f$. In particular, f and f' are chain homotopic if and only if they are *homologous* when viewed as 0-cycles of the complex $[C,D]_*$, so $\mathrm{Hom}_{\mathrm{hCh}(\mathcal{A})}(C_*, D_*)$ can be identified with the 0th homology group of $[C,D]_*$.

Moreover, the mapping complexes of Construction 2.5.0.10 are equipped with maps

$$\circ : [D,E]_m \times [C,D]_n \to [C,E]_{m+n},$$

which refine the composition laws on the categories $\mathrm{Ch}(\mathcal{A})$ and $\mathrm{hCh}(\mathcal{A})$. In §2.5.2, we axiomatize this structure by introducing the notion of a *differential graded category* (Definition 2.5.2.1). By definition, a differential graded category is a category which is enriched over the category $\mathrm{Ch}(\mathbf{Z})$ of graded abelian groups (endowed with the monoidal structure given by the tensor product of chain complexes, which we review in §2.5.1). The category of chain complexes $\mathrm{Ch}(\mathcal{A})$ is a prototypical example of a differential graded category (Example 2.5.2.5), with the enrichment supplied by the mapping complexes of Construction 2.5.0.10.

Let $\mathcal{C}$ be a differential graded category. To every pair of objects $X, Y \in \mathcal{C}$, the enrichment of $\mathcal{C}$ supplies a chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$, whose 0-cycles are the morphisms from X to Y in $\mathcal{C}$. Heuristically, one can think of this data as endowing $\mathcal{C}$ with the structure of a higher category, whose n-morphisms (for $n \geq 2$) are given by the elements of $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{n-1}$ (for varying X and Y). In §2.5.3, we make this heuristic precise by constructing a simplicial set $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$ called the *differential graded nerve* of $\mathcal{C}$ (Definition 2.5.3.7), and proving that it is an ∞-category in the sense of Definition 1.4.0.1 (Theorem 2.5.3.10). In §2.5.4 we show that the homotopy category of $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$ can be obtained directly from $\mathcal{C}$ by identifying homotopic morphisms (Proposition 2.5.4.10); in particular, the homotopy category of $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathrm{Ch}(\mathcal{A}))$ can be identified with the homotopy category of chain complexes $\mathrm{hCh}(\mathcal{A})$ of Construction 2.5.0.9.

The remainder of this section is devoted to studying the relationship between the differential graded nerve $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$ and the homotopy coherent nerve of §2.4. This will require a somewhat lengthy detour through the theory of simplicial abelian groups. In §2.5.5, we will associate to each simplicial set $S_\bullet$ its *normalized chain complex* $\mathrm{N}_*(S; \mathbf{Z})$, given in each degree n by the free abelian group on the set of nondegenerate n-simplices of $S_\bullet$ (Construction 2.5.5.9). The construction $S_\bullet \mapsto \mathrm{N}_*(S; \mathbf{Z})$ determines a functor from the category of simplicial sets to the category $\mathrm{Ch}(\mathbf{Z})$ of chain complexes of abelian groups. In §2.5.6, we show that this functor has a right adjoint $\mathrm{K} : \mathrm{Ch}(\mathbf{Z}) \to \mathrm{Set}_\Delta$, which we will refer to as the *Eilenberg-MacLane functor* (Construction 2.5.6.3). To each chain complex of abelian groups M_*, this functor associates a simplicial abelian group $\mathrm{K}(M_*)$, which we will refer to as the *(generalized) Eilenberg-MacLane space of* M_*. Moreover, the celebrated *Dold-Kan correspondence* (Theorem 2.5.6.1) asserts that the Eilenberg-MacLane functor restricts to an equivalence

$$\mathrm{Ch}(\mathbf{Z})_{\geq 0} \xrightarrow{\sim} \{\text{Simplicial Abelian Groups}\},$$

where $\mathrm{Ch}(\mathbf{Z})_{\geq 0} \subset \mathrm{Ch}(\mathbf{Z})$ denotes the full subcategory spanned by those chain complexes which are concentrated in nonnegative degrees (Definition 2.5.1.1).

Let $S_\bullet$ and $T_\bullet$ be simplicial sets. In §2.5.8, we review the classical *Alexander-Whitney construction*, which supplies a chain map

$$\mathrm{AW} : \mathrm{N}_*(S \times T; \mathbf{Z}) \to \mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(T; \mathbf{Z});$$

here the right hand side denotes the tensor product of the normalized chain complexes $\mathrm{N}_*(S; \mathbf{Z})$ and $\mathrm{N}_*(T; \mathbf{Z})$. Allowing $S_\bullet$ and $T_\bullet$ to vary, these maps determine a lax monoidal structure on the Eilenberg-MacLane functor $\mathrm{K} : \mathrm{Ch}(\mathbf{Z}) \to \mathrm{Set}_\Delta$. Using this structure, we will associate to each differential graded category $\mathcal{C}$ a simplicial category $\mathcal{C}_\bullet^\Delta$ having the same objects, with simplicial mapping sets given by $\mathrm{Hom}_{\mathcal{C}^\Delta}(X, Y) = \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X, Y)_*)$ (Construction 2.5.9.2). In §2.5.9, we construct a comparison map $\mathfrak{Z}$ from the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C}^\Delta)$ to the differential graded nerve $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$ (Proposition 2.5.9.10), and show that it is a trivial Kan fibration (Theorem 2.5.9.18). The proof of this result (and the construction of the map $\mathfrak{Z}$) rely heavily on the *shuffle product* $\triangledown : \mathrm{N}_*(S; \mathbf{Z}) \times \mathrm{N}_*(T; \mathbf{Z}) \to \mathrm{N}_*(S \times T; \mathbf{Z})$ introduced by Eilenberg and MacLane, which we review in §2.5.7.

00NP **Warning 2.5.0.11.** The differential graded nerve construction $\mathcal{C} \mapsto \mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$ can be used to produce many interesting examples of ∞-categories. However, not every ∞-category can be obtained in this way (even up to equivalence). Put differently, ∞-categories of the form $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$ have some special features, which are not shared by general ∞-categories. For example, if $\mathcal{C}$ is a *pretriangulated* differential graded category (Definition [?]), then the differential graded nerve $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$ is a *stable* ∞-category (see Proposition [?]).

2.5.1 Generalities on Chain Complexes

00NQ In this section, we provide a brief review of some of the homological algebra which will be needed throughout §2.5.

Definition 2.5.1.1. Let $\mathcal{A}$ be an additive category, let C_* be a chain complex with values in $\mathcal{A}$, and let n be an integer. We will say that C_* is *concentrated in degrees* $\geq n$ if objects $C_m \in \mathcal{A}$ are zero for $m < n$. Similarly, we say that C_* is *concentrated in degrees* $\leq n$ if the objects C_m are zero for $m > n$. We let $\mathrm{Ch}(\mathcal{A})_{\geq n}$ denote the full subcategory of $\mathrm{Ch}(\mathcal{A})$ spanned by those chain complexes which are concentrated in degrees $\geq n$, and $\mathrm{Ch}(\mathcal{A})_{\leq n}$ the full subcategory spanned by those chain complexes which are concentrated in degrees $\leq n$. 00NR

Example 2.5.1.2. Let $\mathcal{A}$ be an additive category, let $C \in \mathcal{A}$ be an object, and let n be an integer. We will write $C[n]$ for the chain complex given by 00NS

$$C[n]_* = \begin{cases} C & \text{if } * = n \\ 0 & \text{otherwise,} \end{cases}$$

where each differential is the zero morphism. Note that a chain complex M_* is isomorphic to $C[n]$ (for some object $C \in \mathcal{A}$) if and only if it is concentrated both in degrees $\geq n$ and in degrees $\leq n$.

Notation 2.5.1.3 (Cycles and Boundaries)**.** Let $\mathcal{A}$ be an abelian category (Definition [?]) and let C_* be a chain complex with values in $\mathcal{A}$. For each integer n, we let $\mathrm{Z}_n(C)$ denote the kernel of the boundary operator $\partial : C_n \to C_{n-1}$, and $\mathrm{B}_n(C)$ the image of the boundary operator $\partial : C_{n+1} \to C_n$. We regard $\mathrm{Z}_n(C)$ and $\mathrm{B}_n(C)$ as subobjects of C_n. Note that we have $\mathrm{B}_n(C) \subseteq \mathrm{Z}_n(C)$ (this is a reformulation of the identity $\partial^2 = 0$). 00NT

In the special case where $\mathcal{A} = \mathrm{Ab}$ is the category of abelian groups, we will refer to the elements of C_n as *n-chains* of C_*, to the elements of $\mathrm{Z}_n(C)$ as *n-cycles* of C_*, and to the elements of $\mathrm{B}_n(C)$ as *n-boundaries* of C_*.

Definition 2.5.1.4 (Homology)**.** Let $\mathcal{A}$ be an abelian category and let C_* be a chain complex with values in $\mathcal{A}$. For every integer n, we let $\mathrm{H}_n(C)$ denote the quotient $\mathrm{Z}_n(C)/\mathrm{B}_n(C)$. We will refer to $\mathrm{H}_n(C)$ as the *nth homology* of the chain complex C_*. We say that the chain complex C_* is *acyclic* if the homology objects $\mathrm{H}_n(C)$ vanish for every integer n. 00NU

If $\mathcal{A} = \mathrm{Ab}$ is the category of abelian groups and if $x \in \mathrm{Z}_n(C)$ is an n-cycle of C_*, we let $[x]$ denote its image in the homology group $\mathrm{H}_n(C)$: we refer to $[x]$ as the *homology class* of x. We say that a pair of n-cycles $x, x' \in \mathrm{Z}_n(C)$ are *homologous* if $[x] = [x']$: that is, if there exists an $(n+1)$-chain y satisfying $x' = x + \partial(y)$.

Definition 2.5.1.5 (Quasi-Isomorphisms)**.** Let $\mathcal{A}$ be an abelian category, let C_* and D_* be chain complexes with values in $\mathcal{A}$, and let $f : C_* \to D_*$ be a chain map. We say that f is a *quasi-isomorphism* if, for every integer n, the induced map of homology objects $\mathrm{H}_n(C) \to \mathrm{H}_n(D)$ is an isomorphism. 00NV

Remark 2.5.1.6. Let C_* be a chain complex with values in an abelian category $\mathcal{A}$. In practice, the homology objects $\mathrm{H}_*(C)$ are often primary objects of interest, while the chain 00NW complex C_* itself plays an ancillary role. The terminology of Definition 2.5.1.5 emphasizes this perspective: a chain map $f : C_* \to D_*$ which induces an isomorphism on homology should allow us to view the chain complexes C_* and D_* as “the same” for many purposes (this idea is the starting point for Verdier’s theory of *derived categories*, which we will discuss in §[?]).

00NX **Remark 2.5.1.7** (Two-out-of-Three)**.** Let $\mathcal{A}$ be an abelian category and suppose we are given a commutative diagram of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C'_* \arrow[r] \arrow[d, "f'"] & C_* \arrow[r] \arrow[d, "f"] & C''_* \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & D'_* \arrow[r] & D_* \arrow[r] & D''_* \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact. If any two of the chain maps f, f', and f'' are quasi-isomorphisms, then so is the third. This follows by comparing the long exact homology sequences associated to the upper and lower rows (see Construction [?]).

00NY **Proposition 2.5.1.8.** *Let C_* and D_* be chain complexes with values in an abelian category $\mathcal{A}$, and let $f, f' : C_* \to D_*$ be a pair of chain maps. If f and f' are chain homotopic, then they induce the same map from $\mathrm{H}_n(C)$ to $\mathrm{H}_n(D)$ for every integer n.*

Proof. Let $h = \{h_m\}_{m \in \mathbf{Z}}$ be a chain homotopy from f to f', so that $f'_n - f_n = \partial_D \circ h_n + h_{n-1} \circ \partial_C$. It follows that, when restricted to the subobject $\mathrm{Z}_n(C) \subseteq C_n$, the difference $f'_n - f_n = \partial_D \circ h_n$ factors through the subobject $\mathrm{B}_n(D) \subseteq \mathrm{Z}_n(D)$, so the induced maps $\mathrm{H}_n(f), \mathrm{H}_n(f') : \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ are the same. □

00NZ **Corollary 2.5.1.9.** *Let $f : C_* \to D_*$ be a chain map between chain complexes with values in an abelian category $\mathcal{A}$. If f is a chain homotopy equivalence, then it is a quasi-isomorphism.*

For later use, we record the following elementary fact:

00P0 **Proposition 2.5.1.10.** *Let P_* be a chain complex taking values in an abelian category $\mathcal{A}$. Assume that P_* is acyclic, concentrated in degrees ≥ 0, and that each P_n is a projective object of $\mathcal{A}$. Then P_* is a projective object of the category $\mathrm{Ch}(\mathcal{A})$. In other words, every epimorphism of chain complexes $f : M_* \twoheadrightarrow P_*$ admits a section.*

Proof. Our assumption that P_* is acyclic guarantees that for every integer $n \geq 0$, we have a short exact sequence

$$0 \to \mathrm{Z}_n(P) \to P_n \xrightarrow{\partial} \mathrm{Z}_{n-1}(P) \to 0.$$

It follows by induction on n that each of these exact sequences splits and that each $\mathrm{Z}_n(P)$ is also a projective object of $\mathcal{A}$. We can therefore choose a direct sum decomposition $P_n \simeq \mathrm{Z}_n(P) \oplus Q_n$, where the differential on P_* restricts to isomorphisms $\partial : Q_n \simeq \mathrm{Z}_{n-1}(P)$. Since each Q_n is projective and f is an epimorphism in each degree, we can choose maps $u_n : Q_n \to M_n$ for which the composition $f_n \circ u_n$ equal to the identity on Q_n. The maps u_n then extend uniquely to a map of chain complexes $s = \{s_n\}_{n \in \mathbf{Z}}$, characterized by the requirement that each composition

$$Q_{n+1} \oplus Q_n \xrightarrow{\partial \oplus \mathrm{id}} \mathrm{Z}_n(P) \oplus Q_n = P_n \xrightarrow{s_n} M_n$$

is the sum of the maps ∂u_{n+1} and u_n. □

We now specialize our attention to the category $\mathrm{Ch}(\mathbf{Z})$ of chain complexes of abelian groups, which we will endow with a monoidal structure.

Notation 2.5.1.11. Let C_* and D_* be graded abelian groups. We define a new graded abelian group $(C \boxtimes D)_* = C_* \boxtimes D_*$ by the formula `00P1`

$$(C \boxtimes D)_n = \bigoplus_{n = n' + n''} C_{n'} \otimes D_{n''}.$$

Here the direct sum is taken over the set $\{(n', n'') \in \mathbf{Z} \times \mathbf{Z} : n = n' + n''\}$ of all decompositions of n as a sum of two integers n' and n'', and $C_{n'} \otimes D_{n''}$ denotes the tensor product of $C_{n'}$ with $D_{n''}$ (formed in the category of abelian groups). For every pair of elements $x \in C_m$ and $y \in D_n$, we let $x \boxtimes y$ denote the image of the pair (x, y) under the canonical map

$$C_m \times D_n \to C_m \otimes D_n \hookrightarrow (C \boxtimes D)_{m+n}.$$

Proposition 2.5.1.12. *Let (C_*, ∂) and (D_*, ∂) be chain complexes. Then there is a unique homomorphism of graded abelian groups* `00P2`

$$\partial : (C \boxtimes D)_* \to (C \boxtimes D)_{*-1}$$

satisfying the identity

$$\partial(x \boxtimes y) = (\partial(x) \boxtimes y) + (-1)^m (x \boxtimes \partial(y))$$

for $x \in C_m$ and $y \in D_n$. Moreover, this homomorphism satisfies $\partial^2 = 0$, so we can regard the pair $((C \boxtimes D)_*, \partial)$ as a chain complex.

Proof. For every pair of integers $m, n \in \mathbf{Z}$, the construction

$$(x, y) \mapsto (\partial x \boxtimes y) + (-1)^m (x \boxtimes \partial y)$$

determines a bilinear map $C_m \times D_n \to (C \boxtimes D)_{m+n-1}$. Invoking the universal property of tensor products and direct sums, we deduce that there is a unique map $\partial : (C \boxtimes D)_* \to (C \boxtimes D)_{*-1}$ with the desired properties. The identity $\partial^2 = 0$ follows from the calculation

$$\begin{aligned}\partial^2(x \boxtimes y) &= \partial((\partial x \boxtimes y) + (-1)^m (x \boxtimes \partial y)) \\ &= (\partial^2 x \boxtimes y) + (-1)^{m-1}(\partial x \boxtimes \partial y) + (-1)^m (\partial x \boxtimes \partial y) + (-1)^{2m}(x \boxtimes \partial^2 y) \\ &= 0.\end{aligned}$$

□

00P3 **Notation 2.5.1.13.** In the situation of Proposition 2.5.1.12, we will refer to $((C \boxtimes D)_*, \partial)$ as the *tensor product* of the chain complexes (C_*, ∂) and (D_*, ∂).

00P4 **Warning 2.5.1.14** (The Koszul Sign Rule)**.** Let (C_*, ∂) and (D_*, ∂) be chain complexes. There is a unique isomorphism of graded abelian groups $\tau : C_* \boxtimes D_* \to D_* \boxtimes C_*$ satisfying $\tau(x \boxtimes y) = y \boxtimes x$ for all $x \in C_m$, $y \in C_n$. Beware that τ is usually not a chain map: we have

$$\partial\tau(x \boxtimes y) = \partial(y \boxtimes x) = (\partial y \boxtimes x) + (-1)^n (y \boxtimes \partial x)$$

$$\tau(\partial(x \boxtimes y)) = \tau((\partial x \boxtimes y) + (-1)^m (x \boxtimes \partial y)) = (-1)^m (\partial y \boxtimes x) + (y \boxtimes \partial x).$$

This can be remedied by modifying the isomorphism τ: there is another isomorphism of graded abelian groups

$$\sigma : C_* \boxtimes D_* \simeq D_* \boxtimes C_* \qquad \sigma(x \boxtimes y) = (-1)^{mn}(y \boxtimes x).$$

The isomorphism of σ is a chain map (hence an isomorphism of chain complexes) by virtue of the calculation

$$\begin{aligned}\partial\sigma(x \boxtimes y) &= \partial((-1)^{mn} y \boxtimes x) \\ &= (-1)^{mn}(\partial y \boxtimes x) + (-1)^{mn+n}(y \boxtimes \partial x) \\ &= (-1)^m \sigma(x \boxtimes \partial y) + \sigma(\partial x \boxtimes y) \\ &= \sigma(\partial(x \boxtimes y)).\end{aligned}$$

00P5 **Exercise 2.5.1.15** (Universal Property of the Tensor Product)**.** Let (C_*, ∂), (D_*, ∂), and (E_*, ∂) be chain complexes. We will say that a collection of bilinear maps

$$\{f_{m,n} : C_m \times D_n \to E_{m+n}\}_{m,n \in \mathbf{Z}}$$

satisfies the Leibniz rule if, for every pair of elements $x \in C_m$ and $y \in D_n$, the identity

$$\partial f_{m,n}(x, y) = f_{m-1,n}(\partial x, y) + (-1)^m f_{m,n-1}(x, \partial y)$$

holds in the abelian group E_{m+n-1}. Show that there is a canonical bijection from the collection of chain maps $f : C_* \boxtimes D_* \to E_*$ to the collection of systems of bilinear maps $\{f_{m,n} : C_m \times D_n \to E_{m+n}\}_{m,n \in \mathbf{Z}}$ satisfying the Leibniz rule, given by the construction $f_{m,n}(x, y) = f(x \boxtimes y)$.

Remark 2.5.1.16 (Associativity Isomorphisms)**.** Let (C_*, ∂), (D_*, ∂), and (E_*, ∂) be chain complexes of abelian groups. Then there is a unique isomorphism of graded abelian groups 00P6

$$\alpha : C_* \boxtimes (D_* \boxtimes E_*) \to (C_* \boxtimes D_*) \boxtimes E_*$$

satisfying the identity $\alpha(x \boxtimes (y \boxtimes z)) = (x \boxtimes y) \boxtimes z$. Moreover, α is an isomorphism of chain complexes: this follows from the observation that $\alpha(\partial(x \boxtimes (y \boxtimes z)))$ and $\partial\alpha(x \boxtimes (y \boxtimes z))$ are both given by the sum

$$(\partial x \boxtimes y) \boxtimes z + (-1)^m (x \boxtimes \partial y) \boxtimes z + (-1)^{m+n}((x \boxtimes y) \boxtimes \partial z)$$

for $x \in C_m$, $y \in D_n$, $z \in E_p$.

Construction 2.5.1.17 (The Monoidal Structure on Chain Complexes)**.** Let $\mathrm{Ch}(\mathbf{Z})$ denote the category of chain complexes of abelian groups (Definition 2.5.0.3). We define a monoidal structure on $\mathrm{Ch}(\mathbf{Z})$ as follows: 00P7

- The tensor product functor $\boxtimes : \mathrm{Ch}(\mathbf{Z}) \times \mathrm{Ch}(\mathbf{Z}) \to \mathrm{Ch}(\mathbf{Z})$ carries each pair of chain complexes (C_*, ∂) and (D_*, ∂) to the tensor product chain complex $(C_* \boxtimes D_*, \partial)$ of Proposition 2.5.1.12, and carries a pair of chain maps $f : C_* \to C'_*$, $g : D_* \to D'_*$ to the tensor product map

$$(f \boxtimes g) : C_* \boxtimes D_* \to C'_* \boxtimes D'_* \qquad (f \boxtimes g)(x \boxtimes y) = f(x) \boxtimes g(y).$$

- For every triple of chain complexes $C = (C_*, \partial)$, $D = (D_*, \partial)$, and $E = (E_*, \partial)$, the associativity constraint

$$\alpha_{C,D,E} : C_* \boxtimes (D_* \boxtimes E_*) \simeq (C_* \boxtimes D_*) \boxtimes E_*$$

  is the isomorphism of Remark 2.5.1.16.

- The unit object of $\mathrm{Ch}(\mathbf{Z})$ is the chain complex $\mathbf{Z}[0]$ of Example 2.5.1.2, and the unit constraint $\upsilon : \mathbf{Z}[0] \boxtimes \mathbf{Z}[0] \simeq \mathbf{Z}[0]$ is the isomorphism classified by the bilinear map

$$\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \qquad (m, n) \mapsto mn.$$

Remark 2.5.1.18. Let (C_*, ∂) and (D_*, ∂) be chain complexes. The tensor product chain complex $(C_* \boxtimes D_*, \partial)$ of Proposition 2.5.1.12 is characterized up to (unique) isomorphism 00P8 by the universal property of Exercise 2.5.1.15. However, the construction of this tensor product complex (and, by extension, the monoidal structure on Ch(**Z**)) depends on auxiliary choices. These choices are ultimately irrelevant in the sense that they do not change the *isomorphism class* of the monoidal category Ch(**Z**) or, equivalently, of the classifying simplicial set $B_\bullet \operatorname{Ch}(\mathbf{Z})$ of Example 2.3.1.18. This simplicial set can be described concretely (without auxiliary choices): its n-simplices can be identified with systems of chain complexes $\{C(j,i)_*\}_{0 \leq i < j \leq n}$ together with bilinear maps

$$C(k,j)_q \times C(j,i)_p \to C(k,i)_{q+p} \qquad (y,z) \mapsto yz$$

for $0 \leq i < j < k \leq n$ which satisfy the Leibniz rule $\partial(yz) = (\partial y)z + (-1)^q y(\partial z)$ together with the associative law $x(yz) = (xy)z$ for $x \in C(\ell,k)_r$, $y \in C(k,j)_q$, $z \in C(j,i)_p$ with $0 \leq i < j < k < \ell \leq n$.

2.5.2 Differential Graded Categories

00P9 Let Ch(**Z**) denote the category of chain complexes of abelian groups, equipped with the monoidal structure described in Construction 2.5.1.17. A *differential graded category* is a category enriched over Ch(**Z**) (in the sense of Definition 2.1.7.1). For the convenience of the reader, we spell out this definition in detail.

00PA **Definition 2.5.2.1** (Differential Graded Categories)**.** A *differential graded category* $\mathcal{C}$ consists of the following data:

(1) A collection $\operatorname{Ob}(\mathcal{C})$, whose elements we refer to as *objects of* $\mathcal{C}$. We will often abuse notation by writing $X \in \mathcal{C}$ to indicate that X is an element of $\operatorname{Ob}(\mathcal{C})$.

(2) For every pair of objects $X, Y \in \operatorname{Ob}(\mathcal{C})$, a chain complex $(\operatorname{Hom}_{\mathcal{C}}(X,Y)_*, \partial)$. For each integer n, we refer to the elements of $\operatorname{Hom}_{\mathcal{C}}(X,Y)_n$ as *morphisms of degree* n *from* X *to* Y.

(3) For every triple of objects $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ and every pair of integers $m, n \in \mathbf{Z}$, a function

$$c_{Z,Y,X} : \operatorname{Hom}_{\mathcal{C}}(Y,Z)_n \times \operatorname{Hom}_{\mathcal{C}}(X,Y)_m \to \operatorname{Hom}_{\mathcal{C}}(X,Z)_{m+n},$$

which we will refer to as the *composition law*. Given a pair of morphisms $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)_m$ and $g \in \operatorname{Hom}_{\mathcal{C}}(Y,Z)_n$, we will often denote the image $c_{Z,Y,X}(g,f) \in \operatorname{Hom}_{\mathcal{C}}(X,Z)_{m+n}$ by $g \circ f$ or gf.

(4) For every object $X \in \operatorname{Ob}(\mathcal{C})$, a morphism $\operatorname{id}_X \in \operatorname{Hom}_{\mathcal{C}}(X,X)_0$, which we will refer to as the *identity morphism*.

These data are required to satisfy the following conditions:

- The composition law on $\mathcal{C}$ is associative in the following sense: for every triple of elements $f \in \mathrm{Hom}_{\mathcal{C}}(W,X)_{\ell}$, $g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_m$, and $h \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)_n$, we have an equality $h \circ (g \circ f) = (h \circ g) \circ f$ (in the abelian group $\mathrm{Hom}_{\mathcal{C}}(W,Z)_{\ell+m+n}$).
- The composition law on $\mathcal{C}$ is unital on both sides: for every element $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_n$, we have $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the composition maps $\mathrm{Hom}_{\mathcal{C}}(Y,Z)_n \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_m \to \mathrm{Hom}_{\mathcal{C}}(X,Z)_{m+n}$ are bilinear and satisfy the Leibniz rule of Exercise 2.5.1.15. In other words, we have

$$g \circ (f + f') = (g \circ f) + (g \circ f') \qquad (g + g') \circ f = (g \circ f) + (g' \circ f)$$

$$\partial(g \circ f) = (\partial g) \circ f + (-1)^n g \circ (\partial f).$$

Remark 2.5.2.2. Let $\mathcal{C}$ be a differential graded category. For each object $X \in \mathrm{Ob}(\mathcal{C})$, the identity morphism id_X is a 0-cycle of the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,X)_*$: that is, it satisfies $\partial(\mathrm{id}_X) = 0$. This follows from the calculation 00PB

$$\partial(\mathrm{id}_X) = \partial(\mathrm{id}_X \circ \mathrm{id}_X) = \partial(\mathrm{id}_X) \circ \mathrm{id}_X + \mathrm{id}_X \circ \partial(\mathrm{id}_X) = \partial(\mathrm{id}_X) + \partial(\mathrm{id}_X).$$

Remark 2.5.2.3. Let $\mathcal{C}$ be a differential graded category containing a pair of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_m$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)_n$. It follows from the Leibniz rule 00PC

$$\partial(g \circ f) = (\partial g) \circ f + (-1)^n g \circ (\partial f)$$

that if f and g are cycles (that is, if they satisfy $\partial f = 0$ and $\partial g = 0$), then $g \circ f$ is also a cycle. In particular, we have a bilinear composition map

$$\mathrm{Z}_n(\mathrm{Hom}_{\mathcal{C}}(Y,Z)) \times \mathrm{Z}_m(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \to \mathrm{Z}_{m+n}(\mathrm{Hom}_{\mathcal{C}}(X,Z)).$$

Construction 2.5.2.4 (The Underlying Category of a Differential Graded Category)**.** To every differential graded category $\mathcal{C}$, we can associate an ordinary category $\mathcal{C}^{\circ}$ as follows: 00PD

- The objects of $\mathcal{C}^{\circ}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}^{\circ}) = \mathrm{Ob}(\mathcal{C})$, a morphism from X to Y in $\mathcal{C}^{\circ}$ is a 0-cycle of the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$.
- For each object $X \in \mathrm{Ob}(\mathcal{C}^{\circ}) = \mathrm{Ob}(\mathcal{C})$, the identity morphism from X to itself in $\mathcal{C}^{\circ}$ is the identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)_0$ (which is a cycle by virtue of Remark 2.5.2.2).
- Composition of morphisms in $\mathcal{C}^{\circ}$ is given by the composition law on $\mathcal{C}$ (which preserves 0-cycles by virtue of Remark 2.5.2.3).

We will refer to $\mathcal{C}^{\circ}$ as the *underlying category* of the differential graded category $\mathcal{C}$ (note that $\mathcal{C}^{\circ}$ can also be obtained by applying the general procedure described in Example 2.1.7.5).

00PE **Example 2.5.2.5** (Chain Complexes)**.** Let $\mathcal{A}$ be an additive category. We define a differential graded category $\mathrm{Ch}(\mathcal{A})$ as follows:

- The objects of $\mathrm{Ch}(\mathcal{A})$ are chain complexes with values in $\mathcal{A}$ (Definition 2.5.0.1).
- If C_* and D_* are chain complexes with values in $\mathcal{A}$, then $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, D_*)_*$ is the chain complex of abelian groups $[C, D]_*$ defined in Construction 2.5.0.10.
- If C_*, D_*, and E_* are chain complexes with values in $\mathcal{A}$, then the composition law
$$\circ : [D, E]_e \times [C, D]_d \to [C, E]_{d+e}$$
is given by the formula $\{g_n\}_{n\in\mathbf{Z}} \circ \{f_n\}_{n\in\mathbf{Z}} = \{g_{n+d} \circ f_n\}_{n\in\mathbf{Z}}$.

Note that if C_* and D_* are chain complexes with values in $\mathcal{A}$, then a collection of maps $f = \{f_n : C_n \to D_n\}_{n\in\mathbf{Z}}$ is a 0-cycle of the chain complex $[C, D]_*$ if and only if it is a chain map from C_* to D_*. Consequently, applying Construction 2.5.2.4 to the differential graded category $\mathrm{Ch}(\mathcal{A})$ yields the ordinary category of chain complexes and chain maps. In other words, this construction supplies a $\mathrm{Ch}(\mathbf{Z})$-enrichment of the category $\mathrm{Ch}(\mathcal{A})$ introduced in Definition 2.5.0.3.

00PF **Example 2.5.2.6** (Differential Graded Algebras)**.** A *differential graded algebra* is a (not necessarily commutative) graded ring $A_* = \{A_n\}_{n\in\mathbf{Z}}$ equipped with a differential $\partial : A_* \to A_{*-1}$ satisfying $\partial^2 = 0$ and the Leibniz rule $\partial(x \cdot y) = (\partial x) \cdot y + (-1)^m x \cdot (\partial y)$ for $x \in A_m$ and $y \in A_n$. If $\mathcal{C}$ is a differential graded category containing an object X, then the composition law on $\mathcal{C}$ endows the chain complex $\mathrm{End}_{\mathcal{C}}(X)_* = \mathrm{Hom}_{\mathcal{C}}(X, X)_*$ with the structure of a differential graded algebra. Conversely, for every differential graded algebra (A_*, ∂), there is a unique differential graded category $\mathcal{C}$ with $\mathrm{Ob}(\mathcal{C}) = \{X\}$. In other words, the construction $\mathcal{C} \mapsto \mathrm{End}_{\mathcal{C}}(X)_*$ induces a bijective correspondence

$$\begin{tikzcd}
\{\text{Differential graded categories } \mathcal{C} \text{ with } \mathrm{Ob}(\mathcal{C}) = \{X\}\} \arrow[d, "\sim"] \\
\{\text{Differential graded algebras}\}.
\end{tikzcd}$$

00PG **Example 2.5.2.7.** Let $B_\bullet \mathrm{Ch}(\mathbf{Z})$ denote the classifying simplicial set of the monoidal category of chain complexes. For each nonnegative integer $n \geq 0$, we can use the analysis of

Remark 2.5.1.18 to identify n-simplices of $B_\bullet \operatorname{Ch}(\mathbf{Z})$ with differential graded categories $\mathcal{C}$ satisfying $\operatorname{Ob}(\mathcal{C}) = \{0, 1, \cdots, n\}$ and

$$\operatorname{Hom}_{\mathcal{C}}(i,j)_* = \begin{cases} \mathbf{Z}[0] & \text{if } i = j \\ 0 & \text{if } i > j. \end{cases}$$

Definition 2.5.2.8 (Differential Graded Functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be differential graded categories. A *differential graded functor* F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data: 00PH

- For each object $X \in \operatorname{Ob}(\mathcal{C})$, an object $F(X) \in \operatorname{Ob}(\mathcal{D})$.
- For each pair of objects $X, Y \in \operatorname{Ob}(\mathcal{C})$, a chain map $F_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X,Y)_* \to \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))_*$.

These data are required to satisfy the following conditions:

- For every object $X \in \operatorname{Ob}(\mathcal{C})$, the chain map
$$F_{X,X} : \operatorname{Hom}_{\mathcal{C}}(X,X)_* \to \operatorname{Hom}_{\mathcal{D}}(F(X), F(X))_*$$
carries the identity morphism id_X to the identity morphism $\mathrm{id}_{F(X)}$.
- For every triple of objects $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$ and pair of morphisms $f \in \operatorname{Hom}_{\mathcal{C}}(X,Y)_m$, $g \in \operatorname{Hom}_{\mathcal{C}}(Y,Z)_n$, we have $F_{X,Z}(g \circ f) = F_{Y,Z}(g) \circ F_{X,Y}(f)$.

We let $\operatorname{Cat}^{\mathrm{dg}}$ denote the category whose objects are (small) differential graded categories and whose morphisms are differential graded functors.

Remark 2.5.2.9. Let $\mathcal{C}$ and $\mathcal{D}$ be differential graded categories. Then differential graded functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.5.2.8) can be identified with $\operatorname{Ch}(\mathbf{Z})$-enriched functors from $\mathcal{C}$ to $\mathcal{D}$ (in the sense of Definition 2.1.7.10). 00PJ

2.5.3 The Differential Graded Nerve

We now explain how to associate to each differential graded category $\mathcal{C}$ an ∞-category $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$, which we will refer to as the *differential graded nerve* of $\mathcal{C}$. We begin by describing the simplices of $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$. 00PK

Construction 2.5.3.1. Let $\mathcal{C}$ be a differential graded category. For $n \geq 0$, we let $\mathrm{N}^{\mathrm{dg}}_n(\mathcal{C})$ denote the collection of all ordered pairs $(\{X_i\}_{0 \leq i \leq n}, \{f_I\})$, where: 00PL

- Each X_i is an object of the differential graded category $\mathcal{C}$.

- For every subset $I = \{i_0 > i_1 > \cdots > i_k\} \subseteq [n]$ having at least two elements, f_I is an element of the abelian group $\mathrm{Hom}_{\mathcal{C}}(X_{i_k}, X_{i_0})_{k-1}$ which satisfies the identity

$$\partial f_I = \sum_{a=1}^{k-1} (-1)^a (f_{\{i_0 > i_1 > \cdots > i_a\}} \circ f_{\{i_a > \cdots > i_k\}} - f_{I \setminus \{i_a\}})$$

00PM **Example 2.5.3.2** (Vertices of the Differential Graded Nerve)**.** Let $\mathcal{C}$ be a differential graded category. Then $\mathrm{N}^{\mathrm{dg}}_0(\mathcal{C})$ can be identified with the collection $\mathrm{Ob}(\mathcal{C})$ of objects of $\mathcal{C}$.

00PN **Example 2.5.3.3** (Edges of the Differential Graded Nerve)**.** Let $\mathcal{C}$ be a differential graded category. Then $\mathrm{N}^{\mathrm{dg}}_1(\mathcal{C})$ can be identified with the collection of all triples (X_0, X_1, f) where X_0 and X_1 are objects of $\mathcal{C}$ and f is a 0-cycle in the chain complex $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1)_\bullet$. In other words, $\mathrm{N}^{\mathrm{dg}}_1(\mathcal{C})$ is the collection of all morphisms in the underlying category $\mathcal{C}^\circ$ of Construction 2.5.2.4.

00PP **Example 2.5.3.4** (2-Simplices of the Differential Graded Nerve)**.** Let $\mathcal{C}$ be a differential graded category. Then an element of $\mathrm{N}^{\mathrm{dg}}_2(\mathcal{C})$ is given by the following data:

- A triple of objects $X_0, X_1, X_2 \in \mathrm{Ob}(\mathcal{C})$.
- A triple of 0-cycles

$$f_{10} \in \mathrm{Hom}_{\mathcal{C}}(X_0, X_1)_0 \qquad f_{20} \in \mathrm{Hom}_{\mathcal{C}}(X_0, X_2)_0 \qquad f_{21} \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)_0.$$

- A 1-chain $f_{210} \in \mathrm{Hom}_{\mathcal{C}}(X_0, X_2)_1$ satisfying the identity

$$\partial(f_{210}) = f_{20} - (f_{21} \circ f_{10}).$$

Here the 1-chain f_{210} can be regarded as a witness to the assertion that the 0-cycles f_{20} and $f_{21} \circ f_{10}$ are homologous: that is, they represent the same element of the homology group $\mathrm{H}_0(\mathrm{Hom}_{\mathcal{C}}(X_0, X_2))$. We can present this data graphically by the diagram

$$\begin{tikzcd}
& X_1 \arrow[dr, "f_{21}"] \arrow[d, Rightarrow, "f_{210}"] & \\
X_0 \arrow[ur, "f_{10}"] \arrow[rr, "f_{20}"'] & {} & X_2.
\end{tikzcd}$$

We now explain how to organize the collection $\{\mathrm{N}^{\mathrm{dg}}_n(\mathcal{C})\}_{n \geq 0}$ into a simplicial set.

Proposition 2.5.3.5. *Let $\mathcal{C}$ be a differential graded category. Let $m, n \geq 0$ be nonnegative integers and let $\alpha : [n] \to [m]$ be a nondecreasing function. Then the construction* 00PQ

$$(\{X_i\}_{0 \leq i \leq m}, \{f_I\}) \mapsto (\{X_{\alpha(j)}\}_{0 \leq j \leq n}, \{g_J\}),$$

$$g_J = \begin{cases} f_{\alpha(J)} & \text{if } \alpha|_J \text{ is injective} \\ \mathrm{id}_{X_i} & \text{if } J = \{j_0 > j_1\} \text{ with } \alpha(j_0) = i = \alpha(j_1) \\ 0 & \text{otherwise.} \end{cases}$$

determines a map of sets $\alpha^* : \mathrm{N}^{\mathrm{dg}}_m(\mathcal{C}) \to \mathrm{N}^{\mathrm{dg}}_n(\mathcal{C})$.

Proof. Let $(\{X_i\}_{0 \leq i \leq m}, \{f_I\})$ be an element of $\mathrm{N}^{\mathrm{dg}}_m(\mathcal{C})$. For each subset $J \subseteq [n]$ with at least two elements, define g_J as in the statement of Proposition 2.5.3.5. We wish to show that $(\{X_{\alpha(j)}\}_{0 \leq j \leq n}, \{g_J\})$ is an element of $\mathrm{N}^{\mathrm{dg}}_n(\mathcal{C})$. For this, we must show that for each subset

$$J = \{j_0 > j_1 > \cdots > j_{k-1} > j_k\} \subseteq [n]$$

having at least two elements, we have an equality

$$\partial g_J = \sum_{0<a<k} (-1)^a (g_{\{j_0 > j_1 > \cdots > j_a\}} \circ g_{\{j_a > \cdots > j_k\}} - g_{J \setminus \{j_a\}}). \tag{2.3}$$ 00PR

We distinguish three cases:

- Suppose that the restriction $\alpha|_J$ is injective. In this case, we can rewrite (2.3) as an equality

  $$\partial f_{\alpha(J)} = \sum_{0<a<k} (-1)^a (f_{\{\alpha(j_0) > \cdots > \alpha(j_a)\}} \circ f_{\{\alpha(j_a) > \cdots > \alpha(j_k)\}} - f_{\alpha(J) \setminus \{\alpha(j_a)\}}),$$

  which follows from our assumption that $(\{X_i\}_{0 \leq i \leq m}, \{f_I\})$ is an element of $\mathrm{N}^{\mathrm{dg}}_m(\mathcal{C})$.

- Suppose that $J = \{j_0 > j_1\}$ is a two-element set satisfying $\alpha(j_0) = i = \alpha(j_1)$ for some $0 \leq i \leq m$. In this case, we can rewrite (2.3) as an equality $\partial(\mathrm{id}_{X_i}) = 0$, which follows from Remark 2.5.2.2.

- Suppose that $J = \{j_0 > j_1 > \cdots > j_{k-1} > j_k\}$ has at least three elements and that $\alpha|_J$ is not injective, so that $g_J = 0$. We now distinguish three (possibly overlapping) cases:

  - The map α is not injective because $\alpha(j_0) = i = \alpha(j_1)$ for some $0 \leq i \leq m$. In this case, the expressions $g_{J \setminus \{j_a\}}$ and $g_{\{j_0 > \cdots > j_a\}}$ vanish for $1 < a < k$. We can therefore rewrite (2.3) as an an equality

    $$g_{J \setminus \{j_1\}} = g_{\{j_0 > j_1\}} \circ g_{\{j_1 > \cdots > j_k\}},$$

    which follows from the identities $g_{J \setminus \{j_1\}} = g_{\{j_1 > \cdots > j_k\}}$ and $g_{\{j_0 > j_1\}} = \mathrm{id}_{X_i}$.

- The map α is not injective because $\alpha(j_{k-1}) = i = \alpha(j_k)$ for some $0 \leq i \leq m$. In this case, the expressions $g_{J \setminus \{j_a\}}$ and $g_{\{j_a > \cdots > j_k\}}$ vanish for $0 < a < k-1$. We can therefore rewrite (2.3) as an an equality

  $$g_{J \setminus \{j_{k-1}\}} = g_{\{j_0 > \cdots > j_{k-1}\}} \circ g_{\{j_{k-1} > j_k\}},$$

  which follows from the identities $g_{J \setminus \{j_{k-1}\}} = g_{\{j_0 > \cdots > j_{k-1}\}}$ and $g_{\{j_{k-1} > j_k\}} = \mathrm{id}_{X_i}$.
- The map α is not injective because we have $\alpha(j_b) = \alpha(j_{b+1})$ for some $0 < b < k-1$. In this case, the chains $g_{J \setminus \{j_a\}}$ vanish for $a \notin \{b, b+1\}$, and the compositions $g_{\{j_0 > \cdots > j_a\}} \circ g_{\{j_a > \cdots > j_k\}}$ vanish for all $0 < a < k$. We can therefore rewrite (2.3) as an an equality $g_{J \setminus \{j_b\}} = g_{J \setminus \{j_{b+1}\}}$, which is clear.

□

00PS **Exercise 2.5.3.6.** Let $\mathcal{C}$ be a differential graded category. Suppose we are given a pair of nondecreasing functions $\alpha : [k] \to [m]$ and $\beta : [m] \to [n]$. Show that the function $(\beta \circ \alpha)^*$ of Proposition 2.5.3.5 coincides with the composition $\alpha^* \circ \beta^*$.

00PT **Definition 2.5.3.7.** Let $\mathcal{C}$ be a differential graded category. We let $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ denote the simplicial set whose value on an object $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ is the set $\mathrm{N}^{\mathrm{dg}}_{n}(\mathcal{C})$ of Construction 2.5.3.1, and whose value on a nondecreasing function $\alpha : [n] \to [m]$ is the function $\alpha^* : \mathrm{N}^{\mathrm{dg}}_{m}(\mathcal{C}) \to \mathrm{N}^{\mathrm{dg}}_{n}(\mathcal{C})$ of Proposition 2.5.3.5. We will refer to $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ as the *differential graded nerve of* $\mathcal{C}$.

00PU **Remark 2.5.3.8** (Comparison with the Nerve)**.** Let $\mathcal{C}$ be a differential graded category and let $\mathcal{C}^{\circ}$ denote its underlying ordinary category (Construction 2.5.2.4). Suppose that σ is an n-simplex of the nerve $\mathrm{N}_{\bullet}(\mathcal{C}^{\circ})$, consisting of objects $\{X_i\}_{0 \leq i \leq n}$ and 0-cycles $\{f_{ji} \in \mathrm{Hom}_{\mathcal{C}}(X_i, X_j)_0\}$ satisfying $f_{ii} = \mathrm{id}_{X_i}$ and $f_{ki} = f_{kj} \circ f_{ji}$ for $0 \leq i \leq j \leq k \leq n$. We can then construct an n-simplex $U(\sigma)$ of the differential graded nerve $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$, given by

$$U(\sigma) = (\{X_i\}_{0 \leq i \leq n}, \{f_I\}) \qquad f_I = \begin{cases} f_{ji} & \text{if } I = \{j > i\} \\ 0 & \text{otherwise.} \end{cases}$$

The construction $\sigma \mapsto U(\sigma)$ determines a map of simplicial sets $U : \mathrm{N}_{\bullet}(\mathcal{C}^{\circ}) \to \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$. This map is a monomorphism, whose image is the simplicial subset of $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ spanned by those n-simplices $(\{X_i\}_{0 \leq i \leq n}, \{f_I\})$ with the property that $f_I = 0$ for $|I| > 2$.

00PV **Remark 2.5.3.9.** Let $\mathcal{C}$ be a differential graded category and let $K_{\bullet}$ be a simplicial set. To give a map of simplicial sets $f : K_{\bullet} \to \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$, one must supply the following data:

- For each vertex x of $K_{\bullet}$, an object $f(x)$ of the differential graded category $\mathcal{C}$.
- For each $k > 0$ and each k-simplex $\sigma : \Delta^k \to K_{\bullet}$ with initial vertex $x = \sigma(0)$ and final vertex $y = \sigma(k)$, a $(k-1)$-chain $f(\sigma) \in \mathrm{Hom}_{\mathcal{C}}(f(x), f(y))_{k-1}$.

Moreover, this data must satisfy the following conditions:

- If e is a degenerate edge of $K_\bullet$ connecting a vertex x to itself, then $f(e)$ is the identity morphism $\mathrm{id}_{f(x)} \in \mathrm{Hom}_{\mathcal{C}}(f(x), f(x))_0$.
- If σ is a degenerate simplex of $K_\bullet$ having dimension ≥ 2, then $f(\sigma) = 0$.
- Let $k > 0$ and let $\sigma : \Delta^k \to K_\bullet$ be an k-simplex of $K_\bullet$. For $0 < b < k$, let $\sigma_{\leq b} : \Delta^b \hookrightarrow K_\bullet$ denote the composition of σ with the inclusion map $\Delta^b \hookrightarrow \Delta^k$ (which is the identity on vertices), and let $\sigma_{\geq b} : \Delta^{k-b} \hookrightarrow K_\bullet$ denote the composition of σ with the map $\Delta^{k-b} \hookrightarrow \Delta^k$ given on vertices by $i \mapsto i + b$. Then we have

$$\partial f(\sigma) = \sum_{b=1}^{k-1} (-1)^{k-b} (f(\sigma_{\geq b}) \circ f(\sigma_{\leq b}) - f(d^k_b \sigma))$$

Theorem 2.5.3.10. *Let $\mathcal{C}$ be a differential graded category. Then the simplicial set $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$ is an ∞-category.* 00PW

Proof. Suppose we are given $0 < j < n$ and a map of simplicial sets $\sigma_0 : \Lambda^n_j \to \mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$. Using Remark 2.5.3.9, we see that σ_0 can be identified with the data of a pair $(\{X_i\}_{0 \leq i \leq n}, \{f_I\})$, where $\{X_i\}_{0 \leq i \leq n}$ is a collection of objects of $\mathcal{C}$ and $f_I \in \mathrm{Hom}_{\mathcal{C}}(X_{i_0}, X_{i_k})_{k-1}$ is defined for every subset $I = \{i_0 > i_1 > \cdots > i_k\} \subseteq [n]$ for which $k > 0$ and $[n] \neq I \neq [n] \setminus \{j\}$, satisfying the identity

$$\partial f_I = \sum_{a=1}^{k-1} (-1)^a (f_{\{i_0 > i_1 > \cdots > i_a\}} \circ f_{\{i_a > \cdots > i_k\}} - f_{I \setminus \{i_a\}}) \tag{2.4}$$ 00PX

We wish to show that σ_0 can be extended to an n-simplex of $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$. To give such an extension, we must supply chains $f_{[n]} \in \mathrm{Hom}_{\mathcal{C}}(X_0, X_n)_{n-1}$ and $f_{[n] \setminus \{j\}} \in \mathrm{Hom}_{\mathcal{C}}(X_0, X_n)_{n-2}$ which satisfy (2.4) in the cases $I = [n]$ and $I = [n] \setminus \{j\}$. We claim that there is a unique such extension which also satisfies $f_{[n]} = 0$. Applying (2.4) in the case $I = [n]$, we deduce that $f_{[n] \setminus \{j\}}$ is necessarily given by

$$f_{[n] \setminus \{j\}} = \sum_{0 < b < n} (-1)^{b-j} (f_{\{n > \cdots > b\}} \circ f_{\{b > \cdots > 0\}}) - \sum_{0 < b < n, b \neq j} (-1)^{b-j} f_{[n] \setminus \{b\}}.$$

To complete the proof, it will suffice to verify that this prescription also satisfies (2.4) in the case $I = [n] \setminus \{j\}$. In what follows, for $0 \leq a < b \leq n$, let us write $[ba]$ for the set

$\{b > b-1 > \cdots > a\}$. We now compute

$$\begin{aligned}
(-1)^j \partial f_{[n]\setminus\{j\}} &= \sum_{0<b<n} (-1)^b \partial(f_{[nb]} f_{[b0]}) - \sum_{0<b<n, b\neq j} (-1)^b \partial f_{[n]\setminus\{b\}} \\
&= \sum_{0<b<n} (-1)^b (\partial f_{[nb]}) f_{[b]} - \sum_{0<b<n} (-1)^n f_{[nb]} (\partial f_{[b]}) \\
&\quad - \sum_{0<b<n, b\neq j} (-1)^b \partial f_{[n]\setminus\{b\}} \\
&= \sum_{0<b<c<n} (-1)^{n-c+b} f_{[nc]} f_{[cb]} f_{[b0]} - \sum_{0<b<c<n} (-1)^{n-c+b} (f_{[nb]\setminus\{c\}} f_{[b]}) - \\
&\quad \sum_{0<a<b<n} (-1)^{n+b-a} f_{[nb]} f_{[ba]} f_{[a0]} + \sum_{0<a<b<n} (-1)^{n+b-a} f_{[nb]} f_{[b0]\setminus\{a\}} - \\
&\quad \sum_{0<b<c<n, b\neq j} (-1)^{b+n-c} f_{[nc]} f_{[c0]\setminus\{b\}} + \sum_{0<b<c<n, b\neq j} (-1)^{b+n-c} f_{[n0]\setminus\{b,c\}} + \\
&\quad \sum_{0<a<b<n, b\neq j} (-1)^{b+n-a} f_{[na]\setminus\{b\}} f_{[a0]} - \sum_{0<a<b<n, b\neq j} (-1)^{b+n-a} f_{[n0]\setminus\{a,b\}}.
\end{aligned}$$

Here the first and third terms cancel, the seventh term cancels with the second except for those summands with $c = j$, the fifth term cancels with the fourth except for those summands with $a = j$, and the sixth term cancels the eighth except for those terms with $c = j$ and $a = j$, respectively. Multiplying by $(-1)^j$, we can rewrite this identity as

$$\begin{aligned}
\partial f_{[n]\setminus\{j\}} &= \sum_{0<b<j} (-1)^{n-1-b} (f_{[nb]\setminus\{j\}} \circ f_{[b0]}) + \sum_{j<b<n} (-1)^{n-b} (f_{[nb]} \circ f_{[b0]\setminus\{j\}}) \\
&\quad - \sum_{0<b<j} (-1)^{n-1-b} f_{[n]\setminus\{b,j\}} - \sum_{j<b<n} (-1)^{n-b} f_{[n]\setminus\{b,j\}},
\end{aligned}$$

which recovers equation (2.4) in the case $I = [n] \setminus \{j\}$. □

00PY **Remark 2.5.3.11.** The theory of differential graded categories can be regarded as a special case of the more general theory of A_∞-*categories* (see [22]). Definition 2.5.3.7 and Theorem 2.5.3.10 have been extended to the setting of A_∞-categories by Faonte; we refer the reader to [20] for details.

2.5.4 The Homotopy Category of a Differential Graded Category

00SW Let $\mathcal{C}$ be a differential graded category, and let $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$ denote its differential graded nerve (Definition 2.5.3.7). Then $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$ is an ∞-category (Theorem 2.5.3.10). Moreover:

- The objects of the ∞-category $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$ are the objects of $\mathcal{C}$ (Example 2.5.3.2).
- If X and Y are objects of $\mathcal{C}$, then a morphism from X to Y in the ∞-category $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$ can be identified with a 0-cycle in the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ (Example

2.5.3.3), or equivalently with a morphism from X to Y in the underlying category $\mathcal{C}^{\circ}$ of Construction 2.5.2.4.

We now explain how to describe the homotopy category of $\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$ directly in terms of the differential graded category $\mathcal{C}$ (Proposition 2.5.4.10).

Definition 2.5.4.1. Let $\mathcal{C}$ be a differential graded category containing a pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, and let f and f' be 0-cycles of the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$. A *homotopy* from f to f' is a 1-chain $h \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_1$ satisfying $\partial(h) = f' - f$. We will say that f and f' are *homotopic* if there exists a homotopy from f to f': that is, if we have an equality $[f] = [f']$ in the homology group $\mathrm{H}_0(\mathrm{Hom}_{\mathcal{C}}(X,Y))$. 00PZ

Example 2.5.4.2. Let $\mathcal{A}$ be an additive category, let C_* and D_* be chain complexes with values in $\mathcal{A}$, and let $f, f' : C_* \to D_*$ be chain maps, which we regard as 0-cycles in the mapping complex $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, D_*)_*$ in the differential graded category $\mathrm{Ch}(\mathcal{A})$ of Example 2.5.2.5. Let $h = \{h_n : C_n \to D_{n+1}\}_{n \in \mathbf{Z}}$ be a collection of morphisms, which we regard as a 1-chain of $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(C_*, D_*)_*$. Then h is a homotopy from f to f' (in the sense of Definition 2.5.4.1) if and only if it is a chain homotopy from f to f' (in the sense of Definition 2.5.0.5). In particular, f and f' are homotopic morphisms of the differential graded category $\mathrm{Ch}(\mathcal{A})$ (in the sense of Definition 2.5.4.1) if and only if they are chain homotopic (in the sense of Definition 2.5.0.5). 00Q0

Remark 2.5.4.3. Let $\mathcal{C}$ be a differential graded category containing a pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, and let f and g be 0-cycles of the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$. Then giving a homotopy from f to g in the sense of Definition 2.5.4.1 is equivalent to giving a homotopy from f to g as morphisms in the ∞-category $\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$ (Definition 1.4.3.1): this follows from Example 2.5.3.4. In particular, f and g are homotopic in the sense of Definition 2.5.4.1 if and only if they are homotopic in the sense of Definition 1.4.3.1. 00Q1

Remark 2.5.4.4. Let $\mathcal{C}$ be a differential graded category containing objects X, Y, and Z, and suppose we are given 0-cycles $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_0$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)_0$, and $h \in \mathrm{Hom}_{\mathcal{C}}(X,Z)_0$. Then Example 2.5.3.4 supplies an equivalence between the following data: 00Q2

- The datum of a homotopy from $g \circ f$ to h, in the sense of Definition 2.5.4.1.
- The datum of a 2-simplex of $\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$ witnessing h as a composition of f and g, in the sense of Definition 1.4.4.1.

In particular, h is homotopic to the composition $g \circ f$ (in the differential graded category $\mathcal{C}$) if and only if it is a composition of g and f (in the ∞-category $\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$).

Proposition 2.5.4.5. *Let $\mathcal{C}$ be a differential graded category containing a pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$. Let f and g be 0-cycles of the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ which are homotopic. Then:* 00Q3

(a) *For any object* $W \in \mathrm{Ob}(\mathcal{C})$ *and any* 0*-cycle* $u \in \mathrm{Hom}_{\mathcal{C}}(W,X)_0$*, the composite cycles* $f \circ u$ *and* $g \circ u$ *are homotopic.*

(b) *For any object* $Z \in \mathrm{Ob}(\mathcal{C})$ *and any* 0*-cycle* $v \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)_0$*, the composite cycles* $v \circ f$ *and* $v \circ g$ *are homotopic.*

Proof. By virtue of Remarks 2.5.4.3 and 2.5.4.4, we can regard Proposition 2.5.4.5 as a special case of Proposition 1.4.4.7. However, it is easy to prove directly. If $h \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_1$ is a homotopy from f to g and u is a 0-cycle in $\mathrm{Hom}_{\mathcal{C}}(W,X)_0$, then the calculation

$$\begin{aligned}\partial(h \circ u) &= ((\partial h) \circ u) - (h \circ (\partial u))\\ &= (\partial h) \circ u\\ &= (g-f) \circ u\\ &= (g \circ u) - (f \circ u)\end{aligned}$$

shows that $(h \circ u) \in \mathrm{Hom}_{\mathcal{C}}(W,Y)_1$ is a homotopy from $f \circ u$ to $g \circ u$. This proves (a), and (b) follows from a similar argument. □

00Q4 **Construction 2.5.4.6** (The Homotopy Category of a Differential Graded Category)**.** Let $\mathcal{C}$ be a differential graded category. We define a category $\mathrm{h}\mathcal{C}$ as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathrm{h}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$, we define

  $$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y) = \mathrm{H}_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)).$$

  If f is a 0-cycle of the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$, let $[f]$ denote its image in the homology group $\mathrm{H}_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)$.
- For each object $X \in \mathrm{Ob}(\mathrm{h}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$, the identity morphism from X to itself in the category $\mathrm{h}\mathcal{C}$ is given by $[\mathrm{id}_X]$, where id_X is the identity morphism from X to itself in $\mathcal{C}$.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathrm{h}\mathcal{C}) = \mathrm{Ob}(\mathcal{C})$, the composition law

  $$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Z)$$

  is characterized by the formula $[g] \circ [f] = [g \circ f]$ for $f \in \mathrm{Z}_0(\mathrm{Hom}_{\mathcal{C}}(X,Y))$ and $g \in \mathrm{Z}_0(\mathrm{Hom}_{\mathcal{C}}(Y,Z))$ (this composition law is well-defined by virtue of Proposition 2.5.4.5).

We will refer to $\mathrm{h}\mathcal{C}$ as the *homotopy category* of the differential graded category $\mathcal{C}$.

Remark 2.5.4.7. Passage from a differential graded category $\mathcal{C}$ to its homotopy category $\mathrm{h}\mathcal{C}$ can be regarded as a special case of Remark 2.1.7.4, applied to the lax monoidal functor 00Q5

$$\mathrm{Ch}(\mathbf{Z}) \to \mathrm{Set} \qquad (C_*, d) \mapsto \mathrm{H}_0(C)$$

with tensor constraints given by

$$\mu_{C,D} : \mathrm{H}_0(C) \times \mathrm{H}_0(D) \to \mathrm{H}_0(C \boxtimes D) \qquad ([x],[y]) \mapsto [x \boxtimes y].$$

Remark 2.5.4.8. Let $\mathcal{C}$ be a differential graded category, with underlying category $\mathcal{C}^\circ$ (Construction 2.5.2.4) and homotopy category $\mathrm{h}\mathcal{C}$ (Construction 2.5.4.6). There is an evident functor $\mathcal{C}^\circ \to \mathrm{h}\mathcal{C}$ which is the identity on objects, given on morphisms by the construction 00Q6

$$\mathrm{Hom}_{\mathcal{C}^\circ}(X,Y) = \mathrm{Z}_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \twoheadrightarrow \mathrm{H}_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y) \qquad f \mapsto [f].$$

Example 2.5.4.9 (The Homotopy Category of Chain Complexes)**.** Let $\mathcal{A}$ be an additive category, and let $\mathrm{Ch}(\mathcal{A})$ be the differential graded category of chain complexes with values in $\mathcal{A}$ (Example 2.5.2.5). Then the homotopy category of $\mathrm{Ch}(\mathcal{A})$ in the sense of Construction 2.5.4.6 agrees with the homotopy category $\mathrm{hCh}(\mathcal{A})$ introduced in Construction 2.5.0.9. 00Q7

Proposition 2.5.4.10. *Let $\mathcal{C}$ be a differential graded category and let $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$ denote the differential graded nerve of $\mathcal{C}$. Then the homotopy category $\mathrm{hN}^{\mathrm{dg}}_\bullet(\mathcal{C})$ (Definition 1.4.5.3) is canonically isomorphic to the homotopy category $\mathrm{h}\mathcal{C}$ (Construction 2.5.4.6).* 00Q8

Proof. Combine Remarks 2.5.4.3 and 2.5.4.4. □

2.5.5 Digression: The Homology of Simplicial Sets

Among the most useful invariants studied in algebraic topology are the *singular homology groups* $\mathrm{H}_*(X;\mathbf{Z})$ of a topological space X. These are defined as the homology groups of the *singular chain complex* 00Q9

$$\cdots \xrightarrow{\partial} \mathrm{C}_3(X;\mathbf{Z}) \xrightarrow{\partial} \mathrm{C}_2(X;\mathbf{Z}) \xrightarrow{\partial} \mathrm{C}_1(X;\mathbf{Z}) \xrightarrow{\partial} \mathrm{C}_0(X;\mathbf{Z}),$$

where $\mathrm{C}_n(X;\mathbf{Z})$ denotes the free abelian group generated by the set $\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X)$ of singular n-simplices of X, and the boundary operator ∂ is given by the formula

$$\partial : \mathrm{C}_n(X;\mathbf{Z}) \to \mathrm{C}_{n-1}(X;\mathbf{Z}) \qquad \partial(\sigma) = \sum_{i=0}^{n} (-1)^i d^n_i(\sigma).$$

We can therefore view the passage from the topological space X to its homology $\mathrm{H}_*(X;\mathbf{Z})$ as proceeding in four stages:

- We first extract from the topological space X its singular simplicial set $\mathrm{Sing}_\bullet(X)$ (Construction 1.2.2.2).

- We then replace $\mathrm{Sing}_\bullet(X)$ by the simplicial abelian group $\mathbf{Z}[\mathrm{Sing}_\bullet(X)]$, carrying each object $[n] \in \mathbf{\Delta}^{\mathrm{op}}$ to the free abelian group $\mathbf{Z}[\mathrm{Sing}_n(X)]$ generated by the set $\mathrm{Sing}_n(X)$.
- We next regard the abelian groups $\{\mathbf{Z}[\mathrm{Sing}_n(X)]\}_{n \geq 0}$ as the terms of a chain complex $(C_*(X;\mathbf{Z}), \partial)$, where the differential ∂ is given by the alternating sum of the face operators of the simplicial abelian group $\mathbf{Z}[\mathrm{Sing}_\bullet(X)]$.
- For each integer n, we define $\mathrm{H}_n(X;\mathbf{Z})$ to be the nth homology group of the chain complex $(C_*(X;\mathbf{Z}), \partial)$ (Definition 2.5.1.4).

In other words, the functor $X \mapsto \mathrm{H}_n(X;\mathbf{Z})$ factors as a composition

$$\mathrm{Top} \xrightarrow{\mathrm{Sing}_\bullet} \mathrm{Set}_\Delta \xrightarrow{\mathbf{Z}[-]} \mathrm{Ab}_\Delta \xrightarrow{\mathrm{C}_*} \mathrm{Ch}(\mathbf{Z}) \xrightarrow{\mathrm{H}_n} \mathrm{Ab},$$

where Ab_Δ denotes the category of simplicial abelian groups and $\mathrm{C}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathbf{Z})$ is given by the following:

`00QA` **Construction 2.5.5.1** (The Moore Complex)**.** Let $A_\bullet$ be a semisimplicial abelian group (Definition 1.1.1.2). For each $n \geq 1$, we define a group homomorphism $\partial : A_n \to A_{n-1}$ by the formula

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i d_i^n(\sigma),$$

where $d_i^n : A_n \to A_{n-1}$ is the ith face operator (Construction 1.1.1.4). For $n \geq 2$ and $\sigma \in A_n$, we compute

$$\begin{aligned}
\partial^2(\sigma) &= \partial\Big(\sum_{i=0}^{n} (-1)^i d_i^n(\sigma)\Big) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n-1} (-1)^{i+j} (d_j^{n-1} d_i^n)(\sigma) \\
&= 0
\end{aligned}$$

where the final equality follows from the identity $d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$ for $0 \leq i < j \leq n$ (see Remark 1.1.1.7). We let $\mathrm{C}_*(A)$ denote the chain complex of abelian groups given by

$$\mathrm{C}_n(A) = \begin{cases} A_n & \text{if } n \geq 0 \\ 0 & \text{otherwise,} \end{cases}$$

where the differential is given by ∂. We will refer to $\mathrm{C}_*(A)$ as the *Moore complex* of the semisimplicial abelian group $A_\bullet$.

If $A_\bullet$ is a simplicial abelian group, we let $\mathrm{C}_*(A)$ denote the Moore complex of the semisimplicial abelian group underlying $A_\bullet$ (Remark 1.1.1.3).

Definition 2.5.5.2 (Homology of Simplicial Sets)**.** Let $S_\bullet$ be a simplicial set and let $\mathbf{Z}[S_\bullet]$ denote the simplicial abelian group freely generated by $S_\bullet$. We let $\mathrm{C}_*(S;\mathbf{Z})$ denote the Moore complex of $\mathbf{Z}[S_\bullet]$. We will refer to $\mathrm{C}_*(S;\mathbf{Z})$ as the *chain complex of* $S_\bullet$. For each integer n, we denote the nth homology group of $\mathrm{C}_*(S;\mathbf{Z})$ by $\mathrm{H}_n(S;\mathbf{Z})$ and refer to it as the *nth homology group of S* (with coefficients in $\mathbf{Z}$). `00QB`

Example 2.5.5.3. Let X be a topological space. Then the singular chain complex $\mathrm{C}_*(X;\mathbf{Z})$ is the chain complex of the singular simplicial set $\mathrm{Sing}_\bullet(X)$. In particular, the homology groups of the simplicial set $\mathrm{Sing}_\bullet(X)$ are the usual singular homology groups of the topological space X. `00QC`

Example 2.5.5.4. Let $S_\bullet = \Delta^0$ be the standard 0-simplex. Then $S_\bullet$ is a simplicial set having a single simplex of each dimension. Consequently, the chain complex $\mathrm{C}_*(S;\mathbf{Z})$ is given by $\mathbf{Z}$ in each nonnegative degree. For $n > 0$, the differential $\mathbf{Z} \simeq \mathrm{C}_n(S;\mathbf{Z}) \xrightarrow{\partial} \mathrm{C}_{n-1}(S;\mathbf{Z}) \simeq \mathbf{Z}$ is given by multiplication by the integer `00QD`

$$\sum_{i=0}^{n}(-1)^i = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even,} \end{cases}$$

as indicated in the diagram

$$\cdots \to \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{1} \mathbf{Z} \xrightarrow{0} \mathbf{Z}.$$

It follows that the homology groups of $S_\bullet$ are given by

$$\mathrm{H}_n(S;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that although the *homology* of the simplicial set $S_\bullet = \Delta^0$ is concentrated in degree zero, the chain complex $\mathrm{C}_*(S;\mathbf{Z})$ is not. Essentially, this is because $S_\bullet$ has degenerate simplices in each dimension $n > 0$ which do not contribute to its homology. This is a special case of a more general phenomenon.

Notation 2.5.5.5. Let $A_\bullet$ be a simplicial abelian group. For each $n \geq 0$, let $\mathrm{D}_n(A)$ denote the subgroup of $\mathrm{C}_n(A) = A_n$ generated by the images of the degeneracy operators $\{s_i^{n-1} : A_{n-1} \to A_n\}_{0 \leq i \leq n-1}$. By convention, we set $\mathrm{D}_n(A) = 0$ for $n < 0$. `00QE`

Proposition 2.5.5.6. *Let $A_\bullet$ be a simplicial abelian group. For every positive integer n, the boundary operator $\partial : \mathrm{C}_n(A) \to \mathrm{C}_{n-1}(A)$ carries the subgroup $\mathrm{D}_n(A)$ into $\mathrm{D}_{n-1}(A)$. Consequently, we can regard $\mathrm{D}_*(A)$ as a subcomplex of the Moore complex $\mathrm{C}_*(A)$.* `00QF`

Proof. Choose an element $\sigma \in \mathrm{D}_n(A)$; we wish to show that $\partial(\sigma)$ belongs to $\mathrm{D}_{n-1}(A)$. Without loss of generality, we may assume that $\sigma = s_i^{n-1}(\tau)$ for some $0 \leq i \leq n-1$ and some $\tau \in A_{n-1}$. We now compute

$$\begin{aligned}
\partial(\sigma) &= \sum_{j=0}^{n}(-1)^j d_j^n(\sigma) \\
&= (\sum_{j=0}^{i-1}(-1)^j d_j^n s_i^{n-1}\tau) + (-1)^i d_i^n s_i^{n-1}\tau + (-1)^{i+1} d_{i+1}^n s_i^{n-1}\tau + (\sum_{j=i+2}^{n}(-1)^j d_j^n s_i^{n-1}\tau) \\
&= (\sum_{j<i}(-1)^j s_{i-1}^{n-2} d_j^{n-1}\tau) + (-1)^i\tau + (-1)^{i+1}\tau + (\sum_{j=i+2}^{n}(-1)^j s_i^{n-2} d_{j-1}^{n-1}\tau) \\
&\in \mathrm{im}(s_{i-1}^{n-2}) + \mathrm{im}(s_i^{n-2}) \\
&\subseteq \mathrm{D}_{n-1}(A).
\end{aligned}$$

□

00QG **Construction 2.5.5.7** (The Normalized Moore Complex: First Construction)**.** Let $A_\bullet$ be a simplicial abelian group. We let $\mathrm{N}_*(A)$ denote the chain complex given by the quotient $\mathrm{C}_*(A)/\mathrm{D}_*(A)$, where $\mathrm{C}_*(A)$ is the Moore complex of Construction 2.5.5.1 and $\mathrm{D}_*(A) \subseteq \mathrm{C}_*(A)$ is the subcomplex of Proposition 2.5.5.6. We will refer to $\mathrm{N}_*(A)$ as the *normalized Moore complex* of the simplicial abelian group $A_\bullet$.

Put more informally, the normalized Moore complex $\mathrm{N}_*(A)$ of a simplicial abelian group $A_\bullet$ is obtained the Moore complex $\mathrm{C}_*(A)$ by forming the quotient by degenerate simplices of $A_\bullet$.

00SX **Remark 2.5.5.8.** By taking Construction 2.5.5.7 as our definition of the chain complex $\mathrm{N}_*(A)$, we have adopted the perspective that $\mathrm{N}_*(A)$ is a *quotient* of the Moore complex $\mathrm{C}_*(A)$. However, it can also be realized as a *subcomplex* of the Moore complex $\mathrm{C}_*(A)$: see Construction 2.5.6.15 and Proposition 2.5.6.17.

00QH **Construction 2.5.5.9** (The Normalized Chain Complex of a Simplicial Set)**.** Let $S_\bullet$ be a simplicial set and let $\mathbf{Z}[S_\bullet]$ be the simplicial abelian group freely generated by $S_\bullet$. We let $\mathrm{N}_*(S;\mathbf{Z})$ denote the normalized Moore complex of $\mathbf{Z}[S_\bullet]$. This chain complex can be described more concretely as follows:

- For each integer $n \geq 0$, we can identify $\mathrm{N}_n(S)$ with the free abelian group generated by the set S_n^{nd} of *nondegenerate* n-simplices of $S_\bullet$.
- The boundary map $\partial : \mathrm{N}_n(S) \to \mathrm{N}_{n-1}(S)$ is given by the formula

$$\partial(\sigma) = \sum_{i=0}^{n}(-1)^i \begin{cases} d_i^n(\sigma) & \text{if } d_i^n(\sigma) \text{ is nondegenerate} \\ 0 & \text{otherwise.} \end{cases}$$

We will refer to $\mathrm{N}_*(S; \mathbf{Z})$ as the *normalized chain complex* of the simplicial set $S_\bullet$.

Example 2.5.5.10. Let $S_\bullet = \Delta^0$ be the standard 0-simplex. Then the normalized chain complex $\mathrm{N}_*(S; \mathbf{Z})$ can be identified with abelian group $\mathbf{Z}$, regarded as a chain complex concentrated in degree zero. Note that the calculation of Example 2.5.5.4 shows that the quotient map $\mathrm{C}_*(S; \mathbf{Z}) \twoheadrightarrow \mathrm{N}_*(S; \mathbf{Z})$ induces an isomorphism on homology. 00QJ

Example 2.5.5.10 is a special case of the following:

Proposition 2.5.5.11. *For every simplicial abelian group $A_\bullet$, the quotient map $\mathrm{C}_*(A) \twoheadrightarrow \mathrm{N}_*(A)$ is a quasi-isomorphism of chain complexes: that is, it induces an isomorphism on homology groups.* 00QK

Remark 2.5.5.12. In the situation of Proposition 2.5.5.11, an even stronger statement holds: the quotient map $\mathrm{C}_*(A) \twoheadrightarrow \mathrm{N}_*(A)$ is a chain homotopy equivalence (Definition 2.5.0.5). 00QL

We will give the proof of Proposition 2.5.5.11 in §2.5.6 (see Proposition 2.5.6.21).

Example 2.5.5.13. Let $S_\bullet$ be a simplicial set. It follows from Proposition 2.5.5.11 that the quotient map $\mathrm{C}_*(S; \mathbf{Z}) \twoheadrightarrow \mathrm{N}_*(S; \mathbf{Z})$ induces an isomorphism on homology. In particular, the homology groups $\mathrm{H}_*(S; \mathbf{Z})$ of the simplicial set $S_\bullet$ (in the sense of Definition 2.5.5.2) can be computed by means of the normalized chain complex $\mathrm{N}_*(S; \mathbf{Z})$. This has various practical advantages. For example, if $S_\bullet$ is a simplicial set of dimension $\leq d$, then the chain complex $\mathrm{N}_*(S; \mathbf{Z})$ is concentrated in degrees $\leq d$. It follows that the homology groups $\mathrm{H}_*(S; \mathbf{Z})$ are also concentrated in degrees $\leq d$, which is not immediately obvious from the definition (note that the chain complex $\mathrm{C}_*(S; \mathbf{Z})$ is *never* concentrated in degrees $\leq d$, except in the trivial case where $S_\bullet$ is empty). 00QM

Example 2.5.5.14. Let $S_\bullet = \mathrm{N}_\bullet(Q)$ be the nerve of a partially ordered set Q. Suppose that Q has a least element e, which determines a map of simplicial sets $i : \Delta^0 \to S_\bullet$ which is right inverse to the projection map $q : S_\bullet \to \Delta^0$. Passing to normalized chain complexes, we obtain chain maps 00QN

$$\widehat{i} : \mathbf{Z}[0] \simeq \mathrm{N}_*(\Delta^0; \mathbf{Z}) \hookrightarrow \mathrm{N}_*(S_\bullet; \mathbf{Z}) \qquad \widehat{q} : \mathrm{N}_*(S_\bullet; \mathbf{Z}) \to \mathrm{N}_*(\Delta^0; \mathbf{Z}) \simeq \mathbf{Z}[0].$$

We claim that $\widehat{i}$ and $\widehat{q}$ are chain homotopy inverse to one another. In one direction, this is clear: the composition $\widehat{q} \circ \widehat{i}$ is equal to the identity. We complete the proof by constructing a chain homotopy from the composite map $\widehat{i} \circ \widehat{q}$ to the identity id on $\mathrm{N}_*(S_\bullet; \mathbf{Z})$. This chain homotopy is given by a collection of maps $h_m : \mathrm{N}_m(S; \mathbf{Z}) \to \mathrm{N}_{m+1}(S; \mathbf{Z})$, given on nondegenerate simplices by the construction

$$(q_0 < q_1 < \cdots < q_m) \mapsto \begin{cases} 0 & \text{if } q_0 = e \\ (e < q_0 < q_1 < \cdots < q_m) & \text{otherwise.} \end{cases}$$

In particular, if Q is a partially ordered set with a least element, then the homology groups of the nerve $S_\bullet = \mathrm{N}_\bullet(Q)$ are given by

$$\mathrm{H}_*(S;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } * = 0 \\ 0 & \text{otherwise.} \end{cases}$$

00QP **Variant 2.5.5.15** (Relative Chain Complexes)**.** Let $S_\bullet$ be a simplicial set and let $S'_\bullet \subseteq S_\bullet$ be a simplicial subset. Then we can identify the free simplicial abelian group $\mathbf{Z}[S'_\bullet]$ with a simplicial subgroup of $\mathbf{Z}[S_\bullet]$. We let $\mathrm{C}_*(S, S';\mathbf{Z})$ and $\mathrm{N}_*(S, S';\mathbf{Z})$ denote the Moore complex and normalized Moore complex of the simplicial abelian group $\mathbf{Z}[S_\bullet]/\mathbf{Z}[S'_\bullet]$. By virtue of Proposition 2.5.5.11, these complexes have the same homology groups, which we denote by $\mathrm{H}_*(S, S';\mathbf{Z})$ and refer to as the *relative homology groups* of the pair $(S'_\bullet \subseteq S_\bullet)$.

2.5.6 The Dold-Kan Correspondence

00QQ Let Ab denote the category of abelian groups, and $\mathrm{Ab}_\Delta = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Ab})$ the category of simplicial abelian groups. The formation of normalized Moore complexes (Construction 2.5.5.7) determines a functor $\mathrm{N}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathbf{Z})$. Our goal in this section is to prove the following fundamental result, which was discovered independently by Dold ([14]) and Kan ([36]):

00QR **Theorem 2.5.6.1** (The Dold-Kan Correspondence)**.** *The normalized Moore complex functor determines an equivalence of categories* $\mathrm{N}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathbf{Z})_{\geq 0}$.

00QS **Remark 2.5.6.2.** Theorem 2.5.6.1 admits many generalizations. For example, if $\mathcal{A}$ is an abelian category (Definition [?]), then a variant of Construction 2.5.5.9 supplies an equivalence of categories

$$\mathrm{N}_* : \{\text{Simplicial objects of } \mathcal{A}\} \to \mathrm{Ch}(\mathcal{A})_{\geq 0},$$

where $\mathrm{Ch}(\mathcal{A})_{\geq 0}$ denotes the category of (nonnegatively graded) chain complexes with values in $\mathcal{A}$ (see Theorem [?]). For more general categories $\mathcal{A}$, one can think of the category of simplicial objects $\mathcal{A}_\Delta = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{A})$ as a *replacement* for the category of chain complexes $\mathrm{Ch}(\mathcal{A})_{\geq 0}$, which is better behaved in "non-additive" situations.

We begin by constructing a right adjoint to the normalized Moore complex functor.

00QT **Construction 2.5.6.3** (The Eilenberg-MacLane Functor)**.** Let n be a nonnegative integer and let $\mathrm{N}_*(\Delta^n;\mathbf{Z})$ denote the normalized chain complex of the standard n-simplex (Construction 2.5.5.9). For every chain complex M_*, we let $\mathrm{K}_n(M_*)$ denote the collection of chain maps from $\mathrm{N}_*(\Delta^n;\mathbf{Z})$ into M_* (which we regard as an abelian group under addition). Note that the construction $[n] \mapsto \mathrm{N}_*(\Delta^n;\mathbf{Z})$ determines a functor from the simplex category $\mathbf{\Delta}$ to the category of chain complexes, so we can regard $[n] \mapsto \mathrm{K}_n(M_*)$ as a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to the category of abelian groups. We denote this simplicial abelian group by $\mathrm{K}(M_*)$, and refer to it as the *Eilenberg-MacLane space associated to M_**.

Remark 2.5.6.4. Let M_* be a chain complex. We will generally not distinguish in notation between the simplicial abelian group $\mathrm{K}(M_*)$ and its underlying simplicial set. Note that $\mathrm{K}(M_*)$ is automatically a Kan complex (Proposition 1.2.5.9), which motivates our usage of the term "space". `00QU`

Example 2.5.6.5. Let M_* be a chain complex. Then we have canonical isomorphisms `00QV`

$$\mathrm{K}_0(M_*) \simeq \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(\Delta^0; \mathbf{Z}), M_*) \simeq \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathbf{Z}[0], M_*) \simeq \mathrm{Z}_0(M).$$

In other words, we can identify vertices of the simplicial set $\mathrm{K}(M_*)$ with 0-cycles of the chain complex M_*.

Example 2.5.6.6. Let M_* be a chain complex, and let $x, y \in M_0$ be a pair of 0-cycles, which we identify with vertices of the simplicial set $\mathrm{K}(M_*)$. The following conditions are equivalent: `00QW`

(a) The vertices x and y belong to the same connected component of the simplicial set $\mathrm{K}(M_*)$ (Definition 1.2.1.8).

(b) There exists an edge e of the simplicial set $\mathrm{K}(M_*)$ connecting x to y (so that $d^1_1(e) = x$ and $d^1_0(e) = y$).

(c) The cycles x and y are homologous: that is, there exists an element $u \in M_1$ satisfying $\partial(u) = x - y$.

The equivalence of (a) $\Leftrightarrow$ (b) follows from the fact that $\mathrm{K}(M_*)$ is a Kan complex (see Remark 1.4.6.13), while the equivalence (b) $\Leftrightarrow$ (c) follows immediately from the construction of the simplicial set $\mathrm{K}(M_*)$. It follows that the set of connected components $\pi_0(\mathrm{K}(M_*))$ can be identified with the 0th homology group $\mathrm{H}_0(M)$.

Example 2.5.6.7. Let G be an abelian group and let $G[0]$ denote the chain complex given by the single group G, concentrated in degree 0. To supply an n-simplex of the simplicial set $\mathrm{K}(G[0])$, one must give a chain map $\sigma : \mathrm{N}_*(\Delta^n; \mathbf{Z}) \to G[0]$. By definition, a homomorphism of graded abelian groups from $\mathrm{N}_*(\Delta^n; \mathbf{Z})$ to $G[0]$ is given by a tuple $\{g_i\}_{0 \leq i \leq n}$ of elements of G, indexed by the set $[n] = \{0 < 1 < \cdots < n\}$ of vertices of Δ^n. Under this identification, the chain maps can be identified with those tuples $\{g_i\}_{0 \leq i \leq n}$ which are *constant*: that is, which satisfy $g_i = g_j$ for all $i, j \in [n]$. It follows that the Eilenberg-MacLane space $\mathrm{K}(G[0])$ can be identified with the constant simplicial abelian group $\underline{G}$. `00QZ`

00R0 **Example 2.5.6.8.** Let G be an abelian group and let $G[1]$ denote the chain complex consisting of the single abelian group G, concentrated in degree 1. To supply an n-simplex of the simplicial set $\mathrm{K}(G[1])$, one must give a chain map $\sigma : \mathrm{N}_*(\Delta^n; \mathbf{Z}) \to G[1]$. By definition, a homomorphism of graded abelian groups from $\mathrm{N}_*(\Delta^n; \mathbf{Z})$ to $G[1]$ is given by a system $\{a_{i,j}\}_{0 \leq i < j \leq n}$ of elements of G, indexed by the set of all nondegenerate edges of Δ^n. Under this identification, the chain maps can be identified with those systems $\{g_{i,j}\}_{0 \leq i < j \leq n}$ satisfying $g_{i,j} + g_{j,k} = g_{i,k}$ for $0 \leq i < j < k \leq n$. It follows that the Eilenberg-MacLane space $\mathrm{K}(G[1])$ can be identified with the classifying simplicial set $B_\bullet G$ of Construction 1.3.2.5.

We now consider a particularly important special case of Construction 2.5.6.3.

00QX **Construction 2.5.6.9** (Eilenberg-MacLane Spaces)**.** Let G be an abelian group, let n be a nonnegative integer, and let $G[n]$ denote the chain complex consisting of the single abelian group G, concentrated in degree n (Example 2.5.1.2). We will denote the simplicial abelian group $\mathrm{K}(G[n])$ by $\mathrm{K}(G, n)$ and refer to it as the *nth Eilenberg-MacLane space of G*.

For small values of n, it will be useful to consider allow more general coefficients.

- If G is any group (not necessarily abelian), we let $\mathrm{K}(G, 1)$ denote the classifying simplicial set $B_\bullet(G)$ (Construction 1.3.2.5).
- If G is any set, we let $\mathrm{K}(G, 0)$ denote the constant simplicial set $\underline{G}$ (Construction 1.1.5.2).

By virtue of Examples 2.5.6.7 and 2.5.6.8, this recovers the first definition in the case where G is an abelian group.

00R1 **Notation 2.5.6.10.** Let M_* be a chain complex. Then every n-simplex σ of the simplicial set $\mathrm{K}(M_*)$ can be identified with a map of chain complexes $\mathrm{N}_*(\Delta^n; \mathbf{Z}) \to M_*$, which carries the generator of $\mathrm{N}_n(\Delta^n; \mathbf{Z})$ to an n-chain $\widetilde{v}(\sigma) \in M_n$. Moreover:

- Since σ is a morphism of chain complexes, we have

$$\partial(\widetilde{v}(\sigma)) = \sum_{i=0}^{n} (-1)^i \widetilde{v}(d_i^n \sigma).$$

  In other words, the construction $\sigma \mapsto \widetilde{v}(\sigma)$ determines a chain map from the Moore complex $\mathrm{C}_*(\mathrm{K}(M_*))$ to the chain complex M_*.

- If σ is a degenerate n-simplex of $\mathrm{K}(M_*)$, then the map of chain complexes $\sigma : \mathrm{N}_*(\Delta^n; \mathbf{Z}) \to M_*$ factors through $\mathrm{N}_*(\Delta^m; \mathbf{Z})$ for some $m < n$, and therefore annihilates the generator of $\mathrm{N}_n(\Delta^n; \mathbf{Z})$. It follows that $\widetilde{v}$ factors (uniquely) as a composition

$$\mathrm{C}_*(\mathrm{K}(M_*)) \twoheadrightarrow \mathrm{N}_*(\mathrm{K}(M_*)) \xrightarrow{v} M_*.$$

We will refer to the chain map $v : \mathrm{N}_*(\mathrm{K}(M_*)) \to M_*$ as the *counit map.*

Proposition 2.5.6.11. *Let M_* be a chain complex and let $v : \mathrm{N}_*(\mathrm{K}(M_*)) \to M_*$ be the counit map of Notation 2.5.6.10. Then, for any simplicial abelian group $A_\bullet$, the composite map* 00R2

$$\theta : \mathrm{Hom}_{\mathrm{Ab}_\Delta}(A_\bullet, \mathrm{K}(M_*)) \to \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(A), \mathrm{N}_*(\mathrm{K}(M_*))) \xrightarrow{v\circ} \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(A), M_*)$$

is an isomorphism of abelian groups.

Proof. Let us say that a simplicial abelian group $A_\bullet$ is *free* if it can be written as a (possibly infinite) direct sum of simplicial abelian groups of the form $\mathbf{Z}[\Delta^n]$. Note that every simplicial abelian group $A_\bullet$ admits a surjection $P_\bullet \twoheadrightarrow A_\bullet$, where $P_\bullet$ is free (for example, we can take $P_\bullet$ to be the direct sum $\bigoplus_\sigma \mathbf{Z}[\Delta^{\dim(\sigma)}]$ where σ ranges over all the simplices of $A_\bullet$). Applying this observation twice, we observe that every simplicial abelian group $A_\bullet$ admits a resolution

$$Q_\bullet \to P_\bullet \twoheadrightarrow A_\bullet \to 0,$$

which determines a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Ab}_\Delta}(A_\bullet, \mathrm{K}(M_*)) \arrow[r] \arrow[d, "\theta"] & \mathrm{Hom}_{\mathrm{Ab}_\Delta}(P_\bullet, \mathrm{K}(M_*)) \arrow[r] \arrow[d, "\theta'"] & \mathrm{Hom}_{\mathrm{Ab}_\Delta}(Q_\bullet, \mathrm{K}(M_*)) \arrow[d, "\theta''"] \\
0 \arrow[r] & \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(A), M_*) \arrow[r] & \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(P), M_*) \arrow[r] & \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(Q), M_*).
\end{tikzcd}$$

Consequently, to prove that θ is an isomorphism, it will suffice to show that θ' and θ'' are isomorphisms. In other words, we may assume without loss of generality that the simplicial abelian group $A_\bullet$ is free. Decomposing $A_\bullet$ as a direct sum, we can further reduce to the case $A_\bullet = \mathbf{Z}[\Delta^n]$, in which case the result follows immediately from the definitions. □

Corollary 2.5.6.12. *The normalized Moore complex functor* $\mathrm{N}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathbf{Z})$ *admits a right adjoint* $\mathrm{K} : \mathrm{Ch}(\mathbf{Z}) \to \mathrm{Ab}_\Delta$*, given on objects by Construction 2.5.6.3.* 00R3

Note that we can also regard $M_* \mapsto \mathrm{K}(M_*)$ as a functor from chain complexes to simplicial *sets* (by neglecting the group structure on $\mathrm{K}(M_*)$. This simplicial set also has a universal property:

Corollary 2.5.6.13. *The normalized chain complex functor* 00R4

$$\mathrm{N}_*(-; \mathbf{Z}) : \mathrm{Set}_\Delta \to \mathrm{Ch}(\mathbf{Z})$$

admits a right adjoint, given on objects by the functor $M_* \mapsto \mathrm{K}(M_*)$ of Construction 2.5.6.3.

00R5 **Remark 2.5.6.14.** When regarded as a functor from $\mathrm{Ch}(\mathbf{Z})$ to the category of simplicial *sets*, the functor $M_* \mapsto \mathrm{K}(M_*)$ fits into the paradigm of Variant 1.2.2.8: it is the functor $\mathrm{Sing}^Q_\bullet$ associated to the cosimplicial chain complex

$$Q : \mathbf{\Delta} \to \mathrm{Ch}(\mathbf{Z}) \qquad [n] \mapsto \mathrm{N}_*(\Delta^n; \mathbf{Z}).$$

To deduce Theorem 2.5.6.1, it is convenient to use a different description of the normalized Moore complex.

00R6 **Construction 2.5.6.15** (The Normalized Moore Complex: Second Construction)**.** Let $A_\bullet$ be a simplicial abelian group. For each $n \geq 0$, we let $\widetilde{\mathrm{N}}_n(A)$ denote the subgroup of $\mathrm{C}_n(A) = A_n$ consisting of those elements x which satisfy $d^n_i(x) = 0$ for $1 \leq i \leq n$. Note that if x satisfies this condition, then we have

$$\partial(x) = \sum_{i=0}^{n} (-1)^i d^n_i(x) = d^n_0(x).$$

Moreover, the identity $d^{n-1}_i d^n_0(x) = d^{n-1}_0 d^n_{i+1}(x) = 0$ shows that $\partial(x) = d^n_0(x)$ belongs to the subgroup $\widetilde{\mathrm{N}}_{n-1}(A) \subseteq \mathrm{C}_{n-1} = A_{n-1}$. We can therefore regard $\widetilde{N}_*(A)$ as a subcomplex of the Moore complex $\mathrm{C}_*(A)$.

In the situation of Construction 2.5.6.15, we will abuse terminology by referring to the chain complex $\widetilde{\mathrm{N}}_*(A)$ as the normalized Moore complex of $A_\bullet$. This abuse is justified by the observation that the chain complexes $\widetilde{\mathrm{N}}_*(A)$ is canonically isomorphic to the normalized Moore complex $\mathrm{N}_*(A)$ of Construction 2.5.5.7 (Proposition 2.5.6.17 below). We will deduce this from the following more precise statement:

00R7 **Lemma 2.5.6.16.** *Let $A_\bullet$ be a simplicial abelian group and let n be a nonnegative integer. Then the map*

$$f : \bigoplus_{\alpha : [n] \twoheadrightarrow [m]} \widetilde{\mathrm{N}}_m(A) \to A_n \qquad \{x_\alpha\} \mapsto \sum \alpha^*(x_\alpha)$$

is an isomorphism of abelian groups. Here the direct sum is indexed by surjective nondecreasing maps $\alpha : [n] \to [m]$ for $0 \leq m \leq n$, and $\alpha^* : A_m \to A_n$ denotes the associated group homomorphism.

Proof. We first prove that f is surjective. The proof proceeds by induction on n. By virtue of our inductive hypothesis, the image of f contains the subgroups $\widetilde{\mathrm{N}}_n(A), \mathrm{D}_n(A) \subseteq \mathrm{C}_n(A) = A_n$. It will therefore suffice to show that the composite map

$$\widetilde{\mathrm{N}}_n(A) \hookrightarrow \mathrm{C}_n(A) \twoheadrightarrow \mathrm{C}_n(A)/\mathrm{D}_n(A)$$

is surjective. Fix an element $\overline{x} \in \mathrm{C}_n(A)/\mathrm{D}_n(A)$. For each $x \in \mathrm{C}_n(A)$ representing $\overline{x}$, let i_x be the smallest nonnegative integer such that $d^n_j(x)$ vanishes for $i_x < j \leq n$. Without loss of generality, we may assume that x is chosen so that $i = i_x$ is as small as possible. We wish to prove that $i = 0$ (so that x belongs to $\mathrm{N}_n(A)$). Assume otherwise, and set $y = x - (s^{n-1}_{i-1} \circ d^n_i)(x)$. Then y is congruent to x modulo $\mathrm{D}_n(A)$, and for $i \leq j \leq n$ we have

$$\begin{aligned} d^n_j(y) &= d^n_j(x) - (d^n_j \circ s^{n-1}_{i-1} \circ d^n_i)(x) \\ &= d^n_j(x) - \begin{cases} d^n_i(x) & \text{if } i = j \\ (s^{n-2}_{i-1} \circ d^{n-1}_{j-1} \circ d^n_i)(x) & \text{if } i < j. \end{cases} \\ &= d^n_j(x) - \begin{cases} d^n_i(x) & \text{if } i = j \\ (s^{n-2}_{i-1} \circ d^{n-1}_{i} \circ d^n_j)(x) & \text{if } i < j. \end{cases} \\ &= 0. \end{aligned}$$

It follows that $i_y < i = i_x$, contradicting our choice of x.

We now prove that f is injective. Suppose otherwise, so that there exists a nonzero element

$$\{x_\alpha\} \in \bigoplus_{\alpha : [n] \twoheadrightarrow [m]} \widetilde{\mathrm{N}}_m(A)$$

which is annihilated by f. Then there exists some surjective map $\beta : [n] \twoheadrightarrow [k]$ such that x_β is nonzero. Assume that k has been chosen as small as possible. Moreover, we may assume that β is *maximal* among nondecreasing maps $[n] \twoheadrightarrow [k]$ such that $x_\beta \neq 0$: in other words, that for any other map $\alpha : [n] \twoheadrightarrow [k]$ satisfying $\beta(i) \leq \alpha(i)$ for $0 \leq i \leq n$, we either have $\beta = \alpha$ or $x_\alpha = 0$. Let $\gamma : [k] \to [n]$ be the map given by $\gamma(j) = \min\{i \in [n] : \beta(i) = j\}$. Then γ is a nondecreasing map satisfying $\beta \circ \gamma = \mathrm{id}_{[k]}$ and $\gamma(0) = 0$. We then have

$$\begin{aligned} \gamma^* f(\{x_\alpha\}) &= \gamma^*(\sum_{\alpha : [n] \twoheadrightarrow [m]} \alpha^*(x_\alpha)) \\ &= \sum_{\alpha : [n] \twoheadrightarrow [m]} (\alpha \circ \gamma)^*(x_\alpha). \end{aligned}$$

We now inspect the summands appearing on the right hand side:

- Let $\alpha : [n] \twoheadrightarrow [m]$ be a surjective nondecreasing function, and suppose that the composite map $[k] \xrightarrow{\gamma} [n] \xrightarrow{\alpha} [m]$ is not surjective. Then we can choose $0 \leq i \leq m$ such that i does not belong the image of $\alpha \circ \gamma$. Then the homomorphism $(\alpha \circ \gamma)^* : A_m \to A_k$ factors through the face operator $d^m_i : A_m \to A_{m-1}$. Note that we must have $i > 0$ (since $\gamma(0) = 0$ and $\alpha(0) = 0$), so that x_α is annihilated by d^m_i (by virtue of our assumption that x_α belongs to the subgroup $\mathrm{N}_m(A) \subseteq A_m$) and therefore also by $(\alpha \circ \gamma)^*$.

- Let $\alpha : [n] \twoheadrightarrow [m]$ be a surjective nondecreasing function, and suppose that the composite map $[k] \xrightarrow{\gamma} [n] \xrightarrow{\alpha} [m]$ is surjective but not injective. In this case, we must have $m < k$, so that x_α vanishes by virtue of the minimality assumption on k.

- Let $\alpha : [n] \twoheadrightarrow [m]$ be a surjective map, and suppose that the composite map $[k] \xrightarrow{\gamma} [n] \xrightarrow{\alpha} [m]$ is bijective, so that $m = k$ and $\alpha \circ \gamma$ is the identity on $[k]$. For $0 \leq i \leq n$, we have $(\gamma \circ \beta)(i) \leq i$ (by the definition of γ), so that
$$\beta(i) = ((\alpha \circ \gamma) \circ \beta)(i) = (\alpha \circ (\gamma \circ \beta))(i) \leq \alpha(i).$$
Invoking our maximality assumption on β, we conclude that either $\alpha = \beta$ or x_α vanishes.

Combining these observations, we obtain an equality
$$x_\beta = \sum_{\alpha : [n] \twoheadrightarrow [m]} (\alpha \circ \gamma)^*(x_\alpha) = \gamma^* f(\{x_\alpha\}) = 0,$$
contradicting our choice of β. □

00R9 **Proposition 2.5.6.17.** *Let $A_\bullet$ be a simplicial abelian group. Then the composite map $\widetilde{\mathrm{N}}_*(A) \hookrightarrow \mathrm{C}_*(A) \twoheadrightarrow \mathrm{N}_*(A)$ is an isomorphism of chain complexes. In other words, the Moore complex $\mathrm{C}_*(A)$ splits as a direct sum of the subcomplex $\widetilde{\mathrm{N}}_*(A)$ of Construction 2.5.6.15 and the subcomplex $\mathrm{D}_*(A)$ of Proposition 2.5.5.6.*

Proof. The surjectivity of the composite map $\widetilde{\mathrm{N}}_*(A) \hookrightarrow \mathrm{C}_*(A) \twoheadrightarrow \mathrm{N}_*(A)$ follows from Lemma 2.5.6.16. Moreover, it follows by induction that the subgroup $\mathrm{D}_n(A) \subseteq A_n$ is generated by the images of the maps
$$\widetilde{\mathrm{N}}_m(A) \hookrightarrow A_m \xrightarrow{\alpha^*} A_n$$
where $\alpha : [n] \twoheadrightarrow [m]$ is a nondecreasing surjection and $m < n$, so that the injectivity also follows from Lemma 2.5.6.16. □

00R8 **Remark 2.5.6.18.** Let $f : A_\bullet \to B_\bullet$ be a morphism of simplicial abelian groups. By virtue of Proposition 2.5.6.17 and Lemma 2.5.6.16, the following assertions are equivalent:

- For every integer $n \geq 0$, the map of abelian groups $A_n \to B_n$ is surjective (respectively split surjective, injective, split injective).
- For every integer $n \geq 0$, the map of abelian groups $\mathrm{N}_n(A) \to \mathrm{N}_n(B)$ is surjective (respectively split surjective, injective, split injective).

00RA **Warning 2.5.6.19.** Let $A_\bullet$ be a simplicial abelian group, and let $A_\bullet^{\mathrm{op}}$ be the opposite simplicial abelian group (obtained by precomposing the functor $A_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Ab}$ with the order-reversal involution $\mathrm{Op} : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}^{\mathrm{op}}$ of Notation 1.4.2.1). Then there is a canonical isomorphism of Moore complexes $\psi : \mathrm{C}_*(A^{\mathrm{op}}) \simeq \mathrm{C}_*(A)$, given by $\psi(x) = (-1)^n x$ for $x \in A_n$. This isomorphism carries the subcomplex $\mathrm{D}_*(A^{\mathrm{op}})$ generated by the degenerate simplices of $A_\bullet^{\mathrm{op}}$ to the subcomplex $\mathrm{D}_*(A)$ generated by the degenerate simplices of $A_\bullet$, and therefore descends to an isomorphism of normalized Moore complexes $\mathrm{N}_*(A^{\mathrm{op}}) \simeq \mathrm{N}_*(A)$, where we view $\mathrm{N}_*(A)$ and $\mathrm{N}_*(A^{\mathrm{op}})$ as *quotients* of $\mathrm{C}_*(A)$ and $\mathrm{C}_*(A^{\mathrm{op}})$ (as in Construction 2.5.5.7). Beware that the isomorphism ψ does *not* carry the subcomplex $\widetilde{\mathrm{N}}_*(A^{\mathrm{op}}) \subseteq \mathrm{C}_*(A^{\mathrm{op}})$ of Construction 2.5.6.15 to the subcomplex $\widetilde{\mathrm{N}}_*(A) \subseteq \mathrm{C}_*(A)$. Instead, it carries $\widetilde{\mathrm{N}}_*(A^{\mathrm{op}})$ to a different subcomplex of $\mathrm{C}_*(A)$, given in degree n by those elements $x \in \mathrm{C}_n(A) = A_n$ satisfying $d^n_i(x) = 0$ for $0 \leq i < n$, and with differential given by $x \mapsto (-1)^n d^n_n(x)$. This subcomplex is yet another incarnation of the normalized Moore complex of $A_\bullet$, which is canonically isomorphic to $\widetilde{\mathrm{N}}_*(A)$ but not identical as a subcomplex of $\mathrm{C}_*(A)$.

Stated more informally: the definition of the normalized Moore complex $\mathrm{N}_*(A)$ as a *quotient* of $\mathrm{C}_*(A)$ (via Construction 2.5.5.7) is compatible with passage from a simplicial abelian group $A_\bullet$ to its opposite $A^{\mathrm{op}}_\bullet$, but the realization as a *subcomplex* of $\mathrm{C}_*(A)$ (via Construction 2.5.6.15) is not.

Remark 2.5.6.20. Let $A_\bullet$ be a simplicial abelian group. Then Warning 2.5.6.19 supplies a canonical isomorphism of normalized Moore complexes $\mathrm{N}_*(A) \simeq \mathrm{N}_*(A^{\mathrm{op}})$. By virtue of Theorem 2.5.6.1, this isomorphism can be lifted uniquely to an isomorphism of simplicial abelian groups $\varphi : A_\bullet \simeq A^{\mathrm{op}}_\bullet$. The isomorphism φ is characterized by the requirement that for every n-simplex $x \in A_n$, we have $\varphi(x) \equiv (-1)^n x$ modulo degenerate simplices of $A_\bullet$. 00RB

We now use Proposition 2.5.6.17 to deduce Proposition 2.5.5.11, which was stated without proof in §2.5.5. The statement can be reformulated as follows:

Proposition 2.5.6.21. *Let $A_\bullet$ be a simplicial abelian group. Then:* 00RC

(*a*) *The quotient map $\mathrm{C}_*(A) \twoheadrightarrow \mathrm{N}_*(A)$ induces an isomorphism on homology.*

(*b*) *The inclusion map $\widetilde{\mathrm{N}}_*(A) \hookrightarrow \mathrm{C}_*(A)$ induces an isomorphism on homology.*

(*c*) *The subcomplex $\mathrm{D}_*(A) \subseteq \mathrm{C}_*(A)$ of Notation 2.5.5.5 is acyclic: that is, its homology groups are trivial.*

Proof. By virtue of Proposition 2.5.6.17, assertions (*a*), (*b*), and (*c*) are equivalent. It will therefore suffice to prove (*b*). Note that the map $\widetilde{\mathrm{N}}_*(A) \hookrightarrow \mathrm{C}_*(A)$ is the inclusion of a direct summand (Proposition 2.5.6.17) and is therefore automatically injective on homology. To show that it also induces a surjective map, it will suffice to show that every n-cycle $x \in \mathrm{C}_n(A)$ is homologous to an element of the subgroup $\widetilde{\mathrm{N}}_n(A)$. Let i denote the smallest nonnegative integer for which the faces $d^n_j(x)$ vanish for $i < j \leq n$; our proof will proceed by induction on i. If $i = 0$, then x belongs to $\widetilde{\mathrm{N}}_n(A)$, and there is nothing to prove. Otherwise, let $y \in \mathrm{C}_n(A)$ denote the boundary given by $\partial(s_i^n(x))$. We then compute

$$\begin{aligned}
y &= \partial(s_i^n(x)) \\
&= \sum_{j=0}^{n+1}(-1)^j(d_j^{n+1} \circ s_i^n)(x) \\
&= (\sum_{j=0}^{i-1}(-1)^j(s_{i-1}^{n-1} \circ d_j^n)(x)) + (-1)^i x + (-1)^{i+1}x + (\sum_{j=i+2}^{n}(-1)^j(s_i^{n-1} \circ d_{j-1}^n)(x)) \\
&= s_{i-1}^{n-1}(\sum_{j=0}^{i-1}(-1)^j d_j^n(x)) \\
&= s_{i-1}^{n-1}((\sum_{j=0}^{i-1}(-1)^j d_j^n(x)) + (\sum_{j=i+1}^{n}(-1)^j d_j^n(x))) \\
&= s_{i-1}^{n-1}(\partial(x) - (-1)^i d_i^n(x)) \\
&= (-1)^{i+1}(s_{i-1}^{n-1} \circ d_i^n)(x).
\end{aligned}$$

Set $x' = x + (-1)^i y$. For $j \geq i$ we compute

$$\begin{aligned}
d_j^n(x') &= d_j^n(x) + (-1)^i d_j^n(y) \\
&= d_j^n(x) - (d_j^n \circ s_{i-1}^{n-1} \circ d_i^n)(x) \\
&= \begin{cases} d_j^n(x) - d_i^n(x) & \text{if } j = i \\ d_j^n(x) - (s_{i-1}^{n-2} \circ d_i^{n-1} \circ d_j^n)(x) & \text{if } j > i \end{cases} \\
&= 0.
\end{aligned}$$

Our inductive hypothesis then guarantees that x' is homologous to an element of the subgroup $\widetilde{\mathrm{N}}_n(A)$. Since x is homologous to x', it follows that x is also homologous to an element of the subgroup $\widetilde{\mathrm{N}}_n(A)$. □

00RD **Warning 2.5.6.22.** Let $A_\bullet$ be a semisimplicial abelian group. Then we can still apply Construction 2.5.6.15 to define a subcomplex $\widetilde{\mathrm{N}}_*(A)$ of the Moore complex $\mathrm{C}_*(A)$ (note that the definition of $\widetilde{\mathrm{N}}_*(A)$ refers only to the face operators of $A_\bullet$). However, it is generally not true that the inclusion map $\widetilde{\mathrm{N}}_*(A) \hookrightarrow \mathrm{C}_*(A)$ induces an isomorphism on homology unless $A_\bullet$ can be promoted to a simplicial abelian group.

We now turn to the proof of the Dold-Kan correspondence. The main ingredient is the following consequence of Proposition 2.5.6.17:

00RE **Proposition 2.5.6.23.** *Let M_* be a chain complex and let $v : \mathrm{N}_*(\mathrm{K}(M_*)) \to M_*$ be the counit map of Notation 2.5.6.10. Then:*

- *The map* $v_0 : \mathrm{N}_0(\mathrm{K}(M_*)) \to M_0$ *is a monomorphism, whose image is the set* $\mathrm{Z}_0(M)$ *of* 0*-cycles in* M_**.*

- *For* $n > 0$*, the map* $v_n : \mathrm{N}_n(\mathrm{K}(M_*)) \to M_n$ *is an isomorphism.*

Proof. The first assertion follows from Example 2.5.6.5. To prove the second, fix $n > 0$ and let f denote the composite map

$$\widetilde{\mathrm{N}}_n(\mathrm{K}(M_*)) \hookrightarrow \mathrm{C}_n(\mathrm{K}(M_*)) \twoheadrightarrow \mathrm{N}_n(\mathrm{K}(M_*)) \xrightarrow{v_n} M_n.$$

By virtue of Proposition 2.5.6.17, it will suffice to show that f is an isomorphism. By definition, we can identify $\mathrm{C}_n(\mathrm{K}(M_*)) = \mathrm{K}_n(M_*)$ with the set of all chain maps $\sigma : \mathrm{N}_*(\Delta^n; \mathbf{Z}) \to M_*$. Unwinding the definitions, we see that σ belongs to the subgroup $\widetilde{\mathrm{N}}_n(\mathrm{K}(M_*)) \subseteq \mathrm{C}_n(\mathrm{K}(M_*))$ if and only if it annihilates the subcomplex $\mathrm{N}_*(\Lambda^n_0; \mathbf{Z})$, where $\Lambda^n_0 \subset \Delta^n$ is the 0-horn defined in Construction 1.2.4.1. We can therefore identify $\widetilde{\mathrm{N}}_n(\mathrm{K}(M_*))$ with the abelian group $\mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(K_*, M_*)$, where K_* denotes the quotient of $\mathrm{N}_*(\Delta^n; \mathbf{Z})$ by the subcomplex $\mathrm{N}_*(\Lambda^n_0; \mathbf{Z})$. Note that there are exactly two nondegenerate simplices of Δ^n which do not belong to Λ^n_0; let us denote them by τ and τ' (where τ is of dimension n and τ' of dimension $n-1$). Moreover, the differential on $\mathrm{N}_*(\Delta^n; \mathbf{Z})$ satisfies $\partial(\tau) \equiv \tau' \pmod{\mathrm{N}_*(\Lambda^n_0; \mathbf{Z})}$. We conclude by observing that, under the preceding identification, the homomorphism $f : \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(K_*, M_*) \to M_n$ is given by evaluation on τ, and is therefore an isomorphism. □

Proof of Theorem 2.5.6.1. By virtue of Corollary 2.5.6.12, it will suffice to show that the construction $M_* \mapsto \mathrm{K}(M_*)$ induces an equivalence of categories $\mathrm{K} : \mathrm{Ch}(\mathbf{Z})_{\geq 0} \to \mathrm{Ab}_\Delta$. We first show that the functor K is fully faithful when restricted to $\mathrm{Ch}(\mathbf{Z})_{\geq 0}$. Let M_* and M'_* be chain complexes which are concentrated in degrees ≥ 0; we wish to show that the canonical map

$$\varphi : \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(M_*, M'_*) \to \mathrm{Hom}_{\mathrm{Ab}_\Delta}(\mathrm{K}(M_*), \mathrm{K}(M'_*))$$

is an isomorphism. Let $\theta : \mathrm{Hom}_{\mathrm{Ab}_\Delta}(\mathrm{K}(M_*), \mathrm{K}(M'_*)) \simeq \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(\mathrm{K}(M_*)), M'_*)$ be the isomorphism of Proposition 2.5.6.11. Unwinding the definitions, we see that $\theta \circ \varphi$ is given by precomposition with the counit map $v : \mathrm{N}_*(\mathrm{K}(M_*)) \to M_*$ of Notation 2.5.6.10, and is therefore an isomorphism by virtue of Proposition 2.5.6.23 (together with our assumption that M_* is concentrated in degrees ≥ 0). It follows that φ is also an isomorphism, as desired.

We now prove that the functor $\mathrm{K} : \mathrm{Ch}(\mathbf{Z})_{\geq 0} \to \mathrm{Ab}_\Delta$ is essentially surjective. Let $A_\bullet$ be a simplicial abelian group and let $M_* = \mathrm{N}_*(A)$ be its normalized Moore complex. Then there is a unique map of simplicial abelian groups $u : A_\bullet \to \mathrm{K}(M_*)$ for which the isomorphism

$$\theta : \mathrm{Hom}_{\mathrm{Ab}_\Delta}(A_\bullet, \mathrm{K}(M_*)) \to \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(A), M_*)$$

of Proposition 2.5.6.11 carries u to the identity map $\mathrm{id} : \mathrm{N}_*(A) \to M_*$. By construction, the induced map of normalized Moore complexes $\mathrm{N}_*(u) : \mathrm{N}_*(A) \to \mathrm{N}_*(\mathrm{K}(M_*))$ is right inverse to the counit map $v : \mathrm{N}_*(\mathrm{K}(M_*)) \to M_*$, which is an isomorphism by virtue of Proposition 2.5.6.23. Combining this observation with Proposition 2.5.6.17, we deduce that u induces an isomorphism of chain complexes $\widetilde{\mathrm{N}}_*(A) \to \widetilde{\mathrm{N}}_*(\mathrm{K}(M_*))$, and is therefore an isomorphism by virtue of Lemma 2.5.6.16. It follows that $A_\bullet \simeq \mathrm{K}(M_*)$ belongs to the essential image of the functor K, as desired. □

2.5.7 The Shuffle Product

`00RF` Let $\mathrm{Ab}_\Delta = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Ab})$ denote the category of simplicial abelian groups. We will regard Ab_Δ as a monoidal category with respect to the "levelwise" tensor product of (Example 2.1.2.16): if $A_\bullet$ and $B_\bullet$ are simplicial abelian groups, then their tensor product $A_\bullet \otimes B_\bullet$ is the simplicial abelian group given by the construction $([n] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto A_n \otimes B_n$. The category of chain complexes $\mathrm{Ch}(\mathbf{Z})$ is also equipped with a monoidal structure (Construction 2.5.1.17); we denote the tensor product of chain complexes X_* and Y_* by $X_* \boxtimes Y_*$ or $(X \boxtimes Y)_*$; given chains $x \in X_p$ and $y \in Y_q$, we will write $x \boxtimes y$ for the image of (x, y) in the abelian group $(X \boxtimes Y)_{p+q}$. According to Theorem 2.5.6.1, the normalized Moore complex functor $A_\bullet \mapsto \mathrm{N}_*(A)$ determines a fully faithful embedding $\mathrm{N}_* : \mathrm{Ab}_\Delta \hookrightarrow \mathrm{Ch}(\mathbf{Z})$. Beware that this functor does not commute with the formation of tensor products. Nevertheless, we have the following result:

`00RG` **Proposition 2.5.7.1.** *There exists a collection of maps*

$$\triangledown : \mathrm{N}_p(A) \times \mathrm{N}_q(B) \to \mathrm{N}_{p+q}(A \otimes B) \qquad (a, b) \mapsto a \triangledown b,$$

defined for every pair of simplicial abelian groups $A_\bullet$ and $B_\bullet$ and every pair of integers $p, q \in \mathbf{Z}$, and uniquely determined by the following properties:

- *Each of the maps $\triangledown : \mathrm{N}_p(A) \times \mathrm{N}_q(B) \to \mathrm{N}_{p+q}(A \otimes B)$ is bilinear and satisfies the Leibniz rule $\partial(a \triangledown b) = (\partial a) \triangledown b + (-1)^p a \triangledown (\partial b)$ (and therefore induces a chain map $\mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \to \mathrm{N}_*(A \otimes B)$; see Exercise 2.5.1.15).*

- *The operation $\triangledown$ depends functorially on $A_\bullet$ and $B_\bullet$. That is, if $f : A_\bullet \to A'_\bullet$ and $g : B_\bullet \to B'_\bullet$ are homomorphisms of simplicial abelian groups, then the diagram*

$$\begin{tikzcd}
\mathrm{N}_p(A) \times \mathrm{N}_q(B) \arrow[r, "\triangledown"] \arrow[d, "\mathrm{N}_p(f) \times \mathrm{N}_q(g)"] & \mathrm{N}_{p+q}(A \otimes B) \arrow[d, "\mathrm{N}_{p+q}(f \otimes g)"] \\
\mathrm{N}_p(A') \times \mathrm{N}_q(B') \arrow[r, "\triangledown"] & \mathrm{N}_{p+q}(A' \otimes B')
\end{tikzcd}$$

commutes.

- *For $a \in A_0$ and $b \in B_0$, we have $a \nabla b = a \otimes b$ (where we identify a, b, and $a \otimes b$ with the corresponding elements of* $\mathrm{N}_0(A)$, $\mathrm{N}_0(B)$, *and* $\mathrm{N}_0(A \otimes B)$, *respectively).*

For simplicial abelian groups $A_\bullet$ and $B_\bullet$ and integer $p, q \in \mathbf{Z}$, we will refer to the map

$$\nabla : \mathrm{N}_p(A) \times \mathrm{N}_q(B) \to \mathrm{N}_{p+q}(A \otimes B)$$

of Proposition 2.5.7.1 as the *shuffle product.* We begin by giving an explicit construction of this map, following Eilenberg and MacLane (see [18]).

Notation 2.5.7.2 ((p,q)-Shuffles)**.** Let p and q be nonnegative integers. A (p,q)-*shuffle* is a strictly increasing map of partially ordered sets $\sigma : [p+q] \to [p] \times [q]$, which we will often identify with a nondegenerate $(p+q)$-simplex of the cartesian product $\Delta^p \times \Delta^q$. 00RH

If σ is a (p,q)-shuffle, we let $\sigma_- : [p+q] \to [p]$ and $\sigma_+ : [p+q] \to [q]$ denote the nondecreasing maps given by the components of σ (so that $\sigma(i) = (\sigma_-(i), \sigma_+(i))$ for $0 \leq i \leq p+q$). Let I_- denote the set of integers $1 \leq i \leq p+q$ satisfying $\sigma_-(i-1) < \sigma_-(i)$ (or equivalently $\sigma_+(i-1) = \sigma_+(i)$), and let I_+ the set of integers $1 \leq i \leq p+q$ satisfying $\sigma_+(i-1) < \sigma_+(i)$ (or equivalently $\sigma_-(i-1) = \sigma_-(i)$). We let $(-1)^\sigma$ denote the product

$$\prod_{(i,j) \in I_- \times I_+} \begin{cases} 1 & \text{if } i < j \\ -1 & \text{if } i > j. \end{cases}$$

We will refer to $(-1)^\sigma$ as the *sign* of the (p,q)-shuffle σ.

Construction 2.5.7.3 (The Unnormalized Shuffle Product)**.** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups, and suppose we are given elements $a \in A_p$ and $b \in B_q$. We let $a \bar{\nabla} b$ denote the sum 00RJ

$$\sum_\sigma (-1)^\sigma \sigma_-^*(a) \otimes \sigma_+^*(b) \in (A \otimes B)_{p+q}$$

Here the sum is taken over all (p,q)-shuffles $\sigma = (\sigma_-, \sigma_+)$ (Notation 2.5.7.2), and we write $\sigma_-^* : A_p \to A_{p+q}$ and $\sigma_+^* : B_q \to B_{p+q}$ for the structure morphisms of the simplicial abelian groups $A_\bullet$ and $B_\bullet$, respectively. We will refer to $a \bar{\nabla} b$ as the *unnormalized shuffle product* of a and b.

We now summarize some essential properties of Construction 2.5.7.3.

Remark 2.5.7.4 (Unitality of the Shuffle Product)**.** Let $\mathbf{Z}[\Delta^0]$ be the constant simplicial abelian group taking the value $\mathbf{Z}$, and let us identify the integer 1 with the corresponding 0-simplex of $\mathbf{Z}[\Delta^0]$. Then, for any simplicial abelian group $A_\bullet$, the canonical isomorphisms $A_\bullet \simeq (A \otimes \mathbf{Z}[\Delta^0])_\bullet$ and $A_\bullet \simeq (\mathbf{Z}[\Delta^0] \otimes A)_\bullet$ are given by $a \mapsto a \bar{\nabla} 1$ and $a \mapsto 1 \bar{\nabla} a$, respectively. 00RK

00RL **Remark 2.5.7.5** (Commutativity of the Shuffle Product)**.** Let $\sigma : [p+q] \to [p] \times [q]$ be a (p,q)-shuffle, and let $\sigma' : [p+q] \to [q] \times [p]$ denote the composition of σ with the isomorphism $[p] \times [q] \simeq [q] \times [p]$ given by permuting the factors. Then σ' is a (q,p)-shuffle, whose sign is given by $(-1)^{\sigma'} = (-1)^{pq} \cdot (-1)^{\sigma}$. Consequently, if $A_\bullet$ and $B_\bullet$ are simplicial abelian groups containing simplices $a \in A_p$ and $b \in B_q$, then the canonical isomorphism $(A \otimes B)_{p+q} \simeq (B \otimes A)_{p+q}$ carries $a \overline{\nabla} b$ to $(-1)^{pq}(b \overline{\nabla} a)$.

00RM **Remark 2.5.7.6** (Associativity of the Shuffle Product)**.** Let $A_\bullet$, $B_\bullet$, and $C_\bullet$ be simplicial abelian groups containing simplices $a \in A_p$, $b \in B_q$, and $c \in C_r$. Then the canonical isomorphism $(A \otimes (B \otimes C))_{p+q+r} \simeq ((A \otimes B) \otimes C)_{p+q+r}$ carries $a \overline{\nabla}(b \overline{\nabla} c)$ to $(a \overline{\nabla} b) \overline{\nabla} c$. Both of these iterated shuffle products can be described concretely as the sum

$$\sum_{\sigma} (-1)^{\sigma} \sigma_{-}^{*}(a) \otimes \sigma_{0}^{*}(b) \otimes \sigma_{+}^{*}(c),$$

where the sum is taken over all strictly increasing maps $\sigma = (\sigma_{-}, \sigma_{0}, \sigma_{+}) : [p+q+r] \to [p] \times [q] \times [r]$, and $(-1)^{\sigma}$ denotes the product

$$\prod_{1 \leq i < j \leq p+q+r} \begin{cases} -1 & \text{if } \sigma_{-}(j-1) < \sigma_{-}(j) \text{ and } \sigma_{-}(i-1) = \sigma_{-}(i) \\ -1 & \text{if } \sigma_{+}(j-1) = \sigma_{+}(j) \text{ and } \sigma_{+}(i-1) < \sigma_{+}(i) \\ 1 & \text{otherwise.} \end{cases}$$

00RN **Proposition 2.5.7.7.** *Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the unnormalized shuffle product $\overline{\nabla} : A_p \times B_q \to (A \otimes B)_{p+q}$ satisfies the Leibniz rule*

$$\partial(a \overline{\nabla} b) = (\partial a) \overline{\nabla} b + (-1)^{p} a \overline{\nabla} (\partial b).$$

Proof. Without loss of generality, we may assume that $(p,q) \neq (0,0)$ and that the simplicial abelian groups $A_\bullet \simeq \mathbf{Z}[\Delta^p]$ and $B_\bullet \simeq \mathbf{Z}[\Delta^q]$ are freely generated by a and b, respectively. In this case, we can identify $(A \otimes B)_{p+q-1}$ with the free abelian group generated by the set of $(p+q-1)$-simplices of $\Delta^p \times \Delta^q$, which we view as nondecreasing functions $\tau : [p+q-1] \to [p] \times [q]$. For every such simplex τ, let c, c_{-}, and c_{+} denote the coefficients of τ appearing in $\partial(a \overline{\nabla} b)$, $(\partial a) \overline{\nabla} b$, and $a \overline{\nabla} (\partial b)$, respectively. We wish to prove that $c = c_{-} + (-1)^{p} c_{+}$. We may assume without loss of generality that the map τ is injective (otherwise, we have $c = c_{-} = c_{+} = 0$). Let us identify τ with a pair (τ_{-}, τ_{+}), where $\tau_{-} : [p+q-1] \to [p]$ and $\tau_{+} : [p+q-1] \to [q]$ are nondecreasing functions. We now distinguish three cases:

(1) Suppose that the map $\tau_{-} : [p+q-1] \to [p]$ is not surjective (that is, τ belongs to the simplicial subset $(\partial \Delta^p) \times \Delta^q \subseteq \Delta^p \times \Delta^q$). Then $p > 0$ and there exists a unique integer $0 \leq i \leq p$ which does not belong to the image of τ_{-}. We proceed under the assumption that $i < p$ (the case $i > 0$ follows by a similar argument, with minor changes in notation). We then make the following observations:

- There is a unique (p,q)-shuffle σ and integer $0 \leq j \leq p+q$ satisfying $\tau = d^{p+q}_j(\sigma)$. Here j is the smallest integer satisfying $\tau_-(j) = i+1$, and σ is given by the formula
$$\sigma(k) = \begin{cases} (\tau_-(k), \tau_+(k)) & \text{if } k < j \\ (i, \tau_+(j)) & \text{if } k = j \\ (\tau_-(k-1), \tau_+(k-1)) & \text{if } k > j. \end{cases}$$
It follows that $c = (-1)^j \cdot (-1)^{\sigma}$.

- There is a unique $(p-1,q)$-shuffle σ' and integer $0 \leq a \leq p$ such that τ is given by the composition
$$[p+q-1] \xrightarrow{\sigma'} [p-1] \times [q] \xrightarrow{\delta^a_p \times \mathrm{id}} [p] \times [q];$$
here $\delta^a_p : [p-1] \hookrightarrow [p]$ denotes the unique monomorphism whose image does not contain a (Construction 1.1.1.4). These conditions guarantee that $a = i$ and that σ' is given by the formula
$$\sigma'(k) = \begin{cases} (\tau_-(k), \tau_+(k)) & \text{if } k < j \\ (\tau_-(k)-1, \tau_+(k)) & \text{if } k \geq j. \end{cases}$$
Consequently, we have $c_- = (-1)^i \cdot (-1)^{\sigma'}$.

- There does not exist a $(p,q-1)$-shuffle σ'' and an integer $0 \leq b \leq q$ for which τ is equal to the composition
$$[p+q-1] \xrightarrow{\sigma''} [p] \times [q-1] \xrightarrow{\mathrm{id} \times \delta^b_q} [p] \times [q].$$
Consequently, the coefficient c_+ vanishes.

We are therefore reduced to verifying the identity $(-1)^j \cdot (-1)^{\sigma} = (-1)^i \cdot (-1)^{\sigma'}$, which is an immediate consequence of the definitions.

(2) Suppose that the map $\tau_+ : [p+q-1] \to [q]$ is not surjective (that is, τ belongs to the simplicial subset $\Delta^q \times (\partial \Delta^q) \subseteq \Delta^p \times \Delta^q$). The argument in this case proceeds as in (1), with minor adjustments in notation.

(3) The functions τ_- and τ_+ are both surjective. In this case, we have $c_- = c_+ = 0$. Note that there is a unique integer $1 \leq j \leq p+q-1$ satisfying $\tau_-(j-1) < \tau_-(j)$ and $\tau_+(j-1) < \tau_+(j)$. From this, it is easy to see that if σ is a (p,q)-shuffle satisfying

$d^{p+q}_k(\sigma) = \tau$ for some $0 \leq k \leq p+q$, then we must have $k = j$. Moreover, there are exactly two (p,q)-shuffles σ satisfying $d^{p+q}_j(\sigma) = \tau$, given by the formulae

$$\sigma(i) = \begin{cases} \tau(i) & \text{if } i < j \\ (\tau_-(j-1), \tau_+(j)) & \text{if } i = j \\ \tau(i-1) & \text{if } i > j \end{cases} \qquad \sigma(i) = \begin{cases} \tau(i) & \text{if } i < j \\ (\tau_-(j), \tau_+(j-1)) & \text{if } i = j \\ \tau(i-1) & \text{if } i > j. \end{cases}$$

Since these (p,q)-shuffles have opposite sign, we conclude that $c = 0 = c_- + (-1)^p c_+$, as desired.

□

We now adapt the shuffle product to the setting of normalized Moore complexes. For every simplicial abelian group $A_\bullet$, let $\mathrm{D}_*(A) \subseteq \mathrm{C}_*(A)$ be the subcomplex generated by the degenerate simplices of $A_\bullet$ (see Proposition 2.5.5.6).

00RP **Proposition 2.5.7.8.** *Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the unnormalized shuffle product*

$$\bar{\nabla} : \mathrm{C}_p(A) \times \mathrm{C}_q(B) \to \mathrm{C}_{p+q}(A \otimes B)$$

carries the subsets $\mathrm{D}_p(A) \times \mathrm{C}_q(B)$ and $\mathrm{C}_p(A) \times \mathrm{D}_q(B)$ into the subgroup $\mathrm{D}_{p+q}(A \otimes B) \subseteq \mathrm{C}_{p+q}(A \otimes B)$.

Proof. Let $a \in A_p$ and $b \in B_q$ be simplices of $A_\bullet$ and $B_\bullet$, respectively. We wish to show that if either a belongs to $\mathrm{D}_p(A)$ or b belongs to $\mathrm{D}_q(B)$, then the unnormalized shuffle product $a \bar{\nabla} b$ belongs to $\mathrm{D}_{p+q}(A \otimes B)$. Without loss of generality, we may assume that a belongs to $\mathrm{D}_p(A)$. Decomposing a into summands, we can further assume that $a = s^{p-1}_i(a')$ for some $0 \leq i \leq p-1$ and some $a' \in A_{p-1}$. Let $\sigma = (\sigma_-, \sigma_+)$ be a (p,q)-shuffle. Then there exists a unique integer $0 \leq j < p+q$ satisfying $\sigma_-(j) = i$ and $\sigma_-(j+1) = i+1$. It then follows that both $\sigma^*_-(a)$ and $\sigma^*_+(b)$ are fixed points of the composite maps

$$A_{p+q} \xrightarrow{d^{p+q}_j} A_{p+q-1} \xrightarrow{s^{p+q-1}_j} A_{p+q} \qquad B_{p+q} \xrightarrow{d^{p+q}_j} B_{p+q-1} \xrightarrow{s^{p+q-1}_j} B_{p+q},$$

so that $\sigma^*_-(a) \otimes \sigma^*_+(b)$ is a degenerate simplex of $(A \otimes B)_\bullet$. Allowing σ to vary, we deduce that the shuffle product

$$\sum_\sigma (-1)^\sigma \sigma^*_-(a) \otimes \sigma^*_+(b)$$

belongs to $\mathrm{D}_{p+q}(A \otimes B)$. □

00RQ **Construction 2.5.7.9** (The Shuffle Product)**.** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. It follows from Proposition 2.5.7.8 that for every pair of integers $p, q \in \mathbf{Z}$, there is a unique bilinear map $\triangledown : \mathrm{N}_p(A) \times \mathrm{N}_q(B) \to \mathrm{N}_{p+q}(A \otimes B)$ for which the diagram

$$\begin{tikzcd}
\mathrm{C}_p(A) \times \mathrm{C}_q(B) \arrow[r, "\bar{\triangledown}"] \arrow[d] & \mathrm{C}_{p+q}(A \otimes B) \arrow[d] \\
\mathrm{N}_p(A) \times \mathrm{N}_q(B) \arrow[r, "\triangledown"] & \mathrm{N}_{p+q}(A \otimes B)
\end{tikzcd}$$

commutes. We will refer to $\triangledown : \mathrm{N}_p(A) \times \mathrm{N}_q(B) \to \mathrm{N}_{p+q}(A \otimes B)$ as the *shuffle product map*. Given elements $a \in \mathrm{N}_p(A)$ and $b \in \mathrm{N}_q(B)$, we will write $a \triangledown b$ for the image of the pair (a, b) under the shuffle product map, which we refer to as the *shuffle product of a and b.*

We now summarize some properties of the properties of Construction 2.5.7.9, which follow immediately from the corresponding results for the unnormalized shuffle product (Remarks 2.5.7.4, 2.5.7.5, 2.5.7.6, and Proposition 2.5.7.7).

Proposition 2.5.7.10. *Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then:* `00RR`

(1) *The canonical isomorphisms $\mathrm{N}_*(A) \simeq \mathrm{N}_*(A \otimes \mathbf{Z}[\Delta^0])$ and $\mathrm{N}_*(A) \simeq \mathrm{N}_*(\mathbf{Z}[\Delta^0] \otimes A)$ are given by $a \mapsto a \triangledown 1$ and $a \mapsto 1 \triangledown a$, respectively; here we identify the integer 1 with its image in $\mathrm{N}_*(\Delta^0; \mathbf{Z}) \simeq \mathbf{Z}$.*

(2) *For $a \in \mathrm{N}_p(A)$ and $b \in \mathrm{N}_q(B)$, we have $a \triangledown b = (-1)^{pq}(b \triangledown a)$; here we abuse notation by identifying $a \triangledown b$ with its image under the canonical isomorphism $\mathrm{N}_{p+q}(A \otimes B) \simeq \mathrm{N}_{p+q}(B \otimes A)$.*

(3) *Let $C_\bullet$ be another simplicial abelian group, and suppose we are given elements $a \in \mathrm{N}_p(A)$, $b \in \mathrm{N}_q(B)$, and $c \in \mathrm{N}_r(C)$. Then $a \triangledown (b \triangledown c) = (a \triangledown b) \triangledown c$; here we abuse notation by identifying $a \triangledown (b \triangledown c)$ with its image under the canonical isomorphism $\mathrm{N}_{p+q+r}(A \otimes (B \otimes C)) \simeq \mathrm{N}_{p+q+r}((A \otimes B) \otimes C)$.*

(4) *The shuffle product $\triangledown : \mathrm{N}_p(A) \times \mathrm{N}_q(B) \to \mathrm{N}_{p+q}(A \otimes B)$ satisfies the Leibniz rule*

$$\partial(a \triangledown b) = (\partial a) \triangledown b + (-1)^p a \triangledown (\partial b).$$

Notation 2.5.7.11 (The Eilenberg-Zilber Homomorphism)**.** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. It follows from assertion (4) of Proposition 2.5.7.10 that there is a unique chain map `00RS`

$$\mathrm{EZ} : \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \to \mathrm{N}_*(A \otimes B)$$

satisfying $\mathrm{EZ}(a \boxtimes b) = a \triangledown b$ (see Exercise 2.5.1.15). We will refer to EZ as the *Eilenberg-Zilber homomorphism* (see Remark 2.5.7.16). It follows from assertions (1) and (3) of Proposition 2.5.7.10 that the collection of chain maps

$$\{\mathrm{EZ} : \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \to \mathrm{N}_*(A \otimes B)\}_{A_\bullet, B_\bullet \in \mathrm{Ab}_\Delta}$$

determine a lax monoidal structure (Definition 2.1.5.8) on the normalized Moore complex functor $\mathrm{N}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathbf{Z})$, with unit given by the canonical isomorphism of chain complexes $\mathbf{Z}[0] \simeq \mathrm{N}_*(\mathbf{Z}[\Delta^0])$ (in fact, it is even a lax *symmetric* monoidal structure in the sense of Definition [?]: this follows from assertion (2) of Proposition 2.5.7.10).

00RT **Example 2.5.7.12.** Let $S_\bullet$ and $T_\bullet$ be simplicial sets, and let $\mathbf{Z}[S_\bullet]$ and $\mathbf{Z}[T_\bullet]$ denote the free simplicial abelian groups generated by $S_\bullet$ and $T_\bullet$, respectively. Then the tensor product $\mathbf{Z}[S_\bullet] \otimes \mathbf{Z}[T_\bullet]$ can be identified with the free simplicial abelian group $\mathbf{Z}[S_\bullet \times T_\bullet]$ generated by the cartesian product $S_\bullet \times T_\bullet$. Invoking Construction 2.5.7.9, we obtain shuffle product maps

$$\triangledown : \mathrm{N}_p(S; \mathbf{Z}) \times \mathrm{N}_q(T; \mathbf{Z}) \to \mathrm{N}_{p+q}(S \times T; \mathbf{Z})$$

which induce a map of chain complexes $\mathrm{EZ} : \mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(T; \mathbf{Z}) \to \mathrm{N}_*(S \times T; \mathbf{Z})$. Allowing $S_\bullet$ and $T_\bullet$ to vary, these chain maps furnish a lax (symmetric) monoidal structure on the functor

$$\mathrm{N}_*(-; \mathbf{Z}) : \mathrm{Set}_\Delta \to \mathrm{Ch}(\mathbf{Z}) \qquad S_\bullet \mapsto \mathrm{N}_*(S; \mathbf{Z}).$$

00RU **Remark 2.5.7.13.** The Eilenberg-Zilber homomorphism of Example 2.5.7.12 admits a topological interpretation. Recall that, for every simplicial set $S_\bullet$, the topological space $|S_\bullet|$ is a CW complex (Remark 1.2.3.12). More precisely, $|S_\bullet|$ admits a CW decomposition with one cell e_σ for each nondegenerate simplex $\sigma : \Delta^n \to S_\bullet$, where e_σ is defined as the image of the composite map

$$|\Delta^n|^\circ \hookrightarrow |\Delta^n| \xrightarrow{|\sigma|} |S_\bullet|;$$

here $|\Delta^n|^\circ = \{(t_0, \ldots, t_n) \in \mathbf{R}_{>0} : t_0 + \cdots + t_n = 1\}$ denotes the interior of the topological n-simplex. The chain complex $\mathrm{N}_*(S; \mathbf{Z})$ of Construction 2.5.5.9 can then be identified with the *cellular chain complex* associated to this cell decomposition of $|S_\bullet|$.

When $S_\bullet = S'_\bullet \times S''_\bullet$ factors as a product of two other simplicial sets $S'_\bullet$ and $S''_\bullet$, the topological space $|S_\bullet|$ admits a *different* CW structure, whose cells are given by $\varphi^{-1}(e_{\sigma'} \times e_{\sigma''})$; here φ denotes the canonical map $|S_\bullet| \to |S'_\bullet| \times |S''_\bullet|$, and σ' and σ'' range over the collection of nondegenerate simplices of $S'_\bullet$ and $S''_\bullet$, respectively. The cellular chain complex associated to this cell decomposition can be identified with the tensor product complex $\mathrm{N}_*(S'_\bullet; \mathbf{Z}) \boxtimes \mathrm{N}_*(S''_\bullet; \mathbf{Z})$.

It is not difficult to see that if $\sigma' \in S'_p$ and $\sigma'' \in S''_q$ are nondegenerate simplices of $S'_\bullet$ and $S''_\bullet$, respectively, then the subset $\varphi^{-1}(e_{\sigma'} \times e_{\sigma''}) \subseteq |S_\bullet|$ can be written as a finite union of cells of the form e_σ (where σ is a nondegenerate simplex of $S_\bullet$). Writing $[\sigma']$ and $[\sigma'']$ for the corresponding generators of $\mathrm{N}_p(S'; \mathbf{Z})$ and $\mathrm{N}_q(S''; \mathbf{Z})$, the shuffle product is given by

$$[\sigma'] \triangledown [\sigma''] = \sum_{\sigma} \pm [\sigma] \in \mathrm{N}_{p+q}(S),$$

where the sum is taken over all nondegenerate $(p+q)$-simplices σ of $S_\bullet$ satisfying $e_\sigma \subseteq \varphi^{-1}(e_{\sigma'} \times e_{\sigma''})$; note that every such simplex σ can be written uniquely as a composition

$$\Delta^{p+q} \xrightarrow{\tau} \Delta^p \times \Delta^q \xrightarrow{\sigma' \times \sigma''} S'_\bullet \times S''_\bullet = S_\bullet$$

where τ is a (p,q)-shuffle in the sense of Notation 2.5.7.2. Moreover, the sign $(-1)^\tau$ also admits a topological interpretation: it is the degree of the open embedding $\varphi|_{e_\sigma} : e_\sigma \hookrightarrow e_{\sigma'} \times e_{\sigma''}$ (with respect to certain standard orientations of the cells e_σ, $e_{\sigma'}$, and $e_{\sigma''}$).

Theorem 2.5.7.14. *Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the Eilenberg-Zilber homomorphism* `00RV`

$$\mathrm{EZ} : \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \to \mathrm{N}_*(A \otimes B)$$

is a quasi-isomorphism: that is, it induces an isomorphism on homology.

Corollary 2.5.7.15. *Let $S_\bullet$ and $T_\bullet$ be simplicial sets. Then the Eilenberg-Zilber homomorphism* `00RW`

$$\mathrm{EZ} : \mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(T; \mathbf{Z}) \to \mathrm{N}_*(S \times T; \mathbf{Z})$$

is a quasi-isomorphism.

Remark 2.5.7.16. Corollary 2.5.7.15 is essentially due to Eilenberg and Zilber. More precisely, in [19], Eilenberg and Zilber proved that there exists a collection of quasi-isomorphisms $G_{S,T} : \mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(T; \mathbf{Z}) \to \mathrm{N}_*(S \times T; \mathbf{Z})$ depending functorially on the simplicial sets $S_\bullet$ and $T_\bullet$. The proof given in [19] uses the method of acyclic models and does not provide a concrete description of the maps $G_{S,T}$. However, it is not difficult to see that such a collection of chain maps $\{G_{S,T}\}$ must coincide up to sign with the Eilenberg-Zilber homomorphisms of Example 2.5.7.12 (see Exercise 2.5.7.18 below). `00RX`

Variant 2.5.7.17. Let $S_\bullet$ and $T_\bullet$ be simplicial sets containing simplicial subsets $S'_\bullet$ and $T'_\bullet$, respectively. Applying Theorem 2.5.7.14 to the simplicial abelian groups $\mathbf{Z}[S_\bullet]/\mathbf{Z}[S'_\bullet]$ and $\mathbf{Z}[T_\bullet]/\mathbf{Z}[T'_\bullet]$, we obtain a quasi-isomorphism `00RY`

$$\mathrm{EZ} : \mathrm{N}_*(S, S'; \mathbf{Z}) \boxtimes \mathrm{N}_*(T, T'; \mathbf{Z}) \to \mathrm{N}_*(S \times T, (S' \times T) \cup (S \times T'); \mathbf{Z}),$$

Proof of Theorem 2.5.7.14. Let us first regard the simplicial abelian group $A_\bullet$ as fixed. Let $M_* \in \mathrm{Ch}(\mathbf{Z})_{\geq 0}$ be a chain complex of abelian groups which is concentrated in degrees ≥ 0, and let $\mathrm{K}(M_*)$ be the associated Eilenberg-MacLane space (Construction 2.5.6.9). We will say that M_* is *good* if the Eilenberg-Zilber map

$$\mathrm{N}_*(A) \boxtimes M_* \simeq \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(\mathrm{K}(M_*)) \xrightarrow{\mathrm{EZ}} \mathrm{N}_*(A \otimes \mathrm{K}(M_*))$$

is a quasi-isomorphism. By virtue of Theorem 2.5.6.1, it will suffice to show that every object $M_* \in \mathrm{Ch}(\mathbf{Z})_{\geq 0}$ is good. Writing M_* as a filtered direct limit of bounded subcomplexes, we may assume that M_* is concentrated in degrees $\leq n$ for some integer $n \geq 0$. We proceed by induction on n. Let T denote the abelian group M_n, so that we have a short exact sequence of chain complexes

$$0 \to M'_* \to M_* \to T[n] \to 0,$$

where M'_* is concentrated in degrees $\leq n-1$. Note that this sequence is degreewise split, so that the associated exact sequence of simplicial abelian groups

$$0 \to \mathrm{K}(M'_*) \to \mathrm{K}(M_*) \to \mathrm{K}(T[n]) \to 0$$

is also degreewise split (see Remark 2.5.6.18). We therefore have a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{N}_*(A) \boxtimes M'_* \arrow[r] \arrow[d] & \mathrm{N}_*(A) \boxtimes M_* \arrow[r] \arrow[d] & \mathrm{N}_*(A) \boxtimes T[n] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{N}_*(A \otimes \mathrm{K}(M'_*)) \arrow[r] & \mathrm{N}_*(A \otimes \mathrm{K}(M_*)) \arrow[r] & \mathrm{N}_*(A \otimes \mathrm{K}(T[n])) \arrow[r] & 0,
\end{tikzcd}$$

where the left vertical map is a quasi-isomorphism by virtue of our inductive hypothesis. Invoking Remark 2.5.1.7, we see that M_* is good if and only if the chain complex $T[n]$ is good. In particular, the condition that M_* is good depends only the abelian group $T = M_n$. We may therefore assume without loss of generality that M_* factors as a tensor product $\mathrm{N}_*(\Delta^n; \mathbf{Z}) \boxtimes T[0]$. We are therefore reduced to proving Theorem 2.5.7.14 in the special case where $B_\bullet$ factors as a tensor product of $\mathbf{Z}[\Delta^n]$ with the abelian group T.

Applying the same argument with the roles of $A_\bullet$ and $B_\bullet$ reversed, we can also assume that $A_\bullet$ factors as the tensor product of $\mathbf{Z}[\Delta^m]$ with another abelian group T'. In this case, we are reduced to proving that the Eilenberg-Zilber map

$$\mathrm{EZ} : \mathrm{N}_*(\Delta^m; \mathbf{Z}) \boxtimes \mathrm{N}_*(\Delta^n; \mathbf{Z}) \to \mathrm{N}_*(\Delta^m \times \Delta^n; \mathbf{Z})$$

becomes a quasi-isomorphism after tensoring both sides with the abelian group $T' \otimes T$. In fact, we claim that this map is chain homotopy equivalence. To prove this, let u and v denote the initial vertices of Δ^m and Δ^n, respectively, and write $[u]$ and $[v]$ for the corresponding generators of $\mathrm{N}_0(\Delta^m; \mathbf{Z})$ and $\mathrm{N}_0(\Delta^n; \mathbf{Z})$. Then the shuffle product $[u] \triangledown [v]$ is given by $[w]$, where $w = (u, v)$ is the vertex of $\Delta^m \times \Delta^n$ corresponding to the least element of the partially ordered set $[m] \times [n]$. We have a commutative diagram of chain complexes

$$\begin{tikzcd}
\mathbf{Z}[0] \boxtimes \mathbf{Z}[0] \arrow[r, "\sim"] \arrow[d, "{[u] \boxtimes [v]}"] & \mathbf{Z}[0] \arrow[d, "{[w]}"] \\
\mathrm{N}_*(\Delta^m; \mathbf{Z}) \boxtimes \mathrm{N}_*(\Delta^n; \mathbf{Z}) \arrow[r] & \mathrm{N}_*(\Delta^m \times \Delta^n; \mathbf{Z})
\end{tikzcd}$$

where the vertical maps are chain homotopy equivalences (Example 2.5.5.14) and the upper horizontal map is an isomorphism, so the lower horizontal map is a chain homotopy equivalence as well. □

Proof of Proposition 2.5.7.1. It follows immediately from the definitions that the shuffle product maps

$$\triangledown : \mathrm{N}_p(A) \times \mathrm{N}_q(B) \to \mathrm{N}_{p+q}(A \otimes B)$$

depend functorially on $A_\bullet$ and $B_\bullet$ and satisfy $a \triangledown b = a \otimes b$ when $p = q = 0$, and the Leibniz rule follows from Proposition 2.5.7.10. To complete the proof of Proposition 2.5.7.1, we will show that the shuffle product is the *unique* operation with these properties. To this end, suppose we are given another collection of bilinear maps

$$\triangledown' : \mathrm{N}_p(A) \times \mathrm{N}_q(B) \to \mathrm{N}_{p+q}(A \otimes B)$$

which depend functorially on $A_\bullet$ and $B_\bullet$ and satisfy the Leibniz rule. In the special case $A_\bullet = B_\bullet = \mathbf{Z}[\Delta^0]$, we can identify $\mathrm{N}_0(A)$, $\mathrm{N}_0(B)$, and $\mathrm{N}_0(A \otimes B)$ with the group $\mathbf{Z}$ of integers, so that $1 \triangledown' 1 = n$ for some integer n. We will complete the proof by showing that for *every* pair of simplicial abelian groups $A_\bullet$ and $B_\bullet$ and every pair of elements $a \in \mathrm{N}_p(A)$, $b \in \mathrm{N}_q(B)$, we have $a \triangledown' b = n(a \triangledown b)$ (in particular, if $a \triangledown' b = a \otimes b$ whenever $p = q = 0$, we must have $n = 1$ and therefore $\triangledown' = \triangledown$).

Without loss of generality, we may assume that $p, q \geq 0$. We will proceed by induction on $p + q$. Choose a lift of a to an element of $\mathrm{C}_p(A)$, which we identify with a map of simplicial abelian groups $\mathbf{Z}[\Delta^p] \to A_\bullet$. Invoking our assumption that $\triangledown'$ is functorial, we can assume without loss of generality that $A_\bullet = \mathbf{Z}[\Delta^p]$ and that a is the generator of $\mathrm{N}_p(\Delta^p; \mathbf{Z})$ corresponding to the unique nondegenerate p-simplex of Δ^p. Similarly, we may assume that $B_\bullet = \mathbf{Z}[\Delta^q]$ and that $b \in \mathrm{N}_q(\Delta^q; \mathbf{Z})$ is the generator given by the unique nondegenerate q-simplex of Δ^q.

Let $\overline{a}$ and $\overline{b}$ denote the images of a and b in the relative chain complexes $\mathrm{N}_*(\Delta^p, \partial \Delta^p; \mathbf{Z}) \simeq \mathbf{Z}[p]$ and $\mathrm{N}_*(\Delta^q, \partial \Delta^q; \mathbf{Z}) \simeq \mathbf{Z}[q]$. Let $\partial(\Delta^p \times \Delta^q) \subseteq \Delta^p \times \Delta^q$ denote the union of the simplicial subsets $(\partial \Delta^p) \times \Delta^q$ and $\Delta^p \times (\partial \Delta^q)$, so that we have an isomorphism of simplicial abelian groups

$$(\mathbf{Z}[\Delta^p]/\mathbf{Z}[\partial \Delta^p]) \otimes (\mathbf{Z}[\Delta^q]/\mathbf{Z}[\partial \Delta^q]) \simeq \mathbf{Z}[\Delta^p \times \Delta^q]/\mathbf{Z}[\partial(\Delta^p \times \Delta^q)].$$

By virtue of Theorem 2.5.7.14, the Eilenberg-Zilber homomorphism

$$\mathrm{EZ} : \mathrm{N}_*(\Delta^p, \partial \Delta^p; \mathbf{Z}) \boxtimes \mathrm{N}_*(\Delta^q, \partial \Delta^q; \mathbf{Z}) \to \mathrm{N}_*(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q; \mathbf{Z})$$

is a quasi-isomorphism. In particular, the $(p+q)$-cycles of the chain complex $\mathrm{N}_*(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q; \mathbf{Z})$ form a cyclic group generated by the shuffle product $\overline{a} \triangledown \overline{b}$. Since the operation $\triangledown'$ satisfies the Leibniz rule, the chain $\overline{a} \triangledown' \overline{b} \in \mathrm{N}_{p+q}(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q; \mathbf{Z})$ is a cycle, and therefore satisfies $\overline{a}\nabla'\overline{b} = m(\overline{a}\nabla\overline{b})$ for some integer m. Using the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{N}_p(\Delta^p; \mathbf{Z}) \times \mathrm{N}_q(\Delta^q; \mathbf{Z}) \arrow[r, "\nabla'"] \arrow[d, "\sim"] & \mathrm{N}_{p+q}(\Delta^p \times \Delta^q; \mathbf{Z}) \arrow[d, "\sim"] \\
\mathrm{N}_p(\Delta^p, \partial\Delta^p; \mathbf{Z}) \times \mathrm{N}_q(\Delta^q, \partial\Delta^q; \mathbf{Z}) \arrow[r, "\nabla"] & \mathrm{N}_{p+q}(\Delta^p \times \Delta^q, \partial(\Delta^p \times \Delta^q); \mathbf{Z})
\end{tikzcd}$$

and the observation that the vertical maps are isomorphisms, we conclude that $a\nabla' b = m(a\nabla b)$. We will complete the proof by showing that $m = n$. In the case $p = q = 0$, this follows from the definition of the integer n. If $p + q > 0$, we invoke our inductive hypothesis to compute

$$\begin{aligned}
m\partial(a\nabla b) &= \partial(a\nabla' b) \\
&= (\partial a)\nabla' b + (-1)^p a\nabla'(\partial b) \\
&= n((\partial a)\nabla b + (-1)^p a\nabla(\partial b)) \\
&= n\partial(a\nabla b).
\end{aligned}$$

Since $\partial(a\nabla b)$ is a nonzero element of the free abelian group $\mathrm{N}_{p+q-1}(\Delta^p \times \Delta^q; \mathbf{Z})$, we must have $m = n$ as desired. □

00RZ **Exercise 2.5.7.18.** For every pair of simplicial sets $S_\bullet$ and $T_\bullet$, let

$$G_{S,T} : \mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(T; \mathbf{Z}) \to \mathrm{N}(S \times T; \mathbf{Z})$$

be a chain map. Assume that the maps $G_{S,T}$ depend functorially on $S_\bullet$ and $T_\bullet$: that is, for all maps of simplicial sets $f : S_\bullet \to S'_\bullet$ and $g : T_\bullet \to T'_\bullet$, the diagram of chain complexes

$$\begin{tikzcd}
\mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(T; \mathbf{Z}) \arrow[r, "G_{S,T}"] \arrow[d, "\mathrm{N}_*(f;\mathbf{Z}) \boxtimes \mathrm{N}_*(g;\mathbf{Z})"] & \mathrm{N}_*(S \times T; \mathbf{Z}) \arrow[d, "\mathrm{N}_*(f \times g;\mathbf{Z})"] \\
\mathrm{N}_*(S'; \mathbf{Z}) \boxtimes \mathrm{N}_*(T'; \mathbf{Z}) \arrow[r, "G_{S',T'}"] & \mathrm{N}_*(S' \times T'; \mathbf{Z})
\end{tikzcd}$$

is commutative. Adapt the proof Proposition 2.5.7.1 to show that there exists an integer n (not depending on $S_\bullet$ and $T_\bullet$) such that $G_{S,T} = n\mathrm{EZ}$, where EZ is the Eilenberg-Zilber homomorphism of Example 2.5.7.12.

2.5.8 The Alexander-Whitney Construction

Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups, having normalized Moore complexes $\mathrm{N}_*(A)$ and $\mathrm{N}_*(B)$ (Construction 2.5.5.7). In §2.5.7, we introduced the Eilenberg-Zilber homomorphism 00S0

$$\mathrm{EZ} : \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \to \mathrm{N}_*(A \otimes B)$$

and showed that it induces an isomorphism on homology groups (Theorem 2.5.7.14). The Eilenberg-Zilber homomorphism is usually not an isomorphism of chain complexes. However, it always exhibits the tensor product complex $\mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$ as a *direct summand* of the normalized Moore complex $\mathrm{N}_*(A \otimes B)$. More precisely, there exist chain maps $\mathrm{AW} : \mathrm{N}_*(A \otimes B) \to \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$, depending functorially on $A_\bullet$ and $B_\bullet$, for which the composite map

$$\mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \xrightarrow{\mathrm{EZ}} \mathrm{N}_*(A \otimes B) \xrightarrow{\mathrm{AW}} \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$$

is equal to the identity. Our goal in this section is to construct these maps and to establish their basic properties.

Notation 2.5.8.1. Let n be a nonnegative integer. For $0 \leq p \leq n$, we define strictly increasing functions 00S1

$$\iota_{\leq p} : [p] \hookrightarrow [n] \qquad \iota_{\geq p} : [n-p] \hookrightarrow [n]$$

by the formulae $\iota_{\leq p}(i) = i$ and $\iota_{\geq p}(j) = j + p$. If $A_\bullet$ is a simplicial abelian group, we let $\iota_{\leq p}^* : A_n \to A_p$ and $\iota_{\geq p}^* : A_n \to A_{n-p}$ denote the associated group homomorphisms.

Construction 2.5.8.2 (The Alexander-Whitney Construction: Unnormalized Version)**.** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups with Moore complexes $\mathrm{C}_*(A)$ and $\mathrm{C}_*(B)$, respectively. We define a map of graded abelian groups $\overline{\mathrm{AW}} : \mathrm{C}_*(A \otimes B) \to \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B)$ by the formula 00S2

$$\overline{\mathrm{AW}}(a \otimes b) = \sum_{0 \leq p \leq n} \iota_{\leq p}^*(a) \boxtimes \iota_{\geq p}^*(b)$$

for $a \in A_n$ and $b \in B_n$. We will refer to $\overline{\mathrm{AW}}$ as the *unnormalized Alexander-Whitney homomorphism.*

Proposition 2.5.8.3. *Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the unnormalized Alexander-Whitney homomorphism $\overline{\mathrm{AW}} : \mathrm{C}_*(A \otimes B) \to \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B)$ is a chain map.* 00S3

Proof. Let x be an element of the abelian group $\mathrm{C}_n(A \otimes B) = A_n \otimes B_n$; we wish to show that $\partial(\overline{\mathrm{AW}}(x)) = \overline{\mathrm{AW}}(\partial x)$. Without loss of generality, we may assume that $n > 0$ and that x has the form $a \otimes b$, for some elements $a \in A_n$ and $b \in B_n$. In this case, we compute

$$\overline{\mathrm{AW}}(\partial(a \otimes b)) \quad = \quad \sum_{i=0}^{n} (-1)^i \overline{\mathrm{AW}}(d_i^n a \otimes d_i^n b)$$

$$\begin{aligned}
&= \sum_{i=0}^{n}\sum_{p=0}^{n-1}(-1)^i \iota^*_{\leq p}(d^n_i a) \boxtimes \iota^*_{\geq p}(d^n_i b) \\
&= \sum_{i=0}^{n}\sum_{p=0}^{i-1}(-1)^i \iota^*_{\leq p}(d^n_i a) \boxtimes \iota^*_{\geq p}(d^n_i b) + \sum_{i=0}^{n}\sum_{p=i}^{n-1}(-1)^i \iota^*_{\leq p}(d^n_i a) \boxtimes \iota^*_{\geq p}(d^n_i b) \\
&= \sum_{i=0}^{n}\sum_{p=0}^{i-1}(-1)^i \iota^*_{\leq p}(a) \boxtimes d^{n-p}_{i-p}\iota^*_{\geq p}(b) + \sum_{i=0}^{n}\sum_{q=i+1}^{n}(-1)^i d^q_i \iota^*_{\leq q}(a) \boxtimes \iota^*_{\geq q}(b) \\
&= \sum_{i=0}^{n}\sum_{p=0}^{i}(-1)^i \iota^*_{\leq p}(a) \boxtimes d^{n-p}_{i-p}\iota^*_{\geq p}(b) + \sum_{i=0}^{n}\sum_{q=i}^{n}(-1)^i d^q_i \iota^*_{\leq q}(a) \boxtimes \iota^*_{\geq q}(b) \\
&= \sum_{p=0}^{n}(-1)^p \iota^*_{\leq p}(a) \boxtimes (\sum_{j=0}^{n-p}(-1)^j d^{n-p}_j \iota^*_{\geq p}(b)) + \sum_{q=0}^{n}(\sum_{i=0}^{q}(-1)^i d^q_i \iota^*_{\leq q}(a)) \boxtimes \iota^*_{\geq q}(b) \\
&= \sum_{p=0}^{n}(-1)^p \iota^*_{\leq p}(a) \boxtimes \partial\iota^*_{\geq p}(b) + \sum_{q=0}^{n}\partial\iota^*_{\leq q}(a) \boxtimes \iota^*_{\geq q}(b) \\
&= \partial(\sum_{p=0}^{n} \iota^*_{\leq p}(a) \boxtimes \iota^*_{\geq p}(b)) \\
&= \partial(\overline{\mathrm{AW}}(a \otimes b)).
\end{aligned}$$

□

00S4 **Proposition 2.5.8.4.** *The collection of unnormalized Alexander-Whitney homomorphisms* $\overline{\mathrm{AW}} : \mathrm{C}_*(A \otimes B) \to \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B)$ *determine a colax monoidal structure on the Moore complex functor* $\mathrm{C}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathbf{Z})$ *(see Variant 2.1.5.11).*

Proof. We first show that the unnormalized Alexander-Whitney homomorphisms determine a nonunital colax monoidal structure on the functor C_* (Variant 2.1.4.16). By construction, the homomorphism $\overline{\mathrm{AW}} : \mathrm{C}_*(A \otimes B) \to \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B)$ is natural in $A_\bullet$ and $B_\bullet$. It will therefore suffice to show that, for every triple of simplicial abelian groups $A_\bullet$, $B_\bullet$, and $C_\bullet$, the diagram of chain complexes

$$\begin{tikzcd}
\mathrm{C}_*(A \otimes (B \otimes C)) \arrow[r, "\sim"] \arrow[d, "\overline{\mathrm{AW}}"] & \mathrm{C}_*((A \otimes B) \otimes C) \arrow[d, "\overline{\mathrm{AW}}"] \\
\mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B \otimes C) \arrow[d, "\mathrm{id} \boxtimes \overline{\mathrm{AW}}"] & \mathrm{C}_*(A \otimes B) \boxtimes \mathrm{C}_*(C) \arrow[d, "\overline{\mathrm{AW}} \boxtimes \mathrm{id}"] \\
\mathrm{C}_*(A) \boxtimes (\mathrm{C}_*(B) \boxtimes \mathrm{C}_*(C)) \arrow[r, "\sim"] & (\mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B)) \boxtimes \mathrm{C}_*(C)
\end{tikzcd}$$

commutes, where the horizontal maps are given by the associativity constraints of the monoidal categories Ab_{Δ} and $\mathrm{Ch}(\mathbf{Z})$, respectively. Unwinding the definitions, we see that both the clockwise and counterclockwise composition are given by the construction

$$a \otimes (b \otimes c) \mapsto \sum_{0 \leq p \leq q \leq n} (\iota^{*}_{\leq p}(a) \boxtimes \rho^{*}(b)) \boxtimes \iota^{*}_{\geq q}(c)$$

for $a \in A_n$, $b \in B_n$, and $c \in C_n$, where ρ denotes the nondecreasing map $[q-p] \hookrightarrow [n]$ given by $\rho(i) = i + p$.

Note that the unit object of the category of simplicial abelian groups is the constant functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Ab}$ taking the value $\mathbf{Z}$, which we can identify with the free simplicial abelian group $\mathbf{Z}[\Delta^0]$ generated by the simplicial set Δ^0. The image of this object under the functor $\overline{\mathrm{AW}}$ is the unnormalized chain complex $\mathrm{C}_*(\Delta^0; \mathbf{Z})$. On the other hand, the unit object of $\mathrm{Ch}(\mathbf{Z})$ is the chain complex $\mathbf{Z}[0]$, which we will identify with the *normalized* chain complex $\mathrm{N}_*(\Delta^0; \mathbf{Z})$. We will complete the proof of Proposition 2.5.8.4 by showing that the quotient map $\epsilon : \mathrm{C}_*(\Delta^0; \mathbf{Z}) \twoheadrightarrow \mathrm{N}_*(\Delta^0; \mathbf{Z})$ is a counit for the nonunital colax monoidal structure constructed above (in the sense of Variant 2.1.5.11). To prove this, we must show that for every simplicial abelian group $A_\bullet$, both of the composite maps

$$\mathrm{C}_*(A) \simeq \mathrm{C}_*(A \otimes \mathbf{Z}[\Delta^0]) \xrightarrow{\overline{\mathrm{AW}}} \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(\Delta^0; \mathbf{Z}) \xrightarrow{\mathrm{id} \boxtimes \epsilon} \mathrm{C}_*(A) \boxtimes \mathbf{Z}[0] \simeq \mathrm{C}_*(A)$$

$$\mathrm{C}_*(A) \simeq \mathrm{C}_*(\mathbf{Z}[\Delta^0] \otimes A) \xrightarrow{\overline{\mathrm{AW}}} \mathrm{C}_*(\Delta^0; \mathbf{Z}) \boxtimes \mathrm{C}_*(A) \xrightarrow{\epsilon \boxtimes \mathrm{id}} \mathbf{Z}[0] \boxtimes \mathrm{C}_*(A) \simeq \mathrm{C}_*(A)$$

are equal to the identity. This follows immediately from the construction (using the fact that ϵ vanishes on every element of $\mathrm{C}_*(\Delta^0; \mathbf{Z})$ of positive degree). □

We now adapt the Alexander-Whitney construction to the setting of normalized Moore complexes. Recall that, for every simplicial abelian group $A_\bullet$, the degenerate simplices of $A_\bullet$ generate a subcomplex $\mathrm{D}_*(A) \subseteq \mathrm{C}_*(A)$ (Proposition 2.5.5.6) which is a direct summand of $\mathrm{C}_*(A)$ (Proposition 2.5.6.17). It follows that, if $B_\bullet$ is another simplicial abelian group, then we can view $\mathrm{C}_*(A) \boxtimes \mathrm{D}_*(B)$ and $\mathrm{D}_*(A) \boxtimes \mathrm{C}_*(B)$ as direct summands of $\mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B)$.

Proposition 2.5.8.5. *Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups, and let $K_* \subseteq \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B)$ be the subcomplex generated by $\mathrm{C}_*(A) \boxtimes \mathrm{D}_*(B)$ and $\mathrm{D}_*(A) \boxtimes \mathrm{C}_*(B)$. Then K_* contains the image of the composite map* 00S5

$$\mathrm{D}_*(A \otimes B) \hookrightarrow \mathrm{C}_*(A \otimes B) \xrightarrow{\overline{\mathrm{AW}}} \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B).$$

Proof. Let x be an n-simplex of the tensor product $A_\bullet \otimes B_\bullet$, let $0 \leq i \leq n$, and let $s^n_i(x)$ denote the associated degenerate $(n+1)$-simplex of $A_\bullet \otimes B_\bullet$. We wish to show that $\overline{\mathrm{AW}}(s^n_i(x))$

belongs to K_*. Without loss of generality, we may assume that $x = a \otimes b$ for n-simplices $a \in A_n$ and $b \in B_n$. In this case, we have

$$\overline{\mathrm{AW}}(s_i^n(x)) = \overline{\mathrm{AW}}(s_i^n(a) \otimes s_i^n(b)) = \sum_{p=0}^{n+1} \iota^*_{\leq p}(s_i^n(a)) \boxtimes \iota^*_{\geq p}(s_i^n(b)).$$

It will therefore suffice to show that each summand $\iota^*_{\leq p}(s_i^n(a)) \boxtimes \iota^*_{\geq p}(s_i^n(b))$ belongs to K_*. This is clear: the simplex $\iota^*_{\leq p}(s_i^n(a))$ is degenerate if $p > i$, and the simplex $\iota^*_{\geq p}(s_i^n(b))$ is degenerate for $p \leq i$. □

00S6 **Construction 2.5.8.6** (The Alexander-Whitney Construction: Normalized Version)**.** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. It follows from Proposition 2.5.8.5 that there is a unique chain map $\mathrm{AW} : \mathrm{N}_*(A \otimes B) \to \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$ for which the diagram

$$\begin{tikzcd}
\mathrm{C}_*(A \otimes B) \arrow[r, "\overline{\mathrm{AW}}"] \arrow[d] & \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B) \arrow[d] \\
\mathrm{N}_*(A \otimes B) \arrow[r, "\mathrm{AW}"] & \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B).
\end{tikzcd}$$

We will refer to AW as the *Alexander-Whitney homomorphism.*

We have the following normalized variant of Proposition 2.5.8.4 (which follows immediately from Proposition 2.5.8.4 itself):

00S7 **Proposition 2.5.8.7.** *The collection of Alexander-Whitney homomorphisms*

$$\mathrm{AW} : \mathrm{N}_*(A \otimes B) \to \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$$

determine a colax monoidal structure on the normalized Moore complex functor $\mathrm{N}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathbf{Z})$.

00S8 **Warning 2.5.8.8.** Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then we have a canonical isomorphism of simplicial abelian groups $A_\bullet \otimes B_\bullet \simeq B_\bullet \otimes A_\bullet$, given degreewise by the construction $a \otimes b \mapsto b \otimes a$. Likewise, there is a canonical isomorphism of chain complexes $\mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \simeq \mathrm{N}_*(B) \boxtimes \mathrm{N}_*(A)$ given by the Koszul sign rule (see Warning 2.5.1.14). Beware that these isomorphisms are not compatible with the Alexander-Whitney construction: that is, the diagram

$$\begin{tikzcd}
\mathrm{N}_*(A \otimes B) \arrow[r] \arrow[d, "\mathrm{AW}"] & \mathrm{N}_*(B \otimes A) \arrow[d, "\mathrm{AW}"] \\
\mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \arrow[r] & \mathrm{N}_*(B) \boxtimes \mathrm{N}_*(A)
\end{tikzcd}$$

usually does not commute. Instead, the composite map

$$\mathrm{N}_*(A \otimes B) \simeq \mathrm{N}_*(B \otimes A) \xrightarrow{\mathrm{AW}} \mathrm{N}_*(B) \boxtimes \mathrm{N}_*(A) \simeq \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$$

can be identified with the Alexander-Whitney homomorphism associated to the *opposite* simplicial abelian groups $A_\bullet^{\mathrm{op}}$ and $B_\bullet^{\mathrm{op}}$. In other words, the colax monoidal structure of Proposition 2.5.8.7 is not a colax *symmetric* monoidal structure (see Definition [?]). The same remark applies to the unnormalized Alexander-Whitney construction $\overline{\mathrm{AW}}$ of Construction 2.5.8.2.

Proposition 2.5.8.9. *Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the composition* 00S9

$$\mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \xrightarrow{\mathrm{EZ}} \mathrm{N}_*(A \otimes B) \xrightarrow{\mathrm{AW}} \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$$

is the identity map.

Proof. Fix element $a \in \mathrm{N}_p(A)$ and $b \in \mathrm{N}_q(B)$ having shuffle product $a \nabla b \in \mathrm{N}_{p+q}(A \otimes B)$. We wish to show that the Alexander-Whitney homomorphism AW satisfies $\mathrm{AW}(a \nabla b) = a \boxtimes b$. Lift a and b to elements $\overline{a} \in \mathrm{C}_p(A) = A_p$ and $\overline{b} \in \mathrm{C}_q(B) = B_q$, respectively. Unwinding the definitions, we see that $\mathrm{AW}(a \nabla b)$ is given by the image of

$$\begin{aligned} \overline{\mathrm{AW}}(\overline{a} \overline{\nabla} \overline{b}) &= \overline{\mathrm{AW}}(\sum_{\sigma} (-1)^{\sigma} (\sigma_-^* \overline{a}) \otimes (\sigma_+^*(\overline{b}))) \\ &= \sum_{r=0}^{p+q} \sum_{\sigma} (-1)^{\sigma} (\iota_{\leq r}^* \sigma_-^*)(\overline{a}) \boxtimes (\iota_{\geq r}^* \sigma_+^*)(\overline{b}) \end{aligned}$$

under the quotient map $\mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B) \twoheadrightarrow \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$; here the sum is taken over all (p,q)-shuffles $\sigma = (\sigma_-, \sigma_+)$ (see Notation 2.5.7.2). Note that the simplex $(\iota_{\leq r}^* \sigma_-^*)(\overline{a}) \in A_r$ is degenerate unless $\sigma_-(r) = r$ (which implies that $r \leq p$). Similarly, the simplex $(\iota_{\geq r}^* \sigma_+^*)(\overline{b}) \in B_{n-r}$ is degenerate unless $\sigma_+(r) = r - p$ (which guarantees that $r \geq p$). We may therefore ignore every term in the sum except for the one with $r = p$ and $\sigma(i) = \begin{cases} (i, 0) & \text{if } i \leq p \\ (p, i-p) & \text{if } i \geq p, \end{cases}$ for which the corresponding summand is equal to $\overline{a} \boxtimes \overline{b}$ (and therefore has image $a \boxtimes b$ in $\mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$). □

Warning 2.5.8.10. Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the unnormalized shuffle product $\overline{\nabla}$ of Construction 2.5.7.3 induces a chain map $\overline{\mathrm{EZ}} : \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B) \to \mathrm{C}_*(A \otimes B)$. However, the analogue of Proposition 2.5.8.9 for unnormalized Moore complexes is false: that is, the composite map 00SA

$$\mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B) \xrightarrow{\overline{\mathrm{EZ}}} \mathrm{C}_*(A \otimes B) \xrightarrow{\overline{\mathrm{AW}}} \mathrm{C}_*(A) \boxtimes \mathrm{C}_*(B)$$

is usually not equal to the identity.

00SB **Corollary 2.5.8.11.** *Let $A_\bullet$ and $B_\bullet$ be simplicial abelian groups. Then the Alexander-Whitney homomorphism*

$$\mathrm{AW} : \mathrm{N}_*(A \otimes B) \to \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$$

is a quasi-isomorphism: that is, it induces an isomorphism on homology.

Proof. By virtue of Proposition 2.5.8.9, the Alexander-Whitney homomorphism is a left inverse to the Eilenberg-Zilber map $\mathrm{EZ} : \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B) \to \mathrm{N}_*(A \otimes B)$, which is a quasi-isomorphism by virtue of Theorem 2.5.7.14. □

2.5.9 Comparison with the Homotopy Coherent Nerve

00SC Throughout this section, we maintain the notational convention of §2.5.8, denoting the tensor product of chain complexes X_* and Y_* by $X_* \boxtimes Y_*$. According to Proposition 2.5.8.7, the Alexander-Whitney homomorphisms

$$\mathrm{AW} : \mathrm{N}_*(A \otimes B) \to \mathrm{N}_*(A) \boxtimes \mathrm{N}_*(B)$$

determine a colax monoidal structure on the normalized Moore complex functor $\mathrm{N}_* : \mathrm{Ab}_\Delta \to \mathrm{Ch}(\mathbf{Z})$. Applying Remark 2.1.5.12, we deduce that the right adjoint functor $\mathrm{K} : \mathrm{Ch}(\mathbf{Z}) \to \mathrm{Ab}_\Delta$ inherits the structure of a lax monoidal functor. Composing with the (lax monoidal) forgetful functor $\mathrm{Ab}_\Delta \to \mathrm{Set}_\Delta$, we obtain the following:

00SD **Proposition 2.5.9.1.** *The functor $\mathrm{K} : \mathrm{Ch}(\mathbf{Z}) \to \mathrm{Set}_\Delta$ admits a lax monoidal structure, which associates to each pair of chain complexes X_* and Y_* a map of simplicial sets*

$$\mu_{X_*,Y_*} : \mathrm{K}(X_*) \times \mathrm{K}(Y_*) \to \mathrm{K}(X_* \boxtimes Y_*)$$

which can be described concretely as follows:

- *Let σ and τ be n-simplices of $\mathrm{K}(X_*)$ and $\mathrm{K}(Y_*)$, respectively, which we identify with chain maps*
$$\sigma : \mathrm{N}_*(\Delta^n; \mathbf{Z}) \to X_* \qquad \tau : \mathrm{N}_*(\Delta^n; \mathbf{Z}) \to Y_*.$$
Then $\mu_{X_*,Y_*}(\sigma, \tau) \in \mathrm{K}_n(X_* \boxtimes Y_*)$ is the composite map
$$\mathrm{N}_*(\Delta^n; \mathbf{Z}) \hookrightarrow \mathrm{N}_*(\Delta^n \times \Delta^n; \mathbf{Z}) \xrightarrow{\mathrm{AW}} \mathrm{N}_*(\Delta^n; \mathbf{Z}) \boxtimes \mathrm{N}_*(\Delta^n; \mathbf{Z}) \xrightarrow{\sigma \boxtimes \tau} X_* \boxtimes Y_*.$$

Applying the general construction described in Remark 2.1.7.4 to the lax monoidal functor $\mathrm{K} : \mathrm{Ch}(\mathbf{Z}) \to \mathrm{Set}_\Delta$, we obtain the following:

00SE **Construction 2.5.9.2.** Let $\mathcal{C}$ be a differential graded category. We define a simplicial category $\mathcal{C}_\bullet^\Delta$ as follows:

- The objects of $\mathcal{C}^{\Delta}_{\bullet}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}^{\Delta}_{\bullet}) = \mathrm{Ob}(\mathcal{C})$, the simplicial set $\mathrm{Hom}_{\mathcal{C}^{\Delta}}(X,Y)_{\bullet}$ is the generalized Eilenberg-MacLane space $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_{*})$. More concretely, the n-simplices of $\mathrm{Hom}_{\mathcal{C}^{\Delta}}(X,Y)_{\bullet}$ are chain maps $\sigma : \mathrm{N}_{*}(\Delta^{n}; \mathbf{Z}) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_{*}$.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C}^{\Delta}_{\bullet}) = \mathrm{Ob}(\mathcal{C})$ and every nonnegative integer $n \geq 0$, the composition law

$$\mathrm{Hom}_{\mathcal{C}^{\Delta}}(Y,Z)_{n} \times \mathrm{Hom}_{\mathcal{C}^{\Delta}}(X,Y)_{n} \to \mathrm{Hom}_{\mathcal{C}^{\Delta}}(X,Z)_{n}$$

carries a pair (σ, τ) to the n-simplex of $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Z)_{*})$ given by the composite map

$$\begin{aligned} \mathrm{N}_{*}(\Delta^{n}; \mathbf{Z}) & \hookrightarrow \mathrm{N}_{*}(\Delta^{n} \times \Delta^{n}; \mathbf{Z}) \\ & \xrightarrow{\mathrm{AW}} \mathrm{N}_{*}(\Delta^{n}; \mathbf{Z}) \boxtimes \mathrm{N}_{*}(\Delta^{n}; \mathbf{Z}) \\ & \xrightarrow{\sigma \boxtimes \tau} \mathrm{Hom}_{\mathcal{C}}(Y,Z)_{*} \boxtimes \mathrm{Hom}_{\mathcal{C}}(X,Y)_{*} \\ & \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X,Z)_{*}. \end{aligned}$$

We will refer to $\mathcal{C}^{\Delta}_{\bullet}$ as the *underlying simplicial category* of the differential graded category $\mathcal{C}$.

Remark 2.5.9.3. Let $\mathcal{C}$ be a differential graded category and let $\mathcal{C}^{\circ}$ denote its underlying category (in the sense of Construction 2.5.2.4). Then $\mathcal{C}^{\circ}$ is isomorphic to the underlying ordinary category $\mathcal{C}^{\Delta}_{0}$ of the simplicial category $\mathcal{C}^{\Delta}$ (in the sense of Example 2.4.1.4). Both of these categories can be described concretely as follows: 00SF

- The objects of $\mathcal{C}^{\circ} \simeq \mathcal{C}^{\Delta}_{0}$ are the objects of $\mathcal{C}$.
- For objects $X, Y \in \mathcal{C}$, the morphisms from X to Y in the category $\mathcal{C}^{\circ} \simeq \mathcal{C}^{\Delta}_{0}$ are given by 0-cycles in the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{*}$.

Remark 2.5.9.4. Let $\mathcal{C}$ be a differential graded category. Then the underlying simplicial category $\mathcal{C}^{\Delta}_{\bullet}$ is locally Kan (Definition 2.4.1.8). This follows from the observation that each of the simplicial sets $\mathrm{Hom}_{\mathcal{C}^{\Delta}}(X,Y)_{\bullet} = \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_{*})$ has the structure of a simplicial abelian group, and is therefore automatically a Kan complex (Proposition 1.2.5.9). 00SG

Remark 2.5.9.5. Let $\mathcal{C}$ be a differential graded category, let X and Y be objects of $\mathcal{C}$, and let $f, g : X \to Y$ be morphisms from X to Y in the underlying category $\mathcal{C}^{\circ}$ (that is, 0-cycles of the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{*}$). Then f and g are homotopic as morphisms of the differential graded category $\mathcal{C}$ (in the sense of Definition 2.5.4.1) if and only if they are homotopic as morphisms of the simplicial category $\mathcal{C}^{\Delta}_{\bullet}$ (Remark 2.4.1.9); see Example 2.5.6.6. It follows that the isomorphism of underlying categories $\mathcal{C}^{\circ} \simeq \mathcal{C}^{\Delta}_{0}$ of Remark 2.5.9.3 induces an isomorphism from the homotopy $\mathrm{h}\mathcal{C}$ (given by Construction 2.5.4.6) to the homotopy category $\mathrm{h}\mathcal{C}^{\Delta}$ (given by Construction 2.4.6.1). 00SH

Our goal in this section is to establish a refinement of Remark 2.5.9.5. Let $\mathcal{C}$ be a differential graded category and let $\mathcal{C}_{\bullet}^{\Delta}$ denote the underlying simplicial category. Then $\mathcal{C}_{\bullet}^{\Delta}$ is locally Kan (Remark 2.5.9.4), so the homotopy coherent nerve $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}^{\Delta})$ is an ∞-category (Theorem 2.4.5.1). Similarly, the differential graded nerve $\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$ is an ∞-category (Theorem 2.5.3.10). The ∞-categories $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}^{\Delta})$ and $\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$ are generally not isomorphic as simplicial sets. However, we will construct a comparison map $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}^{\Delta}) \to \mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$ and show that it is a trivial Kan fibration (and therefore an *equivalence* of ∞-categories; see Proposition 4.5.3.11). We begin with some auxiliary remarks.

00SJ **Construction 2.5.9.6** (The Fundamental Chain of a Cube)**.** Let I be a finite set of cardinality n, and let $\square^I = \prod_{i \in I} \Delta^1$ denote the associated cube (Notation 2.4.5.2), which we will identify with the nerve of the partially ordered set of all subsets of I. Using this identification, we obtain a bijective correspondence

$$\{\text{Linear orderings of } I\} \simeq \{\text{Nondegenerate } n\text{-simplices of } \square^I\},$$

which carries a linear ordering $\{i_1 < i_2 < \cdots < i_n\}$ to the chain of subsets

$$\emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset \{i_1, \ldots, i_{n-1}\} \subset I.$$

In particular, the symmetric group Σ_I of permutations of I acts simply transitively on the set of nondegenerate n-simplices of $\square^I$.

Fix a linear ordering of I, corresponding to a nondegenerate n-simplex $\sigma : \Delta^n \to \square^I$. We let $[\square^I]$ denote the alternating sum $\sum_{\pi \in \Sigma_I} (-1)^{\pi} \pi(\sigma)$, which we regard as an n-chain of the normalized chain complex $\mathrm{N}_*(\square^I; \mathbf{Z})$. We will refer to $[\square^I]$ as the *fundamental chain* of the cube $\square^I$. We will be particularly interested in the special case where I is the set $\{1, 2, \cdots, n\}$, endowed with its usual ordering; in this case, we denote the cube $\square^I$ by $\square^n$ and its fundamental chain $[\square^I]$ by $[\square^n]$.

00SK **Remark 2.5.9.7.** Let n be a nonnegative integer. Then the fundamental chain $[\square^n]$ of Construction 2.5.9.6 is given by the iterated shuffle product

$$[\Delta^1] \triangledown [\Delta^1] \triangledown \cdots \triangledown [\Delta^1] \in \mathrm{N}_n(\Delta^1 \times \Delta^1 \times \cdots \times \Delta^1; \mathbf{Z}) \simeq \mathrm{N}_n(\square^n; \mathbf{Z})$$

(see §2.5.7); here $[\Delta^1]$ denotes the generator of the group $\mathrm{N}_1(\Delta^1; \mathbf{Z}) \simeq \mathbf{Z}$ (which is also the fundamental chain of the 1-dimensional cube $\square^1$).

00SL **Warning 2.5.9.8.** The simplicial set $\square^I$ and its normalized chain complex $\mathrm{N}_*(\square^I; \mathbf{Z})$ depend only on the choice of the finite set I. However, the fundamental chain $[\square^I]$ of Construction 2.5.9.6 is *a priori* ambiguous up to a sign. One can resolve this ambiguity by choosing a linear ordering on the set I (as in Construction 2.5.9.6), which will be sufficient for our purposes in this section. However, less is needed: one needs only an *orientation* on the set I (or equivalently an orientation of the topological manifold-with-boundary $|\square^I| \simeq [0,1]^I$).

Notation 2.5.9.9. Let $\mathcal{C}$ be a differential graded category and let $\mathcal{C}^{\Delta}_{\bullet}$ denote the underlying simplicial category (Construction 2.5.9.2). Let $n \geq 0$ be a nonnegative integer and let σ be a nondegenerate $(n+1)$-simplex of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta})$, which we will identify with a simplicial functor $\sigma : \mathrm{Path}[n+1]_{\bullet} \to \mathcal{C}^{\Delta}_{\bullet}$. Set $X = \sigma(0)$ and $Y = \sigma(n+1)$, and $I = \{1, 2, \cdots, n\}$, so that Remark 2.4.5.4 supplies a morphism of simplicial sets 00SM

$$\square^{I} \simeq \mathrm{Hom}_{\mathrm{Path}[n+1]}(0, n+1)_{\bullet} \to \mathrm{Hom}_{\mathcal{C}^{\Delta}}(X, Y)_{\bullet} = \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X, Y)_{*}),$$

which we can identify with a chain map $\mathrm{N}_{*}(\square^{I}; \mathbf{Z}) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)_{*}$. For any choice of ordering of I, this map carries the fundamental chain $[\square^{I}]$ of Construction 2.5.9.6 to an element of the abelian group $\mathrm{Hom}_{\mathcal{C}}(X, Y)_{n}$, which we will denote by $\sigma([\square^{n}])$.

Proposition 2.5.9.10. *Let $\mathcal{C}$ be a differential graded category. Then there is a unique functor of ∞-categories $\mathfrak{Z} : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta}) \to \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ with the following properties:* 00SN

- *On 0-simplices the functor $\mathfrak{Z}$ is the identity: that is, it carries each object of the simplicial category $\mathcal{C}^{\Delta}$ to the corresponding object of the differential graded category $\mathcal{C}$.*
- *Let $n \geq 0$ and let σ be an $(n+1)$-simplex of $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta})$. Set $X = \sigma(0)$, $Y = \sigma(n+1)$, and $I = \{1, 2, \cdots, n\}$, which we endow with the* opposite *of its usual ordering. Then the value of $\mathfrak{Z}(\sigma)$ on $\{n+1 > n > \cdots > 0\}$ is the chain $\sigma([\square^{I}]) \in \mathrm{Hom}_{\mathcal{C}}(X, Y)_{n}$ (see Notation 2.5.9.9).*

Warning 2.5.9.11. In the formulation of Proposition 2.5.9.10, the ordering on the set $I = \{1, 2, \cdots, n\}$ is dictated by the "prefix" convention that the composition of a string of morphisms 01NN

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n$$

is denoted by $f_n \circ \cdots \circ f_1$, in which the indices appear (from left to right) in the opposite of their numerical order. Note that reversing the order on I changes the definition of the fundamental chain $[\square^{I}]$ by a factor of $(-1)^{n(n-1)/2}$ (see Warning 2.5.9.8).

The proof of Proposition 2.5.9.10 will require an elementary property of Construction 2.5.9.6.

Notation 2.5.9.12. Let I be a finite linearly ordered set of cardinality $n > 0$ and let $\square^{I}$ denote the corresponding simplicial cube. For each element $i \in I$, the linear ordering on I restricts to linear ordering on the subset $I \setminus \{i\}$, which determines a fundamental chain 00SP

$$[\square^{I \setminus \{i\}}] \in \mathrm{N}_{n-1}(\square^{I \setminus \{i\}}; \mathbf{Z}).$$

We will write $[\{0\} \times \square^{I \setminus \{i\}}] \in \mathrm{N}_{n-1}(\square^{I}; \mathbf{Z})$ for the image of the fundamental chain $[\square^{I \setminus \{i\}}]$ under the inclusion of simplicial sets

$$\square^{I \setminus \{i\}} \simeq \{0\} \times \square^{I \setminus \{i\}} \hookrightarrow \Delta^{1} \times \square^{I \setminus \{i\}} \simeq \square^{I}.$$

Similarly, we write $[\{1\} \times \square^{I \setminus \{i\}}] \in \mathrm{N}_{n-1}(\square^I; \mathbf{Z})$ for the image of the fundamental chain $[\square^{I \setminus \{i\}}]$ under the inclusion

$$\square^{I \setminus \{i\}} \simeq \{1\} \times \square^{I \setminus \{i\}} \hookrightarrow \Delta^1 \times \square^{I \setminus \{i\}} \simeq \square^I .$$

00SQ **Lemma 2.5.9.13.** *Let n be a nonnegative integer and let I denote the set $\{1, 2, \cdots, n\}$, endowed with its usual ordering. Then we have an equality*

$$\partial[\square^I] = \sum_{i=1}^{n} (-1)^i ([\{0\} \times \square^{I \setminus \{i\}}] - [\{1\} \times \square^{I \setminus \{i\}}])$$

in the abelian group $\mathrm{N}_{n-1}(\square^I; \mathbf{Z})$.

00SR **Remark 2.5.9.14.** Lemma 2.5.9.13 is a homological incarnation of the following topological assertion: the geometric realization $|\, \square^I \,| \simeq [0,1]^I$ is a manifold, whose boundary can be written as a union of the faces $\{0\} \times [0,1]^{I \setminus \{i\}}$ and $\{1\} \times [0,1]^{I \setminus \{i\}}$.

Proof of Lemma 2.5.9.13. Using the description of $[\square^I]$ as a shuffle product (Remark 2.5.9.7) and the fact that the shuffle product satisfies the Leibniz rule (Proposition 2.5.7.10), we compute

$$\begin{aligned}
\partial[\square^I] &= \partial([\Delta^1] \triangledown \cdots \triangledown [\Delta^1]) \\
&= \sum_{i=1}^{n} (-1)^{i-1} [\square^{i-1}] \triangledown \partial([\Delta^1]) \triangledown [\square^{n-i}] \\
&= \sum_{i=1}^{n} (-1)^{i} [\square^{i-1}] \triangledown (d^1_1[\Delta^1] - d^1_0[\Delta^1]) \triangledown [\square^{n-i}] \\
&= \sum_{i=1}^{n} (-1)^i ([\{0\} \times \square^{I \setminus \{i\}}] - [\{1\} \times \square^{I \setminus \{i\}}]).
\end{aligned}$$

□

00SS **Remark 2.5.9.15.** Let n be a nonnegative integer. It follows from Lemma 2.5.9.13 that the boundary $\partial[\square^n]$ belongs to the subcomplex $\mathrm{N}_*(\partial \square^n; \mathbf{Z}) \subset \mathrm{N}_*(\square^n; \mathbf{Z})$. In other words, the image of the fundamental chain $[\square^n]$ in the *relative* chain complex

$$\mathrm{N}_*(\square^n, \partial \square^n; \mathbf{Z}) = \mathrm{N}_*(\square^n; \mathbf{Z}) / \mathrm{N}_*(\partial \square^n; \mathbf{Z})$$

is a cycle. In fact, one can be more precise: the construction $1 \mapsto [\square^n]$ determines a quasi-isomorphism of chain complexes $u_n : \mathbf{Z}[n] \rightarrow \mathrm{N}_*(\square^n, \partial \square^n; \mathbf{Z})$. To prove this, we proceed by induction on n: the case $n = 0$ is trivial, and the inductive step follows by identifying u with the composition

$$\begin{array}{rcl}
\mathbf{Z}[n] & \simeq & \mathbf{Z}[1] \boxtimes \mathbf{Z}[n-1] \\
 & \xrightarrow{\mathrm{id} \boxtimes u_{n-1}} & \mathbf{Z}[1] \boxtimes \mathrm{N}_*(\square^{n-1}, \partial \square^{n-1}; \mathbf{Z}) \\
 & \simeq & \mathrm{N}_*(\square^1, \partial \square^1; \mathbf{Z}) \boxtimes \mathrm{N}_*(\square^{n-1}, \partial \square^{n-1}; \mathbf{Z}) \\
 & \xrightarrow{\mathrm{EZ}} & \mathrm{N}_*(\square^n, \partial \square^n; \mathbf{Z})
\end{array}$$

where EZ denotes the Eilenberg-Zilber map of Variant 2.5.7.17 (which is a quasi-isomorphism, by virtue of Theorem 2.5.7.14). Note that this property characterizes the fundamental chain $[\square^n]$ up to sign (since the quotient map $\mathrm{N}_*(\square^n; \mathbf{Z}) \twoheadrightarrow \mathrm{N}_*(\square^n, \partial \square^n; \mathbf{Z})$ is an isomorphism in degree n).

Lemma 2.5.9.16. *Let I be a finite linearly ordered set which is a union of disjoint subsets $I_-, I_+ \subseteq I$ satisfying $i_- < i_+$ for each $i_- \in I_-$ and $i_+ \in I_+$. Then the Alexander-Whitney homomorphism $\mathrm{AW} : \mathrm{N}_*(\square^I; Z) \rightarrow \mathrm{N}_*(\square^{I_-}; \mathbf{Z}) \times \mathrm{N}_*(\square^{I_+}; \mathbf{Z})$ satisfies* 00ST

$$\mathrm{AW}([\square^I]) = [\square^{I_-}] \boxtimes [\square^{I_+}].$$

Proof. Using Remark 2.5.9.7 (and the graded-commutativity of the shuffle product; see Proposition 2.5.7.10), we observe that the shuffle product map

$$\triangledown : \mathrm{N}_*(\square^{I_-}; \mathbf{Z}) \times \mathrm{N}_*(\square^{I_+}; \mathbf{Z}) \rightarrow \mathrm{N}_*(\square^{I_-} \times \square^{I_+}; \mathbf{Z}) \simeq \mathrm{N}_*(\square^I; \mathbf{Z})$$

satisfies $[\square^I] = [\square^{I_-}] \triangledown [\square^{I_+}]$. Applying the Alexander-Whitney homomorphism and invoking Proposition 2.5.8.9, we obtain the identity

$$\mathrm{AW}([\square^I]) = \mathrm{AW}([\square^{I_-}] \triangledown [\square^{I_+}]) = [\square^{I_-}] \boxtimes [\square^{I_+}].$$

$\square$

Proof of Proposition 2.5.9.10. Fix an integer $n \geq 0$, and let σ be an $(n+1)$-simplex of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta})$, which we will identify with a simplicial functor $\sigma : \mathrm{Path}[n+1]_{\bullet} \rightarrow \mathcal{C}^{\Delta}_{\bullet}$. Set $X = \sigma(0)$, $Y = \sigma(n+1)$, and let I denote the set $\{1, 2, \cdots, n\}$, endowed with the *opposite* of its usual ordering. By virtue of Remark 2.5.3.9, it will suffice to verify the following three assertions:

(a) If $n = 0$ and σ is the degenerate edge of $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta})$ determined by the object $X \in \mathcal{C}$, then $\sigma([\square^I]) = \mathrm{id}_X$.

(b) If $n > 0$ and σ is degenerate, then $\sigma([\square^I]) = 0$.

(c) Let $n \geq 0$. For $1 \leq i \leq n$, let $I_{<i}$ denote the set $\{1, 2, \cdots, i-1\}$ and let $I_{>i}$ denote the set $\{i+1, i+2, \cdots, n\}$, which we endow with the reverse of their usual orderings. Then we have

$$\partial \sigma([\square^I]) \;=\; \sum_{i=1}^{n} (-1)^{n+1-i} (\sigma_{\geq i}([\square^{I_{>i}}]) \sigma_{\leq i}([\square^{I_{<i}}]) - d_i^{n+1}(\sigma)([\square^{I \setminus \{i\}}]).$$

Assertion (a) is immediate from the definition. To prove (b), we observe that σ determines a map of simplicial sets

$$\operatorname{Hom}_{\operatorname{Path}[n+1]}(0, n+1)_{\bullet} \to \operatorname{Hom}_{\mathcal{C}^{\Delta}}(X, Y)_{\bullet} \simeq \mathrm{K}(\operatorname{Hom}_{\mathcal{C}}(X, Y)),$$

which we can identify with a chain map $u : \mathrm{N}_*(\operatorname{Hom}_{\operatorname{Path}[n+1]}(0, n+1); \mathbf{Z}) \to \operatorname{Hom}_{\mathcal{C}}(X, Y)_*$. If σ is degenerate, then (as a simplicial functor) it factors as a composition

$$\operatorname{Path}[n+1]_{\bullet} \to \operatorname{Path}[n]_{\bullet} \to \mathcal{C}^{\Delta}_{\bullet},$$

where ρ is a simplicial functor satisfying $\rho(0) = 0$ and $\rho(n+1) = n$. For $n > 0$, it follows that the chain map u factors through the complex $\mathrm{N}_*(\operatorname{Hom}_{\operatorname{Path}[n]}(0, n); \mathbf{Z}) \simeq \mathrm{N}_*(\square^{n-1}; \mathbf{Z})$. Since $\square^{n-1}$ is a simplicial set of dimension $\leq n-1$, the chain complex $\mathrm{N}_*(\square^{n-1}; \mathbf{Z})$ vanishes in degrees $\geq n$ (see Example 2.5.5.13). In particular, the map u vanishes in degree n, so that $\sigma([\square^I]) = 0$.

We now prove (c). Using Lemma 2.5.9.13 (and taking into account the order reversal on the set I), we obtain the identity

$$\partial \sigma([\square^I]) = \sum_{i=1}^{n} (-1)^{n+1-i} (\sigma([\{0\} \times \square^{I \setminus \{i\}}]) - \sigma([\{1\} \times \square^{I \setminus \{i\}}]).$$

It will therefore suffice to show that, for each $1 \leq i \leq n$, we have equalities

$$\sigma([\{0\} \times \square^{I \setminus \{i\}}]) = \sigma_{\geq i}([\square^{I_{>i}}]) \circ \sigma_{\leq i}([\square^{I_{<i}}])$$

$$\sigma([\{1\} \times \square^{I \setminus \{i\}}]) = d_i^{n+1}(\sigma)([\square^{I \setminus \{i\}}])$$

in the abelian group $\operatorname{Hom}_{\mathcal{C}}(X, Y)_{n-1}$. The second of these identities follows immediately from the definition of $d_i(\sigma)$. To prove the first, we note that the inclusion $\{0\} \times \square^{I \setminus \{i\}} \hookrightarrow \square^I \simeq \operatorname{Hom}_{\operatorname{Path}[n+1]}(0, n+1)_{\bullet}$ factors as a composition

$$\begin{aligned} \{0\} \times \square^{I \setminus \{i\}} &\simeq \square^{I_{>i}} \times \square^{I_{<i}} \\ &\simeq \operatorname{Hom}_{\operatorname{Path}[n+1]}(i, n+1)_{\bullet} \times \operatorname{Hom}_{\operatorname{Path}[n+1]}(0, i)_{\bullet} \\ &\xrightarrow{\circ} \operatorname{Hom}_{\operatorname{Path}[n+1]}(0, n+1)_{\bullet}. \end{aligned}$$

Set $Z = \sigma(i)$. Using the fact that σ is a simplicial functor (and the definition of the simplicial category $\mathcal{C}^{\Delta}_{\bullet}$), we see that $\sigma([\{0\} \times \square^{I \setminus \{i\}}])$ is the image of the fundamental chain $[\square^{I \setminus \{i\}}]$ under the composite map

$$\begin{aligned} \mathrm{N}_*([\square^{I\setminus\{i\}}];\mathbf{Z}) & \xrightarrow{\mathrm{AW}} \mathrm{N}_*([\square^{I_{>i}}];\mathbf{Z}) \boxtimes \mathrm{N}_*([\square^{I_{<i}}];\mathbf{Z}) \\ & \xrightarrow{\sigma_{\geq i} \boxtimes \sigma_{\leq i}} \mathrm{Hom}_{\mathcal{C}}(Z,Y)_* \boxtimes \mathrm{Hom}_{\mathcal{C}}(X,Z)_* \\ & \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X,Z)_*. \end{aligned}$$

The desired result now follows from the identity $\mathrm{AW}([\square^{I\setminus\{i\}}]) = [\square^{I_{>i}}] \boxtimes [\square^{I_{<i}}]$ supplied by Lemma 2.5.9.16. □

Exercise 2.5.9.17. Let $\mathcal{C}$ be a differential graded category, and let $\mathfrak{Z} : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta}) \to \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ be the functor of ∞-categories supplied by Proposition 2.5.9.10. Show that $\mathfrak{Z}$ is bijective on simplices of dimension $n \leq 2$ (for the case $n = 2$, this is essentially the content of Remark 2.5.4.4). 00SU

The functor $\mathfrak{Z} : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta}) \to \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ is generally not bijective on simplices of dimension $n \geq 3$. Nevertheless, we have the following:

Theorem 2.5.9.18. *Let $\mathcal{C}$ be a differential graded category and let $\mathfrak{Z} : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta}) \to \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ be the functor of ∞-categories supplied by Proposition 2.5.9.10. Then $\mathfrak{Z}$ is a trivial Kan fibration of simplicial sets.* 00SV

Proof. Fix an integer $n \geq 0$ and a diagram of simplicial sets

$$\begin{tikzcd} \partial\Delta^{n+1} \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta}) \arrow[d, "\mathfrak{Z}"] \\ \Delta^{n+1} \arrow[r, "\tau"] \arrow[ur, dashed, "\sigma"] & \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C}); \end{tikzcd}$$

we wish to show that the map σ_0 admits an extension $\sigma : \Delta^{n+1} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta})$ as indicated, rendering the diagram commutative. Let us abuse notation by identifying σ_0 with a simplicial functor from $\mathrm{Path}[\partial\Delta^{n+1}]_{\bullet}$ to $\mathcal{C}^{\Delta}$. Set $X = \sigma_0(0)$, $Y = \sigma_0(n+1)$, and $I = \{1, 2, \cdots, n\}$, so that σ_0 determines a morphism of simplicial sets

$$u_0 : \partial\square^I \simeq \mathrm{Hom}_{\mathrm{Path}[\partial\Delta^{n+1}]}(0, n+1) \to \mathrm{Hom}_{\mathcal{C}^{\Delta}}(X,Y)_{\bullet} = \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y))$$

(see Proposition 2.4.6.12), which we will identify with a chain map $f_0 : \mathrm{N}_*(\partial\square^I;\mathbf{Z}) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_*$. By virtue of Corollary 2.4.6.13, choosing an extension of σ_0 to a map $\sigma : \Delta^{n+1} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta})$ is equivalent to choosing an extension of u_0 to a map of simplicial sets $u : \square^I \to \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y))$, or an extension of f_0 to a chain map $f : \mathrm{N}_*(\square^I; \mathbf{Z}) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_*$.

Endow $I = \{1, \cdots, n\}$ with the opposite of its usual ordering and let $[\square^I]$ denote the fundamental chain of Construction 2.5.9.6. Note that the boundary $\partial[\square^n]$ belongs to the subcomplex $\mathrm{N}_*(\partial\square^I; \mathbf{Z}) \subset \mathrm{N}_*(\square^I; \mathbf{Z})$ (see Lemma 2.5.9.13). Unwinding the definitions, we see that τ supplies a chain $z \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_n$ satisfying $\partial(z) = f_0(\partial[\square^I]) \in \mathrm{Hom}_{\mathcal{C}}(X,Y)_{n-1}$. Let M_* denote the subcomplex of $\mathrm{N}_*(\square^I; \mathbf{Z})$ generated by $\mathrm{N}_*(\partial\square^I; \mathbf{Z})$ together with the fundamental chain $[\square^I]$, so that f_0 extends uniquely to a chain map $f_1 : M_* \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ satisfying $f_1([\square^I]) = z$. Unwinding the definitions, we see that if $f : \mathrm{N}_*(\square^I; \mathbf{Z}) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ is a map of chain complexes extending f_0, then the corresponding extension $\sigma : \Delta^{n+1} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta})$ of σ_0 satisfies $\mathfrak{Z} \circ \sigma = \tau$ if and only if $f|_{M_*} = f_1$. We will complete the proof by showing that M_* is a direct summand of $\mathrm{N}_*(\square^I; \mathbf{Z})$ (so that any map $f_1 : M_* \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ can be extended to $\mathrm{N}_*(\square^I; \mathbf{Z})$). To prove this, note that we have an exact sequence of chain complexes

$$0 \to \mathbf{Z}[n] \xrightarrow{[\square^I]} \mathrm{N}_*(\square^I, \partial\square^I; \mathbf{Z}) \to \mathrm{N}_*(\square^I; \mathbf{Z})/M_* \to 0,$$

where the first map is a quasi-isomorphism (Variant 2.5.7.17). It follows that the chain complex $\mathrm{N}_*(\square^I; \mathbf{Z})/M_*$ is acyclic and free in each degree, so that the exact sequence

$$0 \to M_* \to \mathrm{N}_*(\square^I; \mathbf{Z}) \to \mathrm{N}_*(\square^I; \mathbf{Z})/M_* \to 0$$

splits by virtue of Proposition 2.5.1.10. $\square$

Chapter 3

Kan Complexes

Recall that a *Kan complex* is a simplicial set X with the property that, for $n > 0$ and $0 \leq i \leq n$, any morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ can be extended to an n-simplex of X (Definition 1.2.5.1). Kan complexes play an important role in the theory of ∞-categories, for three different (but closely related) reasons: `00SY`

(*a*) Every Kan complex is an ∞-category (Example 1.4.0.3). Conversely, every ∞-category $\mathcal{C}$ contains a largest Kan complex $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ (obtained from $\mathcal{C}$ by removing all non-invertible morphisms; see Construction 4.4.3.1), which is an important invariant of $\mathcal{C}$. Consequently, understanding the homotopy theory of Kan complexes can be regarded as a first step towards understanding ∞-categories in general.

(*b*) Let $\mathcal{C}$ be an ∞-category. To every pair of objects $X, Y \in \mathcal{C}$, one can associate a Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ which we will refer to as the *space of maps from X to Y* (see Construction 4.6.1.1). These mapping spaces are essential to the structure of $\mathcal{C}$. For example, we will see later that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ admits a homotopy inverse if and only if it is essentially surjective at the level of homotopy categories and induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for every pair of objects $X, Y \in \mathcal{C}$ (see Theorem 4.6.2.20).

(*c*) The collection of all Kan complexes can be organized into an ∞-category, which we will denote by $\mathcal{S}$ and refer to as the *∞-category of spaces* (Construction 5.5.1.1). The ∞-category $\mathcal{S}$ plays a central role in the general theory of ∞-categories, analogous to the role of Set in classical category theory. This can be articulated in several different ways:

- To any ∞-category $\mathcal{C}$, one can associate a functor $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ called the *Yoneda embedding*, which is given informally (and up to homotopy equivalence) by the construction $C \mapsto \mathrm{Hom}_{\mathcal{C}}(\bullet, C)$ (see Definition 8.3.3.9). Like the classical

Yoneda embedding, the functor h is fully faithful: that is, it induces an equivalence on mapping spaces (Theorem 8.3.3.13).

- The ∞-category $\mathcal{S}$ has a pointed variant $\mathcal{S}_*$, whose objects are *pointed* Kan complexes (Construction 5.5.3.1). This ∞-category is equipped with a forgetful functor $\mathcal{S}_* \to \mathcal{S}$, given on objects by the construction $(X, x) \mapsto X$. This forgetful functor is an example of a *left fibration* of ∞-categories (see Definition 4.2.1.1). In fact, it is a *universal* left fibration in the following sense: for any ∞-category $\mathcal{C}$, the construction

$$(F : \mathcal{C} \to \mathcal{S}) \mapsto (u : \mathcal{C} \times_{\mathcal{S}} \mathcal{S}_* \to \mathcal{C})$$

  induces a bijection from the set of isomorphism classes of functors $F : \mathcal{C} \to \mathcal{S}$ to the set of equivalence classes of left fibrations $\overline{\mathcal{C}} \to \mathcal{C}$ having essentially small fibers (Corollary 5.6.0.6).

- The ∞-category $\mathcal{S}$ admits small colimits (Corollary 7.4.5.6). Moreover, if $\mathcal{C}$ is any other ∞-category which admits small colimits, then evaluation on the Kan complex $\Delta^0 \in \mathcal{S}$ induces an equivalence of ∞-categories

$$\operatorname{Fun}'(\mathcal{S}, \mathcal{C}) \to \mathcal{C} \qquad F \mapsto F(\Delta^0),$$

  where $\operatorname{Fun}'(\mathcal{S}, \mathcal{C})$ denotes the full subcategory of $\operatorname{Fun}(\mathcal{S}, \mathcal{C})$ spanned by those functors which preserve small colimits (Example 8.4.0.4). In other words, the ∞-category $\mathcal{S}$ is *freely* generated under small colimits by the Kan complex Δ^0.

Our goal in this chapter is to give an exposition of the homotopy theory of Kan complexes. We begin in §3.1 by developing the basic vocabulary of simplicial homotopy theory. In particular, we introduce the notions of *Kan fibration* (Definition 3.1.1.1), *anodyne morphism* (Definition 3.1.2.1), and *(weak) homotopy equivalence* between simplicial sets (Definitions 3.1.6.1 and 3.1.6.12), and establish some of their basic formal properties.

Recall that, to any Kan complex X, we can associate a set $\pi_0(X)$ of *connected components* of X (Definition 1.2.1.8). In §3.2, we associate to each base point $x \in X$ a sequence of groups $\{\pi_n(X, x)\}_{n>0}$, which we refer to as the *homotopy groups of* X (Construction 3.2.2.4 and Theorem 3.2.2.10), and establish some of their essential properties. In particular, we prove a simplicial analogue of Whitehead's theorem: a morphism of Kan complexes $f : X \to Y$ is a homotopy equivalence if and only if it induces a bijection $\pi_0(X) \to \pi_0(Y)$ and isomorphisms $\pi_n(X, x) \to \pi_n(Y, f(x))$, for every choice of base point $x \in X$ and every positive integer n (Theorem 3.2.7.1).

A general simplicial set X need not be a Kan complex. However, one can always find a weak homotopy equivalence $f : X \to Y$, where Y is a Kan complex; in this case, we refer to Y as a *fibrant replacement* for X (in the case where X is an ∞-category, one can think of Y

as another ∞-category obtained from X by formally adjoining inverses of all morphisms: see Proposition 6.3.1.20). The existence of fibrant replacements has an easy formal proof (a special case of Quillen's *small object argument*; see §3.1.7), which gives very little information about the structure of the Kan complex Y. In §3.3, we outline another approach (due to Kan) which associates to each simplicial set X a Kan complex $\mathrm{Ex}^{\infty}(X) = \varinjlim_{n \geq 0} \mathrm{Ex}^n(X)$ which is defined using combinatorics of iterated subdivision (Construction 3.3.6.1). The functor $X \mapsto \mathrm{Ex}^{\infty}(X)$ has many useful properties: for example, it preserves Kan fibrations (Proposition 3.3.6.6) and commutes with finite limits (Proposition 3.3.6.4). As an application, we show that a Kan fibration of simplicial sets $f : X \to Y$ is a weak homotopy equivalence if and only if it is a trivial Kan fibration (Proposition 3.3.7.6), and that a monomorphism of simplicial sets $i : A \hookrightarrow B$ is a weak homotopy equivalence if and only if it is anodyne (Corollary 3.3.7.7).

Let Set_{Δ} denote the category of simplicial sets, and let $\mathrm{Kan} \subset \mathrm{Set}_{\Delta}$ denote the full subcategory spanned by the Kan complexes. We let hKan denote the *homotopy category of Kan complexes* (Construction 3.1.5.10), which can be obtained from Kan by identifying morphisms which are homotopic. Beware that the category hKan is somewhat ill-behaved: for example, it admits neither pullbacks or pushouts. In §3.4, we address this point by introducing the notions of *homotopy pullback* and *homotopy pushout* diagrams of simplicial sets (which can be regarded as homotopy-theoretic counterparts for the classical categorical notion of pullback and pushout diagrams), and establishing their basic properties. We will later see that these diagrams can be interpreted as pullback and pushout squares in the ∞-category $\mathcal{S}$ (see Examples 7.6.4.2 and 7.6.4.3), rather than its homotopy category $\mathrm{hKan} \simeq \mathrm{h}\mathcal{S}$.

Recall that, for every topological space Y, the singular simplicial set $\mathrm{Sing}_{\bullet}(Y)$ is a Kan complex (Proposition 1.2.5.8). In §3.6, we show that every Kan complex arises in this way, at least up to homotopy equivalence. More precisely, we show that the unit map $u_X : X \to \mathrm{Sing}_{\bullet}(|X|)$ is a homotopy equivalence for any Kan complex X (and a weak homotopy equivalence for any simplicial set X; see Theorem 3.6.4.1). Using this fact, we show that the geometric realization functor $X \mapsto |X|$ induces a fully faithful embedding of homotopy categories $\mathrm{hKan} \hookrightarrow \mathrm{hTop}$, whose essential image consists of those topological spaces having the homotopy type of a CW complex (Theorem 3.6.0.1). In other words, the (combinatorially defined) homotopy theory of Kan complexes studied in this section is essentially equivalent to the (topologically defined) homotopy theory of CW complexes.

3.1 The Homotopy Theory of Kan Complexes

Let X and Y be simplicial sets, and suppose we are given a pair of maps $f_0, f_1 : X \to Y$. 00SZ A *homotopy* from f_0 to f_1 is a morphism of simplicial sets $h : \Delta^1 \times X \to Y$ satisfying

$f_0 = h|_{\{0\}\times X}$ and $f_1 = h|_{\{1\}\times X}$ (Definition 3.1.5.2). Beware that, for general simplicial sets, this terminology can be misleading: for example, the existence of a homotopy from f_0 to f_1 need not imply the existence of a homotopy from f_1 to f_0. However, the situation is better in the case if we assume that $Y_\bullet$ is a Kan complex. In general, we can identify morphisms from X to Y as vertices of the simplicial set $\mathrm{Fun}(X,Y)$ of Construction 1.5.3.1, and homotopies with edges of the simplicial set $\mathrm{Fun}(X,Y)$. In §3.1.3, we will show that when Y is a Kan complex, then $\mathrm{Fun}(X,Y)$ is also a Kan complex (Corollary 3.1.3.4).

Our approach to Corollary 3.1.3.4 is somewhat indirect. We begin in §3.1.1 by introducing the notion of a *Kan fibration* between simplicial sets. Roughly speaking, a Kan fibration $f : X \to S$ can be viewed as a family of Kan complexes parametrized by S: in particular, if f is a Kan fibration, then each fiber $X_s = \{s\} \times_S X$ is a Kan complex (Remark 3.1.1.9). In §3.1.3, we will deduce Corollary 3.1.3.4 as a consequence of a more general stability result for Kan fibrations under exponentiation (Theorem 3.1.3.1). Our proof of this result will make use of the Gabriel-Zisman calculus of *anodyne morphisms*, which we review in §3.1.2.

We say that a morphism of Kan complexes $f : X \to Y$ is a *homotopy equivalence* if its image in the homotopy category hKan is an isomorphism: that is, if f admits a homotopy inverse $g : Y \to X$. This definition makes sense for more general simplicial sets (Definition 3.1.6.1), but is of somewhat limited utility. When working with simplicial sets which are not Kan complexes, it is often better to consider the more liberal notion of *weak homotopy equivalence* (Definition 3.1.6.12), which we introduce and study in §3.1.6. In §3.1.7, we show that every simplicial set $X_\bullet$ admits an anodyne morphism $f : X_\bullet \to Q_\bullet$ where $Q_\bullet$ is a Kan complex (Corollary 3.1.7.2), using a simple incarnation of Quillen's "small object argument."

3.1.1 Kan Fibrations

00T0 Recall that a simplicial set X is said to be a *Kan complex* if it has the extension property with respect to every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $n > 0$ (Definition 1.2.5.1). For many purposes, it is useful to consider a relative version of this notion, which applies to a morphism between simplicial sets.

00T1 **Definition 3.1.1.1.** Let $f : X \to S$ be a morphism of simplicial sets. We say that f is a *Kan fibration* if, for each $n > 0$ and each $0 \leq i \leq n$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & S
\end{tikzcd}$$

admits a solution (as indicated by the dotted arrow). That is, for every map of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ and every n-simplex $\overline{\sigma} : \Delta^n \to S$ extending $f \circ \sigma_0$, we can extend σ_0 to an

n-simplex $\sigma : \Delta^n \to X$ satisfying $f \circ \sigma = \overline{\sigma}$.

Example 3.1.1.2. Let X be a simplicial set. Then the projection map $X \to \Delta^0$ is a Kan fibration if and only if X is a Kan complex. `00T2`

Example 3.1.1.3. Any isomorphism of simplicial sets is a Kan fibration. `00T3`

Example 3.1.1.4. Let S be a simplicial set and let $S' \subseteq S$ be a simplicial subset. Then the inclusion map $S' \hookrightarrow S$ is a Kan fibration if and only if S' is a summand of S (see Definition 1.2.1.1). `01GE`

Remark 3.1.1.5. The collection of Kan fibrations is closed under retracts. That is, given a diagram of simplicial sets `00T4`

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

where both horizontal compositions are the identity, if f' is a Kan fibration, then so is f.

Remark 3.1.1.6. The collection of Kan fibrations is closed under pullback. That is, given a pullback diagram of simplicial sets `00T5`

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r] & S
\end{tikzcd}$$

where f is a Kan fibration, f' is also a Kan fibration.

Remark 3.1.1.7. Let $f : X \to S$ be a map of simplicial sets. Suppose that, for every simplex $\sigma : \Delta^n \to S$, the projection map $\Delta^n \times_S X \to \Delta^n$ is a Kan fibration. Then f is a Kan fibration. Consequently, if we are given a pullback diagram of simplicial sets `00ZW`

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

where g is surjective and f' is a Kan fibration, then f is also a Kan fibration.

00X6 **Remark 3.1.1.8.** The collection of Kan fibrations is closed under filtered colimits. That is, if $\{f_\alpha : X_\alpha \to S_\alpha\}$ is any filtered diagram in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$ having colimit $f : X \to S$, and each f_α is a Kan fibration of simplicial sets, then f is also a Kan fibration of simplicial sets.

00T6 **Remark 3.1.1.9.** Let $f : X \to S$ be a Kan fibration of simplicial sets. Then, for every vertex $s \in S$, the fiber $\{s\} \times_S X$ is a Kan complex (this follows from Remark 3.1.1.6 and Example 3.1.1.2).

00T7 **Remark 3.1.1.10.** Let $f : X \to Y$ and $g : Y \to Z$ be Kan fibrations. Then the composite map $(g \circ f) : X \to Z$ is a Kan fibration.

01C5 **Remark 3.1.1.11.** Let $f : X \to Y$ be a Kan fibration of simplicial sets. If Y is a Kan complex, then X is also a Kan complex (this follows by applying Remark 3.1.1.10 in the case $Z = \Delta^0$, by virtue of Example 3.1.1.2).

3.1.2 Anodyne Morphisms

00UG By definition, a morphism of simplicial sets $f : X \to S$ is a Kan fibration if it is weakly right orthogonal to every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \leq i \leq n$ and $n > 0$ (Definition 1.5.4.3). If this condition is satisfied, then f is weakly right orthogonal to a much larger collection of morphisms.

00UH **Definition 3.1.2.1** (Anodyne Morphisms)**.** Let T be the smallest collection of morphisms in the category Set_Δ with the following properties:

- For each $n > 0$ and each $0 \leq i \leq n$, the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ belongs to T.
- The collection T is weakly saturated (Definition 1.5.4.12). That is, T is closed under pushouts, retracts, and transfinite composition.

We say that a morphism of simplicial sets $i : A \to B$ is *anodyne* if it belongs to the collection T.

00UJ **Remark 3.1.2.2.** The class of anodyne morphisms was introduced by Gabriel-Zisman in [23].

00UK **Remark 3.1.2.3.** Every anodyne morphism of simplicial sets $i : A \to B$ is a monomorphism. This follows from the observation that the collection of monomorphisms is weakly saturated (Proposition 1.5.5.14) and that every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is a monomorphism.

00UL **Example 3.1.2.4.** Let $i : A \hookrightarrow B$ be an inner anodyne morphism of simplicial sets (Definition 1.5.6.4). Then i is anodyne. The converse is false in general. For example, the horn inclusions $\Lambda^n_0 \hookrightarrow \Delta^n$ and $\Lambda^n_n \hookrightarrow \Delta^n$ are anodyne (for $n > 0$), but are not inner anodyne.

Example 3.1.2.5. For $0 \leq i \leq n$, the inclusion map $\{i\} \hookrightarrow \Delta^n$ is anodyne. To prove this, let $\operatorname{Spine}[n]$ denote the spine of the n-simplex, so that the inclusion $\operatorname{Spine}[n] \hookrightarrow \Delta^n$ is inner anodyne (Example 1.5.7.7) and therefore anodyne (Example 3.1.2.4). It will therefore suffice to show that the inclusion $\{i\} \hookrightarrow \operatorname{Spine}[n]$ is anodyne, which is clear (it can be written as a composition of pushouts of the inclusions $\{0\} \hookrightarrow \Delta^1$ and $\{1\} \hookrightarrow \Delta^1$). 0218

Remark 3.1.2.6. By construction, the collection of anodyne morphisms is weakly saturated. In particular: 00UM

- Every isomorphism of simplicial sets is anodyne.
- If $i : A \to B$ and $j : B \to C$ are anodyne morphisms of simplicial sets, then the composition $j \circ i$ is anodyne.
- For every pushout diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\
B \arrow[r] & B',
\end{tikzcd}$$

  if i is anodyne, then i' is also anodyne.

- Suppose there exists a commutative diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "i'"] & A \arrow[d, "i"] \\
B \arrow[r] & B' \arrow[r] & B,
\end{tikzcd}$$

  where the horizontal compositions are the identity. If i' is anodyne, then i is anodyne.

Remark 3.1.2.7. Let $f : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent: 00UN

(a) The morphism f is a Kan fibration (Definition 3.1.1.1).

(b) For every square diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where i is anodyne, there exists a dotted arrow rendering the diagram commutative.

The implication $(b) \Rightarrow (a)$ is immediate from the definitions (since the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ are anodyne for $n > 0$). The reverse implication follows from the weak saturation of the collection of morphisms which are weakly left orthogonal to f (Proposition 1.5.4.13).

050K **Remark 3.1.2.8.** Let $f : X \to S$ be a Kan fibration of simplicial sets. If f is surjective on vertices, then it is surjective on n-simplices for every integer $n \geq 0$. This follows from the lifting property of Remark 3.1.2.7, combined with the observation that the inclusion map $\{0\} \hookrightarrow \Delta^n$ is anodyne (Example 3.1.2.5).

We will need the following stability properties for the class of anodyne morphisms:

014D **Proposition 3.1.2.9.** *Let $f : A \hookrightarrow B$ and $f' : A' \hookrightarrow B'$ be monomorphisms of simplicial sets. If either f or f' is anodyne, then the induced map*

$$(A \times B') \coprod_{A \times A'} (B \times A') \hookrightarrow B \times B'$$

is anodyne.

The proof of Proposition 3.1.2.9 will require some preliminaries.

00TG **Lemma 3.1.2.10.** *For every pair of integers $0 < i \leq n$, the horn inclusion $f_0 : \Lambda^n_i \hookrightarrow \Delta^n$ is a retract of the inclusion map $f : (\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Lambda^n_i} (\{1\} \times \Delta^n) \hookrightarrow \Delta^1 \times \Delta^n$.*

Proof. Let A denote the simplicial subset of $\Delta^1 \times \Delta^n$ given by the union of $\Delta^1 \times \Lambda^n_i$ with $\{1\} \times \Delta^n$. To prove Lemma 3.1.2.10, it will suffice to show that there exists a commutative diagram of simplicial sets

$$\begin{tikzcd}
\{0\} \times \Lambda^n_i \arrow[r] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & \Lambda^n_i \arrow[d, "f_0"] \\
\{0\} \times \Delta^n \arrow[r] & \Delta^1 \times \Delta^n \arrow[r, "r"] & \Delta^n
\end{tikzcd}$$

where the left horizontal maps are given by inclusion and the horizontal compositions are the identity maps. To achieve this, it suffices to choose r to be given on vertices by the map of partially ordered sets

$$r : [1] \times [n] \to [n] \qquad r(j, k) = \begin{cases} i & \text{if } j = 1 \text{ and } k \leq i \\ k & \text{otherwise.} \end{cases}$$

□

Lemma 3.1.2.11. *Let X be a simplicial set which is the union of a simplicial subset $Y \subseteq X$ with the image of an n-simplex $\sigma : \Delta^n \to X$, where $n > 0$. Suppose that the inverse image $\sigma^{-1}(Y) \subseteq \Delta^n$ is equal to the horn Λ^n_i for some $0 \leq i \leq n$. Then pullback diagram of simplicial sets* 03X4

03X5

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \Delta^n \arrow[d, "\sigma"] \\ Y \arrow[r] & X \end{tikzcd} \tag{3.1}$$

is also a pushout square.

Proof. Fix an integer $m \geq 0$. We wish to show that σ induces a bijection from the set of m-simplices of Δ^n which are not contained in Λ^n_i to the set of m-simplices of X which are not contained in Y. Surjectivity follows from our assumption that X is the union of Y with the image of σ. To prove injectivity, we proceed by induction on m. Let $\alpha, \beta : \Delta^m \to \Delta^n$ be morphisms which do not factor through Λ^n_i, and suppose that $\sigma \circ \alpha = \tau = \sigma \circ \beta$ for some simplex $\tau : \Delta^m \to X$; we wish to show that $\alpha = \beta$.

Suppose first that the simplex α is degenerate: that is, we have $\alpha(j) = \alpha(k)$ for some $0 \leq j < k \leq m$. Then $d^m_j(\alpha)$ is an $(m-1)$-simplex of Δ^n which is not contained Λ^n_i. It follows that $\sigma \circ d^m_j(\alpha) = d^m_j(\tau) = \sigma \circ d^m_j(\beta)$ is an $(m-1)$-simplex of X which is not contained in Y, so that $d^m_j(\beta)$ is not contained in Λ^n_i. Applying our inductive hypothesis, we deduce that $d^m_j(\alpha) = d^m_j(\beta)$. The same argument shows that $d^m_k(\alpha) = d^m_k(\beta)$, so that $\alpha = \beta$.

We may therefore assume without loss of generality that the simplex α is nondegenerate. By a similar argument, we may assume that β is nondegenerate. The equality $\alpha = \beta$ now follows from the observation that Δ^n contains at most one nondegenerate m-simplex which is not contained in Λ^n_i. □

Lemma 3.1.2.12. *Let n be a nonnegative integer. Then there exists a chain of simplicial subsets* 00TH

$$X(0) \subset X(1) \subset \cdots \subset X(n) \subset X(n+1) = \Delta^1 \times \Delta^n$$

with the following properties:

(a) *The simplicial $X(0)$ is given by the union of $\Delta^1 \times \partial\Delta^n$ with $\{1\} \times \Delta^n$ (and can therefore be described abstractly as the pushout $(\Delta^1 \times \partial\Delta^n) \coprod_{\{1\} \times \partial\Delta^n} (\{1\} \times \Delta^n)$).*

(*b*) *For* $0 \leq i \leq n$*, the inclusion map* $X(i) \hookrightarrow X(i+1)$ *fits into a pushout diagram*

$$\begin{tikzcd}
\Lambda^{n+1}_{i+1} \arrow[r] \arrow[d] & X(i) \arrow[d] \\
\Delta^{n+1} \arrow[r] & X(i+1).
\end{tikzcd}$$

Proof. For $0 \leq i \leq n$, let $\sigma_i : \Delta^{n+1} \to \Delta^1 \times \Delta^n$ denote the map of simplicial sets given on vertices by the formula $\sigma_i(j) = \begin{cases} (0,j) & \text{if } j \leq i \\ (1,j-1) & \text{if } j > i. \end{cases}$ We define simplicial subsets $X(i) \subseteq \Delta^1 \times \Delta^n$ inductively by the formulae

$$X(0) = (\Delta^1 \times \partial \Delta^n) \cup (\{1\} \times \Delta^n) \qquad X(i+1) = X(i) \cup \operatorname{im}(\sigma_i),$$

where $\operatorname{im}(\sigma_i)$ denotes the image of the morphism σ_i. Note that $\Delta^1 \times \Delta^n$ is the union of the simplicial subsets $\{\operatorname{im}(\sigma_i)\}_{0 \leq i \leq n}$, and is therefore equal to $X(n+1)$. This definition satisfies condition (a) by construction. To verify (b), it will suffice to show that for $0 \leq i \leq n$, the inverse image $A = \sigma_i^{-1} X(i)$ is equal to Λ^{n+1}_{i+1} (Lemma 3.1.2.11). Regarding σ_i as an $(n+1)$-simplex of $\Delta^1 \times \Delta^n$, we are reduced to showing that the faces $d^{n+1}_j(\sigma_i)$ belong to $X(i)$ if and only if $j \neq i+1$. One direction is clear: the face $d^{n+1}_j(\sigma_i)$ is contained in $\Delta^1 \times \partial \Delta^n$ for $j \notin \{i, i+1\}$, the face $d^{n+1}_i(\sigma_i) = d^{n+1}_i(\sigma_{i-1})$ is contained in $\operatorname{im}(\sigma_{i-1}) \subseteq X(i)$ for $i > 0$, and $d^{n+1}_0(\sigma_0)$ is contained in $\{1\} \times \Delta^n$. To complete the proof, it suffices to show that the face $d^{n+1}_{i+1}(\sigma_i)$ is *not* contained in $X(i)$, which follows by inspection. □

Proof of Proposition 3.1.2.9. Let us first regard the monomorphism $f' : A' \hookrightarrow B'$ as fixed, and let T be the collection of all maps $f : A \to B$ for which the induced map

$$(A \times B') \coprod_{A \times A'} (B \times A') \hookrightarrow B \times B'$$

is anodyne. We wish to show that every anodyne morphism belongs to T. Since T is weakly saturated, it will suffice to show that every horn inclusion $f : \Lambda^n_i \hookrightarrow \Delta^n$ belongs to T (for $n > 0$). Without loss of generality, we may assume that $0 < i$, so that f is a retract of the map $g : (\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Lambda^n_i} (\{1\} \times \Delta^n) \hookrightarrow \Delta^1 \times \Delta^n$ (Lemma 3.1.2.10). It will therefore suffice to show that g belongs to T. Replacing f' by the monomorphism

$$(\Lambda^n_i \times B') \coprod_{\Lambda^n_i \times A'} (\Delta^n \times B') \hookrightarrow \Delta^n \times A',$$

we are reduced to showing that the inclusion $\{1\} \hookrightarrow \Delta^1$ belongs to T.

Let T' denote the collection of all morphisms of simplicial sets $f'' : A'' \to B''$ for which the map $(\{1\} \times B'') \coprod_{\{1\} \times A''} (\Delta^1 \times A'') \to \Delta^1 \times B''$ is anodyne. We will complete the proof by showing that T' contains all monomorphisms of simplicial sets. By virtue of Proposition 1.5.5.14, it will suffice to show that T'' contains the inclusion map $\partial \Delta^m \hookrightarrow \Delta^m$, for each $m > 0$. In other words, we are reduced to showing that the inclusion $(\{1\} \times \Delta^m) \coprod_{\{1\} \times \partial \Delta^m} (\Delta^1 \times \partial \Delta^m) \hookrightarrow \Delta^1 \times \Delta^m$ is anodyne, which follows from Lemma 3.1.2.12. □

3.1.3 Exponentiation for Kan Fibrations

Let B and X be simplicial sets. In §1.5.3, we showed that if X is an ∞-category, then the simplicial set $\operatorname{Fun}(B, X)$ is an ∞-category (Theorem 1.5.3.7). If X is a Kan complex, we can say more: the simplicial set $\operatorname{Fun}(B, X)$ is also a Kan complex (Corollary 3.1.3.4). This is a consequence of the following stronger result: `00TJ`

Theorem 3.1.3.1. *Let $f : X \to S$ be a Kan fibration of simplicial sets, and let $i : A \hookrightarrow B$ be any monomorphism of simplicial sets. Then the induced map* `00TK`

$$\operatorname{Fun}(B, X) \to \operatorname{Fun}(B, S) \times_{\operatorname{Fun}(A,S)} \operatorname{Fun}(A, X)$$

is a Kan fibration.

Proof. By virtue of Remark 3.1.2.7, it will suffice to show that if $i' : A' \hookrightarrow B'$ is an anodyne morphism of simplicial sets, then every lifting problem of the form

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"] & \operatorname{Fun}(B, X) \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(B, S) \times_{\operatorname{Fun}(A,S)} \operatorname{Fun}(A, X)
\end{tikzcd}$$

admits a solution. Equivalently, we must show that every lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{A \times A'} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution. This follows from Remark 3.1.2.7, since the left vertical map is anodyne (Proposition 3.1.2.9) and the right vertical map is a Kan fibration. □

Let us note some special cases of Theorem 3.1.3.1 (which can be obtained by taking the simplicial set A to be empty, the simplicial set S to be Δ^0, or both).

00TL **Corollary 3.1.3.2.** *Let $f : X \to S$ be a Kan fibration of simplicial sets. Then, for every simplicial set B, composition with f induces a Kan fibration $\operatorname{Fun}(B, X) \to \operatorname{Fun}(B, S)$.*

00TM **Corollary 3.1.3.3.** *Let X be a Kan complex. Then, for every monomorphism of simplicial sets $i : A \hookrightarrow B$, the restriction map $\operatorname{Fun}(B, X) \to \operatorname{Fun}(A, X)$ is a Kan fibration.*

00TN **Corollary 3.1.3.4.** *Let X be a Kan complex and let B be an arbitrary simplicial set. Then the simplicial set $\operatorname{Fun}(B, X)$ is a Kan complex.*

Theorem 3.1.3.1 has an analogue for trivial Kan fibrations:

014E **Theorem 3.1.3.5.** *Let $i : A \hookrightarrow B$ be an anodyne morphism of simplicial sets and let $f : X \to S$ be a Kan fibration. Then the induced map*

$$\operatorname{Fun}(B, X) \to \operatorname{Fun}(B, S) \times_{\operatorname{Fun}(A,S)} \operatorname{Fun}(A, X)$$

is a trivial Kan fibration.

Proof. We proceed as in the proof of Theorem 3.1.3.1. Let $i' : A' \hookrightarrow B'$ be a monomorphism of simplicial sets; we must show that every lifting problem

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"] & \operatorname{Fun}(B, X) \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(B, S) \times_{\operatorname{Fun}(A,S)} \operatorname{Fun}(A, X)
\end{tikzcd}$$

admits a solution. Equivalently, we must show that every lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{A \times A'} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution. This follows from Remark 3.1.2.7, since the left vertical map is anodyne (Proposition 3.1.2.9) and the right vertical map is a Kan fibration. □

Taking $S = \Delta^0$ in the statement of Theorem 3.1.3.5, we obtain the following:

00UT **Corollary 3.1.3.6.** *Let $i : A \hookrightarrow B$ be an anodyne morphism of simplicial sets and let X be a Kan complex. Then the restriction map $\operatorname{Fun}(B, X) \to \operatorname{Fun}(A, X)$ is a trivial Kan fibration.*

To formulate some further consequences of Theorem 3.1.3.1, it will be convenient to introduce some notation.

01AB **Construction 3.1.3.7.** Let B and X be simplicial sets, and let $\operatorname{Fun}(B, X)$ be the simplicial set parametrizing morphisms from B to X (Construction 1.5.3.1).

- Suppose we are given another simplicial set A equipped with a pair of morphisms $i : A \to B$ and $f : A \to X$. In this case, we let $\operatorname{Fun}_{A/}(B, X) \subseteq \operatorname{Fun}(B, X)$ denote the fiber of the precomposition morphism $\operatorname{Fun}(B, X) \xrightarrow{\circ i} \operatorname{Fun}(A, X)$ over the vertex $f \in \operatorname{Fun}(A, X)$.

- Suppose we are given another simplicial set S equipped with a pair of morphism $g : B \to S$ and $q : X \to S$. We let $\operatorname{Fun}_{/S}(B, X) \subseteq \operatorname{Fun}(B, X)$ denote the fiber of the postcomposition morphism $\operatorname{Fun}(B, X) \xrightarrow{q\circ} \operatorname{Fun}(B, S)$ over the vertex $g \in \operatorname{Fun}(B, S)$.

- Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "q"] \\
B \arrow[r, "g"] & S.
\end{tikzcd}$$

  In this case, we let $\operatorname{Fun}_{A/\,/S}(B, X) \subseteq \operatorname{Fun}(B, X)$ denote the simplicial subset given by the intersection $\operatorname{Fun}_{A/}(B, X) \cap \operatorname{Fun}_{/S}(B, X)$.

Remark 3.1.3.8. Let B and X be simplicial sets, and let us identify vertices of $\operatorname{Fun}(B, X)$ with morphisms $\overline{f} : B \to X$ in the category of simplicial sets. Then: 01GF

- Suppose we are given another simplicial set A equipped with a pair of morphisms $i : A \to B$ and $f : A \to X$. Then vertices of the simplicial set $\operatorname{Fun}_{A/}(B, X)$ can be identified with morphisms $\overline{f} : B \to X$ satisfying $f = \overline{f} \circ i$.

- Suppose we are given another simplicial set S equipped with a pair of morphisms $g : B \to S$ and $q : X \to S$. Then vertices of the simplicial set $\operatorname{Fun}_{/S}(B, X)$ can be identified with morphisms $\overline{f} : B \to X$ satisfying $g = q \circ \overline{f}$.

- Suppose we are given a square diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "q"] \\
B \arrow[r, "g"] \arrow[ur, dashed, "\overline{f}"] & S.
\end{tikzcd}$$

  Then vertices of the simplicial set $\operatorname{Fun}_{A/\,/S}(B, X)$ can be identified with solutions of the associated lifting problem: that is, morphisms of simplicial sets $\overline{f} : B \to X$ satisfying $f = \overline{f} \circ i$ and $g = q \circ \overline{f}$.

01GG **Remark 3.1.3.9.** Suppose we are given a diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "q"] \\
B \arrow[r, "g"] \arrow[ur, dashed, "\overline{f}"] & S
\end{tikzcd}$$

which does not commute. Then the simplicial set $\mathrm{Fun}_{A/\,/S}(B,X) = \mathrm{Fun}_{A/}(B,X) \cap \mathrm{Fun}_{/S}(B,X)$ of Construction 3.1.3.7 can still be defined, but is automatically empty.

01GH **Remark 3.1.3.10.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "q"] \\
B \arrow[r, "g"] \arrow[ur, dashed, "\overline{f}"] & S.
\end{tikzcd}$$

Then:

- If $S \simeq \Delta^0$ is a final object of the category of simplicial sets, then we have an equality $\mathrm{Fun}_{A/\,/S}(B,X) = \mathrm{Fun}_{A/}(B,X)$ (as simplicial subsets of $\mathrm{Fun}(B,X)$).
- If $A \simeq \emptyset$ is an initial object of the category of simplicial sets, then we have an equality $\mathrm{Fun}_{A/\,/S}(B,X) = \mathrm{Fun}_{/S}(B,X)$ (as simplicial subsets of $\mathrm{Fun}(B,X)$).
- If $S \simeq \Delta^0$ and $A \simeq \emptyset$ are final and initial objects, respectively, then we have an equality $\mathrm{Fun}_{A/\,/S}(B,X) = \mathrm{Fun}(B,X)$.

01GJ **Remark 3.1.3.11.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "q"] \\
B \arrow[r, "g"'] \arrow[ur, dashed, "\overline{f}"] & S.
\end{tikzcd}$$

Then the simplicial set $\mathrm{Fun}_{A/\,/S}(B,X)$ can be identified with the fiber of the induced map

$$\mathrm{Fun}(B,X) \to \mathrm{Fun}(A,X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B,S)$$

over the vertex given by the pair (f,g).

Example 3.1.3.12. Let $q : X \to S$ be a morphism of simplicial sets. Then, for each vertex $s \in S$, the simplicial set $\mathrm{Fun}_{/S}(\{s\}, X)$ can be identified with the fiber $X_s = \{s\} \times_S X$. 01AC

Proposition 3.1.3.13. *Suppose we are given a commutative diagram of simplicial sets* 01GK

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "q"] \\ B \arrow[r, "g"] \arrow[ur, dashed, "\overline{f}"] & S, \end{tikzcd}$$

where i is a monomorphism and q is a Kan fibration. Then the simplicial set $\mathrm{Fun}_{A/\,/S}(B, X)$ is a Kan complex. If i is anodyne, then the Kan complex $\mathrm{Fun}_{A/\,/S}(B, X)$ is contractible.

Proof. By virtue of Remark 3.1.3.11, the simplicial set $\mathrm{Fun}_{A/\,/S}(B, X)$ can be identified with a fiber of the restriction map

$$\theta : \mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B, X).$$

Theorem 3.1.3.1 guarantees that θ is a Kan fibration, so its fibers are Kan complexes by virtue of Remark 3.1.1.9. If i is anodyne, then θ is a trivial Kan fibration (Theorem 3.1.3.5), so its fibers are contractible Kan complexes (Remark 1.5.5.10). □

Corollary 3.1.3.14. *Let B be a simplicial set, let $A \subseteq B$ be a simplicial subset, and let $f : A \to X$ be a morphism of simplicial sets. If X is a Kan complex, then the simplicial set $\mathrm{Fun}_{A/}(B, X)$ is a Kan complex. If the inclusion $A \hookrightarrow B$ is anodyne, then the Kan complex $\mathrm{Fun}_{A/}(B, X)$ is contractible.* 01GL

Proof. Apply Proposition 3.1.3.13 in the special case $S = \Delta^0$. □

Corollary 3.1.3.15. *Let $q : X \to S$ be a Kan fibration of simplicial sets, and let $g : B \to S$ be any morphism of simplicial sets. Then the simplicial set $\mathrm{Fun}_{/S}(B, X)$ is a Kan complex.* 01GM

Proof. Apply Proposition 3.1.3.13 in the special case $A = \emptyset$. □

3.1.4 Covering Maps

Let X and S be topological spaces. Recall that a continuous function $f : X \to S$ is a *covering map* if every point $s \in S$ has an open neighborhood $U \subseteq S$ for which the inverse image $f^{-1}(U)$ is homeomorphic to a disjoint union of copies of U. This definition has a counterpart in the setting of simplicial sets: 0219

021A **Definition 3.1.4.1.** Let $f : X \to S$ be a morphism of simplicial sets. We say that f is a *covering map* if, for every pair of integers $0 \leq i \leq n$ with $n > 0$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

has a *unique* solution.

021B **Remark 3.1.4.2.** Let $f : X \to S$ be a morphism of simplicial sets. Then f is a covering map if and only if the opposite morphism $f^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a covering map.

021C **Remark 3.1.4.3.** Let $f : X \to S$ be a morphism of simplicial sets, and let $\delta : X \to X \times_S X$ be the relative diagonal of f. Then f is a covering map if and only if both f and δ are Kan fibrations. In particular, every covering map is a Kan fibration.

021D **Remark 3.1.4.4.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r] & S.
\end{tikzcd}$$

If f is a covering map, then f' is also a covering map.

021E **Remark 3.1.4.5.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets. Suppose that g is a covering map. Then f is a covering map if and only if $g \circ f$ is a covering map. In particular, the collection of covering maps is closed under composition.

021F **Remark 3.1.4.6.** Let $f : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:

(a) The morphism f is a covering map (Definition 3.1.4.1).

(b) For every square diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where i is anodyne, there exists a unique dotted arrow rendering the diagram commutative.

This follows by combining Remarks 3.1.2.7 and 3.1.4.3.

Proposition 3.1.4.7. *Let $f : X \to S$ be a covering map of simplicial sets, and let $i : A \hookrightarrow B$ be any monomorphism of simplicial sets. Then the induced map* `021G`

$$\mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)$$

is a covering map.

Proof. By virtue of Remark 3.1.4.6, it will suffice to show that if $i' : A' \hookrightarrow B'$ is an anodyne morphism of simplicial sets, then every lifting problem of the form

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"] & \mathrm{Fun}(B,X) \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(B,S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A,X)
\end{tikzcd}$$

admits a unique solution. Equivalently, we must show that every lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{A \times A'} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a unique solution. This follows from Remark 3.1.4.6, since the left vertical map is anodyne (Proposition 3.1.2.9) and f is a covering map. □

Corollary 3.1.4.8. *Let $f : X \to S$ be a covering map of simplicial sets. Then, for every simplicial set B, composition with f induces a covering map $\mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S)$.* `021H`

Proposition 3.1.4.9. *Let $f : X \to S$ be a covering map of topological spaces. Then the induced map $\mathrm{Sing}_\bullet(f) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(S)$ is a covering map of simplicial sets (in the sense of Definition 3.1.4.1).* `021J`

Proof. Let $\delta : X \to X \times_S X$ be the relative diagonal of f. We first claim δ exhibits X as a summand of $X \times_S X$ in the category of topological spaces (that is, it is a homeomorphism of X onto a closed and open subset of the fiber product $X \times_S X$). To verify this, we can work locally on S and thereby reduce to the case where X is a product of S with a discrete topological space, in which case the result is clear. It follows that the induced map of singular simplicial sets

$$\mathrm{Sing}_\bullet(\delta) : \mathrm{Sing}_\bullet(X) \hookrightarrow \mathrm{Sing}_\bullet(X \times_S X) \simeq \mathrm{Sing}_\bullet(X) \times_{\mathrm{Sing}_\bullet(S)} \mathrm{Sing}_\bullet(X)$$

is also the inclusion of a summand (Remark 1.2.2.4), and is therefore a Kan fibration by virtue of Example 3.1.1.4. Consequently, to show that $\mathrm{Sing}_\bullet(f)$ is a covering map, it will suffice to show that it is a Kan fibration (Remark 3.1.4.3). This is a special case of Corollary 3.6.6.11, since $f : X \to S$ exhibits X as a fiber bundle over S (with discrete fibers). □

021K **Warning 3.1.4.10.** The converse of Proposition 3.1.4.9 is false. For example, let $f : X \to S$ be a continuous function between topological spaces where $S = *$ consists of a single point. In this case, the function f is a covering map if and only if the topology on X is discrete. However, the induced map of simplicial sets $\mathrm{Sing}_\bullet(f) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(S)$ is a covering map if and only if the simplicial set $\mathrm{Sing}_\bullet(X)$ is discrete: that is, if and only if every continuous function $[0, 1] \to X$ is constant (Example 3.1.4.13). Many non-discrete topological spaces satisfy this weaker condition (for example, we could take X to be the Cantor set).

021L **Remark 3.1.4.11.** Let $f : X \to S$ be a morphism of simplicial sets. Then f is a covering map (in the sense of Definition 3.1.4.1) if and only if the induced map of geometric realizations $|X| \to |S|$ is a covering map of topological spaces (see Proposition [?]).

Covering maps of simplicial sets have a very simple local structure:

021M **Proposition 3.1.4.12.** *Let $f : X_\bullet \to S_\bullet$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism f is a covering map.*

(2) *For every map of standard simplices $u : \Delta^m \to \Delta^n$, composition with u induces a bijection $X_n \to X_m \times_{S_m} S_n$.*

(3) *For every n-simplex $\sigma : \Delta^n \to S_\bullet$, the projection map $\Delta^n \times_{S_\bullet} X_\bullet \to \Delta^n$ restricts to an isomorphism on each connected component of $\Delta^n \times_{S_\bullet} X_\bullet$.*

Proof. Assume first that (1) is satisfied; we will prove (2). Let $u : \Delta^m \to \Delta^n$ be a morphism of simplicial sets. Choose a vertex $v : \Delta^0 \to \Delta^m$. It follows from Example 3.1.2.5 that v and $u \circ v$ are anodyne morphisms of simplicial sets. Invoking Remark 3.1.4.6, we conclude that the right square and outer rectangle in the diagram

$$\begin{tikzcd}
X_n \arrow[r, "\circ u"] \arrow[d] & X_m \arrow[r, "\circ v"] \arrow[d] & X_0 \arrow[d] \\
S_n \arrow[r, "\circ u"] & S_m \arrow[r, "\circ v"] & S_0
\end{tikzcd}$$

are pullback diagrams. It follows that the left square is a pullback diagram as well.

We next show that (2) implies (3). Fix a map $\sigma : \Delta^n \to S_\bullet$, and let $T = X_n \times_{S_n} \{\sigma\}$ denote the collection of all n-simplices τ of $X_\bullet$ satisfying $f(\tau) = \sigma$. To prove (3), it will suffice to show that the tautological map

$$g : \coprod_{\tau \in T} \Delta^n \to \Delta^n \times_{S_\bullet} X_\bullet$$

is an isomorphism of simplicial sets. Equivalently, we must show that for every map of simplices $u : \Delta^m \to \Delta^n$, the induced map $T \to X_m \times_{S_m} \{\sigma \circ u\}$ is bijective, which follows immediately from (2).

We now complete the proof by showing that (3) implies (1). Assume that (3) is satisfied. We wish to show that, for every pair of integers $0 \leq i \leq n$ with $n \geq 1$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X_\bullet \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S_\bullet
\end{tikzcd}$$

admits a unique solution. To prove this, we are free to replace f by the projection map $\Delta^n \times_{S_\bullet} X_\bullet \to \Delta^n$, and thereby reduce to the case where $S_\bullet$ is a standard simplex. In this case, assumption (3) guarantees that each connected component of $X_\bullet$ is isomorphic to $S_\bullet$. The desired result now follows from the observation that the simplicial sets Λ^n_i and Δ^n are connected. □

Example 3.1.4.13. Let X be a simplicial set. Then the unique morphism $f : X \to \Delta^0$ is a covering map of simplicial sets if and only if X is discrete (see Definition 1.1.5.10). 021N

Corollary 3.1.4.14. *Let $f : X \to S$ be a monomorphism of simplicial sets. The following conditions are equivalent:* 021P

(1) *The morphism f exhibits X as a summand of S (Definition 1.2.1.1).*

(2) *The morphism f is a covering map.*

(3) *The morphism f is a Kan fibration.*

Proof. The implication (1) ⇒ (2) and (2) ⇒ (3) are immediate. Moreover, if f is a monomorphism, then the relative diagonal $\delta : X \to X \times_S X$ is an isomorphism, so the implication (3) ⇒ (2) follows from Remark 3.1.4.3. We will complete the proof by showing that (2) ⇒ (1). Let $u : \Delta^m \to \Delta^n$ be a morphism of standard simplices and let $\sigma : \Delta^n \to S$ be a simplex of S; we wish to show that σ factors through f if and only if $\sigma \circ u$ factors through f. This follows immediately from the criterion of Proposition 3.1.4.12. □

3.1.5 The Homotopy Category of Kan Complexes

00TR The category of simplicial sets is equipped with a good notion of homotopy.

00TS **Definition 3.1.5.1.** Let X and Y be simplicial sets, and suppose we are given a pair of maps $f, g : X \to Y$, which we identify with vertices of the simplicial set $\mathrm{Fun}(X, Y)$. We will say that f and g are *homotopic* if they belong to the same connected component of the simplicial set $\mathrm{Fun}(X, Y)$ (Definition 1.2.1.8).

Let us now make Definition 3.1.5.1 more concrete.

00TT **Definition 3.1.5.2.** Let X and Y be simplicial sets, and suppose we are given a pair of morphisms $f_0, f_1 : X \to Y$. A *homotopy* from f_0 to f_1 is a morphism $h : \Delta^1 \times X \to Y$ satisfying $f_0 = h|_{\{0\}\times X}$ and $f_1 = h|_{\{1\}\times X}$.

014F **Remark 3.1.5.3** (Homotopy Extension Lifting Property)**.** Let $f : X \to S$ be a Kan fibration of simplicial sets. Suppose we are given a morphism of simplicial sets $u : B \to X$ and a homotopy $\overline{h}$ from $f \circ u$ to another map $\overline{v} : B \to S$. Then we can choose a map of simplicial sets $h : \Delta^1 \times B \to X$ satisfying $f \circ h = \overline{h}$ and $h|_{\{0\}\times B} = u$: in other words, $\overline{h}$ can be lifted to a homotopy h from u to another map $v = h|_{\{1\}\times B}$. Moreover, given any simplicial subset $A \subseteq B$ and any map $h_0 : \Delta^1 \times A \to X$ satisfying $f \circ h_0 = \overline{h}|_{\Delta^1\times A}$ and $h_0|_{\{0\}\times A} = u|_A$, we can arrange that h is an extension of h_0. This follows from Theorem 3.1.3.1, which guarantees that the restriction map

$$\mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)$$

is a Kan fibration (and therefore weakly right orthogonal to the inclusion $\{0\} \hookrightarrow \Delta^1$). For a partial converse, see Corollary 4.2.6.2.

00TU **Proposition 3.1.5.4.** *Let X and Y be simplicial sets, and suppose we are given a pair of morphisms $f, g : X \to Y$. Then:*

- *The morphisms f and g are homotopic if and only if there exists a sequence of morphisms $f = f_0, f_1, \ldots, f_n = g$ from X to Y having the property that, for each $1 \leq i \leq n$, either there exists a homotopy from f_{i-1} to f_i or a homotopy from f_i to f_{i-1}.*

- *Suppose that Y is a Kan complex. Then f and g are homotopic if and only if there exists a homotopy from f to g.*

Proof. The first assertion follows by applying Remark 1.2.1.23 to the simplicial set $\mathrm{Fun}(X, Y)$. If Y is a Kan complex, then $\mathrm{Fun}(X, Y)$ is also a Kan complex (Corollary 3.1.3.4), so the second assertion follows from Proposition 1.2.5.10. □

Example 3.1.5.5. Let X be a simplicial set and let Y be a topological space. Suppose we are given a pair of continuous functions $f_0, f_1 : |X| \to Y$, corresponding to morphisms of simplicial sets $f'_0, f'_1 : X \to \operatorname{Sing}_\bullet(Y)$. Let $h : [0,1] \times |X| \to Y$ be a continuous function satisfying $f_0 = h|_{\{0\}\times|X|}$ and $f_1 = h|_{\{1\}\times|X|}$ (that is, a homotopy from f_0 to f_1 in the category of topological spaces). Then the composite map 00TV

$$|\Delta^1 \times X| \xrightarrow{\theta} |\Delta^1| \times |X| = [0,1] \times |X| \xrightarrow{h} Y$$

classifies a morphism of simplicial sets $h' : \Delta^1 \times X \to \operatorname{Sing}_\bullet(Y)$, which is a homotopy from f'_0 to f'_1 (in the sense of Definition 3.1.5.2). We will show later that θ is a homeomorphism of topological spaces (Corollary 3.6.2.2), so every homotopy from f_0 to f_1 arises in this way. In other words, the construction $h \mapsto h'$ induces a bijection

$$\{\text{(Continuous) homotopies from } f_0 \text{ to } f_1\} \simeq \{\text{(Simplicial) homotopies from } f'_0 \text{ to } f'_1\}.$$

Example 3.1.5.6. Let X and Y be topological spaces, and let $h : [0,1] \times X \to Y$ be a continuous function, which we regard as a homotopy from $f_0 = h|_{\{0\}\times X}$ to $f_1 = h|_{\{1\}\times X}$. Then h determines a homotopy between the induced map of simplicial sets $\operatorname{Sing}_\bullet(f_0), \operatorname{Sing}_\bullet(f_1) : \operatorname{Sing}_\bullet(X) \to \operatorname{Sing}_\bullet(Y)$: this follows by applying Example 3.1.5.5 to the composite map $[0,1] \times |\operatorname{Sing}_\bullet(X)| \to [0,1] \times X \xrightarrow{h} Y$. 00TW

Example 3.1.5.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and suppose we are given a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$, which we identify with morphisms of simplicial sets $\mathrm{N}_\bullet(F), \mathrm{N}_\bullet(G) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})$. By definition, a homotopy from $\mathrm{N}_\bullet(F)$ to $\mathrm{N}_\bullet(G)$ is a map of simplicial sets 00ZX

$$h : \Delta^1 \times \mathrm{N}_\bullet(\mathcal{C}) \simeq \mathrm{N}_\bullet([1] \times \mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})$$

satisfying $h|_{\{0\}\times\mathrm{N}_\bullet(\mathcal{C})} = \mathrm{N}_\bullet(F)$ and $h|_{\{1\}\times\mathrm{N}_\bullet(\mathcal{C})} = \mathrm{N}_\bullet(G)$. By virtue of Proposition 1.3.3.1, this is equivalent to the datum of a functor $H : [1] \times \mathcal{C} \to \mathcal{D}$ satisfying $H|_{\{0\}\times\mathcal{C}} = F$ and $H|_{\{1\}\times\mathcal{C}} = G$. In other words, we have a canonical bijection

$$\begin{tikzcd}
\{\text{Natural transformations from } F \text{ to } G\} . \arrow[d, "\sim"] \\
\{\text{Homotopies from } \mathrm{N}_\bullet(F) \text{ to } \mathrm{N}_\bullet(G)\}
\end{tikzcd}$$

In particular, if there exists a natural transformation from F to G, then $\mathrm{N}_\bullet(F)$ and $\mathrm{N}_\bullet(G)$ are homotopic.

Example 3.1.5.8. Let X be a simplicial set, let M_* be a chain complex of abelian groups, and let $\mathrm{K}(M_*)$ denote the associated Eilenberg-MacLane space (Construction 2.5.6.3). 00ZY

Suppose we are given a pair of morphisms $f, g : X \to \mathrm{K}(M_*)$ in the category of simplicial sets, which we can identify with morphisms $f', g' : \mathrm{N}_*(X; \mathbf{Z}) \to M_*$ in the category of chain complexes (Corollary 2.5.6.12); here $\mathrm{N}_*(X; \mathbf{Z})$ denotes the normalized Moore complex of X (Construction 2.5.5.9). The following conditions are equivalent:

(1) The morphisms f and g are homotopic, in the sense of Definition 3.1.5.1.

(2) The chain maps f' and g' are chain homotopic, in the sense of Definition 2.5.0.5.

To prove this, we note that (1) is equivalent to the assertion that there is a homotopy from f to g (since $\mathrm{K}(M_*)$ is a Kan complex; see Remark 2.5.6.4): that is, a map of simplicial sets $h : \Delta^1 \times X \to \mathrm{K}(M_*)$ satisfying $h|_{\{0\} \times X} = f$ and $h|_{\{1\} \times X} = g$. By virtue of Corollary 2.5.6.12, this is equivalent to the existence of a chain map $h' : \mathrm{N}_*(\Delta^1 \times X; \mathbf{Z}) \to M_*$ which is compatible with f' and g'. For any such chain map h', the composition

$$\mathrm{N}_*(\Delta^1) \boxtimes \mathrm{N}_*(X; \mathbf{Z}) \xrightarrow{\mathrm{EZ}} \mathrm{N}_*(\Delta^1 \times X) \xrightarrow{h'} M_*$$

determines a chain homotopy from f' to g' (where EZ denotes the Eilenberg-Zilber homomorphism of Example 2.5.7.12). More explicitly, this chain homotopy is given by the map of graded abelian groups

$$\mathrm{N}_*(X; \mathbf{Z}) \to M_{*+1} \qquad \sigma \mapsto h'(\tau \triangledown \sigma),$$

where τ is the generator of $\mathrm{N}_1(\Delta^1) \simeq \mathbf{Z}$ and $\triangledown$ is the shuffle product of Construction 2.5.7.9. This proves that (1) implies (2). Conversely, if (2) is satisfied, then there exists a chain map $u : \mathrm{N}_*(\Delta^1) \boxtimes \mathrm{N}_*(X; \mathbf{Z}) \to M_*$ compatible with f' and g', and we can verify (1) by taking h' to be the composite map

$$\mathrm{N}_*(\Delta^1 \times X; \mathbf{Z}) \xrightarrow{\mathrm{AW}} \mathrm{N}_*(\Delta^1) \boxtimes \mathrm{N}_*(X; \mathbf{Z}) \xrightarrow{u} M_*$$

where AW is the Alexander-Whitney homomorphism of Construction 2.5.8.6.

`00TX` **Notation 3.1.5.9.** Let $f : X \to Y$ be a morphism of simplicial sets. We let $[f]$ denote the homotopy class of f: that is, the image of f in the set $\pi_0 \mathrm{Fun}(X, Y)$ of homotopy classes of maps from X to Y.

`00TY` **Construction 3.1.5.10** (The Homotopy Category of Kan Complexes)**.** We define a category hKan as follows:

- The objects of hKan are Kan complexes.
- If X and Y are Kan complexes, then $\mathrm{Hom}_{\mathrm{hKan}}(X, Y) = [X, Y] = \pi_0(\mathrm{Fun}(X, Y))$ is the set of homotopy classes of morphisms from X to Y.

- If X, Y, and Z are Kan complexes, then the composition law

$$\circ : \mathrm{Hom}_{\mathrm{hKan}}(Y, Z) \times \mathrm{Hom}_{\mathrm{hKan}}(X, Y) \to \mathrm{Hom}_{\mathrm{hKan}}(X, Z)$$

  is characterized by the formula $[g] \circ [f] = [g \circ f]$.

We will refer to hKan as the *homotopy category of Kan complexes*.

Remark 3.1.5.11. Let Kan denote the full subcategory of Set_{Δ} spanned by the Kan complexes, and let $\mathcal{C}$ be any category. Then precomposition with the quotient map Kan $\to$ hKan induces an isomorphism from the functor category $\mathrm{Fun}(\mathrm{hKan}, \mathcal{C})$ to the full subcategory of $\mathrm{Fun}(\mathrm{Kan}, \mathcal{C})$ spanned by those functors $F : \mathrm{Kan} \to \mathcal{C}$ which satisfy the following condition: 012T

$(*)$ If X and Y are Kan complexes and $u_0, u_1 : X \to Y$ are homotopic morphisms, then $F(u_0) = F(u_1)$ in $\mathrm{Hom}_{\mathcal{C}}(F(X), F(Y))$.

Remark 3.1.5.12. Let $\mathcal{C}$ be a locally Kan simplicial category (Definition 2.4.1.8). Then the homotopy category $\mathrm{h}\mathcal{C}$ of Construction 2.4.6.1 inherits the structure of an hKan-enriched category, which can be described concretely as follows: 02L1

- For every pair of objects $X, Y \in \mathcal{C}$, the mapping object $\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Y)$ is the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet}$, regarded as an object of hKan.

- For every pair of objects $X, Y, Z \in \mathcal{C}$, the composition law

$$\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Z)$$

  is the homotopy class of the composition map $\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \to \mathrm{Hom}_{\mathcal{C}}(X, Z)_{\bullet}$.

Note that the passage from the category Kan to its homotopy category hKan can be viewed as a special case of Construction 2.4.6.1, where we view Kan as a simplicial category with morphism spaces given by $\mathrm{Hom}_{\mathrm{Kan}}(X, Y)_{\bullet} = \mathrm{Fun}(X, Y)$. Applying Construction 2.4.6.16 to this simplicial category, we obtain the following variant:

Construction 3.1.5.13 (The Homotopy 2-Category of Kan Complexes)**.** We define a strict 2-category $\mathrm{h}_2\mathrm{Kan}$ as follows: 02BQ

- The objects of $\mathrm{h}_2\mathrm{Kan}$ are Kan complexes.

- If X and Y are Kan complexes, then $\underline{\mathrm{Hom}}_{\mathrm{h}_2\mathrm{Kan}}(X, Y)$ is the fundamental groupoid of the Kan complex $\mathrm{Fun}(X, Y)$.

- If X, Y, and Z are Kan complexes, then the composition law on $\mathrm{h}_2\mathrm{Kan}$ is given by

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathrm{h}_2\mathrm{Kan}}(Y,Z) \times \underline{\mathrm{Hom}}_{\mathrm{h}_2\mathrm{Kan}}(X,Y) &= \pi_{\leq 1}(\mathrm{Fun}(Y,Z)) \times \pi_{\leq 1}(\mathrm{Fun}(X,Y)) \\ &\simeq \pi_{\leq 1}(\mathrm{Fun}(Y,Z) \times \mathrm{Fun}(X,Y)) \\ &\xrightarrow{\circ} \pi_{\leq 1}(\mathrm{Fun}(X,Z)) \\ &= \underline{\mathrm{Hom}}_{\mathrm{h}_2\mathrm{Kan}}(X,Z).\end{aligned}$$

We will refer to $\mathrm{h}_2\mathrm{Kan}$ as the *homotopy 2-category of Kan complexes*.

02BR **Remark 3.1.5.14.** We can describe the strict 2-category $\mathrm{h}_2\mathrm{Kan}$ more informally as follows:

- The objects of $\mathrm{h}_2\mathrm{Kan}$ are Kan complexes.
- The morphisms of $\mathrm{h}_2\mathrm{Kan}$ are morphisms of Kan complexes $f : X \to Y$.
- If $f_0, f_1 : X \to Y$ are morphisms of Kan complexes, then a 2-morphism $f_0 \Rightarrow f_1$ in $\mathrm{h}_2\mathrm{Kan}$ is an equivalence class of homotopies $h : \Delta^1 \times X \to Y$ from $f_0 = h|_{\{0\}\times X}$ to $f_1 = h|_{\{1\}\times X}$, where we regard h and h' as equivalent if they are homotopic relative to $\partial\Delta^1 \times X$.

02BS **Remark 3.1.5.15.** Every 2-morphism in the 2-category $\mathrm{h}_2\mathrm{Kan}$ is invertible: that is, $\mathrm{h}_2\mathrm{Kan}$ is a $(2,1)$-category in the sense of Definition 2.2.8.5. Moreover, the homotopy category of $\mathrm{h}_2\mathrm{Kan}$ (in the sense of Construction 2.2.8.12) can be identified with the category hKan of Construction 3.1.5.10 (see Remark 2.4.6.18).

3.1.6 Homotopy Equivalences and Weak Homotopy Equivalences

00U1 Let $f : X \to Y$ be a morphism of Kan complexes. We will say that f is a *homotopy equivalence* if the homotopy class $[f]$ is an isomorphism in the homotopy category hKan of Construction 3.1.5.10. This definition can be extended to more general simplicial sets in multiple ways.

00U2 **Definition 3.1.6.1.** Let $f : X \to Y$ be a morphism of simplicial sets. We will say that a morphism $g : Y \to X$ is a *simplicial homotopy inverse* of f if the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity morphisms id_X and id_Y, respectively (in the sense of Definition 3.1.5.1). In the case where X and Y are Kan complexes, we will say that g is a *homotopy inverse of* f if it is a simplicial homotopy inverse to f. We say that $f : X \to Y$ is a *homotopy equivalence* if it admits a simplicial homotopy inverse g.

02BT **Warning 3.1.6.2.** Let $f : X \to Y$ be a morphism of simplicial sets. Many authors refer to a morphism $g : Y \to X$ as a *homotopy inverse* to f if the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity morphisms id_X and id_Y, respectively. However, when X and

Y are ∞-categories, it is natural to consider a different (and more restrictive) notion of homotopy inverse, which requires that $g \circ f$ and $f \circ g$ be *isomorphic* to id_X and id_Y as objects of the ∞-categories $\mathrm{Fun}(X, X)$ and $\mathrm{Fun}(Y, Y)$, respectively (see Definition 4.5.1.10 and Warning 4.5.1.14). For this reason, we will use the term *simplicial homotopy inverse* in the setting of Definition 3.1.6.1 (unless X and Y are Kan complexes, in which case the distinction disappears).

Example 3.1.6.3. Let $f : X \to Y$ be a homotopy equivalence of topological spaces. Then the induced map of singular simplicial sets $\mathrm{Sing}_\bullet(f) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(Y)$ is a homotopy equivalence (see Example 3.1.5.6). 00U3

Remark 3.1.6.4. Let $f : X \to Y$ be a morphism of simplicial sets. The condition that f is a homotopy equivalence depends only on the homotopy class $[f] \in \pi_0(\mathrm{Fun}(X, Y))$. Moreover, if f is a homotopy equivalence, then its simplicial homotopy inverse $g : Y \to X$ is determined uniquely up to homotopy. 00U4

Remark 3.1.6.5. Let $f : X \to Y$ be a morphism of Kan complexes. If f is a homotopy equivalence, then the induced map of fundamental groupoids $\pi_{\leq 1}(f) : \pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$ is an equivalence of categories. In particular, f induces a bijection $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$. 00V1

Remark 3.1.6.6. Let $f : X \to Y$ be a morphism of simplicial sets. The following conditions are equivalent: 00U5

- The morphism f is a homotopy equivalence.
- For every simplicial set Z, composition with f induces a bijection $\pi_0(\mathrm{Fun}(Y, Z)) \to \pi_0(\mathrm{Fun}(X, Z))$.
- For every simplicial set W, composition with f induces a bijection $\pi_0(\mathrm{Fun}(W, X)) \to \pi_0(\mathrm{Fun}(W, Y))$.

In particular (taking $W = \Delta^0$), if f is a homotopy equivalence, then the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.

Remark 3.1.6.7 (Two-out-of-Three)**.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets. If any two of the morphisms f, g, and $g \circ f$ are homotopy equivalences, then so is the third. 00U7

Remark 3.1.6.8. Let $\{f_i : X_i \to Y_i\}_{i \in I}$ be a collection of homotopy equivalences of simplicial sets indexed by a set I, and let $f : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ be their product. Then: 0193

- If I is finite, then f is a homotopy equivalence. This follows from Remark 3.1.6.6 and Corollary 1.2.1.27.

- If each of the simplicial sets X_i and Y_i is a Kan complex, then f is a homotopy equivalence. This follows from Remark 3.1.6.6 and Corollary 1.2.5.11.
- The morphism f need not be a homotopy equivalence in general (see Warning 1.2.1.28).

We now give some more examples of homotopy equivalences.

`00U8` **Proposition 3.1.6.9.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories, and suppose that F admits either a left or a right adjoint. Then the induced map $\mathrm{N}_\bullet(F) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})$ is a homotopy equivalence of simplicial sets.*

Proof. Without loss of generality, we may assume that F admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$. Then there exist natural transformations $u : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $v : F \circ G \to \mathrm{id}_{\mathcal{D}}$ witnessing an adjunction between F and G, so that $\mathrm{N}_\bullet(F)$ is a simplicial homotopy inverse of $\mathrm{N}_\bullet(G)$ by virtue of Example 3.1.5.7. □

`00U9` **Proposition 3.1.6.10.** *Let $f : X \to S$ be a trivial Kan fibration of simplicial sets. Then f is a homotopy equivalence.*

Proof. By virtue of Corollary 1.5.5.5, the morphism f admits a section $s : S \to X$ such that $s \circ f$ is homotopic to the identity map id_X. □

`03DY` **Example 3.1.6.11.** Let S be a simplicial set and let $\mathrm{N}_*(S; \mathbf{Z})$ for the normalized chain complex of S (Construction 2.5.5.9). Let M_* be a chain complex of abelian groups, let $\mathrm{K}(M_*)$ denote the associated (generalized) Eilenberg-MacLane space, and let

$$H_* = \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(S, \mathbf{Z}), M_*)_*$$

denote the chain complex of maps from $\mathrm{N}_*(S; \mathbf{Z})$ to M_*. Then there is a map of Kan complexes

$$\lambda : \mathrm{K}(H_*) \to \mathrm{Fun}(S, \mathrm{K}(M_*)),$$

which classifies the map of chain complexes

$$\begin{aligned} \mathrm{N}_*(S \times \mathrm{K}(H_*); \mathbf{Z}) & \xrightarrow{\mathrm{AW}} \mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(\mathrm{K}(H_*); \mathbf{Z}) \\ & \to \mathrm{N}_*(S) \boxtimes H_* \\ & \xrightarrow{\mathrm{ev}} M_* \end{aligned}$$

where AW is the Alexander-Whitney map (see Construction 2.5.8.6). The morphism λ is a homotopy equivalence of Kan complexes. To prove this, it will suffice to show that for every simplicial set T, composition with λ induces a bijection

$$\lambda_T : \pi_0(\mathrm{Fun}(T, \mathrm{K}(H_*)) \to \pi_0(\mathrm{Fun}(S \times T, \mathrm{K}(M_*))).$$

Using Example 3.1.5.8 (and the definition of the chain complex H_*), we can identify the source of λ_T with the set of chain homotopy classes of maps the tensor product $\mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(T; \mathbf{Z})$ into M_*, and the target of λ_T with the set of chain homotopy classes of maps from $\mathrm{N}_*(S \times T; \mathbf{Z})$ into M_*. Under these identifications, we see that λ_T is induced by precomposition with the Alexander-Whitney map

$$\mathrm{AW} : \mathrm{N}_*(S \times T; \mathbf{Z}) \to \mathrm{N}_*(S; \mathbf{Z}) \boxtimes \mathrm{N}_*(T; \mathbf{Z}).$$

This map is a quasi-isomorphism (Corollary 2.5.8.11), and therefore admit a chain homotopy inverse (since the source and target of AW are nonnegatively graded complexes of free abelian groups; see Remark [?]).

Definition 3.1.6.12. Let $f : X \to Y$ be a morphism of simplicial sets. We will say that f is a *weak homotopy equivalence* if, for every Kan complex Z, precomposition with f induces a bijection $\pi_0(\mathrm{Fun}(Y, Z)) \to \pi_0(\mathrm{Fun}(X, Z))$. `00UA`

Proposition 3.1.6.13. *Let $f : X \to Y$ be a morphism of simplicial sets. If f is a homotopy equivalence, then it is a weak homotopy equivalence. The converse holds if X and Y are Kan complexes.* `00UB`

Proof. The first assertion follows from Remark 3.1.6.6. For the second, assume that f is a weak homotopy equivalence. If X is a Kan complex, then precomposition with f induces a bijection $\pi_0(\mathrm{Fun}(Y, X)) \to \pi_0(\mathrm{Fun}(X, X))$. We can therefore choose a map of simplicial sets $g : Y \to X$ such that $g \circ f$ is homotopic to the identity on X. It follows that $f \circ g \circ f$ is homotopic to $f = \mathrm{id}_Y \circ f$. Invoking the injectivity of the map $\pi_0(\mathrm{Fun}(Y, Y)) \xrightarrow{\circ f} \pi_0(\mathrm{Fun}(X, Y))$, we conclude that $f \circ g$ is homotopic to id_Y, so that g is a homotopy inverse to f. □

Proposition 3.1.6.14. *Let $f : A \hookrightarrow B$ be an anodyne morphism of simplicial sets. Then f is a weak homotopy equivalence.* `00UP`

Remark 3.1.6.15. We will later prove a (partial) converse to Proposition 3.1.6.14: if a monomorphism of simplicial sets $f : A \hookrightarrow B$ is a weak homotopy equivalence, then f is anodyne (see Corollary 3.3.7.7). `00UZ`

Proof of Proposition 3.1.6.14. Let $i : A \hookrightarrow B$ be an anodyne morphism of simplicial sets; we wish to show that i is a weak homotopy equivalence. Let X be any Kan complex. It follows from Corollary 3.1.3.6 that the restriction map $\theta : \mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X)$ is a trivial Kan fibration. In particular, θ is a homotopy equivalence (Proposition 3.1.6.10), and therefore induces a bijection on connected components $\pi_0(\mathrm{Fun}(B, X)) \to \pi_0(\mathrm{Fun}(A, X))$ (Remark 3.1.6.6). □

Remark 3.1.6.16 (Two-out-of-Three)**.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets. If any two of the morphisms f, g, and $g \circ f$ are weak homotopy equivalences, then so is the third. `00UD`

03PJ **Proposition 3.1.6.17.** *Let $f : X \to Y$ be a morphism of simplicial sets, and let Z be a Kan complex. If f is a weak homotopy equivalence, then composition with f induces a homotopy equivalence* $\mathrm{Fun}(Y, Z) \to \mathrm{Fun}(X, Z)$.

Proof. By virtue of Remark 3.1.6.6, it will suffice to show that for every simplicial set A, the induced map $\theta : \mathrm{Fun}(A, \mathrm{Fun}(Y, Z)) \to \mathrm{Fun}(A, \mathrm{Fun}(X, Z))$ induces a bijection on connected components. This follows by observing that θ can be identfied with the map $\mathrm{Fun}(Y, \mathrm{Fun}(A, Z)) \to \mathrm{Fun}(X, \mathrm{Fun}(A, Z))$ given be precomposition with f (since Corollary 3.1.3.4 guarantees that the simplicial set $\mathrm{Fun}(A, Z)$ is a Kan complex). □

00ZZ **Proposition 3.1.6.18.** *Let $f : X \to Y$ be a weak homotopy equivalence of simplicial sets. Then the induced map of normalized chain complexes* $\mathrm{N}_*(X; \mathbf{Z}) \to \mathrm{N}_*(Y; \mathbf{Z})$ *is a chain homotopy equivalence. In particular, f induces an isomorphism of homology groups* $\mathrm{H}_*(X; \mathbf{Z}) \to \mathrm{H}_*(Y; \mathbf{Z})$.

Proof. Let M_* be a chain complex of abelian groups. We wish to show that precomposition with $\mathrm{N}_*(f; \mathbf{Z})$ induces a bijection

$$\begin{tikzcd}
\{\text{Chain homotopy classes of maps } \mathrm{N}_*(Y; \mathbf{Z}) \to M_*\} \arrow[d, "\theta"] \\
\{\text{Chain homotopy classes of maps } \mathrm{N}_*(X; \mathbf{Z}) \to M_*\}.
\end{tikzcd}$$

Let $\mathrm{K}(M_*)$ denote the Eilenberg-MacLane space associated to M_* (Construction 2.5.6.3). Using Example 3.1.5.8, we can identify θ with the map

$$\pi_0(\mathrm{Fun}(Y, \mathrm{K}(M_*))) \to \pi_0(\mathrm{Fun}(X, \mathrm{K}(M_*)))$$

given by precomposition with f. This map is bijective because f is a weak homotopy equivalence (by assumption) and $\mathrm{K}(M_*)$ is a Kan complex (Remark 2.5.6.4). □

0100 **Remark 3.1.6.19.** There is a partial converse to Proposition 3.1.6.18. If $f : X \to Y$ is a morphism between *simply-connected* simplicial sets and the induced map $\mathrm{H}_*(X; \mathbf{Z}) \to \mathrm{H}_*(Y; \mathbf{Z})$ is an isomorphism, one can show that f is a weak homotopy equivalence. Beware that this is not necessarily true if X and Y are not simply connected (see §[?] for further discussion).

00X7 **Remark 3.1.6.20** (Coproducts of Weak Homotopy Equivalences)**.** Let $\{f(i) : X(i) \to Y(i)\}_{i \in I}$ be a collection of weak homotopy equivalences of simplicial sets indexed by a set I.

For every Kan complex Z, we have a commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Fun}(\coprod_{i \in I} Y(i), Z) \arrow[r] \arrow[d, "\sim"] & \mathrm{Fun}(\coprod_{i \in I} X(i), Z) \arrow[d, "\sim"] \\
\prod_{i \in I} \mathrm{Fun}(Y(i), Z) \arrow[r] & \prod_{i \in I} \mathrm{Fun}(X(i), Z),
\end{tikzcd}$$

where the vertical maps are isomorphisms. Passing to the connected components (and using the fact that the functor $Q \mapsto \pi_0(Q)$ preserves products when restricted to Kan complexes; see Corollary 1.2.5.11), we deduce that the map $\pi_0(\mathrm{Fun}(\coprod_{i \in I} Y(i), Z)) \to \pi_0(\mathrm{Fun}(\coprod_{i \in I} X(i), Z))$ is bijective. Allowing Z to vary, we conclude that the induced map $\coprod_{i \in I} X(i) \to \coprod_{i \in I} Y(i)$ is also a weak homotopy equivalence.

Exercise 3.1.6.21. Let G be the directed graph depicted in the diagram 00UE

$$\begin{tikzcd}
0 \arrow[r] & 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & \cdots
\end{tikzcd}$$

and let G denote the associated 1-dimensional simplicial set (see Warning 1.2.1.28). Show that the projection map $G \to \Delta^0$ is a weak homotopy equivalence, but not a homotopy equivalence.

Warning 3.1.6.22. Let X and Y be simplicial sets. The existence of a weak homotopy equivalence $f : X \to Y$ does not guarantee the existence of a weak homotopy equivalence $g : Y \to X$. 00UF

Proposition 3.1.6.23. *Let $f : X \to Y$ and $f' : X' \to Y'$ be weak homotopy equivalences of simplicial sets. Then the induced map $(f \times f') : X \times X' \to Y \times Y'$ is also a weak homotopy equivalence.* 02L2

Proof. By virtue of Remark 3.1.6.16, it will suffice to show that the morphisms $f \times \mathrm{id}_{X'}$ and $\mathrm{id}_Y \times f'$ are weak homotopy equivalences. We will give the proof for $f \times \mathrm{id}_{X'}$; the analogous statement for $\mathrm{id}_Y \times f'$ follows by a similar argument. Let Z be a Kan complex; we wish to show that precomposition with f induces a bijection

$$\begin{aligned}
\pi_0(\mathrm{Fun}(X \times X', Z)) &\simeq \pi_0(\mathrm{Fun}(X, \mathrm{Fun}(X', Z))) \\
&\to \pi_0(\mathrm{Fun}(Y, \mathrm{Fun}(X', Z))) \\
&\simeq \pi_0(\mathrm{Fun}(Y \times X', Z)).
\end{aligned}$$

This follows from our assumption that f is a weak homotopy equivalence, since the simplicial set $\mathrm{Fun}(X', Z)$ is a Kan complex (Corollary 3.1.3.4). □

02L3 **Warning 3.1.6.24.** The collection of weak homotopy equivalences is not closed under the formation of infinite products. For example, if $q : G \to \Delta^0$ is the weak homotopy equivalence described in Exercise 3.1.6.21, then a product of infinitely many copies of q with itself is not a weak homotopy equivalence (since a product of infinitely many copies of G is not a connected simplicial set: see Warning 1.2.1.28).

3.1.7 Fibrant Replacement

00UU The formalism of Kan complexes is extremely useful as a combinatorial foundation for homotopy theory. However, when studying the homotopy theory of Kan complexes, it is often necessary to contemplate more general simplicial sets. For example, if $f_0, f_1 : S \to T$ are morphisms of Kan complexes, then a homotopy from f_0 to f_1 is defined as a morphism of simplicial sets $h : \Delta^1 \times S \to T$; here neither Δ^1 nor the product $\Delta^1 \times S$ is a Kan complex (except in the trivial case $S = \emptyset$; see Exercise 1.2.5.2). When working with a simplicial set X which is not a Kan complex, it is often convenient to *replace* X by a Kan complex having the same weak homotopy type. This can always be achieved: more precisely, one can always find a weak homotopy equivalence $X \to Q$, where Q is a Kan complex (Corollary 3.1.7.2). Our goal in this section is to prove a “fiberwise” version of this result, which can be stated as follows:

00UV **Proposition 3.1.7.1.** *Let $f : X \to Y$ be a morphism of simplicial sets. Then f can be factored as a composition $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$, where f'' is a Kan fibration and f' is anodyne (hence a weak homotopy equivalence, by virtue of Proposition 3.1.6.14). Moreover, the simplicial set $Q(f)$ (and the morphisms f' and f'') can be chosen to depend functorially on f, in such a way that the functor*

$$\mathrm{Fun}([1], \mathrm{Set}_\Delta) \to \mathrm{Set}_\Delta \qquad (f : X \to Y) \to Q(f)$$

commutes with filtered colimits.

Before giving the proof of Proposition 3.1.7.1, let us note some of its consequences. Applying Proposition 3.1.7.1 in the special case $Y = \Delta^0$, we obtain the following:

00UW **Corollary 3.1.7.2.** *Let X be a simplicial set. Then there exists an anodyne morphism $f : X \hookrightarrow Q$, where Q is a Kan complex.*

04GC **Warning 3.1.7.3.** In the situation of Corollary 3.1.7.2, the Kan complex Q is not uniquely determined. However, the homotopy type of Q depends only on X. If Q' is another Kan complex equipped with a map $f' : X \to Q'$, then we can write $f' = g \circ f$ for some map of Kan complexes $g : Q \to Q'$ (Remark 3.1.2.7). If f' is a weak homotopy equivalence, then g is also a weak homotopy equivalence (Remark 3.1.6.16) and therefore a homotopy equivalence (Proposition 3.1.6.13).

Remark 3.1.7.4. In the situation of Corollary 3.1.7.2, the Kan complex Q (and the anodyne morphism f) can be chosen to depend functorially on X. This follows from the proof of Proposition 3.1.7.1 given below, but there are other (arguably more elegant) ways to achieve the same result. For example, we can take Q to be the simplicial set $\mathrm{Ex}^{\infty}(X)$ of Construction 3.3.6.1 (see Propositions 3.3.6.9 and 3.3.6.7), or the singular simplicial set $\mathrm{Sing}_{\bullet}(|X|)$ (see Proposition 1.2.5.8 and Theorem 3.6.4.1). These constructions also have non-aesthetic advantages: for example, the functors $X \mapsto \mathrm{Ex}^{\infty}(X)$ and $X \mapsto \mathrm{Sing}_{\bullet}(|X|)$ both preserve finite limits. 00UX

Corollary 3.1.7.5. *Let $f : X \to Y$ be a morphism of simplicial sets. The following conditions are equivalent:* 00X8

(1) *The morphism f is anodyne.*

(2) *The morphism f is weakly left orthogonal to all Kan fibrations. That is, if $g : Z \to S$ is a Kan fibration of simplicial sets, then every lifting problem*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
Y \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution.

Proof. The implication (1) ⇒ (2) follows from Remark 3.1.2.7. To deduce the converse, we first apply Proposition 3.1.7.1 to write f as a composition $X \xrightarrow{f'} Q \xrightarrow{f''} Y$, where f' is anodyne and f'' is a Kan fibration. If f satisfies condition (2), then the lifting problem

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "f"'] & Q \arrow[d, "f''"] \\
Y \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution. It follows that f is a retract of f' (in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$). Since the collection of anodyne morphisms is closed under retracts, it follows that f is anodyne. □

Recall that the homotopy category hKan of Construction 3.1.5.10 is defined as a quotient of the category of Kan complexes Kan (by identifying morphisms which are homotopic). However, it can also be described as a *localization* of Kan, obtained by inverting the class of homotopy equivalences (see §6.3).

012U **Proposition 3.1.7.6.** *Let $\mathcal{C}$ be a category and let $F : \mathrm{Kan} \to \mathcal{C}$ be a functor. The following conditions are equivalent:*

- $(*)$ *If X and Y are Kan complexes and $u_0, u_1 : X \to Y$ are homotopic morphisms, then $F(u_0) = F(u_1)$ in $\mathrm{Hom}_{\mathcal{C}}(F(X), F(Y))$.*
- $(*')$ *For every homotopy equivalence of Kan complexes $u : X \to Y$, the induced map $F(u) : F(X) \to F(Y)$ is an isomorphism in the category $\mathcal{C}$.*

Proof. The implication $(*) \Rightarrow (*')$ is immediate (note that a morphism of Kan complexes $u : X \to Y$ is a homotopy equivalence if and only if its homotopy class $[u]$ is an isomorphism in the homotopy category hKan). For the converse, assume that $(*')$ is satisfied, let X and Y be Kan complexes, and let $u_0, u_1 : X \to Y$ be a pair of homotopic morphisms. Let us regard u_0 and u_1 as vertices of the Kan complex $\mathrm{Fun}(X, Y)$. Since u_0 and u_1 are homotopic, there exists an edge $e : \Delta^1 \to \mathrm{Fun}(X, Y)$ satisfying $e(0) = u_0$ and $e(1) = u_1$. By virtue of Proposition 3.1.7.1, this morphism factors as a composition $\Delta^1 \xrightarrow{e'} Q \xrightarrow{e''} \mathrm{Fun}(X, Y)$, where e' is anodyne and e'' is a Kan fibration. Since $\mathrm{Fun}(X, Y)$ is a Kan complex (Corollary 3.1.3.4), it follows that Q is also a Kan complex. Let us identify e'' with a morphism of Kan complexes $h : Q \times X \to Y$. Let $i_0 : X \hookrightarrow Q \times X$ be the product of the identity map id_X with the inclusion $\{e'(0)\} \hookrightarrow Q$, and define $i_1 : X \hookrightarrow Q \times X$ similarly. Since e' is anodyne, the restrictions $e'|_{\{0\}}$ and $e'|_{\{1\}}$ are anodyne. In particular, they are weak homotopy equivalences (Proposition 3.1.6.14) and therefore homotopy equivalences (Proposition 3.1.6.13), since Q is a Kan complex. It follows that i_0 and i_1 are also homotopy equivalences, so that $F(i_0)$ and $F(i_1)$ are isomorphisms (by virtue of assumption $(*')$). Using the fact that i_0 and i_1 are left inverse to the projection map $\pi : Q \times X \to X$, we see that $F(\pi)$ is an isomorphism in $\mathcal{C}$ and that we have

$$F(u_0) = F(h) \circ F(i_0) = F(h) \circ F(\pi)^{-1} = F(h) \circ F(i_1) = F(u_1),$$

as desired. □

012V **Corollary 3.1.7.7.** *Let $\mathcal{C}$ be a category, let $\mathcal{E} \subseteq \mathrm{Fun}(\mathrm{Kan}, \mathcal{C})$ be the full subcategory spanned by those functors $F : \mathrm{Kan} \to \mathcal{C}$ which carry homotopy equivalences of Kan complexes to isomorphisms in the category $\mathcal{C}$. Then precomposition with the quotient map* $\mathrm{Kan} \to \mathrm{hKan}$ *induces an isomorphism of categories* $\mathrm{Fun}(\mathrm{hKan}, \mathcal{C}) \to \mathcal{E}$.

Proof. Combine Remark 3.1.5.11 with Proposition 3.1.7.6. □

012W **Variant 3.1.7.8.** Let $\mathcal{C}$ be a category, and let $\mathcal{E}' \subseteq \mathrm{Fun}(\mathrm{Set}_{\Delta}, \mathcal{C})$ be the full subcategory spanned by those functors $F : \mathrm{Set}_{\Delta} \to \mathcal{C}$. which carry weak homotopy equivalences of simplicial sets to isomorphisms in the category $\mathcal{C}$. Then:

(a) For every functor $F \in \mathcal{E}'$, the restriction $F|_{\mathrm{Kan}}$ factors (uniquely) as a composition $\mathrm{Kan} \twoheadrightarrow \mathrm{hKan} \xrightarrow{\overline{F}} \mathcal{C}$.

(b) The construction $F \mapsto \overline{F}$ induces an equivalence of categories $\mathcal{E}' \to \mathrm{Fun}(\mathrm{hKan}, \mathcal{C})$.

Remark 3.1.7.9. Corollary 3.1.7.7 and Variant 3.1.7.8 can be stated more informally as follows: `012X`

- The homotopy category hKan can be obtained from the category Kan of Kan complexes by formally adjoining inverses to all homotopy equivalences.
- The homotopy category hKan can be obtained from the category Set_{Δ} of simplicial sets by formally adjoining inverses to all weak homotopy equivalences.

Either of these assertions characterizes the homotopy category hKan up to equivalence (in fact, Corollary 3.1.7.7 even characterizes hKan up to *isomorphism*).

Proof of Variant 3.1.7.8. Let $\mathcal{E} \subseteq \mathrm{Fun}(\mathrm{Kan}, \mathcal{C})$ be the full subcategory spanned by those functors $F : \mathrm{Kan} \to \mathcal{C}$ which carry homotopy equivalences of Kan complexes to isomorphisms in $\mathcal{C}$. By virtue of Corollary 3.1.7.7, it will suffice to show that the restriction functor $F \mapsto F|_{\mathrm{Kan}}$ induces an equivalence of categories $\mathcal{E}' \to \mathcal{E}$. Using Proposition 3.1.7.1, we can choose a functor $Q : \mathrm{Set}_{\Delta} \to \mathrm{Kan}$ and a natural transformation $u : \mathrm{id}_{\mathrm{Set}_{\Delta}} \to Q$ with the property that, for every simplicial set X, the induced map $u_X : X \to Q(X)$ is anodyne. For every morphism of simplicial sets $f : X \to Y$, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u_X"] & Y \arrow[d, "u_Y"] \\
Q(X) \arrow[r, "Q(f)"] & Q(Y),
\end{tikzcd}$$

where the vertical maps are weak homotopy equivalences (Proposition 3.1.6.14). It follows that if f is a weak homotopy equivalence, then $Q(f)$ is also a weak homotopy equivalence (Remark 3.1.6.16) and therefore a homotopy equivalence (Proposition 3.1.6.13). In other words, the functor Q carries weak homotopy equivalences of simplicial sets to homotopy equivalences of Kan complexes. It follows that precomposition with Q induces a functor $\theta : \mathcal{E} \to \mathcal{E}'$. We claim that θ is homotopy inverse to the restriction functor $\mathcal{E}' \to \mathcal{E}$. This follows from the following pair of observations:

- For every functor $F : \mathrm{Set}_{\Delta} \to \mathcal{C}$, u induces a natural transformation $F \to F|_{\mathrm{Kan}} \circ Q$, which depends functorially on F and is an isomorphism for $F \in \mathcal{E}'$.

- For every functor $F_0 : \mathrm{Kan} \to \mathcal{C}$, u induces a natural transformation $F_0 \to (F_0 \circ Q)|_{\mathrm{Kan}}$, which depends functorially on F_0 and is an isomorphism for $F_0 \in \mathcal{E}$.

□

We now turn to the proof of Proposition 3.1.7.1. We will use an easy version of Quillen's "small object argument" (which we will revisit in greater generality in §[?]).

Proof of Proposition 3.1.7.1. Let $f : X \to Y$ be a morphism of simplicial sets. We construct a sequence of simplicial sets $\{X(m)\}_{m \geq 0}$ and morphisms $f(m) : X(m) \to Y$ by recursion. Set $X(0) = X$ and $f(0) = f$. Assuming that $f(m) : X(m) \to Y$ has been defined, let $S(m)$ denote the set of all commutative diagrams σ :

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X(m) \arrow[d, "f(m)"] \\
\Delta^n \arrow[r, "u_\sigma"] & Y,
\end{tikzcd}$$

where $0 \leq i \leq n$, $n > 0$, and the left vertical map is the inclusion. For every such commutative diagram σ, let $C_\sigma = \Lambda^n_i$ denote the upper left hand corner of the diagram σ, and $D_\sigma = \Delta^n$ the lower left hand corner. Form a pushout diagram

$$\begin{tikzcd}
\coprod_{\sigma \in S(m)} C_\sigma \arrow[r] \arrow[d] & X(m) \arrow[d] \\
\coprod_{\sigma \in S(m)} D_\sigma \arrow[r] & X(m+1)
\end{tikzcd}$$

and let $f(m+1) : X(m+1) \to Y$ be the unique map whose restriction to $X(m)$ is equal to $f(m)$ and whose restriction to each D_σ is equal to u_σ. By construction, we have a direct system of anodyne morphisms

$$X = X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots$$

Set $Q(f) = \varinjlim_m X(m)$. Then the natural map $f' : X \to Q(f)$ is anodyne (since the collection of anodyne maps is closed under transfinite composition), and the system of morphisms $\{f(m)\}_{m \geq 0}$ can be amalgamated to a single map $f'' : Q(f) \to Y$ satisfying $f = f'' \circ f'$. It is clear from the definition that the construction $f \mapsto Q(f)$ is functorial and commutes with filtered colimits. To complete the proof, it will suffice to show that f'' is a Kan fibration: that is, that every lifting problem σ :

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "v"] \arrow[d] & Q(f) \arrow[d, "f''"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution (provided that $n > 0$). Let us abuse notation by identifying each $X(m)$ with its image in $Q(f)$. Since Λ^n_i is a finite simplicial set, its image under v is contained in $X(m)$ for some $m \gg 0$. In this case, we can identify σ with an element of the set $S(m)$, so that the lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "v"] \arrow[d] & X(m+1) \arrow[d, "f(m+1)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution by construction. □

Example 3.1.7.10 (Path Fibrations)**.** If $f : X \to Y$ is a morphism of Kan complexes, then we can give a much more explicit proof of Proposition 3.1.7.1. Let $P(f)$ denote the fiber product $X \times_{\mathrm{Fun}(\{0\},Y)} \mathrm{Fun}(\Delta^1, Y)$. Then f factors as a composition $X \xrightarrow{f'} P(f) \xrightarrow{f''} Y$, where f'' is given by evaluation at the vertex $\{1\} \subseteq \Delta^1$ and f' is obtained by amalgamating the identity morphism id_X with the composition $X \xrightarrow{f} Y \xrightarrow{\delta} \mathrm{Fun}(\Delta^1, Y)$. Moreover: 0101

- The morphism f' is a section of the projection map $P(f) \to X$, which is a pullback of the evaluation map $\mathrm{Fun}(\Delta^1, Y) \to \mathrm{Fun}(\{0\}, Y)$ and therefore a trivial Kan fibration (Corollary 3.1.3.6). It follows that f' is a weak homotopy equivalence. Since it is also a monomorphism, it is anodyne (see Corollary 3.3.7.7).

- The morphism f'' factors as a composition

  $$P(f) = X \times_{\mathrm{Fun}(\{0\},Y)} \mathrm{Fun}(\Delta^1, Y) \xrightarrow{u} X \times \mathrm{Fun}(\{1\}, Y) \xrightarrow{v} Y,$$

  where u is a pullback of the restriction map $\mathrm{Fun}(\Delta^1, Y) \to \mathrm{Fun}(\partial\Delta^1, Y)$ (and therefore a Kan fibration by virtue of Corollary 3.1.3.3) and v is a pullback of the projection map $X \to \Delta^0$ (and therefore a Kan fibration by virtue of our assumption that X is a Kan complex). It follows that f'' is also a Kan fibration.

The proof of Proposition 3.1.7.1 can be repurposed to obtain many analogous results.

00UY **Exercise 3.1.7.11.** Let $f : X \to Y$ be a morphism of simplicial sets. Show that f can be factored as a composition $X \xrightarrow{f'} P(f) \xrightarrow{f''} Y$, where f' is a monomorphism and f'' is a trivial Kan fibration. Moreover, this factorization can be chosen to depend functorially on f (as an object of the arrow category $\operatorname{Fun}([1], \mathrm{Set}_{\Delta})$).

050L **Variant 3.1.7.12.** Let $f : X \to Y$ be a morphism of simplicial sets, and let n be a nonnegative integer. Arguing as in the proof of Proposition 3.1.7.1, we see that f admits a factorization $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$ with the following properties:

(a) The morphism f' can be realized as a transfinite pushout of horn inclusions $\Lambda^m_i \hookrightarrow \Delta^m$ for $0 \leq i \leq m$ and $m > n$.

(b) For $0 \leq i \leq m$ and $m > n$, every lifting problem

$$\begin{tikzcd}
\Lambda^m_i \arrow[r] \arrow[d] & Q(f) \arrow[d, "f''"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution.

It follows from (a) that morphism f' is a monomorphism which is bijective on k-simplices for $k < n$.

Now suppose that the morphism f satisfies the following additional condition:

($*$) For $0 \leq i \leq m$ and $0 < m \leq n$, every lifting problem

$$\begin{tikzcd}
\Lambda^m_i \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution.

Since f' is bijective on k-simplices for $k < n$, it follows that the morphism f'' also satisfies condition ($*$). Combining this with assumption (b), we conclude that f'' is a Kan fibration.

050M **Example 3.1.7.13.** Let $n \geq 0$ be an integer and let X be a simplicial set which satisfies the following condition:

($*$) For $0 < m \leq n$, every horn $\Lambda^m_i \to X$ can be extended to an m-simplex of X.

Applying Variant 3.1.7.12 to the projection map $X \to \Delta^0$, we conclude that X admits an anodyne map $f : X \hookrightarrow Q$ which is bijective on k-simplices for $k < n$, where Q is a Kan complex.

3.2 Homotopy Groups

Our goal in this section is to address the following: 00V2

Question 3.2.0.1. Let $f : X \to Y$ be a morphism of Kan complexes. Under what conditions does f admit a homotopy inverse $g : Y \to X$? 00V3

Let us begin with a partial answer to Question 3.2.0.1. For every Kan complex X, let $\pi_{\leq 1}(X)$ denote the fundamental groupoid of X (Definition 1.4.6.12). For each vertex $x \in X$, we let $\pi_1(X, x)$ denote the automorphism group $\mathrm{Aut}_{\pi_{\leq 1}(X)}(x) = \mathrm{Hom}_{\pi_{\leq 1}(X)}(x, x)$; we will refer to $\pi_1(X, x)$ as the *fundamental group of* X (with respect to the base point x). Every morphism of Kan complexes $f : X \to Y$ induces a functor $\pi_{\leq 1}(f) : \pi_{\leq 1}(X) \to \pi_{\leq 1}(Y)$. Moreover, if f is a homotopy equivalence, then $\pi_{\leq 1}(f)$ is an equivalence of categories (Remark 3.1.6.5). In other words, every homotopy equivalence $f : X \to Y$ satisfies the following pair of conditions:

(W_0) The map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of sets: that is, f induces a bijection from the set of connected components of X to the set of connected components of Y.

(W_1) For every choice of vertex $x \in X$ having image $y = f(x) \in Y$, the induced map of fundamental groups $\pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism.

However, these observations do not supply a complete answer to Question 3.2.0.1: conditions (W_0) and (W_1) are *necessary* for f to be a homotopy equivalence, but they are not *sufficient*. In this section, we will remedy the situation by introducing a hierarchy of additional invariants. To each Kan complex X and each vertex $x \in X$, we will associate a sequence of sets $\{\pi_n(X, x)\}_{n \geq 0}$, which enjoy the following features:

- For every nonnegative integer n, $\pi_n(X, x)$ is defined as the set of homotopy classes of pointed maps from the quotient $\Delta^n / \partial \Delta^n$ to X (Construction 3.2.2.4). Here it is important to work in the homotopy theory of *pointed* simplicial sets, which we review in §3.2.1.

- When $n = 0$, we can identify $\pi_n(X, x)$ with the set $\pi_0(X)$ of connected components of X: in particular, it does not depend on the choice of base point x (Example 3.2.2.6).

- For $n > 0$, the set $\pi_n(X, x)$ comes equipped with a natural group structure (Theorem 3.2.2.10), which we will construct in §3.2.3. For this reason, we will refer to $\pi_n(X, x)$ as the *nth homotopy group of* X (with respect to the base point x). Moreover, the group $\pi_n(X, x)$ is abelian for $n \geq 2$.

- When $n = 1$, we can identify $\pi_1(X, x)$ with the fundamental group of X as defined earlier: that is, with the automorphism group of x as an object of the homotopy category $\pi_{\leq 1}(X)$ (Example 3.2.2.12).

- Let $f : X \to S$ be a Kan fibration between Kan complexes, let $x \in X$ be a vertex having image $s = f(x) \in S$, and let $X_s = \{s\} \times_S X$ denote the fiber of f over the vertex s. Then there is a long exact sequence of homotopy groups

$$\cdots \to \pi_{n+1}(S, s) \xrightarrow{\partial} \pi_n(X_s, x) \to \pi_n(X, x) \to \pi_n(S, s) \xrightarrow{\partial} \pi_{n-1}(X_s, x) \to \cdots$$

  We construct this sequence in §3.2.5, and prove its exactness in §3.2.6 (Theorem 3.2.6.1).

- Let $f : X \to Y$ be a morphism of Kan complexes. In §3.2.7, we show that f is a homotopy equivalence if and only if it induces a bijection $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ and an isomorphism of homotopy groups $\pi_n(X, x) \to \pi_n(Y, f(x))$, for every choice of base point $x \in X$ and every positive integer n (Theorem 3.2.7.1). This is a simplicial counterpart of a classical result of Whitehead ([60]). In §3.2.8, we apply this result to deduce some closure properties for the class of homotopy equivalences (Propositions 3.2.8.1 and 3.2.8.3).

3.2.1 Pointed Kan Complexes

00V4 In §3.1.5, we showed that the collection of Kan complexes can be organized into a category hKan whose morphisms are given by homotopy classes of maps (Construction 3.1.5.10). In this section, we describe a variant of this construction for Kan complexes which are equipped with a specified base point. We begin by introducing a slight generalization of Definition 3.1.5.1.

050N **Definition 3.2.1.1.** Let X and Y be simplicial sets, and let $K \subseteq X$ be a simplicial subset. We say that morphisms $f_0, f_1 : X \to Y$ are *homotopic relative to* K if the following conditions are satisfied:

- The morphisms f_0 and f_1 have the same restriction to K: that is, there is a morphism $\overline{f} : K \to Y$ satisfying $f_0|_K = \overline{f} = f_1|_K$.

- The morphisms f_0 and f_1 belong to the same connected component of the simplicial set $\{\overline{f}\} \times_{\mathrm{Fun}(K,Y)} \mathrm{Fun}(X, Y)$.

050P **Example 3.2.1.2.** Let $f_0, f_1 : X \to Y$ be morphisms of simplicial sets. Then f_0 and f_1 are homotopic (in the sense of Definition 3.1.5.1) if and only if they are homotopic relative to the empty subset $\emptyset \subset X$ (in the sense of Definition 3.2.1.1).

Definition 3.2.1.3. Let $f_0, f_1 : X \to Y$ be a pair of morphisms of simplicial sets and let $h : \Delta^1 \times X \to Y$ be a homotopy from f_0 to f_1. If $K \subseteq X$ is a simplicial subset, we say that h is *constant along* K if the restriction $h|_{\Delta^1 \times K}$ factors through the projection map $\Delta^1 \times K \twoheadrightarrow K$. `050Q`

Proposition 3.2.1.4. *Let $f, g : X \to Y$ be morphisms of simplicial sets and let $K \subseteq X$ be a simplicial subset. Then:* `00V9`

- *The morphisms f and g are homotopic relative to K if and only if there exists a sequence of morphisms $f = f_0, f_1, \ldots, f_n = g$ from X to Y having the property that, for each $1 \leq i \leq n$, there exists either a homotopy from f_{i-1} to f_i which is constant along K, or a homotopy from f_i to f_{i-1} which is constant along K.*

- *Suppose that Y is a Kan complex. Then f and g are homotopic relative to K if and only if there exists a homotopy from f to g which is constant along K.*

Proof. Set $\overline{f} = f|_K$. Without loss of generality, we may assume that $\overline{f}$ is also equal to $g|_K$. The first assertion follows by applying Remark 1.2.1.23 to the simplicial set $Z = \{\overline{f}\} \times_{\operatorname{Fun}(K,Y)} \operatorname{Fun}(X, Y)$. If Y is a Kan complex, then the restriction map $\operatorname{Fun}(X, Y) \to \operatorname{Fun}(K, Y)$ is a Kan fibration (Corollary 3.1.3.3), so that Z is a Kan complex (Remark 3.1.1.9). The second assertion now follows from Proposition 1.2.5.10. $\square$

We will be primarily interested in applying Definition 3.2.1.3 in the special case where $K = \{x\}$ is a vertex of X.

Definition 3.2.1.5. A *pointed simplicial set* is a pair (X, x), where X is a simplicial set and x is a vertex of X. If X is a Kan complex, then we refer to the pair (X, x) as a *pointed Kan complex.* If (X, x) and (Y, y) are pointed Kan complexes, then a *pointed map* from (X, x) to (Y, y) is a morphism of Kan complexes $f : X \to Y$ satisfying $f(x) = y$. We let Kan_* denote the category whose objects are pointed Kan complexes and whose morphisms are pointed maps. `00V5`

Remark 3.2.1.6. We will often abuse terminology by identifying a pointed simplicial set (X, x) with the underlying simplicial set X. In this case, we will refer to x as the *base point* of X. `00V6`

Definition 3.2.1.7. Let (X, x) and (Y, y) be simplicial sets. We say that pointed maps $f_0, f_1 : (X, x) \to (Y, y)$ are *pointed homotopic* if they are homotopic relative to the simplicial subset $\{x\} \subseteq X$, in the sense of Definition 3.2.1.3. A *pointed homotopy* from f_0 to f_1 is a homotopy $h : \Delta^1 \times X \to Y$ which is constant along $\{x\}$ (Definition 3.2.1.3): that is, which carries $\Delta^1 \times \{x\}$ to the degenerate edge id_y. `00V7`

00VA **Example 3.2.1.8.** Let (X, x) be a pointed simplicial set and let (Y, y) be a pointed topological space. Suppose we are given a pair of continuous functions $f_0, f_1 : |X| \to Y$ carrying x to y, which we can identify with pointed morphisms $f'_0, f'_1 : X \to \operatorname{Sing}_\bullet(Y)$. Let $h : [0,1] \times |X| \to Y$ be a continuous function satisfying $f_0 = h|_{\{0\}\times|X|}$, $f_1 = h|_{\{1\}\times|X|}$, and $h(t, x) = y$ for $0 \leq t \leq 1$ (that is, h is a pointed homotopy from f_0 to f_1 in the category of topological spaces). Then the composite map

$$|\Delta^1 \times X| \xrightarrow{\theta} |\Delta^1| \times |X| = [0,1] \times |X| \xrightarrow{h} Y$$

classifies a morphism of simplicial sets $h' : \Delta^1 \times X \to \operatorname{Sing}_\bullet(Y)$, which is a pointed homotopy from f'_0 to f'_1 (in the sense of Definition 3.2.1.7). By virtue of Corollary 3.6.2.2, the map θ is a homeomorphism, so every pointed homotopy from f_0 to f_1 arises in this way. In other words, the construction $h \mapsto h'$ induces a bijection

$$\begin{tikzcd}
\{\text{(Continuous) pointed homotopies from } f_0 \text{ to } f_1\} \arrow[d, "\sim"] \\
\{\text{(Simplicial) pointed homotopies from } f'_0 \text{ to } f'_1\}.
\end{tikzcd}$$

00VB **Example 3.2.1.9.** Let (X, x) and (Y, y) be pointed topological spaces, and let $h : [0,1] \times X \to Y$ be a continuous function satisfying $h(t, x) = y$ for $0 \leq t \leq 1$, which we regard as a pointed homotopy from $f_0 = h|_{\{0\}\times X}$ to $f_1 = h|_{\{1\}\times X}$. Then h determines a homotopy between the induced map of simplicial sets $\operatorname{Sing}_\bullet(f_0), \operatorname{Sing}_\bullet(f_1) : \operatorname{Sing}_\bullet(X) \to \operatorname{Sing}_\bullet(Y)$: this follows by applying Example 3.2.1.8 to the composite map $[0,1] \times |\operatorname{Sing}_\bullet(X)| \to [0,1] \times X \xrightarrow{h} Y$.

00VC **Notation 3.2.1.10.** Let (X, x) and (Y, y) be pointed simplicial sets. We let $[X, Y]_*$ denote the set $\pi_0(\operatorname{Fun}(X, Y) \times_{\operatorname{Fun}(\{x\},Y)} \{y\})$ of pointed homotopy classes of morphisms from (X, x) to (Y, y). If $f : X \to Y$ is a morphism of pointed simplicial sets, we denote its pointed homotopy class by $[f] \in [X, Y]_*$.

00VD **Warning 3.2.1.11.** Notation 3.2.1.10 has the potential to create confusion. If (X, x) and (Y, y) are pointed simplicial sets and $f : X \to Y$ is a morphism satisfying $f(x) = y$, then we use the notation $[f]$ to represent both the homotopy class of f as a map of simplicial sets (that is, the image of f in the set $\pi_0(\operatorname{Fun}(X, Y))$), and the pointed homotopy class of f as a map of pointed simplicial sets (that is, the image of f in the set $[X, Y]_* = \pi_0(\operatorname{Fun}(X, Y) \times_{\operatorname{Fun}(\{x\},Y)} \{y\})$). Beware that these usages are not the same: in general, it is possible for a pair of pointed morphisms $f, g : X \to Y$ to be homotopic without being pointed homotopic.

00VE **Construction 3.2.1.12** (The Homotopy Category of Pointed Kan Complexes)**.** We define a category hKan_* as follows:

- The objects of hKan_* are pointed Kan complexes (X, x).
- If (X, x) and (Y, y) are Kan complexes, then $\mathrm{Hom}_{\mathrm{hKan}}((X, x), (Y, y)) = [X, Y]_*$ is the set of pointed homotopy classes of morphisms from (X, x) to (Y, y).
- If (X, x), (Y, y), and (Z, z) are Kan complexes, then the composition law

  $$\circ : \mathrm{Hom}_{\mathrm{hKan}}((Y, y), (Z, z)) \times \mathrm{Hom}_{\mathrm{hKan}}((X, x), (Y, y)) \to \mathrm{Hom}_{\mathrm{hKan}}((X, x), (Z, z))$$

  is characterized by the formula $[g] \circ [f] = [g \circ f]$.

We will refer to hKan_* as the *homotopy category of pointed Kan complexes.*

Note that there is a forgetful functor $\mathrm{hKan}_* \to \mathrm{hKan}$, given on objects by the construction $(X, x) \mapsto X$. This forgetful functor is conservative:

Proposition 3.2.1.13. *Let $f : (X, x) \to (Y, y)$ be a morphism of pointed Kan complexes. The following conditions are equivalent:* `00WW`

(1) *The underlying morphism of simplicial sets $f : X \to Y$ is a homotopy equivalence (Definition 3.1.6.1): that is, there exists a morphism of simplicial sets $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity maps id_X and id_Y, respectively.*

(2) *The map f is a pointed homotopy equivalence: that is, there exists a morphism of pointed simplicial sets $g : (Y, y) \to (X, x)$ such that $g \circ f$ and $f \circ g$ are pointed homotopic to the identity maps id_X and id_Y, respectively.*

We will deduce Proposition 3.2.1.13 from the following slightly more precise result:

Lemma 3.2.1.14. *Let $f : (X, x) \to (Y, y)$ be a morphism of pointed Kan complexes, and suppose that the homotopy class $[f]$ admits a left inverse in the homotopy category* hKan*. Then $[f]$ also admits a left homotopy inverse in the pointed homotopy category* hKan_*. `04GD`

Proof. Let $g : Y \to X$ be a left homotopy inverse of f. Then there exists a homotopy $\alpha : \Delta^1 \times X \to X$ from the identity morphism $\mathrm{id}_X = \alpha|_{\{0\} \times X}$ to $g \circ f = \alpha|_{\{1\} \times X}$. Then the restriction $\alpha|_{\Delta^1 \times \{x\}}$ determines an edge $e : x \to g(y)$ of X. Since X is a Kan complex, we can use Remark 3.1.5.3 to construct another map $g' : Y \to X$ and a homotopy $\beta : \Delta^1 \times Y \to X$ from $g' = \beta|_{\{0\} \times Y}$ to $g = \beta|_{\{1\} \times Y}$, such that $\beta|_{\{y\} \times \Delta^1}$ is the edge e. Precomposing β with $\mathrm{id}_{\Delta^1} \times f$, we obtain a homotopy β_f from $g' \circ f$ to $g \circ f$. Let $\sigma = s^1_0(e)$ denote the degenerate 2-simplex of X depicted in the diagram

$$\begin{tikzcd}
 & x \arrow[dr, "e"] & \\
x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "e"] & & g(y).
\end{tikzcd}$$

Corollary 3.1.3.3 guarantees that the evaluation map $\mathrm{ev}_x : \mathrm{Fun}(X, X) \to \mathrm{Fun}(\{x\}, X) \simeq X$ is a Kan fibration, so we can lift σ to a 2-simplex of $\mathrm{Fun}(X, X)$ depicted in the diagram

$$\begin{tikzcd}
 & g' \circ f \arrow[dr, "\beta_f"] & \\
\mathrm{id}_X \arrow[ur, "\gamma"] \arrow[rr, "\alpha"] & & g \circ f.
\end{tikzcd}$$

By construction, γ is a pointed homotopy from id_X to the composition $g' \circ f$, so that the homotopy class $[g']$ is a left inverse to $[f]$ in the pointed homotopy category hKan_*. □

Proof of Proposition 3.2.1.13. Let $f : (X, x) \to (Y, y)$ be a morphism of pointed Kan complexes which is a homotopy equivalence; we wish to show that f is a pointed homotopy equivalence (the reverse implication follows immediately from the definitions). Using Lemma 3.2.1.14, we deduce that there is a morphism of pointed Kan complexes $g : (Y, y) \to (X, x)$ such that the homotopy class $[g]$ is a left inverse of $[f]$ in the pointed homotopy category hKan_*. Since f is a homotopy equivalence, it follows that g is also a homotopy equivalence. Applying Lemma 3.2.1.14 again, we conclude that $[g]$ admits a left inverse in the pointed homotopy category hKan_*. In particular, $[g]$ is an isomorphism in hKan_*, so its right inverse $[f]$ is also an isomorphism. □

`02LD` **Proposition 3.2.1.15.** *Let $f : (X, x) \to (Y, y)$ be a morphism of pointed simplicial sets, where Y is a Kan complex. The following conditions are equivalent:*

(1) *The morphism f is nullhomotopic as an unpointed map. That is, there exists a vertex $z \in Y$ and a homotopy from f to the constant map $\underline{z} : X \to Y$ taking the value z (see Definition 3.2.4.5).*

(2) *The morphism f is nullhomotopic as a pointed map: that is, there exists a vertex $y \in Y$ and a pointed homotopy from f to the constant map $\underline{y} : X \to Y$.*

Proof. The implication (2) ⇒ (1) is immediate from the definition. To prove the converse, suppose that there exists a a homotopy $h : \Delta^1 \times X \to Y$ satisfying $h|_{\{0\} \times X} = f$ and $h|_{\{1\} \times X} = \underline{z}$ for some vertex $z \in Y$. Let $e : y \to z$ be the edge of Y given by the restriction $h|_{\Delta^1 \times \{x\}}$ and let $\sigma = s^1_0(e)$ denote the degenerate 2-simplex of Y depicted in the diagram

$$\begin{tikzcd}
 & y \arrow[dr, "e"] & \\
y \arrow[ur, "\mathrm{id}_y"] \arrow[rr, "e"] & & z.
\end{tikzcd}$$

Let $\underline{e} : \underline{y} \to \underline{z}$ denote the image of e in $\mathrm{Fun}(X, Y)$. Since Y is a Kan complex, the restriction map $q : \mathrm{Fun}(X, Y) \to \mathrm{Fun}(\{x\}, Y) \simeq Y$ is a Kan fibration (Corollary 3.1.3.3). It follows that the lifting problem

$$\begin{tikzcd}
\Lambda^2_2 \arrow[r, "(\bullet, h, \underline{e})"] \arrow[d] & \mathrm{Fun}(X, Y) \arrow[d, "q"] \\
\Delta^2 \arrow[r, "\sigma"] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

admits a solution which carries the edge $\mathrm{N}_\bullet(\{0 < 1\}) \subseteq \Delta^2$ to a pointed homotopy from f to $\underline{y}$. $\square$

3.2.2 The Homotopy Groups of a Kan Complex

Let X be a topological space and let $x \in X$ be a point. For every positive integer n, we let $\pi_n(X, x)$ denote the set of homotopy classes of pointed maps $(S^n, x_0) \to (X, x)$, where S^n denotes a sphere of dimension n and $x_0 \in S^n$ is a chosen base point. The set $\pi_n(X, x)$ can be endowed with the structure of a group, which we refer to as the *nth homotopy group of X* (with respect to the base point x). Note that the sphere S^n can be realized as the quotient space $|\Delta^n|/|\partial\Delta^n|$, obtained from the topological simplex $|\Delta^n|$ by collapsing its boundary to the point q. We can therefore identify pointed maps $(S^n, x_0) \to (X, x)$ with maps of simplicial sets $f : \Delta^n \to \mathrm{Sing}_\bullet(X)$ which carry the boundary $\partial\Delta^n$ to the simplicial subset $\{x\} \subseteq \mathrm{Sing}_\bullet(X)$. In [35], Kan elaborated on this observation to give a direct construction of the homotopy group $\pi_n(X, x)$ in terms of the simplicial set $\mathrm{Sing}_\bullet(X)$ (and the vertex x). Moreover, his construction can be applied directly to any Kan complex. 00VJ

Notation 3.2.2.1. Let B be a simplicial set and let $A \subseteq B$ be a simplicial subset. We let B/A denote the pushout $B \coprod_A \{q\}$, formed in the category of simplicial sets. We regard B/A as a pointed simplicial set, with base point given by the vertex q. 00VK

Remark 3.2.2.2. Let B be a simplicial set and let A be a simplicial subset. Then the simplicial set B/A can be described more informally as follows: it is obtained from B by collapsing the simplicial subset $A \subseteq B$ to a single vertex q. Beware that this informal description is a bit misleading when $A = \emptyset$: in this case, the natural map $B \to B/A$ is not surjective (instead, B/A can be described as the coproduct $B_+ = B \coprod \{q\}$, obtained from B by adding a new base point). 00VL

Example 3.2.2.3. For $n \geq 0$, the geometric realization $|\Delta^n/\partial\Delta^n|$ can be obtained from the topological n-simplex $|\Delta^n|$ by collapsing the boundary $|\partial\Delta^n|$ to a point (or by adding a new base point, in the degenerate case $n = 0$). It follows that $|\Delta^n/\partial\Delta^n|$ is homeomorphic to a sphere of dimension n. 00VM

00VN **Construction 3.2.2.4.** Let (X, x) be a pointed Kan complex and let n be a nonnegative integer. We let $\pi_n(X, x)$ denote the set $[\Delta^n / \partial \Delta^n, X]_*$ of pointed homotopy classes of maps from $\Delta^n / \partial \Delta^n$ to X (Notation 3.2.1.10). For $n > 0$, we will refer to $\pi_n(X, x)$ as the *nth homotopy group of X with respect to the base point x* (see Theorem 3.2.2.10 below). In the special case $n = 1$, we refer to $\pi_1(X, x)$ as the *fundamental group of X with respect to the base point x*.

00VP **Notation 3.2.2.5.** Let (X, x) be a pointed Kan complex and let n be a nonnegative integer. Then the set of pointed morphisms $\Delta^n / \partial \Delta^n \to X$ can be identified with the set of n-simplices $\sigma : \Delta^n \to X$ having the property that $\sigma|_{\partial \Delta^n}$ is equal to the constant map $\partial \Delta^n \to \{x\} \subseteq X$. In this case, we write $[\sigma]$ for the image of σ in the set $\pi_n(X, x)$. Note that, if τ is another n-simplex of X for which $\tau|_{\partial \Delta^n}$ is the constant map $\partial \Delta^n \to \{x\} \subseteq X$, then the equality $[\sigma] = [\tau]$ holds in $\pi_n(X, x)$ if and only if there exists a homotopy $h : \Delta^1 \times \Delta^n \to X$ such that $\sigma = h|_{\{0\} \times \Delta^n}$, $\tau = h|_{\{1\} \times \Delta^n}$, and $h|_{\Delta^1 \times \partial \Delta^n}$ is the constant map taking the value x.

00VQ **Example 3.2.2.6.** Let (X, x) be a pointed Kan complex. Then $\pi_0(X, x)$ can be identified with the set $\pi_0(X)$ of connected components of X (Definition 1.2.1.8). Beware that, unlike the higher homotopy groups $\{\pi_n(X, x)\}_{n \geq 1}$, there is no naturally defined group structure on $\pi_0(X, x)$.

00VR **Example 3.2.2.7.** Let X be a topological space and let $x \in X$ be a base point, which we identify with a vertex of the singular simplicial set $\mathrm{Sing}_\bullet(X)$. For every positive integer n, we can identify $\pi_n(\mathrm{Sing}_\bullet(X), x)$ with the set $\pi_n(X, x)$ of (pointed) homotopy classes of maps from the sphere $S^n \simeq |\Delta^n / \partial \Delta^n|$ into X.

00VS **Example 3.2.2.8.** Let X be a Kan complex, let x be a vertex of X, and let $e, e' : x \to x$ be edges of X which begin and end at the vertex x. Then the equality $[e] = [e']$ holds in the fundamental group $\pi_1(X, x)$ if and only if e is homotopic to e' as a morphism in the ∞-category X (in the sense of Definition 1.4.3.1); see Corollary 1.4.3.7.

00VT **Remark 3.2.2.9.** Let n be a nonnegative integer. By virtue of Corollary 3.1.7.2, there exists an anodyne morphism $f : \Delta^n / \partial \Delta^n \to Q$, where Q is a Kan complex. Let $q \in Q$ denote the image of the base point of $\Delta^n / \partial \Delta^n$. If (X, x) is a pointed Kan complex, then precomposition with f induces a trivial Kan fibration $\mathrm{Fun}(Q, X) \to \mathrm{Fun}(\Delta^n / \partial \Delta^n, X)$ (Theorem 3.1.3.5), hence also a trivial Kan fibration

$$\mathrm{Fun}(Q, X) \times_{\mathrm{Fun}(\{q\}, X)} \{x\} \to \mathrm{Fun}(\Delta^n / \partial \Delta^n, X) \times_{\mathrm{Fun}(\{q_0\}, X)} \{x\}.$$

Passing to connected components, we see that f induces a bijection $\mathrm{Hom}_{\mathrm{hKan}_*}(Q, X) \simeq \pi_n(X, x)$. In other words, the functor $(X, x) \mapsto \pi_n(X, x)$ is corepresentable (in the pointed homotopy category hKan_*) by the pointed Kan complex (Q, q) (which can be regarded as a combinatorial incarnation of the n-sphere).

Theorem 3.2.2.10. *Let (X, x) be a pointed Kan complex and let n be a positive integer. Then there is a unique group structure on the set $\pi_n(X, x)$ with the following properties:* `00VU`

- (a) *Let $e : \Delta^n \to \{x\} \to X$ be the constant map. Then the homotopy class $[e]$ is the identity element of $\pi_n(X, x)$.*
- (b) *Let $f : \partial \Delta^{n+1} \to X$ be a morphism of simplicial sets, corresponding to a tuple $(\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ of n-simplices of X (see Proposition 1.1.4.13). Assume that each restriction $\sigma_i|_{\partial \Delta^n}$ is equal to the constant map $\partial \Delta^n \to \{x\} \subseteq X$. Then f extends to a map $\Delta^{n+1} \to X$ if and only if the product*

$$[\sigma_0]^{-1}[\sigma_1][\sigma_2]^{-1}[\sigma_3] \cdots [\sigma_{n+1}]^{(-1)^n}$$

is equal to the identity element of $\pi_n(X, x)$.

Moreover, if $n \geq 2$, then the group $\pi_n(X, x)$ is abelian.

We will give the proof of Theorem 3.2.2.10 in §3.2.3.

Exercise 3.2.2.11. Show that when $n > 0$ is odd, condition (a) of Theorem 3.2.2.10 follows from condition (b) (beware that this is not true when n is even). `00VV`

Example 3.2.2.12. In the special case $n = 1$, we can rewrite condition (b) of Theorem 3.2.2.10 as follows: `00VW`

- Let f, g, and h be edges of X which begin and end at the vertex x. Then the equality $[h] = [g][f]$ holds (in the fundamental group $\pi_1(X, x)$) if and only if there exists a 2-simplex σ of X which witnesses h as a composition of f and g (in the sense of Definition 1.4.4.1), as indicated in the diagram

$$\begin{tikzcd}
 & x \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"] & & x.
\end{tikzcd}$$

It follows that the fundamental group $\pi_1(X, x)$ can be identified with the automorphism group of x as an object of the fundamental groupoid $\pi_{\leq 1}(X) = \mathrm{h}X$.

Example 3.2.2.13. Let $\mathcal{G}$ be a groupoid and let x be an object of $\mathcal{G}$, which we identify with a vertex of the Kan complex $X = \mathrm{N}_\bullet(\mathcal{G})$ (see Proposition 1.3.5.2). Then: `050R`

- The set $\pi_0(X) = \pi_0(X, x)$ can be identified with the collection of isomorphism classes of objects of $\mathcal{G}$.

- The fundamental group $\pi_1(X, x)$ can be identified with the automorphism group $\mathrm{Aut}_{\mathcal{G}}(x)$ of x as an object of $\mathcal{G}$.
- The homotopy groups $\pi_n(X, x)$ are trivial for $n \geq 2$, since an n-simplex $\sigma : \Delta^n \to X$ is determined by the restriction $\sigma|_{\partial \Delta^n}$ (see Exercise 1.3.1.5).

00VX **Warning 3.2.2.14.** Let (X, x) be a pointed Kan complex, so that $\pi_1(X, x)$ can be identified with the set $\mathrm{Hom}_{\pi_{\leq 1}(X)}(x, x)$ of homotopy classes of paths from x to itself. We have adopted the convention that the multiplication on $\pi_1(X, x)$ is given by composition in the homotopy category $\mathrm{h}X$. In other words, if $f, g : x \to x$ are edges which begin and end at x, then the product $[g][f] \in \pi_1(X, x)$ is the homotopy class of a path which can be described informally as traversing the path f first, followed by the path g. Beware that the opposite convention is also common in the literature (note that his issue is irrelevant for the higher homotopy groups $\{\pi_n(X, x)\}_{n \geq 2}$, since they are abelian).

00VY **Remark 3.2.2.15.** Let (X, x) be a pointed Kan complex. For $n \geq 2$, the homotopy group $\pi_n(X, x)$ is abelian. We will generally emphasize this by using additive notation for the group structure on $\pi_n(X, x)$: that is, we denote the group law by

$$+ : \pi_n(X, x) \times \pi_n(X, x) \to \pi_n(X, x) \qquad (\xi, \xi') \mapsto \xi + \xi'.$$

With this convention, we can restate property (b) of Theorem 3.2.2.10 as follows:

(b) Let $f : \partial \Delta^{n+1} \to X$ be a morphism of simplicial sets, corresponding to a tuple $(\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ of n-simplices of X. Then f extends to an $(n+1)$-simplex of X if and only if the sum $\sum_{i=0}^{n+1} (-1)^i [\sigma_i]$ vanishes in $\pi_n(X, x)$.

00VZ **Remark 3.2.2.16** (Functoriality)**.** Let $f : X \to Y$ be a morphism of Kan complexes, let x be a vertex of X, and set $y = f(x)$. For each $n \geq 1$, the morphism f induces a homomorphism $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y)$, characterized by the formula $\pi_n(f)([\sigma]) = [f(\sigma)]$ for each n-simplex σ of X for which $\sigma|_{\partial \Delta^n}$ is the constant map $\partial \Delta^n \to \{x\} \hookrightarrow X$. We can therefore regard the construction $(X, x) \mapsto \pi_n(X, x)$ as a functor from the category of pointed Kan complexes to the category of groups. Moreover, this functor preserves filtered colimits.

04GE **Remark 3.2.2.17** (Homotopy Invariance)**.** In the situation of Remark 3.2.2.16, suppose that $f : X \to Y$ is a homotopy equivalence. It follows from Proposition 3.2.1.13 that the homotopy class $[f]$ determines an isomorphism from (X, x) to (Y, y) in the pointed homotopy category hKan_*. In particular, the induced map $\pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism of groups for all $n > 0$ (and a bijection of sets for $n = 0$).

04GF **Example 3.2.2.18** (Independence of Base Point)**.** Let X be a Kan complex and let $e : x \to y$ be an edge of X. Then evaluation at the vertices $0, 1 \in \Delta^1$ determines a diagram of pointed

Kan complexes $(X, x) \xleftarrow{\mathrm{ev}_0} (\mathrm{Fun}(\Delta^1, X), e) \xrightarrow{\mathrm{ev}_1} (X, y)$, where the underlying maps are trivial Kan fibrations (Corollary 3.1.3.6). Applying For each $n > 0$, Remark 3.2.2.17 then supplies isomorphisms of homotopy groups

$$\pi_n(X, x) \xleftarrow{\sim} \pi_n(\mathrm{Fun}(\Delta^1, X), e) \xrightarrow{\sim} \pi_n(X, y).$$

Warning 3.2.2.19. Let X be a Kan complex and let $n > 0$ be an integer. It follows from Example 3.2.2.18 that if two vertices $x, y \in X$ belong to the same connected component of X, then the homotopy groups $\pi_n(X, x)$ and $\pi_n(X, y)$ are isomorphic. Beware that, in general, there is no *canonical* isomorphism between $\pi_n(X, x)$ and $\pi_n(X, y)$: the isomorphism constructed in Example 3.2.2.18 depends on (the homotopy class) of the chosen edge $e : x \to y$. `04GG`

Remark 3.2.2.20. Let X be a Kan complex and let x be a vertex of X. Then x can also be regarded as a vertex of the opposite simplicial set X^{op}, which is also a Kan complex. For $n \geq 1$, we have an evident bijection $\varphi : \pi_n(X, x) \simeq \pi_n(X^{\mathrm{op}}, x)$. If $n \geq 2$, then this bijection is an isomorphism of abelian groups. Beware that, in the case $n = 1$, it is generally not an isomorphism of groups: instead, it is an anti-isomorphism (that is, it satisfies the identity $\varphi(\xi\xi') = \varphi(\xi')\varphi(\xi)$ for $\xi, \xi' \in \pi_1(X, x)$; see Warning 3.2.2.14 above). `00W0`

Remark 3.2.2.21. Let (X, x) be a pointed Kan complex and let n be a positive integer. Suppose that $\sigma, \sigma' : \Delta^n \to X$ are n-simplices of X for which $\sigma|_{\partial \Delta^n}$ and $\sigma'|_{\partial \Delta^n}$ are equal to the constant map $\partial \Delta^n \to \{x\} \subseteq X$. It follows from Theorem 3.2.2.10 that the equality $[\sigma] = [\sigma']$ holds (in the homotopy group $\pi_n(X, x)$) if and only if there exists an $(n+1)$-simplex τ of X such that $d_0^{n+1}(\tau) = \sigma$, $d_1^{n+1}(\tau) = \sigma'$, and $d_i^{n+1}(\tau)$ is the constant map $\Delta^n \to \{x\} \subseteq X$ for $2 \leq i \leq n+1$. `00W1`

Exercise 3.2.2.22 (Homotopy of Eilenberg-MacLane Spaces)**.** Let M_* be a chain complex of abelian groups and let $X = \mathrm{K}(M_*)$ be the associated Eilenberg-MacLane space (Construction 2.5.6.3). Let $x \in X$ be the vertex corresponding to the zero element, and let n be a positive integer. Note that a pointed map from $\Delta^n / \partial \Delta^n$ to X can be identified with a map of chain complexes $\mathrm{N}_*(\Delta^n, \partial \Delta^n; \mathbf{Z}) \simeq \mathbf{Z}[n] \to M_*$: in other words, it can be identified with an n-cycle of the chain complex M_*, which we will denote by $\overline{\sigma}$. `00W2`

(1) Let $\sigma, \sigma' : \Delta^n \to X$ be n-simplices whose restriction to $\partial \Delta^n$ is equal to the constant map $\partial \Delta^n \to \{x\} \hookrightarrow X$. Show that $[\sigma] = [\sigma']$ in $\pi_n(X, x)$ if and only if $\overline{\sigma}$ and $\overline{\sigma}'$ are homologous as n-cycles of M_* (use Remark 3.2.2.21).

(2) Show that the $[\sigma] \mapsto [\overline{\sigma}]$ induces an isomorphism from $\pi_n(X, x)$ to the homology group $\mathrm{H}_n(M)$.

In particular, if A is an abelian group and $m \geq 0$ is an integer, then the homotopy groups of the Eilenberg-MacLane space $X = \mathrm{K}(A, m)$ are given by

$$\pi_n(X, x) = \begin{cases} A & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

3.2.3 The Group Structure on $\pi_n(X, x)$

00W3 Let (X, x) be a pointed Kan complex and let $n \geq 2$ be an integer, which we regard as fixed throughout this section. Our goal is to give a proof of Theorem 3.2.2.10, which supplies a group structure on the set $\pi_n(X, x) = [\Delta^n / \partial \Delta^n, X]_*$ (note that the case $n = 1$ of Theorem 3.2.2.10 is subsumed in our construction of the homotopy category $\pi_{\leq 1}(X) = \mathrm{h}X$, by virtue of Example 3.2.2.12).

00W4 **Notation 3.2.3.1.** Let Σ denote the collection of all n-simplices $\sigma : \Delta^n \to X$ having the property that the restriction $\sigma|_{\partial \Delta^n}$ is equal to the constant map $\partial \Delta^n \to \{x\} \subseteq X$. We let $e \in \Sigma$ denote the constant map $\Delta^n \to \{x\} \subseteq X$. Note that an $(n+2)$-tuple $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ of elements of Σ can be identified with a map of simplicial sets $f : \partial \Delta^{n+1} \to X$, having the property that the restriction of f to the $(n-1)$-skeleton of $\partial \Delta^{n+1}$ is equal to the constant map $\mathrm{sk}_{n-1}(\partial \Delta^{n+1}) \to \{x\} \subseteq X$ (see Proposition 1.1.4.13). We will say that a tuple $\vec{\sigma}$ *bounds* if f can be extended to an $(n+1)$-simplex of X: that is, if there exists an $(n+1)$-simplex τ of X satisfying $\sigma_i = d_i^{n+1}(\tau)$ for $0 \leq i \leq n+1$.

The construction $\sigma \mapsto [\sigma]$ determines a surjective map $\Sigma \twoheadrightarrow \pi_n(X, x)$. We will say that a pair of elements $\sigma, \sigma' \in \Sigma$ are *homotopic* if $[\sigma] = [\sigma']$ (that is, if there is a homotopy from σ to σ' which is constant along the boundary $\partial \Delta^n$).

00W5 **Lemma 3.2.3.2.** *Let $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ be an $(n+2)$-tuple of elements of Σ. The condition that $\vec{\sigma}$ bounds depends only on the sequence of homotopy classes $\{[\sigma_i] \in \pi_n(X, x)\}_{0 \leq i \leq n+1}$. In other words, if $\vec{\sigma}' = (\sigma'_0, \sigma'_1, \ldots, \sigma'_{n+1})$ is another $(n+2)$-tuple of elements of Σ satisfying $[\sigma'_i] = [\sigma_i]$ for $0 \leq i \leq n+1$ and $\vec{\sigma}$ bounds, then $\vec{\sigma}'$ also bounds.*

Proof. Let us identify $\vec{\sigma}$ and $\vec{\sigma}'$ with morphisms of simplicial sets $f, f' : \partial \Delta^{n+1} \to X$ (carrying the $(n-1)$-skeleton of $\partial \Delta^{n+1}$ to the vertex x). For $0 \leq i \leq n+1$, the equality $[\sigma_i] = [\sigma'_i]$ allows us choose a homotopy $h_i : \Delta^1 \times \Delta^n \to X$ from σ_i to σ'_i which carries $\Delta^1 \times \partial \Delta^n$ to the vertex $\{x\} \subseteq X$. These maps can be amalgamated to a homotopy h from f to f': that is, an edge joining f to f' in the simplicial set $\mathrm{Fun}(\partial \Delta^{n+1}, X)$. If $\vec{\sigma}$ bounds, then f can be extended to an $(n+1)$-simplex $\tau : \Delta^{n+1} \to X$. Since X is a Kan complex, the restriction map $\mathrm{Fun}(\Delta^{n+1}, X) \to \mathrm{Fun}(\partial \Delta^{n+1}, X)$ is a Kan fibration (Corollary 3.1.3.3), so h can be extended to a homotopy $\widetilde{h}$ from τ to another map $\tau' : \Delta^{n+1} \to X$ satisfying $\tau'|_{\partial \Delta^{n+1}} = f'$. It follows that the tuple $\vec{\sigma}'$ also bounds. $\square$

Remark 3.2.3.3. Let $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})$ be an $(n+2)$-tuple of elements of $\pi_n(X, x)$, so that we can write $\eta_i = [\sigma_i]$ for some n-simplex $\sigma_i \in \Sigma$. We will say that the tuple of homotopy classes $\vec{\eta}$ *bounds* if the tuple of simplices $\vec{\sigma} = (\sigma_0, \sigma_1, \ldots, \sigma_{n+1})$ bounds, in the sense of Notation 3.2.3.1. By virtue of Lemma 3.2.3.2, this condition is independent of the choice of $\vec{\sigma}$. `00W6`

With this terminology, Theorem 3.2.2.10 asserts (in the case $n \geq 2$) that there is a unique abelian group structure on the set $\pi_n(X, x)$ with the following pair of properties:

(a) The identity element of $\pi_n(X, x)$ is the homotopy class $[e]$.

(b) An $(n+2)$-tuple $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})$ bounds if and only if the sum $\sum_{i=0}^{n+1} (-1)^i \eta_i$ vanishes in $\pi_n(X, x)$.

Lemma 3.2.3.4. *Let $0 \leq i \leq n+1$, and suppose we are given a collection of homotopy classes $\{\eta_j \in \pi_n(X, x)\}_{0 \leq j \leq n+1, j \neq i}$. Then there is a unique element $\eta_i \in \pi_n(X, x)$ for which the tuple $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})$ bounds.* `00W7`

Proof. For $j \neq i$, choose an element $\sigma_j \in \Sigma$ satisfying $[\sigma_j] = \eta_j$. Then the tuple of n-simplices $(\sigma_0, \ldots, \sigma_{i-1}, \bullet, \sigma_{i+1}, \ldots, \sigma_{n+1})$ determines a map of simplicial sets $f_0 : \Lambda^{n+1}_i \to X$ (see Proposition 1.2.4.7). Since X is a Kan complex, we can extend f_0 to an $(n+1)$-simplex τ of X. Then $\eta_i = [d^{n+1}_i(\tau)]$ has the property that the tuple $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})$ bounds. This proves existence. To prove uniqueness, suppose we are given another element $\eta'_i \in \pi_n(X, x)$ for which the tuple $(\eta_0, \ldots, \eta_{i-1}, \eta'_i, \eta_{i+1}, \ldots, \eta_{n+1})$ bounds. Write $\eta'_i = [\sigma'_i]$ for some $\sigma'_i \in \Sigma$, so that we can choose a simplex $\tau' : \Delta^{n+1} \to X$ satisfying

$$d^{n+1}_j(\tau') = \begin{cases} \sigma'_i & \text{if } j = i \\ \sigma_j & \text{otherwise.} \end{cases}$$

Since the inclusion $\Lambda^{n+1}_i \hookrightarrow \Delta^{n+1}$ is anodyne, so the restriction map $\operatorname{Fun}(\Delta^{n+1}, X) \to \operatorname{Fun}(\Lambda^{n+1}_i, X)$ is a trivial Kan fibration (Corollary 3.1.3.6). It follows that there exists a homotopy from τ to τ' which is constant along the subset $\Lambda^{n+1}_i \subseteq \Delta^{n+1}$, so that $\eta_i = [d^{n+1}_i(\tau)] = [d^{n+1}_i(\tau')] = \eta'_i$. □

As a special case of Lemma 3.2.3.4, we obtain several potential candidates for the composition law on $\pi_n(X, x)$:

Lemma 3.2.3.5. *Fix $1 \leq i \leq n$. Then there is a unique function $m_i : \pi_n(X, x) \times \pi_n(X, x) \to \pi_n(X, x)$ with the following property:* `00W8`

$(*)$ *Let η_{i-1}, η_i, and η_{i+1} be elements of $\pi_n(X, x)$. Then the $(n+2)$-tuple*

$$([e], \ldots, [e], \eta_{i-1}, \eta_i, \eta_{i+1}, [e], \ldots, [e])$$

bounds if and only if $\eta_i = m_i(\eta_{i-1}, \eta_{i+1})$.

00W9 **Example 3.2.3.6.** Let σ be an element of Σ, and let $1 \leq i \leq n$. Then the degenerate $(n+1)$-simplex $\tau = s_i^n(\sigma)$ satisfies $d_j^{n+1}(\tau) = \begin{cases} \sigma & \text{if } j \in \{i, i+1\} \\ e & \text{otherwise.} \end{cases}$ It follows that the multiplication map $m_i : \pi_n(X,x) \times \pi_n(X,x) \to \pi_n(X,x)$ of Lemma 3.2.3.5 satisfies the identity $m_i([e],[\sigma]) = [\sigma]$. A similar argument shows that $m_i([\sigma],[e]) = [\sigma]$.

00WA **Lemma 3.2.3.7.** *Let $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})$ be an $(n+2)$-tuple of elements of $\pi_n(X,x)$, let $1 \leq i \leq n$ be an integer, and let α be another element of $\pi_n(X,x)$. If $\vec{\eta}$ bounds, then the tuple $(\eta_0, \ldots, \eta_{i-2}, m_i(\alpha, \eta_{i-1}), m_i(\alpha, \eta_i), \eta_{i+1}, \ldots, \eta_{n+1})$ also bounds.*

Proof. For $0 \leq i \leq n+1$, choose an element $\sigma_i \in \Sigma$ satisfying $[\sigma_i] = \eta_i$. Since $\vec{\eta}$ bounds, we can choose an $(n+1)$-simplex $\overline{\sigma}$ of X satisfying $\sigma_i = d_i^{n+1}(\overline{\sigma})$ for $0 \leq i \leq n+1$. Choose $\tau \in \Sigma$ satisfying $[\tau] = \alpha$. Since X is a Kan complex, we can choose $(n+1)$-simplices $\rho, \rho' : \Delta^{n+1} \to X$ satisfying the identities

$$d_j^{n+1}(\rho) = \begin{cases} e & \text{if } 0 \leq j < i-1 \\ \tau & \text{if } j = i-1 \\ \sigma_{i-1} & \text{if } j = i+1 \\ e & \text{of } i+1 < j \leq n+1. \end{cases} \qquad d_j^{n+1}(\rho') = \begin{cases} e & \text{if } 0 \leq j < i-1 \\ \tau & \text{if } j = i-1 \\ \sigma_i & \text{if } j = i+1 \\ e & \text{of } i+1 < j \leq n+1. \end{cases}$$

The definition of m_i supplies identities $m_i(\alpha, \eta_{i-1}) = [d_i^{n+1}(\rho)]$ and $m_i(\alpha, \eta_i) = [d_i^{n+1}(\rho')]$. The tuple $(s_i^n(\sigma_0), \ldots, s_i^n(\sigma_{i-2}), \rho, \rho', \bullet, \overline{\sigma}, s_{i+1}^n(\sigma_{i+2}), \ldots, s_{i+1}^n(\sigma_{n+1}))$ therefore determines a map of simplicial sets $\Lambda_{i+1}^{n+2} \to X$ (Proposition 1.2.4.7). Since X is a Kan complex, this map can be extended to an $(n+2)$-simplex of X. Let $\overline{\sigma}'$ denote the $(i+1)$st face of this simplex. By construction, we have

$$d_j^{n+1}(\overline{\sigma}') = \begin{cases} d_i^{n+1}(\rho) & \text{if } j = i-1 \\ d_i^{n+1}(\rho') & \text{if } j = i \\ \sigma_j & \text{otherwise,} \end{cases}$$

so that $\overline{\sigma}'$ witnesses that the tuple $(\eta_0, \ldots, \eta_{i-2}, m_i(\alpha, \eta_{i-1}), m_i(\alpha, \eta_i), \eta_{i+1}, \ldots, \eta_{n+1})$ bounds. □

00WB **Lemma 3.2.3.8.** *Let α, β, and γ be elements of $\pi_n(X,x)$. For $2 \leq i \leq n$, we have $m_i(\alpha, m_{i-1}(\beta, \gamma)) = m_{i-1}(\beta, m_i(\alpha, \gamma))$.*

Proof. Applying Lemma 3.2.3.7 to the tuple $([e], \ldots, [e], \beta, m_{i-1}(\beta, \gamma), \gamma, [e], \ldots, [e])$, we deduce that the tuple $([e], \ldots, [e], \beta, m_i(\alpha, m_{i-1}(\beta, \gamma)), m_i(\alpha, \gamma), [e], \ldots, [e])$ bounds, which is equivalent to the asserted identity. □

Lemma 3.2.3.9. *Let α and β be elements of $\pi_n(X,x)$. For $2 \leq i \leq n$, we have $m_i(\alpha,\beta) = m_{i-1}(\beta,\alpha)$.* `00WC`

Proof. Combining Lemma 3.2.3.8 with Example 3.2.3.6, we obtain

$$m_i(\alpha,\beta) = m_i(\alpha, m_{i-1}(\beta,[e])) = m_{i-1}(\beta, m_i(\alpha,[e])) = m_{i-1}(\beta,\alpha).$$

□

Proof of Theorem 3.2.2.10. For every pair of elements $\alpha, \beta \in \pi_n(X,x)$, let $\alpha\beta$ denote the homotopy class $m_1(\alpha,\beta)$, where $m_1 : \pi_n(X,x) \times \pi_n(X,x) \to \pi_n(X,x)$ is the multiplication map of Lemma 3.2.3.5. We first note that this multiplication is associative: for every triple of elements $\alpha, \beta, \gamma \in \pi_n(X,x)$, Lemmas 3.2.3.9 and 3.2.3.8 yield identities

$$\begin{aligned} \alpha(\beta\gamma) &= m_1(\alpha, m_1(\beta,\gamma)) \\ &= m_1(\alpha, m_2(\gamma,\beta)) \\ &= m_2(\gamma, m_1(\alpha,\beta)) \\ &= m_1(m_1(\alpha,\beta),\gamma) \\ &= (\alpha\beta)\gamma. \end{aligned}$$

Example 3.2.3.6 shows that $[e]$ is a two-sided identity with respect to multiplication. For every element $\alpha \in \pi_n(X,x)$, Lemma 3.2.3.4 implies that we can choose an element $\beta \in \pi_n(X,x)$ for which the tuple $(\alpha, [e], \beta, [e], [e], \ldots, [e])$ bounds, so that $\alpha\beta = m_1(\alpha,\beta) = [e]$. This shows that α has a right inverse, and a similar argument shows that α has a left inverse. It follows that multiplication determines a group structure on the set $\pi_n(X,x)$, having $[e]$ as the identity element.

We now verify that the multiplication on $\pi_n(X,x)$ satisfies condition (b) of Theorem 3.2.2.10. Suppose we are given an $(n+1)$-tuple $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n+1})$ of elements of $\pi_n(X,x)$. We wish to show that $\vec{\eta}$ bounds if and only if the product $\eta_0^{-1}\eta_1\eta_2^{-1}\cdots\eta_{n+1}^{(-1)^n}$ is equal to the identity element of $\pi_n(X,x)$. If $\vec{\eta} = ([e],[e],\ldots,[e])$, there is nothing to prove. Otherwise, there exists some smallest positive integer i such that $\eta_{i-1} \neq [e]$. We proceed by descending induction on i. If $i > n$, we must show that $([e],[e],\ldots,[e],\eta_n,\eta_{n+1})$ bounds if and only if $\eta_n = \eta_{n+1}$, which follows from Example 3.2.3.6. Let us therefore assume that $1 \leq i \leq n$. Define $\vec{\eta}' = (\eta'_0, \eta'_1, \ldots, \eta'_{n+1})$ by the formula

$$\eta'_j = \begin{cases} m_i(\eta_{i-1}^{-1}, \eta_j) & \text{if } j = i-1 \text{ or } j = i \\ \eta_j & \text{otherwise.} \end{cases}$$

Invoking Lemma 3.2.3.9 repeatedly, we obtain

$$\eta'_{i-1} = m_i(\eta_{i-1}^{-1}, \eta_{i-1}) = \begin{cases} \eta_{i-1}^{-1}\eta_{i-1} & \text{if } i \text{ is odd} \\ \eta_{i-1}\eta_{i-1}^{-1} & \text{if } i \text{ is even} \end{cases} = [e]$$

$$\eta_i' = m_i(\eta_{i-1}^{-1}, \eta_i) = \begin{cases} \eta_{i-1}^{-1}\eta_i & \text{if } i \text{ is odd} \\ \eta_i \eta_{i-1}^{-1} & \text{if } i \text{ is even} \end{cases}.$$

We therefore have an equality

$$\eta_0^{-1}\eta_1\eta_2^{-1}\cdots\eta_{n+1}^{(-1)^n} = \eta_0'^{-1}\eta_1'\eta_2'^{-1}\cdots\eta_{n+1}'^{(-1)^n}.$$

Invoking our inductive hypothesis, we conclude that this product vanishes if and only if the tuple $\vec{\eta}'$ bounds. By virtue of Lemma 3.2.3.7, this is equivalent to the assertion that $\vec{\eta}$ bounds.

We now complete the proof of Theorem 3.2.2.10 by showing that the multiplication on $\pi_n(X, x)$ is commutative. Fix a pair of elements $\sigma, \sigma' \in \Sigma$. Then the tuples of n-simplices $(\sigma, e, \sigma', \bullet, e, e, \ldots, e)$ and $(\sigma', e, \sigma, \bullet, e, e, \ldots, e)$ determine maps of simplicial sets $f, f' : \Lambda^{n+1}_3 \to X$ (Proposition 1.2.4.7). Since X is a Kan complex, we can extend f and f' to $(n+1)$-simplices of X, which we will denote by τ and τ', respectively. It follows from the preceding arguments that the faces $d^{n+1}_3(\tau)$ and $d^{n+1}_3(\tau')$ are representatives of the products $[\sigma'][\sigma]$ and $[\sigma][\sigma']$ in $\pi_n(X, x)$, respectively. Let $\overline{e} : \Delta^{n+1} \to X$ denote the constant map taking the value x. Then the tuple of $(n+1)$-simplices $(\tau, s^n_0(\sigma), s^n_1(\sigma), \tau', \bullet, \overline{e}, \overline{e}, \ldots, \overline{e})$ determines a map of simplicial sets $g : \Lambda^{n+2}_4 \to X$ (Proposition 1.2.4.7). Since X is a Kan complex, we can extend g to an $(n+2)$-simplex of X. Then the fourth face of this extension witnesses that the tuple of n-simplices $(d^{n+1}_3(\tau), e, e, d^{n+1}_3(\tau'), e, \ldots, e)$ bounds, so that we have an equality $[\sigma'][\sigma] = [d^{n+1}_3(\tau)] = [d^{n+1}_3(\tau')] = [\sigma][\sigma']$ in the homotopy group $\pi_n(X, x)$. □

3.2.4 Contractibility

02L4 We now study the class of *contractible* simplicial sets.

04GH **Definition 3.2.4.1.** Let X be a simplicial set. We say that X is *contractible* if the projection map $X \to \Delta^0$ is a homotopy equivalence (Definition 3.1.6.1).

00XA **Example 3.2.4.2.** Let $\mathcal{C}$ be a category. If $\mathcal{C}$ has an initial object or a final object, then the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ is contractible (this is a special case of Proposition 3.1.6.9). In particular, for every integer $n \geq 0$, the standard simplex Δ^n is contractible.

Though the condition of contractibility makes sense for any simplicial set X, we will be primarily interested in the special case where X is a Kan complex. In this case, Definition 3.2.4.1 agrees with Definition 1.5.5.8:

04GJ **Theorem 3.2.4.3.** *Let X be a Kan complex. The following conditions are equivalent:*

(1) *The Kan complex X is contractible.*

(2) *The Kan complex X is connected and the homotopy groups $\pi_n(X, x)$ vanish for each $n > 0$ and every choice of base point $x \in X$.*

(3) *The projection map $X \to \Delta^0$ is a trivial Kan fibration of simplicial sets.*

Remark 3.2.4.4. In the formulation of Theorem 3.2.4.3, we can replace (2) by the following *a priori* weaker condition: 04GK

(2′) The Kan complex X is connected and, for some choice of base point $x \in X$, the homotopy groups $\pi_n(X, x)$ vanish for each $n > 0$.

See Example 3.2.2.18.

For the proof of Theorem 3.2.4.3, it will be convenient to introduce some terminology.

Definition 3.2.4.5. Let $f : X \to Y$ be a morphism of simplicial sets. We will say that f is *nullhomotopic* if there exists a vertex $y \in Y$ for which f is homotopic to the constant morphism $X \to \{y\} \hookrightarrow Y$. 02L7

Example 3.2.4.6. Let X be a simplicial set, and let $\emptyset$ denote the empty simplicial set. Then there is a unique morphism of simplicial sets $\emptyset \hookrightarrow X$, which is nullhomotopic if and only if X is nonempty (note that, by the convention of Definition 3.2.4.5, the identity map $\emptyset \to \emptyset$ is *not* considered to be nullhomotopic). 02L8

Example 3.2.4.7. Let (X, x) be a pointed Kan complex, let $n > 0$ be a positive integer, and let $\sigma : \Delta^n / \partial \Delta^n \to (X, x)$ be a morphism of pointed simplicial sets. Then σ is nullhomotopic (in the sense of Definition 3.2.4.5) if and only if the pointed homotopy class $[\sigma]$ is equal to the identity element in the homotopy group $\pi_n(X, x)$. See Proposition 3.2.1.15. 02LE

Exercise 3.2.4.8. Let X be a simplicial set. Show that the following conditions are equivalent: 050S

(1) The simplicial set X is contractible: that is, the projection map $X \to \Delta^0$ is a homotopy equivalence (Definition 3.2.4.1).

(2) The identity morphism $\mathrm{id}_X : X \to X$ is nullhomotopic.

(3) Every morphism of simplicial sets $f : X \to Y$ is nullhomotopic.

(4) Every morphism of simplicial sets $g : Z \to X$ is nullhomotopic.

In particular, these conditions are satisfied in the special case where $X = \Delta^n$ is a standard simplex.

050T **Remark 3.2.4.9.** Let $f : X \to Y$ be a morphism of simplicial sets, and let $f' : X \to Y$ be a morphism which is homotopic to f. Then f is nullhomotopic if and only if f' is nullhomotopic.

02L9 **Remark 3.2.4.10.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets. If either f or g is nullhomotopic, then the composition $g \circ f$ is nullhomotopic.

050V **Lemma 3.2.4.11.** *Let X be a Kan complex and let $0 \leq i \leq n$ be integers with $n > 0$. Then every morphism $\sigma_0 : \Lambda^n_i \to X$ is nullhomotopic.*

Proof. Since X is a Kan complex, we can extend σ_0 to an n-simplex $\sigma : \Delta^n \to X$. By virtue of Remark 3.2.4.10, it will suffice to show that σ is nullhomotopic, which is a special case of Exercise 3.2.4.8. □

050W **Variant 3.2.4.12.** Let X be a Kan complex and let $n \geq 0$ be an integer. Then a morphism of simplicial sets $\sigma_0 : \partial \Delta^n \to X$ is nullhomotopic if and only if it can be extended to an n-simplex of X. The "if" direction follows immediately from Exercise 3.2.4.8 (and does not require the assumption that X is a Kan complex). For the converse, suppose that σ_0 is homotopic to a constant map $\sigma'_0 : \partial \Delta^n \to \{x\} \hookrightarrow X$. Since σ'_0 can be extended to a map $\sigma' : \Delta^n \to \{x\} \hookrightarrow X$, it follows from the homotopy extension lifting property (Remark 3.1.5.3) that σ_0 can also be extended to an n-simplex of X.

050X **Lemma 3.2.4.13.** *Let X be a Kan complex and let $n \geq 2$ be an integer. The following conditions are equivalent:*

- (a_n) *Every morphism $\partial \Delta^n \to X$ can be extended to an n-simplex of X (that is, it is nullhomotopic).*
- (b_n) *For every vertex $x \in X$, the homotopy group $\pi_{n-1}(X, x)$ is trivial.*

Proof. We first show that (a_n) implies (b_n). Fix a vertex $x \in X$ and an $(n-1)$-simplex $\sigma : \Delta^{n-1} \to X$ such that $\sigma|_{\partial \Delta^{n-1}}$ is the constant map taking the value x. Amalgamating σ with the constant map $\Lambda^n_n \to X$, we obtain a morphism $\tau : \partial \Delta^n \to X$. If condition (1) is satisfied, then τ can be extended to an n-simplex of X. Theorem 3.2.2.10 then guarantees that the (pointed) homotopy class $[\sigma]$ is the identity element of the group $\pi_{n-1}(X, x)$.

We now show that (b_n) implies (a_n). Let $\tau : \partial \Delta^n \to X$ be any morphism of simplicial sets. Using Lemma 3.2.4.11, we see that the restriction $\tau|_{\Lambda^n_n}$ is nullhomotopic. Applying the homotopy lifting property (Remark 3.1.5.3), we conclude that τ is homotopic to a morphism $\tau' : \partial \Delta^n \to X$ for which $\tau'|_{\Lambda^n_n}$ is the constant map taking the value x, for some vertex $x \in X$. In particular, τ' is constant when restricted to the $(n-2)$-skeleton of Δ^n. If the homotopy group $\pi_{n-1}(X, x)$ is trivial, then Theorem 3.2.2.10 guarantees that τ' can be extended to an n-simplex of X. Applying Remark 3.1.5.3 again, we conclude that τ can also be extended to an n-simplex of X. □

050Y **Variant 3.2.4.14.** In the situation of Lemma 3.2.4.13, the extension condition (a_n) also makes sense for $n = 0$ and $n = 1$. Here condition (a_0) is equivalent to the requirement that $\pi_0(X)$ has at least one element (that is, X is nonempty), and condition (a_1) is equivalent to the requirement that $\pi_0(X)$ has at most one element (that is, X is either empty or connected). In particular, X satisfies conditions (a_0) and (a_1) if and only if it is connected.

Proof of Theorem 3.2.4.3. Let X be a Kan complex. We wish to show that the following conditions are equivalent:

(1) The Kan complex X is contractible: that is, the projection map $X \to \Delta^0$ is a homotopy equivalence.

(2) The Kan complex X is connected and the homotopy groups $\pi_n(X, x)$ vanish for every integer $n > 0$ and every vertex $x \in X$.

(3) The projection map $X \to \Delta^0$ is a trivial Kan fibration: that is, every morphism of simplicial sets $\partial \Delta^m \to X$ can be extended to an m-simplex of X.

The implication (1) ⇒ (2) follows from Remark 3.2.2.17, and the implication (3) ⇒ (1) is a special case of Proposition 3.1.6.10. The equivalence of (2) and (3) follows from Lemma 3.2.4.13 and Variant 3.2.4.14. □

When working with simplicial sets which are not Kan complexes, it will generally be convenient to work with the following variant of Definition 3.2.4.1.

04GW **Definition 3.2.4.15.** Let X be a simplicial set. We say that X is *weakly contractible* if the projection map $X \to \Delta^0$ is a weak homotopy equivalence (Definition 3.1.6.12).

00WZ **Remark 3.2.4.16.** Let X be a simplicial set. If X is contractible, then it is weakly contractible. The converse holds if X is a Kan complex (Proposition 3.1.6.13). Beware that the converse is false in general (Exercise 3.1.6.21).

02L5 **Remark 3.2.4.17.** Let $f : X \to Y$ be a weak homotopy equivalence of simplicial sets. Then X is weakly contractible if and only if Y is weakly contractible (see Remark 3.1.6.16). If f is a homotopy equivalence, then X is contractible if and only if Y is contractible (see Remark 3.1.6.7).

050U **Remark 3.2.4.18.** Let X be a simplicial set. Then X is weakly contractible if and only if, for every Kan complex Y, every morphism of simplicial sets $f : X \to Y$ is nullhomotopic. To prove this, we may assume that X is nonempty (otherwise the identity morphism id_X is not nullhomotopic; see Example 3.2.4.6). Then, for any Kan complex Y, the diagonal map $\delta_Y : Y \to \mathrm{Fun}(X, Y)$ admits a left inverse (given by evaluation at any vertex $x \in X$), and is automatically injective on connected components. It follows that X is weakly contractible if and only if, for every Kan complex Y, the morphism δ_Y is also surjective on connected components: that is, every morphism $f : X \to Y$ is homotopic to a constant map.

02L6 **Example 3.2.4.19.** Let n be a positive integer. For $0 \leq i \leq n$, the horn Λ^n_i is weakly contractible. This follows from Remark 3.2.4.17, since the inclusion map $\Lambda^n_i \hookrightarrow \Delta^n$ is a weak homotopy equivalence (Proposition 3.1.6.14) and the simplex Δ^n is contractible (Example 3.2.4.2).

3.2.5 The Connecting Homomorphism

00WD Let S be a Kan complex, and let $f : X \to S$ be a Kan fibration of simplicial sets (so that X is also a Kan complex). Fix a vertex $x \in X$, let $s = f(x)$ be its image in S, and let X_s denote the fiber $\{s\} \times_S X$ (so that X_s is also a Kan complex, and we can regard x as a vertex of X_s). In §3.2.6, we will show that the homotopy groups of X, S, and X_s are related by a long exact sequence

$$\cdots \to \pi_{n+1}(S, s) \xrightarrow{\partial} \pi_n(X_s, x) \to \pi_n(X, x) \to \pi_n(S, s) \xrightarrow{\partial} \pi_{n-1}(X_s, x) \to \cdots$$

(see Theorem 3.2.6.1 below). In this section, we set the stage by constructing the maps $\partial : \pi_{n+1}(S, s) \to \pi_n(X_s, x)$ which appear in this sequence.

00WE **Definition 3.2.5.1.** Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes and let $n \geq 0$ be a nonnegative integer. Suppose we are given a pair of maps $\sigma : \Delta^n \to X_s$ and $\tau : \Delta^{n+1} \to S$, having the property that $\sigma|_{\partial \Delta^n}$ and $\tau|_{\partial \Delta^{n+1}}$ are the constant maps taking the values x and s, respectively. We will say that σ *is incident to* τ if there exists a simplex $\widetilde{\tau} : \Delta^{n+1} \to X$ satisfying $\tau = f(\widetilde{\tau})$, $\sigma = d^{n+1}_0(\widetilde{\tau})$, and $\widetilde{\tau}|_{\Lambda^{n+1}_0} : \Lambda^{n+1}_0 \to X$ is the constant map taking the value x.

00WF **Proposition 3.2.5.2.** *Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes and let $n \geq 0$ be a nonnegative integer. Then there exists a unique function $\partial : \pi_{n+1}(S, s) \to \pi_n(X_s, x)$ with the following property:*

(∗) *Let $\sigma : \Delta^n \to X_s$ and $\tau : \Delta^{n+1} \to S$ be simplices having the property that $\sigma|_{\partial \Delta^n}$ and $\tau|_{\partial \Delta^{n+1}}$ are the constant maps taking the values x and s, respectively. Then σ is incident to τ (in the sense of Definition 3.2.5.1) if and only if $\partial([\tau]) = [\sigma]$.*

00WG **Construction 3.2.5.3** (The Connecting Homomorphism)**.** Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes. For each $n \geq 0$, we will refer to the map $\partial : \pi_{n+1}(S, s) \to \pi_n(X_s, x)$ of Proposition 3.2.5.2 as the *connecting homomorphism* (for $n \geq 1$, it is a group homomorphism: see Proposition 3.2.5.4 below).

Proof of Proposition 3.2.5.2. Let $\tau : \Delta^{n+1} \to S$ be an $(n+1)$-simplex for which $\tau|_{\partial \Delta^{n+1}}$ is the constant map taking the value s. To prove Proposition 3.2.5.2, it will suffice to prove the following:

(1) There exists an n-simplex $\sigma : \Delta^n \to X_s$ such that $\sigma|_{\partial\Delta^n}$ is the constant map taking the value x and σ is incident to τ.

(2) Let $\sigma' : \Delta^n \to X_s$ and $\tau' : \Delta^{n+1} \to S$ have the property that $\sigma'|_{\partial\Delta^n}$ and $\tau'|_{\partial\Delta^{n+1}}$ are the constant maps taking the values x and s, respectively, and suppose that $[\tau] = [\tau']$ in $\pi_{n+1}(S, s)$. Then σ' is incident to τ' if and only if $[\sigma] = [\sigma']$ in $\pi_n(X_s, x)$.

Assertion (1) follows from the solvability of the lifting problem

$$\begin{tikzcd}
\Lambda^{n+1}_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^{n+1} \arrow[r, "\tau"] \arrow[ur, dashed, "\widetilde{\tau}"] & S,
\end{tikzcd}$$

where the upper horizontal map is constant taking the value x. Let σ' and τ' be as in (2), and let $\widetilde{\tau}'_0 : \partial\Delta^{n+1} \to X_s$ be the map given by the tuple of n-simplices $(\sigma', e, \ldots, e)$ (see Proposition 1.1.4.13) where $e : \Delta^n \to X_s$ denotes the constant map taking the value x. If $[\sigma] = [\sigma']$ in $\pi_n(X_s, x)$, then we can choose a homotopy from σ to σ' (in the Kan complex X_s) which is constant along the boundary $\partial\Delta^n$, and therefore a homotopy $\widetilde{h}_0$ from $\widetilde{\tau}|_{\partial\Delta^{n+1}}$ to $\widetilde{\tau}'_0$ (also in the Kan complex X_s) which is constant along the simplicial subset $\Lambda^{n+1}_0 \subset \partial\Delta^{n+1}$. Let $h : \Delta^1 \times \Delta^{n+1} \to S$ be a homotopy from τ to τ' which is constant on $\partial\Delta^{n+1}$. Since f is a Kan fibration, the homotopy extension lifting problem

$$\begin{tikzcd}
(\Delta^1 \times \partial\Delta^{n+1}) \coprod_{\{0\} \times \partial\Delta^{n+1}} (\{0\} \times \Delta^{n+1}) \arrow[r, "(\widetilde{h}_0, \widetilde{\tau})"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^1 \times \Delta^{n+1} \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & S
\end{tikzcd}$$

admits a solution $\widetilde{h} : \Delta^1 \times \Delta^{n+1} \to X$ (Remark 3.1.5.3), which we can regard as a homotopy from $\widetilde{\tau}$ to another $(n+1)$-simplex $\widetilde{\tau}' : \Delta^{n+1} \to X$. By construction, this $(n+1)$-simplex witnesses that σ' is incident to τ'.

For the converse, suppose that σ' is incident to τ', so that there exists an $(n+1)$-simplex $\widetilde{\tau}' : \Delta^{n+1} \to X$ satisfying $d^{n+1}_0(\widetilde{\tau}') = \sigma'$, $f(\widetilde{\tau}') = \tau'$, and $\widetilde{\tau}'|_{\Lambda^{n+1}_0}$ is the constant map taking the value x. Since f is a Kan fibration, the lifting problem

$$\begin{tikzcd}
(\Delta^1 \times \Lambda^{n+1}_0) \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_0} (\partial\Delta^1 \times \Delta^{n+1}) \arrow[r, "(\overline{e},(\widetilde{\tau},\widetilde{\tau}'))"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^1 \times \Delta^{n+1} \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & S
\end{tikzcd}$$

admits a solution, where $\overline{e} : \Delta^1 \times \Lambda^{n+1}_0 \to X$ is the constant map taking the value x. Then $\widetilde{h}$ is a homotopy from $\widetilde{\tau}$ to $\widetilde{\tau}'$ (in the Kan complex X) which is constant along the horn $\Lambda^{n+1}_0 \subseteq \Delta^{n+1}$, and it restricts to a homotopy from $\sigma = d^{n+1}_0(\widetilde{\tau})$ to $\sigma' = d^{n+1}_0(\widetilde{\tau}')$ (in the Kan complex X_s) which is constant along the boundary $\partial\Delta^n$. It follows that $[\sigma] = [\sigma']$ in $\pi_n(X_s, x)$. □

00WH **Proposition 3.2.5.4.** *Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes, and let $n \geq 1$ be a positive integer, and let $\partial : \pi_{n+1}(S, s) \to \pi_n(X_s, x)$ be as in Proposition 3.2.5.2. Then ∂ is a group homomorphism.*

Proof. To avoid confusion in the case $n = 1$, let us use multiplicative notation for the group structures on both $\pi_{n+1}(S, s)$ and $\pi_n(X_s, x)$. It is easy to see that the constant map $\Delta^n \to \{x\} \subseteq X_s$ is incident to the constant map $\Delta^{n+1} \to \{s\} \subseteq S$, so the map ∂ carries the identity element of $\pi_{n+1}(S, s)$ to the identity element of $\pi_n(X_s, x)$. To complete the proof, it will suffice to show that if $(\eta_0, \eta_1, \ldots, \eta_{n+1})$ is an $(n+2)$-tuple of elements of $\pi_{n+1}(S, s)$ for which the product $\eta_0^{-1}\eta_1\eta_2^{-1}\cdots\eta_{n+1}^{(-1)^n}$ vanishes in $\pi_{n+1}(S, s)$, then the product $\partial(\eta_0)^{-1}\partial(\eta_1)\partial(\eta_2)^{-1}\cdots\partial(\eta_{n+1})^{(-1)^n}$ vanishes in $\pi_n(X_s, x)$. To prove this, choose simplices $\tau_i : \Delta^{n+1} \to S$ for which each restriction $\tau_i|_{\partial\Delta^{n+1}}$ is the constant map taking the value s and $[\tau_i] = \eta_i$. Using our assumption that f is a Kan fibration, we can lift each τ_i to a simplex $\widetilde{\tau}_i : \Delta^{n+1} \to X$ carrying the horn Λ^{n+1}_0 to the vertex $x \in X$, so that $\partial(\eta_i) = [d^{n+1}_0(\widetilde{\tau}_i)]$. Since $\pi_{n+1}(S, s)$ is abelian, the vanishing of the product $\eta_0^{-1}\eta_1\eta_2^{-1}\cdots\eta_{n+1}^{(-1)^n}$ guarantees that we can choose an $(n+2)$-simplex $\rho : \Delta^{n+2} \to S$ such that $d^{n+2}_0(\rho)$ is the constant map taking the value s and $d^{n+2}_i(\rho) = \tau_{i-1}$ for $1 \leq i \leq n+2$. Let $\widetilde{\rho}_0 : \Lambda^{n+2}_0 \to X$ be the map given by the tuple of $(n+1)$-simplices $(\bullet, \widetilde{\tau}_0, \widetilde{\tau}_1, \ldots, \widetilde{\tau}_{n+1})$ (see Proposition 1.2.4.7). Since f is a Kan fibration, the lifting problem

$$\begin{tikzcd}
\Lambda^{n+2}_0 \arrow[r, "\widetilde{\rho}_0"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^{n+2} \arrow[r, "\rho"] \arrow[ur, dashed, "\widetilde{\rho}"] & S
\end{tikzcd}$$

admits a solution. Then $\sigma = d_0^{n+2}(\widetilde{\rho})$ is an $(n+1)$-simplex of X_s satisfying $d_i^{n+1}(\sigma) = d_0^{n+1}(\widetilde{\tau}_i)$ for $0 \leq i \leq n+1$, and therefore witnesses that the product

$$[d_0^{n+1}(\sigma)]^{-1}[d_1^{n+1}(\sigma)][d_2^{n+1}(\sigma)]^{-1}\cdots[d_{n+1}^{n+1}(\sigma)]^{(-1)^n} = \partial(\eta_0)^{-1}\partial(\eta_1)\partial(\eta_2)^{-1}\cdots\partial(\eta_{n+1})^{(-1)^n}$$

vanishes in the homotopy group $\pi_n(X_s, x)$. □

In the special case $n = 0$, we do not have a group structure on the set $\pi_0(X_s, x)$, so we cannot assert that the connecting map $\partial : \pi_1(S, s) \to \pi_0(X_s, x)$ is a group homomorphism. Nevertheless, the map ∂ is compatible with the group structure on $\pi_1(S, s)$ in the following sense:

Variant 3.2.5.5. Let $f : X \to S$ be a Kan fibration between Kan complexes, let s be a vertex of S, and set $X_s = \{s\} \times_S X$. Then there is a unique left action $a : \pi_1(S, s) \times \pi_0(X_s) \to \pi_0(X_s)$ of the fundamental group $\pi_1(S, s)$ on $\pi_0(X_s)$ with the following property; 00WJ

(∗) For each element $\eta \in \pi_1(S, s)$ and each vertex x of X_s, we have $a(\eta, [x]) = \partial_x(\eta)$, where $\partial_x : \pi_1(S, s) \to \pi_0(X_s, x) = \pi_0(X_s)$ is given by Proposition 3.2.5.2.

Proof. We first show that the function a is well-defined: that is, that the map $\partial_x : \pi_1(S, s) \to \pi_0(X_s)$ depends only on the image of x in $\pi_0(X_s)$. Fix an element $\eta \in \pi_1(S, s)$, which we can write as the homotopy class of an edge $v : s \to s$ in the Kan complex S. Let x and x' be vertices belonging to the same connected component of X_s, so that there exists an edge $u : x' \to x$ of X satisfying $f(u) = \mathrm{id}_s$. We wish to show that $\partial_x(\eta) = \partial_{x'}(\eta)$ in $\pi_0(X_s)$. Since f is a Kan fibration, we can lift v to an edge $\widetilde{v} : x \to y$ in X. Using the fact that f is a Kan fibration, we can solve the lifting problem

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "{(\widetilde{v}, \bullet, u)}"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^2 \arrow[r, "s^1_0(v)"] \arrow[ur, dashed, "\sigma"] & S
\end{tikzcd}$$

to obtain a 2-simplex σ of X depicted in the diagram

$$\begin{tikzcd}
& x \arrow[dr, "\widetilde{v}"] & \\
x' \arrow[ur, "u"] \arrow[rr, "\widetilde{v}'"] & & y.
\end{tikzcd}$$

The edges $\widetilde{v}$ and $\widetilde{v}'$ then witness the identities $\partial_x(\eta) = [y] = \partial_{x'}(\eta)$ in $\pi_0(X_s)$.

We now complete the proof by showing that the function $a : \pi_1(S,s) \times \pi_0(X_s) \to \pi_0(X_s)$ determines a left action of $\pi_1(S,s)$ on $\pi_0(X_s)$. Note that the identity element of $\pi_1(S,s)$ is given by the homotopy class of the degenerate edge $\mathrm{id}_s : s \to s$ of S. For each $x \in X_s$, we can lift id_s to the edge $\mathrm{id}_x : x \to x$ of X, which witnesses the identity $a([\mathrm{id}_s],[x]) = \partial_x([\mathrm{id}_s]) = [x]$ in $\pi_0(X_s)$. To complete the argument, it will suffice to show that for every pair of edges $g, g' : s \to s$ of S and every vertex $x \in X_s$, we have an equality $a([g'][g],[x]) = a([g'], a([g],[x]))$ in $\pi_0(X_s)$. Since f is a Kan fibration, we can lift g to an edge $\widetilde{g} : x \to y$ in X, and g' to an edge $\widetilde{g}' : y \to z$ in X. Since X is a Kan complex, the map $(\widetilde{g}', \bullet, \widetilde{g}) : \Lambda^2_1 \to X$ can be completed to a 2-simplex σ of X, as depicted in the diagram

$$\begin{tikzcd}
 & y \arrow[dr, "\widetilde{g}'"] & \\
x \arrow[ur, "\widetilde{g}"] \arrow[rr, "\widetilde{g}''"] & & z.
\end{tikzcd}$$

The edges $\widetilde{g}$, $\widetilde{g}'$, and $\widetilde{g}''$ then witness the identities $a([g],[x]) = [y]$, $a([g'],[y]) = [z]$, and $a([g'][g],[x]) = [z]$ (respectively), so that we have an equality

$$a([g'][g],[x]) = [z] = a([g'],[y]) = a([g]', a([g],[x]))$$

as desired. □

00WK **Warning 3.2.5.6.** Let $f : (X,x) \to (S,s)$ be a Kan fibration between pointed Kan complexes. Then x and s can also be regarded as vertices of the opposite simplicial sets X_s^{op} and S^{op}, respectively, and we have canonical bijections $\pi_{n+1}(S,s) \simeq \pi_{n+1}(S^{\mathrm{op}},s)$ and $\pi_n(X_s,x) \simeq \pi_n(X_s^{\mathrm{op}},x)$, respectively. However, these bijections are not necessarily compatible with the connecting homomorphisms of Construction 3.2.5.3. The diagram

$$\begin{tikzcd}
\pi_{n+1}(S,s) \arrow[r, "\sim"] \arrow[d, "\partial"] & \pi_{n+1}(S^{\mathrm{op}},s) \arrow[d, "\partial"] \\
\pi_n(X_s,x) \arrow[r, "\sim"] & \pi_n(X_s^{\mathrm{op}},x)
\end{tikzcd}$$

commutes when n is odd, but *anticommutes* if $n \geq 2$ is even. This phenomenon is also visible in the case $n = 0$: in this case, the connecting maps $\partial : \pi_1(S^{\mathrm{op}},s) \to \pi_0(X_s^{\mathrm{op}},x)$ determine a *left* action of the fundamental group $\pi_1(S^{\mathrm{op}},s)$ on $\pi_0(X_s^{\mathrm{op}},x) \simeq \pi_0(X_s,x)$, which can be interpreted as a *right* action of the group $\pi_1(S,s)$ on $\pi_0(X_s,x)$ (see Remark 3.2.2.20). To recover the left action of Variant 3.2.5.5, we must compose with the anti-homomorphism $\pi_1(S,s) \to \pi_1(S,s)$ given by $\eta \mapsto \eta^{-1}$.

3.2.6 The Long Exact Sequence of a Fibration

If (X, x) is a pointed Kan complex, then we regard each $\pi_n(X, x)$ as a pointed set, with base point given by the homotopy class of the constant map $\Delta^n \to \{x\} \subseteq X$ (if $n \geq 1$, then this is the identity element with respect to the group structure on $\pi_n(X, x)$). Recall that a diagram of pointed sets 00WL

$$\cdots \to (G_{n+1}, e_{n+1}) \xrightarrow{f_n} (G_n, e_n) \xrightarrow{f_{n-1}} (G_{n-1}, e_{n-1}) \to \cdots$$

is said to be *exact* if the image of each f_n is equal to the fiber $f_{n-1}^{-1}\{e_{n-1}\} = \{g \in G_n : f_{n-1}(g) = e_{n-1}\}$. Our goal in this section is to prove the following:

Theorem 3.2.6.1. *Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes. Then the sequence of pointed sets* 00WM

$$\cdots \to \pi_2(S, s) \xrightarrow{\partial} \pi_1(X_s, x) \to \pi_1(X, x) \to \pi_1(S, s) \xrightarrow{\partial} \pi_0(X_s, x) \to \pi_0(X, x) \to \pi_0(S, s)$$

is exact; here $\partial : \pi_{n+1}(S, s) \to \pi_n(X_s, x)$ denotes the connecting homomorphism of Construction 3.2.5.3.

Theorem 3.2.6.1 really amounts to three separate assertions, which we will formulate and prove individually (Propositions 3.2.6.2, 3.2.6.4, and 3.2.6.6).

Proposition 3.2.6.2. *Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes and let $n \geq 0$ be an integer. Then the sequence of pointed sets* 00WN

$$\pi_n(X_s, x) \to \pi_n(X, x) \to \pi_n(S, s)$$

is exact.

In the special case $n = 0$, the content of Proposition 3.2.6.2 can be formulated without reference to the base point $x \in X$:

Corollary 3.2.6.3. *Let $f : X \to S$ be a Kan fibration between Kan complexes, let s be a vertex of S, and set $X_s = \{s\} \times_S X$. Then the image of the map $\pi_0(X_s) \to \pi_0(X)$ is equal to the fiber of the map $\pi_0(f) : \pi_0(X) \to \pi_0(S)$ over the connected component $[s] \in \pi_0(S)$ determined by the vertex s. In other words, a vertex $x \in X$ satisfies $[f(x)] = [s]$ in $\pi_0(S)$ if and only if the connected component of x has nonempty intersection with the fiber X_s.* 00WP

Proof of Proposition 3.2.6.2. Fix an n-simplex $\sigma : \Delta^n \to X$ such that $\sigma|_{\partial \Delta^n}$ is the constant map carrying $\partial \Delta^n$ to the base point $x \in X$. We wish to show that the homotopy class $[\sigma]$ belongs to the image of the map $\pi_n(X_s, x) \to \pi_n(X, x)$ if and only if the image $[f(\sigma)]$ is equal to the base point of $\pi_n(S, s)$. The "only if" direction is clear, since the composite map

$X_s \hookrightarrow X \xrightarrow{f} S$ is equal to the constant map taking the value s. For the converse, suppose that $[f(\sigma)]$ is the base point of $\pi_n(S,s)$. Then there exists a homotopy $h : \Delta^1 \times \Delta^n \to S$ from $f(\sigma)$ to the constant map $\sigma'_0 : \Delta^n \to \{s\} \subseteq S$, which is constant when restricted to the boundary $\partial\Delta^n$. Since f is a Kan fibration, we can lift h to a homotopy $\widetilde{h} : \Delta^1 \times \Delta^n \to X$ from σ to another n-simplex $\sigma' : \Delta^n \to X$, where $\widetilde{h}$ is constant along the boundary $\partial\Delta^n$ and $f(\sigma') = \sigma'_0$ (Remark 3.1.5.3). Then σ' represents a homotopy class $[\sigma'] \in \pi_n(X_s, x)$, and the homotopy $\widetilde{h}$ witnesses that $[\sigma]$ is equal to the image of $[\sigma']$ in $\pi_n(X,x)$. □

`00WQ` **Proposition 3.2.6.4.** *Let $f : (X,x) \to (S,s)$ be a Kan fibration between pointed Kan complexes and let $n \geq 0$ be an integer. Then the sequence of pointed sets $\pi_{n+1}(S,s) \xrightarrow{\partial} \pi_n(X_s,x) \to \pi_n(X,x)$ is exact, where ∂ is the connecting homomorphism of Construction 3.2.5.3.*

In the special case $n = 0$, Proposition 3.2.6.4 can also be formulated without reference to the base point $x \in X$.

`00WR` **Corollary 3.2.6.5.** *Let $f : X \to S$ be a Kan fibration between Kan complexes, let s be a vertex of S, and set $X_s = \{s\} \times_S X$. Then two elements of $\pi_0(X_s)$ have the same image in $\pi_0(X)$ if and only if they belong to the same orbit of the action of the fundamental group $\pi_1(S,s)$ (see Variant 3.2.5.5). In other words, the inclusion of Kan complexes $X_s \hookrightarrow X$ induces a monomorphism of sets $(\pi_1(S,s)\backslash\pi_0(X_s)) \hookrightarrow \pi_0(X)$.*

Proof. Combine Variant 3.2.5.5 with Proposition 3.2.6.4. □

Proof of Proposition 3.2.6.4. Fix an n-simplex $\sigma : \Delta^n \to X_s$ such that $\sigma|_{\partial\Delta^n}$ is the constant map carrying $\partial\Delta^n$ to the base point $x \in X_s$. By construction, the homotopy class $[\sigma] \in \pi_n(X_s,x)$ belongs to the image of the connecting homomorphism $\partial : \pi_{n+1}(S,s) \to \pi_n(X_s,x)$ if and only if there exists an $(n+1)$-simplex $\tau : \Delta^{n+1} \to S$ such that $\tau|_{\partial\Delta^{n+1}}$ is the constant map taking the value s and σ is incident to τ, in the sense of Definition 3.2.5.1. This condition is equivalent to the existence of an $(n+1)$-simplex $\widetilde{\tau} : \Delta^{n+1} \to X$ satisfying $d_0^{n+1}(\widetilde{\tau}) = \sigma$ and $d_i^{n+1}(\widetilde{\tau})$ is equal to the constant map $e : \Delta^n \to \{x\} \subseteq X$ for $1 \leq i \leq n+1$. In other words, it is equivalent to the assertion that the tuple of n-simplices of X $(\sigma, e, e, \ldots, e)$ bounds, in the sense of Notation 3.2.3.1. For $n \geq 1$, this is equivalent to the vanishing of the image of $[\sigma]$ in the homotopy group $\pi_n(X,x)$ (Theorem 3.2.2.10). When $n = 0$, it is equivalent to the equality $[\sigma] = [x]$ in $\pi_0(X)$ by virtue of Remark 1.4.6.13. □

`00WS` **Proposition 3.2.6.6.** *Let $f : (X,x) \to (S,s)$ be a Kan fibration between pointed Kan complexes and let $n \geq 0$ be an integer. Then the sequence of pointed sets $\pi_{n+1}(X,x) \xrightarrow{\pi_{n+1}(f)} \pi_{n+1}(S,s) \xrightarrow{\partial} \pi_n(X_s,x)$ is exact, where ∂ is the connecting homomorphism of Construction 3.2.5.3.*

Corollary 3.2.6.7. *Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes. Then the image of the induced map $\pi_1(f) : \pi_1(X, x) \to \pi_1(S, s)$ is equal to the stabilizer of $[x] \in \pi_0(X_s)$ (with respect to the action of $\pi_1(S, s)$ on $\pi_0(X_s)$ supplied by Variant 3.2.5.5.* 00WT

Proof. Combine Variant 3.2.5.5 with Proposition 3.2.6.6. □

Proof of Proposition 3.2.6.6. Fix an $(n + 1)$-simplex $\tau : \Delta^{n+1} \to S$ for which $\tau|_{\partial \Delta^{n+1}}$ is the constant map taking the value s. By construction, the connecting homomorphism $\partial : \pi_{n+1}(S, s) \to \pi_n(X_s, x)$ carries $[\tau]$ to the base point of $\pi_n(X_s, x)$ if and only if the constant map $e : \Delta^n \to \{x\} \hookrightarrow X_s$ is incident to τ, in the sense of Definition 3.2.5.1. This is equivalent to the requirement that τ can be lifted to a map $\widetilde{\tau} : \Delta^{n+1} \to X$ for which $\widetilde{\tau}|_{\partial \Delta^{n+1}}$ is the constant map taking the value x, which clearly implies that that $[\tau]$ belongs to the image of the map $\pi_{n+1}(f) : \pi_{n+1}(X, x) \to \pi_{n+1}(S, s)$. To prove the reverse implication, suppose that $[\tau]$ belongs to the image of $\pi_{n+1}(f)$, so that we can write $[\tau] = [f(\widetilde{\tau}')]$ for some map $\widetilde{\tau}' : \Delta^{n+1} \to X$ for which $\widetilde{\tau}'|_{\partial \Delta^{n+1}}$ is the constant map taking the value x. It follows that there is a homotopy $h : \Delta^1 \times \Delta^{n+1} \to S$ from $f(\widetilde{\tau}')$ to τ which is constant along the boundary $\partial \Delta^{n+1}$. Since f is a Kan fibration, we can lift h to a map $\widetilde{h} : \Delta^1 \times \Delta^{n+1} \to X$ such that $h|_{\{0\} \times \Delta^{n+1}} = \widetilde{\tau}'$ and $h|_{\Delta^1 \times \partial \Delta^{n+1}}$ is the constant map taking the value x (Remark 3.1.5.3). The restriction $\widetilde{\tau} = h|_{\{1\} \times \Delta^{n+1}}$ then satisfies $f(\widetilde{\tau}) = \tau$ and $\widetilde{\tau}|_{\partial \Delta^{n+1}}$ is the constant map taking the value x. □

Corollary 3.2.6.8. *Let $f : (X, x) \to (S, s)$ be a Kan fibration between pointed Kan complexes and let $n > 0$ be an integer. Then the homotopy group $\pi_n(X_s, x)$ vanishes if and only if f satisfies both of the following conditions:* 050Z

- *The group homomorphism $\pi_n(f) : \pi_n(X, x) \to \pi_n(S, s)$ is injective.*
- *The group homomorphism $\pi_{n+1}(f) : \pi_{n+1}(X, x) \to \pi_{n+1}(S, s)$ is surjective.*

Variant 3.2.6.9. Let $f : (X, x) \to (S, s)$ be a Kan fibration between Kan complexes. Then the fiber X_s is connected if and only if f satisfies both of the following conditions: 0510

- The connected component $[s] \in \pi_0(S)$ has a unique preimage under the map $\pi_0(f) : \pi_0(X) \to \pi_0(S)$ (given by $[x] \in \pi_0(X)$).
- The map of fundamental groups $\pi_1(X, x) \to \pi_1(S, s)$ is surjective.

3.2.7 Whitehead's Theorem for Kan Complexes

Let $f : X \to Y$ be a continuous function between nonempty topological spaces. If X and Y are CW complexes, then a classical theorem of Whitehead (see [60]) asserts that f is a homotopy equivalence if and only if it induces a bijection $\pi_0(X) \simeq \pi_0(Y)$ and, for every base point $x \in X$, the induced map of homotopy groups $\pi_n(X, x) \to \pi_n(Y, f(x))$ is an 00WU isomorphism for $n > 0$ (Corollary 3.6.3.10). Our goal in this section is to prove an analogous statement in the setting of Kan complexes.

00WV **Theorem 3.2.7.1.** *Let $f : X \to Y$ be a morphism of Kan complexes. Then f is a homotopy equivalence if and only if it satisfies the following pair of conditions:*

(a) *The map of sets $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.*

(b) *For every vertex $x \in X$ having image $y = f(x)$ in Y and every positive integer n, the map of homotopy groups $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism.*

We begin by proving Theorem 3.2.7.1 in the case of a Kan fibration.

00X1 **Proposition 3.2.7.2.** *Let $f : X \to Y$ be a Kan fibration between Kan complexes. The following conditions are equivalent:*

(1) *The morphism f is a trivial Kan fibration.*

(2) *The morphism f is a homotopy equivalence.*

(3) *The map of sets $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection and, for every vertex $x \in X$ and every integer $n > 0$, the map of homotopy groups $\pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism.*

(4) *For each vertex $y \in Y$, the fiber $X_y = \{y\} \times_Y X$ is a contractible Kan complex.*

Proof. The implication (1) ⇒ (2) follows from Proposition 3.1.6.10 and the implication (2) ⇒ (3) from Remark 3.2.2.17. Using Corollary 3.2.6.8 and Variant 3.2.6.9, we can reformulate (3) as follows:

(3′) The map $\pi_0(f)$ is surjective and, for every vertex $x \in X$ having image $y = f(x)$, the homotopy groups $\pi_n(X_y, x)$ vanish for $n > 0$.

The equivalence of (3′) ⇔ (4) follows from Theorem 3.2.4.3. We will complete the proof by showing that (4) implies (1). Assume that condition (4) is satisfied; we wish to show that every lifting problem

$$\begin{tikzcd}
\partial\Delta^m \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^m \arrow[r, "\sigma"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution. Since σ is nullhomotopic (Exercise 3.2.4.8), we can use the homotopy extension lifting property to to reduce to the special case where σ is the constant map $\Delta^m \to \{y\} \hookrightarrow Y$ for some vertex $y \in Y$. In this case, the desired result follows from the contractibility of the fiber X_y (Theorem 3.2.4.3). □

Remark 3.2.7.3. For the equivalences (1) ⇔ (2) ⇔ (4) of Proposition 3.2.7.2, it is not necessary to assume that X and Y are Kan complexes. See Proposition 3.3.7.6. `00X3`

Proof of Theorem 3.2.7.1. Let $f : X \to Y$ be a morphism of Kan complexes. Suppose that $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection and that the induced map $\pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for every base point $x \in X$ and every positive integer n; we wish to show that f is a homotopy equivalence (the converse follows from Remark 3.2.2.17). Using Proposition 3.1.7.1 (or Example 3.1.7.10), we can factor f as a composition $X \xrightarrow{f'} X' \xrightarrow{f''} Y$, where f' is anodyne and f'' is a Kan fibration. Then X' is Kan complex (Remark 3.1.1.11), so that f' is a homotopy equivalence (Proposition 3.1.6.13). It will therefore suffice to show that the Kan fibration f'' is a homotopy equivalence, which follows from Proposition 3.2.7.2. □

Corollary 3.2.7.4. *Let C_* and D_* be chain complexes of abelian groups and let $f : C_* \to D_*$ be a morphism of chain complexes. The following conditions are equivalent:* `03DZ`

(1) *The induced map of generalized Eilenberg-MacLane spaces $\mathrm{K}(C_*) \to \mathrm{K}(D_*)$ is a homotopy equivalence (see Construction 2.5.6.3).*

(2) *For every integer $n \geq 0$, the induced map of homology groups $\mathrm{H}_n(C) \to \mathrm{H}_n(D)$ is an isomorphism.*

Proof. Remark 2.5.6.4 guarantees that the simplicial sets $\mathrm{K}(C_*)$ and $\mathrm{K}(D_*)$ are Kan complexes. By virtue of Theorem 3.2.7.1, (1) is equivalent to the following pair of assertions:

($1'$) The chain map f induces a bijection $\pi_0(\mathrm{K}(C_*)) \to \pi_0(\mathrm{K}(D_*))$.

($1''$) For every vertex x of $\mathrm{K}(C_*)$ having image $y \in \mathrm{K}(D_*)$ and every integer $n > 0$, the induced of homotopy groups

$$\pi_n(\mathrm{K}(C_*), x) \to \pi_n(\mathrm{K}(D_*), y)$$

is an isomorphism.

Note that we have a commutative diagram of pointed Kan complexes

$$\begin{tikzcd}
(\mathrm{K}(C_*), 0) \arrow[r, "\mathrm{K}(f)"] \arrow[d, "\sim"] & (\mathrm{K}(D_*), 0) \arrow[d, "\sim"] \\
(\mathrm{K}(C_*), x) \arrow[r, "\mathrm{K}(f)"] & (\mathrm{K}(D_*), y),
\end{tikzcd}$$

where the vertical isomorphisms are given by translation by x and y, respectively (using the group structure on the Kan complexes $\mathrm{K}(C_*)$ and $\mathrm{K}(D_*)$. Consequently, to verify ($1''$), we may assume without loss of generality that $x = 0$. Applying Exercise 3.2.2.22, we see that ($1'$) and ($1''$) can be reformulated as follows:

(2′) The chain map f induces an isomorphism $\mathrm{H}_0(C) \to \mathrm{H}_0(D)$.

(2″) For every integer $n > 0$, the chain map f induces an isomorphism $\mathrm{H}_n(C) \to \mathrm{H}_n(D)$.

□

3.2.8 Closure Properties of Homotopy Equivalences

00XB We now apply Whitehead's theorem (Theorem 3.2.7.1) to establish some stability properties for the collection of homotopy equivalences between Kan complexes (and weak homotopy equivalences between arbitrary simplicial sets).

00XC **Proposition 3.2.8.1.** *Suppose we are given a commutative diagram of Kan complexes*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
S \arrow[r, "h"] & S',
\end{tikzcd}$$

where f and f' are Kan fibrations and h is a homotopy equivalence. Then the following conditions are equivalent:

(1) *The morphism g is a homotopy equivalence.*

(2) *For each vertex $s \in S$ having image $s' = h(s)$ in S', the map of fibers $g_s : X_s \to X'_{s'}$ is a homotopy equivalence.*

00XD **Remark 3.2.8.2.** In the situation of Proposition 3.2.8.1, the assumption that S and S' are Kan complexes can be eliminated at the cost of working with weak homotopy equivalences in place of homotopy equivalences: see Proposition 3.3.7.1.

Proof of Proposition 3.2.8.1. Assume first that (1) is satisfied. Let s be a vertex of S having image $s' = h(s)$ in S'; we wish to show that the induced map $g_s : X_s \to X'_{s'}$ is a homotopy equivalence. By virtue of Remark 3.1.6.6, it will suffice to show that for every simplicial set W, the induced map $\mathrm{Fun}(W, X_s) \to \mathrm{Fun}(W, X'_{h(s)})$ is bijective on connected components. Replacing X by $\mathrm{Fun}(W, X)$ (and making similar replacements for X', S, and S'), we may reduce to the problem of showing that g_s induces a bijection $\pi_0(X_s) \to \pi_0(X'_{s'})$. Let us regard $\pi_0(X_s)$ and $\pi_0(X'_{s'})$ as endowed with actions of the fundamental groups $\pi_1(S, s)$ and $\pi_1(S', s')$, respectively (Variant 3.2.5.5). Using our assumption that g and h are homotopy equivalences, we conclude that the induced maps

$$\pi_0(X) \to \pi_0(X') \qquad \pi_0(S) \to \pi_0(S') \qquad \pi_1(S, s) \to \pi_1(S', s')$$

are bijective. Applying Corollaries 3.2.6.3 and 3.2.6.5, we conclude that g_s induces a bijection $\pi_1(S,s)\backslash\pi_0(X_s) \to \pi_1(S',s')\backslash\pi_0(X'_{s'})$. It will therefore suffice to show that, for every vertex $x \in X_s$, the stabilizer in $\pi_1(S,s)$ of the connected component $[x] \in \pi_0(X_s)$ maps isomorphically to the stabilizer in $\pi_1(S',s')$ of the connected component $[g(x)] \in \pi_0(X'_{s'})$. This follows from Corollary 3.2.6.7, since g induces an isomorphism $\pi_1(X,x) \to \pi_1(X', g(x))$.

We now show that (2) ⇒ (1). Assume that, for each vertex $s \in S$ having image $s' = h(s)$ in S', the induced map $g_s : X_s \to X'_{s'}$ is a homotopy equivalence. We wish to show that g is a homotopy equivalence. We first show that the map $\pi_0(g) : \pi_0(X) \to \pi_0(X')$ is bijective. Our assumption that h is a homotopy equivalence guarantees that the map $\pi_0(h) : \pi_0(S) \to \pi_0(S')$ is bijective. It will therefore suffice to show that, for each vertex $s \in S$ having image $s' = h(s)$, the induced map $\pi_0(X) \times_{\pi_0(S)} \{[s]\} \to \pi_0(X') \times_{\pi_0(S')} \{[s']\}$ is bijective. Using Corollaries 3.2.6.3 and 3.2.6.5, we can identify this with the map of quotients $(\pi_1(S,s)\backslash\pi_0(X_s)) \to (\pi_1(S',s')\backslash\pi_0(X'_{s'}))$. The desired result now follows from the bijectivity of the map $\pi_0(g_s) : \pi_0(X_s) \to \pi_0(X'_{s'})$ and of the group homomorphism $\pi_1(S,s) \to \pi_1(S',s')$.

To complete the proof that g is a homotopy equivalence, it will suffice (by virtue of Theorem 3.2.7.1) to show that for every vertex $x \in X$ having image $x' = g(x)$ and every positive integer n, the group homomorphism $\pi_n(X,x) \to \pi_n(X',x')$ is an isomorphism. Setting $s = f(x)$ and $s' = f(x')$, we have a commutative diagram of exact sequences

$$\begin{tikzcd}
\pi_{n+1}(S,s) \arrow[r] \arrow[d, "\sim"] & \pi_n(X_s,x) \arrow[r] \arrow[d, "\sim"] & \pi_n(X,x) \arrow[r] \arrow[d] & \pi_n(S,s) \arrow[r] \arrow[d, "\sim"] & \pi_{n-1}(X_s,x) \arrow[d, "\sim"] \\
\pi_{n+1}(S',s') \arrow[r] & \pi_n(X'_{s'},x') \arrow[r] & \pi_n(X',x') \arrow[r] & \pi_n(S',s') \arrow[r] & \pi_{n-1}(X'_{s'},x').
\end{tikzcd}$$

Our assumptions that g_s and h are homotopy equivalences guarantee that the outer vertical maps are bijective, and elementary diagram chase shows that that the middle vertical map is an isomorphism. □

Proposition 3.2.8.3. *Let $\mathcal{W}$ denote the full subcategory of* $\operatorname{Fun}([1], \mathrm{Set}_\Delta)$ *spanned by those morphisms of simplicial sets $f : X \to Y$ which are weak homotopy equivalences. Then $\mathcal{W}$ is closed under the formation of filtered colimits in* $\operatorname{Fun}([1], \mathrm{Set}_\Delta)$. `00XE`

Proof. Suppose we are given a filtered diagram $\{f_\alpha : X_\alpha \to Y_\alpha\}$ in $\mathcal{W}$, so that each f_α is a weak homotopy equivalence of simplicial sets. We wish to show that the induced map $f : (\varinjlim_\alpha X_\alpha) \to (\varinjlim_\alpha Y_\alpha)$ is also a weak homotopy equivalence. Using Proposition 3.1.7.1, we can choose a diagram of morphisms $\{u_\alpha : Y_\alpha \hookrightarrow Y'_\alpha\}$ with the following properties:

- Each of the maps u_α is anodyne, and the induced map $u : (\varinjlim_\alpha Y_\alpha) \to (\varinjlim_\alpha Y'_\alpha)$ is anodyne.

- Each of the simplicial sets Y'_α is a Kan complex, and (therefore) the colimit $\varinjlim_\alpha Y'_\alpha$ is also a Kan complex.

Since every anodyne morphism is a weak homotopy equivalence (Proposition 3.1.6.14), we can replace $\{f_\alpha : X_\alpha \to Y_\alpha\}$ by the diagram of composite maps $\{(u_\alpha \circ f_\alpha) : X_\alpha \to Y'_\alpha\}$, and therefore reduce to the case where each Y_α is a Kan complex.

Let us regard the system of morphisms $\{f_\alpha\}$ as a morphism from the filtered diagram of simplicial sets $\{X_\alpha\}$ to the filtered diagram $\{Y_\alpha\}$. Applying Proposition 3.1.7.1 again, we see that this diagram admits a factorization $\{X_\alpha\} \xrightarrow{\{g_\alpha\}} \{X'_\alpha\} \xrightarrow{\{h_\alpha\}} \{Y_\alpha\}$ with the following properties:

- Each of the morphisms g_α is anodyne, and the induced map $g : (\varinjlim_\alpha X_\alpha) \to (\varinjlim_\alpha X'_\alpha)$ is anodyne.
- Each of the morphisms h_α is a Kan fibration, and (therefore) the induced map $(\varinjlim_\alpha X'_\alpha) \to (\varinjlim_\alpha Y_\alpha)$ is also a Kan fibration.

Arguing as before, we can replace $\{f_\alpha : X_\alpha \to Y_\alpha\}$ by the diagram of morphisms $\{h_\alpha : X'_\alpha \to Y_\alpha\}$, and thereby reduce to the case where each f_α is a Kan fibration. In this case, Proposition 3.2.7.2 guarantees that each f_α is a trivial Kan fibration. It follows that the colimit map $f : (\varinjlim_\alpha X_\alpha) \to (\varinjlim_\alpha Y_\alpha)$ is also a trivial Kan fibration, and therefore a (weak) homotopy equivalence by virtue of Proposition 3.1.6.10. □

03PK **Corollary 3.2.8.4.** *The collection of weakly contractible simplicial sets is closed under the formation of filtered colimits.*

03PL **Corollary 3.2.8.5.** *Let S be a nonempty linearly ordered set. Then the nerve $\mathrm{N}_\bullet(S)$ is weakly contractible.*

Proof. By virtue of Corollary 3.2.8.4, we may assume without loss of generality that S is finite. In this case, there is an isomorphism $S \simeq [n]$ for some integer $n \geq 0$, so that $\mathrm{N}_\bullet(S)$ is isomorphic to the standard simplex Δ^n. □

3.3 The Ex^{∞} Functor

00XF Let $f : X \to S$ be a Kan fibration of simplicial sets. If S is a Kan complex, then X is also a Kan complex. Moreover, for every vertex $x \in X$ having image $s = f(x) \in S$, Theorem 3.2.6.1 supplies an exact sequence of homotopy groups

$$\cdots \to \pi_2(S,s) \xrightarrow{\partial} \pi_1(X_s,x) \to \pi_1(X,x) \to \pi_1(S,s) \xrightarrow{\partial} \pi_0(X_s,x) \to \pi_0(X,x) \to \pi_0(S,s).$$

If S is not a Kan complex, then the results of §3.2.6 do not apply directly. However, one can obtain similar information by *replacing* f by a Kan fibration $f' : X' \to S'$ between Kan complexes, using the following result:

Theorem 3.3.0.1. *Let $f : X \to S$ be a Kan fibration of simplicial sets. Then there exists a commutative diagram of simplicial sets* `00XG`

$$\begin{tikzcd}
X \arrow[r, "g'"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
S \arrow[r, "g"] & S'
\end{tikzcd}$$

with the following properties:

(a) *The simplicial sets S' and X' are Kan complexes.*

(b) *The morphisms g and g' are weak homotopy equivalences.*

(c) *The morphism f' is a Kan fibration.*

(d) *For every vertex $s \in S$, the induced map $g'_s : X_s \to X'_{g(s)}$ is a homotopy equivalence of Kan complexes.*

Note that we can *almost* deduce Theorem 3.3.0.1 formally from the results of §3.1.7. Given a Kan fibration $f : X \to S$, we can always choose an anodyne map $g : S \to S'$, where S' is a Kan complex (Corollary 3.1.7.2). Applying Proposition 3.1.7.1, we deduce that $g \circ f$ factors as a composition $X \xrightarrow{g'} X' \xrightarrow{f'} S'$, where f' is a Kan fibration and g' is anodyne. The resulting commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g'"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
S \arrow[r, "g"] & S'
\end{tikzcd}$$

then satisfies conditions (a), (b), and (c) of Theorem 3.3.0.1. However, it is not so obvious that this diagram also satisfies condition (d). To guarantee this, it is convenient to adopt a different approach to the results of §3.1.7. Following Kan ([33]), we will introduce a functor $\mathrm{Ex}^{\infty} : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ and a natural transformation of functors $\rho^{\infty} : \mathrm{id}_{\mathrm{Set}_{\Delta}} \to \mathrm{Ex}^{\infty}$ with the following properties:

(a') For every simplicial set S, the simplicial set $\mathrm{Ex}^{\infty}(S)$ is a Kan complex (Proposition 3.3.6.9).

(b') For every simplicial set S, the morphism $\rho^{\infty}_S : S \to \mathrm{Ex}^{\infty}(S)$ is a weak homotopy equivalence (Proposition 3.3.6.7).

(c') For every Kan fibration of simplicial sets $f : X \to S$, the induced map $\mathrm{Ex}^{\infty}(f) : \mathrm{Ex}^{\infty}(X) \to \mathrm{Ex}^{\infty}(S)$ is a Kan fibration (Proposition 3.3.6.6).

(d') The functor $\mathrm{Ex}^{\infty} : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ commutes with finite limits (Proposition 3.3.6.4). In particular, for every morphism of simplicial sets $f : X \to S$ and every vertex $s \in S$, the canonical map $\mathrm{Ex}^{\infty}(X_s) \to \{s\} \times_{\mathrm{Ex}^{\infty}(S)} \mathrm{Ex}^{\infty}(X)$ is an isomorphism (Corollary 3.3.6.5).

It follows from these assertions that for any Kan fibration $f : X \to S$, the diagram of simplicial sets

$$\begin{tikzcd}
X \arrow[r, "\rho_X^{\infty}"] \arrow[d, "f"] & \mathrm{Ex}^{\infty}(X) \arrow[d, "\mathrm{Ex}^{\infty}(f)"] \\
S \arrow[r, "\rho_S^{\infty}"] & \mathrm{Ex}^{\infty}(S)
\end{tikzcd}$$

satisfies the requirements of Theorem 3.3.0.1.

Most of this section is devoted to the definition of the functor Ex^{∞} (and the natural transformation ρ^{∞}) and the verification of assertions (a') through (d'). The construction is rooted in classical geometric ideas. Let n be a nonnegative integer, let

$$|\Delta^n| = \{(t_0, t_1, \ldots, t_n) \in [0,1]^{n+1} : t_0 + t_1 + \cdots + t_n = 1\}$$

denote the topological simplex of dimension n. This topological space admits a triangulation whose vertices are the *barycenters* of its faces. More precisely, there is a canonical homeomorphism of topological spaces $|\mathrm{Sd}(\Delta^n)| \xrightarrow{\sim} |\Delta^n|$, where $\mathrm{Sd}(\Delta^n)$ denotes the nerve of the partially ordered set of faces of Δ^n (Proposition 3.3.2.3). For every topological space Y, composition with this homeomorphism induces a bijection

$$\varphi_n : \mathrm{Sing}_n(Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sd}(\Delta^n), \mathrm{Sing}_{\bullet}(Y)).$$

Motivated by this observation, we define a functor $X \mapsto \mathrm{Ex}(X) = \mathrm{Ex}_{\bullet}(X)$ from the category of simplicial sets to itself by the formula $\mathrm{Ex}_n(X) = \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sd}(\Delta^n), X)$ (Construction 3.3.2.5). The preceding discussion can then be summarized by noting that, when $X = \mathrm{Sing}_{\bullet}(Y)$ is the singular simplicial set of a topological space Y, the bijections $\{\varphi_n\}_{n \geq 0}$ determine an isomorphism of *semisimplicial* sets $\varphi : X \to \mathrm{Ex}(X)$ (Example 3.3.2.9). Beware that φ is generally not an isomorphism of simplicial sets: that is, it need not be compatible with degeneracy operators.

In §3.3.3, we show that the functor $\mathrm{Ex} : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ admits a left adjoint (Corollary 3.3.3.4). We denote the value of this left adjoint on a simplicial set X by $\mathrm{Sd}(X)$, and refer to it as the *subdivision of X*. It is essentially immediate from the definition that, in the special case where $X = \Delta^n$ is a standard simplex, we recover the simplicial set $\mathrm{Sd}(\Delta^n)$ defined above. More generally, we will say that a simplicial set X is *braced* if the collection of nondegenerate simplices of X is closed under face operators (Definition 3.3.1.1). If this condition is satisfied, then the subdivision $\mathrm{Sd}(X)$ can be identified with the nerve of the category $\mathbf{\Delta}_X^{\mathrm{nd}}$ of nondegenerate simplices of X (Proposition 3.3.3.16). Moreover, we also have a canonical homeomorphism of topological spaces $|\operatorname{Sd}(X)| \to |X|$, which carries each vertex of $\mathrm{N}_\bullet(\mathbf{\Delta}_X^{\mathrm{nd}})$ to the barycenter of the corresponding simplex of $|X|$ (Proposition 3.3.3.6).

In §3.3.4, we associate to every simplicial set X a pair of comparison maps

$$\lambda_X : \mathrm{Sd}(X) \to X \qquad \rho_X : X \to \mathrm{Ex}(X);$$

we refer to λ_X as the *last vertex map* of X (Construction 3.3.4.3). In the special case $X = \Delta^n$, the source and target of λ_X are both weakly contractible, so λ_X is automatically a weak homotopy equivalence. From this observation, it follows from a simple formal argument that λ_X is a weak homotopy equivalence for *every* simplicial set X (Proposition 3.3.4.8). In §3.3.5, we exploit this to show that the functor Ex carries Kan fibrations to Kan fibrations (Corollary 3.3.5.4), and that the comparison map $\rho_X : X \to \mathrm{Ex}(X)$ is a weak homotopy equivalence for every simplicial set X (Theorem 3.3.5.1). Consequently, the functor $\mathrm{Ex} : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ satisfies analogues of properties (b'), (c'), and (d') above.

Unfortunately, the functor $\mathrm{Ex} : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ does not satisfy the analogue of condition (a'): in general, a simplicial set of the form $\mathrm{Ex}(X)$ need not satisfy the Kan extension condition. However, one can show that it satisfies a slightly weaker condition: for any morphism of simplicial sets $f_0 : \Lambda^n_i \to \mathrm{Ex}(X)$, the composite map $\Lambda^n_i \xrightarrow{f_0} \mathrm{Ex}(X) \xrightarrow{\rho_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X)$ can be extended to an n-simplex of the simplicial set $\mathrm{Ex}^2(X) = \mathrm{Ex}(\mathrm{Ex}(X))$. We apply this observation in §3.3.6 to deduce that the direct limit

$$\mathrm{Ex}^\infty(X) = \varinjlim(X \xrightarrow{\rho_X} \mathrm{Ex}(X) \xrightarrow{\rho_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \xrightarrow{\rho_{\mathrm{Ex}^2(X)}} \mathrm{Ex}^3(X) \to \cdots)$$

is a Kan complex (Proposition 3.3.6.9). Moreover, properties (b'), (c'), and (d') for the functor $X \mapsto \mathrm{Ex}^\infty(X)$ are immediate consequences of the analogous properties of the functor $X \mapsto \mathrm{Ex}(X)$.

We close this section by outlining some applications of the functor Ex^∞. In §3.3.7 we prove that, in the situation of Theorem 3.3.0.1, assertion (d) is a formal consequence of (b) and (c) (Proposition 3.3.7.1). Using this, we show that a Kan fibration of simplicial sets $f : X \to S$ is a weak homotopy equivalence if and only if it is a trivial Kan fibration (Proposition 3.3.7.6), and that a monomorphism of simplicial sets $g : X \hookrightarrow Y$ is a weak homotopy equivalence if and only if it is anodyne (Corollary 3.3.7.7). In §3.3.8 we prove a refinement of Theorem 3.3.0.1, which guarantees that every Kan fibration $f : X \to S$ is actually *isomorphic* to the pullback of a Kan fibration $f' : X' \to S'$ between Kan complexes (Theorem 3.3.8.1).

3.3.1 Digression: Braced Simplicial Sets

00XT Let $\mathbf{\Delta}$ denote the simplex category (Definition 1.1.0.2), and let $\mathbf{\Delta}_{\mathrm{inj}}$ denote the subcategory of $\mathbf{\Delta}$ spanned by the injective maps (Definition 1.1.1.2). Composition with the inclusion functor $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ determines a forgetful functor from the category $\mathrm{Set}_{\Delta} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ of simplicial sets to the category $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}, \mathrm{Set})$ of semisimplicial sets (Remark 1.1.1.3). Our goal in this section is to show that this functor admits a faithful left adjoint, which we will denote by $S_{\bullet} \mapsto S_{\bullet}^{+}$. We begin by describing the essential image of this left adjoint.

00XU **Definition 3.3.1.1.** Let $X_{\bullet}$ be a simplicial set. We will say that $X_{\bullet}$ is *braced* if, for every nondegenerate simplex $\sigma \in X_n$ of dimension $n > 0$, the faces $\{d^n_i(\sigma)\}_{0 \leq i \leq n}$ are also nondegenerate.

00XV **Exercise 3.3.1.2.** Let $\mathcal{C}$ be a category. Show that the nerve $\mathrm{N}_{\bullet}(\mathcal{C})$ is braced if and only if $\mathcal{C}$ satisfies the following condition:

$(*)$ For every pair of morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$ satisfying $g \circ f = \mathrm{id}_X$, we have $X = Y$ and $f = g = \mathrm{id}_X$.

In particular, for any partially ordered set Q, the nerve $\mathrm{N}_{\bullet}(Q)$ is braced.

00XW **Example 3.3.1.3.** Every simplicial set of dimension ≤ 1 is braced.

00XX **Notation 3.3.1.4.** Let $X_{\bullet}$ be a simplicial set. For each nonnegative integer n, we let $X^{\mathrm{nd}}_n \subseteq X_n$ denote the collection of nondegenerate n-simplices of $X_{\bullet}$. If $X_{\bullet}$ is braced (Definition 3.3.1.1), then the face operators $\{d^n_i : X_n \to X_{n-1}\}_{0 \leq i \leq n}$ carry X^{nd}_n into X^{nd}_{n-1}. In this case, the construction $[n] \mapsto X^{\mathrm{nd}}_n$ determines a semisimplicial set, which we will denote by $X^{\mathrm{nd}}_{\bullet}$.

The terminology of Definition 3.3.1.1 is motivated by the heuristic that a braced simplicial set $X_{\bullet}$ is "supported" by the semisimplicial subset $X^{\mathrm{nd}}_{\bullet} \subseteq X_{\bullet}$. This heuristic is supported by the following:

00XY **Proposition 3.3.1.5.** *Let $X_{\bullet}$ and $Y_{\bullet}$ be simplicial sets, and suppose that $X_{\bullet}$ is braced. Then the restriction map*

$$\begin{tikzcd}
\{\text{Morphisms of simplicial sets } f : X_{\bullet} \to Y_{\bullet}\} \arrow[d] \\
\{\text{Morphisms of semisimplicial sets } f_0 : X^{\mathrm{nd}}_{\bullet} \to Y_{\bullet}\}
\end{tikzcd}$$

is a bijection.

Proof. Fix a morphism of semisimplicial sets $f_0 : X_\bullet^{\mathrm{nd}} \to Y_\bullet$; we wish to show that f_0 extends uniquely to a morphism of simplicial sets from $X_\bullet$ to $Y_\bullet$. Let σ be an n-simplex of $X_\bullet$. By virtue of Proposition 1.1.3.8, we can write σ uniquely as $\alpha^*(\tau)$, where $\alpha : [n] \twoheadrightarrow [m]$ is a nondecreasing surjection and τ is a nondegenerate m-simplex of $X_\bullet$. Define $f(\sigma) = \alpha^* f_0(\tau) \in Y_n$. It is clear that any extension of f_0 to a morphism of simplicial sets $X_\bullet \to Y_\bullet$ must be given by the construction $\sigma \mapsto f(\sigma)$. It will therefore suffice to show that the construction $\sigma \mapsto f(\sigma)$ is a morphism of simplicial sets.

Let σ, τ, and α be as above, and fix a nondecreasing map $\beta : [n'] \to [n]$. We wish to prove that $f(\beta^*\sigma) = \beta^* f(\sigma)$ in the set $Y_{n'}$. Note that $(\alpha \circ \beta) : [n'] \to [m]$ factors uniquely as a composition $[n'] \xrightarrow{\alpha'} [m'] \xrightarrow{\beta'} [m]$, where α' is surjective and β' is injective. Since $X_\bullet$ is braced, $\beta'^*(\tau)$ is a nondegenerate m'-simplex of $X_\bullet$. We now compute

$$\begin{aligned} f(\beta^*\sigma) &= f(\beta^*\alpha^*\tau) \\ &= f(\alpha'^*\beta'^*\tau) \\ &= \alpha'^* f_0(\beta'^*\tau) \\ &= \alpha'^*\beta'^* f_0(\tau) \\ &= \beta^*\alpha^* f_0(\tau) \\ &= \beta^* f(\sigma). \end{aligned}$$

where the second and fifth equality follow from the identity $\alpha \circ \beta = \beta' \circ \alpha'$, the third and sixth equality follow from the definition of f, and the fourth equality from the fact that f_0 is a morphism of semisimplicial sets. □

We now show that every semisimplicial set $S_\bullet$ can be obtained from the procedure of Notation 3.3.1.4.

Construction 3.3.1.6. Let $S_\bullet$ be a semisimplicial set. For each $n \geq 0$, we let S_n^+ denote the collection of pairs (α, τ) where $\alpha : [n] \twoheadrightarrow [m]$ is a nondecreasing surjection of linearly ordered sets and τ is an element of S_m. 00XZ

Let $\beta : [n'] \to [n]$ be a morphism in the category $\mathbf{\Delta}$. For every element $(\alpha, \tau) \in S_n^+$, the composite map $\alpha \circ \beta : [n'] \to [m]$ factors uniquely as a composition $[n'] \xrightarrow{\alpha'} [m'] \xrightarrow{\beta'} [m]$, where α' is surjective and β' is injective. We define a map $\beta^* : S_n^+ \to S_{n'}^+$ by the formula $\beta^*(\alpha, \tau) = (\alpha', \beta'^*(\tau)) \in S_{n'}^+$.

Proposition 3.3.1.7. *Let $S_\bullet$ be a semisimplicial set. Then:* 00Y0

(1) *The assignments*

$$([n] \in \mathbf{\Delta}) \mapsto S_n^+ \qquad (\beta : [n'] \to [n]) \mapsto (\beta^* : S_n^+ \to S_{n'}^+)$$

of Construction 3.3.1.6 define a simplicial set $S_\bullet^+$.

(2) *The construction* $(\tau \in S_n) \mapsto ((\mathrm{id}_{[m]}, \tau) \in S_n^+)$ *determines a monomorphism of semisimplicial sets* $\iota : S_\bullet \hookrightarrow S_\bullet^+$.

(3) *The simplicial set* $S_\bullet^+$ *is braced, and* ι *induces an isomorphism from* $S_\bullet$ *to the semisimplicial subset* $(S_\bullet^+)^{\mathrm{nd}} \subseteq S_\bullet^+$.

Proof. It follows immediately that for each $n \geq 0$, the function $\mathrm{id}_{[n]}^* : S_n^+ \to S_n^+$ is the identity map. To prove (1), it will suffice to show that for every pair of composable morphisms $[n''] \xrightarrow{\gamma} [n'] \xrightarrow{\beta} [n]$ in $\mathbf{\Delta}$, we have an equality $\gamma^* \circ \beta^* = (\beta \circ \gamma)^*$ of functions from S_n^+ to $S_{n''}^+$. Fix an element $(\alpha, \tau) \in S_n^+$, where $\alpha : [n] \to [m]$ is a surjective nondecreasing function and τ is an element of S_m. There is a unique commutative diagram

$$\begin{tikzcd}
{[n'']} \arrow[r, "\gamma"] \arrow[d, "\alpha''"] & {[n']} \arrow[r, "\beta"] \arrow[d, "\alpha'"] & {[n]} \arrow[d, "\alpha"] \\
{[m'']} \arrow[r, "\gamma'"] & {[m']} \arrow[r, "\beta'"] & {[m]}
\end{tikzcd}$$

in the category $\mathbf{\Delta}$, where the vertical maps are surjective and the lower horizontal maps are injective. We then compute

$$\begin{aligned}
(\gamma^* \circ \beta^*)(\alpha, \tau) &= \gamma^*(\alpha', \beta'^* \tau) \\
&= (\alpha'', \gamma'^* \beta'^* \tau) \\
&= (\alpha'', (\beta' \circ \gamma')^* \tau) \\
&= (\beta \circ \gamma)^*(\alpha, \tau),
\end{aligned}$$

which completes the proof of (1).

Assertion (2) is immediate from the definition. Note that if $\beta : [n'] \to [n]$ is a nondecreasing surjection, then the map $\beta^* : S_n^+ \to S_{n'}^+$ is given by the formula $\beta^*(\alpha, \tau) = (\alpha \circ \beta, \tau)$. It follows that an n-simplex $\sigma = (\alpha, \tau)$ of $S_\bullet^+$ is nondegenerate if and only if $\alpha : [n] \twoheadrightarrow [m]$ is a bijection: that is, if and only if σ belongs to the image of ι. Since the image of ι is closed under face operators (by virtue of (2)), we conclude that $S_\bullet^+$ is braced and that ι induces an isomorphism of semisimplicial sets $S_\bullet \simeq (S_\bullet^+)^{\mathrm{nd}}$. □

00Y1 **Corollary 3.3.1.8.** *Let* $\mathrm{Set}_\Delta^{\mathrm{br}} \subseteq \mathrm{Set}_\Delta$ *denote the (non-full) subcategory whose objects are braced simplicial sets and whose morphisms are maps* $f : X_\bullet \to Y_\bullet$ *which carry nondegenerate simplices of* $X_\bullet$ *to nondegenerate simplices of* $Y_\bullet$. *Then the construction* $X_\bullet \mapsto X_\bullet^{\mathrm{nd}}$ *induces an equivalence of categories* $\mathrm{Set}_\Delta^{\mathrm{br}} \to \{\text{Semisimplicial sets}\}$, *with homotopy inverse given by the construction* $S_\bullet \to S_\bullet^+$.

Proof. Let $X_\bullet$ and $Y_\bullet$ be braced simplicial sets. It follows from Proposition 3.3.1.5 that the restriction functor $\mathrm{Hom}_{\mathrm{Set}_\Delta}(X_\bullet, Y_\bullet) \to \mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}, \mathrm{Set})}(X_\bullet^{\mathrm{nd}}, Y_\bullet)$ is a bijection. Moreover, the image of $\mathrm{Hom}_{\mathrm{Set}_\Delta^{\mathrm{br}}}(X_\bullet, Y_\bullet)$ under this bijection is the collection of morphisms of semisimplicial sets from $X_\bullet^{\mathrm{nd}}$ to $Y_\bullet^{\mathrm{nd}} \subseteq Y_\bullet$. This proves full-faithfulness, and the essential surjectivity follows from Proposition 3.3.1.7. □

Corollary 3.3.1.9. *Let $S_\bullet$ be a semisimplicial set. Then, for every simplicial set $Y_\bullet$, composition with the map $\iota : S_\bullet \hookrightarrow S_\bullet^+$ induces a bijection* 00Y2

$$\begin{tikzcd}
\{\text{Morphisms of simplicial sets } f : S_\bullet^+ \to Y_\bullet\} \arrow[d] \\
\{\text{Morphisms of semisimplicial sets } f_0 : S_\bullet \to Y_\bullet\}.
\end{tikzcd}$$

Proof. Combine Proposition 3.3.1.5 with Proposition 3.3.1.7. □

Corollary 3.3.1.10. *The forgetful functor* 00Y3

$$\{\text{Simplicial sets}\} \to \{\text{Semisimplicial sets}\}$$

has a left adjoint, given on objects by the construction $S_\bullet \mapsto S_\bullet^+$.

Corollary 3.3.1.11. *Let $X_\bullet$ be a braced simplicial set. Then the inclusion of semisimplicial sets $g_0 : X_\bullet^{\mathrm{nd}} \hookrightarrow X_\bullet$ extends uniquely to an isomorphism $g : (X_\bullet^{\mathrm{nd}})^+ \simeq X_\bullet$.* 00Y4

Proof. It follows from Corollary 3.3.1.9 that g_0 extends uniquely to a map of simplicial sets $g : (X_\bullet^{\mathrm{nd}})^+ \to X_\bullet$. To show that g is an isomorphism, it will suffice to show that for every simplicial set $Y_\bullet$, composition with g induces a bijection

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Set}_\Delta}(X_\bullet, Y_\bullet) &\to \mathrm{Hom}_{\mathrm{Set}_\Delta}((X_\bullet^{\mathrm{nd}})^+, Y_\bullet) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}, \mathrm{Set})}(X_\bullet^{\mathrm{nd}}, Y_\bullet), \end{aligned}$$

which is precisely the content of Proposition 3.3.1.5. □

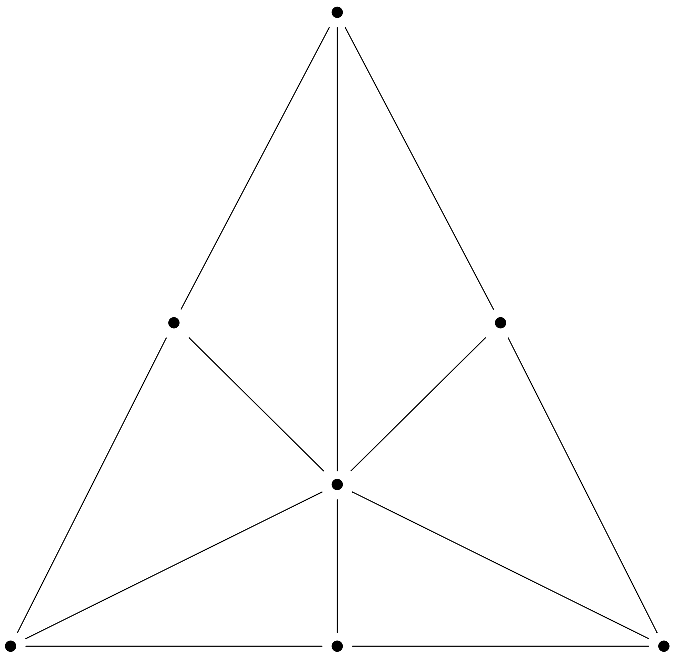
Example 3.3.1.12. Let M be a nonunital monoid and let $M^+ = M \cup \{e\}$ denote the monoid obtained from M by adjoining a unit element (Remark 1.3.2.11). Let $B_\bullet(M^+)$ denote the classifying simplicial set of Construction 1.3.2.5, and let $B_\bullet(M)$ be the semisimplicial set introduced in Variant 1.3.2.12. The inclusion map $M \hookrightarrow M^+$ induces a monomorphism of semisimplicial sets $\iota : B_\bullet M \hookrightarrow B_\bullet(M^+)$, whose image consists of the nondegenerate simplices of $B_\bullet(M^+)$. It follows that the simplicial set $B_\bullet(M^+)$ is braced and that ι extends to an isomorphism of simplicial sets $(B_\bullet M)^+ \xrightarrow{\sim} B_\bullet(M^+)$ (Corollary 3.3.1.11). 04HM

3.3.2 The Subdivision of a Simplex

00XH Let $n \geq 0$ be a nonnegative integer and let

$$|\Delta^n| = \{(t_0, t_1, \ldots, t_n) \in [0,1]^{n+1} : t_0 + t_1 + \cdots + t_n = 1\}$$

be the topological n-simplex. For every nonempty subset $S \subseteq [n] = \{0 < 1 < \cdots < n\}$, let $|\Delta^S|$ denote the corresponding face of $|\Delta^n|$, given by the collection of tuples $(t_0, \ldots, t_n) \in |\Delta^n|$ satisfying $t_i = 0$ for $i \notin S$. Let b_S denote the *barycenter* of the simplex $|\Delta^S|$: that is, the point $(t_0, \ldots, t_n) \in |\Delta^S| \subseteq |\Delta^n|$ given by $t_i = \begin{cases} \frac{1}{|S|} & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$ The collection of barycenters $\{b_S\}_{\emptyset \neq S \subseteq [n]}$ can be regarded as the vertices of a triangulation of $|\Delta^n|$, which we indicate in the case $n = 2$ by the following diagram:



In this section, we show that this triangulation arises from the identification of $|\Delta^n|$ with the geometric realization of another simplicial set (Proposition 3.3.2.3).

00XJ **Notation 3.3.2.1.** Let Q be a partially ordered set. We let $\mathrm{Chain}[Q]$ denote the collection of all nonempty, finite, linearly ordered subsets of Q. We regard $\mathrm{Chain}[Q]$ as a partially ordered set, where the partial order is given by inclusion. In the special case where $Q = [n] = \{0 < 1 < \ldots < n\}$ for some nonnegative integer n, we denote the partially ordered set $\mathrm{Chain}[Q]$ by $\mathrm{Chain}[n]$.

Remark 3.3.2.2 (Functoriality)**.** Let $f : Q \to Q'$ be a nondecreasing map between partially ordered sets. Then f induces a map $\mathrm{Chain}[f] : \mathrm{Chain}[Q] \to \mathrm{Chain}[Q']$, which carries each nonempty linearly ordered subset $S \subseteq Q$ to its image $f(S) \subseteq Q'$. By means of this construction, we can regard $Q \mapsto \mathrm{Chain}[Q]$ as functor from the category of partially ordered sets to itself. 00XK

Proposition 3.3.2.3. *Let $n \geq 0$ be an integer. Then there is a unique homeomorphism of topological spaces* 00XL

$$f : |\,\mathrm{N}_\bullet(\mathrm{Chain}[n])| \to |\Delta^n|$$

with the following properties:

(1) *For every nonempty subset $S \subseteq [n]$, the map f carries S (regarded as a vertex of $\mathrm{N}_\bullet(\mathrm{Chain}[n])$) to the barycenter $b_S \in |\Delta^S| \subseteq |\Delta^n|$.*

(2) *For every m-simplex $\sigma : \Delta^m \to \mathrm{N}_\bullet(\mathrm{Chain}[n])$, the composite map*

$$|\Delta^m| \xrightarrow{|\sigma|} |\,\mathrm{N}_\bullet(\mathrm{Chain}[n])| \xrightarrow{f} |\Delta^n|$$

is affine: that is, it extends to an $\mathbf{R}$-linear map from $\mathbf{R}^{m+1} \supseteq |\Delta^m|$ to $\mathbf{R}^{n+1} \supseteq |\Delta^n|$.

Proof. Note that an affine map $|\Delta^m| \to |\Delta^n|$ is uniquely determined by its values on the vertices of the topological m-simplex $|\Delta^m|$. From this observation, it is easy to deduce that there is a unique continuous function $f : |\,\mathrm{N}_\bullet(\mathrm{Chain}[n])| \to |\Delta^n|$ which satisfies conditions (1) and (2) of Proposition 3.3.2.3. We will complete the proof by showing that f is a homeomorphism. Since the domain and codomain of f are compact Hausdorff spaces, it will suffice to show that f is a bijection. Unwinding the definitions, this can be restated as follows:

$(*)$ For every point $(t_0, t_1, \ldots, t_n) \in |\Delta^n|$, there exists a unique chain $S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_m$ of subsets of $[n]$ and positive real numbers $(s_0, s_1, \ldots, s_m)$ satisfying the identities

$$\sum s_i = 1 \qquad (t_0, t_1, \ldots, t_n) = \sum s_i b_{S_i}.$$

We will deduce $(*)$ from the following more general assertion:

$(*')$ For every element $(t_0, t_1, \ldots, t_n) \in \mathbf{R}^{n+1}_{\geq 0}$, there exists a unique (possibly empty) chain $S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_m$ of subsets of $[n]$ and positive real numbers $(s_0, s_1, \ldots, s_m)$ satisfying $(t_0, t_1, \ldots, t_n) = \sum s_i b_{S_i}$.

Note that, if $(t_0, t_1, \ldots, t_n)$ and $(s_0, s_1, \ldots, s_m)$ are as in $(*')$, then we automatically have $\sum_{i=0}^{m} s_i = \sum_{j=0}^{n} t_j$. It follows that assertion $(*)$ is a special case of $(*')$. To prove $(*')$, let $K \subseteq [n]$ be the collection of those integers j for which $t_j \neq 0$. We proceed by induction on the cardinality of $k = |K|$. If $k = 0$ is empty, there is nothing to prove. Otherwise, set $r = \min\{kt_i\}_{i \in K}$. We can then write

$$(t_0, t_1, \ldots, t_n) = rb_K + (t'_0, t'_1, \ldots, t'_n)$$

for a unique sequence of nonnegative real numbers $(t'_0, \ldots, t'_n)$. Applying our inductive hypothesis to the sequence $(t'_0, \ldots, t'_n)$, we deduce that there is a unique chain of subsets $S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_{m-1}$ of $[n]$ and positive real numbers $(s_0, s_1, \ldots, s_{m-1})$ satisfying $(t'_0, t'_1, \ldots, t'_n) = \sum s_i b_{S_i}$. Note that each S_i is contained in K', and therefore properly contained in K. To complete the proof, we extend this sequence by setting $S_m = K$ and $s_m = r$. □

00XM **Remark 3.3.2.4** (Functoriality)**.** Let $\alpha : [m] \to [n]$ be a nondecreasing function between partially ordered sets, so that α induces a nondecreasing map $\mathrm{Chain}[\alpha] : \mathrm{Chain}[m] \to \mathrm{Chain}[n]$ (Remark 3.3.2.2). If α is injective, then the diagram of topological spaces

$$\begin{tikzcd}
|\mathrm{N}_\bullet(\mathrm{Chain}[m])| \arrow[r, "f_m", "\sim"'] \arrow[d, "|\mathrm{N}_\bullet(\mathrm{Chain}[\alpha])|"] & |\Delta^m| \arrow[d, "|\alpha|"] \\
|\mathrm{N}_\bullet(\mathrm{Chain}[n])| \arrow[r, "f_n", "\sim"'] & |\Delta^n|
\end{tikzcd}$$

is commutative, where the horizontal maps are the homeomorphisms supplied by Proposition 3.3.2.3. Beware that if α is not injective, this this diagram does not necessarily commute. For example, the induced map $|\Delta^m| \to |\Delta^n|$ carries the barycenter of $|\Delta^m|$ to the point

$$\left(\frac{|\alpha^{-1}(0)|}{m+1}, \frac{|\alpha^{-1}(1)|}{m+1}, \ldots, \frac{|\alpha^{-1}(n)|}{m+1}\right) \in |\Delta^n|,$$

which need not be the barycenter of any face $|\Delta^n|$.

It will be convenient to repackage Proposition 3.3.2.3 (and Remark 3.3.2.4) as a statement about the singular simplicial set functor $\mathrm{Sing}_\bullet : \mathrm{Top} \to \mathrm{Set}_\Delta$ of Construction 1.2.2.2. We first introduce a bit of notation (which will play an essential role throughout §3.3).

00XN **Construction 3.3.2.5** (The Ex Functor)**.** Let X be a simplicial set. For every nonnegative integer n, we let $\mathrm{Ex}_n(X)$ denote the collection of all morphisms of simplicial sets $\mathrm{N}_\bullet(\mathrm{Chain}[n]) \to X$. By virtue of Remark 3.3.2.2, the construction $([n] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto (\mathrm{Ex}_n(X) \in \mathrm{Set})$ determines a simplicial set which we will denote by $\mathrm{Ex}(X)$. The construction $X \mapsto \mathrm{Ex}(X)$ determines a functor from the category of simplicial sets to itself, which we denote by $\mathrm{Ex} : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$.

Remark 3.3.2.6. The construction $X \mapsto \mathrm{Ex}(X)$ can be regarded as a special case of Variant 1.2.2.8: it is the functor $\mathrm{Sing}^T_\bullet : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ associated to the cosimplicial object T of Set_Δ given by the construction $[n] \mapsto \mathrm{N}_\bullet(\mathrm{Chain}[n])$. 00XP

Remark 3.3.2.7. The functor $X \mapsto \mathrm{Ex}(X)$ preserves filtered colimits of simplicial sets. To prove this, it suffices to observe that each of the simplicial sets $\mathrm{N}_\bullet(\mathrm{Chain}[n])$ has only finitely many nondegenerate simplices (since the partially ordered set $\mathrm{Chain}[n]$) is finite). 00XQ

Example 3.3.2.8. Let $\mathcal{C}$ be a category and let $\mathrm{N}_\bullet(\mathcal{C})$ denote the nerve of $\mathcal{C}$. Then n-simplices of the simplicial set $\mathrm{Ex}(\mathrm{N}_\bullet(\mathcal{C}))$ can be identified with functors from the partially ordered set $\mathrm{Chain}[n]$ into $\mathcal{C}$ (see Proposition 1.3.3.1). 00XR

Example 3.3.2.9. Let X be a topological space and let $\mathrm{Sing}_\bullet(X)$ denote the singular simplicial set of X. For each nonnegative integer n, the n-simplices of $\mathrm{Sing}_\bullet(X)$ are given by continuous functions $|\Delta^n| \to X$, and the n-simplices of $\mathrm{Ex}(\mathrm{Sing}_\bullet(X))$ are given by continuous functions $|\mathrm{N}_\bullet(\mathrm{Chain}[n])| \to X$. The homeomorphism $|\mathrm{N}_\bullet(\mathrm{Chain}[n])| \simeq |\Delta^n|$ of Proposition 3.3.2.3 determines a bijection $\mathrm{Sing}_n(X) \xrightarrow{\sim} \mathrm{Ex}_n(\mathrm{Sing}_\bullet(X))$, and Remark 3.3.2.4 guarantees that these bijections are compatible with the face operators on the simplicial sets $\mathrm{Sing}_\bullet(X)$ and $\mathrm{Ex}(\mathrm{Sing}_\bullet(X))$. In other words, Proposition 3.3.2.3 supplies an isomorphism of *semisimplicial* sets $\varphi : \mathrm{Sing}_\bullet(X) \xrightarrow{\sim} \mathrm{Ex}(\mathrm{Sing}_\bullet(X))$. Beware that φ is generally not an isomorphism of simplicial sets: that is, it usually does not commute with the degeneracy operators on $\mathrm{Sing}_\bullet(X)$ and $\mathrm{Ex}(\mathrm{Sing}_\bullet(X))$. 00XS

Variant 3.3.2.10 (Ex for Semisimplicial Sets)**.** Note that, for every nonnegative integer n, the simplicial set $\mathrm{N}_\bullet(\mathrm{Chain}[n])$ is braced (Exercise 3.3.1.2). If X is a semisimplicial set, we write $\mathrm{Ex}_n(X)$ for the collection of all morphisms of semisimplicial sets $\mathrm{N}_\bullet(\mathrm{Chain}[n])^{\mathrm{nd}} \to X$; here $\mathrm{N}_\bullet(\mathrm{Chain}[n])^{\mathrm{nd}}$ denotes the semisimplicial subset of $\mathrm{N}_\bullet(\mathrm{Chain}[n])$ spanned by the nondegenerate simplices. The construction $[n] \mapsto \mathrm{Ex}_n(X)$ determines a semisimplicial set, which we denote by $\mathrm{Ex}(X)$. 0102

Note that, if X is the underlying semisimplicial set of a simplicial set Y, then $\mathrm{Ex}(X)$ is the underlying semisimplicial set of the simplicial set $\mathrm{Ex}(Y)$ given by Construction 3.3.2.5 (this is a special case of Proposition 3.3.1.5). In other words, the construction $X \mapsto \mathrm{Ex}(X)$ determines a functor from the category of semisimplicial sets to itself which fits into a commutative diagram

$$\begin{tikzcd}
\{\text{Simplicial sets}\} \arrow[r, "\mathrm{Ex}"] \arrow[d] & \{\text{Simplicial sets}\} \arrow[d] \\
\{\text{Semisimplicial sets}\} \arrow[r, "\mathrm{Ex}"] & \{\text{Semisimplicial sets}\}.
\end{tikzcd}$$

3.3.3 The Subdivision of a Simplicial Set

00Y5 Let $n \geq 0$ be a nonnegative integer. In §3.3.2, we showed that the topological n-simplex $|\Delta^n|$ can be identified with the geometric realization of the set of its faces $\mathrm{Chain}[n]$, partially ordered by inclusion (Proposition 3.3.2.3). We now prove a generalization of this result, replacing the standard simplex Δ^n by an arbitrary braced simplicial set X and the nerve $\mathrm{N}_{\bullet}(\mathrm{Chain}[n])$ by another simplicial set $\mathrm{Sd}(X)$, which we will refer to as the *subdivision* of X.

00Y6 **Definition 3.3.3.1** (Subdivision)**.** Let X and Y be simplicial sets. We will say that a morphism of simplicial sets $u : X \to \mathrm{Ex}(Y)$ *exhibits Y as a subdivision of X* if, for every simplicial set Z, composition with u induces a bijection $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(Y, Z) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(X, \mathrm{Ex}(Z))$ (see Construction 3.3.2.5).

00Y7 **Notation 3.3.3.2.** Let X be a simplicial set. It follows immediately from the definitions that if there exists a simplicial set Y and a morphism $u : X \to \mathrm{Ex}(Y)$ which exhibits Y as a subdivision of X, then the simplicial set Y (and the morphism u) are uniquely determined up to isomorphism and depend functorially on X. To emphasize this dependence, we will denote Y by $\mathrm{Sd}(X)$ and refer to it as *the* subdivision of X.

00Y8 **Proposition 3.3.3.3.** *Let X be a simplicial set. Then there exists another simplicial set $\mathrm{Sd}(X)$ and a morphism $u : X \to \mathrm{Ex}(\mathrm{Sd}(X))$ which exhibits $\mathrm{Sd}(X)$ as a subdivision of X, in the sense of Notation 3.3.3.2.*

Proof. By virtue of Remark 3.3.2.6, this is a special case of Proposition 1.2.3.15. □

00Y9 **Corollary 3.3.3.4.** *The functor $\mathrm{Ex} : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ admits a left adjoint, given by the construction $X \mapsto \mathrm{Sd}(X)$.*

00YA **Example 3.3.3.5.** Let n be a nonnegative integer. Then the identity map

$$\mathrm{id} : \mathrm{N}_{\bullet}(\mathrm{Chain}[n]) \to \mathrm{N}_{\bullet}(\mathrm{Chain}[n])$$

determines a map of simplicial sets $u : \Delta^n \to \mathrm{Ex}(\mathrm{N}_{\bullet}(\mathrm{Chain}[n]))$, which exhibits $\mathrm{N}_{\bullet}(\mathrm{Chain}[n])$ as the subdivision of Δ^n. In particular, the subdivision $\mathrm{Sd}(\Delta^2)$ is the 2-dimensional simplicial set indicated in the diagram

$$\begin{tikzcd}
 & & \{1\} \arrow[dl] \arrow[dd] \arrow[dr] & & \\
 & \{0,1\} \arrow[dr] & & \{1,2\} \arrow[dl] & \\
 & & \{0,1,2\} & & \\
\{0\} \arrow[uur] \arrow[urr] \arrow[rr] & & \{0,2\} \arrow[u] & & \{2\}. \arrow[uul] \arrow[ull] \arrow[ll]
\end{tikzcd}$$

Proposition 3.3.3.6. *Let X be a braced simplicial set. Then there is a canonical homeomorphism of topological spaces $f_X : |\operatorname{Sd}(X)| \to |X|$.* 00YB

Proof. For every topological space Y, Example 3.3.2.9 supplies an isomorphism of semisimplicial sets $\operatorname{Sing}_\bullet(Y) \to \operatorname{Ex}(\operatorname{Sing}_\bullet(Y))$. These isomorphisms depend functorially on Y, and can therefore be regarded as an isomorphism of functors $G \circ \operatorname{Sing}_\bullet \xrightarrow{\sim} G \circ \operatorname{Ex} \circ \operatorname{Sing}_\bullet$, where $G : \operatorname{Set}_\Delta \to \operatorname{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}, \operatorname{Set})$ denotes the forgetful functor from simplicial sets to semisimplicial sets. Passing to left adjoints, we conclude that for every semisimplicial set $S_\bullet$, we have a canonical homeomorphism $|\operatorname{Sd}(S^+_\bullet)| \simeq |S^+_\bullet|$, depending functorially on $S_\bullet$. Proposition 3.3.3.6 now follows from Corollary 3.3.1.11 (applied to the semisimplicial set X^{nd}). □

Remark 3.3.3.7. The homeomorphisms $f_X : |\operatorname{Sd}(X)| \simeq |X|$ constructed in the proof of Proposition 3.3.3.6 are characterized by the following properties: 00YC

- In the special case where $X = \Delta^n$ is a standard simplex, f_X is given by the composition

  $$|\operatorname{Sd}(\Delta^n)| \simeq |\operatorname{N}_\bullet(\operatorname{Chain}[n])| \xrightarrow{f} |\Delta^n|,$$

  where the first map is supplied by the identification $\operatorname{Sd}(\Delta^n) \simeq \operatorname{N}_\bullet(\operatorname{Chain}[n])$ of Example 3.3.3.5 and f is the homeomorphism of Proposition 3.3.2.3.

- Let $u : X \to Y$ be a morphism of braced simplicial sets which carries nondegenerate simplices of X to nondegenerate simplices of Y. Then the diagram of topological spaces

$$\begin{tikzcd}
\lvert \mathrm{Sd}(X) \rvert \arrow[r, "f_X"] \arrow[r, "\sim"'] \arrow[d, "\lvert \mathrm{Sd}(u) \rvert"] & \lvert X \rvert \arrow[d, "\lvert u \rvert"] \\
\lvert \mathrm{Sd}(Y) \rvert \arrow[r, "f_Y"] \arrow[r, "\sim"'] & \lvert Y \rvert
\end{tikzcd}$$

  commutes.

00YD **Warning 3.3.3.8.** Let $u : X \to Y$ be a morphism of braced simplicial sets. If u does not carry nondegenerate simplices of X to nondegenerate simplices of Y, then the diagram of topological spaces

$$\begin{tikzcd}
\lvert \mathrm{Sd}(X) \rvert \arrow[r, "f_X"] \arrow[d, "\lvert \mathrm{Sd}(u) \rvert"] \arrow[d, "\sim"'] & \lvert X \rvert \arrow[d, "\lvert u \rvert"] \\
\lvert \mathrm{Sd}(Y) \rvert \arrow[r, "f_Y"] \arrow[r, "\sim"'] & \lvert Y \rvert
\end{tikzcd}$$

does not necessarily commute (this phenomenon occurs already in the case where X and Y are simplices: see Remark 3.3.2.4).

The subdivision construction is closely related to the category of simplices introduced in §1.2.3.

04HN **Construction 3.3.3.9.** Let X be a simplicial set and let $\mathbf{\Delta}_X$ denote the category of simplices of X (Construction 1.1.3.9). Unwinding the definitions, we see that n-simplices σ of the simplicial set $\mathrm{N}_\bullet(\mathbf{\Delta}_X)$ can be identified with diagrams of simplicial sets

$$\Delta^{k_0} \to \Delta^{k_1} \to \cdots \to \Delta^{k_n} \xrightarrow{\tau} X.$$

For $0 \leq i \leq n$, let $S_i \subseteq [k_n]$ be the image of the underlying map of linearly ordered sets $[k_0] \to [k_n]$, and suppose we are given a morphism $u : X \to \mathrm{Ex}(Y)$ which exhibits Y as a subdivision of X. Then u carries τ to a k_n-simplex of $\mathrm{Ex}(Y)$, which we can identify with a morphism $\mathrm{N}_\bullet(\mathrm{Chain}[k_n]) \to Y$ carrying $(S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n)$ to an n-simplex σ' of Y. The construction $\sigma \mapsto \sigma'$ depends functorially on $[n]$, and therefore determines a comparison map $\psi_X : \mathrm{N}_\bullet(\mathbf{\Delta}_X) \to Y = \mathrm{Sd}(X)$.

04HP **Example 3.3.3.10.** Let $X = \Delta^n$ be a standard simplex. Then the comparison map $\psi_X : \mathrm{N}_\bullet(\mathbf{\Delta}_X) \to \mathrm{Sd}(X)$ of Construction 3.3.3.9, can be identified with the nerve of the functor $\mathbf{\Delta}_X \to \mathrm{Chain}[n]$, which carries each morphism $\Delta^m \to \Delta^n$ to the image of the underlying map of linearly ordered sets $[m] \to [n]$.

Notation 3.3.3.11. Let X be a simplicial set and let $\mathbf{\Delta}_X$ be the category of simplices of X (Construction 1.1.3.9). By definition, the objects of $\mathbf{\Delta}_X$ are given by pairs $([n], \sigma)$, where n is a nonnegative integer and σ is an n-simplex of X. We let $\mathbf{\Delta}_X^{\mathrm{nd}}$ denote the full subcategory of $\mathbf{\Delta}_X$ spanned by those pairs $([n], \sigma)$ where σ is a nondegenerate n-simplex of X. We will refer to $\mathbf{\Delta}_X^{\mathrm{nd}}$ as the *category of nondegenerate simplices of X*. 00YE

Example 3.3.3.12. Let S be a semisimplicial set, and let S^+ be the braced simplicial set given by Construction 3.3.1.6. Then the category of nondegenerate simplices $\mathbf{\Delta}_{S^+}^{\mathrm{nd}}$ can be described concretely as follows: 00YF

- The objects of $\mathbf{\Delta}_{S^+}^{\mathrm{nd}}$ are pairs $([n], \sigma)$, where $[n]$ is an object of $\mathbf{\Delta}_{\mathrm{inj}}$ and σ is an element of S_n.
- A morphism from $([n], \sigma)$ to $([n'], \sigma')$ in $\mathbf{\Delta}_{S^+}^{\mathrm{nd}}$ is a strictly increasing function $\alpha : [n] \hookrightarrow [n']$ satisfying $\sigma = \alpha^*(\sigma')$ in the set S_n.

In other words, $\mathbf{\Delta}_{S^+}^{\mathrm{nd}}$ is the *category of elements* of the functor $S : \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \to \mathrm{Set}$ (see Variant 5.2.6.2).

Example 3.3.3.13. Let Q be a partially ordered set, and let $\mathrm{N}_\bullet(Q)$ denote its nerve. By definition, the nondegenerate n-simplices of $\mathrm{N}_\bullet(Q)$ can be identified with the strictly increasing functions $\sigma : \{0 < 1 < \cdots < n\} \to Q$. The construction $([n], \sigma) \mapsto \mathrm{im}(\sigma)$ determines an *isomorphism* from the category of nondegenerate simplices $\mathbf{\Delta}_{\mathrm{N}_\bullet(Q)}^{\mathrm{nd}}$ to the partially ordered set $\mathrm{Chain}[Q]$ of Notation 3.3.2.1. 00YJ

Warning 3.3.3.14. Though the category $\mathbf{\Delta}_X^{\mathrm{nd}}$ is defined for any simplicial set X, it is primarily useful in the case where X is braced (where we can use the description supplied by Example 3.3.3.12). 00YG

Exercise 3.3.3.15. Let X be a simplicial set. Show that X is braced if and only if the inclusion functor $\mathbf{\Delta}_X^{\mathrm{nd}} \hookrightarrow \mathbf{\Delta}_X$ admits a left adjoint. 00YH

Proposition 3.3.3.16. *Let X be a braced simplicial set. Then the comparison map $\psi_X : \mathrm{N}_\bullet(\mathbf{\Delta}_X) \to \mathrm{Sd}(X)$ of Construction 3.3.3.9 restricts to an isomorphism $\psi_X^\circ : \mathrm{N}_\bullet(\mathbf{\Delta}_X^{\mathrm{nd}}) \xrightarrow{\sim} \mathrm{Sd}(X)$.* 00YL

Example 3.3.3.17. Let Q be a partially ordered set. Combining Proposition 3.3.3.16 with Example 3.3.3.13, we obtain a canonical isomorphism $\mathrm{Sd}(\mathrm{N}_\bullet(Q)) \simeq \mathrm{N}_\bullet(\mathrm{Chain}[Q])$. In the special case $Q = [n]$, this recovers the isomorphism $\mathrm{Sd}(\Delta^n) \simeq \mathrm{N}_\bullet(\mathrm{Chain}[n])$ of Example 3.3.3.5. 00YM

The proof of Proposition 3.3.3.16 will make use of the following:

`00YQ` **Lemma 3.3.3.18.** *The functor*

$$\{\text{Semisimplicial Sets}\} \to \{\text{Simplicial Sets}\} \qquad S \mapsto \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{nd}}_{S^+})$$

preserves colimits.

Proof. Let n be a nonnegative integer. For every semisimplicial set S, Example 3.3.3.12 allows us to identify n-simplices of the nerve $\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{nd}}_{S^+})$ with the set of pairs (τ, σ), where τ is a m-simplex of $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}})$ (given by a diagram of increasing functions $[k_0] \hookrightarrow [k_1] \hookrightarrow \cdots \hookrightarrow [k_n]$) and σ is an element of the set S_{k_n}. It follows that the functor $S \mapsto \mathrm{N}_n(\mathbf{\Delta}^{\mathrm{nd}}_{S^+})$ preserves colimits. Allowing n to vary, we conclude that the functor $S \mapsto \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{nd}}_{S^+})$ preserves colimits. □

`04HQ` **Variant 3.3.3.19.** The proof of Lemma 3.3.3.18 also shows that the functor

$$\mathrm{Set}_\Delta \to \mathrm{Set}_\Delta \qquad X \mapsto \mathrm{N}_\bullet(\mathbf{\Delta}_X)$$

preserves colimits. Consequently, the comparison maps $\psi_X : \mathrm{N}_\bullet(\mathbf{\Delta}_X) \to \mathrm{Sd}(X)$ of Construction 3.3.3.9 are uniquely determined by the following properties:

- The construction $X \mapsto \psi_X$ is functorial: that is, it determines a natural transformation from the functor $X \mapsto \mathrm{N}_\bullet(\mathbf{\Delta}_X)$ to the subdivision functor Sd.
- When $X = \Delta^n$ is a standard simplex, ψ_X is the nerve of the functor

  $$\mathbf{\Delta}_X \to \mathrm{Chain}[n] \qquad (\alpha : [m] \to [n] \mapsto \mathrm{im}(\alpha) \subseteq [n])$$

  described in Example 3.3.3.10.

Proof of Proposition 3.3.3.16. Let X be a braced simplicial set. By virtue of Corollary 3.3.1.8, we can assume that $X = S^+$ for some semisimplicial set S. Let φ_S denote the composite map

$$\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{nd}}_{S^+}) \hookrightarrow \mathrm{N}_\bullet(\mathbf{\Delta}_X) \xrightarrow{\psi_X} \mathrm{Sd}(X) = \mathrm{Sd}(S^+);$$

we wish to show that φ_S is an isomorphism. By virtue of Lemma 3.3.3.18, the construction $S \mapsto \varphi_S$ commutes with small colimits. Since every functor $S : \mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}} \to \mathrm{Set}$ can be written as a colimit of representable functors (see §[?]), we may assume without loss of generality that S is the semisimplicial set represented by an object $[n] \in \mathbf{\Delta}_{\mathrm{inj}}$; that is, $X = \Delta^n$ is the standard simplex. In this case, the conclusion follows immediately from the concrete description of ψ_X given in Example 3.3.3.10. □

Remark 3.3.3.20 (Functoriality)**.** Let $u : X \to Y$ be a morphism of braced simplicial sets. Then u induces a morphism between their subdivisions 00YN

$$\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_X^{\mathrm{nd}}) \simeq \mathrm{Sd}(X) \xrightarrow{\mathrm{Sd}(u)} \mathrm{Sd}(Y) \simeq \mathrm{N}_{\bullet}(\boldsymbol{\Delta}_Y^{\mathrm{nd}}),$$

which can be identified with a functor $U : \boldsymbol{\Delta}_X^{\mathrm{nd}} \to \boldsymbol{\Delta}_Y^{\mathrm{nd}}$ (Proposition 1.3.3.1). If u carries nondegenerate simplices of X to nondegenerate simplices of Y, then the functor U is easy to describe: it is given on objects by the formula $U([n], \sigma) = ([n], u(\sigma))$. More generally, U carries an object $([n], \sigma) \in \boldsymbol{\Delta}_X^{\mathrm{nd}}$ to an object $([m], \tau) \in \boldsymbol{\Delta}_Y^{\mathrm{nd}}$, characterized by the requirement that $u(\sigma)$ factors as a composition $\Delta^n \twoheadrightarrow \Delta^m \xrightarrow{\tau} Y$ (see Proposition 1.1.3.8).

Warning 3.3.3.21. In the statement of Proposition 3.3.3.16, the hypothesis that X is braced cannot be omitted. For example, let X be the simplicial set $\Delta^2 \coprod_{\Delta^1} \Delta^0$ obtained from the standard 2-simplex by collapsing a single edge, which we depict informally by the diagram 00YP

$$\begin{tikzcd}
& \bullet \arrow[dl, equal] \arrow[dr] & \\
\bullet \arrow[rr] & & \bullet.
\end{tikzcd}$$

Then the subdivision of X is the 2-dimensional simplicial set depicted in the diagram

$$\begin{tikzcd}
 & & \bullet \arrow[dl, equal] \arrow[dd] \arrow[dr] & & \\
 & \bullet \arrow[ddl, equal] \arrow[dr] & & \bullet \arrow[dl] & \\
 & & \bullet & & \\
\bullet \arrow[urr] \arrow[rr] & & \bullet \arrow[u] & & \bullet . \arrow[ll] \arrow[ull] \arrow[uul]
\end{tikzcd}$$

This simplicial set cannot arise as the nerve of a category, because it contains a nondegenerate 2-simplex σ for which $d^2_2(\sigma)$ is degenerate.

3.3.4 The Last Vertex Map

00YR Let X be a simplicial set and let $\mathrm{Sd}(X)$ denote its subdivision (Notation 3.3.3.2). If X is braced, then Proposition 3.3.3.6 supplies a canonical homeomorphism of topological spaces $|\mathrm{Sd}(X)| \simeq |X|$. Beware that X and $\mathrm{Sd}(X)$ need not be isomorphic as simplicial sets: for example, the standard simplex $X = \Delta^n$ has $n+1$ vertices, while subdivision $\mathrm{Sd}(\Delta^n)$ has $2^{n+1} - 1$ vertices. Nevertheless, we will prove in this section that X and $\mathrm{Sd}(X)$ are weakly homotopy equivalent. More precisely, for every simplicial set X there is a canonical weak homotopy equivalence $\lambda_X : \mathrm{Sd}(X) \to X$, which we refer to as the *last vertex map* (Construction 3.3.4.3).

00YS **Notation 3.3.4.1.** Let Q be a partially ordered set. Every finite, nonempty, linearly ordered subset $S \subseteq Q$ has a largest element, which we will denote by $\mathrm{Max}(S)$. The construction $S \mapsto \mathrm{Max}(S)$ determines a nondecreasing function $\mathrm{Max} : \mathrm{Chain}[Q] \to Q$, where $\mathrm{Chain}[Q]$ is defined as in Notation 3.3.2.1.

Remark 3.3.4.2. Let $f : P \to Q$ be a nondecreasing function between partially ordered sets. Then the diagram of partially ordered sets `00YT`

$$\begin{tikzcd}
\mathrm{Chain}[P] \arrow[r, "\mathrm{Max}"] \arrow[d, "S \mapsto f(S)"] & P \arrow[d, "f"] \\
\mathrm{Chain}[Q] \arrow[r, "\mathrm{Max}"] & Q
\end{tikzcd}$$

is commutative.

Construction 3.3.4.3. Let X be a simplicial set. For every n-simplex $\sigma : \Delta^n \to X$, we let $\rho_X(\sigma)$ denote the composite map `00YU`

$$\mathrm{N}_\bullet(\mathrm{Chain}[n]) \xrightarrow{\mathrm{Max}} \Delta^n \xrightarrow{\sigma} X,$$

which we regard as an n-simplex of the simplicial set $\mathrm{Ex}(X)$ of Construction 3.3.2.5. It follows from Remark 3.3.4.2 that the construction $\sigma \mapsto \rho_X(\sigma)$ determines a map of simplicial sets $\rho_X : X \to \mathrm{Ex}(X)$.

Let $u : X \to \mathrm{Ex}(\mathrm{Sd}(X))$ be a map of simplicial sets which exhibits $\mathrm{Sd}(X)$ as a subdivision of X (Definition 3.3.3.1). Then there is a unique map of simplicial sets $\lambda_X : \mathrm{Sd}(X) \to X$ for which the composition $X \to \mathrm{Ex}(\mathrm{Sd}(X)) \xrightarrow{\mathrm{Ex}(\lambda_X)} \mathrm{Ex}(X)$ is equal to ρ_X. We will refer to λ_X as the *last vertex map* of X.

Remark 3.3.4.4 (Functoriality)**.** The morphisms $\rho_X : X \to \mathrm{Ex}(X)$ and $\lambda_X : \mathrm{Sd}(X) \to X$ depend functorially on the simplicial set X. That is, for every map of simplicial sets $f : X \to Y$, the diagrams `00YV`

$$\begin{tikzcd}
X \arrow[r, "\rho_X"] \arrow[d, "f"] & \mathrm{Ex}(X) \arrow[d, "\mathrm{Ex}(f)"] \\
Y \arrow[r, "\rho_Y"] & \mathrm{Ex}(Y)
\end{tikzcd} \qquad \begin{tikzcd}
\mathrm{Sd}(X) \arrow[r, "\lambda_X"] \arrow[d, "\mathrm{Sd}(f)"] & X \arrow[d, "f"] \\
\mathrm{Sd}(Y) \arrow[r, "\lambda_Y"] & Y
\end{tikzcd}$$

are commutative. We may therefore regard the constructions $X \mapsto \rho_X$ and $X \mapsto \lambda_X$ as natural transformations of functors

$$\rho : \mathrm{id}_{\mathrm{Set}_\Delta} \to \mathrm{Ex} \qquad \lambda : \mathrm{Sd} \to \mathrm{id}_{\mathrm{Set}_\Delta}.$$

Example 3.3.4.5. Let Q be a partially ordered set, so that we can identify the subdivision of $\mathrm{N}_\bullet(Q)$ with the nerve of the partially ordered set $\mathrm{Chain}[Q]$ (Example 3.3.3.17). Under this identification, the last vertex map $\lambda_{\mathrm{N}_\bullet(Q)}$ corresponds to the morphism $\mathrm{N}_\bullet(\mathrm{Chain}[Q]) \to \mathrm{N}_\bullet(Q)$ induced by $\mathrm{Max} : \mathrm{Chain}[Q] \to Q$. `00YW`

00YX **Example 3.3.4.6.** Let X be a discrete simplicial set (Definition 1.1.5.10). Then the maps

$$\rho_X : X \to \mathrm{Ex}(X) \qquad \lambda_X : \mathrm{Sd}(X) \to X$$

are isomorphisms.

00YY **Example 3.3.4.7.** Let X be a braced simplicial set, so that the subdivision $\mathrm{Sd}(X)$ can be identified with the nerve of the category $\mathbf{\Delta}^{\mathrm{nd}}_X$ of nondegenerate simplices of X (Proposition 3.3.3.16). Under this identification, the last vertex map λ_X corresponds to a morphism of simplicial sets $\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{nd}}_X) \to X$. Concretely, if τ is a k-simplex of $\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{nd}}_X)$ corresponding to a diagram

$$\begin{tikzcd}
\Delta^{n_0} \arrow[r] \arrow[drr, "\sigma_0"] & \Delta^{n_1} \arrow[r] \arrow[dr, "\sigma_1"] & \cdots \arrow[r] & \Delta^{n_{k-1}} \arrow[r] \arrow[dl, "\sigma_{k-1}"] & \Delta^{n_k} \arrow[dll, "\sigma_k"] \\
 & & X, & &
\end{tikzcd}$$

then $\lambda_X(\tau)$ is the k-simplex of X given by the composition

$$\Delta^k \xrightarrow{f} \Delta^{n_k} \xrightarrow{\sigma_k} X,$$

where f carries each vertex $\{i\} \subseteq \Delta^k$ to the image of the last vertex $\{n_i\} \subseteq \Delta^{n_i}$ under the map $\Delta^{n_i} \to \Delta^{n_k}$ given by horizontal composition in the diagram.

We can now state the main result of this section:

00YZ **Proposition 3.3.4.8.** *Let X be a simplicial set. Then the last vertex map $\lambda_X : \mathrm{Sd}(X) \to X$ is a weak homotopy equivalence.*

00Z0 **Remark 3.3.4.9.** Proposition 3.3.4.8 has a counterpart for the comparison map $\rho_X : X \to \mathrm{Ex}(X)$, which we will prove in §3.3.5 (see Theorem 3.3.5.1).

Proof of Proposition 3.3.4.8. For each integer $n \geq 0$, let $\mathrm{sk}_n(X)$ denote the n-skeleton of the simplicial set X. Then the last vertex map $\lambda_X : \mathrm{Sd}(X) \to X$ can be realized as a filtered colimit of the last vertex maps $\lambda_{\mathrm{sk}_n(X)} : \mathrm{Sd}(\mathrm{sk}_n(X)) \to \mathrm{sk}_n(X)$. Since the collection of weak homotopy equivalences is closed under the formation of filtered colimits (Proposition 3.2.8.3), it will suffice to show that each of the maps $\lambda_{\mathrm{sk}_n(X)}$ is a weak homotopy equivalence. We may therefore replace X by $\mathrm{sk}_n(X)$, and thereby reduce to the case where X is n-skeletal for some nonnegative integer $n \geq 0$. We proceed by induction on n. If $n = 0$, then the simplicial set X is discrete and λ_X is an isomorphism (Example 3.3.4.6). We will therefore assume that $n > 0$.

Fix a Kan complex Q; we wish to show that composition with $\lambda_X : \mathrm{Sd}(X) \to X$ induces a bijection $\pi_0(\mathrm{Fun}(X, Q)) \to \pi_0(\mathrm{Fun}(\mathrm{Sd}(X), Q))$. In fact, we will show that the map $\mathrm{Fun}(X, Q) \to \mathrm{Fun}(\mathrm{Sd}(X), Q)$ is a weak homotopy equivalence. Let $Y = \mathrm{sk}_{n-1}(X)$ be the $(n-1)$-skeleton of X, so that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(X, Q) \arrow[r, "\theta"] \arrow[d] & \mathrm{Fun}(\mathrm{Sd}(X), Q) \arrow[d] \\
\mathrm{Fun}(Y, Q) \arrow[r] & \mathrm{Fun}(\mathrm{Sd}(Y), Q),
\end{tikzcd}$$

where the lower horizontal map is a homotopy equivalence by virtue of our inductive hypothesis (together with Proposition 3.1.6.17). It will therefore suffice to show that, for every morphism of simplicial sets $f : Y \to Q$, the induced map of fibers

$$\theta_f : \{f\} \times_{\mathrm{Fun}(Y,Q)} \mathrm{Fun}(X, Q) \to \{f\} \times_{\mathrm{Fun}(\mathrm{Sd}(Y),Q)} \mathrm{Fun}(\mathrm{Sd}(X), Q)$$

is a homotopy equivalence (Proposition 3.2.8.1).

Let S denote the collection of nondegenerate n-simplices of X, let $X' = \coprod_{\sigma \in S} \Delta^n$ denote their coproduct, and let $Y' = \coprod_{\sigma \in S} \partial \Delta^n$ denote the boundary of X'. Proposition 1.1.4.12 then supplies a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod\limits_{\sigma \in S} \partial \Delta^n \arrow[r] \arrow[d] & \coprod\limits_{\sigma \in S} \Delta^n \arrow[d] \\
Y \arrow[r] & X,
\end{tikzcd}$$

which we can use to identify θ_f with the induced map

$$\theta'_f : \{f\} \times_{\mathrm{Fun}(Y',Q)} \mathrm{Fun}(X', Q) \to \{f\} \times_{\mathrm{Fun}(\mathrm{Sd}(Y'),Q)} \mathrm{Fun}(\mathrm{Sd}(X'), Q).$$

Invoking Proposition 3.2.8.1 again, we are reduced to showing that the horizontal maps appearing in the diagram

$$\begin{tikzcd}
\mathrm{Fun}(X', Q) \arrow[r, "\theta"] \arrow[d] & \mathrm{Fun}(\mathrm{Sd}(X'), Q) \arrow[d] \\
\mathrm{Fun}(Y', Q) \arrow[r] & \mathrm{Fun}(\mathrm{Sd}(Y'), Q)
\end{tikzcd}$$

are homotopy equivalences. By virtue of Proposition 3.1.6.17, it will suffice to show that the last vertex maps $\lambda_{Y'} : \mathrm{Sd}(Y') \to Y'$ and $\lambda_{X'} : \mathrm{Sd}(X') \to X'$ are weak homotopy equivalences. In the first case, this follows from our inductive hypothesis (since Y' has dimension $< n$). In the second, we can use Remark 3.1.6.20 to reduce to the problem of showing that the last vertex map $\lambda_{\Delta^n} : \mathrm{Sd}(\Delta^n) \to \Delta^n$ is a weak homotopy equivalence. This is clear, since both $\mathrm{Sd}(\Delta^n)$ and Δ^n are contractible by virtue of Example 3.2.4.2 (they can be realized as the nerves of partially ordered sets $\mathrm{Chain}[n]$ and $[n]$, each of which has a largest element). □

3.3.5 Comparison of X with $\mathrm{Ex}(X)$

00Z1 The goal of this section is to prove the following variant of Proposition 3.3.4.8:

00Z2 **Theorem 3.3.5.1.** *Let X be a simplicial set. Then the comparison map $\rho_X : X \to \mathrm{Ex}(X)$ of Construction 3.3.4.3 is a weak homotopy equivalence.*

0103 **Corollary 3.3.5.2.** *Let $f : X \to Y$ be a morphism of simplicial sets. Then f is a weak homotopy equivalence if and only if $\mathrm{Ex}(f)$ is a weak homotopy equivalence.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\rho_X"] & Y \arrow[d, "\rho_Y"] \\
\mathrm{Ex}(X) \arrow[r, "\mathrm{Ex}(f)"] & \mathrm{Ex}(Y),
\end{tikzcd}$$

where the vertical maps are weak homotopy equivalences (Theorem 3.3.5.1). The desired result now follows from the two-out-of-three property (Remark 3.1.6.16). □

The proof of Theorem 3.3.5.1 will make use of the following fact, which we prove at the end of this section:

00Z3 **Proposition 3.3.5.3.** *Let $f : X \to Y$ be an anodyne morphism of simplicial sets. Then the induced map $\mathrm{Sd}(f) : \mathrm{Sd}(X) \to \mathrm{Sd}(Y)$ is also anodyne.*

00Z4 **Corollary 3.3.5.4.** *Let $f : X \to Y$ be a Kan fibration of simplicial sets. Then the induced map $\mathrm{Ex}(f) : \mathrm{Ex}(X) \to \mathrm{Ex}(Y)$ is also a Kan fibration.*

Proof. We must show that every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & \mathrm{Ex}(X) \arrow[d, "\mathrm{Ex}(f)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Ex}(Y)
\end{tikzcd}$$

admits a solution. This follows by applying Remark 3.1.2.7 to the associated lifting problem

$$\begin{tikzcd}
\mathrm{Sd}(\Lambda^n_i) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Sd}(\Delta^n) \arrow[r] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

since the left vertical map is anodyne by virtue of Proposition 3.3.5.3. ☐

Corollary 3.3.5.5. *Let X be a Kan complex. Then the simplicial set $\mathrm{Ex}(X)$ is also a Kan complex.* 00Z5

Proposition 3.3.5.6. *Let X and Y be simplicial sets, where Y is a Kan complex. Then the bijection* 00Z6

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(X), Y) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(X, \mathrm{Ex}(Y))$$

respects homotopy. That is, for every pair of maps $f, g : \mathrm{Sd}(X) \to Y$ having counterparts $f', g' : X \to \mathrm{Ex}(Y)$, then f is homotopic to g if and only if f' is homotopic to g'.

Proof. Assume first that f and g are homotopic, so that there exists a morphism of simplicial sets $h : \Delta^1 \times \mathrm{Sd}(X) \to Y$ satisfying $h|_{\{0\}\times \mathrm{Sd}(X)} = f$ and $h|_{\{1\}\times \mathrm{Sd}(X)} = g$. The composite map

$$\mathrm{Sd}(\Delta^1 \times X) \to \mathrm{Sd}(\Delta^1) \times \mathrm{Sd}(X) \xrightarrow{\lambda_{\Delta^1}\times \mathrm{id}} \Delta^1 \times \mathrm{Sd}(X) \xrightarrow{h} Y$$

then determines a morphism of simplicial sets $h' : \Delta^1 \times X \to \mathrm{Ex}(Y)$, which is immediately seen to be a homotopy from f' to g'.

Conversely, suppose that f' and g' are homotopic. Since $\mathrm{Ex}(Y)$ is a Kan complex (Corollary 3.3.5.5), we can choose a morphism of simplicial sets $h' : \Delta^1 \times X \to \mathrm{Ex}(Y)$ satisfying $h|_{\{0\}\times X} = f'$ and $h'|_{\{1\}\times X} = g'$, which we can identify with a map $u : \mathrm{Sd}(\Delta^1 \times X) \to Y$. Let v denote the composite map $\mathrm{Sd}(\Delta^1 \times X) \to \mathrm{Sd}(X) \xrightarrow{f} Y$, so that u and v have the same restriction to $\mathrm{Sd}(\{0\} \times X)$. Note that the inclusion of simplicial sets $\{0\} \times X \hookrightarrow \Delta^1 \times X$ is anodyne (Proposition 3.1.2.9), so the subdivision $\mathrm{Sd}(\{0\} \times X) \hookrightarrow \mathrm{Sd}(\Delta^1 \times X)$ is also anodyne (Proposition 3.3.5.3). It follows that the restriction map $\mathrm{Fun}(\mathrm{Sd}(\Delta^1 \times X), Y) \to \mathrm{Fun}(\mathrm{Sd}(\{0\} \times X), Y)$ is a trivial Kan fibration, so that u and v belong to the same path component of $\mathrm{Fun}(\mathrm{Sd}(\Delta^1 \times X), Y)$ and are therefore homotopic. It follows that $f = v|_{\mathrm{Sd}(\{1\}\times X)}$ and $g = u|_{\mathrm{Sd}(\{1\}\times X)}$ are also homotopic. ☐

We can now prove a special case of Theorem 3.3.5.1.

Proposition 3.3.5.7. *Let Y be a Kan complex. Then the comparison map $\rho_Y : Y \to \mathrm{Ex}(Y)$ of Construction 3.3.4.3 is a homotopy equivalence.* 00Z7

Proof. Fix a simplicial set X. We wish to show that postcomposition with ρ_Y induces a bijection

$$\begin{tikzcd}
\{\text{Maps of simplicial sets } X \to Y\}/\text{homotopy} \arrow[d] \\
\{\text{Maps of simplicial sets } X \to \mathrm{Ex}(Y)\}/\text{homotopy}.
\end{tikzcd}$$

By virtue of Proposition 3.3.5.6, this is equivalent to the assertion that precomposition with the last vertex map $\lambda_X : \mathrm{Sd}(X) \to X$ induces a bijection

$$\begin{tikzcd}
\{\text{Maps of simplicial sets } X \to Y\}/\text{homotopy} \arrow[d] \\
\{\text{Maps of simplicial sets } \mathrm{Sd}(X) \to Y\}/\text{homotopy},
\end{tikzcd}$$

which follows from the fact that λ_X is a weak homotopy equivalence (Proposition 3.3.4.8). □

To deduce Theorem 3.3.5.1 from Proposition 3.3.5.7, we will need the following:

00Z8 **Proposition 3.3.5.8.** *Let X be a simplicial set, and let $\rho_X : X \to \mathrm{Ex}(X)$ be the comparison map of Construction 3.3.4.3. Then the morphisms $\rho_{\mathrm{Ex}(X)}, \mathrm{Ex}(\rho_X) : \mathrm{Ex}(X) \to \mathrm{Ex}(\mathrm{Ex}(X))$ are homotopic.*

Proof. Let Q be a partially ordered set. Using Example 3.3.3.17, we can identify the subdivisions $\mathrm{Sd}(\mathrm{N}_\bullet(Q))$ and $\mathrm{Sd}(\mathrm{Sd}(\mathrm{N}_\bullet(Q)))$ with the nerves of partially ordered sets $\mathrm{Chain}[Q]$ and $\mathrm{Chain}[\mathrm{Chain}[Q]]$, respectively. Under this identification, a morphism of simplicial sets

$$\mathrm{Sd}(\lambda_{\mathrm{N}_\bullet(Q)}), \lambda_{\mathrm{Sd}(\mathrm{N}_\bullet(Q))} : \mathrm{Sd}(\mathrm{Sd}(\mathrm{N}_\bullet(Q))) \to \mathrm{Sd}(\mathrm{N}_\bullet(Q))$$

corresponds to a nondecreasing functions $\mathrm{Chain}[\mathrm{Chain}[Q]] \to \mathrm{Chain}[Q]$, whose value on a linearly ordered subset $\vec{S} = (S_0 \subset S_1 \subset \cdots \subset S_n)$ of $\mathrm{Chain}[Q]$ is given by

$$\mathrm{Sd}(\lambda_{\mathrm{N}_\bullet(Q)})(\vec{S}) = \{\mathrm{Max}(S_0), \ldots, \mathrm{Max}(S_n)\} \qquad \lambda_{\mathrm{Sd}(\mathrm{N}_\bullet(Q))}(\vec{S}) = S_n.$$

Note that we always have an inclusion $\{\mathrm{Max}(S_0), \ldots, \mathrm{Max}(S_n)\} \subseteq S_n$. It follows that there is a unique map of simplicial sets

$$h_Q : \Delta^1 \times \mathrm{Sd}(\mathrm{Sd}(\mathrm{N}_\bullet(Q))) \to \mathrm{Sd}(\mathrm{N}_\bullet(Q))$$

satisfying $h_Q|_{\{0\} \times \mathrm{Sd}(\mathrm{Sd}(\mathrm{N}_\bullet(Q)))} = \mathrm{Sd}(\lambda_{\mathrm{N}_\bullet(Q)})$ and $h_Q|_{\{1\} \times \mathrm{Sd}(\mathrm{Sd}(\mathrm{N}_\bullet(Q)))} = \lambda_{\mathrm{Sd}(\mathrm{N}_\bullet(Q))}$, depending functorially on Q.

Let σ be an n-simplex of the simplicial set $\mathrm{Ex}(X)$, which we identify with a map $\sigma : \mathrm{Sd}(\Delta^n) \to X$. We let $f(\sigma)$ denote the composite map

$$\Delta^1 \times \mathrm{Sd}(\mathrm{Sd}(\Delta^n)) \xrightarrow{h_{[n]}} \mathrm{Sd}(\Delta^n) \xrightarrow{\sigma} X,$$

which we will identify with an n-simplex of the simplicial set $\mathrm{Fun}(\Delta^1, \mathrm{Ex}(\mathrm{Ex}(X)))$. The construction $\sigma \mapsto f(\sigma)$ then determines a morphism of simplicial sets $f : \mathrm{Ex}(X) \to \mathrm{Fun}(\Delta^1, \mathrm{Ex}(\mathrm{Ex}(X)))$, which we can identify with a map $\Delta^1 \times \mathrm{Ex}(X) \to \mathrm{Ex}(\mathrm{Ex}(X))$. By construction, this map is a homotopy from $\rho_{\mathrm{Ex}(X)}$ to $\mathrm{Ex}(\rho_X)$. □

Proof of Theorem 3.3.5.1. Let X be a simplicial set. We wish to prove that the comparison map $\rho_X : X \to \mathrm{Ex}(X)$ is a weak homotopy equivalence. Fix a Kan complex Y; we must show that composition with ρ_X induces a bijection $\pi_0(\mathrm{Fun}(\mathrm{Ex}(X), Y)) \to \pi_0(\mathrm{Fun}(X, Y))$. This map fits into a diagram

$$\begin{tikzcd}
\pi_0(\mathrm{Fun}(\mathrm{Ex}(X), Y)) \arrow[r, "\circ \rho_X"] \arrow[d, "\rho_Y \circ", "\sim"'] & \pi_0(\mathrm{Fun}(X, Y)) \arrow[d, "\rho_Y \circ", "\sim"'] \arrow[dl, "f \mapsto \mathrm{Ex}(f)"'] \\
\pi_0(\mathrm{Fun}(\mathrm{Ex}(X), \mathrm{Ex}(Y))) \arrow[r, "\circ \rho_X"] & \pi_0(\mathrm{Fun}(X, \mathrm{Ex}(Y))),
\end{tikzcd}$$

where the vertical maps are bijective (Proposition 3.3.5.7) and the lower triangle commutes by the naturality of ρ. To show that the upper horizontal map is bijective, it will suffice to show that the upper triangle also commutes. Fix a map $f : \mathrm{Ex}(X) \to Y$. We then compute

$$\mathrm{Ex}(f \circ \rho_X) = \mathrm{Ex}(f) \circ \mathrm{Ex}(\rho_X) \sim \mathrm{Ex}(f) \circ \rho_{\mathrm{Ex}(X)} = \rho_Y \circ f$$

where the equality on the left follows from functoriality, the equality on the right from the naturality of ρ, and the homotopy in the middle is supplied by Proposition 3.3.5.8. □

We close this section with the proof of Proposition 3.3.5.3.

Lemma 3.3.5.9. *Let J be a nonempty finite set, let $P(J)$ denote the collection of subsets of J (partially ordered by inclusion), and set $P_-(J) = P(J) \setminus \{J\}$. Then the inclusion of simplicial sets* 00Z9

$$\theta : \mathrm{N}_\bullet(P_-(J)) \hookrightarrow \mathrm{N}_\bullet(P(J)) = \square^J$$

is anodyne.

Proof. Fix an element $j \in J$ and set $I = J \setminus \{j\}$, so that the simplicial cube $\square^J$ can be identified with the product $\Delta^1 \times \square^I \simeq \Delta^1 \times \mathrm{N}_\bullet(P(I))$. Under this identification, θ corresponds to the inclusion map

$$(\Delta^1 \times \mathrm{N}_\bullet(P_-(I))) \coprod_{\{0\} \times \mathrm{N}_\bullet(P_-(I))} (\{0\} \times \mathrm{N}_\bullet(P(I))) \hookrightarrow \Delta^1 \times \mathrm{N}_\bullet(P(I)),$$

which is anodyne by virtue of Proposition 3.1.2.9. □

Proof of Proposition 3.3.5.3. Let S be the collection of all morphisms of simplicial sets $f : X \to Y$ for which the induced map $\mathrm{Sd}(f) : \mathrm{Sd}(X) \to \mathrm{Sd}(Y)$ is anodyne. Since the subdivision functor Sd preserves colimits, the collection S is weakly saturated (in the sense of Definition 1.5.4.12). To prove Proposition 3.3.5.3, it will suffice to show that S contains every horn inclusion. Fix a positive integer n and another integer $0 \leq i \leq n$. We will complete the proof by showing that the inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ induces an anodyne map $\mathrm{Sd}(\Lambda^n_i) \to \mathrm{Sd}(\Delta^n)$.

Let $J = [n] \setminus \{i\}$, let $P(J)$ denote the collection of all subsets of J, partially ordered by inclusion. Set $P_-(J) = P(J) \setminus \{J\}$, $P_+(J) = P(J) \setminus \{\emptyset\}$, and $P_\pm(J) = P(J) \setminus \{\emptyset, J\}$. In what follows, we identify $\mathrm{Sd}(\Delta^n)$ with the nerve of the partially ordered set $\mathrm{Chain}[n]$ of nonempty subsets of $[n]$, and $\mathrm{Sd}(\Lambda^n_i)$ with the nerve of the partially ordered subset of $\mathrm{Chain}[n]$ obtained by removing the elements $[n]$ and J (Proposition 3.3.3.16). The construction $J_0 \mapsto J_0 \cup \{i\}$ determines an inclusion of partially ordered sets $P(J) \to \mathrm{Chain}[n]$, hence a monomorphism of simplicial sets

$$g : \square^J = \mathrm{N}_\bullet(P(J)) \hookrightarrow \mathrm{N}_\bullet(\mathrm{Chain}[n]) = \mathrm{Sd}(\Delta^n).$$

Let $Z \subseteq \mathrm{Sd}(\Delta^n)$ be the union of $\mathrm{Sd}(\Lambda^n_i)$ with the image of g. An elementary calculation shows that the inverse image $g^{-1}(\mathrm{Sd}(\Lambda^n_i))$ can be identified with the nerve of the subset $P_-(J) \subseteq P(J)$, so that we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{N}_\bullet(P_-(J)) \arrow[r] \arrow[d] & \mathrm{Sd}(\Lambda^n_i) \arrow[d] \\
\mathrm{N}_\bullet(P(J)) \arrow[r, "g"] & Z.
\end{tikzcd}$$

The left vertical map is anodyne by virtue of Lemma 3.3.5.9, so the right vertical map is anodyne as well. Let $h : [1] \times P_+(J) \to \mathrm{Chain}[n]$ be the map of partially ordered sets given $h(0, J_0) = J_0$ and $h(1, J_0) = J_0 \cup \{i\}$. Then h determines a map of simplicial sets $\Delta^1 \times \mathrm{N}_\bullet(P_+(J)) \to \mathrm{Sd}(\Delta^n)$. An elementary calculation shows that this map of simplicial sets fits into a pushout diagram

$$\begin{tikzcd}
(\{1\} \times \mathrm{N}_\bullet(P_+(J))) \coprod_{\{1\} \times \mathrm{N}_\bullet(P_\pm(J))} (\Delta^1 \times \mathrm{N}_\bullet(P_\pm(J))) \arrow[r] \arrow[d] & Z \arrow[d] \\
\Delta^1 \times \mathrm{N}_\bullet(P_+(J)) \arrow[r, "h"] & \mathrm{Sd}(\Delta^n).
\end{tikzcd}$$

The left vertical map in this diagram is anodyne by virtue of Proposition 3.1.2.9, so the inclusion $Z \hookrightarrow \mathrm{Sd}(\Delta^n)$ is also anodyne. It follows that the composite map $\mathrm{Sd}(\Lambda^n_i) \hookrightarrow Z \hookrightarrow \mathrm{Sd}(\Delta^n)$ is anodyne, as desired. □

3.3.6 The Ex^∞ Functor

Let X be a simplicial set. In §3.1.7, we proved that one can always choose an embedding $j : X \hookrightarrow Q$, where Q is a Kan complex and j is a weak homotopy equivalence (Corollary 3.1.7.2). In [33], Kan gave an explicit construction of such an embedding, based on iteration of the construction $X \mapsto \mathrm{Ex}(X)$. 00ZA

Construction 3.3.6.1 (The Ex^∞ Functor)**.** For every nonnegative integer n, we let Ex^n denote the n-fold iteration of the functor $\mathrm{Ex} : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ of Construction 3.3.2.5, given inductively by the formula 00ZB

$$\mathrm{Ex}^n(X) = \begin{cases} X & \text{if } n = 0 \\ \mathrm{Ex}(\mathrm{Ex}^{n-1}(X)) & \text{if } n > 0. \end{cases}$$

For every simplicial set X, we let $\mathrm{Ex}^\infty(X)$ denote the colimit of the diagram

$$X \xrightarrow{\rho_X} \mathrm{Ex}(X) \xrightarrow{\rho_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \xrightarrow{\rho_{\mathrm{Ex}^2(X)}} \mathrm{Ex}^3(X) \to \cdots,$$

where each $\rho_{\mathrm{Ex}^n(X)}$ denotes the comparison map of Construction 3.3.4.3, and we let $\rho^\infty_X : X \to \mathrm{Ex}^\infty(X)$ denote the tautological map. The construction $X \mapsto \mathrm{Ex}^\infty(X)$ determines a functor Ex^∞ from the category of simplicial sets to itself, and the construction $X \mapsto \rho^\infty_X$ determines a natural transformation of functors $\mathrm{id}_{\mathrm{Set}_\Delta} \to \mathrm{Ex}^\infty$.

Our goal in this section is to record the main properties of Construction 3.3.6.1. In particular, for every simplicial set X, we show that $\mathrm{Ex}^\infty(X)$ is a Kan complex (Proposition 3.3.6.9) and that the comparison map $\rho^\infty_X : X \to \mathrm{Ex}^\infty(X)$ is a weak homotopy equivalence (Proposition 3.3.6.7).

Proposition 3.3.6.2. *Let X be a simplicial set. Then the comparison map $\rho^\infty_X : X \to \mathrm{Ex}^\infty(X)$ is a monomorphism of simplicial sets. Moreover, it is bijective on vertices.* 00ZC

Proof. It will suffice to show that each of the comparison maps $\rho_{\mathrm{Ex}^n(X)} : \mathrm{Ex}^n(X) \to \mathrm{Ex}^{n+1}(X)$ is a monomorphism which is bijective on vertices. Replacing X by $\mathrm{Ex}^n(X)$, we can reduce to the case $n = 0$. Fix $m \geq 0$. On m-simplices, the comparison map ρ_X is given by the map of sets

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Sd}(\Delta^m), X)$$

induced by precomposition with the last vertex map $\lambda_{\Delta^m} : \mathrm{Sd}(\Delta^m) \to \Delta^m$. To complete the proof, it suffices to observe that λ_{Δ^m} is an epimorphism of simplicial sets (in fact, it admits a section $\Delta^m \to \mathrm{Sd}(\Delta^m) \simeq \mathrm{N}_\bullet(\mathrm{Chain}[m])$, given by the chain of subsets $\{0\} \subset \{0,1\} \subset \cdots \subset \{0,1,\ldots,m\}$), and an isomorphism in the special case $m = 0$. □

00ZD **Example 3.3.6.3.** Let X be a discrete simplicial set (Definition 1.1.5.10). Invoking Example 3.3.4.6 repeatedly, we deduce that the transition maps in the diagram

$$X \xrightarrow{\rho_X} \mathrm{Ex}(X) \xrightarrow{\rho_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \xrightarrow{\rho_{\mathrm{Ex}^2(X)}} \mathrm{Ex}^3(X) \to \cdots,$$

are isomorphisms. It follows that the comparison map $\rho_X^\infty : X \to \mathrm{Ex}^\infty(X)$ is also an isomorphism.

00ZE **Proposition 3.3.6.4.** *The functor $X \mapsto \mathrm{Ex}^\infty(X)$ preserves filtered colimits and finite limits.*

Proof. It will suffice to show that, for every nonnegative integer n, the functor $X \mapsto \mathrm{Ex}^n(X)$ preserves filtered colimits and finite limits. Proceeding by induction on n, we can reduce to the case $n = 1$. We now observe that the functor Ex preserves *all* limits of simplicial sets (either by construction, or because it admits a left adjoint $X \mapsto \mathrm{Sd}(X)$), and preserves filtered colimits by virtue of Remark 3.3.2.7. □

00ZF **Corollary 3.3.6.5.** *Let $f : X \to S$ be a morphism of simplicial sets. Let s be a vertex of S, which we will identify (via Proposition 3.3.6.2) with its image in $\mathrm{Ex}^\infty(S)$. Then the canonical map $\mathrm{Ex}^\infty(X_s) \to \mathrm{Ex}^\infty(X)_s$ is an isomorphism of simplicial sets. Here $X_s = \{s\} \times_S X$ denotes the fiber of f over the vertex s, and $\mathrm{Ex}^\infty(X)_s = \{s\} \times_{\mathrm{Ex}^\infty(S)} \mathrm{Ex}^\infty(X)$ is defined similarly.*

Proof. Combine Proposition 3.3.6.4 with Example 3.3.6.3. □

00ZG **Proposition 3.3.6.6.** *Let $f : X \to S$ be a morphism of simplicial sets. If f is a Kan fibration, then the induced map $\mathrm{Ex}^\infty(f) : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(S)$ is also a Kan fibration.*

Proof. Since the collection of Kan fibrations is stable under the formation of filtered colimits (Remark 3.1.1.8), it will suffice to show that each of the maps $\mathrm{Ex}^n(f) : \mathrm{Ex}^n(X) \to \mathrm{Ex}^n(S)$ is a Kan fibration. Proceeding by induction on n, we can reduce to the case $n = 1$, which follows from Corollary 3.3.5.4. □

00ZH **Proposition 3.3.6.7.** *Let X be a simplicial set. Then the comparison map $\rho_X^\infty : X \to \mathrm{Ex}^\infty(X)$ is a weak homotopy equivalence.*

Proof. By virtue of Proposition 3.2.8.3, it will suffice to show that for each $n \geq 0$, the composite map

$$X \xrightarrow{\rho_X} \mathrm{Ex}(X) \xrightarrow{\rho_{\mathrm{Ex}(X)}} \cdots \xrightarrow{\rho_{\mathrm{Ex}^{n-1}(X)}} \mathrm{Ex}^n(X)$$

is a weak homotopy equivalence. Proceeding by induction on n, we are reduced to showing that each of the comparison maps $\rho_{\mathrm{Ex}^{n-1}(X)} : \mathrm{Ex}^{n-1}(X) \to \mathrm{Ex}^n(X)$ is a weak homotopy equivalence, which is a special case of Theorem 3.3.5.1. □

Corollary 3.3.6.8. *Let $f : X \to Y$ be a morphism of simplicial sets. Then f is a weak homotopy equivalence if and only if $\mathrm{Ex}^{\infty}(f)$ is a weak homotopy equivalence.* 00ZJ

Proof. We have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\rho_X^{\infty}"] & Y \arrow[d, "\rho_Y^{\infty}"] \\
\mathrm{Ex}^{\infty}(X) \arrow[r, "\mathrm{Ex}^{\infty}(f)"] & \mathrm{Ex}^{\infty}(Y),
\end{tikzcd}$$

where the vertical maps are weak homotopy equivalences (Proposition 3.3.6.7). The desired result now follows from the two-out-of-three property (Remark 3.1.6.16). □

Proposition 3.3.6.9. *Let X be a simplicial set. Then $\mathrm{Ex}^{\infty}(X)$ is a Kan complex.* 00ZK

Proof. Let X be a simplicial set and suppose we are given a morphism of simplicial sets $f_0 : \Lambda^n_i \to \mathrm{Ex}^{\infty}(X)$, for some $n > 0$ and $0 \leq i \leq n$. We wish to show that f_0 can be extended to an n-simplex of $\mathrm{Ex}^{\infty}(X)$. Since the simplicial set Λ^n_i has finitely many nondegenerate simplices, we can assume that f_0 factors as a composition $\Lambda^n_i \xrightarrow{f'_0} \mathrm{Ex}^m(X) \to \mathrm{Ex}^{\infty}(X)$, for some positive integer m. We will complete the proof by showing that f'_0 can be extended to an n-simplex of $\mathrm{Ex}^{m+1}(X)$: that is, that there exists a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f'_0"] \arrow[d] & \mathrm{Ex}^m(X) \arrow[d, "\rho_{\mathrm{Ex}^m(X)}"] \\
\Delta^n \arrow[r, "f'"] & \mathrm{Ex}^{m+1}(X).
\end{tikzcd}$$

Replacing X by $\mathrm{Ex}^{m-1}(X)$, we can reduce to the case $m = 1$. In this case, f'_0 can be identified with a morphism of simplicial sets $g_0 : \mathrm{Sd}(\Lambda^n_i) \to X$. Unwinding the definitions, we see that the problem of finding a simplex $f' : \Delta^n \to \mathrm{Ex}^2(X)$ with the desired property is equivalent to the problem of finding a morphism $g : \mathrm{Sd}(\mathrm{Sd}(\Delta^n)) \to X$ whose restriction to $\mathrm{Sd}(\mathrm{Sd}(\Lambda^n_i))$ is equal to the composition

$$\mathrm{Sd}(\mathrm{Sd}(\Lambda^n_i)) \xrightarrow{\mathrm{Sd}(\lambda_{\Lambda^n_i})} \mathrm{Sd}(\Lambda^n_i) \xrightarrow{g_0} X.$$

Without loss of generality, we may assume that $X = \mathrm{Sd}(\Lambda^n_i)$ and that g_0 is the identity map. Let $\mathrm{Chain}[n]$ be the collection of all nonempty subsets of $[n]$ (Notation 3.3.2.1) and let $Q \subset \mathrm{Chain}[n]$ be the subset obtained by removing $[n]$ and $[n] \setminus \{i\}$. Using Proposition

3.3.3.16, we can identify $\mathrm{Sd}(\Lambda^n_i)$, $\mathrm{Sd}(\mathrm{Sd}(\Lambda^n_i))$, and $\mathrm{Sd}(\mathrm{Sd}(\Delta^n))$ with the nerves of the partially ordered sets Q, $\mathrm{Chain}[Q]$, and $\mathrm{Chain}[\mathrm{Chain}[n]]$, respectively. To complete the proof, it will suffice to show that there exists a nondecreasing function of partially ordered sets $g : \mathrm{Chain}[\mathrm{Chain}[n]] \to Q$ having the property that, for every element $(S_0 \subset S_1 \subset \cdots \subset S_m)$ of $\mathrm{Chain}[Q]$, we have $g(S_0 \subset S_1 \subset \cdots \subset S_m) = \{\mathrm{Max}(S_0), \mathrm{Max}(S_1), \ldots, \mathrm{Max}(S_m)\} \in Q$. This requirement is satisfied if g is defined by the formula

$$g(S_0 \subset S_1 \subset \cdots \subset S_m) = \{\mathrm{Max}'(S_0), \mathrm{Max}'(S_1), \ldots, \mathrm{Max}'(S_m)\},$$

where $\mathrm{Max}' : \mathrm{Chain}[n] \to [n]$ is the (non-monotone) map of sets given by

$$\mathrm{Max}'(S) = \begin{cases} i & \text{if } S = [n] \text{ or } S = [n] \setminus \{i\} \\ \mathrm{Max}(S) & \text{otherwise.} \end{cases}$$

□

00ZL **Corollary 3.3.6.10.** *Let X be a Kan complex. Then the comparison map $\rho^\infty_X : X \to \mathrm{Ex}^\infty(X)$ is a homotopy equivalence.*

Proof. Since $\mathrm{Ex}^\infty(X)$ is also a Kan complex (Proposition 3.3.6.9), it will suffice to show that ρ^∞_X is a weak homotopy equivalence (Proposition 3.1.6.13), which follows from Proposition 3.3.6.7. □

3.3.7 Application: Characterizations of Weak Homotopy Equivalences

00ZM Let $f : X \to S$ be a Kan fibration between Kan complexes. In §3.2.7, we proved that f is a homotopy equivalence if and only if it is a trivial Kan fibration (Proposition 3.2.7.2). We now apply the machinery of §3.3.6 to extend this result to the case where S is an arbitrary simplicial set. First, we need a slight generalization of Proposition 3.2.8.1.

00ZN **Proposition 3.3.7.1.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d] & X' \arrow[d] \\ S \arrow[r, "v"] & S', \end{tikzcd}$$

where the vertical maps are Kan fibrations and v is a weak homotopy equivalence. The following conditions are equivalent:

(1) *The morphism u is a weak homotopy equivalence.*

(2) *For every vertex $s \in S$, the induced map of fibers $u_t : X_s \to X'_{v(s)}$ is a homotopy equivalence of Kan complexes.*

Proof. Using Corollaries 3.3.6.8 and 3.3.6.5, we can replace (1) and (2) by the following assertions:

($1'$) The morphism $\mathrm{Ex}^\infty(u) : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(X')$ is a weak homotopy equivalence.

($2'$) For every vertex $s \in S$, the induced map of fibers $u_s : \mathrm{Ex}^\infty(X)_s \to \mathrm{Ex}^\infty(X')_{v(s)}$ is a homotopy equivalence of Kan complexes.

The equivalence of ($1'$) and ($2'$) follows by applying Proposition 3.2.8.1 to the diagram

$$\begin{tikzcd}
\mathrm{Ex}^\infty(X) \arrow[r, "\mathrm{Ex}^\infty(u)"] \arrow[d] & \mathrm{Ex}^\infty(X') \arrow[d] \\
\mathrm{Ex}^\infty(S) \arrow[r, "\mathrm{Ex}^\infty(v)"] & \mathrm{Ex}^\infty(S').
\end{tikzcd}$$

Note that every simplicial set appearing in this diagram is a Kan complex (Proposition 3.3.6.9), the vertical maps are Kan fibrations (Proposition 3.3.6.6) and $\mathrm{Ex}^\infty(v)$ is a homotopy equivalence by virtue of Corollary 3.3.6.8. □

Example 3.3.7.2. Let $f : X \to S$ be a Kan fibration of simplicial sets, and let $s \in S$ be a vertex. If S is weakly contractible, then Proposition 3.3.7.1 guarantees that the inclusion map $X_s \hookrightarrow X$ is a weak homotopy equivalence. 0511

Remark 3.3.7.3. Let $f : X \to S$ be a Kan fibration of simplicial sets. If s and t are vertices of S which belong to the same connected component, then the Kan complexes X_s and X_t are homotopy equivalent. To prove this, we may assume without loss of generality that there is an edge of S with source s and target t. Replacing f by the projection map $\Delta^1 \times_S X \to \Delta^1$, we are reduced to the case where $S = \Delta^1$; in this case, the Example 3.3.7.2 guarantees that the inclusion maps $X_s \hookrightarrow X \hookleftarrow X_t$ are weak homotopy equivalences. 0512

Corollary 3.3.7.4. *Let $v : T \to S$ be a weak homotopy equivalence of simplicial sets. For every Kan fibration $f : X \to S$, the projection map $T \times_S X \to X$ is also a weak homotopy equivalence.* 0104

0105 **Corollary 3.3.7.5.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd} Y \arrow[rr, "u"] \arrow[dr] & & X \arrow[dl] \\ & S, & \end{tikzcd}$$

where the vertical maps are Kan fibrations. Then u is a weak homotopy equivalence if and only if every vertex $s \in S$ satisfies the following condition:

($*_s$) *The induced map of fibers $u_s : X_s \to Y_s$ is a homotopy equivalence of Kan complexes.*

00ZP **Proposition 3.3.7.6.** *Let $f : X \to S$ be a Kan fibration of simplicial sets. The following conditions are equivalent:*

(1) *The morphism f is a trivial Kan fibration.*

(2) *The morphism f is a homotopy equivalence.*

(3) *The morphism f is a weak homotopy equivalence.*

(4) *For every vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a contractible Kan complex.*

Proof. The implication (1) ⇒ (2) is Proposition 3.1.6.10, the implication (2) ⇒ (3) follows from Proposition 3.1.6.13, the equivalence (3) ⇔ (4) is a special case of Corollary 3.3.7.5, and the equivalence (4) ⇔ (1) follows from Proposition 3.5.2.1. □

00ZQ **Corollary 3.3.7.7.** *Let $f : X \to Y$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism f is anodyne.*

(2) *The morphism f is both a monomorphism and a weak homotopy equivalence.*

Proof. The implication (1) ⇒ (2) follows from Proposition 3.1.6.14 and Remark 3.1.2.3. To prove the converse, assume that f is a weak homotopy equivalence and apply Proposition 3.1.7.1 to write f as a composition $X \xrightarrow{f'} Q \xrightarrow{f''} Y$, where f' is anodyne and f'' is a Kan fibration. Then f' is a weak homotopy equivalence (Proposition 3.1.6.14), so f'' is a weak homotopy equivalence (Remark 3.1.6.16). Invoking Proposition 3.3.7.6, we conclude that f''

is a trivial Kan fibration. If f is a monomorphism, then the lifting problem

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "f"] & Q \arrow[d, "f''"] \\
Y \arrow[r, "\mathrm{id}_Y"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution. It follows that f is a retract of f' (in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$). Since the collection of anodyne morphisms is closed under retracts, we conclude that f is anodyne. □

3.3.8 Application: Extending Kan Fibrations

In the proof of Proposition 3.3.7.6, we made essential use of the fact that any Kan fibration of simplicial sets $f : X \to S$ is (fiberwise) homotopy equivalent to a pullback $S \times_{S'} X' \to S$, where $f' : X' \to S'$ is a Kan fibration between Kan complexes. This can be achieved by taking f' to be the Kan fibration $\mathrm{Ex}^{\infty}(f) : \mathrm{Ex}^{\infty}(X) \to \mathrm{Ex}^{\infty}(S)$. Using a variant of this construction, one can obtain a more precise result. `00ZR`

Theorem 3.3.8.1. *Let $f : X \to S$ be a Kan fibration between simplicial sets, and let $g : S \hookrightarrow S'$ be an anodyne map. Then there exists a pullback diagram of simplicial sets* `00ZS`

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
S \arrow[r, "g"] & S',
\end{tikzcd}$$

where f' is a Kan fibration.

Remark 3.3.8.2. We refer the reader to [37] for a proof of Theorem 3.3.8.1 which is slightly different from the proof given below (it avoids the use of Kan's Ex^{∞}-functor by appealing instead to the theory of *minimal Kan fibrations*, which we will discuss in §[?]). See also [53] and [50]. `00ZT`

Remark 3.3.8.3. If $f : X \to S$ is a Kan fibration of simplicial sets, then every vertex $s \in S$ determines a Kan complex $X_s = \{s\} \times_S X$. One can think of the construction $s \mapsto X_s$ as supplying a map from S to the "space" of all Kan complexes. Roughly speaking, one can think of Theorem 3.3.8.1 as asserting that this "space" itself behaves like a Kan complex. We will return to this idea in §5.6. `00ZU`

The proof of Theorem 3.3.8.1 is based on the following observation:

`00ZV` **Lemma 3.3.8.4.** *Let $f : Y \to T$ be a Kan fibration of simplicial sets, and suppose we are given simplicial subsets $X \subseteq Y$ and $S \subseteq T$ satisfying the following conditions:*

(a) The morphism f carries X to S, and the restriction $f_0 = f|_X$ is a Kan fibration from X to S.

(b) For every vertex $s \in S$, the inclusion of fibers $X_s \hookrightarrow Y_s$ is a homotopy equivalence of Kan complexes.

Let $Y' \subseteq Y$ denote the simplicial subset spanned by those simplices $\sigma : \Delta^n \to Y$ having the property that the restriction $\sigma|_{S \times_T \Delta^n}$ factors through X. Then the restriction $f|_{Y'} : Y' \to T$ is a Kan fibration.

Proof. Set $Y_S = S \times_T Y \subseteq Y$. It follows from assumption (b) and Corollary 3.3.7.5 that the inclusion $X \hookrightarrow Y_S$ is a weak homotopy equivalence, and is therefore anodyne (Corollary 3.3.7.7). Since f_0 is a Kan fibration, the lifting problem

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d] & X \arrow[d, "f_0"] \\
Y_S \arrow[r, "f|_{Y_S}"] \arrow[ur, dashed, "r"] & S
\end{tikzcd}$$

admits a solution: that is, there exists a retraction r from Y_S to the simplicial subset $X \subseteq Y_S$ which is compatible with projection to S. Since f is a Kan fibration, Theorem 3.1.3.5 guarantees that the map

$$\mathrm{Fun}(Y_S, Y_S) \to \mathrm{Fun}(X, Y_S) \times_{\mathrm{Fun}(X,S)} \mathrm{Fun}(Y_S, S)$$

is a trivial Kan fibration. We can therefore choose a homotopy $H : \Delta^1 \times Y_S \to Y_S$ from $\mathrm{id}_{Y_S} = H|_{\{0\} \times Y_S}$ to $r = H|_{\{1\} \times Y_S}$, such that $f \circ H$ is the constant homotopy from $f|_{Y_S}$ to itself.

Choose an anodyne map of simplicial sets $i : A \hookrightarrow B$. We wish to show that every lifting problem of the form

$$\begin{tikzcd}
A \arrow[r, "g_0"] \arrow[d, "i"] & Y' \arrow[d, "f|_{Y'}"] \\
B \arrow[r, "\overline{g}"] \arrow[ur, dashed, "g"] & T
\end{tikzcd}$$

admits a solution. Since f is a Kan fibration, we can choose a map $g' : B \to Y$ satisfying $g'|_A = g_0$ and $f \circ g = \overline{g}$. Let $B_S \subseteq B$ denote the simplicial subset given by the fiber product $S \times_T B$, and let $g_1 : (A \cup B_S) \to Y$ be the map of simplicial sets characterized by $g_1|_A = g_0$ and $g_1|_{B_S} = r \circ g'|_{B_S}$ (this map is well-defined, since $r \circ g'$ and g_0 agree on the intersection $A \cap B_S$). Note that H induces a homotopy $h_0 : \Delta^1 \times (A \cup B_S) \to Y$ from $g'|_{A \cup B_S}$ to g_1 (compatible with the projection to S). Since f is a Kan fibration, we can lift h_0 to a homotopy $h : \Delta^1 \times B \to Y$ from g' to some map $g : B \to Y$, compatible with the projection to S (Remark 3.1.5.3). It follows from the construction that g takes values in the simplicial subset $Y' \subseteq Y$ and satisfies the requirements $g|_A = g_0$ and $f \circ g = \overline{g}$. □

Proof of Theorem 3.3.8.1. Let $f : X \to S$ be a Kan fibration of simplicial sets. Let us abuse notation by identifying X and S with simplicial subsets of $Y = \mathrm{Ex}^{\infty}(X)$ and $T = \mathrm{Ex}^{\infty}(S)$, respectively (via the monomorphisms $\rho_X^{\infty} : X \hookrightarrow \mathrm{Ex}^{\infty}(X)$ and $\rho_S^{\infty} : S \hookrightarrow \mathrm{Ex}^{\infty}(S)$), and let $Y' \subseteq \mathrm{Ex}^{\infty}(X)$ be the simplicial subset defined in the statement of Lemma 3.3.8.4. Let $g : S \hookrightarrow S'$ be an anodyne morphism of simplicial sets. Since $\mathrm{Ex}^{\infty}(S)$ is a Kan complex (Proposition 3.3.6.9), the morphism $\rho_S^{\infty} : S \to \mathrm{Ex}^{\infty}(S)$ extends to a map of simplicial sets $u : S' \to \mathrm{Ex}^{\infty}(S)$. Set $X' = S' \times_{\mathrm{Ex}^{\infty}(S)} Y'$, so that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "f'"] & Y' \arrow[d] \\
S \arrow[r, "g"] & S' \arrow[r, "u"] & \mathrm{Ex}^{\infty}(S)
\end{tikzcd}$$

where the right square and outer rectangle are pullback diagrams, so the left square is a pullback diagram as well. Since the projection map $Y' \to \mathrm{Ex}^{\infty}(S)$ is a Kan fibration (Lemma 3.3.8.4), it follows that f' is also a Kan fibration. □

3.4 Homotopy Pullback and Homotopy Pushout Squares

Recall that the category of simplicial sets admits arbitrary limits and colimits (Remark 1.1.0.8). In particular, given a diagram of simplicial sets $X_0 \to X \leftarrow X_1$, we can form the fiber product $X_0 \times_X X_1$. Beware that, in general, this construction is not invariant under weak homotopy equivalence: 0106

0107 **Warning 3.4.0.1.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d] & X_1 \arrow[l] \arrow[d] \\
Y_0 \arrow[r] & Y & Y_1 \arrow[l]
\end{tikzcd}$$

for which the vertical maps are weak homotopy equivalences. Then the induced map

$$X_0 \times_X X_1 \to Y_0 \times_Y Y_1$$

need not be a weak homotopy equivalence. For example, the pullback of the upper half of the diagram

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d, equal] & \Delta^1 \arrow[d] & \{1\} \arrow[l] \arrow[d, equal] \\
\{0\} \arrow[r, "\sim"] & \Delta^0 & \{1\}, \arrow[l, "\sim"']
\end{tikzcd}$$

is empty, while the pullback of the lower half is isomorphic to Δ^0.

Under some mild assumptions, the bad behavior described in Warning 3.4.0.1 can be avoided.

0109 **Proposition 3.4.0.2.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d] & X \arrow[d] & X_1 \arrow[l] \arrow[d] \\
Y_0 \arrow[r, "f'"] & Y & Y_1, \arrow[l]
\end{tikzcd}$$

where the vertical maps are weak homotopy equivalences. If f and f' are Kan fibrations, then the induced map $X_0 \times_X X_1 \to Y_0 \times_Y Y_1$ is a weak homotopy equivalence.

Proof. We have a commutative diagram

$$\begin{tikzcd}
X_0 \times_X X_1 \arrow[r] \arrow[d] & Y_0 \times_Y Y_1 \arrow[d] \\
X_1 \arrow[r] & Y_1,
\end{tikzcd}$$

where the vertical maps are Kan fibrations (since they are pullbacks of f and f', respectively). By virtue of Proposition 3.3.7.1, it will suffice to show that for each vertex $x \in X_1$ having image $y \in Y_1$, the induced map of fibers

$$(X_0 \times_X X_1)_x \simeq X_0 \times_X \{x\} \to Y_0 \times_Y \{y\} = (Y_0 \times_Y Y_1)_y$$

is a homotopy equivalence of Kan complexes. This follows by applying Proposition 3.3.7.1 to the diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f"] & Y_0 \arrow[d, "f'"] \\
X \arrow[r] & Y.
\end{tikzcd}$$

□

To address the phenomenon described in Warning 3.4.0.1 more generally, it is convenient to work with a homotopy-invariant replacement for the fiber product.

Construction 3.4.0.3 (The Homotopy Fiber Product)**.** Let $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ be morphisms of simplicial sets, where X is a Kan complex. We let $X_0 \times^{\mathrm{h}}_X X_1$ denote the simplicial set 010B

$$X_0 \times_{\operatorname{Fun}(\{0\},X)} \operatorname{Fun}(\Delta^1, X) \times_{\operatorname{Fun}(\{1\},X)} X_1.$$

We will refer to $X_0 \times^{\mathrm{h}}_X X_1$ as the *homotopy fiber product of X_0 with X_1 over X*.

Warning 3.4.0.4. For any diagram of simplicial sets $X_0 \to X \leftarrow X_1$, the simplicial set $X_0 \times_{\operatorname{Fun}(\{0\},X)} \operatorname{Fun}(\Delta^1, X) \times_{\operatorname{Fun}(\{1\},X)} X_1$ is well-defined. However, we will refer to it as a *homotopy fiber product* (and denote it by $X_0 \times^{\mathrm{h}}_X X_1$) only in the case where X is a Kan complex. In more general situations, we will refer to this simplicial set as the *oriented fiber product* of X_0 with X_1 over X, and denote it by $X_0 \tilde{\times}_X X_1$ (Definition 4.6.4.1). In the setting of ∞-categories, we will adopt a slightly different definition for the homotopy fiber product $X_0 \times^{\mathrm{h}}_X X_1$: see Construction 4.5.2.1. 0327

Example 3.4.0.5. Let $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ be continuous functions between topological spaces. We let $X_0 \times^{\mathrm{h}}_X X_1$ denote the set of all triples (x_0, x_1, p) where x_0 is a point of X_0, x_1 is a point of X_1, and $p : [0,1] \to X$ is a continuous function satisfying $p(0) = f_0(x_0)$ and $p(1) = f_1(x_1)$. We will refer to $X_0 \times^{\mathrm{h}}_X X_1$ as the *homotopy fiber product of X_0 with X_1 over X*. The homotopy fiber product $X_1 \times^{\mathrm{h}}_X X_1$ carries a natural topology, given by viewing it as a subspace of the product $X_0 \times X_1 \times \operatorname{Hom}_{\mathrm{Top}}([0,1], X)$ (where we endow the path space $\operatorname{Hom}_{\mathrm{Top}}([0,1], X)$ with the compact-open topology). We then have a canonical isomorphism of simplicial sets 01KK

$$\operatorname{Sing}_\bullet(X_0 \times^{\mathrm{h}}_X X_1) \simeq \operatorname{Sing}_\bullet(X_0) \times^{\mathrm{h}}_{\operatorname{Sing}_\bullet(X)} \operatorname{Sing}_\bullet(X_1)$$

where the right hand side is the homotopy fiber product of Kan complexes given in Construction 3.4.0.3.

0328 **Remark 3.4.0.6** (Homotopy Fibers)**.** Let $f : X \to Y$ be a morphism of Kan complexes. Then f is a homotopy equivalence if and only if, for each vertex $y \in Y$, the homotopy fiber $X \times^{\mathrm{h}}_{Y} \{y\}$ is a contractible Kan complex. To see this, we observe that f factors as a composition

$$X \xrightarrow{\delta} X \times^{\mathrm{h}}_{Y} Y \xrightarrow{\pi} Y,$$

where δ is a homotopy equivalence and π is a Kan fibration (Example 3.1.7.10). It follows that f is a homotopy equivalence if and only if π is a homotopy equivalence, which is equivalent to the requirement that each fiber $\pi^{-1}\{y\} = X \times^{\mathrm{h}}_{Y} \{y\}$ is contractible (Proposition 3.3.7.6).

In the situation of Construction 3.4.0.3, the diagonal inclusion

$$X \hookrightarrow \mathrm{Fun}(\Delta^1, X) \qquad x \mapsto \mathrm{id}_x$$

induces a monomorphism from the ordinary fiber product $X_0 \times_X X_1$ to the homotopy fiber product $X_0 \times^{\mathrm{h}}_{X} X_1$.

0329 **Proposition 3.4.0.7.** *Let $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ be morphisms of simplicial sets. Assume that X is a Kan complex and that either f_0 or f_1 is a Kan fibration. Then the inclusion map $X_0 \times_X X_1 \to X_0 \times^{\mathrm{h}}_{X} X_1$ is a weak homotopy equivalence.*

Proof. Without loss of generality we may assume that f_0 is a Kan fibration. Since X is a Kan complex, the evaluation map $\mathrm{ev}_0 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{1\}, X)$ is a trivial Kan fibration (Corollary 3.1.3.6), and therefore induces a trivial Kan fibration $q : \mathrm{Fun}(\Delta^1, X) \times_{\mathrm{Fun}(\{1\},X)} X_1 \to X_1$. The diagonal map $\delta : X \hookrightarrow \mathrm{Fun}(\Delta^1, X)$ determines a map $s : X_1 \to \mathrm{Fun}(\Delta^1, X) \times_{\mathrm{Fun}(\{1\},X)} X_1$ which is a section of q, and therefore also a weak homotopy equivalence. The desired result now follows by applying Proposition 3.4.0.2 to the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f_0"] \arrow[d, equal] & X \arrow[d, equal] & X_1 \arrow[l, "f_1"'] \arrow[d, "s"] \\
X_0 \arrow[r, "f_0"] & X & \mathrm{Fun}(\Delta^1, X) \times_{\mathrm{Fun}(\{1\},X)} X_1. \arrow[l]
\end{tikzcd}$$

□

032A **Warning 3.4.0.8.** The conclusion of Proposition 3.4.0.7 is generally false if neither f_0 or f_1 is assumed to be a Kan fibration. For example, suppose that X is a Kan complex containing vertices x and y. If $x \neq y$, then the fiber product $\{x\} \times_X \{y\}$ is empty. However, the homotopy fiber product $\{x\} \times^{\mathrm{h}}_{X} \{y\}$ is not necessarily empty: its vertices can be identified with edges $p : x \to y$ having source x and target y.

In general, the failure of the inclusion map $X_0 \times_X X_1 \hookrightarrow X_0 \times^{\mathrm{h}}_{X} X_1$ to be a weak homotopy equivalence should be viewed as a feature, rather than a bug. From the perspective of homotopy theory, the homotopy fiber product is better behaved than the ordinary fiber product:

Proposition 3.4.0.9. *Suppose we are given a commutative diagram of simplicial sets* `032B`

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d] & X_1 \arrow[l] \arrow[d] \\
Y_0 \arrow[r] & Y & Y_1 \arrow[l]
\end{tikzcd}$$

where X and Y are Kan complexes and the vertical maps are weak homotopy equivalences. Then the induced map $X_0 \times^{\mathrm{h}}_{X} X_1 \to Y_0 \times^{\mathrm{h}}_{Y} Y_1$ is also a weak homotopy equivalence.

Proof. Apply Proposition 3.4.0.2 to the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, X) \arrow[r] \arrow[d] & \mathrm{Fun}(\partial\Delta^1, X) \arrow[d] & X_0 \times X_1 \arrow[l] \arrow[d] \\
\mathrm{Fun}(\Delta^1, Y) \arrow[r] & \mathrm{Fun}(\partial\Delta^1, Y) & Y_0 \times Y_1, \arrow[l]
\end{tikzcd}$$

noting that the left horizontal maps are Kan fibrations by virtue of Corollary 3.1.3.3. □

Warning 3.4.0.10. Let $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ be morphisms of simplicial sets, where X is a Kan complex. The homotopy fiber products $X_0 \times^{\mathrm{h}}_{X} X_1$ and $X_1 \times^{\mathrm{h}}_{X} X_0$ are generally not isomorphic as simplicial sets. Instead, we have a canonical isomorphism `032C`

$$(X_1 \times^{\mathrm{h}}_{X} X_0)^{\mathrm{op}} \simeq X_0^{\mathrm{op}} \times^{\mathrm{h}}_{X^{\mathrm{op}}} X_1^{\mathrm{op}}.$$

However, $X_0 \times^{\mathrm{h}}_{X} X_1$ and $X_1 \times^{\mathrm{h}}_{X} X_0$ have the same weak homotopy type. To see this, we can use Proposition 3.1.7.1 to factor f_0 as a composition $X_0 \xrightarrow{w} X_0' \xrightarrow{f_0'} X$, where w is a weak homotopy equivalence and f_0' is a Kan fibration. Using Propositions 3.4.0.7 and 3.4.0.9, we see that the maps

$$X_0 \times^{\mathrm{h}}_{X} X_1 \to X_0' \times^{\mathrm{h}}_{X} X_1 \hookleftarrow X_0' \times_X X_1 \simeq X_1 \times_X X_0' \hookrightarrow X_1 \times^{\mathrm{h}}_{X} X_0' \leftarrow X_1 \times^{\mathrm{h}}_{X} X_0$$

are weak homotopy equivalences.

For many applications, it will be useful to reformulate the notion of homotopy fiber product by viewing it as a *property* of diagrams, rather than as a construction. Recall that a commutative diagram of simplicial sets

032D

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\ X_1 \arrow[r] & X \end{tikzcd} \tag{3.2}$$

is a *pullback square* if the induced map $\theta : X_{01} \to X_0 \times_X X_1$ is an isomorphism of simplicial sets. If X is a Kan complex, we will say that the diagram (3.2) is a *homotopy pullback square* if the composite map

$$X_{01} \xrightarrow{\theta} X_0 \times_X X_1 \hookrightarrow X_0 \times^{\mathrm{h}}_X X_1$$

is a weak homotopy equivalence of simplicial sets. In §3.4.1, we give an overview of the theory of homotopy pullback diagrams (beginning with an extension to the case where X is not a Kan complex: see Definition 3.4.1.1 and Corollary 3.4.1.6).

The preceding discussion has an analogue for pushout diagrams. Given morphisms of simplicial sets $f_0 : A \to A_0$ and $f_1 : A \to A_1$ having the same source, we define the *homotopy pushout of* A_0 *with* A_1 *along* A to be the iterated coproduct

$$A_0 {\coprod}^{\mathrm{h}}_A A_1 = A_0 \coprod_{(\{0\} \times A)} (\Delta^1 \times A) \coprod_{(\{1\} \times A)} A_1$$

(Construction 3.4.2.2). We say that a commutative diagram of simplicial sets

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "f_1"] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01} \end{tikzcd}$$

is a *homotopy pushout square* if the induced map

$$A_0 {\coprod}^{\mathrm{h}}_A A_1 \twoheadrightarrow A_0 {\coprod}_A A_1 \to A_{01}$$

is a weak homotopy equivalence (Proposition 3.4.2.5). Many of the basic properties of homotopy pullback diagrams have counterparts for homotopy pushout diagrams, which we summarize in §3.4.2.

The notions of homotopy pullback and homotopy pushout diagram were introduced by Mather (in the setting of topological spaces, rather than simplicial sets) and have subsequently proven to be a very useful tool in algebraic topology. In [41], Mather established two fundamental results relating homotopy pullback and homotopy pushout squares, which are now known as the *Mather cube theorems*:

- Suppose we are given a homotopy pushout square of simplicial sets

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D. \end{tikzcd}$$

  If $\overline{D} \to D$ is a Kan fibration, then the induced diagram

$$\begin{tikzcd} A \times_D \overline{D} \arrow[r] \arrow[d] & B \times_D \overline{D} \arrow[d] \\ C \times_D \overline{D} \arrow[r] & \overline{D} \end{tikzcd}$$

  is also a homotopy pushout square (Proposition 3.4.3.2). Stated more informally, the collection of homotopy pushout squares is stable under pullback by Kan fibrations. In §3.4.3 we establish a slightly more general (and homotopy invariant) version of this statement, which is known as *Mather's second cube theorem* (Theorem 3.4.3.3).

- Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd} \overline{C} \arrow[d] & \overline{A} \arrow[l] \arrow[r, "\overline{i}"] \arrow[d] & \overline{B} \arrow[d] \\ C & A \arrow[l] \arrow[r, "i"] & B, \end{tikzcd}$$

  in which both squares are homotopy pullbacks. If i and $\overline{i}$ are monomorphisms, then both squares in the induced diagram

$$\begin{tikzcd} \overline{C} \arrow[r] \arrow[d] & \overline{C} \coprod_{\overline{A}} \overline{B} \arrow[d] & \overline{B} \arrow[l] \arrow[d] \\ C \arrow[r] & C \coprod_{A} B & B \arrow[l] \end{tikzcd}$$

are also homotopy pullback squares (Proposition 3.4.4.3). In §3.4.4 we establish a slightly more general (and homotopy invariant) version of this statement, which is known as *Mather's first cube theorem* (Theorem 3.4.4.4).

The homotopy theory of topological spaces provides a rich supply of examples of homotopy pushout squares. Let X be a topological space which can be written as the union of two open subsets $U, V \subseteq X$. In §3.4.6, we show that the resulting diagram of singular simplicial sets

$$\begin{tikzcd}
\mathrm{Sing}_\bullet(U \cap V) \arrow[r] \arrow[d] & \mathrm{Sing}_\bullet(U) \arrow[d] \\
\mathrm{Sing}_\bullet(V) \arrow[r] & \mathrm{Sing}_\bullet(X)
\end{tikzcd}$$

is a homotopy pushout square (Theorem 3.4.6.1). To carry out the proof, we make use of the fact that the weak homotopy type of a simplicial set X can be recovered from its underlying semisimplicial set (see Proposition 3.4.5.4 and Corollary 3.4.5.5), which we explain in §3.4.5). We conclude in §3.4.7 by applying Theorem 3.4.6.1 to deduce the classical Seifert-van Kampen theorem (Theorem 3.4.7.1) and the excision theorem for singular homology (Theorem 3.4.7.3).

010J **Remark 3.4.0.11.** The notions of homotopy pullback and homotopy pushout diagrams can be regarded as homotopy-invariant replacements for the usual notion of pullback and pushout diagrams, respectively. We will later make this heuristic precise by showing that a commutative diagram in the ordinary category of Kan complexes

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback square (homotopy pushout square) if and only if it is a pullback square (pushout square) when regarded as a diagram in the ∞-category $\mathcal{S}$ of Kan complexes (Construction 5.5.1.1); see Examples 7.6.4.2 and 7.6.4.3.

3.4.1 Homotopy Pullback Squares

010K We begin by formulating the notion of a homotopy pullback square for general simplicial sets.

Definition 3.4.1.1. A commutative diagram of simplicial sets 010L

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "q"] \\
X_1 \arrow[r] & X
\end{tikzcd} \qquad (3.3)$$ 010M

is a *homotopy pullback square* if, for every factorization $q = q' \circ w$ where $w : X_0 \to X_0'$ is a weak homotopy equivalence and $q' : X_0' \to X$ is a Kan fibration, the induced map $X_{01} \to X_0' \times_X X_1$ is a weak homotopy equivalence.

To verify the condition of Definition 3.4.1.1 in general, it suffices to consider a *single* factorization $q = q' \circ w$:

Proposition 3.4.1.2. *Suppose we are given a commutative diagram of simplicial sets* 010N

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "q"] \\
X_1 \arrow[r] & X.
\end{tikzcd} \qquad (3.4)$$ 010P

Suppose that q factors as a composition $X_0 \xrightarrow{w'} X_0' \xrightarrow{q'} X$, where w' is a weak homotopy equivalence and q' is a Kan fibration. Then (3.4) is a homotopy pullback square if and only if the induced map $\rho' : X_{01} \to X_0' \times_X X_1$ is a weak homotopy equivalence.

Proof. Suppose that q admits another factorization $X_0 \xrightarrow{w''} X_0'' \xrightarrow{q''} X$, where w'' is a weak homotopy equivalence and q'' is a Kan fibration. We wish to show that ρ' is a weak homotopy equivalence if and only if the induced map $\rho'' : X_{01} \to X_0'' \times_X X_1$ is a weak homotopy equivalence. To prove this equivalence, we may assume without loss of generality that w' is anodyne (since this can always be arranged using Proposition 3.1.7.1). In this case, the lifting problem

$$\begin{tikzcd}
X_0 \arrow[r, "w''"] \arrow[d, "w'"'] & X_0'' \arrow[d, "q''"] \\
X_0' \arrow[r, "q'"] \arrow[ur, dashed, "u"] & X
\end{tikzcd}$$

admits a solution $u : X_0' \to X_0''$ (Remark 3.1.2.7). Since w' and w'' are weak homotopy equivalences, the equality $w'' = u \circ w'$ guarantees that u is also a weak homotopy equivalence

(Remark 3.1.6.16), so that the map $X'_0 \times_X X_1 \to X''_0 \times_X X_1$ is a weak homotopy equivalence by virtue of Proposition 3.4.0.2. □

010U **Example 3.4.1.3.** Suppose we are given a commutative diagram of simplicial sets

010V
$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "q"] \\
X_1 \arrow[r] & X,
\end{tikzcd} \tag{3.5}$$

where q is a Kan fibration. Applying Proposition 3.4.1.2 to the factorization $q = q \circ \mathrm{id}_{X_0}$, we see that (3.5) is a homotopy pullback square if and only if the induced map $X_{01} \to X_0 \times_X X_1$ is a weak homotopy equivalence. In particular, if (3.5) is a pullback diagram, then it is also a homotopy pullback diagram. Beware that this conclusion is generally false when q is not a Kan fibration.

010S **Example 3.4.1.4.** Suppose we are given a commutative diagram of simplicial sets

010T
$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S' \arrow[r] & S,
\end{tikzcd} \tag{3.6}$$

where q and q' are Kan fibrations. Then (3.6) is a homotopy pullback square if and only if, for each vertex $s' \in S'$ having image $s \in S$, the induced map of fibers $X'_{s'} \to X_s$ is a homotopy equivalence of Kan complexes. This is essentially a restatement of Proposition 3.3.7.1 (by virtue of Proposition 3.4.1.2).

010Q **Corollary 3.4.1.5.** *Suppose we are given a commutative diagram of simplicial sets*

010R
$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d, "q'"] & X_0 \arrow[d, "q"] \\
X_1 \arrow[r] & X,
\end{tikzcd} \tag{3.7}$$

where q is a weak homotopy equivalence. Then (3.7) is a homotopy pullback square if and only if q' is a weak homotopy equivalence.

Proof. Apply Proposition 3.4.1.2 to the factorization $q = \mathrm{id}_X \circ q$. □

Corollary 3.4.1.6. *Suppose we are given a commutative diagram of simplicial sets* 032E

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d, "q'"] & X_0 \arrow[d, "q"] \\
X_1 \arrow[r] & X,
\end{tikzcd} \tag{3.8}$$ 032F

where X is a Kan complex. Then (3.8) is a homotopy pullback square if and only if the induced map

$$\theta : X_{01} \to X_0 \times_X X_1 \hookrightarrow X_0 \times^{\mathrm{h}}_X X_1$$

is a weak homotopy equivalence.

Proof. Using Proposition 3.1.7.1, we can factor q as a composition $X_0 \xrightarrow{w} X_0' \xrightarrow{q'} X$, where w is a weak homotopy equivalence and q' is Kan fibration. We then have a commutative diagram

$$\begin{tikzcd}
X_{01} \arrow[r, "\theta"] \arrow[d, "\rho"] & X_0 \times^{\mathrm{h}}_X X_1 \arrow[d] \\
X_0' \times_X X_1 \arrow[r] & X_0' \times^{\mathrm{h}}_X X_1,
\end{tikzcd}$$

where the bottom horizontal map is a weak homotopy equivalence (Proposition 3.4.0.7) and the right vertical map is also a weak homotopy equivalence (Proposition 3.4.0.9). It follows that θ is a weak homotopy equivalence if and only if ρ is a weak homotopy equivalence. By virtue of Proposition 3.4.1.2, this is equivalent to the requirement that the diagram (3.8) is a homotopy pullback square. □

Remark 3.4.1.7. A commutative diagram of simplicial sets 032G

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback square if and only if the induced diagram of opposite simplicial sets

$$\begin{tikzcd}
X_{01}^{\mathrm{op}} \arrow[r] \arrow[d] & X_0^{\mathrm{op}} \arrow[d] \\
X_1^{\mathrm{op}} \arrow[r] & X^{\mathrm{op}}
\end{tikzcd}$$

is a homotopy pullback square.

010W **Warning 3.4.1.8.** For a general pair of morphisms $f_0 : X_0 \to X$, $f_1 : X_1 \to X$ in the category of simplicial sets, there need not exist a homotopy pullback square

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X.
\end{tikzcd}$$

For example, if $f_0 : \{0\} \hookrightarrow \Delta^1$ and $f_1 : \{1\} \hookrightarrow \Delta^1$ are the inclusion maps, then the existence of a commutative diagram

010X
$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & \{0\} \arrow[d, "f_0"] \\
\{1\} \arrow[r, "f_1"] & \Delta^1
\end{tikzcd} \tag{3.9}$$

guarantees that the simplicial set X_{01} is empty, in which case (3.9) is not a homotopy pullback square.

Note that Definition 3.4.1.1 is *a priori* asymmetric: it involves replacing the map $f_0 : X_0 \to X$ by a Kan fibration, but leaving the map $f_1 : X_1 \to X$ unchanged. However, this turns out to be irrelevant.

010Y **Proposition 3.4.1.9** (Symmetry)**.** *A commutative diagram of simplicial sets*

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X
\end{tikzcd}$$

is a homotopy pullback square if and only if the transposed diagram

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_1 \arrow[d, "f_1"] \\
X_0 \arrow[r, "f_0"] & X
\end{tikzcd}$$

is a homotopy pullback square.

Proof. Using Proposition 3.1.7.1, we can choose factorizations

$$X_0 \xrightarrow{w_0} X'_0 \xrightarrow{f'_0} S \qquad X_1 \xrightarrow{w_1} X'_1 \xrightarrow{f'_1} S$$

of f_0 and f_1, where both f'_0 and f'_1 are Kan fibrations and both w_0 and w_1 are weak homotopy equivalences. We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
X_{01} \arrow[r, "u"] \arrow[d, "v"] & X_0 \times_X X'_1 \arrow[r] \arrow[d, "v'"] & X_0 \arrow[d, "w_0"] \\
X'_0 \times_X X_1 \arrow[r, "u'"] \arrow[d] & X'_0 \times_X X'_1 \arrow[r] \arrow[d] & X'_0 \arrow[d, "f'_0"] \\
X_1 \arrow[r, "w_1"] & X'_1 \arrow[r, "f'_1"] & X.
\end{tikzcd}$$

We wish to show that u is a weak homotopy equivalence if and only if v is a weak homotopy equivalence (see Proposition 3.4.1.2). This follows from the two-out-of-three property (Remark 3.1.6.16), since the morphisms u' and v' are weak homotopy equivalences by virtue of Corollary 3.3.7.4. □

`010Z` **Remark 3.4.1.10.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d, "w'"] & X_0 \arrow[d, "w"] \\
X'_{01} \arrow[r] \arrow[d] & X'_0 \arrow[d] \\
X_1 \arrow[r] & X,
\end{tikzcd}$$

where w and w' are weak homotopy equivalences. Then the lower half of the diagram is a homotopy pullback square if and only if the outer rectangle is a homotopy pullback square (see Corollary 3.4.1.12 for a slight generalization).

`0110` **Proposition 3.4.1.11** (Transitivity)**.** *Suppose we are given a commutative diagram of simplicial sets*

`032H`

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "h"] & Y \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\
U \arrow[r] & T \arrow[r] & S
\end{tikzcd} \qquad (3.10)$$

where the right half of (3.10) is a homotopy pullback square. Then the left half of (3.10) is a homotopy pullback square if and only if the outer rectangle is a homotopy pullback square.

Proof. By virtue of Proposition 3.1.7.1, the morphism f factors as a composition $X \xrightarrow{w_X} X' \xrightarrow{f'} S$, where f' is a Kan fibration and w_X is a weak homotopy equivalence. Set $Y' = T \times_S X'$, so that g factors as a composition $Y \xrightarrow{w_Y} Y' \xrightarrow{g'} T$ where g' is a Kan fibration. Since the right half of (3.10) is a homotopy pullback square, the morphism w_Y is a weak homotopy equivalence. Applying Proposition 3.4.1.2, we see that both conditions are equivalent to the requirement that the induced map $Z \to U \times_T Y' \simeq U \times_S X'$ is a weak homotopy equivalence. $\square$

`0111` **Corollary 3.4.1.12** (Homotopy Invariance)**.** *Suppose we are given a commutative diagram* *of simplicial sets*

$$\begin{tikzcd}
X_{01} \arrow[rr] \arrow[dd] \arrow[dr, "w_{01}"] & & X_0 \arrow[dd] \arrow[dr, "w_0"] & \\
& Y_{01} \arrow[rr] \arrow[dd] & & Y_0 \arrow[dd] \\
X_1 \arrow[rr] \arrow[dr, "w_1"] & & X \arrow[dr, "w"] & \\
& Y_1 \arrow[rr] & & Y,
\end{tikzcd}$$

where the morphisms w_0, w_1, *and* w *are weak homotopy equivalences. Then any two of the following conditions imply the third:*

(1) *The back face*

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback square.

(2) *The front face*

$$\begin{tikzcd}
Y_{01} \arrow[r] \arrow[d] & Y_0 \arrow[d] \\
Y_1 \arrow[r] & Y
\end{tikzcd}$$

is a homotopy pullback square.

(3) *The morphism* $w_{01} : X_{01} \to Y_{01}$ *is a weak homotopy equivalence of simplicial sets.*

Proof. Using Corollary 3.4.1.5, we see that the bottom square in the commutative diagram

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] \arrow[d, "w_1"] & X \arrow[d, "w"] \\
Y_1 \arrow[r] & Y,
\end{tikzcd}$$

is a homotopy pullback square. Applying Propositions 3.4.1.11 and 3.4.1.9, we see that (1) is equivalent to the following:

(1′) The diagram

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
Y_1 \arrow[r] & Y
\end{tikzcd}$$

is a homotopy pullback square.

If condition (3) is satisfied, then the equivalence (1′) ⇔ (2) is a special case of Remark 3.4.1.10. Conversely, if (1′) and (2) are satisfied, then Propositions 3.4.1.11 and 3.4.1.9 guarantee that the upper half of the commutative diagram

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d, "w_{01}"] & X_0 \arrow[d, "w_0"] \\
Y_{01} \arrow[r] \arrow[d] & Y_0 \arrow[d] \\
Y_1 \arrow[r] & Y
\end{tikzcd}$$

is a homotopy pullback square, so that w_{01} is a weak homotopy equivalence by virtue of Corollary 3.4.1.5. □

Suppose we are given a commutative diagram of Kan complexes σ :

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r, "f_1"] & X.
\end{tikzcd}$$

It follows from Corollary 3.4.1.12 that the condition that σ is a homotopy pullback square depends only on the homotopy type of σ as an object of the diagram category $\operatorname{Fun}([1] \times [1], \mathrm{Kan})$. Beware that it does *not* depend only on the image of σ in the homotopy category hKan.

Example 3.4.1.13. Let X be a Kan complex containing a vertex $x \in X$, let ΩX denote the loop space $\{x\} \times^{\mathrm{h}}_{X} \{x\}$, and let P denote the path space $X \times^{\mathrm{h}}_{X} \{x\}$, and let $\iota : \Omega X \hookrightarrow P$ be the inclusion map. We then have a pullback diagram of Kan complexes 01YU

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$$\begin{tikzcd}
\Omega X \arrow[r, "\iota"] \arrow[d] & P \arrow[d, "\mathrm{ev}_0"] \\
\{x\} \arrow[r] & X,
\end{tikzcd} \tag{3.11}$$

where ev_0 is given by evaluation at the vertex $0 \in \Delta^1$. Since ev_0 is a Kan fibration, the diagram (3.11) is also a homotopy pullback square (Example 3.4.1.3). Note that the Kan complex P is contractible, so that ι is homotopic to the constant map $\iota' : \Omega X \to P$ carrying ΩX to the constant path id_x. However, the commutative diagram of Kan complexes

$$\begin{tikzcd}
\Omega X \arrow[r, "\iota'"] \arrow[d] & P \arrow[d, "\mathrm{ev}_0"] \\
\{x\} \arrow[r] & X
\end{tikzcd}$$

is never a homotopy pullback square unless the Kan complex ΩX is contractible (again by Example 3.4.1.3).

Proposition 3.4.1.14 (Summands)**.** *Suppose we are given a homotopy pullback square of* 01GN *simplicial sets*

$$\begin{tikzcd}
X_{01} \arrow[r, "u"] \arrow[d, "v"] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X.
\end{tikzcd}$$

Let $X'_0 \subseteq X_0$*,* $X'_1 \subseteq X_1$*, and* $X' \subseteq X$ *be summands satisfying* $f_0(X'_0) \subseteq X' \supseteq f_1(X'_1)$*, and set* $X'_{01} = u^{-1}(X'_0) \cap v^{-1}(X'_1) \subseteq X_{01}$*. Then the diagram of simplicial sets*

$$\begin{tikzcd}
X'_{01} \arrow[r] \arrow[d] & X'_0 \arrow[d] \\
X'_1 \arrow[r] & X'
\end{tikzcd}$$

is also a homotopy pullback square.

Proof. Consider the diagram of simplicial sets

$$\begin{tikzcd}
v^{-1}(X'_1) \arrow[r] \arrow[d] & X_{01} \arrow[r, "u"] \arrow[d, "v"] & X_0 \arrow[d, "f_0"] \\
X'_1 \arrow[r] & X_1 \arrow[r] & X.
\end{tikzcd}$$

The square on the left is a pullback diagram whose horizontal maps are Kan fibrations (Example 3.1.1.4), and is therefore a homotopy pullback square (Example 3.4.1.3). The square on the right is a homotopy pullback by assumption. Applying Proposition 3.4.1.11, we deduce that bottom half of the commutative diagram

$$\begin{tikzcd}
X'_{01} \arrow[r] \arrow[d] & X'_0 \arrow[d] \\
v^{-1}(X'_1) \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X'_1 \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback square. The top half is a pullback diagram whose vertical maps are Kan fibrations (Example 3.1.1.4), and is therefore also a homotopy pullback square (Example 3.4.1.3). Applying Proposition 3.4.1.11 again, we conclude that the outer rectangle in the diagram

$$\begin{tikzcd}
X'_{01} \arrow[r] \arrow[d] & X'_0 \arrow[r, equal] \arrow[d] & X'_0 \arrow[d] \\
X'_1 \arrow[r] & X' \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback square. Here the square on the right is a pullback diagram of Kan fibrations (Example 3.1.1.4), and therefore a homotopy pullback. Applying Proposition 3.4.1.11 again, we conclude that the left square is a homotopy pullback, as desired. □

3.4.2 Homotopy Pushout Squares

We now formulate a dual version of Definition 3.4.1.1. 0112

Definition 3.4.2.1. A commutative diagram of simplicial sets 0113

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a *homotopy pushout square* if, for every Kan complex X, the diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(A_{01}, X) \arrow[r] \arrow[d] & \operatorname{Fun}(A_0, X) \arrow[d] \\
\operatorname{Fun}(A_1, X) \arrow[r] & \operatorname{Fun}(A, X)
\end{tikzcd}$$

is homotopy pullback square (Definition 3.4.1.1).

We begin by observing that if a diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r, "f_0"] \arrow[d, "f_1"] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a homotopy pushout square, then we can recover the simplicial set A_{01} (up to weak homotopy equivalence) from the morphisms $f_0 : A \to A_0$ and $f_1 : A \to A_1$. To see this, it will be convenient to introduce a dual version of Construction 3.4.0.3.

032K **Construction 3.4.2.2** (Homotopy Pushouts)**.** Let $f_0 : A \to A_0$ and $f_1 : A \to A_1$ be morphisms of simplicial sets. We let $A_0 \coprod^{\mathrm{h}}_A A_1$ denote the iterated pushout

$$A_0 \coprod_{(\{0\} \times A)} (\Delta^1 \times A) \coprod_{(\{1\} \times A)} A_1.$$

We will refer to $A_0 \coprod^{\mathrm{h}}_A A_1$ as the *homotopy pushout of* A_0 *with* A_1 *along* A. Note that the projection map $\Delta^1 \times A \twoheadrightarrow A$ induces a comparison map $A_0 \coprod^{\mathrm{h}}_A A_1 \twoheadrightarrow A_0 \coprod_A A_1$ from the homotopy pushout to the usual pushout, which is an epimorphism of simplicial sets.

032L **Remark 3.4.2.3.** Let $f_0 : A \to A_0$ and $f_1 : A \to A_1$ be morphisms of simplicial sets, and let X be a Kan complex. Then the simplicial set $\operatorname{Fun}(A, X)$ is a Kan complex (Corollary 3.1.3.4), and we have a canonical isomorphism

$$\operatorname{Fun}(A_0 \coprod^{\mathrm{h}}_A A_1, X) \simeq \operatorname{Fun}(A_0, X) \times^{\mathrm{h}}_{\operatorname{Fun}(A,X)} \operatorname{Fun}(A_1, X),$$

where the right hand side is the homotopy fiber product of Construction 3.4.0.3.

032M **Remark 3.4.2.4.** Let $f_0 : A \to A_0$ and $f_1 : A \to A_1$ be morphisms of simplicial sets. Then we have a canonical isomorphism $(A_0 \coprod^{\mathrm{h}}_A A_1)^{\mathrm{op}} \simeq A_1^{\mathrm{op}} \coprod^{\mathrm{h}}_{A^{\mathrm{op}}} A_0^{\mathrm{op}}$.

032N **Proposition 3.4.2.5.** *A commutative diagram of simplicial sets*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a homotopy pushout square if and only if the induced map

$$\theta : A_0 \coprod^{\mathrm{h}}_A A_1 \twoheadrightarrow A_0 \coprod_A A_1 \to A_{01}$$

is a weak homotopy equivalence of simplicial sets.

Proof. Let X be a Kan complex, so that $\operatorname{Fun}(A, X)$ is also a Kan complex (Corollary 3.1.3.4). Applying Corollary 3.4.1.6, we see that the diagram

$$\begin{tikzcd}
\operatorname{Fun}(A_{01}, X) \arrow[r] \arrow[d] & \operatorname{Fun}(A_0, X) \arrow[d] \\
\operatorname{Fun}(A_1, X) \arrow[r] & \operatorname{Fun}(A, X)
\end{tikzcd}$$

is a homotopy pullback square if and only if the composite map

$$\rho_X : \operatorname{Fun}(A_{01}, X) \to \operatorname{Fun}(A_0, X) \times_{\operatorname{Fun}(A,X)} \operatorname{Fun}(A_1, X) \hookrightarrow \operatorname{Fun}(A_0, X) \times^{\mathrm{h}}_{\operatorname{Fun}(A,X)} \operatorname{Fun}(A_1, X)$$

is a homotopy equivalence. Using the isomorphism of Remark 3.4.2.3, we can identify ρ_X with the morphism $\operatorname{Fun}(A_{01}, X) \to \operatorname{Fun}(A_0 \coprod^{\mathrm{h}}_A A_1, X)$ given by precomposition with θ. Proposition 3.4.2.5 now follows by allowing the Kan complex X to vary. □

We now summarize some of the formal properties enjoyed by Definition 3.4.2.1 and Construction 3.4.2.2.

Proposition 3.4.2.6. *A commutative diagram of simplicial sets* 032P

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a homotopy pushout square if and only if the induced diagram of opposite simplicial sets

$$\begin{tikzcd}
A^{\mathrm{op}} \arrow[r] \arrow[d] & A_0^{\mathrm{op}} \arrow[d] \\
A_1^{\mathrm{op}} \arrow[r] & A_{01}^{\mathrm{op}}
\end{tikzcd}$$

is a homotopy pushout square.

Proof. Apply Remark 3.4.1.7. □

Proposition 3.4.2.7 (Symmetry)**.** *A commutative diagram of simplicial sets* 0114

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a homotopy pushout square if and only if the transposed diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_1 \arrow[d] \\
A_0 \arrow[r] & A_{01}
\end{tikzcd}$$

is a homotopy pushout square.

Proof. Apply Proposition 3.4.1.9. □

0115 **Proposition 3.4.2.8** (Transitivity)**.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C',
\end{tikzcd}$$

where the left half is a homotopy pushout square. Then the right half is a homotopy pushout square if and only if the outer rectangle is a homotopy pushout square.

Proof. Apply Proposition 3.4.1.11. □

0116 **Proposition 3.4.2.9** (Homotopy Invariance)**.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] \arrow[dr, "w"] & & A_0 \arrow[dd] \arrow[dr, "w_0"] & \\
& B \arrow[rr] \arrow[dd] & & B_0 \arrow[dd] \\
A_1 \arrow[rr] \arrow[dr, "w_1"] & & A_{01} \arrow[dr, "w_{01}"] & \\
& B_1 \arrow[rr] & & B_{01},
\end{tikzcd}$$

where the morphisms w, w_0, and w_1 are weak homotopy equivalences. Then any two of the following three conditions imply the third:

(1) *The back face*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a homotopy pushout square.

(2) *The front face*

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & B_0 \arrow[d] \\
B_1 \arrow[r] & B_{01}
\end{tikzcd}$$

is a homotopy pushout square.

(3) *The morphism* w_{01} *is a weak homotopy equivalence.*

Proof. Combine Corollary 3.4.1.12 with Proposition 3.1.6.17. □

Proposition 3.4.2.10. *Suppose we are given a commutative diagram of simplicial sets* 0117

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd} \tag{3.12}$$ 0118

where f *is a weak homotopy equivalence. Then (3.12) is a homotopy pushout square if and only if* f' *is a weak homotopy equivalence.*

Proof. For every Kan complex X, we obtain a commutative diagram of simplicial sets

$$\begin{tikzcd}
\operatorname{Fun}(A, X) & \operatorname{Fun}(B, X) \arrow[l, "u"'] \\
\operatorname{Fun}(A', X) \arrow[u] & \operatorname{Fun}(B', X), \arrow[l, "u'"'] \arrow[u]
\end{tikzcd} \tag{3.13}$$ 0119

where u is a homotopy equivalence of Kan complexes (Proposition 3.1.6.17). Applying Corollary 3.4.1.5, we conclude that (3.13) is a homotopy pullback square if and only if u is a homotopy equivalence of Kan complexes. Consequently, (3.12) is a homotopy pushout square if and only if, for every Kan complex X, the composition with f' induces a homotopy equivalence $\operatorname{Fun}(B', X) \to \operatorname{Fun}(A', X)$. By virtue of Proposition 3.1.6.17, this is equivalent to the requirement that f' is a weak homotopy equivalence. □

011A **Proposition 3.4.2.11.** *Suppose we are given a commutative diagram of simplicial sets*

011B
$$\begin{tikzcd}
A \arrow[r, "f_0"] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01},
\end{tikzcd} \tag{3.14}$$

where f_0 is a monomorphism. Then (3.14) is a homotopy pushout square if and only if the induced map $A_0 \coprod_A A_1 \to A_{01}$ is a weak homotopy equivalence.

Proof. For every Kan complex X, we obtain a commutative diagram of simplicial sets

011C
$$\begin{tikzcd}
\operatorname{Fun}(A, X) & \operatorname{Fun}(A_0, X) \arrow[l, "u"'] \\
\operatorname{Fun}(A_1, X) \arrow[u] & \operatorname{Fun}(A_{01}, X), \arrow[l] \arrow[u]
\end{tikzcd} \tag{3.15}$$

where u is a Kan fibration (Corollary 3.1.3.3). It follows that the diagram (3.15) is a homotopy pullback square if and only if the induced map

$$\operatorname{Fun}(A_{01}, X) \to \operatorname{Fun}(A_0, X) \times_{\operatorname{Fun}(A,X)} \operatorname{Fun}(A_1, X) \simeq \operatorname{Fun}(A_0 \coprod_A A_1, X)$$

is a weak homotopy equivalence (Example 3.4.1.3). Consequently, the diagram (3.14) is a homotopy pushout square if and only if, for every Kan complex X, the induced map $\operatorname{Fun}(A_{01}, X) \to \operatorname{Fun}(A_0 \coprod_A A_1, X)$ is a homotopy equivalence of Kan complexes. By virtue of Proposition 3.1.6.17, this is equivalent to the requirement that the morphism $A_0 \coprod_A A_1 \to A$ is a weak homotopy equivalence. □

Example 3.4.2.12. Suppose we are given a pushout diagram of simplicial sets 011D

$$\begin{tikzcd}
A \arrow[r, "f_0"] \arrow[d] & A_1 \arrow[d] \\
A_1 \arrow[r] & A_{01}.
\end{tikzcd} \qquad (3.16)$$ 012Q

If f_0 is a monomorphism, then (3.16) is also a homotopy pushout diagram.

Corollary 3.4.2.13. *Let $f_0 : A \to A_0$ and $f_1 : A \to A_1$ be morphisms of simplicial sets. If either f_0 or f_1 is a monomorphism, then the comparison map $A_0 \coprod^{\mathrm{h}}_A A_1 \twoheadrightarrow A_0 \coprod_A A_1$ is a weak homotopy equivalence.* 032Q

Proof. Combine Example 3.4.2.12 with Proposition 3.4.2.5. □

Corollary 3.4.2.14. *Suppose we are given a commutative diagram of simplicial sets* 032R

$$\begin{tikzcd}
A_0 \arrow[d] & A \arrow[l, "f_0"'] \arrow[r, "f_1"] \arrow[d] & A_1 \arrow[d] \\
B_0 & B \arrow[l, "g_0"'] \arrow[r, "g_1"] & B_1,
\end{tikzcd}$$

where f_0 and g_0 are monomorphisms and the vertical maps are weak homotopy equivalences. Then the induced map

$$A_0 \coprod_A A_1 \to B_0 \coprod_B B_1$$

is a weak homotopy equivalence.

Proof. Combine Example 3.4.2.12 with Proposition 3.4.2.9. □

Corollary 3.4.2.15. *Suppose we are given a commutative diagram of simplicial sets* 032S

$$\begin{tikzcd}
A_0 \arrow[d] & A \arrow[l, "f_0"'] \arrow[r, "f_1"] \arrow[d] & A_1 \arrow[d] \\
B_0 & B \arrow[l, "g_0"'] \arrow[r, "g_1"] & B_1,
\end{tikzcd}$$

where the vertical maps are weak homotopy equivalences. Then the induced map

$$A_0 \coprod\nolimits_A^{\mathrm{h}} A_1 \to B_0 \coprod\nolimits_B^{\mathrm{h}} B_1$$

is also a weak homotopy equivalence.

Proof. Apply Corollary 3.4.2.14 to the diagram

$$\begin{tikzcd}
\Delta^1 \times A \arrow[d] & \partial\Delta^1 \times A \arrow[l] \arrow[r] \arrow[d] & A_0 \coprod A_1 \arrow[d] \\
\Delta^1 \times B & \partial\Delta^1 \times B \arrow[l] \arrow[r] & B_0 \coprod B_1.
\end{tikzcd}$$

□

Let us conclude with an application of these concepts.

032T **Proposition 3.4.2.16.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

with the following property: for every simplex $\sigma : \Delta^k \to S$, the induced map $f_\sigma : \Delta^k \times_S X \to \Delta^k \times_S Y$ is a weak homotopy equivalence of simplicial sets. Then f is a weak homotopy equivalence of simplicial sets.

Proof. We will prove the following stronger assertion: for every morphism of simplicial sets $S' \to S$, the induced map

$$f_{S'} : S' \times_S X \to S' \times_S Y$$

is a weak homotopy equivalence of simplicial sets. By virtue of Proposition 3.2.8.3, (and Remark 1.1.4.4), we may assume without loss of generality that S' has dimension $\leq k$ for some integer $k \geq -1$. We proceed by induction on k. In the case $k = -1$, the simplicial set S' is empty and there is nothing to prove. Assume therefore that $k \geq 0$. Let S'' denote the $(k-1)$-skeleton of S' and let I be the set of nondegenerate k-simplices of S', so that

Proposition 1.1.4.12 supplies a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{i \in I} \partial \Delta^k \arrow[r] \arrow[d] & \coprod_{i \in I} \Delta^k \arrow[d] \\
S'' \arrow[r] & S',
\end{tikzcd}$$

where the horizontal maps are monomorphisms. It follows that the front and back faces of the diagram

$$\begin{tikzcd}
(\coprod_{i \in I} \partial \Delta^k) \times_S X \arrow[rr] \arrow[dr, "u"] \arrow[dd] & & \coprod_{i \in I} (\Delta^k \times_S X) \arrow[dr, "v"] \arrow[dd] & \\
& \coprod_{i \in I} (\partial \Delta^k \times_S Y) \arrow[rr] & & \coprod_{i \in I} (\Delta^k \times_S Y) \arrow[dd] \\
S'' \times_S X \arrow[rr] \arrow[dr, "f_{S''}"] & & S' \times_S X \arrow[dr, "f_{S'}"] & \\
& S'' \times_S Y \arrow[rr] \arrow[from=uu, crossing over] & & S' \times_S Y
\end{tikzcd}$$

are homotopy pushout squares (Proposition 3.4.2.11). Consequently, to show that $f_{S'}$ is a weak homotopy equivalence, it will suffice to show that $f_{S''}$, u, and v are weak homotopy equivalences (Proposition 3.4.2.9). In the first two cases, this follows from our inductive hypothesis. We may therefore replace S' by the coproduct $\coprod_{i \in I} \Delta^k$, and thereby reduce to the case of a coproduct of simplices. Using Remark 3.1.6.20, we can further reduce to the case where $S' \simeq \Delta^k$ is a standard simplex, in which case the desired result follows from our hypothesis on f. □

Corollary 3.4.2.17. *Let $f : X \to S$ be a morphism of simplicial sets. Suppose that, for every k-simplex $\Delta^k \to S$, the fiber product $\Delta^k \times_S X$ is weakly contractible. Then f is a weak homotopy equivalence.* `032U`

Proof. Apply Proposition 3.4.2.16 in the special case $Y = S$. □

3.4.3 Mather's Second Cube Theorem

011H Our goal in this section is to prove a theorem of Mather (Theorem 3.4.3.3), which asserts that the collection of homotopy pushout squares is stable under the formation of homotopy pullback. This is an analogue (and consequence) of a more elementary statement about sets:

011J **Exercise 3.4.3.1.** Suppose we are given a pushout square of sets

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D.
\end{tikzcd}$$

Then, for every map of sets $\overline{D} \to D$, the induced diagram

$$\begin{tikzcd}
A \times_D \overline{D} \arrow[r] \arrow[d] & B \times_D \overline{D} \arrow[d] \\
C \times_D \overline{D} \arrow[r] & \overline{D}
\end{tikzcd}$$

is also a pushout square.

Since limits and colimits in the category of simplicial sets are computed pointwise, Exercise 3.4.3.1 immediately implies that the collection of pushout squares in the category of simplicial sets is stable under the formation of pullback along any morphism of simplicial sets $q : \overline{D} \to D$. This statement has an analogue for homotopy pushout diagrams of simplicial sets, provided that we assume that q is a Kan fibration.

011K **Proposition 3.4.3.2.** *Suppose we are given a homotopy pushout square of simplicial sets*

011L
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D,
\end{tikzcd} \tag{3.17}$$

and let $q : \overline{D} \to D$ *be a Kan fibration of simplicial sets. Then the induced diagram*

$$\begin{tikzcd}
A \times_D \overline{D} \arrow[r] \arrow[d] & B \times_D \overline{D} \arrow[d] \\
C \times_D \overline{D} \arrow[r] & \overline{D}
\end{tikzcd}$$

is also a homotopy pushout square.

Proof. Choose a factorization of f as a composition $A \xrightarrow{f'} B' \xrightarrow{w} B$, where f' is a monomorphism and w is a weak homotopy equivalence (Exercise 3.1.7.11). Set $D' = B' \coprod_A C$. Our assumption that (3.17) is a homotopy pushout square guarantees that the induced map $D' \to D$ is a weak homotopy equivalence (Proposition 3.4.2.11). We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
A \times_D \overline{D} \arrow[r] \arrow[d] & B' \times_D \overline{D} \arrow[r] \arrow[d] & B \times_D \overline{D} \arrow[d] \\
C \times_D \overline{D} \arrow[r] & D' \times_D \overline{D} \arrow[r] & \overline{D}.
\end{tikzcd}$$

The left square in this diagram is a pushout square (by virtue of Exercise 3.4.3.1) and the map $A \times_D \overline{D} \to B' \times_D \overline{D}$ is a monomorphism, so it is a homotopy pushout square (Example 3.4.2.12). It follows from Corollary 3.3.7.4 that the horizontal maps on the right side of the diagram are weak homotopy equivalences, so the right square is also a homotopy pushout (Proposition 3.4.2.10). Applying Proposition 3.4.2.8, we deduce that the outer rectangle is also a homotopy pushout square, as desired. □

We now formulate a homotopy-invariant version of Proposition 3.4.3.2.

Theorem 3.4.3.3 (Mather's Second Cube Theorem [41])**.** *Suppose we are given a cubical* 011M *diagram of simplicial sets*

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] & \\
& \overline{C} \arrow[rr] \arrow[dd] & & \overline{D} \arrow[dd, "q"] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

having the property that the faces

011N

$$\begin{tikzcd}
\overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[d] & \overline{A} \arrow[r] \arrow[d] & \overline{C} \arrow[d] \\
A \arrow[r] & B & A \arrow[r] & C \\
\overline{C} \arrow[r] \arrow[d] & \overline{D} \arrow[d] & \overline{B} \arrow[r] \arrow[d] & \overline{D} \arrow[d] \\
C \arrow[r] & D & B \arrow[r] & D
\end{tikzcd} \tag{3.18}$$

are homotopy pullback squares. If the bottom face

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is a homotopy pushout square, then the top face

$$\begin{tikzcd}
\overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[d] \\
\overline{C} \arrow[r] & \overline{D}
\end{tikzcd}$$

is also a homotopy pushout square.

Proof. Using Proposition 3.1.7.1, we can factor q as a composition $\overline{D} \xrightarrow{w} \overline{D}' \xrightarrow{q'} D$, where w is a weak homotopy equivalence and q' is a Kan fibration. We then obtain another commutative diagram

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dd] \arrow[dr] & & \overline{B} \arrow[dd] \arrow[dr] & \\
& \overline{C} \arrow[rr] \arrow[dd] & & \overline{D} \arrow[dd, "w"] \\
A \times_D \overline{D}' \arrow[rr] \arrow[dr] & & B \times_D \overline{D}' \arrow[dr] & \\
& C \times_D \overline{D}' \arrow[rr] & & \overline{D}',
\end{tikzcd} \tag{3.19}$$

011P

where the bottom face is a homotopy pushout square by virtue of Proposition 3.4.3.2. Since the diagrams (3.18) are homotopy pullback squares, the vertical arrows in (3.19) are weak homotopy equivalences. Applying Proposition 3.4.2.9, we conclude that the top face

$$\begin{tikzcd}
\overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[d] \\
\overline{C} \arrow[r] & \overline{D}
\end{tikzcd}$$

is also a homotopy pushout square. □

3.4.4 Mather's First Cube Theorem

011Q Our goal in this section is to prove a converse of Theorem 3.4.3.3, known as *Mather's first cube theorem.* As before, we begin with an elementary statement about the category of sets.

011R **Exercise 3.4.4.1.** Suppose we are given a commutative diagram of sets

$$\begin{tikzcd}
\overline{C} \arrow[d] & \overline{A} \arrow[l] \arrow[r, "\overline{i}"] \arrow[d] & \overline{B} \arrow[d] \\
C & A \arrow[l] \arrow[r, "i"] & B
\end{tikzcd}$$

where both squares are pullback diagrams, and i is a monomorphism (so that $\overline{i}$ is also a monomorphism). Show that both squares in the resulting diagram

$$\begin{tikzcd}
\overline{C} \arrow[r] \arrow[d] & \overline{C} \coprod_{\overline{A}} \overline{B} \arrow[d] & \overline{B} \arrow[l] \arrow[d] \\
C \arrow[r] & C \coprod_{A} B & B \arrow[l]
\end{tikzcd}$$

are pullback squares.

011S **Warning 3.4.4.2.** The conclusion of Exercise 3.4.4.1 does not necessarily hold if the map i is not injective. For example, let G be a group with multiplication map $m : G \times G \to G$, and let $\pi, \pi' : G \times G \to G$ be the projection maps onto the two factors. Then the diagram of sets

$$\begin{tikzcd}
G \arrow[d] & G \times G \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "m"] & G \arrow[d] \\
* & G \arrow[l] \arrow[r] & *
\end{tikzcd}$$

consists of pullback squares, but the induced diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & G \coprod_{G \times G} G \arrow[d] & G \arrow[l] \arrow[d] \\
* \arrow[r] & * \coprod_{G} * & * \arrow[l]
\end{tikzcd}$$

does not (except in the case where G is trivial).

Exercise 3.4.4.1 has an analogue for homotopy pullback diagrams of simplicial sets.

Proposition 3.4.4.3. *Suppose we are given a commutative diagram of simplicial sets* `011T`

$$\begin{tikzcd}
\overline{C} \arrow[d] & \overline{A} \arrow[l] \arrow[r, "\overline{i}"] \arrow[d] & \overline{B} \arrow[d] \\
C & A \arrow[l] \arrow[r, "i"] & B,
\end{tikzcd}$$

in which both squares are homotopy pullbacks. If i and $\overline{i}$ are monomorphisms, then both squares in the induced diagram

$$\begin{tikzcd}
\overline{C} \arrow[r] \arrow[d] & \overline{C} \coprod_{\overline{A}} \overline{B} \arrow[d] & \overline{B} \arrow[l] \arrow[d] \\
C \arrow[r] & C \coprod_{A} B & B \arrow[l]
\end{tikzcd}$$

are also also homotopy pullbacks.

Proposition 3.4.4.3 is an immediate consequence of Example 3.4.2.12 together with the following homotopy-invariant statement:

Theorem 3.4.4.4 (Mather's First Cube Theorem)**.** *Suppose we are given a cubical diagram of simplicial sets* `011U`

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] & \\
& \overline{C} \arrow[rr] \arrow[dd] & & \overline{D} \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

(3.20) `011V`

having the property that the back and left faces

$$\begin{tikzcd}
\overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[d] & \overline{A} \arrow[r] \arrow[d] & \overline{C} \arrow[d] \\
A \arrow[r] & B & A \arrow[r] & C
\end{tikzcd}$$

are homotopy pullback squares, and the top and bottom faces

$$\begin{tikzcd}
\overline{A} \arrow[r] \arrow[d] & \overline{B} \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[d] \\
\overline{C} \arrow[r] & \overline{D} & C \arrow[r] & D
\end{tikzcd}$$

are homotopy pushout squares. Then the front and right faces

$$\begin{tikzcd}
\overline{C} \arrow[r] \arrow[d] & \overline{D} \arrow[d] & \overline{B} \arrow[r] \arrow[d] & \overline{D} \arrow[d] \\
C \arrow[r] & D & B \arrow[r] & D
\end{tikzcd}$$

are also homotopy pullback squares.

Proof. The proof will proceed in several steps, each of which involves replacing one or more of the terms in (3.20) by a weakly equivalent simplicial set (by virtue of Corollary 3.4.1.12 and Proposition 3.4.2.9, such replacements will not affect the truth of our hypotheses or of the desired conclusion). Let us denote each of the morphisms appearing in the diagram (3.20) by f_{XY}, where $X, Y \in \{\overline{A}, \overline{B}, \overline{C}, \overline{D}, A, B, C, D\}$ are the source and target of f_{XY}, respectively.

- By virtue of Proposition 3.1.7.1, the morphism $f_{\overline{B}B} : \overline{B} \to B$ factors as a composition $\overline{B} \xrightarrow{w} \overline{B}' \xrightarrow{f'_{\overline{B}B}} B$, where w is anodyne and $f'_{\overline{B}B}$ is a Kan fibration. Replacing $\overline{B}$ by $\overline{B}'$ (and $\overline{D}$ by the pushout $\overline{B}' \coprod_{\overline{B}} \overline{D}$), we can reduce to the case where $f_{\overline{B}B}$ is a Kan fibration. Similarly, we can arrange that the map $f_{\overline{C}C} : \overline{C} \to C$ is a Kan fibration.

- Applying Proposition 3.1.7.1 again, we can factor the morphism $g : \overline{A} \to A \times_{(B \times C)} (\overline{B} \times \overline{C})$ as a composition

$$\overline{A} \xrightarrow{w} \overline{A}' \xrightarrow{g'} A \times_{(B \times C)} (\overline{B} \times \overline{C}),$$

where w is anodyne and g' is a Kan fibration. Replacing $\overline{A}$ by $\overline{A}'$, we can reduce to the case where g is a Kan fibration, so that the morphism $f_{\overline{A}A}$ is also a Kan fibration.

- By virtue of Exercise 3.1.7.11, the morphism f_{AB} factors as a composition $A \xrightarrow{f'_{AB}} B' \xrightarrow{w} B$, where f'_{AB} is a monomorphism and w is a trivial Kan fibration. Replacing B by B' (and $\overline{B}$ by the fiber product $B' \times_B \overline{B}'$), we can reduce to the case where f_{AB} is a monomorphism. Similarly, we may assume that f_{AC} is a monomorphism.

- By virtue of Exercise 3.1.7.11, the morphism $f_{\overline{AB}}$ factors as a composition $\overline{A} \xrightarrow{f'_{\overline{AB}}} \overline{B}' \xrightarrow{w} \overline{B}$, where $f'_{\overline{AB}}$ is a monomorphism and w is a trivial Kan fibration. Replacing $\overline{B}$ by $\overline{B}'$, we can reduce to the case where $f_{\overline{AB}}$ is a monomorphism. Similarly, we can assume that $f_{\overline{AC}}$ is a monomorphism.

- The back face

$$\begin{tikzcd}
\overline{A} \arrow[r, "f_{\overline{AB}}"] \arrow[d, "f_{\overline{A}A}"] & \overline{B} \arrow[d, "f_{\overline{B}B}"] \\
A \arrow[r, "f_{AB}"] & B
\end{tikzcd} \tag{3.21}$$

011X

is a homotopy pullback square in which the horizontal maps are monomorphisms and the vertical maps are Kan fibrations. It follows that, for every vertex $a \in A$ having image $b = f_{AB}(a) \in B$, the induced map of fibers $\overline{A}_a \to \overline{B}_b$ is a homotopy equivalence. Let $\overline{B}' \subseteq \overline{B}$ denote the simplicial subset spanned by those simplices $\sigma : \Delta^n \to \overline{B}$ having the property that the restriction $\sigma|_{A \times_B \Delta^n}$ factors through $\overline{A}$. Applying Lemma 3.3.8.4, we deduce that the restriction $f_{\overline{B}B}|_{\overline{B}'} : \overline{B}' \to B$ is also a Kan fibration. Moreover, the inclusion map $\overline{B}' \hookrightarrow \overline{B}$ induces a homotopy equivalence of fibers $\overline{B}'_b \hookrightarrow \overline{B}_b$, for each vertex $b \in B$. It follows that the inclusion $\overline{B}' \hookrightarrow \overline{B}$ is a weak homotopy equivalence (Corollary 3.3.7.5). Replacing $\overline{B}$ by $\overline{B}'$, we can reduce to the case where the diagram (3.21) is a pullback square. Similarly, we can arrange that the diagram

$$\begin{tikzcd}
\overline{A} \arrow[r, "f_{\overline{AC}}"] \arrow[d, "f_{\overline{A}A}"] & \overline{C} \arrow[d, "f_{\overline{C}C}"] \\
A \arrow[r, "f_{AC}"] & C
\end{tikzcd}$$

is a pullback square.

- By assumption, the top and bottom faces

011Y
$$\begin{tikzcd}
\overline{A} \arrow[r] \arrow[d, "f_{\overline{AC}}"] & \overline{B} \arrow[d] & A \arrow[r] \arrow[d, "f_{AC}"] & B \arrow[d] \\
\overline{C} \arrow[r] & \overline{D} & C \arrow[r] & D
\end{tikzcd} \tag{3.22}$$

  are homotopy pushout squares. Since $f_{\overline{AC}}$ and f_{AC} are monomorphisms, it follows that the induced maps

$$\overline{C} \coprod_{\overline{A}} \overline{B} \to \overline{D} \qquad C \coprod_{A} B \to D$$

  are weak homotopy equivalences (Proposition 3.4.2.11). We may therefore replace D by the pushout $C \coprod_A B$ and $\overline{D}$ by the pushout $\overline{C} \coprod_{\overline{A}} \overline{B}$, and thereby reduce to the case where the diagrams (3.22) are pushout squares.

- Applying Exercise 3.4.4.1 levelwise, we deduce that the front and right faces

011Z
$$\begin{tikzcd}
\overline{C} \arrow[r] \arrow[d, "f_{\overline{C}C}"] & \overline{D} \arrow[d, "f_{\overline{D}D}"] & \overline{B} \arrow[r] \arrow[d, "f_{\overline{B}B}"] & \overline{D} \arrow[d, "f_{\overline{D}D}"] \\
C \arrow[r] & D & B \arrow[r] & D
\end{tikzcd} \tag{3.23}$$

  are pullback squares in the category of simplicial sets. In particular, for every simplex $\sigma : \Delta^n \to D$, the projection map $\Delta^n \times_D \overline{D} \to \Delta^n$ is a pullback either of $f_{\overline{B}B}$ or of $f_{\overline{C}C}$, and is therefore a Kan fibration. Applying Remark 3.1.1.7, we conclude that $f_{\overline{D}D} : \overline{D} \to D$ is a Kan fibration. It follows that the diagrams (3.23) are also homotopy pullback squares, as desired.

□

3.4.5 Digression: Weak Homotopy Equivalences of Semisimplicial Sets

0120 Recall that a morphism of simplicial sets $f : X \to Y$ is a *weak homotopy equivalence* if, for every Kan complex Z, precomposition with f induces a bijection $\pi_0(\mathrm{Fun}(Y, Z)) \to \pi_0(\mathrm{Fun}(X, Z))$ (Definition 3.1.6.12). Our goal in this section is to show that this condition depends only on the underlying morphism of semisimplicial sets. To see this, we begin by recalling that the forgetful functor

$$\{\text{Simplicial Sets}\} \to \{\text{Semisimplicial Sets}\}$$

admits a left adjoint, which we denote by $X \mapsto X^{+}$ (Corollary 3.3.1.10).

Definition 3.4.5.1. Let $f : X \to Y$ be a morphism of semisimplicial sets. We will say that f is a *weak homotopy equivalence* if the induced map of simplicial sets $f^+ : X^+ \to Y^+$ is a weak homotopy equivalence, in the sense of Definition 3.1.6.12. 0121

Remark 3.4.5.2. The collection of weak homotopy equivalences of semisimplicial sets is closed under the formation of filtered colimits. This follows immediately from the corresponding assertion for simplicial sets (Proposition 3.2.8.3), since the construction $X \mapsto X^+$ commutes with filtered colimits. 0122

Remark 3.4.5.3. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of semisimplicial sets. If any two of the morphisms f, g, and $g \circ f$ are weak homotopy equivalences, then so is the third (see Remark 3.1.6.16). 0123

When X is a simplicial set, we write $v_X : X^+ \to X$ for the counit map (that is, the unique morphism of simplicial sets whose restriction to $(X^+)^{\mathrm{nd}} \simeq X$ is the identity map). To compare Definition 3.4.5.1 with Definition 3.1.6.12, we need the following:

Proposition 3.4.5.4. *For every simplicial set X, the counit map $v_X : X^+ \to X$ is a weak homotopy equivalence.* 0124

Corollary 3.4.5.5. *Let $f : X \to Y$ be a morphism of simplicial sets. Then f is a weak homotopy equivalence (in the sense of Definition 3.1.6.12) if and only if the underlying morphism of semisimplicial sets is a weak homotopy equivalence (in the sense of Definition 3.4.5.1).* 0125

Proof. We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
X^+ \arrow[r, "f^+"] \arrow[d, "v_X"] & Y^+ \arrow[d, "v_Y"] \\
X \arrow[r, "f"] & Y,
\end{tikzcd}$$

where the vertical maps are weak homotopy equivalences by virtue of Proposition 3.4.5.4. Invoking Remark 3.1.6.16, we deduce that f is a weak homotopy equivalence if and only if f^+ is a weak homotopy equivalence. □

Corollary 3.4.5.6. *For every semisimplicial set X, the inclusion map $\iota : X \hookrightarrow X^+$ is a weak homotopy equivalence of semisimplicial sets.* 0126

Proof. We wish to show that the map $\iota^+ : X^+ \to (X^+)^+$ is a weak homotopy equivalence of simplicial sets. This is clear, since ι^+ is right inverse to the counit map $v_{X^+} : (X^+)^+ \to X^+$, which is a weak homotopy equivalence of simplicial sets by virtue of Proposition 3.4.5.4. □

0127 **Variant 3.4.5.7.** Let X be a simplicial set, and let $\iota : X \hookrightarrow X^+$ be the inclusion map. Then the map $\mathrm{Ex}(\iota) : \mathrm{Ex}(X) \hookrightarrow \mathrm{Ex}(X^+)$ is a weak homotopy equivalence of semisimplicial sets.

Proof. By virtue of Proposition 3.4.5.4, the counit map $v_X : X^+ \to X$ is a weak homotopy equivalence of simplicial sets. Applying Corollary 3.3.5.2, we deduce that the map $\mathrm{Ex}(v_X) : \mathrm{Ex}(X^+) \to \mathrm{Ex}(X)$ is a weak homotopy equivalence of simplicial sets, hence also a weak homotopy equivalence of the underlying semisimplicial sets (Corollary 3.4.5.5). Since the composite map

$$\mathrm{Ex}(X) \xrightarrow{\mathrm{Ex}(\iota)} \mathrm{Ex}(X^+) \xrightarrow{\mathrm{Ex}(v_X)} \mathrm{Ex}(X)$$

is the identity, it follows that $\mathrm{Ex}(\iota)$ is also a weak homotopy equivalence of semisimplicial sets. □

0128 **Corollary 3.4.5.8.** *Let X and Y be simplicial sets and let $f : X \to Y$ be a morphism of semisimplicial sets. Then f is a weak homotopy equivalence of semisimplicial sets if and only if the induced map $\mathrm{Ex}(f) : \mathrm{Ex}(X) \to \mathrm{Ex}(Y)$ is a weak homotopy equivalence of semisimplicial sets.*

Proof. By definition, $f : X \to Y$ is a weak homotopy equivalence of semisimplicial sets if and only if the induced map $f^+ : X^+ \to Y^+$ is a weak homotopy equivalence of simplicial sets. By virtue of Corollary 3.3.5.2, this is equivalent to the assertion that $\mathrm{Ex}(f^+) : \mathrm{Ex}(X^+) \to \mathrm{Ex}(Y^+)$ is a weak homotopy equivalence when viewed as a morphism of simplicial sets, or equivalently when viewed as a morphism of semisimplicial sets (Corollary 3.4.5.5). The desired result now follows by inspecting the commutative diagram of semisimplicial sets

$$\begin{tikzcd}
\mathrm{Ex}(X) \arrow[r, "\mathrm{Ex}(f)"] \arrow[d] & \mathrm{Ex}(Y) \arrow[d] \\
\mathrm{Ex}(X^+) \arrow[r, "\mathrm{Ex}(f^+)"] & \mathrm{Ex}(Y^+),
\end{tikzcd}$$

since the vertical maps are weak homotopy equivalences by virtue of Variant 3.4.5.7. □

We now turn to the proof of Proposition 3.4.5.4. The main ingredient we will need is the following is the following:

0129 **Lemma 3.4.5.9.** *Let $\mathcal{C}$ be a category, and suppose that the collection of non-identity morphisms in $\mathcal{C}$ is closed under composition. Then the counit map $v_{\mathrm{N}_\bullet(\mathcal{C})} : \mathrm{N}_\bullet(\mathcal{C})^+ \to \mathrm{N}_\bullet(\mathcal{C})$ is a homotopy equivalence of simplicial sets.*

Proof. Let $\mathcal{C}^+$ denote the category obtained from $\mathcal{C}$ by formally adjoining a new identity morphism id_X^+ for each object $X \in \mathcal{C}$. More precisely, the category $\mathcal{C}^+$ is defined as follows:

- The objects of $\mathcal{C}^{+}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}^{+}$, we have

$$\mathrm{Hom}_{\mathcal{C}^{+}}(X,Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \text{if } X \neq Y \\ \mathrm{Hom}_{\mathcal{C}}(X,Y) \coprod \{\mathrm{id}_X^{+}\} & \text{if } X = Y. \end{cases}$$

- If $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{C}^{+}$, then the composition $g \circ f$ is equal to g if $f = \mathrm{id}_Y^{+}$, to the morphism f if $g = \mathrm{id}_Y^{+}$, and is otherwise given by the composition law for morphisms in $\mathcal{C}$.

Note that the collection of non-identity morphisms in $\mathcal{C}^{+}$ is closed under composition, so that the nerve $\mathrm{N}_\bullet(\mathcal{C}^{+})$ is a braced simplicial set (Exercise 3.3.1.2). Unwinding the definitions, we see that the semisimplicial subset $\mathrm{N}_\bullet(\mathcal{C}^{+})^{\mathrm{nd}} \subseteq \mathrm{N}_\bullet(\mathcal{C}^{+})$ can be identified with the $\mathrm{N}_\bullet(\mathcal{C})$ (as a semisimplicial set). Using Corollary 3.3.1.11, we obtain a canonical isomorphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C})^{+} \simeq \mathrm{N}_\bullet(\mathcal{C}^{+})$. Under this isomorphism, the counit map $v_{\mathrm{N}_\bullet(\mathcal{C})}$ is induced by the functor $F : \mathcal{C}^{+} \to \mathcal{C}$ which is the identity on objects, and carries each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{C}^{+}}(X,Y)$ to itself.

Let $G : \mathcal{C} \to \mathcal{C}^{+}$ be the functor which is the identity on objects, and which carries a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ to the morphism

$$G(f) = \begin{cases} \mathrm{id}_X^{+} & \text{if } X = Y \text{ and } f = \mathrm{id}_X \\ f & \text{otherwise.} \end{cases} \in \mathrm{Hom}_{\mathcal{C}^{+}}(X,Y);$$

this functor is well-defined by virtue of our assumption that the collection of non-identity morphisms of $\mathcal{C}$ is closed under composition. We will complete the proof by showing that the induced map $\mathrm{N}_\bullet(G) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C}^{+})$ is a simplicial homotopy inverse of $\mathrm{N}_\bullet(F) = v_{\mathrm{N}_\bullet(\mathcal{C})}$. One direction is clear: the composition $\mathcal{C} \xrightarrow{G} \mathcal{C}^{+} \xrightarrow{F} \mathcal{C}$ is the identity functor $\mathrm{id}_{\mathcal{C}}$, so $\mathrm{N}_\bullet(F) \circ \mathrm{N}_\bullet(G)$ is equal to the identity. The composition $\mathcal{C}^{+} \xrightarrow{F} \mathcal{C} \xrightarrow{G} \mathcal{C}^{+}$ is not the identity functor on $\mathcal{C}^{+}$: for each object $X \in \mathcal{C}$, it carries the morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X) \subset \mathrm{Hom}_{\mathcal{C}^{+}}(X,X)$ to the "new" identity morphism id_X^{+}. However, there is a natural transformation $\alpha : G \circ F \to \mathrm{id}_{\mathcal{C}^{+}}$, given by the construction $(X \in \mathcal{C}^{+}) \mapsto \mathrm{id}_X$. It follows that the map of simplicial sets $\mathrm{N}_\bullet(G) \circ \mathrm{N}_\bullet(F)$ is homotopic to the identity (Example 3.1.5.7). □

Proof of Proposition 3.4.5.4. We proceed as in the proof of Proposition 3.3.4.8. For every simplicial set X, the counit map $v_X : X^{+} \to X$ can be realized as a filtered colimit of counit maps $\{v_{\mathrm{sk}_n(X)} : \mathrm{sk}_n(X)^{+} \to \mathrm{sk}_n(X)\}_{n \geq 0}$. Since the collection of weak homotopy equivalences is closed under the formation of filtered colimits (Proposition 3.2.8.3), it will suffice to show that each of the maps $v_{\mathrm{sk}_n(X)}$ is a weak homotopy equivalence. We may therefore replace X by $\mathrm{sk}_n(X)$, and thereby reduce to the case where X is n-skeletal for some nonnegative integer $n \geq 0$. We now proceed by induction on n.

Let $Y = \mathrm{sk}_{n-1}(X)$ be the $(n-1)$-skeleton of X. Let S denote the collection of nondegenerate n-simplices of X, let $X' = \coprod_{\sigma \in S} \Delta^n$ denote their coproduct, and let $Y' = \coprod_{\sigma \in S} \partial \Delta^n$ denote the boundary of X'. Proposition 1.1.4.12 then supplies a pushout diagram of simplicial sets

012A
$$\begin{tikzcd}
Y' \arrow[r] \arrow[d] & X' \arrow[d] \\
Y \arrow[r] & X.
\end{tikzcd} \tag{3.24}$$

Note that both (3.24) and the induced diagram

$$\begin{tikzcd}
Y'^{+} \arrow[r] \arrow[d] & X'^{+} \arrow[d] \\
Y^{+} \arrow[r] & X^{+}
\end{tikzcd}$$

are homotopy pushout squares (this is a special case of Example 3.4.2.12, since the maps $Y' \hookrightarrow X'$ and $Y'^{+} \hookrightarrow X'^{+}$ are monomorphisms). Moreover, our inductive hypothesis guarantees that the maps $v_Y : Y^{+} \to Y$ and $v_{Y'} : Y'^{+} \to Y'$ are weak homotopy equivalences. Applying Proposition 3.4.2.9 to the commutative diagram

$$\begin{tikzcd}
Y'^{+} \arrow[rr] \arrow[dr, "v_{Y'}"] \arrow[ddd] & & Y^{+} \arrow[dr, "v_Y"] \arrow[ddd] & \\
& Y' \arrow[rr] \arrow[ddd] & & Y \arrow[ddd] \\
& & & \\
X'^{+} \arrow[rr] \arrow[dr, "v_{X'}"] & & X^{+} \arrow[dr, "v_X"] & \\
& X' \arrow[rr] & & X,
\end{tikzcd}$$

we are reduced to proving that $v_{X'}$ is a weak homotopy equivalence. Using Remark 3.1.6.20, we can reduce further to the problem of showing that the map $v_X : X^+ \to X$ is a weak homotopy equivalence in the special case $X = \Delta^n$, which follows from Lemma 3.4.5.9. □

3.4.6 Excision

Let X be a topological space which is a union of two open subsets $U, V \subseteq X$. Then the diagram `012B`

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

is a pushout square in the category of topological spaces. Stated more informally, the topological space X can be obtained by gluing U and V along their common open subset $U \cap V$. This observation has a homotopy-theoretic counterpart:

Theorem 3.4.6.1 (Excision)**.** *Let X be a topological space, and let $U, V \subseteq X$ be subsets whose interiors $\mathring{U} \subseteq U$ and $\mathring{V} \subseteq V$ comprise an open covering of X. Then the diagram of singular simplicial sets* `012C`

$$\begin{tikzcd} \operatorname{Sing}_\bullet(U \cap V) \arrow[r] \arrow[d] & \operatorname{Sing}_\bullet(U) \arrow[d] \\ \operatorname{Sing}_\bullet(V) \arrow[r] & \operatorname{Sing}_\bullet(X) \end{tikzcd}$$

is a homotopy pushout square (Definition 3.4.2.1).

Remark 3.4.6.2. In the situation of Theorem 3.4.6.1, the canonical maps $\operatorname{Sing}_\bullet(U) \hookleftarrow \operatorname{Sing}_\bullet(U \cap V) \hookrightarrow \operatorname{Sing}_\bullet(V)$ are monomorphisms. Consequently, the conclusion of Theorem 3.4.6.1 is equivalent to the assertion that the natural map `012D`

$$\operatorname{Sing}_\bullet(U) \coprod_{\operatorname{Sing}_\bullet(U \cap V)} \operatorname{Sing}_\bullet(V) \to \operatorname{Sing}_\bullet(X)$$

is a weak homotopy equivalence of simplicial sets (see Proposition 3.4.2.11).

`012E` **Warning 3.4.6.3.** In the situation of Theorem 3.4.6.1, it is generally not true that the diagram

$$\begin{tikzcd}
\operatorname{Sing}_{\bullet}(U \cap V) \arrow[r] \arrow[d] & \operatorname{Sing}_{\bullet}(U) \arrow[d] \\
\operatorname{Sing}_{\bullet}(V) \arrow[r] & \operatorname{Sing}_{\bullet}(X)
\end{tikzcd}$$

is a pushout square of simplicial sets. Concretely, this is because the image of a continuous function $f : |\Delta^n| \to X$ need not be contained in either U or V.

Our goal in this section is to prove a stronger version Theorem 3.4.6.1, where we allow more general coverings of X.

`012F` **Definition 3.4.6.4.** Let X be a topological space and let $\mathcal{U}$ be a collection of subsets of X. We say that a singular n-simplex $\sigma : |\Delta^n| \to X$ is *$\mathcal{U}$-small* if its image is contained in U, for some $U \in \mathcal{U}$. We let $\operatorname{Sing}_n^{\mathcal{U}}(X)$ denote the subset of $\operatorname{Sing}_n(X)$ consisting of the $\mathcal{U}$-small simplices. Note that the subsets $\{\operatorname{Sing}_n^{\mathcal{U}}(X)\}_{n \geq 0}$ are stable under the face and degeneracy operators of the simplicial set $\operatorname{Sing}_{\bullet}(X)$, and therefore determine a simplicial subset which we will denote by $\operatorname{Sing}_{\bullet}^{\mathcal{U}}(X) \subseteq \operatorname{Sing}_{\bullet}(X)$.

`012G` **Remark 3.4.6.5.** In the situation of Definition 3.4.6.4, the simplicial set $\operatorname{Sing}_{\bullet}^{\mathcal{U}}(X)$ is given by the union $\bigcup_{U \in \mathcal{U}} \operatorname{Sing}_{\bullet}(U)$, where we regard each $\operatorname{Sing}_{\bullet}(U)$ as a simplicial subset of $\operatorname{Sing}_{\bullet}(X)$.

Our main result can now be stated as follows:

`012H` **Theorem 3.4.6.6.** *Let X be a topological space and let $\mathcal{U}$ be a collection of subsets of X satisfying $X = \bigcup_{U \in \mathcal{U}} \mathring{U}$. Then the inclusion map $\operatorname{Sing}_{\bullet}^{\mathcal{U}}(X) \hookrightarrow \operatorname{Sing}_{\bullet}(X)$ is a weak homotopy equivalence.*

Proof of Theorem 3.4.6.1 from Theorem 3.4.6.6. Let X be a topological space and let $\mathcal{U} = \{U, V\}$ be a pair of subsets of X. Then $\operatorname{Sing}_{\bullet}^{\mathcal{U}}(X)$ can be identified with the pushout

$$\operatorname{Sing}_{\bullet}(U) \coprod_{\operatorname{Sing}_{\bullet}(U \cap V)} \operatorname{Sing}_{\bullet}(V),$$

formed in the category of simplicial sets. Theorem 3.4.6.6 then asserts that if $X = \mathring{U} \cup \mathring{V}$, then the inclusion

$$\operatorname{Sing}_{\bullet}(U) \coprod_{\operatorname{Sing}_{\bullet}(U \cap V)} \operatorname{Sing}_{\bullet}(V) \hookrightarrow \operatorname{Sing}_{\bullet}(X)$$

is a weak homotopy equivalence. By virtue of Remark 3.4.6.2, this is equivalent to Theorem 3.4.6.1. □

The proof of Theorem 3.4.6.6 is based on the observation that every singular n-simplex $\sigma : |\Delta^n| \to X$ can be "decomposed" into $\mathcal{U}$-small simplices by repeatedly applying the barycentric subdivision described in Proposition 3.3.2.3. To make this precise, we need the following geometric observation:

Lemma 3.4.6.7. *Let V be a normed vector space over the real numbers and let $K \subseteq V$ be the convex hull of a finite collection of points $v_0, v_1, \ldots, v_n \in V$, given by the image of a continuous function:* 012J

$$f : |\Delta^n| \to V \qquad (t_0, t_1, \ldots, t_n) \mapsto t_0 v_0 + t_1 v_1 + \cdots + t_n v_n.$$

Let σ be any m-simplex of the subdivision $\operatorname{Sd}(\Delta^n)$, let f_σ denote the composite map

$$|\Delta^m| \xrightarrow{|\sigma|} |\operatorname{Sd}(\Delta^n)| \simeq |\Delta^n| \xrightarrow{f} V$$

(where the homeomorphism $|\operatorname{Sd}(\Delta^n)| \leq |\Delta^n|$ is supplied by Proposition 3.3.2.3), and let $K_0 \subseteq K$ be the image of f_σ. Then the diameters of K_0 and K satisfy the inequality $\operatorname{diam}(K_0) \leq \frac{n}{n+1} \operatorname{diam}(K)$.

Proof. Let us denote the norm on the vector space V by $| \bullet |_V$. Fix points $x, y \in |\Delta^m|$; we wish to show that $|f_\sigma(x) - f_\sigma(y)|_V \leq \frac{n}{n+1} \operatorname{diam}(K)$. Note that, if we regard the point x as fixed, then the function $y \mapsto |f_\sigma(x) - f_\sigma(y)|_V$ is convex, and therefore achieves its supremum at some vertex of $|\Delta^m|$. We may therefore assume without loss of generality that y is a vertex of $|\Delta^m|$. Similarly, we may assume that x is a vertex of $|\Delta^m|$. We may also assume that $x \neq y$ (otherwise there is nothing to prove). Exchanging x and y if necessary, it follows that there exist disjoint nonempty subsets $A, B \subseteq \{0, 1, \ldots, n\}$ of cardinality $a = |A|$ and $b = |B|$ satisfying

$$f_\sigma(x) = \sum_{i \in A} \frac{v_i}{a} \qquad f_\sigma(y) = \sum_{i \in A \cup B} \frac{v_i}{a+b}.$$

We then compute

$$\begin{aligned}
|f_\sigma(x) - f_\sigma(y)|_V &= |\sum_{(i,j) \in A \times B} \frac{v_i - v_j}{a(a+b)}|_V \\
&\leq \sum_{(i,j) \in A \times B} \frac{|v_i - v_j|_V}{a(a+b)} \\
&\leq \sum_{(i,j) \in A \times B} \frac{\operatorname{diam}(K)}{a(a+b)} \\
&= \frac{b}{a+b} \operatorname{diam}(K) \\
&\leq \frac{n}{n+1} \operatorname{diam}(K).
\end{aligned}$$

□

Proof of Theorem 3.4.6.6. Let X be a topological space and let $\mathcal{U}$ be a collection of subsets of X satisfying $X = \bigcup_{U \in \mathcal{U}} \mathring{U}$. For each $k \geq 0$, let $Y(k) \subseteq \operatorname{Sing}_\bullet(X)$ denote the semisimplicial subset spanned by those singular n-simplices $f : |\Delta^n| \to X$ having the property that, for every m-simplex σ of the iterated subdivision $\operatorname{Sd}^k(\Delta^n)$, the composite map

$$|\Delta^m| \xrightarrow{|\sigma|} |\operatorname{Sd}^k(\Delta^n)| \simeq |\Delta^n| \xrightarrow{f} X$$

is $\mathcal{U}$-small; here the identification $|\operatorname{Sd}^k(\Delta^n)| \simeq |\Delta^n|$ is given by iteratively applying the barycentric subdivision of Proposition 3.3.2.3. By construction, we have inclusions of semisimplicial sets

$$\operatorname{Sing}^{\mathcal{U}}_\bullet(X) = Y(0) \subseteq Y(1) \subseteq Y(2) \subseteq \cdots \subseteq \operatorname{Sing}_\bullet(X).$$

We first claim that $\operatorname{Sing}_\bullet(X) = \bigcup_{k \geq 0} Y(k)$. Fix a continuous function $f : |\Delta^n| \to X$, regarded as an n-simplex of $\operatorname{Sing}_\bullet(X)$; we wish to show that f belongs to $Y(k)$ for $k \gg 0$. Let us identify the topological n-simplex $|\Delta^n|$ with the subset of Euclidean space $V = \mathbf{R}^{n+1}$ given by the convex hull of the standard basis vectors $\{v_i\}_{0 \leq i \leq n}$. Then the collection of inverse images $\{f^{-1}(U)\}_{U \in \mathcal{U}}$ can be refined to an open covering of $|\Delta^n|$. It follows that there exists a positive real number ϵ with the property that, for every point $v \in |\Delta^n|$, the open ball

$$B_\epsilon(v) = \{w \in |\Delta^n| : |v - w|_V < \epsilon\}$$

is contained in $f^{-1}(U)$, for some $U \in \mathcal{U}$. Choose an integer k satisfying $(\frac{n}{n+1})^k \operatorname{diam}(|\Delta^n|) < \epsilon$. It then follows from iterated application of Lemma 3.4.6.7 that the composite map

$$|\operatorname{Sd}^k(\Delta^n)| \simeq |\Delta^n| \xrightarrow{f} X$$

carries each simplex of $\operatorname{Sd}^k(\Delta^n)$ into a subset $U \subseteq X$ belonging to $\mathcal{U}$, so that f belongs to the semisimplicial subset $Y(k) \subseteq \operatorname{Sing}_\bullet(X)$.

Note that the inclusion $\iota : \operatorname{Sing}^{\mathcal{U}}_\bullet(X) \hookrightarrow \operatorname{Sing}_\bullet(X)$ is a weak homotopy equivalence of simplicial sets if and only if it is a weak homotopy equivalence when regarded as a morphism of semisimplicial sets (Corollary 3.4.5.5). It follows from the preceding argument that, as a morphism of semisimplicial sets, ι can be realized as a filtered colimit of the inclusion maps $\iota(k) : \operatorname{Sing}^{\mathcal{U}}_\bullet(X) = Y(0) \hookrightarrow Y(k)$. Since the collection of weak homotopy equivalences is closed under filtered colimits (Remark 3.4.5.2), it will suffice to show that each $\iota(k)$ is a weak homotopy equivalence. Proceeding by induction on k, we are reduced to showing that each of the inclusion maps $Y(k) \hookrightarrow Y(k+1)$ is a weak homotopy equivalence. Note that the semisimplicial isomorphism $\varphi : \operatorname{Sing}_\bullet(X) \simeq \operatorname{Ex}(\operatorname{Sing}_\bullet(X))$ of Example 3.3.2.9 restricts to a map $\varphi^{\mathcal{U}} : \operatorname{Sing}^{\mathcal{U}}_\bullet(X) \to \operatorname{Ex}(\operatorname{Sing}^{\mathcal{U}}_\bullet(X))$ (which is generally not an isomorphism). Unwinding the definitions, we see that the inclusion $Y(k) \hookrightarrow Y(k+1)$ can be identified with the map

$\mathrm{Ex}^k(\varphi^{\mathcal{U}}) : \mathrm{Ex}^k(\mathrm{Sing}^{\mathcal{U}}_\bullet(X)) \to \mathrm{Ex}^{k+1}(\mathrm{Sing}^{\mathcal{U}}_\bullet(X))$ (see Variant 3.3.2.10). By virtue of Corollary 3.4.5.8, it will suffice to show that $\varphi^{\mathcal{U}}$ is a weak homotopy equivalence.

Fix an integer $n \geq 0$ as above, let $\mathrm{Chain}[n]$ denote the collection of all nonempty subsets of $[n] = \{0 < 1 < \cdots < n\}$. Let σ be an n-simplex of the simplicial set $\Delta^1 \times \mathrm{Sing}^{\mathcal{U}}_\bullet(X)$, which we identify with a pair (ϵ, f) where $\epsilon : [n] \to [1]$ is a nondecreasing function and $f : |\Delta^n| \to X$ is a continuous map of topological spaces. Define a map of sets $g_\epsilon : \mathrm{Chain}[n] \to |\Delta^n|$ by the formula

$$g_\epsilon(S) = \begin{cases} \frac{\sum_{i \in S} v_i}{|S|} & \text{if } \epsilon|_S = 0 \\ v_{\mathrm{Max}(S)} & \text{otherwise.} \end{cases}$$

Then g_ϵ extends to a continuous map

$$\overline{g}_\epsilon : |\,\mathrm{N}_\bullet(\mathrm{Chain}[n])| \to |\Delta^n|$$

which is affine when restricted to each simplex of $|\,\mathrm{N}_\bullet(\mathrm{Chain}[n])| \simeq |\,\mathrm{Sd}(\Delta^n)|$. The composite map

$$|\,\mathrm{Sd}(\Delta^n)| \xrightarrow{\overline{g}_\epsilon} |\Delta^n| \xrightarrow{f} X$$

can be identified with an n-simplex of $\mathrm{Ex}(\mathrm{Sing}^{\mathcal{U}}_\bullet(X))$, which we will denote by $h(\sigma)$. It is not difficult to see that the construction $\sigma \mapsto h(\sigma)$ is compatible with face operators, and therefore determines a morphism of semisimplicial sets $h : \Delta^1 \times \mathrm{Sing}^{\mathcal{U}}_\bullet(X) \to \mathrm{Ex}(\mathrm{Sing}^{\mathcal{U}}_\bullet(X))$. By construction, this morphism fits into a commutative diagram of semisimplicial sets

$$\begin{tikzcd}
\Delta^1 \times \mathrm{Sing}^{\mathcal{U}}_\bullet(X) \arrow[drr, "h"] & & \{0\} \times \mathrm{Sing}^{\mathcal{U}}_\bullet(X) \arrow[ll, "i_0"'] \arrow[d, "\varphi^{\mathcal{U}}"] \\
\{1\} \times \mathrm{Sing}^{\mathcal{U}}_\bullet(X) \arrow[u, "i_1"] \arrow[rr, "\rho"] & & \mathrm{Ex}(\mathrm{Sing}^{\mathcal{U}}_\bullet(X)),
\end{tikzcd}$$

where i_0 and i_1 are the inclusion maps and $\rho = \rho_{\mathrm{Sing}^{\mathcal{U}}_\bullet(X)}$ is the comparison map of Construction 3.3.4.3. Note that the morphisms i_0, i_1, and ρ are weak homotopy equivalences of simplicial sets (Theorem 3.3.5.1), and therefore also weak homotopy equivalences of semisimplicial sets (Corollary 3.4.5.5). Invoking the two-out-of-three property (Remark 3.4.5.3), we conclude that h and $\varphi^{\mathcal{U}}$ are also weak homotopy equivalences of semisimplicial sets. □

3.4.7 The Seifert van-Kampen Theorem

012K

Let X be a topological space containing a pair of subsets $U, V \subseteq X$. If X is covered by the interiors $\mathring{U}$ and $\mathring{V}$, then Theorem 3.4.6.1 guarantees that the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Sing}_{\bullet}(U \cap V) \arrow[r] \arrow[d] & \mathrm{Sing}_{\bullet}(U) \arrow[d] \\
\mathrm{Sing}_{\bullet}(V) \arrow[r] & \mathrm{Sing}_{\bullet}(X)
\end{tikzcd}$$

is a homotopy pushout square. In this section, we apply this assertion to recover several classical results in algebraic topology.

012L **Theorem 3.4.7.1** (Seifert-van Kampen)**.** *Let X be a topological space containing a pair of subsets $U, V \subseteq X$ which satisfy the following conditions:*

(1) *The topological spaces U, V, and $U \cap V$ are path connected.*

(2) *The interiors $\mathring{U} \subseteq U$ and $\mathring{V} \subseteq V$ comprise an open covering of X.*

Then, for every point $x \in U \cap V$, the diagram

$$\begin{tikzcd}
\pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\
\pi_1(V, x) \arrow[r] & \pi_1(X, x)
\end{tikzcd}$$

is a pushout square in the category of groups.

We will deduce Theorem 3.4.7.1 from the following variant of Brown ([7]), which does not require any connectivity hypotheses.

012M **Theorem 3.4.7.2** (Seifert-van Kampen, Groupoid Version)**.** *Let X be a topological space, and let $U, V \subseteq X$ be subsets whose interiors $\mathring{U} \subseteq U$ and $\mathring{V} \subseteq V$ comprise an open covering of X. Then the diagram of fundamental groupoids*

$$\begin{tikzcd}
\pi_{\leq 1}(U \cap V) \arrow[r] \arrow[d] & \pi_{\leq 1}(U) \arrow[d] \\
\pi_{\leq 1}(V) \arrow[r] & \pi_{\leq 1}(X)
\end{tikzcd}$$

is a pushout square in the (ordinary) category Cat*.*

Proof. Let $\mathcal{C}$ be a category; we wish to show that the diagram of sets σ :

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Cat}}(\pi_{\leq 1}(U \cap V), \mathcal{C}) & \mathrm{Hom}_{\mathrm{Cat}}(\pi_{\leq 1}(U), \mathcal{C}) \arrow[l] \\
\mathrm{Hom}_{\mathrm{Cat}}(\pi_{\leq 1}(V), \mathcal{C}) \arrow[u] & \mathrm{Hom}_{\mathrm{Cat}}(\pi_{\leq 1}(X), \mathcal{C}) \arrow[l] \arrow[u]
\end{tikzcd}$$

is a pullback square. Replacing $\mathcal{C}$ by its core $\mathcal{C}^{\simeq}$ (Construction 1.3.5.4), we may assume without loss of generality that $\mathcal{C}$ is a groupoid. Let $\mathrm{N}_{\bullet}(\mathcal{C})$ denote the nerve of $\mathcal{C}$, so that we can identify σ with the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sing}_{\bullet}(U \cap V), \mathrm{N}_{\bullet}(\mathcal{C})) & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sing}_{\bullet}(U), \mathrm{N}_{\bullet}(\mathcal{C})) \arrow[l] \\
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sing}_{\bullet}(V), \mathrm{N}_{\bullet}(\mathcal{C})) \arrow[u] & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Sing}_{\bullet}(X), \mathrm{N}_{\bullet}(\mathcal{C})). \arrow[l] \arrow[u]
\end{tikzcd}$$

Let K denote the pushout $\mathrm{Sing}_{\bullet}(U) \coprod_{\mathrm{Sing}_{\bullet}(U \cap V)} \mathrm{Sing}_{\bullet}(V)$, which we regard as a simplicial subset of $\mathrm{Sing}_{\bullet}(X)$. Unwinding the definitions, we must show that every morphism of simplicial sets $f : K \to \mathrm{N}_{\bullet}(\mathcal{C})$ extends uniquely to a map $\overline{f} : \mathrm{Sing}_{\bullet}(X) \to \mathrm{N}_{\bullet}(\mathcal{C})$. Note that the inclusion $K \hookrightarrow \mathrm{Sing}_{\bullet}(X)$ is a weak homotopy equivalence (Theorem 3.4.6.1) and therefore anodyne (Corollary 3.3.7.7), so the existence of $\overline{f}$ follows from the observation that $\mathrm{N}_{\bullet}(\mathcal{C})$ is a Kan complex (Proposition 1.3.5.2). To prove uniqueness, suppose that we are given a pair of maps $\overline{f}, \overline{f}' : \mathrm{Sing}_{\bullet}(X) \to \mathrm{N}_{\bullet}(\mathcal{C})$ satisfying $\overline{f}|_K = f = \overline{f}'|_K$. It follows that there exists a homotopy $h : \Delta^1 \times \mathrm{Sing}_{\bullet}(X) \to \mathrm{N}_{\bullet}(\mathcal{C})$ which is constant when restricted to $\Delta^1 \times K$. Note that $\overline{f}$ and $\overline{f}'$ can be identified with functors $F, F' : \pi_{\leq 1}(X) \to \mathcal{C}$, and h with a natural transformation of functors $H : F \to F'$. Since every vertex of $\mathrm{Sing}_{\bullet}(X)$ is contained in K, this natural transformation carries each point $x \in X$ to the identity morphism $\mathrm{id}_{\overline{f}(x)} : F(x) \to F(x) = F'(x)$. It follows that the functors F and F' are identical, so that the morphisms $\overline{f}$ and $\overline{f}'$ are the same. □

Proof of Theorem 3.4.7.1. For every group G, let us write BG for the groupoid having a single object with automorphism group G (Remark 1.3.2.4). Fix a point $x \in U \cap V$. To show that the diagram

$$\begin{tikzcd}
\pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\
\pi_1(V, x) \arrow[r] & \pi_1(X, x)
\end{tikzcd}$$

is a pushout square in the category of groups, it will suffice to show that the diagram σ_0:

$$\begin{tikzcd}
B\pi_1(U \cap V, x) \arrow[r] \arrow[d] & B\pi_1(U, x) \arrow[d] \\
B\pi_1(V, x) \arrow[r] & B\pi_1(X, x)
\end{tikzcd}$$

is a pushout square in the (ordinary) category Cat.

For each point $y \in X$, choose a continuous path $p_y : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$. By virtue of our assumption that U, V, and $U \cap V$ are path connected, we can arrange that these paths satisfy the following requirements:

- If $y = x$, then $p_y : [0,1] \to X$ is the constant map taking the value x.
- If y is contained in the intersection $U \cap V$, then the path p_y factors through $U \cap V$.
- If y is contained in U, then the path p_y factors through U.
- If y is contained in V, then the path p_y factors through V.

Note that, for $W \in \{X, U, V, U \cap V\}$, we can identify $B\pi_1(W, x)$ with the full subcategory of $\pi_{\leq 1}(W)$ spanned by the point x. Let $r_W : \pi_{\leq 1}(W) \to B\pi_1(W, x)$ be the functor which carries each point of W to the point x, and each morphism $\alpha \in \mathrm{Hom}_{\pi_{\leq 1}(W)}(y, z)$ to the composition $[p_z]^{-1} \circ \alpha \circ [p_y]$ (where $[p_y]$ and $[p_z]$ denote the homotopy classes of the paths p_y and p_z, regarded as morphisms in the fundamental groupoid $\pi_{\leq 1}(W)$). The functors r_W restrict to the identity on $B\pi_1(W, x)$ and are compatible as W varies, and therefore exhibit σ_0 as a retract of the diagram σ :

$$\begin{tikzcd}
\pi_{\leq 1}(U \cap V) \arrow[r] \arrow[d] & \pi_{\leq 1}(U) \arrow[d] \\
\pi_{\leq 1}(V) \arrow[r] & \pi_{\leq 1}(X)
\end{tikzcd}$$

in the category $\mathrm{Fun}([1] \times [1], \mathrm{Cat})$. Since σ is a pushout square (by virtue of Theorem 3.4.7.2), it follows that σ_0 is also a pushout square. □

If X is a topological space and $U \subseteq X$ is a subspace (not necessarily open), we will write $\mathrm{H}_*(X, U; \mathbf{Z})$ for the *relative homology groups* of the pair (X, U): that is, the homology groups of the quotient chain complex $\mathrm{C}_*(X; \mathbf{Z}) / \mathrm{C}_*(U; \mathbf{Z})$ (see Example 2.5.5.3).

Theorem 3.4.7.3 (Excision for Homology)**.** *Let X be a topological space and let $U, V \subseteq X$ be subsets whose interiors $\mathring{U} \subseteq U$ and $\mathring{V} \subseteq V$ comprise an open covering of X. Then the inclusion $U \hookrightarrow X$ induces an isomorphism of relative homology groups* 012N

$$\mathrm{H}_*(U, U \cap V; \mathbf{Z}) \to \mathrm{H}_*(X, V; \mathbf{Z}).$$

Proof. Let K denote the pushout $\mathrm{Sing}_\bullet(U) \coprod_{\mathrm{Sing}_\bullet(U \cap V)} \mathrm{Sing}_\bullet(V)$. We then have a commutative diagram of short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{C}_*(V; \mathbf{Z}) \arrow[r] \arrow[d, equal] & \mathrm{C}_*(K; \mathbf{Z}) \arrow[r] \arrow[d, "\theta'"] & \mathrm{C}_*(U; \mathbf{Z}) / \mathrm{C}_*(U \cap V; \mathbf{Z}) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & \mathrm{C}_*(V; \mathbf{Z}) \arrow[r] & \mathrm{C}_*(X; \mathbf{Z}) \arrow[r] & \mathrm{C}_*(X; \mathbf{Z}) / \mathrm{C}_*(V; \mathbf{Z}) \arrow[r] & 0.
\end{tikzcd}$$

Consequently, to show that θ is a quasi-isomorphism, it will suffice to show that θ' is a quasi-isomorphism (Remark 2.5.1.7). This is a special case of Proposition 3.1.6.18, since the inclusion $K \hookrightarrow \mathrm{Sing}_\bullet(X)$ is a weak homotopy equivalence of simplicial sets (Theorem 3.4.6.1). □

Remark 3.4.7.4 (The Mayer-Vietoris Sequence)**.** Let X be a topological space, let $U, V \subseteq X$ be subsets whose interiors $\mathring{U} \subseteq U$ and $\mathring{V} \subseteq V$ comprise an open covering of X, and set $K = \mathrm{Sing}_\bullet(U) \coprod_{\mathrm{Sing}_\bullet(U \cap V)} \mathrm{Sing}_\bullet(V)$. Then the inclusion $K \hookrightarrow \mathrm{Sing}_\bullet(X)$ induces a quasi-isomorphism $\mathrm{C}_*(K; \mathbf{Z}) \hookrightarrow \mathrm{C}_*(X; \mathbf{Z})$ (by virtue of Theorem 3.4.6.1 and Proposition 3.1.6.18), and we have a short exact sequence of chain complexes 012P

$$0 \to \mathrm{C}_*(U \cap V; \mathbf{Z}) \to \mathrm{C}_*(U; \mathbf{Z}) \oplus \mathrm{C}_*(V; \mathbf{Z}) \to \mathrm{C}_*(K; \mathbf{Z}) \to 0.$$

Passing to homology groups (see Construction [?]), we obtain a long exact sequence of abelian groups

$$\cdots \to \mathrm{H}_{*+1}(X; \mathbf{Z}) \xrightarrow{\delta} \mathrm{H}_*(U \cap V; \mathbf{Z}) \to \mathrm{H}_*(U; \mathbf{Z}) \oplus \mathrm{H}_*(V; \mathbf{Z}) \to \mathrm{H}_*(X; \mathbf{Z}) \to \cdots$$

which we refer to as the *Mayer-Vietoris sequence* of the covering $\{U, V\}$. The existence of this sequence is essentially equivalent to the statement of Theorem 3.4.7.3.

3.5 Truncations and Postnikov Towers

Let (X, x) be a pointed Kan complex. In §3.2.2, we introduced a sequence of groups 0513

$$\{\pi_m(X, x)\}_{m > 0}$$

called the *homotopy groups* of (X, x). These groups are very useful tools for analyzing the homotopy type of X. For example, the Kan complex X is contractible if and only if it is connected and all of its homotopy groups are trivial (Theorem 3.2.4.3). This is a special case of the following more general result, which classifies Kan complexes having (at most) one nontrivial homotopy group:

0514 **Proposition 3.5.0.1.** *Let X be a connected Kan complex, let $x \in X$ be a vertex, and let n be a positive integer. Suppose that the homotopy groups $\pi_m(X, x)$ vanish for every positive integer $m \neq n$. Then X is homotopy equivalent to an Eilenberg-MacLane space* $\mathrm{K}(G, n)$ *for some group G (which is abelian if $n \geq 2$).*

We will prove Proposition 3.5.0.1 in §3.5.7 (see Corollary 3.5.7.18). To carry out the proof, it will be useful to break the hypothesis of Proposition 3.5.0.1 into two parts. In what follows, we fix a positive integer n.

- We say that a Kan complex X is *n-connective* if it is connected and the homotopy group $\pi_m(X, x)$ vanishes for every integer $0 < m < n$ and every choice of base point $x \in X$.

- We say that a Kan complex X is *n-truncated* if the homotopy group $\pi_m(X, x)$ vanishes for every integer $m > n$ and every choice of base point $x \in X$.

Stated more informally, a Kan complex X is n-connective if its homotopy groups are concentrated in degrees $\geq n$, and n-truncated if its homotopy groups are concentrated in degrees $\leq n$. Each of these conditions admits a number of equivalent formulations, which we study in §3.5.1 and §3.5.7, respectively.

Proposition 3.5.0.1 asserts that if a Kan complex X is both n-connective and n-truncated, then it is homotopy equivalent to an Eilenberg-MacLane space $\mathrm{K}(G, n)$. We will deduce this from a structural analysis of n-truncated Kan complexes in general. We begin by observing that X is n-truncated if and only if, for every integer $m \geq n + 2$, the restriction map

$$\theta_m : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^m, X)$$

is surjective (see Proposition 3.5.7.7). In §3.5.3, §3.5.4, and §3.5.5, we study stronger versions of this condition:

- We say that X is *$(n + 1)$-coskeletal* if, for every integer $m \geq n + 2$, the map θ_m is a bijection (Definition 3.5.3.1).

- We say that X is *weakly n-coskeletal* if it is $(n + 1)$-coskeletal and, in addition, the map θ_{n+1} is injective (Definition 3.5.4.1).

- We say that X is an *n-groupoid* if it is weakly n-coskeletal and every n-simplex $\sigma : \Delta^n \to X$ is determined by its homotopy class relative to $\partial \Delta^n$ (see Definition 3.5.5.1 and Proposition 3.5.5.12).

For any Kan complex X, we have the following implications:

$$\begin{tikzcd}
X \text{ is an } n\text{-groupoid} \arrow[d, Rightarrow] \\
X \text{ is weakly } n\text{-coskeletal} \arrow[d, Rightarrow] \\
X \text{ is } (n+1)\text{-coskeletal} \arrow[d, Rightarrow] \\
X \text{ is } n\text{-truncated.}
\end{tikzcd}$$

None of these implications is reversible. However, they are reversible “up to homotopy” in the following sense: every n-truncated Kan complex is homotopy equivalent to an n-groupoid. More generally, in §3.5.6 we will associate to any Kan complex X an n-groupoid $\pi_{\leq n}(X)$, which we refer to as the *fundamental n-groupoid of X* (Construction 3.5.6.10). It is equipped with a comparison map $f : X \to \pi_{\leq n}(X)$ which is universal among maps from X to n-groupoids (Proposition 3.5.6.5), which is a homotopy equivalence if and only if X is n-truncated (Variant 3.5.7.16).

Remark 3.5.0.2. The preceding characterization of n-truncated Kan complexes has a counterpart for n-connective Kan complexes. A Kan complex X is n-connective if and only if it is homotopy equivalent to a Kan complex Y having a single m-simplex for each $m < n$ (Proposition 3.5.2.9 and Remark 3.5.2.10). We will prove Proposition 3.5.0.1 by showing that, in this case, the fundamental n-groupoid $\pi_{\leq n}(Y)$ is *isomorphic* to an Eilenberg-MacLane space $\mathrm{K}(G, n)$, for some group G. See Proposition 3.5.5.16. 0515

For any Kan complex X, the collection of fundamental n-groupoids $\{\pi_{\leq n}(X)\}$ can be organized into an inverse system

$$\cdots \to \pi_{\leq 3}(X) \to \pi_{\leq 2}(X) \to \pi_{\leq 1}(X) \to \pi_0(X)$$

which we will refer to as the (canonical) Postnikov tower of X (Example 3.5.8.2). In §3.5.8, we show that each of the transition maps $\pi_{\leq n}(X) \to \pi_{\leq n-1}(X)$ is a Kan fibration, whose fiber over a vertex x is homotopy equivalent to the Eilenberg-MacLane space $\mathrm{K}(G, n)$ for $G = \pi_n(X, x)$ (Corollary 3.5.8.9). Stated more informally, every Kan complex X can be built as a successive extension of Eilenbeg-MacLane spaces.

For many applications, it will be useful to work with relative versions of the preceding conditions. Let $f : X \to Y$ be a morphism of Kan complexes. Let us assume for simplicity that f is a Kan fibration, so that the fiber $X_y = \{y\} \times_Y X$ is a Kan complex for each vertex $y \in Y$. We will say that f is *n-connective* if each of the Kan complexes X_y is n-connective (see Definition 3.5.1.13 and Proposition 3.5.1.22), and we say that f is *n-truncated* if each of the Kan complexes X_y is n-truncated (see Definition 3.5.9.1 and Proposition 3.5.9.8). In §3.5.2 and §3.5.9, we study a number of different formulations of these conditions. In particular, we show that both are characterized by lifting properties:

- A Kan fibration $f : X \to Y$ is n-connective if and only if every lifting problem

0516
$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{3.25}$$

  admits a solution, provided that B is a simplicial set of dimension $\leq n$ and A is a simplicial subset of B. (Proposition 3.5.2.1).

- A Kan fibration $f : X \to Y$ is n-truncated if and only if every lifting problem (3.25) has solution, provided that A is a simplicial subset of B which contains the $(n+1)$-skeleton of B (Corollary 3.5.9.23).

3.5.1 Connectivity

0517 Recall that a simplicial set X is *connected* if the set of path components $\pi_0(X)$ has exactly one element (Corollary 1.2.1.15). When X is a Kan complex, we can use the homotopy groups introduced in §3.2 to formulate a hierarchy of stronger connectivity conditions.

04GL **Definition 3.5.1.1.** Let X be a Kan complex and let n be a nonnegative integer. We say that X is *n-connective* if it is nonempty and, for every vertex $x \in X$ and every integer $0 \leq m < n$, the set $\pi_m(X, x)$ consists of a single element.

0518 **Remark 3.5.1.2.** It will sometimes be useful to extend Definition 3.5.1.1 to the case where n is an arbitrary integer. By convention, if $n < 0$, then *every* Kan complex X is n-connective.

Warning 3.5.1.3. The terminology of Definition 3.5.1.1 is not standard. Many authors refer to a Kan complex X as *n-connected* if it is $(n+1)$-connective in the sense of Definition 3.5.1.1. 04GM

Remark 3.5.1.4. Let X be a Kan complex. It follows from Example 3.2.2.18 that the isomorphism class of the homotopy group $\pi_m(X, x)$ depends only on the connected component $[x] \in \pi_0(X)$. Consequently, if $n > 0$, then X is n-connective if and only if it is connected and the homotopy groups $\pi_m(X, x)$ are trivial for $0 < m < n$ for *some* choice of vertex $x \in X$. 04GN

Remark 3.5.1.5 (Homotopy Invariance)**.** Let X and Y be Kan complexes which are homotopy equivalent. Then X is n-connective if and only if Y is n-connective. See Remark 3.2.2.17. 04GP

Variant 3.5.1.6. Let X be a simplicial set and let n be an integer. Using Corollary 3.1.7.2, we can choose an anodyne map $X \hookrightarrow Q$, where Q is a Kan complex. We will say that X is *n-connective* if the Kan complex Q is n-connective, in the sense of Definition 3.5.1.1. By virtue of Remark 3.5.1.5 (and Warning 3.1.7.3), this condition is independent of the choice of Q. 04GQ

Example 3.5.1.7. A simplicial set X is 0-connective if and only if it is nonempty. 04GR

Example 3.5.1.8. A simplicial set X is 1-connective if and only if it is connected (see Corollary 1.2.1.15). 04GS

Example 3.5.1.9. A Kan complex X is 2-connective if and only if it is *simply connected*: that is, X is connected and the fundamental group $\pi_1(X, x)$ vanishes (by virtue of Remark 3.5.1.4, this condition does not depend on the choice of base point $x \in X$). 04GT

Example 3.5.1.10. Let X be a Kan complex which has only a single k-simplex for $0 \leq k \leq n$ (that is, the n-skeleton $\mathrm{sk}_n(X)$ is isomorphic to Δ^0). Then X is $(n+1)$-connective. For a partial converse, see Proposition 3.5.2.9. 0519

Remark 3.5.1.11. Let X be a simplicial set. Then X is weakly contractible if and only if it is n-connective for every integer n. To prove this, we can use Corollary 3.1.7.2 to reduce to the case where X is a Kan complex, in which case it is a reformulation of Proposition 3.5.1.12. 051A

Definition 3.5.1.1 admits a number of alternative formulations.

Proposition 3.5.1.12. *Let X be a Kan complex and let n be a nonnegative integer. The following conditions are equivalent:* 04GV

(1) *The Kan complex X is n-connective.*

(2) *For every integer $0 \leq m \leq n$, every morphism $\partial \Delta^m \to X$ can be extended to an m-simplex of X.*

(3) *Let B be a simplicial set of dimension $\leq n$ and let $A \subseteq B$ be a simplicial subset. Then every morphism $f_0 : A \to X$ admits an extension $f : B \to X$.*

(4) *Let A be a simplicial set of dimension $< n$. Then every morphism $f : A \to X$ is nullhomotopic.*

Proof. The equivalence (1) ⇔ (2) follows from Lemma 3.2.4.13 (and Variant 3.2.4.14), the implication (2) ⇒ (3) follows from Proposition 1.1.4.12, and the implication (4) ⇒ (2) follows from Variant 3.2.4.12. We complete the proof by showing that (3) implies (4). Applying assumption (3) to the inclusion map $\emptyset \subseteq \Delta^0$, we deduce that there exists a vertex $x \in X$. It will therefore suffice to show that if A is a simplicial set of dimension $< n$, then every pair of morphisms $f_0, f_1 : A \to X$ are homotopic (in particular, f_0 is homotopic to the constant map $A \to \{x\}$). This follows by applying (3) to the inclusion map $\partial \Delta^1 \times A \hookrightarrow \Delta^1 \times A$ (see Proposition 1.1.3.6).

□

We now introduce a relative version of Definition 3.5.1.1.

04GZ **Definition 3.5.1.13.** Let $f : X \to Y$ be a morphism of Kan complexes. We say that f is *n-connective* if it satisfies the following conditions:

- If $n \geq 0$, then the underlying map of connected components $\pi_0(X) \to \pi_0(Y)$ is surjective.
- If $n > 0$ and $x \in X$ is a vertex having image $y = f(x)$, the induced map $\pi_m(X, x) \to \pi_m(Y, y)$ is a bijection when $m < n$ and a surjection when $m = n$.

04H0 **Remark 3.5.1.14.** Suppose we are given a diagram of Kan complexes

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

which commutes up to homotopy. If the horizontal maps are homotopy equivalences, then f is n-connective if and only if f' is n-connective.

Variant 3.5.1.15. Let $f : X \to Y$ be a morphism of simplicial sets and let n be a nonnegative integer. Using Proposition 3.1.7.1, we can choose a commutative diagram 04H1

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y \arrow[r] & Y',
\end{tikzcd} \tag{3.26}$$

04H2

where X' and Y' are Kan complexes and the horizontal maps are weak homotopy equivalences. We will say that f is *n-connective* if the morphism of Kan complexes f' is n-connective, in the sense of Definition 3.5.1.13. It follows from Remark 3.5.1.14 that this condition does not depend on the choice of the diagram (3.26).

Example 3.5.1.16. For $n < 0$, every morphism of simplicial sets $f : X \to Y$ is n-connective. 051B

Example 3.5.1.17. A morphism of simplicial sets $f : X \to Y$ is 0-connective if and only if the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective. 051C

Example 3.5.1.18. Let X be a simplicial set and let n be an integer. Then X is n-connective (in the sense of Variant 3.5.1.6) if and only if the projection map $X \to \Delta^0$ is n-connective (in the sense of Variant 3.5.1.15). 04H3

Remark 3.5.1.19. Let $f : X \to Y$ be a morphism of simplicial sets. Then f is a weak homotopy equivalence if and only if it is n-connective for every integer n. To see this, we can assume without loss of generality that X and Y are Kan complexes, in which case it is a restatement of Theorem 3.2.7.1. 04HJ

Remark 3.5.1.20. Let $f, f' : X \to Y$ be morphisms of simplicial sets which are homotopic. Then f is n-connective if and only if f' is n-connective. 04H4

Remark 3.5.1.21 (Monotonicity)**.** Let n be a nonnegative integer and let $f : X \to Y$ be an n-connective morphism of simplicial sets. Then f is also m-connective for every integer $m \leq n$. 04H5

Proposition 3.5.1.22. *Let $f : X \to Y$ be a Kan fibration of simplicial sets and let n be an integer. Then f is n-connective (in the sense of Variant 3.5.1.15) if and only if, for every vertex $y \in Y$, the Kan complex $X_y = \{y\} \times_Y X$ is n-connective (in the sense of Definition 3.5.1.1).* 04H7

Proof. Using Proposition 3.1.7.1, we can choose a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

where the horizontal maps are inner anodyne, Y' is a Kan complex, and f' is a Kan fibration. Without loss of generality, we may assume that the map g is bijective on vertices (for example, we could take $Y' = \mathrm{Ex}^{\infty}(Y)$; see Proposition 3.3.6.2). It follows from Proposition 3.3.7.1 that for each vertex $y \in Y$, the induced map of Kan complexes $X_y \to X'_{g(y)}$ is a homotopy equivalence. In particular, X_y is n-connective if and only if $X'_{g(y)}$ is n-connective (Remark 3.5.1.5). We can therefore replace f by f', and thereby reduce to proving Proposition 3.5.1.22 in the special case where X and Y are Kan complexes.

Without loss of generality, we may assume that $n \geq 0$ (otherwise, the assertion is vacuous). Our proof proceeds by induction on n. In the case $n = 0$, we must show that f induces a surjection $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ if and only if every fiber of f is nonempty, which follows from Corollary 3.2.6.3. Let us therefore assume that $n > 0$ and that f has nonempty fibers. To carry out the inductive step, it will suffice to show that for every vertex $x \in X$ having image $y = f(x)$, the following conditions are equivalent:

(a) The morphism f induces a surjective group homomorphism $\pi_n(X, x) \to \pi_n(Y, y)$, and the kernel of the map $\pi_{n-1}(X, x) \to \pi_{n-1}(Y, y)$ is trivial (by convention, in the case $n = 1$, we define this kernel to be the inverse image of $[y] \in \pi_0(Y)$).

(b) The set $\pi_{n-1}(X_y, x)$ consists of a single element.

This follows from Corollary 3.2.6.8 in the case $n > 1$, and from Variant 3.2.6.9 in the case $n = 1$. □

051D **Remark 3.5.1.23.** In the situation of Proposition 3.5.1.22, it is not necessary to verify that X_y is n-connective for *every* vertex $y \in Y$; it is enough to check this condition at one vertex in each connected component of Y (see Remark 3.3.7.3). In particular, if Y is connected, then it is enough to check that X_y is n-connective for *any* vertex $y \in Y$.

051E **Corollary 3.5.1.24.** *Let n be an integer and suppose we are given a homotopy pullback square of simplicial sets*

04HU
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "g"] & Y'. \end{tikzcd} \tag{3.27}$$

If the morphism f' is n-connective (in the sense of Variant 3.5.1.6), then f is also n-connective. Moreover, the converse holds if g is surjective on connected components.

Proof. Using Proposition 3.1.7.1, we can reduce to the case where f and f' are Kan fibrations. In this case, our assumption that (3.27) is a homotopy pullback square guarantees that for every vertex $y \in Y$, the induced map of fibers $X_y \to X'_{g(y)}$ is a homotopy equivalence of Kan complexes (Example 3.4.1.4). The desired result now follows from criterion of Proposition 3.5.1.22 (together with Remark 3.5.1.23). □

Corollary 3.5.1.25. *Let $f : X \to Y$ be a morphism of Kan complexes and let n be an integer. The following conditions are equivalent:* 051F

(1) *The morphism f is n-connective.*

(2) *For every morphism of Kan complexes $Y' \to Y$, the projection map $Y' \times^{\mathrm{h}}_{Y} X \to Y'$ is n-connective.*

(3) *For every vertex $y \in Y$, the homotopy fiber $\{y\} \times^{\mathrm{h}}_{Y} X$ is n-connective*

Proof. Using Proposition 3.4.0.9, we can reduce to the case where f is a Kan fibration. In this case, we can use Proposition 3.4.0.7 to reformulate conditions (2) and (3) as follows:

(2′) For every morphism of Kan complexes $Y' \to Y$, the projection map $Y' \times_{Y} X \to Y'$ is n-connective.

(3′) For every vertex $y \in Y$, the fiber $\{y\} \times_{Y} X$ is n-connective.

The equivalence (1) ⇔ (3′) now follows from Proposition 3.5.1.22, and the equivalence (1) ⇔ (2′) from Corollary 3.5.1.24 □

Proposition 3.5.1.26. *Let $f : X \to Y$ be a morphism of simplicial sets and let n be an integer. Then:* 04H6

(1) *If the Y is n-connective and f is n-connective, then X is n-connective.*

(2) *If X is n-connective and Y is $(n+1)$-connective, then f is n-connective.*

(3) *If f is n-connective and X is $(n+1)$-connective, then Y is $(n+1)$-connective.*

Proof. Without loss of generality, we may assume that X and Y are Kan complexes. We proceed by induction on n. If $n < 0$, then assertions (1) and (2) are vacuous, and (3) reduces to the assertion that if X is nonempty, then Y is also nonempty. When $n = 0$, we can restate Proposition 3.5.1.26 as follows:

(1_0) If Y is nonempty and $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective, then X is nonempty.

(2_0) If X is nonempty and Y is connected, then the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective.

(3_0) If $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective and X is connected, then Y is connected.

Assume that $n > 0$, and let $x \in X$ be a vertex having image $y = f(x)$. The inductive step is a consequence of the following observations:

(1_n) If the morphism $\pi_{n-1}(f) : \pi_{n-1}(X, x) \to \pi_{n-1}(Y, y)$ is bijective and $\pi_{n-1}(Y, y)$ is a singleton, then $\pi_{n-1}(X, x)$ is also a singleton.

(2_n) If the sets $\pi_{n-1}(X, x)$ and $\pi_n(Y, y)$ are singletons, then $\pi_{n-1}(f)$ is injective and $\pi_n(f)$ is surjective.

(3_n) If $\pi_n(f)$ is surjective and $\pi_n(X, x)$ is a singleton, then $\pi_n(Y, y)$ is a singleton.

□

051G **Corollary 3.5.1.27.** *Let Y be a simplicial set and let $n \geq -1$ be an integer. The following conditions are equivalent:*

(1) *The simplicial set Y is n-connective.*

(2) *The simplicial set Y is nonempty and, for every vertex $y \in Y$, the inclusion map $\{y\} \hookrightarrow Y$ is $(n-1)$-connective.*

(3) *There exists a vertex $y \in Y$ for which the inclusion map $\{y\} \hookrightarrow Y$ is $(n-1)$-connective.*

04H9 **Corollary 3.5.1.28** (Transitivity)**.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets and let n be an integer. Then:*

(1) *If f and g are n-connective, then the composition $(g \circ f) : X \to Z$ is n-connective.*

(2) *If $g \circ f$ is n-connective and g is $(n+1)$-connective, then f is n-connective.*

(3) *If f is n-connective and $(g \circ f)$ is $(n+1)$-connective, then g is $(n+1)$-connective.*

Proof. Using Proposition 3.1.7.1, we can reduce to the case where Z is a Kan complex and the morphisms f and g are Kan fibrations. Using the criterion of Proposition 3.5.1.22, we can further reduce to the case $Z = \Delta^0$. In this case, Corollary 3.5.1.28 is a restatement of Proposition 3.5.1.26. □

04HA **Corollary 3.5.1.29.** *Let $f : X \to Y$ be a Kan fibration of simplicial sets and let n be a nonnegative integer. Then f is n-connective if and only if it satisfies the following pair of conditions:*

(*a*) *The map of connected components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective.*

(*b*) *The diagonal map* $\delta_{X/Y} : X \to X \times_Y X$ *is* $(n-1)$*-connective.*

Proof. Without loss of generality, we may assume that condition (a) is satisfied. Since f is a Kan fibration, every vertex $y \in Y$ has the form $f(x)$ for some vertex $x \in X$ (Corollary 3.2.6.3). It follows that every fiber of f can also be viewed as a fiber of the map $q : X \times_Y X \to X$ given by projection onto the first factor. Using the criterion of Proposition 3.5.1.22, we see that f is n-connective if and only if q is n-connective. The desired result now follows by applying Corollary 3.5.1.28 to the morphisms $X \xrightarrow{\delta_{X/Y}} X \times_Y X \xrightarrow{q} X$, since the composite map $q \circ \delta_{X/Y} = \mathrm{id}_X$ is n-connective. □

Variant 3.5.1.30. Let $f : X \to Y$ be a morphism of Kan complexes which is surjective on connected components and let $n \geq 0$ be an integer. The following conditions are equivalent: `051H`

(1) The morphism f is n-connective.

(2) The induced map

$$\theta : \mathrm{Fun}(\Delta^1, X) = X \times^{\mathrm{h}}_X X \to X \times^{\mathrm{h}}_Y X$$

is $(n-1)$-connective.

(3) For every pair of vertices $x, x' \in X$, the map of path spaces

$$\{x\} \times^{\mathrm{h}}_X \{x'\} \to \{f(x)\} \times^{\mathrm{h}}_Y \{f(x')\}$$

is $(n-1)$-connective.

Proof. Using Proposition 3.1.7.1, we can factor f as a composition $X \xrightarrow{i} \overline{X} \xrightarrow{\overline{f}} Y$, where $\overline{f}$ is a Kan fibration and i is a homotopy equivalence. Replacing $\overline{X}$ by a full simplicial subset if necessary, we may further assume that i is surjective on vertices. It follows from Remark 3.5.1.14 (and Proposition 3.4.0.9) that conditions (1), (2), or (3) is satisfied by f if and only if it is satisfied by $\overline{f}$. Consequently, we may replace f by $\overline{f}$ and thereby reduce to proving Variant 3.5.1.30 in the special case where f is a Kan fibration. In this case, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\delta_{X/Y}"] & \mathrm{Fun}(\Delta^1, X) \arrow[d, "\theta"] \\
X \times_Y X \arrow[r] & X \times^{\mathrm{h}}_Y X
\end{tikzcd}$$

where the horizontal maps are homotopy equivalences (Proposition 3.4.0.7), so the equivalence of (1) and (2) follows from Corollary 3.5.1.29 (together with Remark 3.5.1.14). Since θ is a Kan fibration (Theorem 3.1.3.1), the equivalence of (2) and (3) follows from Proposition 3.5.1.22. □

04HL **Corollary 3.5.1.31.** *Let $f : X \to Y$ be a Kan fibration of simplicial sets. Then f is a weak homotopy equivalence if and only if the relative diagonal $\delta_{X/Y} : X \to X \times_Y X$ is a weak homotopy equivalence and the map of connected components $\pi_0(X) \to \pi_0(Y)$ is a surjection.*

Proof. Combine Remark 3.5.1.19 with Corollary 3.5.1.29. □

04HB **Corollary 3.5.1.32.** *Let n be a nonnegative integer. Then a simplicial set X is n-connective if and only if it is nonempty and the diagonal map $\delta_X : X \to X \times X$ is $(n-1)$-connective.*

Proof. Using Corollary 3.1.7.2 and Proposition 3.1.6.23, we can reduce to the situation where X is a Kan complex. In this case, the follows by applying Corollary 3.5.1.31 in the special case $Y = \Delta^0$. □

02LF **Corollary 3.5.1.33.** *A simplicial set X is weakly contractible if and only if it is nonempty and the diagonal map $\delta_X : X \hookrightarrow X \times X$ is a weak homotopy equivalence.*

Proof. Combine Remark 3.5.1.19 with Corollary 3.5.1.32. □

3.5.2 Connectivity as a Lifting Property

051J Suppose we are given a commutative diagram of simplicial sets

051K
$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y \arrow[r] & Y'.
\end{tikzcd} \tag{3.28}$$

If (3.28) is a homotopy pullback square and f' is n-connective, then Corollary 3.5.1.24 guarantees that f is also n-connective. In this section, we will prove a dual result: if (3.28) is a homotopy pushout square and f is n-connective, then f' is also n-connective (Corollary 3.5.2.7). We will deduce this from the following relative version of Proposition 3.5.1.12:

04HC **Proposition 3.5.2.1.** *Let $f : X \to Y$ be a Kan fibration of simplicial sets and let n be an integer. The following conditions are equivalent:*

(1) *The morphism f is n-connective.*

(2) *For every vertex $y \in Y$, the Kan complex $X_y = \{y\} \times_Y X$ is n-connective.*

(3) *For every simplicial set B of dimension $\leq n$ and every simplicial subset $A \subseteq B$, every lifting problem*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a solution.

(4) *For every integer $0 \leq m \leq n$, every lifting problem*

$$\begin{tikzcd} \partial \Delta^m \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^m \arrow[r, "\sigma"] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a solution.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is Proposition 3.5.1.22, the implication (3) $\Rightarrow$ (4) is immediate, and the converse follows from Proposition 1.1.4.12. Note that any morphism $\sigma : \Delta^m \to Y$ is homotopic to a constant map. Using the homotopy extension lifting property (Remark 3.1.5.3), we see that (4) is equivalent to the following *a priori* weaker assertion:

(4′) For every integer $0 \leq m \leq n$ and every vertex $y \in Y$, every lifting problem

$$\begin{tikzcd} \partial \Delta^m \arrow[r] \arrow[d] & X_y \arrow[d, "f_y"] \\ \Delta^m \arrow[r, "\sigma"] \arrow[ur, dashed] & \{y\} \end{tikzcd}$$

admits a solution.

The equivalence of (2) and (4′) now follows from Proposition 3.5.1.12. □

Corollary 3.5.2.2. *Let n be an integer and let $f : X \to Y$ be a morphism of simplicial sets which is bijective on k-simplices for $k < n$ and surjective for $k = n$. Then f is n-connective.* 051L

Proof. For $n \leq 0$, this follows immediately from Example 3.5.1.17. We will therefore assume that $n > 0$. Our assumptions on f guarantee that for $0 < m \leq n$, every lifting problem

$$\begin{tikzcd}
\Lambda^m_i \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution. Using Variant 3.1.7.12, we can factor f as a composition $X \xrightarrow{i} X' \xrightarrow{f'} Y$, where i is an anodyne morphism which is bijective on k-simplices for $k < n$, and f' is a Kan fibration. Since the collection of n-connective morphisms is closed under composition, it will suffice to show that f' is n-connective. By virtue of Proposition 3.5.1.22, this is equivalent to the assertion that for each vertex $y \in Y$, the fiber $X'_y = \{y\} \times_Y X$ is an n-connective Kan complex. This follows from Example 3.5.1.10. □

051M **Example 3.5.2.3.** Let X be a simplicial set, let n be an integer, and let $\operatorname{sk}_n(X)$ denote the n-skeleton of X. Then the inclusion map $i : \operatorname{sk}_n(X) \hookrightarrow X$ is bijective on m-simplices for $m \leq n$. Applying Corollary 3.5.2.2, we conclude that i is n-connective.

051N **Corollary 3.5.2.4.** *Let $f : X \to Z$ be a Kan fibration of simplicial sets and let n be an integer. The following conditions are equivalent:*

(1) *The morphism f is n-connective.*

(2) *The morphism f factors as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$, where f' is a monomorphism which is bijective on k-simplices for $k \leq n$ and f'' is a trivial Kan fibration.*

(3) *The morphism f factors as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$ where f' is bijective on k-simplices for $k < n$ and surjective for $k = n$, and f'' is n-connective.*

Proof. The implication (2) ⇒ (3) is immediate and the implication (3) ⇒ (1) follows from Corollary 3.5.2.2 (since the collection of n-connective morphisms is closed under composition; see Corollary 3.5.1.28). We will complete the proof by showing that (1) implies (2). Using a variant of Exercise 3.1.7.11, we can choose a factorization of f as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$ with the following properties;

(a) For every integer $m > n$, every lifting problem

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & Y \arrow[d, "f''"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & Z
\end{tikzcd}$$

admits a solution.

(*b*) The morphism f' can be realized as a transfinite pushout of inclusion maps $\partial \Delta^m \hookrightarrow \Delta^m$ for $m > n$.

It follows immediately from (*b*) that the morphism f' is bijective on k-simplices for $0 \leq k \leq n$. We will complete the proof by showing that, if f is n-connective, then f'' is a trivial Kan fibration: that is, every lifting problem

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & Y \arrow[d, "f''"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & Z
\end{tikzcd} \tag{3.29}$$ 051P

admits a solution. For $m > n$, this follows from (*b*). For $m \leq n$, we can identify (3.29) with a lifting problem

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & Z,
\end{tikzcd}$$

which admits a solution by virtue of our assumption that f is an n-connective Kan fibration (Proposition 3.5.2.1). □

Corollary 3.5.2.5. *Let X be a Kan complex and let n be an integer. The following conditions are equivalent:* 051Q

(1) *The Kan complex X is n-connective.*

(2) *There exists a monomorphism of Kan complexes $f : X \hookrightarrow Y$ where Y is contractible and f is bijective on k-simplices for $0 \leq k \leq n$.*

(3) *There exists a morphism of simplicial sets $f : X \to Y$ where Y is n-connective, f is bijective on k-simplices for $k < n$, and f is surjective on n-simplices.*

Proof. Apply Corollary 3.5.2.4 in the special case $Z = \Delta^0$ (together with Example 3.5.1.18). □

Corollary 3.5.2.6. *For every nonnegative integer n, the simplicial set $\partial \Delta^n$ is $(n-1)$-connective.* 051R

Proof. Since the inclusion map $\partial \Delta^n \hookrightarrow \Delta^n$ is bijective on k-simplices for $k < n$, it will suffice to show that the standard simplex Δ^n is $(n-1)$-connective (Corollary 3.5.2.5). This is clear, since Δ^n is contractible (Example 3.2.4.2). □

051S **Corollary 3.5.2.7.** *Let n be an integer, and suppose we are given a homotopy pushout square of simplicial sets*

051T
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y'. \end{tikzcd} \tag{3.30}$$

If f is n-connective, then f' is also n-connective.

Proof. Using Proposition 3.1.7.1, we can factor f as a composition $X \xrightarrow{i} \overline{X} \xrightarrow{\overline{f}} Y$, where i is anodyne and $\overline{f}$ is a Kan fibration. Replacing X by $\overline{X}$ and X' by the pushout $X' \coprod_X \overline{X}$, we are reduced to proving Corollary 3.5.2.7 in the special case where f is a Kan fibration. In this case, we can use Corollary 3.5.2.4 to factor f as a composition $X \xrightarrow{\widetilde{f}} \widetilde{Y} \xrightarrow{q} Y$, where q is a trivial Kan fibration and $\widetilde{f}$ is a monomorphism which is bijective on k-simplices for $k \leq n$. In this case, our assumption that (3.30) is a homotopy pushout square guarantees that the induced map $X' \coprod_X \widetilde{Y} \to Y'$ is a weak homotopy equivalence (Proposition 3.4.2.11). Consequently, to show that f' is n-connective, it will suffice to show that the inclusion map $j : X' \hookrightarrow X' \coprod_X \widetilde{Y}$ is n-connective (Corollary 3.5.1.28). This is a special case of Corollary 3.5.2.2, since j is bijective on k-simplices for $k \leq n$. □

051U **Definition 3.5.2.8.** Let X be a simplicial set and let n be a nonnegative integer. We say that X is *n-reduced* if the n-skeleton $\mathrm{sk}_n(X)$ is isomorphic to the standard 0-simplex Δ^0: that is, X has a single m-simplex for every integer $0 \leq m \leq n$.

051V **Proposition 3.5.2.9.** *Let X be a simplicial set and let $n \geq 0$ be an integer. Then X is $(n+1)$-connective if and only if there exists a weak homotopy equivalence $f : X \to Y$, where Y is n-reduced.*

Proof. Assume first that there exists a weak homotopy equivalence $f : X \to Y$, where Y is n-reduced. Choose a vertex $y \in Y$. Our assumption that Y is n-reduced guarantees that the inclusion map $i : \{y\} \hookrightarrow Y$ is bijective on m-simplices for $m \leq n$. Applying Corollary 3.5.2.2, we deduce that i is n-connective. It follows that Y is $(n+1)$-connective (Corollary 3.5.1.27). Since f is a weak homotopy equivalence, the simplicial set X is also $(n+1)$-connective.

We now prove the converse. Assume that X is $(n+1)$-connective. In particular, X is nonempty; we can therefore choose a vertex $x \in X$. Using Proposition 3.1.7.1, we can factor the inclusion map $\{x\} \hookrightarrow X$ as a composition $\{x\} \xrightarrow{j} E \xrightarrow{g} X$, where j is anodyne and g is a Kan fibration. Since the simplicial set E is weakly contractible, our hypothesis that X is $(n+1)$-connective guarantees that g is n-connective (Proposition 3.5.1.26). Applying Corollary 3.5.2.4, we can factor g as a composition $E \xrightarrow{g'} \widetilde{X} \xrightarrow{g''} X$, where g' is a monomorphism which is bijective on m-simplices for $m \leq n$ and g'' is a trivial Kan fibration. Let s be a section of g'' and let $Y = \widetilde{X}/E$ be the simplicial set obtained from $\widetilde{X}$ by collapsing the image of g', so that we have a pushout square

051W

$$\begin{tikzcd}
E \arrow[r, "g'"] \arrow[d] & \widetilde{X} \arrow[d, "q"] \\
\Delta^0 \arrow[r] & Y.
\end{tikzcd} \quad (3.31)$$

Since g' is a monomorphism, (3.31) is a homotopy pushout square (Example 3.4.2.12). Since E is weakly contractible, it follows that q is a weak homotopy equivalence (Proposition 3.4.2.10). It follows that the composite map $X \xrightarrow{s} \widetilde{X} \xrightarrow{q} Y$ is a weak homotopy equivalence from X to an n-reduced simplicial set Y. □

Remark 3.5.2.10. In the situation of Proposition 3.5.2.9, we can arrange that the simplicial set Y is a Kan complex: this follows from Example 3.1.7.13. 051X

We now record a few other consequences of Proposition 3.5.2.1.

Proposition 3.5.2.11. *Let m and n be integers and let $f : X \to Y$ be an $(m+n)$-connective morphism of Kan complexes. Let B be a simplicial set of dimension $\leq m$, and let $A \subseteq B$ be a simplicial subset. Then the restriction map* 051Y

$$u : \operatorname{Fun}(B, X) \to \operatorname{Fun}(A, X) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(B, Y)$$

is n-connective.

Proof. Without loss of generality, we may assume that $m \geq 0$ (otherwise, our hypothesis guarantees that A and B are empty, so that u is an isomorphism) and that $n \geq 0$ (otherwise, the conclusion that u is n-connective is vacuous). Using Proposition 3.1.7.1, we can factor f as a composition $X \xrightarrow{j} X' \xrightarrow{f'} Y$, where j is anodyne and f' is a Kan fibration. Since Y is a Kan complex, the simplicial set X' is a Kan complex (Remark 3.1.1.11), so j is a homotopy equivalence of Kan complexes. We then have a commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}(B, X) \arrow[r, "u"] \arrow[d, "j\circ"] & \operatorname{Fun}(A, X) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(B, Y) \arrow[d] \\
\operatorname{Fun}(B, X') \arrow[r, "u'"] & \operatorname{Fun}(A, X) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(B, Y)
\end{tikzcd}$$

where the vertical maps are homotopy equivalences (see Proposition 3.4.0.2). Consequently, to show that u is n-connective, it will suffice to show that u' is n-connective. We may therefore replace f by f', and thereby reduce to proving Proposition 3.5.2.11 in the special case where f is a Kan fibration. In this case, u is also a Kan fibration (Theorem 3.1.3.1). By virtue of Proposition 3.5.2.1, it will suffice to show that if B' is a simplicial set of dimension $\leq n$ and $A' \subseteq B'$ is a simplicial subset, then every lifting problem

04HE
$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & \operatorname{Fun}(B,X) \arrow[d, "u"] \\
B' \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(A,X) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(B,Y)
\end{tikzcd} \tag{3.32}$$

admits a solution. Unwinding the definitions, we can rewrite (3.32) as a lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{(A \times A')} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

Since the simplicial set $B \times B'$ has dimension $\leq m+n$ (Proposition 1.1.3.6), the existence of a solution follows from our assumption that f is $(m+n)$-connective (Proposition 3.5.2.1). □

04HF **Corollary 3.5.2.12.** *Let m and n be integers, let B be a simplicial set of dimension $\leq m$, and let X be a Kan complex which is $(m+n)$-connective. Then, for every simplicial subset $A \subseteq B$, the restriction map $\operatorname{Fun}(B,X) \to \operatorname{Fun}(A,X)$ is n-connective.*

Proof. Apply Proposition 3.5.2.11 in the special case $Y = \Delta^0$. □

04HG **Corollary 3.5.2.13.** *Let m and n be integers, let B be a simplicial set of dimension $\leq m$, and let $f : X \to Y$ be a morphism of Kan complexes which is $(m+n)$-connective. Then the induced map $\operatorname{Fun}(B,X) \to \operatorname{Fun}(B,Y)$ is n-connective.*

Proof. Applying Proposition 3.5.2.11 in the special case $A = \emptyset$. □

04HH **Corollary 3.5.2.14.** *Let m and n be integers, let X be a Kan complex which is $(m+n)$-connective, and let B be a simplicial set of dimension $\leq m$. Then the Kan complex $\operatorname{Fun}(B,X)$ is n-connective.*

Proof. Apply Corollary 3.5.2.12 in the special case $A = \emptyset$ (or Corollary 3.5.2.13 in the special case $Y = \Delta^0$). □

3.5.3 Coskeletal Simplicial Sets

Let X be a simplicial set and let n be an integer. Recall that X *has dimension* $\leq n$ if every m-simplex of X is degenerate for $m > n$ (Definition 1.1.3.1). If this condition is satisfied, then X is determined by its simplices of dimension $\leq n$ in the following sense: to give a morphism of simplicial sets $f : X \to Y$, it suffices to specify the value of f on m-simplices for $m \leq n$ (see Proposition 1.1.3.11 for a precise statement). In this section, we introduce a dual condition which instead controls the classification of morphisms $Y \to X$ (Proposition 3.5.3.10). `051Z`

Definition 3.5.3.1. Let n be an integer and let X be a simplicial set. We say that X is *n-coskeletal* if, for every nonnegative integer $m > n$, the restriction map `0520`

$$\theta_m : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^m, X)$$

is a bijection: that is, every morphism of simplicial sets $\partial \Delta^m \to X$ extends uniquely to an m-simplex of X.

Example 3.5.3.2. Let n be a negative integer. Then a simplicial set X is n-coskeletal if and only if it is a final object of Set_Δ: that is, if and only if it is isomorphic to the standard 0-simplex Δ^0. `0521`

Example 3.5.3.3. Let Q be a partially ordered set. Then the nerve $\mathrm{N}_\bullet(Q)$ is 1-coskeletal. In particular, every discrete simplicial set is 1-coskeletal. `0522`

Example 3.5.3.4. Let $\mathcal{C}$ be a category. Then the nerve $\mathrm{N}_\bullet(\mathcal{C})$ is 2-coskeletal. See Exercise 1.3.1.5. `0523`

Example 3.5.3.5. Let $\mathcal{C}$ be a 2-category. Then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ is 3-coskeletal. See Corollary 2.3.1.10. `0524`

Remark 3.5.3.6 (Monotonicity)**.** Let m be an integer and let X be an m-coskeletal simplicial set. Then X is n-coskeletal for every integer $n \geq m$. `0525`

Remark 3.5.3.7. Let n be an integer. Then the collection of n-coskeletal simplicial sets is closed under the formation of limits (in the category Set_Δ). `0526`

Remark 3.5.3.8. Let n be a positive integer. Then a simplicial set X is n-coskeletal if and only if each connected component of X is n-coskeletal (beware that this is false for $n = 0$). `0527`

Proposition 3.5.3.9. *Let (A_*, ∂) be a chain complex of abelian groups, let $X = \mathrm{K}(A_*)$ be the associated Eilenberg-MacLane space, and let n be a nonnegative integer. Then X is n-coskeletal if and only if it satisfies the following conditions:* `0528`

(a) *The abelian groups A_m vanish for $m \geq n+2$.*

(b) *The boundary map $\partial : A_{n+1} \to A_n$ is a monomorphism, whose image is the group of n-cycles $Z_n = \{y \in A_n : \partial y = 0\}$.*

Proof. Fix an integer $m \geq 0$, and let $\sigma : \Delta^m \to \Delta^m$ be the identity map, which we identify with its image in the normalized chain complex $\mathrm{N}_*(\Delta^m; \mathbf{Z})$. Then $\partial(\sigma) = \sum_{i=0}^{m} (-1)^i d_i^m(\sigma)$ is a cycle in the subcomplex $\mathrm{N}_*(\partial \Delta^m; \mathbf{Z})$. Suppose we are given a morphism of simplicial sets $\tau_0 : \partial \Delta^m \to X$, which we identify with a chain map $f_0 : \mathrm{N}_*(\partial \Delta^m; \mathbf{Z}) \to A_*$. Then $y = f_0(\partial \sigma)$ is an $(m-1)$-cycle of A_*. Note that, if $m > 0$, then every $(m-1)$-cycle y of A_* can be obtained in this way (for example, we can take $f_0 : \mathrm{N}_*(\partial \Delta^m; \mathbf{Z}) \to A_*$ to be the map of chain complexes which carries $d_0^m(\sigma)$ to y and every other nondegenerate simplex to zero).

If $\tau : \Delta^m \to X$ is an extension of τ_0 corresponding to a map of chain complexes $f : \mathrm{N}_*(\Delta^m; \mathbf{Z}) \to A_*$, then $x = f(\sigma)$ is an m-chain of A_* satisfying $\partial(x) = y$. This construction induces a bijection

$$\{m\text{-simplices } \tau \text{ with } \tau|_{\partial \Delta^m} = \tau_0\} \xrightarrow{\sim} \{x \in A_m \text{ with } \partial(x) = y\}.$$

It follows that the simplicial set X is n-coskeletal if and only if it satisfies the following condition for each $m > n$:

(b_m) The boundary map $\partial : A_m \to A_{m-1}$ is a monomorphism whose image is the group of $(m-1)$-cycles $Z_{m-1} = \{y \in A_{m-1} : \partial y = 0\}$.

Note that (b_{n+1}) is a restatement of (b). Moreover, if condition (b_m) is satisfied for some integer m, then condition (b_{m+1}) is equivalent to the requirement that the abelian group A_{m+1} is trivial. In particular, (b_m) is satisfied for all $m > n$ if and only if A_* satisfies conditions (a) and (b). □

0529 **Proposition 3.5.3.10.** *Let X be a simplicial set. For every integer n, the following conditions are equivalent:*

(1) *The simplicial set X is n-coskeletal.*

(2) *For every simplicial set S, the restriction map*

$$\theta_S : \mathrm{Hom}_{\mathrm{Set}_\Delta}(S, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_n(S), X)$$

is a bijection.

Proof. For every nonnegative integer $m > n$, the n-skeleton of Δ^m is contained in $\partial \Delta^m$, and therefore coincides with with the n-skeleton of $\partial \Delta^m$. We therefore have a commutative diagram of restriction maps

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^m, X) \arrow[rr] \arrow[dr, "\theta_{\Delta^m}"'] & & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^m, X) \arrow[dl, "\theta_{\partial \Delta^m}"] \\
& \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(\Delta^m), X). &
\end{tikzcd}$$

If condition (2) is satisfied, then the vertical maps are bijections, so the horizontal map is a bijection as well. Allowing m to vary, we deduce that X is n-coskeletal.

We now prove the converse. For every simplicial set S, we can identify $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S, X)$ with the inverse limit of the tower of restriction maps

$$\cdots \rightarrow \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_{n+2}(S), X) \rightarrow \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_{n+1}(S), X) \rightarrow \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_{n}(S), X).$$

Consequently, to prove (2), it will suffice to show that the restriction map

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_{m}(S), X) \rightarrow \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_{m-1}(S), X)$$

is a bijection for $m > n$. Using Proposition 1.1.4.12, we can reduce to the case $S = \Delta^m$, in which case the statement reduces to the assertion that X is n-coskeletal. □

Remark 3.5.3.11. In the situation of Proposition 3.5.3.10, it suffices to consider the special case where $S = \Delta^m$ is a standard simplex. This follows from Corollary 1.1.4.9, since any simplicial set can be realized as a colimit of simplices (Remark 1.1.3.13). 052A

Corollary 3.5.3.12. *Let n be an integer and let K and X be simplicial sets. If X is n-coskeletal, then* $\mathrm{Fun}(K, X)$ *is also n-coskeletal.* 052B

Proof. By virtue of Proposition 3.5.3.10, it will suffice to show that for every simplicial set S, the restriction map $\theta : \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S, \mathrm{Fun}(K, X)) \rightarrow \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(S), \mathrm{Fun}(K, X))$ is a bijection. Using Proposition 1.5.3.2, we can identify θ with the horizontal map in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S \times K, X) \arrow[rr] \arrow[dr] & & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(S) \times K, X) \arrow[dl] \\
& \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(S \times K), X). &
\end{tikzcd}$$

It will therefore suffice to show that the vertical maps in this diagram are bijections, which follows from our assumption that X is n-coskeletal (Proposition 3.5.3.10). □

052C **Corollary 3.5.3.13.** *Let $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set and let n be an integer. Then $X_\bullet$ is n-coskeletal if and only if it satisfies the following condition for each $m \geq 0$:*

- $(*_n)$ *Let $\mathcal{C} = \mathbf{\Delta}^{\leq n}_{\Delta^m}$ denote the category of simplices of Δ^m having dimension $\leq n$ (see Construction 1.1.3.9). Then the tautological map*

$$\theta_m : X_m \to \varprojlim_{([k],\sigma)\in\mathcal{C}^{\mathrm{op}}} X_k$$

is a bijection.

Proof. For each $n \geq 0$, we can identify θ_m with the restriction map

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X_\bullet) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(S_\bullet, X_\bullet),$$

where $S_\bullet$ denotes the colimit $\varinjlim_{([k],\sigma)\in\mathcal{C}} \Delta^k$. Using Corollary 1.1.4.8, we can identify $S_\bullet$ with the n-skeleton $\mathrm{sk}_n(\Delta^m)$, so the desired result follows from Proposition 3.5.3.10 and Remark 3.5.3.11. □

052D **Remark 3.5.3.14.** Corollary 3.5.3.13 can be reformulated using the language of Kan extensions (see Definition 7.3.0.1): it asserts that a simplicial set $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ is n-coskeletal if and only if it is right Kan extended from the full subcategory of $\mathbf{\Delta}^{\mathrm{op}}$ spanned by the objects $\{[k]\}_{k\leq n}$. Compare with Remark 1.1.3.12.

052E **Definition 3.5.3.15.** Let X be a simplicial set and let n be an integer. We will say that a morphism of simplicial sets $f : X \to Y$ *exhibits Y as an n-coskeleton of X* if it satisfies the following pair of conditions:

- The simplicial set Y is n-coskeletal.
- The morphism f is bijective on m-simplices for $m \leq n$.

052F **Proposition 3.5.3.16** (Existence)**.** *Let X be a simplicial set. For every integer n, there exists a simplicial set $\mathrm{cosk}_n(X)$ and a morphism $f : X \to \mathrm{cosk}_n(X)$ which exhibits $\mathrm{cosk}_n(X)$ as an n-coskeleton of X.*

Proof. Let $\mathrm{cosk}_n(X)$ denote the simplicial set given by the construction

$$([m] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_n(\Delta^m), X),$$

and let $f : X \to \mathrm{cosk}_n(X)$ be the morphism of simplicial sets given on m-simplices by the restriction map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_n(\Delta^m), X))$. If $m \leq n$, then $\mathrm{sk}_n(\Delta^m) = \Delta^m$; it follows that f is bijective on m-simplices for $m \leq n$. We will complete the proof by showing that $\operatorname{cosk}_n(X)$ is n-coskeletal. Fix an integer $m > n$; we wish to show that the restriction map

$$\theta : \operatorname{Hom}_{\operatorname{Set}_\Delta}(\Delta^m, \operatorname{cosk}_n(X)) \to \operatorname{Hom}_{\operatorname{Set}_\Delta}(\partial\Delta^m, \operatorname{cosk}_n(X))$$

is a bijection. Writing $\partial\Delta^m$ as a colimit of simplices (Remark 1.1.3.13) and applying Corollary 1.1.4.9, we can identify θ with the restriction map

$$\operatorname{Hom}_{\operatorname{Set}_\Delta}(\operatorname{sk}_n(\Delta^m), X) \to \operatorname{Hom}_{\operatorname{Set}_\Delta}(\operatorname{sk}_n(\partial\Delta^m), X).$$

The desired result now follows from the observation that the n-skeleton of Δ^m is contained in $\partial\Delta^m$. □

Definition 3.5.3.15 can be reformulated as a universal mapping property.

Proposition 3.5.3.17 (Uniqueness)**.** *Let n be an integer and let $f : X \to Y$ be a morphism of simplicial sets, where Y is n-coskeletal. The following conditions are equivalent:* 052G

(1) *The morphism f exhibits Y as an n-coskeleton of X: that is, it is bijective on m-simplices for $m \leq n$.*

(2) *For every n-coskeletal simplicial set Z, composition with f induces an isomorphism of simplicial sets $\operatorname{Fun}(Y, Z) \to \operatorname{Fun}(X, Z)$.*

(3) *For every n-coskeletal simplicial set Z, composition with f induces a bijection*

$$\operatorname{Hom}_{\operatorname{Set}_\Delta}(Y, Z) \to \operatorname{Hom}_{\operatorname{Set}_\Delta}(X, Z).$$

Proof. Assertion (3) is equivalent to the requirement that, for every n-coskeletal simplicial set Z and every simplicial set K, composition with f induces a bijection

$$\operatorname{Hom}_{\operatorname{Set}_\Delta}(K, \operatorname{Fun}(Y, Z)) \to \operatorname{Hom}_{\operatorname{Set}_\Delta}(K, \operatorname{Fun}(X, Z)).$$

By virtue of Corollary 3.5.3.12, we can replace Z by $\operatorname{Fun}(K, Z)$ and thereby reduce to the case $K = \Delta^0$. This proves the equivalence (2) ⇔ (3).

The implication (1) ⇒ (3) follows immediately from Proposition 3.5.3.10. We will complete the proof by showing that (3) implies (1). Using Proposition 3.5.3.16, we can choose a morphism $u : X \to \operatorname{cosk}_n(X)$ which exhibits $\operatorname{cosk}_n(X)$ as an n-coskeleton of X. Then u satisfies condition (3), so f factors (uniquely) as a composition $X \xrightarrow{u} \operatorname{cosk}_n(X) \xrightarrow{g} Y$. We can therefore replace X by $\operatorname{cosk}_n(X)$ and thereby reduce to the case where X is n-coskeletal. In this case, condition (3) implies that f is an isomorphism of (n-coskeletal) simplicial sets, and therefore bijective on m-simplices for $m \leq n$. □

052H **Notation 3.5.3.18.** Let X be a simplicial set and let n be an integer. It follows from Proposition 3.5.3.16 that there exists a morphism of simplicial sets $f : X \to Y$ which exhibits Y as an n-coskeleton of X. Moreover, Proposition 3.5.3.17 guarantees that Y is unique up to (canonical) isomorphism and depends functorially on X. To emphasize this dependence, we will denote Y by $\mathrm{cosk}_n(X)$ and refer to it as *the* n-coskeleton of X. More explicitly, we can take $\mathrm{cosk}_n(X)$ to be the simplicial set constructed in the proof of Proposition 3.5.3.16, given by the construction

$$([m] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(\Delta^m), X).$$

052J **Corollary 3.5.3.19.** *Let n be an integer. Then the inclusion functor*

$$\{n\text{-coskeletal simplicial sets}\} \hookrightarrow \mathrm{Set}_{\Delta}$$

admits a left adjoint, given on objects by the construction $X \mapsto \mathrm{cosk}_n(X)$.

052K **Remark 3.5.3.20.** For every integer n, the coskeleton functor

$$\mathrm{cosk}_n : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$$

preserves small limits and filtered colimits. If $n > 0$, then it also preserves coproducts.

052L **Remark 3.5.3.21.** Let X be a simplicial set, let n be an integer, and let $\mathrm{cosk}_n(X)$ denote the n-coskeleton of X. For every simplicial set S, we have canonical isomorphisms

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(S), X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(S), \mathrm{cosk}_n(X)) \xleftarrow{\sim} \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(S, \mathrm{cosk}_n(X))$$

where the map on the left is bijective because the map $X \to \mathrm{cosk}_n(X)$ is bijective on simplices of dimension $\leq n$, and the map on the right is bijective by virtue of Proposition 3.5.3.10. Note that this observation was implicitly used (in the special case $S = \partial \Delta^m$) in the proof of Proposition 3.5.3.16.

052M **Remark 3.5.3.22.** Let X be a simplicial set and let n be an integer. Then the tautological map $u : X \to \mathrm{cosk}_n(X)$ is bijective on m-simplices for $m \leq n$. Applying Corollary 3.5.2.2, we deduce that u is n-connective.

052N **Proposition 3.5.3.23.** *Let X be a Kan complex and let n be an integer. Then the n-coskeleton $\mathrm{cosk}_n(X)$ is also a Kan complex.*

Proof. Let m be a positive integer. Fix an integer $0 \leq i \leq m$ and a morphism of simplicial sets $\sigma_0 : \Lambda^m_i \to \mathrm{cosk}_n(X)$; we wish to show that σ_0 can be extended to an m-simplex of $\mathrm{cosk}_n(X)$. Using Remark 3.5.3.21, we can identify σ_0 with a morphism of simplicial sets $f_0 : \mathrm{sk}_n(\Lambda^m_i) \to X$; we wish to show that f_0 can be extended to the n-skeleton of Δ^m. If $n < m - 1$, then $\mathrm{sk}_n(\Lambda^m_i) = \mathrm{sk}_n(\Delta^m)$ and there is nothing to prove. We may therefore assume that $n \geq m - 1$, so that $\mathrm{sk}_n(\Lambda^m_i) = \Lambda^m_i$. In this case, our assumption that X is a Kan complex guarantees that f_0 can be extended to an n-simplex of X. □

Example 3.5.3.24. Let A_* be a nonnegatively graded chain complex of abelian groups 052P
and let $X = \mathrm{K}(A_*)$ denote the associated simplicial abelian group. For every integer $n \geq 0$, the coskeleton $\operatorname{cosk}_n(X)$ inherits the structure of a simplicial abelian group. It follows from Theorem 2.5.6.1 that $\operatorname{cosk}_n(X)$ can be identified with the Eilenberg-MacLane space $\mathrm{K}(A'_*)$, for some nonnegatively graded chain complex A'_*. Here A'_* is universal among chain complexes which satisfy conditions (a) and (b) of Proposition 3.5.3.9 and are equipped with a chain map $A_* \to A'_*$. More concretely, we can identify A'_* with the chain complex

$$\cdots \to 0 \to Z_n \hookrightarrow A_n \xrightarrow{\partial} A_{n-1} \xrightarrow{\partial} A_{n-2} \to \cdots ,$$

where Z_n is the group of n-cycles of A_*.

3.5.4 Weakly Coskeletal Simplicial Sets

It will be useful to consider a variant of Definition 3.5.3.1. 052Q

Definition 3.5.4.1. Let n be an integer. We say that a simplicial set X is *weakly n-coskeletal* 052R
if the restriction map

$$\operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \operatorname{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^m, X)$$

is a bijection for $m \geq n+2$ and and injection for $m = n+1$ (provided that $n \geq -1$).

Remark 3.5.4.2. Let n be an integer and let X be a simplicial set. Then: 052S

- If X is n-coskeletal, then it is weakly n-coskeletal.
- If X is weakly n-coskeletal, then it is $(n+1)$-coskeletal.

Example 3.5.4.3. For $n \leq -2$, a simplicial set X is weakly n-coskeletal if and only if it 052T
isomorphic to the standard 0-simplex Δ^0.

Example 3.5.4.4. A simplicial set X is weakly (-1)-coskeletal if and only if it is either 052U
empty or isomorphic to the standard 0-simplex Δ^0.

Example 3.5.4.5. Let Q be a partially ordered set. Then the nerve $\mathrm{N}_\bullet(Q)$ is weakly 052V
0-coskeletal. In particular, every discrete simplicial set is weakly 0-coskeletal.

Example 3.5.4.6. Let $\mathcal{C}$ be a category. Then the nerve $\mathrm{N}_\bullet(\mathcal{C})$ is weakly 1-coskeletal. See 052W
Exercise 1.3.1.5.

Example 3.5.4.7. Let $\mathcal{C}$ be a 2-category. Then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ is weakly 2- 052X
coskeletal. See Corollary 2.3.1.10.

052Y **Exercise 3.5.4.8.** Let A_* be a chain complex of abelian groups and let $n \geq -1$ be an integer. Show that the Eilenberg-MacLane space $\mathrm{K}(A_*)$ is weakly n-coskeletal if and only if it satisfies the following conditions:

- The abelian groups A_m vanish for $m \geq n+2$.
- The differential $\partial : A_{n+1} \to A_n$ is a monomorphism.

Compare with Proposition 3.5.3.9.

052Z **Remark 3.5.4.9.** For every integer n, the collection of weakly n-coskeletal simplicial sets is closed under the formation of limits.

0530 **Remark 3.5.4.10.** Let n be a nonnegative integer. Then a simplicial set X is weakly n-coskeletal if and only if each connected component of X is weakly n-coskeletal.

0531 **Exercise 3.5.4.11.** Let X be a Kan complex and let $n \geq -1$ be an integer. Show that X is weakly n-coskeletal if and only if, for every integer $m > n$, the restriction map $\theta_m : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^m, X)$ is injective.

0532 **Proposition 3.5.4.12.** *Let n be an integer. Then a simplicial set X is weakly n-coskeletal if and only if it satisfies the following conditions:*

(1) *For every simplicial set S, the restriction map*

$$\theta_S : \mathrm{Hom}_{\mathrm{Set}_\Delta}(S, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_n(S), X)$$

is a monomorphism.

(2) *The image of θ_S consists of those morphisms $\mathrm{sk}_n(S) \to X$ which can be extended to the $(n+1)$-skeleton of S.*

Proof. For every integer $m > n$, the n-skeleton $\mathrm{sk}_n(\Delta^m)$ is contained in the boundary $\partial \Delta^m$, and therefore coincides with the n-skeleton of $\partial \Delta^m$. We therefore have a commutative diagram of restriction maps

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \arrow[rr] \arrow[dr, "\theta_{\Delta^m}"'] & & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^m, X) \arrow[dl, "\theta_{\partial \Delta^m}"] \\
& \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_n(\Delta^m), X). &
\end{tikzcd}$$

If condition (1) is satisfied, then the vertical maps are injective, so the upper horizontal map is also injective. Moreover, if $m \geq n+2$, then $\partial \Delta^m$ contains the $(n+1)$-skeleton of Δ^m. In this case, condition (2) guarantees that the vertical maps have the same image, so that the horizontal map is bijective. It follows that X is weakly n-coskeletal.

We now prove the converse. Assume that X is weakly n-coskeletal, and let S be a simplicial set. Then we can identify $\mathrm{Hom}_{\mathrm{Set}_\Delta}(S, X)$ with the inverse limit of the tower of restriction maps

$$\cdots \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_{n+2}(S), X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_{n+1}(S), X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_{n}(S), X).$$

Consequently, to prove (1), it will suffice to show that the restriction map

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_{m+1}(S), X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_{m}(S), X)$$

is injective for $m = n$ and bijective for $m > n$. Using Proposition 1.1.4.12, we can reduce to the case $S = \Delta^{m+1}$ is a standard simplex, in which case the desired result is immediate from the definition. □

Corollary 3.5.4.13. *Let n be an integer and let K and X be simplicial sets. If X is weakly n-coskeletal, then* $\mathrm{Fun}(K, X)$ *is also weakly n-coskeletal.* 0533

Proof. It follows from Corollary 3.5.3.12 that $\mathrm{Fun}(K, X)$ is $(n+1)$-coskeletal. It will therefore suffice to show that, if $n \geq -1$, then the restriction map $\theta : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^{n+1}, \mathrm{Fun}(K, X)) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^{n+1}, \mathrm{Fun}(K, X))$ is injective. Note that θ can be identified with the restriction map

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^{n+1} \times K, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^{n+1} \times K, X).$$

Since $\partial \Delta^{n+1} \times K$ contains the n-skeleton of $\Delta^{n+1} \times K$, the injectivity of this map follows from Proposition 3.5.4.12. □

Definition 3.5.4.14. Let X be a simplicial set and let n be an integer. We will say that a morphism of simplicial sets $f : X \to Y$ *exhibits Y as a weak n-coskeleton of X* if the following conditions are satisfied: 0534

- The simplicial set Y is weakly n-coskeletal.
- The morphism f is bijective on simplices of dimension $\leq n$ and surjective on $(n+1)$-simplices (provided that $n \geq -1$).

Warning 3.5.4.15. The terminology of Definition 3.5.4.14 is potentially confusing. If $f : X \to Y$ is a morphism which exhibits Y as an n-coskeleton of X, then it generally does not exhibit Y as a weak n-coskeleton of X (because f need not be surjective on $(n+1)$-simplices). 0535

Remark 3.5.4.16. Let $f : X \to Y$ be a morphism of simplicial sets which exhibits Y as a weak n-coskeleton of X. Then f is $(n+1)$-connective. See Corollary 3.5.2.2. 0536

0537 **Proposition 3.5.4.17.** *Let X be a simplicial set, let n be an integer, and let $\operatorname{cosk}_n^{\circ}(X)$ denote the image of tautological map $\operatorname{cosk}_{n+1}(X) \to \operatorname{cosk}_n(X)$. Then the composite map*

$$f : X \to \operatorname{cosk}_{n+1}(X) \twoheadrightarrow \operatorname{cosk}_n^{\circ}(X)$$

exhibits $\operatorname{cosk}_n^{\circ}(X)$ as a weak n-coskeleton of X.

Proof. The tautological map $X \to \operatorname{cosk}_{n+1}(X)$ is bijective on m-simplices for $m \leq n+1$, so f is surjective on m-simplices for $m \leq n+1$. Moreover, for $m \leq n$, the composite map $X \xrightarrow{f} \operatorname{cosk}_n^{\circ}(X) \hookrightarrow \operatorname{cosk}_n(X)$ is bijective on m-simplices for $m \leq n$, so that f is injective on m-simplices. To complete the proof, it will suffice to show that $\operatorname{cosk}_n^{\circ}(X)$ is weakly n-coskeletal. Fix an integer $m > n$ and a morphism $\sigma_0 : \partial \Delta^m \to \operatorname{cosk}_n^{\circ}(X)$. Since $\operatorname{cosk}_n(X)$ is n-coskeletal, the morphism σ_0 extends uniquely to an m-simplex σ of $\operatorname{cosk}_n(X)$. To complete the proof, it will suffice to show that if $m \geq n+2$, then σ is contained in $\operatorname{cosk}_n^{\circ}(X)$: that is, that it can be lifted to an m-simplex of $\operatorname{cosk}_{n+1}(X)$. Using Remark 3.5.3.21, we can identify σ with a morphism of simplicial sets $u : \operatorname{sk}_n(\Delta^m) \to X$; we wish to show that u can be extended to the $(n+1)$-skeleton of Δ^m. By virtue of Proposition 1.1.4.12, this is equivalent to the requirement that, for every nondegenerate $(n+1)$-simplex τ of Δ^m, the composite map

$$\partial \Delta^{n+1} \xrightarrow{\tau|_{\partial \Delta^{n+1}}} \operatorname{sk}_n(\Delta^m) \xrightarrow{u} X$$

can be extended to an $(n+1)$-simplex of X. This follows from our assumption that $\sigma_0 \circ \tau$ factors through $\operatorname{cosk}_n^{\circ}(X)$. □

Definition 3.5.4.14 can be reformulated as a universal mapping property.

0538 **Proposition 3.5.4.18.** *Let n be an integer and let $f : X \to Y$ be a morphism of simplicial sets, where Y is weakly n-coskeletal. The following conditions are equivalent:*

(1) *The morphism f exhibits Y as a weak n-coskeleton of X: that is, it is bijective on m-simplices for $m \leq n$ and surjective on $(n+1)$-simplices (provided that $n \geq -1$).*

(2) *For every weakly n-coskeletal simplicial set Z, composition with f induces an isomorphism of simplicial sets $\operatorname{Fun}(Y, Z) \to \operatorname{Fun}(X, Z)$.*

(3) *For every weakly n-coskeletal simplicial set Z, composition with f induces a bijection $\operatorname{Hom}_{\operatorname{Set}_{\Delta}}(Y, Z) \to \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(X, Z)$.*

Proof. Condition (3) is equivalent to the requirement that, for every weakly n-coskeletal simplicial set Z and every simplicial set K, composition with f induces a bijection

$$\operatorname{Hom}_{\operatorname{Set}_{\Delta}}(K, \operatorname{Fun}(Y, Z)) \to \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(K, \operatorname{Fun}(X, Z)).$$

By virtue of Corollary 3.5.4.13, we can replace Z by $\operatorname{Fun}(K, Z)$ and thereby reduce to the case $K = \Delta^0$. This proves the equivalence (2) ⇔ (3).

We next show that (1) implies (3). Assume that condition (1) is satisfied, and let Z be a weakly n-coskeletal simplicial set. We then have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\operatorname{Set}_{\Delta}}(Y, Z) \arrow[r, "\circ f"] \arrow[d] & \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(X, Z) \arrow[d] \\
\operatorname{Hom}_{\operatorname{Set}_{\Delta}}(\operatorname{sk}_n(Y), Z) \arrow[r, "\circ f"] & \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(\operatorname{sk}_n(X), Z).
\end{tikzcd}$$

Since f is bijective on m-simplices for $m \leq n$, the lower horizontal map is a bijection. Using Proposition 3.5.4.12, we see that the vertical maps are injective. Consequently, to prove (3), it will suffice to show that their images agree (under the bijection provided by the the lower horizontal map). This follows from Proposition 3.5.4.12, together with our assumption that f is surjective on $(n+1)$-simplices.

We now show that (3) implies (1). Using Proposition 3.5.4.17, we can choose a morphism $u : X \to \operatorname{cosk}_n^{\circ}(X)$ which exhibits $\operatorname{cosk}_n^{\circ}(X)$ as a weak n-coskeleton of X. Then u satisfies condition (3), so f factors (uniquely) as a composition $X \xrightarrow{u} \operatorname{cosk}_n^{\circ}(X) \xrightarrow{g} Y$. We can therefore replace X by $\operatorname{cosk}_n^{\circ}(X)$ and thereby reduce to the case where X is weakly n-coskeletal. Condition (3) then guarantees that f is an isomorphism of (weakly n-coskeletal) simplicial sets, so condition (1) is automatic. □

Notation 3.5.4.19. Let X be a simplicial set and let n be an integer. It follows from Proposition 3.5.4.17 that there exists a morphism of simplicial sets $f : X \to Y$ which exhibits Y as a weak n-coskeleton of X. Moreover, Proposition 3.5.4.18 guarantees that Y is unique up to (canonical) isomorphism and depends functorially on X. To emphasize this dependence, we will denote Y by $\operatorname{cosk}_n^{\circ}(X)$ and refer to it as *the* weak n-coskeleton of X. More explicitly, we can take $\operatorname{cosk}_n^{\circ}(X)$ to be the image of the restriction map $\operatorname{cosk}_{n+1}(X) \to \operatorname{cosk}_n(X)$ (see the proof of Proposition 3.5.4.17). 0539

Corollary 3.5.4.20. *Let n be an integer. Then the inclusion functor* 053A

$$\{\text{Weakly } n\text{-coskeletal simplicial sets}\} \hookrightarrow \operatorname{Set}_{\Delta}$$

admits a left adjoint, given on objects by the construction $X \mapsto \operatorname{cosk}_n^{\circ}(X)$.

Remark 3.5.4.21. Let X be a simplicial set, let n be an integer, and let $\operatorname{cosk}_n^{\circ}(X)$ denote the weak n-coskeleton of X. It follows from Proposition 3.5.4.12 that, for every simplicial set S, the restriction map 053B

$$\operatorname{Hom}_{\operatorname{Set}_{\Delta}}(S, \operatorname{cosk}_n^{\circ}(X)) \to \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(\operatorname{sk}_n(S), \operatorname{cosk}_n^{\circ}(X)) \xleftarrow{\sim} \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(\operatorname{sk}_n(S), X)$$

is an injection, whose image consists of those morphisms $f : \mathrm{sk}_n(S) \to X$ which can be extended to the $(n+1)$-skeleton of S.

Let X be a simplicial set. For every n, the weak n-coskeleton $\mathrm{cosk}_n^{\circ}(X)$ is $(n+1)$-coskeletal (Remark 3.5.4.2). It follows from Proposition 3.5.3.17 that the tautological map $X \to \mathrm{cosk}_n^{\circ}(X)$ factors (uniquely) through the $(n+1)$-coskeleton of X.

053C **Proposition 3.5.4.22.** *Let X be a simplicial set. For every integer n, the tautological map $q : \mathrm{cosk}_{n+1}(X) \to \mathrm{cosk}_n^{\circ}(X)$ is a trivial Kan fibration.*

Proof. Fix an integer $m \geq 0$; we wish to show that every lifting problem

053D
$$\begin{tikzcd}
\partial\Delta^m \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{cosk}_{n+1}(X) \arrow[d, "q"] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathrm{cosk}_n^{\circ}(X)
\end{tikzcd} \tag{3.33}$$

admits a solution. We consider two cases:

- If $m \leq n+1$, then $\overline{\sigma}$ can be lifted to an m-simplex of X. In particular, there exists an m-simplex σ of $\mathrm{cosk}_{n+1}(X)$ satisfying $q(\sigma) = \overline{\sigma}$. Since q is bijective on k-simplices for $k \leq n$, the commutativity of the diagram (3.33) guarantees that $\sigma|_{\partial\Delta^m} = \sigma_0$.
- If $m \geq n+2$, then σ_0 extends uniquely to an m-simplex σ of $\mathrm{cosk}_{n+1}(X)$. The commutativity of diagram (3.33) guarantees that $q(\sigma)$ and $\overline{\sigma}$ have the same restriction to $\partial\Delta^m$, and therefore coincide (since $\partial\Delta^m$ contains the $(n+1)$-skeleton of Δ^m).

In either case, the m-simplex σ is a solution to the lifting problem (3.33). □

053E **Corollary 3.5.4.23.** *Let X be a simplicial set and let n be an integer. If X is a Kan complex, then the weak n-coskeleton $\mathrm{cosk}_n^{\circ}(X)$ is a Kan complex.*

Proof. Proposition 3.5.4.22 supplies a trivial Kan fibration

$$q : \mathrm{cosk}_{n+1}(X) \twoheadrightarrow \mathrm{cosk}_n^{\circ}(X).$$

Since $\mathrm{cosk}_{n+1}(X)$ is a Kan complex (Proposition 3.5.3.23), it follows that $\mathrm{cosk}_n^{\circ}(X)$ is also a Kan complex (Proposition 1.5.5.11). □

053F **Corollary 3.5.4.24.** *Let X be a simplicial set and let $n \geq -2$ be an integer. The following conditions are equivalent:*

(1) *The comparison map $f : \mathrm{cosk}_{n+2}(X) \to \mathrm{cosk}_n^{\circ}(X)$ is a trivial Kan fibration.*

(2) *Every morphism* $\partial \Delta^{n+2} \rightarrow X$ *can be extended to an* $(n+2)$*-simplex of* X*.*

(3) *The weak* $(n+1)$*-coskeleton* $\mathrm{cosk}^{\circ}_{n+1}(X)$ *coincides with* $\mathrm{cosk}_{n+1}(X)$*.*

Proof. The equivalence of (2) and (3) follows from Remark 3.5.4.21. The implication (3) ⇒ (1) follows from the observation that f factors as a composition

$$\mathrm{cosk}_{n+2}(\mathcal{C}) \twoheadrightarrow \mathrm{cosk}^{\circ}_{n+1}(\mathcal{C}) \hookrightarrow \mathrm{cosk}_{n+1}(\mathcal{C}) \twoheadrightarrow \mathrm{cosk}^{\circ}_{n}(\mathcal{C}),$$

where the outer maps are trivial Kan fibrations (Proposition 3.5.4.22). We will complete the proof by showing that (2) implies (3). Suppose we are given a morphism $\sigma_0 : \partial \Delta^{n+2} \rightarrow \mathcal{C}$. Since $\mathrm{cosk}^{\circ}_{n}(\mathcal{C})$ is $(n+1)$-coskeletal we can extend $F \circ \sigma_0$ to an $(n+2)$-simplex $\overline{\sigma}$ of $\mathrm{cosk}^{\circ}_{n}(\mathcal{C})$. If condition (2) is satisfied, then the lifting problem

$$\begin{tikzcd}
\partial \Delta^{n+2} \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^{n+2} \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & \mathrm{cosk}^{\circ}_{n}(\mathcal{C})
\end{tikzcd}$$

admits a solution; in particular, σ_0 can be extended to an $(n+2)$-simplex of $\mathcal{C}$. □

Example 3.5.4.25. Let A_* be a nonnegatively graded chain complex of abelian groups and let $X = \mathrm{K}(A_*)$ denote the associated simplicial abelian group. For every integer $n \geq -1$, the weak n-coskeleton $\mathrm{cosk}^{\circ}_{n}(X)$ inherits the structure of a simplicial abelian group. It follows from Theorem 2.5.6.1 that $\mathrm{cosk}^{\circ}_{n}(X)$ can be identified with the Eilenberg-MacLane space $\mathrm{K}(A'_*)$, for some nonnegatively graded chain complex A'_*. Here A'_* is universal among chain complexes which satisfy the criterion of Exercise 3.5.4.8 and are equipped with a chain map $A_* \rightarrow A'_*$. More concretely, we can identify A'_* with the chain complex 053G

$$\cdots \rightarrow 0 \rightarrow A_{n+1}/Z_{n+1} \hookrightarrow A_n \xrightarrow{\partial} A_{n-1} \rightarrow \cdots ,$$

where $Z_{n+1} \subseteq A_{n+1}$ denotes the subgroup of $(n+1)$-cycles of A_*.

Proposition 3.5.4.26. *Let* X *be a Kan complex. Then, for every integer* n*, the tautological map* $u : X \rightarrow \mathrm{cosk}^{\circ}_{n}(X)$ *is a Kan fibration.* 053H

Warning 3.5.4.27. Let X be a Kan complex. For every integer n, we have a commutative diagram 053J

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dr, "v"'] & & \mathrm{cosk}^{\circ}_{n}(X) \\
& \mathrm{cosk}_{n+1}(X) \arrow[ur, "q"'] &
\end{tikzcd}$$

where u is a Kan fibration (Proposition 3.5.4.26) and q is a trivial Kan fibration (Proposition 3.5.4.22). Beware that v is usually not a Kan fibration.

Proof of Proposition 3.5.4.26. Fix a pair of integers $0 \leq i \leq m$ with $m > 0$; we wish to show that every lifting problem

053K
$$\begin{tikzcd} \Lambda^m_i \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "u"] \\ \Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \operatorname{cosk}^{\circ}_n(X) \end{tikzcd} \qquad (3.34)$$

admits a solution. We consider two cases:

- If $m \leq n+1$, then we can choose an m-simplex σ of X satisfying $u(\sigma) = \overline{\sigma}$. Since u is bijective on simplices of dimension $\leq n$, the commutativity of the diagram (3.34) guarantees that $\sigma|_{\Lambda^m_i} = \sigma_0$.

- If $m \geq n+2$, then our assumption that X is a Kan complex guarantees that σ_0 can be extended to an m-simplex σ of X. The commutativity of the diagram (3.34) then guarantees that $u(\sigma)$ and $\overline{\sigma}$ have the same restriction to the horn $\Lambda^m_i \subset \Delta^m$. In particular, they have the same restriction to the n-skeleton of Δ^m, so $u(\sigma) = \overline{\sigma}$.

In either case, it follows that σ is a solution to the lifting problem (3.34). □

3.5.5 Higher Groupoids

053L Recall that a *groupoid* is a category $\mathcal{G}$ in which every morphism is an isomorphism (Definition 1.3.5.1). By virtue of Propositions 1.3.3.1 and 1.3.5.2, the construction $\mathcal{G} \mapsto \mathrm{N}_\bullet(\mathcal{G})$ determines a fully faithful functor from the category of groupoids to the category of Kan complexes. Consequently, little information is lost by identifying $\mathcal{G}$ with $\mathrm{N}_\bullet(\mathcal{G})$, and thereby viewing groupoids as special kinds of Kan complexes. In this section, we exploit this perspective to introduce a notion of *n-groupoid* for $n \geq 0$.

053M **Definition 3.5.5.1.** Let n be a nonnegative integer. An *n-groupoid* is a Kan complex X which satisfies the following condition: for every pair of integers $0 \leq i \leq m$ with $m > n$, the restriction map $\operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \operatorname{Hom}_{\mathrm{Set}_\Delta}(\Lambda^m_i, X)$ is a bijection.

053N **Remark 3.5.5.2.** In the situation of Definition 3.5.5.1, the assumption that X is a Kan complex already guarantees that the restriction map $\operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \operatorname{Hom}_{\mathrm{Set}_\Delta}(\Lambda^m_i, X)$ is surjective. Consequently, X is an n-groupoid if and only if, whenever σ and τ are simplices of X having dimension $m > n$ which satisfy $\sigma|_{\Lambda^m_i} = \tau|_{\Lambda^m_i}$ for some $0 \leq i \leq m$, we have $\sigma = \tau$.

Remark 3.5.5.3 (Monotonicity)**.** Let $m \geq 0$ and let X be an m-groupoid. Then X is also an n-groupoid for any $n \geq m$. `053P`

Warning 3.5.5.4. We have now given two *a priori* different definitions for the notion of 2-groupoid: `053Q`

- According to Definition 2.2.8.24, a 2-groupoid is a 2-category $\mathcal{C}$ such that every 1-morphism of $\mathcal{C}$ is an isomorphism and every 2-morphism of $\mathcal{C}$ is an isomorphism.
- According to Definition 3.5.5.1, a 2-groupoid is a simplicial set X satisfying certain extension conditions.

We will show later that definitions are compatible: a simplicial set X is a 2-groupoid in the sense of Definition 3.5.5.1 if and only if it is isomorphic to the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$, where $\mathcal{C}$ is a 2-groupoid in the sense of Definition 2.2.8.24 (in this case, the 2-category $\mathcal{C}$ is uniquely determined up to non-strict isomorphism; see Theorem 2.3.4.1). See Proposition [?].

Remark 3.5.5.5. Let X be a simplicial set and let $n \geq 0$ be an integer. Then X is an n-groupoid if and only if every connected component of X is an n-groupoid. `053R`

Remark 3.5.5.6. Let $\{X_j\}_{j \in \mathcal{J}}$ be a diagram of simplicial sets having limit $X = \varprojlim_{j \in \mathcal{J}} X_j$ and let n be a nonnegative integer. If each X_j is an n-groupoid and X is a Kan complex, then X is also an n-groupoid. In particular, any product of n-groupoids is an n-groupoid (see Example 1.2.5.3). `053S`

Proposition 3.5.5.7. *A simplicial set X is a 0-groupoid if and only if it is discrete: that is, if and only if it is isomorphic to a constant simplicial set $\underline{S}$ for some set S.* `053T`

Proof. By virtue of Remark 3.5.5.5, we may assume without loss of generality that X is connected. Assume that X is a 0-groupoid; we wish to show that the projection map $X \to \Delta^0$ is an isomorphism (the converse follows immediately from the definition). To prove this, it will suffice to show that for every pair of m-simplices $\sigma, \tau : \Delta^m \to X$, we have $\sigma = \tau$. Our proof proceeds by induction on m. If $m > 0$, then our inductive hypothesis guarantees that $\sigma|_{\Lambda^m_0} = \tau|_{\Lambda^m_0}$, so that $\sigma = \tau$ by virtue of our assumption that X is a 0-groupoid. It will therefore suffice to treat the case $m = 0$, so that σ and τ can be identified with vertices $x, y \in X$. Since X is a connected Kan complex, there exists an edge $e : x \to y$ with source x and target y (Proposition 1.2.5.10). Our assumption that X is a 0-groupoid then guarantees that $e = \mathrm{id}_x$, so that $x = y$ as desired. □

Proposition 3.5.5.8. *Let X be a simplicial set. The following conditions are equivalent:* `053U`

(1) *There exists a groupoid $\mathcal{G}$ and an isomorphism of simplicial sets $X \xrightarrow{\sim} \mathrm{N}_{\bullet}(\mathcal{G})$.*

(2) *The simplicial set X is a 1-groupoid (in the sense of Definition 3.5.5.1).*

(3) *The simplicial set X is a Kan complex, and the tautological map $X \to \mathrm{N}_\bullet(\pi_{\leq 1}(X))$ is an isomorphism.*

Proof. We first show that (1) implies (2). For every groupoid $\mathcal{G}$, Proposition 1.3.5.2 guarantees that the simplicial set $\mathrm{N}_\bullet(\mathcal{G})$ is a Kan complex. To show that $\mathrm{N}_\bullet(\mathcal{G})$ is a 1-groupoid, we must prove that if $\sigma, \tau : \Delta^m \to \mathrm{N}_\bullet(\mathcal{G})$ are m-simplices for $m \geq 2$ which have the same restriction to some horn $\Lambda^m_i \subset \Delta^m$, then $\sigma = \tau$. For $m > 2$, this is immediate (since Λ^m_i contains the 1-skeleton of Δ^m). In the case $m = 2$, we can identify m-simplices of $\mathrm{N}_\bullet(\mathcal{G})$ with commutative diagrams

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

in the groupoid $\mathcal{G}$. The desired result then follows from the observation that any two of the morphisms f, g, and h determine the third.

The implication (3) ⇒ (1) is immediate. We will complete the proof by showing that (2) implies (3). Assume that X is a 1-groupoid and let $\mathcal{G} = \pi_{\leq 1}(X)$ be its fundamental groupoid. We wish to show that the tautological map $u : X \to \mathrm{N}_\bullet(\mathcal{G})$ is an isomorphism: that is, it is bijective on m-simplices for $m \geq 0$. The proof proceeds by induction on m. The case $m = 0$ is immediate from the definitions. For $m \geq 2$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \arrow[r, "u\circ"] \arrow[d] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \mathrm{N}_\bullet(\mathcal{G})) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^m_0, X) \arrow[r, "u\circ"] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^m_0, \mathrm{N}_\bullet(\mathcal{G}))
\end{tikzcd}$$

where the vertical maps are bijective (since X and $\mathrm{N}_\bullet(\mathcal{G})$ are 1-groupoids) and the bottom horizontal map is bijective (by virtue of our inductive hypothesis); it follows that the upper horizontal map is bijective as well. It will therefore suffice to treat the case $m = 1$. Let $e, e' : x \to y$ be edges of the simplicial set X having the same source and target; we wish to show that if the homotopy classes $[e]$ and $[e']$ coincide (as morphisms in the category $\mathcal{G} = \pi_{\leq 1}(X)$), then $e = e'$. Let σ be a 2-simplex of X which is a homotopy from e to e':

that is, a 2-simplex whose boundary is depicted in the diagram

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "e"] \arrow[rr, "e'"] & & y \end{tikzcd}$$

(see Definition 1.4.3.1). Let τ be the right-degenerate 2-simplex $s^1_1(e)$. Then σ and τ have the same restriction to the horn $\Lambda^2_1 \subset \Delta^2$. Invoking our assumption that X is a 1-groupoid, we conclude that $\sigma = \tau$. In particular, we have $e' = d^2_1(\sigma) = d^2_1(\tau) = e$. □

Proposition 3.5.5.9. *Let n be a nonnegative integer, let A_* be a chain complex of abelian groups, and let $X = \mathrm{K}(A_*)$ denote the associated Eilenberg-MacLane space (Construction 2.5.6.3). Then X is an n-groupoid if and only if the abelian groups A_m vanish for $m > n$.* 053V

Proof. Fix a pair of integers $0 \leq i \leq m$ with $m > n$. Let $\sigma : \Delta^m \to \Delta^m$ be the identity map, which we identify with its image in the normalized chain complex $\mathrm{N}_*(\Delta^m; \mathbf{Z})$. Then the chain complex $\mathrm{N}_*(\Delta^m; \mathbf{Z})$ splits as a direct sum of $\mathrm{N}_*(\Lambda^m_i; \mathbf{Z})$ with the subcomplex C_* spanned by the σ and $\partial(\sigma)$. We therefore obtain a canonical bijection

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) &\simeq \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(\Delta^m; \mathbf{Z}), A_*) \\ &\simeq \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(\Lambda^m_i; \mathbf{Z}), A_*) \times \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(C_*, A_*) \\ &\simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^m_i, X) \times A_m. \end{aligned}$$

It follows that the restriction map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^m_i, X)$ is a bijection if and only if the abelian group A_m vanishes. The desired result follows by allowing the integers $0 \leq i \leq m$ to vary. □

Proposition 3.5.5.10. *Let n be a nonnegative integer and let X be a Kan complex. Then:* 053W

(1) *If X is weakly $(n-1)$-coskeletal, then it is an n-groupoid.*

(2) *If X is an n-groupoid, then it is weakly n-coskeletal.*

Proof. We first prove (1). Assume that X is weakly $(n-1)$-coskeletal. Suppose we are given an integer $m > n$ and a pair of m-simplices $\sigma, \tau : \Delta^m \to X$ which satisfy $\sigma|_{\Lambda^m_i} = \tau|_{\Lambda^m_i}$ for some $0 \leq i \leq m$; we wish to show that $\sigma = \tau$. Note that $d^m_i(\sigma)$ and $d^m_i(\tau)$ are $(m-1)$-simplices of X which coincide on the boundary $\partial \Delta^{m-1}$. Since X is weakly $(n-1)$-coskeletal, it follows that $d^m_i(\sigma) = d^m_i(\tau)$, so that σ and τ coincide on the boundary $\partial \Delta^m$. Invoking our assumption that X is weakly $(n-1)$-coskeletal again, we conclude that $\sigma = \tau$.

We now prove (2). Suppose that X is an n-groupoid; we wish to show that it is weakly n-coskeletal. By virtue of Exercise 3.5.4.11, it will suffice to show that for every integer $m > n$, the restriction map

$$\theta : \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^m, X)$$

is injective. This is clear: our assumption that X is an n-groupoid guarantees that the composition of θ with the restriction map $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^m, X) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Lambda^m_0, X)$ is a bijection. □

053X **Corollary 3.5.5.11.** *Let n be a nonnegative integer and let X be a Kan complex. If X is $(n-1)$-coskeletal, then it is an n-groupoid. if X is an n-groupoid, then it is $(n+1)$-coskeletal.*

Proof. Combine Proposition 3.5.5.10 with Remark 3.5.4.2. □

053Y **Proposition 3.5.5.12.** *Let $n \geq 0$ be an integer and let X be a Kan complex which is weakly n-coskeletal. Then X is an n-groupoid if and only if it satisfies the following condition:*

- $(*)$ *Let $\sigma_0, \sigma_1 : \Delta^n \to X$ be n-simplices which are homotopic relative to $\partial \Delta^n$ (Definition 3.2.1.1) . Then $\sigma_0 = \sigma_1$.*

Proof. Assume first that X is an n-groupoid and let $\sigma_0, \sigma_1 : \Delta^n \to X$ be n-simplices of X which are homotopic relative to $\partial \Delta^n$. Since X is a Kan complex, there exists a homotopy $h : \Delta^1 \times \Delta^n \to X$ from σ_0 to σ_1 which is constant along $\partial \Delta^n$ (Proposition 3.2.1.4). For $0 \leq i \leq n$, let $\alpha_i : [n+1] \to [1] \times [n]$ denote the nondecreasing function given by the formula

$$\alpha_i(j) = \begin{cases} (0, j) & \text{if } j \leq i \\ (1, j-1) & \text{if } j > i, \end{cases}$$

and let τ_i denote the $(n+1)$-simplex of X given by the composition

$$\Delta^{n+1} \xrightarrow{\alpha_i} \Delta^1 \times \Delta^n \xrightarrow{h} X.$$

Let $\rho_i, \rho'_i : \Delta^n \to X$ be the n-simplices of X given by $\rho_i = d^{n+1}_i(\tau_i)$ and $\rho'_i = d^{n+1}_{i+1}(\tau_i)$; by construction, we have

$$\sigma_0 = \rho'_n \qquad \rho_n = \rho'_{n-1} \qquad \cdots \qquad \rho_1 = \rho'_0 \qquad \rho_0 = \sigma_1.$$

We will complete the proof by showing that $\rho_i = \rho'_i$ for $0 \leq i \leq n$. Using our assumption that the homotopy h is constant along the boundary $\partial \Delta^n$, we see that the degenerate $(n+1)$-simplex $s^n_i(\rho_i)$ coincides with τ_i on the horn $\Lambda^{n+1}_i \subset \Delta^{n+1}$. Invoking our assumption that X is an n-groupoid, we conclude that $\tau_i = s^n_i(\rho_i)$. Applying the face operator d^{n+1}_{i+1}, we obtain $\rho_i = \rho'_i$.

We now prove the converse. Assume that X satisfies condition $(*)$; we wish to show that the Kan complex X is an n-groupoid. Fix a pair of integers $0 \leq i \leq m$ with $m > n$ and a pair of m-simplices $\tau_0, \tau_1 : \Delta^m \to X$ which coincide on the horn $\Lambda^m_i \subset \Delta^m$; we wish to show that $\tau_0 = \tau_1$. Since X is weakly n-coskeletal, it will suffice to prove that τ_0 and τ_1 coincide on the boundary $\partial \Delta^m$: that is, to show that the $(m-1)$-simplices $\sigma_0 = d^m_i(\tau_0)$ and $\sigma_1 = d^m_i(\tau_1)$ coincide. Note that σ_0 and σ_1 have the same restriction to the boundary $\partial \Delta^{m-1}$. Consequently, if $m \geq n+2$, the desired result follows from our assumption that X is weakly n-coskeletal. We may therefore assume that $m = n+1$. By virtue of $(*)$, it will suffice to show that the $(m-1)$-simplices σ_0 and σ_1 are homotopic relative to $\partial \Delta^{m-1}$. In fact, we will prove a stronger claim: the m-simplices τ_0 and τ_1 are homotopic relative to the horn $\Lambda^m_i \subset \Delta^m$. This follows from the observation that the restriction map $\operatorname{Fun}(\Delta^m, X) \to \operatorname{Fun}(\Lambda^m_i, X)$ is a trivial Kan fibration; see Corollary 3.1.3.6. □

Corollary 3.5.5.13 (Exponentation for n-Groupoids)**.** *Let $n \geq 0$ be an integer and let X be an n-groupoid. Then, for any simplicial set K, the simplicial set* $\operatorname{Fun}(K, X)$ *is also an n-groupoid.* 053Z

Proof. It follows from Corollary 3.1.3.4 that $\operatorname{Fun}(K, X)$ is a Kan complex. Since X is weakly n-coskeletal (Proposition 3.5.5.10), it follows that $\operatorname{Fun}(K, X)$ is also weakly n-coskeletal (Corollary 3.5.4.13). We will complete the proof by showing that $\operatorname{Fun}(K, X)$ satisfies condition $(*)$ of Proposition 3.5.5.12. Suppose we are given a pair of n-simplices $\sigma_0, \sigma_1 : \Delta^n \to \operatorname{Fun}(K, X)$ which are homotopic relative to $\partial \Delta^n$; we wish to show that $\sigma_0 = \sigma_1$. Let us identify σ_0 and σ_1 with morphisms $f_0, f_1 : \Delta^n \times K \to X$. Since X is weakly n-coskeletal, it will suffice to show that f_0 and f_1 coincide on m-simplices $\tau = (\tau', \tau'')$ of $\Delta^n \times K$ for $m \leq n$. If τ' factors through the boundary $\partial \Delta^n$, this follows immediately from the equality $\sigma_0|_{\partial \Delta^n} = \sigma_1|_{\partial \Delta^n}$. We may therefore assume without loss of generality that $m = n$ and that $\tau' : \Delta^m \to \Delta^n$ is the identity map. In this case, our assumption that σ_0 and σ_1 are homotopic relative to $\partial \Delta^n$ guarantees that $f_0(\tau)$ and $f_1(\tau)$ are homotopic relative to $\partial \Delta^n$, so that $f_0(\tau) = f_1(\tau)$ by virtue of Proposition 3.5.5.12. □

Corollary 3.5.5.14. *Let n be a nonnegative integer and let $f : X \to Y$ be a morphism of Kan complexes. Assume that Y is an n-groupoid and that f is bijective on m-simplices for $m < n$. The following conditions are equivalent:* 0540

(1) *The morphism f is a Kan fibration.*

(2) *The morphism f is surjective on n-simplices.*

(3) *The morphism f is n-connective.*

Proof. We first show that (1) implies (2). Here we may assume that $n > 0$ (otherwise, the result is a special case of Proposition 3.5.1.22). Let $\overline{\tau}$ be an n-simplex of Y, and set

$\overline{\tau}_0 = \overline{\tau}|_{\Lambda^n_0}$. Since f is bijective on m-simplices for $m < n$, we can lift $\overline{\tau}_0$ to a morphism $\tau_0 : \Lambda^n_0 \to X$. If f is a Kan fibration, then the lifting problem

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\tau_0"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r, "\overline{\tau}"] \arrow[ur, dashed, "\tau"] & Y
\end{tikzcd}$$

admits a solution, given by an n-simplex τ of X satisfying $f(\tau) = \overline{\tau}$.

The implication (2) ⇒ (3) is a special case of Corollary 3.5.2.2. We will complete the proof by showing that (3) implies (1). Assume that f is n-connective and fix a pair of integers $0 \leq i \leq m$ with $m > 0$; we wish to show that every lifting problem

$$\begin{tikzcd}
\Lambda^m_i \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & Y
\end{tikzcd}$$

admits a solution. We consider three cases:

- Suppose that $m < n$. In this case, our assumption that f is bijective on m-simplices guarantees that there is a unique m-simplex σ of X satisfying $f(\sigma) = \overline{\sigma}$. By construction, we have $(f \circ \sigma)|_{\Lambda^m_i} = \overline{\sigma}|_{\Lambda^m_i} = f \circ \sigma_0$. Since f is bijective on simplices of dimension $< m$, it follows that $\sigma_0 = \sigma|_{\Lambda^m_i}$.

- Suppose that $m > n$. In this case, our assumption that X is a Kan complex guarantees that we can extend σ_0 to an m-simplex σ of X. By construction, we have
$$(f \circ \sigma)|_{\Lambda^m_i} = f \circ \sigma_0 = \overline{\sigma}|_{\Lambda^m_i}.$$
Since Y is an n-groupoid, it follows that $f \circ \sigma = \overline{\sigma}$.

- Suppose that $m = n$. Since f is bijective on $(n-1)$-simplices, the morphism σ_0 admits a unique extension $\sigma_1 : \partial\Delta^n \to X$ satisfying $f \circ \sigma_1 = \overline{\sigma}|_{\partial\Delta^n}$. The morphism f factors as a composition
$$X \xrightarrow{i} X \times_{\operatorname{Fun}(\{0\},Y)} \operatorname{Fun}(\Delta^1, Y) \xrightarrow{q} Y,$$
where i is a homotopy equivalence and q is a Kan fibration (see Example 3.1.7.10). Since f is n-connective, the Kan fibration q is also n-connective (Proposition 3.5.1.26). Applying Proposition 3.5.2.1, we conclude that there is an n-simplex σ of X satisfying $\sigma_1 = \sigma|_{\partial\Delta^n}$ and a homotopy from $f(\sigma)$ to $\overline{\sigma}$ which is constant when restricted to $\partial\Delta^n$. Since Y is an n-groupoid, Proposition 3.5.5.12 guarantees that $f(\sigma) = \overline{\sigma}$.

□

Corollary 3.5.5.15. *Let n be a nonnegative integer and let $f : X \to Y$ be a homotopy equivalence of n-groupoids. If f is bijective on m-simplices for $m < n$, then f is an isomorphism.* 0541

Proof. It follows from Corollary 3.5.5.14 that f is a Kan fibration. Applying Proposition 3.2.7.2, we deduce that f is a trivial Kan fibration. In particular, f admits a section $g : Y \hookrightarrow X$. To complete the proof, it will suffice to show that g is an epimorphism of simplicial sets. This follows from Corollary 3.5.5.14, since g is also bijective on m-simplices for $m < n$. □

Proposition 3.5.5.16. *Let X be a Kan complex and let $n \geq 0$ be an integer. The following conditions are equivalent:* 0542

(1) *The Kan complex X is isomorphic to an Eilenberg-MacLane space $\mathrm{K}(G, n)$. Here G is a set if $n = 0$, a group if $n = 1$, and an abelian group if $n \geq 2$ (see Construction 2.5.6.9).*

(2) *The Kan complex X is an n-groupoid having a single m-simplex for each $m < n$.*

Proof. If $n = 0$, the desired result follows from Proposition 3.5.5.7. If $n = 1$, then X is an n-groupoid if and only if it is isomorphic to the nerve $\mathrm{N}_{\bullet}(\mathcal{C})$, where $\mathcal{C}$ is a groupoid (Proposition 3.5.5.8). In this case, the assumption that X has a single vertex is equivalent to the requirement that the category $\mathcal{C}$ contains a single object C, in which case we can identify $\mathrm{N}_{\bullet}(\mathcal{C})$ with the classifying simplicial set $\mathrm{K}(G, 1) = B_{\bullet}G$ for $G = \mathrm{Aut}_{\mathcal{C}}(C)$. We may therefore assume that $n \geq 2$. The implication (1) ⇒ (2) follows from Proposition 3.5.5.9. For the converse, assume that X is an n-groupoid having a single m-simplex for each $m < n$. Let x be the unique vertex of X and set $G = \pi_n(X, x)$. For every n-simplex σ of X, the restriction $\sigma|_{\partial \Delta^n}$ is the constant map taking the value x, so the homotopy class $[\sigma]$ can be regarded as an element of the group G. Our assumption that X is an n-groupoid guarantees that the assignment $\sigma \mapsto [\sigma]$ determines a bijection from the collection of n-simplices of X to the group G, which determines an isomorphism f_0 from the n-skeleton of X to the n-skeleton of $\mathrm{K}(G, n)$. Invoking Theorem 3.2.2.10, we see that a morphism $\tau_0 : \partial \Delta^{n+1} \to X$ can be extended to an $(n+1)$-simplex of X if and only if the composite map

$$\partial \Delta^{n+1} \xrightarrow{\tau_0} \mathrm{sk}_n(X) \xrightarrow{f_0} \mathrm{K}(G, n)$$

can be extended to an $(n+1)$-simplex of $\mathrm{K}(G, n)$. Since $\mathrm{K}(G, n)$ is weakly n-coskeletal, it follows that f_0 extends uniquely to a morphism of simplicial sets $f : X \to \mathrm{K}(G, n)$ (Proposition 3.5.4.12) which is surjective on $(n+1)$-simplices. In particular, f exhibits $\mathrm{K}(G, n)$ as a weak n-coskeleton of X (Definition 3.5.4.14). Since X is weakly n-coskeletal (Proposition 3.5.5.10), we conclude that f is an isomorphism. □

3.5.6 Higher Fundamental Groupoids

0543 Let X be a Kan complex. Recall that the *fundamental groupoid* $\pi_{\leq 1}(X)$ is a category whose objects are the vertices of X, and whose morphisms are given by homotopy classes of paths (Definition 1.4.6.12). We now consider a higher-dimensional version of this construction.

0544 **Definition 3.5.6.1.** Let X be a Kan complex and let $n \geq 0$ be an integer. We say that a morphism of Kan complexes $f : X \to Y$ *exhibits Y as a fundamental n-groupoid of X* if the following conditions are satisfied:

(a) The simplicial set Y is an n-groupoid (Definition 3.5.5.1).

(b) The morphism f is bijective on m-simplices for $m < n$.

(c) The morphism f is surjective on n-simplices.

(d) If σ and σ' are n-simplices of X satisfying $f(\sigma) = f(\sigma')$, then σ and σ' are homotopic relative to $\partial \Delta^n$ (see Definition 3.2.1.3).

0545 **Remark 3.5.6.2.** Let n be a nonnegative integer and let $f : X \to Y$ be a morphism of Kan complexes which exhibits Y as a fundamental n-groupoid of X. Then f is a Kan fibration (Corollary 3.5.5.14). In particular, since it is surjective on vertices, it is surjective on m-simplices for every integer m (see Remark 3.1.2.8).

0546 **Example 3.5.6.3.** Let X be a Kan complex and let $\pi_{\leq 1}(X)$ denote the fundamental groupoid of X (Definition 1.4.6.12). Then the nerve $\mathrm{N}_\bullet(\pi_{\leq 1}(X))$ is a 1-groupoid (Proposition 3.5.5.8). By construction, the tautological map $u : X \to \mathrm{N}_\bullet(\pi_{\leq 1}(X))$ is bijective on vertices and surjective on edges. Moreover, two edges of X have the same image in $\mathrm{N}_\bullet(\pi_{\leq 1}(X))$ if and only if they are homotopic relative to $\partial \Delta^1$ (see Corollary 1.4.3.7). It follows that u exhibits $\mathrm{N}_\bullet(\pi_{\leq 1}(X))$ as a fundamental 1-groupoid of X.

0547 **Example 3.5.6.4.** Let n be a nonnegative integer and let X be an n-groupoid. Then the identity map $\mathrm{id}_X : X \to X$ exhibits X as a fundamental n-groupoid of itself.

Fundamental n-groupoids can be characterized by a universal mapping property.

0548 **Proposition 3.5.6.5.** *Let n be a nonnegative integer and let $f : X \to Y$ be a morphism of Kan complexes which exhibits Y as a fundamental n-groupoid of X. Then, for every n-groupoid Z, composition with f induces an isomorphism of simplicial sets* $\mathrm{Fun}(Y, Z) \to \mathrm{Fun}(X, Z)$.

Proof. Let K be a simplicial set; we will show that precomposition with f induces a bijection $\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Fun}(Y, Z)) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Fun}(X, Z))$. Replacing Z by the n-groupoid $\mathrm{Fun}(K, Z)$ (Corollary 3.5.5.13), we are reduced to proving that the natural map

$$\theta : \mathrm{Hom}_{\mathrm{Set}_\Delta}(Y, Z) \xrightarrow{\circ f} \mathrm{Hom}_{\mathrm{Set}_\Delta}(X, Z)$$

is a bijection.

Let $h : X \to Z$ be a morphism of Kan complexes; we wish to show that there is a unique morphism $g : Y \to Z$ satisfying $g \circ f = h$. We first claim that there is a unique morphism $g_0 : \mathrm{sk}_n(Y) \to Z$ satisfying $g_0 \circ \mathrm{sk}_n(f) = h|_{\mathrm{sk}_n(X)}$. Our assumption that f exhibits Y as a fundamental n-groupoid of X guarantees that $\mathrm{sk}_n(f)$ is surjective. It will therefore suffice to show that if σ and σ' are m-simplices of X for $m \leq n$ satisfying $f(\sigma) = f(\sigma')$, then $h(\sigma) = h(\sigma')$. If $m < n$, then $\sigma = \sigma'$ and the result is clear. In the case $m = n$, the assumption that $f(\sigma) = f(\sigma')$ guarantees that σ and σ' are homotopic relative to $\partial \Delta^n$, in which case the desired result follows from Proposition 3.5.5.12.

It follows from Remark 3.5.6.2 that every $(n+1)$-simplex τ of Y can be lifted to an $(n+1)$-simplex $\widetilde{\tau}$ of X. In particular, $g_0 \circ \tau|_{\partial \Delta^{n+1}}$ extends to an $(n+1)$-simplex of Z, given by $h(\widetilde{\tau})$. Since Z is weakly n-coskeletal (Proposition 3.5.5.10), Proposition 3.5.4.12 guarantees that g_0 extends uniquely to a morphism $g : Y \to Z$, which automatically satisfies the equation $g \circ f = h$. □

Notation 3.5.6.6. Let X be a Kan complex and let $n \geq 0$ be an integer. We will see later that there exists a morphism of Kan complexes $f : X \to Y$ which exhibits Y as a fundamental n-groupoid of X (Theorem 3.5.6.17). It follows from Proposition 3.5.6.5 that Y is unique up to (canonical) isomorphism and depends functorially on X. To emphasize this dependence, we will typically denote the simplicial set Y by $\pi_{\leq n}(X)$, and refer to it as *the* fundamental n-groupoid of X. 0549

Warning 3.5.6.7. Let X be a Kan complex. We have now assigned two different meanings to the notation $\pi_{\leq 1}(X)$: 054A

- The fundamental groupoid of X (Definition 1.4.6.12), which is a category.
- The fundamental 1-groupoid of X (Notation 3.5.6.6), which is a simplicial set.

However, the danger of confusion is slight: by virtue of Example 3.5.6.3, the fundamental 1-groupoid of X is isomorphic to the nerve of the fundamental groupoid of X.

Example 3.5.6.8. Let X be a Kan complex, let $\pi_0(X)$ be the set of connected components of X. Then the fundamental 0-groupoid $\pi_{\leq 0}(X)$ can be identified with the constant simplicial set $\underline{\pi_0(X)}$. More precisely, the tautological map $X \to \underline{\pi_0(X)}$ exhibits $\underline{\pi_0(X)}$ as a fundamental 0-groupoid of X (see Proposition 3.5.5.7). 054B

Example 3.5.6.9. Let $n \geq 0$ be an integer and let $X = \mathrm{K}(A_*)$ be the Eilenberg-MacLane space associated to a chain complex of abelian groups 054C

$$\cdots \to A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \xrightarrow{\partial} A_{n-2} \to \cdots$$

(see Construction 2.5.6.3). Let $B_n \subseteq A_n$ denote the image of the differential $\partial : A_{n+1} \to A_n$, and let A'_* denote the chain complex

$$\cdots \to 0 \to A_n/B_n \to A_{n-1} \xrightarrow{\partial} A_{n-2} \to \cdots$$

Since A'_* is concentrated in degrees $\leq n$, the Eilenberg-MacLane space $\mathrm{K}(A'_*)$ is n-groupoid, which we can identify with the fundamental n-groupoid $\pi_{\leq n}(X)$. More precisely, the quotient map $A_* \twoheadrightarrow A'_*$ is an isomorphism in degrees $< n$ and induces an isomorphism on homology in degrees $\leq n$, so the induced map of Kan complexes $\mathrm{K}(A_*) \twoheadrightarrow \mathrm{K}(A'_*)$ exhibits $\mathrm{K}(A'_*)$ as a fundamental n-groupoid of X.

Let X be a Kan complex. Our goal for the remainder of this section is to give an explicit construction of a fundamental n-groupoid of X, for each $n \geq 0$.

054D **Construction 3.5.6.10.** Let n be a nonnegative integer, let X be a Kan complex. and let $\operatorname{cosk}^{\circ}_n(X)$ denote the weak n-coskeleton of X (Notation 3.5.4.19). For every integer $m \geq 0$, we will identify m-simplices of $\operatorname{cosk}^{\circ}_n(X)$ with morphisms $\sigma : \operatorname{sk}_n(\Delta^m) \to X$ which can be extended to the $(n+1)$-skeleton of Δ^m (see Remark 3.5.4.21). Given two such morphisms $\sigma, \sigma' : \operatorname{sk}_n(\Delta^m) \to X$, we write $\sigma \sim_m \sigma'$ if σ and σ' are homotopic relative to $\operatorname{sk}_{n-1}(\Delta^m)$. The construction

$$([m] \in \boldsymbol{\Delta}^{\mathrm{op}}) \mapsto \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(\Delta^m, \operatorname{cosk}^{\circ}_n(X))/\sim_m$$

determines a simplicial set, which we will denote by $\pi_{\leq n}(X)$. By construction, we have an epimorphism of simplicial sets $q : \operatorname{cosk}^{\circ}_n(X) \twoheadrightarrow \pi_{\leq n}(X)$, which determines a comparison map $X \to \pi_{\leq n}(X)$.

054E **Remark 3.5.6.11.** In the situation of Construction 3.5.6.10, the relation $\sigma \sim_m \sigma'$ implies that $\sigma = \sigma'$ whenever $m < n$. It follows that the tautological map $v : X \to \pi_{\leq n}(X)$ is bijective on simplices of dimension $< n$, and surjective on simplices of dimension n.

054F **Proposition 3.5.6.12.** *Let X be a Kan complex and let n be a nonnegative integer. Then, for every simplicial set A, the comparison map*

$$\theta : \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(A, \operatorname{cosk}^{\circ}_n(X)) \to \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(A, \pi_{\leq n}(X))$$

is surjective. Moreover, if $f_0, f_1 : A \to \operatorname{cosk}^{\circ}_n(X)$ are morphisms of simplicial sets which correspond to maps $u_0, u_1 : \operatorname{sk}_n(A) \to X$, then $\theta(f_0) = \theta(f_1)$ if and only if u_0 and u_1 are homotopic relative to $\operatorname{sk}_{n-1}(A)$.

Proof. We first prove that θ is a surjection. Fix a morphism $g : A \to \pi_{\leq n}(X)$. Using Remark 3.5.6.11 (and Proposition 1.1.4.12), we see that $g|_{\operatorname{sk}_n(A)}$ can be lifted to a morphism of simplicial sets $u : \operatorname{sk}_n(A) \to X$. We will show that u can be extended to the $(n+1)$-skeleton of A (and is therefore classified by a morphism $f : A \to \operatorname{cosk}^{\circ}_n(X)$ satisfying

$\theta(f) = g$; see Remark 3.5.4.21). By virtue of Proposition 1.1.4.12 and Variant 3.2.4.12, this is equivalent to the assertion that for every $(n+1)$-simplex σ of A having restriction $\sigma_0 = \sigma|_{\partial\Delta^{n+1}}$, the composition $(g \circ \sigma_0) : \partial\Delta^{n+1} \to X$ is nullhomotopic. Choose a lift of $g(\sigma)$ to an $(n+1)$-simplex of $\operatorname{cosk}^{\circ}_n(X)$, which we identify with a nullhomotopic map $\tau_0 : \partial\Delta^{n+1} \to X$. By construction, $g \circ \sigma_0$ and τ_0 coincide after composing with the comparison map $v : X \to \pi_{\leq n}(X)$. Using Proposition 1.1.4.12 again, we see that $g \circ \sigma_0$ and τ_0 are homotopic relative to the $\operatorname{sk}_{n-1}(\Delta^{n+1})$, so that $g \circ \sigma_0$ is also nullhomotopic.

Now suppose that we are given a pair of morphisms $f_0, f_1 : A \to \operatorname{cosk}^{\circ}_n(X)$ satisfying $\theta(f_0) = \theta(f_1)$. We wish to show that the associated maps $u_0, u_1 : \operatorname{sk}_n(A) \to X$ are homotopic relative to $\operatorname{sk}_{n-1}(A)$ (the converse is immediate from the definitions). Using Remark 3.5.6.11, we deduce that u_0 and u_1 coincide on $\operatorname{sk}_{n-1}(A)$. By virtue of Proposition 1.1.4.12, we are reduced to showing that for every nondegenerate n-simplex σ of A, the compositions $u_0 \circ \sigma$ and $u_1 \circ \sigma$ are homotopic relative to $\partial\Delta^n$. This follows from our assumption that the maps $\theta(f_0), \theta(f_1) : A \to \pi_{\leq n}(X)$ coincide on the simplex σ. □

Remark 3.5.6.13. Let X be a Kan complex, let $n \geq 0$ be an integer, and let A be a simplicial set. Stated more informally, Proposition 3.5.6.12 asserts that $\operatorname{Hom}_{\operatorname{Set}_\Delta}(A, \pi_{\leq n}(X))$ can be viewed as a subquotient of the set $\operatorname{Hom}_{\operatorname{Set}_\Delta}(\operatorname{sk}_n(A), X)$: 054G

- A morphism $u : \operatorname{sk}_n(A) \to X$ determines a map from A to $\pi_{\leq n}(X)$ if and only if u can be extended to the $(n+1)$-skeleton of A.
- Two such morphisms $u_0, u_1 : \operatorname{sk}_n(A) \to X$ determine the same map from A to $\pi_{\leq n}(X)$ if and only if they are homotopic relative to the $(n-1)$-skeleton of A.

Corollary 3.5.6.14. *Let X be a Kan complex and let $n \geq 0$ be an integer. Then the comparison map $q : \operatorname{cosk}^{\circ}_n(X) \twoheadrightarrow \pi_{\leq n}(X)$ of Construction 3.5.6.10 is a trivial Kan fibration.* 054H

Proof. Fix an integer $m \geq 0$; we wish to show that every lifting problem

054J
$$\begin{tikzcd}
\partial\Delta^m \arrow[r, "\sigma_0"] \arrow[d] & \operatorname{cosk}^{\circ}_n(X) \arrow[d, "q"] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & \pi_{\leq n}(X)
\end{tikzcd} \tag{3.35}$$

admits a solution.

Let σ be any m-simplex of $\operatorname{cosk}^{\circ}_n(X)$ satisfying $q(\sigma) = \overline{\sigma}$. By virtue of Remark 3.5.6.11, the commutativity of the diagram (3.35) guarantees that σ_0 and σ coincide on the $(n-1)$-skeleton of $\partial\Delta^m$. Consequently, if $m \leq n$, then σ is a solution to the the lifting problem (3.35). We will therefore assume that $m > n$. In this case, the boundary $\partial\Delta^m$ contains the

n-skeleton of Δ^m. It will therefore suffice to show that σ_0 can be extended to an m-simplex σ' of $\operatorname{cosk}_n^\circ(X)$: the commutativity of the diagram (3.35) guarantees that any such extension satisfies the identity $q(\sigma') = \overline{\sigma}$ (Proposition 3.5.6.12). If $m \geq n+2$, then the existence of σ' is automatic (since $\operatorname{cosk}_n^\circ(X)$ is $(n+1)$-coskeletal). It will therefore suffice to treat the case $m = n+1$. In this case, we can identify σ_0 with a morphism $\tau_0 : \partial\Delta^{n+1} \to X$, and we wish to show that τ_0 is nullhomotopic. Note that $\sigma|_{\partial\Delta^m}$ determines a morphism $\tau_1 : \partial\Delta^{n+1} \to X$. Moreover, the commutativity of the diagram (3.35) guarantees that τ_0 and τ_1 are homotopic relative to $\operatorname{sk}_{n-1}(\Delta^{n+1})$ (Proposition 3.5.6.12). It will therefore suffice to show that τ_1 is nullhomotopic, which follows from the existence of σ. ☐

054K **Corollary 3.5.6.15.** *Let X be a Kan complex, let n be a nonnegative integer, and let $\pi_{\leq n}(X)$ be as in Construction 3.5.6.10. Then the quotient map $\operatorname{cosk}_{n+1}(X) \twoheadrightarrow \pi_{\leq n}(X)$ is a trivial Kan fibration of simplicial sets.*

Proof. Combine Corollary 3.5.6.14 with Proposition 3.5.4.22. ☐

054L **Corollary 3.5.6.16.** *Let X be a Kan complex, let n be a nonnegative integer, and let $\pi_{\leq n}(X)$ be as in Construction 3.5.6.10. Then $\pi_{\leq n}(X)$ is an n-groupoid.*

Proof. By virtue of Proposition 3.5.3.23, the coskeleton $\operatorname{cosk}_{n+1}(X)$ is a Kan complex. Combining Corollary3.5.6.15 with Proposition 1.5.5.11, we conclude that $\pi_{\leq n}(X)$ is a Kan complex. To complete the proof, it will suffice to show that if σ and τ are m-simplices of $\pi_{\leq n}(X)$ for some $m > n$ which satisfy $\sigma|_{\Lambda^m_i} = \tau|_{\Lambda^m_i}$ for some $0 \leq i \leq m$, then $\sigma = \tau$. Choose maps $\widetilde{\sigma}, \widetilde{\tau} : \operatorname{sk}_n(\Delta^m) \to X$ representing σ and τ. Using Proposition 3.5.6.12, we can choose a homotopy from $\widetilde{\sigma}|_{\operatorname{sk}_n(\Lambda^m_i)}$ to $\widetilde{\tau}|_{\operatorname{sk}_n(\Lambda^m_i)}$ which is constant when restricted to the skeleton $\operatorname{sk}_{n-1}(\Lambda^m_i) = \operatorname{sk}_{n-1}(\Delta^m)$. If $m \geq n+2$, then h is also a homotopy from $\widetilde{\sigma}|_{\operatorname{sk}_n(\Delta^m)}$ to $\widetilde{\tau}|_{\operatorname{sk}_n(\Delta^m)}$, so that $\sigma = \tau$ as desired. In the case $m = n+1$, the morphisms $\widetilde{\sigma}$ and $\widetilde{\tau}$ can be extended to morphisms $\overline{\sigma}, \overline{\tau} : \Delta^{n+1} \to X$. Using Corollary 3.1.3.6, we can extend h to a homotopy $\overline{h}$ from $\overline{\sigma}$ to $\overline{\tau}$. Restricting this homotopy to the n-skeleton of Δ^m, we again conclude that $\sigma = \tau$. ☐

054M **Theorem 3.5.6.17.** *Let X be a Kan complex. For every $n \geq 0$, the comparison map $v : X \to \pi_{\leq n}(X)$ of Construction 3.5.6.10 exhibits $\pi_{\leq n}(X)$ as a fundamental n-groupoid of X.*

Proof. It follows from Remark 3.5.6.11 that v is bijective on m-simplices for $m < n$ and surjective on n-simplices. By construction, if σ and σ' are n-simplices of X, then $v(\sigma) = v(\sigma')$ if and only if σ and σ' are homotopic relative to $\partial\Delta^n$. It will therefore suffice to show that $\pi_{\leq n}(X)$ is an n-groupoid, which follows from Corollary 3.5.6.16. ☐

3.5.7 Truncated Kan Complexes

`054N` Let X be a Kan complex. According to Proposition 3.5.2.1, X is n-connective if and only if, for every nonnegative integer $m \leq n$, every map $\partial \Delta^m \to X$ can be extended to an m-simplex of X. We now study a dual version of this condition.

`054P` **Definition 3.5.7.1.** Let n be an integer. We say that a Kan complex X is *n-truncated* if, for every integer $m \geq n+2$, every morphism of simplicial sets $\partial \Delta^m \to X$ can be extended to an m-simplex of X.

`054Q` **Example 3.5.7.2.** Let n be an integer. Recall that a Kan complex X is *$(n+1)$-coskeletal* if, for every integer $m \geq n+2$, every morphism of simplicial sets $\partial \Delta^m \to X$ extends *uniquely* to an m-simplex of X (Definition 3.5.3.1). If this condition is satisfied, then X is n-truncated. In particular, every n-groupoid is n-truncated (Corollary 3.5.5.11). See Proposition 3.5.7.15 (or Variant 3.5.7.16) for a partial converse.

`054R` **Example 3.5.7.3.** For $n \leq -2$, a Kan complex X is n-truncated if and only if it is contractible (see Theorem 3.2.4.3).

`054S` **Example 3.5.7.4.** A Kan complex X is (-1)-truncated if and only if it is either empty or contractible.

`054T` **Example 3.5.7.5.** A Kan complex X is 0-truncated if and only if it satisfies any of the following equivalent conditions:

- Every connected component of X is contractible.
- The projection map $X \twoheadrightarrow \pi_0(X)$ is a trivial Kan fibration of simplicial sets.
- The projection map $X \twoheadrightarrow \pi_0(X)$ is a homotopy equivalence.
- The Kan complex X is homotopy equivalent to a discrete simplicial set.

`054U` **Remark 3.5.7.6.** Let n be an integer. Then the collection of n-truncated Kan complexes is closed under products.

`054V` **Proposition 3.5.7.7.** *Let X be a Kan complex and let $n \geq 0$ be an integer. Then X is n-truncated if and only if it satisfies the following condition for every integer $m > n$:*

$(*_m)$ *For every vertex $x \in X$, the homotopy group $\pi_m(X, x)$ is trivial.*

Proof. Apply Lemma 3.2.4.13. □

`054W` **Remark 3.5.7.8.** Proposition 3.5.7.7 is also true in the case $n = -1$, provided that restate condition $(*_m)$ as follows:

$(*'_m)$ For every vertex $x \in X$, the set $\pi_m(X, x)$ consists of a single element.

Note that $(*'_0)$ is equivalent to the assertion that every pair of vertices of X belong to the same connected component: that is, every morphism $\partial \Delta^1 \to X$ can be extended to a 1-simplex of X.

054X **Remark 3.5.7.9.** In the situation of Proposition 3.5.7.7, it is not necessary to verify the vanishing of the group $\pi_m(X, x)$ for *every* choice of vertex $x \in X$; it is enough to check this at one point from each connected component of X (see Example 3.2.2.18). In particular, if X is connected, then it is enough to check that this condition holds for *any* choice of vertex $x \in X$.

054Y **Example 3.5.7.10.** Let $n \geq -1$ be an integer and let A_* be a chain complex of abelian groups. Then the Eilenberg-MacLane space $\mathrm{K}(A_*)$ is n-truncated if and only if the homology groups $\mathrm{H}_m(A)$ vanish for $m > n$ (see Exercise 3.2.2.22).

054Z **Remark 3.5.7.11.** Let $n \geq 0$ be a nonnegative integer. Then a Kan complex X is n-truncated if and only if every connected component of X is n-truncated.

0550 **Corollary 3.5.7.12** (Homotopy Invariance)**.** *Let $n \geq -2$ and let X and Y be Kan complexes which are homotopy equivalent. Then X is n-truncated if and only if Y is n-truncated.*

Proof. For $n \geq 0$ this follows from the criterion of Proposition 3.5.7.7. The case $n < 0$ follows from Examples 3.5.7.4 and 3.5.7.3. □

0551 **Corollary 3.5.7.13.** *Let n be an integer and let $f : X \to Y$ be a morphism of n-truncated Kan complexes. Then f is a homotopy equivalence if and only if it is $(n+1)$-connective.*

Proof. If $n \leq -2$, then X and Y are contractible and there is nothing to prove. For $n \geq -1$, the desired result follows by combining Proposition 3.5.7.7 (and Remark 3.5.7.8) with Theorem 3.2.7.1. □

0552 **Corollary 3.5.7.14.** *Let $f : X \to Y$ be a morphism of Kan complexes which is n-connective for some integer n. Then the induced map $\mathrm{cosk}_n(f) : \mathrm{cosk}_n(X) \to \mathrm{cosk}_n(Y)$ is a homotopy equivalence.*

Proof. We have a commutative diagram of Kan complexes

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & \mathrm{cosk}_n(X) \arrow[d, "\mathrm{cosk}_n(f)"] \\
Y \arrow[r] & \mathrm{cosk}_n(Y),
\end{tikzcd}$$

where the horizontal maps are n-connective (Remark 3.5.3.22). Applying Corollary 3.5.1.28, we deduce that $\operatorname{cosk}_n(f)$ is n-connective. Since $\operatorname{cosk}_n(X)$ and $\operatorname{cosk}_n(Y)$ are $(n-1)$-truncated (Example 3.5.7.2), Corollary 3.5.7.13 guarantees that $\operatorname{cosk}_n(f)$ is a homotopy equivalence. □

Proposition 3.5.7.15. *Let X be a Kan complex and let n be an integer. The following conditions are equivalent:* 0553

(1) *The Kan complex X is n-truncated.*

(2) *There exists an n-truncated Kan complex Y which is homotopy equivalent to X.*

(3) *There exists an $(n+1)$-coskeletal Kan complex Y which is homotopy equivalent to X.*

(4) *The tautological map $X \to \operatorname{cosk}_{n+1}(X)$ is a homotopy equivalence.*

Proof. The implication (2) ⇒ (1) follows from Corollary 3.5.7.12, the implication (3) ⇒ (2) from Example 3.5.7.2, and the implication (4) ⇒ (3) from the observation that $\operatorname{cosk}_{n+1}(X)$ is a Kan complex (Proposition 3.5.3.23). We will complete the proof by showing that (1) implies (4). Assume that X is n-truncated; we wish to show that the tautological map $u : X \to \operatorname{cosk}_{n+1}(X)$ is a homotopy equivalence. Since $\operatorname{cosk}_{n+1}(X)$ is also n-truncated (Example 3.5.7.2), it will suffice to show that u is $(n+1)$-connective (Corollary 3.5.7.13). This is a special case of Remark 3.5.3.22. □

Variant 3.5.7.16. Let X be a Kan complex and let $n \geq 0$. The following conditions are equivalent: 0554

(1) The Kan complex X is n-truncated.

(2) There exists a homotopy equivalence $X \to Y$, where Y is an n-groupoid.

(3) The tautological map $X \to \pi_{\leq n}(X)$ is a homotopy equivalence.

Proof. The implication (3) ⇒ (2) follows from Corollary 3.5.6.16 and the implication (2) ⇒ (1) follows from Example 3.5.7.2. The implication (3) ⇒ (1) follows from Proposition 3.5.7.15, since the tautological map $X \to \pi_{\leq n}(X)$ factors as a composition

$$X \to \operatorname{cosk}_{n+1}(X) \xrightarrow{q} \pi_{\leq n}(X),$$

where q is a trivial Kan fibration (Corollary 3.5.6.15). □

Example 3.5.7.17. Let X be a Kan complex. The following conditions are equivalent: 0555

- The Kan complex X is 1-truncated.
- For every vertex $x \in X$, the homotopy groups $\pi_n(X, x)$ are trivial for $n \geq 2$.

- There exists a groupoid $\mathcal{G}$ and a homotopy equivalence $X \xrightarrow{\sim} \mathrm{N}_\bullet(\mathcal{G})$.
- The tautological map $X \to \pi_{\leq 1}(X)$ is a homotopy equivalence.

We now give the proof of Proposition 3.5.0.1:

0556 **Corollary 3.5.7.18.** *Let X be a Kan complex and let $n \geq 0$ be an integer. The following conditions are equivalent:*

(1) *There exists a homotopy equivalence of Kan complexes $X \to \mathrm{K}(G, n)$. Here G is a set if $n = 0$, a group if $n = 1$, and an abelian group if $n \geq 2$ (see Construction 2.5.6.9).*

(2) *The Kan complex X is n-truncated and n-connective.*

Proof. We will show that (2) ⇒ (1) (the reverse implication is clear). Assume that X is n-truncated and n-connective. By virtue of Proposition 3.5.2.9 (and Remark 3.5.2.10), we can assume that X has a single m-simplex for each $m < n$. Our assumption that X is n-truncated guarantees that the tautological map $X \to \pi_{\leq n}(X)$ is a homotopy equivalence (Variant 3.5.7.16). It will therefore suffice to show that $\pi_{\leq n}(X)$ is an Eilenberg-MacLane space $\mathrm{K}(G, n)$, which follows from the criterion of Proposition 3.5.5.16. □

0557 **Definition 3.5.7.19.** Let $f : X \to Y$ be a morphism of Kan complexes and let n be an integer. We say that f *exhibits Y as an n-truncation of Y* if Y is n-truncated and f is $(n+1)$-connective. We say that Y *is an n-truncation of X* if there exists a morphism $f : X \to Y$ which exhibits Y as an n-truncation of X.

0558 **Remark 3.5.7.20.** Let $f : X \to Y$ be a morphism of Kan complexes and let $n \geq 0$. Then f exhibits Y as an n-truncation of X if and only if the following conditions are satisfied:

- The morphism f induces a bijection from $\pi_0(X)$ to $\pi_0(Y)$.
- For every vertex $x \in X$ having image $y = f(x)$, the map of homotopy groups $\pi_m(X, x) \to \pi_m(Y, y)$ is a bijection for $0 < m \leq n$.
- For each vertex $y \in Y$ and every integer $m > n$, the homotopy group $\pi_m(Y, y)$ vanishes.

See Proposition 3.5.7.7.

0559 **Remark 3.5.7.21.** Let $f : X \to Y$ be a morphism of Kan complexes. The condition that f exhibits Y as an n-truncation of X depends only on the homotopy class $[f]$ (see Remark 3.5.1.20).

055A **Remark 3.5.7.22.** Let $f : X \to Y$ and be a morphism of Kan complexes, and let $g : Y \to Z$ be a homotopy equivalence of Kan complexes. Then f exhibits Y as an n-truncation of X if and only if $g \circ f$ exhibits Z as an n-truncation of X. See Corollaries 3.5.1.28 and 3.5.7.12.

Example 3.5.7.23. Let X be a Kan complex and let n be an integer. The coskeleton $\operatorname{cosk}_{n+1}(X)$ is a Kan complex (Proposition 3.5.3.23) which is $(n+1)$-coskeletal, and therefore n-truncated (Example 3.5.7.2). Remark 3.5.3.22 guarantees that the tautological map $f : X \to \operatorname{cosk}_{n+1}(X)$ is $(n+1)$-connective. It follows that f exhibits $\operatorname{cosk}_{n+1}(X)$ as an n-truncation of X. 055B

Example 3.5.7.24. Let X be a Kan complex and let n be an integer. Then the tautological map $X \to \operatorname{cosk}^{\circ}_{n}(X)$ exhibits the weak coskeleton $\operatorname{cosk}^{\circ}_{n}(X)$ as an n-truncation of X. This follows from Example 3.5.7.23 and Remark 3.5.7.22, since the quotient map $\operatorname{cosk}_{n+1}(X) \twoheadrightarrow \operatorname{cosk}^{\circ}_{n}(X)$ is a trivial Kan fibration (Proposition 3.5.4.22). Alternatively, it can be deduced directly from Remark 3.5.4.16. 055C

Example 3.5.7.25. Let X be a Kan complex and let n be a nonnegative integer. Then the tautological map $f : X \to \pi_{\leq n}(X)$ exhibits the fundamental n-groupoid $\pi_{\leq n}(X)$ as an n-truncation of X. This follows from Example 3.5.7.23 and Remark 3.5.7.22, since the quotient map $\operatorname{cosk}_{n+1}(X) \twoheadrightarrow \pi_{\leq n}(X)$ is a trivial Kan fibration (Corollary 3.5.6.15). 055D

Example 3.5.7.26. Let $f : X \to Y$ be a morphism of Kan complexes. Then f exhibits Y as a (-1)-truncation of X if and only if one of the following two conditions is satisfied: 055E

- Both X and Y are empty.
- The Kan complex X is nonempty and Y is contractible.

See Example 3.5.7.4.

Example 3.5.7.27. Let $f : X \to Y$ be a morphism of Kan complexes. For $n \leq -2$, f exhibits Y as an n-truncation of X if and only if Y is contractible. See Example 3.5.7.3. 055F

Example 3.5.7.28. Let X be a Kan complex and let n be an integer. Then the projection map $X \to \Delta^0$ exhibits Δ^0 as an n-truncation of X if and only if X is $(n+1)$-connective. 055G

Let X be a Kan complex and let n be an integer. It follows from Example 3.5.7.23 that there exists a morphism of Kan complexes $f : X \to Y$ which exhibits Y as an n-truncation of X. We now show that this property characterizes Y up to homotopy equivalence. This is a consequence of the following universal mapping property:

Proposition 3.5.7.29. *Let n be an integer and let $f : X \to Y$ be a morphism of Kan complexes, where Y is n-truncated. The following conditions are equivalent:* 055H

(1) *The morphism f exhibits Y as an n-truncation of X: that is, f is $(n+1)$-connective.*

(2) *For every n-truncated Kan complex Z, composition with f induces a homotopy equivalence of Kan complexes* $\operatorname{Fun}(Y, Z) \to \operatorname{Fun}(X, Z)$.

(3) *For every n-truncated Kan complex Z, composition with the homotopy class $[f]$ induces a bijection $\pi_0(\mathrm{Fun}(Y, Z)) \to \pi_0(\mathrm{Fun}(X, Z))$.*

Proof. We first show that (1) implies (2). Let Z be an n-truncated simplicial set; we wish to show that composition with f induces a homotopy equivalence $\theta : \mathrm{Fun}(Y, Z) \to \mathrm{Fun}(X, Z)$. By virtue of Proposition 3.5.7.15, we may assume without loss of generality that Z is $(n+1)$-coskeletal. In this case, we can use Proposition 3.5.3.17 to identify θ with the map

$$\mathrm{Fun}(\mathrm{cosk}_{n+1}(Y), Z) \to \mathrm{Fun}(\mathrm{cosk}_{n+1}(X), Z)$$

given by precomposition with $\mathrm{cosk}_{n+1}(f)$. If f is $(n+1)$-connective, then Corollary 3.5.7.14 guarantees that $\mathrm{cosk}_{n+1}(f)$ is a homotopy equivalence, so that θ is also a homotopy equivalence.

The implication (2) ⇒ (3) follows from Remark 3.1.6.5. We will complete the proof by showing that (3) implies (1). Let $u : X \to \mathrm{cosk}_{n+1}(X)$ be the tautological map. Then u exhibits $\mathrm{cosk}_{n+1}(X)$ as an n-truncation of X (Example 3.5.7.23). In particular, $\mathrm{cosk}_{n+1}(X)$ is an n-truncated Kan complex, so condition (3) guarantees that there exists a morphism $g : Y \to \mathrm{cosk}_{n+1}(X)$ such that $g \circ f$ is homotopic to u. For every n-truncated Kan complex Z, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{hKan}}(\mathrm{cosk}_{n+1}(X), Z)) \arrow[rr, "\circ[g]"] \arrow[dr, "\circ[u]"'] & & \mathrm{Hom}_{\mathrm{hKan}}(Y, Z) \arrow[dl, "\circ[f]"] \\
& \mathrm{Hom}_{\mathrm{hKan}}(X, Z), &
\end{tikzcd}$$

where the vertical maps are bijective. It follows that g is a homotopy equivalence, so that f exhibits Y as an n-truncation of X by virtue of Remarks 3.5.7.22 and 3.5.7.21. □

055J **Corollary 3.5.7.30.** *Let n be an integer and let $\mathrm{hKan}^{\leq n}$ denote the full subcategory of the homotopy category hKan spanned by the n-truncated Kan complexes. Then the inclusion map $\mathrm{hKan}^{\leq n} \hookrightarrow \mathrm{hKan}$ admits a left adjoint, given by the construction $X \mapsto \mathrm{cosk}_{n+1}(X)$.*

3.5.8 The Postnikov Tower of a Kan Complex

055K If X is a Kan complex, then its truncations can be arranged into a diagram.

055L **Definition 3.5.8.1.** Let X be a Kan complex. Suppose we are given an inverse system of Kan complexes $Y = \{Y(n)\}_{n \geq 0}$, which we display as

$$\cdots \to Y(3) \to Y(2) \to Y(1) \to Y(0).$$

We say that a morphism of simplicial sets $u : X \to \varprojlim_n Y(n)$ *exhibits* Y *as a Postnikov tower of* X if, for every integer $n \geq 0$, the induced map $u_n : X \to Y(n)$ exhibits $Y(n)$ as an n-truncation of Y: that is, $Y(n)$ is n-truncated and u_n is $(n+1)$-connective (see Definition 3.5.7.19). We say that Y *is a Postnikov tower of* X if there exists a morphism $u : X \to \varprojlim_n Y(n)$ which exhibits Y as a Postnikov tower of X.

Example 3.5.8.2 (The Canonical Tower)**.** Let X be a Kan complex. For every integer $n \geq 0$, let u_n denote the tautological map from X to its fundamental n-groupoid $\pi_{\leq n}(X)$. Since $\pi_{\leq n}(X)$ is also an $(n+1)$-groupoid (Remark 3.5.5.3), Proposition 3.5.6.5 guarantees that u_n factors uniquely as a composition $X \xrightarrow{u_{n+1}} \pi_{\leq n+1}(X) \xrightarrow{r_n} \pi_{\leq n}(X)$. We therefore obtain an inverse system of Kan complexes 055M

$$\cdots \to \pi_{\leq 3}(X) \xrightarrow{r_2} \pi_{\leq 2}(X) \xrightarrow{r_1} \pi_{\leq 1}(X) \xrightarrow{r_0} \pi_{\leq 0}(X),$$

Since each u_n is bijective on m-simplices for $m < n$, the induced map $u : X \to \varprojlim_n \pi_{\leq n}(X)$ is an isomorphism of simplicial sets. It follows from Example 3.5.7.25 that u exhibits the inverse system $\{\pi_{\leq n}(X)\}_{n \geq 0}$ as a Postnikov tower of X, which we will refer to as the *canonical Postnikov tower of* X.

Remark 3.5.8.3 (Uniqueness)**.** Let X be a Kan complex and let $Y = \{Y(n)\}_{n \geq 0}$ be a Postnikov tower of X. Then Y is homotopy equivalent to the canonical Postnikov tower of Example 3.5.8.2. More precisely, let $u : X \to \varprojlim_n Y(n)$ be a morphism of simplicial sets which exhibits Y as a Postnikov tower of X, given by a compatible system of morphisms $u_n : X \to Y(n)$. We then have a commutative diagram of towers 055N

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{\leq 3}(X) \arrow[r] \arrow[d, "\pi_{\leq 3}(u_3)"] & \pi_{\leq 2}(X) \arrow[r] \arrow[d, "\pi_{\leq 2}(u_2)"] & \pi_{\leq 1}(X) \arrow[r] \arrow[d, "\pi_{\leq 1}(u_1)"] & \pi_{\leq 0}(X) \arrow[d, "\pi_{\leq 0}(u_0)"] \\
\cdots \arrow[r] & \pi_{\leq 3}(Y(3)) \arrow[r] & \pi_{\leq 2}(Y(2)) \arrow[r] & \pi_{\leq 1}(Y(1)) \arrow[r] & \pi_{\leq 0}(Y(0)) \\
\cdots \arrow[r] & Y(3) \arrow[r] \arrow[u] & Y(2) \arrow[r] \arrow[u] & Y(1) \arrow[r] \arrow[u] & Y(0), \arrow[u]
\end{tikzcd}$$

where the upper vertical maps are homotopy equivalences by virtue of our assumption that u_n is $(n+1)$-connective (see Corollaries 3.5.7.14 and 3.5.6.15), and the lower vertical maps are homotopy equivalences by virtue of our assumption that each $Y(n)$ is n-truncated (Variant 3.5.7.16).

055P **Example 3.5.8.4** (The Coskeletal Tower)**.** Let X be a Kan complex. For every integer n, let $v_n : X \to \operatorname{cosk}_n(X)$ denote the tautological map from X to its n-coskeleton. Since the $\operatorname{cosk}_n(X)$ is $(n+1)$-coskeletal, the morphism v_n factors (uniquely) as a composition $X \xrightarrow{v_{n+1}} \operatorname{cosk}_{n+1}(X) \xrightarrow{q_n} \operatorname{cosk}_n(X)$. We therefore obtain an inverse system of Kan complexes

$$\cdots \to \operatorname{cosk}_4(X) \xrightarrow{q_3} \operatorname{cosk}_3(X) \xrightarrow{q_2} \operatorname{cosk}_2(X) \xrightarrow{q_1} \operatorname{cosk}_1(X)$$

which we will refer to as the *coskeletal tower of* X. Since v_n is bijective on m-simplices for $m \leq n$, the induced map $v : X \to \varprojlim_n \operatorname{cosk}_n(X)$ is an isomorphism of simplicial sets. It follows from Example 3.5.7.23 that v exhibits the coskeletal tower as a Postnikov tower of X (that is, each of the morphisms $v_{n+1} : X \to \operatorname{cosk}_{n+1}(X)$ exhibits $\operatorname{cosk}_{n+1}(X)$ as an n-truncation of X).

055Q **Example 3.5.8.5** (The Weakly Coskeletal Tower)**.** Let X be a Kan complex. For every integer $n \geq 0$, let $v_n^{\circ} : X \to \operatorname{cosk}_n^{\circ}(X)$ denote the tautological map from X to its weak n-coskeleton (Notation 3.5.4.19). Since the $\operatorname{cosk}_n^{\circ}(X)$ is $(n+1)$-coskeletal, the morphism v_n° factors (uniquely) as a composition $X \xrightarrow{v_{n+1}^{\circ}} \operatorname{cosk}_{n+1}^{\circ}(X) \xrightarrow{q_n^{\circ}} \operatorname{cosk}_n^{\circ}(X)$. We therefore obtain an inverse system of Kan complexes

$$\cdots \to \operatorname{cosk}_3^{\circ}(X) \xrightarrow{q_2^{\circ}} \operatorname{cosk}_2^{\circ}(X) \xrightarrow{q_1^{\circ}} \operatorname{cosk}_1^{\circ}(X) \xrightarrow{q_0^{\circ}} \operatorname{cosk}_0^{\circ}(X)$$

which we will refer to as the *weakly coskeletal tower of* X. Since v_n° is bijective on m-simplices for $m < n$, the induced map $v^{\circ} : X \to \varprojlim_n \operatorname{cosk}_n^{\circ}(X)$ is an isomorphism of simplicial sets. It follows from Example 3.5.7.24 that v° exhibits the weakly coskeletal tower as a Postnikov tower of X (that is, each of the morphisms $v_n^{\circ} : X \to \operatorname{cosk}_n^{\circ}(X)$ exhibits $\operatorname{cosk}_n^{\circ}(X)$ as an n-truncation of X).

055R **Remark 3.5.8.6.** Let X be a Kan complex. Then the Postnikov towers described in Examples 3.5.8.2, 3.5.8.4, and 3.5.8.5 are related by a commutative diagram

055S
$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{cosk}_4(X) \arrow[r] \arrow[d] & \operatorname{cosk}_3(X) \arrow[r] \arrow[d] & \operatorname{cosk}_2(X) \arrow[r] \arrow[d] & \operatorname{cosk}_1(X) \arrow[d] \\
\cdots \arrow[r] & \operatorname{cosk}_3^{\circ}(X) \arrow[r] \arrow[d] & \operatorname{cosk}_2^{\circ}(X) \arrow[r] \arrow[d] & \operatorname{cosk}_1^{\circ}(X) \arrow[r] \arrow[d] & \operatorname{cosk}_0^{\circ}(X) \arrow[d] \\
\cdots \arrow[r] & \pi_{\leq 3}(X) \arrow[r] & \pi_{\leq 2}(X) \arrow[r] & \pi_{\leq 1}(X) \arrow[r] & \pi_{\leq 0}(X),
\end{tikzcd} \tag{3.36}$$

where the vertical maps are trivial Kan fibrations (see Proposition 3.5.4.22 and Corollary 3.5.6.14).

Proposition 3.5.8.7. *Let X be a Kan complex, let $n \geq 0$ be an integer, and let v°_{n-1} denote the tautological map from X to its weak $(n-1)$-coskeleton $\mathrm{cosk}^{\circ}_{n-1}(X)$. Then:* 055T

(1) *The morphism v°_{n-1} factors uniquely as a composition $X \to \pi_{\leq n}(X) \xrightarrow{f_n} \mathrm{cosk}^{\circ}_{n-1}(X)$.*

(2) *The morphism f_n exhibits $\mathrm{cosk}^{\circ}_{n-1}(X)$ as a weak $(n-1)$-coskeleton of the fundamental n-groupoid $\pi_{\leq n}(X)$.*

(3) *The morphism f_n is a Kan fibration.*

(4) *Let $x \in X$ be a vertex and set $G = \pi_n(X, x)$. Then the fiber $\{x\} \times_{\mathrm{cosk}^{\circ}_{n-1}(X)} \pi_{\leq n}(X)$ is isomorphic to the Eilenberg-MacLane space $\mathrm{K}(G, n)$ of Construction 2.5.6.9.*

Proof. Since the weak coskeleton $\mathrm{cosk}^{\circ}_{n-1}(X)$ is an n-groupoid (Proposition 3.5.5.10), assertion (1) is a special case of Proposition 3.5.6.5. Note that, since v°_{n-1} is surjective on n-simplices, the morphism f_n has the same property. Consequently, to prove (2), it will suffice to show that f_n is bijective on m-simplices for $m < n$ (see Definition 3.5.4.14). This is clear, since the morphisms v°_{n-1} and $u_n : X \to \pi_{\leq n}(X)$ are bijective on m-simplices.

Assertion (3) follows by combining (2) with Proposition 3.5.4.26. It remains to prove (4). Fix a vertex $x \in X$, and let us abuse notation by identifying x with its images in $\pi_{\leq n}(X)$ and $\mathrm{cosk}^{\circ}_{n-1}(X)$. Let Y denote the fiber $f_n^{-1}\{x\}$. It follows from (3) that Y is a Kan complex. Applying Remark 3.5.5.6, we see that Y is an n-groupoid. Since f_n is bijective on m-simplices for $m < n$, the simplicial set Y has a single m-simplex for $m < n$. Applying Proposition 3.5.5.16, we obtain an isomorphism $Y \xrightarrow{\sim} \mathrm{K}(G, n)$ where G is a set if $n = 0$, a group if $n = 1$, and an abelian group if $n \geq 2$. To complete the proof, it suffices to observe that the tautological maps

$$G \simeq \pi_n(Y, x) \to \pi_n(\pi_{\leq n}(X), x) \leftarrow \pi_n(X, x)$$

are isomorphisms: for the map on the right, this follows from Example 3.5.7.25, and for the map on the left it follows from the long exact sequence of Theorem 3.2.6.1. □

Remark 3.5.8.8. Let X be a Kan complex. Then the morphisms $f_n : \pi_{\leq n}(X) \to \mathrm{cosk}^{\circ}_{n-1}(X)$ of Proposition 3.5.8.7 fit into a commutative diagram 055U

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{cosk}^{\circ}_{3}(X) \arrow[r] \arrow[d] & \mathrm{cosk}^{\circ}_{2}(X) \arrow[r] \arrow[d] & \mathrm{cosk}^{\circ}_{1}(X) \arrow[r] \arrow[d] & \mathrm{cosk}^{\circ}_{0}(X) \arrow[d] \\
\cdots \arrow[r] \arrow[ur] & \pi_{\leq 3}(X) \arrow[r] \arrow[ur, "f_3"] & \pi_{\leq 2}(X) \arrow[r] \arrow[ur, "f_2"] & \pi_{\leq 1}(X) \arrow[r] \arrow[ur, "f_1"] & \pi_{\leq 0}(X),
\end{tikzcd}$$

which intertwines the intertwine the canonical Postnikov tower of Example 3.5.8.2 with the weakly coskeletal tower of Example 3.5.8.5.

055V **Corollary 3.5.8.9.** *Let X be a Kan complex. Then the transition maps in the canonical Postnikov tower*

$$\cdots \to \pi_{\leq 3}(X) \to \pi_{\leq 2}(X) \to \pi_{\leq 1}(X) \to \pi_{\leq 0}(X)$$

are Kan fibrations. Moreover, for every vertex $x \in X$ and every integer $n > 0$, there is a canonical homotopy equivalence

$$\mathrm{K}(G, n) \to \{x\} \times_{\pi_{\leq n-1}(X)} \pi_{\leq n}(X),$$

for $G = \pi_n(X, x)$.

Proof. For $n > 0$, the transition map $\pi_{\leq n}(X) \to \pi_{\leq n-1}(X)$ factors as a composition

$$\pi_{\leq n}(X) \xrightarrow{f_n} \operatorname{cosk}^{\circ}_{n-1}(X) \xrightarrow{g} \pi_{\leq n-1}(X),$$

where f_n is the Kan fibration of Proposition 3.5.8.7 and g is the trivial Kan fibration of Corollary 3.5.6.14. We therefore obtain a pullback diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{K}(G, n) \arrow[r] \arrow[d] & \{x\} \times_{\pi_{\leq n-1}(X)} \pi_{\leq n}(X) \arrow[d] \\
\{x\} \arrow[r] & \{x\} \times_{\pi_{\leq n-1}(X)} \operatorname{cosk}^{\circ}_{n-1}(X)
\end{tikzcd}$$

where the vertical maps are Kan fibrations and the lower right corner is contractible. In particular, the lower horizontal map is a homotopy equivalence. Applying Corollary 3.4.1.5, we deduce that the upper horizontal map is also a homotopy equivalence. □

055W **Variant 3.5.8.10.** Let X be a Kan complex. Then the transition maps in the weakly coskeletal tower

$$\cdots \to \operatorname{cosk}^{\circ}_3(X) \to \operatorname{cosk}^{\circ}_2(X) \to \operatorname{cosk}^{\circ}_1(X) \to \operatorname{cosk}^{\circ}_0(X)$$

are Kan fibrations (whose fibers are homotopy equivalent to Eilenberg-MacLane spaces).

055X **Warning 3.5.8.11.** Let X be a Kan complex. Then the transition maps in the coskeletal tower

$$\cdots \to \operatorname{cosk}_4(X) \to \operatorname{cosk}_3(X) \to \operatorname{cosk}_2(X) \to \operatorname{cosk}_1(X)$$

are generally not Kan fibrations. See Warning 3.5.4.27.

3.5.9 Truncated Morphisms

We now formulate a relative version of Definition 3.5.7.1. `055Y`

Definition 3.5.9.1. Let $f : X \to Y$ be a morphism of Kan complexes and let $n \geq -1$ be an integer. We say that f is *n-truncated* if, for every vertex $x \in X$ having image $y = f(x)$, the induced map `055Z`

$$\pi_m(f) : \pi_m(X, x) \to \pi_m(Y, y)$$

is injective for $m = n+1$ and bijective for $m > n+1$. If $n \leq -2$, we say that f is *n-truncated* if it is (-1)-truncated and the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective.

Example 3.5.9.2. For $n \leq -2$, a morphism of Kan complexes $f : X \to Y$ is n-truncated if and only if it is a homotopy equivalence. This is a reformulation of Theorem 3.2.7.1. `0560`

Example 3.5.9.3. A morphism of Kan complexes $f : X \to Y$ is (-1)-truncated if and only if it induces a homotopy equivalence from X to a summand of Y. `0561`

Example 3.5.9.4. Let X be a Kan complex and let n be an integer. Then X is n-truncated (in the sense of Definition 3.5.7.1) if and only if the projection map $X \to \Delta^0$ is n-truncated (in the sense of Definition 3.5.9.1). For $n \geq 0$, this is a restatement of Proposition 3.5.7.7. `0562`

Remark 3.5.9.5 (Homotopy Invariance)**.** Let $f, f' : X \to Y$ be morphisms of Kan complexes which are homotopic. Then f is n-truncated if and only if f' is n-truncated. `0563`

Remark 3.5.9.6 (Monotonicity)**.** Let $f : X \to Y$ be a morphism of Kan complexes which is m-truncated for some integer m. Then f is also n-truncated for every integer $n \geq m$. `0564`

Remark 3.5.9.7 (Symmetry)**.** Let $f : X \to Y$ be a morphism of Kan complexes. Then f is n-truncated if and only if the opposite morphism $f^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is n-truncated. See Remark 3.2.2.20. `0565`

Proposition 3.5.9.8. *Let $f : X \to Y$ be a Kan fibration between Kan complexes and let n be an integer. Then f is n-truncated (in the sense of Definition 3.5.9.1) if and only if, for each vertex $y \in Y$, the Kan complex $X_y = \{y\} \times_Y X$ is n-truncated (in the sense of Definition 3.5.7.1).* `0566`

Proof. For $n \geq 0$, this follows from Corollary 3.2.6.8. This extends to the case $n = -1$ by virtue of Variant 3.2.6.9, and to the case $n \leq -2$ by virtue of Corollary 3.2.6.3. □

Remark 3.5.9.9. In the situation of Proposition 3.5.9.8, it is not necessary to verify that the fiber X_y is n-truncated for *every* vertex $y \in Y$; it is enough to check this condition at one vertex from each connected component of Y (see Remark 3.3.7.3). In particular, if Y is connected, then it is enough to check that the fiber X_y is n-truncated for *any* choice of vertex $y \in Y$. `0567`

0568 **Variant 3.5.9.10.** Suppose we are given a commutative diagram of Kan complexes

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\
& Z, &
\end{tikzcd}$$

where the vertical maps are Kan fibrations. Then f is n-truncated if and only if, for every vertex $z \in Z$, the induced map $f_z : X_z \to Y_z$ is n-truncated. To prove this, we can use Proposition 3.1.7.1 to reduce to the case where f is a Kan fibration. In this case, the desired result follows from the criterion of Proposition 3.5.9.8 (since a Kan complex can be realized as a fiber of f if and only if it can be realized as a fiber of f_z for some vertex $z \in Z$).

0569 **Corollary 3.5.9.11.** *Let n be an integer and suppose we are given a homotopy pullback diagram of Kan complexes*

056A
$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y \arrow[r, "g"] & Y'.
\end{tikzcd} \tag{3.37}$$

If f' is n-truncated, then f is also n-truncated. The converse holds if $\pi_0(g)$ is surjective.

Proof. Using Proposition 3.1.7.1, we can reduce to the case where f and f' are Kan fibrations. In this case, our assumption that (3.37) is a homotopy pullback square guarantees that for each vertex $y \in Y$, the induced map of fibers $X_y \to X'_{g(y)}$ is a homotopy equivalence (Example 3.4.1.4) In particular, X_y is n-truncated if and only if $X'_{g(y)}$ is n-truncated (Corollary 3.5.7.12). The desired result now follows from the criterion of Proposition 3.5.9.8 (together with Remark 3.5.9.9). □

056B **Corollary 3.5.9.12.** *Let $f : X \to Y$ be a morphism of Kan complexes and let n be an integer. The following conditions are equivalent:*

(1) *The morphism f is n-truncated.*

(2) *For every morphism of Kan complexes $Y' \to Y$, the projection map $Y' \times^{\mathrm{h}}_{Y} X \to Y'$ is n-truncated.*

(3) *For every vertex $y \in Y$, the homotopy fiber $\{y\} \times^{\mathrm{h}}_{Y} X$ is n-truncated.*

Proof. Using Proposition 3.4.0.9, we can reduce to the case where f is a Kan fibration. In this case, we can use Proposition 3.4.0.7 to reformulate conditions (2) and (3) as follows:

(2′) For every morphism of Kan complexes $Y' \to Y$, the projection map $Y' \times_Y X \to Y'$ is n-truncated.

(3′) For every vertex $y \in Y$, the fiber $\{y\} \times_Y X$ is n-truncated.

The equivalence (1) ⇔ (3′) now follows from Proposition 3.5.9.8, and the equivalence (1) ⇔ (2′) from Corollary 3.5.9.11. □

Proposition 3.5.9.13. *Let $f : X \to Y$ be a morphism of Kan complexes and let n be an integer. Then:* `056C`

(a) If Y is n-truncated and f is n-truncated, then X is n-truncated.

(b) If X is n-truncated and Y is $(n+1)$-truncated, then f is n-truncated.

(c) If X is n-truncated, f is $(n-1)$-truncated, and $\pi_0(f)$ is surjective, then Y is n-truncated.

Proof. For every integer $m \geq 0$ and every vertex $x \in X$ having image $y = f(x)$, we make the following observations:

(a_m) If the morphism $\pi_m(f) : \pi_m(X, x) \to \pi_m(Y, y)$ is injective and $\pi_m(Y, y)$ is a singleton, then $\pi_m(X, x)$ is also a singleton.

(b_m) If the sets $\pi_m(X, x)$ and $\pi_{m+1}(Y, y)$ are singletons, then $\pi_m(f)$ is injective and $\pi_{m+1}(f)$ is surjective.

(c_m) If $\pi_m(f)$ is surjective and $\pi_m(X, x)$ is a singleton, then $\pi_m(Y, y)$ is a singleton.

If $n \geq -1$, then Proposition 3.5.9.13 follows by combining these observations with Proposition 3.5.7.7 (together with Remarks 3.5.7.8 and 3.5.7.9), by allowing m to range over integers $> n$ and allowing the vertex x to vary. The case $n \leq -2$ then follows from the following additional observations:

(a_{-1}) If the morphism $\pi_0(f)$ is surjective and $\pi_0(Y)$ is nonempty, then $\pi_0(X)$ is also nonempty.

(b_{-1}) If $\pi_0(X)$ is nonempty and $\pi_0(Y)$ has at most one element, then $\pi_0(f)$ is surjective.

(c_{-1}) If $\pi_0(X)$ is nonempty, then $\pi_0(Y)$ is also nonempty.

□

Corollary 3.5.9.14 (Transitivity)**.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of Kan complexes and let n be an integer. Then:* `056D`

(a) *If the morphisms f and g are n-truncated, then the composition $(g \circ f) : X \to Z$ is n-truncated.*

(b) *If $(g \circ f)$ is n-truncated and g is $(n+1)$-truncated, then f is n-truncated.*

(c) *If $(g \circ f)$ is n-truncated, f is $(n-1)$-truncated, and $\pi_0(f)$ is surjective, then g is n-truncated.*

Proof. Using Proposition 3.1.7.1, we can reduce to the case where Z is a Kan complex and the morphisms f and g are Kan fibrations. Using the criterion of Proposition 3.5.9.8, we can further reduce to the case $Z = \Delta^0$. In this case, Corollary 3.5.9.14 is a restatement of Proposition 3.5.9.13. □

056E **Proposition 3.5.9.15.** *Let X be a Kan complex, let n be an integer, and let k be a nonnegative integer. If X is n-truncated, then the diagonal map $\delta : X \to \mathrm{Fun}(\partial \Delta^k, X)$ is $(n-k)$-truncated. The converse holds if $k \leq n+2$.*

Proof. We proceed by induction on k. If $k = 0$, the result is a reformulation of Example 3.5.9.4. Let us therefore assume that $k > 0$. Note that δ factors as a composition $X \hookrightarrow \mathrm{Fun}(\Delta^k, X) \xrightarrow{R_k} \mathrm{Fun}(\partial \Delta^k, X)$, where the first map is a homotopy equivalence (see Example 3.2.4.2) and R_k is a Kan fibration (Corollary 3.1.3.3). Consequently, δ is $(n-k)$-truncated if and only if the morphism R_k is $(n-k)$-truncated. To carry out the inductive step, it will suffice to prove the following:

(∗) If $R_{k-1} : \mathrm{Fun}(\Delta^{k-1}, X) \to \mathrm{Fun}(\partial \Delta^{k-1}, X)$ is m-truncated, then $R_k : \mathrm{Fun}(\Delta^k, X) \to \mathrm{Fun}(\partial \Delta^k, X)$ is $(m-1)$-truncated. The converse holds for $m \geq -1$.

Assume first that R_{k-1} is m-truncated. Note that we have a pullback diagram of restriction maps

056F
$$\begin{tikzcd}
\mathrm{Fun}(\partial \Delta^k, X) \arrow[r, "T"] \arrow[d] & \mathrm{Fun}(\Lambda^k_k, X) \arrow[d] \\
\mathrm{Fun}(\Delta^{k-1}, X) \arrow[r, "R_{k-1}"] & \mathrm{Fun}(\partial \Delta^{k-1}, X).
\end{tikzcd} \tag{3.38}$$

Applying the criterion of Proposition 3.5.9.8, we conclude that T is also m-truncated. Note that the composition $(T \circ R_k) : \mathrm{Fun}(\Delta^k, X) \to \mathrm{Fun}(\Lambda^k_k, X)$ is given by precomposition with the horn inclusion $\Lambda^k_k \hookrightarrow \Delta^k$, and is therefore a trivial Kan fibration (Corollary 3.1.3.6). In particular, $T \circ R_k$ is $(m-1)$-truncated, so Corollary 3.5.9.14 guarantees that R_k is $(m-1)$-truncated by virtue of Corollary 3.5.9.14.

We now prove the converse. Assume that R_k is $(m-1)$-truncated and that $m \geq -1$; we wish to show that R_{k-1} is m-truncated. Let $\mathrm{Fun}'(\partial \Delta^k, X)$ denote the summand of

$\operatorname{Fun}(\partial\Delta^k, X)$ whose vertices are nullhomotopic maps $\partial\Delta^k \to X$, and let $T' : \operatorname{Fun}'(\partial\Delta^k, X) \to \operatorname{Fun}(\Lambda^k_k, X)$ be the restriction map. As above, the composition $T' \circ R_k$ is a trivial Kan fibration, and therefore m-truncated Applying Corollary 3.5.9.14, we conclude that T' is m-truncated.

Fix a morphism $\sigma_0 : \partial\Delta^{k-1} \to X$, and set $Y = \{\sigma_0\} \times_{\operatorname{Fun}(\partial\Delta^{k-1}, X)} \operatorname{Fun}(\Delta^{k-1}, X)$; by virtue of Proposition 3.5.9.8, it will suffice to show that Y is m-truncated. We first consider the case $m \geq 0$. By virtue of Remark 3.5.7.11, it will suffice to show that every connected component $Z \subseteq Y$ is m-truncated. Fix a vertex of Z, corresponding to a map $\sigma : \Delta^{k-1} \to X$ extending σ_0. Choose an extension of σ to a k-simplex $\tau : \Delta^k \to X$ (for example, we can take τ to be the degenerate k-simplex $s^{k-1}_{k-1}(\sigma)$), and set $\tau_0 = \tau|_{\Lambda^k_k}$. Since (3.38) is a pullback square, it induces an isomorphism from Y to the fiber $T^{-1}\{\tau_0\} = \{\tau_0\} \times_{\operatorname{Fun}(\Lambda^k_k, X)} \operatorname{Fun}(\partial\Delta^k, X)$. By construction, this isomorphism identifies Z to a connected component of the fiber $T'^{-1}\{\tau_0\} = \times_{\operatorname{Fun}(\Lambda^k_k, X)} \operatorname{Fun}(\partial\Delta^k, X)$. Our assumption that T' is m-truncated guarantees that this fiber $T'^{-1}\{\tau_0\}$ is m-truncated (Proposition 3.5.9.8), so that Z is also m-truncated (Remark 3.5.7.11).

We now treat the case $m = -1$: in this case, we wish to show that Y is either empty or contractible. Let us assume that Y is nonempty: that is, σ_0 can be extended to a $(k-1)$-simplex $\sigma : \Delta^{k-1} \to X$. Define τ and τ_0 as above, so that we can identify Y with the fiber $T^{-1}\{\tau_0\}$. We will complete the proof by showing that T is a trivial Kan fibration. Since T is a Kan fibration, it will suffice to show that it is a homotopy equivalence (Proposition 3.2.7.2). Since $T \circ R_k$ is a homotopy equivalence, we are reduced to showing that R_k is a homotopy equivalence. This is a reformulation of our hypothesis that R_k is $(m-1)$-truncated (see Example 3.5.9.2). □

Corollary 3.5.9.16. *Let $f : X \to Y$ be a Kan fibration between Kan complexes, let n be an integer, and let k be a nonnegative integer. If f is n-truncated, then the relative diagonal map* 056G

$$\delta : X \to Y \times_{\operatorname{Fun}(\partial\Delta^k, Y)} \operatorname{Fun}(\partial\Delta^k, X)$$

is $(n-k)$-truncated. The converse holds if $k \geq n+2$.

Proof. We have a commutative diagram of Kan complexes

$$\begin{tikzcd}
X \arrow[rr, "\delta"] \arrow[dr, "f"'] & & Y \times_{\operatorname{Fun}(\partial\Delta^k, Y)} \operatorname{Fun}(\partial\Delta^k, X) \arrow[dl] \\
& Y &
\end{tikzcd}$$

where the vertical maps are Kan fibrations. Using Variant 3.5.9.10, we see that δ is $(n-k)$-truncated if and only if, for each vertex $y \in Y$, the induced map of fibers

$$X_y \to \{y\} \times_{\operatorname{Fun}(\partial\Delta^k, Y)} \operatorname{Fun}(\partial\Delta^k, X) \simeq \operatorname{Fun}(\partial\Delta^k, X_y)$$

is $(n-k)$-truncated. The desired result now follows from Proposition 3.5.9.15. □

056H **Corollary 3.5.9.17.** *Let $f : X \to Y$ be a Kan fibration between Kan complexes and let $n \geq -1$. Then f is n-truncated if and only if the relative diagonal $\delta_{X/Y} : X \to X \times_Y X$ is $(n-1)$-truncated.*

Proof. Apply Corollary 3.5.9.16 in the case $k = 1$. □

056J **Example 3.5.9.18.** Let X be a Kan complex. Then the diagonal map $\delta_X : X \hookrightarrow X \times X$ factors as a composition

$$X \xrightarrow{u} \operatorname{Fun}(\Delta^1, X) \xrightarrow{q} \operatorname{Fun}(\partial\Delta^1, X) \simeq X \times X,$$

where u is a homotopy equivalence and q is a Kan fibration (Corollary 3.1.3.3). Combining Corollary 3.5.9.17 with Proposition 3.5.9.8, we see that the following conditions are equivalent for every integer $n \geq -1$:

- The Kan complex X is n-truncated.
- The diagonal morphism $\delta_X : X \hookrightarrow X \times X$ is $(n-1)$-truncated.
- For every pair of vertices $x, y \in X$, the path space

  $$\{x\} \times^{\mathrm{h}}_X \{y\} = \{(x,y)\} \times_{\operatorname{Fun}(\partial\Delta^1, X)} \operatorname{Fun}(\Delta^1, X)$$

  is $(n-1)$-truncated.

056K **Variant 3.5.9.19.** Let $f : X \to Y$ be a morphism between Kan complexes, let n be an integer, and let k be a nonnegative integer. If f is n-truncated, then the restriction map

$$u : \operatorname{Fun}(\Delta^k, X) \to \operatorname{Fun}(\Delta^k, Y) \times_{\operatorname{Fun}(\partial\Delta^k, Y)} \operatorname{Fun}(\partial\Delta^k, X)$$

is $(n-k)$-truncated. The converse holds if $k \geq n+2$.

Proof. Using Proposition 3.1.7.1, we can factor f as a composition $X \xrightarrow{i} X' \xrightarrow{f'} Y$, where i is anodyne and f' is a Kan fibration. Then X' is a Kan complex (Remark 3.1.1.11), so i is a homotopy equivalence. We then have a commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}(\Delta^k, X) \arrow[r, "u"] \arrow[d] & \operatorname{Fun}(\Delta^k, Y) \times_{\operatorname{Fun}(\partial\Delta^k, Y)} \operatorname{Fun}(\partial\Delta^k, X) \arrow[d] \\
\operatorname{Fun}(\Delta^k, X') \arrow[r] & \operatorname{Fun}(\Delta^k, Y) \times_{\operatorname{Fun}(\partial\Delta^k, Y)} \operatorname{Fun}(\partial\Delta^k, X')
\end{tikzcd}$$

where the vertical maps are homotopy equivalences. It follows that u is $(n-k)$-truncated if and only if u' is $(n-k)$-truncated. We may therefore replace f by f' and thereby reduce to proving Variant 3.5.9.19 in the special case where f is a Kan fibration. In this case, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[d] & Y \times_{\operatorname{Fun}(\partial \Delta^k, Y)} \operatorname{Fun}(\partial \Delta^k, X) \arrow[d] \\
\operatorname{Fun}(\Delta^k, X) \arrow[r, "u"] & \operatorname{Fun}(\Delta^k, Y) \times_{\operatorname{Fun}(\partial \Delta^k, Y)} \operatorname{Fun}(\partial \Delta^k, X)
\end{tikzcd}$$

where the vertical maps are homotopy equivalences (by virtue of the contractibility of Δ^k). It follows that u is $(n-k)$-truncated if and only if δ is $(n-k)$-truncated, so that Variant 3.5.9.19 is a reformulation of Corollary 3.5.9.16. □

Corollary 3.5.9.20. *Let $f : X \to Y$ be a morphism of Kan complexes and let $n \geq -2$. The following conditions are equivalent:* 056L

(1) *The morphism f is n-truncated.*

(2) *The restriction map*

$$\theta : \operatorname{Fun}(\Delta^{n+2}, X) \to \operatorname{Fun}(\Delta^{n+2}, Y) \times_{\operatorname{Fun}(\partial \Delta^{n+2}, Y)} \operatorname{Fun}(\partial \Delta^{n+2}, X)$$

is a homotopy equivalence.

(3) *The diagram of Kan complexes*

$$\begin{tikzcd}
\operatorname{Fun}(\Delta^{n+2}, X) \arrow[r] \arrow[d, "f"] & \operatorname{Fun}(\partial \Delta^{n+2}, X) \arrow[d] \\
\operatorname{Fun}(\Delta^{n+2}, Y) \arrow[r] & \operatorname{Fun}(\partial \Delta^{n+2}, Y)
\end{tikzcd}$$

is a homotopy pullback square.

(4) *The diagram of Kan complexes*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & \operatorname{Fun}(\partial \Delta^{n+2}, X) \arrow[d] \\
Y \arrow[r] & \operatorname{Fun}(\partial \Delta^{n+2}, Y)
\end{tikzcd}$$

is a homotopy pullback square.

Proof. The equivalence (1) ⇔ (2) follows by applying Variant 3.5.9.19 in the special case $k = n + 2$. The equivalence (2) ⇔ (3) follows from Example 3.4.1.3, and the equivalence (3) ⇔ (4) from Corollary 3.4.1.12. □

`056M` **Remark 3.5.9.21.** In the situation of Corollary 3.5.9.20, suppose that f is a Kan fibration. Then the morphism θ is also a Kan fibration (Theorem 3.1.3.1). Consequently, f is n-truncated if and only if θ is a trivial Kan fibration (Proposition 3.2.7.2).

`056N` **Corollary 3.5.9.22.** *Let X be a Kan complex and let $n \geq -2$ be an integer. Then X is n-truncated if and only if the diagonal map $X \to \operatorname{Fun}(\partial \Delta^{n+2}, X)$ is a homotopy equivalence.*

`056P` **Corollary 3.5.9.23.** *Let $f : X \to Y$ be a Kan fibration between Kan complexes and let n be an integer. The following conditions are equivalent:*

(1) *The morphism f is n-truncated.*

(2) *For every nonnegative integer $m \geq n + 2$, the induced map*

$$\theta : \operatorname{Fun}(\Delta^m, X) \to \operatorname{Fun}(\partial \Delta^m, X) \times_{\operatorname{Fun}(\partial \Delta^m, Y)} \operatorname{Fun}(\Delta^m, Y)$$

is a trivial Kan fibration.

(3) *For every nonnegative integer $m \geq n + 2$, every lifting problem*

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

has a solution.

(4) *For every simplicial set B and every simplicial subset $A \subseteq B$ which contains the $(n+1)$-skeleton $\operatorname{sk}_{n+1}(B)$, every lifting problem*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution.

Proof. If f is n-truncated, then it is also n'-truncated for every integer $n' \geq n$ (Remark 3.5.9.6). Consequently, the implication (1) ⇒ (2) follows from Remark 3.5.9.21. The implication (2) ⇒ (3) is immediate from the definitions, and the implication (3) ⇒ (1) follows from Proposition 3.5.9.8. The equivalence (3) ⇔ (4) follows from Proposition 1.1.4.12. □

Proposition 3.5.9.24. *Let $f : X \to Y$ be an n-truncated Kan fibration between Kan complexes, let k be an integer, and let $j : A \to B$ be a morphism of simplicial sets which is $(k-1)$-connective. Then the induced map* 056Q

$$\theta : \mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B, Y)$$

is $(n-k)$-truncated.

Proof. Using Proposition 3.1.7.1, we can factor j as a composition $A \xrightarrow{i} A' \xrightarrow{j} B$, where i is anodyne and j is a Kan fibration. In this case, θ factors as a composition

$$\mathrm{Fun}(B, X) \xrightarrow{\theta'} \mathrm{Fun}(A', X) \times_{\mathrm{Fun}(A',Y)} \mathrm{Fun}(B, Y) \xrightarrow{\rho} \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B, Y),$$

where ρ is a trivial Kan fibration (Theorem 3.1.3.5). It will therefore suffice to show that θ' is $(n-k)$-truncated. Using Corollary 3.5.2.4 (or Exercise 3.1.7.11, in the case $k = 0$), we can factor j' as a composition $A' \xrightarrow{\widetilde{j}} \widetilde{B} \xrightarrow{q} B$, where $\widetilde{j}$ is a monomorphism which is bijective on simplices of dimension $\leq k-1$ and q is a trivial Kan fibration. In this case, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(B, X) \arrow[r, "\theta'"] \arrow[d] & \mathrm{Fun}(A', X) \times_{\mathrm{Fun}(A',Y)} \mathrm{Fun}(B, Y) \arrow[d] \\
\mathrm{Fun}(\widetilde{B}, X) \arrow[r, "\widetilde{\theta}"] & \mathrm{Fun}(A', X) \times_{\mathrm{Fun}(A',Y)} \mathrm{Fun}(\widetilde{B}, Y)
\end{tikzcd}$$

where the vertical maps are homotopy equivalences. Consequently, to prove that θ is $(n-k)$-truncated, it will suffice to show that $\widetilde{\theta}$ is $(n-k)$-truncated. We may therefore replace j by $\widetilde{j}$ and thereby reduce to proving Proposition 3.5.9.24 in the special case where j is a monomorphism which is bijective on simplices of dimension $\leq k-1$.

If j is a monomorphism, then θ is a Kan fibration (Theorem 3.1.3.1). Consequently, to show that θ is $(n-k)$-connective, it will suffice to show that every lifting problem

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & \operatorname{Fun}(B, X) \arrow[d, "\theta"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(A, X) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(B, Y)
\end{tikzcd}$$

has a solution, provided that $m \geq n-k+2$ (Corollary 3.5.9.23) Unwinding the definitions, we can rewrite this as a lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"] & \operatorname{Fun}(\Delta^m, X) \arrow[d, "\theta'"] \\
B \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(\partial \Delta^m, X) \times_{\operatorname{Fun}(\partial \Delta^m, Y)} \operatorname{Fun}(\Delta^m, Y).
\end{tikzcd}$$

Since f is a Kan fibration, θ' is also a Kan fibration (Theorem 3.1.3.1), and our assumption that f is n-truncated guarantees that θ' is $(n-m)$-truncated (Variant 3.5.9.19). In particular, θ' is $(k-2)$-truncated (Remark 3.5.9.6), so the existence of the desired solution follows from Corollary 3.5.9.23. □

056R **Corollary 3.5.9.25.** *Let X be an n-truncated Kan complex, let k be an integer, and let $j : A \to B$ be a $(k-1)$-connective morphism of simplicial sets. Then the induced map $\operatorname{Fun}(B, X) \to \operatorname{Fun}(A, X)$ is $(n-k)$-connective.*

Proof. Apply Proposition 3.5.9.24 in the special case $Y = \Delta^0$ (see Example 3.5.9.4). □

056S **Corollary 3.5.9.26.** *Let $f : X \to Y$ be an n-truncated morphism between Kan complexes. Then, for every simplicial set B, the induced map $\operatorname{Fun}(B, X) \to \operatorname{Fun}(B, Y)$ is n-truncated.*

Proof. Using Proposition 3.1.7.1, we can reduce to the case where f is a Kan fibration. In this case, the desired result follows by applying Proposition 3.5.9.24 in the special case $A = \emptyset$ (and the integer k is equal to zero). □

056T **Corollary 3.5.9.27.** *Let X be an n-truncated Kan complex. Then, for any simplicial set B, the Kan complex $\operatorname{Fun}(B, X)$ is also n-truncated.*

Proof. Apply Corollary 3.5.9.25 in the special case $A = \emptyset$ (or Corollary 3.5.9.26 in the special case $Y = \Delta^0$). □

Corollary 3.5.9.28. *Let $n \geq -2$ be an integer and let $f : X \to Y$ be a morphism of Kan complexes. The following conditions are equivalent:* 056U

(1) *The morphism f is n-truncated.*

(2) *For every $(n+1)$-connective morphism of simplicial sets $A \to B$, the diagram of Kan complexes*

$$\begin{tikzcd}
\operatorname{Fun}(B, X) \arrow[r] \arrow[d] & \operatorname{Fun}(A, X) \arrow[d] \\
\operatorname{Fun}(B, Y) \arrow[r] & \operatorname{Fun}(A, Y)
\end{tikzcd}$$

is a homotopy pullback square.

(3) *The diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \operatorname{Fun}(\partial \Delta^{n+2}, X) \arrow[d] \\
Y \arrow[r] & \operatorname{Fun}(\partial \Delta^{n+2}, Y)
\end{tikzcd}$$

is a homotopy pullback square.

Proof. The equivalence (1) ⇔ (3) follows from Corollary 3.5.9.20, and the implication (2) ⇒ (3) from Corollary 3.5.2.6. It will therefore suffice to show that (1) implies (2). Using Proposition 3.1.7.1, we can reduce to the case where f is a Kan fibration. In this case, the map $\operatorname{Fun}(A, X) \to \operatorname{Fun}(A, Y)$ is also a Kan fibration (Corollary 3.1.3.2), so condition (2) is equivalent to the requirement that the map $\theta : \operatorname{Fun}(B, X) \to \operatorname{Fun}(A, X) \times_{\operatorname{Fun}(A,Y)} \operatorname{Fun}(B, Y)$ is a homotopy equivalence (Example 3.4.1.3). This follows from Proposition 3.5.9.24 (and Example 3.5.9.2). □

Corollary 3.5.9.29. *Let X be a Kan complex and let $n \geq -2$ be an integer. The following conditions are equivalent:* 056V

(1) *The Kan complex X is n-truncated.*

(2) *For $m \geq n+2$, every morphism $f : \partial \Delta^m \to X$ is nullhomotopic.*

(3) *If A is an $(n+1)$-connective simplicial set, then every morphism $f : A \to X$ is nullhomotopic.*

(4) *If A is an $(n+1)$-connective simplicial set, then the diagonal map $X \to \mathrm{Fun}(A, X)$ is a homotopy equivalence.*

(5) *For every $(n+1)$-connective morphism of simplicial sets $A \to B$, the induced map $\mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X)$ is a homotopy equivalence.*

Proof. The equivalence of (1) ⇔ (2) follows from Variant 3.2.4.12. The implication (3) ⇒ (2) follows from Corollary 3.5.2.6 and the implications (5) ⇒ (4) ⇒ (3) are immediate (see Example 3.5.1.18). To complete the proof, it will suffice to show that (1) implies (5). This follows by applying Corollary 3.5.9.28 in the special case $Y = \Delta^0$. □

3.6 Comparison with Topological Spaces

012Y Let Set_Δ denote the category of simplicial sets and let Top denote the category of topological spaces. In §1.2.2 and §1.2.3, we constructed a pair of adjoint functors

$$\begin{tikzcd} \mathrm{Set}_\Delta \arrow[r, shift left, "|\bullet|"] & \mathrm{Top} . \arrow[l, shift left, "\mathrm{Sing}_\bullet"] \end{tikzcd}$$

Our goal in this section is to prove that, after passing to homotopy categories, these functors are not far from being (mutually inverse) equivalences:

012Z **Theorem 3.6.0.1.** *The geometric realization functor $| \bullet | : \mathrm{Set}_\Delta \to \mathrm{Top}$ induces an equivalence from the homotopy category* hKan *to the full subcategory of* hTop *spanned by those topological spaces X which have the homotopy type of a CW complex.*

Theorem 3.6.0.1 is essentially due to Milnor (see [44]). We give a proof in §3.6.5, which has three main steps. The first of these is of a technical nature: we must show that geometric realization is well-defined at the level of homotopy categories (see Construction 3.6.5.1). Let X and Y be Kan complexes, and suppose that we are given a pair of morphisms $f_0, f_1 : X \to Y$. If f_0 is homotopic to f_1 (in the category of Kan complexes), then there exists a morphism of simplicial sets $h : \Delta^1 \times X \to Y$ satisfying $f_0 = h|_{\{0\} \times X}$ and $f_1 = h|_{\{1\} \times X}$. Passing to geometric realizations, we obtain a continuous function $|h| : |\Delta^1 \times X| \to |Y|$. We would like to interpret $|h|$ as a homotopy from $|f_0|$ to $|f_1|$ (in the category of topological spaces). For this, we need to know that the comparison map

$$|\Delta^1 \times X| \to |\Delta^1| \times |X| \simeq [0, 1] \times |X|$$

is a homeomorphism. In §3.6.2, we prove a more general assertion: for any pair of simplicial sets A and B, the comparison map $|A \times B| \to |A| \times |B|$ is a bijection (Theorem 3.6.2.1), which is a homeomorphism if either A or B is *finite* (that is, if either A or B has only finitely many nondegenerate simplices; see Corollary 3.6.2.2).

The second step in the proof of Theorem 3.6.0.1 is to show that the geometric realization functor $| \bullet | : \mathrm{hKan} \to \mathrm{hTop}$ is fully faithful (Proposition 3.6.5.2). This is equivalent to the assertion that for any Kan complex X, the unit map $u_X : X \to \mathrm{Sing}_\bullet(|X|)$ is a homotopy equivalence. More generally, we show in §3.6.4 that for any simplicial set X, the unit map $u_X : X \to \mathrm{Sing}_\bullet(|X|)$ is a weak homotopy equivalence (Theorem 3.6.4.1). Our strategy is to reduce to the case where the simplicial set X is finite, and to proceed by induction on the number of nondegenerate simplices of X. The inductive step will make use of excision (Theorem 3.4.6.1) to analyze the homotopy type of the Kan complex $\mathrm{Sing}_\bullet(|X|)$.

To complete the proof of Theorem 3.6.0.1, we must show that if Y is a topological space, then the counit map $v_Y : |\mathrm{Sing}_\bullet(Y)| \to Y$ is a homotopy equivalence if and only if Y has the homotopy type of a CW complex (Proposition 3.6.5.3). It follows formally from the preceding step that the map v_Y is always a *weak* homotopy equivalence: that is, it induces a bijection on path components and an isomorphism on homotopy groups for any choice of base point (Corollary 3.6.4.2). We will complete the proof using a result of Whitehead which asserts that any weak homotopy equivalence between CW complexes is a homotopy equivalence (see Proposition 3.6.3.8 and Corollary 3.6.3.10), which we prove in §3.6.3.

3.6.1 Digression: Finite Simplicial Sets

We now introduce a finiteness condition on simplicial sets. 0130

Definition 3.6.1.1. We say that a simplicial set X is *finite* if it satisfies the following pair of conditions: 0131

- For every integer $n \geq 0$, the set of n-simplices $X_n \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X)$ is finite.
- The simplicial set X is finite-dimensional (Definition 1.1.3.1): that is, there exists an integer m such that every nondegenerate simplex has dimension $\leq m$.

Example 3.6.1.2. For each integer $n \geq 0$, the standard n-simplex Δ^n is finite. 0132

Remark 3.6.1.3. Let X be a finite simplicial set. Then any simplicial subset $Y \subseteq X$ is also finite. In particular, any retract of X is finite. 0133

Remark 3.6.1.4. If X and Y are finite simplicial sets, then the coproduct $X \coprod Y$ is also finite. 0134

Remark 3.6.1.5. Let $f : X \twoheadrightarrow Y$ be an epimorphism of simplicial sets. If X is finite, then Y is also finite. 0135

Remark 3.6.1.6. Let X and Y be finite simplicial sets. Then the product $X \times Y$ is finite (see Proposition 1.1.3.6). 0136

0137 **Proposition 3.6.1.7.** *Let X be a simplicial set. The following conditions are equivalent:*

- (a) *The simplicial set X has only finitely many nondegenerate simplices.*
- (b) *There exists an epimorphism of simplicial sets $f : Y \twoheadrightarrow X$, where $Y \simeq \coprod_{i \in I} \Delta^{n_i}$ is a finite coproduct of standard simplices.*
- (c) *The simplicial set X is finite (Definition 3.6.1.1).*

Proof. If X is finite, then it has dimension $\leq n$ for some integer $n \gg 0$. It follows that every nondegenerate simplex of X has dimension $\leq n$. Since X has only finitely many (nondegenerate) simplices of each dimension, it follows that X has only finitely many nondegenerate simplices. This proves that $(c) \Rightarrow (a)$. The implication $(b) \Rightarrow (c)$ follows from Example 3.6.1.2 together with Remarks 3.6.1.4 and 3.6.1.5. We will complete the proof by showing that (a) implies (b). Let $\{\sigma_i : \Delta^{n_i} \to X\}_{i \in I}$ be the collection of all nondegenerate simplices of X, and amalgamate the morphisms σ_i to a single map $f : Y = \coprod_{i \in I} \Delta^{n_i} \to X$. By construction, every nondegenerate simplex of X belongs to the image of f and therefore every simplex of f belongs to the image of f (see Proposition 1.1.3.8). It follows that f is an epimorphism of simplicial sets. If condition (a) is satisfied, then the set I is finite, so that $f : Y \twoheadrightarrow X$ satisfies the requirements of (b). □

0138 **Remark 3.6.1.8.** Every simplicial set X can be realized as a union $\bigcup_{X' \subseteq X} X'$, where X' ranges over the collection of finite simplicial subsets of X (to prove this, we observe that every n-simplex σ is contained in a finite simplicial subset $X' \subseteq X$: in fact, we can take X' to be the image of $\sigma : \Delta^n \to X$). Moreover, the collection of finite simplicial subsets of X is closed under finite unions. It follows that realization $X \simeq \bigcup_{X' \subseteq X} X'$ exhibits X as a *filtered* colimit of its finite simplicial subsets.

02LG **Proposition 3.6.1.9.** *Let X be a simplicial set. Then X is finite if and only if it is a compact object of the category* Set_{Δ}*: that is, if and only if the corepresentable functor*

$$\mathrm{Set}_{\Delta} \to \mathrm{Set} \qquad Y \mapsto \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(X, Y)$$

commutes with filtered colimits.

Proof. By virtue of Remark 3.6.1.8, we can write X as a filtered colimit of finite simplicial subsets $Y \subseteq X$. If X is a compact object of Set_{Δ}, then the identity map $\mathrm{id}_X : X \to X$ factors through some finite simplicial subset $Y \subseteq X$. It follows that $Y = X$, so that X is a finite simplicial set. To prove the converse, assume that X is finite. Using Proposition 3.6.1.7, we can choose an epimorphism of simplicial sets $U \twoheadrightarrow X$, where U is a finite coproduct of standard simplices. In particular, U is also a finite simplicial set (Example 3.6.1.2 and Remark 3.6.1.4). The fiber product $U \times_X U$ can be regarded as a simplicial subset of $U \times U$, and is therefore also finite (Remarks 3.6.1.6 and 3.6.1.3). Applying Proposition 3.6.1.7 again, we can choose an epimorphism of simplicial sets $V \twoheadrightarrow U \times_X U$, where V is a finite coproduct of standard simplices. It follows that X can be realized as the coequalizer of a pair of maps $f_0, f_1 : V \to U$. Consequently, to show that X is compact, it will suffice to show that U and V are compact. Since the collection of compact objects of Set_{Δ} is closed under the formation of finite coproducts and coequalizers, we are reduced to showing that each standard simplex Δ^n is a compact object of Set_{Δ}. This is an immediate consequence of Proposition 1.1.0.12 and Remark 1.1.0.8. □

Corollary 3.6.1.10. *Let X be a finite simplicial set. Then the functor* 03X6

$$\mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta} \qquad Y \mapsto \mathrm{Fun}(X, Y)$$

commutes with filtered colimits.

Proof. Since colimits in the category of simplicial sets are computed levelwise (Remark 1.1.0.8), it will suffice to that the functor

$$\mathrm{Set}_{\Delta} \to \mathrm{Set} \qquad Y \mapsto \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathrm{Fun}(X, Y)) \simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n \times X, Y)$$

commutes with filtered colimits for each $n \geq 0$. This is a special case of Proposition 3.6.1.9, since the product $\Delta^n \times X$ is also a finite simplicial set (Remark 3.6.1.6 and Example 3.6.1.2. □

Let X be a simplicial set having geometric realization $|X|$. For every simplicial subset $X' \subseteq X$, the inclusion of X' into X induces a homeomorphism from $|X'|$ onto a closed subset of $|X|$. In what follows, we will abuse notation by identifying $|X'|$ with its image in $|X|$.

Proposition 3.6.1.11. *Let X be a simplicial set. Then a subset $K \subseteq |X|$ is compact if and only if it is closed and contained in $|X'| \subseteq |X|$, for some finite simplicial subset $X' \subseteq X$.* 0139

Corollary 3.6.1.12. *A simplicial set X is finite if and only if the topological space $|X|$ is compact.* 013A

The proof of Proposition 3.6.1.11 is based on the following observation:

Lemma 3.6.1.13. *Let X be a simplicial set and let S be a subset of the geometric realization $|X|$. Suppose that, for every nondegenerate n-simplex σ of X, the inverse image of S under the composite map $|\Delta^n| \to |X|$ contains only finitely many points of the interior $|\overset{\circ}{\Delta}{}^n| \subseteq |\Delta^n|$. Then S is closed.* 013B

Proof. The geometric realization $|X|$ can be described as the colimit $\varinjlim_{\sigma:\Delta^n \to X} |\Delta^n|$, indexed by the category of simplices of X (see Construction 1.1.3.9). Consequently, to show that the subset $S \subseteq |X|$ is closed, it will suffice to show that the inverse image $|\sigma|^{-1}(S) \subseteq |\Delta^n|$ is closed, for every n-simplex $\sigma : \Delta^n \to X$. We proceed by induction on n. Using Proposition 1.1.3.8, we can reduce to the case where σ is nondegenerate. In this case, our inductive hypothesis guarantees that $|\sigma|^{-1}(S)$ has closed intersection with the boundary $|\,\partial \Delta^n\,| \subseteq |\Delta^n|$. Since $|\sigma|^{-1}(S)$ contains only finitely many points in the interior of $|\Delta^n|$, it is closed. □

Proof of Proposition 3.6.1.11. Let X be a simplicial set. If $X' \subseteq X$ is a finite simplicial subset, then the geometric realization $|X'|$ is a continuous image of a finite disjoint union $\coprod_{i \in I} |\Delta^{n_i}|$ (Proposition 3.6.1.7), and is therefore compact. It follows that any closed subset $K \subseteq |X'|$ is also compact. Conversely suppose that $K \subseteq |X|$ is compact. Since $|X|$ is Hausdorff, the set K is closed. We wish to show that K is contained in $|X'|$ for some finite simplicial subset $X' \subseteq X$. Suppose otherwise. Then we can choose an infinite collection of nondegenerate simplices $\{\sigma_j : \Delta^{n_j} \to X\}_{j \in J}$ for which each of the corresponding cells $|\mathring{\Delta}^{n_j}| \hookrightarrow |X|$ contains some point $x_j \in K$. Applying Lemma 3.6.1.13, we deduce that for every subset $J' \subseteq J$, the set $\{x_j\}_{j \in J'}$ is closed in $|X|$. In particular, $\{x_j\}_{j \in J}$ is an infinite closed subset of K endowed with the discrete topology, contradicting our assumption that $K \subseteq |X|$ is compact. □

3.6.2 Exactness of Geometric Realization

013C Our goal in this section is to study the exactness properties of the geometric realization functor $X \mapsto |X|$ of Definition 1.2.3.1. Our main result can be stated as follows:

013D **Theorem 3.6.2.1.** *The geometric realization functor*

$$\mathrm{Set}_{\Delta} \to \mathrm{Set} \qquad X \mapsto |X|$$

preserves finite limits. In particular, for every diagram of simplicial sets $X \to Z \leftarrow Y$, the induced map $|X \times_Z Y| \to |X| \times_{|Z|} |Y|$ is a bijection.

Before giving the proof of Theorem 3.6.2.1, let us collect some consequences.

013E **Corollary 3.6.2.2.** *Let X and Y be simplicial sets. Then the canonical map $\theta_{X,Y}$: $|X \times Y| \to |X| \times |Y|$ is a bijection. If either X or Y is finite, then θ is a homeomorphism.*

Proof. The first assertion follows immediately from Theorem 3.6.2.1. If X and Y are both finite, then the product $X \times Y$ is also finite (Remark 3.6.1.6), so that the geometric realizations $|X|$, $|Y|$, and $|X \times Y|$ are compact Hausdorff spaces (Corollary 3.6.1.12). In this case, $\theta_{X,Y}$ is a continuous bijection between compact Hausdorff spaces, and therefore a homeomorphism.

Now suppose that X is finite and Y is arbitrary. Let $M = \mathrm{Hom}_{\mathrm{Top}}(|X|, |X \times Y|)$ denote the set of all continuous functions from $|X|$ to $|X \times Y|$, endowed with the compact-open topology. For every finite simplicial subset $Y' \subseteq Y$, the composite map

$$|X| \times |Y'| \xrightarrow{\theta^{-1}_{X,Y'}} |X \times Y'| \hookrightarrow |X \times Y|,$$

determines a continuous function $\rho_{Y'} : |Y'| \to M$. Writing the geometric realization $|Y|$ as a colimit $\varinjlim_{Y' \subseteq Y} |Y'|$ (see Remark 3.6.1.8), we can amalgamate the functions $f_{Y'}$ to a single continuous function $\rho : |Y| \to M$. Our assumption that X guarantees that the topological space $|X|$ is compact and Hausdorff, so the evaluation map

$$\mathrm{ev} : |X| \times M \to |X \times Y| \qquad (x, f) \mapsto f(x)$$

is continuous (see Theorem [?]). We complete the proof by observing that the bijection $\theta^{-1}_{X,Y}$ is a composition of continuous functions

$$|X| \times |Y| \xrightarrow{\mathrm{id} \times \rho} |X| \times M \xrightarrow{\mathrm{ev}} |X \times Y|,$$

and is therefore continuous. □

Warning 3.6.2.3. Let X and Y be simplicial sets. If neither X or Y is assumed to be finite, then the comparison map $\theta_{X,Y} : |X \times Y| \to |X| \times |Y|$ need not be a homeomorphism. For an explicit counterexample, we refer the reader to Section 5 of [15]. 013F

Remark 3.6.2.4. Let X and Y be simplicial sets having at most countably many simplices of each dimension. Then the comparison map $\theta_{X,Y} : |X \times Y| \to |X| \times |Y|$ is a homeomorphism. For a proof, we refer the reader to [44]. 013G

Example 3.6.2.5. Let X be a simplicial set and let Y be a topological space, and let $\mathrm{Hom}_{\mathrm{Top}}(|X|, Y)_{\bullet}$ be the simplicial set defined in Example 2.4.1.5. For each $n \geq 0$, precomposition with the homeomorphism $|X \times \Delta^n| \to |X| \times |\Delta^n|$ induces a bijection 013H

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Top}}(|X|, Y)_n &= \mathrm{Hom}_{\mathrm{Top}}(|X| \times |\Delta^n|, Y) \\ &\simeq \mathrm{Hom}_{\mathrm{Top}}(|X \times \Delta^n|, Y) \\ &\simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(X \times \Delta^n, \mathrm{Sing}_{\bullet}(Y)) \\ &= \mathrm{Fun}(X, \mathrm{Sing}_{\bullet}(Y))_n. \end{aligned}$$

These bijections are compatible with face and degeneracy operators, and therefore determine an isomorphism of simplicial sets $\mathrm{Hom}_{\mathrm{Top}}(|X|, Y)_{\bullet} \to \mathrm{Fun}(X, \mathrm{Sing}_{\bullet}(Y))$.

We now turn to the proof of Theorem 3.6.2.1. Our proof will make use of an explicit description of the underlying set of a geometric realization $|X|$ (see Remark 3.6.2.10) which given by Drinfeld in [16] (and also appears in unpublished work of Besser and Grayson).

013J **Construction 3.6.2.6.** Let S be a finite subset of the unit interval $[0,1]$, and assume that $0,1 \in S$. For each $n \geq 0$, we let $|\Delta^n|_S$ denote the subset of the topological n-simplex

$$|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathbf{R}^{n+1}_{\geq 0} : t_0 + t_1 + \cdots + t_n = 1\}$$

consisting of those tuples $(t_0, t_1, \ldots, t_n)$ having the property that each of the partial sums $t_0 + t_1 + \cdots t_i$ belongs to S. Note that these subsets are stable under the coface and codegeneracy operators of the cosimplicial topological space $|\Delta^\bullet|$, so we can regard the construction $[n] \mapsto |\Delta^n|_S$ as a cosimplicial set.

By virtue of Proposition 1.2.3.15, the functor

$$\mathrm{Set} \to \mathrm{Set}_\Delta \qquad (Y \mapsto ([n] \mapsto \mathrm{Hom}_{\mathrm{Set}}(|\Delta^n|_S, Y)))$$

admits a left adjoint, which we will denote by $|\bullet|_S : \mathrm{Set}_\Delta \to \mathrm{Set}$ and refer to as the *S-partial geometric realization.* Concretely, this functor carries a simplicial set X to the colimit $|X|_S = \varinjlim_{\Delta^n \to X} |\Delta^n|_S$, where the colimit is indexed by the category of simplices $\mathbf{\Delta}_X$ of Construction 1.1.3.9.

013K **Remark 3.6.2.7.** For each integer $n \geq 0$, the topological n-simplex $|\Delta^n|$ can be identified with the filtered direct limit $\varinjlim_S |\Delta^n|_S$, where S ranges over the collection of all finite subsets of $[0,1]$ which contain the endpoints 0 and 1 (which we regard as a partially ordered set with respect to inclusion). We therefore obtain a canonical isomorphism of cosimplicial sets $\varinjlim_S |\Delta^\bullet|_S \xrightarrow{\sim} |\Delta^\bullet|$. It follows that, for every simplicial set X, the canonical map $\varinjlim_S |X|_S \to |X|$ is a bijection.

013L **Notation 3.6.2.8.** Let $\mathrm{Lin}_{\neq\emptyset}$ denote the category whose objects are nonempty finite linearly ordered sets, and whose morphisms are nondecreasing functions. Note that, if S is a finite subset of the unit interval $[0,1]$, then the complement $[0,1] \setminus S$ has finitely many connected components. Moreover, there is a unique linear ordering on the set $\pi_0([0,1] \setminus S)$ for which the quotient map

$$([0,1] \setminus S) \to \pi_0([0,1] \setminus S)$$

is nondecreasing. We can therefore regard $\pi_0([0,1] \setminus S)$ as an object of the category $\mathrm{Lin}_{\neq\emptyset}$.

013M **Proposition 3.6.2.9.** *Let S be a finite subset of the unit interval $[0,1]$ which contains 0 and 1. Then the cosimplicial set*

$$|\Delta^\bullet|_S : \mathbf{\Delta} \to \mathrm{Set} \qquad [n] \mapsto |\Delta^n|_S$$

is a corepresentable functor. More precisely, there exists a functorial bijection $|\Delta^n|_S \simeq \mathrm{Hom}_{\mathrm{Lin}_{\neq\emptyset}}(\pi_0([0,1] \setminus S), [n])$.

Proof. Let $S = \{0 = s_0 < s_1 < \cdots < s_k = 1\}$ be a finite subset of the unit interval $[0,1]$ which contains 0 and 1. Let n be a nonnegative integer and let $(t_0, \ldots, t_n)$ be a point of $|\Delta^n|_S$. For every real number $u \in [0,1] \setminus S$, there exists a unique integer $0 \leq i \leq n$ satisfying

$$t_0 + t_1 + \cdots + t_{i-1} < u < t_0 + t_1 + \cdots + t_i.$$

The construction $u \mapsto i$ defines a continuous nondecreasing function $([0,1] \setminus S) \to [n]$. This observation induces a bijection

$$\begin{aligned} |\Delta^n|_S &\simeq \{\text{Continuous nondecreasing functions } f : [0,1] \setminus S \to [n]\} \\ &\simeq \operatorname{Hom}_{\mathrm{Lin}_{\neq \emptyset}}(\pi_0([0,1] \setminus S), [n]). \end{aligned}$$

Explicitly, the inverse bijection carries a continuous nondecreasing function $f : [0,1] \setminus S \to [n]$ to the sequence

$$(\mu(f^{-1}\{0\}), \mu(f^{-1}\{1\}), \cdots, \mu(f^{-1}\{n\})),$$

where

$$\mu(f^{-1}\{i\}) = \sum_{(s_{j-1}, s_j) \subseteq f^{-1}\{i\}} (s_j - s_{j-1})$$

denotes the measure of the inverse image $f^{-1}\{i\}$. □

Proof of Theorem 3.6.2.1. Let $U : \mathrm{Top} \to \mathrm{Set}$ denote the forgetful functor. We wish to show that the composite functor

$$\mathrm{Set}_{\Delta} \xrightarrow{|\bullet|} \mathrm{Top} \xrightarrow{U} \mathrm{Set}$$

preserves finite limits. By virtue of Remark 3.6.2.7, we can write this composite functor as a filtered colimit of functors of the form $X \mapsto |X|_S$, where S ranges over all finite subsets of the unit interval $[0,1]$ which contain 0 and 1. It will therefore suffice to show that each of the functors $X \mapsto |X|_S$ preserves finite limits. Using Proposition 3.6.2.9, see that $X \mapsto |X|_S$ can be identified with the evaluation functor $X \mapsto X_m$, where m is chosen so that there is an isomorphism of linearly ordered sets $[m] \simeq \pi_0([0,1] \setminus S)$. □

Remark 3.6.2.10. Let X be a simplicial set, which we view as a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to the category of sets. Then X admits a canonical extension to a functor $\mathrm{Lin}_{\neq \emptyset}^{\mathrm{op}} \to \mathrm{Set}$, given on objects by the construction $(I = \{i_0 < i_1 < \cdots < i_n\}) \mapsto X_n$. Let us write $X(I)$ for the value of this extension on an object $I \in \mathrm{Lin}_{\neq \emptyset}$. Arguing as in the proof of Theorem 3.6.2.1, we obtain a canonical bijection 013N

$$\varinjlim_{S} X([0,1] \setminus S) \simeq \varinjlim_{S} |X|_S \xrightarrow{\sim} X,$$

where the (filtered) colimit is taken over the collection of all finite subsets $S \subseteq [0,1]$ containing 0 and 1.

3.6.3 Weak Homotopy Equivalences in Topology

013P Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous function. Recall that f is a *homotopy equivalence* if there exists a continuous function $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity maps id_X and id_Y, respectively. In other words, f is a homotopy equivalence if its homotopy class $[f]$ is invertible when regarded as a morphism in the homotopy category of topological spaces hTop (see Example 2.4.6.6). For some purposes, it is convenient to consider a somewhat weaker condition.

013Q **Definition 3.6.3.1.** Let X and Y be topological spaces. We say that a continuous function $f : X \to Y$ is a *weak homotopy equivalence* if the induced map of singular simplicial sets $\mathrm{Sing}_\bullet(f) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(Y)$ is a homotopy equivalence (Definition 3.1.6.1).

013R **Remark 3.6.3.2.** Let $f : X \to Y$ be a continuous function between topological spaces. Then f is a weak homotopy equivalence of topological spaces if and only if $\mathrm{Sing}_\bullet(f)$ is a weak homotopy equivalence of simplicial sets. This is a special case of Proposition 3.1.6.13, since the simplicial sets $\mathrm{Sing}_\bullet(X)$ and $\mathrm{Sing}_\bullet(Y)$ are Kan complexes (Proposition 1.2.5.8).

013S **Example 3.6.3.3.** Let X and Y be topological spaces, and let $f : X \to Y$ be a homotopy equivalence. Then f is a weak homotopy equivalence. This is a reformulation of Example 3.1.6.3.

013T **Remark 3.6.3.4.** Let $f : X \to Y$ be a continuous function between topological spaces. Then f is a weak homotopy equivalence if and only if it satisfies the following pair of conditions:

- The induced map of path components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.
- For every point $x \in X$ and every $n \geq 1$, the map of homotopy groups $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism.

This follows by applying Theorem 3.2.7.1 to the map of Kan complexes $\mathrm{Sing}_\bullet(f) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(Y)$ (see Example 3.2.2.7).

013U **Example 3.6.3.5.** We say that a topological space X is *weakly contractible* if the projection map $f : X \to *$ is a weak homotopy equivalence (in other words, X is weakly contractible if the singular simplicial set $\mathrm{Sing}_\bullet(X)$ is a contractible Kan complex). Using Remark 3.6.3.4, we see that X is weakly contractible if and only if it is path connected (that is, the set $\pi_0(X)$ is a singleton) and the homotopy groups $\pi_n(X, x)$ are trivial for $n > 0$ and any choice of base point $x \in X$ (assuming that X is path connected, this condition is independent of the choice of base point).

Remark 3.6.3.6. Recall that a topological space X is *contractible* if the projection map $X \to *$ is a homotopy equivalence. Equivalently, X is contractible if the identity map $\mathrm{id}_X : X \to X$ is homotopic to the constant function $X \to \{x\} \hookrightarrow X$, for some base point $x \in X$. It follows from Example 3.6.3.3 that every contractible topological space is weakly contractible. In particular, for each $n \geq 0$, the standard simplex $|\Delta^n|$ is weakly contractible. 013V

Example 3.6.3.7. Let X be a topological space with the property that every continuous path $p : [0,1] \to X$ is constant (this condition is satisfied, for example, if X is totally disconnected). Let X' denote the topological space whose underlying set coincides with X, but endowed with the discrete topology. Then the identity map $f : X' \to X$ induces an isomorphism of singular simplicial sets $\mathrm{Sing}_{\bullet}(X') \to \mathrm{Sing}_{\bullet}(X)$, and is therefore a weak homotopy equivalence of topological spaces. However, f is a homotopy equivalence if and only if the topology on X is discrete (since any homotopy inverse of f must coincide with the identity map $f^{-1} : X \to X'$). 013W

Example 3.6.3.7 illustrates that the notions of homotopy equivalence and weak homotopy equivalence are not the same in general. However, they agree for sufficiently nice topological spaces.

Proposition 3.6.3.8. *Let $f : X \to Y$ be a weak homotopy equivalence of topological spaces. Assume that both X and Y have the homotopy type of a CW complex (that is, there exist homotopy equivalences $X' \to X$ and $Y' \to Y$, where X' and Y' are CW complexes). Then f is a homotopy equivalence.* 013X

Warning 3.6.3.9. In the formulation of Proposition 3.6.3.8, the hypothesis that X and Y have the homotopy type of a CW complex cannot be omitted. For any topological space Y, the counit map $v : |\mathrm{Sing}_{\bullet}(Y)| \to Y$ is a weak homotopy equivalence (Corollary 3.6.4.2), whose domain is a CW complex (Remark 1.2.3.12). If Y satisfies the conclusion of Proposition 3.6.3.8, then v is a homotopy equivalence, so Y has the homotopy type of a CW complex. 013Y

Corollary 3.6.3.10 (Whitehead's Theorem for Topological Spaces)**.** *Let X and Y be topological spaces having the homotopy type of CW complexes, and let $f : X \to Y$ be a continuous function. Then f is a homotopy equivalence if and only if it satisfies the following pair of conditions:* 013Z

- *The induced map of path components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.*
- *For every point $x \in X$ and every $n \geq 1$, the map of homotopy groups $\pi_n(f) : \pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism.*

Proof. Combine Remark 3.6.3.4 with Proposition 3.6.3.8 (and Example 3.6.3.3). □

We will deduce Proposition 3.6.3.8 from the following:

0140 **Lemma 3.6.3.11.** *Let $f : X \to Y$ be a weak homotopy equivalence of topological spaces, let K be a CW complex, and let $g : K \to Y$ be a continuous function. Then there exists a continuous function $\overline{g} : K \to X$ such that g is homotopic to $f \circ \overline{g}$.*

Proof. For each $n \geq -1$, let $\mathrm{sk}_n(K)$ denote the n-skeleton of K (with respect to some fixed cell decomposition), so that $\mathrm{sk}_{-1}(K) = \emptyset$. To prove Lemma 3.6.3.11, it will suffice to construct a compatible sequence of continuous functions $\overline{g}_n : \mathrm{sk}_n(K) \to X$ and homotopies $h_n : [0,1] \times \mathrm{sk}_n(K) \to Y$ from $\overline{g}_n$ to $g|_{\mathrm{sk}_n(K)}$. We proceed by recursion. Assume that $n \geq 0$ and that the pair $(\overline{g}_{n-1}, h_{n-1})$ has already been constructed. Let S denote the collection of n-cells of K. For each $s \in S$, let $b_s : |\,\partial\Delta^n\,| \to \mathrm{sk}_{n-1}(K)$ denote the corresponding attaching map. To construct the pair $(\overline{g}_n, h_n)$, it will suffice to show that each composition $\overline{g}_{n-1} \circ b_s$ can be extended to a continuous map $u_s : |\Delta^n| \to X$ and that each composition $h_{n-1} \circ (b_s \times \mathrm{id}_{[0,1]})$ can be extended to a homotopy from u_s to $g|_{|\Delta^n|}$. Unwinding the definitions, we can rephrase this as a lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Sing}_\bullet(X) \times_{\mathrm{Fun}(\{0\}, \mathrm{Sing}_\bullet(Y))} \mathrm{Fun}(\Delta^1, \mathrm{Sing}_\bullet(Y)) \arrow[d, "\theta"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\{1\}, \mathrm{Sing}_\bullet(Y))
\end{tikzcd}$$

in the category of simplicial sets. Here the morphism θ is the path fibration of Example 3.1.7.10 (associated to the map of Kan complexes $\mathrm{Sing}_\bullet(f) : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(Y)$). Our assumption that f is a weak homotopy equivalence guarantees that $\mathrm{Sing}_\bullet(f)$ is a homotopy equivalence of Kan complexes, so that θ is also a homotopy equivalence. Applying Proposition 3.2.7.2, we deduce that θ is a trivial Kan fibration, so that the lifting problem admits a solution as desired. □

Proof of Proposition 3.6.3.8. In what follows, we denote the homotopy class of a continuous function $f : X \to Y$ by $[f]$. Let $f : X \to Y$ be a weak homotopy equivalence of topological spaces, and suppose that there exists a homotopy equivalence $u : Y' \to Y$, where Y' is a CW complex. Using Lemma 3.6.3.11, we deduce that $[u] = [f] \circ [\overline{u}]$ for some continuous function $\overline{u} : Y' \to X$. Let $v : Y \to Y'$ be a homotopy inverse to u and set $g = \overline{u} \circ v$. Then

$$[f] \circ [g] = [f \circ \overline{u}] \circ [v] = [u] \circ [v] = [\mathrm{id}_Y],$$

so g is a right homotopy inverse to f. Since f is a weak homotopy equivalence, it follows that g is also a weak homotopy equivalence. If X also has the homotopy type of a CW complex, then we can apply the same reasoning to deduce that g admits a right homotopy inverse $f' : X \to Y$. Then

$$[g] \circ [f] = [g] \circ [f] \circ [\mathrm{id}_X] = [g] \circ [f] \circ [g] \circ [f'] = [g] \circ [\mathrm{id}_Y] \circ [f'] = [g] \circ [f'] = [\mathrm{id}_X].$$

It follows that g is also a left homotopy inverse to f, so that f is a homotopy equivalence (with homotopy inverse g). □

3.6.4 The Unit Map $u : X \to \mathrm{Sing}_\bullet(|X|)$

Our goal in this section is to prove the following result: 0141

Theorem 3.6.4.1 (Milnor). *Let X be a simplicial set. Then the unit map $u_X : X \to \mathrm{Sing}_\bullet(|X|)$ is a weak homotopy equivalence of simplicial sets.* 0142

Theorem 3.6.4.1 was proved by Milnor in [44]. It is closely related to the following earlier result of Giever ([25]):

Corollary 3.6.4.2. *Let X be a topological space. Then the counit map $v_X : |\mathrm{Sing}_\bullet(X)| \to X$ is a weak homotopy equivalence of topological spaces.* 0143

Proof. We must show that $\mathrm{Sing}_\bullet(v_X) : \mathrm{Sing}_\bullet(|\mathrm{Sing}_\bullet(X)|) \to \mathrm{Sing}_\bullet(X)$ is a homotopy equivalence of Kan complexes. This is clear, since $\mathrm{Sing}_\bullet(v_X)$ is left inverse to the unit map $u_{\mathrm{Sing}_\bullet(X)} : \mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(|\mathrm{Sing}_\bullet(X)|)$, which is a weak homotopy equivalence by virtue of Theorem 3.6.4.1 (and therefore a homotopy equivalence, since both $\mathrm{Sing}_\bullet(X)$ and $\mathrm{Sing}_\bullet(|\mathrm{Sing}_\bullet(X)|)$ are Kan complexes). □

Corollary 3.6.4.3. *Let $f : X \to Y$ be a morphism of simplicial sets. The following conditions are equivalent:* 03X7

(1) *The morphism f is a weak homotopy equivalence, in the sense of Definition 3.1.6.12.*

(2) *The induced map of topological spaces $|X| \to |Y|$ is a weak homotopy equivalence, in the sense of Definition 3.6.3.1.*

(3) *The induced map of topological spaces $|X| \to |Y|$ is a homotopy equivalence.*

Proof. We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u_X"] & Y \arrow[d, "u_Y"] \\
\mathrm{Sing}_\bullet(|X|) \arrow[r, "\mathrm{Sing}_\bullet(|f|)"] & \mathrm{Sing}_\bullet(|Y|),
\end{tikzcd}$$

where the vertical maps are weak homotopy equivalences by virtue of Theorem 3.6.4.1. The equivalence (1) ⇔ (2) now follows from Remark 3.1.6.16. The implication (3) ⇒ (2) follows from Example 3.6.3.3, and the reverse implication is a special case of Proposition 3.6.3.8 (since the topological spaces $|X|$ and $|Y|$ are CW complexes; see Remark 1.2.3.12). □

03X8 **Example 3.6.4.4.** A simplicial set X is weakly contractible if and only if the geometric realization $|X|$ is a contractible topological space.

Proof of Theorem 3.6.4.1. Let X be a simplicial set. By virtue of Remark 3.6.1.8, we can write X as a filtered colimit of finite simplicial subsets $X' \subseteq X$. It follows from Proposition 3.6.1.11 that, for any compact topological space K, every continuous function $f : K \to |X|$ factors through $|X'| \subseteq |X|$ for some finite simplicial subset $X' \subseteq X$. Applying this observation in the case $K = |\Delta^n|$, we conclude that the natural map $\varinjlim_{X' \subseteq X} \operatorname{Sing}_\bullet(|X'|) \to \operatorname{Sing}_\bullet(|X|)$ is an isomorphism of simplicial sets. It follows that the unit map $u_X : X \to \operatorname{Sing}_\bullet(|X|)$ can be realized as filtered colimit of unit maps $u_{X'} : X' \to \operatorname{Sing}_\bullet(|X'|)$. Since the collection of weak homotopy equivalences is closed under filtered colimits (Proposition 3.2.8.3), it will suffice to show that each of the morphisms $u_{X'}$ is a weak homotopy equivalence. Replacing X by X', we are reduced to proving Theorem 3.6.4.1 under the additional assumption that the simplicial set X is finite.

We now proceed by induction on the dimension of X. If X is empty, then u_X is an isomorphism and the result is obvious. Otherwise, let $n \geq 0$ be the dimension of X. We proceed by induction on the number of nondegenerate n-simplices of X. Using Proposition 1.1.4.12, we can choose a pushout diagram

0144
$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\
X' \arrow[r] & X,
\end{tikzcd} \tag{3.39}$$

where X' is a simplicial subset of X with a smaller number of nondegenerate n-simplices. Since the inclusion $\partial \Delta^n \hookrightarrow \Delta^n$ is a monomorphism, the diagram (3.39) is also a homotopy pushout square (Proposition 3.4.2.11). By virtue of our inductive hypotheses, the unit morphisms $u_{X'}$ and $u_{\partial \Delta^n}$ are weak homotopy equivalences. Since the simplicial sets Δ^n and $\operatorname{Sing}_\bullet(|\Delta^n|)$ are contractible (Remark 3.2.4.16), the unit map u_{Δ^n} is also a (weak) homotopy equivalence. Invoking Proposition 3.4.2.9, we see that u_X is a homotopy equivalence if and only if the diagram of simplicial sets

$$\begin{tikzcd}
\operatorname{Sing}_{\bullet}(|\,\partial\Delta^n\,|) \arrow[r] \arrow[d] & \operatorname{Sing}_{\bullet}(|\Delta^n|) \arrow[d] \\
\operatorname{Sing}_{\bullet}(|X'|) \arrow[r] & \operatorname{Sing}_{\bullet}(|X|),
\end{tikzcd} \tag{3.40}$$ 0145

is also homotopy pushout square.

Let $V = |\Delta^n| \setminus |\,\partial\Delta^n\,|$ be the interior of the topological n-simplex, and fix a point $v \in V$ having image $x \in |X|$. We then have a commutative diagram of simplicial sets

$$\begin{tikzcd}
 & \operatorname{Sing}_{\bullet}(V \setminus \{v\}) \arrow[r] \arrow[d] & \operatorname{Sing}_{\bullet}(V) \arrow[d] \\
\operatorname{Sing}_{\bullet}(|\,\partial\Delta^n\,|) \arrow[r] \arrow[d] & \operatorname{Sing}_{\bullet}(|\Delta^n| \setminus \{v\}) \arrow[r] \arrow[d] & \operatorname{Sing}_{\bullet}(|\Delta^n|) \arrow[d] \\
\operatorname{Sing}_{\bullet}(|X'|) \arrow[r] & \operatorname{Sing}_{\bullet}(|X| \setminus \{x\}) \arrow[r] & \operatorname{Sing}_{\bullet}(|X|).
\end{tikzcd} \tag{3.41}$$ 0146

Note that the left horizontal maps and the upper vertical maps are homotopy equivalences, since they are obtained from homotopy equivalences of topological spaces

$$|X'| \hookrightarrow |X| \setminus \{x\} \qquad |\,\partial\Delta^n\,| \hookrightarrow |\Delta^n| \setminus \{v\} \hookleftarrow V \setminus \{v\} \qquad |\Delta^n| \hookleftarrow V$$

(see Example 3.6.3.3). It follows that the upper square and left square in diagram (3.41) are homotopy pushout squares (Proposition 3.4.2.10). Moreover, the outer rectangle on the right is a homotopy pushout square by virtue of Theorem 3.4.6.1. Applying Proposition 3.4.1.11, we deduce that the lower right square and bottom rectangle are also homotopy pushout squares. □

3.6.5 Comparison of Homotopy Categories

Our goal in this section is to carry out the proof of Theorem 3.6.0.1. We begin with an elementary application of the results of §3.6.2. 0147

Construction 3.6.5.1 (Geometric Realization as a Simplicial Functor)**.** Let X and Y be simplicial sets and let σ be an n-simplex of the simplicial set $\operatorname{Fun}(X, Y)$, which we identify 0148 with a morphism $\Delta^n \times X \to Y$. By virtue of Corollary 3.6.2.2, the geometric realization of σ can be identified with a continuous function

$$|\sigma| : |\Delta^n| \times |X| \to |Y|,$$

which we can view as an n-simplex of the simplicial set $\mathrm{Hom}_{\mathrm{Top}}(|X|, |Y|)_\bullet$ parametrizing continuous functions from X to Y (see Example 2.4.1.5). This construction is compatible with face and degeneracy operators, and therefore determines a morphism of simplicial sets $\mathrm{Fun}(X, Y) \to \mathrm{Hom}_{\mathrm{Top}}(|X|, |Y|)_\bullet$. Allowing X and Y to vary, we obtain a simplicial structure on the geometric realization functor $| \bullet | : \mathrm{Set}_\Delta \to \mathrm{Top}$.

0149 **Proposition 3.6.5.2.** *Let X and Y be simplicial sets. If Y is a Kan complex, then the comparison map*

$$\theta : \mathrm{Fun}(X, Y) \to \mathrm{Hom}_{\mathrm{Top}}(|X|, |Y|)_\bullet$$

of Construction 3.6.5.1 is a homotopy equivalence of Kan complexes.

Proof. Using Example 3.6.2.5, we can identify θ with the morphism

$$\mathrm{Fun}(X, Y) \to \mathrm{Fun}(X, \mathrm{Sing}_\bullet(|Y|))$$

given by postcomposition with the unit map $u_Y : Y \to \mathrm{Sing}_\bullet(|Y|)$. By virtue of Theorem 3.6.4.1, the map u_Y is a weak homotopy equivalence. Since Y and $\mathrm{Sing}_\bullet(|Y|)$ are Kan complexes, we conclude that u_Y is a homotopy equivalence (Proposition 3.1.6.13). It follows that θ is also a homotopy equivalence (it admits a homotopy inverse, given by postcomposition with any homotopy inverse to u_Y). □

014A **Proposition 3.6.5.3.** *Let X be a topological space. The following conditions are equivalent:*

(1) *The counit map $|\mathrm{Sing}_\bullet(X)| \to X$ is a homotopy equivalence of topological spaces.*

(2) *There exists a Kan complex Y and a homotopy equivalence of topological spaces $|Y| \to X$.*

(3) *There exists a simplicial set Y and a homotopy equivalence of topological spaces $|Y| \to X$.*

(4) *There exists a homotopy equivalence of topological spaces $X' \to X$, where X' is a CW complex.*

Proof. The implication (1) ⇒ (2) follows from the observation that $\mathrm{Sing}_\bullet(X)$ is a Kan complex (Proposition 1.2.5.8), the implication (2) ⇒ (3) is trivial, and the implication (3) ⇒ (4) follows from Remark 1.2.3.12. To complete the proof, it will suffice to show that if X has the homotopy type of a CW complex, then the counit map $v : |\mathrm{Sing}_\bullet(X)| \to X$ is a homotopy equivalence. By virtue of Proposition 3.6.3.8, it will suffice to show that v is a weak homotopy equivalence, which follows from Corollary 3.6.4.2. □

Corollary 3.6.5.4. *Let $f : Y \to Z$ be a continuous function between topological spaces. The following conditions are equivalent:* `04HV`

(1) *The function f is a weak homotopy equivalence (Definition 3.6.3.1).*

(2) *For every simplicial set S, the induced map $\mathrm{Fun}(S, \mathrm{Sing}_\bullet(Y)) \to \mathrm{Fun}(S, \mathrm{Sing}_\bullet(Z))$ is a homotopy equivalence of Kan complexes.*

(3) *For every simplicial set S, the induced map $\mathrm{Hom}_{\mathrm{Top}}(|S|, Y)_\bullet \to \mathrm{Hom}_{\mathrm{Top}}(|S|, Z)_\bullet$ is a homotopy equivalence of Kan complexes.*

(4) *For every topological space X which has the homotopy type of a CW complex, the induced map $\mathrm{Hom}_{\mathrm{Top}}(X, Y)_\bullet \to \mathrm{Hom}_{\mathrm{Top}}(X, Z)_\bullet$ is a homotopy equivalence of Kan complexes.*

Proof. The equivalence (1) ⇔ (2) follows from Proposition 3.1.6.17, the equivalence (2) ⇔ (3) from Example 3.6.2.5, and the equivalence (3) ⇔ (4) from Proposition 3.6.5.3. □

Proof of Theorem 3.6.0.1. Using Construction 3.6.5.1, we see that the geometric realization functor $| \bullet | : \mathrm{Set}_\Delta \to \mathrm{Top}$ induces a functor of homotopy categories $| \bullet | : \mathrm{hKan} \to \mathrm{hTop}$. It follows from Proposition 3.6.5.2 that this functor is fully faithful, and from Proposition 3.6.5.3 that its essential image consists of those topological spaces X which have the homotopy type of a CW complex. □

Remark 3.6.5.5. Proposition 3.6.5.2 implies a stronger version of Theorem 3.6.0.1: the simplicially enriched functor $| \bullet | : \mathrm{Kan} \to \mathrm{Top}$ induces a fully faithful embedding of ∞-categories $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Top})$ (see Remark 5.5.1.9). `014B`

Using Theorem 3.6.4.1, we can also give a purely topological characterization of the homotopy category hKan (which does not make reference to the theory of simplicial sets).

Corollary 3.6.5.6. *Let $\mathcal{C}$ be a category, and let $\mathcal{E}' \subseteq \mathrm{Fun}(\mathrm{Top}, \mathcal{C})$ be the full subcategory spanned by those functors $F : \mathrm{Top} \to \mathcal{C}$. which carry weak homotopy equivalences of topological spaces to isomorphisms in the category $\mathcal{C}$. Then:* `014C`

(a) *For every functor $F \in \mathcal{E}'$, the composite functor*

$$\mathrm{Kan} \xrightarrow{|\bullet|} \mathrm{Top} \xrightarrow{F} \mathcal{C}$$

factors uniquely as a composition $\mathrm{Kan} \twoheadrightarrow \mathrm{hKan} \xrightarrow{\overline{F}} \mathcal{C}$.

(b) *The construction $F \mapsto \overline{F}$ induces an equivalence of categories $\mathcal{E}' \to \mathrm{Fun}(\mathrm{hKan}, \mathcal{C})$.*

We can state Corollary 3.6.5.6 more informally as follows: the homotopy category hKan of Kan complexes can be obtained from the category of topological spaces Top by formally adjoining inverses to all weak homotopy equivalences.

Proof of Corollary 3.6.5.6. Let $\mathcal{E} \subseteq \operatorname{Fun}(\operatorname{Kan}, \mathcal{C})$ be the full subcategory spanned by those functors $F : \operatorname{Kan} \to \mathcal{C}$ which carry homotopy equivalences of Kan complexes to isomorphisms in $\mathcal{C}$. By virtue of Corollary 3.1.7.7, it will suffice to show that precomposition with the geometric realization functor $| \bullet | : \operatorname{Kan} \to \operatorname{Top}$ induces an equivalence of categories $\mathcal{E}' \to \mathcal{E}$. We claim that this functor has a homotopy inverse $\mathcal{E} \to \mathcal{E}'$, given by precomposition with the functor $\operatorname{Sing}_{\bullet} : \operatorname{Top} \to \operatorname{Kan}$. This follows from the following pair of observations:

- For every functor $F : \operatorname{Top} \to \mathcal{C}$, the counit map $\overline{F} \circ \operatorname{Sing}_{\bullet} \to F$ is an isomorphism when F belongs to $\mathcal{E}'$ (since, for every topological space X, the counit map $|\operatorname{Sing}_{\bullet}(X)| \to X$ is a weak homotopy equivalence; see Corollary 3.6.4.2).
- For every functor $F_0 : \operatorname{Kan} \to \mathcal{C}$, the unit map $F_0 \to \overline{F_0 \circ \operatorname{Sing}_{\bullet}}$ is an isomorphism (since, for every simplicial set Y, the unit map $Y \to \operatorname{Sing}_{\bullet}(|Y|)$ is a weak homotopy equivalence of simplicial sets, and therefore induces a homotopy equivalence of topological spaces $|Y| \to |\operatorname{Sing}_{\bullet}(|Y|)|$).

□

3.6.6 Serre Fibrations

021Q We now study the counterpart of Definition 3.1.1.1 in the setting of topological spaces.

021R **Definition 3.6.6.1.** Let $q : X \to S$ be a continuous function between topological spaces. We say that q is a *Serre fibration* if, for every integer $n \geq 0$, every lifting problem

$$\begin{tikzcd}
\{0\} \times |\Delta^n| \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
{[0,1] \times |\Delta^n|} \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution.

021S **Example 3.6.6.2.** For every topological space X, the projection map $X \to \{*\}$ is a Serre fibration.

021T **Remark 3.6.6.3.** Suppose we are given a pullback diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S' \arrow[r] & S
\end{tikzcd}$$

in the category of topological spaces. If q is a Serre fibration, then q' is also a Serre fibration.

Remark 3.6.6.4. Let $f : X \to Y$ and $g : Y \to Z$ be Serre fibrations. Then the composition $(g \circ f) : X \to Z$ is a Serre fibration. `021U`

Proposition 3.6.6.5. *Let $q : X \to S$ be a continuous function between topological spaces. Then q is a Serre fibration if and only if the induced map of singular simplicial sets* $\mathrm{Sing}_{\bullet}(q) : \mathrm{Sing}_{\bullet}(X) \to \mathrm{Sing}_{\bullet}(S)$ *is a Kan fibration.* `021V`

Remark 3.6.6.6. In the special case where S is a point, Proposition 3.6.6.5 reduces to the assertion that for every topological space X, the singular simplicial set $\mathrm{Sing}_{\bullet}(X)$ is a Kan complex, which was established earlier as Proposition 1.2.5.8. `021W`

Proof of Proposition 3.6.6.5. Assume first that the map $\mathrm{Sing}_{\bullet}(q) : \mathrm{Sing}_{\bullet}(X) \to \mathrm{Sing}_{\bullet}(S)$ is a Kan fibration of simplicial sets. It follows that, for each $n \geq 0$, $\mathrm{Sing}_{\bullet}(q)$ is weakly right orthogonal to the inclusion map $\{0\} \times \Delta^n \hookrightarrow \Delta^1 \times \Delta^n$ (which is anodyne, by virtue of Proposition 3.1.2.9). It follows that the continuous function q is weakly right orthogonal to the map of geometric realizations $|\{0\} \times \Delta^n| \hookrightarrow |\Delta^1 \times \Delta^n|$, which can be identified with the inclusion $\{0\} \times |\Delta^n| \hookrightarrow [0,1] \times |\Delta^n|$ (see Corollary 3.6.2.2). Allowing n to vary, we deduce that q is a Serre fibration.

We now prove the converse. Suppose that q is a Serre fibration; we wish to show that the induced map of simplicial sets $\mathrm{Sing}_{\bullet}(q) : \mathrm{Sing}_{\bullet}(X) \to \mathrm{Sing}_{\bullet}(S)$ is weakly right orthogonal to the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for every pair of integers $0 \leq i \leq n$ with $n > 0$. Equivalently, we wish to show that q is weakly right orthogonal to the inclusion of geometric realizations $\iota : |\Lambda^n_i| \hookrightarrow |\Delta^n|$. We proceed by refining the proof of Proposition 1.2.5.8. Define a continuous function $c : |\Delta^n| \to [0,1]$ by the formula $c(t_0, t_1, \cdots, t_n) = \min\{t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n\}$. Let $h : [0,1] \times |\Delta^n| \to |\Delta^n|$ be the continuous function given by the formula

$$h(s, (t_0, \cdots, t_n)) = (t_0 - \lambda, \cdots, t_{i-1} - \lambda, t_i + n\lambda, t_{i+1} - \lambda, \cdots, t_n - \lambda)$$

$$\lambda = \max\{0, c(t_0, \cdots, t_n) - s\}$$

By construction, the composition

$$|\Delta^n| \xrightarrow{(c,\mathrm{id})} [0,1] \times |\Delta^n| \xrightarrow{h} |\Delta^n|$$

is the identity map. Moreover, the function (c, id) carries the horn $|\Lambda^n_i| \subset |\Delta^n|$ to the closed subset $\{0\} \times |\Delta^n| \subseteq |\Delta^n|$, and the function h carries $\{0\} \times |\Delta^n|$ to the horn $|\Lambda^n_i| \subset |\Delta^n|$. It follows that h and (c, id) exhibit ι as a retract of the inclusion map $\iota' : \{0\} \times |\Delta^n| \hookrightarrow [0,1] \times |\Delta^n|$ in the category of topological spaces. Consequently, to show that q is weakly right orthogonal to ι, it will suffice to show that it is weakly right orthogonal to ι' (Proposition 1.5.4.9), which follows immediately from our assumption that q is a Serre fibration. □

021X **Exercise 3.6.6.7.** Show that, for every pair of integers $0 \leq i \leq n$ with $n > 0$, there exists a homeomorphism of topological spaces

$$h : [0,1] \times |\Delta^{n-1}| \simeq |\Delta^n|$$

which restricts to a homeomorphism of $\{0\} \times |\Delta^{n-1}|$ with the horn $|\Lambda^n_i| \subset |\Delta^n|$. Use this homeomorphism to give a more direct proof of Proposition 3.6.6.5.

021Y **Corollary 3.6.6.8** (The Homotopy Extension Lifting Property)**.** *Let $q : X \to S$ be a continuous function between topological spaces. The following conditions are equivalent:*

(1) *The morphism q is a Serre fibration.*

(2) *For every simplicial set B, every lifting problem*

$$\begin{tikzcd}
\{0\} \times |B| \arrow[r] \arrow[d, hook] & X \arrow[d, "q"] \\
{[0,1] \times |B|} \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution.

(3) *For every monomorphism of simplicial sets $A \hookrightarrow B$, every lifting problem*

021Z
$$\begin{tikzcd}
{([0,1] \times |A|) \coprod_{(\{0\} \times |A|)} (\{0\} \times |B|)} \arrow[r] \arrow[d, hook] & X \arrow[d, "q"] \\
{[0,1] \times |B|} \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd} \tag{3.42}$$

admits a solution.

Proof. The implication (3) ⇒ (2) ⇒ (1) are immediate from the definition. We will complete the proof by showing that (1) implies (3). Using Corollary 3.6.2.2, we observe that every lifting problem of the form (3.42) can be rewritten as a lifting problem

$$\begin{tikzcd}
{(\Delta^1 \times_A) \coprod_{(\{0\} \times A)} (\{0\} \times B)} \arrow[r] \arrow[d, "\iota"] & \operatorname{Sing}_\bullet(X) \arrow[d, "\operatorname{Sing}_\bullet(q)"] \\
\Delta^1 \times B \arrow[r] \arrow[ur, dashed] & \operatorname{Sing}_\bullet(S)
\end{tikzcd}$$

in the category of simplicial sets. If q is Serre fibration, then $\mathrm{Sing}_{\bullet}(q)$ is a Kan fibration (Proposition 3.6.6.5), so the existence of the desired lifting follows from the observation that ι is an anodyne morphism (Proposition 3.1.2.9). $\square$

Remark 3.6.6.9. A continuous function $q : X \to S$ is a *Hurewicz fibration* if, for every topological space Y, every lifting problem 0220

$$\begin{tikzcd}
\{0\} \times Y \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
{[0,1] \times Y} \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution. Equivalently, q is a Hurewicz fibration if the evaluation map

$$\mathrm{Hom}_{\mathrm{Top}}([0,1], X) \to \mathrm{Hom}_{\mathrm{Top}}(\{0\}, X) \times_{\mathrm{Hom}_{\mathrm{Top}}(\{0\}, S)} \mathrm{Hom}_{\mathrm{Top}}([0,1], S)$$

admits a continuous section, where we endow $\mathrm{Hom}_{\mathrm{Top}}([0,1], X)$ and $\mathrm{Hom}_{\mathrm{Top}}([0,1], S)$ with their compact-open topologies. Every Hurewicz fibration is a Serre fibration. However, the converse is false.

The lifting condition of Definition 3.6.6.1 can be tested locally:

Proposition 3.6.6.10. *Let $q : X \to S$ be a continuous function between topological spaces. Suppose that, for every point $s \in S$, there exists an open subset $U \subseteq S$ containing the point s for which the induced map $q_U : U \times_S X \to U$ is a Serre fibration. Then q is a Serre fibration.* 0221

Proof. Let $\mathcal{U}$ be the collection of all open subsets $U \subseteq S$ for which the map q_U is a Serre fibration. Suppose we are given a finite simplicial set B and a simplicial subset $A \subseteq B$. We will say that a lifting problem

$$\begin{tikzcd}
([0,1] \times |A|) \coprod_{(\{0\} \times |A|)} (\{0\} \times |B|) \arrow[r] \arrow[d, hook, "\iota"] & X \arrow[d, "q"] \\
{[0,1] \times |B|} \arrow[r, "h"] \arrow[ur, dashed] & S
\end{tikzcd}$$

is *$\mathcal{U}$-small* if, for every element $s \in [0,1]$ and every simplex $\sigma : \Delta^k \to B$, the image of the composite map

$$\{s\} \times |\Delta^k| \xrightarrow{\sigma} [0,1] \times |B| \xrightarrow{h} S$$

is contained in some open set belonging to the cover $\mathcal{U}$. We first claim that every $\mathcal{U}$-small lifting problem admits a solution. Proceeding by induction on the number of simplices of B which do not belong to A, we can reduce to the case where B is a standard simplex and A is its boundary. In this case, it follows from our $\mathcal{U}$-smallness assumption and the compactness of the product $[0,1] \times |B|$ that there exists some integer $m \gg 0$ with the property that, for each $1 \leq k \leq m$, the composite map

$$[\frac{k-1}{m}, \frac{k}{m}] \times |B| \hookrightarrow [0,1] \times |B| \xrightarrow{h} S$$

has image contained in some open set $U_k \in \mathcal{U}$. Writing ι as a composition of inclusion maps

$$([0,1] \times |A|) \coprod_{([0,\frac{k-1}{m}] \times |A|)} ([0, \frac{k-1}{m}] \times |B|) \hookrightarrow ([0,1] \times |A|) \coprod_{([0,\frac{k}{m}] \times |A|)} ([0, \frac{k}{m}] \times |B|),$$

we are reduced to solving a finite sequence of lifting problems

$$\begin{tikzcd}
([\frac{k-1}{m}, \frac{k}{m}] \times |A|) \coprod_{(\{\frac{k-1}{m}\} \times |A|)} (\{\frac{k-1}{m}\} \times |B|) \arrow[r] \arrow[d, hook] & U_k \times_S X \arrow[d, "q_{U_k}"] \\
{[\frac{k-1}{m}, \frac{k}{m}] \times |B|} \arrow[r] \arrow[ur, dashed] & U_k,
\end{tikzcd}$$

which is possible by virtue of our assumption that q_{U_k} is a Serre fibration (Corollary 3.6.6.8).

Fix an integer $n \geq 0$; we wish to show that every lifting problem

0222
$$\begin{tikzcd}
\{0\} \times |\Delta^n| \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
{[0,1] \times |\Delta^n|} \arrow[r, "h"] \arrow[ur, dashed] & S
\end{tikzcd} \tag{3.43}$$

admits a solution. Fix an integer $t \geq 0$, and $B = \mathrm{Sd}^t(\Delta^n)$ denote the t-fold subdivision of Δ^n. Then Proposition 3.3.3.6 supplies a homeomorphism $|B| \simeq |\Delta^n|$, which we can use to rewrite (3.43) as a lifting problem

0223
$$\begin{tikzcd}
\{0\} \times |B| \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
{[0,1] \times |B|} \arrow[r, "h'"] \arrow[ur, dashed] & S
\end{tikzcd} \tag{3.44}$$

It follows from Lemma 3.4.6.7 that the lifting problem (3.44) is $\mathcal{U}$-small for $t \gg 0$, and therefore admits a solution by the first step of the proof. □

Corollary 3.6.6.11. *Let $q : X \to S$ be a continuous function between topological spaces. Suppose that q is a fiber bundle: that is, for every point $s \in S$, there exists an open set $U \subseteq S$ containing s and a homeomorphism $U \times_S X \simeq U \times Y$ for some topological space Y (compatible with the projection to U). Then q is a Serre fibration.* 0224

Proof. By virtue of Proposition 3.6.6.10, it suffices to check this locally on S and we may therefore assume that there exists a pullback diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "q"] & Y \arrow[d] \\
S \arrow[r] & \{*\}
\end{tikzcd}$$

for some topological space Y. Using Remark 3.6.6.3, we are reduced to showing that the projection map $Y \to \{*\}$ is a Serre fibration, which follows from Example 3.6.6.2. □

Chapter 4

The Homotopy Theory of ∞-Categories

01GP Let $q : X \to S$ be a morphism of simplicial sets. Recall that q is a *Kan fibration* if and only if it is weakly right orthogonal to every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $n > 0$ and $0 \leq i \leq n$. The theory of Kan fibrations can be viewed as a relativization of the theory of Kan complexes, which plays an essential role in the classical homotopy theory of simplicial sets (as in Chapter 3). In this chapter, we study several weaker notions of fibration, which will play an analogous role in the study of ∞-categories:

- We say that a morphism of simplicial sets $q : X \to S$ is an *inner fibration* if it is weakly right orthogonal to every *inner* horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$, $0 < i < n$ (Definition 4.1.1.1). If this condition is satisfied, then for each vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is an ∞-category (Remark 4.1.1.6). Consequently, the theory of inner fibrations can be regarded as a relative version of the theory of ∞-categories, which we study in §4.1.

- We say that a morphism of simplicial sets $q : X \to S$ is a *left fibration* if it is weakly right orthogonal to the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \leq i < n$, and a *right fibration* if it is weakly right orthogonal to the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i \leq n$ (Definition 4.2.1.1). If either of these conditions are satisfied, then the fiber $X_s = \{s\} \times_S X$ is a Kan complex for each vertex $s \in S$ (Corollary 4.4.2.3). We will see later that the construction $s \mapsto X_s$ is *covariantly* functorial when q is a left fibration, and *contravariantly* functorial when q is a right fibration (see §5.2.2). In §4.2, we develop some basic formal properties of left and right fibrations; we will carry out a more detailed analysis in Chapter 5.

- We say that a morphism of simplicial sets $q : X \to S$ is an *isofibration* if it is weakly right orthogonal to every inclusion of simplicial sets $A \hookrightarrow B$ which is a categorical equivalence (Definition 4.5.5.5). This condition is primarily useful in the case where X and S are ∞-categories, in which case it is equivalent to the requirement that q is an inner fibration which satisfies a lifting property with respect to isomorphisms (Proposition 4.5.5.1). We study isofibrations between ∞-categories in §4.4, and between general simplicial sets in §4.5.5).

If $q : X \to S$ is a morphism of simplicial sets, we have the following diagram of implications:

$$\begin{tikzcd}
 & \text{q is a trivial Kan fibration} \arrow[d, Rightarrow] & \\
 & \text{q is a Kan fibration} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{q is a left fibration} \arrow[dr, Rightarrow] & & \text{q is a right fibration} \arrow[dl, Rightarrow] \\
 & \text{q is an isofibration} \arrow[d, Rightarrow] & \\
 & \text{q is an inner fibration.} &
\end{tikzcd}$$

Beware that, in general, none of these implications is reversible.

In §4.3, we consider some prototypical examples of left and right fibrations which arise frequently in practice. Let $\mathcal{C}$ be an ∞-category. To each object $X \in \mathcal{C}$, one can associate a simplicial set $\mathcal{C}_{/X}$, whose n-simplices are given by maps $\sigma : \Delta^{n+1} \to \mathcal{C}$ which carry the final vertex of Δ^{n+1} to the object $X \in \mathcal{C}$. In particular, vertices of $\mathcal{C}_{/X}$ can be identified with morphisms $f : Y \to X$ in $\mathcal{C}$ having target X, and edges of $\mathcal{C}_{/X}$ can be identified with commutative diagrams

$$\begin{tikzcd}
Z \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\
 & X. &
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$ (see Notation 4.3.5.6 for a precise definition). The simplicial set $\mathcal{C}_{/X}$ is itself an ∞-category, which we will refer to as the *slice* ∞*-category of* $\mathcal{C}$ *over the object* X. Moreover, the evident forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ (given on objects by the construction $(f : Y \to X) \mapsto Y$) is a right fibration (Proposition 4.3.6.1). A dual version of this construction produces another ∞-category $\mathcal{C}_{X/}$ whose objects are morphisms $f : X \to Y$ in the ∞-category $\mathcal{C}$, which we refer to as the *coslice* ∞*-category of* $\mathcal{C}$ *under the object* X. The slice and coslice constructions (and generalizations thereof) provide a rich supply of right and left fibrations between simplicial sets, and will play an essential role throughout this book.

Recall that an *equivalence of categories* is a functor $F : \mathcal{C} \to \mathcal{D}$ which admits a homotopy inverse: that is, for which there exists another functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ and $F \circ G$ are isomorphic to the identity functors $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$, respectively. In §4.5, we study the ∞-categorical counterpart of this notion. We say that a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ is a *categorical equivalence* if, for every ∞-category $\mathcal{E}$, precomposition with F induces a bijection

$$\{\text{Isomorphism classes of diagrams } \mathcal{D} \to \mathcal{E}\} \to \{\text{Isomorphism classes of diagrams } \mathcal{C} \to \mathcal{E}\};$$

see Definition 4.5.3.1. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then this is equivalent to the requirement that F admits a homotopy inverse $G : \mathcal{D} \to \mathcal{C}$ in the sense described above (see Example 4.5.3.3). In this case, we say that F is an *equivalence of* ∞*-categories* (Definition 4.5.1.10).

A functor $F : \mathcal{C} \to \mathcal{D}$ between ordinary categories is an equivalence if and only if it satisfies the following pair of conditions:

(1) The functor F is *fully faithful*. That is, for every pair of objects $X, Y \in \mathcal{C}$, the functor F induces a bijection $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.

(2) The functor F is *essentially surjective*: that is, every object $Y \in \mathcal{D}$ is isomorphic to $F(X)$, for some object $X \in \mathcal{C}$.

This characterization is quite useful: in practice, it is often easier to verify conditions (1) and (2) than to explicitly describe a homotopy inverse of the functor F (which might require some auxiliary choices). In §4.6, we establish an analogue of this characterization in the ∞-categorical setting. To every pair of objects X and Y of an ∞-category $\mathcal{C}$, we associate a Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ which we refer to as the *space of morphisms from* X *to* Y (Construction 4.6.1.1). We say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* if it induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for every pair of objects $X, Y \in \mathcal{C}$ (Definition 4.6.2.1). In §4.6.2, we show that F is an equivalence of ∞-categories if and only if it is fully faithful and essentially surjective at the level of homotopy categories (Theorem 4.6.2.20).

4.1 Inner Fibrations

Recall that a simplicial set X is an ∞-*category* if, for every pair of integers $0 < i < n$, every morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ can be extended to an n-simplex of X (Definition 1.4.0.1). The goal of this section is to introduce and study a relative version of this condition. We say that a morphism of simplicial sets $q : X \to S$ is an *inner fibration* if it is weakly right orthogonal to the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i < n$ (Definition 4.1.1.1). In the special case $S = \Delta^0$, this reduces to the assumption that X is an ∞-category (Example 4.1.1.2). More generally, we will see in §4.1.1 that a morphism $q : X \to S$ is an inner fibration if and only if the inverse image of every simplex of S is an ∞-category (Remark 4.1.1.13). 01B8

Let $\mathcal{C}$ be an ∞-category. We will say that a simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ is a *subcategory* of $\mathcal{C}$ if the inclusion map $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an inner fibration (Definition 4.1.2.2). In this case, $\mathcal{C}'$ is also an ∞-category, whose homotopy category $\mathrm{h}\mathcal{C}'$ can be identified with a subcategory of $\mathrm{h}\mathcal{C}$ (in the sense of classical category theory). In §4.1.2, we show that every subcategory of $\mathrm{h}\mathcal{C}$ can be obtained (uniquely) in this way: more precisely, the construction $\mathcal{C}' \mapsto \mathrm{h}\mathcal{C}'$ induces a bijection from the set of subcategories of $\mathcal{C}$ to the set of subcategories of $\mathrm{h}\mathcal{C}$ (Proposition 4.1.2.10).

Recall that a morphism of simplicial sets $i : A \hookrightarrow B$ is said to be *inner anodyne* if it can be constructed from inner horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ using pushouts, retracts, and transfinite composition (Definition 1.5.6.4). It follows immediately from the definitions that a morphism of simplicial sets $q : X \to S$ is an inner fibration if and only if it is weakly right orthogonal to all inner anodyne morphisms (Proposition 4.1.3.1). In §4.1.3, we use a version of Quillen's small object argument (Proposition 4.1.3.2) to show that, conversely, a morphism $i : A \hookrightarrow B$ is inner anodyne if and only if it is weakly left orthogonal to every inner fibration (Corollary 4.1.3.4).

If $\mathcal{C}$ is an ∞-category and K is an arbitrary simplicial set, then the simplicial set $\mathrm{Fun}(K, \mathcal{C})$ is also an ∞-category (Theorem 1.5.3.7). In §4.1.4, we establish a relative form of this result: if $q : X \to S$ is an inner fibration of simplicial sets, then postcomposition with q induces another inner fibration $\mathrm{Fun}(K, X) \to \mathrm{Fun}(K, S)$ (Corollary 4.1.4.3). This is a special case of a more general result (Proposition 4.1.4.1), which is essentially equivalent to the stability of inner anodyne morphisms under the formation of pushout-products (see Lemma 1.5.7.5).

4.1.1 Inner Fibrations of Simplicial Sets

We now introduce the primary objects of interest in this section. 01B9

`01BA` **Definition 4.1.1.1.** Let $q : X \to S$ be a morphism of simplicial sets. We say that q is an *inner fibration* if, for every pair of integers $0 < i < n$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & S
\end{tikzcd}$$

admits a solution (as indicated by the dotted arrow). That is, for every map of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ and every n-simplex $\overline{\sigma} : \Delta^n \to S$ extending $q \circ \sigma_0$, we can extend σ_0 to an n-simplex $\sigma : \Delta^n \to X$ satisfying $q \circ \sigma = \overline{\sigma}$.

`01BB` **Example 4.1.1.2.** Let X be a simplicial set. Then the projection map $X \to \Delta^0$ is an inner fibration if and only if X is an ∞-category.

`01CB` **Remark 4.1.1.3.** Let $q : X \to S$ be a morphism of simplicial sets. Then q is an inner fibration if and only if the opposite morphism $q^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is an inner fibration.

`01BD` **Remark 4.1.1.4.** The collection of inner fibrations is closed under retracts. That is, given a diagram of simplicial sets

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "q"] & X' \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

where both horizontal compositions are the identity, if q' is an inner fibration, then so is q.

`01BE` **Remark 4.1.1.5.** The collection of inner fibrations is closed under pullback. That is, given a pullback diagram of simplicial sets

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S' \arrow[r, "f"] & S
\end{tikzcd}$$

where q is an inner fibration, the morphism q' is also an inner fibration. Conversely, if q' is an inner fibration and f is surjective, then q is an inner fibration.

Remark 4.1.1.6. Let $q : X \to S$ be an inner fibration of simplicial sets. Then, for every vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is an ∞-category (this follows from Remark 4.1.1.5 and Example 4.1.1.2). `01BF`

Remark 4.1.1.7. The collection of inner fibrations is closed under filtered colimits. That is, if $\{q_\alpha : X_\alpha \to S_\alpha\}$ is a filtered diagram in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$ having colimit $q : X \to S$, and each q_α is an inner fibration of simplicial sets, then q is also an inner fibration of simplicial sets. `01BG`

Remark 4.1.1.8. Let $p : X \to Y$ and $q : Y \to Z$ be inner fibrations of simplicial sets. Then the composite map $(q \circ p) : X \to Z$ is an inner fibration of simplicial sets. `01BH`

Remark 4.1.1.9. Let $q : X \to Y$ be an inner fibration of simplicial sets. If Y is an ∞-category, then X is also an ∞-category (this follows by combining Remark 4.1.1.8 with Example 4.1.1.2). `01BJ`

Proposition 4.1.1.10. *Let $\mathcal{C}$ be a category, and let $q : X \to \mathrm{N}_\bullet(\mathcal{C})$ be a morphism of simplicial sets. Then q is an inner fibration if and only if X is an ∞-category.* `01BK`

Proof. If q is an inner fibration, then Remark 4.1.1.9 guarantees that X is an ∞-category. Conversely, suppose that X is an ∞-category and that we are given a lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathrm{N}_\bullet(\mathcal{C})
\end{tikzcd}$$

for integers $0 < i < n$. Our assumption that X is an ∞-category guarantees that σ_0 can be extended to an n-simplex $\sigma : \Delta^n \to X$. The equality $q \circ \sigma = \overline{\sigma}$ is automatic by virtue of Proposition 1.3.4.1. □

Corollary 4.1.1.11. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then the induced map $\mathrm{N}_\bullet(F) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})$ is an inner fibration of simplicial sets.* `01NP`

Example 4.1.1.12. Let $\mathcal{C}$ be an ∞-category and let $\mathrm{h}\mathcal{C}$ denote its homotopy category (Construction 1.4.5.1). Then the canonical map $\mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ is an inner fibration. `01CC`

Remark 4.1.1.13. Let $q : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent: `01BL`

(1) The morphism q is an inner fibration.

(2) For every simplex $\sigma : \Delta^n \to S$, the projection map $\Delta^n \times_S X \to \Delta^n$ is an inner fibration.

(3) For every simplex $\sigma : \Delta^n \to S$, the fiber product $\Delta^n \times_S X$ is an ∞-category.

The equivalence (1) $\Leftrightarrow$ (2) is immediate from the definition, and the equivalence (2) $\Leftrightarrow$ (3) follows from Proposition 4.1.1.10.

0225 **Remark 4.1.1.14.** Suppose we are given an inverse system of simplicial sets

$$\cdots \to X(4) \to X(3) \to X(2) \to X(1) \to X(0),$$

where each of the transition maps $X(n) \to X(n-1)$ is an inner fibration. Then each of the projection maps $\varprojlim_n X(n) \to X(m)$ is an inner fibration. In particular, if any of the simplicial sets $X(m)$ is an ∞-category, then the inverse limit $\varprojlim_n X(n)$ is also an ∞-category.

4.1.2 Subcategories of ∞-Categories

01CD Let $\mathcal{C}$ be a category, and let $\mathrm{Ob}(\mathcal{C})$ be the set of objects of $\mathcal{C}$. Suppose that we are given a subset $\mathrm{Ob}'(\mathcal{C}) \subseteq \mathrm{Ob}(\mathcal{C})$ and, for every pair of objects $X, Y \in \mathrm{Ob}'(\mathcal{C})$, a subset $\mathrm{Hom}'_{\mathcal{C}}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X,Y)$ satisfying the following conditions:

- For every object $X \in \mathrm{Ob}'(\mathcal{C})$, the identity morphism id_X belongs to $\mathrm{Hom}'_{\mathcal{C}}(X,X)$.
- For every triple of objects $X, Y, Z \in \mathrm{Ob}'(\mathcal{C})$ and every pair of morphisms $f \in \mathrm{Hom}'_{\mathcal{C}}(X,Y)$, $g \in \mathrm{Hom}'_{\mathcal{C}}(Y,Z)$, the composition $g \circ f$ belongs to $\mathrm{Hom}'_{\mathcal{C}}(X,Z)$.

In this case, we can construct a category $\mathcal{C}'$ by setting $\mathrm{Ob}(\mathcal{C}') = \mathrm{Ob}'(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}'_{\mathcal{C}}(X,Y)$ for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}')$ (where the composition of morphisms in $\mathcal{C}'$ agrees with their composition in $\mathcal{C}$). In this situation, we refer to $\mathcal{C}'$ as the *subcategory of $\mathcal{C}$ spanned by the objects* $\mathrm{Ob}'(\mathcal{C})$ *and the morphisms* $\{\mathrm{Hom}'_{\mathcal{C}}(X,Y)\}_{X,Y \in \mathrm{Ob}'(\mathcal{C})}$.

01CE **Remark 4.1.2.1.** Let $\mathcal{C}$ be a category. We will say that a category $\mathcal{C}'$ is a *subcategory of $\mathcal{C}$* if it arises from the construction described above (for some collection of objects $\mathrm{Ob}'(\mathcal{C})$ and collections of morphisms $\{\mathrm{Hom}'_{\mathcal{C}}(X,Y)\}_{X,Y \in \mathrm{Ob}'(\mathcal{C})}$). Phrased differently, a category $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if the following conditions are satisfied:

- The set of objects $\mathrm{Ob}(\mathcal{C}')$ is a subset of the set of objects $\mathrm{Ob}(\mathcal{C})$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C}') \subseteq \mathrm{Ob}(\mathcal{C})$, the set of morphisms $\mathrm{Hom}_{\mathcal{C}'}(X,Y)$ is a subset of the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.
- There is a functor $\mathcal{C}' \to \mathcal{C}$ which is the identity on objects and morphisms.

We write $\mathcal{C}' \subseteq \mathcal{C}$ to indicate that $\mathcal{C}'$ is a subcategory of $\mathcal{C}$.

We now generalize the notion of subcategory to the setting of ∞-categories.

Definition 4.1.2.2. Let $\mathcal{C}$ be an ∞-category. A *subcategory of* $\mathcal{C}$ is a simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ for which the inclusion map $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an inner fibration. `01CF`

Remark 4.1.2.3. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory. Then $\mathcal{C}'$ is also an ∞-category (Remark 4.1.1.9). `01CG`

Example 4.1.2.4. Let $\mathcal{C}$ be an ordinary category and let $\mathrm{N}_\bullet(\mathcal{C})$ be its nerve. For every subcategory $\mathcal{C}' \subseteq \mathcal{C}$, the nerve $\mathrm{N}_\bullet(\mathcal{C}')$ can be viewed as a subcategory of $\mathrm{N}_\bullet(\mathcal{C})$ (the inclusion map $\mathrm{N}_\bullet(\mathcal{C}') \hookrightarrow \mathrm{N}_\bullet(\mathcal{C})$ is automatically an inner fibration, by virtue of Proposition 4.1.1.10). We will see in a moment that every subcategory of $\mathrm{N}_\bullet(\mathcal{C})$ arises in this way (Corollary 4.1.2.11). In other words, when restricted to (the nerves of) ordinary categories, Definition 4.1.2.2 reduces to the classical notion of subcategory. `01CH`

Warning 4.1.2.5. The terminology of Definition 4.1.2.2 has the potential to cause confusion. If $\mathcal{C}$ is an ∞-category and $\mathcal{C}' \subseteq \mathcal{C}$ is a subcategory, then $\mathcal{C}'$ need not be (isomorphic to the nerve of) an ordinary category. Our use of the term "subcategory" (rather than the more technically correct "sub-∞-category") is intended to avoid awkward language. `01CJ`

Remark 4.1.2.6 (Pullbacks of Subcategories)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and let $\mathcal{D}'$ be a subcategory of $\mathcal{D}$. Then the inverse image $F^{-1}(\mathcal{D}') \subseteq \mathcal{C}$ is a subcategory of $\mathcal{C}$ (see Remark 4.1.1.5). `01CK`

Remark 4.1.2.7. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory. Suppose that $\mathcal{C}$ contains a 2-simplex σ : `01CL`

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

which witnesses h as a composition of g and f (Definition 1.4.4.1). If f and g belong to the subcategory $\mathcal{C}'$, then the 2-simplex σ also belongs to the subcategory $\mathcal{C}'$ (since the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is weakly right orthogonal to the horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$). In particular, if f and g belong to $\mathcal{C}'$, then h also belongs to $\mathcal{C}'$.

Remark 4.1.2.8. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory. Suppose we are given a pair of morphisms $f, g : X \to Y$ in $\mathcal{C}$ having the same source and target. If f and g are homotopic as morphisms in the ∞-category $\mathcal{C}$ and f belongs to the subcategory $\mathcal{C}'$, then g also belongs to the subcategory $\mathcal{C}'$ and the morphisms f and g are homotopic in the ∞-category $\mathcal{C}'$. This a special case of Remark 4.1.2.7 (note that f and g are homotopic if and only if g is a composition of f with the identity morphism id_Y; see Definition 1.4.3.1). `01CM`

01CN **Remark 4.1.2.9.** Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory, let $\sigma : \Delta^n \to \mathcal{C}$ be an n-simplex of $\mathcal{C}$ for $n > 0$. The following conditions are equivalent:

(1) The n-simplex σ is contained in the subcategory $\mathcal{C}'$.

(2) For every pair of integers $0 \leq i < j \leq n$, the edge

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{i < j\}) \hookrightarrow \Delta^n \xrightarrow{\sigma} \mathcal{C}$$

is contained in the subcategory $\mathcal{C}'$.

(3) For every integer $1 \leq j \leq n$, the edge

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{j-1 < j\}) \hookrightarrow \Delta^n \xrightarrow{\sigma} \mathcal{C}$$

is contained in the subcategory $\mathcal{C}'$.

The implications (1) ⇒ (2) ⇒ (3) are immediate from the definitions, and the implication (3) ⇒ (1) follows from fact that the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}$ is weakly right orthogonal to the inner anodyne morphism $\mathrm{Spine}[n] \hookrightarrow \Delta^n$ (see Example 1.5.7.7 and Proposition 4.1.3.1).

01CP **Proposition 4.1.2.10.** *Let $\mathcal{C}$ be an ∞-category, let $\mathrm{h}\mathcal{C}$ be its homotopy category, and let $F : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ be the canonical map. Then the construction $(\mathcal{D} \subseteq \mathrm{h}\mathcal{C}) \mapsto (F^{-1}(\mathrm{N}_\bullet(\mathcal{D})) \subseteq \mathcal{C})$ induces a bijection*

$$\{\text{Subcategories of the ordinary category } \mathrm{h}\mathcal{C}\} \simeq \{\text{Subcategories of the } \infty\text{-category } \mathcal{C}\}$$

Proof. We first observe that if $\mathcal{D}$ is a subcategory of the homotopy category $\mathrm{h}\mathcal{C}$, then the nerve $\mathrm{N}_\bullet(\mathcal{D})$ is a subcategory of the ∞-category $\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ (Example 4.1.2.4), so that $F^{-1}(\mathrm{N}_\bullet(\mathcal{D}))$ is a subcategory of the ∞-category $\mathcal{C}$ (Remark 4.1.2.6). Moreover, the subcategory $\mathcal{D}$ is uniquely determined by its inverse image $F^{-1}(\mathrm{N}_\bullet(\mathcal{D}))$: this follows from the fact that $F : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ is an epimorphism of simplicial sets (Remark 1.5.7.10). To complete the proof, it will suffice to show that every subcategory $\mathcal{C}' \subseteq \mathcal{C}$ arises in this way. Note that the inclusion map $\mathcal{C}' \hookrightarrow \mathcal{C}$ induces a functor of homotopy categories $G : \mathrm{h}\mathcal{C}' \hookrightarrow \mathrm{h}\mathcal{C}$, which is obviously injective at the level of objects. In addition, for every pair of objects $X, Y \in \mathrm{h}\mathcal{C}'$, the functor G induces a monomorphism $\mathrm{Hom}_{\mathrm{h}\mathcal{C}'}(X,Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y)$: this follows from the observation that a pair of morphisms $f, g : X \to Y$ are homotopic in the ∞-category $\mathcal{C}'$ if and only if they are homotopic in the ∞-category $\mathcal{C}$ (Remark 4.1.2.8). It follows that the functor G induces an isomorphism from $\mathrm{h}\mathcal{C}'$ onto a subcategory $\mathcal{D} \subseteq \mathrm{h}\mathcal{C}$. We therefore have an inclusion $\mathcal{C}' \subseteq F^{-1}(\mathrm{N}_\bullet(\mathcal{D}))$. To complete the proof, it will suffice to show that this inclusion is an equality. In other words, we must show that an n-simplex $\sigma : \Delta^n \to \mathcal{C}$ is contained in $\mathcal{C}'$ if and only if the induced map $[n] \to \mathrm{h}\mathcal{C}$ factors through the subcategory $\mathcal{D} \subseteq \mathrm{h}\mathcal{C}$. Without loss of generality, we may assume that $n > 0$ (the case $n = 0$ is trivial).

Using Remark 4.1.2.9, we can reduce to the case where $n = 1$, so that σ can be identified with a morphism $g : X \to Y$ in the ∞-category $\mathcal{C}$. Our assumption that $F(\sigma)$ belongs to $\mathrm{N}_{\bullet}(\mathcal{D})$ guarantees that g is homotopic to a morphism $f : X \to Y$ which belongs to the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ (and, in particular, that the objects X and Y belong to $\mathcal{C}'$). Invoking Remark 4.1.2.8, we conclude that g also belongs to the subcategory $\mathcal{C}'$, as desired. □

Corollary 4.1.2.11. *Let $\mathcal{C}$ be an ordinary category. Then the construction $\mathcal{C}' \mapsto \mathrm{N}_{\bullet}(\mathcal{C}')$ induces a bijection* 01CQ

$$\{\text{Subcategories of the ordinary category } \mathcal{C}\} \simeq \{\text{Subcategories of the } \infty\text{-category } \mathrm{N}_{\bullet}(\mathcal{C})\}$$

Proof. Combine Proposition 4.1.2.10 with Example 1.4.5.4. □

Corollary 4.1.2.12. *Let $\mathcal{C}$ be an ∞-category, let S be a collection of objects of $\mathcal{C}$, and let T be a collection of morphisms of $\mathcal{C}$. The following conditions are equivalent:* 01CR

- *There exists a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ whose objects are the elements of S and whose morphisms are the elements of T.*
- *The collections S and T satisfy the following conditions:*
  - (1) *For each object $X \in S$, the identity morphism id_X belongs to T.*
  - (2) *For each morphism $f : X \to Y$ of $\mathcal{C}$ which belongs to T, the objects X and Y belong to S.*
  - (3) *If $f : X \to Y$ is an morphism of $\mathcal{C}$ which belongs to T and $g : X \to Y$ is a morphism of $\mathcal{C}$ which is homotopic to f, then g also belongs to T.*
  - (4) *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of $\mathcal{C}$ which belong to T, then some composition $(g \circ f) : X \to Z$ also belongs to T.*

Moreover, if these conditions are satisfied, then the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is uniquely determined by S and T.

Proof. The necessity of conditions (1) and (2) is immediate, and the necessity of (3) and (4) follow from Remark 4.1.2.8 and Remark 4.1.2.7. Conversely, suppose that conditions (1) through (4) are satisfied. Using (1), (2), and (4), we deduce that there exists a subcategory $\mathcal{D} \subseteq \mathrm{h}\mathcal{C}$ whose objects are the elements of S and whose morphisms are the homotopy classes of morphisms belonging to T. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the inverse image of the subcategory $\mathrm{N}_{\bullet}(\mathcal{D}) \subseteq \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$. It follows immediately from the definition that an object of $\mathcal{C}$ belongs to the subcategory $\mathcal{C}'$ if and only if it is an element of S, and from (3) that a morphism of $\mathcal{C}$ belongs to the subcategory $\mathcal{C}'$ if and only if it is an element of T. The uniqueness of the subcategory $\mathcal{C}'$ follows from Proposition 4.1.2.10. □

01CS **Definition 4.1.2.13.** Let $\mathcal{C}$ be an ∞-category. Suppose we are given a collection S of objects of $\mathcal{C}$ and a collection T of morphisms of $\mathcal{C}$ satisfying the assumptions of Corollary 4.1.2.12, so that there exists a unique subcategory $\mathcal{C}' \subseteq \mathcal{C}$ whose objects are the elements of S and whose morphisms are the elements of T. In this case, we will refer to $\mathcal{C}'$ as the *subcategory of* $\mathcal{C}$ *spanned by the objects of* S *and the morphisms of* T.

01CT **Remark 4.1.2.14.** Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{C}' \subseteq \mathcal{C}$ be the subcategory spanned by the collection of objects S of $\mathcal{C}$ and a collection of morphisms T of $\mathcal{C}$. Then a morphism of simplicial sets $f : K \to \mathcal{C}$ factors through the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ if and only if it carries each vertex of K to an element of S and each edge of K to an element of T.

Let $\mathcal{C}$ be an ordinary category. Recall that a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is *full* if, for every pair of objects $X, Y \in \mathcal{C}'$, the inclusion map $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is bijective. This definition has an obvious counterpart in the ∞-categorical setting.

01CU **Definition 4.1.2.15.** Let $\mathcal{C}$ be a simplicial set. We say that a simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ is *full* if it satisfies the following condition:

- Let $\sigma : \Delta^n \to \mathcal{C}$ be a simplex of $\mathcal{C}$ having the property that, for each integer $0 \leq i \leq n$, the vertex $\sigma(i) \in \mathcal{C}$ belongs to $\mathcal{C}'$. Then σ belongs to $\mathcal{C}'$.

If this condition is satisfied, then the inclusion map $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an inner fibration. In particular, if $\mathcal{C}$ is an ∞-category, then $\mathcal{C}'$ is a subcategory of $\mathcal{C}$; in this case, we will say that $\mathcal{C}'$ is a *full subcategory of* $\mathcal{C}$.

01CV **Proposition 4.1.2.16.** *Let* $\mathcal{C}$ *be a simplicial set and let* S *be a collection of vertices of* $\mathcal{C}$. *Then there exists a unique full simplicial subset* $\mathcal{C}' \subseteq \mathcal{C}$ *having vertex set* S.

Proof. Take $\mathcal{C}'$ to be the simplicial subset of $\mathcal{C}$ consisting of those simplices $\sigma : \Delta^n \to \mathcal{C}$ having the property that, for $0 \leq i \leq n$, the vertex $\sigma(i)$ belongs to S. □

01CW **Definition 4.1.2.17.** Let $\mathcal{C}$ be a simplicial set and let S be a collection of vertices of $\mathcal{C}$. By virtue of Proposition 4.1.2.16, there exists a unique full simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ having vertex set S. We will refer to $\mathcal{C}'$ as the *full simplicial subset of* $\mathcal{C}$ *spanned by* S. If $\mathcal{C}$ is an ∞-category, we will refer to $\mathcal{C}'$ as the *full subcategory of* $\mathcal{C}$ *spanned by* S.

01CX **Remark 4.1.2.18.** Let $\mathcal{C}$ be a simplicial set and let $\mathcal{C}' \subseteq \mathcal{C}$ be the full simplicial subset of $\mathcal{C}$ spanned by a set of vertices S of $\mathcal{C}$. Then a morphism of simplicial sets $f : K \to \mathcal{C}$ factors through the simplicial subset $\mathcal{C}' \subseteq \mathcal{C}$ if and only if, for every vertex $x \in K$, the image $f(x) \in \mathcal{C}$ belongs to S.

01CY **Remark 4.1.2.19.** Let $\mathcal{C}$ be an ordinary category. Then the construction $\mathcal{C}' \mapsto \mathrm{N}_{\bullet}(\mathcal{C}')$ induces a bijection

$$\{\text{Full subcategories of } \mathcal{C}\} \simeq \{\text{Full subcategories of } \mathrm{N}_{\bullet}(\mathcal{C})\}.$$

4.1.3 Inner Anodyne Morphisms

By definition, a morphism of simplicial sets $q : X \to S$ is an inner fibration if it is weakly right orthogonal to every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$. From this, one can immediately deduce a stronger lifting property. 01BM

Proposition 4.1.3.1. *Let $q : X \to S$ be a morphism of simplicial sets. Then q is an inner fibration if and only if it satisfies the following condition:* 01BN

$(*)$ *For every square diagram of simplicial sets*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

where i is inner anodyne, there exists a dotted arrow rendering the diagram commutative.

Proof. The "if" direction is immediate from the definition, since the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is inner anodyne for $0 < i < n$. The reverse implication follows from Proposition 1.5.4.13. □

Proposition 4.1.3.2. *Let $f : X \to Y$ be a morphism of simplicial sets. Then f can be factored as a composition $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$, where f'' is an inner fibration and f' is inner anodyne. Moreover, the simplicial set $Q(f)$ (and the morphisms f' and f'') can be chosen to depend functorially on f, in such a way that the functor* 01BP

$$\mathrm{Fun}([1], \mathrm{Set}_{\Delta}) \to \mathrm{Set}_{\Delta} \qquad (f : X \to Y) \to Q(f)$$

commutes with filtered colimits.

Proof. We proceed as in the proof of Proposition 3.1.7.1. We construct a sequence of simplicial sets $\{X(m)\}_{m \geq 0}$ and morphisms $f(m) : X(m) \to Y$ by recursion. Set $X(0) = X$ and $f(0) = f$. Assuming that $f(m) : X(m) \to Y$ has been defined, let $S(m)$ denote the set of all commutative diagrams σ :

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X(m) \arrow[d, "f(m)"] \\ \Delta^n \arrow[r, "u_\sigma"] & Y, \end{tikzcd}$$

where $0 < i < n$ and the left vertical map is the inclusion. For every such commutative diagram σ, let $C_\sigma = \Lambda^n_i$ denote the upper left hand corner of the diagram σ, and $D_\sigma = \Delta^n$ the lower left hand corner. Form a pushout diagram

$$\begin{tikzcd}
\coprod_{\sigma \in S(m)} C_\sigma \arrow[r] \arrow[d] & X(m) \arrow[d] \\
\coprod_{\sigma \in S(m)} D_\sigma \arrow[r] & X(m+1)
\end{tikzcd}$$

and let $f(m+1) : X(m+1) \to Y$ be the unique map whose restriction to $X(m)$ is equal to $f(m)$ and whose restriction to each D_σ is equal to u_σ. By construction, we have a direct system of inner anodyne morphisms

$$X = X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots$$

Set $Q(f) = \varinjlim_m X(m)$. Then the natural map $f' : X \to Q(f)$ is inner anodyne (since the collection of inner anodyne maps is closed under transfinite composition), and the system of morphisms $\{f(m)\}_{m \geq 0}$ can be amalgamated to a single map $f'' : Q(f) \to Y$ satisfying $f = f'' \circ f'$. It is clear from the definition that the construction $f \mapsto Q(f)$ is functorial and commutes with filtered colimits. To complete the proof, it will suffice to show that f'' is a inner fibration: that is, that every lifting problem σ :

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "v"] \arrow[d] & Q(f) \arrow[d, "f''"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution (provided that $0 < i < n$). Let us abuse notation by identifying each $X(m)$ with its image in $Q(f)$. Since Λ^n_i is a finite simplicial set, its image under v is contained in $X(m)$ for some $m \gg 0$. In this case, we can identify σ with an element of the set $S(m)$, so that the lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "v"] \arrow[d] & X(m+1) \arrow[d, "f(m+1)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution by construction. $\square$

Applying Proposition 4.1.3.2 in the special case $Y = \Delta^0$, we obtain the following:

Corollary 4.1.3.3. *Let X be a simplicial set. Then there exists an inner anodyne morphism $f : X \to Q(X)$, where $Q(X)$ is an ∞-category. Moreover, the ∞-category $Q(X)$ (and the morphism f) can be chosen to depend functorially on X, in such a way that the functor $X \mapsto Q(X)$ commutes with filtered colimits.* `01BQ`

Using Proposition 4.1.3.2, we obtain the following counterpart of Proposition 4.1.3.1:

Corollary 4.1.3.4. *Let $i : A \to B$ be a morphism of simplicial sets. Then i is inner anodyne if and only if it satisfies the following condition:* `01BR`

$(*)$ *For every square diagram of simplicial sets*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "q"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where q is an inner fibration, there exists a dotted arrow rendering the diagram commutative.

Proof. The "if" direction follows from Proposition 4.1.3.1. For the converse, suppose that condition $(*)$ is satisfied. Using Proposition 4.1.3.2, we can factor i as a composition $A \xrightarrow{i'} Q \xrightarrow{q} B$, where i' is inner anodyne and q is an inner fibration. If the lifting problem

$$\begin{tikzcd}
A \arrow[r, "i'"] \arrow[d, "i"] & Q \arrow[d, "q"] \\
B \arrow[r, "\mathrm{id}"] \arrow[ur, dashed, "r"] & B
\end{tikzcd}$$

admits a solution, then the morphism r exhibits i as a retract of i' (in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$). Since the collection of inner anodyne morphisms is closed under retracts, it follows that i is inner anodyne. □

4.1.4 Exponentiation for Inner Fibrations

Recall that, if $\mathcal{C}$ is an ∞-category and B is an arbitrary simplicial set, then the simplicial set $\mathrm{Fun}(B, \mathcal{C})$ is also an ∞-category (Theorem 1.5.3.7). We now record a relative version of this result. `01BS`

01BT **Proposition 4.1.4.1.** *Let* $q : X \to S$ *be an inner fibration of simplicial sets, and let* $i : A \hookrightarrow B$ *be a monomorphism of simplicial sets. Then the restriction map*

$$\rho : \mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B, S)$$

is also an inner fibration of simplicial sets.

Proof. By virtue of Proposition 4.1.3.1, it will suffice to show that every lifting problem

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"] & \mathrm{Fun}(B, X) \arrow[d, "\rho"] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)
\end{tikzcd}$$

admits a solution, provided that i' is inner anodyne. Equivalently, we must show that every lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{A \times A'} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution. This follows from Proposition 4.1.3.1, since the left vertical map is inner anodyne (Lemma 1.5.7.5) and q is an inner fibration. □

01BU **Corollary 4.1.4.2.** *Let* $\mathcal{C}$ *be an* ∞*-category and let* $i : A \hookrightarrow B$ *be a monomorphism of simplicial sets. Then the restriction functor* $\mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$ *is an inner fibration.*

Proof. Apply Proposition 4.1.4.1 in the special case $S = \Delta^0$. □

01BV **Corollary 4.1.4.3.** *Let* $q : X \to S$ *be an inner fibration of simplicial sets and let* B *be an arbitrary simplicial set. Then composition with* q *induces an inner fibration* $\mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S)$.

Proof. Apply Proposition 4.1.4.1 in the special case $A = \emptyset$. □

We now record an analogous generalization of Proposition 1.5.7.6.

01BW **Proposition 4.1.4.4.** *Let* $q : X \to S$ *be an inner fibration of simplicial sets, and let* $i : A \hookrightarrow B$ *be an inner anodyne morphism of simplicial sets. Then the restriction map*

$$\rho : \mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B, S)$$

is a trivial Kan fibration.

Proof. We wish to show that every lifting problem

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"] & \mathrm{Fun}(B, X) \arrow[d, "\rho"] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)
\end{tikzcd}$$

admits a solution, provided that i' is a monomorphism of simplicial sets. Equivalently, we must show that every lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{A \times A'} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution. This follows from Proposition 4.1.3.1, since the left vertical map is inner anodyne (Lemma 1.5.7.5) and q is an inner fibration. □

Proposition 4.1.4.4 admits the following converse (generalizing Theorem 1.5.6.1):

Proposition 4.1.4.5. *Let $q : X \to S$ be a morphism of simplicial sets. Then q is an inner fibration if and only if the induced map* 01BX

$$\rho : \mathrm{Fun}(\Delta^2, X) \to \mathrm{Fun}(\Lambda^2_1, X) \times_{\mathrm{Fun}(\Lambda^2_1, S)} \mathrm{Fun}(\Delta^2, S)$$

is a trivial Kan fibration.

Proof. The "only if" direction follows from Proposition 4.1.4.4. For the converse, we observe that ρ is a trivial Kan fibration if and only if q is weakly right orthogonal to the inclusion map

$$(\Delta^m \times \Lambda^2_1) \coprod_{\partial \Delta^m \times \Lambda^2_1} (\partial \Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$$

for every nonnegative integer m. Since the collection of inner anodyne morphisms is generated (as a weakly saturated class) by such inclusions (Lemma 1.5.6.9), it follows that q is weakly right orthogonal to all inner anodyne morphisms (Proposition 1.5.4.13) and is therefore an inner fibration (Proposition 4.1.3.1). □

Proposition 4.1.4.6. *Suppose we are given a commutative diagram* 01GQ

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "q"] \\
B \arrow[r, "g"] \arrow[ur, dashed, "\overline{f}"] & S
\end{tikzcd}$$

of simplicial sets, where i is a monomorphism and q is an inner fibration. Then the simplicial set $\mathrm{Fun}_{A//S}(B,X)$ *of Construction 3.1.3.7 is an ∞-category. Moreover, if i is inner anodyne, then* $\mathrm{Fun}_{A//S}(B,X)$ *is a contractible Kan complex.*

Proof. By virtue of Remark 3.1.3.11, the simplicial set $\mathrm{Fun}_{A//S}(B,X)$ can be identified with a fiber of the restriction map

$$\theta : \mathrm{Fun}(B,X) \to \mathrm{Fun}(A,X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B,S).$$

Proposition 4.1.4.1 asserts that θ is an inner fibration of simplicial sets, so its fibers are ∞-categories (Remark 4.1.1.6). If i is inner anodyne, then Proposition 4.1.4.4 guarantees that θ is a trivial Kan fibration, so its fibers are contractible Kan complexes. □

01GR **Corollary 4.1.4.7.** *Let B be a simplicial set, let $A \subseteq B$ be a simplicial subset, and let $f : A \to \mathcal{C}$ be a morphism of simplicial sets. If $\mathcal{C}$ is an ∞-category, then the simplicial set* $\mathrm{Fun}_{A/}(B,\mathcal{C})$ *is an ∞-category. Moreover, if the inclusion $A \hookrightarrow B$ is inner anodyne, then* $\mathrm{Fun}_{A/}(B,\mathcal{C})$ *is a contractible Kan complex.*

Proof. Apply Proposition 4.1.4.6 in the special case $S = \Delta^0$. □

01GS **Corollary 4.1.4.8.** *Let $q : X \to S$ be an inner fibration of simplicial sets and let $g : B \to S$ be any morphism of simplicial sets. Then the simplicial set* $\mathrm{Fun}_{/S}(B,X)$ *is an ∞-category.*

Proof. Apply Proposition 4.1.4.6 in the special case $A = \emptyset$. □

4.1.5 Inner Covering Maps

0226 We now study a special class of inner fibrations.

0227 **Definition 4.1.5.1.** Let $f : X \to S$ be a morphism of simplicial sets. We say that f is an *inner covering map* if, for every pair of integers $0 < i < n$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] & S
\end{tikzcd}$$

has a *unique* solution.

0228 **Example 4.1.5.2.** Every covering map of simplicial sets (in the sense of Definition 3.1.4.1) is an inner covering map. In particular, if $f : X \to S$ is a covering map of topological spaces, then the induced map $\mathrm{Sing}_{\bullet}(f) : \mathrm{Sing}_{\bullet}(X) \to \mathrm{Sing}_{\bullet}(S)$ is an inner covering of simplicial sets (Proposition 3.1.4.9).

Example 4.1.5.3. Let X be a simplicial set. Then the projection map $f : X \to \Delta^0$ is an inner covering map if and only if X is isomorphic to the nerve of a category (this is a restatement of Proposition 1.3.4.1). 0229

Remark 4.1.5.4. Let $f : X \to S$ be a morphism of simplicial sets. Then f is an inner covering map if and only if the opposite morphism $f^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is an inner covering map. 022A

Remark 4.1.5.5. Let $f : X \to S$ be a morphism of simplicial sets, and let $\delta : X \to X \times_S X$ be the relative diagonal of f. Then f is an inner covering map if and only if both f and δ are inner fibrations. In particular, every inner covering map is an inner fibration. 022B

Example 4.1.5.6. Let $f : X \hookrightarrow S$ be a monomorphism of simplicial sets, so that the relative diagonal $\delta : X \hookrightarrow X \times_S X$ is an isomorphism. Then f is an inner fibration if and only if it is an inner covering. In particular, if $\mathcal{C}$ is an ∞-category and $\mathcal{C}_0 \subseteq \mathcal{C}$ is subcategory, then the inclusion map $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is an inner covering. 022C

Remark 4.1.5.7. Suppose we are given a pullback diagram of simplicial sets 022D

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r] & S.
\end{tikzcd}$$

If f is an inner covering map, then f' is also an inner covering map.

Remark 4.1.5.8. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets. Suppose that g is an inner covering map. Then f is an inner covering map if and only if $g \circ f$ is an inner covering map. In particular, the collection of inner covering maps is closed under composition. 022E

Remark 4.1.5.9. Let $f : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent: 022F

(a) The morphism f is an inner covering map (Definition 4.1.5.1).

(b) For every square diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where i is inner anodyne, there exists a unique dotted arrow rendering the diagram commutative.

022G **Proposition 4.1.5.10.** *Let $\mathcal{C}$ be a category, and let $f : X \to \mathrm{N}_\bullet(\mathcal{C})$ be a morphism of simplicial sets. Then f is an inner covering map if and only if X is isomorphic to the nerve of a category.*

Proof. Combine Remark 4.1.5.8 with Example 4.1.5.2. □

022H **Corollary 4.1.5.11.** *Let $f : X \to S$ be a morphism of simplicial sets. Then f is an inner covering if and only if, for every simplex $\sigma : \Delta^n \to S$, the fiber product $\Delta^n \times_S X$ is isomorphic to the nerve of a category.*

Proof. Suppose f is an inner covering. For every simplex $\sigma : \Delta^n \to S$, it follows from Remark 4.1.5.7 that the projection map $\Delta^n \times_S X \to \Delta^n$ is also an inner covering map, so that $\Delta^n \times_S X$ is isomorphic to the nerve of a category by virtue of Proposition 4.1.5.10. Conversely, to show that f is an inner covering map, it will suffice to show that every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] & S
\end{tikzcd}$$

has a unique solution for $0 < i < n$. If the fiber product $\Delta^n \times_S X$ is the nerve of a category, then the existence and uniqueness of the desired solution follow from (and uniqueness) of the desired solution follow from Proposition 1.3.4.1. □

022J **Exercise 4.1.5.12.** Let $f : X \to S$ be an inner covering map of simplicial sets and let $i : A \hookrightarrow B$ be any monomorphism of simplicial sets. Show that the restriction map

$$\theta : \mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)$$

is also an inner covering map. If i is inner anodyne, show that θ is an isomorphism.

4.2 Left and Right Fibrations

014H Let $q : X \to S$ be a morphism of simplicial sets. Recall that q is a *Kan fibration* if and only if it is weakly right orthogonal to every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $n > 0$ and $0 \leq i \leq n$ (Definition 3.1.1.1). In particular, if q is a Kan fibration, then it is weakly right orthogonal to both of the inclusion maps $\{0\} \hookrightarrow \Delta^1 \hookleftarrow \{1\}$. Concretely, this translates into the following pair of assertions:

(Left Path Lifting Property): Let $q : X \to S$ be a Kan fibration of simplicial sets, let x be a vertex of X, and let $\overline{e} : q(x) \to \overline{y}$ be an edge of S originating at the vertex $q(x)$. Then there exists an edge $e : x \to y$ in X which originates at the vertex x and satisfies $q(e) = \overline{e}$.

(Right Path Lifting Property): Let $q : X \to S$ be a Kan fibration of simplicial sets, let y be a vertex of X, and let $\overline{e} : \overline{x} \to q(y)$ be an edge of S terminating at the vertex $q(y)$. Then there exists an edge $e : x \to y$ in X which terminates at the vertex y and satisfies $q(e) = \overline{e}$.

In §4.2.1, we introduce stronger versions of these lifting properties. We say that a morphism of simplicial sets $q : X \to S$ is a *left fibration* if it is weakly right orthogonal to the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 \leq i < n$, and a *right fibration* if it is weakly right orthogonal to the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i \leq n$ (Definition 4.2.1.1). Setting $n = 1$, we see that every left fibration satisfies the left path lifting property, and that every right fibration satisfies the right path lifting property. Moreover, this assertion has a partial converse. Note that evaluation at the vertices of Δ^1 induces morphisms of simplicial sets

$$\mathrm{ev}_0 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S) \simeq X \times_S \mathrm{Fun}(\Delta^1, S)$$

$$\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{1\}, X) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\Delta^1, S) \simeq X \times_S \mathrm{Fun}(\Delta^1, S).$$

In §4.2.6, we show that f is a left fibration if and only if the evaluation map ev_0 is a trivial Kan fibration, and that f is a right fibration if and only if ev_1 is a trivial Kan fibration (Proposition 4.2.6.1). The "only if" direction of this assertion is a special case of general stability properties of left and right fibrations under exponentiation, which we prove in §4.2.5 (Propositions 4.2.5.1 and 4.2.5.4). Our proofs will make use of some basic facts about *left anodyne* and *right anodyne* morphisms of simplicial sets, which we establish in §4.2.4.

The notions of left and right fibration have antecedents in classical category theory. In §4.2.2, we show that the induced map of simplicial sets $\mathrm{N}_\bullet(U) : \mathrm{N}_\bullet(\mathcal{E}) \to \mathrm{N}_\bullet(\mathcal{C})$ is a right fibration if and only if U is a *fibration in groupoids* (see Definition 4.2.2.1). We will be particularly interested in the special case where U is a fibration in groupoids for which each fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is a discrete category. In §4.2.3, we show that this is equivalent to the condition that the induced map of simplicial sets $\mathrm{N}_\bullet(U)$ is a *right covering* of simplicial sets (Proposition 4.2.3.16): that is, it satisfies a *unique* lifting property for horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ with $0 < i \leq n$ (Definition 4.2.3.8).

4.2.1 Left and Right Fibrations of Simplicial Sets

We begin by introducing some terminology. 01GT

00T9 **Definition 4.2.1.1.** Let $f : X \to S$ be a morphism of simplicial sets. We will say that f is a *left fibration* if, for every pair of integers $0 \leq i < n$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & S
\end{tikzcd}$$

has a solution (as indicated by the dotted arrow). That is, for every map of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ and every n-simplex $\overline{\sigma} : \Delta^n \to S$ extending $f \circ \sigma_0$, we can extend σ_0 to an n-simplex $\sigma : \Delta^n \to X$ satisfying $f \circ \sigma = \overline{\sigma}$.

We say that f is a *right fibration* if, for every pair of integers $0 < i \leq n$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & S
\end{tikzcd}$$

has a solution.

014K **Example 4.2.1.2.** Any isomorphism of simplicial sets is both a left fibration and a right fibration.

00TA **Remark 4.2.1.3.** Let $f : X \to S$ be a morphism of simplicial sets. Then f is a left fibration if and only if the opposite morphism $f^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a right fibration.

01GU **Remark 4.2.1.4.** Let $f : X \to S$ be a morphism of simplicial sets. If f is either a left fibration or a right fibration, then it is an inner fibration. In this case, if S is an ∞-category, then X is also an ∞-category (Remark 4.1.1.9).

014L **Example 4.2.1.5.** A morphism of simplicial sets $f : X \to S$ is a Kan fibration if and only if it is both a left fibration and a right fibration.

00TD **Warning 4.2.1.6.** In the statement of Example 4.2.1.5, both hypotheses are necessary: a left fibration of simplicial sets need not be a right fibration and vice versa. For example, the inclusion map $\{1\} \hookrightarrow \Delta^1$ is a left fibration, but not a right fibration (and therefore not a Kan fibration).

Remark 4.2.1.7. The collection of left and right fibrations is closed under retracts. That is, suppose we are given a diagram of simplicial sets `014M`

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

where both horizontal compositions are the identity. If f' is a left fibration, then f is a left fibration. If f' is a right fibration, then f is a right fibration.

Remark 4.2.1.8. The collections of left and right fibrations are closed under pullback. That is, suppose we are given a pullback diagram of simplicial sets `014N`

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r] & S.
\end{tikzcd}$$

If f is a left fibration, then f' is also a left fibration. If f is a right fibration, then f' is a right fibration.

Remark 4.2.1.9. Let $f : X \to S$ be a map of simplicial sets. Suppose that, for every simplex $\sigma : \Delta^n \to S$, the projection map $\Delta^n \times_S X \to \Delta^n$ is a left fibration (right fibration). Then f is a left fibration (right fibration). Consequently, if we are given a pullback diagram of simplicial sets `014P`

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

where g is surjective and f' is a left fibration (right fibration), then f is also a left fibration (right fibration).

Remark 4.2.1.10. The collections of left and right fibrations are closed under filtered colimits. That is, suppose we are given a filtered diagram $\{f_\alpha : X_\alpha \to S_\alpha\}$ in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$, having colimit $f : X \to S$. If each f_α is a left fibration , then f is also a left fibration. If each f_α is a right fibration, then f is also a right fibration. `014Q`

014R **Remark 4.2.1.11.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets. If both f and g are left fibrations, then the composite map $(g \circ f) : X \to Z$ is a left fibration. If both f and g are right fibrations, then $g \circ f$ is a right fibration.

4.2.2 Fibrations in Groupoids

0156 We now introduce a category-theoretic counterpart of Definition 4.2.1.1.

0157 **Definition 4.2.2.1.** Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. We say that U is a *fibration in groupoids* if the following conditions are satisfied:

(A) For every object $Y \in \mathcal{E}$ and every morphism $\overline{f} : \overline{X} \to U(Y)$ in $\mathcal{C}$, there exists a morphism $f : X \to Y$ in $\mathcal{E}$ with $\overline{X} = U(X)$ and $\overline{f} = U(f)$.

(B) For every morphism $g : Y \to Z$ in $\mathcal{E}$ and every object $X \in \mathcal{E}$, the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(X,Y) \arrow[r, "g\circ"] \arrow[d, "U"] & \mathrm{Hom}_{\mathcal{E}}(X,Z) \arrow[d, "U"] \\
\mathrm{Hom}_{\mathcal{C}}(U(X),U(Y)) \arrow[r, "U(g)\circ"] & \mathrm{Hom}_{\mathcal{C}}(U(X),U(Z))
\end{tikzcd}$$

is a pullback square.

In this case, we will also say that $\mathcal{E}$ is *fibered in groupoids over* $\mathcal{C}$.

015L **Warning 4.2.2.2.** The requirement that a functor $U : \mathcal{E} \to \mathcal{C}$ is a fibration in groupoids is not invariant under equivalence. For example, an equivalence of categories need not be a fibration in groupoids.

0159 **Remark 4.2.2.3.** Condition (B) of Definition 4.2.2.1 can be rephrased as follows: given any commutative diagram

$$\begin{tikzcd}
& \overline{Y} \arrow[dr, "\overline{g}"] & \\
\overline{X} \arrow[ur, "\overline{f}"] \arrow[rr, "\overline{h}"] & & \overline{Z}
\end{tikzcd}$$

in the category $\mathcal{C}$ and any partially defined lift

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, dashed, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

to a diagram in $\mathcal{E}$ (so that $U(g) = \overline{g}$ and $U(h) = \overline{h}$), there exists a unique extension as indicated (that is, a unique morphism $f : X \to Y$ in $\mathcal{C}$ satisfying $U(f) = \overline{f}$).

Variant 4.2.2.4. Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. We say that U is a *opfibration in groupoids* if the following conditions are satisfied: `015A`

(A') For every object $X \in \mathcal{E}$ and every morphism $\overline{f} : U(X) \to \overline{Y}$ in $\mathcal{C}$, there exists a morphism $f : X \to Y$ in $\mathcal{E}$ with $\overline{Y} = U(Y)$ and $\overline{f} = U(f)$.

(B') For every morphism $g : X \to Y$ in $\mathcal{E}$ and every object $Z \in \mathcal{E}$, the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(Y, Z) \arrow[r, "\circ g"] \arrow[d, "U"] & \mathrm{Hom}_{\mathcal{E}}(X, Z) \arrow[d, "U"] \\
\mathrm{Hom}_{\mathcal{C}}(U(Y), U(Z)) \arrow[r, "\circ U(g)"] & \mathrm{Hom}_{\mathcal{C}}(U(X), U(Z))
\end{tikzcd}$$

is a pullback square.

In this case, we will also say that $\mathcal{E}$ is *opfibered in groupoids over $\mathcal{C}$*.

Warning 4.2.2.5. Some authors use the term *cofibration in groupoids* to refer to what we call an opfibration in groupoids. We will avoid the use of the word "cofibration" in this context, since it appears often in homotopy theory with a very different meaning. `015B`

Remark 4.2.2.6. Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. Then U is an opfibration in groupoids if and only if the opposite functor $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a fibration in groupoids. `01SV`

Example 4.2.2.7. Let $\mathcal{E}$ be a category, let $[0]$ denote the category having a single object and a single morphism, and let $U : \mathcal{E} \to [0]$ be the unique functor. The following conditions are equivalent: `015C`

- The functor U is a fibration in groupoids.
- The functor U is an opfibration in groupoids.
- The category $\mathcal{E}$ is a groupoid.

Remark 4.2.2.8. Suppose we are given a pullback diagram `015D`

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r] & \mathcal{C}
\end{tikzcd}$$

in the ordinary category Cat (so that the category $\mathcal{E}'$ is *isomorphic* to the fiber product $\mathcal{C}' \times_{\mathcal{C}} \mathcal{E}$). If U is a fibration in groupoids, then so is U'. Similarly, if U is an opfibration in groupoids, then so is U'.

The notion of a fibration in groupoids can be regarded as a special case of the notion of a right fibration between simplicial sets:

015H **Proposition 4.2.2.9.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. Then:*

(1) *The functor U is a fibration in groupoids if and only if the induced map $\mathrm{N}_\bullet(U) : \mathrm{N}_\bullet(\mathcal{E}) \to \mathrm{N}_\bullet(\mathcal{C})$ is a right fibration of simplicial sets.*

(2) *A functor U is an opfibration in groupoids if and only if the induced map $\mathrm{N}_\bullet(U) : \mathrm{N}_\bullet(\mathcal{E}) \to \mathrm{N}_\bullet(\mathcal{C})$ is a left fibration of simplicial sets.*

Proof. We will prove (1); the proof of (2) is similar. Assume first that U is a fibration in groupoids; we wish to show that for every pair of integers $0 < i \leq n$, every lifting problem

015J
$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{N}_\bullet(\mathcal{E}) \arrow[d, "\mathrm{N}_\bullet(U)"] \\ \Delta^n \arrow[r, "\tau"] \arrow[ur, dashed] & \mathrm{N}_\bullet(\mathcal{C}) \end{tikzcd} \tag{4.1}$$

admits a solution. If $0 < i < n$, then σ_0 admits a unique extension $\sigma : \Delta^n \to \mathrm{N}_\bullet(\mathcal{E})$ (Proposition 1.3.4.1). Moreover, since $\mathrm{N}_\bullet(U) \circ \sigma$ and τ coincide on the simplicial subset $\Lambda^n_i \subseteq \Delta^n$, they automatically coincide (again by Proposition 1.3.4.1). We may therefore assume without loss of generality that $i = n$. We consider four cases:

- If $n = 1$, then the existence of a solution to the lifting problem (4.1) is equivalent to condition (A) of Definition 4.2.2.1, and is therefore ensured by our assumption that U is a fibration in groupoids.

- If $n = 2$, then the existence of a solution to the lifting problem (4.1) follows from condition (B) of Definition 4.2.2.1 (see Remark 4.2.2.3), and is again ensured by our assumption that U is a fibration in groupoids.

- If $n = 3$, then the morphism σ_0 encodes a collection of objects $\{X_j\}_{0 \leq j \leq 3}$ and morphisms $\{f_{kj} : X_j \to X_k\}_{0 \leq j < k \leq 3}$ in the category $\mathcal{E}$, which satisfy the identities

$$f_{30} = f_{31} \circ f_{10} \qquad f_{30} = f_{32} \circ f_{20} \qquad f_{31} = f_{32} \circ f_{21}.$$

To extend σ_0 to a 3-simplex σ of $\mathrm{N}_\bullet(\mathcal{C})$, we must show that $f_{20} = f_{21} \circ f_{10}$ (note that any such extension automatically satisfies $\tau = \mathrm{N}_\bullet(U) \circ \sigma$, since the horn Λ^3_3 contains the 1-skeleton of Δ^3). Invoking our assumption that U is a fibration in groupoids, we deduce that the map

$$\mathrm{Hom}_{\mathcal{E}}(X_0, X_2) \to \mathrm{Hom}_{\mathcal{E}}(X_0, X_3) \times \mathrm{Hom}_{\mathcal{C}}(F(X_0), F(X_2)) \qquad u \mapsto (f_{32} \circ u, F(u))$$

is injective. Using the calculation

$$f_{32} \circ f_{20} = f_{30} = f_{31} \circ f_{10} = (f_{32} \circ f_{21}) \circ f_{10} = f_{32} \circ (f_{21} \circ f_{10}),$$

we are reduced to proving that $U(f_{20})$ is equal to $U(f_{21} \circ f_{10}) = U(f_{21}) \circ U(f_{10})$, which follows from the existence of the 3-simplex τ.

- If $n \geq 4$, then the horn Λ^n_i contains the 2-skeleton of Δ^n. It follows that σ_0 admits a unique extension to a map $\sigma : \Delta^n \to \mathrm{N}_\bullet(\mathcal{E})$, which automatically satisfies $\tau = \mathrm{N}_\bullet(U) \circ \sigma$.

We now prove the converse. Assume that $\mathrm{N}_\bullet(U)$ is a right fibration of simplicial sets; we wish to show that U is a fibration in groupoids. As above, we note that condition (A) of Definition 4.2.2.1 follows from the solvability of the lifting problem (4.1) in the special case $i = n = 1$. To verify condition (B), we must show that for every diagram

$$\begin{tikzcd}
 & Y \arrow[d, "g"] \\
X \arrow[r, "h"] & Z
\end{tikzcd}$$

in the category $\mathcal{E}$ and every compatible extension

$$\begin{tikzcd}
 & U(Y) \arrow[dr, "U(g)"] & \\
U(X) \arrow[ur, "\overline{f}"] \arrow[rr, "U(h)"] & & U(Z)
\end{tikzcd}$$

in the category $\mathcal{C}$, there exists a unique morphism $f : X \to Y$ in $\mathcal{E}$ satisfying $U(f) = \overline{f}$ and $g \circ f = h$. The existence of f follows from the solvability of the lifting problem (4.1) in the special case $i = n = 2$. To prove uniqueness, suppose we are given a pair of morphisms $f, f'' : X \to Y$ in $\mathcal{E}$ satisfying $U(f) = \overline{f} = U(f')$ and $g \circ f = h = g \circ f'$. Consider the not-necessarily-commutative diagram

$$\begin{tikzcd}
 & Y \arrow[r, "\mathrm{id}_Y"] \arrow[drr, "g"] & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[urr, "f'"] \arrow[rrr, "h"] & & & Z
\end{tikzcd}$$

in the category $\mathcal{E}$. Every triangle in this diagram commutes with the possible exception of the upper left, so it determines a map of simplicial sets $\sigma_0 : \Lambda^3_3 \to \mathrm{N}_\bullet(\mathcal{E})$. Moreover, the equation $U(u) = U(u')$ guarantees that $\mathrm{N}_\bullet(F) \circ \sigma_0$ extends to a 3-simplex τ of $\mathrm{N}_\bullet(\mathcal{D})$. Invoking the solvability of the lifting problem (4.1) in the case $i = n = 3$, we conclude that σ_0 can be extended to a 3-simplex of $\mathcal{C}$, which witnesses the identity $f' = \mathrm{id}_Y \circ f = f$. □

4.2.3 Left and Right Covering Maps

022K Recall that a Kan fibration of simplicial sets $f : X \to S$ is a *covering map* if, for every pair of integers $0 \leq i \leq n$ with $n \geq 1$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] & S
\end{tikzcd}$$

admits a *unique* solution (Definition 3.1.4.1). In this section, we study counterparts of this definition in the setting of left and right fibrations.

01QH **Definition 4.2.3.1.** Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. We say that U is a *left covering functor* if it satisfies the following condition:

- For every object $X \in \mathcal{E}$ and every morphism $\overline{f} : U(X) \to \overline{Y}$ in the category $\mathcal{C}$, there is a *unique* pair (Y, f), where Y is an object of $\mathcal{E}$ with $U(Y) = \overline{Y}$ and $f : X \to Y$ is a morphism in $\mathcal{E}$ with $U(f) = \overline{f}$.

We say that U is a *right covering functor* if it satisfies the following dual condition:

- For every object $Y \in \mathcal{E}$ and every morphism $\overline{f} : \overline{X} \to U(Y)$ in the category $\mathcal{C}$, there is a *unique* pair (X, f), where X is an object of $\mathcal{E}$ satisfying $U(X) = \overline{X}$ and $f : X \to Y$ is a morphism in $\mathcal{E}$ satisfying $U(f) = \overline{f}$.

022L **Remark 4.2.3.2.** Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. Then U is a right covering functor and only if the opposite functor $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a left covering functor.

01QL **Example 4.2.3.3.** We define a category Set_* as follows:

- The objects of Set_* are pairs (X, x), where X is a set and $x \in X$ is an element.
- A morphism from (X, x) to (Y, y) in Set_* is a function $f : X \to Y$ satisfying $f(x) = y$.

We will refer to Set_* as the *category of pointed sets*. The construction $(X, x) \mapsto X$ determines a left covering functor $\mathrm{Set}_* \to \mathrm{Set}$ (for a more general assertion, see Remark 4.3.1.6).

Example 4.2.3.4. Let $[0]$ denote the category having a single object and a single morphism. For any category $\mathcal{E}$, there is a unique functor $U : \mathcal{E} \to [0]$. The following conditions are equivalent: `01QM`

- The functor U is a left covering functor.
- The functor U is a right covering functor.
- The category $\mathcal{E}$ is discrete: that is, every morphism in $\mathcal{E}$ is an identity morphism.

Remark 4.2.3.5. Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. The following conditions are equivalent: `022M`

- The functor U is an isomorphism of categories.
- The functor U is a left covering functor which induces a bijection $\operatorname{Ob}(\mathcal{E}) \to \operatorname{Ob}(\mathcal{C})$.
- The functor U is a right covering functor which induces a bijection $\operatorname{Ob}(\mathcal{E}) \to \operatorname{Ob}(\mathcal{C})$.

Remark 4.2.3.6. Suppose we are given a pullback diagram of categories `01QN`

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r] & \mathcal{C}.
\end{tikzcd}$$

If U is a left covering functor, then U' is a left covering functor. If U is a right covering functor, then U' is a right covering functor.

Proposition 4.2.3.7. *Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. Then:* `01RS`

- *The functor U is a right covering map (in the sense of Definition 4.2.3.1) if and only if it is a fibration in groupoids (Definition 4.2.2.1) and, for every object $C \in \mathcal{C}$, the fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is a discrete category.*
- *The functor U is left covering map (in the sense of Definition 4.2.3.1) if and only if it is an opfibration in groupoids (Variant 4.2.2.4) and, for every object $C \in \mathcal{C}$, the fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is a discrete category.*

Proof. We will prove the first assertion; the second follows by a similar argument. Assume first that U is a right covering map. Then, for each object $C \in \mathcal{C}$, the projection map $\mathcal{E}_C \to \{C\}$ is also a right covering map (Remark 4.2.3.6), so that $\mathcal{E}_C$ is a discrete category by virtue of Example 4.2.3.4. We wish to show that U is a fibration in groupoids. Suppose that we are given an object Y of the category $\mathcal{E}$ and a morphism $\overline{f} : \overline{X} \to U(Y)$ in $\mathcal{C}$. By virtue of our assumption that U is a right covering map, we can lift $\overline{f}$ uniquely to a morphism $f : X \to Y$ in the category $\mathcal{E}$. Suppose that we are given a diagram

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ W \arrow[rr, "h"] & & Y \end{tikzcd}$$

in the category $\mathcal{E}$ and a morphism $\overline{g} : U(W) \to U(Y)$ in $\mathcal{C}$ satisfying $U(h) = U(f) \circ \overline{g}$; we wish to show that there is a unique morphism $g : W \to X$ in $\mathcal{E}$ satisfying $U(g) = \overline{g}$ and $h = f \circ g$. Invoking our assumption that U is a right covering map, we deduce that there is a unique pair (W', g'), where W' is an object of $\mathcal{E}$ satisfying $U(X') = U(X)$ and $g' : W' \to X$ is a morphism satisfying $U(g') = \overline{g}$. To complete the proof, it will suffice to show that $W' = W$ and $f \circ g' = h$. This follows from the assumption that U is a right covering map, $U(W') = U(W)$ and $U(f \circ g') = U(f) \circ U(g') = U(f) \circ \overline{g} = U(h)$.

We now prove the converse. Assume that U is a fibration in groupoids and that, for every object $C \in \mathcal{C}$, the fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is a discrete category. We wish to show that U is a right covering map. Fix an object $Y \in \mathcal{E}$ and a morphism $\overline{f} : \overline{X} \to U(Y)$ in the category $\mathcal{C}$. Since U is a fibration in groupoids, we can choose an object $X \in \mathcal{E}$ satisfying $U(X) = \overline{X}$ and a morphism $f : X \to Y$ satisfying $U(f) = \overline{f}$. To complete the proof, it will suffice to show that if X' is *any* object of $\mathcal{E}$ satisfying $U(X') = \overline{X}$ and $f' : X' \to Y$ is any morphism satisfying $U(f') = \overline{f}$, then $X' = X$ and $f' = f$. Since U is a fibration in groupoids, we see that there is a unique commutative diagram

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X' \arrow[ur, "e"] \arrow[rr, "f'"] & & Y \end{tikzcd}$$

in the category $\mathcal{E}$ satisfying $U(e) = \mathrm{id}_{\overline{X}}$. In this case, our assumption that the fiber $\mathcal{E}_{\overline{X}}$ is a discrete category guarantees that e is an identity morphism. It follows that $X = X'$ and $f' = f \circ e = f \circ \mathrm{id}_X = f$, as desired. □

We now reformulate Definition 4.2.3.1 in the language of simplicial sets.

022N **Definition 4.2.3.8.** Let $f : X \to S$ be a morphism of simplicial sets. We say that f is a *left covering map* if, for every pair of integers $0 \leq i < n$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a *unique* solution. We say that f is a *right covering map* if the analogous condition holds for $0 < i \leq n$.

Remark 4.2.3.9. Let $f : X \to S$ be a morphism of simplicial sets. Then f is a left covering map and only if the opposite morphism $f^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a right covering map. 022P

Remark 4.2.3.10. Let $f : X \to S$ be a morphism of simplicial sets. Then f is a covering map (in the sense of Definition 3.1.4.1) if and only if f is both a left covering map and a right covering map (in the sense of Definition 4.2.3.8). 022Q

Remark 4.2.3.11. Let $f : X \to S$ be a morphism of simplicial sets, and let $\delta : X \to X \times_S X$ be the relative diagonal of f. Then f is a left covering map (Definition 4.2.3.8) if and only if both f and δ are left fibrations. Similarly, f is a right covering map if and only if both f and δ are right fibrations. In particular, every left covering map is a left fibration, and every right covering map is a right fibration. 022R

Example 4.2.3.12. Let $f : X \to S$ be a monomorphism of simplicial sets. Then f is a left covering map if and only if it is a left fibration, and a right covering map if and only if it is a right fibration. 022S

Remark 4.2.3.13. Let $f : X \to S$ be a morphism of simplicial sets. If f is either a left covering map or a right covering map, then it is an inner covering map (see Definition 4.1.5.1). 022T

Remark 4.2.3.14. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets, and suppose that g is a left covering map. Then f is a left covering map if and only if $g \circ f$ is a left covering map. Similarly, if g is a right covering map, then f is a right covering map if and only if $g \circ f$ is a right covering map. In particular, the collections of left and right covering maps are closed under composition. 022U

Remark 4.2.3.15. Suppose we are given a pullback diagram of simplicial sets 022V

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r] & S.
\end{tikzcd}$$

If f is a left covering map, then f' is a left covering map. If f is a right covering map then f' is a right covering map.

Conversely, suppose that $f : X \to S$ is a morphism of simplicial sets having the property that, for every n-simplex $\Delta^n \to S$, the projection map $\Delta^n \times_S X \to \Delta^n$ is a left covering map. Then f is left covering map. If every projection map $\Delta^n \times_S X \to \Delta^n$ is a right covering map, then f is a right covering map.

Definition 4.2.3.1 can be regarded as a special case of Definition 4.2.3.8:

022W **Proposition 4.2.3.16.** *Let $\mathcal{C}$ be a category and let $f : X \to \mathrm{N}_\bullet(\mathcal{C})$ be a morphism of simplicial sets. Then:*

- *The morphism f is a left covering map (in the sense of Definition 4.2.3.8) if and only if X is isomorphic to the nerve of a category $\mathcal{E}$ and the induced map $F : \mathcal{E} \to \mathcal{C}$ is a left covering functor (in the sense of Definition 4.2.3.1).*

- *The morphism f is a right covering map if and only if X is isomorphic to the nerve of a category $\mathcal{E}$ and the induced map $F : \mathcal{E} \to \mathcal{C}$ is a right covering functor.*

Proof. We will prove the first assertion; the proof of the second is similar. Assume first that f is a left covering map. Then f is also an inner covering map (Remark 4.2.3.13). By virtue of Proposition 4.1.5.10, we can assume without loss of generality that $X = \mathrm{N}_\bullet(\mathcal{E})$ is the nerve of a category $\mathcal{E}$, so that $f : X \to \mathrm{N}_\bullet(\mathcal{C})$ can be realized as the nerve of a functor $F : \mathcal{E} \to \mathcal{C}$ (Proposition 1.3.3.1). We wish to show that F is a left covering functor: that is, for every object $Y \in \mathcal{E}$ and every morphism $\overline{u} : F(Y) \to \overline{Z}$ in $\mathcal{C}$, there exists a unique morphism $u : Y \to Z$ of $\mathcal{E}$ satisfying $F(Z) = \overline{Z}$ and $F(u) = \overline{u}$. In other words, we wish to show that the lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r, "Y"] \arrow[d, hook] & \mathrm{N}_\bullet(\mathcal{E}) \arrow[d, "\mathrm{N}_\bullet(F)"] \\
\Delta^1 \arrow[r, "\overline{u}"] \arrow[ur, dashed, "u"] & \mathrm{N}_\bullet(\mathcal{C})
\end{tikzcd}$$

has a unique solution, which again follows from our assumption that f is a left covering map.

We now prove the converse. Assume that f arises as the nerve of a left covering functor

$F : \mathcal{E} \to \mathcal{C}$. We wish to show that, for every pair of integers $0 \leq i < n$, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d, hook] & \mathrm{N}_\bullet(\mathcal{E}) \arrow[d, "\mathrm{N}_\bullet(F)"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathrm{N}_\bullet(\mathcal{C})
\end{tikzcd}$$

has a unique solution. Note that the functor F is an opfibration in groupoids (Proposition 4.2.3.7), so that $\mathrm{N}_\bullet(F)$ is a left fibration of simplicial sets (Proposition 4.2.2.9). This proves the existence of the lift σ. To prove uniqueness, suppose that σ and σ' are n-simplices of $\mathrm{N}_\bullet(\mathcal{E})$ satisfying $\sigma|_{\Lambda^n_i} = \sigma'|_{\Lambda^n_i}$ and $f(\sigma) = f(\sigma')$; we wish to show that $\sigma = \sigma'$. Fix integers $0 \leq j < k \leq n$, so that σ carries the edge $\mathrm{N}_\bullet(\{j < k\}) \subseteq \Delta^n$ to a morphism $u : Y \to Z$ of $\mathcal{E}$, and σ' carries $\mathrm{N}_\bullet(\{j < k\}) \subseteq \Delta^n$ to a morphism $u' : Y' \to Z'$ of $\mathcal{E}$. Since the vertex j belongs to $\Lambda^n_i \subseteq \Delta^n$, we must have $Y = Y'$. The equality $f(\sigma) = f(\sigma')$ guarantees that $F(u)$ and $F(u')$ are the same morphism of $\mathcal{C}$. Applying our assumption that F is a left covering functor, we conclude that $Z = Z'$ and $u = u'$. □

Remark 4.2.3.17. Let $f : X \to S$ be a morphism of simplicial sets which is either a left covering map or a right covering map. For each vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a discrete simplicial set. To prove this, we can use Remark 4.2.3.15 to reduce to the case where $S = \{s\}$ is a 0-simplex, in which case it follows by combining Proposition 4.2.3.16 with Example 4.2.3.4. 022X

Corollary 4.2.3.18. *Let $f : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:* 022Y

(1) *The morphism f is a left covering map of simplicial sets.*

(2) *For every category $\mathcal{C}$ and every morphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C}) \to S$, the pullback $\mathrm{N}_\bullet(\mathcal{C}) \times_S X$ is isomorphic to the nerve of a category $\mathcal{E}$, and f induces a left covering functor $F : \mathcal{E} \to \mathcal{C}$.*

(3) *For every n-simplex $\Delta^n \to S$, the fiber product $\Delta^n \times_S X$ is isomorphic to the nerve of a category $\mathcal{E}$ and the induced map $\mathcal{E} \to [n]$ is a left covering functor.*

Proof. Combine Proposition 4.2.3.16 with Remark 4.2.3.15. □

Proposition 4.2.3.19. *Let $f : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:* 022Z

(1) *The morphism f is an isomorphism.*

(2) *The morphism f is a left covering map and induces a bijection from the set of vertices of X to the set of vertices of S.*

(3) *The morphism f is a right covering map and induces a bijection from the set of vertices of X to the set of vertices of S.*

Proof. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are immediate. We will show that (2) $\Rightarrow$ (1); the proof that (3) $\Rightarrow$ (1) is similar. Assume that f is a left covering map which is bijective at the level of vertices; we wish to show that every n-simplex $\sigma : \Delta^n \to S$ can be lifted uniquely to an n-simplex of X. Replacing f by the projection map $\Delta^n \times_S X \to \Delta^n$, we may assume that $S = \Delta^n$ is a standard simplex (Remark 4.2.3.15). In this case, Proposition 4.2.3.16 guarantees that we can identify f with the nerve of a left covering map of categories $F : \mathcal{E} \to [n]$, so the desired result follows from Remark 4.2.3.5. $\square$

0230 **Corollary 4.2.3.20.** *Let $f : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism f is a covering map.*

(2) *The morphism f is a left covering map and a Kan fibration.*

(3) *The morphism f is a right covering map and a Kan fibration.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Remarks 4.2.3.10 and 3.1.4.3. We will prove that (2) $\Rightarrow$ (1) (the equivalence of (1) and (3) follows by a similar argument). Assume that f is a left covering map and a Kan fibration; we wish to show that f is a covering map. By virtue of Remark 3.1.4.3, it will suffice to show that the relative diagonal $\delta : X \to X \times_S X$ is a Kan fibration. Note that δ is a left fibration (Remark 4.2.3.11) and therefore a left covering map (Example 4.2.3.12). Let $D \subseteq X \times_S X$ denote the smallest summand which contains the image of δ. We will complete the proof by showing that δ induces an isomorphism from X to D (see Corollary 3.1.4.14). By virtue of Proposition 4.2.3.19, it will suffice to show that the map $\delta : X \to D$ is bijective on vertices. Equivalently, we must show that if $(e, e') : (x, x') \to (y, y')$ is any edge of the simplicial $X \times_S X$, then $x = x'$ if and only if $y = y'$. If $x = x'$, then our assumption that f is a covering map immediately guarantees that $e = e'$, so that $y = y'$. For the converse, suppose that $y = y'$, and set $s = f(x) = f(x')$. Invoking our assumption that f is a Kan fibration, we conclude that there exists a 2-simplex $\sigma : \Delta^2 \to X$ whose boundary is indicated in the diagram

$$\begin{tikzcd}
& x' \arrow[dr, "e'"] & \\
x \arrow[ur, "u"] \arrow[rr, "e"] & & y,
\end{tikzcd}$$

where $f(u) = \mathrm{id}_s$. Since f is a left covering map, the fiber $X_s = \{s\} \times_S X$ is discrete (Remark 4.2.3.17). It follows that u is a degenerate 1-simplex of X, so that $x = x'$ as desired. □

4.2.4 Left Anodyne and Right Anodyne Morphisms

To study left and right fibrations between simplicial sets, it is useful to consider the following counterpart of Definitions 3.1.2.1 and 1.5.6.4: 014S

Definition 4.2.4.1 (Left Anodyne Morphisms)**.** Let T_L be the smallest collection of morphisms in the category Set_Δ with the following properties: 014T

- For every pair of integers $0 \leq i < n$, the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ belongs to T_L.
- The collection T_L is weakly saturated (Definition 1.5.4.12). That is, T_L is closed under pushouts, retracts, and transfinite composition.

We say that a morphism of simplicial sets $f : A \to B$ is *left anodyne* if it belongs to T_L.

Variant 4.2.4.2 (Right Anodyne Morphisms)**.** Let T_R be the smallest collection of morphisms in the category Set_Δ with the following properties: 014U

- For every pair of integers $0 < i \leq n$, the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ belongs to T_R.
- The collection T_R is weakly saturated (Definition 1.5.4.12). That is, T_R is closed under pushouts, retracts, and transfinite composition.

We say that a morphism of simplicial sets $f : A \to B$ is *right anodyne* if it belongs to T_R.

Remark 4.2.4.3. Let $f : A \to B$ be a morphism of simplicial sets. Then f is left anodyne if and only if the opposite morphism $f^{\mathrm{op}} : A^{\mathrm{op}} \to B^{\mathrm{op}}$ is right anodyne. 014V

Remark 4.2.4.4. Let $f : A \to B$ be a morphism of simplicial sets. If f is either left or right anodyne, then it is anodyne (Definition 3.1.2.1). In particular, any left or right anodyne morphism of simplicial sets is a monomorphism (Remark 3.1.2.3) and a weak homotopy equivalence (Proposition 3.1.6.14). Conversely, if f is inner anodyne (Definition 1.5.6.4), then it is both left anodyne and right anodyne. That is, we have inclusions 014W

$$\begin{array}{ccc} \{\text{Inner anodyne morphisms}\} & \subset & \{\text{Left anodyne morphisms}\} \\ \cap & & \cap \\ \{\text{Right anodyne morphisms}\} & \subset & \{\text{Anodyne morphisms}\}. \end{array}$$

All of these inclusions are strict (see Example 4.2.4.6).

Proposition 4.2.4.5. *Let $q : X \to S$ be a morphism of simplicial sets. Then:* 014X

(1) *The morphism q is a left fibration if and only if, for every square diagram of simplicial sets*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

where i is left anodyne, there exists a dotted arrow rendering the diagram commutative.

(2) *The morphism q is a right fibration if and only if, for every square diagram of simplicial sets*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

where i is right anodyne, there exists a dotted arrow rendering the diagram commutative.

Proof. The "only if" directions are immediate from the definitions, and the "if" directions follow from Proposition 1.5.4.13. □

014Y **Example 4.2.4.6.** The inclusion map $i_0 : \{0\} \hookrightarrow \Delta^1$ is left anodyne (and therefore anodyne). However, it is not right anodyne (and therefore not inner anodyne). This follows from Proposition 4.2.4.5, since the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r, "\mathrm{id}"] \arrow[d, "i_0"] & \{0\} \arrow[d, "i_0"] \\ \Delta^1 \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \Delta^1 \end{tikzcd}$$

does not admit a solution (note that the inclusion map $i_0 : \{0\} \hookrightarrow \Delta^1$ is a right fibration; see Warning 4.2.1.6).

014Z **Proposition 4.2.4.7.** *Let $f : X \to Y$ be a morphism of simplicial sets. Then f can be factored as a composition $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$, where f'' is a left fibration and f' is left anodyne. Moreover, the simplicial set $Q(f)$ (and the morphisms f' and f'') can be chosen to depend functorially on f, in such a way that the functor*

$$\mathrm{Fun}([1], \mathrm{Set}_\Delta) \to \mathrm{Set}_\Delta \qquad (f : X \to Y) \to Q(f)$$

commutes with filtered colimits.

Proof. We proceed as in the proof of Proposition 3.1.7.1. We construct a sequence of simplicial sets $\{X(m)\}_{m \geq 0}$ and morphisms $f(m) : X(m) \rightarrow Y$ by recursion. Set $X(0) = X$ and $f(0) = f$. Assuming that $f(m) : X(m) \rightarrow Y$ has been defined, let $S(m)$ denote the set of all commutative diagrams σ :

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X(m) \arrow[d, "f(m)"] \\
\Delta^n \arrow[r, "u_\sigma"] & Y,
\end{tikzcd}$$

where $0 \leq i < n$ and the left vertical map is the inclusion. For every such commutative diagram σ, let $C_\sigma = \Lambda^n_i$ denote the upper left hand corner of the diagram σ, and $D_\sigma = \Delta^n$ the lower left hand corner. Form a pushout diagram

$$\begin{tikzcd}
\coprod_{\sigma \in S(m)} C_\sigma \arrow[r] \arrow[d] & X(m) \arrow[d] \\
\coprod_{\sigma \in S(m)} D_\sigma \arrow[r] & X(m+1)
\end{tikzcd}$$

and let $f(m+1) : X(m+1) \rightarrow Y$ be the unique map whose restriction to $X(m)$ is equal to $f(m)$ and whose restriction to each D_σ is equal to u_σ. By construction, we have a direct system of left anodyne morphisms

$$X = X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots$$

Set $Q(f) = \varinjlim_m X(m)$. Then the natural map $f' : X \rightarrow Q(f)$ is left anodyne (since the collection of left anodyne maps is closed under transfinite composition), and the system of morphisms $\{f(m)\}_{m \geq 0}$ can be amalgamated to a single map $f'' : Q(f) \rightarrow Y$ satisfying $f = f'' \circ f'$. It is clear from the definition that the construction $f \mapsto Q(f)$ is functorial and commutes with filtered colimits. To complete the proof, it will suffice to show that f'' is a left fibration: that is, that every lifting problem σ :

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "v"] \arrow[d] & Q(f) \arrow[d, "f''"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution (provided that $0 \leq i < n$). Let us abuse notation by identifying each $X(m)$ with its image in $Q(f)$. Since Λ^n_i is a finite simplicial set, its image under v is contained in $X(m)$ for some $m \gg 0$. In this case, we can identify σ with an element of the set $S(m)$, so that the lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "v"] \arrow[d] & X(m+1) \arrow[d, "f(m+1)"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution by construction. □

0150 **Variant 4.2.4.8.** Let $f : X \to Y$ be a morphism of simplicial sets. Then f can be factored as a composition $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$, where f'' is a right fibration and f' is right anodyne. Moreover, the simplicial set $Q(f)$ (and the morphisms f' and f'') can be chosen to depend functorially on f, in such a way that the functor

$$\mathrm{Fun}([1], \mathrm{Set}_\Delta) \to \mathrm{Set}_\Delta \qquad (f : X \to Y) \to Q(f)$$

commutes with filtered colimits.

Using Proposition 4.2.4.7 (and Variant 4.2.4.8), we obtain the following converse of Proposition 4.2.4.5:

0151 **Corollary 4.2.4.9.** *Let $i : A \to B$ be a morphism of simplicial sets. Then:*

(1) *The morphism i is left anodyne if and only if, for every square diagram of simplicial sets*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where f is left fibration, there exists a dotted arrow rendering the diagram commutative.

(2) *The morphism i is right anodyne if and only if, for every square diagram of simplicial sets*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

where f is right fibration, there exists a dotted arrow rendering the diagram commutative.

Proof. We will prove (1); the proof of (2) is similar. Using Proposition 4.2.4.7, we can factor i as a composition $A \xrightarrow{i'} Q \xrightarrow{f} B$, where i' is left anodyne and f is a left fibration. If the lifting problem

$$\begin{tikzcd} A \arrow[r, "i'"] \arrow[d, "i"'] & Q \arrow[d, "f"] \\ B \arrow[r, "\mathrm{id}"] \arrow[ur, dashed, "r"] & B \end{tikzcd}$$

admits a solution, then the map r exhibits i as a retract of i' (in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$). Since the collection of anodyne morphisms is closed under retracts, it follows that i is anodyne. This proves the "if" direction of (1); the reverse implication follows from Proposition 4.2.4.5. □

For later use, we record the following:

Proposition 4.2.4.10. *For every pair of integers $0 \leq j < n$, the horn Λ^n_j admits a finite filtration* 05HW

$$\{0\} = X(0) \subset X(1) \subset \cdots \subset X(k) = \Lambda^n_j,$$

where each of the inclusion maps $X(a-1) \hookrightarrow X(a)$ can be realized as a pushout of a horn inclusion $\Lambda^m_i \hookrightarrow \Delta^m$ for $0 \leq i < m < n$. In particular, the inclusion $\{0\} \hookrightarrow \Lambda^n_j$ is left anodyne.

Proof. Let us say that a monomorphism of simplicial sets $A \hookrightarrow B$ is *good* if it can be written as a composition of finitely many morphisms, each of which is a pushout of a horn inclusion $\Lambda^m_i \hookrightarrow \Delta^m$ for $0 \leq i < m < n$. We wish to show that the inclusion map $\{0\} \hookrightarrow \Lambda^n_j$ is good. Our proof proceeds by induction on n. It follows from our inductive hypothesis that the inclusion map $\{0\} \hookrightarrow \Lambda^j_0$ is good; in particular, the inclusion map $\{0\} \hookrightarrow \Delta^j$ is good. For every integer $d \geq 0$, let $Y(d) \subseteq \Delta^n$ be the simplicial subset whose nondegenerate simplices have vertex set $J \subseteq [n]$ satisfying one of the following conditions:

- The set J is contained in $[j] = \{0 < 1 < \cdots < j\}$.
- The set $J \setminus \{j\}$ has cardinality $< d$.

We have inclusion maps

$$\Delta^j = Y(0) \subseteq Y(1) \subseteq Y(2) \subseteq \cdots \subseteq Y(n-1) = \Lambda^n_j.$$

It will therefore suffice to show that for $0 < d < n$, the inclusion map $Y(d-1) \hookrightarrow Y(d)$ is good. Let S be the collection of all nondegenerate d-simplices of $Y(d)$. By construction, for each $\sigma \in S$, there is a unique integer $0 \leq i_{\sigma} < d$ satisfying $\sigma(i_{\sigma}) = j$. Unwinding the definitions, we see that there is a pushout square

$$\begin{tikzcd}
\coprod_{\sigma \in S} \Lambda^{d}_{i_{\sigma}} \arrow[r] \arrow[d, phantom] & Y(d-1) \arrow[d] \\
\coprod_{\sigma \in S} \Delta^{d} \arrow[r] & Y(d),
\end{tikzcd}$$

so that the inclusion $Y(d-1) \hookrightarrow Y(d)$ can be written as a finite pushout of morphisms of the form $\Lambda^{d}_{i} \hookrightarrow \Delta^{d}$ for $i < d$. □

05HX **Corollary 4.2.4.11.** *For every integer $n \geq 0$, the inclusion map $\iota : \{0\} \hookrightarrow \Delta^{n}$ is left anodyne.*

For a more general statement, see Example 4.3.7.11.

Proof of Corollary 4.2.4.11. We may assume $n > 0$ (otherwise the result is trivial). Choose an integer $0 \leq i < n$, so that the horn inclusion $\Lambda^{n}_{i} \hookrightarrow \Delta^{n}$ is left anodyne. It will therefore suffice to show that the inclusion $\{0\} \hookrightarrow \Lambda^{n}_{i}$ is left anodyne, which follows from Proposition 4.2.4.10. □

0231 **Corollary 4.2.4.12.** *Let $q : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism q is a left covering map, in the sense of Definition 4.2.3.8.*

(2) *Every lifting problem*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "q"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a unique solution, provided that the morphism i is inner anodyne.

(3) *For every integer $n \geq 0$, the diagram of sets*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^{n}, X) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\{0\}, X) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^{n}, S) \arrow[r] & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\{0\}, S)
\end{tikzcd}$$

is a pullback square.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 4.2.4.5 and Remark 4.2.3.11, and the implication (2) $\Rightarrow$ (3) follows from Corollary 4.2.4.11. We will complete the proof by showing that (3) implies (1). Assume that condition (3) is satisfied; we wish to show that, for $0 \leq j < n$, the left half of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, X) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_j, X) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\{0\}, X) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, S) \arrow[r] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^n_j, S) \arrow[r] & \mathrm{Hom}_{\mathrm{Set}_\Delta}(\{0\}, S)
\end{tikzcd}$$

is a pullback square. We proceed by induction on n. Assumption (3) guarantees that the outer rectangle is a pullback, so we are reduced to showing that the square on the right is a pullback. This follows by combining our inductive hypothesis with Proposition 4.2.4.10. □

4.2.5 Exponentiation for Left and Right Fibrations

We now establish a stability property for left and right fibrations under exponentiation. 0152

Proposition 4.2.5.1. *Let $f : X \to S$ and $i : A \hookrightarrow B$ be morphisms of simplicial sets, where i is a monomorphism, and let* 00TP

$$\rho : \mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)$$

be the induced map. If f is a left fibration, then ρ is a left fibration. If f is a right fibration, then ρ is a right fibration.

Corollary 4.2.5.2. *Let $f : X \to S$ be a morphism of simplicial sets, let B be an arbitrary simplicial set, and let $\rho : \mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S)$ be the morphism induced by composition with f. If f is a left fibration, then ρ is a left fibration. If f is a right fibration, then ρ is a right fibration.* 00TQ

Proposition 4.2.5.1 is essentially equivalent to the following stability property of left and right anodyne morphisms:

Proposition 4.2.5.3. *Let $f : A \hookrightarrow B$ and $f' : A' \hookrightarrow B'$ be monomorphisms of simplicial sets. If f is left anodyne, then the induced map* 0153

$$\theta : (A \times B') \coprod_{A \times A'} (B \times A') \hookrightarrow B \times B'$$

is left anodyne. If f is right anodyne, then θ is right anodyne.

Proof. We will prove the second assertion (the first follows by a similar argument). We proceed as in the proof of Proposition 3.1.2.9. Let us first regard the monomorphism $f' : A' \hookrightarrow B'$ as fixed, and let T be the collection of all maps $f : A \to B$ for which the induced map

$$\theta_{f,f'} : (A \times B') \coprod_{A \times A'} (B \times A') \hookrightarrow B \times B'$$

is right anodyne. We wish to show that every right anodyne morphism belongs to T. Since T is weakly saturated, it will suffice to show that every horn inclusion $f : \Lambda^n_i \hookrightarrow \Delta^n$ belongs to T for $0 < i \leq n$. In this case, Lemma 3.1.2.10 guarantees that f is a retract of the morphism $g : (\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Lambda^n_i} (\{1\} \times \Delta^n) \hookrightarrow \Delta^1 \times \Delta^n$. It will therefore suffice to show that g belongs to T. Replacing f' by the monomorphism $(\Lambda^n_i \times B') \coprod_{\Lambda^n_i \times A'} (\Delta^n \times A') \hookrightarrow \Delta^n \times B'$, we are reduced to showing that the inclusion $\{1\} \hookrightarrow \Delta^1$ belongs to T.

Let T' denote the collection of all morphisms of simplicial sets $f'' : A'' \to B''$ for which the map $(\{1\} \times B'') \coprod_{\{1\} \times A''} (\Delta^1 \times A'') \to \Delta^1 \times B''$ is right anodyne. We will complete the proof by showing that T' contains all monomorphisms of simplicial sets. By virtue of Proposition 1.5.5.14, it will suffice to show that T'' contains the inclusion map $\partial \Delta^m \hookrightarrow \Delta^m$, for each $m > 0$. In other words, we are reduced to showing that the inclusion $(\{1\} \times \Delta^m) \coprod_{\{1\} \times \partial \Delta^m} (\Delta^1 \times \partial \Delta^m) \hookrightarrow \Delta^1 \times \Delta^m$ is right anodyne, which follows from Lemma 3.1.2.12. □

Proof of Proposition 4.2.5.1. Let $f : X \to S$ be a left fibration of simplicial sets and let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets. We wish to show that the restriction map

$$\rho : \operatorname{Fun}(B, X) \to \operatorname{Fun}(B, S) \times_{\operatorname{Fun}(A,S)} \operatorname{Fun}(A, X)$$

is also a left fibration (the dual assertion about right fibrations follows by passing to opposite simplicial sets). By virtue of Proposition 4.2.4.5, this is equivalent to the assertion that every lifting problem

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"] & \operatorname{Fun}(B, X) \arrow[d, "\rho"] \\
B' \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(B, S) \times_{\operatorname{Fun}(A,S)} \operatorname{Fun}(A, X)
\end{tikzcd}$$

admits a solution, provided that i' is left anodyne. Equivalently, we must show that every lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{A \times A'} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution. This follows from Proposition 4.2.4.5, since the left vertical map is left anodyne (Proposition 4.2.5.3) and the right vertical map is a left fibration. ☐

Proposition 4.2.5.3 has another application, which will be useful in the next section:

Proposition 4.2.5.4. *Let $f : X \to S$ and $i : A \to B$ be morphisms of simplicial sets, and let* 0154

$$\rho : \mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)$$

be the induced map. If f is a left fibration and i is left anodyne, then ρ is a trivial Kan fibration. If f is a right fibration and i is right anodyne, then ρ is a trivial Kan fibration.

Proof. We proceed as in the proof of Proposition 4.2.5.1. Assume that f is a left fibration and that i is left anodyne; we will show that ρ is a trivial Kan fibration (the dual assertion for right fibrations follows by a similar argument). Fix a monomorphism of simplicial sets $i' : A' \hookrightarrow B'$; we wish to show that every lifting problem

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"] & \mathrm{Fun}(B, X) \arrow[d, "\rho"] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)
\end{tikzcd}$$

admits a solution. Equivalently, we must show that every lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{A \times A'} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution. This follows from Proposition 4.2.4.5, since the left vertical map is left anodyne (Proposition 4.2.5.3) and the right vertical map is a left fibration. ☐

Exercise 4.2.5.5. Let $f : X \to S$ be a left covering morphism of simplicial sets. Show that, for any left anodyne morphism $i : A \hookrightarrow B$, the induced map 032V

$$\rho : \mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(A, X)$$

is an isomorphism of simplicial sets.

4.2.6 The Homotopy Extension Lifting Property

We now show that left and right fibrations can be characterized by homotopy lifting properties. 00T8

00TE **Proposition 4.2.6.1.** *Let $f : X \to S$ be a morphism of simplicial sets. Then:*

- *The morphism f is a left fibration if and only if the evaluation map*
$$\mathrm{ev}_0 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S)$$
is a trivial Kan fibration.
- *The morphism f is a right fibration if and only if the evaluation map*
$$\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{1\}, X) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\Delta^1, S)$$
is a trivial Kan fibration.

Proof. We prove the second assertion; the first follows by passing to opposite simplicial sets. If f is a right fibration, then the evaluation map ev_1 is a trivial Kan fibration by virtue of Proposition 4.2.5.4 (since the inclusion $\{1\} \hookrightarrow \Delta^1$ is right anodyne). Conversely, suppose that ev_1 is a trivial Kan fibration. Then every lifting problem

$$\begin{tikzcd}
(\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Lambda^n_i} (\{1\} \times \Delta^n) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution. In other words, f is weakly right orthogonal to the inclusion map
$$u : (\Delta^1 \times \Lambda^n_i) \coprod_{\{1\} \times \Lambda^n_i} (\{1\} \times \Delta^n) \hookrightarrow \Delta^1 \times \Delta^n.$$

If $0 < i \leq n$, then the horn inclusion $u_0 : \Lambda^n_i \hookrightarrow \Delta^n$ is a retract of u (Lemma 3.1.2.10). It follows that f is also weakly left orthogonal to u_0 (Proposition 1.5.4.9): that is, every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & S
\end{tikzcd}$$

admits a solution. □

0155 **Corollary 4.2.6.2.** *Let $f : X \to S$ be a morphism of simplicial sets. Then f is a Kan fibration if and only if both of the evaluation maps*
$$\mathrm{ev}_0 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S)$$
$$\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{1\}, X) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\Delta^1, S)$$
are trivial Kan fibrations.

Proof. Combine Proposition 4.2.6.1 with Example 4.2.1.5. □

Remark 4.2.6.3 (The Homotopy Extension Lifting Property)**.** Let $f : X \to S$ be a morphism of simplicial sets. Unwinding the definitions, we see that the following conditions are equivalent: 00TB

- The morphism f is a left fibration.
- For every monomorphism of simplicial sets $i : A \hookrightarrow B$, every lifting problem

$$\begin{tikzcd}
A \arrow[rr] \arrow[d, "i"] & & \mathrm{Fun}(\Delta^1, X) \arrow[d, "\mathrm{ev}_0"] \\
B \arrow[r] \arrow[urr, dashed] & \mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\}, S)} \mathrm{Fun}(\Delta^1, S) &
\end{tikzcd}$$

  admits a solution (indicated by the dotted arrow in the diagram).

- For every monomorphism of simplicial sets $i : A \hookrightarrow B$, every lifting problem

$$\begin{tikzcd}
(\Delta^1 \times A) \coprod_{\{0\} \times A} (\{0\} \times B) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\Delta^1 \times B \arrow[r, "\overline{h}"] \arrow[ur, dashed, "h"] & S
\end{tikzcd}$$

  admits a solution (indicated by the dotted arrow in the diagram).

- Let $u : B \to X$ be a map of simplicial sets and let $\overline{h} : \Delta^1 \times B \to S$ be a map satisfying $\overline{h}|_{\{0\} \times B} = f \circ u$: that is, $\overline{h}$ is a homotopy from $f \circ u$ to another map $\overline{v} = \overline{h}|_{\{1\} \times B}$. Then we can choose a map of simplicial sets $h : \Delta^1 \times B \to X$ satisfying $f \circ h = \overline{h}$ and $h|_{\{0\} \times B} = u$: in other words, $\overline{h}$ can be lifted to a homotopy h from u to another map $v = h|_{\{1\} \times B}$. Moreover, given any simplicial subset $A \subseteq B$ and any map $h_0 : \Delta^1 \times A \to X$ satisfying $f \circ h_0 = \overline{h}|_{\Delta^1 \times A}$ and $h_0|_{\{0\} \times A} = u|_A$, we can arrange that h is an extension of h_0.

In the special case where $B = \Delta^0$ and $A = \emptyset$, each of these assertions reduces to the left path lifting property of f.

Exercise 4.2.6.4. Let $f : X \to S$ be a morphism of simplicial sets. Show that: 0232

- The morphism f is a left covering map if and only if the evaluation map

$$\mathrm{ev}_0 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{0\}, X) \times_{\mathrm{Fun}(\{0\}, S)} \mathrm{Fun}(\Delta^1, S)$$

  is an isomorphism of simplicial sets

- The morphism f is a right covering map if and only if the evaluation map

$$\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{1\}, X) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\Delta^1, S)$$

  is an isomorphism of simplicial sets.

- The morphism f is a covering map if and only if both ev_0 and ev_1 are isomorphisms.

4.3 The Slice and Join Constructions

015M Let $F : \mathcal{K} \to \mathcal{C}$ be a functor between categories. A *cone over* F is an object $C \in \mathcal{C}$ together with a collection of morphisms $\{\alpha_K : C \to F(K)\}_{K \in \mathcal{K}}$ with the following property: for every morphism $\beta : K \to K'$ of the category $\mathcal{K}$, the diagram

$$\begin{tikzcd} & C \arrow[dl, "\alpha_K"'] \arrow[dr, "\alpha_{K'}"] & \\ F(K) \arrow[rr, "F(\beta)"] & & F(K') \end{tikzcd}$$

commutes. The collection of cones $(C, \{\alpha_K\}_{K \in \mathcal{K}})$ can be organized into a category, which we will denote by $\mathcal{C}_{/F}$ and refer to as the *slice category of* $\mathcal{C}$ *over* F (Construction 4.3.1.8). This construction plays an important role in category theory: for example, a *limit* of the diagram F is (by definition) a final object of the category $\mathcal{C}_{/F}$.

Our goal in this section is to generalize the construction $(F : \mathcal{K} \to \mathcal{C}) \mapsto \mathcal{C}_{/F}$ to the setting of ∞-categories. Our first step is to show that the slice category $\mathcal{C}_{/F}$ can be characterized by a universal property. In §4.3.2, we associate to every pair of categories $\mathcal{D}$ and $\mathcal{K}$ a new category $\mathcal{D} \star \mathcal{K}$, which we refer to as the *join of* $\mathcal{D}$ *and* $\mathcal{K}$ (Definition 4.3.2.1). This is a new category which contains $\mathcal{D}$ and $\mathcal{K}$ as full subcategories, having a unique morphism from each object of $\mathcal{D}$ to each object of $\mathcal{K}$ (and no morphisms in the opposite direction). We then show the datum of a functor $\mathcal{D} \to \mathcal{C}_{/F}$ is equivalent to the datum of a functor $\overline{F} : \mathcal{D} \star \mathcal{K} \to \mathcal{C}$ satisfying $\overline{F}|_{\mathcal{K}} = F$ (Proposition 4.3.2.10).

In §4.3.3, we extend the join construction to the setting of ∞-categories. To every pair of simplicial sets X and Y, we associate a new simplicial set $X \star Y$ (Construction 4.3.3.13), which contains X and Y as (disjoint) simplicial subsets. This construction has the following features:

- For every pair of categories $\mathcal{C}$ and $\mathcal{D}$, there is a canonical isomorphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C}) \star \mathrm{N}_\bullet(\mathcal{D}) \simeq \mathrm{N}_\bullet(\mathcal{C} \star \mathcal{D})$ (Example 4.3.3.14). Consequently, the join operation on simplicial sets can be regarded as a generalization of the join operation on categories.

- For every pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the join $\mathcal{C} \star \mathcal{D}$ is an ∞-category (Corollary 4.3.3.25).
- For every pair of simplicial sets X and Y, the join $X \star Y$ is equipped with a continuous bijection

$$|X \star Y| \simeq |X| \coprod_{(|X| \times \{0\} \times |Y|)} (|X| \times [0,1] \times |Y|) \coprod_{(|X| \times \{1\} \times |Y|)} |Y|,$$

  which is a homeomorphism if either X or Y is finite (Proposition 4.3.4.11 and Corollary 4.3.4.12).

Let $f : K \to X$ be any morphism of simplicial sets. In §4.3.5, we introduce a new simplicial set $X_{/f}$, which we will refer to as the *slice of X over f* (Construction 4.3.5.1). The simplicial set $X_{/f}$ is characterized (up to isomorphism) by the following universal mapping property: for any simplicial set Y, the datum of a morphism of simplicial sets $Y \to X_{/f}$ is equivalent to the datum of a morphism of simplicial sets $\overline{f} : Y \star K \to X$ satisfying $\overline{f}|_K = f$ (Proposition 4.3.5.13). Moreover, we will show that it has the following additional properties:

- If $F : \mathcal{K} \to \mathcal{C}$ is a functor between categories and $\mathrm{N}_\bullet(F) : \mathrm{N}_\bullet(\mathcal{K}) \to \mathrm{N}_\bullet(\mathcal{C})$ is the associated map of simplicial sets, then there is a canonical isomorphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C})_{/\,\mathrm{N}_\bullet(F)} \simeq \mathrm{N}_\bullet(\mathcal{C}_{/F})$ (Example 4.3.5.7). Consequently, the slice operation on simplicial sets can be regarded as a generalization of the slice operation on categories.
- If $\mathcal{C}$ is an ∞-category and $f : K \to \mathcal{C}$ is a morphism of simplicial sets, then the simplicial set $\mathcal{C}_{/f}$ is also an ∞-category. Moreover, the evident forgetful functor $\mathcal{C}_{/f} \to \mathcal{C}$ is a right fibration of ∞-categories (Proposition 4.3.6.1).
- If $q : X \to S$ is a left fibration of simplicial sets and $f : K \to X$ is any morphism of simplicial sets, then the natural map $X_{/f} \to X \times_S S_{/(q \circ f)}$ is a Kan fibration of simplicial sets (Corollary 4.3.7.3).
- If $q : X \to S$ is a right fibration of simplicial sets and $x \in X$ is a vertex (which we identify with a map of simplicial sets $\Delta^0 \to X$) having image $s \in S$, then the induced map $X_{/x} \to S_{/s}$ is a trivial Kan fibration of simplicial sets (Corollary 4.3.7.13).

4.3.1 Slices of Categories

We begin by discussing the slice construction in a special case. 015N

Construction 4.3.1.1 (Slice Categories over Objects)**.** Let $\mathcal{C}$ be a category containing an object S. We define a category $\mathcal{C}_{/S}$ as follows: 015P

- The objects of $\mathcal{C}_{/S}$ are pairs (X, f), where X is an object of $\mathcal{C}$ and $f : X \to S$ is a morphism in $\mathcal{C}$.
- If (X, f) and (Y, g) are objects of $\mathcal{C}_{/S}$, then a morphism from (X, f) to (Y, g) in the category $\mathcal{C}_{/S}$ is a morphism $u : X \to Y$ in the category $\mathcal{C}$ satisfying $f = g \circ u$. In other words, morphisms from (X, f) to (Y, g) are given by commutative diagrams

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

  in the category $\mathcal{C}$.
- Composition of morphisms in the category $\mathcal{C}_{/S}$ is given by composition of morphisms in the category $\mathcal{C}$.

We will refer to $\mathcal{C}_{/S}$ as the *slice category of* $\mathcal{C}$ *over* S.

015Q **Example 4.3.1.2.** Let Set denote the category of sets, and let $S \in \mathrm{Set}$ be a set. Then the construction

$$(f : X \to S) \mapsto \{X_s = f^{-1}\{s\}\}_{s \in S}$$

induces an equivalence of categories $\mathrm{Set}_{/S} \to \prod_{s \in S} \mathrm{Set}$.

015R **Remark 4.3.1.3.** Let $\mathcal{C}$ be a category which admits finite limits and let $*$ denote a final object of $\mathcal{C}$. For any object $S \in \mathcal{C}$, one can adapt the construction of Example 4.3.1.2 to define a functor

$$F : \mathcal{C}_{/S} \to \prod_{s : * \to S} \mathcal{C} \qquad F(X \to S) = \{* \times_S X\}_{s : * \to S}.$$

Motivated by this observation, it is often useful to think of objects of the slice category $\mathcal{C}_{/S}$ as “families” of objects of $\mathcal{C}$ which are parametrized by S. Beware that the functor F is usually not an equivalence of categories.

015S **Variant 4.3.1.4** (Coslice Categories under Objects)**.** Let $\mathcal{C}$ be a category containing an object S. We define a category $\mathcal{C}_{S/}$ as follows:

- The objects of $\mathcal{C}_{S/}$ are pairs (X, f), where X is an object of $\mathcal{C}$ and $f : S \to X$ is a morphism in $\mathcal{C}$.

- If (X, f) and (Y, g) are objects of $\mathcal{C}_{S/}$, then a morphism from (X, f) to (Y, g) in the category $\mathcal{C}_{S/}$ is a morphism $u : X \to Y$ in the category $\mathcal{C}$ satisfying $g = f \circ u$. In other words, morphisms from (X, f) to (Y, g) are given by commutative diagrams

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "u"] & & Y \end{tikzcd}$$

  in the category $\mathcal{C}$.

- Composition of morphisms in the category $\mathcal{C}_{S/}$ is given by composition of morphisms in the category $\mathcal{C}$.

We will refer to $\mathcal{C}_{S/}$ as the *coslice category of* $\mathcal{C}$ *under* S.

Remark 4.3.1.5. Variant 4.3.1.4 is formally dual to Construction 4.3.1.1. More precisely, if S is an object of a category $\mathcal{C}$, then we have a canonical isomorphism of categories `015T`

$$(\mathcal{C}_{/S})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{S/},$$

where we view S also as an object of the opposite category $\mathcal{C}^{\mathrm{op}}$.

Remark 4.3.1.6. Let $\mathcal{C}$ be a category and let S be an object of $\mathcal{C}$. Then the forgetful functor $\mathcal{C}_{/S} \to \mathcal{C}$ is a right covering map, in the sense of Definition 4.2.3.1. Similarly, the forgetful functor $\mathcal{C}_{S/} \to \mathcal{C}$ is a left covering map. `0233`

Remark 4.3.1.7 (Slice Categories as Oriented Fiber Products)**.** Let $\mathcal{C}$ be a category and let $\mathrm{Fun}([1], \mathcal{C})$ denote the arrow category of $\mathcal{C}$, so that the elements $0, 1 \in [1]$ determine evaluation functors `015U`

$$\mathrm{ev}_0 : \mathrm{Fun}([1], \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C} \qquad \mathrm{ev}_1 : \mathrm{Fun}([1], \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}\,.$$

For each object $S \in \mathcal{C}$, the slice category $\mathcal{C}_{/S}$ can be identified with the fiber of the evaluation functor ev_1 over S, and the coslice category $\mathcal{C}_{S/}$ can be identified with the fiber of the evaluation functor ev_0 over S. That is, we have pullback diagrams

$$\begin{tikzcd} \mathcal{C}_{S/} \arrow[r] \arrow[d] & \mathrm{Fun}([1], \mathcal{C}) \arrow[d, "\mathrm{ev}_0"] & \mathcal{C}_{/S} \arrow[r] \arrow[d] & \mathrm{Fun}([1], \mathcal{C}) \arrow[d] \\ \{S\} \arrow[r] & \mathcal{C} & \{S\} \arrow[r] & \mathcal{C}\,. \end{tikzcd}$$

In other words, we can identify $\mathcal{C}_{/S}$ with the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \{S\}$ of Notation 2.1.4.19 (here we identify the object S with the constant functor $[0] \to \mathcal{C}$ taking the value S), and $\mathcal{C}_{S/}$ with the oriented fiber product $\{S\} \tilde{\times}_{\mathcal{C}} \mathcal{C}$.

For many applications it is useful to consider a generalization of Construction 4.3.1.1, which associates a slice category $\mathcal{C}_{/F}$ to an arbitrary diagram $F : \mathcal{K} \to \mathcal{C}$ (instead of a single object $S \in \mathcal{C}$).

015V **Construction 4.3.1.8** (Slice Categories over Diagrams)**.** Let $\mathcal{K}$ and $\mathcal{C}$ be categories. For each object $C \in \mathcal{C}$, we let $\underline{C} : \mathcal{K} \to \mathcal{C}$ denote the associated constant functor (carrying each object of $\mathcal{K}$ to the object C and each morphism of $\mathcal{K}$ to the identity morphism id_C). The construction $C \mapsto \underline{C}$ determines a functor $\mathcal{C} \to \mathrm{Fun}(\mathcal{K}, \mathcal{C})$.

For every functor $F : \mathcal{K} \to \mathcal{C}$, we let $\mathcal{C}_{/F}$ denote the fiber product $\mathcal{C} \times_{\mathrm{Fun}(\mathcal{K},\mathcal{C})} \mathrm{Fun}(\mathcal{K}, \mathcal{C})_{/F}$, where $\mathrm{Fun}(\mathcal{K}, \mathcal{C})_{/F}$ is the slice category of Construction 4.3.1.1. Similarly, we let $\mathcal{C}_{F/}$ denote the fiber product $\mathcal{C} \times_{\mathrm{Fun}(\mathcal{K},\mathcal{C})} \mathrm{Fun}(\mathcal{K}, \mathcal{C})_{F/}$, where $\mathrm{Fun}(\mathcal{K}, \mathcal{C})_{F/}$ denotes the coslice category of Variant 4.3.1.4. We will refer to $\mathcal{C}_{/F}$ as the *slice category of* $\mathcal{C}$ *over* F, and to $\mathcal{C}_{F/}$ as the *coslice category of* $\mathcal{C}$ *under* F.

015W **Remark 4.3.1.9.** The slice and coslice constructions of Construction 4.3.1.8 are mutually dual. More precisely, if $F : \mathcal{K} \to \mathcal{C}$ is a functor between categories and $F^{\mathrm{op}} : \mathcal{K}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is the induced functor between opposite categories, then we have canonical isomorphisms

$$(\mathcal{C}_{/F})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{F^{\mathrm{op}}/} \qquad (\mathcal{C}_{F/})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{/F^{\mathrm{op}}}.$$

015X **Example 4.3.1.10.** Let $[0]$ denote the category having a single object and a single morphism. For any category $\mathcal{C}$, the diagonal map

$$\delta : \mathcal{C} \to \mathrm{Fun}([0], \mathcal{C}) \qquad S \mapsto \underline{S}$$

is an isomorphism of categories. It follows that, for any object $S \in \mathcal{C}$, we have canonical isomorphisms

$$\mathcal{C}_{/S} \simeq \mathcal{C}_{/\underline{S}} \qquad \mathcal{C}_{S/} \simeq \mathcal{C}_{\underline{S}/}.$$

Consequently, we can view Construction 4.3.1.1 and Variant 4.3.1.4 as special cases of Construction 4.3.1.8.

015Y **Remark 4.3.1.11.** Let $F : \mathcal{K} \to \mathcal{C}$ be a functor between categories. Remark 4.3.1.7, we see that the slice and coslice categories of Construction 4.3.1.8 are can be realized as oriented fiber products: more precisely, we have canonical isomorphisms

$$\mathcal{C}_{/F} \simeq \mathcal{C} \tilde{\times}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})} \{F\} \qquad \mathcal{C}_{F/} \simeq \{F\} \tilde{\times}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})} \mathcal{C}.$$

Remark 4.3.1.12. Let $\mathcal{C}$ be a category and let $F : \mathcal{K} \to \mathcal{C}$ be a diagram in $\mathcal{C}$. If F admits a limit $S = \varprojlim_{I \in \mathcal{K}} F(I)$, then the slice category $\mathcal{C}_{/F}$ is isomorphic to $\mathcal{C}_{/S}$. Similarly, if F admits a colimit $S' = \varinjlim_{I \in \mathcal{K}} F(I)$, then the coslice category $\mathcal{C}_{F/}$ is isomorphic to $\mathcal{C}_{S/}$. In §7.1, we will use this observation to extend the theory of limits and colimits to the setting of ∞-categories. 015Z

4.3.2 Joins of Categories

Our next goal is to characterize the slice categories of Construction 4.3.1.8 by a universal mapping property. 0160

Definition 4.3.2.1 (Joins of Categories)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We define a category $\mathcal{C} \star \mathcal{D}$ as follows: 0161

- The set of objects $\mathrm{Ob}(\mathcal{C} \star \mathcal{D})$ is the disjoint union of $\mathrm{Ob}(\mathcal{C})$ with $\mathrm{Ob}(\mathcal{D})$.
- Given a pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C} \star \mathcal{D})$, we have

$$\mathrm{Hom}_{\mathcal{C} \star \mathcal{D}}(X, Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \text{if } X, Y \in \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Hom}_{\mathcal{D}}(X, Y) & \text{if } X, Y \in \mathrm{Ob}(\mathcal{D}) \\ * & \text{if } X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D}) \\ \emptyset & \text{if } X \in \mathrm{Ob}(\mathcal{D}), Y \in \mathrm{Ob}(\mathcal{C}). \end{cases}$$

- Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathcal{C} \star \mathcal{D}$. If $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, then $g \circ f \in \mathrm{Hom}_{\mathcal{C} \star \mathcal{D}}(X, Z)$ is given by the composition of morphisms in $\mathcal{C}$. If $X, Y, Z \in \mathrm{Ob}(\mathcal{D})$, then $g \circ f$ is given by composition of morphisms in $\mathcal{D}$. Otherwise, we let $g \circ f$ denote the unique morphism from X to Z (note that in this case, we necessarily have $X \in \mathrm{Ob}(\mathcal{C})$ and $Z \in \mathrm{Ob}(\mathcal{D})$).

We will refer to $\mathcal{C} \star \mathcal{D}$ as the *join of* $\mathcal{C}$ *with* $\mathcal{D}$.

Remark 4.3.2.2. In the situation of Definition 4.3.2.1, we will generally abuse notation by identifying $\mathcal{C}$ and $\mathcal{D}$ with full subcategories of the join $\mathcal{C} \star \mathcal{D}$. 0162

Remark 4.3.2.3. Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{D} \to \mathcal{D}'$ be functors. Then F and G induce a functor 0163

$$(F \star G) : \mathcal{C} \star \mathcal{D} \to \mathcal{C}' \star \mathcal{D}',$$

which is uniquely determined by the requirement that it coincides with F on the full subcategory $\mathcal{C} \subseteq \mathcal{C} \star \mathcal{D}$ and with G on the full subcategory $\mathcal{D} \subseteq \mathcal{C} \star \mathcal{D}$. We can therefore regard the join construction as a functor

$$\star : \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat} \qquad (\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \star \mathcal{D},$$

where Cat denotes the category of (small) categories.

0164 **Example 4.3.2.4.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. If $\mathcal{D}$ is empty, then the inclusion map $\mathcal{C} \hookrightarrow \mathcal{C} \star \mathcal{D}$ is an isomorphism of categories.

0165 **Example 4.3.2.5** (Cones)**.** Let $[0]$ denote the category having a single object and a single morphism, and let $\mathcal{C}$ be an arbitrary category. We let $\mathcal{C}^{\triangleleft}$ denote the join $[0] \star \mathcal{C}$, and $\mathcal{C}^{\triangleright}$ the join $\mathcal{C} \star [0]$. We refer to $\mathcal{C}^{\triangleleft}$ as the *left cone of* $\mathcal{C}$, and to $\mathcal{C}^{\triangleright}$ as the *right cone on* $\mathcal{C}$.

More informally, we can describe the left cone $\mathcal{C}^{\triangleleft}$ as the category obtained from $\mathcal{C}$ by adjoining a new object X_0 satisfying

$$\operatorname{Hom}_{\mathcal{C}^{\triangleleft}}(X_0, Y) = * \qquad \operatorname{Hom}_{\mathcal{C}^{\triangleleft}}(X_0, X_0) = * \qquad \operatorname{Hom}_{\mathcal{C}^{\triangleleft}}(Y, X_0) = \emptyset$$

for $Y \in \mathcal{C}$. Note that X_0 is an initial object of the category $\mathcal{C}^{\triangleleft}$, which we will refer to as the *cone point* of $\mathcal{C}^{\triangleleft}$. Similarly, the right cone $\mathcal{C}^{\triangleright}$ is obtained from $\mathcal{C}$ by adjoining a new object which we refer to as the *cone point* of $\mathcal{C}^{\triangleright}$ (and which is a final object of $\mathcal{C}^{\triangleright}$).

0166 **Remark 4.3.2.6.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories. Then there is a canonical isomorphism of iterated joins

$$\alpha : \mathcal{C} \star (\mathcal{D} \star \mathcal{E}) \simeq (\mathcal{C} \star \mathcal{D}) \star \mathcal{E},$$

characterized by the requirement that it restricts to the identity on $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ (which we can regard as full subcategories of both $\mathcal{C} \star (\mathcal{D} \star \mathcal{E})$ and $(\mathcal{C} \star \mathcal{D}) \star \mathcal{E}$, by means of Remark 4.3.2.2).

0167 **Remark 4.3.2.7.** Let Cat denote the category of (small) categories. Then Cat admits a monoidal structure, where the tensor product is given by the join functor

$$\star : \mathrm{Cat} \times \mathrm{Cat} \to \mathrm{Cat} \qquad (\mathcal{C}, \mathcal{D}) \mapsto \mathcal{C} \star \mathcal{D}$$

of Remark 4.3.2.3, and the associativity constraints are the isomorphisms of Remark 4.3.2.6. The unit for this monoidal structure is the empty category $\emptyset \in \mathrm{Cat}$ (Example 4.3.2.4).

0168 **Warning 4.3.2.8.** The join operation of Definition 4.3.2.1 is not commutative. For example, if $\mathcal{C}$ is a category, then the left cone $\mathcal{C}^{\triangleleft}$ need not be isomorphic (or even equivalent) to the right cone $\mathcal{C}^{\triangleright}$. However, we do have canonical isomorphisms

$$(\mathcal{C} \star \mathcal{D})^{\mathrm{op}} \simeq \mathcal{D}^{\mathrm{op}} \star \mathcal{C}^{\mathrm{op}},$$

depending functorially on $\mathcal{C}$ and $\mathcal{D}$.

We now relate the join construction of Definition 4.3.2.1 with the slice categories of Construction 4.3.1.8. We begin with a simple observation.

0169 **Lemma 4.3.2.9.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{C} \star \mathcal{D}$ and $\iota_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{C} \star \mathcal{D}$ denote the inclusion maps. Then:*

(1) *The inclusion functor $\iota_{\mathcal{C}}$ factors uniquely as a composition*

$$\mathcal{C} \xrightarrow{\overline{\iota}_{\mathcal{C}}} (\mathcal{C} \star \mathcal{D})_{/\iota_{\mathcal{D}}} \to \mathcal{C} \star \mathcal{D}.$$

(2) *The inclusion functor $\iota_{\mathcal{D}}$ factors uniquely as a composition*

$$\mathcal{D} \xrightarrow{\overline{\iota}_{\mathcal{D}}} (\mathcal{C} \star \mathcal{D})_{\iota_{\mathcal{C}}/} \to \mathcal{C} \star \mathcal{D}.$$

Proof. Let $\pi_{\mathcal{C}} : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_{\mathcal{D}} : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ denote the projection maps. Using Remark 4.3.1.11, we see that both (1) and (2) are equivalent to the assertion that there is a unique natural transformation u from $\iota_{\mathcal{C}} \circ \pi_{\mathcal{C}}$ to $\iota_{\mathcal{D}} \circ \pi_{\mathcal{D}}$ (as functors from the product category $\mathcal{C} \times \mathcal{D}$ to the join category $\mathcal{C} \star \mathcal{D}$). Concretely, this natural transformation carries each object $(C, D) \in \mathcal{C} \times \mathcal{D}$ to the unique element of $\mathrm{Hom}_{\mathcal{C} \star \mathcal{D}}(C, D)$. □

Proposition 4.3.2.10. *Let $\mathcal{C}$ be a category and let $G : \mathcal{D} \to \mathcal{E}$ be a functor between categories. For every functor $U : \mathcal{C} \star \mathcal{D} \to \mathcal{E}$ extending G, let $\overline{F}(U)$ denote the composite functor* 016A

$$\mathcal{C} \xrightarrow{\overline{\iota}_{\mathcal{C}}} (\mathcal{C} \star \mathcal{D})_{/\iota_{\mathcal{D}}} \xrightarrow{U} \mathcal{E}_{/(U \circ \iota_{\mathcal{D}})} = \mathcal{E}_{/G}.$$

Then the construction $U \mapsto \overline{F}(U)$ induces a bijection

$$\{\text{Functors } U : \mathcal{C} \star \mathcal{D} \to \mathcal{E} \text{ satisfying } U|_{\mathcal{D}} = G\} \to \{\text{Functors } \overline{F} : \mathcal{C} \to \mathcal{E}_{/G}\}.$$

Example 4.3.2.11. Let $G : \mathcal{D} \to \mathcal{E}$ be a functor of categories. Applying Proposition 4.3.2.10 in the case $\mathcal{C} = [0]$, we see that objects of the slice category $\mathcal{E}_{/G}$ can be identified with functors $U : \mathcal{D}^{\triangleleft} \to \mathcal{E}$ satisfying $U|_{\mathcal{D}} = G$. 016B

Example 4.3.2.12. Let $\mathcal{C}$ and $\mathcal{E}$ be categories and let S be an object of $\mathcal{E}$. Applying Proposition 4.3.2.10 in the case $\mathcal{D} = [0]$, we see that functors from $\mathcal{C}$ to the slice category $\mathcal{E}_{/S}$ can be identified with functors $U : \mathcal{C}^{\triangleright} \to \mathcal{E}$ which carry the cone point of $\mathcal{C}^{\triangleright}$ to the object S. 016C

In the situation of Proposition 4.3.2.10, we can use Remark 4.3.1.11 to identify functors $\overline{F} : \mathcal{C} \to \mathcal{E}_{/G}$ with ordered pairs (F, v), where $F : \mathcal{C} \to \mathcal{E}$ is a functor (given by the composition of $\overline{F}$ with the forgetful functor $\mathcal{E}_{/G} \to \mathcal{E}$) and v is a natural transformation from $F \circ \pi_{\mathcal{C}}$ to $G \circ \pi_{\mathcal{D}}$ (regarded as functors from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$). Note that, in the case where $\overline{F} = \overline{F}(U)$ is obtained from a functor $U : \mathcal{C} \star \mathcal{D} \to \mathcal{E}$, we have $F = U|_{\mathcal{C}}$. We can therefore reformulate Proposition 4.3.2.10 in a more symmetric fashion:

Proposition 4.3.2.13. *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories, and suppose we are given functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$. Let $u : \iota_{\mathcal{C}} \circ \pi_{\mathcal{C}} \to \iota_{\mathcal{D}} \circ \pi_{\mathcal{D}}$ be the natural transformation appearing* 016D *in the proof of Lemma 4.3.2.9. Then evaluation on u induces a bijection*

$$\begin{tikzcd}
\{\text{Functors } U : \mathcal{C} \star \mathcal{D} \to \mathcal{E} \text{ with } U|_{\mathcal{C}} = F \text{ and } U|_{\mathcal{D}} = G\} . \arrow[d] \\
\{\text{Natural transformations from } F \circ \pi_{\mathcal{C}} \text{ to } G \circ \pi_{\mathcal{D}} \}
\end{tikzcd}$$

Proof. Let v be a natural transformation from $F \circ \pi_{\mathcal{C}}$ to $G \circ \pi_{\mathcal{D}}$, carrying each object $(C, D) \in \mathcal{C} \times \mathcal{D}$ to a morphism $v_{C,D} : F(C) \to G(D)$ in the category $\mathcal{E}$. We wish to show that there is a unique functor $U : \mathcal{C} \star \mathcal{D} \to \mathcal{E}$ satisfying $U|_{\mathcal{C}} = F$, $H|_{\mathcal{D}} = G$, and $U(u_{C,D}) = v_{C,D}$ for $(C, D) \in \mathcal{C} \times \mathcal{D}$. These requirements uniquely determine the value of U on all objects and morphisms of the category $\mathcal{C} \star \mathcal{D}$. To complete the proof, it will suffice to show that U is compatible with composition: that is, for every pair of morphisms $s : X \to Y$ and $t : Y \to Z$ in $\mathcal{C} \star \mathcal{D}$, we have $U(t \circ s) = U(t) \circ U(s)$. We consider four cases:

- If X, Y, and Z belong to $\mathcal{C}$, then we have $U(t \circ s) = F(t \circ s) = F(t) \circ F(s) = U(t) \circ U(s)$.
- If X and Y belong to $\mathcal{C}$ and Z belongs to $\mathcal{D}$, then we have $U(t \circ s) = v_{X,Z} = v_{Y,Z} \circ F(s) = U(t) \circ U(s)$, where the second equality follows from the naturality of v in the first variable.
- If Y and Z belong to $\mathcal{D}$ and X belongs to $\mathcal{C}$, then we have $U(t \circ s) = v_{X,Z} = G(t) \circ v_{X,Y} = U(t) \circ U(s)$, where the second equality follows from the naturality of v in the second variable.
- If X, Y, and Z belong to $\mathcal{D}$, then we have $U(t \circ s) = G(t \circ s) = G(t) \circ G(s) = U(t) \circ U(s)$.

□

016E **Remark 4.3.2.14.** Stated more informally, Proposition 4.3.2.13 asserts that the join $\mathcal{C} \star \mathcal{D}$ is universal among categories $\mathcal{E}$ which are equipped with a pair of functors $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$ and a natural transformation $v : (F \circ \pi_{\mathcal{C}}) \to (G \circ \pi_{\mathcal{D}})$. More precisely, there is a pushout square

$$\begin{tikzcd}
(\mathcal{C} \times \{0\} \times \mathcal{D}) \coprod (\mathcal{C} \times \{1\} \times \mathcal{D}) \arrow[r] \arrow[d] & \mathcal{C} \times [1] \times \mathcal{D} \arrow[d] \\
(\mathcal{C} \times \{0\}) \coprod (\{1\} \times \mathcal{D}) \arrow[r] & \mathcal{C} \star \mathcal{D}
\end{tikzcd}$$

in the (ordinary) category Cat, where the right vertical map encodes the natural transformation $u : \iota_{\mathcal{C}} \circ \pi_{\mathcal{C}} \to \iota_{\mathcal{D}} \circ \pi_{\mathcal{D}}$ appearing in the proof of Lemma 4.3.2.9.

Example 4.3.2.15 (The Universal Property of a Cone)**.** Let $\mathcal{C}$ be a category. Applying Remark 4.3.2.14 in the special case $\mathcal{D} = [0]$, we obtain a pushout diagram of categories 016F

$$\begin{tikzcd}
\mathcal{C} \times \{1\} \arrow[r] \arrow[d] & \mathcal{C} \times [1] \arrow[d] \\
{[0]} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

where the bottom horizontal map carries the unique object of $[0]$ to the cone point of $\mathcal{C}^{\triangleright}$. This is essentially a reformulation of Examples 4.3.2.11 and 4.3.2.12. Stated more informally, the right cone $\mathcal{C}^{\triangleright}$ is obtained from the product $[1] \times \mathcal{C}$ by "collapsing" the full subcategory $\{1\} \times \mathcal{C}$ to the cone point. Similarly, the left cone of a category $\mathcal{D}$ is characterized by the existence of a pushout diagram

$$\begin{tikzcd}
\{0\} \times \mathcal{D} \arrow[r] \arrow[d] & {[1]} \times \mathcal{D} \arrow[d] \\
{[0]} \arrow[r] & \mathcal{D}^{\triangleleft}.
\end{tikzcd}$$

For completeness, we record the dual of Proposition 4.3.2.10, which supplies a universal property of coslice categories (and is also a reformulation of Proposition 4.3.2.13).

Corollary 4.3.2.16. *Let $\mathcal{D}$ be a category and let $F : \mathcal{C} \to \mathcal{E}$ be a functor between categories. For every functor $U : \mathcal{C} \star \mathcal{D} \to \mathcal{E}$ extending F, let $\overline{G}(U)$ denote the composite functor* 016G

$$\mathcal{D} \xrightarrow{\overline{\iota}_{\mathcal{D}}} (\mathcal{C} \star \mathcal{D})_{\iota_{\mathcal{C}}/} \xrightarrow{U} \mathcal{E}_{(U \circ \iota_{\mathcal{C}})/} = \mathcal{E}_{F/}.$$

Then the construction $U \mapsto \overline{G}(U)$ induces a bijection

$$\{\text{Functors } U : \mathcal{C} \star \mathcal{D} \to \mathcal{E} \text{ satisfying } U|_{\mathcal{C}} = F\} \to \{\text{Functors } \overline{G} : \mathcal{D} \to \mathcal{E}_{F/}\}.$$

Corollary 4.3.2.17. 016H

- *For any category $\mathcal{D}$, the join functor*

$$\mathrm{Cat} \to \mathrm{Cat}_{\mathcal{D}/} \qquad \mathcal{C} \mapsto \mathcal{C} \star \mathcal{D}$$

  admits a right adjoint, given on objects by the slice construction $(G : \mathcal{D} \to \mathcal{E}) \mapsto \mathcal{E}_{/G}$.

- *For any category $\mathcal{C}$, the join functor*

$$\mathrm{Cat} \to \mathrm{Cat}_{\mathcal{C}/} \qquad \mathcal{D} \mapsto \mathcal{C} \star \mathcal{D}$$

admits a right adjoint, given on objects by the coslice construction $(F : \mathcal{C} \to \mathcal{E}) \mapsto \mathcal{E}_{F/}$.

016J **Remark 4.3.2.18.** Let $G : \mathcal{D} \to \mathcal{E}$ be a functor between categories. According to Remark 4.3.1.11, the slice category $\mathcal{E}_{/G}$ can be identified with the iterated fiber product

$$(\mathrm{Fun}([0], \mathcal{E}) \times_{\mathrm{Fun}(\{0\} \times \mathcal{D}, \mathcal{E})} \mathrm{Fun}([1] \times \mathcal{D}, \mathcal{E})) \times_{\mathrm{Fun}(\{1\} \times \mathcal{D}, \mathcal{E})} \{G\}.$$

Using Example 4.3.2.15, we can identify the left factor with the functor category $\mathrm{Fun}(\mathcal{D}^{\triangleleft}, \mathcal{E})$. We therefore obtain a pullback diagram of categories

$$\begin{tikzcd}
\mathcal{E}_{/G} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{D}^{\triangleleft}, \mathcal{E}) \arrow[d] \\
\{G\} \arrow[r] & \mathrm{Fun}(\mathcal{D}, \mathcal{E}),
\end{tikzcd}$$

which recovers Example 4.3.2.11 at the level of objects.

Similarly, if $F : \mathcal{C} \to \mathcal{E}$ is a functor of categories, then the coslice category $\mathcal{E}_{F/}$ fits into a pullback square

$$\begin{tikzcd}
\mathcal{E}_{F/} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{E}) \arrow[d] \\
\{F\} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{E}).
\end{tikzcd}$$

4.3.3 Joins of Simplicial Sets

016K Our next goal is to extend the join operation of Definition 4.3.2.1 to the setting of ∞-categories (and more general simplicial sets). We begin with a slightly more general discussion. Let Lin denote the category whose objects are finite linearly ordered sets and whose morphisms are nondecreasing functions. The functor category $\mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ is equivalent to the category of *augmented* simplicial sets (see §[?]), and contains a full subcategory which is equivalent to the category of simplicial sets (see Proposition 4.3.3.11 below).

016L **Notation 4.3.3.1.** Let J be a linearly ordered set. We say that a subset $I \subseteq J$ is an *initial segment* of J if it is closed downwards: that is, if, for every pair of elements $i \leq j$ in J, we have $(j \in I) \Rightarrow (i \in I)$. We will write $I \sqsubseteq J$ to indicate that I is an initial segment of J.

Construction 4.3.3.2 (Joins of Augmented Simplicial Sets)**.** For every pair of functors $X, Y : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$, we let $(X \star Y) : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$ denote a new functor given on objects by the formula 016M

$$(X \star Y)(J) = \coprod_{I \sqsubseteq J} (X(I) \times Y(J \setminus I)).$$

Here the coproduct is indexed by the collection of all initial segments $I \sqsubseteq J$.

More formally, the functor $(X \star Y) : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$ can be described as follows:

- For every finite linearly ordered set J, $(X \star Y)(J)$ is the collection of all triples (I, x, y), where I is an initial segment of J, x is an element of $X(I)$, and y is an element of $Y(J \setminus I)$.
- If $\alpha : J' \to J$ is a nondecreasing function, then the induced map $(X \star Y)(\alpha) : (X \star Y)(J) \to (X \star Y)(J')$ is given by the construction

$$(I, x, y) \mapsto (\alpha^{-1}(I), X(\alpha|_{\alpha^{-1}(I)})(x), Y(\alpha|_{\alpha^{-1}(J \setminus I)})(y)).$$

We will refer to $X \star Y$ as the *join of* X *and* Y.

Example 4.3.3.3. Let $E : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$ denote the functor given by 016N

$$E(I) = \begin{cases} * & \text{if } I = \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

For every functor $X : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$, we have canonical bijections

$$(X \star E)(J) = \coprod_{I \sqsubseteq J} (X(I) \times E(J \setminus I)) \simeq X(J) \times E(\emptyset) \simeq X(J)$$

$$(E \star X)(J) = \coprod_{I \sqsubseteq J} (E(I) \times X(J \setminus I)) \simeq E(\emptyset) \times X(J) \simeq X(J).$$

These bijections depend functorially on J, and therefore determine isomorphisms of functors

$$X \star E \simeq X \simeq E \star X.$$

Remark 4.3.3.4 (Functoriality)**.** Construction 4.3.3.2 determines a functor 016P

$$\star : \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \times \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \qquad (X, Y) \mapsto X \star Y.$$

Note that this functor preserves colimits separately in each variable.

016Q **Remark 4.3.3.5** (Associativity)**.** Let X, Y, and Z be functors from $\mathrm{Lin}^{\mathrm{op}}$ to the category of sets. For every finite linearly ordered set K, we have a canonical bijection

$$\begin{aligned}
(X \star (Y \star Z))(K) &= \coprod_{I \sqsubseteq K} (X(I) \times (Y \star Z)(K \setminus I)) \\
&= \coprod_{I \sqsubseteq K} (X(I) \times \coprod_{J \sqsubseteq K \setminus I} (Y(J) \times Z(K \setminus (I \cup J)))) \\
&\simeq \coprod_{I \sqsubseteq K} \coprod_{J \sqsubseteq K \setminus I} (X(I) \times Y(J) \times Z(K \setminus (I \cup J))) \\
&\simeq \coprod_{J' \sqsubseteq K} \coprod_{I \sqsubseteq J'} (X(I) \times Y(J' \setminus I) \times Z(K \setminus J')) \\
&\simeq \coprod_{J' \sqsubseteq K} (\coprod_{I \sqsubseteq J'} (X(I) \times Y(J' \setminus I) \times Z(K \setminus J')) \\
&= \coprod_{J \sqsubseteq K} ((X \star Y)(J') \star Z(K \setminus J')) \\
&= ((X \star Y) \star Z)(K).
\end{aligned}$$

These bijections depend functorially on $K \in \mathrm{Lin}^{\mathrm{op}}$, and therefore supply an isomorphism of functors $\alpha_{X,Y,Z} : X \star (Y \star Z) \simeq (X \star Y) \star Z$.

016R **Remark 4.3.3.6.** The join operation of Construction 4.3.3.2 determines a functor

$$\star : \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \times \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}).$$

This functor determines a monoidal structure on the category $\mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$, whose associativity constraints are the isomorphisms $\alpha_{X,Y,Z}$ of Remark 4.3.3.5 and whose unit object is the functor E of Example 4.3.3.3.

016S **Example 4.3.3.7.** For every category $\mathcal{C}$, let $h_{\mathcal{C}} : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$ denote the functor represented by $\mathcal{C}$, given by the formula

$$h_{\mathcal{C}}(J) = \{\text{Functors from } J \text{ to } \mathcal{C}\}.$$

For every pair of categories $\mathcal{C}$ and $\mathcal{D}$ and every finite linearly ordered set J, we have a canonical bijection

$$\begin{aligned}
(h_{\mathcal{C}} \star h_{\mathcal{D}})(J) &= \coprod_{I \sqsubseteq J} h_{\mathcal{C}}(I) \times h_{\mathcal{D}}(J \setminus I) \\
&= \coprod_{I \sqsubseteq J} \{\text{Functors } I \to \mathcal{C}\} \times \{\text{Functors } (J \setminus I) \to \mathcal{D}\} \\
&\simeq \{\text{Functors } J \to \mathcal{C} \star \mathcal{D}\} \\
&= h_{\mathcal{C} \star \mathcal{D}}(J).
\end{aligned}$$

These bijections depend functorially on J, and therefore determine an isomorphism $h_{\mathcal{C}} \star h_{\mathcal{D}} \simeq h_{\mathcal{C} \star \mathcal{D}}$ in the category $\mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$; here $\mathcal{C} \star \mathcal{D}$ denotes the join of the categories $\mathcal{C}$ and $\mathcal{D}$, in the sense of Definition 4.3.2.1.

Remark 4.3.3.8. Let $\mathcal{C}$ be a small monoidal category. Then the presheaf category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ inherits a monoidal structure given by *Day convolution* (see §[?]), which is characterized up to equivalence by the following properties: `016T`

(1) The Yoneda embedding

$$h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \qquad C \mapsto \mathrm{Hom}_{\mathcal{C}}(\bullet, C)$$

can be promoted to a monoidal functor.

(2) The tensor product on $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ preserves small colimits separately in each variable.

Let us specialize to the case where $\mathcal{C} = \mathrm{Lin}$ is the category of finite linearly ordered sets. Note that Lin can be identified with a full subcategory of Cat which is closed under the formation of joins (and contains the unit object $\emptyset \in \mathrm{Cat}$), and therefore inherits the structure of a monoidal category (where the tensor product is given by joins). With respect to this monoidal structure, the Yoneda embedding $h : \mathrm{Lin} \to \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ satisfies condition (1) (Example 4.3.3.7), and the join functor on $\mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ satisfies (2) by virtue of Remark 4.3.3.4. It follows that the join operation on $\mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ is given by Day convolution (with respect to the join operation on the category Lin).

We now adapt Construction 4.3.3.2 to the setting of simplicial sets.

Notation 4.3.3.9. Let $\mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors $X : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$ for which the set $X(\emptyset)$ is a singleton (that is, the full subcategory spanned by those functors which preserve final objects). `016U`

Remark 4.3.3.10. For every pair of functors $X, Y : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$, we have a canonical bijection $(X \star Y)(\emptyset) = X(\emptyset) \times Y(\emptyset)$. In particular, if X and Y belong to the subcategory $\mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \subseteq \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$, then the join $X \star Y$ also belongs to $\mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$. Moreover, $\mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ contains the unit object E of Example 4.3.3.3. It follows that $\mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ inherits the structure of a monoidal category (with respect to the join operation of Construction 4.3.3.2). `016V`

Recall that the simplex category $\mathbf{\Delta}$ of Definition 1.1.0.2 is the full subcategory of Lin spanned by objects of the form $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$.

Proposition 4.3.3.11. *The restriction functor* `016W`

$$\mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Set}_{\Delta} \qquad X \mapsto X|_{\mathbf{\Delta}^{\mathrm{op}}}$$

is an equivalence of categories.

Proof. Let S be a one-element set, and let $\mathrm{Fun}'_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ spanned by those functors $X : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$ satisfying $X(\emptyset) = S$. Since the inclusion functor $\mathrm{Fun}'_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \hookrightarrow \mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ is an equivalence of categories, it will suffice to show that the restriction functor

$$\mathrm{Fun}'_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Set}_{\Delta} \qquad X \mapsto X|_{\boldsymbol{\Delta}^{\mathrm{op}}}$$

is an equivalence of categories. Let $\mathrm{Lin}_{\neq\emptyset}$ denote the full subcategory of Lin spanned by the nonempty finite linearly ordered sets, so that the category Lin can be identified with the left cone $\mathrm{Lin}^{\triangleleft}_{\neq\emptyset}$ of Example 4.3.2.5. Using Proposition 4.3.2.13 (and the fact that the forgetful functor $\mathrm{Set}_{/*} \to \mathrm{Set}$ is an isomorphism), we deduce that the restriction functor $\mathcal{C} \to \mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}_{\neq\emptyset}, \mathrm{Set})$ is an isomorphism of categories. We are therefore reduced to showing that the restriction functor $\mathrm{Fun}(\mathrm{Lin}^{\mathrm{op}}_{\neq\emptyset}, \mathrm{Set}) \to \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathrm{Set}) = \mathrm{Set}_{\Delta}$ is an equivalence of categories. This is clear, since the inclusion $\boldsymbol{\Delta} \hookrightarrow \mathrm{Lin}_{\neq\emptyset}$ is an equivalence (Remark 1.1.0.3). □

016X **Remark 4.3.3.12.** The inclusion functor $\boldsymbol{\Delta} \hookrightarrow \mathrm{Lin}_{\neq\emptyset}$ has a *unique* left inverse $R : \mathrm{Lin}_{\neq\emptyset} \to \boldsymbol{\Delta}$, given on objects by the formula $R(I) = [n]$ when I has cardinality $n+1$. It follows that the equivalence $\mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Set}_{\Delta}$ of Proposition 4.3.3.11 admits an explicit right inverse, which carries a simplicial set $X : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ to the functor $X^+ : \mathrm{Lin}^{\mathrm{op}} \to \mathrm{Set}$ given by the formula

$$X^+(I) = \begin{cases} X(R(I)) & \text{if } I \text{ is nonempty} \\ * & \text{otherwise.} \end{cases}$$

016Y **Construction 4.3.3.13** (Joins of Simplicial Sets)**.** Let X and Y be simplicial sets. We let $X \star Y$ denote the simplicial set given by the restriction $(X^+ \star Y^+)|_{\boldsymbol{\Delta}^{\mathrm{op}}}$. Here $X^+, Y^+ \in \mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set})$ are given by Remark 4.3.3.12, and $X^+ \star Y^+$ denotes the join of Construction 4.3.3.2. We will refer to $X \star Y$ as the *join of X and Y*. The construction $X, Y \mapsto X \star Y$ determines a functor $\star : \mathrm{Set}_{\Delta} \times \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$, which we will refer to as the *join functor.* It is characterized (up to isomorphism) by the fact that the diagram

$$\begin{tikzcd}
\mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \times \mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \arrow[r, "\star"] \arrow[d] & \mathrm{Fun}_*(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \arrow[d] \\
\mathrm{Set}_{\Delta} \times \mathrm{Set}_{\Delta} \arrow[r, "\star"] & \mathrm{Set}_{\Delta}
\end{tikzcd}$$

commutes up to isomorphism, where the vertical maps are the equivalences supplied by Proposition 4.3.3.11.

Example 4.3.3.14. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Using Example 4.3.3.7, we obtain a canonical isomorphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C}) \star \mathrm{N}_\bullet(\mathcal{D}) \simeq \mathrm{N}_\bullet(\mathcal{C} \star \mathcal{D})$, where $\mathcal{C} \star \mathcal{D}$ denotes the join of the categories $\mathcal{C}$ and $\mathcal{D}$. 0175

In particular, for integers $p, q \geq 0$, there is a unique isomorphism of simplicial sets

$$\Delta^p \star \Delta^q \simeq \Delta^{p+1+q},$$

which is given on vertices of Δ^p by the construction $i \mapsto i$ and on vertices of Δ^q by $j \mapsto p + 1 + j$.

Example 4.3.3.15. For every simplicial set X, we have canonical isomorphisms $X \star \emptyset \simeq X \simeq \emptyset \star X$ (compare with Example 4.3.3.3). 0174

Remark 4.3.3.16. For every pair of simplicial sets X and Y, we have canonical monomorphisms 016Z

$$X \simeq X \star \emptyset \hookrightarrow X \star Y \hookleftarrow \emptyset \star Y \simeq Y.$$

We will often abuse notation by identifying X and Y with the simplicial subsets of $X \star Y$ given by the images of these monomorphisms.

Remark 4.3.3.17. Let X and Y be simplicial sets. For each n-simplex $\sigma : \Delta^n \to X \star Y$, exactly one of the following conditions holds: 0170

- The morphism σ factors through X (where we identify X with a simplicial subset of $X \star Y$ as in Remark 4.3.3.16).
- The morphism σ factors through Y (where we identify Y with a simplicial subset of $X \star Y$ as in Remark 4.3.3.16).
- The morphism σ factors as a composition

  $$\Delta^n = \Delta^{p+1+q} \simeq \Delta^p \star \Delta^q \xrightarrow{\sigma_- \star \sigma_+} X \star Y,$$

  for integers $p, q \geq 0$ satisfying $p + 1 + q = n$ and simplices $\sigma_- : \Delta^p \to X$ and $\sigma_+ : \Delta^q \to Y$ of X and Y, respectively. Moreover, in this case, the simplices σ_- and σ_+ (and the integers $p, q \geq 0$) are uniquely determined.

Remark 4.3.3.18. Let $i : X \hookrightarrow X'$ and $j : Y \hookrightarrow Y'$ be monomorphisms of simplicial sets. From the description of Remark 4.3.3.17, we see that the join $(i \star j) : X \star Y \to X' \star Y'$ is also a monomorphism of simplicial sets. 0171

Remark 4.3.3.19. Let $X_\bullet$ and $Y_\bullet$ be simplicial sets. By virtue of Remark 4.3.3.17, the join $(X \star Y)_\bullet$ can be described explicitly by the formula 0172

$$(X \star Y)_n = X_n \amalg (\coprod_{p+1+q=n} X_p \times Y_q) \amalg Y_n.$$

In these terms, the face and degeneracy operators $\{d_i^n : (X \star Y)_n \to (X \star Y)_{n-1}\}_{0 \leq i \leq n}$ and $\{s_i^n : (X \star Y)_n \to (X \star Y)_{n+1}\}$ are given on the first and third summand by the analogous operators for $X_\bullet$ and $Y_\bullet$, and on elements $(\sigma, \tau) \in X_p \times Y_q$ by the formula

$$d_i^n(\sigma,\tau) = \begin{cases} (d_i^p(\sigma), \tau) & \text{if } i \leq p \\ (\sigma, d_{i-1-p}^q(\tau)) & \text{if } i > p. \end{cases} \qquad s_i^n(\sigma,\tau) = \begin{cases} (s_i^p(\sigma), \tau) & \text{if } i \leq p \\ (\sigma, s_{i-1-p}^q(\tau)) & \text{if } i > p. \end{cases}$$

056W **Remark 4.3.3.20.** Let X and Y be simplicial sets, and let $\sigma : \Delta^m \to X \star Y$ be an m-simplex which factors as a composition

$$\Delta^m \simeq \Delta^{m_-} \star \Delta^{m_+} \xrightarrow{\sigma_- \star \sigma_+} X \star Y$$

for some integers $m_-, m_+ \geq 0$ satisfying $m_- + 1 + m_+ = m$. Then σ is nondegenerate if and only if both σ_- and σ_+ are nondegenerate. It follows that, for every integer n, the n-skeleton of $X \star Y$ is given by the union

$$\mathrm{sk}_n(X) \cup \Big(\bigcup_{p+1+q=n} \mathrm{sk}_p(X) \star \mathrm{sk}_q(Y) \Big) \cup \mathrm{sk}_n(Y).$$

In particular, we have an equality

$$\dim(X \star Y) = \dim(X) + 1 + \dim(Y),$$

provided that we adopt the convention that an empty simplicial set has dimension -1.

02LH **Remark 4.3.3.21.** Let X and Y be finite simplicial sets. Then the join $X \star Y$ is also finite.

0173 **Remark 4.3.3.22.** For every pair of simplicial sets X and Y, we have a canonical isomorphism $(X \star Y)^{\mathrm{op}} \simeq Y^{\mathrm{op}} \star X^{\mathrm{op}}$.

0234 **Remark 4.3.3.23.** Let X, Y, and K be simplicial sets. Unwinding the definitions, we see that morphisms from K to $X \star Y$ can be identified with triples (π, f_-, f_+), where

$$\pi : K \to \Delta^1 \qquad f_- : \{0\} \times_{\Delta^1} K \to X \qquad f_+ : \{1\} \times_{\Delta^1} K \to Y$$

are morphisms of simplicial sets (note that, when K is a simplex, this recovers the description of Remark 4.3.3.19).

02QU **Proposition 4.3.3.24.** *Let $u : X \to X'$ and $v : Y \to Y'$ be inner fibrations of simplicial sets. Then the join $(u \star v) : X \star Y \to X' \star Y'$ is also an inner fibration of simplicial sets.*

02QV **Corollary 4.3.3.25.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Then the join $\mathcal{C} \star \mathcal{D}$ is an ∞-category.*

Proof. Since $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, the projection maps $u : \mathcal{C} \to \Delta^0$ and $v : \mathcal{D} \to \Delta^0$ are inner fibrations (Example 4.1.1.2). Applying Proposition 4.3.3.24, we deduce that the join

$$(u \star v) : \mathcal{C} \star \mathcal{D} \to \Delta^0 \star \Delta^0 \simeq \Delta^1$$

is also an inner fibration. Since Δ^1 is an ∞-category, it follows that $\mathcal{C} \star \mathcal{D}$ is an ∞-category (Remark 4.1.1.9). □

Proof of Proposition 4.3.3.24. Let $u : X \to X'$ and $v : Y \to Y'$ be inner fibrations of simplicial sets and let $0 < i < n$ be integers; we wish to show that every lifting problem

02QW

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & X \star Y \arrow[d, "u \star v"] \\
\Delta^n \arrow[r, "\sigma'"] \arrow[ur, dashed, "\sigma"] & X' \star Y'
\end{tikzcd} \tag{4.2}$$

admits a solution. If σ' factors through either X' or Y', this follows immediately from our assumption that u and v are inner fibrations. We may therefore assume without loss of generality that σ' factors as a composition

$$\Delta^n = \Delta^{p+1+q} \simeq \Delta^p \star \Delta^q \xrightarrow{\sigma'_- \star \sigma'_+} X' \star Y'$$

for some pair of integers $p, q \geq 0$ satisfying $p + 1 + q = n$ and simplices $\sigma'_- : \Delta^p \to X'$ and $\sigma'_+ : \Delta^q \to Y'$. Let ι_- denote the inclusion map

$$\Delta^p \hookrightarrow \Delta^p \star \Delta^q \simeq \Delta^{p+1+q} = \Delta^n,$$

and define $\iota_+ : \Delta^q \hookrightarrow \Delta^n$ similarly. Note that both ι_- and ι_+ factor through the inner horn $\Lambda^n_i \subseteq \Delta^n$. Set $\sigma_- = \sigma_0 \circ \iota_-$ and $\sigma_+ = \sigma_0 \circ \iota_+$. Unwinding the definitions, we see that the composite map

$$\Delta^n = \Delta^{p+1+q} \simeq \Delta^p \star \Delta^q \xrightarrow{\sigma_- \star \sigma_+} X \star Y$$

determines an n-simplex σ of $X \star Y$ which is a solution to the lifting problem (4.2).

□

Construction 4.3.3.26. Let X be a simplicial set. We will denote the join $\Delta^0 \star X$ by $X^{\triangleleft}$ and refer to it as the *left cone of* X. Similarly, we denote the join $X \star \Delta^0$ by $X^{\triangleright}$ and refer to it as the *right cone of* X. We will often abuse notation by using Remark 4.3.3.16 to identify X with its image in the cones $X^{\triangleleft}$ and $X^{\triangleright}$. Moreover, Remark 4.3.3.16 also supplies morphisms of simplicial sets $X^{\triangleleft} \hookleftarrow \Delta^0 \hookrightarrow X^{\triangleright}$, which we can identify with vertices which we refer to as the *cone points* of $X^{\triangleleft}$ and $X^{\triangleright}$, respectively. 0177

0178 **Example 4.3.3.27.** Let $\mathcal{C}$ be a category. Then Example 4.3.3.14 supplies canonical isomorphisms

$$\mathrm{N}_{\bullet}(\mathcal{C})^{\triangleleft} \simeq \mathrm{N}_{\bullet}(\mathcal{C}^{\triangleleft}) \qquad \mathrm{N}_{\bullet}(\mathcal{C})^{\triangleright} \simeq \mathrm{N}_{\bullet}(\mathcal{C}^{\triangleright}),$$

where $\mathcal{C}^{\triangleleft}$ and $\mathcal{C}^{\triangleright}$ denote the left and right cones of $\mathcal{C}$ (see Example 4.3.2.5).

0179 **Example 4.3.3.28.** Let $n \geq 0$, and let Δ^n denote the standard n-simplex. Using Example 4.3.3.27, we see that there is a unique isomorphism of simplicial sets $(\Delta^n)^{\triangleright} \simeq \Delta^{n+1}$, which carries each vertex $i \in \{0, 1, \ldots, n\}$ to itself and the cone point of $(\Delta^n)^{\triangleright}$ to the final vertex $n+1$. This isomorphism carries the simplicial subset $(\partial \Delta^n)^{\triangleright} \subseteq (\Delta^n)^{\triangleright}$ to the horn $\Lambda^{n+1}_{n+1} \subseteq \Delta^{n+1}$. Similarly, the left cone $(\partial \Delta^n)^{\triangleleft}$ is isomorphic to the horn Λ^{n+1}_0.

017A **Remark 4.3.3.29.** For every simplicial set X, Remark 4.3.3.22 supplies a canonical isomorphism $(X^{\triangleleft})^{\mathrm{op}} \simeq (X^{\mathrm{op}})^{\triangleright}$, carrying the cone point of $X^{\triangleleft}$ to the cone point of $(X^{\mathrm{op}})^{\triangleright}$.

056X **Remark 4.3.3.30.** Let X be a simplicial set. Then, for every nonnegative integer n, the n-skeleton of the cone $X^{\triangleright}$ fits into a pushout diagram

$$\begin{tikzcd}
\mathrm{sk}_{n-1}(X) \arrow[r] \arrow[d] & \mathrm{sk}_{n-1}(X)^{\triangleright} \arrow[d] \\
\mathrm{sk}_{n}(X) \arrow[r] & \mathrm{sk}_{n}(X)^{\triangleright};
\end{tikzcd}$$

see Remark 4.3.3.20. In particular, $X^{\triangleright}$ has dimension $\leq n$ if and only if X has dimension $\leq n-1$.

017B **Remark 4.3.3.31.** Let X be a simplicial set. Then the construction $Y \mapsto X \star Y$ determines a functor

$$\mathrm{Set}_{\Delta} \to (\mathrm{Set}_{\Delta})_{X/}$$

which preserves small colimits. It follows that the composite functor

$$\mathrm{Set}_{\Delta} \to (\mathrm{Set}_{\Delta})_{X/} \to \mathrm{Set}_{\Delta} \qquad Y \mapsto X \star Y$$

preserves filtered colimits and pushouts. Beware that it does not preserve colimits in general (for example, it carries the initial object $\emptyset \in \mathrm{Set}_{\Delta}$ to the simplicial set X, which need not be initial).

017C **Remark 4.3.3.32** (Associativity)**.** Let X, Y, and Z be simplicial sets. Then Remark 4.3.3.5 supplies a canonical isomorphism of simplicial sets $\alpha_{X,Y,Z} : X \star (Y \star Z) \simeq (X \star Y) \star Z$. These isomorphisms are associativity constraints for a monoidal structure on the category of simplicial sets, which is characterized (up to isomorphism) by the requirement that the equivalence $\mathrm{Fun}_{*}(\mathrm{Lin}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Set}_{\Delta}$ of Proposition 4.3.3.11 can be promoted to a monoidal functor.

Warning 4.3.3.33. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Then the join $\mathcal{C} \star \mathcal{D}$ of Definition 4.3.2.1 is characterized (up to isomorphism) by the existence of a pushout diagram 017D

$$\begin{tikzcd}
(\{0\} \times \mathcal{C} \times \mathcal{D}) \coprod (\{1\} \times \mathcal{C} \times \mathcal{D}) \arrow[r] \arrow[d] & {[1]} \times \mathcal{C} \times \mathcal{D} \arrow[d] \\
(\{0\} \times \mathcal{C}) \coprod (\{1\} \times \mathcal{D}) \arrow[r] & \mathcal{C} \star \mathcal{D}
\end{tikzcd}$$

in the category Cat (see Remark 4.3.2.14). Beware that, in the setting of simplicial sets, the analogous statement is not quite true. To every pair of simplicial sets X and Y, one can associate a commutative diagram of simplicial sets

$$\begin{tikzcd}
(\{0\} \times X \times Y) \coprod (\{1\} \times X \times Y) \arrow[r] \arrow[d] & \Delta^1 \times X \times Y \arrow[d] \\
(\{0\} \times X) \coprod (\{1\} \times Y) \arrow[r] & X \star Y
\end{tikzcd}$$

(see Construction 4.5.8.1), which is almost never a pushout square. Nevertheless, the pushout can be regarded as a good approximation to the join $X \star Y$: see Proposition 4.5.8.2 and Theorem 4.5.8.8.

4.3.4 Joins of Topological Spaces

The join operation on simplicial sets admits a topological interpretation. 017E

Construction 4.3.4.1. Let X and Y be topological spaces, and let $[0,1] = |\Delta^1|$ denote the unit interval. We let $X \star Y$ denote the topological space given by the iterated pushout 017F

$$X \coprod_{(X \times \{0\} \times Y)} (X \times [0,1] \times Y) \coprod_{(X \times \{1\} \times Y)} Y.$$

We will refer to $X \star Y$ as the *join of X and Y*.

Remark 4.3.4.2. Let X and Y be topological spaces. Then the join $X \star Y$ of Construction 4.3.4.1 is equipped with a pair of maps $\iota_X : X \hookrightarrow X \star Y$ and $\iota_Y : Y \hookrightarrow X \star Y$. It is not difficult to see that these maps are closed embeddings: that is, they induce homeomorphisms from X and Y onto closed subsets of $X \star Y$. We will generally abuse notation by identifying X and Y with their images under ι_X and ι_Y, respectively. 017G

017H **Remark 4.3.4.3.** Let X, Y, and Z be topological spaces. Then the datum of a continuous function $X \star Y \to Z$ is equivalent to the datum of a triple (f_X, f_Y, h), where $f_X : X \to Z$ and $f_Y : Y \to Z$ are continuous functions and $h : X \times [0,1] \times Y \to Z$ is a homotopy from $f_X \circ \pi_X$ to $f_Y \circ \pi_Y$; here $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the projection maps.

017J **Remark 4.3.4.4** (Symmetry)**.** Let X and Y be topological spaces. Then there is a canonical homeomorphism $X \star Y \simeq Y \star X$, which is induced by the homeomorphism

$$X \times [0,1] \times Y \to Y \times [0,1] \times X \qquad (x,t,y) \mapsto (y, 1-t, x).$$

017K **Example 4.3.4.5** (Cones)**.** Let $*$ denote the topological space consisting of a single point. For any topological space X, we write $X^{\triangleleft}$ for the join $* \star X$, and $X^{\triangleright}$ for the join $X \star *$, given more concretely by the formulae

$$X^{\triangleleft} = * \coprod_{(\{0\} \times X)} ([0,1] \times X) \qquad X^{\triangleright} = (X \times [0,1]) \coprod_{(X \times \{1\})} *.$$

We will refer to both $X^{\triangleleft}$ and $X^{\triangleright}$ as the *cone on X* (note that they are canonically homeomorphic, by virtue of Remark 4.3.4.4).

017L **Remark 4.3.4.6.** Let X be a locally compact Hausdorff space. Then the functor

$$\mathrm{Top} \to \mathrm{Top}_{X/} \qquad Y \mapsto X \star Y$$

preserves colimits. This follows from the fact that the functors $Y \mapsto X \times Y$ and $Y \mapsto X \times [0,1] \times Y$ preserve colimits.

017M **Example 4.3.4.7.** For each integer $n \geq 0$, let $|\Delta^n| = \{(u_0, \ldots, u_n) \in \mathbf{R}_{\geq 0} : u_0 + \cdots + u_n = 1\}$ denote the topological n-simplex. For $p, q \geq 0$, we have maps $|\Delta^p| \xrightarrow{\iota} |\Delta^{p+1+q}| \xleftarrow{\iota'} |\Delta^q|$ given by the formulae

$$\iota(u_0, \ldots, u_p) = (u_0, \ldots, u_p, 0, \ldots, 0) \qquad \iota'(v_0, \ldots, v_q) = (0, \ldots, 0, v_0, \ldots, v_q).$$

There is a "straight-line" homotopy $h : |\Delta^p| \times [0,1] \times |\Delta^q| \to |\Delta^{p+1+q}|$ from $\iota \circ \pi_{|\Delta^p|}$ to $\iota' \circ \pi_{|\Delta^q|}$, given concretely by the formula

$$h((u_0, \ldots, u_p), t, (v_0, \ldots, v_q)) = ((1-t)u_0, (1-t)u_1, \ldots, (1-t)u_p, tv_0, \ldots, tv_q).$$

By virtue of Remark 4.3.4.3, the triple (ι, ι', h) can be identified with a continuous function $H_{p,q} : |\Delta^p| \star |\Delta^q| \to |\Delta^{p+1+q}|$.

017N **Proposition 4.3.4.8.** *Let p and q be nonnegative integers. Then the function $H_{p,q}$: $|\Delta^p| \star |\Delta^q| \to |\Delta^{p+1+q}|$ of Example 4.3.4.7 is a homeomorphism of topological spaces.*

Proof. Since $|\Delta^p| \star |\Delta^q|$ is compact and $|\Delta^{p+1+q}|$ is Hausdorff, the continuous function $H_{p,q}$ is automatically closed. To complete the proof, it will suffice to show that $H_{p,q}$ is bijective. Fix a point x of $|\Delta^{p+1+q}|$, given by a sequence of nonnegative real numbers $(\overline{u}_0, \ldots, \overline{u}_m, \overline{v}_0, \overline{v}_1, \ldots, \overline{v_n})$ satisfying

$$\overline{u}_0 + \cdots + \overline{u}_m + \overline{v}_0 + \cdots + \overline{v}_n = 0.$$

Set $t = \overline{v}_0 + \cdots + \overline{v_n}$. If $t = 0$, the set $H_{p,q}^{-1}\{x\}$ consists of a single point of $|\Delta^p|$ (regarded as a subset of $|\Delta^p| \star |\Delta^q|$), given by the sequence $(\overline{u}_0, \ldots, \overline{u}_m)$. If $t = 1$, the set $H_{p,q}^{-1}\{x\}$ consists of a single point of $|\Delta^q|$ (regarded as a subset of $|\Delta^p| \star |\Delta^q|$), given by the sequence $(\overline{v}_0, \ldots, \overline{v}_m)$. In the case $0 < t < 1$, the set $H_{p,q}^{-1}\{x\}$ consists of a single point of $|\Delta^p| \star |\Delta^q|$, given as the image of the triple

$$(\frac{\overline{u}_0}{1-t}, \ldots, \frac{\overline{u}_m}{1-t}), t, (\frac{\overline{v}_0}{t}, \ldots, \frac{\overline{v}_n}{t})) \in |\Delta^p| \times [0,1] \times |\Delta^n|.$$

□

We now compare the join operation on topological spaces (given by Construction 4.3.4.1) to the join operation on simplicial sets (given by Construction 4.3.3.13).

Construction 4.3.4.9. Let X and Y be simplicial sets, and let $\sigma : \Delta^n \to X \star Y$ be a morphism. We define a continuous function $f(\sigma) : |\Delta^n| \to |X| \star |Y|$ as follows (see Remark 4.3.3.17): 017P

- If σ factors through X, we let $f(\sigma)$ denote the composition

  $$|\Delta^n| \xrightarrow{|\sigma|} |X| \xrightarrow{\iota_{|X|}} |X| \star |Y|,$$

  where the second map is the inclusion of Remark 4.3.4.2.

- If σ factors through Y, we let $f(\sigma)$ denote the composition

  $$|\Delta^n| \xrightarrow{|\sigma|} |Y| \xrightarrow{\iota_{|Y|}} |X| \star |Y|,$$

  where the second map is the inclusion of Remark 4.3.4.2.

- If σ factors as a composition

  $$\Delta^n = \Delta^{p+1+q} \simeq \Delta^p \star \Delta^q \xrightarrow{\sigma_- \star \sigma_+} X \star Y,$$

  then we let $f(\sigma)$ denote the composite map

  $$|\Delta^n| = |\Delta^{p+1+q}| \xrightarrow{H_{p,q}^{-1}} |\Delta^p| \star |\Delta^q| \xrightarrow{|\sigma_-| \star |\sigma_+|} |X| \star |Y|,$$

  where $H_{p,q}$ denotes the homeomorphism of Proposition 4.3.4.8.

The construction $\sigma \mapsto f(\sigma)$ is compatible with face and degeneracy operators, and therefore determines a morphism of simplicial sets $f : X \star Y \to \mathrm{Sing}_\bullet(|X| \star |Y|)$. We will identify f with a continuous function $T_{X,Y} : |X \star Y| \to |X| \star |Y|$, which we will refer to as the *join comparison map.*

017Q **Example 4.3.4.10.** Let $X = \Delta^p$ and $Y = \Delta^q$ be standard simplices. Then the join comparison map $T_{X,Y} : |\Delta^p \star \Delta^q| \to |\Delta^p| \star |\Delta^q|$ fits into a commutative diagram

$$\begin{tikzcd}
{|\Delta^p \star \Delta^q|} \arrow[rr, "T_{X,Y}"] \arrow[dr, "|\rho|"'] & & {|\Delta^p| \star |\Delta^q|} \arrow[dl, "H_{p,q}"] \\
& {|\Delta^{p+1+q}|,} &
\end{tikzcd}$$

where $\rho : \Delta^p \star \Delta^q \simeq \Delta^{p+1+q}$ denotes the isomorphism of simplicial sets appearing in Example 4.3.3.14 and $H_{p,q}$ is the homeomorphism of Proposition 4.3.4.8. In particular, $T_{X,Y}$ is a homeomorphism.

017R **Proposition 4.3.4.11.** *Let X and Y be simplicial sets. If either X or Y is finite, then the join comparison map $T_{X,Y} : |X \star Y| \to |X| \star |Y|$ of Construction 4.3.4.9 is a homeomorphism.*

Proof. Without loss of generality, we may assume that X is finite. Then the geometric realization $|X|$ is a compact Hausdorff space (Corollary 3.6.1.12). Using Remarks 4.3.4.6 and 4.3.3.31, we see that the functors

$$\mathrm{Set}_\Delta \to \mathrm{Top}_{|X|/} \qquad Y \mapsto |X \star Y|, Y \mapsto |X| \star |Y|$$

preserve colimits. Consequently, if we regard X as fixed, then the collection of simplicial sets Y for which $T_{X,Y}$ is a homeomorphism is closed under colimits. Since every simplicial set can be realized as a colimit of standard simplices (Remark 1.1.3.13), it will suffice to prove Proposition 4.3.4.11 in the special case where $Y = \Delta^q$ is a standard simplex. In this case, Y is also finite. Repeating the preceding argument (with the roles of X and Y reversed), we are reduced to proving that $T_{X,Y}$ is a homeomorphism in the case where $X = \Delta^p$ is also a standard simplex. In this case, the desired result follows from Example 4.3.4.10. □

017S **Corollary 4.3.4.12.** *Let X be a simplicial set. Then the join comparison maps $T_{\Delta^0,X}$ and T_{X,Δ^0} supply homeomorphisms of topological spaces*

$$|X^{\triangleleft}| \simeq |X|^{\triangleleft} \qquad |X^{\triangleright}| \simeq |X|^{\triangleright}.$$

Here $X^{\triangleleft}$ and $X^{\triangleright}$ denote the left and right cones on X in the category of simplicial sets (Construction 4.3.3.26), while $|X|^{\triangleleft}$ and $|X|^{\triangleright}$ denote the cone $|X|$ in the category of topological spaces (Example 4.3.4.5).

The join comparison map $T_{X,Y} : |X \star Y| \to |X| \star |Y|$ need not be a homeomorphism in general. However, we do have the following:

Corollary 4.3.4.13. *Let X and Y be simplicial sets. Then the join comparison map $T_{X,Y} : |X \star Y| \to |X| \star |Y|$ is a bijection.* `017T`

Proof. As a map of sets, we can realize $T_{X,Y}$ as a filtered colimit of join comparison maps $T_{X',Y}$, where X' ranges over the finite simplicial subsets of X (Remark 3.6.1.8). Each of these maps is bijective (even a homeomorphism), by virtue of Proposition 4.3.4.11. □

Warning 4.3.4.14. Let X and Y be simplicial sets, and let $X \diamond Y$ denote the simplicial set given by the iterated coproduct `017U`

$$X \coprod_{(X \times \{0\} \times Y)} (X \times \Delta^1 \times Y) \coprod_{(X \times \{1\} \times Y)} Y$$

(see Notation 4.5.8.3). Since the formation of geometric realization commutes with the formation of colimits, we have an evident comparison map of topological spaces

$$|X \diamond Y| \to |X| \star |Y|.$$

This map is always bijective, and is a homeomorphism if either X or Y is finite (see Corollary 3.6.2.2). In this case, Corollary 4.3.4.13 supplies a homeomorphism of geometric realizations $|X \diamond Y| \simeq |X \star Y|$. Beware that this homeomorphism does *not* arise from a morphism of simplicial sets. In the case $X = \Delta^p$ and $Y = \Delta^q$, it arises from the homotopy

$$h : |\Delta^p| \times |\Delta^1| \times |\Delta^q| \to |\Delta^{p+1+q}|$$

$$h((u_0, \ldots, u_p), t, (v_0, \ldots, v_q)) = ((1-t)u_0, (1-t)u_1, \ldots, (1-t)u_p, tv_0, \ldots, tv_q).$$

appearing in Example 4.3.4.7, which is not piecewise-linear with respect to the natural triangulation of the polysimplex $|\Delta^p| \times |\Delta^1| \times |\Delta^q|$.

4.3.5 Slices of Simplicial Sets

Let $\mathcal{C}$ be a category. In §4.3.1, we associated to every diagram $F : \mathcal{K} \to \mathcal{C}$ a *slice category* $\mathcal{C}_{/F}$ and a *coslice category* $\mathcal{C}_{F/}$ (Construction 4.3.1.8). We now introduce a generalization of this construction, where we replace (the nerves of) $\mathcal{C}$ and $\mathcal{K}$ by arbitrary simplicial sets. As our starting point, we recall that the construction $\mathcal{C} \mapsto \mathcal{C}_{/F}$ can be characterized as the right adjoint of the join functor `017V`

$$\mathrm{Cat} \to \mathrm{Cat}_{\mathcal{K}/} \qquad \mathcal{E} \mapsto \mathcal{E} \star \mathcal{K}$$

(see Corollary 4.3.2.17).

017W **Construction 4.3.5.1** (Slice Simplicial Sets)**.** Let $f : K \to X$ be a morphism of simplicial sets. We define a simplicial set $X_{/f}$ as follows:

- For each $n \geq 0$, an n-simplex of $X_{/f}$ is a map of simplicial sets $\overline{f} : \Delta^n \star K \to X$ satisfying $\overline{f}|_K = f$.
- For every nondecreasing function $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, the associated map

  $$\alpha^* : \{n\text{-simplices of } X_{/f}\} \to \{m\text{-simplices of } X_{/f}\}$$

  carries an n-simplex $\overline{f} : \Delta^n \star K \to X$ to the composite map

  $$\Delta^m \star K \xrightarrow{\alpha \star \mathrm{id}_K} \Delta^n \star K \xrightarrow{\overline{f}} X.$$

We will refer to $X_{/f}$ as the *slice simplicial set of X over f*.

017X **Remark 4.3.5.2.** Let $f : K \to X$ be a morphism of simplicial sets, and let $\overline{f} : \Delta^n \star K \to X$ be an n-simplex of the slice simplicial set $X_{/f}$. Then the restriction $\overline{f}|_{\Delta^n}$ is an n-simplex of X. The construction $\overline{f} \mapsto \overline{f}|_{\Delta^n}$ determines a morphism of simplicial sets $X_{/f} \to X$, which we will refer to as the *projection map* or the *forgetful functor* (in the case where X is an ∞-category). We will often abuse notation by identifying a vertex of $X_{/f}$ with its image in X.

017Y **Variant 4.3.5.3** (Coslice Simplicial Sets)**.** Let $f : K \to X$ be a morphism of simplicial sets. We define a simplicial set $X_{f/}$ as follows:

- For each $n \geq 0$, an n-simplex of $X_{f/}$ is a map of simplicial sets $\overline{f} : K \star \Delta^n \to X$ satisfying $\overline{f}|_K = f$.
- For every nondecreasing function $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$, the associated map

  $$\alpha^* : \{n\text{-simplices of } X_{f/}\} \to \{m\text{-simplices of } X_{f/}\}$$

  carries an n-simplex $\overline{f} : K \star \Delta^n \to X$ to the composite map

  $$K \star \Delta^m \xrightarrow{\mathrm{id}_K \star \alpha} K \star \Delta^n \xrightarrow{\overline{f}} X.$$

We will refer to $X_{f/}$ as the *coslice simplicial set of X under f*. As in Remark 4.3.5.2, it is equipped with a projection map $X_{f/} \to X$.

017Z **Remark 4.3.5.4.** Construction 4.3.5.1 and Variant 4.3.5.3 are opposite to one another. More precisely, if $f : K \to X$ is a morphism of simplicial sets and $f^{\mathrm{op}} : K^{\mathrm{op}} \to X^{\mathrm{op}}$ denotes the induced map of opposite simplicial sets, then we have a canonical isomorphism of simplicial sets $(X_{/f})^{\mathrm{op}} \simeq (X^{\mathrm{op}})_{f^{\mathrm{op}}/}$.

Remark 4.3.5.5. Let $f : K \to X$ be a morphism of simplicial sets. Then vertices of the slice simplicial set $X_{/f}$ are morphisms of simplicial sets $\overline{f} : K^{\triangleleft} \to X$ satisfying $\overline{f}|_K = f$. Similarly, vertices of the coslice simplicial set $X_{f/}$ are morphisms of simplicial sets $\overline{f} : K^{\triangleright} \to X$ satisfying $\overline{f}|_K = f$. Here $K^{\triangleleft}$ and $K^{\triangleright}$ denote the left and right cone of K (Construction 4.3.3.26). 0180

Notation 4.3.5.6 (Slicing over Vertices)**.** Let X be a simplicial set containing a vertex x, and let $f_x : \Delta^0 \to X$ be the map carrying the unique vertex of Δ^0 to x. We will generally abuse notation by not distinguishing between the vertex x and the morphism f_x. For example, we will denote the slice simplicial set $X_{/f_x}$ by $X_{/x}$, and the coslice simplicial set $X_{f_x/}$ by $X_{x/}$. 0181

Example 4.3.5.7. Let $F : \mathcal{K} \to \mathcal{C}$ be a functor between categories, and let $f = \mathrm{N}_\bullet(F)$ denote the induced morphism of simplicial sets from $\mathrm{N}_\bullet(\mathcal{K})$ to $\mathrm{N}_\bullet(\mathcal{C})$. For each $n \geq 0$, we have canonical bijections 0182

$$\begin{aligned}
\{n\text{-simplices of } \mathrm{N}_\bullet(\mathcal{C})_{/f}\} &\simeq \{\text{Morphisms } \overline{f} : \Delta^n \star \mathrm{N}_\bullet(\mathcal{K}) \to \mathrm{N}_\bullet(\mathcal{C}) \text{ with } \overline{f}|_{\mathrm{N}_\bullet(\mathcal{K})} = f\} \\
&\simeq \{\text{Morphisms } \overline{f} : \mathrm{N}_\bullet([n]) \star \mathrm{N}_\bullet(\mathcal{K}) \to \mathrm{N}_\bullet(\mathcal{C}) \text{ with } \overline{f}|_{\mathrm{N}_\bullet(\mathcal{K})} = f\} \\
&\simeq \{\text{Morphisms } \overline{f} : \mathrm{N}_\bullet([n] \star \mathcal{K}) \to \mathrm{N}_\bullet(\mathcal{C}) \text{ with } \overline{f}|_{\mathrm{N}_\bullet(\mathcal{K})} = f\} \\
&\simeq \{\text{Functors } \overline{F} : [n] \star \mathcal{K} \to \mathcal{C} \text{ with } \overline{F}|_{\mathcal{K}} = F\} \\
&\simeq \{\text{Functors } [n] \to \mathcal{C}_{/F}\} \\
&\simeq \{n\text{-simplices of } \mathrm{N}_\bullet(\mathcal{C}_{/F})\ \}.
\end{aligned}$$

Here the third bijection comes from Example 4.3.3.14, the fourth from Proposition 1.3.3.1, and the fifth from Proposition 4.3.2.10. These bijections depend functorially on $[n] \in \mathbf{\Delta}$, and therefore determine an isomorphism of simplicial sets $\mathrm{N}_\bullet(\mathcal{C})_{/f} \simeq \mathrm{N}_\bullet(\mathcal{C}_{/F})$. Similarly, we have a canonical isomorphism $\mathrm{N}_\bullet(\mathcal{C})_{f/} \simeq \mathrm{N}_\bullet(\mathcal{C}_{F/})$. For a more general statement, see Corollary 4.3.5.17.

Example 4.3.5.8. Let $\mathcal{C}$ be a category containing an object X, which we also view as a vertex of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$. Specializing Example 4.3.5.7 (and invoking Example 4.3.1.10), we obtain canonical isomorphisms 0183

$$\mathrm{N}_\bullet(\mathcal{C})_{/X} \simeq \mathrm{N}_\bullet(\mathcal{C}_{/X}) \qquad \mathrm{N}_\bullet(\mathcal{C})_{X/} \simeq \mathrm{N}_\bullet(\mathcal{C}_{X/}).$$

Example 4.3.5.9. Let K be a simplicial set, let Y be a topological space, and let $f : K \to \mathrm{Sing}_\bullet(Y)$ be a morphism of simplicial sets, which we will identify with a continuous function $F : |K| \to Y$. For each $n \geq 0$, we have canonical bijections 0184

$$\begin{aligned}
\{n\text{-simplices of } \mathrm{Sing}_\bullet(Y)_{/f}\} &\simeq \{\text{Morphisms } \overline{f} : \Delta^n \star K \to \mathrm{Sing}_\bullet(Y) \text{ with } \overline{f}|_{\mathrm{N}_\bullet(\mathcal{K})} = f\} \\
&\simeq \{\text{Continuous maps } \overline{F} : |\Delta^n \star K| \to Y \text{ with } \overline{Y}|_{|K|} = f\} \\
&\simeq \{\text{Continuous maps } \overline{F} : |\Delta^n| \star |K| \to Y \text{ with } \overline{F}|_{|K|} = F\}
\end{aligned}$$

Here the third bijection is provided by Proposition 4.3.4.11. Using the fact that these bijections depend functorially on $[n] \in \mathbf{\Delta}$ and invoking the universal property $|\Delta^n| \star |K|$ (see Remark 4.3.4.3), we obtain an isomorphism of $\operatorname{Sing}_\bullet(Y)_{/f}$ with the iterated fiber product

$$\operatorname{Sing}_\bullet(Y) \times_{\operatorname{Fun}(\{0\}\times K, \operatorname{Sing}_\bullet(Y))} \operatorname{Fun}(\Delta^1 \times K, \operatorname{Sing}_\bullet(Y)) \times_{\operatorname{Fun}(\{1\}\times K, \operatorname{Sing}_\bullet(Y))} \{f\}.$$

0185 **Example 4.3.5.10.** Let Y be a topological space equipped with a base point y. Let $P = \{p : [0,1] \to Y\}$ denote the collection of all continuous functions from the unit interval $[0,1]$ to Y, and let $P_y = \{p \in P : p(1) = y\}$ denote the subset of P consisting of those continuous paths which end at the point y. We regard P as a topological space by equipping it with the compact-open topology, so the singular simplicial set $\operatorname{Sing}_\bullet(P)$ can be identified with $\operatorname{Fun}(\Delta^1, \operatorname{Sing}_\bullet(Y))$ (see Warning 2.4.2.18). Identifying y with a vertex of the singular simplicial set $\operatorname{Sing}_\bullet(Y)$, Example 4.3.5.9 supplies an isomorphism of simplicial sets

$$\operatorname{Sing}_\bullet(Y)_{/y} \simeq \operatorname{Sing}_\bullet(P) \times_{\operatorname{Sing}_\bullet(Y)} \{y\} = \operatorname{Sing}_\bullet(P_y).$$

In particular, since the topological space P_y is contractible, the simplicial set $\operatorname{Sing}_\bullet(Y)_{/y}$ is a contractible Kan complex (this is a special case of a general phenomenon: see Corollary 4.3.7.14).

0187 **Warning 4.3.5.11.** Recall that, if $F : \mathcal{K} \to \mathcal{C}$ is a functor between categories, then the slice category $\mathcal{C}_{/F}$ can be defined as the oriented fiber product $\mathcal{C} \tilde{\times}_{\operatorname{Fun}(\mathcal{K},\mathcal{C})} \{F\}$ (see Remark 4.3.1.11). In the setting of simplicial sets, our definition is somewhat different. Nevertheless, to any morphism of simplicial sets $F : K \to \mathcal{C}$, one can associate a comparison map

$$\delta_{/F} : \mathcal{C}_{/F} \to \mathcal{C} \times_{\operatorname{Fun}(\{0\}\times K, \mathcal{C})} \operatorname{Fun}(\Delta^1 \times K, \mathcal{C}) \times_{\operatorname{Fun}(\{1\}\times K, \mathcal{C})} \{F\}$$

which we will refer to as the *slice diagonal morphism* (see Construction 4.6.4.13). This map has the following features:

- When $\mathcal{C}$ is (the nerve of) an ordinary category, the morphism $\delta_{/F}$ is an isomorphism of simplicial sets.
- When $\mathcal{C}$ is an ∞-category, the morphism $\delta_{/F}$ is an equivalence of ∞-categories (Theorem 4.6.4.17).
- When $\mathcal{C} = \operatorname{Sing}_\bullet(X)$ is the singular simplicial set of a topological space X, the morphism $\delta_{/F}$ does *not* coincide with the isomorphism constructed in Example 4.3.5.9 (however, they are naturally homotopic).
- The morphism $\delta_{/F}$ is usually not an isomorphism of simplicial sets (see Warning 4.3.3.33).

The slice simplicial sets of Construction 4.3.5.1 can be characterized by a universal property.

Construction 4.3.5.12. Let $f : K \to X$ be a morphism of simplicial sets. We define a morphism of simplicial sets $c : X_{/f} \star K \to X$ as follows: 0188

- The restriction of c to the simplicial subset $X_{/f} \subseteq X_{/f} \star K$ is equal to the projection map $X_{/f} \to X$ of Remark 4.3.5.2.
- The restriction of c to the simplicial subset $K \subseteq X_{/f} \star K$ is equal to f.
- Let $\sigma : \Delta^n \to X_{/f} \star K$ be an n-simplex which does not belong to $X_{/f}$ or K, so that σ factors (uniquely) as a composition
$$\Delta^n \simeq \Delta^{p+1+q} \simeq \Delta^p \star \Delta^q \xrightarrow{\sigma_- \star \sigma_+} X_{/f} \star K$$
for $p + 1 + q = n$ (see Remark 4.3.3.17). Using the definition of the simplicial set $X_{/f}$, we can identify σ_- with a morphism of simplicial sets $\overline{f} : \Delta^p \star K \to X$ satisfying $\overline{f}|_K = f$. We then define $c(\sigma)$ to be the n-simplex of X given by the composite map
$$\Delta^n \simeq \Delta^{p+1+q} \simeq \Delta^p \star \Delta^q \xrightarrow{\mathrm{id} \star \sigma_+} \Delta^p \star K \xrightarrow{\overline{f}} X.$$

We will refer to c as the *slice contraction morphism*. Applying a similar construction to the opposite simplicial sets, we obtain a morphism $c' : K \star X_{f/} \to X$ which we will refer to as the *coslice contraction morphism*.

Proposition 4.3.5.13. *Let $f : K \to X$ be a morphism of simplicial sets, and let $c : X_{/f} \star K \to X$ be the slice contraction morphism of Construction 4.3.5.12. Then, for any simplicial set Y, postcomposition with c induces a bijection* 0189

$$\theta_Y : \mathrm{Hom}_{\mathrm{Set}_\Delta}(Y, X_{/f}) \to \{\text{Morphisms } \overline{f} : Y \star K \to X \text{ satisfying } \overline{f}|_K = f\}$$

Similarly, postcomposition with the coslice contraction morphism $c' : K \star X_{f/} \to X$ induces a bijection

$$\theta'_Y : \mathrm{Hom}_{\mathrm{Set}_\Delta}(Y, X_{f/}) \to \{\text{Morphisms } \overline{f} : K \star Y \to X \text{ satisfying } \overline{f}|_K = f\}.$$

Proof. In the case where Y is a standard simplex, both assertions follow immediately from the definition of the simplicial sets $X_{/f}$ and $X_{f/}$. Since every simplicial set can be realized as a colimit of simplices (Remark 1.1.3.13), it will suffice to show that the constructions $Y \mapsto \theta_Y$ and $Y \mapsto \theta'_Y$ carry colimits of simplicial sets to limits in the arrow category $\mathrm{Fun}([1], \mathrm{Set})$. This follows from the observation that the functors

$$\mathrm{Set}_\Delta \to (\mathrm{Set}_\Delta)_{K/} \qquad Y \mapsto Y \star K, Y \mapsto K \star Y$$

preserve small colimits (see Remark 4.3.3.31). $\square$

018A **Corollary 4.3.5.14.** *Let K be a simplicial set. Then the join functor*

$$\mathrm{Set}_{\Delta} \to (\mathrm{Set}_{\Delta})_{K/} \qquad Y \mapsto Y \star K$$

admits a right adjoint, given on objects by the slice construction $(f : K \to X) \mapsto X_{/f}$*. Similarly, the join functor*

$$\mathrm{Set}_{\Delta} \to (\mathrm{Set}_{\Delta})_{K/} \qquad Y \mapsto K \star Y$$

admits a right adjoint, given on objects by the coslice construction $(f : K \to X) \mapsto X_{f/}$*.*

018B **Example 4.3.5.15.** Let X be a simplicial set containing a vertex x. Let Y be a simplicial set, and let v and v' denote the cone points of $Y^{\triangleright}$ and $Y^{\triangleleft}$, respectively. Then Proposition 4.3.5.13 supplies bijections

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(Y, X_{/x}) \simeq \{\text{Morphisms } f : Y^{\triangleright} \to X \text{ with } f(v) = x\}$$

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(Y, X_{x/}) \simeq \{\text{Morphisms } f : Y^{\triangleleft} \to X \text{ with } f(v') = x\}.$$

056Y **Remark 4.3.5.16** (Slices of Coskeleta)**.** Let X and Y be simplicial sets. For every integer $n \geq 0$, Remark 4.3.3.30 supplies a pullback diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(Y^{\triangleright}), X) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_n(Y), X) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_{n-1}(Y)^{\triangleright}, X) \arrow[r] & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{sk}_{n-1}(Y), X).
\end{tikzcd}$$

Restricting the left side of the diagram to morphisms which carry the cone point of $Y^{\triangleright}$ to some fixed vertex $x \in x$ and invoking the universal properties of Example 4.3.5.15 and Remark 3.5.3.21, we obtain a pullback diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(Y, \mathrm{cosk}_n(X)_{/x}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(Y, \mathrm{cosk}_n(X)) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(Y, \mathrm{cosk}_{n-1}(X_{/x})) \arrow[r] & \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(Y, \mathrm{cosk}_{n-1}(X)).
\end{tikzcd}$$

This diagram depends functorially on Y, and therefore arises from a canonical isomorphism

$$\mathrm{cosk}_n(X)_{/x} \xrightarrow{\sim} \mathrm{cosk}_{n-1}(X_{/x}) \times_{\mathrm{cosk}_{n-1}(X)} \mathrm{cosk}_n(X).$$

Example 4.3.5.7 can be adapted to describe *any* slice or coslice of a simplicial set having the form $\mathrm{N}_{\bullet}(\mathcal{C})$.

Corollary 4.3.5.17. *Let $\mathcal{C}$ be a category and let K be a simplicial set equipped with a morphism $f : K \to \mathrm{N}_\bullet(\mathcal{C})$. Let $u : K \to \mathrm{N}_\bullet(\mathcal{K})$ be a morphism of simplicial sets which exhibits $\mathcal{K}$ as a homotopy category of K (see Definition 1.3.6.1), so that f factors uniquely as a composition $K \xrightarrow{u} \mathrm{N}_\bullet(\mathcal{K}) \xrightarrow{\mathrm{N}_\bullet(F)} \mathrm{N}_\bullet(\mathcal{C})$ for some functor $F : \mathcal{K} \to \mathcal{C}$. Then u induces isomorphisms of simplicial sets* 018C

$$\theta : \mathrm{N}_\bullet(\mathcal{C}_{/F}) \simeq \mathrm{N}_\bullet(\mathcal{C})_{/\mathrm{N}_\bullet(F)} \to \mathrm{N}_\bullet(\mathcal{C})_{/f} \qquad \theta' : \mathrm{N}_\bullet(\mathcal{C}_{F/}) \simeq \mathrm{N}_\bullet(\mathcal{C})_{\mathrm{N}_\bullet(F)/} \to \mathrm{N}_\bullet(\mathcal{C})_{f/}.$$

Proof. We will prove that θ is an isomorphism; the proof for θ' is similar. Fix an n-simplex σ of $\mathrm{N}_\bullet(\mathcal{C})_{/f}$, which we identify with a morphism of simplicial sets $\overline{f} : \Delta^n \star K \to \mathrm{N}_\bullet(\mathcal{C})$ satisfying $\overline{f}|_K = f$. Let $\overline{f}_0 = \overline{f}|_{\Delta^n}$. Using Proposition 4.3.5.13, we can identify $\overline{f}$ with a morphism of simplicial sets $g : K \to \mathrm{N}_\bullet(\mathcal{C})_{\overline{f}_0/}$. We wish to show that σ can be lifted uniquely to an n-simplex of $\mathrm{N}_\bullet(\mathcal{C})_{/\mathrm{N}_\bullet(F)}$. Equivalently, we wish to show that g admits a unique factorization

$$K \xrightarrow{u} \mathrm{N}_\bullet(\mathcal{K}) \xrightarrow{\overline{g}} \mathrm{N}_\bullet(\mathcal{C})_{\overline{f}_0/}$$

for which the composite map $\mathrm{N}_\bullet(\mathcal{K}) \xrightarrow{\overline{g}} \mathrm{N}_\bullet(\mathcal{C})_{\overline{f}_0/} \to \mathrm{N}_\bullet(\mathcal{C})$ is equal to $\mathrm{N}_\bullet(F)$. This follows our assumption that u exhibits $\mathcal{K}$ as a homotopy category of K, since the simplicial set $\mathrm{N}_\bullet(\mathcal{C})_{\overline{f}_0/}$ is isomorphic to the nerve of a category (see Example 4.3.5.7). □

Corollary 4.3.5.18. *Let A and B be simplicial sets, and let $u : A \to \mathrm{N}_\bullet(\mathcal{A})$ and $v : B \to \mathrm{N}_\bullet(\mathcal{B})$ be morphisms which exhibit $\mathcal{A}$ and $\mathcal{B}$ as the homotopy categories of A and B, respectively. Then the composite map* 018D

$$A \star B \xrightarrow{u \star v} \mathrm{N}_\bullet(\mathcal{A}) \star \mathrm{N}_\bullet(\mathcal{B}) \simeq \mathrm{N}_\bullet(\mathcal{A} \star \mathcal{B})$$

exhibits $\mathcal{A} \star \mathcal{B}$ as the homotopy category of $A \star B$.

Proof. Let $\mathcal{C}$ be a category, and suppose we are given a map of simplicial sets $f : A \star B \to \mathrm{N}_\bullet(\mathcal{C})$. Applying Corollary 4.3.5.17 to the morphism $f|_A$, we deduce that f factors uniquely as a composition

$$A \star B \xrightarrow{u \star \mathrm{id}} \mathrm{N}_\bullet(\mathcal{A}) \star B \xrightarrow{f'} \mathrm{N}_\bullet(\mathcal{C}).$$

Similarly, f' factors uniquely as a composition

$$\mathrm{N}_\bullet(\mathcal{A}) \star B \xrightarrow{\mathrm{id} \star v} \mathrm{N}_\bullet(\mathcal{A}) \star \mathrm{N}_\bullet(\mathcal{B}) \xrightarrow{f''} \mathrm{N}_\bullet(\mathcal{C}).$$

Combining these observations (together with Example 4.3.3.14 and Proposition 1.3.3.1), we conclude that f factors uniquely as a composition

$$A \star B \xrightarrow{u \star v} \mathrm{N}_\bullet(\mathcal{A}) \star \mathrm{N}_\bullet(\mathcal{B}) \simeq \mathrm{N}_\bullet(\mathcal{A} \star \mathcal{B}) \xrightarrow{\mathrm{N}_\bullet(F)} \mathrm{N}_\bullet(\mathcal{C})$$

for some functor $F : \mathcal{A} \star \mathcal{B} \to \mathcal{C}$. □

4.3.6 Slices of ∞-Categories

018E Recall that, if $\mathcal{C}$ is a category containing an object S, then the forgetful functors

$$\mathcal{C}_{S/} \to \mathcal{C} \qquad \mathcal{C}_{/S} \to \mathcal{C}$$

are left and right covering maps, respectively (Remark 4.3.1.6). In this section, we will prove an ∞-categorical counterpart of this assertion:

018F **Proposition 4.3.6.1** (Joyal [32])**.** *Let K be a simplicial set, let $\mathcal{C}$ be an ∞-category, and let $f : K \to \mathcal{C}$ be a diagram. Then the projection map $\mathcal{C}_{f/} \to \mathcal{C}$ is a left fibration of simplicial sets, and the projection map $\mathcal{C}_{/f} \to \mathcal{C}$ is a right fibration of simplicial sets. In particular, the simplicial sets $\mathcal{C}_{f/}$ and $\mathcal{C}_{/f}$ are ∞-categories (see Remark 4.2.1.4).*

018G **Remark 4.3.6.2.** In the special case where $\mathcal{C}$ is (the nerve of) an ordinary category, Proposition 4.3.6.1 follows from Corollary 4.3.5.17; in fact, both of the simplicial sets $\mathcal{C}_{f/}$ and $\mathcal{C}_{/f}$ are (the nerves of) ordinary categories.

We begin with some elementary remarks.

018H **Construction 4.3.6.3.** Let $f : A \hookrightarrow A'$ and $g : B \hookrightarrow B'$ be monomorphisms of simplicial sets. Using Remark 4.3.3.18, we see that the induced maps

$$A \star B' \xrightarrow{f \star \mathrm{id}_{B'}} A' \star B' \xleftarrow{\mathrm{id}_{A'} \star g} A' \star B$$

are also monomorphisms. Moreover, the intersection of their images is the image of the monomorphism $(f \star g) : A \star B \hookrightarrow A' \star B'$. We therefore obtain a monomorphism of simplicial sets

$$(A \star B') \coprod_{(A \star B)} (A' \star B) \hookrightarrow A' \star B',$$

which we will refer to as the *pushout-join* of f and g.

We will deduce Proposition 4.3.6.1 from the following property of Construction 4.3.6.3:

018J **Proposition 4.3.6.4** (Joyal [32])**.** *] Let $f : A \hookrightarrow A'$ and $g : B \hookrightarrow B'$ be monomorphisms of simplicial sets. If f is right anodyne or g is left anodyne, then the pushout-join*

$$(A \star B') \coprod_{(A \star B)} (A' \star B) \hookrightarrow A' \star B'$$

is an inner anodyne morphism of simplicial sets.

0235 **Example 4.3.6.5.** Let $f : A \hookrightarrow A'$ be a right anodyne morphism of simplicial sets. Applying Proposition 4.3.6.4 to the inclusion $\emptyset \hookrightarrow \Delta^0$, we deduce that the natural map $A^{\triangleright} \coprod_A A' \hookrightarrow A'^{\triangleright}$ is inner anodyne. Similarly, if $g : B \hookrightarrow B'$ is left anodyne, the induced map $B' \coprod_B B^{\triangleleft} \to B'^{\triangleleft}$ is inner anodyne.

Corollary 4.3.6.6. *Let $f : A \hookrightarrow B$ be an inner anodyne morphism of simplicial sets. Then, for every simplicial set K, the induced map $g : A \star K \hookrightarrow B \star K$ is also inner anodyne.* `02VK`

Proof. The morphism g factors as a composition

$$A \star K \xrightarrow{g'} B \coprod_{A} (A \star K) \xrightarrow{g''} B \star K.$$

The morphism g' is inner anodyne since it is a pushout of f, and the morphism g'' is inner anodyne by virtue of Proposition 4.3.6.4. It follows that $g = g'' \circ g'$ is also inner anodyne. $\square$

Example 4.3.6.7. Let $f : A \hookrightarrow B$ be an inner anodyne morphism of simplicial sets. Then the inclusion maps $f^{\triangleright} : A^{\triangleright} \hookrightarrow B^{\triangleright}$ and $i^{\triangleleft} : A^{\triangleleft} \hookrightarrow B^{\triangleleft}$ are inner anodyne. `02VL`

Proposition 4.3.6.4 implies the following stronger version of Proposition 4.3.6.1:

Proposition 4.3.6.8. *Let $q : X \to S$ be an inner fibration of simplicial sets, let $f : K \to X$ be any morphism of simplicial sets, let K_0 be a simplicial subset of K, and set $f_0 = f|_{K_0}$. Then the restriction map* `01BZ`

$$X_{/f} \to X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}$$

is a right fibration, and the restriction map

$$X_{f/} \to X_{f_0/} \times_{S_{(q \circ f_0)/}} S_{(q \circ f)/}$$

is a left fibration.

Proof. We will prove the first assertion; the second follows by a similar argument. By virtue of Proposition 4.2.4.5, it will suffice to show that for every right anodyne morphism $i : A \hookrightarrow A'$, every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & X_{/f} \arrow[d] \\ A' \arrow[r] \arrow[ur, dashed] & X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)} \end{tikzcd}$$

admits a solution. Unwinding the definitions, this is equivalent to solving an associated lifting problem

$$\begin{tikzcd} (A \star K) \coprod_{A \star K_0} (A' \star K_0) \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ A' \star K \arrow[r] \arrow[ur, dashed] & S, \end{tikzcd}$$

where the left vertical morphism is the pushout-join of Construction 4.3.6.3. Proposition 4.3.6.4 guarantees that this morphism is inner anodyne, so that the desired extension exists by virtue of our assumption that q is an inner fibration (Proposition 4.1.3.1). □

01C0 **Corollary 4.3.6.9.** *Let $q : X \to S$ be an inner fibration of simplicial sets and let $f : K \to X$ be any morphism of simplicial sets. Then the restriction map*

$$X_{/f} \to X \times_S S_{/(q \circ f)}$$

is a right fibration, and the restriction map

$$X_{f/} \to X \times_S S_{(q \circ f)/}$$

is a left fibration.

Proof. Apply Proposition 4.3.6.8 in the special case $K_0 = \emptyset$. □

01NS **Corollary 4.3.6.10.** *Let $q : X \to S$ be an inner fibration of simplicial sets and let $f : K \to X$ be any morphism of simplicial sets. Then the induced maps*

$$X_{/f} \to S_{/(q \circ f)} \qquad X_{f/} \to S_{(q \circ f)/}$$

are inner fibrations.

018K **Corollary 4.3.6.11.** *Let $\mathcal{C}$ be an ∞-category, let $f : K \to \mathcal{C}$ be a morphism of simplicial sets, and let $f_0 = f|_{K_0}$ be the restriction of f to a simplicial subset $K_0 \subseteq K$. Then the restriction map $\mathcal{C}_{/f} \to \mathcal{C}_{/f_0}$ is a right fibration, and the restriction map $\mathcal{C}_{f/} \to \mathcal{C}_{f_0/}$ is a left fibration.*

Proof. Apply Proposition 4.3.6.8 to the inner fibration $q : \mathcal{C} \to \Delta^0$. □

Proof of Proposition 4.3.6.1. Apply Corollary 4.3.6.11 in the special case $K_0 = \emptyset$. □

Proposition 4.3.6.4 also yields the following:

01C1 **Proposition 4.3.6.12.** *Let $q : X \to S$ be an inner fibration of simplicial sets, let $f : K \to X$ be any morphism of simplicial sets, let K_0 be a simplicial subset of K, and set $f_0 = f|_{K_0}$. If the inclusion $K_0 \hookrightarrow K$ is left anodyne, then the restriction map $X_{/f} \to X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}$ is a trivial Kan fibration. If the inclusion $K_0 \hookrightarrow K$ is right anodyne, then the restriction map $X_{f/} \to X_{f_0/} \times_{S_{(q \circ f_0)/}} S_{(q \circ f)/}$ is a trivial Kan fibration.*

Proof. We will prove the first assertion; the second follows by a similar argument. Assume that the inclusion $K_0 \hookrightarrow K$ is left anodyne. We wish to show that, for every monomorphism of simplicial sets $i : A \hookrightarrow A'$, every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X_{/f} \arrow[d] \\
A' \arrow[r] \arrow[ur, dashed] & X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}
\end{tikzcd}$$

admits a solution. Unwinding the definitions, this is equivalent to solving an associated lifting problem

$$\begin{tikzcd}
(A \star K) \coprod_{A \star K_0} (A' \star K_0) \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
A' \star K \arrow[r] \arrow[ur, dashed] & S,
\end{tikzcd}$$

where the left vertical morphism is the pushout-join of Construction 4.3.6.3. Since the left vertical map is inner anodyne (Proposition 4.3.6.4), the desired solution exists by virtue of our assumption that q is an inner fibration (Proposition 4.1.3.1). □

Corollary 4.3.6.13. *Let $\mathcal{C}$ be an ∞-category, let $f : K \to \mathcal{C}$ be a morphism of simplicial sets, and let $f_0 = f|_{K_0}$ be the restriction of f to a simplicial subset $K_0 \subseteq K$. If the inclusion $K_0 \hookrightarrow K$ is left anodyne, then the restriction map $\mathcal{C}_{/f} \to \mathcal{C}_{/f_0}$ is a trivial Kan fibration. If the inclusion $K_0 \hookrightarrow K$ is right anodyne, then the restriction map $\mathcal{C}_{f/} \to \mathcal{C}_{f_0/}$ is a trivial Kan fibration.* `018L`

Proof. Apply Proposition 4.3.6.12 to the inner fibration $q : \mathcal{C} \to \Delta^0$. □

Example 4.3.6.14 (Composition Functors)**.** Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism in $\mathcal{C}$, which we identify with a diagram $\Delta^1 \to \mathcal{C}$. The inclusions $\{0\} \hookrightarrow \Delta^1 \hookleftarrow \{1\}$ then induce restriction functors $\mathcal{C}_{X/} \xleftarrow{e_0} \mathcal{C}_{f/} \xrightarrow{e_1} \mathcal{C}_{Y/}$. It follows from Corollary 4.3.6.13 that e_1 is a trivial Kan fibration, and therefore admits a section $s : \mathcal{C}_{Y/} \to \mathcal{C}_{f/}$ (which is unique up to isomorphism). The composition $e_0 \circ s$ can then be viewed as a functor from $\mathcal{C}_{Y/}$ to $\mathcal{C}_{X/}$, which we will refer to as *precomposition with f*. Concretely, this functor takes an object $g : Y \to Z$ of the ∞-category $\mathcal{C}_{Y/}$ to an object $h : X \to Z$ of the ∞-category $\mathcal{C}_{X/}$, which is `04Q4` characterized (up to isomorphism) by the requirement that there exists a 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z, \end{tikzcd}$$

so that h is a composition of f with g in the sense of Definition 1.4.4.1. Applying the same construction in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$, we obtain a functor $\mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ which we will refer to as *postcomposition with* f; concretely, it carries an object $e : W \to X$ of the ∞-category $\mathcal{C}_{/X}$ to an object $W \to Y$ of $\mathcal{C}_{/Y}$ which is a composition of e with f.

We now turn to the proof of Proposition 4.3.6.4.

018N **Lemma 4.3.6.15** (Joyal [32]). *Let $p, q \geq 0$ be nonnegative integers. Then:*

- *Assume $p > 0$. Then, for $0 \leq i \leq p$, the pushout-join monomorphism*

$$(\Lambda^p_i \star \Delta^q) \coprod_{(\Lambda^p_i \star \partial \Delta^q)} (\Delta^p \star \partial \Delta^q) \hookrightarrow \Delta^p \star \Delta^q$$

  of Construction 4.3.6.3 is isomorphic to the horn inclusion $\Lambda^{p+1+q}_i \hookrightarrow \Delta^{p+1+q}$.

- *Assume $q > 0$. Then, for $0 \leq j \leq q$, the pushout-join monomorphism*

$$(\partial \Delta^p \star \Delta^q) \coprod_{(\partial \Delta^p \star \Lambda^q_j)} (\Delta^p \star \Lambda^q_j) \hookrightarrow \Delta^p \star \Delta^q$$

  of Construction 4.3.6.3 is isomorphic to the horn inclusion $\Lambda^{p+1+q}_{p+1+j} \hookrightarrow \Delta^{p+1+q}$.

Proof. We will prove the first assertion; the second follows by symmetry. We begin by observing that there is a unique isomorphism of simplicial sets $u : \Delta^p \star \Delta^q \simeq \Delta^{p+1+q}$ (Example 4.3.3.14). Let σ be an n-simplex of the join $\Delta^p \star \Delta^q$; we wish to show that $u(\sigma)$ belongs to the horn Λ^{p+1+q}_i if and only if σ belongs to the union of the simplicial subsets

$$\Lambda^p_i \star \Delta^q \subseteq \Delta^p \star \Delta^q \supseteq \Delta^p \star \partial \Delta^q .$$

We consider three cases (see Remark 4.3.3.17):

- The simplex σ belongs to the simplicial subset $\Delta^p \subseteq \Delta^p \star \Delta^q$. In this case, σ is contained in $\Delta^p \star \partial \Delta^q$ and $u(\sigma)$ is contained in Λ^{p+1+q}_i.
- The simplex σ belongs to the simplicial subset $\Delta^q \subseteq \Delta^p \star \Delta^q$. In this case, σ is contained in $\Lambda^p_i \star \Delta^q$ and $u(\sigma)$ is contained in Λ^{p+1+q}_i (since $p > 0$).

- The simplex σ factors as a composition

  $$\Delta^n = \Delta^{p'+1+q'} \simeq \Delta^{p'} \star \Delta^{q'} \xrightarrow{\sigma_- \star \sigma_+} \Delta^p \star \Delta^q.$$

  Let us abuse notation by identifying σ_- and σ_+ with nondecreasing functions $[p'] \to [p]$ and $[q'] \to [q]$, and $u(\sigma)$ with the nondecreasing function $[n] \to [p+1+q]$ given by their join. In this case, σ fails to belong to the union $(\Lambda^p_i \star \Delta^q) \cup (\Delta^p \star \partial \Delta^q)$ if and only if both of the following conditions are satisfied:

  - The image of the nondecreasing function $\sigma_- : [p'] \to [p]$ contains $[p] \setminus \{i\}$.
  - The nondecreasing function $\sigma_+ : [q'] \to [q]$ is surjective.

  Together, these are equivalent to the assertion that the image of the nondecreasing function $u(\sigma) : [n] \to [p+1+q]$ contains $[p+1+q] \setminus \{i\}$: that is, it fails to belong to the horn $\Lambda^{p+1+q}_i \subseteq \Delta^{p+1+q}$.

□

Proof of Proposition 4.3.6.4. For every pair of morphisms of simplicial sets $f : A \to A'$ and $g : B \to B'$, let

$$\theta_{f,g} : (A \star B') \coprod_{(A \star B)} (A' \star B) \to A' \star B'$$

denote their pushout join. We will show that, if f is right anodyne and g is a monomorphism, then $\theta_{f,g}$ is inner anodyne (the analogous assertion for the case where g is left anodyne follows by a similar argument). Let us first regard f as fixed, and let T be the collection of all morphisms g of simplicial sets for which $\theta_{f,g}$ is inner anodyne. Then T is weakly saturated (in the sense of Definition 1.5.4.12). We wish to prove that T contains every monomorphism of simplicial sets. By virtue of Proposition 1.5.5.14, we are reduced to proving that the morphism $\theta_{f,g}$ is inner anodyne in the special case where g is the boundary inclusion $\partial \Delta^q \hookrightarrow \Delta^q$ for some $q \geq 0$.

Let us now regard $g : \partial \Delta^q \hookrightarrow \Delta^q$ as fixed, and let S denote the collection of all morphisms of simplicial sets for which $\theta_{f,g}$ is inner anodyne. To complete the proof, we must show that S contains every right anodyne morphism of simplicial sets. As before, we note that S is weakly saturated. It will therefore suffice to show that S contains every horn inclusion $\Lambda^p_i \hookrightarrow \Delta^p$ for $0 < i \leq p$ (see Variant 4.2.4.2). In other words, we are reduced to checking that the pushout-join

$$\theta_{f,g} : (\Lambda^p_i \star \Delta^q) \coprod_{(\Lambda^p_i \star \partial \Delta^q)} (\Delta^p \star \partial \Delta^q) \hookrightarrow \Delta^p \star \Delta^q$$

is inner anodyne. This is clear, since $\theta_{f,g}$ can be identified with the inner horn inclusion $\Lambda^{p+1+q}_i \hookrightarrow \Delta^{p+1+q}$ by virtue of Lemma 4.3.6.15. □

Using Lemma 4.3.6.15, we can also establish a converse to Proposition 4.3.6.1:

0236 **Corollary 4.3.6.16.** *Let $\mathcal{C}$ be a simplicial set. The following conditions are equivalent:*

(1) *The simplicial set $\mathcal{C}$ is an ∞-category.*

(2) *For every vertex X of $\mathcal{C}$, the projection map $\mathcal{C}_{X/} \to \mathcal{C}$ is a left fibration of simplicial sets.*

(3) *For every vertex Y of $\mathcal{C}$, the projection map $\mathcal{C}_{/Y} \to \mathcal{C}$ is a right fibration of simplicial sets.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) are special cases of Proposition 4.3.6.1. We will complete the proof by showing that (3) implies (1); the proof that (2) implies (1) is similar. Assume that (3) is satisfied, and suppose that we are given a map $\sigma_0 : \Lambda^n_i \to \mathcal{C}$, where $0 < i < n$; we wish to show that σ_0 can be extended to an n-simplex σ of $\mathcal{C}$. Setting $Y = \sigma_0(n)$ and using the isomorphism $\Lambda^n_i \simeq \Delta^{n-1} \coprod_{\Lambda^{n-1}_i} (\Lambda^{n-1}_i)^{\triangleright}$ supplied by Lemma 4.3.6.15, we are reduced to solving a lifting problem of the form

$$\begin{tikzcd}
\Lambda^{n-1}_i \arrow[r] \arrow[d] & \mathcal{C}_{/Y} \arrow[d] \\
\Delta^{n-1} \arrow[r] \arrow[ur, dashed] & \mathcal{C}.
\end{tikzcd}$$

Since $0 < i \leq n-1$, the desired solution exists by virtue of our assumption that the projection map $\mathcal{C}_{/Y} \to \mathcal{C}$ is a right fibration. □

For future use, let us record a variant of Lemma 4.3.6.15:

02BU **Variant 4.3.6.17.** Let p and q be nonnegative integers. Then the pushout-join monomorphism

$$(\partial\Delta^p \star \Delta^q) \coprod_{(\partial\Delta^p \star \partial\Delta^q)} (\Delta^p \star \partial\Delta^q) \hookrightarrow \Delta^p \star \Delta^q$$

of Construction 4.3.6.3 is isomorphic to the boundary inclusion $\partial\Delta^{p+1+q} \hookrightarrow \Delta^{p+1+q}$.

Proof. We proceed as in the proof of Lemma 4.3.6.15. Let $u : \Delta^p \star \Delta^q \simeq \Delta^{p+1+q}$ be the isomorphism supplied by Example 4.3.3.14, and let σ be an n-simplex of the join $\Delta^p \star \Delta^q$. We wish to show that $u(\sigma)$ belongs to the boundary $\partial\Delta^{p+1+q}$ if and only if σ belongs to the union of the simplicial subsets

$$\partial\Delta^p \star \Delta^q \subseteq \Delta^p \star \Delta^q \supseteq \Delta^p \star \partial\Delta^q .$$

We consider three cases (see Remark 4.3.3.17):

- The simplex σ belongs to the simplicial subset $\Delta^p \subseteq \Delta^p \star \Delta^q$. In this case, σ is contained in $\Delta^p \star \partial\Delta^q$ and $u(\sigma)$ is contained in $\partial\Delta^{p+1+q}$.

- The simplex σ belongs to the simplicial subset $\Delta^q \subseteq \Delta^p \star \Delta^q$. In this case, σ is contained in $\partial\Delta^p \star \Delta^q$ and $u(\sigma)$ is contained in $\partial\Delta^{p+1+q}$.

- The simplex σ factors as a composition

  $$\Delta^n = \Delta^{p'+1+q'} \simeq \Delta^{p'} \star \Delta^{q'} \xrightarrow{\sigma_- \star \sigma_+} \Delta^p \star \Delta q.$$

  In this case, σ belongs to the union $(\partial\Delta^p \star \Delta^q) \cup (\Delta^p \star \partial\Delta^q)$ if and only if either σ_- or σ_+ fails to be surjective at the level of vertices. This is equivalent to the requirement that the map $u(\sigma) : \Delta^n \to \Delta^{p+1+q}$ fails to be surjective at the level of vertices: that is, it is a simplex of the boundary $\partial\Delta^{p+1+q}$.

□

4.3.7 Slices of Left and Right Fibrations

In this section, we collect some further applications of Lemma 4.3.6.15. 018P

Proposition 4.3.7.1. *Let $f : A \hookrightarrow A'$ and $g : B \hookrightarrow B'$ be monomorphisms of simplicial sets, and let* 018Q

$$\theta_{f,g} : (A \star B') \coprod_{(A \star B)} (A' \star B) \hookrightarrow A' \star B'$$

be the pushout-join of Construction 4.3.6.3. If f is anodyne, then $\theta_{f,g}$ is left anodyne. If g is anodyne, then $\theta_{f,g}$ is right anodyne.

Proof. We will prove the first assertion; the proof of the second is similar. We proceed as in the proof of Proposition 4.3.6.4. Let us first regard the anodyne morphism f as fixed, and let T be the collection of all morphisms g of simplicial sets for which $\theta_{f,g}$ is left anodyne. Then T weakly saturated (in the sense of Definition 1.5.4.12). We wish to prove that T contains every monomorphism of simplicial sets. By virtue of Proposition 1.5.5.14, we are reduced to proving that the morphism $\theta_{f,g}$ is left anodyne in the special case where g is the boundary inclusion $\partial\Delta^q \hookrightarrow \Delta^q$ for some $q \geq 0$.

Let us now regard $g : \partial\Delta^q \hookrightarrow \Delta^q$ as fixed, and let S denote the collection of all morphisms of simplicial sets for which $\theta_{f,g}$ is left anodyne. To complete the proof, we must show that S contains every anodyne morphism of simplicial sets. As before, we note that S is weakly saturated. It will therefore suffice to show that S contains every horn inclusion $\Lambda^p_i \hookrightarrow \Delta^p$ when $p > 0$ and $0 \leq i \leq p$. In other words, we are reduced to checking that the pushout-join

$$\theta_{f,g} : (\Lambda^p_i \star \Delta^q) \coprod_{(\Lambda^p_i \star \partial\Delta^q)} (\Delta^p \star \partial\Delta^q) \hookrightarrow \Delta^p \star \Delta^q$$

is left anodyne. This is clear, since $\theta_{f,g}$ can be identified with the horn inclusion $\Lambda^{p+1+q}_i \hookrightarrow \Delta^{p+1+q}$ by virtue of Lemma 4.3.6.15. $\square$

018R **Proposition 4.3.7.2.** *Let $f : K \to X$ and $q : X \to S$ be morphisms of simplicial sets, let $K_0 \subseteq K$ be a simplicial subset, and set $f_0 = f|_{K_0}$. Then:*

- *If q is a left fibration, then the induced map*

$$X_{/f} \to X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}$$

is a Kan fibration.

- *If q is a right fibration, then the induced map*

$$X_{f/} \to X_{f_0/} \times_{S_{(q \circ f_0)/}} S_{(q \circ f)/}$$

is a Kan fibration.

Proof. We will prove the first assertion; the proof of the second is similar. Assume that q is a left fibration; we wish to show that the map $X_{/f} \to X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}$ is a Kan fibration. Equivalently, we wish to show that every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X_{/f} \arrow[d] \\
A' \arrow[r] \arrow[ur, dashed] & X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}
\end{tikzcd}$$

admits a solution, provided that the left vertical map $A \to A'$ is anodyne. Unwinding the definitions, we see that this can be rephrased as a lifting problem

$$\begin{tikzcd}
(A \star K) \coprod_{(A \star K_0)} (A' \star K_0) \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
A' \star K \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

This problem admits a solution, since the vertical map on the left is left anodyne (Proposition 4.3.7.1) and q is a left fibration. $\square$

018S **Corollary 4.3.7.3.** *Let $f : K \to X$ and $q : X \to S$ be morphisms of simplicial sets. Then:*

- *If q is a left fibration, then the induced map*

$$X_{/f} \to X \times_S S_{/(q \circ f)}$$

is a Kan fibration.

- *If q is a right fibration, then the induced map*

$$X_{f/} \to X \times_S S_{(q \circ f)/}$$

is a Kan fibration.

Proof. Apply Proposition 4.3.7.2 in the special case $K_0 = \emptyset$. □

Corollary 4.3.7.4. *Let X be a Kan complex, let $f : K \to X$ be a morphism of simplicial sets, let $K_0 \subseteq K$ be a simplicial subset, and set $f_0 = f|_{K_0}$. Then the restriction maps* 018T

$$X_{/f} \to X_{/f_0} \qquad X_{f/} \to X_{f_0/}$$

are Kan fibrations.

Proof. Apply Proposition 4.3.7.2 in the special case $S = \Delta^0$. □

Corollary 4.3.7.5. *Let X be a Kan complex and let $f : K \to X$ be a morphism of simplicial sets. Then the projection maps* 018U

$$X_{/f} \to X \qquad X_{f/} \to X$$

are Kan fibrations. In particular, the simplicial sets $X_{/f}$ and $X_{f/}$ are Kan complexes.

Proof. Apply Corollary 4.3.7.4 in the special case $K_0 = \emptyset$ (or Corollary 4.3.7.3 in the special case $S = \Delta^0$). □

Proposition 4.3.7.6. *Let $f : K \to X$ and $q : X \to S$ be morphisms of simplicial sets, let $K_0 \subseteq K$ be a simplicial subset, and set $f_0 = f|_{K_0}$. Then:* 018V

- *If q is a right fibration and the inclusion $K_0 \hookrightarrow K$ is anodyne, then the induced map*

$$X_{/f} \to X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}$$

is a trivial Kan fibration.

- *If q is a left fibration and the inclusion $K_0 \hookrightarrow K$ is anodyne, then the induced map*

$$X_{f/} \to X_{f_0/} \times_{S_{(q \circ f_0)/}} S_{(q \circ f)/}$$

is a trivial Kan fibration.

Proof. We will prove the first assertion; the proof of the second is similar. Assume that q is a right fibration and that the inclusion $K_0 \hookrightarrow K$ is anodyne. We wish to show that the map $X_{/f} \to X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}$ is a trivial Kan fibration. Equivalently, we wish to show that every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X_{/f} \arrow[d] \\
A' \arrow[r] \arrow[ur, dashed] & X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}
\end{tikzcd}$$

admits a solution, provided that the left vertical map $A \to A'$ is a monomorphism. Unwinding the definitions, we see that this can be rephrased as a lifting problem

$$\begin{tikzcd}
(A \star K) \coprod_{(A \star K_0)} (A' \star K_0) \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
A' \star K \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

This problem admits a solution, since the vertical map on the left is right anodyne (Proposition 4.3.7.1) and q is a right fibration. □

018W **Corollary 4.3.7.7.** *Let X be a Kan complex, let $f : K \to X$ be a morphism of simplicial sets, let $K_0 \subseteq K$ be a simplicial subset for which the inclusion $K_0 \hookrightarrow K$ is anodyne, and set $f_0 = f|_{K_0}$. Then the restriction maps*

$$X_{/f} \to X_{/f_0} \qquad X_{f/} \to X_{f_0/}$$

are trivial Kan fibrations.

Proof. Apply Proposition 4.3.7.6 in the special case $S = \Delta^0$. □

We now record some variants of the preceding results.

0196 **Lemma 4.3.7.8.** *Let $f : A \hookrightarrow B$ be a monomorphism of simplicial sets. Then the inclusion $f^{\triangleright} : A^{\triangleright} \hookrightarrow B^{\triangleright}$ is right anodyne, and the inclusion $A^{\triangleleft} \hookrightarrow B^{\triangleleft}$ is left anodyne.*

Proof. We will prove the first assertion (the second follows by a similar argument). Let T be the collection of all morphisms f of simplicial sets for which $f^{\triangleright}$ is right anodyne. We wish to show that every monomorphism belongs to T. Since the collection T is weakly saturated, it will suffice to show that every boundary inclusion $f : \partial \Delta^n \hookrightarrow \Delta^n$ belongs to T (Proposition 1.5.5.14). In this case, we can identify $f^{\triangleright}$ with the with the horn inclusion $\Lambda^{n+1}_{n+1} \hookrightarrow \Delta^{n+1}$ (see Example 4.3.3.28). □

Lemma 4.3.7.8 immediately implies the following stronger assertion:

Proposition 4.3.7.9. *Let X and Y be simplicial sets. If X is weakly contractible and Y' is a simplicial subset of Y, then the inclusion $X \star Y' \hookrightarrow X \star Y$ is left anodyne. If Y is weakly contractible and X' is a simplicial subset of X, then the inclusion $X' \star Y \hookrightarrow X \star Y$ is right anodyne.* 0197

Proof. We will prove the first assertion; the second follows by a similar argument. Fix a vertex $x \in X$, so that the inclusion morphism $\iota : X \star Y' \hookrightarrow X \star Y$ factors as a composition

$$X \star Y' \xrightarrow{\iota'} (X \star Y') \coprod_{(\{x\} \star Y')} (\{x\} \star Y) \xrightarrow{\iota''} X \star Y.$$

The morphism ι' is a pushout of the inclusion $Y'^{\lhd} \hookrightarrow Y^{\lhd}$, and is left anodyne by virtue of Lemma 4.3.7.8. It will therefore suffice to show that ι'' is left anodyne. This is a special case of Proposition 4.3.7.1, since the inclusion map $\{x\} \hookrightarrow X$ is a weak homotopy equivalence (by virtue of our assumption that X is weakly contractible) and therefore anodyne (by virtue of Corollary 3.3.7.7). □

Example 4.3.7.10. Let X and Y be simplicial sets. If X is weakly contractible, then Proposition 4.3.7.9 guarantees that the inclusion $\iota_X : X \hookrightarrow X \star Y$ is left anodyne. If Y is weakly contractible, then Proposition 4.3.7.9 guarantees that the inclusion $\iota_Y : Y \hookrightarrow X \star Y$ is right anodyne. 0198

Example 4.3.7.11. Let X be a simplicial set, and let v denote the cone point of the simplicial set $X^{\rhd}$. Then the inclusion $\{v\} \hookrightarrow X^{\rhd}$ is right anodyne. In particular, it is a weak homotopy equivalence. 02LJ

Proposition 4.3.7.12. *Let $q : X \to S$ and $f : K \to X$ be morphisms of simplicial sets. Then:* 0199

- *If q is a right fibration and K is weakly contractible, then the induced map $X_{/f} \to S_{/(q \circ f)}$ is a trivial Kan fibration.*
- *If q is a left fibration and K is weakly contractible, then the induced map $X_{f/} \to S_{(q \circ f)/}$ is a trivial Kan fibration.*

Proof. We will prove the first assertion; the second follows by a similar argument. To show that the morphism $X_{/f} \to S_{/(q \circ f)}$ is a trivial Kan fibration, we must prove that every lifting problem

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & X_{/f} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S_{/q \circ f}
\end{tikzcd}$$

admits a solution. Unwinding the definitions, we can rephrase this as a lifting problem

$$\begin{tikzcd}
(\partial \Delta^n) \star K \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
(\Delta^n) \star K \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

This lifting problem admits a solution, since q is assumed to be a right fibration and the left vertical map is right anodyne (Proposition 4.3.7.9). ☐

02LK **Corollary 4.3.7.13.** *Let $q : X \to S$ be a morphism of simplicial sets, and let $x \in X$ be a vertex having image $s = q(x)$ in S. Then:*

- *If q is a right fibration, then the induced map $X_{/x} \to S_{/s}$ is a trivial Kan fibration.*
- *If q is a left fibration, then the induced map $X_{x/} \to S_{s/}$ is a trivial Kan fibration.*

018Y **Corollary 4.3.7.14.** *Let X be a Kan complex containing a vertex x. Then the simplicial sets $X_{/x}$ and $X_{x/}$ are contractible Kan complexes.*

Proof. Apply Corollary 4.3.7.13 in the special case $S = \Delta^0$. ☐

018Z **Proposition 4.3.7.15.** *Let $f : K \to X$ and $q : X \to S$ be morphisms of simplicial sets, let $K_0 \subseteq K$ be a simplicial subset, and set $f_0 = f|_{K_0}$. If q is a trivial Kan fibration, then the induced maps*

$$X_{/f} \to X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)} \qquad X_{f/} \to X_{f_0/} \times_{S_{(q \circ f_0)/}} S_{(q \circ f)/}$$

are also trivial Kan fibrations.

Proof. To show that the map $X_{/f} \to X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}$ is a trivial Kan fibration, we must show that every lifting problem every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X_{/f} \arrow[d] \\
A' \arrow[r] \arrow[ur, dashed] & X_{/f_0} \times_{S_{/(q \circ f_0)}} S_{/(q \circ f)}
\end{tikzcd}$$

admits a solution, provided that the left vertical map $A \to A'$ is a monomorphism. Unwinding the definitions, we see that this can be rephrased as a lifting problem

$$\begin{tikzcd}
(A \star K) \coprod_{(A \star K_0)} (A' \star K_0) \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
A' \star K \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

This problem admits a solution, since the vertical map on the left is a monomorphism (Proposition 4.3.7.1) and q is a trivial Kan fibration. □

Corollary 4.3.7.16. *Let $q : X \to S$ be a trivial Kan fibrations of simplicial sets and let $f : K \to X$ be any morphism of simplicial sets. Then the induced maps* 0190

$$X_{/f} \to X \times_S S_{/(q \circ f)} \qquad X_{f/} \to X \times_S S_{(q \circ f)/}$$

are trivial Kan fibrations.

Proof. Apply Proposition 4.3.7.15 in the special case $K_0 = \emptyset$. □

Corollary 4.3.7.17. *Let $q : X \to S$ be a trivial Kan fibration of simplicial sets and let $f : K \to X$ be any morphism of simplicial sets. Then the induced maps* 01NT

$$X_{/f} \to S_{/(q \circ f)} \qquad X_{f/} \to S_{(q \circ f)/}$$

are trivial Kan fibrations.

Corollary 4.3.7.18. *Let X be a contractible Kan complex, let $f : K \to X$ be a morphism of simplicial sets, let K_0 be a simplicial subset of K, and set $f_0 = f|_{K_0}$. Then the restriction maps* 0191

$$X_{/f} \to X_{/f_0} \qquad X_{f/} \to X_{f_0/}$$

are trivial Kan fibrations.

Proof. Apply Proposition 4.3.7.15 in the special case $S = \Delta^0$. □

Corollary 4.3.7.19. *Let X be a contractible Kan complex and let $f : K \to X$ be a morphism of simplicial sets. Then the projection maps* 0192

$$X_{/f} \to X \qquad X_{f/} \to X$$

are trivial Kan fibrations. In particular, $X_{/f}$ and $X_{f/}$ are also contractible Kan complexes.

Proof. Apply Corollary 4.3.7.16 in the special case $S = \Delta^0$ (or Corollary 4.3.7.18 in the special case $K_0 = \emptyset$). □

4.4 Isomorphisms and Isofibrations

01EM Let $\mathcal{C}$ be an ∞-category. Recall that a morphism $u : X \to Y$ in $\mathcal{C}$ is an *isomorphism* if the homotopy class $[u]$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$ (Definition 1.4.6.1). Our goal in this section is to study the notion of isomorphism in more detail.

Our first goal is to show that the class of isomorphisms can be characterized by a lifting property. Let $u : X \to Y$ be an isomorphism in an ∞-category $\mathcal{C}$, and let $f : X \to Z$ be any other morphism in $\mathcal{C}$. Then the composition $[f] \circ [u]^{-1} \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Z)$ can be written as the homotopy class of some morphism $g : Y \to Z$ in $\mathcal{C}$. The equality of homotopy classes $[f] = [g] \circ [u]$ is witnessed by some 2-simplex σ which we depict as a diagram

$$\begin{tikzcd}
& Y \arrow[dr, dashed, "g"] & \\
X \arrow[ur, "u"] \arrow[rr, "f"] & & Z.
\end{tikzcd}$$

Phrased differently, u and f determine a morphism of simplicial sets $\sigma_0 : \Lambda^2_0 \to \mathcal{C}$, and the preceding argument shows that σ_0 can be extended to a 2-simplex of $\mathcal{C}$. In §4.4.2, we extend this argument to simplices of higher dimension. Suppose that we are given an integer $n \geq 2$ and a morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to \mathcal{C}$. If $0 < i < n$, then σ_0 can be extended to an n-simplex of $\mathcal{C}$ by virtue of our assumption that $\mathcal{C}$ is an ∞-category. In the extreme cases $i = 0$ and $i = n$, such an extension need not exist. However, we will show that it exists in the case $i = 0$ when σ_0 carries the initial edge $\mathrm{N}_\bullet(\{0 < 1\}) \subseteq \Lambda^n_i$ to an isomorphism in $\mathcal{C}$, or in the case $i = n$ when σ_0 carries the final edge $\mathrm{N}_\bullet(\{n-1 < n\}) \subseteq \Lambda^n_i$ to an isomorphism in $\mathcal{C}$ (Theorem 4.4.2.6).

Theorem 4.4.2.6 has a number of useful consequences. For example, it implies that an ∞-category $\mathcal{C}$ is a Kan complex if and only if *every* morphism of $\mathcal{C}$ is an isomorphism (Proposition 4.4.2.1). More generally, it implies that any ∞-category $\mathcal{C}$ contains a largest Kan complex, which we will denote by $\mathcal{C}^{\simeq}$ and refer to as the *core of* $\mathcal{C}$ (Construction 1.3.5.4). The construction $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$ supplies a link between the theory of ∞-categories and the classical homotopy theory of Kan complexes, which will play an important role throughout this book.

Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. Then, for every object $D \in \mathcal{D}$, the fiber $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ is an ∞-category (Remark 4.1.1.6). Beware that, in general, this construction behaves poorly with respect to isomorphisms. For example, if the fiber $\mathcal{C}_D$ is nonempty and $D' \in \mathcal{D}$ is an object which is isomorphic to D, then the fiber $\mathcal{C}_{D'}$ could be empty. One can rule out this sort of behavior by imposing an additional assumption on the functor F. In §4.4.1, we introduce the notion of an *isofibration* of ∞-categories (Definition 4.4.1.4). Roughly speaking, an isofibration between ∞-categories is an inner fibration which also satisfies a path lifting property for isomorphisms. This condition guarantees that passage to the fiber is a homotopy invariant operation. For example, if $F : \mathcal{C} \to \mathcal{D}$ is an isofibration of ∞-categories, then it restricts to a Kan fibration of cores $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ (Proposotion 4.4.3.7).

Let B be a simplicial set containing a simplicial subset A. Recall that, for every ∞-category $\mathcal{C}$, the restriction functor $\theta : \operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(A, \mathcal{C})$ is an inner fibration (Corollary 4.1.4.2). In §4.4.5, we prove that θ is an isofibration (Corollary 4.4.5.3; see Proposition 4.4.5.1 for a stronger relative statement). The proof is based on the following recognition principle, which we establish in §4.4.4: if $\mathcal{C}$ is an ∞-category and $u : F \to G$ is a morphism in an ∞-category of the form $\operatorname{Fun}(X, \mathcal{C})$, then u is an isomorphism in $\operatorname{Fun}(X, \mathcal{C})$ if and only if, for every vertex $x \in X$, the induced map $u_x : F(x) \to G(x)$ is an isomorphism in the ∞-category $\mathcal{C}$ (Theorem 4.4.4.4). In other words, if each u_x admits a homotopy inverse $v_x : G(x) \to F(x)$, then we can choose the morphisms $\{v_x\}_{x \in X}$ (and homotopies witnessing the identifications $v_x \circ u_x \simeq \mathrm{id}_{F(x)}$ and $u_x \circ v_x \simeq \mathrm{id}_{G(x)}$) to depend functorially on $x \in X$.

4.4.1 Isofibrations of ∞-Categories

Let us begin by reviewing a bit of classical category theory. `01ER`

Definition 4.4.1.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We say that F is an *isofibration* if it satisfies the following condition: `01EN`

$(*)$ For every object $C \in \mathcal{C}$ and every isomorphism $u : D \to F(C)$ in the category $\mathcal{D}$, there exists an isomorphism $\overline{u} : \overline{D} \to C$ in the category $\mathcal{C}$ satisfying $F(\overline{u}) = u$.

Example 4.4.1.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. If F is a fibration in groupoids (or an opfibration in groupoids), then F is an isofibration. For a more general statement, see Example 4.4.1.11. `01EP`

The notion of isofibration is self-dual:

Proposition 4.4.1.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then F is an isofibration if and only if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is an isofibration.* `01EQ`

Proof. Assume that F is an isofibration; we will show that F^{op} is also an isofibration (the reverse implication follows by the same argument). Fix an object $C \in \mathcal{C}$ and an isomorphism $u : F(C) \to D$ in the category $\mathcal{D}$. Since F is an isofibration, the inverse isomorphism $u^{-1} : D \to F(C)$ can be lifted to an isomorphism $v : \overline{D} \to C$ in the category $\mathcal{C}$. Then $v^{-1} : C \to \overline{D}$ satisfies $F(v^{-1}) = u$. □

We now introduce an ∞-categorical counterpart of Definition 4.4.1.1.

Definition 4.4.1.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We say that F is an *isofibration* if it is an inner fibration (Definition 4.1.1.1) which satisfies the following additional condition: `01ES`

$(*)$ For every object $C \in \mathcal{C}$ and every isomorphism $u : D \to F(C)$ in the category $\mathcal{D}$, there exists an isomorphism $\overline{u} : \overline{D} \to C$ in the category $\mathcal{C}$ satisfying $F(\overline{u}) = u$.

`01ET` **Example 4.4.1.5.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then F is an isofibration (in the sense of Definition 4.4.1.1) if and only if the induced map of simplicial sets $\mathrm{N}_\bullet(F) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})$ is an isofibration of ∞-categories. This follows from the observation that $\mathrm{N}_\bullet(F)$ is automatically an inner fibration (see Proposition 4.1.1.10).

`01GV` **Example 4.4.1.6.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}$ be an ordinary category. By virtue of Proposition 4.1.1.10, every functor $F : \mathcal{C} \to \mathrm{N}_\bullet(\mathcal{D})$ is automatically an inner fibration. If every isomorphism in $\mathcal{D}$ is an identity morphism, then F is also an isofibration. In particular, every functor of ∞-categories $\mathcal{C} \to \Delta^n$ is automatically an isofibration.

`01EU` **Proposition 4.4.1.7.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration between ∞-categories. Then F is an isofibration of ∞-categories (in the sense of Definition 4.4.1.4) if and only if the induced functor of homotopy categories $f : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an isofibration of ordinary categories (in the sense of Definition 4.4.1.1).*

Proof. Assume first that F is an isofibration and let $C \in \mathcal{C}$ be an object, and let $[u] : D \to F(C)$ be an isomorphism in the homotopy category $\mathrm{h}\mathcal{D}$, given by the homotopy class of some morphism $u : D \to F(C)$ in the ∞-category $\mathcal{D}$. Then u is an isomorphism, so our assumption that F is an isofibration guarantees that we can lift u to an isomorphism $\overline{u} : \overline{D} \to C$ in the ∞-category $\mathcal{C}$. The homotopy class $[\overline{u}]$ is then an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$ satisfying $f([\overline{u}]) = [u]$. Allowing C and $[u]$ to vary, we conclude that f is an isofibration of ordinary categories.

Now suppose that f is an isofibration, let $C \in \mathcal{C}$ be an object, and let $u : D \to F(C)$ be an isomorphism in the ∞-category $\mathcal{D}$. Then the homotopy class $[u] : D \to F(C)$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{D}$. Invoking our assumption that f is an isofibration, we conclude that there exists an isomorphism $[v] : \overline{D} \to C$ in the homotopy category $\mathrm{h}\mathcal{C}$ satisfying $f([v]) = [u]$. Then $[v]$ can be realized as the homotopy class of some morphism $v : \overline{D} \to C$ in the ∞-category $\mathcal{C}$, which is automatically an isomorphism. The equation $f([v]) = [u]$ guarantees that there exists a homotopy from $F(v)$ to u in the ∞-category $\mathcal{D}$, given by a 2-simplex σ :

$$\begin{tikzcd}
 & F(C) \arrow[dr, "\mathrm{id}_{F(C)}"] & \\
D \arrow[ur, "F(v)"] \arrow[rr, "u"] & & F(C).
\end{tikzcd}$$

Since F is an inner fibration, it is weakly right orthogonal to the inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$. We can therefore lift σ to a 2-simplex $\overline{\sigma}$:

$$\begin{tikzcd}
 & C \arrow[dr, "\mathrm{id}_C"] & \\
\overline{D} \arrow[ur, "v"] \arrow[rr, "\overline{u}"] & & C.
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Since v and id_C are isomorphisms, it follows that $\overline{u}$ is an isomorphism (Remark 1.4.6.3). Allowing C and u to vary, we conclude that F is an isofibration of ∞-categories. □

Corollary 4.4.1.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F is an isofibration if and only if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is an isofibration.* 01EV

Proof. Combine Proposition 4.4.1.7, Proposition 4.4.1.3, and Remark 4.1.1.3. □

Corollary 4.4.1.9. *Let $\mathcal{C}$ be an ∞-category. Then the tautological map $U : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ is an isofibration of ∞-categories.* 056Z

Proof. It follows from Proposition 4.1.1.10 that U is an inner fibration. Since U induces an isomorphism of homotopy categories, Proposition 4.4.1.7 guarantees that U is an isofibration. □

Remark 4.4.1.10. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be isofibrations of ∞-categories. Then the composition $G \circ F$ is also an isofibration of ∞-categories (for a more general statement, see Remark 4.5.5.13). 01GW

Example 4.4.1.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a right fibration between ∞-categories. Then F is an inner fibration (Remark 4.2.1.4), and any isomorphism $u : D \to F(C)$ can be lifted to a morphism $\overline{u} : \overline{D} \to C$ in $\mathcal{C}$, which is automatically an isomorphism by virtue of Proposition 4.4.2.11. It follows that F is an isofibration. Similarly, any left fibration of ∞-categories is an isofibration. For a more general statement, see Corollary 5.6.7.5. 01EW

Example 4.4.1.12 (Replete Subcategories)**.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a subcategory (Definition 4.1.2.2). The following conditions are equivalent: 01EX

(1) The inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an isofibration.

(2) If $u : X \to Y$ is an isomorphism in $\mathcal{C}$ and the object Y belongs to the subcategory $\mathcal{C}'$, then the isomorphism u also belongs to the subcategory $\mathcal{C}'$ (and, in particular, the object X also belongs to $\mathcal{C}'$).

(3) If $u : X \to Y$ is an isomorphism in $\mathcal{C}$ and the object X belongs to the subcategory $\mathcal{C}'$, then the isomorphism u also belongs to the subcategory $\mathcal{C}'$ (and, in particular, the object Y also belongs to $\mathcal{C}'$).

If these conditions are satisfied, then we say that the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is *replete*.

01GX **Exercise 4.4.1.13.** Let X be a Kan complex, and let $Y \subseteq X$ be a simplicial subset. Show that Y is a summand of X (Definition 1.2.1.1) if and only if it is a replete full subcategory of X.

01EY **Example 4.4.1.14.** Let $\mathcal{C}$ be an ∞-category, and let $\mathrm{Isom}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by the isomorphisms in $\mathcal{C}$. Then the subcategory $\mathrm{Isom}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ is replete. Unwinding the definitions, this amounts to the observation that for every commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "v"] & Y \arrow[d, "v'"] \\ X' \arrow[r, "u'"] & Y' \end{tikzcd}$$

in the ∞-category $\mathcal{C}$ where u, v, and v' are isomorphisms, the morphism u' is also an isomorphism. This follows immediately from the two-out-of-three property of Remark 1.4.6.3.

4.4.2 Isomorphisms and Lifting Properties

019C Recall that a morphism of simplicial sets $X \to S$ is a Kan fibration if and only if it is both a left fibration and a right fibration (Example 4.2.1.5). In the special case $S = \Delta^0$, either one of these conditions is individually sufficient.

019D **Proposition 4.4.2.1** (Joyal [32]). *Let X be a simplicial set. The following conditions are equivalent:*

(*a*) *The projection map $X \to \Delta^0$ is a Kan fibration.*

(*b*) *The simplicial set X is a Kan complex.*

(*c*) *The simplicial set X is an ∞-category and the homotopy category* $\mathrm{h}X$ *is a groupoid.*

(*d*) *The simplicial set X is an ∞-category and every morphism in X is an isomorphism.*

(*e*) *The projection map $X \to \Delta^0$ is a left fibration.*

(*f*) *The projection map $X \to \Delta^0$ is a right fibration.*

Corollary 4.4.2.2 (Duskin [17]). *Let $\mathcal{C}$ be a 2-category. Then $\mathcal{C}$ is a 2-groupoid (in the sense of Definition 2.2.8.24) if and only if the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is a Kan complex.* 02BV

Proof. The 2-category $\mathcal{C}$ is a 2-groupoid if and only if it is a $(2,1)$-category and the homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid (Remark 2.2.8.25). The first condition is equivalent to the requirement that $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.3.2.1). If this condition is satisfied, then Corollary 2.3.4.6 supplies an isomorphism $\mathrm{h}\mathcal{C} \simeq \mathrm{h}\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$. The desired equivalence now follows from Proposition 4.4.2.1. □

Corollary 4.4.2.3. *Let $q : X \to S$ be morphism of simplicial sets which is either a left or a right fibration. Then, for every vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a Kan complex.* 019E

Proof. Combine Proposition 4.4.2.1 with Remark 4.2.1.8. □

Corollary 4.4.2.4. *Suppose we are given a commutative diagram of simplicial sets* 01GY

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "q"] \\
B \arrow[r, "g"] \arrow[ur, dashed, "\overline{f}"] & S,
\end{tikzcd}$$

where i is a monomorphism. Then:

- *If q is either a left or right fibration, then the simplicial set $\mathrm{Fun}_{A/\,/S}(B,X)$ of Construction 3.1.3.7 is a Kan complex.*
- *If q is a left fibration and i is left anodyne, then the Kan complex $\mathrm{Fun}_{A/\,/S}(B,X)$ is contractible.*
- *If q is a right fibration and i is right anodyne, then the Kan complex $\mathrm{Fun}_{A/\,/S}(B,X)$ is contractible.*

Proof. Without loss of generality, we may assume that q is a left fibration. By virtue of Remark 3.1.3.11, the simplicial set $\mathrm{Fun}_{A/\,/S}(B,X)$ can be identified with a fiber of the restriction map

$$\theta : \mathrm{Fun}(B,X) \to \mathrm{Fun}(A,X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B,S).$$

Proposition 4.2.5.1 asserts that θ is a left fibration of simplicial sets, so its fibers are Kan complexes (Corollary 4.4.2.3). If i is left anodyne, then θ is a trivial Kan fibration (Proposition 4.2.5.4), so its fibers are contractible Kan complexes. □

01GZ **Corollary 4.4.2.5.** *Let $q : X \to S$ and $g : B \to S$ be morphisms of simplicial sets. If q is either a left fibration or a right fibration, then the simplicial set $\mathrm{Fun}_{/S}(B, X)$ is a Kan complex.*

Proof. Apply Corollary 4.4.2.4 in the special case $A = \emptyset$. □

Our proof of Proposition 4.4.2.1 is based on the following characterization of isomorphisms in an ∞-category $\mathcal{C}$:

019F **Theorem 4.4.2.6** (Joyal)**.** *Let $\mathcal{C}$ be an ∞-category and let $u : X \to Y$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:*

(1) The morphism u is an isomorphism.

(2) Let $n \geq 2$ and let $\sigma_0 : \Lambda^n_0 \to \mathcal{C}$ be a morphism of simplicial sets for which the initial edge

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{0 < 1\}) \hookrightarrow \Lambda^n_0 \xrightarrow{\sigma_0} \mathcal{C}$$

is equal to u. Then σ_0 can be extended to an n-simplex $\sigma : \Delta^n \to \mathcal{C}$.

(3) Let $n \geq 2$ and let $\sigma_0 : \Lambda^n_n \to \mathcal{C}$ be a morphism of simplicial sets for which the final edge

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{n-1 < n\}) \hookrightarrow \Lambda^n_n \xrightarrow{\sigma_0} \mathcal{C}$$

is equal to u. Then σ_0 can be extended to an n-simplex $\sigma : \Delta^n \to \mathcal{C}$.

Proof of Proposition 4.4.2.1 from Theorem 4.4.2.6. Let X be a simplicial set. By definition, the projection map $X \to \Delta^0$ is a left fibration if and only if, for every pair of integers $0 \leq i < n$, every morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to X$ can be extended to an n-simplex $\sigma : \Delta^n \to X$. This condition is automatically satisfied when $n = 1$ (we can identify σ_0 with a vertex $x \in X$, and take σ to be the degenerate edge id_x), and is satisfied for $0 < i < n$ if and only if X is an ∞-category. Assuming that X is an ∞-category, it is satisfied for $i = 0$ if and only if every morphism in X is an isomorphism (by virtue of Theorem 4.4.2.6). This proves the equivalence $(d) \Leftrightarrow (e)$, and the equivalence $(d) \Leftrightarrow (f)$ follows by applying the same reasoning to the opposite simplicial set X^{op}. In particular, (e) and (f) are equivalent to one another, and therefore equivalent to (a) (see Example 4.2.1.5). The equivalences $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$ are immediate from the definitions. □

The proof of Theorem 4.4.2.6 will require some preliminaries.

019G **Definition 4.4.2.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We will say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *conservative* if it satisfies the following condition:

- Let $u : X \to Y$ be a morphism in $\mathcal{C}$. If $F(u) : F(X) \to F(Y)$ is an isomorphism in the ∞-category $\mathcal{D}$, then u is an isomorphism.

Example 4.4.2.8. Let $\mathcal{C}$ be an ∞-category. Then the canonical map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ is conservative. 019H

Example 4.4.2.9. Let $\mathcal{D}$ be an ∞-category, and let $\mathcal{C} \subseteq \mathcal{D}$ be a replete subcategory (Example 4.4.1.12). Then the inclusion map $\mathcal{C} \hookrightarrow \mathcal{D}$ is conservative. That is, if $u : X \to Y$ is a morphism of $\mathcal{C}$ which is an isomorphism in $\mathcal{D}$, then u is an isomorphism in $\mathcal{C}$. To prove this, we observe that if $v : Y \to X$ is a homotopy inverse of u in the ∞-category $\mathcal{D}$, then the morphism v also belongs to $\mathcal{C}$ (by virtue of our assumption that $\mathcal{C}$ is a replete subcategory of $\mathcal{D}$) and is also a homotopy inverse to u in $\mathcal{C}$. 02QX

Remark 4.4.2.10. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors between ∞-categories, where G is conservative. Then F is conservative if and only if the composition $(G \circ F) : \mathcal{C} \to \mathcal{E}$ is conservative. 019J

Proposition 4.4.2.11. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. If F is a left or a right fibration, then F is conservative.* 019K

Proof. Without loss of generality, we may assume that F is a left fibration. Let $u : X \to Y$ be a morphism in $\mathcal{C}$, and suppose that $F(u)$ is an isomorphism in $\mathcal{D}$. Let $\overline{v} : F(Y) \to F(X)$ is a homotopy inverse to $F(u)$, so that there exists a 2-simplex $\overline{\sigma}$ of $\mathcal{D}$ as depicted in the following diagram:

$$\begin{tikzcd}
 & F(Y) \arrow[dr, "\overline{v}"] & \\
F(X) \arrow[ur, "F(u)"] \arrow[rr, "F(\mathrm{id}_X)"] & & F(X).
\end{tikzcd}$$

Invoking our assumption that F is a left fibration, we can lift $\overline{\sigma}$ to a diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "v"] & \\
X \arrow[ur, "u"] \arrow[rr, "\mathrm{id}_X"] & & X
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. This lift supplies a morphism $v : Y \to X$ and witnesses id_X as a composition of v with u, so that v is a left homotopy inverse to u. Moreover, the image $F(v) = \overline{v}$ is an isomorphism in $\mathcal{D}$. Repeating the preceding argument (with $u : X \to Y$ replaced by $v : Y \to X$), we deduce that there exists a morphism $w : X \to Y$ which is left homotopy inverse to v. It follows that u and w are homotopic, so that v is a homotopy inverse to u (Remark 1.4.6.6). In particular, u is an isomorphism. □

02BW **Corollary 4.4.2.12.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a conservative functor of ∞-categories and let $q : K \to \mathcal{C}$ be a diagram in $\mathcal{C}$. Then the induced functors*

$$F_{q/} : \mathcal{C}_{q/} \to \mathcal{D}_{(F \circ q)/} \qquad F_{/q} : \mathcal{C}_{/q} \to \mathcal{D}_{/(F \circ q)}$$

are also conservative.

Proof. We will show that the functor $F_{/q}$ is conservative; the conservativity of $F_{q/}$ follows by a similar argument. Let $\pi : \mathcal{C}_{/q} \to \mathcal{C}$ and $\pi' : \mathcal{D}_{/(F \circ q)} \to \mathcal{D}$ denote the projection maps. Then π and π' are right fibrations of ∞-categories (Proposition 4.3.6.1), and therefore conservative (Proposition 4.4.2.11). Since F is conservative, from Remark 4.4.2.10 that the functor $F \circ \pi = \pi' \circ F_{/q}$ is also conservative. Applying Remark 4.4.2.10 again, we conclude that $F_{/q}$ is conservative. □

01H0 **Proposition 4.4.2.13.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let u be an isomorphism in $\mathcal{C}$, let $n \geq 2$ be an integer, and suppose we are given a lifting problem*

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "q"] \\ \Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{D}. \end{tikzcd}$$

If the composite map

$$\Delta^1 \simeq \mathrm{N}_{\bullet}(\{n-1 < n\}) \hookrightarrow \Lambda^n_n \xrightarrow{\sigma_0} \mathcal{C}$$

is equal to u, then there exists an n-simplex $\sigma : \Delta^n \to \mathcal{C}$ rendering the diagram commutative.

Proof. Using Lemma 4.3.6.15, we can identify the horn Λ^n_n with the pushout

$$(\Delta^{n-2} \star \{1\}) \coprod_{(\partial \Delta^{n-2} \star \{1\})} (\partial \Delta^{n-2} \star \Delta^1) \subseteq \Delta^{n-2} \star \Delta^1 \simeq \Delta^n.$$

Set $f = \sigma_0|_{\Delta^{n-2}}$ and $f_0 = \sigma_0|_{\partial \Delta^{n-2}}$, and let $\mathcal{E}$ denote the fiber product $\mathcal{C}_{f_0/} \times_{\mathcal{D}_{(q \circ f_0)/}} \mathcal{D}_{(q \circ f)/}$. Note that there is an evident projection map $\theta : \mathcal{E} \to \mathcal{C}$, given by the composition

$$\mathcal{E} \xrightarrow{\theta'} \mathcal{C}_{f_0/} \xrightarrow{\theta''} \mathcal{C}.$$

The morphism θ'' is a left fibration (Proposition 4.3.6.1), and the morphism θ' is a pullback of the restriction map $\mathcal{D}_{(q \circ f)/} \to \mathcal{D}_{(q \circ f_0)/}$ and is therefore also a left fibration (Corollary 4.3.6.13). It follows that $\theta : \mathcal{E} \to \mathcal{C}$ is a left fibration (Remark 4.2.1.11), and in particular $\mathcal{E}$ is an ∞-category (Remark 4.1.1.9).

Note that the restriction of σ_0 to $\Delta^{n-2} \star \{1\}$ can be identified with an object Y of the coslice ∞-category $\mathcal{C}_{f/}$. Let

$$\rho : \mathcal{C}_{f/} \to \mathcal{C}_{f_0/} \times_{\mathcal{D}_{(q \circ f_0)/}} \mathcal{D}_{(q \circ f)/} = \mathcal{E}$$

be the left fibration of Proposition 4.3.6.8, and set $\overline{Y} = \rho(Y) \in \mathcal{E}$. Then the restriction $\sigma_0|_{\partial \Delta^{n-2} \star \Delta^1}$ and $\overline{\sigma}$ determine a morphism $\overline{v} : \overline{X} \to \overline{Y}$ in the ∞-category $\mathcal{E}$. Unwinding the definitions, we see that choosing an n-simplex $\sigma : \Delta^n \to \mathcal{C}$ satisfying the requirements of Proposition 4.4.2.13 is equivalent to choosing a morphism $v : X \to Y$ in $\mathcal{C}_{f/}$ satisfying $\rho(v) = \overline{v}$. Since ρ is a left fibration, it is an isofibration (Example 4.4.1.11). Consequently, to prove the existence of v, it will suffice to show that $\overline{v}$ is an isomorphism in the ∞-category $\mathcal{E}$. Since θ is a left fibration, this follows from our assumption that $u = \theta(\overline{v})$ is an isomorphism in the ∞-category $\mathcal{C}$ (Proposition 4.4.2.11). □

Proof of Theorem 4.4.2.6. The implication (1) ⇒ (3) is a special case of Proposition 4.4.2.13. We will complete the proof by showing that (3) ⇒ (1) (a similar argument shows that (1) and (2) are equivalent). Let $u : X \to Y$ be a morphism in an ∞-category $\mathcal{C}$, and consider the map $\sigma_0 : \Lambda^2_2 \to \mathcal{C}$ depicted in the diagram

$$\begin{tikzcd}
& X \arrow[dr, "u"] & \\
Y \arrow[ur, dashed, "v"] \arrow[rr, "\mathrm{id}_Y"] & & Y.
\end{tikzcd}$$

If u satisfies condition (3), then we can complete σ_0 to a 2-simplex σ of $\mathcal{C}$, which witnesses the morphism $v = d^2_2(\sigma)$ as a right homotopy inverse of u. The tuple $(\sigma, s^1_0(u), s^1_1(u), \bullet)$ then determines a morphism of simplicial sets $\tau_0 : \Lambda^3_3 \to \mathcal{C}$ (see Proposition 1.2.4.7). Invoking assumption (3) again, we can extend τ_0 to a 3-simplex τ of $\mathcal{C}$. The face $d^3_3(\tau)$ then witnesses that v is also a left homotopy inverse to u, so that u is an isomorphism as desired. □

We close this section by recording another useful consequence of Proposition 4.4.2.13:

Proposition 4.4.2.14. *Let $q : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:* 01NU

(1) *The morphism q is a trivial Kan fibration.*

(2) *The morphism q is a left fibration and, for every vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a contractible Kan complex.*

(3) *The morphism q is a right fibration and, for every vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a contractible Kan complex.*

We will deduce Proposition 4.4.2.14 from the following more precise assertion:

01A7 **Lemma 4.4.2.15.** *Let $q : X \to S$ be a left fibration of simplicial sets, let $s \in S$ be a vertex having the property that the Kan complex $X_s = \{s\} \times_S X$ is contractible, and let $\overline{\sigma} : \Delta^n \to S$ be an n-simplex of S satisfying $\overline{\sigma}(n) = s$. Then every lifting problem*

$$\begin{tikzcd}
\partial \Delta^n \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & S
\end{tikzcd}$$

admits a solution.

Proof. When $n = 0$, the desired result follows from the fact that the fiber X_s is nonempty. We may therefore assume without loss of generality that $n > 0$. Replacing q by the projection map $\Delta^n \times_S X \to \Delta^n$, we may further reduce to the special case where $S = \Delta^n$ and $\overline{\sigma}$ is the identity map. In this case, our assumption that q is a left fibration guarantees that X is an ∞-category (Remark 4.1.1.9).

Let $\overline{h} : \Delta^1 \times \Delta^n \to \Delta^n$ be the morphism given on vertices by $h(i, j) = \begin{cases} j & \text{if } i = 0 \\ n & \text{if } i = 1. \end{cases}$ Since the inclusion $\{0\} \times \partial \Delta^n \hookrightarrow \Delta^1 \times \partial \Delta^n$ is left anodyne (Proposition 4.2.5.3), our assumption that q is a left fibration guarantees the existence of a morphism $h' : \Delta^1 \times \partial \Delta^n \to X$ satisfying $h'|_{\{0\} \times \partial \Delta^n} = \sigma_0$ and $q \circ h' = \overline{h}|_{\Delta^1 \times \partial \Delta^n}$. We will complete the proof by showing that h' can be extended to a map $h : \Delta^1 \times \Delta^n \to X$ satisfying $q \circ h = \overline{h}$ (in this case, our original lifting problem admits the solution $\sigma = h|_{\{0\} \times \Delta^n}$).

Let $Y(0) \subset Y(1) \subset Y(2) \subset \cdots \subset Y(n+1) = \Delta^1 \times \Delta^n$ denote the filtration constructed in the proof of Lemma 3.1.2.12. Then $Y(0)$ can be described as the pushout

$$(\Delta^1 \times \partial \Delta^n) \coprod_{(\{1\} \times \partial \Delta^n)} (\{1\} \times \Delta^n).$$

Using our assumption that the fiber X_s is a contractible Kan complex, we see that h' can be extended to a morphism of simplicial sets $h_0 : Y(0) \to X$ satisfying $q \circ h_0 = \overline{h}|_{Y(0)}$. We claim that h_0 can be extended to a compatible sequence of maps $h_i : Y(i) \to X$ satisfying $q \circ h_i = \overline{h}|_{Y(i)}$. To prove this, we recall that each $Y(i+1)$ can be realized as a pushout of the horn inclusion $\Lambda^{n+1}_{i+1} \hookrightarrow \Delta^{n+1}$, so that the construction of h_{i+1} from h_i can be rephrased as a lifting problem

$$\begin{tikzcd}
\Lambda^{n+1}_{i+1} \arrow[r, "f_i"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^{n+1} \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

For $0 \leq i < n$, this lifting problem is automatically solvable by virtue of our assumption that q is a left fibration. In the case $i = n$, the edge

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{n, n+1\}) \hookrightarrow \Lambda^{n+1}_{n+1} \xrightarrow{f_i} X$$

is an edge of the Kan complex X_s, and is therefore an isomorphism in the ∞-category X (Proposition 1.4.6.10). In this case, the existence of the desired extension follows from Theorem 4.4.2.6. We complete the proof by taking $h = h_{n+1}$. □

Proof of Proposition 4.4.2.14. The implication (1) ⇒ (2) is immediate, and the converse follows from Lemma 4.4.2.15. The equivalence (1) ⇔ (3) follows by a similar argument. □

4.4.3 The Core of an ∞-Category

Let $\mathcal{C}$ be a category. Recall that the *core* of $\mathcal{C}$ is the subcategory $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ comprised of all objects of $\mathcal{C}$ and all isomorphisms between them (Construction 1.3.5.4). In this section, we generalize this construction to the setting of ∞-categories. 01CZ

Construction 4.4.3.1. Let $\mathcal{C}$ be an ∞-category. We let $\mathcal{C}^{\simeq}$ denote the simplicial subset of $\mathcal{C}$ comprised of those simplices $\sigma : \Delta^n \to \mathcal{C}$ which carry each edge of Δ^n to an isomorphism in $\mathcal{C}$. We will refer to $\mathcal{C}^{\simeq}$ as the *core of* $\mathcal{C}$. 01D0

Remark 4.4.3.2. Let $\mathcal{C}$ be an ∞-category, let $\mathrm{h}\mathcal{C}$ be its homotopy category, and let $\mathrm{h}\mathcal{C}^{\simeq}$ denote the core of $\mathrm{h}\mathcal{C}$. Then the core $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ fits into a pullback diagram of simplicial sets 01D1

$$\begin{tikzcd}
\mathcal{C}^{\simeq} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathrm{N}_\bullet(\mathrm{h}\mathcal{C}^{\simeq}) \arrow[r] & \mathrm{N}_\bullet(\mathrm{h}\mathcal{C}).
\end{tikzcd}$$

Example 4.4.3.3. Let $\mathcal{C}$ be an ordinary category, and let $\mathcal{C}^{\simeq}$ denote its core (in the sense of Construction 1.3.5.4). Then the core of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ (in the sense of Construction 4.4.3.1) can be identified with the nerve of $\mathcal{C}^{\simeq}$. That is, we have a canonical isomorphism $\mathrm{N}_\bullet(\mathcal{C})^{\simeq} \simeq \mathrm{N}_\bullet(\mathcal{C}^{\simeq})$. 01D2

02BX **Example 4.4.3.4.** Let $\mathcal{C}$ be a $(2,1)$-category, so that the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.3.2.1). Then the core $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})^{\simeq}$ can be identified with the Duskin nerve of the 2-groupoid $\mathcal{C}^{\simeq}$ (Construction 2.2.8.27). That is, we have a canonical isomorphism $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})^{\simeq} \simeq \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}^{\simeq})$.

01D3 **Remark 4.4.3.5** (Functoriality)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F carries the core $\mathcal{C}^{\simeq}$ into the core $\mathcal{D}^{\simeq}$ (see Remark 1.5.1.6), and therefore restricts to a morphism of simplicial sets $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$.

01D4 **Proposition 4.4.3.6.** *Let $\mathcal{C}$ be an ∞-category. Then the core $\mathcal{C}^{\simeq}$ is a replete subcategory of $\mathcal{C}$ (Example 4.4.1.12): that is, the inclusion $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ is an isofibration of ∞-categories*

Proof. Combining Example 4.1.2.4, Remark 4.1.2.6, and Remark 4.4.3.2, we deduce that the inclusion map $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ is an inner fibration; in particular, $\mathcal{C}^{\simeq}$ is an ∞-category. The repleteness is immediate from the definition (since $\mathcal{C}^{\simeq}$ contains every isomorphism in $\mathcal{C}$). ☐

01EZ **Proposition 4.4.3.7.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories. Then the induced map $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a Kan fibration.*

Proof. Fix integers $n > 0$ and $0 \leq i \leq n$; we wish to show that every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C}^{\simeq} \arrow[d, "F^{\simeq}"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{D}^{\simeq}
\end{tikzcd}$$

admits a solution. In the case $n = 1$, this follows either from our definition of isofibration (in the case $i = 1$) or from Corollary 4.4.1.8 (in the case $i = 0$). We may therefore assume that $n \geq 2$. We claim that σ_0 can be extended to an n-simplex $\sigma : \Delta^n \to \mathcal{C}$ satisfying $F(\sigma) = \overline{\sigma}$. If $0 < i < n$, this follows from the fact that F is an inner fibration. The extremal cases $i = 0$ and $i = n$ follow from Proposition 4.4.2.13 (applied to the inner fibration $F : \mathcal{C} \to \mathcal{D}$ and its opposite $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$). To complete the proof, it will suffice to show that σ carries each edge of Δ^n to an isomorphism in $\mathcal{C}$. For $n > 2$, this is automatic (since the horn Λ^n_i contains every edge of Δ^n). In the case $n = 2$ it follows from the two-out-of-three property for isomorphisms in $\mathcal{C}$ (Remark 1.4.6.3). ☐

01NV **Corollary 4.4.3.8.** *Let $q : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, where $\mathcal{D}$ is a Kan complex. The following conditions are equivalent:*

(1) *The morphism q is a Kan fibration.*

(2) *The morphism q is a left fibration.*

(3) *The morphism q is a right fibration.*

(4) *The morphism q is a conservative isofibration of ∞-categories.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) follow from Example 4.2.1.5, Proposition 4.4.2.11, and Example 4.4.1.11 (and do not require the assumption that $\mathcal{D}$ is a Kan complex). We will complete the proof by showing that (4) $\Rightarrow$ (1). Our assumption that $\mathcal{D}$ is a Kan complex guarantees that every morphism in $\mathcal{D}$ is an isomorphism. Since q is conservative, it follows that every morphism in $\mathcal{C}$ is an isomorphism. We can therefore identify q with the induced map $q^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$, which is Kan fibration by virtue of Proposition 4.4.3.7. ☐

Corollary 4.4.3.9. *Let $q : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, where $\mathcal{D}$ is a Kan complex. The following conditions are equivalent:* 0237

(1) *The morphism q is a covering map.*

(2) *The morphism q is a left covering map.*

(3) *The morphism q is a right covering map.*

Proof. Combine Corollaries 4.4.3.8 and 4.2.3.20. ☐

Corollary 4.4.3.10. *Let $F : X \to Y$ be an isofibration between Kan complexes. Then F is a Kan fibration.* 01NW

Corollary 4.4.3.11. *Let $\mathcal{C}$ be an ∞-category. Then the core $\mathcal{C}^{\simeq}$ is a Kan complex.* 01H1

Proof. Apply Proposition 4.4.3.7 to the isofibration $\mathcal{C} \to \Delta^0$. ☐

Exercise 4.4.3.12. Deduce Corollary 4.4.3.11 directly from the criterion of Proposition 4.4.2.1. 01H2

Corollary 4.4.3.13. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a replete subcategory. Then the Kan complex $\mathcal{C}_0^{\simeq}$ is a summand of the Kan complex $\mathcal{C}^{\simeq}$.* 04Q5

Proof. The assumption that $\mathcal{C}_0$ is replete guarantees that the inclusion map $\iota : \mathcal{C}_0 \subseteq \mathcal{C}$ is an isofibration (Example 4.4.1.12). Applying Proposition 4.4.3.7, we deduce that the inclusion $\mathcal{C}_0^{\simeq} \hookrightarrow \mathcal{C}^{\simeq}$ is a Kan fibration, so that $\mathcal{C}_0^{\simeq}$ is a summand of $\mathcal{C}^{\simeq}$ by virtue of Example 3.1.1.4. ☐

Corollary 4.4.3.14. *Let $\mathcal{C}$ be an ∞-category and let $u : X \to Y$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:* 01D6

(1) *The morphism u is an isomorphism.*

(2) *There exists a Kan complex $\mathcal{E}$, a morphism $\overline{u} : \overline{X} \to \overline{Y}$ in $\mathcal{E}$, and a functor $F : \mathcal{E} \to \mathcal{C}$ satisfying $F(\overline{u}) = u$.*

(3) *There exists a contractible Kan complex $\mathcal{E}$, a morphism $\overline{u} : \overline{X} \to \overline{Y}$ in $\mathcal{E}$, and a functor $F : \mathcal{E} \to \mathcal{C}$ satisfying $F(\overline{u}) = u$.*

Proof. If u is an isomorphism, then it belongs to the image of the inclusion functor $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$. Since the core $\mathcal{C}^{\simeq}$ is a Kan complex, this proves that (1) ⇒ (2). Conversely, if we can write $u = F(\overline{u})$ for some functor $F : \mathcal{E} \to \mathcal{C}$ where $\mathcal{E}$ is a Kan complex, then Remark 1.5.1.6 guarantees that u is an isomorphism in $\mathcal{C}$ (since $\overline{u}$ is automatically an isomorphism in $\mathcal{E}$, by virtue of Proposition 1.4.6.10). This proves that (2) ⇒ (1).

The implication (3) ⇒ (2) is immediate. We will complete the proof by showing that (2) implies (3). Let $\mathcal{E}$ be a Kan complex, let $F : \mathcal{E} \to \mathcal{C}$ be a functor, and let $\overline{u}$ be an edge of $\mathcal{E}$ satisfying $F(\overline{u}) = u$. Let us identify $\overline{u}$ with a morphism of simplicial sets $\Delta^1 \to \mathcal{E}$. By virtue of Proposition 3.1.7.1, this morphism factors as a composition $\Delta^1 \xrightarrow{v} \mathcal{E}' \xrightarrow{q} \mathcal{E}$, where v is anodyne and q is a Kan fibration. Since $\mathcal{E}$ is a Kan complex and q is a Kan fibration, the simplicial set $\mathcal{E}'$ is a Kan complex (Remark 3.1.1.11). Because Δ^1 is weakly contractible and v is a weak homotopy equivalence, the Kan complex $\mathcal{E}'$ is contractible. We can then write $u = F'(v)$ where $F' = F \circ q$. □

01H3 **Corollary 4.4.3.15.** *Let $\mathcal{C}$ be an ∞-category containing objects X and Y. The following conditions are equivalent:*

(1) *The objects X and Y are isomorphic.*

(2) *There exists a connected Kan complex $\mathcal{E}$, a pair of vertices $\overline{X}, \overline{Y} \in \mathcal{E}$, and a morphism $f : \mathcal{E} \to \mathcal{C}$ satisfying $f(\overline{X}) = X$ and $f(\overline{Y}) = Y$.*

(3) *There exists a contractible Kan complex $\mathcal{E}$, a pair of vertices $\overline{X}, \overline{Y} \in \mathcal{E}$, and a morphism $f : \mathcal{E} \to \mathcal{C}$ satisfying $f(\overline{X}) = X$ and $f(\overline{Y}) = Y$.*

01D7 **Notation 4.4.3.16.** Let $\mathcal{C}$ be an ∞-category. We let $\pi_0(\mathcal{C}^{\simeq})$ denote the set of connected components of the Kan complex $\mathcal{C}^{\simeq}$. Note that $\pi_0(\mathcal{C}^{\simeq})$ can be identified with the set of *isomorphism classes* of objects of $\mathcal{C}$ (that is, the quotient of the set of objects of $\mathcal{C}$ by the equivalence relation of isomorphism).

If $\mathcal{C}$ is an ∞-category, then the Kan complex $\mathcal{C}^{\simeq}$ can be characterized by a universal property:

01D8 **Proposition 4.4.3.17.** *Let $\mathcal{C}$ be an ∞-category and let X be a Kan complex. Then composition with the inclusion $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ induces a bijection* $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(X, \mathcal{C}^{\simeq}) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(X, \mathcal{C})$.

Proof. Let $F : X \to \mathcal{C}$ be a morphism of simplicial sets. To show that F factors through the core $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$, we must show that for every edge $u : x \to y$ of the Kan complex X, the image $F(u)$ is an isomorphism in $\mathcal{C}$. This follows from Remark 1.5.1.6, since u is automatically an isomorphism in the ∞-category X (Proposition 1.4.6.10). □

Corollary 4.4.3.18. *Let $\mathcal{C}$ be an ∞-category. Then the core $\mathcal{C}^{\simeq}$ is the largest Kan complex which is contained in $\mathcal{C}$.* `01D9`

Proof. Combine Corollary 4.4.3.11 with Proposition 4.4.3.17. □

Corollary 4.4.3.19 (Pullbacks of Isofibrations)**.** *Suppose we are given a pullback diagram of simplicial sets* `01H4`

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "F"] \arrow[d, "q'"] & \mathcal{C} \arrow[d, "q"] \\
\mathcal{D}' \arrow[r, "F'"] & \mathcal{D},
\end{tikzcd}$$

where q is an isofibration of ∞-categories and $\mathcal{D}'$ is an ∞-category. Then:

(1) *The simplicial set $\mathcal{C}'$ is an ∞-category.*

(2) *The diagram of Kan complexes* `01H5`

$$\begin{tikzcd}
\mathcal{C}'^{\simeq} \arrow[r] \arrow[d, "q'^{\simeq}"] & \mathcal{C}^{\simeq} \arrow[d, "q^{\simeq}"] \\
\mathcal{D}'^{\simeq} \arrow[r] & \mathcal{D}^{\simeq}
\end{tikzcd} \tag{4.3}$$

is a pullback square and a homotopy pullback square.

(3) *A morphism $u : X \to Y$ in the ∞-category $\mathcal{C}'$ is an isomorphism if and only if $F(u)$ is an isomorphism in the ∞-category $\mathcal{C}$ and $q'(u)$ is an isomorphism in the ∞-category $\mathcal{D}'$.*

(4) *The morphism q' is an isofibration of ∞-categories.*

Proof. Since q is an isofibration, it is an inner fibration. It follows that the morphism q' is also an inner fibration (Remark 4.1.1.5). Since $\mathcal{D}'$ is an ∞-category, the simplicial set $\mathcal{C}'$ is also an ∞-category (Remark 4.1.1.9). This proves (1).

Let $\mathcal{E}$ denote the fiber product $\mathcal{C}^{\simeq} \times_{\mathcal{D}^{\simeq}} \mathcal{D}'^{\simeq}$, which we regard as a simplicial subset of $\mathcal{C}' = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$. It follows from Proposition 4.4.3.7 that q restricts to a Kan fibration

$q^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$. The projection map $\mathcal{E} \to \mathcal{D}'^{\simeq}$ is a pullback of $q^{\simeq}$, and is therefore also a Kan fibration. Since $\mathcal{D}'^{\simeq}$ is a Kan complex (Corollary 4.4.3.11), it follows that $\mathcal{E}$ is a Kan complex (Remark 3.1.1.11). Applying Corollary 4.4.3.18, we deduce that $\mathcal{E}$ is contained in the core $\mathcal{C}'^{\simeq} \subseteq \mathcal{C}'$, which proves that the diagram (4.3) is a pullback square. Since $q^{\simeq}$ is a Kan fibration, it is also a homotopy pullback square (Example 3.4.1.3). This proves assertion (2), and assertion (3) is an immediate consequence.

To complete the proof of (4), it will suffice to show that the morphism q' satisfies condition $(*)$ of Definition 4.4.1.4. Let Y' be an object of $\mathcal{C}'$ and let $\overline{u}' : \overline{X}' \to q'(Y')$ be an isomorphism in the ∞-category $\mathcal{D}'$; we wish to show that $\overline{u'}$ can be written as $q'(u')$ for some isomorphism $u' : X' \to Y'$ in the ∞-category $\mathcal{C}'$. By virtue of (3), this is equivalent to showing that $F'(\overline{u}')$ can be written as $q(u)$ for some isomorphism $u : X \to F(Y)$ in the ∞-category $\mathcal{C}$, which follows from our assumption that q is an isofibration. □

01H6 **Corollary 4.4.3.20.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories, and let $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ be the fiber of q over an object $D \in \mathcal{D}$. Then the canonical map $(\mathcal{C}_D)^{\simeq} \to \{D\} \times_{\mathcal{D}^{\simeq}} \mathcal{C}^{\simeq}$ is an isomorphism. In other words, the inclusion functor $\mathcal{C}_D \hookrightarrow \mathcal{C}$ is conservative.*

Proof. Apply Corollary 4.4.3.19 in the special case $\mathcal{D}' = \{D\}$. □

04HW **Corollary 4.4.3.21.** *Let $q : \mathcal{C} \to \mathcal{D}$ be a conservative isofibration of ∞-categories. Then, for each object $D \in \mathcal{D}$, the fiber $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ is a Kan complex.*

Proof. Since q is an inner fibration, the simplicial set $\mathcal{C}_D$ is an ∞-category (Remark 4.1.1.6). It will therefore suffice to show that every morphism f in $\mathcal{C}_D$ is an isomorphism (Proposition 4.4.2.1). By virtue of Corollary 4.4.3.20, this is equivalent to the requirement that f is an isomorphism in the ∞-category $\mathcal{C}$. This follows from our assumption that q is conservative, since $q(f) = \mathrm{id}_D$ is an isomorphism in the ∞-category $\mathcal{D}$. □

We close this section by establishing a relative version of Proposition 4.4.3.17. Let $\mathcal{C}$ be an ∞-category, and let X be an arbitrary simplicial set. Then the simplicial set $\mathrm{Fun}(X, \mathcal{C})$ is an ∞-category (Theorem 1.5.3.7), and the simplicial set $\mathrm{Fun}(X, \mathcal{C}^{\simeq})$ is a Kan complex (Corollary 3.1.3.4). The inclusion $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ induces a monomorphism of simplicial sets $\mathrm{Fun}(X, \mathcal{C}^{\simeq}) \hookrightarrow \mathrm{Fun}(X, \mathcal{C})$, which automatically factors through the core $\mathrm{Fun}(X, \mathcal{C})^{\simeq}$ (Corollary 4.4.3.18).

01DA **Proposition 4.4.3.22.** *Let $\mathcal{C}$ be an ∞-category and let X be a Kan complex. Then the canonical map*

$$\theta : \mathrm{Fun}(X, \mathcal{C}^{\simeq}) \hookrightarrow \mathrm{Fun}(X, \mathcal{C})^{\simeq}$$

is an isomorphism of simplicial sets.

Remark 4.4.3.23. Proposition 4.4.3.17 can be regarded as a special case of Proposition 4.4.3.22: it is equivalent to the assertion that, for every ∞-category $\mathcal{C}$ and every Kan complex X, the canonical map $\mathrm{Fun}(X, \mathcal{C}^{\simeq}) \hookrightarrow \mathrm{Fun}(X, \mathcal{C})^{\simeq}$ is bijective on vertices. `01DB`

Warning 4.4.3.24. The conclusion of Proposition 4.4.3.22 generally does not hold if X is not a Kan complex. `01DC`

Proof of Proposition 4.4.3.22. Let $\sigma : Y \to \mathrm{Fun}(X, \mathcal{C})^{\simeq}$ be a morphism of simplicial sets, which we identify with a diagram $F : X \times Y \to \mathcal{C}$. To show that σ factors through the monomorphism θ, it will suffice to show that F factors through the core $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$. Equivalently, we wish to show that for every edge $(u, v) : (x, y) \to (x', y')$ in the product simplicial set $X \times Y$, the morphism $F(u, v) : F(x, y) \to F(x', y')$ is an isomorphism in the ∞-category $\mathcal{C}$. Note that $F(u, v)$ can be identified with a composition of morphisms

$$F(x, y) \xrightarrow{F(u, \mathrm{id}_y)} F(x', y) \xrightarrow{F(\mathrm{id}_{x'}, v)} F(x', y')$$

in the ∞-category $\mathcal{C}$. Since the collection of isomorphisms in $\mathcal{C}$ is closed under composition (Remark 1.4.6.3), it will suffice to show that $F(u, \mathrm{id}_y)$ and $F(\mathrm{id}_{x'}, v)$ are isomorphsms in $\mathcal{C}$. In the first case, this follows from Proposition 4.4.3.17 (applied to the morphism $F|_{X \times \{y\}}$), since X is a Kan complex. In the second case, it follows from our assumption that σ factors through the core $\mathrm{Fun}(X, \mathcal{C})^{\simeq} \subseteq \mathrm{Fun}(X, \mathcal{C})$ (and therefore carries the edge $v : y \to y'$ to an isomorphism in the diagram ∞-category $\mathrm{Fun}(X, \mathcal{C})$). □

4.4.4 Natural Isomorphisms

Recall that, if X is an arbitrary simplicial set and $\mathcal{C}$ is an ∞-category, then the simplicial set $\mathrm{Fun}(X, \mathcal{C})$ is also an ∞-category (Theorem 1.5.3.7). In this section, we study isomorphisms in ∞-categories of the form $\mathrm{Fun}(X, \mathcal{C})$. `01DF`

Definition 4.4.4.1. Let $\mathcal{C}$ be an ∞-category, let X be a simplicial set, and suppose we are given a pair of diagrams $f, f' : X \to \mathcal{C}$. A *natural transformation from f to f'* is a morphism $u : f \to f'$ in the ∞-category $\mathrm{Fun}(X, \mathcal{C})$. A *natural isomorphism from f to f'* is a natural transformation $u : f \to f'$ which is an isomorphism in the ∞-category $\mathrm{Fun}(X, \mathcal{C})$ (Definition 1.4.6.1). We say that f and f' are *naturally isomorphic* if there exists a natural isomorphism from f to f'. `01DG`

Remark 4.4.4.2. In the situation of Definition 4.4.4.1, a natural transformation from f to f' is simply a *homotopy* from f to f', in the sense of Definition 3.1.5.2: that is, a map of simplicial sets $h : \Delta^1 \times X \to \mathcal{C}$ satisfying $h|_{\{0\} \times X} = f$ and $h|_{\{1\} \times X} = f'$. However, the terminology of Definition 4.4.4.1 is intended to signal a shift in emphasis. We will generally reserve use of the term *homotopy* between diagrams $f, f' : X \to \mathcal{C}$ for the case where $\mathcal{C}$ is a Kan complex, and use the term *natural transformation* when $\mathcal{C}$ is a more general ∞-category. `01DH`

01DJ **Example 4.4.4.3.** Let $\mathcal{C}$ be an ordinary category, and suppose we are given a pair of diagrams $f, f' : X \to \mathrm{N}_\bullet(\mathcal{C})$. Then a natural transformation from f to f' can be identified with a collection of morphisms $\{u_x : f(x) \to f'(x)\}_{x \in X}$ with the following property: for every edge $e : x \to y$ of the simplicial set X, the diagram

$$\begin{tikzcd}
f(x) \arrow[r, "u_x"] \arrow[d, "f(e)"] & f'(x) \arrow[d, "f'(e)"] \\
f(y) \arrow[r, "u_y"] & f'(y)
\end{tikzcd}$$

commutes (in the category $\mathcal{C}$).

In particular, if $\mathcal{C}$ and $\mathcal{D}$ are ordinary categories and we are given a pair of functors $f, f' : \mathcal{D} \to \mathcal{C}$, then giving a natural transformation from f to f' (in the sense of classical category theory) is equivalent to giving a natural transformation from $\mathrm{N}_\bullet(f) : \mathrm{N}_\bullet(\mathcal{D}) \to \mathrm{N}_\bullet(\mathcal{C})$ to $\mathrm{N}_\bullet(f') : \mathrm{N}_\bullet(\mathcal{D}) \to \mathrm{N}_\bullet(\mathcal{C})$.

Let $\mathcal{C}$ be an ∞-category and let X be an arbitrary simplicial set. For every vertex $x \in X$, evaluation at x determines a functor

$$\mathrm{ev}_x : \mathrm{Fun}(X, \mathcal{C}) \to \mathrm{Fun}(\{x\}, \mathcal{C}) \simeq \mathcal{C}\,.$$

In particular, if $u : f \to f'$ is an isomorphism in the ∞-category $\mathrm{Fun}(X, \mathcal{C})$, then $\mathrm{ev}_x(u) : f(x) \to f'(x)$ is an isomorphism in the ∞-category $\mathcal{C}$. Our goal in this section is to prove the converse:

01DK **Theorem 4.4.4.4.** *Let $\mathcal{C}$ be an ∞-category, let $f, f' : X \to \mathcal{C}$ be diagrams in $\mathcal{C}$ indexed by a simplicial set X, and let $u : f \to f'$ be a natural transformation. Then u is a natural isomorphism if and only if, for every vertex $x \in X$, the induced map $\mathrm{ev}_x(u) : f(x) \to f'(x)$ is an isomorphism in the ∞-category $\mathcal{C}$.*

01DL **Remark 4.4.4.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and suppose we are given a pair of functors $F, G : \mathcal{C} \to \mathcal{D}$, which restrict to functors between their cores $F^{\simeq}, G^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ (see Remark 4.4.3.5). Let u be a natural transformation from F to G, which we identify with a map of simplicial sets $u : \Delta^1 \times \mathcal{C} \to \mathcal{D}$. If u is a natural isomorphism, then it restricts to a map of simplicial sets $u_0 : \Delta^1 \times \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$, which we can regard as a homotopy from $F^{\simeq}$ to $G^{\simeq}$. In particular, if the functors F and G are naturally isomorphic, then the morphisms $F^{\simeq}$ and $G^{\simeq}$ are homotopic.

01DM **Corollary 4.4.4.6.** *Let $\mathcal{C}$ be an ∞-category. Then the functor*

$$(\mathrm{Set}_\Delta)^{\mathrm{op}} \to \mathrm{Set}_\Delta \qquad X \mapsto \mathrm{Fun}(X, \mathcal{C})^{\simeq}$$

preserves limits (that is, it carries colimits in the category of simplicial sets to limits of Kan complexes).

The proof of Theorem 4.4.4.4 will use the following combinatorial assertion:

Lemma 4.4.4.7. *Let $m \geq 0$ and $n \geq 2$ be integers. Then there exists a sequence of simplicial subsets* 01DN

$$X(0) \subset X(1) \subset X(2) \subset \cdots \subset X(t) = \Delta^m \times \Delta^n$$

with the following properties:

(1) *The simplicial subset $X(0) \subseteq \Delta^m \times \Delta^n$ is the union of $\Delta^m \times \Lambda^n_0$ and $\partial \Delta^m \times \Delta^n$.*

(2) *For each $0 < s \leq t$, there exist integers $q \geq 2$ and $0 \leq p < q$ and a pushout diagram of simplicial sets*

$$\begin{tikzcd}
\Lambda^q_p \arrow[r] \arrow[d] & X(s-1) \arrow[d] \\
\Delta^q \arrow[r, "\sigma"] & X(s).
\end{tikzcd}$$

Moreover, if $p = 0$, then the map $\sigma : \Delta^q \to X(s) \subseteq \Delta^m \times \Delta^n$ satisfies $\sigma(0) = (0,0)$ and $\sigma(1) = (0,1)$.

Proof. Let σ be a nondegenerate q-simplex of the product $\Delta^m \times \Delta^n$, given by a chain

$$(i_0, j_0) < (i_1, j_1) < \cdots < (i_q, j_q).$$

We will say that σ is *free* if the composite maps

$$\Delta^q \xrightarrow{\sigma} \Delta^m \times \Delta^n \twoheadrightarrow \Delta^m \qquad \Delta^q \xrightarrow{\sigma} \Delta^m \times \Delta^n \twoheadrightarrow \Delta^n$$

are surjective and there exists an integer $0 \leq p < q$ such that $(i_p, j_p) = (p, 0)$ and $(i_{p+1}, j_{p+1}) = (p, 1)$. If this condition is satisfied, then the integer p is uniquely determined; we will refer to p as the *index* of σ and denote it by $p(\sigma)$. We also denote the dimension q of σ by $q(\sigma)$.

Let $\{\sigma_1, \sigma_2, \cdots, \sigma_t\}$ be an enumeration of the collection of all free simplices of the product $\Delta^m \times \Delta^n$. Without loss of generality, we may assume that that this enumeration satisfies the following pair of conditions:

- For $1 \leq s \leq s' \leq t$, we have $q(\sigma_s) \leq q(\sigma_{s'})$.
- If $1 \leq s \leq s' \leq t$ are integers satisfying $q(\sigma_s) = q(\sigma_{s'})$, then $p(\sigma_s) \geq p(\sigma_{s'})$.

Let $X(0)$ denote the union $(\Delta^m \times \Lambda^n_0) \cup (\partial \Delta^m \times \Delta^n) \subseteq \Delta^m \times \Delta^n$. For $0 < s \leq t$, we let $X(s)$ denote the smallest simplicial subset of $\Delta^m \times \Delta^n$ which contains $X(0)$ together with the simplices $\{\sigma_1, \sigma_2, \ldots, \sigma_s\}$. We will show that the sequence

$$X(0) \subset X(1) \subset \cdots \subset X(t)$$

satisfies the requirements of Lemma 4.4.4.7.

We first claim that $X(t) = \Delta^m \times \Delta^n$. Let σ be an arbitrary nondegenerate q-simplex of $\Delta^m \times \Delta^n$, which we will identify with a sequence

$$(i_0, j_0) < (i_1, j_1) < \cdots < (i_q, j_q)$$

of elements of the partially ordered set $[m] \times [n]$. We wish to show that σ is contained in $X(t)$. Without loss of generality, we may assume that the sequence $(i_0, i_1, \ldots, i_q)$ contains every element of the set $[m] = \{0 < 1 < \cdots < m\}$. (otherwise, σ is contained in the simplicial subset $\partial \Delta^m \times \Delta^n \subseteq X(0) \subseteq X(t)$). Similarly, we may assume that that the sequence $(j_0, j_1, \ldots, j_q)$ contains every element of the set $\{1 < 2 < \cdots < n\}$ (otherwise, σ is is contained in the simplicial subset $\Delta^m \times \Lambda^n_0 \subseteq X(0) \subseteq X(t)$). In particular, the sequence σ contains $(p, 1)$, for some integer $0 \leq p \leq n$. Let us assume that p is chosen as small as possible. In this case, there are two possibilities:

- The sequence σ also contains the pair $(p, 0)$. In this case, σ is a free simplex of $\Delta^m \times \Delta^n$, and therefore belongs to $X(t)$.
- The sequence σ does not contain $(p, 0)$, and therefore has the form

  $$(0,0) < (1,0) < \cdots < (p-1, 0) < (p, 1) < (i_{p+1}, j_{p+1}) < \cdots < (i_q, j_q).$$

  We can then identify σ with the pth face of the $(q+1)$-simplex σ' given by the sequence

  $$(0,0) < (1,0) < \cdots < (p-1, 0) < (p, 0) < (p, 1) < (i_{p+1}, j_{p+1}) < \cdots < (i_q, j_q).$$

  The simplex σ' is free and therefore belongs to $X(t)$, so that σ belongs to $X(t)$ as well.

We now complete the proof by verifying requirement (2) of Lemma 4.4.4.7. Fix an integer $0 < s \leq t$ and let $\sigma = \sigma_s$ be the corresponding free simplex of $\Delta^m \times \Delta^n$. Let $q = q(\sigma)$ be the dimension of σ and let $p = p(\sigma)$ be the index of σ, so that $0 \leq p < q$ and σ has the form

$$(0,0) < (1,0) < \cdots < (p, 0) < (p, 1) < (i_{p+2}, j_{p+2}) < \cdots < (i_q, j_q).$$

By construction, the simplicial subset $X(s) \subseteq \Delta^m \times \Delta^n$ is the union of $X(s-1)$ with the image of σ. Let $K \subseteq \Delta^1$ denote the inverse image $\sigma^{-1} X(s-1)$. We will show that K is equal to the horn $\Lambda^q_p \subseteq \Delta^q$, so that the pullback diagram of simplicial sets

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & X(s-1) \arrow[d] \\
\Delta^q \arrow[r, "\sigma"] & X(s)
\end{tikzcd}$$

is also a pushout square (Lemma 3.1.2.11).

We first show that the horn Λ^q_p is contained in K. For this, it will suffice to show that for every integer $0 \leq p' \leq q$ satisfying $p' \neq p$, the face $\tau = d^q_{p'}(\sigma)$ is contained in $X(s-1)$. We consider three cases:

- For $p' < p$, the simplex τ is given by the sequence

  $$(0,0) < \cdots < (p'-1,0) < (p'+1,0) < \cdots < (p,0) < (p,1) < \cdots < (i_q, j_q),$$

  which is contained in the simplicial subset $\partial \Delta^m \times \Delta^n \subseteq X(0) \subseteq X(s-1)$.

- For $p' = p+1$, the simplex τ is given by the sequence

  $$(0,0) < (1,0) < \cdots < (p,0) < (i_{p+2}, j_{p+2}) < \cdots < (i_q, j_q).$$

  If $j_{p+2} \geq 2$, then τ belongs to the simplicial subset $\Delta^m \times \Lambda^n_0 \subseteq X(0) \subseteq X(s-1)$. Otherwise, we must have $(i_{p+2}, j_{p+2}) = (p+1, 1)$, so that τ occurs as a face of the free simplex σ' given by the sequence

  $$(0,0) < (1,0) < \cdots < (p,0) < (p+1,0) < (p+1,1) < \cdots < (i_q, j_q),$$

  which has dimension q and index $p+1$. By construction, σ' belongs to the set $\{\sigma_1, \sigma_2, \ldots, \sigma_{s-1}\}$, and is therefore contained in the simplicial subset $X(s-1) \subseteq \Delta^m \times \Delta^n$.

- For $p' > p+1$, the simplex τ is given by the sequence

  $$(0,0) < \cdots < (p,0) < (p,1) < \cdots < (i_{p'-1}, j_{p'-1}) < (i_{p'+1}, j_{p'+1}) < \cdots < (i_q, j_q).$$

  It follows that τ is either contained in the simplicial subset $X(0) = (\Delta^m \times \Lambda^n_0) \cup (\partial \Delta^m \times \Delta^n)$ or that it is a free simplex of $\Delta^m \times \Delta^n$ having dimension $q-1$. In the latter case, τ must belong to the set $\{\sigma_1, \ldots, \sigma_{s-1}\}$, and is therefore contained in the simplicial subset $X(s-1) \subseteq \Delta^m \times \Delta^n$.

To show that the inclusion $\Lambda^q_p \subseteq K$ is an equality, it will suffice to show that K does not contain the pth face of Δ^q. Let $\tau = d^q_p(\sigma)$ be the pth face of σ, given by the sequence

$$(0,0) < (1,0) < \cdots < (p-1,0) < (p,1) < (i_{p+1}, j_{p+1}) < \cdots < (i_q, j_q).$$

We wish to show that τ is not contained in $X(s-1)$. Assume otherwise. Since τ is not contained in $X(0)$, we conclude that τ is contained in some free simplex $\sigma' \in \{\sigma_1, \sigma_2, \ldots, \sigma_{s-1}\}$. Note that $\tau \neq \sigma'$ (since τ is not free), so we have inequalities

$$q - 1 = q(\tau) < q(\sigma') \leq q(\sigma) = q.$$

It follows that σ' is a free q-simplex of $\Delta^m \times \Delta^n$ which contains τ and is not equal to σ, and is therefore necessarily given by the sequence

$$(0,0) < (1,0) < \cdots < (p-1,0) < (p-1,1) < (p,1) < (i_{p+1}, j_{p+1}) < \cdots < (i_q, j_q).$$

We therefore have $p(\sigma') = p - 1 < p = p(\sigma)$, which contradicts our assumption regarding the choice of enumeration $\{\sigma_1, \sigma_2, \ldots, \sigma_t\}$. □

01DP **Lemma 4.4.4.8.** *Let $r : Y \to S$ be an inner fibration of simplicial sets, let $\overline{F} : B \to S$ be any morphism of simplicial sets, let A be a simplicial subset of B, let $n \geq 2$ be an integer. Let $\pi : B \times \Delta^n \to B$ be the projection map and suppose we are given a lifting problem*

0239
$$\begin{tikzcd} (A \times \Delta^n) \coprod_{(A \times \Lambda^n_0)} (B \times \Lambda^n_0) \arrow[r, "F_0"] \arrow[d] & Y \arrow[d, "r"] \\ B \times \Delta^n \arrow[r, "\overline{F} \circ \pi"] \arrow[ur, dashed, "F"] & S. \end{tikzcd} \qquad (4.4)$$

Assume that, for every vertex $b \in B$, the edge

$$\Delta^1 \simeq \{b\} \times \mathrm{N}_\bullet(\{0,1\}) \hookrightarrow B \times \Lambda^n_0 \xrightarrow{F_0} \{\overline{F}(b)\} \times_S Y$$

is an isomorphism in the ∞-category $Y_b = \{\overline{F}(b)\} \times_S X$. Then the lifting problem (4.4) admits a solution $F : B \times \Delta^n \to Y$.

Proof. Let P denote the collection of all pairs (K, F_K), where $K \subseteq B$ is a simplicial subset containing A and $F_K : K \times \Delta^n \to Y$ is a morphism of simplicial sets satisfying $F_K|_{A \times \Delta^n} = F_0|_{A \times \Delta^n}$, $F_K|_{K \times \Lambda^n_0} = F_0|_{K \times \Lambda^n_0}$, and $r \circ F_K = (\overline{F} \circ \pi)|_{K \times \Delta^n}$. We regard P as partially ordered set, where $(K, F_K) \leq (K', F_{K'})$ if $K \subseteq K'$ and $F_K = F_{K'}|_{K \times \Delta^n}$. The partially ordered set P satisfies the hypotheses of Zorn's lemma, and therefore has a maximal element $(K_{\max}, F_{K_{\max}})$. We will complete the proof by showing that $K_{\max} = B$. Assume otherwise. Then there exists some nondegenerate m-simplex $\tau : \Delta^m \to B$ whose image is not contained in $K_{\max}$. Choosing m as small as possible, we can assume that τ carries the boundary $\partial \Delta^m$ into $K_{\max}$. Let $K' \subseteq B$ be the union of $K_{\max}$ with the image of τ, so that we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & K_{\max} \arrow[d] \\
\Delta^m \arrow[r] & K'.
\end{tikzcd}$$

We will complete the proof by showing that the lifting problem

$$\begin{tikzcd}
(K_{\max} \times \Delta^n) \coprod_{(K_{\max} \times \Lambda^n_0)} (K' \times \Lambda^n_0) \arrow[r, "(F_{K_{\max}}, F_0|_{K' \times \Lambda^n_0})"] \arrow[d] & Y \arrow[d, "r"] \\
K' \times \Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution (contradicting the maximality of the pair $(K_{\max}, F_{K_{\max}})$). To prove this, we can replace the inclusion $K_{\max} \hookrightarrow K'$ by $\partial \Delta^m \hookrightarrow \Delta^m$. We are therefore reduced to proving Lemma 4.4.4.8 in the special case where $B = \Delta^m$ is a simplex and $A = \partial \Delta^m$ is its boundary. Replacing r by the projection map $\Delta^m \times_S Y \to \Delta^m$, we may further assume that S is an ∞-category.

Choose a sequence of simplicial subsets

$$X(0) \subset X(1) \subset X(2) \subset \cdots \subset X(t) = \Delta^m \times \Delta^n$$

satisfying the requirements of Lemma 4.4.4.7, so that F_0 can be identified with a morphism $X(0) \to Y$. We will show that, for $0 \leq s \leq t$, there exists a morphism of simplicial sets $F_s : X(s) \to Y$ satisfying $F_s|_{X(0)} = F_0$ and $r \circ F_s = (\overline{F} \circ \pi)|_{X(s)}$ (taking $s = t$, this will complete the proof of Lemma 4.4.4.8). We proceed by induction on s, the case $s = 0$ being vacuous. Assume that $s > 0$ and that we have already constructed a morphism $F_{s-1} : X(s-1) \to Y$ satisfying $F_{s-1}|_{X(0)} = F_0$ and $r \circ F_{s-1} = (\overline{F} \circ \pi)|_{X(s-1)}$ By construction, there exists integers $q \geq 2$, $0 \leq p < q$, and a pushout diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^q_p \arrow[r, "\sigma_0"] \arrow[d] & X(s-1) \arrow[d] \\
\Delta^q \arrow[r, "\sigma"] & X(s).
\end{tikzcd}$$

Moreover, in the special case $p = 0$, we can assume that $\sigma(0) = (0, 0)$ and $\sigma(1) = (0, 1)$, so that the composite map

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{0 < 1\}) \hookrightarrow \Lambda^q_p \xrightarrow{\sigma_0} X(s-1) \xrightarrow{F_{s-1}} Y$$

corresponds to an isomorphism in Y. To construct the desired extension $F_s : X(s) \to Y$, it will suffice to solve a lifting problem of the form

$$\begin{tikzcd}
\Lambda^q_p \arrow[r] \arrow[d] & Y \arrow[d, "r"] \\
\Delta^q \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

In the case $0 < p < q$, this lifting problem admits a solution by virtue of our assumption that r is an inner fibration of simplicial sets. In the special case $p = 0$, it follows from Proposition 4.4.2.13. □

Theorem 4.4.4.4 is a special case of the following more general assertion:

023A **Proposition 4.4.4.9.** *Let $q : X \to S$ be an inner fibration of simplicial sets, let $\overline{F} : B \to S$ be a morphism of simplicial sets, and let $u : F \to F'$ be a morphism in the ∞-category $\mathrm{Fun}_{/S}(B, X)$. The following conditions are equivalent:*

(1) *The morphism u is an isomorphism in the ∞-category $\mathrm{Fun}_{/S}(B, X)$.*

(2) *For every vertex $b \in B$, the morphism $u_b : F(b) \to F'(b)$ is an isomorphism in the ∞-category $X_b = \{\overline{F}(b)\} \times_S X$.*

Proof. For each vertex $b \in B$, evaluation at b determines a functor of ∞-categories $\mathrm{Fun}_{/S}(B, X) \to X_b$. Consequently, the implication (1) ⇒ (2) follows from Remark 1.5.1.6. The converse implication follows by combining Lemma 4.4.4.8 (in the special case $A = \emptyset$) with the criterion of Theorem 4.4.2.6. □

Proof of Theorem 4.4.4.4. Apply Proposition 4.4.4.9 in the case $S = \Delta^0$. □

4.4.5 Exponentiation for Isofibrations

01F0 We now show that the formation of ∞-categories of functors behaves well with respect to isofibrations.

Proposition 4.4.5.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories, let B be a simplicial set, and let $A \subseteq B$ be a simplicial subset. Then the restriction map* 01F1

$$F' : \operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(A, \mathcal{C}) \times_{\operatorname{Fun}(A,\mathcal{D})} \operatorname{Fun}(B, \mathcal{D})$$

is an isofibration of ∞-categories.

Remark 4.4.5.2. Proposition 4.4.5.1 generalizes to isofibrations between arbitrary simplicial sets: see Proposition 4.5.5.14. 01F2

We will give the proof of Proposition 4.4.5.1 at the end of this section.

Corollary 4.4.5.3. *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, and let $A \subseteq B$ be a simplicial subset. Then the restriction map $\operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(A, \mathcal{C})$ is an isofibration of ∞-categories.* 01F3

Proof. Apply Proposition 4.4.5.1 in the special case $\mathcal{D} = \Delta^0$. □

Corollary 4.4.5.4. *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, and let $A \subseteq B$ be a simplicial subset. Then the restriction functor $\operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(A, \mathcal{C})$ induces a Kan fibration of simplicial sets $\operatorname{Fun}(B, \mathcal{C})^{\simeq} \to \operatorname{Fun}(A, \mathcal{C})^{\simeq}$.* 01F4

Proof. Combine Corollary 4.4.5.3 with Proposition 4.4.3.7. □

Corollary 4.4.5.5. *Let $\mathcal{C}$ be an ∞-category, and let $\operatorname{Isom}(\mathcal{C})$ denote the full subcategory of $\operatorname{Fun}(\Delta^1, \mathcal{C})$ spanned by the isomorphisms. Then the restriction map* 02QY

$$\operatorname{Isom}(\mathcal{C}) \to \operatorname{Fun}(\partial \Delta^1, \mathcal{C}) \simeq \mathcal{C} \times \mathcal{C} \qquad (f : X \to Y) \mapsto (X, Y)$$

is an isofibration of ∞-categories.

Proof. Combine Corollary 4.4.5.3 with Example 4.4.1.14. □

Corollary 4.4.5.6. *Let $q : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories. For every simplicial set B, the induced map $\operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(B, \mathcal{D})$ is also an isofibration of ∞-categories.* 01H7

Proof. Apply Proposition 4.4.5.1 in the special case $A = \emptyset$. □

Corollary 4.4.5.7. *Let $q : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories. For every simplicial set B, the induced map $\operatorname{Fun}(B, \mathcal{C})^{\simeq} \to \operatorname{Fun}(B, \mathcal{D})^{\simeq}$ is a Kan fibration of Kan complexes.* 02QZ

Proof. Combine Corollary 4.4.5.6 with Proposition 4.4.3.7. □

The main ingredient needed in our proof of Proposition 4.4.5.1 is the following isomorphism extension result:

01NX **Proposition 4.4.5.8.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let B be a simplicial set, let $A \subseteq B$ be a simplicial subset which contains every vertex of B, and suppose we are given a lifting problem*

$$\begin{tikzcd}
(\Delta^1 \times A) \coprod_{(\{1\} \times A)} (\{1\} \times B) \arrow[r, "h_0"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^1 \times B \arrow[r, "\overline{h}"] \arrow[ur, dashed, "h"] & \mathcal{D}
\end{tikzcd}$$

with the following property:

(∗) *For every simplex $\tau : \Delta^n \to B$ which is not contained in A having final vertex $b = \tau(n)$, the edge*

$$\Delta^1 \simeq \Delta^1 \times \{b\} \xrightarrow{h_0} \mathcal{C}$$

is an isomorphism in $\mathcal{C}$.

Then h_0 can be extended to a diagram $h : \Delta^1 \times B \to \mathcal{C}$ satisfying $\overline{h} = F \circ h$.

Proof. We proceed as in the proof of Lemma 4.4.4.8, with some minor modifications. Let P denote the collection of all pairs (K, h_K), where $K \subseteq B$ is a simplicial subset containing A and $h_K : \Delta^1 \times K \to \mathcal{C}$ is a morphism of simplicial sets satisfying

$$h_K|_{\Delta^1 \times A} = h_0|_{\Delta^1 \times A} \qquad h_K|_{\{1\} \times K} = h_0|_{\{1\} \times K}.$$

We regard P as partially ordered set, where $(K, h_K) \leq (K', h_{K'})$ if $K \subseteq K'$ and $h_K = h_{K'}|_{\Delta^1 \times K}$. The partially ordered set P satisfies the hypotheses of Zorn's lemma, and therefore has a maximal element $(K_{\max}, h_{K_{\max}})$. We will complete the proof by showing that $K_{\max} = B$. Assume otherwise. Then there exists some nondegenerate n-simplex $\tau : \Delta^n \to B$ whose image is not contained in $K_{\max}$. Choosing n as small as possible, we can assume that τ carries the boundary $\partial \Delta^n$ into $K_{\max}$. Note that, since A contains every vertex of B, we must have $n > 0$. Let $K' \subseteq B$ be the union of $K_{\max}$ with the image of τ, so that we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & K_{\max} \arrow[d] \\
\Delta^n \arrow[r] & K'.
\end{tikzcd}$$

We will complete the proof by showing that the lifting problem

$$\begin{tikzcd}
(\Delta^1 \times K_{\max}) \coprod_{(\{1\} \times K_{\max})} (\{1\} \times K') \arrow[r, "{(h_K, h_0|_{\{1\} \times K'})}"] \arrow[d] & \mathcal{C} \arrow[d] \\
\Delta^1 \times K' \arrow[r] \arrow[ur, dashed] & \Delta^0
\end{tikzcd}$$

admits a solution, where the dotted arrow carries each edge $\Delta^1 \times \{x\}$ to an isomorphism in $\mathcal{C}$ (contradicting the maximality of the pair $(K_{\max}, h_{K_{\max}})$). To prove this, we can replace the inclusion $K_{\max} \hookrightarrow K'$ by $\partial \Delta^n \hookrightarrow \Delta^n$. We are therefore reduced to proving Lemma 4.4.4.8 in the special case where $B = \Delta^n$ is a simplex and $A = \partial \Delta^n$ is its boundary.

Let

$$(\Delta^1 \times \partial \Delta^n) \cup (\{1\} \times \Delta^n) = X(0) \subset X(1) \subset X(2) \subset \cdots \subset X(n+1) = \Delta^1 \times \Delta^n$$

be the sequence of simplicial subsets appearing in the proof of Lemma 3.1.2.12, so that h_0 can be identified with a morphism of simplicial sets from $X(0)$ to $\mathcal{C}$. We will show that, for $0 \leq i \leq n+1$, there exists a morphism of simplicial sets $h_i : X(i) \to \mathcal{C}$ satisfying $h_i|_{X(0)} = h_0$ and $F \circ h_i = \overline{h}|_{X(i)}$ (taking $i = n+1$, this will complete the proof of Proposition 4.4.5.8). We proceed by induction on i, the case $i = 0$ being vacuous. Assume that $i > 0$ and that we have already constructed a morphism $h_{i-1} : X(i-1) \to \mathcal{C}$ satisfying $h_{i-1}|_{X(0)} = h_0$ and $F \circ h_{i-1} = \overline{h}|_{X(i-1)}$. By virtue of Lemma 3.1.2.12, we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^{n+1}_i \arrow[r, "\sigma_0"] \arrow[d] & X(i-1) \arrow[d] \\
\Delta^{n+1} \arrow[r, "\sigma"] & X(i).
\end{tikzcd}$$

Consequently, to prove the existence of h_i, it suffices to solve the lifting problem

$$\begin{tikzcd}
\Lambda^{n+1}_i \arrow[r, "h_{i-1} \circ \sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^{n+1} \arrow[r, "\overline{h} \circ \sigma"] \arrow[ur, dashed] & \mathcal{D}.
\end{tikzcd}$$

For $0 < i < n+1$, the existence of the desired solution follows from our assumption that F is an inner fibration. In the case $i = n+1$, the existence follows from Proposition 4.4.2.13, since the map $\sigma : \Delta^{n+1} \to \Delta^1 \times \Delta^n$ carries the final edge $\mathrm{N}_\bullet(\{n < n+1\}) \subseteq \Delta^{n+1}$ to the edge $\Delta^1 \times \{n\} \subseteq \Delta^1 \times \Delta^n$, which h_0 carries to an isomorphism in the ∞-category $\mathcal{C}$ by virtue of assumption $(*)$. $\square$

01NY **Corollary 4.4.5.9.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories, let B be a simplicial set, let $A \subseteq B$ be a simplicial subset, and suppose we are given a lifting problem*

$$\begin{tikzcd}
(\Delta^1 \times A) \coprod_{(\{1\} \times A)} (\{1\} \times B) \arrow[r, "h_0"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^1 \times B \arrow[r, "\overline{h}"] \arrow[ur, dashed, "h"] & \mathcal{D}
\end{tikzcd}$$

with the following properties:

- *For every vertex $a \in A$, the edge*

$$\Delta^1 \simeq \Delta^1 \times \{a\} \xrightarrow{h_0} \mathcal{C}$$

is an isomorphism in $\mathcal{C}$.

- *For every vertex $b \in B$, the edge*

$$\Delta^1 \simeq \Delta^1 \times \{b\} \xrightarrow{\overline{h}} \mathcal{D}$$

is an isomorphism in $\mathcal{D}$.

Then h_0 can be extended to a diagram $h : \Delta^1 \times B \to \mathcal{C}$ satisfying $\overline{h} = F \circ h$. Moreover, we can arrange that for every vertex $b \in B$, the edge $\Delta^1 \simeq \Delta^1 \times \{b\} \xrightarrow{h} \mathcal{C}$ is an isomorphism in the ∞-category $\mathcal{C}$ (so that h can be regarded as an isomorphism in the diagram ∞-category $\mathrm{Fun}(B, \mathcal{C})$*, by virtue of Theorem 4.4.4.4).*

Proof. Let A' be the union of A with the 0-skeleton $\mathrm{sk}_0(B)$, regarded as a simplicial subset of B. For each vertex $b \in B$ which does not belong to A, our assumption that F is an isofibration allows us to choose an edge $e_b : \Delta^1 \to \mathcal{C}$ which is an isomorphism in the ∞-category $\mathcal{C}$ satisfying $e_b(1) = h_0(1, b)$ and $F \circ e_b = \overline{h}|_{\Delta^1 \times \{b\}}$. The morphism h_0 and the edges e_b can then be amalgamated to a map $h_0' : (\Delta^1 \times A') \coprod_{(\{1\} \times A')} (\{1\} \times B) \to \mathcal{C}$. The desired result now follows by applying Proposition 4.4.5.8 to the commutative diagram

$$\begin{tikzcd}
(\Delta^1 \times A') \coprod_{(\{1\} \times A')} (\{1\} \times B) \arrow[r, "h_0'"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^1 \times B \arrow[r, "\overline{h}"] & \mathcal{D}.
\end{tikzcd}$$

□

Specializing Corollary 4.4.5.9 to the case $\mathcal{D} = \Delta^0$, we obtain the following:

Corollary 4.4.5.10. *Let $\mathcal{C}$ be an ∞-category, let $\mathrm{Isom}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the full subcategory spanned by the isomorphisms, and let $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Isom}(\mathcal{C}) \to \mathcal{C}$ be the functors given by evaluation at the vertices $0, 1 \in \Delta^1$. Then the functors ev_0 and ev_1 are trivial Kan fibrations.* 02BY

Proof of Proposition 4.4.5.1. Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories, let B be a simplicial set, and let $A \subseteq B$ be a simplicial subset. We wish to show that the restriction map

$$F' : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{D})} \mathrm{Fun}(B, \mathcal{D})$$

is an isofibration of ∞-categories. We first note that the projection map

$$\mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{D})} \mathrm{Fun}(B, \mathcal{D}) \to \mathrm{Fun}(A, \mathcal{C})$$

is a pullback of the inner fibration $\mathrm{Fun}(B, \mathcal{D}) \to \mathrm{Fun}(A, \mathcal{D})$ (see Corollary 4.1.4.2). Since $\mathrm{Fun}(A, \mathcal{C})$ is an ∞-category (Theorem 1.5.3.7), it follows that $\mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{D})} \mathrm{Fun}(B, \mathcal{D})$ is also an ∞-category (Remark 4.1.1.9). It follows from Proposition 4.1.4.1 that F' is an inner fibration. It will therefore suffice to show that, for every object $Y \in \mathrm{Fun}(B, \mathcal{C})$, every isomorphism $u : X \to F'(Y)$ in the ∞-category $\mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{D})} \mathrm{Fun}(B, \mathcal{D})$ can be lifted to an isomorphism $\overline{u} : \overline{X} \to Y$ in the ∞-category $\mathrm{Fun}(B, \mathcal{C})$. This follows immediately from Corollary 4.4.5.9. □

Replacing Corollary 4.4.5.9 by Proposition 4.4.5.8 in the preceding argument, we obtain the following:

Variant 4.4.5.11. Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let B be a simplicial set, and let $A \subseteq B$ be a simplicial subset which contains every vertex of B. Then the induced map 01NZ

$$F' : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{D})} \mathrm{Fun}(B, \mathcal{D})$$

is an isofibration of ∞-categories.

4.5 Equivalence

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ satisfying the identities $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$. This condition is somewhat unnatural, since it refers to equalities between *objects* of the functor categories $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$. For most purposes, it is better to adopt a looser definition. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories* 01DE if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ for which the composite functors $G \circ F$ and $F \circ G$ are *isomorphic* to the identity functors $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$, respectively. In category theory, the notion of equivalence between categories plays a much more central role than the notion of isomorphism between categories, and virtually all important concepts are invariant under equivalence.

In §4.5.1, we extend the notion of equivalence to the ∞-categorical setting. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, we will say that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of ∞-categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ for which the composite maps $G \circ F$ and $F \circ G$ are isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$, when viewed as objects of the ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, respectively (Definition 4.5.1.10). Phrased differently, a functor F is an equivalence of ∞-categories if it is an isomorphism when viewed as a morphism of the category hQCat, whose objects are ∞-categories and whose morphisms are isomorphism classes of functors (Construction 4.5.1.1).

In the study of ∞-categories, it can be technically convenient to work with simplicial sets which do *not* satisfy the weak Kan extension condition. For example, it is often harmless to replace the standard n-simplex Δ^n by its spine $\mathrm{Spine}[n] \subseteq \Delta^n$: for any ∞-category $\mathcal{C}$, the restriction map $\mathrm{Fun}(\Delta^n, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Spine}[n], \mathcal{C})$ is a trivial Kan fibration (see Example 1.5.7.7). In §4.5.3, we formalize this observation by introducing the notion of *categorical equivalence* between simplicial sets. By definition, a morphism of simplicial sets $f : X \to Y$ is a categorical equivalence if, for every ∞-category $\mathcal{C}$, the induced functor of ∞-categories $\mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ is bijective on isomorphism classes of objects (Definition 4.5.3.1). If X and Y are ∞-categories, this reduces to the condition that f is an equivalence of ∞-categories in the sense of §4.5.1 (Example 4.5.3.3). However, we will encounter many other examples of categorical equivalences between simplicial sets which are *not* ∞-categories: for example, every inner anodyne morphism of simplicial sets is a categorical equivalence (Corollary 4.5.3.14).

Throughout this book, we will generally emphasize concepts which are invariant under categorical equivalence. In practice, this requires us to take some care when manipulating elementary constructions, such as fiber products. If $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ are functors of ∞-categories, then the fiber product $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1$ (formed in the category of simplicial sets) need not be an ∞-category. Moreover, the construction $(F_0, F_1) \mapsto \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1$ does not preserve categorical equivalence in general. In §4.5.2, we remedy the situation by enlarging the fiber product $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1$ to the *homotopy fiber product* $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$, given by the formula

$$\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 = \mathcal{C}_0 \times_{\mathrm{Fun}(\{0\}, \mathcal{C})} \mathrm{Isom}(\mathcal{C}) \times_{\mathrm{Fun}(\{1\}, \mathcal{C})} \mathcal{C}_1$$

(see Construction 4.5.2.1). The homotopy fiber product $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is always an ∞-category (Remark 4.5.2.2), and the construction $(F_0, F_1) \mapsto \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is invariant under equivalence

(Corollary 4.5.2.20). We will say that a commutative diagram of ∞-categories

032W

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}
\end{tikzcd} \tag{4.5}$$

is a *categorical pullback square* if it induces an equivalence of ∞-categories $\mathcal{C}_{01} \to \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ (Definition 4.5.2.8). This is closely related to the notion of homotopy pullback diagram introduced in §3.4.1:

- A commutative diagram of Kan complexes is a homotopy pullback square if and only if it is a categorical pullback square (Proposition 4.5.2.10).
- The diagram of ∞-categories (4.5) is a categorical pullback square if and only if, for every simplicial set X, the induced diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(X, \mathcal{C}_{01})^{\simeq} \arrow[r] \arrow[d] & \operatorname{Fun}(X, \mathcal{C}_0)^{\simeq} \arrow[d] \\
\operatorname{Fun}(X, \mathcal{C}_1)^{\simeq} \arrow[r] & \operatorname{Fun}(X, \mathcal{C})^{\simeq}
\end{tikzcd}$$

  is a homotopy pullback square (Proposition 4.5.2.14).

In §4.5.4 we study the dual notion of *categorical pushout square* (Definition 4.5.4.1), which is an ∞-categorical counterpart of the theory of homotopy pushout squares developed in §3.4.2.

Recall that every ∞-category $\mathcal{C}$ contains a largest Kan complex, which we denote by $\mathcal{C}^{\simeq}$ and refer to as the *core* of $\mathcal{C}$ (Construction 4.4.3.1). The construction $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$ can often be used to reformulate questions about ∞-categories in terms of the classical homotopy theory of Kan complexes. It is not difficult to show that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is an equivalence if and only if, for every simplicial set X, the induced map $\operatorname{Fun}(X, \mathcal{C})^{\simeq} \xrightarrow{F\circ} \operatorname{Fun}(X, \mathcal{D})^{\simeq}$ is a homotopy equivalence of Kan complexes (Proposition 4.5.1.22). In §4.5.7, we show that it suffices to verify this condition in the special case $X = \Delta^1$ (Theorem 4.5.7.1). As an application, we show that the collection of categorical equivalences is stable under the formation of filtered colimits (Corollary 4.5.7.2).

In §4.5.8, we study an important class of categorical equivalences emerging from the theory of joins developed in §4.3. Recall that, if $\mathcal{C}$ and $\mathcal{D}$ are categories, then the join $\mathcal{C} \star \mathcal{D}$

is isomorphic to the iterated pushout

$$\mathcal{C} \coprod_{(\mathcal{C} \times \{0\} \times \mathcal{D})} (\mathcal{C} \times [1] \times \mathcal{D}) \coprod_{(\mathcal{C} \times \{1\} \times \mathcal{D})} \mathcal{D},$$

formed in the category Cat of (small) categories (Remark 4.3.2.14). In the setting of ∞-categories, the situation is more subtle (Warning 4.3.3.33). For any simplicial sets X and Y, there is a natural comparison map

$$c_{X,Y} : X \coprod_{(X \times \{0\} \times Y)} (X \times \Delta^1 \times Y) \coprod_{(X \times \{1\} \times Y)} Y \to X \star Y$$

(Notation 4.5.8.3), which is almost never an isomorphism. Nevertheless, we show in §4.5.8 that $c_{X,Y}$ is always a categorical equivalence of simplicial sets (Theorem 4.5.8.8).

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Recall that F is an inner fibration if and only if every lifting problem

01H8
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & \mathcal{C} \arrow[d, "F"] \\ B \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd} \tag{4.6}$$

admits a solution, provided that the morphism $i : A \hookrightarrow B$ is inner anodyne (Proposition 4.1.3.1). In §4.5.5, we show that F is an isofibration if and only if the following stronger condition holds: the lifting problem (4.6) admits a solution whenever the map $i : A \hookrightarrow B$ is both a monomorphism and a categorical equivalence (Proposition 4.5.5.1). Using this characterization, we extend the notion of isofibration to simplicial sets which are not necessarily ∞-categories (Definition 4.5.5.5).

4.5.1 Equivalences of ∞-Categories

01DQ The collection of ∞-categories can be organized into a category, in which the morphisms are given by isomorphism classes of functors.

01DR **Construction 4.5.1.1** (The Homotopy Category of ∞-Categories)**.** We define a category hQCat as follows:

- The objects of hQCat are ∞-categories.
- If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then $\mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C}, \mathcal{D}) = \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq})$ is the set of isomorphism classes of objects of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (or, equivalently, of the homotopy category $\mathrm{hFun}(\mathcal{C}, \mathcal{D})$). If $F : \mathcal{C} \to \mathcal{D}$ is a functor, we denote its isomorphism class by $[F] \in \mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C}, \mathcal{D})$.

- If $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are ∞-categories, then the composition law

$$\circ : \mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{D}, \mathcal{E}) \times \mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C}, \mathcal{E})$$

  is characterized by the formula $[G] \circ [F] = [G \circ F]$.

We will refer to hQCat as the *homotopy category of ∞-categories.*

Remark 4.5.1.2. We will later study a refinement of Construction 4.5.1.1. The collection of (small) ∞-categories can itself be organized into a (large) ∞-category $\mathcal{QC}$, whose homotopy category can be identified with the ordinary category hQCat of Construction 4.5.1.1. See Construction 5.5.4.1. 01DS

Remark 4.5.1.3. Let **Cat** denote the (strict) 2-category of categories (Example 2.2.0.4) and let hCat denote its homotopy category (Construction 2.2.8.12). Then the construction $\mathcal{C} \mapsto \mathrm{N}_{\bullet}(\mathcal{C})$ determines a fully faithful functor from hCat to the homotopy category hQCat of Construction 4.5.1.1. This functor admits a left adjoint, which carries an ∞-category $\mathcal{C}$ to its homotopy category h$\mathcal{C}$. 01P0

Remark 4.5.1.4. Let hKan denote the homotopy category of Kan complexes (Construction 3.1.5.10). Then we can regard hKan as a full subcategory of the ∞-category hQCat (Construction 4.5.1.1), spanned by those ∞-categories which are Kan complexes. This follows from the observation that if Y is a Kan complex, then a pair of morphisms $f, g : X \to Y$ are isomorphic as objects of the ∞-category $\mathrm{Fun}(X, Y)$ if and only if they are homotopic (Proposition 3.1.5.4). 01DT

The inclusion functor hKan $\hookrightarrow$ hQCat has both left and right adjoints.

Proposition 4.5.1.5. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}^{\simeq}$ denote its core (Construction 4.4.3.1). For every Kan complex X, composition with the inclusion map $\iota : \mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ induces a bijection* 01DU

$$\mathrm{Hom}_{\mathrm{hKan}}(X, \mathcal{C}^{\simeq}) = \mathrm{Hom}_{\mathrm{hQCat}}(X, \mathcal{C}^{\simeq}) \to \mathrm{Hom}_{\mathrm{hQCat}}(X, \mathcal{C}).$$

Proof. By virtue of Proposition 4.4.3.22, postcomposition with ι induces an isomorphism of Kan complexes $\mathrm{Fun}(X, \mathcal{C}^{\simeq}) \to \mathrm{Fun}(X, \mathcal{C})^{\simeq}$. Proposition 4.5.1.5 follows by passing to connected components. □

Corollary 4.5.1.6. *The inclusion functor* hKan $\hookrightarrow$ hQCat *of Remark 4.5.1.4 admits a right adjoint, given on objects by the construction $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$.* 01DV

Remark 4.5.1.7. The right adjoint hQCat $\to$ hKan of Corollary 4.5.1.6 can be described more explicitly as follows: 01DW

- To each ∞-category $\mathcal{C}$, it associates the Kan complex $\mathcal{C}^{\simeq}$ of Construction 4.4.3.1.

- To each morphism $[F] : \mathcal{C} \to \mathcal{D}$ in the homotopy category hQCat (given by the isomorphism class of a functor $F : \mathcal{C} \to \mathcal{D}$), it associates the homotopy class $[F^{\simeq}]$ of the underlying map of cores $F^{\simeq} = F|_{\mathcal{C}^{\simeq}}$ (note that the homotopy class of $F^{\simeq}$ depends only on the isomorphism class of F, by virtue of Remark 4.4.4.5).

01DX **Proposition 4.5.1.8.** *The inclusion functor* hKan ↪ hQCat *of Remark 4.5.1.4 admits a left adjoint.*

Proof. Let $\mathcal{C}$ be an ∞-category. We wish to show that there exists a Kan complex X and a morphism $u : \mathcal{C} \to X$ with the following property: for every Kan complex Y, precomposition with u induces a bijection

$$\mathrm{Hom}_{\mathrm{hKan}}(X, Y) = \mathrm{Hom}_{\mathrm{hQCat}}(X, Y) \to \mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C}, Y).$$

Unwinding the definitions, we see that this is a reformulation of the requirement that u is a weak homotopy equivalence of simplicial sets. The existence of u now follows from Corollary 3.1.7.2. □

01H9 **Remark 4.5.1.9.** The left adjoint hQCat → hKan of Proposition 4.5.1.8 admits a category-theoretic interpretation: it carries an ∞-category $\mathcal{C}$ to the localization $\mathcal{C}[W^{-1}]$ obtained by formally inverting the collection W of *all* morphisms in $\mathcal{C}$ (see Proposition 6.3.1.20).

01DY **Definition 4.5.1.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that a functor $G : \mathcal{D} \to \mathcal{C}$ is *homotopy inverse* to F if the isomorphism class $[G]$ is an inverse to $[F]$ in the homotopy category hQCat: that is, if $G \circ F$ and $F \circ G$ are isomorphic to the identity functors $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$ as objects of the ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, respectively. We will say that F is an *equivalence of ∞-categories* if $[F]$ is an isomorphism in the homotopy category hQCat: that is, if F admits a homotopy inverse $G : \mathcal{D} \to \mathcal{C}$. We say that ∞-categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists an equivalence from $\mathcal{C}$ to $\mathcal{D}$.

01E0 **Example 4.5.1.11.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be an isomorphism of simplicial sets. Then F is an equivalence of ∞-categories. In particular, for every ∞-category $\mathcal{C}$, the identity functor $\mathrm{id}_{\mathcal{C}}$ is an equivalence of ∞-categories.

01E1 **Example 4.5.1.12.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then the induced map $\mathrm{N}_{\bullet}(F) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{D})$ is an equivalence of ∞-categories if and only if F is an equivalence of categories.

01E2 **Example 4.5.1.13.** Let $f : X \to Y$ be a morphism of Kan complexes. Then f is a homotopy equivalence if and only if it is an equivalence of ∞-categories (see Remark 4.5.1.4). In this case, a morphism $g : Y \to X$ is a homotopy inverse to f in the sense of Definition 4.5.1.10 if and only if it is a homotopy inverse to f, in the sense of Definition 3.1.6.1.

Warning 4.5.1.14. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F is an equivalence of ∞-categories (in the sense of Definition 4.5.1.10), then it is a homotopy equivalence of simplicial sets (in the sense of Definition 3.1.6.1). More precisely, if $G : \mathcal{D} \to \mathcal{C}$ is a homotopy inverse to the functor F (in the sense of Definition 4.5.1.10), then G is also a simplicial homotopy inverse to F (in the sense of Definition 3.1.6.1). Beware that the converse assertion is false in general. For example, the projection map $\Delta^1 \to \Delta^0$ is a homotopy equivalence of simplicial sets (with homotopy inverse given by the inclusion $\Delta^0 \simeq \{0\} \hookrightarrow \Delta^1$), but not an equivalence of ∞-categories. 01E3

Remark 4.5.1.15. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors which are isomorphic when regarded as objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Then F is an equivalence of ∞-categories if and only if G is an equivalence of ∞-categories. 02R0

Remark 4.5.1.16. Let X be an arbitrary simplicial set. Then the construction $\mathcal{C} \mapsto \mathrm{Fun}(X, \mathcal{C})$ determines a functor from the homotopy category hQCat to itself. In particular, if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then the induced map $\mathrm{Fun}(X, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{D})$ is also an equivalence of ∞-categories. 01HA

Remark 4.5.1.17. Let $\{F_i : \mathcal{C}_i \to \mathcal{D}_i\}_{i \in I}$ be a collection of functors between ∞-categories indexed by a set I. If each F_i is an equivalence of ∞-categories, then the product functor $\prod_{i \in I} \mathcal{C}_i \to \prod_{i \in I} \mathcal{D}_i$ is also an equivalence of ∞-categories. 023B

Remark 4.5.1.18 (Two-out-of-Three)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors between ∞-categories. If any two of the functors F, G, and $G \circ F$ is an equivalence of ∞-categories, then so is the third. In particular, the collection of equivalences is closed under composition. 01E5

Remark 4.5.1.19. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. If F is an equivalence of ∞-categories, then the induced map of cores $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a homotopy equivalence of Kan complexes. This follows from Corollary 4.5.1.6 (and Remark 4.5.1.7): if the isomorphism class $[F]$ is an invertible morphism in the homotopy category hQCat, then the homotopy class $[F^{\simeq}]$ is an invertible morphism in the homotopy category hKan. 01E6

Remark 4.5.1.20. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories. Then the induced functor $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an equivalence of ordinary categories. In particular, a morphism u in the ∞-category $\mathcal{C}$ is an isomorphism if and only if $F(u)$ is an isomorphism in the ∞-category $\mathcal{D}$. 01HB

Remark 4.5.1.21. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories. If $\mathcal{D}$ is a Kan complex, then $\mathcal{C}$ is a Kan complex. To prove this, it suffices to show that every morphism $u : X \to Y$ in $\mathcal{C}$ is an isomorphism (Proposition 4.4.2.1). By virtue of Remark 4.5.1.20, this is equivalent to the assertion that $F(u) : F(X) \to F(Y)$ is an isomorphism in $\mathcal{D}$, which is automatic when $\mathcal{D}$ is a Kan complex (Proposition 1.4.6.10). Similarly, if $\mathcal{C}$ is a Kan complex, then $\mathcal{D}$ is a Kan complex (this follows by applying the same argument to an inverse equivalence $\mathcal{D} \to \mathcal{C}$). 01HC

032X **Proposition 4.5.1.22.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:*

(1) *The functor F is an equivalence of ∞-categories.*

(2) *For every simplicial set X, composition with F induces an equivalence of ∞-categories $\mathrm{Fun}(X, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{D})$.*

(3) *For every simplicial set X, composition with F induces a homotopy equivalence of Kan complexes $\mathrm{Fun}(X, \mathcal{C})^{\simeq} \to \mathrm{Fun}(X, \mathcal{D})^{\simeq}$.*

(4) *For every ∞-category $\mathcal{B}$, composition with F induces a homotopy equivalence of Kan complexes $\mathrm{Fun}(\mathcal{B}, \mathcal{C})^{\simeq} \to \mathrm{Fun}(\mathcal{B}, \mathcal{D})^{\simeq}$.*

(5) *For every ∞-category $\mathcal{B}$, composition with F induces a bijection of sets $\pi_0(\mathrm{Fun}(\mathcal{B}, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{B}, \mathcal{D})^{\simeq})$.*

Proof. The implication (1) ⇒ (2) follows from Remark 4.5.1.16, the implication (2) ⇒ (3) from Remark 4.5.1.19, the implication (3) ⇒ (4) is immediate, and the implication (4) ⇒ (5) follows from Remark 3.1.6.5, and the implication (5) ⇒ (1) follows from Yoneda's lemma (applied to the homotopy category hQCat). □

We close this section by introducing a refinement of Construction 4.5.1.1:

02BZ **Construction 4.5.1.23** (The Homotopy 2-Category of ∞-Categories)**.** We define a strict 2-category $\mathrm{h}_2\mathbf{QCat}$ as follows:

- The objects of $\mathrm{h}_2\mathbf{QCat}$ are ∞-categories.
- If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then $\underline{\mathrm{Hom}}_{\mathrm{h}_2\mathbf{QCat}}(\mathcal{C}, \mathcal{D}) = \mathrm{hFun}(\mathcal{C}, \mathcal{D})$ is the homotopy category of the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.
- If $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are ∞-categories, then the composition law on $\mathrm{h}_2\mathbf{QCat}$ is given by

$$\begin{aligned}
\underline{\mathrm{Hom}}_{\mathrm{h}_2\mathbf{QCat}}(\mathcal{D}, \mathcal{E}) \times \underline{\mathrm{Hom}}_{\mathrm{h}_2\mathbf{QCat}}(\mathcal{C}, \mathcal{D}) &= (\mathrm{hFun}(\mathcal{D}, \mathcal{E})) \times (\mathrm{hFun}(\mathcal{C}, \mathcal{D})) \\
&\simeq \mathrm{h}(\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{D})) \\
&\xrightarrow{\circ} \mathrm{hFun}(\mathcal{C}, \mathcal{E}) \\
&= \underline{\mathrm{Hom}}_{\mathrm{h}_2\mathbf{QCat}}(\mathcal{C}, \mathcal{E}).
\end{aligned}$$

We will refer to $\mathrm{h}_2\mathbf{QCat}$ as the *homotopy 2-category of ∞-categories.* We let $\mathrm{h}_2\mathrm{QCat}$ denote the pith of $\mathrm{h}_2\mathbf{QCat}$, in the sense of Construction 2.2.8.9; we will refer to $\mathrm{h}_2\mathrm{QCat}$ as the *homotopy $(2,1)$-category of ∞-categories.*

Remark 4.5.1.24. We can describe the strict 2-category $\mathrm{h}_2\mathbf{QCat}$ more informally as follows: 02C0

- The objects of $\mathrm{h}_2\mathbf{QCat}$ are ∞-categories.
- The morphisms of $\mathrm{h}_2\mathbf{QCat}$ are functors $F : \mathcal{C} \to \mathcal{D}$.
- If $F_0, F_1 : \mathcal{C} \to \mathcal{D}$ are functors between ∞-categories, then a 2-morphism $F_0 \Rightarrow F_1$ in $\mathrm{h}_2\mathbf{QCat}$ is a homotopy class of natural transformations from F_0 to F_1.

The strict 2-category $\mathrm{h}_2\mathrm{QCat}$ can be described in a similar way, except that its 2-morphisms are homotopy classes of natural isomorphisms (rather than general natural transformations).

Remark 4.5.1.25. The homotopy category hQCat of Construction 4.5.1.1 can be identified with the homotopy category of the 2-category $\mathrm{h}_2\mathbf{QCat}$ (in the sense of Construction 2.2.8.12); see Remark 2.4.6.18). 02C1

Remark 4.5.1.26. Let $\mathbf{Cat}$ denote the (strict) 2-category of categories (see Example 2.2.0.4). The construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ defines an isomorphism from $\mathbf{Cat}$ to the full subcategory of $\mathrm{h}_2\mathbf{QCat}$ spanned by those objects of the form $\mathrm{N}_\bullet(\mathcal{C})$, where $\mathcal{C}$ is a (small) category. 02C2

Remark 4.5.1.27. Let $\mathrm{h}_2\mathrm{Kan}$ denote the homotopy 2-category of Kan complexes (Construction 3.1.5.13). Then $\mathrm{h}_2\mathrm{Kan}$ can be identified with the full subcategory of $\mathrm{h}_2\mathbf{QCat}$ spanned by the Kan complexes. Since $\mathrm{h}_2\mathrm{Kan}$ is a $(2,1)$-category, this subcategory is contained in the pith $\mathrm{h}_2\mathrm{QCat} = \mathrm{Pith}(\mathrm{h}_2\mathbf{QCat})$; we can therefore also view $\mathrm{h}_2\mathrm{Kan}$ as a full subcategory of $\mathrm{h}_2\mathrm{QCat}$. 02C3

4.5.2 Categorical Pullback Squares

Recall that a commutative diagram of Kan complexes 032Y

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "q"] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

is a *homotopy pullback square* if the induced map

$$X_{01} \to X_0 \times_X X_1 \hookrightarrow X_0 \times^{\mathrm{h}}_X X_1$$

is a homotopy equivalence, where $X_0 \times^{\mathrm{h}}_X X_1$ is the homotopy fiber product of Construction 3.4.0.3 (see Corollary 3.4.1.6). In this section, we study an analogous condition in the setting of ∞-categories. We begin with a variant of Construction 3.4.0.3.

032Z **Construction 4.5.2.1** (The Homotopy Fiber Product of ∞-Categories)**.** Let $\mathcal{C}$ be an ∞-category, and let $\operatorname{Isom}(\mathcal{C}) \subseteq \operatorname{Fun}(\Delta^1, \mathcal{C})$ denote the full subcategory spanned by the isomorphisms in $\mathcal{C}$ (Example 4.4.1.14). If $\mathcal{C}_0$ and $\mathcal{C}_1$ are ∞-categories equipped with functors $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$, we let $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ denote the iterated pullback

$$\mathcal{C}_0 \times_{\operatorname{Fun}(\{0\},\mathcal{C})} \operatorname{Isom}(\mathcal{C}) \times_{\operatorname{Fun}(\{1\},\mathcal{C})} \mathcal{C}_1 .$$

We will refer to $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ as the *homotopy fiber product of* $\mathcal{C}_0$ *with* $\mathcal{C}_1$ *over* $\mathcal{C}$. Note that the diagonal map $\mathcal{C} \to \operatorname{Isom}(\mathcal{C}) \subseteq \operatorname{Fun}(\Delta^1, \mathcal{C})$ induces a comparison map $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1 \hookrightarrow \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$, which is a monomorphism of simplicial sets.

0330 **Remark 4.5.2.2.** Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories. It follows from Corollary 4.4.5.5 that the projection map $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \to \mathcal{C}_0 \times \mathcal{C}_1$ is an isofibration. In particular, the homotopy fiber product $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is an ∞-category. By construction, the objects of $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ can be identified with triples (C_0, C_1, e), where C_0 is an object of $\mathcal{C}_0$, C_1 is an object of $\mathcal{C}$, and $e : F_0(C_0) \to F_1(C_1)$ is an isomorphism in the ∞-category $\mathcal{C}$.

0331 **Example 4.5.2.3.** Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories. If $\mathcal{C}$ is a Kan complex, then every morphism in $\mathcal{C}$ is an isomorphism (Proposition 1.4.6.10): that is, we have $\operatorname{Isom}(\mathcal{C}) = \operatorname{Fun}(\Delta^1, \mathcal{C})$. It follows that the homotopy fiber product $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ of Construction 4.5.2.1 coincides with the homotopy fiber product introduced in Construction 3.4.0.3.

04HX **Example 4.5.2.4.** Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ordinary categories. Then the homotopy fiber product $\mathrm{N}_\bullet(\mathcal{C}_0) \times^{\mathrm{h}}_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet(\mathcal{C}_1)$ can be identified with the nerve of a category $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$, which can be described concretely as follows:

- The objects of $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ are triples (C_0, C_1, e), where C_0 is an object of $\mathcal{C}_0$, C_1 is an object of $\mathcal{C}$, and $e : F_0(C_0) \to F_1(C_1)$ is an isomorphism in $\mathcal{C}$.
- A morphism from (C_0, C_1, e) to (C_0', C_1', e') is a pair (f_0, f_1), where $f_0 : C_0 \to C_0'$ is a morphism in the category $\mathcal{C}_0$, $f_1 : C_1 \to C_1'$ is a morphism in the category $\mathcal{C}_1$, and the diagram

$$\begin{tikzcd}
C_0 \arrow[r, "f_0"] \arrow[d, "e", "\sim"'] & C_0' \arrow[d, "e'", "\sim"'] \\
C_1 \arrow[r, "f_1"] & C_1'
\end{tikzcd}$$

  commutes in the category $\mathcal{C}$.

We will refer to $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ as the *homotopy fiber product of* $\mathcal{C}_0$ *with* $\mathcal{C}_1$ *over* $\mathcal{C}$.

Remark 4.5.2.5. Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories. Then there is a canonical isomorphism of simplicial sets 0332

$$(\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1)^{\mathrm{op}} \simeq \mathcal{C}_1^{\mathrm{op}} \times^{\mathrm{h}}_{\mathcal{C}^{\mathrm{op}}} \mathcal{C}_0^{\mathrm{op}}.$$

Remark 4.5.2.6. Let $\mathcal{C}$ be an ∞-category and let X be a simplicial set. Using Theorem 4.4.4.4, we see that the natural identification $\mathrm{Fun}(X, \mathrm{Fun}(\Delta^1, \mathcal{C})) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Fun}(X, \mathcal{C}))$ restricts to an isomorphism $\mathrm{Fun}(X, \mathrm{Isom}(\mathcal{C})) \simeq \mathrm{Isom}(\mathrm{Fun}(X, \mathcal{C}))$. If $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ are functors of ∞-categories, we obtain a canonical isomorphism 0333

$$\mathrm{Fun}(X, \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1) \simeq \mathrm{Fun}(X, \mathcal{C}_0) \times^{\mathrm{h}}_{\mathrm{Fun}(X,\mathcal{C})} \mathrm{Fun}(X, \mathcal{C}_1).$$

Remark 4.5.2.7. Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories. Applying Corollary 4.4.3.19 to the pullback diagram 0334

$$\begin{tikzcd}
\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \arrow[r] \arrow[d] & \mathrm{Isom}(\mathcal{C}) \arrow[d] \\
\mathcal{C}_0 \times \mathcal{C}_1 \arrow[r] & \mathcal{C} \times \mathcal{C},
\end{tikzcd}$$

we deduce that the diagram of cores

$$\begin{tikzcd}
(\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1)^{\simeq} \arrow[r] \arrow[d] & \mathrm{Isom}(\mathcal{C})^{\simeq} \arrow[d] \\
\mathcal{C}_0^{\simeq} \times \mathcal{C}_1^{\simeq} \arrow[r] & \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}
\end{tikzcd}$$

is also a pullback square: that is, we have a canonical isomorphism of Kan complexes

$$(\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1)^{\simeq} \simeq \mathcal{C}_0^{\simeq} \times^{\mathrm{h}}_{\mathcal{C}^{\simeq}} \mathcal{C}_1^{\simeq}.$$

Definition 4.5.2.8. A commutative diagram of ∞-categories 0335

0336

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}.
\end{tikzcd} \tag{4.7}$$

is a *categorical pullback square* if the composite map

$$\mathcal{C}_{01} \to \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1 \hookrightarrow \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$$

is an equivalence of ∞-categories.

0337 **Remark 4.5.2.9.** Suppose we are given a categorical pullback diagram of ∞-categories

$$\begin{tikzcd} \mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\ \mathcal{C}_1 \arrow[r] & \mathcal{C}. \end{tikzcd}$$

Then, for every simplicial set X, the induced diagram

$$\begin{tikzcd} \mathrm{Fun}(X, \mathcal{C}_{01}) \arrow[r] \arrow[d] & \mathrm{Fun}(X, \mathcal{C}_0) \arrow[d] \\ \mathrm{Fun}(X, \mathcal{C}_1) \arrow[r] & \mathrm{Fun}(X, \mathcal{C}) \end{tikzcd}$$

is also a categorical pullback square. This follows by combining Remarks 4.5.2.6 and 4.5.1.16.

0338 **Proposition 4.5.2.10.** *A commutative diagram of Kan complexes*

0339
$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "q"] \\ X_1 \arrow[r] & X \end{tikzcd} \tag{4.8}$$

is a categorical pullback square if and only if it is a homotopy pullback square.

Proof. Combine Corollary 3.4.1.6 with Examples 4.5.2.3 and Example 4.5.1.13. □

046L **Variant 4.5.2.11.** Suppose we are given a commutative diagram of ∞-categories

046M
$$\begin{tikzcd} \mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d, "q"] \\ \mathcal{C}_1 \arrow[r] & \mathcal{C}, \end{tikzcd} \tag{4.9}$$

where $\mathcal{C}$ is a Kan complex. If (4.9) is a categorical pullback square, then it is also a homotopy pullback square.

Proof. By assumption, the induced map $\mathcal{C}_{01} \to \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is an equivalence of ∞-categories, and therefore a weak homotopy equivalence of simplicial sets (Remark 4.5.3.4). The desired result now follows from the criterion of Corollary 3.4.1.6. □

In more general situations, the notions of homotopy pullback square and categorical pullback square are distinct:

Exercise 4.5.2.12. Show that the diagram of ∞-categories 033A

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \{0\} \arrow[d] \\
\{1\} \arrow[r] & \Delta^1
\end{tikzcd}$$

is a categorical pullback square which is not a homotopy pullback square.

Exercise 4.5.2.13. Show that the diagram of ∞-categories 033B

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \Delta^1 \arrow[d] \\
\Delta^1 \arrow[r] & \Delta^1
\end{tikzcd}$$

is a homotopy pullback square which is not a categorical pullback square.

Proposition 4.5.2.14. *A commutative diagram of ∞-categories* 033C

033D

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}
\end{tikzcd} \tag{4.10}$$

is a categorical pullback square if and only if, for every simplicial set X, the diagram of Kan *complexes*

033E
$$\begin{tikzcd}
\operatorname{Fun}(X, \mathcal{C}_{01})^{\simeq} \arrow[r] \arrow[d] & \operatorname{Fun}(X, \mathcal{C}_{0})^{\simeq} \arrow[d] \\
\operatorname{Fun}(X, \mathcal{C}_{1})^{\simeq} \arrow[r] & \operatorname{Fun}(X, \mathcal{C})^{\simeq}
\end{tikzcd} \tag{4.11}$$

is a homotopy pullback square.

Proof. By definition, the diagram (4.10) is a categorical pullback square if and only if the induced map $\theta : \mathcal{C}_{01} \to \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is an equivalence of ∞-categories. Using the criterion of Proposition 4.5.1.22, we see that this is equivalent to the requirement that θ induces a homotopy equivalence $\theta_X : \operatorname{Fun}(X, \mathcal{C}_{01})^{\simeq} \to \operatorname{Fun}(X, \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1)^{\simeq}$ for every simplicial set X. Using Remarks 4.5.2.6 and 4.5.2.7, we can identify θ_X with the map

$$\operatorname{Fun}(X, \mathcal{C}_{01})^{\simeq} \to \operatorname{Fun}(X, \mathcal{C}_0)^{\simeq} \times^{\mathrm{h}}_{\operatorname{Fun}(X,\mathcal{C})^{\simeq}} \operatorname{Fun}(X, \mathcal{C}_1)^{\simeq}$$

determined by the commutative diagram (4.11). The desired result now follows from the criterion of Corollary 3.4.1.6. □

033F **Remark 4.5.2.15.** In the situation of Proposition 4.5.2.14, it suffices to verify that the diagram (4.11) is a homotopy pullback square in the case where X is an ∞-category. In fact, we will later see that it suffices to consider the case where $X = \Delta^1$ (Corollary 4.5.7.4).

We now apply Proposition 4.5.2.14 to deduce some formal properties of the notion of categorical pullback square.

033G **Proposition 4.5.2.16.** *A commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_{0} \arrow[d] \\
\mathcal{C}_{1} \arrow[r] & \mathcal{C}
\end{tikzcd}$$

is a categorical pullback square if and only if the induced diagram of opposite ∞-categories

$$\begin{tikzcd}
\mathcal{C}_{01}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathcal{C}_{0}^{\mathrm{op}} \arrow[d] \\
\mathcal{C}_{1}^{\mathrm{op}} \arrow[r] & \mathcal{C}^{\mathrm{op}}
\end{tikzcd}$$

is a categorical pullback square.

Proof. Combine Proposition 4.5.2.14 with Remark 3.4.1.7. □

Proposition 4.5.2.17 (Symmetry)**.** *A commutative diagram of ∞-categories* 033H

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}
\end{tikzcd}$$

is a categorical pullback square if and only if the transposed diagram

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_1 \arrow[d] \\
\mathcal{C}_0 \arrow[r] & \mathcal{C}
\end{tikzcd}$$

is a categorical pullback square.

Proof. Combine Propositions 4.5.2.14 and 3.4.1.9. □

Proposition 4.5.2.18 (Transitivity)**.** *Suppose we are given a commutative diagram of ∞-categories* 033J

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C}'' \arrow[d] \\
\mathcal{D} \arrow[r] & \mathcal{D}' \arrow[r] & \mathcal{D}'',
\end{tikzcd}$$

where the square on the right is a categorical pullback. Then the square on the left is a categorical pullback if and only if the outer rectangle is a categorical pullback.

Proof. Combine Propositions 4.5.2.14 and 3.4.1.11. □

Proposition 4.5.2.19 (Homotopy Invariance)**.** *Suppose we are given a commutative diagram* 033K *of ∞-categories*

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[rr] \arrow[dd] \arrow[dr, "F_{01}"] & & \mathcal{C}_0 \arrow[dd] \arrow[dr, "F_0"] & \\
& \mathcal{D}_{01} \arrow[rr] \arrow[dd] & & \mathcal{D}_0 \arrow[dd] \\
\mathcal{C}_1 \arrow[rr] \arrow[dr, "F_1"] & & \mathcal{C} \arrow[dr, "F"] & \\
& \mathcal{D}_1 \arrow[rr] & & \mathcal{D},
\end{tikzcd}$$

where F_0, F_1, and F are equivalences of ∞-categories. Then any two of the following conditions imply the third:

(1) *The back face*

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}
\end{tikzcd}$$

is a categorical pullback square.

(2) *The front face*

$$\begin{tikzcd}
\mathcal{D}_{01} \arrow[r] \arrow[d] & \mathcal{D}_0 \arrow[d] \\
\mathcal{D}_1 \arrow[r] & \mathcal{D}
\end{tikzcd}$$

is a categorical pullback square.

(3) *The functor F_{01} is an equivalence of ∞-categories.*

Proof. Using Proposition 4.5.1.22, we see that (3) is equivalent to the following:

(3′) For every simplicial set X, the functor F_{01} induces a homotopy equivalence of Kan complexes $\operatorname{Fun}(X, \mathcal{C}_{01})^{\simeq} \to \operatorname{Fun}(X, \mathcal{D}_{01})^{\simeq}$.

The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3′) now follow by combining Proposition 4.5.2.14 with Corollary 3.4.1.12. $\square$

Corollary 4.5.2.20. *Suppose we are given a commutative diagram of ∞-categories* 033L

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] & \mathcal{C}_1 \arrow[l] \arrow[d] \\
\mathcal{D}_0 \arrow[r] & \mathcal{D} & \mathcal{D} \arrow[l]
\end{tikzcd}$$

where the vertical maps are equivalences of ∞-categories. Then the induced map $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \to \mathcal{D}_0 \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D}_1$ is an equivalence of ∞-categories.

Proposition 4.5.2.21. *Suppose we are given a commutative diagram of ∞-categories* 033M

033N
$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D}' \arrow[r] & \mathcal{D}.
\end{tikzcd} \qquad (4.12)$$

where F is an equivalence of ∞-categories. Then (4.12) is a categorical pullback square if and only if F' is an equivalence of ∞-categories.

Proof. Combine Proposition 4.5.1.22, Proposition 4.5.2.14, and Corollary 3.4.1.5. $\square$

Corollary 4.5.2.22. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let* 0570

$$\delta : \mathcal{C} \to \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D} = \mathcal{C} \times_{\mathrm{Fun}(\{0\},\mathcal{D})} \mathrm{Fun}(\Delta^1, \mathcal{D})$$

be map induced by the diagonal embedding $c : \mathcal{D} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{D})$. Then δ is fully faithful, and its essential image is the homotopy fiber product $\mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D}$ of Construction 4.5.2.1.

Proof. Let us identify the objects of $\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ with triples (C, D, u), where C is an object of $\mathcal{C}$, D is an object of $\mathcal{D}$, and $u : F(C) \to D$ is a morphism in $\mathcal{D}$. By definition, $\mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D}$ is the full subcategory of $\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ spanned by those triples (C, D, u) where u is an isomorphism in $\mathcal{D}$. The functor δ is given on objects by the formula $\delta(C) = (C, F(C), \mathrm{id}_{F(C)})$, and therefore factors through $\mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D}$. To complete the proof, it will suffice to show that the functor $\delta : \mathcal{C} \to \mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D}$ is an equivalence of ∞-categories. Equivalently, we wish to show that the

diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{id}"] \arrow[d, "F"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D} \arrow[r, "\mathrm{id}"] & \mathcal{D}
\end{tikzcd}$$

is a categorical pullback square, which is a special case of Proposition 4.5.2.21. □

02VM **Corollary 4.5.2.23.** *Let $f : K \to \mathcal{D}$ be a morphism of simplicial sets, where $\mathcal{D}$ is an ∞-category. Then f factors as a composition $K \xrightarrow{j} \mathcal{C} \xrightarrow{U} \mathcal{D}$, where U is an isofibration of ∞-categories and j is both a monomorphism and a categorical equivalence.*

Proof. Using Proposition 4.1.3.2, we can factor f as a composition $K \xrightarrow{i} \mathcal{K} \xrightarrow{F} \mathcal{D}$, where i is inner anodyne and F is an inner fibration. Note that the simplicial set $\mathcal{K}$ is an ∞-category (Remark 4.1.1.9), and that i is a categorical equivalence of simplicial sets (Corollary 4.5.3.14). We may therefore replace f by F, and thereby reduce to the special case where $K = \mathcal{K}$ is an ∞-category. Let $\mathcal{C}$ denote the homotopy fiber product $\mathcal{K} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D}$. Then F factors as a composition

$$\mathcal{K} \xrightarrow{\delta} \mathcal{K} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D} \xrightarrow{U} \mathcal{D},$$

where the diagonal embedding δ is an equivalence of ∞-categories (Corollary 4.5.2.22) and U is an isofibration (see Remark 4.5.2.2). □

0479 **Remark 4.5.2.24.** Let $F : \mathcal{C} \to \mathcal{E}$ be an inner fibration of ∞-categories. Applying Corollary 4.5.2.23, we can factor F as a composition $\mathcal{C} \xrightarrow{F'} \mathcal{D} \xrightarrow{F''} \mathcal{E}$, where F'' is an isofibration and F' is an equivalence of ∞-categories. For each object $E \in \mathcal{E}$, the equivalence F' restricts to a functor $F'_E : \mathcal{C}_E \to \mathcal{D}_E$. Beware that F'_E need not be an equivalence of ∞-categories. However, it is always fully faithful: see Proposition 4.6.2.8.

037Y **Remark 4.5.2.25.** In the situation of Corollary 4.5.2.23, it is not necessary to assume that $\mathcal{D}$ is an ∞-category: every morphism of simplicial sets $f : X \to Z$ admits a factorization $X \xrightarrow{f'} Y \xrightarrow{f''} Z$, where f'' is an isofibration and f' both a monomorphism and a categorical equivalence (Proposition [?]). However, the proof is somewhat more difficult.

033P **Proposition 4.5.2.26.** *Suppose we are given a commutative diagram of ∞-categories*

033Q

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{D}' \arrow[r] & \mathcal{D},
\end{tikzcd} \tag{4.13}$$

where U is an isofibration. Then (4.13) is a categorical pullback square if and only if the induced map $\theta : \mathcal{C}' \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}'$ is an equivalence of ∞-categories.

Proof. For every simplicial set X, Corollary 4.4.5.7 guarantees that the induced map $\mathrm{Fun}(X, \mathcal{C})^{\simeq} \to \mathrm{Fun}(X, \mathcal{D})^{\simeq}$ is a Kan fibration. Combining Proposition 4.5.2.14 with Example 3.4.1.3, we see that (4.13) is a categorical pullback square if and only if it induces a homotopy equivalence

$$\rho_X : \mathrm{Fun}(X, \mathcal{C}')^{\simeq} \to \mathrm{Fun}(X, \mathcal{C})^{\simeq} \times_{\mathrm{Fun}(X,\mathcal{D})^{\simeq}} \mathrm{Fun}(X, \mathcal{D}')^{\simeq},$$

for every simplicial set X. Using Corollary 4.4.3.19, we can identify ρ_X with the map $\mathrm{Fun}(X, \mathcal{C}')^{\simeq} \to \mathrm{Fun}(X, \mathcal{C} \times_{\mathcal{D}} \mathcal{D}')^{\simeq}$ given by postcomposition with θ. The desired result now follows from the criterion of Proposition 4.5.1.22. □

Corollary 4.5.2.27. *Suppose we are given a pullback diagram of ∞-categories* 033R

033S

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{D}' \arrow[r] & \mathcal{D}.
\end{tikzcd} \tag{4.14}$$

If U is an isofibration, then (4.14) is a categorical pullback square.

Corollary 4.5.2.28. *Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories. If either F_0 or F_1 is an isofibration, then the comparison map* 033T

$$\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1 \hookrightarrow \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \qquad (C_0, C_1) \mapsto (C_0, C_1, \mathrm{id})$$

is an equivalence of ∞-categories.

Proof. This is a restatement of Corollary 4.5.2.27. □

Corollary 4.5.2.29. *Suppose we are given a pullback diagram of ∞-categories* 01HJ

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "F'"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{D}' \arrow[r, "F"] & \mathcal{D},
\end{tikzcd}$$

where U is an isofibration. If F is an equivalence of ∞-categories, then F' is also an equivalence of ∞-categories.

Proof. Combine Corollary 4.5.2.27 with Proposition 4.5.2.21. □

03PM **Corollary 4.5.2.30.** *Suppose we are given a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "U"] \arrow[d] & \mathcal{C} \arrow[d] & \mathcal{C}_1 \arrow[l] \arrow[d] \\
\mathcal{D}_0 \arrow[r, "V"] & \mathcal{D} & \mathcal{D}_1, \arrow[l]
\end{tikzcd}$$

where the vertical maps are equivalences of ∞-categories. If U and V are isofibrations, then the induced map $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1 \to \mathcal{D}_0 \times_{\mathcal{D}} \mathcal{D}_1$ is an equivalence of ∞-categories.

Proof. Combine Corollaries 4.5.2.20 and 4.5.2.28. □

033U **Corollary 4.5.2.31.** *Suppose we are given a categorical pullback square of ∞-categories*

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[r] \arrow[d, "U"] & \widetilde{\mathcal{D}} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D},
\end{tikzcd}$$

where U and V are isofibrations. Let $C \in \mathcal{C}$ be an object having image $D = F(C)$. Then the induced map

$$\widetilde{\mathcal{C}}_C = \{C\} \times_{\mathcal{C}} \widetilde{\mathcal{C}} \to \{D\} \times_{\mathcal{D}} \widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_D$$

is an equivalence of ∞-categories.

Proof. Apply Corollary 4.5.2.30 in the special case $\mathcal{C}_1 = \{C\}$ and $\mathcal{D}_1 = \{D\}$. □

01HK **Corollary 4.5.2.32.** *Suppose we are given a diagram of ∞-categories*

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[r, "\widetilde{F}"] \arrow[d, "U"] & \widetilde{\mathcal{D}} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D},
\end{tikzcd}$$

where U and V are isofibrations and the functors F and $\widetilde{F}$ are equivalences of ∞-categories. Let $C \in \mathcal{C}$ be an object having image $D = F(C)$. Then the induced map

$$\widetilde{\mathcal{C}}_C = \{C\} \times_{\mathcal{C}} \widetilde{\mathcal{C}} \to \{D\} \times_{\mathcal{D}} \widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}_D$$

is an equivalence of ∞-categories.

Proof. Combine Proposition 4.5.2.21 with Corollary 4.5.2.31. □

Warning 4.5.2.33. Suppose we are given a commutative diagram of Kan complexes 01HL

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S' \arrow[r, "f"] & S,
\end{tikzcd}$$

where q and q' are Kan fibrations and f is a homotopy equivalence. By virtue of Proposition 3.2.8.1, the following conditions are equivalent:

(1) The morphism f' is a homotopy equivalence of Kan complexes.

(2) For each vertex $s' \in S'$ having image $s = f(s') \in S$, the induced map of fibers $X'_{s'} \to X_s$ is a homotopy equivalence of Kan complexes.

Corollary 4.5.2.32 can be regarded as a generalization of the implication (1) ⇒ (2), where we allow ∞-categories in place of Kan complexes and isofibrations in place of Kan fibrations. Beware that the implication (2) ⇒ (1) does not generalize. For example, we have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\partial \Delta^1 \arrow[r] \arrow[d] & \Delta^1 \arrow[d, "\mathrm{id}"] \\
\Delta^1 \arrow[r, "\mathrm{id}"] & \Delta^1,
\end{tikzcd}$$

where the vertical maps are isofibrations, the bottom horizontal map is an isomorphism, and the upper horizontal map restricts to an isomorphism on each fiber, but is nevertheless not an equivalence of ∞-categories.

Corollary 4.5.2.34. *Let $U : \mathcal{E} \to \mathcal{C}$ be an isofibration of ∞-categories, let $B \to \mathcal{C}$ be a diagram, and let $f : A \to B$ be a categorical equivalence of simplicial sets. Then precomposition with f induces an equivalence of ∞-categories $\mathrm{Fun}_{/\mathcal{C}}(B, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{C}}(A, \mathcal{E})$.* 033V

Proof. Apply Corollary 4.5.2.32 to the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathcal{E}) \arrow[r, "\circ f"] \arrow[d, "U\circ"] & \mathrm{Fun}(A, \mathcal{E}) \arrow[d, "U\circ"] \\
\mathrm{Fun}(B, \mathcal{C}) \arrow[r, "\circ f"] & \mathrm{Fun}(A, \mathcal{C});
\end{tikzcd}$$

note that the vertical maps are isofibrations (Corollary 4.4.5.6) and the horizontal maps are equivalences of ∞-categories (Proposition 4.5.3.8). ☐

`02C5` **Corollary 4.5.2.35.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories, let $A \subseteq B$ be simplicial sets, and suppose we are given a diagram $A \to \mathcal{C}$. Then postcomposition with F induces an equivalence of ∞-categories $\operatorname{Fun}_{A/}(B, \mathcal{C}) \to \operatorname{Fun}_{A/}(B, \mathcal{D})$.*

Proof. Apply Corollary 4.5.2.32 to the commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}(B, \mathcal{C}) \arrow[r, "F\circ"] \arrow[d] & \operatorname{Fun}(B, \mathcal{D}) \arrow[d] \\
\operatorname{Fun}(A, \mathcal{C}) \arrow[r, "F\circ"] & \operatorname{Fun}(A, \mathcal{D});
\end{tikzcd}$$

note that the vertical maps are isofibrations by virtue of Corollary 4.4.5.3 and the horizontal maps are equivalences by virtue of Remark 4.5.1.16. ☐

`033W` **Remark 4.5.2.36** (Categorical Pullback Squares of Simplicial Sets)**.** Suppose we are given a commutative diagram of simplicial sets

`033X`
$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X.
\end{tikzcd} \tag{4.15}$$

Applying Proposition 4.1.3.2 repeatedly, we can enlarge 4.15 to a cubical diagram

$$\begin{tikzcd}
X_{01} \arrow[rr] \arrow[dd] \arrow[dr] & & X_0 \arrow[dd] \arrow[dr] & \\
& \mathcal{C}_{01} \arrow[rr] \arrow[dd] & & \mathcal{C}_0 \arrow[dd] \\
X_1 \arrow[rr] \arrow[dr] & & X \arrow[dr] & \\
& \mathcal{C}_1 \arrow[rr] & & \mathcal{C},
\end{tikzcd}$$

where the diagonal maps are inner anodyne and the front face

033Y

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}
\end{tikzcd} \tag{4.16}$$

is a diagram of ∞-categories. Let us say that that the diagram of simplicial sets (4.15) is a *categorical pullback square* if the diagram of ∞-categories (4.16) is a categorical pullback square, in the sense of Definition 4.5.2.8. Using Proposition 4.5.2.19, it is not difficult to show that this condition depends only on the original diagram (for a more general statement, see Proposition 7.5.5.13). Beware that this more general notion of categorical pullback diagram can be badly behaved: for example, it does not satisfy the analogue of Proposition 4.5.2.26 (see Warning 4.5.5.12).

4.5.3 Categorical Equivalence

Recall that a morphism of simplicial sets $f : X \to Y$ is a *weak homotopy equivalence* if, for every Kan complex Z, precomposition with f induces a bijection $\pi_0(\mathrm{Fun}(Y, Z)) \to \pi_0(\mathrm{Fun}(X, Z))$ (Definition 3.1.6.12). If this condition is satisfied, then one should regard X and Y as indistinguishable from the perspective of classical homotopy theory. However, from the ∞-categorical perspective, the relation of weak homotopy equivalence is somewhat too coarse: it is possible for a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ to be a weak homotopy equivalence (or even a homotopy equivalence) without being an equivalence of ∞-categories (Warning 4.5.1.14). For this reason, it will be convenient to introduce a finer notion of equivalence. 01E7

Definition 4.5.3.1. Let $f : X \to Y$ be a morphism of simplicial sets. We say that f is a *categorical equivalence* if, for every ∞-category $\mathcal{C}$, the induced functor $\mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ induces a bijection on isomorphism classes $\pi_0(\mathrm{Fun}(Y, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(X, \mathcal{C})^{\simeq})$. 01E8

Example 4.5.3.2. Every isomorphism of simplicial sets is a categorical equivalence. 01E9

Example 4.5.3.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is a categorical equivalence (in the sense of Definition 4.5.3.1) if and only if it is an equivalence of ∞-categories (in the sense of Definition 4.5.1.10). Both conditions are equivalent to the assertion that for every ∞-category $\mathcal{E}$, precomposition with F induces a bijection $\mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{D}, \mathcal{E}) \to \mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C}, \mathcal{E})$. 01EA

Remark 4.5.3.4. Let $f : X \to Y$ be a categorical equivalence of simplicial sets. Then f is a weak homotopy equivalence (since every Kan complex is an ∞-category). Beware that the converse is generally false. 01EB

01ED **Remark 4.5.3.5** (Two-out-of-Three)**.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of simplicial sets. If any two of the morphisms f, g, and $g \circ f$ is a categorical equivalence, then so is the third. In particular, the collection of categorical equivalences is closed under composition.

02R1 **Remark 4.5.3.6.** The collection of categorical equivalences is closed under retracts. That is, if there exists a commutative diagram of simplicial sets

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

where the horizontal compositions are the identity and f' is a categorical equivalence, then f is also a categorical equivalence.

01EE **Remark 4.5.3.7.** Let $f : X \to Y$ be a categorical equivalence of simplicial sets. Then, for any simplicial set K, the induced map $f_K : X \times K \to Y \times K$ is also a categorical equivalence of simplicial sets. To prove this, we must show that for every ∞-category $\mathcal{C}$, the restriction map $\theta : \mathrm{Fun}(Y \times K, \mathcal{C}) \to \mathrm{Fun}(X \times K, \mathcal{C})$ induces a bijection on isomorphism classes of objects. This follows from our assumption that f is a categorical equivalence, since θ can be identified with the map $\mathrm{Fun}(Y, \mathrm{Fun}(K, \mathcal{C})) \to \mathrm{Fun}(X, \mathrm{Fun}(K, \mathcal{C}))$ given by precomposition with f.

01EF **Proposition 4.5.3.8.** *Let $f : X \to Y$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism $f : X \to Y$ is a categorical equivalence. That is, for every ∞-category $\mathcal{C}$, precomposition with f induces a bijection*

$$\pi_0(\mathrm{Fun}(Y, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(X, \mathcal{C})^{\simeq}).$$

(2) *For every ∞-category $\mathcal{C}$, precomposition with f induces a homotopy equivalence of Kan complexes* $\mathrm{Fun}(Y, \mathcal{C})^{\simeq} \to \mathrm{Fun}(X, \mathcal{C})^{\simeq}$*.*

(3) *For every ∞-category $\mathcal{C}$, precomposition with f induces an equivalence of ∞-categories* $\mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$*.*

Proof. The implication (2) ⇒ (1) follows from Remark 3.1.6.5, and the implication (3) ⇒ (2) follows from Remark 4.5.1.19. We will complete the proof by showing that (1) implies (3). Assume that f is a categorical equivalence of simplicial sets, let $\mathcal{C}$ be an ∞-category, and let $f^* : \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ denote the functor given by precomposition with f. We wish to show that $[f^*]$ is an isomorphism in the homotopy category hQCat. For this, it will suffice to show that for any ∞-category $\mathcal{D}$, the induced map

$$\theta : \pi_0(\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(Y, \mathcal{C}))^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(X, \mathcal{C}))^{\simeq})$$

is bijective. We conclude by observing that θ can be identified with the map

$$\pi_0(\mathrm{Fun}(Y, \mathrm{Fun}(\mathcal{D}, \mathcal{C}))^{\simeq}) \to \pi_0(\mathrm{Fun}(X, \mathrm{Fun}(\mathcal{D}, \mathcal{C}))^{\simeq})$$

given by precomposition with f. □

Corollary 4.5.3.9. *Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let $f, f' : K \to \mathcal{C}$ be diagrams which are isomorphic (when viewed as objects of the ∞-category* $\mathrm{Fun}(K, \mathcal{C})$*). Then f is a categorical equivalence if and only if f' is a categorical equivalence.* 033Z

Corollary 4.5.3.10. *Let $\{f_i : X_i \to Y_i\}_{i \in I}$ be a collection of categorical equivalences indexed by a set I. Then the coproduct map* 023C

$$f : \coprod_{i \in I} X_i \to \coprod_{i \in I} Y_i$$

is also a categorical equivalence.

Proof. By virtue of Proposition 4.5.3.8, it will suffice to show that for every ∞-category $\mathcal{C}$, precomposition with f induces an equivalence of ∞-categories

$$F : \mathrm{Fun}(\coprod_{i \in I} Y_i, \mathcal{C}) \to \mathrm{Fun}(\coprod_{i \in I} X_i, \mathcal{C}).$$

Note that F factors as a product of functors $F_i : \mathrm{Fun}(Y_i, \mathcal{C}) \to \mathrm{Fun}(X_i, \mathcal{C})$, each of which is induced by precomposition with f_i. Since each f_i is a categorical equivalence, Proposition 4.5.3.8 guarantees that each F_i is an equivalence of ∞-categories. Applying Remark 4.5.1.17, we conclude that F is an equivalence of ∞-categories. □

Proposition 4.5.3.11. *Let $f : X \to Y$ be a trivial Kan fibration of simplicial sets. Then f is a categorical equivalence.* 01EG

Proof. Let $\mathcal{C}$ be an ∞-category. We wish to show that precomposition with f induces a bijection

$$f^* : \pi_0(\mathrm{Fun}(Y, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(X, \mathcal{C})^{\simeq}).$$

Let $s : Y \to X$ be a section of f (so that $f \circ s = \mathrm{id}_Y$). Then precomposition with s induces a function $s^* : \pi_0(\mathrm{Fun}(X, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(Y, \mathcal{C})^{\simeq})$ for which the composition $s^* \circ f^*$ is equal to the identity on the set $\pi_0(\mathrm{Fun}(Y, \mathcal{C})^{\simeq})$. We will complete the proof by showing that the composition $f^* \circ s^*$ is isomorphic to the identity on $\pi_0(\mathrm{Fun}(X, \mathcal{C})^{\simeq})$. Fix a map of simplicial sets $g : X \to \mathcal{C}$; we wish to show that g is isomorphic to the composite map

$$X \xrightarrow{f} Y \xrightarrow{s} X \xrightarrow{g} \mathcal{C}$$

as an object of the ∞-category $\mathrm{Fun}(X, \mathcal{C})$.

Since f is a trivial Kan fibration, the composition $s \circ f$ is fiberwise homotopic to the identity map id_X: that is, we can choose a morphism of simplicial sets $h : \Delta^1 \times X \to X$ which is compatible with the projection to Y and which satisfies $h|_{\{0\} \times X} = s \circ f$ and $h|_{\{1\} \times X} = \mathrm{id}_X$. The composition $g \circ h$ can then be regarded as a natural transformation $u : (g \circ s \circ f) \to g$. We will complete the proof by showing that u is an isomorphism in the ∞-category $\mathrm{Fun}(X, \mathcal{C})$. By virtue of Theorem 4.4.4.4, it will suffice to prove that for each vertex $x \in X$, the composite map

$$\Delta^1 \simeq \Delta^1 \times \{x\} \hookrightarrow \Delta^1 \times X \xrightarrow{h} X \xrightarrow{g} \mathcal{C}$$

describes an invertible morphism in $\mathcal{C}$. Setting $y = f(x)$, we note that this composite map factors through the (contractible) Kan complex X_y, so the desired result follows from Proposition 1.4.6.10. □

01EH **Corollary 4.5.3.12.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a trivial Kan fibration. Then F is an equivalence of ∞-categories.*

Proof. Combine Proposition 4.5.3.11 with Example 4.5.3.3. □

02C4 **Corollary 4.5.3.13.** *Let $\mathcal{C}$ be an ∞-category, and let $\mathrm{Isom}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by the isomorphisms of $\mathcal{C}$ (Example 4.4.1.14). Then the diagonal embedding*

$$\delta : \mathcal{C} \hookrightarrow \mathrm{Isom}(\mathcal{C}) \qquad C \mapsto \mathrm{id}_C$$

is an equivalence of ∞-categories.

Proof. Let $\mathrm{ev}_0 : \mathrm{Isom}(\mathcal{C}) \to \mathcal{C}$ denote the evaluation map

$$\mathrm{Isom}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}\,.$$

Then $\mathrm{ev}_0 \circ \delta$ is the identity functor $\mathrm{id}_{\mathcal{C}}$. Corollary 4.4.5.10 guarantees that ev_0 is a trivial Kan fibration, and therefore an equivalence of ∞-categories (Corollary 4.5.3.12). Applying the two-out-of-three property (Remark 4.5.1.18), we conclude that δ is also an equivalence of ∞-categories. □

01EJ **Corollary 4.5.3.14.** *Let $f : A \hookrightarrow B$ be an inner anodyne morphism of simplicial sets. Then f is a categorical equivalence.*

Proof. By virtue of Proposition 4.5.3.8, it will suffice to show that for every ∞-category $\mathcal{C}$, the restriction map $f^* : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$ is an equivalence of ∞-categories. This follows from Corollary 4.5.3.12, since f^* is a trivial Kan fibration (Proposition 1.5.7.6). □

01EK **Warning 4.5.3.15.** Let $f : A \to B$ be a morphism of simplicial sets. By virtue of Corollary 3.3.7.7, the morphism f is anodyne if and only if it is both a monomorphism and a weak homotopy equivalence. Beware that the analogous assertion for inner anodyne morphisms is false. If f is inner anodyne, then it is both a monomorphism (Remark 1.5.6.5) and a categorical equivalence (Corollary 4.5.3.14). However, the converse fails: a monomorphism $A \hookrightarrow B$ which is a categorical equivalence need not be inner anodyne. For example, an inner anodyne morphism of simplicial sets is automatically bijective on vertices (Exercise 1.5.6.6). However, there can be other obstructions as well: see Example 4.5.3.16.

01HD **Example 4.5.3.16** ([8]). Let $X = \Delta^2 \coprod_{\mathrm{N}_{\bullet}(\{1<2\})} \Delta^0$ be the simplicial set obtained from the standard 2-simplex by collapsing the final edge to a point, which we represent by the diagram

$$\begin{tikzcd}
 & \bullet \arrow[dr, equal] & \\
\bullet \arrow[ur, "e'"] \arrow[rr, "e"] & & \bullet.
\end{tikzcd}$$

Then X has exactly two nondegenerate edges $e, e' : \Delta^1 \to X$, as indicated in the diagram. We now make the following observations:

- There is a pushout diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r] \arrow[d] & \Delta^2 \arrow[d] \\
\Delta^1 \arrow[r, "e'"] & X.
\end{tikzcd}$$

  Consequently, the morphism $e' : \Delta^1 \to X$ is inner anodyne, and therefore a categorical equivalence (Corollary 4.5.3.14).

- There is a unique morphism of simplicial sets $r : X \to \Delta^1$ satisfying $r \circ e' = \mathrm{id}_{\Delta^1}$; the composite map $\Delta^2 \twoheadrightarrow X \xrightarrow{r} \Delta^1$ is given on vertices by $0 \mapsto 0$, $1 \mapsto 1$, and $2 \mapsto 1$. Since e' is a categorical equivalence, it follows that r is also a categorical equivalence (Remark 4.5.3.5).

- The composite map $\Delta^1 \xrightarrow{e} X \xrightarrow{r} \Delta^1$ is equal to the identity map id_{Δ^1}. Since r is a categorical equivalence, it follows that e is also a categorical equivalence. Moreover, e is also a monomorphism of simplicial sets which is bijective on vertices.

- The morphism $e : \Delta^1 \hookrightarrow X$ is an inner fibration. This follows from Remark 4.1.1.5, since we have a pullback diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^2_2 \arrow[r] \arrow[d] & \Delta^1 \arrow[d, "e"] \\
\Delta^2 \arrow[r] & X,
\end{tikzcd}$$

where the horizontal maps are surjective and the inclusion $\Lambda^2_2 \hookrightarrow \Delta^2$ is an inner fibration (since can be realized as the nerve of a morphism between partially ordered sets).

- The morphism e is not inner anodyne, since the lifting problem

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\mathrm{id}"] \arrow[d, "e"] & \Delta^1 \arrow[d, "e"] \\
X \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & X
\end{tikzcd}$$

has no solution.

01EL **Remark 4.5.3.17** (Axioms for Categorical Equivalence)**.** The collection of categorical equivalences of simplicial sets has the following properties:

(A) If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then a functor $F : \mathcal{C} \to \mathcal{D}$ is a categorical equivalence if and only if it is an equivalence of ∞-categories (Example 4.5.3.3).

(B) Every inner anodyne morphism of simplicial sets is a categorical equivalence (Corollary 4.5.3.14).

(C) If $f : X \to Y$ and $g : Y \to Z$ have the property that two of the morphisms f, g, and $g \circ f$ are categorical equivalences, then so is the third (Remark 4.5.3.5).

In fact, the collection of categorical equivalences is characterized by assertions (A), (B) and (C). Let $f : X \to Y$ be a morphism of simplicial sets. Invoking Proposition 4.1.3.1 twice, we can construct a commutative diagram of simplicial sets

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & \mathcal{C} \arrow[d, "F"] \\
Y \arrow[r, "v"] & \mathcal{D},
\end{tikzcd}$$

where u and v are inner anodyne and $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories. It follows from (A), (B) and (C) that the morphism f is a categorical equivalence if and only if the functor F is an equivalence of ∞-categories.

4.5.4 Categorical Pushout Squares

Recall that a commutative diagram of simplicial sets 01F6

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a *homotopy pushout square* if, for every Kan complex X, the diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(A, X) & \operatorname{Fun}(A_0, X) \arrow[l] \\
\operatorname{Fun}(A_1, X) \arrow[u] & \operatorname{Fun}(A_{01}, X) \arrow[l] \arrow[u]
\end{tikzcd}$$

is a homotopy pullback square (Definition 3.4.2.1). In this section, we study a stronger version of this condition.

Definition 4.5.4.1. A commutative diagram of simplicial sets 01F7

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a *categorical pushout square* if, for every ∞-category $\mathcal{C}$, the diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(A, \mathcal{C})^{\simeq} & \operatorname{Fun}(A_0, \mathcal{C})^{\simeq} \arrow[l] \\
\operatorname{Fun}(A_1, \mathcal{C})^{\simeq} \arrow[u] & \operatorname{Fun}(A_{01}, \mathcal{C})^{\simeq} \arrow[l] \arrow[u]
\end{tikzcd}$$

is a homotopy pullback square.

01F8 **Remark 4.5.4.2.** Every categorical pushout square of simplicial sets is also a homotopy pushout square of simplicial sets (since every Kan complex X is an ∞-category which satisfies $\operatorname{Fun}(K, X)^{\simeq} = \operatorname{Fun}(K, X)$ for every simplicial set K).

02VN **Remark 4.5.4.3.** Suppose we are given a categorical pushout square of simplicial sets

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}.
\end{tikzcd}$$

Then, for every simplicial set K, the induced diagram

$$\begin{tikzcd}
A \times K \arrow[r] \arrow[d] & A_0 \times K \arrow[d] \\
A_1 \times K \arrow[r] & A_{01} \times K
\end{tikzcd}$$

is also a categorical pushout square. That is, for every ∞-category $\mathcal{C}$, the diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(A \times K, \mathcal{C})^{\simeq} & \operatorname{Fun}(A_0 \times K, \mathcal{C})^{\simeq} \arrow[l] \\
\operatorname{Fun}(A_1 \times K, \mathcal{C})^{\simeq} \arrow[u] & \operatorname{Fun}(A_{01} \times K, \mathcal{C})^{\simeq} \arrow[l] \arrow[u]
\end{tikzcd}$$

is a homotopy pullback square. This follows by applying the requirement Definition 4.5.4.1 to the ∞-category $\operatorname{Fun}(K, \mathcal{C})$.

Proposition 4.5.4.4. *A commutative diagram of simplicial sets* 0340

0341

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd} \tag{4.17}$$

is a categorical pushout square if and only if it satisfies the following condition:

$(*)$ *For every ∞-category $\mathcal{C}$, the diagram of ∞-categories*

0342

$$\begin{tikzcd}
\operatorname{Fun}(A, \mathcal{C}) & \operatorname{Fun}(A_0, \mathcal{C}) \arrow[l] \\
\operatorname{Fun}(A_1, \mathcal{C}) \arrow[u] & \operatorname{Fun}(A_{01}, \mathcal{C}) \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.18}$$

is a categorical pullback square.

Proof. Fix an ∞-category $\mathcal{C}$. If the diagram of ∞-categories (4.18) is a categorical pullback square, then the diagram of cores

$$\begin{tikzcd}
\operatorname{Fun}(A, \mathcal{C})^{\simeq} & \operatorname{Fun}(A_0, \mathcal{C})^{\simeq} \arrow[l] \\
\operatorname{Fun}(A_1, \mathcal{C})^{\simeq} \arrow[u] & \operatorname{Fun}(A_{01}, \mathcal{C})^{\simeq} \arrow[l] \arrow[u]
\end{tikzcd}$$

is a homotopy pullback square (Proposition 4.5.2.14). Allowing $\mathcal{C}$ to vary, we see that if $(*)$ is satisfied, then (4.17) is a categorical pushout square. For the converse, assume that (4.17) is a categorical pullback square. For every simplicial set X, the simplicial set $\operatorname{Fun}(X, \mathcal{C})$ is an ∞-category (Theorem 1.5.3.7), so the diagram of Kan complexes

0343

$$\begin{tikzcd}
\operatorname{Fun}(A, \operatorname{Fun}(X, \mathcal{C}))^{\simeq} & \operatorname{Fun}(A_0, \operatorname{Fun}(X, \mathcal{C}))^{\simeq} \arrow[l] \\
\operatorname{Fun}(A_1, \operatorname{Fun}(X, \mathcal{C}))^{\simeq} \arrow[u] & \operatorname{Fun}(A_{01}, \operatorname{Fun}(X, \mathcal{C}))^{\simeq} \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.19}$$

is a homotopy pullback square. Identifying (4.19) with the diagram

$$\begin{tikzcd}
\operatorname{Fun}(X, \operatorname{Fun}(A, \mathcal{C}))^{\simeq} & \operatorname{Fun}(X, \operatorname{Fun}(A_0, \mathcal{C}))^{\simeq} \arrow[l] \\
\operatorname{Fun}(X, \operatorname{Fun}(A_1, \mathcal{C}))^{\simeq} \arrow[u] & \operatorname{Fun}(X, \operatorname{Fun}(A_{01}, \mathcal{C}))^{\simeq} \arrow[l] \arrow[u]
\end{tikzcd}$$

and allowing X to vary, we conclude that the diagram (4.18) is a categorical pullback square (Proposition 4.5.2.14). ☐

02VP **Corollary 4.5.4.5.** *Suppose we are given a categorical pushout square of simplicial sets*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
A' \arrow[r] & B',
\end{tikzcd}$$

where the horizontal maps are monomorphisms. Let $\mathcal{C}$ be an ∞-category. For every diagram $A' \to \mathcal{C}$, the restriction map $\operatorname{Fun}_{A'/}(B', \mathcal{C}) \to \operatorname{Fun}_{A/}(B, \mathcal{C})$ is an equivalence of ∞-categories.

Proof. Proposition 4.5.4.4 guarantees that the diagram

$$\begin{tikzcd}
\operatorname{Fun}(B', \mathcal{C}) \arrow[r] \arrow[d] & \operatorname{Fun}(B, \mathcal{C}) \arrow[d] \\
\operatorname{Fun}(A', \mathcal{C}) \arrow[r] & \operatorname{Fun}(A, \mathcal{C})
\end{tikzcd}$$

is a categorical pullback square, and Corollary 4.4.5.3 guarantees that the vertical maps are isofibrations. The desired result now follows from Corollary 4.5.2.31. ☐

0344 **Proposition 4.5.4.6.** *A commutative diagram of simplicial sets*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a categorical pushout square if and only if the induced diagram of opposite simplicial sets

$$\begin{tikzcd}
A^{\mathrm{op}} \arrow[r] \arrow[d] & A_0^{\mathrm{op}} \arrow[d] \\
A_1^{\mathrm{op}} \arrow[r] & A_{01}^{\mathrm{op}}
\end{tikzcd}$$

is a categorical pushout square.

Proof. Apply Remark 3.4.1.7. □

Proposition 4.5.4.7 (Symmetry)**.** *A commutative diagram of simplicial sets* 01F9

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a categorical pushout square if and only if the transposed diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_1 \arrow[d] \\
A_0 \arrow[r] & A_{01}
\end{tikzcd}$$

is a categorical pushout square.

Proof. Apply Proposition 3.4.1.9. □

Proposition 4.5.4.8 (Transitivity)**.** *Suppose we are given a commutative diagram of simplicial sets* 01FA

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C',
\end{tikzcd}$$

where the left square is a categorical pushout. Then the right square is a categorical pushout if and only if the outer rectangle is a categorical pushout.

Proof. Apply Proposition 3.4.1.11. □

01FB **Proposition 4.5.4.9** (Homotopy Invariance)**.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, "w"] \arrow[dd] & & A_0 \arrow[dr, "w_0"] \arrow[dd] & \\
& B \arrow[rr] \arrow[dd] & & B_0 \arrow[dd] \\
A_1 \arrow[rr] \arrow[dr, "w_1"] & & A_{01} \arrow[dr, "w_{01}"] & \\
& B_0 \arrow[rr] & & B_{01},
\end{tikzcd}$$

where the morphisms w, w_0, and w_1 are categorical equivalences. Then any two of the following three conditions imply the third:

(1) *The back face*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a categorical pushout square.

(2) *The front face*

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & B_0 \arrow[d] \\
B_1 \arrow[r] & B_{01}
\end{tikzcd}$$

is a categorical pushout square.

(3) *The morphism w_{01} is a categorical equivalence of simplicial sets.*

Proof. Combine Corollary 3.4.1.12 with Proposition 4.5.3.8. □

Proposition 4.5.4.10. *Suppose we are given a commutative diagram of simplicial sets* 01FC

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r, "f'"] & A_{01} \end{tikzcd} \quad (4.20)$$ 01FD

where f is a categorical equivalence. Then (4.20) is a categorical pushout square if and only if f' is a categorical equivalence.

Proof. For every ∞-category $\mathcal{C}$, we obtain a commutative diagram of simplicial sets

$$\begin{tikzcd} \mathrm{Fun}(A, \mathcal{C})^{\simeq} & \mathrm{Fun}(A_0, \mathcal{C})^{\simeq} \arrow[l, "u"'] \\ \mathrm{Fun}(A_1, \mathcal{C})^{\simeq} \arrow[u] & \mathrm{Fun}(A_{01}, \mathcal{C})^{\simeq}, \arrow[l, "u'"'] \arrow[u] \end{tikzcd} \quad (4.21)$$ 01FE

where u is a homotopy equivalence of Kan complexes (Proposition 4.5.3.8). Applying Corollary 3.4.1.5, we conclude that (4.21) is a homotopy pullback square if and only if u' is a homotopy equivalence of Kan complexes. Consequently, (4.20) is a categorical pushout square if and only if, for every ∞-category $\mathcal{C}$, the composition with f' induces a homotopy equivalence $\mathrm{Fun}(A_{01}, \mathcal{C})^{\simeq} \to \mathrm{Fun}(A_1, \mathcal{C})^{\simeq}$. By virtue of Proposition 4.5.3.8, this is equivalent to the requirement that f' is a categorical equivalence. □

Proposition 4.5.4.11. *Suppose we are given a commutative diagram of simplicial sets* 01FF

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01}, \end{tikzcd} \quad (4.22)$$ 01FG

where f is a monomorphism. Then (4.22) is a categorical pushout square if and only if the induced map $\rho : A_0 \coprod_A A_1 \to A_{01}$ is a categorical equivalence of simplicial sets.

Proof. For every ∞-category $\mathcal{C}$, we obtain a commutative diagram of ∞-categories

01FH
$$\begin{tikzcd}
\mathrm{Fun}(A, \mathcal{C}) & \mathrm{Fun}(A_0, \mathcal{C}) \arrow[l, "u"'] \\
\mathrm{Fun}(A_1, \mathcal{C}) \arrow[u] & \mathrm{Fun}(A_{01}, \mathcal{C}), \arrow[l] \arrow[u]
\end{tikzcd} \tag{4.23}$$

where u is an isofibration (Corollary 4.4.5.3). It follows that the diagram (4.23) is a categorical pullback square if and only if the induced map

$$\theta_{\mathcal{C}} : \mathrm{Fun}(A_{01}, \mathcal{C}) \to \mathrm{Fun}(A_0, \mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{C})} \mathrm{Fun}(A_1, \mathcal{C}) \simeq \mathrm{Fun}(A_0 \coprod_A A_1, \mathcal{C})$$

is an equivalence of ∞-categories (Proposition 4.5.2.26). Using Proposition 4.5.4.4, we see that this condition is satisfied for every ∞-category $\mathcal{C}$ if and only if (4.22) is a categorical pushout square. The desired result now follows from Proposition 4.5.3.8. □

01FJ **Example 4.5.4.12.** Suppose we are given a pushout diagram of simplicial sets

01FK
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}.
\end{tikzcd} \tag{4.24}$$

If f is a monomorphism, then (4.24) is also a categorical pushout square.

01HE **Remark 4.5.4.13.** Suppose we are given a pushout diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & A_0 \arrow[d, "g'"] \\
A_1 \arrow[r] & A_{01},
\end{tikzcd}$$

where f is a monomorphism. If g is a categorical equivalence, then g' is also a categorical equivalence. This follows by combining Example 4.5.4.12 with Proposition 4.5.4.10.

Corollary 4.5.4.14. *Suppose we are given a commutative diagram of simplicial sets* 0345

$$\begin{tikzcd}
A_0 \arrow[d] & A \arrow[l, "f_0"'] \arrow[r, "f_1"] \arrow[d] & A_1 \arrow[d] \\
B_0 & B \arrow[l, "g_0"'] \arrow[r, "g_1"] & B_1,
\end{tikzcd}$$

where f_0 and g_0 are monomorphisms and the vertical maps are categorical equivalences. Then the induced map

$$A_0 \coprod_A A_1 \to B_0 \coprod_B B_1$$

is a categorical equivalence.

Proof. Combine Example 4.5.4.12 with Proposition 4.5.4.9. □

Corollary 4.5.4.15. *Let $i : A \to B$ and $i' : A' \to B'$ be morphisms of simplicial sets. Assume that i is a monomorphism and that either i or i' is a categorical equivalence. Then the induced map* 01FL

$$(A \times B') \coprod_{(A \times A')} (B \times A') \to B \times B'$$

is a categorical equivalence.

Proof. By virtue of Proposition 4.5.4.11, it will suffice to show that the diagram

01FM

$$\begin{tikzcd}
A \times A' \arrow[r] \arrow[d] & B \times A' \arrow[d] \\
A \times B' \arrow[r] & B \times B'
\end{tikzcd} \tag{4.25}$$

is a categorical pushout square. This follows from the criterion of Proposition 4.5.4.10: if i is a categorical equivalence, then the horizontal maps in the diagram (4.25) are categorical equivalences (Remark 4.5.3.7). Similarly, if i' is a categorical equivalence, then the vertical maps in the diagram (4.25) are categorical equivalences. □

4.5.5 Isofibrations of Simplicial Sets

We now characterize isofibrations between ∞-categories by means of a lifting property. 01FU

Proposition 4.5.5.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F is an isofibration if and only if it satisfies the following condition:* 01FR

$(*)$ *Let B be a simplicial set and let $A \subseteq B$ be a simplicial subset for which the inclusion $A \hookrightarrow B$ is a categorical equivalence. Then every lifting problem*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
B \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution.

We begin by proving a weak form of Proposition 4.5.5.1.

01FS **Lemma 4.5.5.2.** *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, and let $A \subseteq B$ be a simplicial subset with the property that the inclusion $A \hookrightarrow B$ is a categorical equivalence. Then every diagram $f_0 : A \to \mathcal{C}$ can be extended to a diagram $f : B \to \mathcal{C}$.*

Proof. By virtue of Corollary 4.4.5.4, the restriction map $\theta : \mathrm{Fun}(B, \mathcal{C})^{\simeq} \to \mathrm{Fun}(A, \mathcal{C})^{\simeq}$ is a Kan fibration. Since the inclusion $A \hookrightarrow B$ is a categorical equivalence, the map θ is a homotopy equivalence of Kan complexes (Proposition 4.5.3.8). Invoking Proposition 3.3.7.6, we conclude that θ is a trivial Kan fibration. In particular, it is surjective on vertices. □

01FT **Lemma 4.5.5.3.** *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, let $A \subseteq B$ be a simplicial subset, and suppose we are given a pair of diagrams $f, g : B \to \mathcal{C}$ together with a natural transformation $u_0 : f|_A \to f'|_A$. If the inclusion $A \hookrightarrow B$ is a categorical equivalence, then u_0 can be lifted to a natural transformation $u : f \to g$. Moreover, if u_0 is a natural isomorphism, then u is automatically a natural isomorphism.*

Proof. The existence of the natural transformation u follows by applying Lemma 4.5.5.2 to the inclusion of simplicial sets

$$(\Delta^1 \times A) \coprod_{(\partial \Delta^1 \times A)} (\partial \Delta^1 \times B) \hookrightarrow \Delta^1 \times B,$$

which is a categorical equivalence by virtue of Corollary 4.5.4.15. We will complete the proof by showing that if u_0 is a natural isomorphism, then u is a natural isomorphism.

Let us identify u with a morphism of simplicial sets $v : B \to \mathrm{Fun}(\Delta^1, \mathcal{C})$, and let $\mathrm{Isom}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by the isomorphisms in $\mathcal{C}$. Since u_0 is a natural isomorphism, the restriction $v|_A$ factors through the full subcategory $\mathrm{Isom}(\mathcal{C})$. Invoking Lemma 4.5.5.2, we conclude that $v|_A$ extends to a diagram $v' : B \to \mathrm{Isom}(\mathcal{C})$. Since the inclusion $A \hookrightarrow B$ is a categorical equivalence, the equality $v|_A = v'|_A$ guarantees that v and v' are isomorphic as objects of the ∞-category $\mathrm{Fun}(B, \mathrm{Fun}(\Delta^1, \mathcal{C}))$. Since the full subcategory $\mathrm{Isom}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ is replete (Example 4.4.1.14), we conclude that v also factors through $\mathrm{Isom}(\mathcal{C})$, so that u is a natural isomorphism by virtue of Theorem 4.4.4.4. □

Proof of Proposition 4.5.5.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Assume first that F satisfies condition $(*)$ of Proposition 4.5.5.1; we will prove that F is an isofibration. For $0 < i < n$, the inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is a categorical equivalence (Corollary 4.5.3.14), so condition $(*)$ guarantees that F is an inner fibration. Fix an object $C \in \mathcal{C}$ and an isomorphism $u : D \to F(C)$ in the ∞-category $\mathcal{D}$; we wish to show that u can be lifted to an isomorphism $\overline{u} : \overline{D} \to C$ in the ∞-category $\mathcal{C}$. By virtue of Corollary 4.4.3.14, we can assume that $u = G(v)$ for some functor $G : \mathcal{E} \to \mathcal{D}$, where $\mathcal{E}$ is a contractible Kan complex and $v : X \to Y$ is a morphism in $\mathcal{E}$. Since the inclusion $\{Y\} \hookrightarrow \mathcal{E}$ is a categorical equivalence (Example 4.5.1.13), condition $(*)$ guarantees the existence of a solution to the lifting problem

$$\begin{tikzcd}
\{Y\} \arrow[r, "Y \mapsto C"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{E} \arrow[r, "G"] \arrow[ur, dashed, "\overline{G}"] & \mathcal{D}.
\end{tikzcd}$$

Then $\overline{u} = \overline{G}(v)$ is an isomorphism of $\mathcal{C}$ having the desired property.

Now suppose that the functor $F : \mathcal{C} \to \mathcal{D}$ is an isofibration; we wish to show that condition $(*)$ is satisfied. Let B be a simplicial set and $A \subseteq B$ a simplicial subset for which the inclusion $A \hookrightarrow B$ is a categorical equivalence. We wish to show that every lifting problem

$$\begin{tikzcd}
A \arrow[r, "f_0"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
B \arrow[r, "\overline{f}"] \arrow[ur, dashed, "f"] & \mathcal{D}
\end{tikzcd}$$

admits a solution. Invoking Lemma 4.5.5.2, we see that f_0 can be extended to a morphism of simplicial sets $f' : B \to \mathcal{C}$. Let $\overline{f}'$ denote the composition $B \xrightarrow{f'} \mathcal{C} \xrightarrow{F} \mathcal{D}$, so that $\overline{f}|_A = \overline{f}'|_A$. Invoking Lemma 4.5.5.3, we conclude that there exists an isomorphism $\overline{u} : \overline{f} \to \overline{f}'$ in the diagram ∞-category $\mathrm{Fun}(B, \mathcal{D})$ whose image in $\mathrm{Fun}(A, \mathcal{D})$ is the identity transformation $\mathrm{id}_{\overline{f}|_A}$. Applying Corollary 4.4.5.9, we deduce that $\overline{u}$ can be lifted to an isomorphism $u : f \to f'$ in the diagram ∞-category $\mathrm{Fun}(B, \mathcal{C})$ whose image in $\mathrm{Fun}(A, \mathcal{C})$ is the identity transformation id_{f_0}. The diagram $f : B \to \mathcal{C}$ then satisfies $f|_A = f_0$ and $F \circ f = \overline{f}$, as desired. □

Proposition 4.5.5.1 has a converse:

02R2 **Proposition 4.5.5.4.** *Let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets. Then i is a categorical equivalence if and only if the following condition is satisfied:*

$(*)$ *Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories. Then every lifting problem*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ B \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

has a solution.

Proof. Assume that condition $(*)$ is satisfied; we will show that the morphism $i : A \hookrightarrow B$ is a categorical equivalence of simplicial sets (the converse follows from Proposition 4.5.5.1). Fix an ∞-category $\mathcal{E}$; we wish to show that precomposition with i induces a bijection $\theta : \pi_0(\mathrm{Fun}(B, \mathcal{E})^{\simeq}) \to \pi_0(\mathrm{Fun}(A, \mathcal{E})^{\simeq})$. The surjectivity of θ follows by applying condition $(*)$ to the isofibration $\mathcal{E} \to \Delta^0$, and the injectivity of θ follows by applying θ to the isofibration $\mathrm{Isom}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}$ of Corollary 4.4.5.5. □

We now use the characterization of Proposition 4.5.5.1 to generalize the notion of isofibration to arbitrary simplicial sets.

01FV **Definition 4.5.5.5.** Let $q : X \to S$ be a morphism of simplicial sets. We will say that q is an *isofibration* if it satisfies the following condition:

$(*)$ Let B be a simplicial set and let $A \subseteq B$ be a simplicial subset for which the inclusion $A \hookrightarrow B$ is a categorical equivalence. Then every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

admits a solution.

01FW **Remark 4.5.5.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We have now given two *a priori* different definitions of an isofibration from $\mathcal{C}$ to $\mathcal{D}$:

- According to Definition 4.4.1.4, an isofibration $F : \mathcal{C} \to \mathcal{D}$ is an inner fibration with the property that every isomorphism $u : D \to F(C)$ in the ∞-category $\mathcal{D}$ can be lifted to an isomorphism $\overline{u} : \overline{D} \to C$ in the ∞-category $\mathcal{C}$.

- According to Definition 4.5.5.5, an isofibration $F : \mathcal{C} \to \mathcal{D}$ is a morphism of simplicial sets which has the right lifting property with respect to all monomorphisms $A \hookrightarrow B$ which are categorical equivalences.

However, these definitions are equivalent: this is the content of Proposition 4.4.5.1.

Remark 4.5.5.7. Let $q : X \to S$ be an isofibration of simplicial sets. Then q is an inner fibration: that is, it has the right lifting property with respect to every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i < n$ (such inclusions are categorical equivalences, by virtue of Corollary 4.5.3.14). In particular, for each vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is an ∞-category (Remark 4.1.1.6). Moreover, if S is an ∞-category, then X is also an ∞-category (Remark 4.1.1.9). 01FX

Example 4.5.5.8. Let $q : X \to S$ be a Kan fibration of simplicial sets. Then q is an isofibration. To prove this, we note that if a monomorphism of simplicial sets $i : A \hookrightarrow B$ is a categorical equivalence, then it is a weak homotopy equivalence (Remark 4.5.3.4) and therefore anodyne (Corollary 3.3.7.7), so that q has the right lifting property with respect to i (Remark 3.1.2.7). 01FY

Remark 4.5.5.9. Let $q : X \to S$ be a morphism of simplicial sets. Then q is an isofibration if and only if the opposite morphism $q^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is an isofibration. 01FZ

Remark 4.5.5.10. The collection of isofibrations is closed under retracts. That is, given a diagram of simplicial sets 01G0

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "q"] & X' \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S \arrow[r] & S' \arrow[r] & S
\end{tikzcd}$$

where both horizontal compositions are the identity, if q' is an isofibration, then so is q.

Remark 4.5.5.11. The collection of isofibrations is closed under pullback. That is, given a pullback diagram of simplicial sets 01G1

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S' \arrow[r] & S
\end{tikzcd}$$

where q is an isofibration, the morphism q' is also an isofibration.

01J1 **Warning 4.5.5.12.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S' \arrow[r, "f"] & S,
\end{tikzcd}$$

where q is an isofibration. If f is an equivalence of ∞-categories, then f' is also an equivalence of ∞-categories (Corollary 4.5.2.29). Beware that if f is merely assumed to be a categorical equivalence of simplicial sets, then it is not necessarily true that f' is a categorical equivalence of simplicial sets.

01G2 **Remark 4.5.5.13.** Let $p : X \to Y$ and $q : Y \to Z$ be isofibrations of simplicial sets. Then the composite map $(q \circ p) : X \to Z$ is an isofibration of simplicial sets.

We have the following generalization of Proposition 4.4.5.1:

01G3 **Proposition 4.5.5.14.** *Let $q : X \to S$ be an isofibration of simplicial sets and let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets. Then the restriction map*

$$q' : \operatorname{Fun}(B, X) \to \operatorname{Fun}(A, X) \times_{\operatorname{Fun}(A,S)} \operatorname{Fun}(B, S)$$

is also an isofibration of simplicial sets.

Proof. Let B' be a simplicial set and let $A' \subseteq B'$ be a simplicial subset for which the inclusion $A' \hookrightarrow B'$ is a categorical equivalence. We wish to show that every lifting problem

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & \operatorname{Fun}(B, X) \arrow[d, "q'"] \\
B' \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(A, X) \times_{\operatorname{Fun}(A,S)} \operatorname{Fun}(B, S)
\end{tikzcd}$$

admits a solution. Unwinding the definitions, we are reduced to the problem of solving an associated lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{(A \times A')} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

The left vertical map in this diagram is a categorical equivalence by virtue of Corollary 4.5.4.15, so the existence of the desired solution follows from our assumption that q is an isofibration. □

Corollary 4.5.5.15. *Let $q : X \to S$ be an isofibration of simplicial sets. For every simplicial set B, the induced map $\mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S)$ is also an isofibration.* `01G4`

Proof. Apply Proposition 4.5.5.14 in the special case $A = \emptyset$. □

Corollary 4.5.5.16. *Let $q : X \to S$ be an isofibration of simplicial sets. Suppose we are given a morphism of simplicial sets $B \to S$ and a simplicial subset $A \subseteq B$. Then the restriction map $\theta : \mathrm{Fun}_{/S}(B, X) \to \mathrm{Fun}_{/S}(A, X)$ is an isofibration of ∞-categories.* `02VQ`

Proof. The morphism θ is a pullback of the isofibration $\mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B, S)$ of Proposition 4.5.5.14, and is therefore also an isofibration (Remark 4.5.5.11). We conclude by observing that since q is an inner fibration (Remark 4.5.5.7), the simplicial sets $\mathrm{Fun}_{/S}(B, X)$ and $\mathrm{Fun}_{/S}(A, X)$ are ∞-categories (Proposition 4.1.4.6). □

Remark 4.5.5.17. Suppose we are given a lifting problem in the category of simplicial sets `02VR`

`02VS`

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "q"] \\
B \arrow[r, "\overline{f}"] \arrow[ur, dashed] & S,
\end{tikzcd} \tag{4.26}$$

where q is an isofibration and i is a monomorphism. It follows from Corollary 4.5.5.16 that, if we regard the morphisms q, i, and $\overline{f}$ as fixed, then the existence of a solution to the lifting problem (4.26) depends only on the isomorphism class of f as an object of the ∞-category $\mathrm{Fun}_{/S}(A, X)$.

Proposition 4.5.5.18. *Let $q : X \to S$ be an isofibration of simplicial sets and let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets. If i is a categorical equivalence, then the restriction map* `01G5`

$$q' : \mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B, S)$$

is a trivial Kan fibration.

Proof. Let B' be a simplicial set and let $A' \subseteq B'$ be a simplicial subset. We wish to show that every lifting problem

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & \mathrm{Fun}(B, X) \arrow[d, "q'"] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,S)} \mathrm{Fun}(B, S)
\end{tikzcd}$$

admits a solution. Unwinding the definitions, we are reduced to the problem of solving an associated lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{(A \times A')} (B \times A') \arrow[r] \arrow[d] & X \arrow[d, "q"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & S.
\end{tikzcd}$$

The left vertical map in this diagram is a categorical equivalence by virtue of Corollary 4.5.4.15, so the existence of the desired solution follows from our assumption that q is an isofibration. □

01G6 **Corollary 4.5.5.19.** *Let $\mathcal{C}$ be an ∞-category and let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets. If i is a categorical equivalence, then the restriction functor* $\mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$ *is a trivial Kan fibration of simplicial sets.*

Proof. Apply Proposition 4.5.5.18 in the special case $S = \Delta^0$. □

01G7 **Proposition 4.5.5.20.** *Let $q : X \to S$ be a morphism of simplicial sets. Then q is a trivial Kan fibration if and only if it is both an isofibration and a categorical equivalence.*

Proof. If q is a trivial Kan fibration, then it is an isofibration by virtue of Example 4.5.5.8 and a categorical equivalence by virtue of Proposition 4.5.3.11. Conversely, suppose that q is both an isofibration and a categorical equivalence. Using Exercise 3.1.7.11, we can write q as a composition $X \xrightarrow{q'} Y \xrightarrow{q''} S$, where q' is a monomorphism and q'' is a trivial Kan fibration. Then q'' is a categorical equivalence (Proposition 4.5.3.11), so that q' is also a categorical equivalence (Remark 4.5.3.5). Invoking our assumption that q is an isofibration, we conclude that the lifting problem

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "q'"] & X \arrow[d, "q"] \\
Y \arrow[r, "q''"] \arrow[ur, dashed, "r"] & S
\end{tikzcd}$$

admits a solution. It follows that q is a retract of the morphism q'', and is therefore also a trivial Kan fibration. □

4.5.6 Isofibrant Diagrams

0346 Let $\mathcal{C}$ be a small category. Every diagram of simplicial sets $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ has a limit in the category Set_{Δ}, given concretely by the formula

$$\varprojlim(\mathscr{F})(C)_n = \varprojlim_{C \in \mathcal{C}} \mathscr{F}(C)_n;$$

see Remark 1.1.0.8. Beware that, when using simplicial sets as a framework for higher category theory, this operation is badly behaved in general:

- If each of the simplicial sets $\mathscr{F}(C)$ is an ∞-category, then the limit $\varprojlim(\mathscr{F})$ need not be an ∞-category.
- If $\alpha : \mathscr{F} \rightarrow \mathscr{G}$ be a natural transformation between functors $\mathscr{F}, \mathscr{G} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ which is a levelwise categorical equivalence (Definition 4.5.6.1), then the induced map $\varprojlim(\alpha) : \varprojlim(\mathscr{F}) \rightarrow \varprojlim(\mathscr{G})$ need not be a categorical equivalence.

In this section, we will introduce the class of *isofibrant* diagrams $\mathscr{F} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ (Definition 4.5.6.3), and show that it does not suffer from these defects:

- If $\mathscr{F} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ is an isofibrant diagram of simplicial sets, then the limit $\varprojlim(\mathscr{F})$ is an ∞-category (Corollary 4.5.6.13).
- If $\alpha : \mathscr{F} \rightarrow \mathscr{G}$ is a levelwise categorical equivalence between isofibrant diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$, then the induced map $\varprojlim(\alpha) : \varprojlim(\mathscr{F}) \rightarrow \varprojlim(\mathscr{G})$ is an equivalence of ∞-categories (Corollary 4.5.6.17).

We begin by introducing some terminology.

Definition 4.5.6.1. Let $\mathcal{C}$ be a category and let $\alpha : \mathscr{F} \rightarrow \mathscr{G}$ be a natural transformation between diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$. We say that α is a *levelwise categorical equivalence* if, for every object $C \in \mathcal{C}$, the induced map $\alpha_C : \mathscr{F}(C) \rightarrow \mathscr{G}(C)$ is a categorical equivalence of simplicial sets. `0347`

Remark 4.5.6.2. Definition 4.5.6.1 is a special case of a general convention. If P is a property of morphisms of simplicial sets and $\alpha : \mathscr{F} \rightarrow \mathscr{G}$ is a natural transformation between diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$, then we say that α *has the property P levelwise* if, for every object $C \in \mathcal{C}$, the morphism of simplicial sets $\alpha_C : \mathscr{F}(C) \rightarrow \mathscr{G}(C)$ has the property P. For example, we say that α is a *levelwise weak homotopy equivalence* if, for every object $C \in \mathcal{C}$, the morphism $\alpha_C : \mathscr{F}(C) \rightarrow \mathscr{G}(C)$ is a weak homotopy equivalence of simplicial sets. `0348`

Definition 4.5.6.3. Let $\mathcal{C}$ be a small category. We say that a diagram $\mathscr{F} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ is *isofibrant* if it satisfies the following condition: `0349`

$(*)$ Let $\mathscr{E} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ be a functor and let $\mathscr{E}_0 \subseteq \mathscr{E}$ be a subfunctor for which the inclusion $\mathscr{E}_0 \hookrightarrow \mathscr{E}$ is a levelwise categorical equivalence. Then every natural transformation $\alpha_0 : \mathscr{E}_0 \rightarrow \mathscr{F}$ admits an extension $\alpha : \mathscr{E} \rightarrow \mathscr{F}$.

Example 4.5.6.4. Let $\mathcal{C} = \{X\}$ be a category having a single object and a single morphism. Then a diagram $\mathscr{F} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ is determined by the simplicial set $\mathscr{F}(X)$. The diagram $\mathscr{F}$ is isofibrant (in the sense of Definition 4.5.6.3) if and only if the simplicial set $\mathscr{F}(X)$ is an ∞-category. `034A`

034B **Remark 4.5.6.5.** Let $\mathcal{C}$ be a small category and $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be an isofibrant diagram. Then, for each object $C \in \mathcal{C}$, the simplicial set $\mathscr{F}(C)$ is an ∞-category. That is, for $0 < i < n$, every morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to \mathscr{F}(C)$ can be extended to an n-simplex of $\mathscr{F}(C)$. This follows by applying condition $(*)$ of Definition 4.5.6.3 to the functor

$$\mathscr{E} : \mathcal{C} \to \mathrm{Set}_{\Delta} \qquad \mathscr{E}(D) = \Delta^n \times \mathrm{Hom}_{\mathcal{C}}(C, D),$$

together with the subfunctor $\mathscr{E}_0 \subseteq \mathscr{E}$ given by $\mathscr{E}_0(D) = \Lambda^n_i \times \mathrm{Hom}_{\mathcal{C}}(C, D)$.

In some cases, Definition 4.5.6.3 can be formulated more concretely.

034D **Proposition 4.5.6.6.** *Let $(Q, \leq)$ be a well-founded partially ordered set (see Definition 4.7.1.1). Then a diagram of simplicial sets $\mathscr{F} : Q^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ is isofibrant if and only if, for each element $q \in Q$, the map*

$$\theta_q : \mathscr{F}(q) \to \varprojlim_{p<q} \mathscr{F}(p)$$

is an isofibration of simplicial sets.

034F **Example 4.5.6.7** (Isofibrant Squares)**.** A square diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E}_{01} \arrow[r, "F_1'"] \arrow[d, "F_0'"] & \mathcal{E}_0 \arrow[d, "F_0"] \\
\mathcal{E}_1 \arrow[r, "F_1"] & \mathcal{E}
\end{tikzcd}$$

is isofibrant (when regarded as a functor $[1] \times [1] \to \mathrm{Set}_{\Delta}$) if and only if it satisfies the following conditions:

- The functors $F_0 : \mathcal{E}_0 \to \mathcal{E}$ and $F_1 : \mathcal{E}_1 \to \mathcal{E}$ are isofibrations of ∞-categories.
- The functor $(F_1', F_0') : \mathcal{E}_{01} \to \mathcal{E}_0 \times_{\mathcal{E}} \mathcal{E}_1$ is an isofibration of ∞-categories.

034E **Example 4.5.6.8** (Isofibrant Towers)**.** Let $\mathscr{F} : \mathbf{Z}^{\mathrm{op}}_{\geq 0} \to \mathrm{Set}_{\Delta}$ be a diagram, which we identify with a tower of simplicial sets

$$\cdots \to \mathscr{F}(3) \to \mathscr{F}(2) \to \mathscr{F}(1) \to \mathscr{F}(0).$$

Then $\mathscr{F}$ is isofibrant (in the sense of Definition 4.5.6.3) if and only if each of the simplicial sets $\mathscr{F}(n)$ is an ∞-category and each of the transition functors $\mathscr{F}(n+1) \to \mathscr{F}(n)$ is an isofibration of ∞-categories.

Example 4.5.6.9 (The Postnikov Tower)**.** Let X be a Kan complex. Then the tower of fundamental n-groupoids 0571

$$\cdots \to \pi_{\leq 3}(X) \to \pi_{\leq 2}(X) \to \pi_{\leq 1}(X) \to \pi_0(X)$$

is an isofibrant diagram of Kan complexes (Corollary 3.5.8.9).

Variant 4.5.6.10. If X is a Kan complex, then the weakly coskeletal tower 0572

$$\cdots \to \operatorname{cosk}^{\circ}_3(X) \to \operatorname{cosk}^{\circ}_2(X) \to \operatorname{cosk}^{\circ}_1(X) \to \operatorname{cosk}^{\circ}_0(X)$$

of Example 3.5.8.5 is an isofibrant diagram (Variant 3.5.8.10). Beware that the coskeletal tower

$$\cdots \to \operatorname{cosk}_4(X) \to \operatorname{cosk}_3(X) \to \operatorname{cosk}_2(X) \to \operatorname{cosk}_1(X)$$

is generally not isofibrant (Warning 3.5.8.11).

Proof of Proposition 4.5.6.6. Suppose first that $\mathscr{F} : Q^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ is an isofibrant diagram. We will show that, for each element $q \in Q$, the induced map $\theta_q : \mathscr{F}(q) \to \varprojlim_{p<q} \mathscr{F}(p)$ is an isofibration of simplicial sets (for this step, we do not need to assume that Q is well-founded). Fix a simplicial set B and a simplicial subset $A \subseteq B$ for which the inclusion map $A \hookrightarrow B$ is a categorical equivalence; we wish to show that every lifting problem

034G

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathscr{F}(q) \arrow[d, "\theta_q"] \\ B \arrow[r] \arrow[ur, dashed] & \varprojlim_{p<q} \mathscr{F}(p) \end{tikzcd} \tag{4.27}$$

admits a solution. Define $\mathscr{B} : Q^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ by the formula $\mathscr{B}(p) = \begin{cases} B & \text{if } p \leq q \\ \emptyset & \text{otherwise,} \end{cases}$ and let $\mathscr{B}_0 \subseteq \mathscr{B}$ be the subfunctor given by the formula

$$\mathscr{B}_0(p) = \begin{cases} B & \text{if } p < q \\ A & \text{if } p = q \\ \emptyset & \text{otherwise.} \end{cases}$$

The lifting problem (4.27) can be identified with a natural transformation of functors $\alpha_0 : \mathscr{B}_0 \to \mathscr{F}$. Since the inclusion $\mathscr{B}_0 \hookrightarrow \mathscr{B}$ is a levelwise categorical equivalence and $\mathscr{F}$ is isofibrant, we can extend α_0 to a natural transformation $\alpha : \mathscr{B} \to \mathscr{F}$, which determines a solution to the lifting problem (4.27).

Now suppose that the partially ordered set $(Q, \leq)$ is well-founded and that for each $q \in Q$, the morphism θ_q is an isofibration of simplicial sets. We wish to show that the diagram $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ is isofibrant. Let $\mathscr{E} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor, let $\mathscr{E}_0 \subseteq \mathscr{E}$ be a subfunctor for which the inclusion $\mathscr{E}_0 \hookrightarrow \mathscr{E}$ is a levelwise categorical equivalence, and let $\alpha_0 : \mathscr{E}_0 \to \mathscr{F}$ be a natural transformation; we wish to show that α_0 can be extended to a natural transformation $\alpha : \mathscr{E} \to \mathscr{F}$.

For every downward-closed subset $P \subseteq Q$, let $\mathscr{E}^P \subseteq \mathscr{E}$ denote the subfunctor given by $\mathscr{E}^P(q) = \begin{cases} \mathscr{E}(q) & \text{if } q \in P \\ \emptyset & \text{otherwise,} \end{cases}$, and set $\mathscr{E}_0^P = \mathscr{E}^P \cap \mathscr{E}_0$. Let S denote the collection of pairs (P, α^P), where $P \subseteq Q$ is a downward-closed subset and $\alpha^P : \mathscr{E}^P \to \mathscr{F}$ is a natural transformation satisfying $\alpha^P|_{\mathscr{E}_0^P} = \alpha_0|_{\mathscr{E}_0^P}$. We regard S as a partially ordered set, where $(P, \alpha^P) \leq (P', \alpha^{P'})$ if P is contained in P' and α^P is equal to the restriction $\alpha^{P'}|_{\mathscr{E}^P}$. The partially ordered set S satisfies the hypotheses of Zorn's lemma, and therefore contains a maximal element (P, α^P). To complete the proof, it will suffice to show that $P = Q$, so that $\alpha^P : \mathscr{E} \to \mathscr{F}$ is an extension of α_0. Assume otherwise. Since Q is well-founded, the complement $Q \setminus P$ contains a minimal element q. Set $P' = P \cup \{q\}$. Since θ_q is an isofibration of simplicial sets, the lifting problem

$$\begin{tikzcd}
\mathscr{E}_0(q) \arrow[rr, "\alpha_0"] \arrow[d] & & \mathscr{F}(q) \arrow[d, "\theta_q"] \\
\mathscr{E}(q) \arrow[r] \arrow[urr, dashed] & \varprojlim_{p<q} \mathscr{E}(p) \arrow[r, "\alpha^P"] & \varprojlim_{p<q} \mathscr{F}(p)
\end{tikzcd}$$

admits a solution in the category of simplicial sets. This solution determines a natural transformation $\alpha^{P'} : \mathscr{E}^{P'} \to \mathscr{F}$ satisfying $\alpha^{P'}|_{\mathscr{E}^P} = \alpha^P$ and $\alpha^{P'}|_{\mathscr{E}_0^{P'}} = \alpha_0|_{\mathscr{E}_0^{P'}}$, contradicting the maximality of the pair (P, α^P). $\square$

We now record some useful properties of isofibrant diagrams of simplicial sets. Fix a small category $\mathcal{C}$, and let us regard $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$ as equipped with the simplicial enrichment described in Example 2.4.2.2. For every simplicial set K, we let $\underline{K}$ denote the constant functor $\mathcal{C} \to \mathrm{Set}_{\Delta}$ taking the value K.

034H **Proposition 4.5.6.11.** *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be an isofibrant diagram of simplicial sets. For every functor $\mathscr{E} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ and every subfunctor $\mathscr{E}_0 \subseteq \mathscr{E}$, the restriction map*

$$\theta : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{F})_{\bullet} \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{E}_0, \mathscr{F})_{\bullet}$$

is an isofibration of simplicial sets. If the inclusion $\mathscr{E}_0 \hookrightarrow \mathscr{E}$ is a levelwise categorical equivalence, then θ is a trival Kan fibration.

Proof. Let B be a simplicial set and let $A \subseteq B$ be a simplicial subset. We wish to show that every lifting problem

034J

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E},\mathscr{F})_{\bullet} \arrow[d, "\theta"] \\
B \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}_0,\mathscr{F})_{\bullet}
\end{tikzcd} \quad (4.28)$$

admits a solution, provided that either the inclusion map $A \hookrightarrow B$ is a categorical equivalence or the inclusion $\mathscr{E}_0 \hookrightarrow \mathscr{E}$ is a levelwise categorical equivalence. Unwinding the definitions, we see that the diagram (4.28) determines a natural transformation

$$\alpha_0 : (\underline{A} \times \mathscr{E}) \coprod_{(\underline{A} \times \mathscr{E}_0)} (\underline{B} \times \mathscr{E}_0) \to \mathscr{F},$$

and that solutions to (4.28) can be identified with extensions of α_0 to a natural transformation $\alpha : \underline{B} \times \mathscr{E} \to \mathscr{F}$. By virtue of our assumption that $\mathscr{F}$ is isofibrant, we are reduced to proving that the inclusion map

$$(\underline{A} \times \mathscr{E}) \coprod_{(\underline{A} \times \mathscr{E}_0)} (\underline{B} \times \mathscr{E}_0) \hookrightarrow \underline{B} \times \mathscr{E}$$

is a levelwise categorical equivalence, which follows from Corollary 4.5.4.15. □

Corollary 4.5.6.12. *Let $\mathcal{C}$ be a small category and let $\mathscr{E}, \mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be diagrams of simplicial sets. If $\mathscr{F}$ is isofibrant, then the simplicial set $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E},\mathscr{F})_{\bullet}$ is an ∞-category.* 034K

Proof. Apply Proposition 4.5.6.11 in the special case $\mathscr{E}_0 = \emptyset$. □

Corollary 4.5.6.13. *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be an isofibrant diagram of simplicial sets. Then the limit $\varprojlim(\mathscr{F})$ is an ∞-category.* 034L

Proof. Apply Corollary 4.5.6.12 in the special case $\mathscr{E} = \underline{\Delta^0}$. □

Proposition 4.5.6.14. *Let $\mathcal{C}$ be a small category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be an isofibrant diagram, and let $\alpha : \mathscr{E} \to \mathscr{E}'$ be a natural transformation between diagrams $\mathscr{E}, \mathscr{E}' : \mathcal{C} \to \mathrm{Set}_{\Delta}$. If α is a levelwise categorical equivalence, then precomposition with α induces an equivalence of ∞-categories* 034M

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}',\mathscr{F})_{\bullet} \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E},\mathscr{F})_{\bullet}.$$

Proof. Using Exercise 3.1.7.11, we can choose a contractible Kan complex X containing a pair of vertices $x, y \in X$ with $x \neq y$. Evaluation at the vertices x and y determine trivial Kan fibrations of ∞-categories

$$\mathrm{ev}_x, \mathrm{ev}_y : \mathrm{Fun}(X, \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E},\mathscr{F})_{\bullet}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E},\mathscr{F})_{\bullet}.$$

Form a pullback diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "T"] \arrow[d, "U"] & \mathrm{Fun}(X, \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{F})_{\bullet}) \arrow[d, "\mathrm{ev}_x"] \\
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}', \mathscr{F})_{\bullet} \arrow[r, "\circ\alpha"] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{F})_{\bullet},
\end{tikzcd}$$

so that U is also a trivial Kan fibration and therefore an equivalence of ∞-categories. It will therefore suffice to show that $\mathrm{ev}_x \circ T$ is an equivalence of ∞-categories. Since the functors ev_x and ev_y are isomorphic, this is equivalent to the requirement that $\mathrm{ev}_y \circ T$ is an equivalence of ∞-categories. In fact, the functor $\mathrm{ev}_y \circ T$ is a trivial Kan fibration: this follows by applying Proposition 4.5.6.11 to the levelwise categorical equivalence

$$\underline{\{y\}} \times \mathscr{E} \hookrightarrow (\underline{X} \times \mathscr{E}) \coprod_{(\underline{\{x\}} \times \mathscr{E})} \mathscr{E}'.$$

□

034N **Corollary 4.5.6.15.** *Let $\mathcal{C}$ be a small category and let $\alpha : \mathscr{E} \to \mathscr{F}$ be a levelwise categorical equivalence of isofibrant diagrams $\mathscr{E}, \mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. Then α admits a homotopy inverse: that is, there is a natural transformation $\beta : \mathscr{F} \to \mathscr{E}$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are isomorphic to $\mathrm{id}_{\mathscr{F}}$ and $\mathrm{id}_{\mathscr{E}}$ as objects of the ∞-categories $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{F})_{\bullet}$ and $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{E})_{\bullet}$, respectively.*

Proof. Since $\mathscr{E}$ is isofibrant, Proposition 4.5.6.14 guarantees that the functor

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{E})_{\bullet} \xrightarrow{\circ\alpha} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{E})_{\bullet}$$

is an equivalence of ∞-categories. In particular, there exists a natural transformation $\beta : \mathscr{F} \to \mathscr{E}$ such that $\beta \circ \alpha$ is isomorphic to $\mathrm{id}_{\mathscr{E}}$ (when viewed as an object of the ∞-category $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{E})_{\bullet}$). To complete the proof, it will suffice to show that β is also a right homotopy inverse to α: that is, the composition $\alpha \circ \beta$ is isomorphic to $\mathrm{id}_{\mathscr{F}}$ (when viewed as an object of the ∞-category $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{F})_{\bullet}$).

For each object $C \in \mathcal{C}$, the functor $\beta_C : \mathscr{F}(C) \to \mathscr{E}(C)$ is a left homotopy inverse of the functor $\alpha_C : \mathscr{E}(C) \to \mathscr{F}(C)$. Since α_C is an equivalence of ∞-categories, it follows that β_C is also an equivalence of ∞-categories. Allowing C to vary, we conclude that β is a levelwise categorical equivalence. We can therefore repeat the preceding argument to obtain a natural transformation $\gamma : \mathscr{E} \to \mathscr{F}$ such that $\gamma \circ \beta$ is isomorphic to $\mathrm{id}_{\mathscr{F}}$. We then have isomorphisms

$$\alpha \simeq (\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha) \simeq \gamma$$

in the ∞-category $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{F})_{\bullet}$, so that $\alpha \circ \beta$ is also isomorphic to $\mathrm{id}_{\mathscr{F}}$. □

Corollary 4.5.6.16. *Let $\mathcal{C}$ be a small category, let $\mathscr{E} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a levelwise categorical equivalence between diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. If $\mathscr{F}$ and $\mathscr{G}$ are isofibrant, then composition with α induces an equivalence of ∞-categories* 034P

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{F})_{\bullet} \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{G})_{\bullet}.$$

Corollary 4.5.6.17. *Let $\mathcal{C}$ be a small category, and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a levelwise categorical equivalence between isofibrant diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. Then the induced map $\varprojlim(\alpha) : \varprojlim(\mathscr{F}) \to \varprojlim(\mathscr{G})$ is an equivalence of ∞-categories.* 034Q

Proof. Apply Corollary 4.5.6.16 in the special case $\mathscr{E} = \underline{\Delta^0}$. ☐

Example 4.5.6.18. Suppose we are given a commutative diagram of ∞-categories 034R

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{C}(3) \arrow[r] \arrow[d] & \mathcal{C}(2) \arrow[r] \arrow[d] & \mathcal{C}(1) \arrow[r] \arrow[d] & \mathcal{C}(0) \arrow[d] \\
\cdots \arrow[r] & \mathcal{D}(3) \arrow[r] & \mathcal{D}(2) \arrow[r] & \mathcal{D}(1) \arrow[r] & \mathcal{D}(0),
\end{tikzcd}$$

where the horizontal maps are isofibrations and the vertical maps are equivalences of ∞-categories. Then the induced map $\varprojlim \mathcal{C}(n) \to \varprojlim \mathcal{D}(n)$ is an equivalence of ∞-categories. This follows by combining Example 4.5.6.8, Corollary 4.5.6.13, and Corollary 4.5.6.17.

Proposition 4.5.6.19. *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be an isofibrant diagram. Suppose that, for every object $C \in \mathcal{C}$, the simplicial set $\mathscr{F}(C)$ is a Kan complex. Then, for every diagram $\mathscr{E} : \mathcal{C} \to \mathrm{Set}_{\Delta}$, the simplicial set $X = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{F})_{\bullet}$ is a Kan complex.* 034S

Proof. By virtue of Corollary 4.5.6.12, the simplicial set X is an ∞-category. Define $\mathscr{F}^{\Delta^1} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ by the formula $\mathscr{F}^{\Delta^1}(C) = \mathrm{Fun}(\Delta^1, \mathscr{F}(C))$. Then $\mathscr{F}^{\Delta^1}$ is also an isofibrant diagram. Moreover, our assumption that each $\mathscr{F}(C)$ is a Kan complex guarantees that the diagonal embedding $\mathscr{F} \hookrightarrow \mathscr{F}^{\Delta^1}$ is a levelwise categorical equivalence. Applying Corollary 4.5.6.16, we deduce that the diagonal map $X \hookrightarrow \mathrm{Fun}(\Delta^1, X)$ is an equivalence of ∞-categories. In particular, every morphism of X is isomorphic (as an object of the ∞-category $\mathrm{Fun}(\Delta^1, X)$) to an identity morphism, and is therefore an isomorphism (Example 4.4.1.14). Applying Proposition 4.4.2.1, we deduce that X is a Kan complex. ☐

Corollary 4.5.6.20. *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be an isofibrant diagram. Suppose that, for every object $C \in \mathcal{C}$, the simplicial set $\mathscr{F}(C)$ is a Kan complex. Then the simplicial set $\varprojlim(\mathscr{F})$ is a Kan complex.* 034T

Proof. Apply Proposition 4.5.6.19 in the special case $\mathscr{E} = \underline{\Delta^0}$. ☐

034U **Corollary 4.5.6.21.** *Let $\mathcal{C}$ be a small category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be an isofibrant diagram, and define $\mathscr{F}^{\simeq} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ by the formula $\mathscr{F}^{\simeq}(C) = \mathscr{F}(C)^{\simeq}$. Then $\mathscr{F}^{\simeq}$ is also an isofibrant diagram. Moreover, the inclusion map $\varprojlim(\mathscr{F}^{\simeq}) \hookrightarrow \varprojlim(\mathscr{F})$ restricts to an isomorphism of $\varprojlim(\mathscr{F}^{\simeq})$ with the core of the ∞-category $\varprojlim(\mathscr{F})$.*

Proof. We first show that the diagram $\mathscr{F}^{\simeq}$ is isofibrant. Let $\mathscr{E} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor and let $\mathscr{E}_0 \subseteq \mathscr{E}$ be a subfunctor for which the inclusion $\mathscr{E}_0 \hookrightarrow \mathscr{E}$ is a levelwise categorical equivalence. Suppose we are given a natural transformation $\alpha_0 : \mathscr{E}_0 \to \mathscr{F}^{\simeq}$. Our assumption that $\mathscr{F}$ is isofibrant guarantees that α_0 can be extended to a natural transformation $\alpha : \mathscr{E} \to \mathscr{F}$. We claim that α automatically factors through the functor $\mathscr{F}^{\simeq}$: that is, for every object $C \in \mathcal{C}$, the map $\alpha_C : \mathscr{E}(C) \to \mathscr{F}(C)$ factors through the core of $\mathscr{F}(C)$. This follows from the observation that the lifting problem

$$\begin{tikzcd}
\mathscr{E}_0(C) \arrow[r, "\alpha_0"] \arrow[d] & \mathscr{F}(C)^{\simeq} \arrow[d] \\
\mathscr{E}(C) \arrow[r, "\alpha"] \arrow[ur, dashed] & \mathscr{F}(C)
\end{tikzcd}$$

has a (unique) solution, since the inclusion $\mathscr{F}(C)^{\simeq} \hookrightarrow \mathscr{F}(C)$ is an isofibration (Proposition 4.4.3.6).

We now prove the second assertion. Let X denote the core of the ∞-category $\varprojlim(\mathscr{F})$. For every object $C \in \mathcal{C}$, the projection map $\varprojlim(\mathscr{F}) \to \mathscr{F}(C)$ carries X into the core of $\mathscr{F}(C)$. It follows that X is contained in the inverse limit $\varprojlim(\mathscr{F}^{\simeq})$. The reverse inclusion follows from Corollary 4.4.3.18, since the simplicial set $\varprojlim(\mathscr{F}^{\simeq})$ is a Kan complex (Corollary 4.5.6.20). ☐

0238 **Corollary 4.5.6.22.** *Suppose we are given an inverse system of ∞-categories*

$$\cdots \to \mathcal{C}(3) \to \mathcal{C}(2) \to \mathcal{C}(1) \to \mathcal{C}(0)$$

where each of the transition functors $\mathcal{C}(n) \to \mathcal{C}(n-1)$ is an isofibration. Then the limit $\mathcal{C} = \varprojlim_n \mathcal{C}(n)$ is an ∞-category, whose core $\mathcal{C}^{\simeq}$ is the inverse limit $\varprojlim_n \mathcal{C}(n)^{\simeq}$. In other words, a morphism of $\mathcal{C}$ is an isomorphism if and only if its image in each $\mathcal{C}(n)$ is an isomorphism.

Proof. Combine Example 4.5.6.8, Corollary 4.5.6.13, and Corollary 4.5.6.21. ☐

We close this section by establishing a variant of Proposition 4.5.6.6 for simplicial objects of the category Set_{Δ}.

Proposition 4.5.6.23. *Let $\mathbf{\Delta}$ be the simplex category (Definition 1.1.0.2) and let $\mathscr{F} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets. Then $\mathscr{F}$ is isofibrant if and only if it satisfies the following condition for each $n \geq 0$:* 05HY

($*_n$) *The comparison map*

$$\theta_n : \mathscr{F}([n]) \to \varprojlim_{[m] \hookrightarrow [n]} \mathscr{F}([m])$$

is an isofibration of simplicial sets. Here the limit is indexed by the (partially ordered) collection of strictly increasing functions $[m] \hookrightarrow [n]$ which are not bijective.

Remark 4.5.6.24. For small values of n, condition ($*_n$) of Proposition 4.5.6.23 can be stated more concretely: 05HZ

- For $n = 0$, it asserts that the simplicial set $\mathscr{F}([0])$ is an ∞-category.
- For $n = 1$, it asserts that the face operators of $\mathscr{F}$ determine an isofibration of simplicial sets
$$(d^1_0, d^1_0) : \mathscr{F}([1]) \to \mathscr{F}([0]) \times \mathscr{F}([0]).$$

If both of these conditions are satisfied, then the face operators $d^1_0, d^1_1 : \mathscr{F}([1]) \to \mathscr{F}([0])$ are isofibrations of ∞-categories.

Remark 4.5.6.25. For every simplicial set S, let $\underline{S} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ denote the functor which carries each object $[m] \in \mathbf{\Delta}^{\mathrm{op}}$ to the set $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^m, S)$, regarded as a constant simplicial set. Unwinding the definitions, we see that the morphism θ_n appearing in Proposition 4.5.6.23 can be identified with the restriction map 05J0

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}_{\Delta})}(\underline{\Delta^n}, \mathscr{F})_{\bullet} \to \mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}_{\Delta})}(\underline{\partial \Delta^n}, \mathscr{F})_{\bullet}.$$

Consequently, if $\mathscr{F}$ is isofibrant, then θ_n is an isofibration of ∞-categories (Proposition 4.5.6.11 and Corollary 4.5.6.12). If $\mathscr{F}$ is an isofibrant diagram of Kan complexes, then θ_n is a Kan fibration between Kan complexes (Corollary 4.5.6.20).

Proof of Proposition 4.5.6.23. Let $\mathscr{F} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ be a diagram which satisfies condition ($*_n$) of Proposition 4.5.6.23 for every integer $n \geq 0$; we wish to show that $\mathscr{F}$ is isofibrant (the converse follows from Remark 4.5.6.25). Let $\mathscr{E} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets and let $\mathscr{E}_0 \subseteq \mathscr{E}$ be a subfunctor having the property that, for every integer $n \geq 0$, the inclusion map $\mathscr{E}_0([n]) \hookrightarrow \mathscr{E}([n])$ is a categorical equivalence. We wish to show that every natural transformation $\alpha_0 : \mathscr{E}_0 \to \mathscr{F}$ can be extended to a map $\alpha : \mathscr{E} \to \mathscr{F}$.

For each integer $n \geq 0$, let $\mathscr{E}_n \subseteq \mathscr{E}$ denote the smallest subfunctor which contains $\mathscr{E}_0$ and satisfies $\mathscr{E}_n([m]) = \mathscr{E}([m])$ for $m < n$. Then $\mathscr{E}$ can be written as the union of an increasing sequence of subfunctors

$$\mathscr{E}_0 \subseteq \mathscr{E}_1 \subseteq \mathscr{E}_2 \subseteq \cdots$$

To complete the proof, it will suffice to show that every natural transformation $\alpha_n : \mathscr{E}_n \to \mathscr{F}$ admits an extension $\alpha_{n+1} : \mathscr{E}_{n+1} \to \mathscr{F}$. Using Proposition 1.1.4.12, we see that the inclusion $\mathscr{E}_n \hookrightarrow \mathscr{E}_{n+1}$ is a pushout of the inclusion map

$$\iota_n : (\underline{\mathscr{E}_n([n])} \times \underline{\Delta^n}) \coprod_{(\underline{\mathscr{E}_n([n])} \times \underline{\partial \Delta^n})} (\underline{\mathscr{E}([n])} \times \underline{\partial \Delta^n}) \hookrightarrow \underline{\mathscr{E}([n])} \times \underline{\Delta^n}.$$

Consequently, to prove the existence of α_{n+1}, we are reduced to solving a lifting problem

$$\begin{tikzcd}
\mathscr{E}_n([n]) \arrow[r] \arrow[d] & \mathscr{F}([n]) \arrow[d, "\theta_n"] \\
\mathscr{E}([n]) \arrow[r] \arrow[ur, dashed] & \varprojlim_{[m] \hookrightarrow [n]} \mathscr{F}([m]).
\end{tikzcd}$$

By virtue of assumption $(*_n)$, it will suffice to show that the inclusion map $\mathscr{E}_n([n]) \hookrightarrow \mathscr{E}([n])$ is a categorical equivalence of simplicial sets.

In fact, we will prove a stronger assertion: the inclusion map $\rho_n : \mathscr{E}_n \hookrightarrow \mathscr{E}$ is a levelwise categorical equivalence for every integer $n \geq 0$. Our proof proceeds by induction on n, the case $n = 0$ being trivial. To carry out the inductive step, let us assume that ρ_n is a levelwise categorical equivalence for some $n \geq 0$; in particular, the inclusion map $\mathscr{E}_n([n]) \hookrightarrow \mathscr{E}([n])$ is a categorical equivalence. Since the collection of categorical equivalences is closed under the formation of coproducts (Corollary 4.5.3.10), it follows that the natural transformation ι_n is a levelwise categorical equivalence. The inclusion map $\mathscr{E}_n \hookrightarrow \mathscr{E}_{n+1}$ is a pushout of ι_n, and is therefore also a levelwise categorical equivalence (Remark 4.5.4.13). Since the collection of (levelwise) categorical equivalences satisfies the two-out-of-three property (Remark 4.5.3.5), it follows that ρ_{n+1} is also a categorical equivalence. □

4.5.7 Detecting Equivalences of ∞-Categories

01HF Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. If F is an equivalence of ∞-categories, then the induced map $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a homotopy equivalence of Kan complexes (Remark 4.5.1.19). The converse assertion is not true in general. For example, the inclusion map $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ induces an isomorphism on cores, but is never an equivalence of ∞-categories unless $\mathcal{C}$ is a Kan complex. However, we have the following slightly weaker result:

01HG **Theorem 4.5.7.1.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is an equivalence of ∞-categories if and only if the induced map of Kan complexes $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq} \to \mathrm{Fun}(\Delta^1, \mathcal{D})^{\simeq}$ is a homotopy equivalence.*

Proof. For every simplicial set X, let $\theta_X : \mathrm{Fun}(X, \mathcal{C})^{\simeq} \to \mathrm{Fun}(X, \mathcal{D})^{\simeq}$ denote the map given by postcomposition with the functor F. Let us say that X is *good* if the morphism θ_X is a homotopy equivalence. By virtue of Proposition 4.5.1.22, the functor F is an equivalence of ∞-categories if and only if every simplicial set X is good. In particular, if F is an equivalence of ∞-categories, then Δ^1 is good. To prove the converse, we make the following observations:

(a) Let X be the colimit of a diagram of monomorphisms

$$X(0) \hookrightarrow X(1) \hookrightarrow X(2) \hookrightarrow \cdots$$

We then obtain a commutative diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(X(0), \mathcal{C})^{\simeq} \arrow[d, "\theta_{X(0)}"] & \operatorname{Fun}(X(1), \mathcal{C})^{\simeq} \arrow[l] \arrow[d, "\theta_{X(1)}"] & \operatorname{Fun}(X(2), \mathcal{C})^{\simeq} \arrow[l] \arrow[d, "\theta_{X(2)}"] & \cdots \arrow[l] \\
\operatorname{Fun}(X(0), \mathcal{D})^{\simeq} & \operatorname{Fun}(X(1), \mathcal{D})^{\simeq} \arrow[l] & \operatorname{Fun}(X(2), \mathcal{D})^{\simeq} \arrow[l] & \cdots, \arrow[l]
\end{tikzcd}$$

where the horizontal maps are Kan fibrations (Corollary 4.4.5.4). Moreover, the induced map of inverse limits can be identified with the map $\theta_X : \operatorname{Fun}(X, \mathcal{C})^{\simeq} \to \operatorname{Fun}(X, \mathcal{D})^{\simeq}$ (Corollary 4.4.4.6). If each $X(n)$ is good, then the vertical maps appearing in the diagram are homotopy equivalences, so that θ_X is also a homotopy equivalence (Example 4.5.6.18). It follows that X is also good.

(b) Let X be a simplicial set which is given as a coproduct $\coprod_{\alpha} X(\alpha)$ of a collection of simplicial sets $X(\alpha)$. Then θ_X can be identified with the product of the maps $\theta_{X(\alpha)}$ (Corollary 4.4.4.6). Consequently, if each of the summands $X(\alpha)$ is good, then X is also good (Remark 3.1.6.8).

(c) Let $u : X \to Y$ be an inner anodyne morphism of simplicial sets. Then we have a commutative diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(X, \mathcal{C})^{\simeq} \arrow[r] \arrow[d] & \operatorname{Fun}(Y, \mathcal{C})^{\simeq} \arrow[d] \\
\operatorname{Fun}(X, \mathcal{D})^{\simeq} \arrow[r] & \operatorname{Fun}(Y, \mathcal{D})^{\simeq},
\end{tikzcd}$$

where the horizontal maps are homotopy equivalences (Proposition 4.5.3.8). It follows that X is good if and only if Y is good.

(d) Suppose we are given a categorical pushout square of simplicial sets

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \arrow[d] \\
Y \arrow[r] & Y'.
\end{tikzcd}$$

If X, X', and Y are good, then Y' is also good (see Corollary 3.4.1.12).

(e) Let X be a retract of a simplicial set Y. If Y is good, then X is also good.

Now suppose that the simplicial set Δ^1 is good. We will show that every simplicial set X is good, so that F is an equivalence of ∞-categories by virtue of Proposition 4.5.1.22. Writing X as the direct limit of its skeleta $\{\mathrm{sk}_n(X)\}_{n \geq 0}$ and using (a), we can reduce to the case where X has dimension $\leq n$ for some integer n. We proceed by induction on n. The case $n = -1$ is trivial (in this case, the simplicial set X is empty and the morphism θ_X is an isomorphism). We may therefore assume that $n \geq 0$. Let S be the collection of nondegenerate n-simplices of X, so that Proposition 1.1.4.12 supplies a pushout diagram

$$\begin{tikzcd}
\coprod_{\sigma \in S} \partial \Delta^n \arrow[r] \arrow[d] & \coprod_{\sigma \in S} \Delta^n \arrow[d] \\
\mathrm{sk}_{n-1}(X) \arrow[r] & X.
\end{tikzcd}$$

Since the horizontal maps in this diagram are monomorphisms, it is also a categorical pushout square (Example 4.5.4.12). Moreover, our inductive hypothesis guarantees that the simplicial sets $\mathrm{sk}_{n-1}(X)$ and $\coprod_{\sigma \in S} \partial \Delta^n$ are good. Applying (d), we are reduced to showing that the coproduct $\coprod_{\sigma \in S} \Delta^n$ is good. Using (b), we are reduced to showing that the standard simplex Δ^n is good. If $n \geq 2$, then the inner horn inclusion $\Lambda^n_1 \hookrightarrow \Delta^n$ is a categorical equivalence, so that the desired result follows from our inductive hypothesis together with (c). We are therefore reduced to showing that the standard simplices Δ^0 and Δ^1 are good. In the second case this follows from our assumption (4), and in the first case it follows from (e) (since the 0-simplex Δ^0 is a retract of Δ^1). □

`01HH` **Corollary 4.5.7.2.** *Let $\mathcal{W}$ denote the full subcategory of* $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$ *spanned by those morphisms of simplicial sets $f : X \to Y$ which are categorical equivalences. Then $\mathcal{W}$ is closed under the formation of filtered colimits in* $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$.

Proof. By virtue of Corollary 4.1.3.3, there exists a functor $Q : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ which commutes with filtered colimits and a natural transformation of functors $u : \mathrm{id}_{\mathrm{Set}_{\Delta}} \to Q$ with the property that, for every simplicial set X, the simplicial set $Q(X)$ is an ∞-category and the morphism $u_X : X \to Q(X)$ is inner anodyne. For every morphism of simplicial sets $f : X \to Y$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u_X"] & Y \arrow[d, "u_Y"] \\ Q(X) \arrow[r, "Q(f)"] & Q(Y) \end{tikzcd}$$

where the vertical maps are categorical equivalences (Corollary 4.5.3.14). It follows from Remark 4.5.3.5 that f is a categorical equivalence if and only if the functor $Q(f)$ is an equivalence of ∞-categories. Using the criterion of Theorem 4.5.7.1, we see that f is a categorical equivalence if and only if the induced map $\mathrm{Fun}(\Delta^1, Q(X))^{\simeq} \to \mathrm{Fun}(\Delta^1, Q(Y))^{\simeq}$ is a homotopy equivalence of Kan complexes. The desired result now follows by observing that the construction $X \mapsto \mathrm{Fun}(\Delta^1, Q(X))^{\simeq}$ commutes with filtered colimits, since the collection of homotopy equivalences between Kan complexes is closed under filtered colimits (Proposition 3.2.8.3). □

Corollary 4.5.7.3. *Suppose we are given a commutative diagram of simplicial sets* 023D

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

with the following property: for every simplex $\sigma : \Delta^k \to S$, the induced map $f_{\sigma} : \Delta^k \times_S X \to \Delta^k \times_S Y$ is a categorical equivalence of simplicial sets. Then f is a categorical equivalence of simplicial sets.

Proof. We will prove the following stronger assertion: for every morphism of simplicial sets $S' \to S$, the induced map

$$f_{S'} : S' \times_S X \to S' \times_S Y$$

is a categorical equivalence of simplicial sets. By virtue of Corollary 4.5.7.2 (and Remark 1.1.4.4), we may assume without loss of generality that S' has dimension $\leq k$ for some integer $k \geq -1$. We proceed by induction on k. In the case $k = -1$, the simplicial set

S' is empty and there is nothing to prove. Assume therefore that $k \geq 0$. Let S'' denote the $(k-1)$-skeleton of S' and let I be the set of nondegenerate d-simplices of S', so that Proposition 1.1.4.12 supplies a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{i \in I} \partial \Delta^k \arrow[r] \arrow[d] & \coprod_{i \in I} \Delta^k \arrow[d] \\
S'' \arrow[r] & S',
\end{tikzcd}$$

where the horizontal maps are monomorphisms. It follows that the front and back faces of the diagram

$$\begin{tikzcd}
(\coprod_{i \in I} \partial \Delta^k) \times_S X \arrow[rr] \arrow[dr, "u"] \arrow[dd] & & \coprod_{i \in I} (\Delta^k \times_S X) \arrow[dr, "v"] \arrow[dd] & \\
& \coprod_{i \in I} (\partial \Delta^k \times_S Y) \arrow[rr] \arrow[dd] & & \coprod_{i \in I} (\Delta^k \times_S Y) \arrow[dd] \\
S'' \times_S X \arrow[rr] \arrow[dr, "f_{S''}"] & & S' \times_S X \arrow[dr, "f_{S'}"] & \\
& S'' \times_S Y \arrow[rr] & & S' \times_S Y
\end{tikzcd}$$

are categorical pushout squares (Proposition 4.5.4.11). Consequently, to show that $f_{S'}$ is a categorical equivalence, it will suffice to show that $f_{S''}$, u, and v are categorical equivalences (Proposition 4.5.4.9). In the first two cases, this follows from our inductive hypothesis. We may therefore replace S' by the coproduct $\coprod_{i \in I} \Delta^k$, and thereby reduce to the case of a coproduct of simplices. Using Corollary 4.5.3.10, we can further reduce to the case where $S' \simeq \Delta^k$ is a standard simplex, in which case the desired result follows from our hypothesis on f. □

Corollary 4.5.7.4. *A commutative diagram of ∞-categories* 034V

034W
$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d, "U"] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}.
\end{tikzcd} \tag{4.29}$$

is a categorical pullback square if and only if the induced diagram of Kan complexes

034X
$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathcal{C}_{01})^{\simeq} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}_0)^{\simeq} \arrow[d] \\
\mathrm{Fun}(\Delta^1, \mathcal{C}_1)^{\simeq} \arrow[r] & \mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}
\end{tikzcd} \tag{4.30}$$

is a homotopy pullback square.

Proof. We proceed as in the proof of Proposition 4.5.2.14. By definition, the diagram (4.29) is a categorical pullback square if and only if the induced map $\theta : \mathcal{C}_{01} \to \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is an equivalence of ∞-categories. Using the criterion of Theorem 4.5.7.1, we see that this is equivalent to the requirement that θ induces a homotopy equivalence of Kan complexes $\rho : \mathrm{Fun}(\Delta^1, \mathcal{C}_{01})^{\simeq} \to \mathrm{Fun}(\Delta^1, \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1)^{\simeq}$. Using Remarks 4.5.2.6 and 4.5.2.7, we can identify ρ with the map

$$\mathrm{Fun}(\Delta^1, \mathcal{C}_{01})^{\simeq} \to \mathrm{Fun}(\Delta^1, \mathcal{C}_0)^{\simeq} \times^{\mathrm{h}}_{\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}} \mathrm{Fun}(\Delta^1, \mathcal{C}_1)^{\simeq}$$

determined by the commutative diagram (4.30). The desired result now follows from the criterion of Corollary 3.4.1.6. □

4.5.8 Application: Universal Property of the Join

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $\mathcal{C} \star \mathcal{D}$ denote their join (Definition 4.3.2.1). Proposition 4.3.2.13 (and Remark 4.3.2.14) supplies a pushout diagram of categories. 01HM

$$\begin{tikzcd}
(\mathcal{C} \times \{0\} \times \mathcal{D}) \coprod (\mathcal{C} \times \{1\} \times \mathcal{D}) \arrow[r] \arrow[d] & \mathcal{C} \times [1] \times \mathcal{D} \arrow[d] \\
(\mathcal{C} \times \{0\}) \coprod (\{1\} \times \mathcal{D}) \arrow[r] & \mathcal{C} \star \mathcal{D}.
\end{tikzcd}$$

Passing to nerves, we obtain a commutative diagram of simplicial sets

$$\begin{tikzcd}
(\mathrm{N}_\bullet(\mathcal{C}) \times \{0\} \times \mathrm{N}_\bullet(\mathcal{D})) \coprod (\mathrm{N}_\bullet(\mathcal{C}) \times \{1\} \times \mathrm{N}_\bullet(\mathcal{D})) \arrow[r] \arrow[d] & \mathrm{N}_\bullet(\mathcal{C}) \times \Delta^1 \times \mathrm{N}_\bullet(\mathcal{D}) \arrow[d] \\
(\mathrm{N}_\bullet(\mathcal{C}) \times \{0\}) \coprod (\{1\} \times \mathrm{N}_\bullet(\mathcal{D})) \arrow[r] & \mathrm{N}_\bullet(\mathcal{C}) \star \mathrm{N}_\bullet(\mathcal{D}).
\end{tikzcd}$$

Beware that this diagram is generally not a pushout square. However, we will show in this section that it is nevertheless a *categorical* pushout square, in the sense of Definition 4.5.4.1. Moreover, an analogous statement holds if we replace $\mathrm{N}_\bullet(\mathcal{C})$ and $\mathrm{N}_\bullet(\mathcal{D})$ by arbitrary simplicial sets X and Y.

01HN **Construction 4.5.8.1.** Let X and Y be simplicial sets, let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the projection maps, and let $\iota_X : X \hookrightarrow X \star Y$ and $\iota_Y : Y \hookrightarrow X \star Y$ denote the inclusion maps. Then there is a unique map of simplicial sets $c : X \times \Delta^1 \times Y \to X \star Y$ with the property that $c|_{X \times \{0\} \times Y} = \iota_X \circ \pi_X$ and $c|_{X \times \{1\} \times Y} = \iota_Y \circ \pi_Y$. Concretely, if $\sigma = (\sigma_X, \sigma_{\Delta^1}, \sigma_Y)$ is an n-simplex of the product $X \times \Delta^1 \times Y$, then $c(\sigma)$ is the n-simplex of $X \star Y$ given by the composition

$$\Delta^n \simeq (\sigma_{\Delta^1}^{-1}\{0\}) \star (\sigma_{\Delta^1}^{-1}\{1\}) \xrightarrow{\sigma_X \star \sigma_Y} X \star Y.$$

We will refer to $c : X \times \Delta^1 \times Y \to X \star Y$ as the *collapse map.*

01HP **Proposition 4.5.8.2.** *Let X and Y be simplicial sets, and let $c : X \times \Delta^1 \times Y \to X \star Y$ denote the collapse map of Construction 4.5.8.1. Then the commutative diagram of simplicial sets*

01HQ
$$\begin{tikzcd}
(X \times \{0\} \times Y) \coprod (X \times \{1\} \times Y) \arrow[r] \arrow[d, "\pi_X \coprod \pi_Y"] & X \times \Delta^1 \times Y \arrow[d, "c"] \\
(X \times \{0\}) \coprod (\{1\} \times Y) \arrow[r, "(\iota_X, \iota_Y)"] & X \star Y
\end{tikzcd} \tag{4.31}$$

is a categorical pushout square.

It will be convenient to state Proposition 4.5.8.2 in a slightly different form.

01HR **Notation 4.5.8.3** (The Blunt Join)**.** Let X and Y be simplicial sets. We let $X \diamond Y$ denote the simplicial set given by the iterated pushout

$$X \coprod_{(X \times \{0\} \times Y)} (X \times \Delta^1 \times Y) \coprod_{(X \times \{1\} \times Y)} Y,$$

so that we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
X \times \partial\Delta^1 \times Y \arrow[r] \arrow[d, "\pi_X \coprod \pi_Y"] & X \times \Delta^1 \times Y \arrow[d] \\
(X \times \{0\}) \coprod (\{1\} \times Y) \arrow[r] & X \diamond Y.
\end{tikzcd}$$

We will refer $X \diamond Y$ as the *blunt join* of X and Y. The commutative diagram (4.31) determines a morphism of simplicial sets $c_{X,Y} : X \diamond Y \to X \star Y$, which we will refer to as the *comparison map*.

Example 4.5.8.4. Let X and Y be simplicial sets. If X is empty, then the blunt join $X \diamond Y$ can be identified with Y. If Y is empty, then the blunt join $X \diamond Y$ can be identified with X. In either case, the comparison map $c_{X,Y} : X \diamond Y \to X \star Y$ is an isomorphism of simplicial sets. 01HS

Exercise 4.5.8.5. Let X and Y be simplicial sets. Show that the comparison map $c_{X,Y} : X \diamond Y \to X \star Y$ of Notation 4.5.8.3 is an epimorphism of simplicial sets: that is, it is surjective at the level of n-simplices for each $n \geq 0$. 02GG

Remark 4.5.8.6 (Functoriality)**.** The blunt join construction $(X, Y) \mapsto X \diamond Y$ determines a functor $\diamond : \mathrm{Set}_\Delta \times \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$. Moreover: 01HT

- For fixed X, the functor

$$\mathrm{Set}_\Delta \to \mathrm{Set}_\Delta \qquad Y \mapsto X \diamond Y$$

  preserves monomorphisms, filtered colimits and pushout diagrams.

- For fixed Y, the functor

$$\mathrm{Set}_\Delta \to \mathrm{Set}_\Delta \qquad X \mapsto X \diamond Y$$

  preserves monomorphisms, filtered colimits, and pushout diagrams.

Remark 4.5.8.7. Let $f : X \to X'$ and $g : Y \to Y'$ be categorical equivalences of simplicial sets. Then the induced map $(f \diamond g) : X \diamond Y \to X' \diamond Y'$ is also a categorical equivalence. This 01HU follows by applying Proposition 4.5.4.9 to the diagram

$$\begin{tikzcd}
X \times \partial\Delta^1 \times Y \arrow[rr] \arrow[dr] \arrow[dd] & & X \times \Delta^1 \times Y \arrow[dr] \arrow[dd] & \\
& X' \times \partial\Delta^1 \times Y' \arrow[rr] \arrow[dd] & & X' \times \Delta^1 \times Y' \arrow[dd] \\
(X \times \{0\}) \coprod (\{1\} \times Y) \arrow[rr] \arrow[dr] & & X \diamond Y \arrow[dr] & \\
& (X' \times \{0\}) \coprod (\{1\} \times Y') \arrow[rr] & & X' \diamond Y'.
\end{tikzcd}$$

By virtue of Proposition 4.5.4.11, Proposition 4.5.8.2 can be restated as follows:

01HV **Theorem 4.5.8.8.** *Let X and Y be simplicial sets. Then the comparison map $c_{X,Y} : X \diamond Y \to X \star Y$ of Notation 4.5.8.3 is a categorical equivalence of simplicial sets.*

01HW **Corollary 4.5.8.9.** *Let $f : X \to X'$ and $g : Y \to Y'$ be categorical equivalences of simplicial sets. Then the induced map $(f \star g) : X \star Y \to X' \star Y'$ is also a categorical equivalence of simplicial sets.*

Proof. We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
X \diamond Y \arrow[r, "f \diamond g"] \arrow[d, "c_{X,Y}"] & X' \diamond Y' \arrow[d, "c_{X',Y'}"] \\
X \star Y \arrow[r, "f \star g"] & X' \star Y',
\end{tikzcd}$$

where $f \diamond g$, $c_{X,Y}$, and $c_{X',Y'}$ are categorical equivalences (Remark 4.5.8.7 and Theorem 4.5.8.8). Invoking the two-out-of-three property (Remark 4.5.3.5), we conclude that $f \star g$ is also a categorical equivalence. □

The proof of Theorem 4.5.8.8 will require some preliminaries. We begin by reducing to the special case where $X = \Delta^1$.

Lemma 4.5.8.10. *Let Y be a simplicial set, and suppose that the comparison map $c_{\Delta^1,Y} : \Delta^1 \diamond Y \to \Delta^1 \star Y$ is a categorical equivalence. Then, for every simplicial set X, the comparison map $c_{X,Y} : X \diamond Y \to X \star Y$ is a categorical equivalence.* `01HX`

Proof. Throughout the proof, we regard the simplicial set Y as fixed. Let us say that a simplicial set X is *good* if $c_{X,Y}$ is a categorical equivalence. We begin with some elementary observations:

(a) The collection of good simplicial sets is closed under the formation of filtered colimits (since the collection of categorical equivalences is closed under filtered colimits, by virtue of Corollary 4.5.7.2).

(b) Suppose we are given a pushout diagram of simplicial sets

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & X(0) \arrow[d] \\
X(1) \arrow[r] & X(01),
\end{tikzcd}$$

where f is a monomorphism. If X, $X(0)$, and $X(1)$ are good, then $X(01)$ is good. This follows by applying Proposition 4.5.4.9 to the commutative diagram

$$\begin{tikzcd}
X \diamond Y \arrow[rr] \arrow[dr, "c_{X,Y}"] \arrow[dd] & & X(0) \diamond Y \arrow[dr, "c_{X(0),Y}"] \arrow[dd] & \\
& X \star Y \arrow[rr] \arrow[dd] & & X(0) \star Y \arrow[dd] \\
X(1) \diamond Y \arrow[rr] \arrow[dr, "c_{X(1),Y}"] & & X(01) \diamond Y \arrow[dr, "c_{X(01),Y}"] & \\
& X(1) \star Y \arrow[rr] & & X(01) \star Y,
\end{tikzcd}$$

noting that the front and back squares are categorical pushouts by virtue of Example 4.5.4.12.

(c) Let $f : X \to X'$ be an inner anodyne morphism of simplicial sets. Then X is good if and only if X' is good. To prove this, we observe that there is a commutative diagram of simplicial sets

$$\begin{tikzcd}
X \diamond Y \arrow[r, "c_{X,Y}"] \arrow[d, "f \diamond \mathrm{id}_Y"] & X \star Y \arrow[d, "f \star \mathrm{id}_Y"] \\
X' \diamond Y \arrow[r, "c_Y"] & X' \star Y.
\end{tikzcd}$$

By the two-out-of-three property (Remark 4.5.3.5), it will suffice to show that the morphisms $f \diamond \mathrm{id}_Y$ and $f \star \mathrm{id}_Y$ are categorical equivalences. In the first case, this follows from Remark 4.5.8.7. For the second, we observe that $f \star \mathrm{id}_Y$ is actually inner anodyne, since it factors as a composition

$$X \star Y \xrightarrow{u} X' \coprod_X (X \star Y) \xrightarrow{v} X' \star Y,$$

where u is a pushout of f (hence inner anodyne because f is inner anodyne) and v is inner anodyne by virtue of Proposition 4.3.6.4.

We wish to show that if the 1-simplex Δ^1 is good, then every simplicial set X is good. Writing X as the filtered colimit of its finite simplicial subsets (Remark 3.6.1.8), we can use (a) to reduce to the case where X is finite. We now proceed by induction on the dimension of X. If $X = \emptyset$, then $c_{X,Y}$ is an isomorphism (Example 4.5.8.4). Otherwise, the simplicial set X has dimension $n \geq 0$. We now proceed by induction on the number of nondegenerate n-simplices of X. Using Proposition 1.1.4.12, we can choose a pushout diagram of simplicial sets

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\
X' \arrow[r] & X
\end{tikzcd}$$

where $X' \subsetneq X$ is a simplicial subset having one fewer nondegenerate n-simplex. It then follows from our inductive hypothesis that $\partial \Delta^n$ and X' are good. By virtue of (b), it will suffice to show that Δ^n is good. This holds for $n = 1$ by assumption, and also for $n = 0$ because Δ^0 is a retract of Δ^1. We may therefore assume that $n \geq 2$, so that the horn inclusion $\Lambda^n_1 \hookrightarrow \Delta^n$ is inner anodyne. Our inductive hypothesis guarantees that Λ^n_1 is good, so that Δ^n is good by virtue of (c). □

01HY **Lemma 4.5.8.11.** *The comparison map* $c_{\Delta^1,\Delta^0} : \Delta^1 \diamond \Delta^0 \to \Delta^1 \star \Delta^0$ *is a categorical equivalence.*

Proof. Unwinding the definitions, we can identify the blunt join $\Delta^1 \diamond \Delta^0$ with the simplicial set $(\Delta^1 \times \Delta^1) \coprod_{\Delta^1 \times \{1\}} \Delta^0$, which we represent informally by the diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d, equal] \\
\bullet \arrow[r] & \bullet.
\end{tikzcd}$$

Let X denote the simplicial set $\Delta^2 \coprod_{\mathrm{N}_\bullet(\{1<2\})} \Delta^0$ obtained from the standard 2-simplex by collapsing the final edge to a point. We then have an inclusion map $\iota : X \hookrightarrow \Delta^1 \diamond \Delta^0$ (corresponding to the triangle in the upper right of the preceding diagram), which fits into a pushout diagram of simplicial sets

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d] & \Delta^1 \arrow[d, "u"] \\
\Delta^1 \diamond \Delta^0 \arrow[r, "c_{\Delta^1, \Delta^0}"] & \Delta^1 \star \Delta^0;
\end{tikzcd}$$

here u classifies to the "long edge" of the 2-simplex $\Delta^1 \star \Delta^0 \simeq \Delta^2$. Since the vertical maps are monomorphisms and r is a categorical equivalence (see Example 4.5.3.16), it follows that c_{Δ^1, Δ^0} is also a categorical equivalence (Remark 4.5.4.13). □

Proposition 4.5.8.12. *Let X be a simplicial set. Then the comparison map $c_{X,\Delta^0} : X \diamond \Delta^0 \to X \star \Delta^0 = X^{\triangleright}$ is a categorical equivalence of simplicial sets.* 01HZ

Proof. Combine Lemmas 4.5.8.10 and 4.5.8.11. □

Remark 4.5.8.13. We will later prove a generalization of Proposition 4.5.8.12; see Proposition 5.2.4.5. 023E

Corollary 4.5.8.14. *Let $f : A \hookrightarrow B$ be a right anodyne morphism of simplicial sets. Then the induced map* 01J0

$$\theta : B \coprod_A (A \diamond \Delta^0) \hookrightarrow B \diamond \Delta^0$$

is a categorical equivalence of simplicial sets.

Proof. Proposition 4.3.6.4 guarantees that the natural map $B \coprod_A A^{\triangleright} \hookrightarrow B^{\triangleright}$ is inner anodyne, and therefore a categorical equivalence (Corollary 4.5.3.14). Using Proposition 4.5.4.11, we conclude that the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & A^{\triangleright} \arrow[d, "f^{\triangleright}"] \\
B \arrow[r] & B^{\triangleright}
\end{tikzcd}$$

is categorical pushout square. It then follows from Theorem 4.5.8.8 and Proposition 4.5.4.9 that the equivalent diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & A \diamond \Delta^0 \arrow[d, "f \diamond \mathrm{id}_{\Delta^0}"] \\
B \arrow[r] & B \diamond \Delta^0
\end{tikzcd}$$

is also categorical pushout square, so that θ is a categorical equivalence by virtue of Proposition 4.5.4.11. □

Proof of Theorem 4.5.8.8. Let X and Y be arbitrary simplicial sets; we wish to show that the comparison map $c_{X,Y} : X \diamond Y \to X \star Y$ is a categorical equivalence. By virtue of Lemma 4.5.8.10, we may assume without loss of generality that $X = \Delta^1$. Note that the map the $c_{X,Y}$ fits into a commutative diagram of simplicial sets

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] & (X \diamond \Delta^0) \times Y \arrow[r, "c_{X,\Delta^0} \times \mathrm{id}_Y"] \arrow[d] & X^{\triangleright} \times Y \arrow[d] \\
X \arrow[r] & X \diamond Y \arrow[r, "c_{X,Y}"] & X \star Y.
\end{tikzcd}$$

Note that the morphism $c_{X,\Delta^0} \times \mathrm{id}_Y$ is a categorical equivalence by virtue of Proposition 4.5.8.12 and Remark 4.5.3.7. Consequently, to show that $c_{X,Y}$ is a categorical equivalence, it will suffice to show that the square on the right is a categorical pushout (Proposition 4.5.4.10). Note that left part of the diagram is a pushout square in which the horizontal maps are monomorphisms, hence also a categorical pushout square (Proposition 4.5.4.11). We are therefore reduced to showing that the outer rectangle is a categorical pushout square (Proposition 4.5.4.8).

Specializing now to the case $X = \Delta^1$, we wish to show that the lower part of the

commutative diagram

$$\begin{tikzcd}
\{1\} \times Y \arrow[r] \arrow[d] & \{1\}^{\triangleright} \times Y \arrow[d] \\
\Delta^1 \times Y \arrow[r] \arrow[d] & (\Delta^1)^{\triangleright} \times Y \arrow[d] \\
\Delta^1 \arrow[r] & \Delta^1 \star Y
\end{tikzcd}$$

is a categorical pushout square. We first claim that the upper square is a categorical pushout: by virtue of Proposition 4.5.4.11, this is equivalent to the assertion that the induced map

$$\theta : (\Delta^1 \times Y) \coprod_{\{1\} \times Y} (\{1\}^{\triangleright} \times Y) \to (\Delta^1)^{\triangleright} \times Y$$

is a categorical equivalence. This follows from Remark 4.5.3.7, since θ factors as a product of the identity map id_Y with the inner horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$. To complete the proof, it will suffice to show that the outer rectangle is a categorical pushout square. Using the criterion of Proposition 4.5.4.11, we are reduced to showing that the map

$$\rho : \Delta^1 \coprod_{\{1\} \times Y} (\{1\}^{\triangleright} \times Y) \to \Delta^1 \star Y$$

is a categorical equivalence. Unwinding the definitions, we can identify ρ with the composition

$$\Delta^1 \coprod_{\{1\}} (\{1\} \diamond Y) \xrightarrow{\rho'} \Delta^1 \coprod_{\{1\}} (\{1\} \star Y) \xrightarrow{\rho''} \Delta^1 \star Y.$$

Here the map ρ' is a categorical equivalence by virtue of Proposition 4.5.8.12 (together with Remark 4.5.4.13), and the map ρ'' is inner anodyne by virtue of Proposition 4.3.6.4. □

4.5.9 Relative Exponentiation

Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets. For every vertex $B \in \mathcal{B}$, let $\mathcal{C}_B = \{B\} \times_{\mathcal{B}} \mathcal{C}$ denote the corresponding fiber of U. If $\mathcal{D}$ is an ∞-category, then Theorem 1.5.3.7 guarantees that the simplicial set $\mathrm{Fun}(\mathcal{C}_B, \mathcal{D})$ is also an ∞-category. Our goal in this section is to study the dependence of this construction on the vertex $B \in \mathcal{B}$. We begin by introducing a relative version of Construction 1.5.3.1. `02TM`

046N **Construction 4.5.9.1.** Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets and let $\mathcal{D}$ be another simplicial set. For every integer $n \geq 0$, we let $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})_n$ denote the collection of pairs (σ, f), where σ is an n-simplex of $\mathcal{B}$ and $f : \Delta^n \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ is a morphism of simplicial sets. Note that every nondecreasing function $\alpha : [m] \to [n]$ induces a map

$$\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})_n \to \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})_m \qquad (\sigma, f) \mapsto (\alpha^*(\sigma), f'),$$

where f' denotes the composite map

$$\Delta^m \times_{\mathcal{B}} \mathcal{C} \xrightarrow{\alpha \times \mathrm{id}} \Delta^n \times_{\mathcal{B}} \mathcal{C} \xrightarrow{f} \mathcal{D}.$$

This construction is compatible with composition, and therefore endows $\{\operatorname{Fun}(\mathcal{C} / \mathcal{B}), \mathcal{D})_n\}_{n \geq 0}$ with the structure of a simplicial set $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})$.

Note that the assignment $(\sigma, f) \mapsto \sigma$ determines a morphism of simplicial sets $\pi : \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathcal{B}$. Moreover, there is an evaluation map $\mathrm{ev} : \mathcal{C} \times_{\mathcal{B}} \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathcal{D}$, which carries an n-simplex $(\widetilde{\sigma}, (\sigma, f))$ of the fiber product $\mathcal{C} \times_{\mathcal{B}} \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})$ to the n-simplex of $\mathcal{D}$ given by the composite map $\Delta^n \xrightarrow{\mathrm{id} \times \widetilde{\sigma}} \Delta^n \times_{\mathcal{B}} \mathcal{C} \xrightarrow{f} \mathcal{D}$.

02TT **Example 4.5.9.2.** Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets and let $U : \mathcal{C} \to \Delta^0$ denote the projection map. Then the simplicial set $\operatorname{Fun}(\mathcal{C} / \Delta^0, \mathcal{D})$ of Construction 4.5.9.1 can be identified with the simplicial set $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ of Construction 1.5.3.1.

046P **Example 4.5.9.3.** Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets and let $U : \mathcal{C} \to \mathcal{C}$ be the identity morphjism. Then the simplicial set $\operatorname{Fun}(\mathcal{C} / \mathcal{C}, \mathcal{D})$ of Construction 4.5.9.1 can be identified with the product $\mathcal{C} \times \mathcal{D}$.

02TS **Example 4.5.9.4.** Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets. Then the projection map $\pi : \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \Delta^0) \to \mathcal{B}$ is an isomorphism.

The direct image $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$ of Construction 4.5.9.1 can be characterized by a universal mapping property:

02TP **Proposition 4.5.9.5.** *Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets and let $\mathcal{D}$ be a simplicial set. For every morphism of simplicial sets $\mathcal{B}' \to \mathcal{B}$, postcomposition with the evaluation map* $\mathrm{ev} : \mathcal{C} \times_{\mathcal{B}} \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E}) \to \mathcal{D}$ *induces a bijection*

$$\operatorname{Hom}_{(\mathrm{Set}_{\Delta})_{/\mathcal{B}}}(\mathcal{B}', \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})) \to \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\mathcal{C} \times_{\mathcal{B}} \mathcal{B}', \mathcal{D}).$$

Proof. Writing $\mathcal{B}'$ as a colimit of simplices, we may reduce to the case where $\mathcal{B}' = \Delta^n$, so that σ is an n-simplex of $\mathcal{B}$. In this case, the desired result follows immediately from the definition of the simplicial set $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$. □

Remark 4.5.9.6. In the situation of Proposition 4.5.9.5, postcomposition with the evaluation map $\mathrm{ev} : \mathcal{C} \times_{\mathcal{B}} \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \to \mathcal{D}$ induces an isomorphism of simplicial sets 02TQ

$$\mathrm{Fun}_{/\mathcal{B}}(\mathcal{B}', \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C} \times_{\mathcal{B}} \mathcal{B}', \mathcal{D}).$$

The bijectivity of this map on n-simplices follows by applying Proposition 4.5.9.5 to the product $\mathcal{B}' \times \Delta^n$.

Remark 4.5.9.7. Suppose we are given a pullback diagram of simplicial sets 02TU

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "U'"] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{B}' \arrow[r] & \mathcal{B}.
\end{tikzcd}$$

For every simplicial set $\mathcal{D}$, we have a canonical isomorphism of simplicial sets $\mathrm{Fun}(\mathcal{C}' \,/\, \mathcal{B}', \mathcal{D}) \simeq \mathcal{B}' \times_{\mathcal{B}} \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$.

Remark 4.5.9.8. Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets, let $\mathcal{D}$ be a simplicial set, and let $\pi : \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \to \mathcal{B}$ be the projection map of Construction 4.5.9.1. For every vertex $B \in \mathcal{B}$, Remark 4.5.9.7 and Example 4.5.9.2 supply an isomorphism of simplicial sets 02TV

$$\pi^{-1}\{B\} = \{B\} \times_{\mathcal{B}} \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \simeq \mathrm{Fun}(\{B\} \times_{\mathcal{B}} \mathcal{C}, \mathcal{D}).$$

Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets and let $\mathcal{D}$ be an ∞-category. It follows from Remark 4.5.9.8 Theorem 1.5.3.7 that every fiber of the projection map $\pi : \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \to \mathcal{B}$ is an ∞-category. Beware that π is generally not an inner fibration.

Exercise 4.5.9.9. Let $\mathcal{B} = \Delta^2$ be the standard 2-simplex and let $\mathcal{C} = \mathrm{N}_{\bullet}(\{0 < 2\})$ be the long edge of $\mathcal{C}$. Show that $\mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \Delta^1)$ is not an ∞-category. 02TW

To avoid the behavior described in Exercise 4.5.9.9, we need to impose an additional condition on the morphism $U : \mathcal{C} \to \mathcal{B}$.

Definition 4.5.9.10. Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets. We will say that U is *exponentiable* if it satisfies the following condition: 02TX

$(*)$ For every diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C}'' \arrow[r, "F"] \arrow[d] & \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{B}'' \arrow[r, "\overline{F}"] & \mathcal{B}' \arrow[r] & \mathcal{B}
\end{tikzcd}$$

in which both squares are pullbacks, if $\overline{F}$ is a categorical equivalence, then F is also a categorical equivalence.

02TY **Remark 4.5.9.11.** We will be primarily interested in the special case of Definition 4.5.9.10 where U is an isofibration of simplicial sets. In this case, Definition 4.5.9.10 can be considerably simplified: to show that an inner fibration of simplicial sets $U : \mathcal{C} \to \mathcal{B}$ is exponentiable, it suffices to verify condition $(*)$ in the special case where $\overline{F} : \mathcal{B}'' \to \mathcal{B}'$ is the inner horn $\Lambda^2_1 \hookrightarrow \Delta^2$ (see Corollary 9.3.6.30).

05J1 **Remark 4.5.9.12.** Let $U : \mathcal{C} \to \mathcal{B}$ be an inner fibration of ∞-categories. If U is exponentiable, then it is is an isofibration. To prove this, fix an object $Y \in \mathcal{C}$ having image $\overline{Y} = U(Y)$ and an isomorphism $\overline{e} : \overline{X} \to \overline{Y}$ in the ∞-category $\mathcal{B}$; we wish to show that $\overline{e}$ can be lifted to an isomorphism $e : X \to Y$ in the ∞-category $\mathcal{C}$. Using Corollary 4.4.3.14, we can reduce to the case where $\mathcal{B}$ is a contractible Kan complex, so that the inclusion map $\{\overline{X}\} \hookrightarrow \mathcal{B}$ is an equivalence of ∞-categories. Our assumption that U is exponentiable then guarantees that the inclusion map $\{\overline{X}\} \times_{\mathcal{B}} \mathcal{C} \hookrightarrow \mathcal{C}$ is also an equivalence of ∞-categories. In particular, there exists an object $X \in \mathcal{C}$ which is isomorphic to Y such that $U(X) = \overline{X}$. Choose an isomorphism $e' : X \to Y$ in $\mathcal{E}$, and set $\overline{e}' = U(e')$. Our assumption that $\mathcal{B}$ is a contractible Kan complex then guarantees that we can choose a 2-simplex $\overline{\sigma}$ of $\mathcal{B}$ with boundary indicated in the diagram

$$\begin{tikzcd}
 & \overline{X} \arrow[dr, "\overline{e}'"] & \\
\overline{X} \arrow[ur, "\mathrm{id}"] \arrow[rr, "\overline{e}"] & & \overline{Y}.
\end{tikzcd}$$

Since U is an inner fibration, we can lift σ to a 2-simplex σ of $\mathcal{C}$ with boundary indicated in the diagram

$$\begin{tikzcd}
 & X \arrow[dr, "e'"] & \\
X \arrow[ur, "\mathrm{id}"] \arrow[rr, dashed] & & Y.
\end{tikzcd}$$

It follows that $e = d^2_1(\sigma)$ is an isomorphism from X to Y in the ∞-category $\mathcal{C}$ satisfying $U(e) = \overline{e}$.

02TZ **Remark 4.5.9.13.** Let $U : \mathcal{C} \to \mathcal{B}$ and $V : \mathcal{D} \to \mathcal{C}$ be exponentiable morphisms of simplicial sets. Then the composition $(U \circ V) : \mathcal{D} \to \mathcal{B}$ is also exponentiable.

Remark 4.5.9.14. Suppose we are given a pullback diagram of simplicial sets 02U0

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "U'"] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{B}' \arrow[r] & \mathcal{B}.
\end{tikzcd}$$

If U is exponentiable, then U' is also exponentiable.

Remark 4.5.9.15. The collection of exponentiable morphisms of simplicial sets is closed under retracts. That is, if we are given a commutative diagram of simplicial sets 02U1

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "U"] & \mathcal{C}' \arrow[r] \arrow[d, "U'"] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{B} \arrow[r] & \mathcal{B}' \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where U' is exponentiable and both horizontal compositions are the identity, then U is also exponentiable.

Example 4.5.9.16. Let $\mathcal{C}$ be any simplicial set. Then the projection map $\mathcal{C} \to \Delta^0$ is exponentiable (this is a reformulation of Remark 4.5.3.7). 02U2

Example 4.5.9.17. The inclusion map $\mathrm{N}_\bullet(\{0 < 2\}) \hookrightarrow \Delta^2$ is an isofibration of ∞-categories which is not exponentiable. Note that there is a pullback diagram of simplicial sets 02U4

$$\begin{tikzcd}
\{0\} \coprod \{2\} \arrow[r] \arrow[d] & \mathrm{N}_\bullet(\{0 < 2\}) \arrow[d] \\
\Lambda^2_1 \arrow[r] & \Delta^2
\end{tikzcd}$$

where the lower horizontal map is a categorical equivalence, but the upper horizontal map is not.

The terminology of Definition 4.5.9.10 is motivated by the following:

Proposition 4.5.9.18. *Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets. The following conditions are equivalent:* 046Q

(1) *The morphism U is exponentiable (Definition 4.5.9.10).*

(2) *For every isofibration of simplicial sets $V : \mathcal{D} \to \mathcal{E}$, the induced map*

$$\mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \xrightarrow{V \circ} \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})$$

is also an isofibration of simplicial sets.

(3) *For every isofibration of ∞-categories $V : \mathcal{D} \to \mathcal{E}$, the induced map*

$$\mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \xrightarrow{V \circ} \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})$$

is also an isofibration.

Proof. We first show that (1) implies (2). Assume that U is exponentiable, let $V : \mathcal{D} \to \mathcal{E}$ be an isofibration of simplicial sets, and let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets which is a categorical equivalence; we wish to show that every lifting problem

02U6
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \arrow[d, "V \circ"] \\
B \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})
\end{tikzcd} \tag{4.32}$$

admits a solution. Note that the lower horizontal map determines a morphism of simplicial sets $B \to \mathcal{B}$. Invoking the universal property of Proposition 4.5.9.5, we can rewrite (4.32) as a lifting problem

02XZ
$$\begin{tikzcd}
A \times_{\mathcal{B}} \mathcal{C} \arrow[r] \arrow[d, "j"] & \mathcal{D} \arrow[d, "V"] \\
B \times_{\mathcal{B}} \mathcal{C} \arrow[r] \arrow[ur, dashed] & \mathcal{E} .
\end{tikzcd} \tag{4.33}$$

Because U is exponentiable, the left vertical map is a categorical equivalence of simplicial sets. Our assumption that V is an isofibration then guarantees the existence of a solution.

The implication (2) ⇒ (3) is immediate. We will complete the proof by showing that (3) implies (1). Assume that condition (3) is satisfied and suppose that we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C}'' \arrow[r, "F"] \arrow[d] & \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{B}'' \arrow[r, "\overline{F}"] & \mathcal{B}' \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where both squares are pullbacks and $\overline{F}$ is a categorical equivalence; we wish to show that F is also a categorical equivalence. By virtue of Exercise 3.1.7.11, there exists a monomorphism of simplicial sets $\iota : \mathcal{B}'' \hookrightarrow Q$, where Q is a contractible Kan complex. Replacing $\overline{F}$ by the morphism $(\iota, \overline{F}) : \mathcal{B}'' \hookrightarrow Q \times \mathcal{B}'$ (and F by the morphism $(\iota, F) : \mathcal{C}'' \hookrightarrow Q \times \mathcal{C}'$), we can reduce to the case where $\overline{F}$ is a monomorphism of simplicial sets, so that F is also a monomorphism of simplicial sets. To show that F is a categorical equivalence, it will suffice to show that if $V : \mathcal{D} \to \mathcal{E}$ is an isofibration of ∞-categories, then every lifting problem

02U7

$$\begin{tikzcd}
\mathcal{C}'' \arrow[r] \arrow[d, "F"] & \mathcal{D} \arrow[d, "V"] \\
\mathcal{C}' \arrow[r] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd} \tag{4.34}$$

admits a solution (Proposition 4.5.5.4). Invoking the universal property of direct images (Proposition 4.5.9.5), we can rewrite (4.34) as a lifting problem

$$\begin{tikzcd}
\mathcal{B}'' \arrow[r] \arrow[d, "\overline{F}"] & \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \arrow[d, "V \circ"] \\
\mathcal{B}' \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E}).
\end{tikzcd}$$

Condition (3) guarantees that the right vertical map is an isofibration, so that the solution exists by virtue of our assumption that $\overline{F}$ is a categorical equivalence. □

Corollary 4.5.9.19. *Let $U : \mathcal{C} \to \mathcal{B}$ be an exponentiable morphism of simplicial sets. For every ∞-category $\mathcal{D}$, the projection map $\pi : \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \to \mathcal{B}$ is an isofibration of simplicial sets.* 02U8

Proof. Apply Proposition 4.5.9.18 in the special case $\mathcal{E} = \Delta^0$ (see Example 4.5.9.4). □

4.6 Morphism Spaces

Let $\mathcal{C}$ be an ∞-category containing a pair of objects X and Y. Recall that a *morphism* from X to Y is an edge f of $\mathcal{C}$ satisfying $d_1^1(f) = X$ and $d_0^1(f) = Y$ (Definition 1.4.1.1). Morphisms from X to Y can be identified with vertices of a simplicial set $\operatorname{Hom}_{\mathcal{C}}(X, Y)$, given by the iterated fiber product 01J3

$$\{X\} \times_{\operatorname{Fun}(\{0\}, \mathcal{C})} \operatorname{Fun}(\Delta^1, \mathcal{C}) \times_{\operatorname{Fun}(\{1\}, \mathcal{C})} \{Y\}.$$

In §4.6.1, we show that the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a Kan complex (Proposition 4.6.1.10), which we refer to as the *space of morphisms from X to Y* (Construction 4.6.1.1).

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that F is *fully faithful* if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ is a homotopy equivalence of Kan complexes (Definition 4.6.2.1). We say that F is *essentially surjective* if it induces a surjection $\pi_0(\mathcal{C}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq})$ on isomorphism classes of objects. In §4.6.2, we show that F is an equivalence of ∞-categories if and only if it is both fully faithful and essentially surjective (Theorem 4.6.2.20). This is essentially a reformulation of the criterion of Theorem 4.5.7.1. Nevertheless, it can be quite useful: the mapping spaces $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ are often more amenable to calculation than the Kan complex $\mathrm{Fun}(\Delta^1,\mathcal{C})^{\simeq}$.

In practice, it is often useful to work with a variant of Construction 4.6.1.1. Let $\mathcal{C}$ be an ∞-category containing a pair of objects X and Y. We define simplicial sets $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ by the formulae

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) = \mathcal{C}_{X/} \times_{\mathcal{C}} \{Y\} \qquad \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y) = \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Y}.$$

We will refer to $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ as the *left-pinched space of morphisms from X to Y*, and to $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ as the *right-pinched space of morphisms from X to Y*. These simplicial sets are also Kan complexes, which can often be described very explicitly:

- Let $\mathcal{C}$ be a $(2,1)$-category containing objects X and Y, and let $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ denote the Duskin nerve of $\mathcal{C}$ (Construction 2.3.1.1). Then there are canonical isomorphisms of simplicial sets

  $$\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X,Y) \simeq \mathrm{N}_{\bullet}(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)) \simeq \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X,Y)^{\mathrm{op}};$$

  see Example 4.6.5.13.

- Let $\mathcal{C}$ be a differential graded category containing objects X and Y, and let $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ denote the differential graded nerve of $\mathcal{C}$ (Definition 2.5.3.7). Then there is a canonical isomorphism of simplicial sets

  $$\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})}(X,Y) \simeq \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*),$$

  where $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*)$ denotes the Eilenberg-MacLane space associated to the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ (Example 4.6.5.15).

- Let $\mathcal{C}$ be a locally Kan simplicial category containing a pair of objects X and Y, and let $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ denote the homotopy coherent nerve of $\mathcal{C}$ (Definition 2.4.3.5). Then there are canonical homotopy equivalences

  $$\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y) \leftarrow \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \to \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)^{\mathrm{op}};$$

see Theorem 4.6.8.5. This is a special case of a more general result (where the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ is assumed to be an ∞-category rather than a Kan complex), which we prove in §4.6.8.

In §4.6.5, we construct comparison maps

$$\iota^{\mathrm{L}}_{X,Y} : \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y) \qquad \iota^{\mathrm{R}}_{X,Y} : \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y),$$

which we refer to as the *pinch inclusion maps*, and show that they are homotopy equivalences of Kan complexes (Proposition 4.6.5.10). This follows from a more general statement about the relationship between (co)slice ∞-categories and oriented fiber products (Theorem 4.6.4.17), which we formulate and prove in §4.6.4). Our proof will make use of a general detection principle for natural isomorphisms of diagrams (Theorem 4.6.3.8), which we explain in §4.6.3.

Let $\mathcal{C}$ be an ∞-category. In §4.6.9, we associate to every triple of objects $X, Y, Z \in \mathcal{C}$ a morphism of Kan complexes

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z),$$

which is well-defined up to homotopy (Construction 4.6.9.9). We show that this composition law is unital and associative up to homotopy (Propositions 4.6.9.11 and 4.6.9.12), and therefore determines an enrichment of the homotopy category $\mathrm{h}\mathcal{C}$ over the homotopy category of Kan complexes hKan (Construction 4.6.9.13 and Remark 4.6.9.14).

4.6.1 Morphism Spaces

Let $\mathcal{C}$ be a category. To every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, one can associate a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of *morphisms from X to Y*. Our goal in this section is to explain a counterpart of this construction in the setting of ∞-categories. 01J4

Construction 4.6.1.1. Let $\mathcal{C}$ be a simplicial set containing a pair of vertices X and Y. We let $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ denote the simplicial set given by the fiber product 01J5

$$\{X\} \times_{\mathrm{Fun}(\{0\},\mathcal{C})} \mathrm{Fun}(\Delta^1,\mathcal{C}) \times_{\mathrm{Fun}(\{1\},\mathcal{C})} \{Y\}.$$

We will typically be interested in this construction only in the case where $\mathcal{C}$ is an ∞-category; if this condition is satisfied, we will refer to $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ as the *space of morphisms from X to Y*.

Remark 4.6.1.2. Let $\mathcal{C}$ be an ∞-category containing a pair of objects X and Y. Recall that a *morphism* from X to Y is an edge $e : \Delta^1 \to \mathcal{C}$ satisfying $e(0) = X$ and $e(1) = Y$ (Definition 1.4.1.1). It follows that morphisms from X to Y can be identified with vertices of the morphism space $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of Construction 4.6.1.1. 01J6

03X9 **Variant 4.6.1.3** (Endomorphism Spaces)**.** Let $\mathcal{C}$ be an ∞-category containing an object X. We let $\mathrm{End}_{\mathcal{C}}(X)$ denote the simplicial set

$$\mathrm{Hom}_{\mathcal{C}}(X, X) = \{X\} \times_{\mathcal{C}} \mathrm{Fun}(\Delta^1 / \operatorname{\partial}\Delta^1, \mathcal{C}).$$

We will refer to $\mathrm{End}_{\mathcal{C}}(X)$ as the *space of endomorphisms of* X. Note that vertices of the simplicial set $\mathrm{End}_{\mathcal{C}}(X)$ can be identified with endomorphisms of X, in the sense of Definition 1.4.1.5

01J7 **Example 4.6.1.4.** Let $\mathcal{C}$ be an ordinary category containing objects X and Y, which we will identify with objects of the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$. Then the morphism space $\mathrm{Hom}_{\mathrm{N}_{\bullet}(\mathcal{C})}(X, Y)$ of Construction 4.6.1.1 can be identified with the constant simplicial set having the value $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (see Example 4.6.4.6). In particular, when $X = Y$ we can identify the simplicial set $\mathrm{End}_{\mathrm{N}_{\bullet}(\mathcal{C})}(X) = \mathrm{Hom}_{\mathrm{N}_{\bullet}(\mathcal{C})}(X, X)$ with the endomorphism monoid $\mathrm{End}_{\mathcal{C}}(X)$ of Example 1.3.2.2.

01J8 **Example 4.6.1.5.** Let X be a topological space containing a pair of points x and y, which we regard as objects of the ∞-category $\mathrm{Sing}_{\bullet}(X)$. Then we have a canonical isomorphism of Kan complexes

$$\mathrm{Hom}_{\mathrm{Sing}_{\bullet}(X)}(x, y) \simeq \mathrm{Sing}_{\bullet}(P_{x,y}),$$

where $P_{x,y}$ denotes the topological space of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$ (equipped with the compact-open topology). Setting $x = y$, we obtain an isomorphism $\mathrm{End}_{\mathrm{Sing}_{\bullet}(X)}(x) = \mathrm{Sing}_{\bullet}(\Omega(X))$, where $\Omega(X)$ is the based loop space of X. See Example 3.4.0.5.

01J9 **Example 4.6.1.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, so that the join $\mathcal{C} \star \mathcal{D}$ is also an ∞-category (Corollary 4.3.3.25). Then the morphism spaces in $\mathcal{C} \star \mathcal{D}$ are described by the formula

$$\mathrm{Hom}_{\mathcal{C} \star \mathcal{D}}(X, Y) \simeq \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \text{if } X, Y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{D}}(X, Y) & \text{if } X, Y \in \mathcal{D} \\ \Delta^0 & \text{if } X \in \mathcal{C}, Y \in \mathcal{D} \\ \emptyset & \text{if } X \in \mathcal{D}, Y \in \mathcal{C}\,. \end{cases}$$

01JA **Example 4.6.1.7.** Let $\mathcal{C}$ be a simplicial set containing vertices X and Y. Let K be a simplicial set, and let $\underline{X}, \underline{Y} : K \to \mathcal{C}$ be the constant maps taking the values X and Y, respectively. Then there is a canonical isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathrm{Fun}(K, \mathcal{C})}(\underline{X}, \underline{Y}) \simeq \mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(X, Y)).$$

01JB **Remark 4.6.1.8.** Let $\mathcal{C}$ be a simplicial set containing vertices X and Y, which we also regard as vertices of the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$. Then there is a canonical isomorphism of simplicial sets $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X)^{\mathrm{op}}$.

Remark 4.6.1.9. Let $\{\mathcal{C}_i\}_{i \in I}$ be a collection of ∞-categories having a product $\mathcal{C} = \prod_{i \in I} \mathcal{C}_i$. Let X and Y be objects of $\mathcal{C}$, which we identify with collections $\{X_i \in \mathcal{C}_i\}_{i \in I}$ and $\{Y_i \in \mathcal{C}_i\}_{i \in I}$, respectively. Then there is a canonical isomorphism of simplicial sets 034Y

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}_i}(X_i, Y_i).$$

Proposition 4.6.1.10. *Let $\mathcal{C}$ be an ∞-category. For every pair of objects $X, Y \in \mathcal{C}$, the morphism space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a Kan complex.* 01JC

Proposition 4.6.1.10 is a special case of the following more general assertion:

Proposition 4.6.1.11. *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, let $A \subseteq B$ be a simplicial subset which contains every vertex of B, and let $f : A \to \mathcal{C}$ be a diagram. Then the fiber product $\mathrm{Fun}(B, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{C})} \{f\}$ is a Kan complex.* 01P1

Proof. Corollary 4.4.5.3 guarantees the restriction map $\theta : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$ is an isofibration, so that the fiber $\mathrm{Fun}(B, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{C})} \{f\}$ is an ∞-category. To show that it is a Kan complex, it will suffice to show that every morphism u in $\mathrm{Fun}(B, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{C})} \{f\}$ is an isomorphism (Proposition 4.4.2.1). By virtue of Corollary 4.4.3.20, this is equivalent to the assertion that the image of u in the ∞-category $\mathrm{Fun}(B, \mathcal{C})$ is an isomorphism. This follows from Theorem 4.4.4.4, since for every vertex $b \in B$, the evaluation functor

$$\mathrm{ev}_b : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(\{b\}, \mathcal{C}) \simeq \mathcal{C}$$

factors through $\mathrm{Fun}(A, \mathcal{C})$ and therefore carries u to the identity morphism $\mathrm{id}_{f(b)}$. □

Remark 4.6.1.12. Let $\mathcal{C}$ be an ∞-category containing a pair of morphisms $f, g : X \to Y$ having the same source and target. Then f and g are homotopic (Definition 1.4.3.1) if and only if they belong to the same connected component of the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$: this follows from the characterization of Corollary 1.4.3.7. Consequently, we obtain a bijection $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \simeq \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y))$. 01JD

Example 4.6.1.13 (Loop Spaces)**.** Let (X, x) be a pointed Kan complex. The Kan complex $\mathrm{Hom}_X(x, x)$ is often denoted by $\Omega(X)$ and referred to as the *based loop space* of X. Note that it can be identified with the fiber over x of the evaluation map 01JE

$$q : \{x\} \times_{\mathrm{Fun}(\{0\}, X)} \mathrm{Fun}(\Delta^1, X) \to \mathrm{Fun}(\{1\}, X) = X.$$

By virtue of Example 3.1.7.10, this map is a Kan fibration whose domain is a contractible Kan complex. It follows that the long exact sequence of Theorem 3.2.6.1 yields isomorphisms $\pi_n(\mathrm{Hom}_X(x, x), \mathrm{id}_x) \simeq \pi_{n+1}(X, x)$ for $n \geq 0$.

046R **Remark 4.6.1.14** (Morphism Spaces in Homotopy Fiber Products)**.** Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories. Let $X_0, Y_0 \in \mathcal{C}_0$ and $X_1, Y_1 \in \mathcal{C}_1$ be objects having the same images $F_0(X_0) = X = F_1(X_1)$ and $F_0(Y_0) = Y = F_1(Y_1)$ in $\mathcal{C}$, so that $X_{01} = (X_0, X_1, \mathrm{id}_X)$ and $Y_{01} = (Y_0, Y_1, \mathrm{id}_Y)$ can be viewed as objects of the homotopy fiber product $\mathcal{C}_{01} = \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ (see Construction 4.5.2.1). Then the mapping space $\mathrm{Hom}_{\mathcal{C}_{01}}(X_{01}, Y_{01})$ can be identified with the homotopy fiber product of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}_0}(X_0, Y_0) \times^{\mathrm{h}}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \mathrm{Hom}_{\mathcal{C}_1}(X_1, Y_1).$$

It will sometimes be convenient to work with a relative version of Construction 4.6.1.1.

01P2 **Construction 4.6.1.15.** Let $q : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, let X and Y be vertices of $\mathcal{C}$, and let $e : q(X) \to q(Y)$ be an edge of the simplicial set $\mathcal{D}$. We let $\mathrm{Hom}_{\mathcal{C}}(X, Y)_e$ denote the fiber product $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times_{\mathrm{Hom}_{\mathcal{D}}(q(X), q(Y))} \{e\}$, which we regard as a simplicial subset of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

01P3 **Example 4.6.1.16.** In the situation of Construction 4.6.1.15, suppose that the simplicial $\mathrm{Hom}_{\mathcal{D}}(q(X), q(Y))$ is isomorphic to Δ^0 (this condition is satisfied, for example, if $\mathcal{D}$ is the nerve of a partially ordered set). Then the inclusion map $\mathrm{Hom}_{\mathcal{C}}(X, Y)_e \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an isomorphism.

01P4 **Example 4.6.1.17.** Let $q : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let X and Y be vertices of $\mathcal{C}$ having the same image $D = q(X) = q(Y)$ in $\mathcal{D}$. Then we have a canonical isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)_{\mathrm{id}_D} \simeq \mathrm{Hom}_{\mathcal{C}_D}(X, Y),$$

where $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ denotes the fiber of q over the vertex D.

01P5 **Remark 4.6.1.18.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{C}' \arrow[d] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}'.
\end{tikzcd}$$

Let X and Y be vertices of $\mathcal{C}$, and let $e : q(X) \to q(Y)$ be an edge of the simplicial set $\mathcal{D}$. Then composition with F induces an isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)_e \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))_{\overline{F}(e)}.$$

Remark 4.6.1.19. Let $q : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, let X and Y be vertices of $\mathcal{C}$, and let $e : q(X) \to q(Y)$ be an edge of $\mathcal{D}$. Form a pullback diagram of simplicial sets 01P6

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "q"] \\
\Delta^1 \arrow[r, "e"] & \mathcal{D},
\end{tikzcd}$$

so that X lifts uniquely to a vertex $\widetilde{X} \in \mathcal{C}'$ lying over the vertex $0 \in \Delta^1$, and Y lifts uniquely to a vertex $\widetilde{Y} \in \mathcal{C}'$ lying over the vertex $1 \in \Delta^1$. Remark 4.6.1.18 and Example 4.6.1.16 supply isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)_e \simeq \mathrm{Hom}_{\mathcal{C}'}(\widetilde{X},\widetilde{Y})_{\mathrm{id}_{\Delta^1}} = \mathrm{Hom}_{\mathcal{C}'}(\widetilde{X},\widetilde{Y}).$$

Proposition 4.6.1.20. *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets, let X and Y be vertices of $\mathcal{C}$, and let $e : q(X) \to q(Y)$ be an edge of $\mathcal{D}$. Then the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_e$ is a Kan complex.* 01P7

Proof. Form a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "q'"] & \mathcal{C} \arrow[d, "q"] \\
\Delta^1 \arrow[r, "e"] & \mathcal{D}.
\end{tikzcd}$$

Since q is an inner fibration, the morphism q' is also an inner fibration (Remark 4.1.1.5), so that $\mathcal{C}'$ is an ∞-category (Remark 4.1.1.9). Remark 4.6.1.19 then supplies an isomorphism of $\mathrm{Hom}_{\mathcal{C}}(X,Y)_e$ with a simplicial set of the form $\mathrm{Hom}_{\mathcal{C}'}(\widetilde{X},\widetilde{Y})$, which is a Kan complex by virtue of Proposition 4.6.1.10. □

In the special case where $\mathcal{D}$ is an ∞-category, we can prove a slightly stronger assertion:

Proposition 4.6.1.21. *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let X and Y be objects of $\mathcal{C}$. Then the induced map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(q(X),q(Y))$ is a Kan fibration of simplicial sets.* 01P8

Remark 4.6.1.22. Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let X and Y be objects of $\mathcal{C}$, and let $e : q(X) \to q(Y)$ be a morphism in $\mathcal{D}$. By construction, we have a 01P9 pullback diagram of simplicial sets

01PA
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y)_e \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[d] \\
\{e\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(q(X),q(Y)).
\end{tikzcd} \tag{4.35}$$

It follows from Proposition 4.6.1.21 that the vertical maps in this diagram are Kan fibrations, so that (4.35) is also a homotopy pullback square. Stated more informally, we have a homotopy fiber sequence

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)_e \to \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(q(X),q(Y)).$$

Proposition 4.6.1.21 is an immediate consequence of the following more general assertion:

01PC **Proposition 4.6.1.23.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let B be a simplicial set, let $A \subseteq B$ be a simplicial subset which contains every vertex of B, and let $f : A \to \mathcal{C}$ be a diagram. Then the induced map*

$$\mathrm{Fun}(B,\mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{C})} \{f\} \to \mathrm{Fun}(B,\mathcal{D}) \times_{\mathrm{Fun}(A,\mathcal{D})} \{q \circ f\}$$

is a Kan fibration of simplicial sets.

Proof. It follows from Proposition 4.6.1.10 that the simplicial sets $\mathrm{Fun}(B,\mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{C})} \{f\}$ and $\mathrm{Fun}(B,\mathcal{D}) \times_{\mathrm{Fun}(A,\mathcal{D})} \{q \circ f\}$ are Kan complexes. It will therefore suffice to show that θ is an isofibration (Corollary 4.4.3.10). This follows from the observation that θ is a pullback of the restriction map

$$\mathrm{Fun}(B,\mathcal{C}) \to \mathrm{Fun}(B,\mathcal{D}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(A,\mathcal{C}),$$

which is an isofibration by virtue of Variant 4.4.5.11. □

Proof of Proposition 4.6.1.21. Apply Proposition 4.6.1.23 in the special case $B = \Delta^1$ and $A = \partial \Delta^1$. □

01PB **Exercise 4.6.1.24.** Let $q : \mathcal{C} \to \mathcal{D}$ be an isofibration of simplicial sets, and let X and Y vertices of $\mathcal{C}$. Show that the induced map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(q(X),q(Y))$ is a Kan fibration.

4.6.2 Fully Faithful and Essentially Surjective Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it satisfies the following pair of conditions: `01JG`

(1) The functor F is *fully faithful*: that is, for every pair of objects $X, Y \in \mathcal{C}$, the induced map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is bijective.

(2) The functor F is *essentially surjective*: that is, for every object $X \in \mathcal{D}$, there exists an object $Y \in \mathcal{C}$ and an isomorphism $X \simeq F(Y)$ in the category $\mathcal{D}$.

Our goal in this section is to give an analogous characterization of equivalences in the setting of ∞-categories (Theorem 4.6.2.20). We begin by formulating ∞-categorical analogues of conditions (1) and (2).

Definition 4.6.2.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that F is *fully faithful* if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map of morphism spaces $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a homotopy equivalence of Kan complexes. `01JH`

Example 4.6.2.2. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory (Definition 4.1.2.15). Then the inclusion map $\iota : \mathcal{C}' \hookrightarrow \mathcal{C}$ is fully faithful. In fact, for every pair of objects $X, Y \in \mathcal{C}'$, the inclusion ι induces an isomorphism of simplicial sets $\mathrm{Hom}_{\mathcal{C}'}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$. `01JJ`

Example 4.6.2.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then F is fully faithful if and only if the induced map $\mathrm{N}_{\bullet}(F) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{D})$ is fully faithful (in the sense of Definition 4.6.2.1). Consequently, we can regard Definition 4.6.2.1 as a generalization of the classical notion of fully faithful functor. `01JK`

Remark 4.6.2.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, so that F induces a functor of homotopy categories $f : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$. If F is fully faithful, then f is also fully faithful (see Remark 4.6.1.12). Beware that the converse is generally false. `01JL`

Remark 4.6.2.5 (Transitivity)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, where G is fully faithful. Then F is fully faithful if and only if $G \circ F$ is fully faithful. In particular, the collection of fully faithful functors is closed under composition. `01JM`

Remark 4.6.2.6. Suppose we are given a commutative diagram of ∞-categories `046S`

`046T`

$$\begin{tikzcd} \mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\ \mathcal{C}_1 \arrow[r] & \mathcal{C} \end{tikzcd} \qquad (4.36)$$

Combining Remark 4.6.1.14 with Corollary 3.4.1.6, we see that the following conditions are equivalent:

(1) The diagram (4.36) induces a fully faithful functor from $\mathcal{C}_{01}$ to the homotopy fiber product $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$.

(2) For every object $X_{01} \in \mathcal{C}_{01}$ having images $X_0 \in \mathcal{C}_0$, $X_1 \in \mathcal{C}_1$, $X \in \mathcal{C}$ and every object $Y_{01} \in \mathcal{C}_{01}$ having images $Y_0 \in \mathcal{C}_0$, $Y_1 \in \mathcal{C}_1$, $Y \in \mathcal{C}$, the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}_{01}}(X_{01}, Y_{01}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}_0}(X_0, Y_1) \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}_1}(X_1, Y_1) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X, Y)
\end{tikzcd}$$

is a homotopy pullback square.

In particular, if (4.36) is a categorical pullback diagram, then it satisfies condition (2).

046U **Remark 4.6.2.7.** Suppose we are given a categorical pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{D} \arrow[d] \\
\mathcal{C}' \arrow[r, "F'"] & \mathcal{D}'.
\end{tikzcd}$$

If F' is fully faithful, then F is fully faithful (see Remark 4.6.2.6 and Corollary 3.4.1.5).

01PD **Proposition 4.6.2.8.** *Suppose we are given a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{C}' \arrow[d, "q'"] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}'.
\end{tikzcd}$$

Assume that the functors q and q' are inner fibrations and that the functors F and $\overline{F}$ are fully faithful. Then, for every object $D \in \mathcal{D}$, the induced functor $F_D : \mathcal{C}_D \to \mathcal{C}'_{\overline{F}(D)}$ is fully faithful.

Proof. Let X and Y be objects of the ∞-category $\mathcal{C}_D$. We then have a cubical diagram of

Kan complexes

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}_D}(X,Y) \arrow[rr] \arrow[dr] \arrow[dd] & & \operatorname{Hom}_{\mathcal{C}}(X,Y) \arrow[dr] \arrow[dd] & \\
& \operatorname{Hom}_{\mathcal{C}'_{\overline{F}}(D)}(F(X),F(Y)) \arrow[rr] \arrow[dd] & & \operatorname{Hom}_{\mathcal{C}'}(F(X),F(Y)) \arrow[dd] \\
\{\mathrm{id}_D\} \arrow[rr] \arrow[dr] & & \operatorname{Hom}_{\mathcal{D}}(D,D) \arrow[dr] & \\
& \{\mathrm{id}_{\overline{F}(D)}\} \arrow[rr] & & \operatorname{Hom}_{\mathcal{D}'}(\overline{F}(D),\overline{F}(D)).
\end{tikzcd}$$

The front and back faces of this diagram are homotopy pullback squares (Remark 4.6.1.22), the comparison maps

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X),F(Y)) \qquad \operatorname{Hom}_{\mathcal{D}}(D,D) \to \operatorname{Hom}_{\mathcal{D}}(\overline{F}(D),\overline{F}(D))$$

are homotopy equivalences by virtue of our assumptions that F and $\overline{F}$ are fully faithful, and the map of singletons $\{\mathrm{id}_D\} \to \{\mathrm{id}_{\overline{F}(D)}\}$ is an isomorphism. Applying Corollary 3.4.1.12, we conclude that the comparison map $\operatorname{Hom}_{\mathcal{C}_D}(X,Y) \to \operatorname{Hom}_{\mathcal{C}'_{\overline{F}(D)}}(F(X),F(Y))$ is also a homotopy equivalence. $\square$

Proposition 4.6.2.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories. Then F is conservative (Definition 4.4.2.7). That is, if $u : X \to Y$ is a morphism in $\mathcal{C}$ for which $F(u)$ is an isomorphism in the ∞-category $\mathcal{D}$, then u is an isomorphism in the ∞-category $\mathcal{C}$.* `01JN`

Proof. Let $\overline{v} : F(Y) \to F(X)$ be a homotopy inverse to $F(u)$. Since F is fully faithful, the natural map $\operatorname{Hom}_{\mathcal{C}}(Y,X) \to \operatorname{Hom}_{\mathcal{D}}(F(Y),F(X))$ is a homotopy equivalence. We may therefore assume without loss of generality that $\overline{v} = F(v)$, for some morphism $v : Y \to X$ in the ∞-category $\mathcal{C}$. Let $v \circ u$ be a composition of u and v in the ∞-category $\mathcal{C}$. Since $F(u)$ is homotopy inverse to $F(v)$, the morphism $F(v \circ u)$ is homotopic to $\mathrm{id}_{F(C)} = F(\mathrm{id}_C)$. Since the map $\operatorname{Hom}_{\mathcal{C}}(X,X) \to \operatorname{Hom}_{\mathcal{D}}(F(X),F(X))$ is a homotopy equivalence, it follows that $v \circ u$ is homotopic to id_C: that is, v is a left homotopy inverse to u. A similar argument (with the roles of u and v reversed) shows that v is also a right homotopy inverse to u. It follows that u is an isomorphism. $\square$

01JP **Corollary 4.6.2.10.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories. Then the induced map of cores $\mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is also fully faithful.*

Proof. Fix objects $X, Y \in \mathcal{C}^{\simeq}$. Our assumption that F is fully faithful guarantees that the induced map $\theta : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a homotopy equivalence of Kan complexes. By virtue of Proposition 4.6.2.9, θ restricts to a homotopy equivalence from the summand of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ spanned by the isomorphisms from X to Y to the summand of $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ spanned by the isomorphisms from $F(X)$ to $F(Y)$. Unwinding the definitions, we conclude that $F^{\simeq}$ induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}^{\simeq}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}^{\simeq}}(F(X), F(Y))$. □

01JQ **Definition 4.6.2.11.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The *essential image* of F is the full subcategory of $\mathcal{D}$ spanned by those objects $D \in \mathcal{D}$ for which there exists an object $C \in \mathcal{C}$ and an isomorphism $F(C) \simeq D$. We say that F is *essentially surjective* if its essential image is the entire ∞-category $\mathcal{D}$: that is, if the map of sets $\pi_0(\mathcal{C}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq})$ is surjective.

01JR **Remark 4.6.2.12.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $\mathcal{D}' \subseteq \mathcal{D}$ be the essential image of F. Then $\mathcal{D}'$ is a replete full subcategory of $\mathcal{D}$, and F can be regarded as an essentially surjective functor from $\mathcal{C}$ to $\mathcal{D}'$. Moreover, the essential image $\mathcal{D}'$ is uniquely determined by these properties.

01JS **Remark 4.6.2.13.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F is essentially surjective if and only if the induced functor of homotopy categories $f : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is essentially surjective (in the sense of classical category theory).

01JT **Remark 4.6.2.14.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F is essentially surjective if and only if the induced map of Kan complexes $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is essentially surjective.

01JU **Example 4.6.2.15.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories. Then F is essentially surjective if and only if the induced map $\mathrm{N}_{\bullet}(F) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{D})$ is an essentially surjective functor of ∞-categories (in the sense of Definition 4.6.2.11).

01JV **Example 4.6.2.16.** Let $f : X \to Y$ be a morphism of Kan complexes. Then f is essentially surjective (in the sense of Definition 4.6.2.11) if and only if the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a surjection.

01JW **Remark 4.6.2.17** (Transitivity)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. If F and G are essentially surjective, then the composition $G \circ F$ is essentially surjective. Conversely, if $G \circ F$ is essentially surjective, then G is essentially surjective.

Remark 4.6.2.18. Suppose we are given a categorical pullback diagram of ∞-categories 0573

$$\begin{tikzcd} \mathcal{C}' \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r] & \mathcal{D} . \end{tikzcd}$$

If F is essentially surjective, then F' is essentially surjective. This follows from Proposition 4.5.2.14 and Corollary 3.5.1.24.

Remark 4.6.2.19. Suppose we are given a commutative diagram of ∞-categories 01PE

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{C}' \arrow[d, "q'"] \\ \mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}' \end{tikzcd}$$

satisfying the following conditions:

(a) The functor q is an inner fibration and q' is an isofibration.

(b) The functor $\overline{F}$ is essentially surjective.

(c) For each object $D \in \mathcal{D}$, the induced functor $F_D : \mathcal{C}_D \to \mathcal{C}'_{\overline{F}(D)}$ is essentially surjective.

Then the functor F is essentially surjective. To prove this, consider an arbitrary object $Z \in \mathcal{C}'$. Assumption (b) guarantees that there exists an object $D \in \mathcal{D}$ and an isomorphism $\overline{u} : \overline{F}(D) \to q'(Z)$ in the ∞-category $\mathcal{D}'$. Assumption (a) guarantees that we can lift $\overline{u}$ to an isomorphism $u : Y \to Z$ in the ∞-category $\mathcal{C}'$, where Y belongs to the fiber $\mathcal{C}'_{\overline{F}(D)}$. Applying ($c$), we can choose an object $X \in \mathcal{C}_D$ and an isomorphism $v : F(X) \to Y$ in the ∞-category $\mathcal{C}'_{\overline{F}(D)}$. It follows that Z is isomorphic to $F(X)$ in the ∞-category $\mathcal{C}'$.

Theorem 4.6.2.20. *Let $F : \mathcal{C} \to \mathcal{D}$ be functor of ∞-categories. Then F is an equivalence of ∞-categories if and only if it is fully faithful and essentially surjective.* 01JX

We begin by considering the special case of Theorem 4.6.2.20 where $\mathcal{C}$ and $\mathcal{D}$ are Kan complexes.

Lemma 4.6.2.21. *Let $f : X \to Y$ be a morphism of Kan complexes which is fully faithful and essentially surjective. Then f is a homotopy equivalence.* 01JY

Proof. Since f is essentially surjective, the underlying map of connected components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective. We claim that it is also injective. To prove this, suppose that x and x' are vertices of X such that $f(x)$ and $f(x')$ belong to the same connected component of Y. Then the morphism space $\mathrm{Hom}_Y(f(x), f(x'))$ is nonempty. Since f is fully faithful, it induces a homotopy equivalence $\mathrm{Hom}_X(x, x') \to \mathrm{Hom}_Y(f(x), f(x'))$. It follows that $\mathrm{Hom}_X(x, x')$ is nonempty, so that x and x' belong to the same connected component of X. This completes the proof that $\pi_0(f)$ is a bijection.

By virtue of Whitehead's theorem (Theorem 3.2.7.1), it will suffice to show that for every vertex $x \in X$ having image $y = f(x) \in Y$ and every integer $n \geq 0$, the induced map $\theta : \pi_{n+1}(X, x) \to \pi_{n+1}(Y, y)$ is an isomorphism. Using Example 4.6.1.13, we can identify θ with the natural map $\pi_n(\mathrm{Hom}_X(x, x), \mathrm{id}_x) \to \pi_n(\mathrm{Hom}_Y(y, y), \mathrm{id}_y)$, which is bijective by virtue of our assumption that f induces a homotopy equivalence $\mathrm{Hom}_X(x, x) \to \mathrm{Hom}_Y(y, y)$. □

Proof of Theorem 4.6.2.20. Assume first that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories. Then F induces a homotopy equivalence of Kan complexes $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ (Remark 4.5.1.19). Passing to connected components, we conclude that the induced map $\pi_0(\mathcal{C}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq})$ is bijective. In particular, F is essentially surjective. We have a commutative diagram of Kan complexes

01JZ
$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq} \arrow[r, "\theta"] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{D})^{\simeq} \arrow[d] \\
\mathrm{Fun}(\partial\Delta^1, \mathcal{C})^{\simeq} \arrow[r, "\theta_0"] & \mathrm{Fun}(\partial\Delta^1, \mathcal{D})^{\simeq},
\end{tikzcd} \tag{4.37}$$

where the horizontal maps are homotopy equivalences (Theorem 4.5.7.1) and the vertical maps are Kan fibrations (Corollary 4.4.5.4). Applying Proposition 3.2.8.1, we conclude that for every vertex $(X, Y) \in \mathrm{Fun}(\partial\Delta^1, \mathcal{C})^{\simeq}$, the induced map of fibers

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(X, Y) &= \{(X, Y)\} \times_{\mathrm{Fun}(\partial\Delta^1, \mathcal{C})^{\simeq}} \mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq} \\
&\to \{(X, Y)\} \times_{\mathrm{Fun}(\partial\Delta^1, \mathcal{D})^{\simeq}} \mathrm{Fun}(\Delta^1, \mathcal{D})^{\simeq} \\
&= \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))
\end{aligned}$$

is a homotopy equivalence. It follows that F is fully faithful.

Now suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories which is fully faithful and essentially surjective. Using Corollary 4.6.2.10 and Remark 4.6.2.14, we see that the induced map $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is also fully faithful and essentially surjective, and is therefore a homotopy equivalence of Kan complexes (Lemma 4.6.2.21). It follows that the morphism θ_0 in (4.37) is a homotopy equivalence of Kan complexes. Combining our assumption that F is fully faithful with Proposition 3.2.8.1, we conclude that θ is also a homotopy equivalence. Applying Theorem 4.5.7.1, we conclude that F is an equivalence of ∞-categories. □

Corollary 4.6.2.22. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $\mathcal{D}' \subseteq \mathcal{D}$ be the essential image of F. Then F is fully faithful if and only if it induces an equivalence of ∞-categories $\mathcal{C} \to \mathcal{D}'$.* 01K0

Corollary 4.6.2.23. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories. Then, for every simplicial set K, the induced map $\operatorname{Fun}(K, \mathcal{C}) \xrightarrow{F\circ} \operatorname{Fun}(K, \mathcal{D})$ is also fully faithful.* 04Q6

Proof. Using Corollary 4.6.2.22, we can replace $\mathcal{C}$ by its essential image and thereby reduce to the case where $F : \mathcal{C} \hookrightarrow \mathcal{D}$ is the inclusion of a full subcategory. In this case, the induced map $\operatorname{Fun}(K, \mathcal{C}) \xrightarrow{F\circ} \operatorname{Fun}(K, \mathcal{D})$ is also the inclusion of a full subcategory, and therefore automatically fully faithful (Example 4.6.2.2). □

Corollary 4.6.2.24. *Let $f : X \to Y$ be a morphism of Kan complexes. Then f is fully faithful (when regarded as a functor of ∞-categories) if and only if it induces a homotopy equivalence from X to a summand of Y.* 01K1

Proof. Combine Corollary 4.6.2.22 with Exercise 4.4.1.13. □

4.6.3 Digression: Categorical Mapping Cylinders

Let $\mathcal{C}$ be an ∞-category, and let $f_0, f_1 : B \to \mathcal{C}$ be diagrams in $\mathcal{C}$ indexed by a simplicial set B. Recall that f_0 and f_1 are *naturally isomorphic* if they are isomorphic as objects of the diagram ∞-category $\operatorname{Fun}(B, \mathcal{C})$ (Definition 4.4.4.1). Our goal in this section is to establish a detection criterion for natural isomorphisms. 01K2

Proposition 4.6.3.1. *Let $\mathcal{C}$ be an ∞-category and let $f_0, f_1 : B \to \mathcal{C}$ be a pair of diagrams. The following conditions are equivalent:* 01K3

(1) *The diagrams f_0 and f_1 are isomorphic when regarded as objects of the ∞-category $\operatorname{Fun}(B, \mathcal{C})$.*

(2) *There exists a factorization of the fold map $(\mathrm{id}_B, \mathrm{id}_B) : B \coprod B \to B$ as a composition*

$$B \coprod B \xrightarrow{(s_0, s_1)} \overline{B} \xrightarrow{\pi} B,$$

where π is a categorical equivalence, and a diagram $\overline{f} : \overline{B} \to \mathcal{C}$ satisfying $f_0 = \overline{f} \circ s_0$ and $f_1 = \overline{f} \circ s_1$.

(3) *For every factorization of the fold map $(\mathrm{id}_B, \mathrm{id}_B) : B \coprod B \to B$ as a composition*

$$B \coprod B \xrightarrow{(s_0, s_1)} \overline{B} \xrightarrow{\pi} B,$$

where s_0 and s_1 have disjoint images, there exists a diagram $\overline{f} : \overline{B} \to \mathcal{C}$ satisfying $f_0 = \overline{f} \circ s_0$ and $f_1 = \overline{f} \circ s_1$.

We will deduce Proposition 4.6.3.1 from a more general statement (Theorem 4.6.3.8), which we prove at the end of this section.

01K4 **Remark 4.6.3.2.** Proposition 4.6.3.1 has an interpretation in the language of model categories. Let us regard the category Set_{Δ} of simplicial sets as equipped with the Joyal model structure of Remark [?]. Conditions (2) and (3) of Proposition 4.6.3.1 are equivalent to the requirement that the morphisms $f_0, f_1 : B \to \mathcal{C}$ are homotopic with respect to the Joyal model structure (in the sense of Definition [?]). Proposition 4.6.3.1 asserts that this is equivalent to the requirement that f_0 and f_1 are naturally isomorphic (in the sense of Definition 4.4.4.1).

Let us introduce a bit of terminology which is useful for exploiting Proposition 4.6.3.1.

01K5 **Definition 4.6.3.3.** Let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets. A *categorical mapping cylinder for B relative to A* is a simplicial set $\overline{B}$ equipped with a morphism $\pi : \overline{B} \to B$ together with a pair of sections $s_0, s_1 : B \to \overline{B}$ having the following properties:

(1) The morphism $\pi : \overline{B} \to B$ is a categorical equivalence of simplicial sets.

(2) The morphisms $s_0, s_1 : B \to \overline{B}$ satisfy $s_0 \circ i = s_1 \circ i$, and the induced map $(s_0, s_1) : (B \coprod_A B) \to \overline{B}$ is a monomorphism.

If these conditions are satisfied in the special case $A = \emptyset$, we will simply refer to $\overline{B}$ (together with the morphisms π, s_0, and s_1) as a *categorical mapping cylinder for B*.

01K6 **Remark 4.6.3.4.** In the situation of Definition 4.6.3.3, condition (2) is equivalent to the requirement that the diagram of simplicial sets

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i"] & B \arrow[d, "s_0"] \\
B \arrow[r, "s_1"] & \overline{B}
\end{tikzcd}$$

commutes and is a pullback square (note that the morphisms s_0 and s_1 are automatically monomorphisms, since they are right inverse to the map $\pi : \overline{B} \to B$).

01K7 **Remark 4.6.3.5.** Let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets, and let $(\mathrm{id}_B, \mathrm{id}_B) : (B \coprod_A B) \to B$ be the fold map. Unwinding the definitions, we see that a categorical mapping cylinder for B relative to A can be identified with a factorization of $(\mathrm{id}_B, \mathrm{id}_B)$ as a composition

$$B \coprod_A B \xrightarrow{\iota} \overline{B} \xrightarrow{\pi} B,$$

where ι is a monomorphism of simplicial sets and π is a categorical equivalence. Such factorizations always exist: by virtue of Exercise 3.1.7.11, we can even arrange that π is a trivial Kan fibration of simplicial sets (hence a categorical equivalence by virtue of Proposition 4.5.3.11).

Example 4.6.3.6. Let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets, and let Q be a contractible Kan complex containing vertices $x_0, x_1 \in Q$ with $x_0 \neq x_1$. Set $\overline{B} = A \coprod_{(Q \times A)} (Q \times B)$. The commutative diagram 01K8

$$\begin{tikzcd}
Q \times A \arrow[r] \arrow[d] & Q \times B \arrow[d] \\
A \arrow[r, "i"] & B
\end{tikzcd}$$

is a categorical pushout square (since the vertical maps are categorical equivalences), and therefore induces a categorical equivalence $\overline{\pi} : \overline{B} \to B$ (Proposition 4.5.4.11). Let $s_0 : B \to \overline{B}$ be the section of π given by the composition

$$B \simeq \{x_0\} \times B \hookrightarrow Q \times B \to \overline{B},$$

and define $s_1 : B \to \overline{B}$ similarly. Then the quadruple $(\overline{B}, \pi, s_0, s_1)$ is a categorical mapping cylinder of B relative to A.

Corollary 4.6.3.7. *Let $\mathcal{C}$ be an ∞-category, and let $f_0, f_1 : B \to \mathcal{C}$ be a pair of diagrams indexed by a simplicial set B. Let* 01K9

$$(B \coprod B) \xrightarrow{(s_0,s_1)} \overline{B} \xrightarrow{\pi} B$$

be a categorical mapping cylinder for B (Definition 4.6.3.3). The following conditions are equivalent:

- (a) *The diagrams f_0 and f_1 are isomorphic when regarded as objects of the ∞-category* $\operatorname{Fun}(B, \mathcal{C})$.
- (b) *There exists a diagram $\overline{f} : \overline{B} \to \mathcal{C}$ satisfying $f_0 = \overline{f} \circ s_0$ and $f_1 = \overline{f} \circ s_1$.*

In particular, condition (b) does not depend on the choice of categorical mapping cylinder.

Proof. The implication $(a) \Rightarrow (b)$ follows from the implication $(1) \Rightarrow (3)$ of Proposition 4.6.3.1, and the implication $(b) \Rightarrow (a)$ from the implication $(2) \Rightarrow (1)$ of Proposition 4.6.3.1. □

We will deduce Proposition 4.6.3.1 from a more general relative statement.

`01KA` **Theorem 4.6.3.8.** *Let $q : X \to S$ be an isofibration of simplicial sets, let $g : B \to S$ be a morphism of simplicial sets, and let $f_0, f_1 : B \to X$ be morphisms satisfying $q \circ f_0 = g = q \circ f_1$. Let $A \subseteq B$ be a simplicial subset satisfying $f_0|_A = f_1|_A$. The following conditions are equivalent:*

(1) *The diagrams f_0 and f_1 are isomorphic when regarded as objects of the ∞-category $\mathrm{Fun}_{A//S}(B, X)$ (see Proposition 4.1.4.6).*

(2) *There exists a factorization of the fold map $(\mathrm{id}_B, \mathrm{id}_B) : B \coprod_A B \to B$ as a composition*

$$B \coprod_A B \xrightarrow{(s_0,s_1)} \overline{B} \xrightarrow{\pi} B,$$

where π is a categorical equivalence and the lifting problem

$$\begin{tikzcd}
B \coprod_A B \arrow[r, "(f_0,f_1)"] \arrow[d, "(s_0,s_1)"'] & X \arrow[d, "q"] \\
\overline{B} \arrow[r, "g\circ\pi"] \arrow[ur, dashed, "\overline{f}"] & S
\end{tikzcd}$$

admits a solution.

(3) *For every factorization of the fold map $(\mathrm{id}_B, \mathrm{id}_B) : B \coprod_A B \to B$ as a composition*

$$B \coprod_A B \xrightarrow{(s_0,s_1)} \overline{B} \xrightarrow{\pi} B,$$

where the map $(s_0, s_1) : B \coprod_A B \to \overline{B}$ is a monomorphism, the lifting problem

$$\begin{tikzcd}
B \coprod_A B \arrow[r, "(f_0,f_1)"] \arrow[d, "(s_0,s_1)"'] & X \arrow[d, "q"] \\
\overline{B} \arrow[r, "g\circ\pi"] \arrow[ur, dashed, "\overline{f}"] & S
\end{tikzcd}$$

admits a solution.

Proof. By virtue of Corollary 4.4.3.15, condition (1) is satisfied if and only if there exists a morphism of simplicial sets $u : Q \to \mathrm{Fun}_{A//S}(B, X)$, where Q is a contractible Kan complex, and a pair of vertices $x_0, x_1 \in Q$ satisfying $u(x_0) = f_0$ and $u(x_1) = f_1$. Moreover, we may assume (modifying Q if necessary) that the vertices x_0 and x_1 are distinct. In this case, let $\overline{B} = A \coprod_{(Q \times A)} (Q \times B)$ and let

$$B \coprod_A B \xrightarrow{(s_0, s_1)} \overline{B} \xrightarrow{\pi} B$$

be the categorical mapping cylinder described in Example 4.6.3.6. Unwinding the definitions, we see that morphisms $u : Q \to \operatorname{Fun}_{A/\,/S}(B, X)$ satisfying $u(x_0) = f_0$ and $u(x_1) = f_1$ can be identified with solutions to the lifting problem

$$\begin{tikzcd}
B \coprod_A B \arrow[r, "(f_0, f_1)"] \arrow[d, "(s_0, s_1)"'] & X \arrow[d, "q"] \\
\overline{B} \arrow[r, "g \circ \pi"] \arrow[ur, dashed] & S.
\end{tikzcd}$$

This proves that (3) ⇒ (1) ⇒ (2).

We will complete the proof by showing that (2) ⇒ (3). Assume that (2) is satisfied, so that the fold map $(\mathrm{id}_B, \mathrm{id}_B) : B \coprod_A B \to B$ factors as a composition

$$B \coprod_A B \xrightarrow{(s_0, s_1)} \overline{B} \xrightarrow{\pi} B$$

where π is a categorical equivalence and there exists a morphism $\overline{f} : \overline{B} \to X$ for which the diagram

$$\begin{tikzcd}
B \coprod_A B \arrow[r, "(f_0, f_1)"] \arrow[d, "(s_0, s_1)"'] & X \arrow[d, "q"] \\
\overline{B} \arrow[r, "g \circ \pi"] \arrow[ur, "\overline{f}"] & S
\end{tikzcd}$$

commutes. Using Exercise 3.1.7.11, we can factor π as a composition

$$\overline{B} \xrightarrow{j} \overline{B}' \xrightarrow{\pi'} B,$$

where j is a monomorphism and π' is a trivial Kan fibration. Then π' is also a categorical equivalence (Proposition 4.5.3.11), so the morphism j is a categorical equivalence (Remark 4.5.3.5). Our assumption that q is an isofibration guarantees that the lifting problem

$$\begin{tikzcd}
\overline{B} \arrow[r, "\overline{f}"] \arrow[d, "j"'] & X \arrow[d, "q"] \\
\overline{B}' \arrow[r, "g \circ \pi'"] \arrow[ur, dashed, "\overline{f}'"] & S
\end{tikzcd}$$

admits a solution $\overline{f}' : \overline{B}' \to X$.

We now show that condition (3) is satisfied. Suppose that we are given another factorization of the fold map $(\mathrm{id}_B, \mathrm{id}_B) : (\mathrm{id}_B, \mathrm{id}_B) : B \coprod_A B \to B$ as a composition

$$B \coprod_A B \xrightarrow{\iota} \overline{B}'' \xrightarrow{\pi''} B,$$

where ι is a monomorphism. We wish to show that the lifting problem

$$\begin{tikzcd}
B \coprod_A B \arrow[r, "(f_0, f_1)"] \arrow[d, "\iota"'] & X \arrow[d, "q"] \\
\overline{B}'' \arrow[r, "g \circ \pi''"] \arrow[ur, dashed, "\overline{f}''"] & S
\end{tikzcd}$$

admits a solution $\overline{f}'' : \overline{B}'' \to X$. We first observe that the lifting problem

$$\begin{tikzcd}
B \coprod_A B \arrow[r, "j \circ (s_0, s_1)"] \arrow[d, "\iota"'] & \overline{B}' \arrow[d, "\pi'"] \\
\overline{B}'' \arrow[r, "\pi''"] \arrow[ur, dashed, "v"] & B
\end{tikzcd}$$

admits a solution $v : \overline{B}'' \to \overline{B}'$, since ι is a monomorphism and π' is a trivial Kan fibration. We now conclude the proof by setting $\overline{f}'' = \overline{f}' \circ v$. □

01KB **Corollary 4.6.3.9.** *Let $\mathcal{C}$ be an ∞-category, let $f_0, f_1 : B \to \mathcal{C}$ be a pair of diagrams indexed by a simplicial set B, and let $A \subseteq B$ be a simplicial subset satisfying $f_0|_A = f_1|_A$. The following conditions are equivalent:*

(1) *The diagrams f_0 and f_1 are isomorphic when regarded as objects of the ∞-category $\mathrm{Fun}_{A/}(B, \mathcal{C})$.*

(2) *There exists a factorization of the fold map $(\mathrm{id}_B, \mathrm{id}_B) : B \coprod_A B \to B$ as a composition*

$$B \coprod_A B \xrightarrow{(s_0, s_1)} \overline{B} \xrightarrow{\pi} B,$$

where π is a categorical equivalence, and a morphism $\overline{f} : \overline{B} \to \mathcal{C}$ satisfying $f_0 = \overline{f} \circ s_0$ and $f_1 = \overline{f} \circ s_1$.

(3) *For every factorization of the fold map* $(\mathrm{id}_B, \mathrm{id}_B) : B \coprod_A B \to B$ *as a composition*

$$B \coprod_A B \xrightarrow{(s_0,s_1)} \overline{B} \xrightarrow{\pi} B$$

where the map $(s_0, s_1) : B \coprod_A B \to \overline{B}$ *is a monomorphism, there exists a morphism* $\overline{f} : \overline{B} \to \mathcal{C}$ *satisfying* $f_0 = \overline{f} \circ s_0$ *and* $f_1 = \overline{f} \circ s_1$.

Proof. Apply Theorem 4.6.3.8 in the special case where $S = \Delta^0$. □

Proof of Proposition 4.6.3.1. Apply Corollary 4.6.3.9 in the special case $A = \emptyset$. □

For later use, we record a relative version of Corollary 4.6.3.7.

Corollary 4.6.3.10. *Let* $q : X \to S$ *be an isofibration of simplicial sets, let* $g : B \to S$ *be a morphism of simplicial sets, and let* $f_0, f_1 : B \to X$ *be morphisms satisfying* $q \circ f_0 = g = q \circ f_1$. *Let* A *be a simplicial subset of* B *satisfying* $f_0|_A = f_1|_A$, *and let* 01KC

$$(B \coprod_A B) \xrightarrow{(s_0,s_1)} \overline{B} \xrightarrow{\pi} B$$

be a categorical mapping cylinder of B *relative to* A. *The following conditions are equivalent:*

(*a*) *The diagrams* f_0 *and* f_1 *are isomorphic when regarded as objects of the* ∞*-category* $\mathrm{Fun}_{A//S}(B, X)$.

(*b*) *The lifting problem*

$$\begin{tikzcd} B \coprod_A B \arrow[r, "(f_0,f_1)"] \arrow[d, "(s_0,s_1)"'] & X \arrow[d, "q"] \\ \overline{B} \arrow[r, "g\circ\pi"] \arrow[ur, dashed, "\overline{f}"] & S \end{tikzcd}$$

admits a solution.

In particular, condition (*b*) *does not depend on the choice of categorical mapping cylinder.*

Proof. The implication (*a*) ⇒ (*b*) follows from the implication (1) ⇒ (3) of Theorem 4.6.3.8, and the implication (*b*) ⇒ (*a*) from the implication (2) ⇒ (1) of Theorem 4.6.3.8. □

Corollary 4.6.3.11. *Let* $\mathcal{C}$ *be an* ∞*-category and let* $f_0, f_1 : B \to \mathcal{C}$ *be a pair of diagrams indexed by a simplicial set* B. *Let* A *be a simplicial subset of* B *satisfying* $f_0|_A = f_1|_A$, *and let* 01KD

$$(B \coprod_A B) \xrightarrow{(s_0,s_1)} \overline{B} \xrightarrow{\pi} B$$

be a categorical mapping cylinder of B *relative to* A. *The following conditions are equivalent:*

(*a*) *The diagrams f_0 and f_1 are isomorphic when regarded as objects of the ∞-category $\mathrm{Fun}_{A/}(B, \mathcal{C})$.*

(*b*) *There exists a diagram $\overline{f} : \overline{B} \to \mathcal{C}$ satisfying $f_0 = \overline{f} \circ s_0$ and $f_1 = \overline{f} \circ s_1$.*

In particular, condition (*b*) *does not depend on the choice of categorical mapping cylinder.*

Proof. Apply Corollary 4.6.3.10 in the special case $S = \Delta^0$. □

4.6.4 Oriented Fiber Products

01KE Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories. To every pair of functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$, one can associate the *oriented fiber product* $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$, whose objects are triples (C, D, η) where C is an object of $\mathcal{C}$, D is an object of $\mathcal{D}$, and $\eta : F(C) \to G(D)$ is a morphism in the category $\mathcal{E}$ (Notation 2.1.4.19). This construction has a counterpart in the setting of ∞-categories.

01KF **Definition 4.6.4.1** (The Oriented Fiber Product)**.** Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets. We let $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ denote the simplicial set given by the iterated fiber product

$$\mathcal{C} \times_{\mathrm{Fun}(\{0\}, \mathcal{E})} \mathrm{Fun}(\Delta^1, \mathcal{E}) \times_{\mathrm{Fun}(\{1\}, \mathcal{E})} \mathcal{D}.$$

We will refer to $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ as the *oriented fiber product* of $\mathcal{C}$ with $\mathcal{D}$ over $\mathcal{E}$.

As our notation suggests, we will be primarily interested in the special case of Definition 4.6.4.1 where the simplicial sets $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are ∞-categories.

01KG **Proposition 4.6.4.2.** *Let $\mathcal{E}$ be an ∞-category, and suppose we are given morphisms of simplicial sets $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$. Then the projection map $\theta : \mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ is an isofibration of simplicial sets.*

Proof. By construction, we have a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D} \arrow[r] \arrow[d, "\theta"] & \mathrm{Fun}(\Delta^1, \mathcal{E}) \arrow[d, "\theta_0"] \\
\mathcal{C} \times \mathcal{D} \arrow[r] & \mathrm{Fun}(\partial \Delta^1, \mathcal{E}).
\end{tikzcd}$$

Since $\mathcal{E}$ is an ∞-category, the restriction map θ_0 is an isofibration of ∞-categories (Corollary 4.4.5.3). Invoking Remark 4.5.5.11, we conclude that θ is an isofibration of simplicial sets. □

01KH **Corollary 4.6.4.3.** *Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Then the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ is also an ∞-category.*

Proof. By virtue of Proposition 4.6.4.2, the projection map $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ is an isofibration. Since $\mathcal{C} \times \mathcal{D}$ is an ∞-category, it follows that $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ is also an ∞-category (Remark 4.5.5.7). □

Remark 4.6.4.4 (Homotopy Invariance)**.** Suppose we are given a commutative diagram of ∞-categories 03PN

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] & \mathcal{D} \arrow[l] \arrow[d] \\
\mathcal{C}' \arrow[r] & \mathcal{E}' & \mathcal{D}', \arrow[l]
\end{tikzcd}$$

where the vertical maps are equivalences of ∞-categories. Then the induced map

$$\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D} \to \mathcal{C}' \tilde{\times}_{\mathcal{E}'} \mathcal{D}'$$

is also an equivalence of ∞-categories. This follows by applying Corollary 4.5.2.30 to the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}(\partial\Delta^1, \mathcal{E}) \arrow[d] & \mathcal{C} \times \mathcal{D} \arrow[l] \arrow[d] \\
\mathrm{Fun}(\Delta^1, \mathcal{E}') \arrow[r] & \mathrm{Fun}(\partial\Delta^1, \mathcal{E}') & \mathcal{C}' \times \mathcal{D}'. \arrow[l]
\end{tikzcd}$$

Remark 4.6.4.5. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Then we can identify objects of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ with triples (C, D, e), where C is an object of $\mathcal{C}$, D is an object of $\mathcal{D}$, and $e : F(C) \to G(D)$ is a morphism in the ∞-category $\mathcal{E}$. Note that the homotopy fiber product $\mathcal{C} \times^{\mathrm{h}}_{\mathcal{E}} \mathcal{D}$ of Construction 4.5.2.1 can be identified with the full subcategory of $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ spanned by those triples (C, D, e) where the morphism e is an isomorphism. 034Z

Example 4.6.4.6. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors between ordinary categories, and let $\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}$ denote the oriented fiber product of Notation 2.1.4.19. Since the nerve construction is compatible with the formation of inverse limits and functor categories, we have a canonical isomorphism of simplicial sets 01KJ

$$\mathrm{N}_{\bullet}(\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D}) \simeq (\mathrm{N}_{\bullet}(\mathcal{C}) \tilde{\times}_{\mathrm{N}_{\bullet}(\mathcal{E})} \mathrm{N}_{\bullet}(\mathcal{D})).$$

Consequently, Definition 4.6.4.1 can be viewed as a generalization of the classical oriented fiber product.

Example 4.6.4.7. Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets. Then the oriented fiber product $\mathcal{C} \tilde{\times}_{\Delta^0} \mathcal{D}$ can be identified with the cartesian product $\mathcal{C} \times \mathcal{D}$. 02C6

01KL **Remark 4.6.4.8.** Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets, and let $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{op}}$ and $G^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{op}}$ be the opposite morphisms. Then we have a canonical isomorphism of simplicial sets

$$(\mathcal{C} \tilde{\times}_{\mathcal{E}} \mathcal{D})^{\mathrm{op}} \simeq (\mathcal{D}^{\mathrm{op}} \tilde{\times}_{\mathcal{E}^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}).$$

01KN **Remark 4.6.4.9.** Let $F : K \to \mathcal{C}$ be a morphism of simplicial sets, which we identify with a vertex of the simplicial set $\mathrm{Fun}(K, \mathcal{C})$. For any simplicial set J, we have canonical isomorphisms

$$\mathrm{Fun}(J, \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}) \simeq \mathrm{Fun}_{K/}(J \diamond K, \mathcal{C}) \qquad \mathrm{Fun}(J, \{F\} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \mathcal{C}) \simeq \mathrm{Fun}_{K/}(K \diamond J, \mathcal{C}),$$

where $J \diamond K$ and $K \diamond J$ denote the blunt joins introduced in Notation 4.5.8.3. Restricting to vertices, we obtain bijections

$$\{\text{Morphisms } J \to \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}\} \simeq \{\text{Morphisms } \overline{F} : J \diamond K \to \mathcal{C} \text{ with } \overline{F}|_K = F\}$$

$$\{\text{Morphisms } J \to \{F\} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \mathcal{C}\} \simeq \{\text{Morphisms } \overline{F}' : K \diamond J \to \mathcal{C} \text{ with } \overline{F}'|_K = F\}.$$

01KP **Example 4.6.4.10.** Let $\mathcal{C}$ be a simplicial set containing vertices X and Y, which we identify with morphisms of simplicial sets $X, Y : \Delta^0 \to \mathcal{C}$. Then the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of Construction 4.6.1.1 is the oriented fiber product $\{X\} \tilde{\times}_{\mathcal{C}} \{Y\}$.

The following result is a relative version of Proposition 4.6.1.10:

01KQ **Proposition 4.6.4.11.** *Let $\mathcal{C}$ be an ∞-category containing an object X. Then the projection map $\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \to \mathcal{C}$ is a left fibration and the projection map $\mathcal{C} \tilde{\times}_{\mathcal{C}} \{X\} \to \mathcal{C}$ is a right fibration.*

Proof. We will prove the second assertion; the first follows by a similar argument. Let $A \hookrightarrow B$ be a right anodyne morphism of simplicial sets; we wish to show that every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{C} \tilde{\times}_{\mathcal{C}} \{X\} \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & \mathcal{C}
\end{tikzcd}$$

admits a solution. Unwinding the definitions, we are reduced to showing that a map of simplicial sets

$$\sigma_0 : B \coprod_A (A \diamond \{X\}) \to \mathcal{C}$$

can be extended to a map $\sigma : B \diamond \{X\} \to \mathcal{C}$ (see Notation 4.5.8.3). By virtue of Lemma 4.5.5.2, it will suffice to show that the inclusion map

$$\iota : B \coprod_A (A \diamond \{X\}) \hookrightarrow B \diamond \{X\}$$

is a categorical equivalence of simplicial sets, which follows from Corollary 4.5.8.14. □

Corollary 4.6.4.12. *Let $\mathcal{D}$ be an ∞-category containing an object X, and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then the projection map $\{X\} \tilde{\times}_{\mathcal{D}} \mathcal{C} \to \mathcal{C}$ is a left fibration, and the projection map $\mathcal{C} \tilde{\times}_{\mathcal{D}} \{X\} \to \mathcal{C}$ is a right fibration.* 01KR

Proof. Unwinding the definition, we have pullback diagrams

$$\begin{tikzcd}
\{X\} \tilde{\times}_{\mathcal{D}} \mathcal{C} \arrow[r] \arrow[d] & \{X\} \tilde{\times}_{\mathcal{D}} \mathcal{D} \arrow[d] & \mathcal{C} \tilde{\times}_{\mathcal{D}} \{X\} \arrow[r] \arrow[d] & \mathcal{D} \tilde{\times}_{\mathcal{D}} \{X\} \arrow[d] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D} & \mathcal{C} \arrow[r, "F"] & \mathcal{D} .
\end{tikzcd}$$

The desired result now follows by combining Proposition 4.6.4.11 with Remark 4.2.1.8. □

If $F : \mathcal{K} \to \mathcal{C}$ is a functor between ordinary categories, then Remark 4.3.1.11 supplies canonical isomorphisms

$$\mathcal{C}_{/F} \simeq \mathcal{C} \tilde{\times}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})} \{F\} \qquad \mathcal{C}_{F/} \simeq \{F\} \tilde{\times}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})} \mathcal{C} .$$

Our next goal is to establish a similar result in the setting of ∞-categories. Here the situation is a bit more subtle: if $F : K \to \mathcal{C}$ is a diagram in an ∞-category $\mathcal{C}$, then the simplicial sets $\mathcal{C}_{/F}$ and $\mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}$ are generally not isomorphic. However, we will show that they are equivalent as ∞-categories.

Construction 4.6.4.13 (The Slice Diagonal Morphism)**.** Let $F : K \to \mathcal{C}$ be a morphism of simplicial sets, and let $c : \mathcal{C}_{/F} \diamond K \to \mathcal{C}_{/F} \star K$ be the comparison morphism of Notation 4.5.8.3. By virtue of Remark 4.6.4.9, the composite map 02GH

$$\mathcal{C}_{/F} \diamond K \xrightarrow{c} \mathcal{C}_{/F} \star K \to \mathcal{C}$$

determines a morphism of simplicial sets $\delta_{/F} : \mathcal{C}_{/F} \to \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}$, which we will refer to as the *slice diagonal morphism*. Similarly, the composition

$$K \diamond \mathcal{C}_{F/} \to K \star \mathcal{C}_{F/} \to \mathcal{C}$$

determines a morphism of simplicial sets $\delta_{F/} : \mathcal{C}_{F/} \to \{F\} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \mathcal{C}$, which we will refer as the *coslice diagonal morphism.*

02GJ **Remark 4.6.4.14.** Let $F : K \to \mathcal{C}$ be a morphism of simplicial sets. For every simplicial set J, composition with the slice diagonal $\delta_{/F}$ of Construction 4.6.4.13 determines a map of sets

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(J, \mathcal{C}_{/F}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(J, \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}).$$

Under the bijection of Remark 4.6.4.9, this identifies with the map

$$\mathrm{Hom}_{(\mathrm{Set}_\Delta)_{K/}}(J \star K, \mathcal{C}) \to \mathrm{Hom}_{(\mathrm{Set}_\Delta)_{K/}}(J \diamond K, \mathcal{C})$$

given by precomposition with the comparison map $c_{J,K} : J \diamond K \twoheadrightarrow J \star K$ of Notation 4.5.8.3.

01KT **Remark 4.6.4.15.** Let $F : K \to \mathcal{C}$ be a morphism of simplicial sets. Then the slice and coslice diagonal morphisms

$$\mathcal{C}_{/F} \to \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\} \qquad \mathcal{C}_{F/} \to \{F\} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \mathcal{C}$$

are monomorphisms of simplicial sets. This follows from Remark 4.6.4.14, together with the observation that for every simplicial set J, the comparison maps

$$c_{J,K} : J \diamond K \twoheadrightarrow J \star K \qquad c_{K,J} : K \diamond J \twoheadrightarrow K \star J$$

are epimorphisms (see Exercise 4.5.8.5)

02GK **Exercise 4.6.4.16.** Let $f : K \to \mathcal{C}$ be a morphism of simplicial sets. Then f can be identified with a vertex of the simplicial set $\mathrm{Fun}(K, \mathcal{C})$, which (to avoid confusion) we will temporarily denote by F. Applying Construction 4.6.4.13 to the inclusion map $\{F\} \hookrightarrow \mathrm{Fun}(K, \mathcal{C})$, we obtain a monomorphism of simplicial sets $\mathrm{Fun}(K, \mathcal{C})_{/F} \hookrightarrow \mathrm{Fun}(K, \mathcal{C}) \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}$, which induces a monomorphism

$$u : \mathcal{C} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K, \mathcal{C})_{/F} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}.$$

Show that the slice diagonal morphism $\delta_{/f} : \mathcal{C}_{/f} \to \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}$ of Construction 4.6.4.13 factors (uniquely) through u. In particular, $\delta_{/f}$ determines a morphism of simplicial sets $\mathcal{C}_{/f} \to \mathcal{C} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K, \mathcal{C})_{/F}$. Similarly, the coslice diagonal morphism $\delta_{f/}$ induces a morphism of simplicial sets $\mathcal{C}_{f/} \to \mathrm{Fun}(K, \mathcal{C})_{F/} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathcal{C}$.

We can now formulate our main result, which we prove at the end of this section.

01KU **Theorem 4.6.4.17.** *Let $\mathcal{C}$ be an ∞-category and let $F : K \to \mathcal{C}$ be a diagram. Then the slice and coslice diagonal maps*

$$\delta_{/F} : \mathcal{C}_{/F} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\} \qquad \delta_{F/} : \mathcal{C}_{F/} \hookrightarrow \{F\} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \mathcal{C}$$

are equivalences of ∞-categories.

Corollary 4.6.4.18. *Let $\mathcal{C}$ be an ∞-category and let $C \in \mathcal{C}$ be an object. Then the slice and coslice diagonal maps* 02VT

$$\delta_{/C} : \mathcal{C}_{/C} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathcal{C}} \{C\} \qquad \delta_{C/} : \mathcal{C}_{C/} \hookrightarrow \{C\} \tilde{\times}_{\mathcal{C}} \mathcal{C}$$

are equivalences of ∞-categories.

Corollary 4.6.4.19. *Let $G : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories and let $F : K \to \mathcal{C}$ be a diagram in $\mathcal{C}$. Then the induced functors* 02GL

$$G' : \mathcal{C}_{/F} \to \mathcal{D}_{/(G \circ F)} \qquad G'' : \mathcal{C}_{F/} \to \mathcal{D}_{(G \circ F)/}$$

are equivalences of ∞-categories.

Proof. We will show that G' is an equivalence of ∞-categories; the analogous statement for G'' follows by a similar argument. Note that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{/F} \arrow[r, "G'"] \arrow[d] & \mathcal{D}_{/(G \circ F)} \arrow[d] \\
\mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\} \arrow[r, "\overline{G}'"] & \mathcal{D} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{D})} \{G \circ F\},
\end{tikzcd}$$

where the vertical maps are equivalences of ∞-categories by virtue of Theorem 4.6.4.17. It will therefore suffice to show that $\overline{G}'$ is an equivalence of ∞-categories, which is a special case of Remark 4.6.4.4. □

Corollary 4.6.4.20. *Let $G : \mathcal{C} \to \mathcal{D}$ be fully faithful functor of ∞-categories and let $F : K \to \mathcal{C}$ be a morphism of simplicial sets. Then the induced functors* 02C8

$$G' : \mathcal{C}_{/F} \to \mathcal{D}_{/(G \circ F)} \qquad G'' : \mathcal{C}_{F/} \to \mathcal{D}_{(G \circ F)/}$$

are also fully faithful.

Proof. Let $\mathcal{C}' \subseteq \mathcal{D}$ be the essential image of G (Definition 4.6.2.11), so that G induces an equivalence of ∞-categories $\mathcal{C} \to \mathcal{C}'$ (Corollary 4.6.2.22). By virtue of Corollary 4.6.4.19, the functors G' and G'' restrict to equivalences

$$\mathcal{C}_{/F} \to \mathcal{C}'_{/(G \circ F)} \qquad \mathcal{C}_{F/} \to \mathcal{C}'_{(G \circ F)/}$$

We may therefore replace $\mathcal{C}$ by $\mathcal{C}'$ and thereby reduce to the case where $G : \mathcal{C} \hookrightarrow \mathcal{D}$ is the inclusion of a full subcategory. In this case, the functors G' and G'' are also the inclusions of full subcategories, hence fully faithful (Example 4.6.2.2). □

We now turn to the proof of Theorem 4.6.4.17. As we will see, it is essentially a reformulation of Theorem 4.5.8.8.

`01KV` **Lemma 4.6.4.21.** *Let $\mathcal{C}$ be an ∞-category, let $F : K \to \mathcal{C}$ be a diagram indexed by a simplicial set K. Suppose we are given a pair of diagrams $e_0, e_1 : J \to \mathcal{C}_{/F}$ indexed by a simplicial set J, which we identify with diagrams $F_0, F_1 : J \star K \to \mathcal{C}$ satisfying $F_0|_K = F = F_1|_K$. The following conditions are equivalent:*

(1) *The diagrams e_0 and e_1 are isomorphic when regarded as objects of the diagram ∞-category* $\mathrm{Fun}(J, \mathcal{C}_{/F})$*.*

(2) *The diagrams F_0 and F_1 are isomorphic when regarded as objects of the ∞-category* $\mathrm{Fun}_{K/}(J \star K, \mathcal{C})$*.*

Proof. Choose a categorical mapping cylinder

$$J \coprod J \xrightarrow{(s_0,s_1)} \overline{J} \xrightarrow{\pi} J$$

for the simplicial set J (Definition 4.6.3.3). Using Corollary 4.5.8.9, we deduce that the resulting diagram

$$(J \star K) \coprod_K (J \star K) \xrightarrow{(s'_0,s'_1)} \overline{J} \star K \xrightarrow{\pi'} J \star K$$

is a categorical mapping cylinder for the join $J \star K$ relative to K. Using the criterion of Corollary 4.6.3.11, we see that (1) and (2) can be reformulated as follows:

(1′) There exists a diagram $\overline{e} : \overline{J} \to \mathcal{C}_{/F}$ satisfying $e_0 = \overline{e} \circ s_0$ and $e_1 = \overline{e} \circ s_1$.

(2′) There exists a diagram $\overline{F}' : \overline{J} \star K \to \mathcal{C}$ satisfying $F_0 = \overline{F} \circ s'_0$ and $F_1 = \overline{F} \circ s'_1$.

The equivalence of (1′) and (2′) follows immediately from the universal property of the slice ∞-category $\mathcal{C}_{/F}$. □

Proof of Theorem 4.6.4.17. Let $\mathcal{C}$ be an ∞-category and let $F : K \to \mathcal{C}$ be a diagram, which we regard as an object of the ∞-category $\mathrm{Fun}(K, \mathcal{C})$. We will show that the slice diagonal morphism

$$\delta_{/F} : \mathcal{C}_{/F} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}$$

is an equivalence of ∞-categories; the corresponding assertion for the coslice diagonal morphism follows by a similar argument. Fix a simplicial set J; we wish to show that the induced map of sets

$$\theta : \pi_0(\mathrm{Fun}(J, \mathcal{C}_{/F})^{\simeq}) \to \pi_0(\mathrm{Fun}(J, \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\})^{\simeq})$$

is a bijection. Using Lemma 4.6.4.21 and Remark 4.6.4.14, we can identify θ with the map of sets

$$\pi_0(\mathrm{Fun}_{K/}(J \star K, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}_{K/}(J \diamond K, \mathcal{C})^{\simeq})$$

induced by precomposition with the comparison map $c_{J,K} : J \diamond K \twoheadrightarrow J \star K$ of Notation 4.5.8.3. It will therefore suffice to show that composition with $c_{J,K}$ induces an equivalence of ∞-categories $\mathrm{Fun}_{K/}(J \star K, \mathcal{C}) \to \mathrm{Fun}_{K/}(J \diamond K, \mathcal{C})$. This follows by applying Corollary 4.5.2.32 to the commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(J \star K, \mathcal{C}) \arrow[r, "\circ c_{J,K}"] \arrow[d] & \mathrm{Fun}(J \diamond K, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}(K, \mathcal{C}) \arrow[r, equal] & \mathrm{Fun}(K, \mathcal{C});
\end{tikzcd}$$

here the vertical maps are isofibrations (Corollary 4.4.5.3) and the upper horizontal map is an equivalence of ∞-categories because the morphism $c_{J,K}$ is a categorical equivalence (Theorem 4.5.8.8). □

4.6.5 Pinched Morphism Spaces

Let $\mathcal{C}$ be an ∞-category. In §4.6.1, we associated to every pair of objects $X, Y \in \mathcal{C}$ a Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, which we refer to as the *space of morphisms from X to Y* (Construction 4.6.1.1). In this section, we discuss a variant of this construction which is often more technically convenient to work with. 01KW

Construction 4.6.5.1. Let $\mathcal{C}$ be a simplicial set containing vertices X and Y. We let $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ denote the fiber product $\mathcal{C}_{X/} \times_{\mathcal{C}} \{Y\}$, and we let $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ denote the fiber product $\{X\} \times_{\mathcal{C}} \mathcal{C}_{/Y}$. We will be primarily interested in these constructions in the situation where $\mathcal{C}$ is an ∞-category. In this case, we refer to $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ as the *left-pinched space of morphisms from X to Y* and to $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ as the *right-pinched space of morphisms from X to Y*. 01KX

Remark 4.6.5.2. Let $\mathcal{C}$ be a simplicial set containing vertices X and Y. For every integer $n \geq 0$, one can identify n-simplices of the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ with $(n+1)$-simplices $\sigma : \Delta^{n+1} \to \mathcal{C}$ for which $\sigma(0) = X$ and the face $d^{n+1}_0(\sigma)$ is the constant map $\Delta^n \to \{Y\} \hookrightarrow \mathcal{C}$. Similarly, one can identify n-simplices of the right-pinched morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ with $(n+1)$-simplices $\sigma' : \Delta^{n+1} \to \mathcal{C}$ for which $\sigma(n+1) = Y$ and the face $d^{n+1}_{n+1}(\sigma)$ is the constant map $\Delta^n \to \{X\} \hookrightarrow \mathcal{C}$. In particular, we have canonical bijections 01KY

$$\{\text{Vertices of } \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)\} \simeq \{\text{Edges } f : X \to Y \text{ in } \mathcal{C}\} \simeq \{\text{Vertices of } \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)\}.$$

01KZ **Remark 4.6.5.3.** Let $\mathcal{C}$ be a simplicial set containing vertices X and Y, which we also regard as vertices of the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$. Then we have canonical isomorphisms of simplicial sets

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}^{\mathrm{op}}}(X,Y) \simeq \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(Y,X)^{\mathrm{op}} \qquad \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}^{\mathrm{op}}}(X,Y) \simeq \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(Y,X)^{\mathrm{op}}.$$

0574 **Remark 4.6.5.4.** Let $\mathcal{C}$ be a simplicial set, let $n \geq 0$ be an integer, and let $\mathrm{cosk}_n(\mathcal{C})$ denote the n-coskeleton of $\mathcal{C}$ (Notation 3.5.3.18). For every pair of vertices $X, Y \in \mathcal{C}$, Remark 4.3.5.16 supplies canonical isomorphisms

$$\mathrm{cosk}_n(\mathcal{C})_{X/} \simeq \mathrm{cosk}_{n-1}(\mathcal{C}_{X/}) \times_{\mathrm{cosk}_{n-1}(\mathcal{C})} \mathrm{cosk}_n(\mathcal{C})$$

$$\mathrm{cosk}_n(\mathcal{C})_{/Y} \simeq \mathrm{cosk}_{n-1}(\mathcal{C}_{/Y}) \times_{\mathrm{cosk}_{n-1}(\mathcal{C})} \mathrm{cosk}_n(\mathcal{C}).$$

Passing to fibers over the vertices Y and X, we obtain isomorphisms of pinched morphism spaces

$$\mathrm{Hom}^{\mathrm{L}}_{\mathrm{cosk}_n(\mathcal{C})}(X,Y) \simeq \mathrm{cosk}_{n-1}(\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)) \quad \mathrm{Hom}^{\mathrm{R}}_{\mathrm{cosk}_n(\mathcal{C})}(X,Y) \simeq \mathrm{cosk}_{n-1}(\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)).$$

In particular, if $\mathcal{C}$ is n-coskeletal, then the pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ are $(n-1)$-coskeletal.

01L0 **Proposition 4.6.5.5.** *Let $\mathcal{C}$ be an ∞-category. For every pair of objects $X, Y \in \mathcal{C}$, the pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ are Kan complexes.*

Proof. By virtue of Proposition 4.3.6.1, the projection map $\mathcal{C}_{X/} \to \mathcal{C}$ is a left fibration. Applying Corollary 4.4.2.3, we deduce that the fiber $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) = \mathcal{C}_{X/} \times_{\mathcal{C}} \{Y\}$ is a Kan complex. A similar argument shows that $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ is a Kan complex. □

01L1 **Remark 4.6.5.6.** Let $\mathcal{C}$ be an ∞-category containing a pair of morphisms $f, g : X \to Y$ having the same source and target. Then the datum of an edge $e : f \to g$ in the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ is equivalent to the datum of a homotopy from f to g, in the sense of Definition 1.4.3.1. In particular, f and g are homotopic if and only if they belong to the same connected component of $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$. We therefore have a canonical bijection $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,Y) \simeq \pi_0(\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y))$.

We now compare the pinched morphism spaces of Construction 4.6.5.1 with the morphism spaces of Construction 4.6.1.1.

01L2 **Construction 4.6.5.7.** Let $\mathcal{C}$ be a simplicial set containing vertices X and Y, and let

$$\delta_{X/} : \mathcal{C}_{X/} \hookrightarrow \{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \qquad \delta_{/Y} : \mathcal{C}_{/Y} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathcal{C}} \{Y\}$$

be the coslice and slice diagonal morphisms of Construction 4.6.4.13. Restricting to the fibers over the objects $Y, X \in \mathcal{C}$, we obtain morphisms of Kan complexes

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) = \mathcal{C}_{X/} \times_{\mathcal{C}} \{Y\} \to \{X\} \tilde{\times}_{\mathcal{C}} \{Y\} = \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

$$\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y) = \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Y} \to \{X\} \tilde{\times}_{\mathcal{C}} \{Y\} = \mathrm{Hom}_{\mathcal{C}}(X,Y),$$

which we will denote by $\iota^{\mathrm{L}}_{X,Y}$ and $\iota^{\mathrm{R}}_{X,Y}$, respectively. We will refer to $\iota^{\mathrm{L}}_{X,Y}$ as the *left-pinch inclusion map* and to $\iota^{\mathrm{R}}_{X,Y}$ as the *right-pinch inclusion map*.

Remark 4.6.5.8. Let $\mathcal{C}$ be a simplicial set containing vertices X and Y. Then the pinch inclusion maps 01L3

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) \xrightarrow{\iota^{\mathrm{L}}_{X,Y}} \mathrm{Hom}_{\mathcal{C}}(X,Y) \xleftarrow{\iota^{\mathrm{R}}_{X,Y}} \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$$

are monomorphisms (see Remark 4.6.4.15).

Remark 4.6.5.9. Let $\mathcal{C}$ be an ∞-category containing objects X and Y. Then the pinch inclusion maps 01L4

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) \xrightarrow{\iota^{\mathrm{L}}_{X,Y}} \mathrm{Hom}_{\mathcal{C}}(X,Y) \xleftarrow{\iota^{\mathrm{R}}_{X,Y}} \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$$

are bijective on vertices: vertices of each simplicial set can be identified with morphisms from X to Y in the ∞-category $\mathcal{C}$ (Remarks 4.6.1.2 and 4.6.5.2). However, they are generally not bijective on edges. Note that edges of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ can be identified with diagrams

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] \arrow[dr, "g"] \arrow[dr, phantom, "\sigma", shift left=6ex] \arrow[dr, phantom, "\tau"', shift right=6ex] & Y \arrow[d, "\mathrm{id}_Y"] \\
X \arrow[r, "f'"] & Y
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Such a diagram belongs to the image of the left-pinch inclusion map $\iota^{\mathrm{L}}_{X,Y}$ if and only $\tau = s_0(g)$ (so that the simplex τ is degenerate, $f' = g$, and the entire diagram is determined by σ). Similarly, the diagram belongs to the image of the right-pinch inclusion map $\iota^{\mathrm{R}}_{X,Y}$ if and only if $\sigma = s_1(g)$ (so that the simplex σ is degenerate, $f = g$, and the entire diagram is determined by τ).

01L5 **Proposition 4.6.5.10.** *Let $\mathcal{C}$ be an ∞-category. For every pair of objects $X, Y \in \mathcal{C}$, the pinch inclusion morphisms*

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) \xrightarrow{\iota^{\mathrm{L}}_{X,Y}} \mathrm{Hom}_{\mathcal{C}}(X,Y) \xleftarrow{\iota^{\mathrm{R}}_{X,Y}} \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$$

are homotopy equivalences of Kan complexes.

Proof. We will prove that the left-pinch inclusion morphism $\iota^{\mathrm{L}}_{X,Y}$ is a homotopy equivalence; the proof for the right-pinch inclusion morphism $\iota^{\mathrm{R}}_{X,Y}$ is similar. Note that we have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C}_{X/} \arrow[r] \arrow[d] & \{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathrm{id}"] & \mathcal{C},
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories (Corollary 4.6.4.18) and the vertical maps are left fibrations (Propositions 4.3.6.1 and 4.6.4.11), hence isofibrations (Example 4.4.1.11). Applying Corollary 4.5.2.32, we deduce that the induced map of fibers

$$\iota^{\mathrm{L}}_{X,Y} : \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) = (\mathcal{C}_{X/}) \times_{\mathcal{C}} \{Y\} \to \{X\} \tilde{\times}_{\mathcal{C}} \{Y\} = \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

is an equivalence of ∞-categories, hence a homotopy equivalence of Kan complexes (Remark 4.5.1.4). □

01PX **Corollary 4.6.5.11.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. The following conditions are equivalent:*

- *The functor F is fully faithful. That is, for every pair of objects $X, Y \in \mathcal{C}$, the functor F induces a homotopy equivalence of Kan complexes $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.*
- *For every pair of objects $X, Y \in \mathcal{C}$, the functor F induces a homotopy equivalence of left-pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}^{\mathrm{L}}_{\mathcal{D}}(F(X), F(Y))$.*
- *For every pair of objects $X, Y \in \mathcal{C}$, the functor F induces a homotopy equivalence of right-pinched morphism spaces $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}^{\mathrm{R}}_{\mathcal{D}}(F(X), F(Y))$.*

01L6 **Example 4.6.5.12.** Let $\mathcal{C}$ be an ordinary category containing objects X and Y. Then the slice and coslice diagonal morphisms

$$\delta_{X/} : \mathrm{N}_{\bullet}(\mathcal{C})_{X/} \to \{X\} \tilde{\times}_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}(\mathcal{C}) \qquad \delta_{/Y} : \mathrm{N}_{\bullet}(\mathcal{C})_{/Y} \to (\mathrm{N}_{\bullet}(\mathcal{C}) \tilde{\times}_{\mathrm{N}_{\bullet}(\mathcal{C})} \{Y\}$$

are isomorphisms (see Remark 4.3.1.7). In particular, we can identify the pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}(\mathcal{C})}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}_{\bullet}(\mathcal{C})}(X,Y)$ with the constant simplicial set $\mathrm{Hom}_{\mathrm{N}_{\bullet}(\mathcal{C})}(X,Y)$ associated to the usual morphism set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Let $\mathcal{C}$ be an ∞-category containing a pair of objects X and Y. By virtue of Proposition 4.6.5.10, the pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ of Construction 4.6.5.1 contain the same homotopy-theoretic information as the morphism space $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of Construction 4.6.1.1. However, they package this information in a more efficient way: an n-simplex of the Kan complex $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ can be identified with a single $(n+1)$-simplex of the ∞-category $\mathcal{C}$ (see Remark 4.6.5.2), but to specify an n-simplex of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ one must supply $n+1$ different $(n+1)$-simplices of $\mathcal{C}$ (see Remark 4.6.5.9 for the case $n=1$).

Example 4.6.5.13 (Pinched Morphism Spaces in the Duskin Nerve)**.** Let $\mathcal{C}$ be a 2-category (Definition 2.2.1.1). For each integer $n \geq 0$, we can use Remark 2.3.1.8 to identify $(n+1)$-simplices σ of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ with the following data: `01L7`

(0) A collection of objects $\{Z_i\}_{0 \leq i \leq n+1}$ of the 2-category $\mathcal{C}$.

(1) A collection of 1-morphisms $\{f_{j,i} : Z_i \to Z_j\}_{0 \leq i \leq j \leq n+1}$ in the 2-category $\mathcal{C}$, satisfying $f_{j,i} = \mathrm{id}_{Z_i}$ when $i = j$.

(2) A collection of 2-morphisms $\{\mu_{k,j,i} : f_{k,j} \circ f_{j,i} \Rightarrow f_{k,i}\}_{0 \leq i \leq j \leq k \leq n+1}$ in the 2-category $\mathcal{C}$, satisfying some additional constraints (see (b) and (c) of Proposition 2.3.1.9).

Fix a pair of objects X and Y. Then σ represents an n-simplex of the right pinched morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X,Y)$ if and only if the above data satisfies the following additional conditions:

- For $0 \leq i \leq n$, the object Z_i is equal to X. For $i = n+1$, the object Z_i is equal to Y.
- For $0 \leq i \leq j \leq n$, the 1-morphism $f_{j,i}$ is equal to the identity 1-morphism id_X.
- For $0 \leq i \leq j \leq k \leq n$, the 2-morphism $\mu_{k,j,i}$ is equal to the unit constraint $\upsilon : \mathrm{id}_X \circ \mathrm{id}_X \Rightarrow \mathrm{id}_X$.

In this case, we can identify (1) with a collection of 1-morphisms $\{g_i : X \to Y\}_{0 \leq i \leq n}$ given by $g_i = f_{n+1,i}$, and (2) with a collection of 2-morphisms $\{\nu_{j,i} : g_j \Rightarrow g_i\}_{0 \leq i \leq j \leq n}$, where $\nu_{j,i}$ is given by the composition

$$g_j \overset{\sim}{\Longrightarrow} g_j \circ \mathrm{id}_X = f_{n+1,j} \circ f_{j,i} \xRightarrow{\mu_{n+1,j,i}} f_{n+1,i} = g_i.$$

Unwinding the definitions, condition (b) translates to the requirement that $\nu_{j,i}$ is an identity 2-morphism when $i = j$, and condition (c) translates to the identity $\nu_{k,j} \circ \nu_{j,i} = \nu_{k,i}$ for $0 \leq i \leq j \leq k \leq n$. In this case, we can identify the pair $(\{g_i\}_{0 \leq i \leq n}, \{\nu_{j,i}\}_{0 \leq i \leq j \leq n})$ with a functor $[n] \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)^{\mathrm{op}}$. These identifications depends functorially on $[n] \in \mathbf{\Delta}$, and therefore determine a canonical isomorphism of simplicial sets

$$\mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X,Y) \simeq \mathrm{N}_{\bullet}(\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)^{\mathrm{op}}).$$

Using similar reasoning, we obtain an isomorphism of simplicial sets

$$\operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X,Y) \simeq \mathrm{N}_{\bullet}(\underline{\operatorname{Hom}}_{\mathcal{C}}(X,Y)).$$

01L8 **Example 4.6.5.14.** Let X be a topological space containing a pair of points x and y, which we regard as objects of the ∞-category $\operatorname{Sing}_{\bullet}(X)$. Using Example 4.3.5.9, we obtain canonical isomorphisms of Kan complexes

$$\operatorname{Hom}^{\mathrm{L}}_{\operatorname{Sing}_{\bullet}(X)}(x,y) \simeq \operatorname{Sing}_{\bullet}(P_{x,y}) \simeq \operatorname{Hom}^{\mathrm{R}}_{\operatorname{Sing}_{\bullet}(X)}(x,y),$$

where $P_{x,y}$ denotes the topological space of continuous paths $p : [0,1] \to X$ satisfying $p(0) = x$ and $p(1) = y$ (equipped with the compact-open topology). Combining this observation with Example 4.6.1.5, we can identify the pinch inclusion maps $\iota^{\mathrm{L}}_{x,y}$ and $\iota^{\mathrm{R}}_{x,y}$ with monomorphisms from the simplicial set $\operatorname{Sing}_{\bullet}(P_{x,y})$ to itself. Beware that these maps are not the identity (though one can show that they are homotopic to the identity).

01L9 **Example 4.6.5.15** (Pinched Morphism Spaces in the Differential Graded Nerve)**.** Let $\mathcal{C}$ be a differential graded category (Definition 2.5.2.1), let $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ denote the differential graded nerve of $\mathcal{C}$ (Definition 2.5.3.7), and let X and Y be objects of $\mathcal{C}$ (which we also view as objects of the ∞-category $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$), and let $\operatorname{Hom}_{\mathcal{C}}(X,Y)_{*}$ denote the chain complex of morphisms from X to Y. For $n \geq 0$, we can identify n-simplices of the left-pinched morphism space $\operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})}(X,Y)$ with $(n+1)$-simplices $\sigma : \Delta^{n+1} \to \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ for which $\sigma(0) = X$ and $d^{n+1}_{0}(\sigma)$ is the constant n-simplex with the value Y (Remark 4.6.5.2). Concretely, such a simplex can be described as a datum $I \mapsto f_I$, defined for each subset $I = \{i_0 > i_1 > i_2 > \cdots > i_k > i_{k+1}\} \subseteq [n+1]$ having at least two elements, with the following properties:

(1) If $i_{k+1} > 0$, then f_I is an element of the abelian group $\operatorname{Hom}_{\mathcal{C}}(Y,Y)_k$, which is equal to id_Y in the case $k = 0$ and vanishes for $k > 0$.

(2) If $i_{k+1} = 0$, then f_I is an element of the abelian group $\operatorname{Hom}_{\mathcal{C}}(X,Y)_k$ which satisfies the identity

$$\partial f_I = \sum_{a=1}^{k} (-1)^{a} (f_{\{i_0 > i_1 > \cdots > i_a\}} \circ f_{\{i_a > \cdots > i_{k+1}\}} - f_{I \setminus \{i_a\}}).$$

Note that, by virtue of (1), we can rewrite this identity as

01PY
$$\partial f_I = \begin{cases} 0 & \text{if } k = 0 \\ \sum_{a=0}^{k} (-1)^{a+1} f_{I \setminus \{i_a\}} & \text{if } k > 0. \end{cases} \tag{4.38}$$

Let $J = \{j_0 < j_1 < \cdots < j_k\}$ be a nonempty subset of $[n]$. For $\{f_I\}$ as above, define $g_J \in \operatorname{Hom}_{\mathcal{C}}(X,Y)_k$ by the formula $g_J = (-1)^{k(k-1)/2} f_{\{j_k+1 > j_{k-1}+1 > \cdots > 0\}}$. We can then rewrite the identity (4.38) as

$$\partial g_J = \sum_{b=0}^{k} (-1)^b g_{J \setminus \{j_b\}}.$$

The construction $J \mapsto g_J$ can then be identified with a morphism from the normalized chain complex $\mathrm{N}_*(\Delta^n)$ of Construction 2.5.5.9 to the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$. This identification depends functorially on n, and therefore determines an isomorphism of simplicial sets

$$\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})}(X,Y) \simeq \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*),$$

where $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*)$ denotes the Eilenberg-MacLane space associated to the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ (Construction 2.5.6.3). In particular, the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})}(X,Y)$ has the structure of a simplicial abelian group.

4.6.6 Digression: Diagrams in Slice ∞-Categories

For some applications, it will be useful to combine the slice and coslice constructions introduced in §4.3. 04HY

Notation 4.6.6.1. Let K_-, K_+, and $\mathcal{C}$ be simplicial sets, and suppose we are given a morphism $f_\pm : K_- \star K_+ \to \mathcal{C}$. Set $f_- = f_\pm|_{K_-}$ and $f_+ = f_\pm|_{K_+}$, and let $\pi : \mathcal{C}_{/f_+} \to \mathcal{C}$ denote the projection map. Then $f_\pm$ determines a morphism of simplicial sets $\widetilde{f}_- : K_- \to \mathcal{C}_{/f_+}$ for which the diagram 04HZ

$$\begin{tikzcd} & \mathcal{C}_{/f_+} \arrow[dr, "\pi"] & \\ K_- \arrow[ur, "\widetilde{f}_-"] \arrow[rr, "f_-"] & & \mathcal{C} \end{tikzcd}$$

is commutative. In this situation, we let $\mathcal{C}_{f_-//f_+}$ denote the coslice simplicial set $(\mathcal{C}_{/f_+})_{\widetilde{f}_-/}$.

Remark 4.6.6.2. In the situation of Notation 4.6.6.1, we can also identify $f_\pm$ with a morphism of simplicial sets $\widetilde{f}_+ : K_+ \to \mathcal{C}_{f_-/}$. Let Y be any simplicial set. Using Proposition 4.3.5.13 we see that the following data are equivalent: 01NR

(1) Morphisms from Y to $(\mathcal{C}_{/f_+})_{\widetilde{f}_-/}$.

(2) Morphisms from Y to $(\mathcal{C}_{f_-/})_{/\widetilde{f}_+}$.

(3) Morphisms $f : K_- \star Y \star K_+ \to \mathcal{C}$ satisfying $f|_{K_- \star K_+} = f_\pm$.

It follows that the simplicial set $\mathcal{C}_{f_-//f_+} = (\mathcal{C}_{/f_+})_{\widetilde{f}_-/}$ can also be identified with the slice simplicial set $(\mathcal{C}_{f_-/})_{/\widetilde{f}_+}$.

04J0 **Warning 4.6.6.3.** In the situation of Notation 4.6.6.1, the simplicial set $\mathcal{C}_{f_-//f_+}$ depends on the morphism $f_\pm : K_- \star K_+ \to \mathcal{C}$, and not only on the morphisms $f_- = f_\pm|_{K_-}$ and $f_+ = f_\pm|_{K_+}$ indicated in the notation.

In the situation of Notation 4.6.6.1, suppose that the simplicial set $\mathcal{C}$ is an ∞-category. Applying Proposition 4.3.6.1 twice, we deduce that the simplicial set $\mathcal{C}_{f_-//f_+}$ is also an ∞-category. We now exploit the relationship between slice constructions and oriented fiber products (Theorem 4.6.4.17) to give an alternative description of $\mathcal{C}_{f_-//f_+}$.

04J1 **Construction 4.6.6.4.** Let $f_\pm : K_- \star K_+ \to \mathcal{C}$ be a morphism of simplicial sets, and set $f_- = f_\pm|_{K_-}$ and $f_+ = f_\pm|_{K_+}$. Let K be another simplicial set, set $M = \operatorname{Fun}(K, \mathcal{C}_{f_-//f_+})$. Let $\mathrm{ev} : M \times K \to \mathcal{C}_{f_-//f_+}$ be the evaluation map, and let $\pi_- : M \times K_- \to K_-$ and $\pi_+ : M \times K_+ \to K_+$ be given by projection onto the second factor. Then the composition

$$\begin{array}{rcl} M \times (K_- \star K \star K_+) & \to & (M \times K_-) \star (M \times K) \star (M \times K_+) \\ & \xrightarrow{\pi_- \star \mathrm{ev} \star \pi_+} & K_- \star \mathcal{C}_{f_-//f_+} \star K_+ \\ & \to & \mathcal{C} \end{array}$$

classifies a morphism of simplicial sets $M \to \operatorname{Fun}(K_- \star K \star K_+, \mathcal{C})$, whose composition with the restriction map $\operatorname{Fun}(K_- \star K \star K_+, \mathcal{C}) \to \operatorname{Fun}(K_- \star K_+, \mathcal{C})$ is the constant map taking the value $f_\pm$. We therefore obtain a comparison map

$$\theta : \operatorname{Fun}(K, \mathcal{C}_{f_-//f_+}) \to \operatorname{Fun}(K_- \star K \star K_+, \mathcal{C}) \times_{\operatorname{Fun}(K_- \star K_+, \mathcal{C})} \{f_\pm\}.$$

04J2 **Theorem 4.6.6.5.** *Let $\mathcal{C}$ be an ∞-category and let $f_\pm : K_- \star K_+ \to \mathcal{C}$ be a diagram. Then, for any simplicial set K, the comparison map*

$$\theta : \operatorname{Fun}(K, \mathcal{C}_{f_-//f_+}) \to \operatorname{Fun}(K_- \star K \star K_+, \mathcal{C}) \times_{\operatorname{Fun}(K_- \star K_+, \mathcal{C})} \{f_\pm\}$$

of Construction 4.6.6.4 is an equivalence of ∞-categories.

04J3 **Example 4.6.6.6.** Let $\mathcal{C}$ be an ∞-category and let $f_\pm : K_- \star K_+ \to \mathcal{C}$ be a diagram. Applying Theorem 4.6.6.5 in the special case $K = \Delta^0$, we obtain an equivalence of ∞-categories

$$\mathcal{C}_{f_-//f_+} \to \operatorname{Fun}(K_- \star \Delta^0 \star K_+, \mathcal{C}) \times_{\operatorname{Fun}(K_- \star K_+, \mathcal{C})} \{f_\pm\}.$$

We begin by proving Theorem 4.6.6.5 in the special case where K_- is empty.

04J4 **Lemma 4.6.6.7.** *Let $\mathcal{C}$ be an ∞-category and let $f_+ : K_+ \to \mathcal{C}$ be a diagram. Then, for every simplicial set K, the comparison map*

$$\theta : \operatorname{Fun}(K, \mathcal{C}_{/f_+}) \to \operatorname{Fun}(K \star K_+, \mathcal{C}) \times_{K_+, \mathcal{C})} \{f_+\}$$

of Construction 4.6.6.4 is an equivalence of ∞-categories.

Proof. Let $c : K \diamond K_+ \to K \star K_+$ be as in Notation 4.5.8.3. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(K \star K_+, \mathcal{C}) \arrow[r, "\circ c"] \arrow[d] & \mathrm{Fun}(K \diamond K_+, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}(K_+, \mathcal{C}) \arrow[r, equal] & \mathrm{Fun}(K_+, \mathcal{C})
\end{tikzcd}$$

where the vertical maps are isofibrations (Corollary 4.4.5.3). Since c is a categorical equivalence of simplicial sets (Theorem 4.5.8.8), the upper horizontal map is an equivalence of ∞-categories. Applying Corollary 4.5.2.32, we conclude that composition with c induces an equivalence of ∞-categories

$$\begin{aligned}
\mathrm{Fun}(K \star K_+, \mathcal{C}) \times_{\mathrm{Fun}(K_+,\mathcal{C})} \{f_+\} \quad &\xrightarrow{\theta'} \quad \mathrm{Fun}(K \diamond K_+, \mathcal{C}) \times_{\mathrm{Fun}(K_+,\mathcal{C})} \{f_+\} \\
&\simeq \quad \mathrm{Fun}(K, \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K_+,\mathcal{C})} \{f_+\}).
\end{aligned}$$

It will therefore suffice to show that $\theta' \circ \theta$ is an equivalence of ∞-categories. We conclude by observing that $\theta' \circ \theta$ is given by postcomposition with the slice diagonal $\mathcal{C}_{/f_+} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K_+,\mathcal{C})} \{f_+\}$, which is an equivalence of ∞-categories by virtue of Theorem 4.6.4.17. □

Example 4.6.6.8. Let $\mathcal{C}$ be an ∞-category and let $f_+ : K_+ \to \mathcal{C}$ be a diagram. Applying Lemma 4.6.6.7 in the special case $K = \Delta^0$, we obtain an equivalence of ∞-categories $\mathcal{C}_{/f_+} \to \mathrm{Fun}(K_+^{\triangleleft}, \mathcal{C}) \times_{\mathrm{Fun}(K_+,\mathcal{C})} \{f_+\}$. 04J5

Proof of Theorem 4.6.6.5. Let $\mathcal{C}$ be an ∞-category and let $f_\pm : K_- \star K_+ \to \mathcal{C}$ be a diagram. Set $f_- = f_\pm|_{K_-}$ and $f_+ = f_\pm|_{K_+}$, so that $f_\pm$ can be identified with a morphism $\widetilde{f}_- : K_- \to \mathcal{C}_{/f_+}$. We then have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Fun}(K, \mathcal{C}_{f_-//f_+}) \arrow[r] \arrow[d] & \mathrm{Fun}(K_- \star K, \mathcal{C}_{/f_+}) \arrow[r] \arrow[d] & \mathrm{Fun}(K_- \star K \star K_+, \mathcal{C}) \arrow[d] \\
\{\widetilde{f}_-\} \arrow[r] & \mathrm{Fun}(K_-, \mathcal{C}_{/f_+}) \arrow[r] \arrow[d] & \mathrm{Fun}(K_- \star K_+, \mathcal{C}) \arrow[d] \\
 & \{f_+\} \arrow[r] & \mathrm{Fun}(K_+, \mathcal{C}),
\end{tikzcd}$$

where the vertical maps are isofibrations (Corollary 4.4.5.3). It follows from Lemma 4.6.6.7 (and Proposition 4.5.2.26) that the lower right square and the outer right rectangle are categorical pullback squares, so the upper right corner is also a categorical pullback square (Proposition 4.5.2.18). Similarly, the dual of Lemma 4.6.6.7 guarantees that the upper left corner is a categorical pullback square. Applying Proposition 4.5.2.18, we conclude that the outer rectangle on the top of the diagram is a categorical pullback square, which is a restatement of Theorem 4.6.6.5 (Proposition 4.5.2.26). □

04J6 **Corollary 4.6.6.9.** *Let $\mathcal{C}$ be an ∞-category and let $f_{\pm} : K_{-} \star K_{+} \to \mathcal{C}$ be a diagram. Then, for any inclusion of simplicial sets $A \hookrightarrow B$, the diagram of ∞-categories*

04J7
$$\begin{tikzcd}
\mathrm{Fun}(B, \mathcal{C}_{f_{-}//f_{+}}) \arrow[r] \arrow[d] & \mathrm{Fun}(K_{-} \star B \star K_{+}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}(A, \mathcal{C}_{f_{-}//f_{+}}) \arrow[r] & \mathrm{Fun}(K_{-} \star A \star K_{+}, \mathcal{C})
\end{tikzcd} \tag{4.39}$$

is a categorical pullback square.

Proof. We can identify (4.39) with the upper half of a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathcal{C}_{f_{-}//f_{+}}) \arrow[r] \arrow[d] & \mathrm{Fun}(K_{-} \star B \star K_{+}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}(A, \mathcal{C}_{f_{-}//f_{+}}) \arrow[r] \arrow[d] & \mathrm{Fun}(K_{-} \star A \star K_{+}, \mathcal{C}) \arrow[d] \\
\{f_{\pm}\} \arrow[r] & \mathrm{Fun}(K_{-} \star K_{+}, \mathcal{C}).
\end{tikzcd}$$

By virtue of Proposition 4.5.2.18, it will suffice to show that the lower half and outer rectangle of the diagram are categorical pullback squares, which follows from Theorem 4.6.6.5. □

We conclude this section by recording a thematically related result, which characterizes slices of functor ∞-categories (rather than functors into a slice ∞-category).

02GN **Variant 4.6.6.10.** Let $\mathcal{C}$ be an ∞-category and let $f : K \to \mathcal{C}$ be a diagram, which we identify with an object F of the ∞-category $\mathrm{Fun}(K, \mathcal{C})$. Then the functors

$$\mathcal{C}_{/f} \to \mathcal{C} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K, \mathcal{C})_{/F} \qquad \mathcal{C}_{f/} \to \mathrm{Fun}(K, \mathcal{C})_{F/} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathcal{C}$$

of Exercise 4.6.4.16 are equivalences of ∞-categories.

Proof. We will show that the slice diagonal $\delta_{/f}$ induces an equivalence of ∞-categories $\mathcal{C}_{/f} \to \mathcal{C} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K, \mathcal{C})_{/F}$; the analogous assertion for coslice ∞-categories follows by a similar argument. By virtue of Theorem 4.6.4.17, it will suffice to show that the inclusion map

$$\mathcal{C} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K, \mathcal{C})_{/F} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}$$

is an equivalence of ∞-categories. By construction, this map fits into a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \times_{\operatorname{Fun}(K,\mathcal{C})} \operatorname{Fun}(K,\mathcal{C})_{/F} \arrow[r] \arrow[d, "\iota"] & \operatorname{Fun}(K,\mathcal{C})_{/F} \arrow[d, "U"] \\
\mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{F\} \arrow[r] \arrow[d] & \operatorname{Fun}(K,\mathcal{C}) \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{F\} \arrow[d, "V"] \\
\mathcal{C} \arrow[r] & \operatorname{Fun}(K,\mathcal{C}),
\end{tikzcd}$$

where the upper square and lower square are both pullback diagrams. Note that the morphisms V and $V \circ U$ are both right fibrations (Propositions 4.6.4.11 and 4.3.6.1), and therefore isofibrations (Example 4.4.1.11). Using Propositions 4.5.2.26 and 4.5.2.18, we see that the upper square is a categorical pullback. Theorem 4.6.4.17 guarantees that U is an equivalence of ∞-categories, so that ι is an equivalence of ∞-categories by virtue of Proposition 4.5.2.21. □

4.6.7 Initial and Final Objects

Let $\mathcal{C}$ be a category. Recall that an object $Y \in \mathcal{C}$ is *initial* if, for every object $Z \in \mathcal{C}$, there is a unique morphism from Y to Z. This definition has an obvious counterpart in the setting of ∞-categories. 02H6

Definition 4.6.7.1. Let $\mathcal{C}$ be an ∞-category. We say that an object $Y \in \mathcal{C}$ is *initial* if, for every object $Z \in \mathcal{C}$, the morphism space $\operatorname{Hom}_{\mathcal{C}}(Y, Z)$ is a contractible Kan complex. We say that Y is *final* if, for every object $X \in \mathcal{C}$, the morphism space $\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a contractible Kan complex. 02H7

Remark 4.6.7.2. Let $\mathcal{C}$ be an ∞-category. Then an object $Y \in \mathcal{C}$ is initial if and only if it is final when viewed as an object of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. 02H8

Example 4.6.7.3. Let $\mathcal{C}$ be a category. An object $Y \in \mathcal{C}$ is initial if and only if it is initial when viewed as an object of the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$. Similarly, an object $Y \in \mathcal{C}$ is final if and only if it is final when viewed as an object of the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$. 02H9

Example 4.6.7.4. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $\mathcal{C} \star \mathcal{D}$ denote their join (Construction 4.3.3.13). Then $\mathcal{C} \star \mathcal{D}$ is also an ∞-category (Corollary 4.3.3.25). It follows from Example 4.6.1.6 that if X is an initial object of $\mathcal{C}$, then it is also initial when regarded as an object of $\mathcal{C} \star \mathcal{D}$. Similarly, if Y is a final object of $\mathcal{D}$, then it is also final when regarded as an object of $\mathcal{C} \star \mathcal{D}$. 02MW

`02MX` **Example 4.6.7.5.** Let $\mathcal{C}$ be an ∞-category. Then the cone point of the ∞-category $\mathcal{C}^{\triangleleft}$ is an initial object. Similarly, the cone point of $\mathcal{C}^{\triangleright}$ is a final object.

`02HA` **Remark 4.6.7.6.** In the formulation of Definition 4.6.7.1, we can replace the Kan complexes $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ by their left-pinched variants $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(Y, Z)$, or by their right-pinched variants $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(Y, Z)$ (see Proposition 4.6.5.10).

`02HB` **Example 4.6.7.7.** Let $\mathcal{C}$ be a locally Kan simplicial category, so that the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.4.5.1). Combining Remark 4.6.7.6 with Theorem 4.6.8.5, we deduce the following:

- An object $Y \in \mathcal{C}$ is initial when viewed as an object of the ∞-category $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ if and only if, for every object $Z \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet}$ is contractible.

- An object $Y \in \mathcal{C}$ final when viewed as an object of the ∞-category $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ if and only if, for every object $X \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet}$ is contractible.

`02HD` **Example 4.6.7.8.** Let $\mathcal{C}$ be a $(2,1)$-category, so that the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.3.2.1). Combining Remark 4.6.7.6) with Example 4.6.5.13, we obtain the following:

- An object $Y \in \mathcal{C}$ is initial when viewed as an object of the ∞-category $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ if and only if, for every object $Z \in \mathcal{C}$, the groupoid $\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z)$ is contractible (that is, there exists a 1-morphism from Y to Z and for every pair of morphisms $f, g : X \to Y$, there is a unique isomorphism $\gamma : f \xRightarrow{\sim} g$).

- An object $Y \in \mathcal{C}$ is final when viewed as an object of the ∞-category $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ if and only if, for every object $X \in \mathcal{C}$, the groupoid $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ is contractible.

`02HE` **Proposition 4.6.7.9.** *Let $\mathcal{C}$ be a differential graded category, so that the differential graded nerve $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.5.3.10). Let Y be an object of $\mathcal{C}$. The following conditions are equivalent:*

(1) *The object Y is initial when viewed as an object of the ∞-category $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$.*

(2) *The object Y is final when viewed as an object of the ∞-category $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$.*

(3) *The identity morphism $\mathrm{id}_Y : Y \to Y$ is nullhomologous: that is, there exists a 1-chain $e \in \mathrm{Hom}_{\mathcal{C}}(Y, Y)_1$ satisfying $\partial(e) = \mathrm{id}_Y$.*

Proof. We will show that (1) ⇔ (3); the proof that (2) ⇔ (3) is similar. If condition (1) is satisfied, then there exists a 2-simplex of $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ with boundary as indicated in the diagram

$$\begin{tikzcd} & Y \arrow[dr, "0"] & \\ Y \arrow[ur, "0"] \arrow[rr, "\mathrm{id}_Y"] & & Y, \end{tikzcd}$$

which we can identify with a 1-chain $e \in \mathrm{Hom}_{\mathcal{C}}(Y,Y)_1$ satisfying $\partial(e) = \mathrm{id}_Y$ (see Example 2.5.3.4). Conversely, suppose that there exists $e \in \mathrm{Hom}_{\mathcal{C}}(Y,Y)_1$ satisfying $\partial(e) = \mathrm{id}_Y$. For every object $Z \in \mathcal{C}$, e determines a chain homotopy from the identity map $\mathrm{id} : \mathrm{Hom}_{\mathcal{C}}(Y,Z)_* \to \mathrm{Hom}_{\mathcal{C}}(Y,Z)_*$ to the zero map. It follows that the homology of chain complex $\mathrm{Hom}_{\mathcal{C}}(Y,Z)_*$ vanishes, so that the Eilenberg-MacLane space $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_*)$ of Construction 2.5.6.3 is a contractible Kan complex. Example 4.6.5.15 supplies an isomorphism of Kan complexes $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{dg}}_{\bullet}}(Y,Z) \simeq \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_*)$. Allowing Z to vary and invoking Remark 4.6.7.6, we conclude that Y is an initial object of the ∞-category $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$. □

Proposition 4.6.7.10. *Let $\mathcal{C}$ be an ∞-category and let Y be an object of $\mathcal{C}$. Then:* 02HF

(1) *The object Y is initial if and only if the projection map $\mathcal{C}_{Y/} \to \mathcal{C}$ is a trivial Kan fibration of simplicial sets.*

(2) *The object Y is final if and only if the projection map $\mathcal{C}_{/Y} \to \mathcal{C}$ is a trivial Kan fibration of simplicial sets.*

Proof. We will give the proof of (1); the proof of (2) is similar. Proposition 4.3.6.1 guarantees that the projection map $q : \mathcal{C}_{Y/} \to \mathcal{C}$ is a left fibration of simplicial sets. Applying Proposition 4.4.2.14, we see that q is a trivial Kan fibration if and only if, for each object $Z \in \mathcal{C}$, the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(Y,Z) = \mathcal{C}_{Y/} \times_{\mathcal{C}} \{Z\}$ is a contractible Kan complex. By virtue of Remark 4.6.7.6, this is equivalent to the assumption that Y is an initial object of $\mathcal{C}$. □

Corollary 4.6.7.11. *Let X be a Kan complex and let $x \in X$ be a vertex. The following conditions are equivalent:* 02HJ

(1) *The vertex x is initial when viewed as an object of the ∞-category X.*

(2) *The vertex x is final when viewed as an object of the ∞-category X.*

(3) *The Kan complex X is contractible.*

In particular, these conditions are independent of the choice of vertex $x \in X$.

Proof. If the Kan complex X is contractible, then the projection map $X_{x/} \to X$ is a trivial Kan fibration (Corollary 4.3.7.19), so the object $x \in X$ is initial by virtue of Proposition 4.6.7.10. Conversely, if the projection map $X_{x/} \to X$ is a trivial Kan fibration, then it is a homotopy equivalence (Proposition 3.1.6.10). Since the Kan complex $X_{x/}$ is contractible (Corollary 4.3.7.14), it follows that X is contractible. This proves the equivalence of (1) and (3); the equivalence of (2) and (3) follows by a similar argument. □

02HG **Corollary 4.6.7.12.** *Let $\mathcal{C}$ be an ∞-category, let $f : K \to \mathcal{C}$ be a diagram, let $U : \mathcal{C}_{/f} \to \mathcal{C}$ be the projection map, and let Y be an initial object of $\mathcal{C}$. Then:*

(1) *There exists an object $\widetilde{Y} \in \mathcal{C}_{/f}$ satisfying $U(\widetilde{Y}) = Y$.*

(2) *If $\widetilde{Y}$ is any object of $\mathcal{C}_{/f}$ satisfying $U(\widetilde{Y}) = Y$, then $\widetilde{Y}$ is an initial object of $\mathcal{C}_{/f}$.*

Proof. Assertion (1) is equivalent to the statement that f can be lifted to a map $\widetilde{f} : K \to \mathcal{C}_{Y/}$. This is clear, since the projection map $\mathcal{C}_{Y/} \to \mathcal{C}$ is a trivial Kan fibration (Proposition 4.6.7.10). To prove (2), fix an object $\widetilde{Y} \in \mathcal{C}_{/f}$ satisfying $U(\widetilde{Y}) = Y$. By virtue of Proposition 4.6.7.10, it will suffice to show that the projection map $(\mathcal{C}_{/f})_{\widetilde{Y}/} \to \mathcal{C}_{/f}$ is a trivial Kan fibration. Equivalently, we wish to show that every lifting problem

02HH
$$\begin{tikzcd}
A \arrow[r] \arrow[d] & (\mathcal{C}_{/f})_{\widetilde{Y}/} \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{/f}
\end{tikzcd} \tag{4.40}$$

admits a solution, provided that the left vertical map is a monomorphism. Unwinding the definitions, we can rewrite (4.40) as a lifting problem

$$\begin{tikzcd}
A \star K \arrow[r] \arrow[d] & \mathcal{C}_{Y/} \arrow[d] \\
B \star K \arrow[r] \arrow[ur, dashed] & \mathcal{C}.
\end{tikzcd}$$

Our assumption that the object $Y \in \mathcal{C}$ is initial guarantees that this lifting problem has a solution (Proposition 4.6.7.10). □

02HK **Corollary 4.6.7.13.** *Let $\mathcal{C}$ be an ∞-category. An object $Y \in \mathcal{C}$ is initial if and only if, for every integer $n \geq 1$ and every morphism of simplicial sets $\sigma : \partial \Delta^n \to \mathcal{C}$ satisfying $\sigma(0) = Y$, there exists an n-simplex $\overline{\sigma} : \Delta^n \to \mathcal{C}$ satisfying $\overline{\sigma}|_{\partial \Delta^n} = \sigma$.*

Proof. Let n be a positive integer. Using the isomorphism

$$\partial \Delta^n \simeq (\emptyset \star \Delta^{n-1}) \coprod_{(\emptyset \star \partial \Delta^{n-1})} (\Delta^0 \star \partial \Delta^{n-1})$$

supplied by Variant 4.3.6.17, we see that a morphism of simplicial sets $\sigma : \partial \Delta^n \to \mathcal{C}$ satisfying $\sigma(0) = Y$ can be identified with a commutative diagram

02HL

$$\begin{tikzcd}
\partial \Delta^{n-1} \arrow[r] \arrow[d] & \mathcal{C}_{Y/} \arrow[d] \\
\Delta^{n-1} \arrow[r] \arrow[ur, dashed] & \mathcal{C},
\end{tikzcd} \qquad (4.41)$$

and that an extension of σ to an n-simplex of $\mathcal{C}$ can be identified with a dotted arrow which renders the diagram commutative. By virtue of Proposition 4.6.7.10, the object Y is initial if and only if every lifting problem of the form (4.41) admits a solution: that is, if and only if the projection map $\mathcal{C}_{Y/} \to \mathcal{C}$ is a trivial Kan fibration of simplicial sets. □

Let $\mathcal{C}$ be an ∞-category which contains an initial object X. This object is rarely unique: every object $Y \in \mathcal{C}$ which is isomorphic to X is also initial (Corollary 4.6.7.15). However, the object X is *essentially unique* in the following sense:

Corollary 4.6.7.14. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}^{\mathrm{init}} \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by the initial objects of $\mathcal{C}$, and let $\mathcal{C}^{\mathrm{fin}} \subseteq \mathcal{C}$ be the full subcategory spanned by the final objects of $\mathcal{C}$. If $\mathcal{C}$ contains an initial object, then $\mathcal{C}^{\mathrm{init}}$ is a contractible Kan complex. If $\mathcal{C}$ contains a final object, then $\mathcal{C}^{\mathrm{fin}}$ is a contractible Kan complex.* 02HM

Proof. Assume that $\mathcal{C}$ contains an initial object, we will show that $\mathcal{C}^{\mathrm{init}}$ is a contractible Kan complex (the analogous assertion for final objects follows by a similar argument). Suppose we are given a morphism of simplicial sets $\sigma : \partial \Delta^n \to \mathcal{C}^{\mathrm{init}}$; we wish to show that σ can be extended to a morphism $\overline{\sigma} : \Delta^n \to \mathcal{C}^{\mathrm{init}}$. If $n = 0$, this follows from our assumption that $\mathcal{C}$ contains an initial object. If $n > 0$, then we can regard σ as a morphism from $\partial \Delta^n$ to $\mathcal{C}$ with the property that $\sigma(i) \in \mathcal{C}$ is initial for $0 \leq i \leq n$. Setting $i = 0$, we conclude that σ can be extended to a morphism $\overline{\sigma} : \Delta^n \to \mathcal{C}$, which automatically factors through the full subcategory $\mathcal{C}^{\mathrm{init}} \subseteq \mathcal{C}$. □

Corollary 4.6.7.15. *Let $\mathcal{C}$ be an ∞-category and let X be an initial object of $\mathcal{C}$. Then an object $Y \in \mathcal{C}$ is initial if and only if it is isomorphic to X.* 02HN

Proof. If X and Y are initial objects of $\mathcal{C}$, then they are contained in the contractible Kan complex $\mathcal{C}^{\mathrm{init}}$ of Corollary 4.6.7.14 and are therefore isomorphic when viewed as objects of

$\mathcal{C}$. Conversely, suppose that X is initial and that there exists an isomorphism $f : X \to Y$ in $\mathcal{C}$; we wish to show that Y is also initial. Fix an object $Z \in \mathcal{C}$; we wish to show that the mapping space $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ is contractible. Let us regard the homotopy category $\mathrm{h}\mathcal{C}$ as enriched over the homotopy category hKan of Kan complexes (see Construction 4.6.9.13). Since f an isomorphism in $\mathcal{C}$, its homotopy class $[f]$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$, so composition with $[f]$ induces an isomorphism $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ in the category hKan. Since the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Z)$ is contractible, it follows that $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ is also contractible. □

02HP **Notation 4.6.7.16.** Let $\mathcal{C}$ be an ∞-category. We will often use the symbol $\emptyset_{\mathcal{C}}$ to denote an initial object of $\mathcal{C}$, provided that such an object exists. In this case, we will sometimes abuse terminology by referring to $\emptyset_{\mathcal{C}}$ as *the* initial object of $\mathcal{C}$. This abuse is justified by Corollary 4.6.7.14, which guarantees that $\emptyset_{\mathcal{C}}$ is uniquely determined up to a contractible space of choices (in particular, it is well-defined up to isomorphism). Similarly, we will often use the symbol $\mathbf{1}_{\mathcal{C}}$ to denote a final object of $\mathcal{C}$, provided that such an object exists, and will sometimes abuse terminology by referring to $\mathbf{1}_{\mathcal{C}}$ as *the* final object of $\mathcal{C}$. When it is unlikely to cause confusion, we will sometimes omit the subscripts and denote the objects $\emptyset_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{C}}$ by $\emptyset$ and $\mathbf{1}$, respectively.

02HQ **Corollary 4.6.7.17.** *Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. Then:*

(1) *If X is initial as an object of the ∞-category $\mathcal{C}$, then it is also initial when viewed as an object of the homotopy category $\mathrm{h}\mathcal{C}$.*

(2) *If $\mathcal{C}$ has an initial object and X is initial as an object of the homotopy category $\mathrm{h}\mathcal{C}$, then X is initial as an object of the ∞-category $\mathcal{C}$.*

Proof. Assertion (1) is immediate from the definition. To prove (2), assume that $\mathcal{C}$ has an initial object Y. Then Y is also initial when viewed as an object of the homotopy category $\mathrm{h}\mathcal{C}$. If X is an initial object of $\mathrm{h}\mathcal{C}$, then X and Y are isomorphic when viewed as objects of $\mathrm{h}\mathcal{C}$, hence also when viewed as objects of the ∞-category $\mathcal{C}$. Invoking Corollary 4.6.7.15, we conclude that X is also an initial object of $\mathcal{C}$. □

02HR **Warning 4.6.7.18.** Let $\mathcal{C}$ be an ∞-category containing an object X which is initial as an object of the homotopy category $\mathrm{h}\mathcal{C}$. Then X need not be initial when viewed as an object of $\mathcal{C}$. For example, if $\mathcal{C}$ is simply connected Kan complex, then every object $X \in \mathcal{C}$ is initial when viewed as an object of the homotopy category $\mathrm{h}\mathcal{C} = \pi_{\leq 1}(\mathcal{C})$. However, X is initial as an object of $\mathcal{C}$ only if $\mathcal{C}$ is contractible (Corollary 4.6.7.11).

02HS **Proposition 4.6.7.19.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories and let Y be an object of $\mathcal{C}$. Then:*

(1) *If $F(Y)$ is an initial object of $\mathcal{D}$, then Y is an initial object of $\mathcal{C}$.*

(2) *If $F(Y)$ is a final object of $\mathcal{D}$, then Y is a final object of $\mathcal{C}$.*

Proof. Let Z be an object of $\mathcal{C}$. If $F(Y)$ is an initial object of the ∞-category $\mathcal{D}$, then the mapping space $\mathrm{Hom}_{\mathcal{D}}(F(Y), F(Z))$ is a contractible Kan complex. Since F is fully faithful, it follows that $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ is also a contractible Kan complex. Allowing Z to vary, we conclude that Y is an initial object of $\mathcal{C}$. This proves (1); the proof of (2) is similar. □

Corollary 4.6.7.20. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories, and let Y be an object of $\mathcal{C}$. Then:* 02HT

(1) *The object Y is initial if and only if $F(Y)$ is an initial object of $\mathcal{D}$.*

(2) *The object Y is final if and only if $F(Y)$ is a final object of $\mathcal{D}$.*

Proof. We will prove (1); the proof of (2) is similar. Note that since F is an equivalence of ∞-categories, it is fully faithful (Theorem 4.6.2.20). If $F(Y)$ is an initial object of $\mathcal{D}$, then Proposition 4.6.7.19 guarantees that the object $Y \in \mathcal{C}$ is initial. To prove the converse, let $G : \mathcal{D} \to \mathcal{C}$ be a homotopy inverse of the functor F. Then $G \circ F$ is isomorphic to the identity functor $\mathrm{id}_{\mathcal{C}}$ as an object of the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, so that $(G \circ F)(Y)$ is isomorphic to Y as an object of the ∞-category $\mathcal{C}$. If Y is an initial object of $\mathcal{C}$, then Corollary 4.6.7.15 guarantees that $(G \circ F)(Y)$ is also an initial object of $\mathcal{C}$. Since the equivalence G is fully faithful (Theorem 4.6.2.20), Proposition 4.6.7.19 guarantees that $F(Y)$ is an initial object of $\mathcal{D}$. □

Corollary 4.6.7.21. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which are equivalent. Then:* 02MY

- *The ∞-category $\mathcal{C}$ has an initial object if and only if the ∞-category $\mathcal{D}$ has an initial object.*
- *The ∞-category $\mathcal{C}$ has a final object if and only if the ∞-category $\mathcal{D}$ has a final object.*

Proposition 4.6.7.22. *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. The following conditions are equivalent:* 02J2

(1) *The morphism f is an isomorphism from X to Y in the ∞-category $\mathcal{C}$ (Definition 1.4.6.1).*

(2) *The morphism f is final when regarded as an object of the slice ∞-category $\mathcal{C}_{/Y}$.*

(2′) *The morphism f is final when regarded as an object of the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \{Y\}$.*

(3) *The morphism f is initial when regarded as an object of the coslice ∞-category $\mathcal{C}_{X/}$.*

(3′) *The morphism f is initial when regarded as an object of the oriented fiber product $\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C}$.*

Proof. The equivalences (2) ⇔ (2′) and (3) ⇔ (3′) follow from Corollaries 4.6.4.18 and 4.6.7.20. We will complete the proof by showing that (1) ⇔ (3); the equivalence (1) ⇔ (2) follows by applying the same argument in the ∞-category $\mathcal{C}^{\mathrm{op}}$. By virtue of Corollary 4.6.7.13, condition (3) is equivalent to the requirement that the restriction map $\mathcal{C}_{f/} \to \mathcal{C}_{X/}$ is a trivial Kan fibration: that is, every lifting problem

02J3

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{f/} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{X/}
\end{tikzcd} \tag{4.42}$$

admits a solution. Using the isomorphism of simplicial sets

$$(\Delta^1 \star \partial \Delta^n) \coprod_{\{0\} \star \partial \Delta^n} (\{0\} \star \Delta^n) \simeq \Lambda^{n+2}_0$$

supplied by Lemma 4.3.6.15, we can identify (4.42) with a lifting problem

$$\begin{tikzcd}
\Lambda^{n+2}_0 \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d] \\
\Delta^{n+2} \arrow[r] \arrow[ur, dashed, "\sigma"] & \Delta^0,
\end{tikzcd}$$

where σ_0 carries the initial edge $\Delta^1 \simeq N_\bullet(\{0 < 1\}) \subseteq \Lambda^{n+2}_0$ to the morphism f. The equivalence (1) ⇔ (3) now follows from the criterion of Theorem 4.4.2.6. □

02J4 **Corollary 4.6.7.23.** *Let $\mathcal{C}$ be an ∞-category and let Y be an object of $\mathcal{C}$. Then:*

(1) *The object Y is final if and only if the projection map $F : \mathcal{C}_{/Y} \to \mathcal{C}$ admits a section G satisfying $G(Y) = \mathrm{id}_Y$.*

(2) *The object Y is initial if and only if the projection map $F' : \mathcal{C}_{Y/} \to \mathcal{C}$ admits a section G' satisfying $G'(Y) = \mathrm{id}_Y$.*

Proof. We will prove (1); the proof of (2) is similar. If Y is a final object, then the projection map $F : \mathcal{C}_{/Y} \to \mathcal{C}$ is a trivial Kan fibration (Proposition 4.6.7.10), so the construction $Y \mapsto \mathrm{id}_Y$ can be extended to a section of F. Conversely, suppose that F admits a section

$G : \mathcal{C} \to \mathcal{C}_{/Y}$ satisfying $G(Y) = \mathrm{id}_Y$. Let X be an object of $\mathcal{C}$: we wish to show that the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is contractible. The functors G and F induce morphisms of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}_{/Y}}(G(X), \mathrm{id}_Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

whose composition is the identity. In particular, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a retract of $\mathrm{Hom}_{\mathcal{C}_{/Y}}(G(X), \mathrm{id}_Y)$. It will therefore suffice to show that the Kan complex $\mathrm{Hom}_{\mathcal{C}_{/Y}}(G(X), \mathrm{id}_Y)$ is contractible. This is clear, since id_Y is a final object of the slice ∞-category $\mathcal{C}_{/Y}$ (Proposition 4.6.7.22). □

Corollary 4.6.7.24. *Let $\mathcal{C}$ be an ∞-category and let Y be an object of $\mathcal{C}$. The following conditions are equivalent:* 02J5

(1) *The object $Y \in \mathcal{C}$ is final.*

(2) *There exists a functor $F : \mathcal{C}^{\triangleright} \to \mathcal{C}$ satisfying $F|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$ and for which the composition*

$$\Delta^1 \simeq \{Y\}^{\triangleright} \hookrightarrow \mathcal{C}^{\triangleright} \xrightarrow{F} \mathcal{C}$$

is the identity morphism id_Y (in particular, F carries the cone point of $\mathcal{C}^{\triangleright}$ to the object Y).

(3) *The inclusion map $\{Y\} \hookrightarrow \mathcal{C}$ is right anodyne.*

Proof. The equivalence (1) ⇔ (2) is a reformulation of Corollary 4.6.7.23. We next show that (2) implies (3). If condition (2) is satisfied, then we have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\{Y\} \arrow[r] \arrow[d] & \{Y\}^{\triangleright} \arrow[r] \arrow[d] & \{Y\} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright} \arrow[r, "F"] & \mathcal{C}
\end{tikzcd}$$

where the horizontal compositions are the identity. Since the inclusion $\{Y\}^{\triangleright} \hookrightarrow \mathcal{C}^{\triangleright}$ is right anodyne (Lemma 4.3.7.8), it follows that the inclusion $\{Y\} \hookrightarrow \mathcal{C}$ is also right anodyne.

We now complete the proof by showing that (3) implies (2). Suppose that the inclusion $\{Y\} \hookrightarrow \mathcal{C}$ is right anodyne; we wish to show that there exists a functor $F : \mathcal{C}^{\triangleright} \to \mathcal{C}$ satisfying $F|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$ and $F|_{\{Y\}^{\triangleright}} = \mathrm{id}_Y$. For this, it will suffice to show that the inclusion map

$$\mathcal{C} \coprod_{\{Y\}} \{Y\}^{\triangleright} \hookrightarrow \mathcal{C}^{\triangleright}$$

is inner anodyne, which is a special case of Proposition 4.3.6.4. □

02P2 **Corollary 4.6.7.25.** *Let $\mathcal{C}$ be an ∞-category which has either an initial object or a final object. Then $\mathcal{C}$ is weakly contractible.*

Proof. We will assume that $\mathcal{C}$ has a final object Y; the case where $\mathcal{C}$ has an initial object follows by a similar argument. Corollary 4.6.7.24 implies that the inclusion map $\{Y\} \hookrightarrow \mathcal{C}$ is right anodyne. In particular, it is anodyne and therefore a weak homotopy equivalence. $\square$

4.6.8 Morphism Spaces in the Homotopy Coherent Nerve

01LA Let $\mathcal{C}$ be a simplicial category and let $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ denote the homotopy coherent nerve of $\mathcal{C}$ (Definition 2.4.3.5). Suppose that $\mathcal{C}$ is locally Kan, so that the simplicial set $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.4.5.1). Our goal in this section is to describe the morphism spaces in the ∞-category $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$. Our main result (Theorem 4.6.8.5) implies that, for every pair of objects $X, Y \in \mathcal{C}$, there is a canonical homotopy equivalence

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y),$$

where $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ denotes the Kan complex of morphisms from X to Y in $\mathcal{C}$, and $\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$ is given by Construction 4.6.1.1.

01LB **Notation 4.6.8.1.** Let K be a simplicial set. We define a simplicial category $\mathcal{E}[K]$ as follows:

- The category $\mathcal{E}[K]$ has exactly two objects, which we will denote by x and y.
- The morphism spaces in $\mathcal{E}[K]$ are given by the formulae

$$\mathrm{Hom}_{\mathcal{E}[K]}(x,x)_{\bullet} = \{\mathrm{id}_x\} \qquad \mathrm{Hom}_K(y,y)_{\bullet} = \{\mathrm{id}_y\}$$

$$\mathrm{Hom}_{\mathcal{E}[K]}(x,y)_{\bullet} = K \qquad \mathrm{Hom}_K(y,x)_{\bullet} = \emptyset.$$

01LC **Remark 4.6.8.2.** The simplicial category $\mathcal{E}[K]$ is characterized by the following universal property: if $\mathcal{C}$ is any simplicial category containing a pair of objects X and Y, then the natural map

$$\begin{tikzcd}
\{\text{Simplicial functors } F : \mathcal{E}[K] \to \mathcal{C} \text{ with } F(x) = X \text{ and } F(y) = Y\} \arrow[d] \\
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(K, \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet})
\end{tikzcd}$$

is a bijection (see Proposition 2.4.5.9).

Construction 4.6.8.3. Fix an integer $n \geq 0$, let $[n]$ denote the linearly ordered set $\{0 < 1 < \cdots < n\}$, and let $\{x\} \star [n]$ denote the linearly ordered set obtained from $[n]$ by adjoining a new least element x. Let $\mathrm{Path}[\{x\} \star [n]]_{\bullet}$ denote the simplicial path category of Notation 2.4.3.1. We define a simplicial functor $\pi : \mathrm{Path}[\{x\} \star [n]]_{\bullet} \to \mathcal{E}[\Delta^n]$ as follows: 01LD

- On objects, the functor π is given by the formula

$$\pi(i) = \begin{cases} x & \text{if } i = x \\ y & \text{if } 0 \leq i \leq n. \end{cases}$$

- For $0 \leq m \leq n$, the morphism of simplicial sets

$$\mathrm{Hom}_{\mathrm{Path}[\{x\} \star [n]]}(x, m)_{\bullet} \to \mathrm{Hom}_{\mathcal{E}[\Delta^n]}(x, y)_{\bullet} = \Delta^n$$

  is given by the map of partially ordered sets

$$\{\text{Subsets } S = \{x < i_0 < \cdots < i_k = m\} \subseteq \{x\} \star [n]\}^{\mathrm{op}} \to [n] \qquad S \mapsto i_0.$$

Let $\mathcal{C}$ be a simplicial category containing a pair of objects X and Y. Then every n-simplex $\sigma \in \mathrm{Hom}_{\mathcal{C}}(X, Y)_n$ determines a simplicial functor $F_{\sigma} : \mathcal{E}[\Delta^n] \to \mathcal{C}$, given on objects by $F_{\sigma}(x) = X$ and $F_{\sigma}(y) = Y$. The composition $F_{\sigma} \circ \pi$ is a simplicial functor from $\mathrm{Path}[\{x\} \star [n]]_{\bullet}$ to $\mathcal{C}$, which (by Proposition 2.4.4.15) we can view as a map of simplicial sets $f_{\sigma} : \{x\} \star \Delta^n \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$. By construction, f_{σ} carries x to X, and the restriction $f_{\sigma}|_{\mathrm{N}_{\bullet}(\{0<1<\cdots<n\})}$ is the constant map taking the value Y. We can therefore identify f_{σ} with an n-simplex $\theta(\sigma)$ of the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Y)$ introduced in Construction 4.6.5.1 (see Remark 4.6.5.2). The construction $\sigma \mapsto \theta(\sigma)$ depends functorially on the object $[n] \in \mathbf{\Delta}$, and therefore determines a map of simplicial sets

$$\theta : \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Y),$$

which we will refer to as the *comparison map.*

Exercise 4.6.8.4. Let $\mathcal{C}$ be a differential graded category containing a pair of objects X and Y, and let $\mathcal{C}^{\Delta}$ denote the associated simplicial category (Construction 2.5.9.2). Show that the isomorphism $K(\mathrm{Hom}_{\mathcal{C}}(X, Y)_*) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})}(X, Y)$ of Example 4.6.5.15 factors as a composition 01PZ

$$K(\mathrm{Hom}_{\mathcal{C}}(X, Y)_*) = \mathrm{Hom}_{\mathcal{C}^{\Delta}}(X, Y)_{\bullet} \xrightarrow{\theta} \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta})}(X, Y) \xrightarrow{\rho} \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})}(X, Y),$$

where θ is the comparison map of Construction 4.6.8.3 and ρ is induced by the trivial Kan fibration $\mathfrak{Z} : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\Delta}) \to \mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$ of Proposition 2.5.9.10. Beware that θ and ρ are generally not isomorphisms.

Our comparison result can now be formulated as follows:

01LE **Theorem 4.6.8.5.** *Let $\mathcal{C}$ be a locally Kan simplicial category containing a pair of objects $X, Y \in \mathcal{C}$. Then the comparison map*

$$\theta : \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(X,Y)$$

of Construction 4.6.8.3 is a homotopy equivalence of Kan complexes.

01LF **Remark 4.6.8.6.** Let $\mathcal{C}$ be a locally Kan simplicial category containing a pair of objects $X, Y \in \mathcal{C}$. Combining Theorem 4.6.8.5 with Proposition 4.6.5.10, we obtain a homotopy equivalence of Kan complexes $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(X,Y)$, given by composing the comparison map θ of Construction 4.6.8.3 with the left-pinch inclusion map of Construction 4.6.5.7.

Before giving the proof of Theorem 4.6.8.5, let us outline some applications.

01Q0 **Definition 4.6.8.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial categories and let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial functor.

- We say that F is *weakly fully faithful* if, for every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, the induced map $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))_\bullet$ is a weak homotopy equivalence of simplicial sets.
- We say that F is *weakly essentially surjective* if the induced functor of homotopy categories $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is essentially surjective (that is, every object of $\mathcal{D}$ is homotopy equivalent to an object of the form $F(X)$, for some $X \in \mathrm{Ob}(\mathcal{C})$).
- We say that F is a *weak equivalence of simplicial categories* if it is weakly fully faithful and weakly essentially surjective.

01Q1 **Corollary 4.6.8.8.** *Let $\mathcal{C}$ and $\mathcal{D}$ be locally Kan simplicial categories, let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial functor, and let $\mathrm{N}^{\mathrm{hc}}_\bullet(F) : \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{D})$ be the induced functor of ∞-categories. Then:*

(1) *The functor $\mathrm{N}^{\mathrm{hc}}_\bullet(F)$ is fully faithful (in the sense of Definition 4.6.2.1) if and only if the simplicial functor F is weakly fully faithful (in the sense of Definition 4.6.8.7).*

(2) *The functor $\mathrm{N}^{\mathrm{hc}}_\bullet(F)$ is essentially surjective (in the sense of Definition 4.6.2.11) if and only if the simplicial functor F is weakly essentially surjective (in the sense of Definition 4.6.8.7).*

(3) *The functor $\mathrm{N}^{\mathrm{hc}}_\bullet(F)$ is an equivalence of ∞-categories (in the sense of Definition 4.5.1.10) if and only if F is a weak equivalence of simplicial categories (in the sense of Definition 4.6.8.7).*

Proof. For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, we have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \arrow[r, "F"] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))_{\bullet} \arrow[d] \\
\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y) \arrow[r, "\mathrm{N}^{\mathrm{hc}}_{\bullet}(F)"] & \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{D})}(F(X),F(Y)),
\end{tikzcd}$$

where the vertical maps are the homotopy equivalences supplied by Remark 4.6.8.6. It follows that the upper horizontal map is a homotopy equivalence if and only if the lower horizontal map is a homotopy equivalence. This proves (1). Assertion (2) follows from Proposition 2.4.6.9. Assertion (3) follows by combining (1) and (2) with the criterion of Theorem 4.6.2.20. □

Theorem 4.6.8.5 is an immediate consequence of the following more general result:

Theorem 4.6.8.9. *Let $\mathcal{C}$ be a simplicial category containing a pair of objects X and Y, and suppose that the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ is an ∞-category. Then the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$ is also an ∞-category, and the comparison map* 01LG

$$\theta : \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$$

of Construction 4.6.8.3 is an equivalence of ∞-categories.

Remark 4.6.8.10. Let $\mathcal{C}$ be a simplicial category containing a pair of objects X and Y, and suppose that the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ is an ∞-category. Applying Theorem 4.6.8.9 to the opposite simplicial category $\mathcal{C}^{\mathrm{op}}$ (and using Remark 4.6.5.3), we obtain an equivalence of ∞-categories 01LH

$$\theta' : \mathrm{Hom}_{\mathcal{C}}(X,Y)^{\mathrm{op}}_{\bullet} \to \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y),$$

which can be described explicitly using a variant of Construction 4.6.8.3.

The remainder of this section is devoted to the proof of Theorem 4.6.8.9.

Construction 4.6.8.11. Let K be a simplicial set. We let $\Sigma(K)$ denote the pushout $(\{x\} \star K) \coprod_{K} \{y\}$ (this is a model for the *unreduced suspension* of K). Let $\mathrm{Path}[\Sigma(K)]_{\bullet}$ denote the simplicial path category of $\Sigma(K)$ (Notation 2.4.4.2). Then $\mathrm{Path}[\Sigma(K)]_{\bullet}$ has exactly two objects, which we denote by x and y. We let $\Phi(K)$ denote the simplicial set $\mathrm{Hom}_{\mathrm{Path}[\Sigma(K)]}(x,y)_{\bullet}$. 01LJ

Example 4.6.8.12. If $K = \Delta^0$, then the suspension $\Sigma(K)$ can be identified with Δ^1. In this case, the simplicial path category $\mathrm{Path}[\Sigma(K)]_{\bullet}$ can be identified with the ordinary category $[1]$, and the simplicial set $\Phi(K)$ is isomorphic to Δ^0. 01LK

01LL **Remark 4.6.8.13.** Let K be a simplicial set, and let $\mathcal{D}$ be another simplicial set containing vertices X and Y. Unwinding the definitions, we have a canonical bijection

$$\begin{tikzcd}
\{\text{Morphisms } K \to \operatorname{Hom}^{\mathrm{L}}_{\mathcal{D}}(X,Y)\} \arrow[d, "\sim"] \\
\{\text{Morphisms } F : \Sigma(K) \to \mathcal{D} \text{ with } F(x) = X \text{ and } F(y) = Y\}.
\end{tikzcd}$$

01LM **Remark 4.6.8.14.** Let K be a simplicial set. Note that, for $n > 0$, every nondegenerate simplex $\sigma : \Delta^n \to \Sigma(K)$ satisfies $\sigma(0) = x$ and $\sigma(n) = y$. Using Theorem 2.4.4.10, we see that for each $m \geq 0$, $\operatorname{Path}[\Sigma(K)]_m$ can be identified with the path category of a directed graph G_m with vertex set $\operatorname{Vert}(G_m) = \{x, y\}$, where each edge of G_m has source x and target y. These path categories are easy to describe: they satisfy

$$\operatorname{Hom}_{\operatorname{Path}[G_m]}(x,x) = \{\mathrm{id}_x\} \qquad \operatorname{Hom}_{\operatorname{Path}[G_m]}(y,y) = \{\mathrm{id}_y\}$$

$$\operatorname{Hom}_{\operatorname{Path}[G_m]}(x,y) = \operatorname{Edge}(G_m) \qquad \operatorname{Hom}_{\operatorname{Path}[G_m]}(y,x) = \emptyset.$$

Allowing m to vary, we conclude that the simplicial category $\operatorname{Path}[\Sigma(K)]_\bullet$ satisfies

$$\operatorname{Hom}_{\operatorname{Path}[\Sigma(K)]}(x,x)_\bullet = \{\mathrm{id}_x\} \quad \operatorname{Hom}_{\operatorname{Path}[\Sigma(K)]}(y,x)_\bullet = \emptyset \quad \operatorname{Hom}_{\operatorname{Path}[\Sigma(K)]}(y,y)_\bullet = \{\mathrm{id}_y\}.$$

That is, $\operatorname{Path}[\Sigma(K)]_\bullet$ can be identified with the simplicial category $\mathcal{E}[\Phi(X)]$ of Notation 4.6.8.1. Moreover, we can identify m-simplices of $\Phi(X)$ with elements of the set $E(\Sigma(K), m)$ defined in Notation 2.4.4.9.

01LN **Remark 4.6.8.15.** Let $u : K \to K'$ be a monomorphism of simplicial sets. Then the induced map $\Phi(u) : \Phi(K) \to \Phi(K')$ is also a monomorphism (this follows immediately from the description given in Remark 4.6.8.14).

01LP **Lemma 4.6.8.16.** *Let K be a simplicial set and let $\mathcal{C}$ be a simplicial category containing objects X and Y. Then the natural map*

$$\begin{tikzcd}
\{\text{Functors } F : \operatorname{Path}[\Sigma(K)]_\bullet \to \mathcal{C} \text{ with } F(x) = X \text{ and } F(y) = Y\} \arrow[d] \\
\{\text{Morphisms } \Phi(K) \to \operatorname{Hom}_{\mathcal{C}}(X,Y)_\bullet\}
\end{tikzcd}$$

is a bijection.

Proof. Combine Remarks 4.6.8.2 and 4.6.8.14. □

Combining Remark 4.6.8.13 with Lemma 4.6.8.16 and invoking the universal property of simplicial path categories, we obtain the following:

Corollary 4.6.8.17. *Let K be a simplicial set and let $\mathcal{C}$ be a simplicial category containing objects X and Y. Then we have a canonical bijection* 01LQ

$$\{\text{Morphisms } K \to \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)\} \simeq \{\text{Morphisms } \Phi(K) \to \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}\}.$$

Remark 4.6.8.18. It follows from Corollary 4.6.8.17 that the left-pinched morphism space $\operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$ depends only on the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$, and not on any other features of the simplicial category $\mathcal{C}$. In particular, there is a canonical isomorphism 03LK

$$\operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y) \to \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Set}_{\Delta})}(\Delta^0, \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}).$$

Corollary 4.6.8.19. *Let A and B be simplicial sets, and let $\mathcal{E}[B]$ be the simplicial category of Notation 4.6.8.1. Then we have a canonical bijection* 01LR

$$\{\text{Morphisms } A \to \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{E}[B])}(x,y)\} \simeq \{\text{Morphisms } \Phi(A) \to B\}.$$

Proof. Apply Corollary 4.6.8.17 in the special case $\mathcal{C} = \mathcal{E}[B]$. □

Corollary 4.6.8.20. *The functor* 01LS

$$\mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta} \qquad B \mapsto \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{E}[B])}(x,y)$$

has a left adjoint, given by the functor $A \mapsto \Phi(A)$ of Construction 4.6.8.11.

Remark 4.6.8.21. The adjunction of Corollary 4.6.8.20 has an interpretation in the framework of Proposition 1.2.3.15. Let $Q^{\bullet}$ denote the cosimplicial object of Set_{Δ} given by the construction $[n] \mapsto \Phi(\Delta^n)$. For every simplicial set B, Corollary 4.6.8.19 supplies a canonical isomorphism of simplicial sets 01LT

$$\operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{E}[B])}(x,y) \simeq \operatorname{Sing}^{Q}_{\bullet}(B),$$

where $\operatorname{Sing}^{Q}_{\bullet}(B)$ is the simplicial set defined in Variant 1.2.2.8. It follows that Φ can be identified with the generalized geometric realization functor $K \mapsto |K|^{Q}$ of Proposition 1.2.3.15.

Corollary 4.6.8.22. *The functor $\Phi : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ of Construction 4.6.8.11 preserves colimits.* 01LU

`01LV` **Construction 4.6.8.23.** Let K be a simplicial set, let $\mathcal{E}[K]$ be the simplicial category of Notation 4.6.8.1, and let

$$\theta : K = \operatorname{Hom}_{\mathcal{E}[K]}(x,y)_{\bullet} \to \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{E}[K])}(x,y)$$

be the comparison map of Construction 4.6.8.3. We let $\rho_K : \Phi(K) \to K$ denote the image of θ under the bijection of Corollary 4.6.8.19.

We will deduce Theorem 4.6.8.9 from the following result, which we prove at the end of this section:

`01LW` **Proposition 4.6.8.24.** *Let K be a simplicial set. Then the morphism $\rho_K : \Phi(K) \to K$ of Construction 4.6.8.23 is a categorical equivalence of simplicial sets.*

`01LX` **Corollary 4.6.8.25.** *Let $u : K \to K'$ be a categorical equivalence of simplicial sets. Then the induced map $\Phi(u) : \Phi(K) \to \Phi(K')$ is also a categorial equivalence of simplicial sets.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
\Phi(K) \arrow[r, "\Phi(u)"] \arrow[d, "\rho_K"] & \Phi(K') \arrow[d, "\rho_{K'}"] \\
K \arrow[r, "u"] & K'
\end{tikzcd}$$

where u is a categorical equivalence by hypothesis and the vertical maps are categorical equivalences by Proposition 4.6.8.24. Using Remark 4.5.3.5, we conclude that $\Phi(u)$ is a categorical equivalence as well. □

`01LY` **Corollary 4.6.8.26.** *Let $\mathcal{C}$ be a simplicial category containing a pair of objects X and Y, and assume that the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ is an ∞-category. Then the simplicial set $\operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$ is also an ∞-category.*

Proof. Let $i : A \hookrightarrow B$ be an inner anodyne morphism of simplicial sets. We wish to show that every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y) \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & \Delta^0
\end{tikzcd}$$

admits a solution. By virtue of Corollary 4.6.8.17, we can rephrase this as a lifting problem

$$\begin{tikzcd}
\Phi(A) \arrow[r] \arrow[d, "\Phi(i)"] & \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \arrow[d] \\
\Phi(B) \arrow[r] \arrow[ur, dashed] & \Delta^0.
\end{tikzcd}$$

Note that $\Phi(i)$ is a monomorphism (Remark 4.6.8.15) and a categorical equivalence (Corollary 4.6.8.25), so the desired result follows from Lemma 4.5.5.2. □

Corollary 4.6.8.27. *Let $\mathcal{C}$ be a simplicial category containing a pair of objects X and Y, and assume that the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ is an ∞-category. Let K be another simplicial set, and suppose we are given a pair of morphisms $f_0, f_1 : K \to \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$, which correspond (under the bijection of Corollary 4.6.8.17) to diagrams $f_0', f_1' : \Phi(K) \to \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$. The following conditions are equivalent:* 01LZ

(1) *The diagrams f_0 and f_1 are isomorphic when regarded as objects of the ∞-category $\operatorname{Fun}(K, \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y))$.*

(2) *The diagrams f_0' and f_1' are isomorphic when regarded as objects of the ∞-category $\operatorname{Fun}(\Phi(K), \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet})$.*

Proof. Choose a categorical mapping cylinder

$$K \coprod K \xrightarrow{(s_0,s_1)} \overline{K} \xrightarrow{\pi} K$$

for the simplicial set K (Definition 4.6.3.3). Using Remark 4.6.8.15, Corollary 4.6.8.22, and Corollary 4.6.8.25, we conclude that the induced diagram

$$\Phi(K) \coprod \Phi(K) \xrightarrow{(\Phi(s_0),\Phi(s_1))} \Phi(\overline{K}) \xrightarrow{\Phi(\pi)} \Phi(K)$$

exhibits $\Phi(\overline{K})$ as a categorical mapping cylinder of K. Using Corollary 4.6.3.7, we see that (1) and (2) are equivalent to the following:

(1′) There exists a diagram $\overline{f} : \overline{K} \to \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$ satisfying $f_0 = \overline{f} \circ s_0$ and $f_1 = \overline{f} \circ s_1$.

(2′) There exists a diagram $\overline{f}' : \Phi(\overline{K}) \to \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ satisfying $f_0' = \overline{f}' \circ \Phi(s_0)$ and $f_1' = \overline{f}' \circ \Phi(s_1)$.

The equivalence of (1′) and (2′) follows from Corollary 4.6.8.17. □

Proof of Theorem 4.6.8.9. Let $\mathcal{C}$ be a simplicial category containing a pair of objects $X, Y \in \mathcal{C}$ for which the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ is an ∞-category. Applying Corollary 4.6.8.26, we deduce that the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$ is also an ∞-category. We wish to show that the comparison map

$$\theta : \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$$

of Construction 4.6.8.3 is an equivalence of ∞-categories. To prove this, it will suffice to show that for every simplicial set K, postcomposition with θ induces a bijection

$$\theta_K : \pi_0(\mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet})^{\simeq}) \to \pi_0(\mathrm{Fun}(K, \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y))^{\simeq}).$$

By virtue of Corollary 4.6.8.27, we can identify $\pi_0(\mathrm{Fun}(K, \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y))^{\simeq})$ with the set $\pi_0(\mathrm{Fun}(\Phi(K), \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet})^{\simeq})$. Under this identification, θ_K corresponds to the map

$$\pi_0(\mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet})^{\simeq}) \to \pi_0(\mathrm{Fun}(\Phi(K), \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet})^{\simeq})$$

given by precomposition with the map $\rho_K : \Phi(K) \to K$ of Construction 4.6.8.23, which is bijective by virtue of the fact that ρ_K is a categorical equivalence of simplicial sets (Proposition 4.6.8.24). □

We now turn to the proof of Proposition 4.6.8.24. Our strategy is to use formal arguments to reduce to the case where the simplicial set K is a standard simplex, which can be analyzed explicitly.

`01M0` **Example 4.6.8.28.** Let m and n be nonnegative integers. By virtue of Remark 4.6.8.14, we can identify m-simplices of the simplicial set $\Phi(\Delta^n)$ with the set $E(\Sigma(\Delta^n), m)$ defined in Notation 2.4.4.9. By definition, the elements of $E(\Sigma(\Delta^n), m)$ are given by pairs $(\sigma, \overrightarrow{I})$, where $\sigma : \Delta^k \to \Sigma(\Delta^n)$ is a nondegenerate simplex of dimension $k > 0$ and $\overrightarrow{I} = (I_0 \supseteq I_1 \supseteq \cdots \supseteq I_m)$ is a chain of subsets of $[k]$ satisfying $I_0 = [k]$ and $I_m = \{0, k\}$.

For each $k > 0$, there is a canonical bijection

$$\{\text{Subsets } S \subseteq [n] \text{ of cardinality } k\} \simeq \{\text{Nondegenerate } k\text{-Simplices of } \Sigma(\Delta^n)\},$$

which carries a subset S to the k-simplex σ_S given by the composite map

$$\Delta^k \simeq \{x\} \star \mathrm{N}_{\bullet}(S) \hookrightarrow \{x\} \star \Delta^n \twoheadrightarrow \Sigma(\Delta^n).$$

For every such subset S, let $\iota_S : \mathrm{N}_{\bullet}(S) \hookrightarrow \Delta^k$ be the inclusion map. Then the construction

$$(\sigma_S, \overrightarrow{I}) \mapsto (\sigma_S^{-1}(I_0) \supseteq \sigma_S^{-1}(I_1) \supseteq \cdots \supseteq \sigma_S^{-1}(I_m))$$

induces a bijection from $E(\Sigma(\Delta^n), m)$ to the collection of chains $\overrightarrow{S} = (S_0 \supseteq S_1 \supseteq \cdots \supseteq S_m)$ of subsets of $[n]$ which satisfy the following pair of conditions:

(a) The set S_m contains exactly one element.

(b) For $0 \leq i \leq m$, the unique element of S_m is the largest element of S_i.

Let us henceforth use this bijection to identify m-simplices of $\Phi(\Delta^n)$ with chains $\overrightarrow{S}$ satisfying (a) and (b). In these terms, the face and degeneracy operators for the simplicial set $\Phi(\Delta^n) = \Phi(\Delta^n)_\bullet$ can be described explicitly as follows:

- For $0 \leq i \leq m$, the degeneracy operator $s_i^m : \Phi(\Delta^n)_m \to \Phi(\Delta^n)_{m+1}$ is given by
$$s_i^m(S_0 \supseteq \cdots \supseteq S_m) = (S_0 \supseteq \cdots \supseteq S_{i-1} \supseteq S_i \supseteq S_i \supseteq S_{i+1} \supseteq \cdots \supseteq S_m)$$
- For $0 \leq i < m$, the face operator $d_i^m : \Phi(\Delta^n)_m \to \Phi(\Delta^n)_{m-1}$ is given by the construction
$$d_i^m(S_0 \supseteq \cdots \supseteq S_m) = (S_0 \supseteq \cdots \supseteq S_{i-1} \supseteq S_{i+1} \supseteq \cdots \supseteq S_m).$$
- For $m > 0$, the face operator $d_m^m : \Phi(\Delta^n)_m \to \Phi(\Delta^n)_{m-1}$ is given by
$$d_m^m(S_0 \supseteq \cdots \supseteq S_m) = (S'_0 \supseteq S'_1 \supseteq \cdots \supseteq S'_{m-1}),$$
where $S'_i = \{j \in S_i : j \leq \min(S_{m-1})\}$.

See Remark 2.4.4.17.

Construction 4.6.8.29. Let m and n be nonnegative integers. Suppose we are given an m-simplex of $\Phi(\Delta^n)$, which we identify with a chain of subsets $\overrightarrow{S} = (S_0 \supseteq \cdots \supseteq S_m)$ satisfying conditions (a) and (b) of Example 4.6.8.28. Let $\tau : [m] \to [1]$ be a nondecreasing function. Let $\overrightarrow{S}' = (S'_0 \supseteq \cdots \supseteq S'_m)$ be the chain of subsets of $[n+1]$ given by the formula 01M1

$$S'_i = \begin{cases} \{s+1 : s \in S_i\} \text{ if } \tau(i) = 1 & \\ \{0\} & \text{if } \tau(m) = 0 \\ \{0\} \cup \{s+1 : s \in S_i\} & \text{otherwise.} \end{cases}$$

The construction $(\overrightarrow{S}, \tau) \mapsto \overrightarrow{S}'$ is compatible with the formation of face and degeneracy operators, and therefore determines a morphism of simplicial sets $\pi : \Phi(\Delta^n) \times \Delta^1 \to \Phi(\Delta^{n+1})$.

Lemma 4.6.8.30. *Let $n \geq 0$ be an integer, and let $\pi : \Phi(\Delta^n) \times \Delta^1 \to \Phi(\Delta^{n+1})$ be the morphism of simplicial sets defined in Construction 4.6.8.29. Then π fits into a pushout diagram of simplicial sets* 01M2

$$\begin{tikzcd}
\Phi(\Delta^n) \times \{0\} \arrow[r] \arrow[d] & \Phi(\Delta^n) \times \Delta^1 \arrow[d, "\pi"] \\
\Delta^0 \arrow[r] & \Phi(\Delta^{n+1}).
\end{tikzcd}$$

01M3 **Remark 4.6.8.31.** It follows from Lemma 4.6.8.30 that the morphism π of Construction 4.6.8.29 induces an isomorphism of simplicial sets $\Delta^0 \diamond \Phi(\Delta^n) \to \Phi(\Delta^{n+1})$, where $\diamond$ denotes the blunt join of Notation 4.5.8.3.

Proof of Lemma 4.6.8.30. Fix an integer $m \geq 0$. By construction, the restriction $\pi|_{\Phi(\Delta^n)\times\{0\}}$ is the constant map which carries each m-simplex of $\Phi(\Delta^n)$ to the element of $\Phi(\Delta^{n+1})$ given by the constant chain $\overrightarrow{S}_0 = (\{0\} \subseteq \{0\} \subseteq \cdots \subseteq \{0\})$. To complete the proof, we must show that for each $m \geq 0$, the map π induces a bijection

$$\begin{tikzcd}
\Phi(\Delta^n)_m \times \{\text{Nondecreasing functions } \tau : [m] \to [1] \text{ with } \tau(m) = 1\} . \arrow[d] \\
\Phi(\Delta^{n+1})_m \setminus \{\overrightarrow{S}_0\}
\end{tikzcd}$$

The inverse bijection can be described explicitly as follows: it carries an m-simplex $(S'_0 \supseteq \cdots \supseteq S'_m) \neq \overrightarrow{S}_0$ of $\Phi(\Delta^{n+1})$ to the pair $(\overrightarrow{S}, \tau)$, where $\overrightarrow{S} = (S_0 \supseteq \cdots \supseteq S_m)$ is the m-simplex of $\Phi(\Delta^n)$ given by

$$S_i = \{s - 1 : s \in S'_i, s > 0\} \qquad \tau(i) = \begin{cases} 0 & \text{if } 0 \in S'_i \\ 1 & \text{if } 0 \notin S'_i. \end{cases}$$

□

Proof of Proposition 4.6.8.24. Let K be a simplicial set. We wish to show that the map $\rho_K : \Phi(K) \to K$ of Construction 4.6.8.23 is a categorical equivalence of simplicial sets. Using Corollary 4.6.8.22, we can write ρ_K as a filtered colimit of morphisms $\rho_{K_\alpha} : \Phi(K_\alpha) \to K_\alpha$, where K_α ranges over the collection of all finite simplicial subsets of K (Remark 3.6.1.8). Since the collection of categorical equivalences is closed under the formation of filtered colimits (Corollary 4.5.7.2), it will suffice to show that each ρ_{K_α} is a categorical equivalence. We may therefore replace K by K_α and thereby reduce to the case where the simplicial set K is finite.

Since K is a finite simplicial set, it has dimension $\leq n$ for some integer $n \geq -1$. We proceed by induction on n. If $n = -1$, then both K and $\Phi(K)$ are empty, and there is nothing to prove. We may therefore assume that $n \geq 0$ and that $\rho_{K'}$ is a categorical equivalence for every simplicial set K' of dimension $< n$. We now proceed by induction on the number m of nondegenerate n-simplices of K. If $m = 0$, then K has dimension $\leq n - 1$ and the desired result holds by virtue of our inductive hypothesis. We may therefore assume that K has at least one nondegenerate n-simplex $\sigma : \Delta^n \to K$. Using Proposition 1.1.4.12, we see that there is a pushout diagram of simplicial sets

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d, "\sigma"] \\
K' \arrow[r] & K,
\end{tikzcd}$$

where S' is a simplicial set of dimension $\leq n$ with exactly $(m-1)$-nondegenerate m-simplices. We then have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Phi(\partial \Delta^n) \arrow[rr] \arrow[dr, "\rho_{\partial \Delta^n}"] \arrow[dd] & & \Phi(\Delta^n) \arrow[dr, "\rho_{\Delta^n}"] \arrow[dd] & \\
& \partial \Delta^n \arrow[rr] \arrow[dd] & & \Delta^n \arrow[dd] \\
\Phi(K') \arrow[rr] \arrow[dr, "\rho_{K'}"] & & \Phi(K) \arrow[dr, "\rho_K"] & \\
& K' \arrow[rr] & & K
\end{tikzcd}$$

where the front and back faces are pushout squares (Corollary 4.6.8.22) in which the horizontal maps are monomorphisms (Remark 4.6.8.15), and are therefore categorical pushout squares (Example 4.5.4.12). Our inductive hypotheses guarantees that the maps $\rho_{K'}$ and $\rho_{\partial \Delta^n}$ are categorical equivalences. Consequently, to show that ρ_K is a categorical equivalence, it will suffice to show that ρ_{Δ^n} is a categorical equivalence (Proposition 4.5.4.9). We may therefore replace K by Δ^n and thereby reduce to the case where K is a standard simplex.

If $n = 0$, then the map $\rho_{\Delta^n} : \Phi(\Delta^n) \to \Delta^n$ is an isomorphism (Example 4.6.8.12). We may therefore assume without loss of generality that $n > 0$, so that Lemma 4.6.8.30 supplies an isomorphism of simplicial sets $\Phi(\Delta^n) \simeq \Delta^0 \diamond \Phi(\Delta^{n-1})$. Using this isomorphism, we can identify ρ_{Δ^n} with the composite map

$$\Delta^0 \diamond \Phi(\Delta^{n-1}) \xrightarrow{\mathrm{id} \diamond \rho_{\Delta^{n-1}}} \Delta^0 \diamond \Delta^{n-1} \xrightarrow{c} \Delta^0 \star \Delta^{n-1} \simeq \Delta^n,$$

where c is the comparison map of Notation 4.5.8.3 (to check this, it suffices to observe that they agree on vertices). Our inductive hypothesis guarantees that $\rho_{\Delta^{n-1}}$ is a categorical equivalence of simplicial sets, so that the induced map $\Delta^0 \diamond \Phi(\Delta^{n-1}) \xrightarrow{\mathrm{id} \diamond \rho_{\Delta^{n-1}}} \Delta^0 \diamond \Delta^{n-1}$ is also a categorical equivalence by virtue of Remark 4.5.8.7. We are therefore reduced to showing that c is a categorical equivalence, which is a special case of Proposition 4.5.8.12. □

4.6.9 Composition of Morphisms

01PF Let $\mathcal{C}$ be an ordinary category. For every triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the composition of morphisms in $\mathcal{C}$ determines a map

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z).$$

Our goal in this section is to construct an analogous operation in the ∞-categorical setting. Here the situation is more subtle: as we saw in §1.4.4, a pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in an ∞-category $\mathcal{C}$ generally do not have a unique composition. Nevertheless, we will show that the mapping spaces of Construction 4.6.1.1 can be endowed with a composition law which is well-defined up to homotopy (and even up to a contractible space of choices). To describe this composition law, it will be convenient to introduce a generalization of Construction 4.6.1.1.

01PG **Notation 4.6.9.1.** Let $\mathcal{C}$ be a simplicial set containing a (nonempty) finite sequence of vertices $X_0, X_1, \ldots, X_n$. We let $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1, \cdots, X_n)$ denote the simplicial set given by the fiber product

$$\mathrm{Fun}(\Delta^n, \mathcal{C}) \times_{\mathrm{Fun}(\{0,1,\cdots,n\}, \mathcal{C})} \{(X_0, X_1, \cdots, X_n)\}.$$

01PH **Example 4.6.9.2.** Let $\mathcal{C}$ be a simplicial set containing vertices X_0 and X_1. Then the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1)$ of Notation 4.6.9.1 agrees with the morphism space $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1)$ of Construction 4.6.1.1. In particular, if $\mathcal{C}$ is an ∞-category, then $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1)$ is a Kan complex (Proposition 4.6.1.10).

01PJ **Example 4.6.9.3.** Let $\mathcal{C}$ be a simplicial set and let X_0 be a vertex of $\mathcal{C}$. Then the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X_0)$ of Notation 4.6.9.1 is isomorphic to Δ^0.

Let $\mathcal{C}$ be a simplicial set containing a sequence of vertices $X_0, X_1, \ldots, X_n$. For every pair of integers $0 \leq i < j \leq n$, precomposition with the edge $\Delta^1 \simeq \mathrm{N}_\bullet(\{i < j\}) \hookrightarrow \Delta^n$ determines a restriction map $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1, \cdots, X_n) \to \mathrm{Hom}_{\mathcal{C}}(X_i, X_j)$.

01PK **Proposition 4.6.9.4.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets, and let $X_0, X_1, \ldots, X_n$ be vertices of $\mathcal{C}$ having images $\overline{X}_0, \overline{X}_1, \ldots, \overline{X}_n \in \mathcal{D}$. Then the restriction* *map*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X_0, \cdots, X_n) \arrow[d, "\theta"] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{X}_0, \cdots, \overline{X}_n) \times_{\prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{D}}(\overline{X}_{i-1}, \overline{X}_i)} \prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{C}}(X_{i-1}, X_i)
\end{tikzcd}$$

is a trivial Kan fibration of simplicial sets.

Proof. Let Spine$[n]$ denote the spine of the standard n-simplex Δ^n (see Example 1.5.7.7). Unwinding the definitions, we see that θ is a pullback of the restriction map

$$\theta' : \mathrm{Fun}(\Delta^n, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Spine}[n], \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{Spine}[n], \mathcal{D})} \mathrm{Fun}(\Delta^n, \mathcal{D}).$$

Since q is an inner fibration and the inclusion Spine$[n] \hookrightarrow \Delta^n$ is inner anodyne (Example 1.5.7.7), the morphism θ' is a trivial Kan fibration (Proposition 4.1.4.4). □

Corollary 4.6.9.5. *Let $\mathcal{C}$ be an ∞-category containing objects $X_0, X_1, \ldots, X_n$. Then the restriction map* 01PL

$$\mathrm{Hom}_{\mathcal{C}}(X_0, X_1, \cdots, X_n) \to \prod_{i=1}^{n} \mathrm{Hom}_{\mathcal{C}}(X_{i-1}, X_i)$$

is a trivial Kan fibration of simplicial sets.

Example 4.6.9.6. Let $\mathcal{C}$ be an ordinary category containing objects $X_0, X_1, \ldots, X_n$, which we also regard as objects of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$. Then the restriction map 01PM

$$\theta : \mathrm{Hom}_{\mathrm{N}_\bullet(\mathcal{C})}(X_0, X_1, \cdots, X_n) \to \prod_{i=1}^{n} \mathrm{Hom}_{\mathrm{N}_\bullet(\mathcal{C})}(X_{i-1}, X_i)$$

is an isomorphism of (discrete) simplicial sets.

Remark 4.6.9.7. It follows from Corollary 4.6.9.5 that the construction 01PN

$$(X_0, X_1, \cdots, X_n) \mapsto \mathrm{Hom}_{\mathcal{C}}(X_0, X_1, \cdots, X_n)$$

endows the collection of objects of $\mathcal{C}$ with the structure of a *Segal category* (see Definition [?]). We will return to this point in §[?].

Corollary 4.6.9.8. *Let $\mathcal{C}$ be an ∞-category. For every sequence of objects $X_0, X_1, \cdots, X_n \in \mathcal{C}$, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1, \cdots, X_n)$ is a Kan complex.* 01PP

Proof. Combine Corollary 4.6.9.5 with Proposition 4.6.1.10. □

01PQ **Construction 4.6.9.9.** Let $\mathcal{C}$ be an ∞-category containing objects X, Y, and Z. By virtue of Corollary 4.6.9.5, the restriction map

$$\theta : \mathrm{Hom}_{\mathcal{C}}(X, Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

is a trivial Kan fibration, so its homotopy class $[\theta]$ is an isomorphism in the homotopy category hKan. We let

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

denote the morphism in hKan obtained by composing $[\theta]^{-1}$ with (the homotopy class of) the restriction map $\mathrm{Hom}_{\mathcal{C}}(X, Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$. We will refer to $\circ$ as the *composition law* on the ∞-category $\mathcal{C}$.

01PR **Remark 4.6.9.10.** Let $\mathcal{C}$ be an ∞-category containing objects X, Y, and Z. Then the composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

of Construction 4.6.9.9 induces a map of sets

$$\pi_0(\mathrm{Hom}_{\mathcal{C}}(Y, Z)) \times \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y)) \to \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Z)).$$

Concretely, this map is given by the construction $([g], [f]) \mapsto [h]$, where h is a composition of f and g in the sense of Definition 1.4.4.1.

01PS **Proposition 4.6.9.11** (Unitality)**.** *Let $\mathcal{C}$ be an ∞-category containing a pair of objects X and Y. Then:*

(1) *The composition*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \{\mathrm{id}_X\} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(X, X) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

is equal to the identity (in the homotopy category of Kan complexes hKan*).*

(2) *The composition*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \{\mathrm{id}_Y\} \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, Y) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

is equal to the identity (in the homotopy category of Kan complexes hKan*).*

Proof. There is a diagram of Kan complexes

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathcal{C}}(X, X, Y) \arrow[d] \arrow[dr] & \\
\mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[ur] \arrow[r, "\mathrm{id} \times \{\mathrm{id}_X\}"] & \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(X, X) \arrow[r, dashed, "\circ"] & \mathrm{Hom}_{\mathcal{C}}(X, Y),
\end{tikzcd}$$

where the left diagonal arrow is induced by the map $\sigma^0_1 : [2] \to [1]$ of Construction 1.1.2.1 and the right diagonal arrow is induced by the map $\delta^1_2 : [1] \to [2]$ of Construction 1.1.1.4. Here the solid arrows are well-defined as morphisms of simplicial sets, while the dotted arrow is well-defined only as a morphism in the homotopy category hKan. We now observe that the triangle on the left is strictly commutative, the triangle on the right commutes up to homotopy (by the construction of the composition law $\circ$). Assertion (1) follows from the observation that the composition of the diagonal arrows is the identity on the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (since $\sigma^0_1 \circ \delta^1_2$ is the identity on the object $[1] \in \mathbf{\Delta}$). Assertion (2) follows by a similar argument. □

Proposition 4.6.9.12 (Associativity)**.** *Let $\mathcal{C}$ be an ∞-category containing objects W, X, Y, and Z. Then the diagram* 01PT

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "\circ"] \arrow[d, "\circ"] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(W,Y) \arrow[r, "\circ"] & \mathrm{Hom}_{\mathcal{C}}(W,Z)
\end{tikzcd} \tag{4.43}$$ 01PU

commutes (in the homotopy category of Kan complexes hKan*).*

Proof. By virtue of Corollary 4.6.9.5, (4.43) is isomorphic to the diagram of restriction maps

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(W,X,Y,Z) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(W,X,Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}}(W,Y,Z) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(W,Z),
\end{tikzcd}$$

which commutes in the category of simplicial sets (and therefore also in the homotopy category hKan). □

Construction 4.6.9.13 (The Enriched Homotopy Category)**.** Let hKan denote the homotopy category of Kan complexes, which we endow with the monoidal structure given by cartesian products. To every ∞-category $\mathcal{C}$, we define an hKan-enriched category $\mathrm{h}\mathcal{C}$ as follows: 01PV

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, the Kan complex $\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Y)$ is the morphism space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of Construction 4.6.1.1.

- For every object $X \in \mathcal{C}$, the unit map $\Delta^0 \to \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, X)$ is the homotopy class of the inclusion $\{\mathrm{id}_X\} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, X)$.
- For every triple of objects $X, Y, Z \in \mathcal{C}$, the composition law

$$\circ : \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(Y, Z) \times \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Y) \to \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Z)$$

  is given by Construction 4.6.9.9.

Note that this definition satisfies the axiomatics of Definition 2.1.7.1 by virtue of Propositions 4.6.9.11 and 4.6.9.12 We will refer to $\mathrm{h}\mathcal{C}$ as the *enriched homotopy category* of the ∞-category $\mathcal{C}$.

01PW **Remark 4.6.9.14.** Let $\mathcal{C}$ be an ∞-category and let $\mathrm{h}\mathcal{C}$ denote the enriched homotopy category of $\mathcal{C}$. Then $\mathrm{h}\mathcal{C}$ has an underlying category (Example 2.1.7.5), which we will also denote by $\mathrm{h}\mathcal{C}$. Concretely, this category can be described as follows:

- The objects of $\mathrm{h}\mathcal{C}$ are the objects of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathrm{hKan}}(\Delta^0, \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Y)) \simeq \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y)).$$

  In other words, $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ can be identified with the set of homotopy classes of morphisms from X to Y in the ∞-category $\mathcal{C}$.

By virtue of Remark 4.6.9.10, the composition of morphisms in the category $\mathrm{h}\mathcal{C}$ agrees with the composition law of Construction 1.4.5.1. In other words, we can identify $\mathrm{h}\mathcal{C}$ with the homotopy category constructed in §1.4.5.

01TP **Notation 4.6.9.15.** Let $\mathcal{C}$ be an ∞-category containing objects X, Y, and Z. For every morphism $f : X \to Y$ in $\mathcal{C}$, the composition law of Construction 4.6.9.9 restricts to a morphism of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \{f\} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z),$$

which is well-defined up to homotopy. Note that this morphism depends only on the homotopy class $[f]$ of the morphism f. We will denote this map by $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ [f]} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ and refer to it as *precomposition with f*. Similarly, for every morphism $g : Y \to Z$, the composition law of Remark 4.6.9.10 determines a homotopy class of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{[g]\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z)$, which we will refer to as *postcomposition with g.*

To describe the precomposition morphism of Notation 4.6.9.15 concretely, it is convenient to replace the morphism spaces $\mathrm{Hom}_{\mathcal{C}}(X, Z)$ and $\mathrm{Hom}_{\mathcal{C}}(Y, Z)$ by their right-pinched variants $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Z) = \mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\}$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(Y, Z) = \mathcal{C}_{Y/} \times_{\mathcal{C}} \{Z\}$, respectively (see Construction 4.6.5.1).

Proposition 4.6.9.16. *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. For every object $Z \in \mathcal{C}$, the diagram of Kan complexes* 02LL

$$\begin{tikzcd}
\mathcal{C}_{Y/} \times_{\mathcal{C}} \{Z\} \arrow[d, "\sim"', "\iota^{\mathrm{R}}_{Y,Z}"] & \mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} \arrow[l, "\sim"] \arrow[r] & \mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\} \arrow[d, "\sim"', "\iota^{\mathrm{R}}_{X,Z}"] \\
\mathrm{Hom}_{\mathcal{C}}(Y,Z) \arrow[rr, "\circ[f]"] & & \mathrm{Hom}_{\mathcal{C}}(X,Z)
\end{tikzcd}$$

commutes up to homotopy, where the vertical maps are the right-pinch inclusion morphisms of Construction 4.6.5.7.

Remark 4.6.9.17. In the situation of Proposition 4.6.9.16, the morphisms 02LM

$$\iota^{\mathrm{R}}_{Y,Z} : \mathcal{C}_{Y/} \times_{\mathcal{C}} \{Z\} \to \mathrm{Hom}_{\mathcal{C}}(Y,Z) \qquad \iota^{\mathrm{R}}_{X,Z} : \mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\} \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

are homotopy equivalences, by virtue of Proposition 4.6.5.10. Moreover, the restriction map $\mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} \to \mathcal{C}_{Y/} \times_{\mathcal{C}} \{Z\}$ is a trivial Kan fibration (Corollary 4.3.6.13). Consequently, the precomposition map $\mathrm{Hom}_{\mathcal{C}}(Y,Z) \xrightarrow{\circ[f]} \mathrm{Hom}_{\mathcal{C}}(X,Z)$ is characterized (up to homotopy) by the conclusion of Proposition 4.6.9.16.

Proof of Proposition 4.6.9.16. It will suffice to show that there exists a morphism of Kan complexes

$$\iota^{\mathrm{R}}_{X,Y,Z} : \mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} \to \{f\} \times_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \mathrm{Hom}_{\mathcal{C}}(X,Y,Z)$$

for which the diagram

$$\begin{tikzcd}
\mathcal{C}_{Y/} \times_{\mathcal{C}} \{Z\} \arrow[d, "\iota^{\mathrm{R}}_{Y,Z}"] & \mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} \arrow[l] \arrow[r] \arrow[d, "\iota^{\mathrm{R}}_{X,Y,Z}"] & \mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\} \arrow[d, "\iota^{\mathrm{R}}_{X,Z}"] \\
\mathrm{Hom}_{\mathcal{C}}(Y,Z) & \{f\} \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \arrow[l] \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(X,Z)
\end{tikzcd}$$

commutes (in the category of simplicial sets).

We first observe that there is a unique morphism of simplicial sets $e : \Delta^2 \times \mathcal{C}_{f/} \to \Delta^1 \star \mathcal{C}_{f/}$ with the property that $e|_{\Delta^1 \times \mathcal{C}_{f/}}$ is given by projection onto the first factor, and $e|_{\{2\} \times \mathcal{C}_{f/}}$ is given by projection onto the second factor. Note that the composite map

$$\Delta^2 \times \mathcal{C}_{f/} \xrightarrow{e} \Delta^1 \star \mathcal{C}_{f/} \to \mathcal{C}$$

can be identified with a morphism of simplicial sets $e' : \mathcal{C}_{f/} \to \mathrm{Fun}(\Delta^2, \mathcal{C})$. Unwinding the definition, we see that e' restricts to a morphism of simplicial subsets

$$\iota^{\mathrm{R}}_{X,Y,Z} : \mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} \to \{f\} \times_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \subseteq \mathrm{Fun}(\Delta^2, \mathcal{C})$$

having the desired properties. □

02R3 **Corollary 4.6.9.18.** *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ and $g : X \to Z$ be morphisms of $\mathcal{C}$, which we identify with objects of the coslice ∞-category $\mathcal{C}_{X/}$. Then the morphism space $\mathrm{Hom}_{\mathcal{C}_{X/}}(f, g)$ can be identified with the homotopy fiber of the composition map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ[f]} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ over the vertex $g \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$.*

Proof. Using Proposition 4.6.9.16, we can replace the composition map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ[f]} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ with the restriction map $\theta : \mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} \to \mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\}$. The morphism θ is a left fibration (Corollary 4.3.6.11). Since the left-pinched morphism space $\mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\} = \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Z)$ is a Kan complex (Proposition 4.6.5.5), it follows that θ is a Kan fibration (Corollary 4.4.3.8). In particular, the homotopy fiber of the composition map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ[f]} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ over the vertex g can be identified with the fiber

$$\theta^{-1}\{g\} \simeq \mathcal{C}_{f/} \times_{\mathcal{C}_{X/}} \{g\} = \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}_{X/}}(f, g),$$

which is homotopy equivalent to $\mathrm{Hom}_{\mathcal{C}_{X/}}(f, g)$ by virtue of Proposition 4.6.5.10. □

Let $\mathcal{C}$ be a locally Kan simplicial category, so that the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.4.5.1). In this case, the composition law of Construction 4.6.9.9 has a direct description:

02LN **Proposition 4.6.9.19.** *Let $\mathcal{C}$ be a locally Kan simplicial category. For every pair of objects $X, Y \in \mathcal{C}$, let $\theta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}}(X, Y)$ denote the homotopy equivalence of Kan complexes supplied by Remark 4.6.8.6. Then, for every triple of objects $X, Y, Z \in \mathcal{C}$, the diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \arrow[r, "\circ"] \arrow[d, "\sim"', "{[\theta_{Y,Z} \times \theta_{X,Y}]}"] & \mathrm{Hom}_{\mathcal{C}}(X, Z)_{\bullet} \arrow[d, "\sim"', "{[\theta_{X,Z}]}"] \\
\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(Y, Z) \times \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Y) \arrow[r, "\circ"] & \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Z)
\end{tikzcd}$$

commutes in the homotopy category hKan*; here the lower horizontal map is the composition law of Construction 4.6.9.9.*

Proof. We will show that there exists a morphism of Kan complexes

$$\theta_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Y, Z)$$

for which the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \arrow[rr, "\circ"] \arrow[d, "\theta_{Y,Z}\times\theta_{X,Y}"] \arrow[dr, "\theta_{X,Y,Z}"] & & \mathrm{Hom}_{\mathcal{C}}(X,Z)_\bullet \arrow[d, "\theta_{X,Z}"] \\
\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(Y,Z) \times \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(X,Y) & \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(X,Y,Z) \arrow[l] \arrow[r] & \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(X,Z)
\end{tikzcd}$$

is commutative.

Fix an integer $n \geq 0$. Let $\mathcal{E}$ denote the simplicial category with object set $\mathrm{Ob}(\mathcal{E}) = \{x, y, z\}$ and morphism spaces given by

$$\mathrm{Hom}_{\mathcal{E}}(x,x)_\bullet = \{\mathrm{id}_x\} \qquad \mathrm{Hom}_{\mathcal{E}}(y,y)_\bullet = \{\mathrm{id}_y\} \qquad \mathrm{Hom}_{\mathcal{E}}(z,z)_\bullet = \{\mathrm{id}_z\}$$

$$\mathrm{Hom}_{\mathcal{E}}(y,x)_\bullet = \emptyset \qquad \mathrm{Hom}_{\mathcal{E}}(z,x)_\bullet = \emptyset \qquad \mathrm{Hom}_{\mathcal{E}}(z,y)_\bullet = \emptyset$$

$$\mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet = \Delta^n \qquad \mathrm{Hom}_{\mathcal{E}}(y,z)_\bullet = \Delta^n,$$

where the composition law $\mathrm{Hom}_{\mathcal{E}}(y,z)_\bullet \times \mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet \to \mathrm{Hom}_{\mathcal{E}}(x,z)_\bullet$ is an isomorphism (so that $\mathrm{Hom}_{\mathcal{E}}(x,z)_\bullet$ can be identified with the product $\Delta^n \times \Delta^n$). Note that there is a unique simplicial functor $F : \mathrm{Path}[\Delta^2 \times \Delta^n]_\bullet \to \mathcal{E}$ satisfying the following conditions:

- On objects, the functor F is given by the formula

$$F(i,j) = \begin{cases} x & \text{if } i = 0 \\ y & \text{if } i = 1 \\ z & \text{if } i = 2. \end{cases}$$

- Let (i,j) and (i',j') be vertices of $\Delta^2 \times \Delta^n$ satisfying $i < i'$ and $j \leq j'$, so that there is a unique indecomposable morphism u from (i,j) to (i',j') in the path category $\mathrm{Path}[\Delta^2 \times \Delta^n]$ (given by the chain $\{(i,j) < (i',j')\}$). If $i = 0$ and $i' = 1$, then $F(u)$ is the vertex j' of $\Delta^n = \mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet$. If $i = 1$ and $i' = 2$, then $F(u)$ is the vertex j' of $\Delta^n = \mathrm{Hom}_{\mathcal{E}}(y,z)_\bullet$. If $i = 0$ and $i' = 2$, then $F(u)$ is the vertex (j', j') of $\Delta^n \times \Delta^n = \mathrm{Hom}_{\mathcal{E}}(x,z)_\bullet$.

Let σ and τ be n-simplices of the Kan complexes $\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$, respectively. Then there is a unique simplicial functor $G_{\sigma,\tau} : \mathcal{E} \to \mathcal{C}$ satisfying the following conditions:

- On objects, the functor $G_{\sigma,\tau}$ is given by $G_{\sigma,\tau}(x) = X$, $G_{\sigma,\tau}(y) = Y$, and $G_{\sigma,\tau}(z) = Z$.

- The induced map $\Delta^n = \mathrm{Hom}_{\mathcal{E}}(x,y)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ is the n-simplex τ.

- The induced map $\Delta^n = \mathrm{Hom}_{\mathcal{E}}(y,z)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet$ is the n-simplex σ.

The composite simplicial functor

$$\mathrm{Path}[\Delta^2 \times \Delta^n]_\bullet \xrightarrow{F} \mathcal{E} \xrightarrow{G_{\sigma,\tau}} \mathcal{C}$$

determines a morphism from $\Delta^2 \times \Delta^n$ to the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$, which can be identified with an n-simplex $\theta_{X,Y,Z}(\sigma,\tau)$ of the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X,Y,Z)_\bullet$. Allowing n to vary, the construction $(\sigma,\tau) \mapsto \theta_{X,Y,Z}(\sigma,\tau)$ determines a morphism of simplicial sets $\theta_{X,Y,Z} : \mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})}(X,Y,Z)$ having the desired properties. □

02LP **Corollary 4.6.9.20.** *Let $\mathcal{C}$ be a locally Kan simplicial category, and let $U : \mathrm{h}\mathcal{C} \to \mathrm{hN}_\bullet^{\mathrm{hc}}(\mathcal{C})$ be the isomorphism of homotopy categories supplied by Proposition 2.4.6.9. Then the homotopy equivalences $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{hc}}}(X,Y)$ of Remark 4.6.8.6 promote U to an isomorphism of* hKan-*enriched categories. Here* $\mathrm{h}\mathcal{C}$ *is endowed with the* hKan-*enrichment of Remark 3.1.5.12 and* $\mathrm{hN}_\bullet^{\mathrm{hc}}(\mathcal{C})$ *is endowed with the* hKan-*enrichment of Construction 4.6.9.13.*

Let $\mathcal{C}$ be a differential graded category. For every pair of objects $X, Y \in \mathcal{C}$, we let $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ denote the chain complex of morphisms from X to Y and $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*)$ the associated Eilenberg-MacLane space (Construction 2.5.6.3). In what follows, let us write

$$\rho_{Y,X} : \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*) \hookrightarrow \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y)$$

for the composition of the isomorphism $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*) \simeq \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y)$ of Example 4.6.5.15 with the pinch inclusion morphism $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y)$ of Construction 4.6.5.7. We then have the following:

03E0 **Proposition 4.6.9.21.** *Let $\mathcal{C}$ be a differential graded category containing objects X, Y, and Z, so that the composition law*

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z)_* \otimes \mathrm{Hom}_{\mathcal{C}}(X,Y)_* \to \mathrm{Hom}_{\mathcal{C}}(X,Z)_*$$

induces a bilinear map of simplicial abelian groups

$$\mu : \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_*) \times \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*) \to \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Z)_*)$$

(see Proposition 2.5.9.1). Then the diagram of Kan complexes

03E1

$$\begin{tikzcd}
\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_*) \times \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*) \arrow[r, "\mu"] \arrow[d, "\rho_{Z,Y} \times \rho_{Y,X}"] & \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Z)_*) \arrow[d, "\rho_{Z,X}"] & \\
\mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(Y,Z) \arrow[r] & \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y) \arrow[r] & \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Z)
\end{tikzcd} \tag{4.44}$$

commutes up to homotopy, where the bottom horizontal map is the composition law of Construction 4.6.9.9.

Remark 4.6.9.22. In the situation of Proposition 4.6.9.21, the morphisms $\rho_{Y,X}$, $\rho_{Z,Y}$, and $\rho_{Z,X}$ are homotopy equivalences (Proposition 4.6.5.10). Consequently, Proposition 4.6.9.21 determines the composition law on the hKan-enriched homotopy category of $\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$. 03E2

Proof of Proposition 4.6.9.21. Let $\mathcal{C}^{\Delta}$ denote the underlying simplicial category of the differential graded category $\mathcal{C}$ (Construction 2.5.9.2). By virtue of Exercise 4.6.8.4, we can identify (4.44) with the outer rectangle of a larger diagram

$$\begin{tikzcd}
\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_*) \times \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*) \arrow[r, "\mu"] \arrow[d] & \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Z)_*) \arrow[d] & \\
\mathrm{Hom}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}^{\Delta})}(Y,Z) \times \mathrm{Hom}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}^{\Delta})}(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}^{\Delta})}(X,Z) \arrow[d] & \\
\mathrm{Hom}_{\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})}(Y,Z) \arrow[r] & \mathrm{Hom}_{\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})}(X,Y) \arrow[r] & \mathrm{Hom}_{\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})}(X,Z),
\end{tikzcd}$$

where middle horizontal map is given by the composition law of the ∞-category $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}^{\Delta})$. We now observe that the upper square commutes up to homotopy by virtue of Proposition 4.6.9.19, and the lower square commutes up to homotopy by the functoriality of Construction 4.6.9.9. □

4.7 Size Conditions on ∞-Categories

Recall that a small category $\mathcal{C}$ consists of the following data: 03PP

- A set $\mathrm{Ob}(\mathcal{C})$, whose elements are referred to as *objects* of $\mathcal{C}$.
- For every pair of objects $X, Y \in \mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, whose elements are referred to as *morphisms from X to Y*.
- For every triple of objects $X, Y, Z \in \mathcal{C}$, a composition law

  $$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

  which is required to be unital and associative.

This definition treats categories as algebraic objects akin to groups (though somewhat more general), which is perfectly adequate for many purposes. However, it is often useful to apply the theory to categories which are not small, such as the category of sets $\mathcal{C} = \mathrm{Set}$. In this case, $\mathrm{Ob}(\mathcal{C})$ is the collection of all sets, and must be treated with a bit of care to avoid paradoxes.

03PQ **Example 4.7.0.1.** When speaking informally, it is common to say that the category Set has all limits and colimits. A more precise statement is that the category Set has all *small* limits and colimits; that is, every diagram $F : \mathcal{J} \to \mathrm{Set}$ indexed by a small category $\mathcal{J}$ has a limit and colimit. Here the size restriction on $\mathcal{J}$ cannot be omitted. For example, if $\{S_j\}_{j \in J}$ is a collection of sets indexed by another set J, then it is permissible to form the coproduct $\coprod_{j \in J} S_j$. However, it is not permissible to form the coproduct $\coprod_{S \in \mathrm{Ob}(\mathrm{Set})} S$ of *all* sets.

In the setting of higher category theory, one encounters similar issues. In §1.4, we defined an ∞-category to be a simplicial set $\mathcal{C}_\bullet$ which satisfies a filling condition for inner horns (Definition 1.4.0.1). By analogy with the discussion above, we might be better to refer to such objects as *small* ∞*-categories*. However, we will often want to apply the ideas developed in this book to ∞-categories $\mathcal{C}_\bullet$ which are not small, because the collections n-simplices $\mathcal{C}_n$ are "too big" to be sets (this situation arises, for example, if $\mathcal{C}_\bullet$ is the nerve of a large category). For the most part, we will ignore the set-theoretic issues which are raised by allowing such objects into our discourse. However, this is not always possible: as Example 4.7.0.1 illustrates, it is sometimes important to track the distinction between "large" and "small."

The first goal of this section is to introduce some language for quantifying the sizes of category-theoretic objects. Let κ be an infinite cardinal. We will say that a set is κ*-small* if its cardinality is strictly smaller than κ (Definition 4.7.3.1). We will say that a simplicial set S is κ*-small* if the collection of nondegenerate simplices of S is κ-small (Definition 4.7.4.1). We summarize the basic properties of κ-small sets and simplicial sets in §4.7.3 and §4.7.4, respectively. Beware that κ-smallness is not a homotopy invariant condition: that is, it is possible for a κ-small ∞-category to be equivalent to an ∞-category which is not κ-small. In §4.7.5, we address this point by introducing the notion of *essential smallness.* If κ is an uncountable cardinal, we say that an ∞-category $\mathcal{C}$ is *essentially* κ*-small* if it is equivalent to a κ-small ∞-category (Definition 4.7.5.1). One can formulate this condition also in the case $\kappa = \aleph_0$, but it is poorly behaved: it is very rare for finite simplicial sets to be ∞-categories (see Warning 4.7.5.6).

The second goal of this section is to provide a concrete criterion which can be used to test if an ∞-category is essentially κ-small. For simplicity, let us assume that κ is an (uncountable) regular cardinal. We say that an ∞-category $\mathcal{C}$ is *locally* κ*-small* if, for every pair of objects $C, D \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(C, D)$ is essentially κ-small (Definition 4.7.8.1). In §4.7.8, we show that $\mathcal{C}$ is essentially κ-small if and only if it locally κ-small and the set of isomorphism classes $\pi_0(\mathcal{C}^{\simeq})$ is κ-small (Proposition 4.7.8.7). We are therefore reduced to the problem of testing essential κ-smallness of Kan complexes. In §4.7.7, we address this problem by showing that a Kan complex X is essentially κ-small if and only if the set $\pi_0(X)$ is κ-small and the homotopy groups $\{\pi_n(X, x)\}_{n > 0}$ are κ-small for every vertex $x \in X$ (Proposition 4.7.7.1).

The proofs of Propositions 4.7.8.7 and 4.7.7.1 will use a common strategy. In both cases, the hard part is to show that if $\mathcal{C}$ is an ∞-category for which certain homotopy-invariant quantities are bounded in size, then $\mathcal{C}$ is equivalent to an ∞-category $\mathcal{C}_0$ for which the collection of simplices is bounded in size. We will prove this using the theory of *minimal models*. We say that an ∞-category $\mathcal{C}_0$ is *minimal* if the datum of a simplex $\sigma : \Delta^n \to \mathcal{C}$ is determined by its homotopy class relative to the boundary $\partial \Delta^n$ (see Definition 4.7.6.4). In §4.7.6, we will prove the following:

- For every ∞-category $\mathcal{C}$, there exists an equivalence of ∞-categories $\mathcal{C}_0 \to \mathcal{C}$, where $\mathcal{C}_0$ is minimal (Proposition 4.7.6.15). Moreover, $\mathcal{C}_0$ is uniquely determined up to isomorphism (Corollary 4.7.6.14).
- If $\mathcal{C}_0$ is a minimal ∞-category, then every equivalence of ∞-categories $\mathcal{C}_0 \to \mathcal{C}$ is a monomorphism of simplicial sets (Lemma 4.7.6.11). Consequently, $\mathcal{C}_0$ is essentially κ-small if and only if it is κ-small (Corollary 4.7.6.12).

Remark 4.7.0.2. Throughout this section, we will need some elementary properties of cardinals and cardinal arithmetic. For the reader's convenience, we briefly review the set-theoretic prerequisites in §4.7.1 and §4.7.2. 03PR

Remark 4.7.0.3. The notion of minimal ∞-category was introduced by Joyal in [30]. In the setting of Kan complexes, the theory of minimal models is much older (see [2]). 03PS

Remark 4.7.0.4. Let κ be an uncountable regular cardinal. We will see later that the essentially κ-small ∞-categories admit a more intrinsic characterization: they are precisely the κ-compact objects of the ∞-category $\mathcal{QC}$ of ∞-categories (see Proposition [?]). 03PT

Remark 4.7.0.5. Throughout this book, we will make reference to a dichotomy between "small" and "large" mathematical objects. We will generally take a somewhat informal view of this dichotomy, taking care only to avoid maneuvers which are obviously illegitimate (see Example 4.7.0.1). However, the reader who wishes to adopt a more scrupulous approach could proceed (within the framework of Zermelo-Fraenkel set theory) as follows: 03PU

- Assume the existence of an uncountable strongly inaccessible cardinal κ (see Definition 4.7.3.20).
- Declare that an ∞-category $\mathcal{C}$ is small (essentially small, locally small) if it is κ-small (essentially κ-small, locally κ-small), and apply similar conventions to other mathematical objects of interest (such as sets and categories).

4.7.1 Ordinals and Well-Orderings

In this section, we review some standard facts about ordinals and well-ordered sets. 03PV

034C **Definition 4.7.1.1.** Let $(S, \leq)$ be a partially ordered set. We say that $(S, \leq)$ is *well-founded* if every nonempty subset $S_0 \subseteq S$ contains a minimal element: that is, an element $s \in S_0$ for which the set $\{t \in S_0 : t < s\}$ is empty.

03PW **Exercise 4.7.1.2.** Let $(S, \leq)$ be a partially ordered set. Show that the following conditions are equivalent:

(1) The partial order $\leq$ is well-founded: that is, every nonempty subset of S contains a minimal element.

(2) The set S does not contain an infinite descending sequence $s_0 > s_1 > s_2 > \cdots$.

03PX **Example 4.7.1.3.** Every finite partially ordered set $(S, \leq)$ is well-founded.

03PY **Example 4.7.1.4.** Let S be any set, and let $\leq$ be the *discrete* partial ordering of S: that is, we have $s \leq t$ if and only if $s = t$. Then $(S, \leq)$ is well-founded.

03PZ **Remark 4.7.1.5.** Let $(S, \leq)$ be a well-founded partially ordered set. Then every subset $S_0 \subseteq S$ is also well-founded (when endowed with the partial order given by the restriction of $\leq$).

03Q0 **Definition 4.7.1.6.** Let $(S, \leq)$ be a linearly ordered set. We say that $(S, \leq)$ is *well-ordered* if it is well-founded when regarded as a partially ordered set: that is, if every nonempty subset $S_0 \subseteq S$ contains a smallest element. In this case, we will refer to the relation $\leq$ as a *well-ordering* of the set S.

03Q1 **Definition 4.7.1.7** (Ordinals)**.** An *ordinal* is an isomorphism class of well-ordered sets. If $(S, \leq)$ is a well-ordered set, then its isomorphism class is an ordinal which we will refer to as the *order type* of S.

03Q2 **Notation 4.7.1.8.** We will typically use lower-case Greek letters to denote ordinals.

03Q3 **Example 4.7.1.9** (Finite Ordinals)**.** Let n be a nonnegative integer. Up to isomorphism, there is a unique linearly ordered set S having exactly n elements, which we can identify with the set $\{0 < 1 < \cdots < n-1\}$. We will abuse notation by identifying n with the order type of the linearly ordered set S. By means of this convention, we can view every nonnegative integer as an ordinal. We say that an ordinal α is *finite* it arises in this way (that is, if it is the order type of a finite linearly ordered set), and *infinite* if it does not.

03Q4 **Example 4.7.1.10.** The set of nonnegative integers $\mathbf{Z}_{\geq 0} = \{0 < 1 < 2 < \cdots\}$ is well-ordered (with respect to its usual ordering). Its order type is an infinite ordinal, which we denote by ω.

By definition, well-ordered sets $(S, \leq)$ and $(T, \leq)$ have the same order type if there is an order-preserving bijection $f : S \xrightarrow{\sim} T$. We will show in a moment that in this case, the bijection f is uniquely determined (Corollary 4.7.1.16). First, let us introduce a bit of additional terminology.

Definition 4.7.1.11. Let $(S, \leq)$ be a linearly ordered set. We say that a subset $S_0 \subseteq S$ is an *initial segment* if it is closed downwards: that is, for every pair of elements $s \leq s'$ of S, if s' is contained in S_0, then s is also contained in S_0. If $(T, \leq)$ is another linearly ordered set, we say that a function $f : S \hookrightarrow T$ is an *initial segment embedding* if it is an isomorphism (of linearly ordered sets) from S to an initial segment of T. `03Q5`

Example 4.7.1.12. Let $(S, \leq)$ be a linearly ordered set. Then the identity morphism $\mathrm{id}_S : S \xrightarrow{\sim} S$ is an initial segment embedding. `03Q6`

Remark 4.7.1.13 (Transitivity)**.** Let $(R, \leq)$, $(S, \leq)$, and $(T, \leq)$ be linearly ordered sets. Suppose that $f : R \hookrightarrow S$ and $g : S \hookrightarrow T$ are initial segment embeddings. Then the composition $(g \circ f) : R \hookrightarrow T$ is also an initial segment embedding. `03Q7`

Proposition 4.7.1.14. *Let $(S, \leq)$ and $(T, \leq)$ be linearly ordered sets, and let $f, f' : S \hookrightarrow T$ be strictly increasing functions. Suppose that S is well-ordered and that f is an initial segment embedding. Then, for each $s \in S$, we have $f(s) \leq f'(s)$.* `03Q8`

Proof. Set $S_0 = \{s \in S : f'(s) < f(s)\}$. We wish to show that S_0 is empty. Assume otherwise. Since S is well-ordered, there is a least element $s \in S_0$. Since f is an initial segment embedding, the inequality $f'(s) < f(s)$ implies that we can write $f'(s) = f(t)$ for some $t < s$. Then $t \notin S_0$, so we must have $f(t) \leq f'(t)$. It follows that $f'(s) \leq f'(t)$, contradicting our assumption that the function f' is strictly increasing. □

Corollary 4.7.1.15 (Rigidity)**.** *Let $(S, \leq)$ and $(T, \leq)$ be linearly ordered sets, and let $f, f' : S \hookrightarrow T$ be initial segment embeddings. If S is well-ordered, then $f = f'$.* `03Q9`

Corollary 4.7.1.16. *Let $(S, \leq)$ and $(T, \leq)$ be well-ordered sets. If there exists an order-preserving bijection $f : S \xrightarrow{\sim} T$, then f is unique.* `03QA`

Corollary 4.7.1.17. *Let $(S, \leq)$ and $(T, \leq)$ be well-ordered sets. Then one of the following conditions is satisfied:* `03QB`

(1) *There exists an initial segment embedding $f : S \hookrightarrow T$.*

(2) *There exists an initial segment embedding $g : T \hookrightarrow S$.*

Proof. For each element $s \in S$, let $S_{\leq s}$ denote the initial segment $\{s' \in S : s' \leq s\}$. Let $S_0 \subseteq S$ denote the collection of elements $s \in S$ for which there exists an initial segment embedding $f_{\leq s} : S_{\leq s} \hookrightarrow T$. Note that, if this condition is satisfied, then the morphism $f_{\leq s}$ is uniquely determined (Corollary 4.7.1.15). Moreover, if $s' \leq s$, then composite map $S_{\leq s'} \subseteq S_{\leq s} \xrightarrow{f_{\leq s}} T$ is also an initial segment embedding; it follows that s' belongs to S_0, and $f|_{\leq s'}$ is the restriction of $f|_{\leq s}$ to $S_{\leq s'}$. Consequently, the construction $s \mapsto f_s(s)$ determines a function $f : S_0 \to T$, which is an isomorphism of S_0 with an initial segment $T_0 \subseteq T$. If $S_0 = S$, then f is an initial segment embedding from S to T. If $T_0 = T$, then $g = f^{-1}$ is an initial segment embedding from T to S. Assume that neither of these conditions is satisfied: that is, the sets $S \setminus S_0$ and $T \setminus T_0$ are both nonempty. Let s be a least element of $S \setminus S_0$, and let t be a least element of $T \setminus T_0$. Then f extends uniquely to an initial segment embedding

$$f_{\leq s} : S_{\leq s} = S_0 \cup \{s\} \xrightarrow{\sim} T_0 \cup \{t\} \subseteq T \qquad s \mapsto t.$$

The existence of $f_{\leq s}$ shows that s belongs to S_0, which is a contradiction. □

03QC **Remark 4.7.1.18.** In the situation of Corollary 4.7.1.17, suppose that conditions (1) and (2) are both satisfied: that is, there exist initial segment embeddings $f : S \hookrightarrow T$ and $g : T \hookrightarrow S$. Then $g \circ f$ is an initial segment embedding of S into itself, and therefore coincides with id_S (Corollary 4.7.1.16). The same argument shows that $f \circ g = \mathrm{id}_T$, so that f and g are mutually inverse bijections. In particular, S and T have the same order type.

03QD **Definition 4.7.1.19.** Let α and β be ordinals, given by the order types of well-ordered sets $(S, \leq)$ and $(T, \leq)$. We write $\alpha \leq \beta$ if there exists an initial segment embedding from $(S, \leq)$ to $(T, \leq)$ (note that this condition depends only on the order types of S and T).

03QE **Proposition 4.7.1.20.** *The relation $\leq$ of Definition 4.7.1.19 determines a linear ordering on the collection of ordinals.*

Proof. The reflexivity of the relation $\leq$ follows from Example 4.7.1.12, and the transitivity follows from Remark 4.7.1.13. Let α and β be ordinals, which we identify with the order types of well-ordered sets $(S, \leq)$ and $(T, \leq)$, respectively. Invoking Corollary 4.7.1.17, we deduce that $\alpha \leq \beta$ or $\beta \leq \alpha$. Moreover, if both conditions are satisfied, then Remark 4.7.1.18 shows that $\alpha = \beta$. □

03QF **Remark 4.7.1.21.** Let $(S, \leq)$ and $(T, \leq)$ be well-ordered sets. The following conditions are equivalent:

(1) There exists an initial segment embedding $f : S \hookrightarrow T$.

(2) There exists a strictly increasing function $f : S \hookrightarrow T$.

The implication (1) ⇒ (2) is immediate from the definitions. To prove the converse, let $f : S \hookrightarrow T$ be a strictly increasing function, and suppose that there is no initial segment embedding from S to T. Invoking Corollary 4.7.1.17, we deduce that there is an initial segment embedding $g : T \hookrightarrow S$. The composition $(g \circ f) : S \hookrightarrow S$ is strictly increasing, and therefore satisfies $(g \circ f)(s) \geq s$ for each $s \in S$ (Proposition 4.7.1.14). Since the image of g is an initial segment $S_0 \subseteq S$, we must have $S_0 = S$. It follows that $g^{-1} : S \xrightarrow{\sim} T$ is an isomorphism of linearly ordered sets, contradicting our assumption.

We now show that, for every ordinal α, there is a preferred candidate for a well-ordered set of order type α: namely, the collection $\mathrm{Ord}_{<\alpha}$ of ordinals smaller than α.

Proposition 4.7.1.22. *Let $(S, \leq)$ be a well-ordered set, and let α denote its order type. Then there is a unique order-preserving bijection $S \to \mathrm{Ord}_{<\alpha}$, which carries each element $s \in S$ to the order type of the well-ordered set $S_{<s} = \{s' \in S : s' < s\}$.* 03QG

Proof. We will prove existence; uniqueness then follows from Corollary 4.7.1.16. For each $s \in S$, let α_s denote the order type of the set $S_{<s}$ (which is well-ordered, by virtue of Remark 4.7.1.5). Note that, since there is an initial segment embedding $S_{<s} \hookrightarrow S$ which is not bijective, we must have $\alpha_s < \alpha$ (Remark 4.7.1.18). Consequently, the construction $s \mapsto \alpha_s$ determines a function $S \to \mathrm{Ord}_{<\alpha}$. If $s < t$ in S, then there is an initial segment embedding from $S_{<s}$ to $S_{<t}$ which is not bijective, so that $\alpha_s < \alpha_t$ (again by Remark 4.7.1.18). To complete the proof, it will suffice to show that the function $s \mapsto \alpha_s$ is surjective. Let β be an ordinal which is strictly smaller than α. Then β is the order type of some initial segment $S_0 \subsetneq S$. Since S is well-ordered, the set $S \setminus S_0$ has a smallest element s. It follows that $S_0 = S_{<s}$, so that $\beta = \alpha_s$. □

Corollary 4.7.1.23. *For every ordinal α, $\mathrm{Ord}_{<\alpha}$ is a well-ordered set of order type α.* 03QH

Corollary 4.7.1.24. *Let S be any nonempty collection of ordinals. Then S has a least element.* 03QJ

Proof. Choose an ordinal $\alpha \in S$. If α is a least element of S, then we are done. Otherwise, we can replace S by the nonempty subset $S_{<\alpha} = \{\beta \in S : \beta < \alpha\}$. Note that $S_{<\alpha}$ is a nonempty subset of $\mathrm{Ord}_{<\alpha}$, and therefore has a smallest element by virtue of Corollary 4.7.1.23. □

Warning 4.7.1.25 (The Burali-Forti Paradox)**.** One can informally summarize Corollary 4.7.1.24 by saying that the collection Ord of all ordinals is well-ordered (with respect to the order relation of Definition 4.7.1.19). Beware that one must treat this statement with some care to avoid paradoxes. The proof of Proposition 4.7.1.22 shows that the order type of Ord is strictly larger than α, for each ordinal $\alpha \in \mathrm{Ord}$. This paradox has a standard remedy: we regard the collection Ord as "too large" to form a set (so that its order type is not regarded as an ordinal). 03QK

03QL **Definition 4.7.1.26.** Let $(S, \leq)$ and $(T, \leq)$ be linearly ordered sets. We say that a function $f : S \to T$ is *cofinal* if it is nondecreasing and, for every element $t \in T$, there exists an element $s \in S$ satisfying $f(s) \geq t$.

03QM **Proposition 4.7.1.27.** *Let $(T, \leq)$ be a linearly ordered set. There exists a well-ordered subset $S \subseteq T$ for which the inclusion map $S \hookrightarrow T$ is cofinal.*

Proof. Let $\{S_q\}_{q \in Q}$ be the collection of all well-ordered subsets of T. We regard Q as a partially ordered set, where $q \leq q'$ if the set S_q is an initial segment of $S_{q'}$. This partial ordering satisfies the hypotheses of Zorn's lemma, and therefore contains a maximal element $S_{\max}$. To complete the proof, it will suffice to show that the inclusion $S_{\max} \hookrightarrow T$ is cofinal. Assume otherwise: then there exists an element $t \in T$ satisfying $s < t$ for each $s \in S_{\max}$. Then $S_{\max}$ is an initial segment of the well-ordered subset $S_{\max} \cup \{t\} \subseteq T$, contradicting the maximality of $S_{\max}$. □

03QN **Definition 4.7.1.28** (Cofinality)**.** Let $(T, \leq)$ be a linearly ordered set. We let $\mathrm{cf}(T)$ denote the smallest ordinal α for which there exists a well-ordered set $(S, \leq)$ of order type α and a cofinal function $f : S \to T$. We refer to $\mathrm{cf}(T)$ as the *cofinality* of the linearly ordered set T.

If β is an ordinal, let $\mathrm{cf}(\beta)$ denote the cofinality $\mathrm{cf}(T)$, where $(T, \leq)$ is any well-ordered set of order type β. We refer to $\mathrm{cf}(\beta)$ as the *cofinality of β*.

03QP **Remark 4.7.1.29.** For any linearly ordered set $(T, \leq)$, the identity map $\mathrm{id} : T \to T$ is cofinal. Consequently, if T is well-ordered set of order type α, then we have $\mathrm{cf}(\alpha) = \mathrm{cf}(T) \leq \alpha$. Beware that the inequality is often strict.

03QQ **Example 4.7.1.30.** Let $(T, \leq)$ be a linearly ordered set. Then $\mathrm{cf}(T) = 0$ if and only if T is empty.

03QR **Example 4.7.1.31.** Let $(T, \leq)$ be a nonempty linearly ordered set. The following conditions are equivalent:

- The cofinality $\mathrm{cf}(T)$ is a positive integer.
- The cofinality $\mathrm{cf}(T)$ is equal to 1.
- The linearly ordered set T contains a largest element.

03QS **Example 4.7.1.32.** Let $(T, \leq)$ be a linearly ordered set. Then the cofinality $\mathrm{cf}(T)$ is equal to ω if and only if T contains an unbounded increasing sequence $\{t_0 < t_1 < t_2 < \cdots\}$.

03QT **Proposition 4.7.1.33.** *Let $(T, \leq)$ be a linearly ordered set. Then the cofinality $\mathrm{cf}(T)$ is the smallest ordinal α with the following property:*

$(*)$ *There exists a well-ordered set* $(S, \leq)$ *of order type* α *and a function* $f : S \to T$ *which is unbounded (that is, every element* $t \in T$ *satisfies* $t \leq f(s)$ *for some* $s \in S$*). Here we do not require* f *to be nondecreasing.*

Proof. It is clear that the cofinality $\mathrm{cf}(T)$ satisfies condition $(*)$. For the converse, assume that $(S, \leq)$ is a well-ordered set of order type α and that $f : S \to T$ is an unbounded function. Let us say that an element $s \in S$ is *good* if, for every element $s' < s$ of S, we have $f(s') < f(s)$. Let S_0 be the collection of good elements of S, and set $f_0 = f|_{S_0}$. By construction, the function f_0 is strictly increasing. Moreover, the order type of S_0 is $\leq \alpha$ (Remark 4.7.1.21). To complete the proof, it will suffice to show that $f_0 : S_0 \hookrightarrow T$ is cofinal. Fix an element $t \in T$, and set $S_{\geq t} = \{s \in S : t \leq f(s)\}$. We wish to show that the intersection $S_{\geq t} \cap S_0$ is nonempty. We first observe that $S_{\geq t}$ is nonempty (by virtue of our assumption that f is unbounded). Since $(S, \leq)$ is well-ordered, the set $S_{\geq t}$ contains a least element s. We claim that s belongs to S_0. Assume otherwise: then there exists some $s' < s$ satisfying $f(s') \geq f(s)$. It follows that s' belongs to $S_{\geq t}$, contradicting the minimality of s. $\square$

We conclude this section by observing that well-orderings exist in abundance.

Theorem 4.7.1.34 (The Well-Ordering Theorem)**.** *Every set* S *admits a well-ordering.* `03QU`

By virtue of Example 4.7.1.4, Theorem 4.7.1.34 is a special case of the following more refined result:

Proposition 4.7.1.35. *Let* $(S, \preceq)$ *be a well-founded partially ordered set. Then there exists a well-ordering* $\leq$ *on* S *which refines* $\preceq$ *in the following sense: for every pair of elements* $s, t \in S$ *satisfying* $s \preceq t$*, we also have* $s \leq t$*.* `03QV`

Proof. Let Q denote the set of ordered pairs $(T, \leq_T)$, where T is a subset of S which is closed downward with respect to $\preceq$ and $\leq_T$ is a well-ordering of T which refines $\preceq$. We regard Q as a partially ordered set, where $(T, \leq_T) \leq (T', \leq_{T'})$ if T is an initial segment of T' (with respect to the ordering $\leq_{T'}$), and the ordering $\leq_T$ coincides with the restriction of $\leq_{T'}$. The partially ordered set Q satisfies the hypotheses of Zorn's lemma, and therefore contains a maximal element $(T_{\mathrm{max}}, \leq_{T_{\mathrm{max}}})$. To complete the proof, it will suffice to show that $T_{\mathrm{max}} = S$. Suppose otherwise. Then the set $S \setminus T_{\mathrm{max}}$ is nonempty, and therefore contains an element s which is minimal with respect to the ordering $\preceq$. Set $T' = T_{\mathrm{max}} \cup \{s\}$, and extend $\leq_{T_{\mathrm{max}}}$ to a linear ordering $\leq_{T'}$ of T' by declaring s to be a largest element. Then $(T', \leq_{T'})$ is an element of Q, contradicting the maximality of the pair $(T_{\mathrm{max}}, \leq_{T_{\mathrm{max}}})$. $\square$

4.7.2 Cardinals and Cardinality

`03QW`

Let S and T be sets. We say that S and T *have the same cardinality* if there exists a bijection $S \xrightarrow{\sim} T$. This is an equivalence relation on the collection of sets, whose equivalence classes are called *cardinals*. Following a standard convention in set theory, it will be convenient to view a cardinal as a special type of ordinal.

`03QX` **Definition 4.7.2.1.** Let S be a set. We let $|S|$ denote the smallest ordinal α for which there exists a well-ordering of S having order type α. We will refer to $|S|$ as the *cardinality* of the set S. A *cardinal* is an ordinal κ which has the form $|S|$, for some set S.

`03QY` **Remark 4.7.2.2.** Let S be a set, and let A be the collection of all ordinals which arise as the order types of well-orderings on S. The collection A is nonempty (Theorem 4.7.1.34), and therefore contains a smallest element (Corollary 4.7.1.24). It follows that the cardinality $|S|$ is well-defined.

`03QZ` **Proposition 4.7.2.3.** *Let S and T be sets. Then $|S| \leq |T|$ if and only if there exists a monomorphism $f : S \hookrightarrow T$.*

Proof. Choose well-orderings $(S, \leq_S)$ and $(T, \leq_T)$ having order types $|S|$ and $|T|$, respectively. If $|S| \leq |T|$, then there is an isomorphism of $(S, \leq_S)$ with an initial segment of $(T, \leq_T)$; this isomorphism in particular gives a monomorphism of sets $S \hookrightarrow T$. For the converse, suppose that there exists a monomorphism $f : S \hookrightarrow T$. Then there is a unique linear ordering $\leq'_S$ on the set S for which f defines a strictly increasing function $(S, \leq'_S) \to (T, \leq_T)$. Then $\leq'_S$ is a well-ordering (Remark 4.7.1.5); let α denote its order type. We then have $|S| \leq \alpha \leq |T|$, where the second inequality follows from Remark 4.7.1.21. □

`03R0` **Corollary 4.7.2.4.** *Let S and T be sets. Then S and T have the same cardinality if and only if there exists a bijection $S \xrightarrow{\sim} T$.*

Proof. Choose well-orderings $(S, \leq_S)$ and $(T, \leq_T)$ having order types $|S|$ and $|T|$, respectively. If $|S| = |T|$, then there is an isomorphism of linearly ordered sets $(S, \leq_S) \simeq (T, \leq_T)$, and therefore a bijection $S \xrightarrow{\sim} T$. The converse follows from Proposition 4.7.2.3. □

`03R1` **Corollary 4.7.2.5.** *Let $(S, \leq)$ be a well-ordered set of order type α. Then the cardinality $\kappa = |S|$ is the largest cardinal which satisfies $\kappa \leq \alpha$.*

Proof. The inequality $\kappa \leq \alpha$ follows immediately from the definition of $|S|$. Let λ be another cardinal satisfying $\lambda \leq \alpha$. Then λ is the order type of an initial segment $S_0 \subseteq S$, so we have $\lambda = |S_0| \leq |S| = \kappa$. □

`03R2` **Remark 4.7.2.6.** Let κ be an ordinal. The following conditions are equivalent:

(1) The ordinal κ is a cardinal. That is, there exists a set S such that $\kappa = |S|$.

(2) For every well-ordered set $(S, \leq)$ of order type κ, we have $\kappa = |S|$.

(3) The set of ordinals $\mathrm{Ord}_{<\kappa}$ has cardinality κ.

See Corollary 4.7.1.23.

Example 4.7.2.7 (Finite Cardinals)**.** Let n be a nonnegative integer. Then a set S has cardinality n (in the sense of Definition 4.7.2.1) if and only if it has exactly n elements: that is, there exists a bijection $S \simeq \{0 < 1 < \cdots < n-1\}$. In particular, n is a cardinal. We will say that a cardinal κ is *finite* if it arises in this way (that is, if it is the cardinality of a finite set); otherwise, we say that κ is *infinite*. 03R3

Proposition 4.7.2.8 (Cantor's Diagonal Argument)**.** *Let S be a set, and let $P(S)$ denote the collection of all subsets of S. Then $|S| < |P(S)|$.* 03R4

Proof. The construction $s \mapsto \{s\}$ determines an injection from S to $P(S)$, which shows that $|S| \leq |P(S)|$. To show that the inequality is strict, it suffices to observe that no function $f : S \to P(S)$ can be surjective, since the set $T = \{s \in S : s \notin f(s)\}$ is an element of $P(S)$ which does not belong to the image of f. □

Remark 4.7.2.9. The collection of cardinals is well-ordered. That is, if S is any nonempty collection of cardinals, then S contains a smallest element (see Corollary 4.7.1.24). 03R5

Example 4.7.2.10 (The First Infinite Cardinal)**.** We let $\aleph_0$ denote the smallest infinite cardinal. Alternatively, $\aleph_0$ can be defined as the ordinal ω of Example 4.7.1.10 (the order type of the linearly ordered set $\{0 < 1 < 2 < \cdots\}$). A set S has cardinality $\aleph_0$ if and only if it is countably infinite. 03R6

Example 4.7.2.11 (Successor Cardinals)**.** Let κ be a cardinal. Proposition 4.7.2.8 implies that there exists another cardinal λ such that $\kappa < \lambda$. By virtue of Remark 4.7.2.9, there is a smallest cardinal with this property. We denote this cardinal by κ^+ and refer to it as the *successor* of κ. 03R7

Example 4.7.2.12 (The First Uncountable Cardinal)**.** We say that a cardinal κ is *uncountable* if it is strictly larger than $\aleph_0$. By virtue of Remark 4.7.2.9, there is a smallest uncountable cardinal, which we denote by $\aleph_1$. In other words, $\aleph_1$ is the successor cardinal $\aleph_0^+$. 03R8

Remark 4.7.2.13 (The Continuum Hypothesis)**.** Let $\mathbf{R}$ be the set of real numbers. Then $|\mathbf{R}|$ is an uncountable cardinal (it is also the cardinality of the power set $P(\mathbf{Z})$). The *continuum hypothesis* is the assertion that $|\mathbf{R}|$ coincides with the smallest uncountable cardinal $\aleph_1$. This was a central question in the early days of set theory (and first of Hilbert's celebrated list of problems for the mathematics of the 20th century). It is now known to be neither provable nor disprovable from the axioms of Zermelo-Fraenkel set theory (assuming that they are consistent), thanks to the work of Gödel ([26]) and Cohen ([9], [10]). 03R9

`03RA` **Proposition 4.7.2.14.** *Let $(T, \leq)$ be a linearly ordered set and let $\kappa = \mathrm{cf}(T)$ be its cofinality (Definition 4.7.1.28). Then κ is a cardinal.*

Proof. Choose a well-ordered set $(S, \leq)$ of order type κ and a cofinal function $f : S \to T$. If κ is not a cardinal, then we can choose another well-ordering $\leq'$ of S having order type $\alpha < \kappa$. Applying Proposition 4.7.1.33, we obtain $\mathrm{cf}(T) \leq \alpha < \kappa$, which is a contradiction. □

4.7.3 Small Sets

`03RB` We now introduce some terminology which will be useful for measuring the sizes of various mathematical objects.

`03RC` **Definition 4.7.3.1.** Let κ be a cardinal. We say that a set S is *κ-small* if the cardinality $|S|$ is strictly smaller than κ.

`03RD` **Example 4.7.3.2.** Let $\aleph_0$ denote the first infinite cardinal (Example 4.7.2.10). Then a set S is $\aleph_0$-small if and only if it is finite.

`03RE` **Example 4.7.3.3.** Let $\aleph_1$ denote the first uncountable cardinal (Example 4.7.2.12). Then a set S is $\aleph_1$-small if and only if it is countable.

`03RF` **Remark 4.7.3.4.** Let κ be a cardinal and let T be a κ-small set. Then:

- Any subset of T is also κ-small (see Proposition 4.7.2.3).
- The set T is λ-small for every cardinal $\lambda \geq \kappa$.
- For every surjective morphism of sets $T \twoheadrightarrow S$, the set S is also κ-small.

`03RG` **Proposition 4.7.3.5.** *Let κ be an infinite cardinal. Then the collection of κ-small sets is closed under finite products.*

Proof. We first note that the collection of finite sets is closed under finite products. It will therefore suffice to show that, for every infinite cardinal λ, the following condition is satisfied:

$(*_\lambda)$ If S and T are sets of cardinality $\leq \lambda$, then the product $S \times T$ has cardinality $\leq \lambda$.

By virtue of Remark 4.7.2.9, we may assume that condition $(*_\mu)$ is satisfied for every cardinal $\mu < \lambda$. Without loss of generality, we may assume that $S = \mathrm{Ord}_{<\lambda} = T$, where $\mathrm{Ord}_{<\lambda}$ denotes the collection of ordinals smaller than λ. Given a pair of elements $(\alpha, \beta), (\alpha', \beta') \in S \times T$, let us write $(\alpha', \beta') \preceq (\alpha, \beta)$ if either $\max(\alpha', \beta') < \max(\alpha, \beta)$, or $\max(\alpha', \beta') = \max(\alpha, \beta)$ and $\alpha' < \alpha$, or $\max(\alpha', \beta') = \max(\alpha, \beta)$ and $\alpha' = \alpha$ and $\beta' \leq \beta$. The relation $\preceq$ defines a well-ordering of the set $S \times T$. To prove $(*_\lambda)$, it will suffice to show this well ordering has order type $\leq \lambda$. Assume otherwise. Then there exists an element $(\alpha, \beta) \in S \times T$ such that λ is the order type of the initial segment $K = \{(\alpha', \beta') \in S \times T : (\alpha', \beta') \prec (\alpha, \beta)\}$. Note that K is a subset of the product $\mathrm{Ord}_{\leq \gamma}$ and $\mathrm{Ord}_{\leq \gamma}$, where $\gamma = \max(\alpha, \beta)$. Our inductive hypothesis guarantees that K has cardinality $< \lambda$, contradicting Corollary 4.7.2.5. □

Corollary 4.7.3.6. *Let κ be an infinite cardinal. Then the collection of κ-small sets is closed under finite coproducts.* 03RH

Proof. Let $\{S_i\}_{i \in I}$ be a finite collection of κ-small sets. Then the disjoint union $\coprod_{i \in I} S_i$ can be identified with a subset of the product $\prod_{i \in I}(S_i \coprod \{i\})$, which is κ-small by virtue of Proposition 4.7.3.5. □

We will need the following generalization of Corollary 4.7.3.6:

Proposition 4.7.3.7. *Let κ and λ be cardinals, where λ is infinite. The following conditions are equivalent:* 03RJ

(1) *The cardinal κ is strictly smaller than the cofinality* $\mathrm{cf}(\lambda)$ *(see Definition 4.7.1.28).*

(2) *Let $\{T_s\}_{s \in S}$ be a collection of λ-small sets indexed by a set S of cardinality $\leq \kappa$. Then the coproduct $\coprod_{s \in S} T_s$ is λ-small.*

Proof. Assume first that condition (1) is satisfied. Let $\{T_s\}_{s \in S}$ be a collection of λ-small sets indexed by a set S of cardinality $\leq \kappa$; we wish to show that the coproduct $T = \coprod_{s \in S} T_s$ is λ-small. Using Theorem 4.7.1.34, we can choose a well-ordering $\leq_S$ on the set S, and a well-ordering $\leq_s$ on the set T_s for each $s \in S$. For elements $t \in T_s$ and $t' \in T_{s'}$, write $t \leq_T t'$ if either $s <_S s'$, or $s = s'$ and $t \leq_s t'$. Then $\leq_T$ is a well-ordering of the set T. If T is not λ-small, then it has an initial segment of order type λ. Passing to subsets, we may assume without loss of generality that T itself has order type λ. Moreover, we may assume without loss of generality that each of the sets T_s is nonempty, and therefore contains a smallest element t_s. We consider two cases:

- Suppose that S contains a largest element s. In this case, we can write T as the disjoint union of the initial segment $T' = \coprod_{s' < s} T_{s'}$ with the set T_s. Since T_s is nonempty, T' has order type smaller than λ, and is therefore λ-small. Applying Corollary 4.7.3.6, we deduce that $T = T' \coprod T_s$ is also λ-small.

- Suppose that S does not have a largest element. In this case, the construction $(s \in S) \mapsto (t_s \in T)$ is a cofinal function from S to T. It follows that the order type of $(S, \leq_S)$ is greater than or equal to the cofinality $\mathrm{cf}(T) = \mathrm{cf}(\lambda)$, contradicting assumption (1).

We now prove the reverse implication. Assume that condition (2) is satisfied. Choose a well-ordering $(S, \leq_S)$ of order type $\mathrm{cf}(\lambda)$ and a cofinal map $f : S \to \mathrm{Ord}_{<\lambda}$. If $\kappa \geq \mathrm{cf}(\lambda)$, then condition (2) implies that the disjoint union $\coprod_{s \in S} \mathrm{Ord}_{<f(s)}$ is λ-small. Since f is cofinal, the tautological map $\coprod_{s \in S} \mathrm{Ord}_{<f(s)} \to \mathrm{Ord}_{<\lambda}$ is surjective. It follows that $\mathrm{Ord}_{<\lambda}$ is λ-small, which is a contradiction. □

03RK **Corollary 4.7.3.8.** *Let λ be an infinite cardinal. Then $\kappa = \mathrm{cf}(\lambda)$ is the smallest cardinal for which there exists a set S of cardinality κ and a collection of λ-small sets $\{T_s\}_{s \in S}$, where the coproduct $\coprod_{s \in S} T_s$ is not λ-small.*

Proof. Proposition 4.7.2.14 guarantees that κ is a cardinal. The characterization is a restatement of Proposition 4.7.3.7. □

03RL **Corollary 4.7.3.9.** *Let λ be an infinite cardinal and let $\kappa = \mathrm{cf}(\lambda)$ be its cofinality. Suppose we are given a collection of λ-small sets $\{T_s\}_{s \in S}$. If the index set S is κ-small, then coproduct $\coprod_{s \in S} T_s$ is λ-small.*

03RM **Definition 4.7.3.10** (Regular Cardinals)**.** Let κ be a cardinal. We say that κ is *regular* if it is infinite and $\mathrm{cf}(\kappa) = \kappa$. Here $\mathrm{cf}(\kappa)$ denotes the cofinality of κ (Definition 4.7.1.28). We say that κ is *singular* if it is infinite but not regular.

03RN **Remark 4.7.3.11.** Let κ be an infinite cardinal. Then κ is regular if and only if the collection of κ-small sets is closed under κ-small coproducts (this is a special case of Corollary 4.7.3.8).

03RP **Example 4.7.3.12.** Let $\aleph_0$ denote the first infinite cardinal (Example 4.7.2.10). Then $\aleph_0$ is regular: that is, the collection of finite sets is closed under finite coproducts.

03RQ **Example 4.7.3.13** (Successor Cardinals)**.** Let κ be an infinite cardinal and let κ^+ be its successor (Example 4.7.2.11). Then a set S is κ^+-small if and only if it has cardinality $\leq \kappa$. It follows that κ^+ is a regular cardinal. That is, if $\{T_s\}_{s \in S}$ is a collection of sets of cardinality $\leq \kappa$ indexed by a set S of cardinality $\leq \kappa$, then the disjoint union $\coprod_{s \in S} T_s$ also has cardinality $\leq \kappa$. To prove this, choose a collection of monomorphisms $\{i_s : T_s \hookrightarrow T\}_{s \in S}$, where T is a set of cardinality κ. We then obtain a monomorphism

$$\coprod_{s \in S} T_s \hookrightarrow S \times T \qquad (x \in T_s) \mapsto (s, i_s(x)),$$

where the set $S \times T$ has cardinality $\leq \kappa$ by virtue of Proposition 4.7.3.5.

03RR **Example 4.7.3.14.** Let $\aleph_1$ denote the first uncountable cardinal (Example 4.7.2.12). Then $\aleph_1$ is regular: that is, the collection of countable sets is closed under the formation of countable disjoint unions. This is a special case of Example 4.7.3.13, since $\aleph_1 = \aleph_0^+$.

03RS **Example 4.7.3.15.** Let $(T, \leq)$ be a nonempty linearly ordered set with no largest element. Then the cofinality $\kappa = \mathrm{cf}(T)$ is a regular cardinal. To see this, choose a well-ordered set $(S, \leq)$ of order type κ and a cofinal function $f : S \to T$. Proposition 4.7.2.14 guarantees that κ is a cardinal, and Example 4.7.1.31 shows that κ is infinite. If it is not regular, then there exists a cofinal map $g : R \to S$, where $(R, \leq)$ is a well-ordered set of order type $\alpha < \kappa$. This contradicts the definition of $\kappa = \mathrm{cf}(T)$, since the composite map $(f \circ g) : R \to T$ is cofinal.

It will be convenient to introduce the following bit of nonstandard terminology:

Definition 4.7.3.16. Let λ be an infinite cardinal. We let $\mathrm{ecf}(\lambda)$ denote the least cardinal κ with the following property: there exists a set S of cardinality κ and a collection of λ-small sets $\{T_s\}_{s \in S}$ for which the product $\prod_{s \in S} T_s$ is not λ-small. We will refer to $\mathrm{ecf}(\lambda)$ as the *exponential cofinality of* λ. `03RT`

Remark 4.7.3.17. Let λ be an infinite cardinal. Then the exponential cofinality $\mathrm{ecf}(\lambda)$ satisfies $\aleph_0 \leq \mathrm{ecf}(\lambda) \leq \mathrm{cf}(\lambda)$. In particular, we have $\mathrm{ecf}(\lambda) \leq \lambda$. The inequality $\aleph_0 \leq \mathrm{ecf}(\lambda)$ is a reformulation of the fact that the collection of λ-small sets is closed under finite products (Proposition 4.7.3.5). To prove the other inequality, choose a set S of cardinality $\mathrm{cf}(\lambda)$ and a collection of λ-small sets $\{T_s\}_{s \in S}$ for which the coproduct $T = \coprod_{s \in S} T_s$ is not λ-small. We now observe that T can be identified with a subset of the product $\prod_{s \in S} (T_s \coprod \{s\})$. Since each of the sets $T_s \coprod \{s\}$ is also λ-small, we obtain $\mathrm{ecf}(\lambda) \leq \mathrm{cf}(\lambda)$. `03RU`

Remark 4.7.3.18. Let κ and λ be infinite cardinals. Then $\kappa \leq \mathrm{ecf}(\lambda)$ if and only if the following condition is satisfied: for every collection of λ-small sets $\{T_s\}_{s \in S}$ indexed by a κ-small set S, the product $\prod_{s \in S} T_s$ is also λ-small. `03RV`

Remark 4.7.3.19. Let κ be an infinite cardinal. Then there are arbitrarily large regular cardinals λ satisfying $\mathrm{ecf}(\lambda) > \kappa$. To see this, it will suffice (by enlarging κ) to show that there exists *some* regular cardinal λ of exponential cofinality $\geq \kappa$. Let S be a set of cardinality κ and let 2^{κ} denote the cardinality of the power set $P(S) = \{S_0 : S_0 \subseteq S\}$. Proposition 4.7.3.5 implies that the product $S \times S$ also has cardinality κ, so that $P(S \times S) \simeq \prod_{s \in S} P(S)$ also has cardinality 2^{κ}. It follows that the collection of sets of cardinality $\leq 2^{\kappa}$ is closed under the formation of products indexed by sets of cardinality $\leq \kappa$, so that $\lambda = (2^{\kappa})^{+}$ has exponential cofinality $> \kappa$. `03UQ`

Definition 4.7.3.20. Let κ be an infinite cardinal. We say that κ is *strongly inaccessible* if $\kappa = \mathrm{ecf}(\kappa)$. In other words, κ is strongly inaccessible if the collection of κ-small sets is closed under the formation of κ-small products. `03RW`

Example 4.7.3.21. Let $\aleph_0$ be the least infinite cardinal. Then $\aleph_0$ is strongly inaccessible. That is, the collection of finite sets is closed under finite products. `03RX`

Remark 4.7.3.22. Let κ be a strongly inaccessible cardinal. Then κ is regular: this follows immediately from the inequality $\mathrm{ecf}(\kappa) \leq \mathrm{cf}(\kappa)$ of Remark 4.7.3.17. `03RY`

Warning 4.7.3.23. The existence of uncountable strongly inaccessible cardinals cannot be proven from the axioms of Zermelo-Fraenkel set theory (assuming those axioms are consistent). `03RZ`

Proposition 4.7.3.24. *Let λ be an infinite cardinal and let $\kappa = \mathrm{ecf}(\lambda)$ be the exponential cofinality of λ. Then κ is a regular cardinal.* `03S0`

Proof. Suppose that κ is not regular: that is, there is a collection of κ-small sets $\{T_s\}_{s \in S}$ indexed by a κ-small set S such that $T = \coprod_{s \in S} T_s$ has cardinality $\geq \kappa$. Choose a collection of λ-small sets $\{U_t\}_{t \in T}$ for which the product $U = \prod_{t \in T} U_t$ is not λ-small. For each $s \in S$, let U_s denote the product $\prod_{t \in T_s} U_t$. Since T_s is $\operatorname{ecf}(\lambda)$-small, the set U_s is λ-small. Since S is also $\operatorname{ecf}(\lambda)$-small, it follows that $U \simeq \prod_{s \in S} U_s$ is also λ-small, which is a contradiction. □

4.7.4 Small Simplicial Sets

03S1 Definition 4.7.3.1 has a counterpart in the setting of simplicial sets.

03S2 **Definition 4.7.4.1.** Let κ be an infinite cardinal. We say that a simplicial set S is *κ-small* if the collection of nondegenerate simplices of S is κ-small.

03S3 **Remark 4.7.4.2.** In the situation of Definition 4.7.4.1, the dimension of the simplices under consideration is not fixed. That is, a simplicial set $S_\bullet$ is κ-small if and only if the disjoint union $\coprod_{m \geq 0} S_m^{\mathrm{nd}}$ is a κ-small set, where $S_m^{\mathrm{nd}} \subseteq S_m$ denotes the set of nondegenerate m-simplices of $S_\bullet$.

03S4 **Remark 4.7.4.3.** Let κ be an infinite cardinal. Then a simplicial set S is κ-small if and only if the opposite simplicial set S^{op} is κ-small.

03S5 **Example 4.7.4.4.** A simplicial set S is $\aleph_0$-small (in the sense of Definition 4.7.4.1) if and only if it is finite (Definition 3.6.1.1).

03S6 **Remark 4.7.4.5** (Coproducts)**.** Let κ be an infinite cardinal and let $\{S_i\}_{i \in I}$ be a collection of κ-small simplicial sets. Suppose that the cardinality of the index set I is smaller than the cofinality $\operatorname{cf}(\kappa)$. Then the coproduct $\coprod_{i \in I} S_i$ is also κ-small (see Corollary 4.7.3.9). In particular:

- The collection of κ-small simplicial sets is closed under finite coproducts.
- If κ is regular, then the collection of κ-small simplicial sets is closed under κ-small coproducts.

03S7 **Remark 4.7.4.6** (Colimits)**.** Let κ be an infinite cardinal and let $\{S_i\}_{i \in \mathcal{I}}$ be a diagram of κ-simplicial sets indexed by a category $\mathcal{I}$. Suppose that the set of objects $\operatorname{Ob}(\mathcal{I})$ has cardinality smaller than the cofinality of κ. Then the colimit $\varinjlim_{i \in \mathcal{I}} S_i$ is also κ-small (since it can be realized as a quotient of the coproduct $\coprod S_i$, which is κ-small by virtue of Remark 4.7.4.5).

03S8 **Remark 4.7.4.7.** Let S be a simplicial set. Then there is a least infinite cardinal κ for which S is κ-small. If S is finite, then $\kappa = \aleph_0$. If S is not finite, then $\kappa = \lambda^+$, where λ is the cardinality of the set of all nondegenerate simplices of S. In particular, κ is always a regular cardinal.

Remark 4.7.4.8. Let κ be an infinite cardinal and let T be a κ-small simplicial set. Then: 03S9

- Every simplicial subset of T is κ-small.
- The simplicial set T is λ-small for each $\lambda \geq \kappa$.
- For every epimorphism of simplicial sets $T \twoheadrightarrow S$, the simplicial set S is also κ-small.

See Remark 4.7.3.4.

Proposition 4.7.4.9. *Let κ be an infinite cardinal and $S_\bullet$ be a simplicial set. Assume that the cofinality of κ is larger than $\aleph_0$ (this condition is satisfied, for example, if κ is uncountable and regular). The following conditions are equivalent:* 03SA

(1) *The simplicial set $S_\bullet$ is κ-small.*

(2) *For every integer $n \geq 0$, the set S_n is κ-small.*

(3) *For every finite simplicial set K, the set $\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, S_\bullet)$ is κ-small.*

Proof. We first show that (1) implies (2). Assume that $S_\bullet$ is κ-small and let $n \geq 0$ be an integer. For each integer $m \geq 0$, let S_m^{nd} denote the set of nondegenerate m-simplices of $S_\bullet$. Using Proposition 1.1.3.8, we can identify S_n with the coproduct $\coprod_{\alpha:[n]\twoheadrightarrow[m]} S_m^{\mathrm{nd}}$, where α ranges over all surjective maps of linearly ordered sets $[n] \twoheadrightarrow [m]$. Our assumption that $S_\bullet$ is κ-small guarantees that each of the sets S_m^{nd} is κ-small, so that S_n is also κ-small (Corollary 4.7.3.6).

We now show that (2) implies (1). Assume that, for each $n \geq 0$, the set S_n is κ-small. Since κ has cofinality $> \aleph_0$ it follows that the coproduct $\coprod_{n \geq 0} S_n$ is also κ-small. In particular, the coproduct $\coprod_{n\geq 0} S_n^{\mathrm{nd}}$ is κ-small: that is, the simplicial set $S_\bullet$ is κ-small.

The implication (3) ⇒ (2) is immediate from the definition. We will complete the proof by showing that (2) ⇒ (3). Assume that, for each $n \geq 0$, the set S_n is κ-small, and let K be a finite simplicial set. By virtue of Proposition 3.6.1.7, there exists an epimorphism $f: K' \twoheadrightarrow K$, where $K' = \coprod_{i \in I} \Delta^{n_i}$ is a disjoint union of finitely many standard simplices. Then precomposition with f induces a monomorphism

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, S_\bullet) \hookrightarrow \mathrm{Hom}_{\mathrm{Set}_\Delta}(K', S_\bullet) \simeq \prod_{i \in I} S_{n_i}.$$

Since the collection of κ-small sets is closed under finite products and passage to subsets (Proposition 4.7.3.5 and Remark 4.7.3.4), it follows that the set $\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, S_\bullet)$ is also κ-small. □

Warning 4.7.4.10. The implications (1) ⇒ (2) ⇔ (3) of Proposition 4.7.4.9 are valid for an arbitrary infinite cardinal κ. However, the implication (2) ⇒ (1) is false if κ has countable cofinality (for example, if $\kappa = \aleph_0$). 03SB

03SC **Corollary 4.7.4.11.** *Let κ be an infinite cardinal. Then the collection of κ-small simplicial sets is closed under finite products.*

Proof. Let $\{S_i\}_{i \in I}$ be a collection of κ-small simplicial sets indexed by a finite set I; we wish to show that the product $S = \prod_{i \in I} S_i$ is κ-small. Without loss of generality, we may assume that κ is the least infinite cardinal for which each of the simplicial sets S_i is κ-small. Then κ is regular (Remark 4.7.4.7). If $\kappa = \aleph_0$, then the desired result follows from Remark 3.6.1.6. We may therefore assume that κ is uncountable. In this case, the desired result follows from the criterion of Proposition 4.7.4.9, since the collection of κ-small sets is closed under finite products (Proposition 4.7.3.5). □

03SD **Corollary 4.7.4.12.** *Let κ be an uncountable cardinal, let S be a κ-small simplicial set, and let K be a finite simplicial set. Then the simplicial set* $\mathrm{Fun}(K, S)$ *is κ-small.*

Proof. Without loss of generality, we may assume that κ is the least uncountable cardinal for which S is κ-small. In particular, κ is regular (Remark 4.7.4.7). By virtue of Proposition 4.7.4.9, it will suffice to show that for every finite simplicial set L, the set $\mathrm{Hom}_{\mathrm{Set}_\Delta}(L, \mathrm{Fun}(K, S)) \simeq \mathrm{Hom}_{\mathrm{Set}_\Delta}(K \times L, S)$ is κ-small. This is a special case of Proposition 4.7.4.9, since the simplicial set $K \times L$ is finite (Remark 3.6.1.6). □

03SE **Warning 4.7.4.13.** The assertion of Corollary 4.7.4.12 is false in the case $\kappa = \aleph_0$. That is, if K and S are finite simplicial sets, then the simplicial set $\mathrm{Fun}(K, S)$ need not be finite.

We close by recording stronger forms of Corollaries 4.7.4.11 abnd 4.7.4.12.

03SF **Corollary 4.7.4.14.** *Let λ be an infinite cardinal and let $\kappa = \mathrm{ecf}(\lambda)$ be its exponential cofinality (Definition 4.7.3.16). Then the collection of λ-small simplicial sets is closed under κ-small products.*

Proof. Let $\{S_i\}_{i \in I}$ be a collection of λ-small simplicial sets indexed by a κ-small set I; we wish to show that the product $S = \prod_{i \in I} S_i$ is λ-small. If $\kappa = \aleph_0$, this follows from Corollary 4.7.4.11. We may therefore assume that κ is uncountable. Then the cofinality $\mathrm{cf}(\lambda)$ is also uncountable (Remark 4.7.3.17). The desired result now follows from the criterion of Proposition 4.7.4.9, since the collection of λ-small sets is closed under κ-small products. □

03SG **Corollary 4.7.4.15.** *Let λ be an uncountable cardinal and let $\kappa = \mathrm{ecf}(\lambda)$ be its exponential cofinality. If S is a λ-small simplicial set and K be a κ-small simplicial set. Then* $\mathrm{Fun}(K, S)$ *is λ-small.*

Proof. Since K is κ-small, we can choose an epimorphism of simplicial sets $\coprod_{i \in I} \Delta^{n_i} \twoheadrightarrow K$, where I is a κ-small set. It follows that $\mathrm{Fun}(K, S)$ can be identified with a simplicial subset of the product $\prod_{i \in I} \mathrm{Fun}(\Delta^{n_i}, S)$. Corollary 4.7.4.12 guarantees that each factor $\mathrm{Fun}(\Delta^{n_i}, S)$ is λ-small, so that the product $\prod_{i \in I} \mathrm{Fun}(\Delta^{n_i}, S)$ is λ-small by virtue of Corollary 4.7.4.14. □

4.7.5 Essential Smallness

Let κ be an infinite cardinal. Beware that the condition that a simplicial set is κ-small is not invariant under categorical equivalence. For this reason, it is useful to consider the following variant of Definition 4.7.4.1: 03SH

Definition 4.7.5.1. Let κ be an uncountable cardinal. We will say that a simplicial set $\mathcal{C}$ is *essentially κ-small* if there exists a categorical equivalence of simplicial sets $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a κ-small ∞-category. 03SJ

Remark 4.7.5.2. Let κ be an uncountable cardinal, and let $F : \mathcal{C} \to \mathcal{D}$ be a categorical equivalence of simplicial sets. Then $\mathcal{C}$ is essentially κ-small if and only if $\mathcal{D}$ is essentially κ-small. 03SK

Remark 4.7.5.3. Let κ be an uncountable cardinal. Then a simplicial set $\mathcal{C}$ is essentially κ-small if and only if the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is essentially κ-small. See Remark 4.7.4.3. 03SL

Variant 4.7.5.4. Let $\mathcal{C}$ be a simplicial set. We say that $\mathcal{C}$ is *essentially small* if there exists a categorical equivalence $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a small ∞-category. 03SM

Proposition 4.7.5.5. *Let κ be an uncountable cardinal and let $\mathcal{C}$ be a κ-small simplicial set. Then there exists an inner anodyne morphism $\mathcal{C} \hookrightarrow \mathcal{D}$, where $\mathcal{D}$ is a κ-small ∞-category. In particular, $\mathcal{C}$ is essentially κ-small.* 03SN

Proof. Without loss of generality, we may assume that κ is the least uncountable cardinal for which $\mathcal{C}$ is κ-small, so that κ is regular (Remark 4.7.4.7). We proceed as in the proof of Proposition 4.1.3.2. We will construct $\mathcal{D}$ as the colimit of a diagram of inner anodyne morphisms

$$\mathcal{C} = \mathcal{C}(0) \hookrightarrow \mathcal{C}(1) \hookrightarrow \mathcal{C}(2) \hookrightarrow \mathcal{C}(3) \hookrightarrow \cdots$$

where each transition map fits into a pushout diagram

$$\begin{tikzcd}
\coprod_{s \in S(n)} \Lambda^{n_s}_{i_s} \arrow[r, "\{u_s\}_{s \in S(n)}"] \arrow[d] & \mathcal{C}(n) \arrow[d] \\
\coprod_{s \in S(n)} \Delta^{n_s} \arrow[r] & \mathcal{C}(n+1);
\end{tikzcd}$$

here the coproducts are indexed by the collection $\{u_s : \Lambda^{n_s}_{i_s} \to \mathcal{C}(n)\}_{s \in S(n)}$ of all inner horns in the simplicial set $\mathcal{C}(n)$. Note that if the simplicial set $\mathcal{C}(n)$ is κ-small, then the set $S(n)$ is also κ-small (Proposition 4.7.4.9), so that $\mathcal{C}(n+1)$ is also κ-small. Since κ is regular and uncountable, it follows that the colimit $\mathcal{C} = \varinjlim \mathcal{C}(n)$ is κ-small (Remark 4.7.4.6). $\square$

03SP **Warning 4.7.5.6.** The statement of Proposition 4.7.5.5 is false in the case $\kappa = \aleph_0$. If S is a finite simplicial set, we generally cannot choose a categorical equivalence $f : S \to \mathcal{D}$, where $\mathcal{D}$ is an ∞-category which is also a finite simplicial set. For example, take $S = \Delta^2 / \partial \Delta^2$, so that the geometric realization $|S|$ is homeomorphic to a sphere of dimension 2. Since every edge of S is degenerate, the homotopy category $\mathrm{h}S$ is a groupoid. Consequently, if f is a categorical equivalence from S to an ∞-category $\mathcal{D}$, then $\mathcal{D}$ is a Kan complex (Proposition 4.4.2.1), which is homotopy equivalent to the singular simplicial set $\mathrm{Sing}_\bullet(|S|)$ (Theorem 3.6.4.1). It follows that $\pi_2(\mathcal{D})$ is an infinite cyclic group (generated by the homotopy class $[f]$), so that the Kan complex $\mathcal{D}$ must contain infinitely many 2-simplices.

03SQ **Remark 4.7.5.7** (Coproducts)**.** Let κ be an uncountable cardinal and let $\{\mathcal{C}_i\}_{i \in I}$ be a collection of essentially κ-small simplicial sets. Suppose that the cardinality of the index set I is smaller than the cofinality $\mathrm{cf}(\kappa)$. Then the coproduct $\coprod_{i \in I} \mathcal{C}_i$ is also essentially κ-small. This follows by combining Remark 4.7.4.5 with Corollary 4.5.3.10. In particular:

- The collection of essentially κ-small simplicial sets is closed under finite coproducts.
- If κ is regular, then the collection of essentially κ-small simplicial sets is closed under κ-small coproducts.

03SR **Remark 4.7.5.8** (Products)**.** Let κ be an uncountable cardinal and let $\{\mathcal{C}_i\}_{i \in I}$ be a finite collection of simplicial sets which are essentially κ-small. Then the product $\prod_{i \in I} \mathcal{C}_i$ is essentially κ-small. This follows by combining Corollary 4.7.4.14, since the collection of categorical equivalences is stable under the formation of finite products (Remark 4.5.3.7).

03SS **Variant 4.7.5.9.** Let κ be an uncountable cardinal and let $\{\mathcal{C}_i\}_{i \in I}$ be a collection of essentially κ-small ∞-categories. Suppose that the cardinality of the index set I has smaller than the exponential cofinality $\mathrm{ecf}(\kappa)$. Then the product $\prod_{i \in I} \mathcal{C}_i$ is also essentially κ-small. This follows by combining Corollary 4.7.4.14 with Remark 4.5.1.17.

03ST **Remark 4.7.5.10.** Let λ be an uncountable cardinal, let $\mathcal{C}$ be an ∞-category which is essentially λ-small, and let K be a simplicial set. Suppose that K is κ-small, where $\kappa = \mathrm{ecf}(\lambda)$ is the exponential cofinality of λ. Then the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ is essentially λ-small. To prove this, we can use Remark 4.5.1.16 to reduce to the case where $\mathcal{C}$ is λ-small, in which case it follows from Corollary 4.7.4.12. Moreover, if κ is uncountable, then it suffices to assume that K is essentially κ-small.

03SU **Proposition 4.7.5.11.** *Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is essentially κ-small. Then any replete subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is also essentially κ-small.*

Proof. Choose an equivalence of ∞-categories $F : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is κ-small. Then the inverse image $\mathcal{D}_0 = F^{-1}(\mathcal{C}_0)$ is κ-small (Remark 4.7.4.8), and the functor F restricts to an equivalence of ∞-categories $\mathcal{D}_0 \to \mathcal{C}_0$ (Corollary 4.5.2.29). $\square$

Corollary 4.7.5.12. *Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is essentially κ-small. Then the core $\mathcal{C}^{\simeq}$ is an essentially κ-small Kan complex.* 03SV

Proof. Since $\mathcal{C}^{\simeq}$ is a replete subcategory of $\mathcal{C}$ (Proposition 4.4.3.6), this is a special case of Proposition 4.7.5.11. □

Corollary 4.7.5.13. *Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is essentially κ-small. Then any full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is essentially κ-small.* 03SW

Proof. Let $\mathcal{C}_1 \subseteq \mathcal{C}$ be the full subcategory spanned by those objects $X \in \mathcal{C}$ which are isomorphic to an object of $\mathcal{C}_0$. Proposition 4.7.5.11 guarantees that $\mathcal{C}_1$ is essentially κ-small. Since the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}_1$ is an equivalence of ∞-categories, it follows that $\mathcal{C}_0$ is also essentially κ-small (Remark 4.7.5.2). □

Proposition 4.7.5.14. *Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories and let κ be an uncountable cardinal. If $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}$ are essentially κ-small, then the oriented fiber product $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ is also essentially κ-small.* 03SX

Proof. Choose equivalences of ∞-categories

$$\mathcal{D}_0 \to \mathcal{C}_0 \qquad \mathcal{C} \to \mathcal{D} \qquad \mathcal{D}_1 \to \mathcal{C}_1,$$

where $\mathcal{D}_0$, $\mathcal{D}_1$, and $\mathcal{D}$ are κ-small. By virtue of Remark 4.6.4.4, the induced maps

$$\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1 \leftarrow \mathcal{D}_0 \tilde{\times}_{\mathcal{C}} \mathcal{D}_1 \to \mathcal{D}_0 \tilde{\times}_{\mathcal{D}} \mathcal{D}_1$$

are equivalences of ∞-categories. It will therefore suffice to show that the ∞-category $\mathcal{D}_0 \tilde{\times}_{\mathcal{D}} \mathcal{D}_1$ is κ-small. This follows from Corollaries 4.7.4.12 and 4.7.4.11, since $\mathcal{D}_0 \tilde{\times}_{\mathcal{D}} \mathcal{D}_1$ can be identified with a simplicial subset of the product $\mathcal{D}_0 \times \operatorname{Fun}(\Delta^1, \mathcal{D}) \times \mathcal{D}_1$. □

Corollary 4.7.5.15. *Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories and let κ be an uncountable cardinal. If $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}$ are essentially κ-small, then the homotopy fiber product $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is essentially κ-small.* 03SY

Proof. Since $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is a full subcategory of the oriented fiber product $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$, this follows from Proposition 4.7.5.14 and Corollary 4.7.5.13. □

Corollary 4.7.5.16. *Let κ be an uncountable cardinal and suppose we are given a categorical pullback diagram of ∞-categories* 03SZ

03T0

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C} .
\end{tikzcd} \tag{4.45}$$

If $\mathcal{C}_0$, $\mathcal{C}$, and $\mathcal{C}_1$ are essentially κ-small, then $\mathcal{C}_{01}$ is essentially κ-small.

Proof. Combine Remark 4.7.5.2 with Corollary 4.7.5.15. □

04J8 **Corollary 4.7.5.17.** *Let λ be an uncountable cardinal, let $\mathcal{C}$ be an ∞-category which is essentially λ-small, and let K be a simplicial set. Suppose that K is κ-small, where $\kappa = \mathrm{ecf}(\lambda)$ is the exponential cofinality of λ. Then, for any diagram $f : K \to \mathcal{C}$, the ∞-categories $\mathcal{C}_{f/}$ and $\mathcal{C}_{/f}$ are essentially λ-small. Moreover, if κ is uncountable, then it suffices to assume that K is essentially κ-small.*

Proof. We will show that the ∞-category $\mathcal{C}_{/f}$ is essentially λ-small; the corresponding assertion for $\mathcal{C}_{f/}$ follows by a similar argument. Theorem 4.6.4.17 supplies an equivalence of ∞-categories $\mathcal{C}_{/f} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{f\}$. By virtue of Proposition 4.7.5.14, it will suffice to show that $\mathrm{Fun}(K, \mathcal{C})$ is essentially λ-small, which follows from Remark 4.7.5.10. □

04J9 **Example 4.7.5.18.** Let λ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is essentially λ-small. Then, for every object $X \in \mathcal{C}$, the ∞-categories $\mathcal{C}_{/X}$ and $\mathcal{C}_{X/}$ are essentially λ-small.

4.7.6 Minimal ∞-Categories

03T1 Let κ be an uncountable cardinal. An ∞-category $\mathcal{D}$ is essentially κ-small if and only if there exists an equivalence $\mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ is a κ-small ∞-category. Our goal in this section is to show that, if this condition is satisfied, then there is a preferred choice for the ∞-category $\mathcal{C}$ which is characterized (up to noncanonical isomorphism) by the requirement that it is *minimal*. We will need some terminology.

0575 **Definition 4.7.6.1.** Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, and let $A \subseteq B$ be a simplicial subset. Suppose we are given a pair of diagrams $f_0, f_1 : B \to \mathcal{C}$. An *isomorphism of f_0 with f_1 relative to A* is an isomorphism $u : f_0 \to f_1$ in the ∞-category $\mathrm{Fun}(B, \mathcal{C})$ for which the image of u in $\mathrm{Fun}(A, \mathcal{C})$ is an identity morphism. We say that f_0 *is isomorphic to f_1 relative to A* if there exists an isomorphism of f_0 with f_1 relative to A.

0576 **Remark 4.7.6.2.** In the situation of Definition 4.7.6.1, two diagrams $f_0, f_1 : B \to \mathcal{C}$ are isomorphic relative to A if and only if they satisfy the following pair of conditions:

- The diagrams f_0 and f_1 have the same restriction to A: that is, we have $f_0|_A = \overline{f} = f_1|_A$ for some diagram $\overline{f} : A \to \mathcal{C}$.

- The diagrams f_0 and f_1 are isomorphic when viewed as objects of the ∞-category $\mathrm{Fun}(B, \mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{C})} \{\overline{f}\}$.

Remark 4.7.6.3. Let $\mathcal{C}$ be an ∞-category, let $f_0, f_1 : B \to \mathcal{C}$ be a pair of diagrams, and let $A \subseteq B$ be a simplicial subset. If f_0 and f_1 are isomorphic relative to A (in the sense of Definition 4.7.6.1), then they are homotopic relative to A (in the sense of Definition 3.2.1.1). The converse holds if the restriction functor $\operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(A, \mathcal{C})$ is conservative. In particular, the converse holds if $\mathcal{C}$ is a Kan complex, or if A contains every vertex of B. 0577

Definition 4.7.6.4. Let $\mathcal{C}$ be an ∞-category and let $n \geq 0$ be an integer. We say that $\mathcal{C}$ is *minimal in dimension* n if it satisfies the following condition: 03T2

$(*_n)$ Let $\sigma_0, \sigma_1 : \Delta^n \to \mathcal{C}$ be n-simplices of $\mathcal{C}$. If σ_0 is isomorphic to σ_1 relative to $\partial \Delta^n$, then $\sigma_0 = \sigma_1$.

We say that $\mathcal{C}$ is *minimal* if it is minimal in dimension n for every integer n.

Example 4.7.6.5. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is minimal in dimension 0 if and only if, for every pair of isomorphic objects $X, Y \in \mathcal{C}$, we have $X = Y$. 03T3

Example 4.7.6.6. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is minimal in dimension 1 if and only if, for every pair of objects $X, Y \in \mathcal{C}$ and every pair of morphisms $f, g : X \to Y$ which are homotopic, we have $f = g$ (see Corollary 1.4.3.7). 03T4

Exercise 4.7.6.7. Let $\mathcal{C}$ be a category. Show that the nerve $\operatorname{N}_\bullet(\mathcal{C})$ is minimal in dimension n for every integer $n > 0$ (see Proposition 4.8.3.1 for a more general statement). Consequently, the ∞-category $\operatorname{N}_\bullet(\mathcal{C})$ is minimal if and only if, for every pair of isomorphic objects $X, Y \in \mathcal{C}$, we have $X = Y$. 03T5

Remark 4.7.6.8. Let $\mathcal{C}$ be a minimal ∞-category, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a simplicial subset. If $\mathcal{C}_0$ is an ∞-category, then it is also minimal. 03T6

Remark 4.7.6.9. Let $\{\mathcal{C}_i\}_{i \in I}$ be a collection of minimal ∞-categories. Then the product $\prod_{i \in I} \mathcal{C}_i$ and the coproduct $\coprod_{i \in I} \mathcal{C}_i$ are also minimal ∞-categories. 03T7

Warning 4.7.6.10. The collection of minimal ∞-categories has poor closure properties: 03T8

- If $\mathcal{C}$ is a minimal ∞-category and K is a simplicial set, then the ∞-category $\operatorname{Fun}(K, \mathcal{C})$ need not be minimal (even in the case $K = \Delta^1$).
- If $\mathcal{C}$ is a minimal ∞-category and $q : K \to \mathcal{C}$ is a diagram, then the ∞-categories $\mathcal{C}_{/q}$ and $\mathcal{C}_{q/}$ need not be minimal (even in the case $K = \Delta^0$).
- If $\mathcal{C}$ is a minimal ∞-category and $\mathcal{D}$ is equivalent to $\mathcal{C}$, then $\mathcal{D}$ need not be minimal.

The goal of this section is to show that every ∞-category $\mathcal{D}$ admits a *minimal model*: that is, a minimal ∞-category $\mathcal{C}$ equipped with an equivalence $F : \mathcal{C} \to \mathcal{D}$. Moreover, the ∞-category $\mathcal{C}$ is uniquely determined up to isomorphism (Corollary 4.7.6.16). Our first step is to show that, in this case, the functor F is automatically a monomorphism.

03T9 **Lemma 4.7.6.11.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories. If $\mathcal{C}$ is minimal, then F is a monomorphism of simplicial sets.*

Proof. Let $\sigma, \sigma' : \Delta^n \to \mathcal{C}$ be n-simplices of $\mathcal{C}$ satisfying $F(\sigma) = F(\sigma')$; we wish to show that $\sigma = \sigma'$. Our proof proceeds by induction on n. Set $\tau = F(\sigma) = F(\sigma')$ and $\sigma_0 = \sigma|_{\partial \Delta^n}$, so that our inductive hypothesis guarantees that $\sigma_0 = \sigma'|_{\partial \Delta^n}$.

Fix a functor $G : \mathcal{D} \to \mathcal{C}$ which is homotopy inverse to F, so that there exists a 2-simplex

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \arrow[rr, "\mathrm{id}"] \arrow[dr, "\alpha"'] & & \mathrm{id}_{\mathcal{C}} \\
& G \circ F \arrow[ur, "\beta"'] &
\end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, where α and β are (mutually inverse) isomorphisms. Precomposing with the morphism $\sigma_0 : \partial \Delta^n \to \mathcal{C}$, we obtain a 2-simplex

03TA
$$\begin{tikzcd}
\sigma_0 \arrow[rr, "\mathrm{id}"] \arrow[dr, "\alpha(\sigma_0)"'] & & \sigma_0 \\
& (G \circ F)(\sigma_0) \arrow[ur, "\beta(\sigma_0)"'] &
\end{tikzcd} \tag{4.46}$$

in the ∞-category $\mathrm{Fun}(\partial \Delta^n, \mathcal{C})$. Since $\mathcal{C}$ is an ∞-category, Theorem 1.5.6.1 guarantees that we can lift (4.46) to a 2-simplex

$$\begin{tikzcd}
\sigma \arrow[rr, "\gamma"] \arrow[dr, "\alpha(\sigma)"'] & & \sigma' \\
& G(\tau). \arrow[ur, "\beta(\sigma')"'] &
\end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\Delta^n, \mathcal{C})$. By construction, γ is an isomorphism relative to $\partial \Delta^n$. Invoking our assumption that $\mathcal{C}$ is minimal, we deduce that $\sigma = \sigma'$. □

03TB **Corollary 4.7.6.12.** *Let $\mathcal{C}$ be a minimal ∞-category and let κ be an uncountable cardinal. Then $\mathcal{C}$ is essentially κ-small if and only if it is κ-small.*

Proof. Suppose that $\mathcal{C}$ is essentially κ-small. Then there exists an equivalence of ∞-categories $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is κ-small. Since $\mathcal{C}$ is minimal, the functor F is a monomorphism of simplicial sets (Lemma 4.7.6.11), so that $\mathcal{C}$ is also κ-small (Remark 4.7.4.8). □

Proposition 4.7.6.13 (Uniqueness)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories. If $\mathcal{C}$ and $\mathcal{D}$ are minimal, then F is an isomorphism of simplicial sets.* 03TC

Proof. Let $G : \mathcal{D} \to \mathcal{C}$ be a homotopy inverse to F. It follows from Lemma 4.7.6.11 that F and G are monomorphisms of simplicial sets. We will complete the proof by showing that the composite map $(F \circ G) : \mathcal{D} \to \mathcal{D}$ is an epimorphism of simplicial sets (so that, in particular, F is an epimorphism). Let σ be an n-simplex of $\mathcal{D}$; we wish to show that σ belongs to the image of $F \circ G$. The proof proceeds by induction on n. Set $\sigma_0 = \sigma|_{\partial \Delta^n}$; our inductive hypothesis then guarantees that we can write $\sigma_0 = (F \circ G)(\tau_0)$ for some morphism $\tau_0 : \partial \Delta^n \to \mathcal{D}$.

Choose a 2-simplex

$$\begin{tikzcd}
F \circ G \arrow[rr, "\mathrm{id}_{F \circ G}"] \arrow[dr, "\alpha"'] & & F \circ G \\
& \mathrm{id}_{\mathcal{D}} \arrow[ur, "\beta"'] &
\end{tikzcd}$$

in the ∞-category $\operatorname{Fun}(\mathcal{D}, \mathcal{D})$, where α and β are isomorphisms. Precomposing with $\tau_0 : \partial \Delta^n \to \mathcal{D}$, we obtain a 2-simplex

03TD

$$\begin{tikzcd}
\sigma_0 \arrow[rr, "\mathrm{id}"] \arrow[dr, "\alpha(\tau_0)"'] & & \sigma_0 \\
& \tau_0 \arrow[ur, "\beta(\tau_0)"'] &
\end{tikzcd} \tag{4.47}$$

in the ∞-category $\operatorname{Fun}(\partial \Delta^n, \mathcal{D})$. Using Corollary 4.4.5.9, we can lift $\alpha(\tau_0)$ to an isomorphism $\widetilde{\alpha} : \sigma \to \tau$ in the ∞-category $\operatorname{Fun}(\Delta^n, \mathcal{D})$. Since $\mathcal{D}$ is an ∞-category, Theorem 1.5.6.1 guarantees that we can lift (4.47) to a 2-simplex

$$\begin{tikzcd}
\sigma \arrow[rr, "\gamma"] \arrow[dr, "\widetilde{\alpha}"'] & & (F \circ G)(\tau) \\
& \tau. \arrow[ur, "\beta(\tau)"'] &
\end{tikzcd}$$

in the ∞-category $\operatorname{Fun}(\Delta^n, \mathcal{D})$. By construction, γ is an isomorphism relative to $\partial \Delta^n$. Our assumption that $\mathcal{D}$ is minimal then guarantees that $\sigma = (F \circ G)(\tau)$ belongs to the image of $F \circ G$. □

03TE **Corollary 4.7.6.14.** *Let $\mathcal{C}$ and $\mathcal{D}$ be minimal ∞-categories. Then $\mathcal{C}$ and $\mathcal{D}$ are equivalent if and only if they are isomorphic.*

We now prove the existence of minimal models.

03TF **Proposition 4.7.6.15** (Existence)**.** *Let $\mathcal{D}$ be an ∞-category. Then there exists an equivalence of ∞-categories $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ is minimal.*

03TG **Corollary 4.7.6.16.** *The construction*

$$\{\text{Minimal } \infty\text{-Categories}\}/\text{Isomorphism} \to \{\infty\text{-Categories}\}/\text{Equivalence}$$

is a bijection.

Proof. Injectivity is a restatement of Corollary 4.7.6.14, and surjectivity follows from Proposition 4.7.6.15. □

03TH **Corollary 4.7.6.17.** *Let $\mathcal{C}$ be a simplicial set. Then there is a least uncountable cardinal κ for which $\mathcal{C}$ is essentially κ-small. Moreover, κ is always a successor cardinal.*

Proof. By virtue of Proposition 4.7.6.15, we may assume that $\mathcal{C}$ is a minimal ∞-category. In this case, the desired result follows by combining Corollary 4.7.6.12 with Remark 4.7.4.7. □

Proof of Proposition 4.7.6.15. Let $\mathcal{D}$ be an ∞-category. If σ and σ' are n-simplices of $\mathcal{D}$, we write $\sigma \sim \sigma'$ if they are isomorphic relative to $\partial\Delta^n$. Note that, if this condition is satisfied, then we must have $\sigma|_{\partial\Delta^n} = \sigma'|_{\partial\Delta^n}$. In particular, if σ and σ' are both degenerate, we must have $\sigma = \sigma'$. Let $R(n)$ denote a collection of n-simplices of $\mathcal{D}$ which contains all degenerate n-simplices, and contains exactly one element of every $\sim$-class. We let $\mathcal{C} \subseteq \mathcal{D}$ denote the simplicial subset consisting of all simplices $\tau : \Delta^m \to \mathcal{D}$ having the property that, for every morphism of linearly ordered sets $\alpha : [n] \to [m]$, the n-simplex $\Delta^n \to \Delta^m \xrightarrow{\tau} \mathcal{D}$ belongs to $R(n)$ (by construction, it suffices to check this in the case where α is injective). To complete the proof, it will suffice to establish the following:

(1) The simplicial set $\mathcal{C}$ is an ∞-category.

(2) The ∞-category $\mathcal{C}$ is minimal.

(3) The inclusion map $\mathcal{C} \hookrightarrow \mathcal{D}$ is an equivalence of ∞-categories.

We begin by proving (1). Suppose we are given integers $0 < i < n$ and a morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to \mathcal{C}$; we wish to show that σ_0 can be extended to an n-simplex σ of $\mathcal{C}$. Since $\mathcal{D}$ is an ∞-category, we can extend σ_0 to an n-simplex $\sigma'' : \Delta^n \to \mathcal{D}$. Let $\overline{\sigma}'' = d^n_i(\sigma'')$ denote the ith face of σ''. Then there is a unique element $\overline{\sigma}' \in R(n-1)$ satisfying $\overline{\sigma}' \sim \overline{\sigma}''$. Choose an isomorphism $\overline{\alpha} : \overline{\sigma}' \to \overline{\sigma}''$ in the ∞-category $\operatorname{Fun}(\Delta^{n-1}, \mathcal{D})$

whose image in $\mathrm{Fun}(\partial\Delta^{n-1}, \mathcal{D})$ is an identity morphism. Then $\overline{\alpha}$ can be lifted uniquely to an isomorphism $\widetilde{\alpha} : \widetilde{\sigma}' \to \sigma''|_{\partial\Delta^n}$ relative to the horn Λ^n_i. Applying Proposition 4.4.5.8, we can lift $\widetilde{\alpha}$ to an isomorphism $\alpha : \sigma' \to \sigma''$ in the ∞-category $\mathrm{Fun}(\Delta^n, \mathcal{D})$. By construction, the restriction $\sigma'|_{\partial\Delta^n}$ factors through $\mathcal{C}$. Let σ be the unique n-simplex of $\mathcal{D}$ which belongs to $R(n)$ and satisfies $\sigma \sim \sigma'$. Then σ is an n-simplex of $\mathcal{C}$ satisfying $\sigma|_{\Lambda^n_i} = \sigma'|_{\Lambda^n_i} = \sigma''|_{\Lambda^n_i} = \sigma_0$. This completes the proof of (1).

We now prove (2). Let σ and σ' be n-simplices of $\mathcal{C}$ which are isomorphic relative to $\partial\Delta^n$. Then, when regarded as n-simplices of $\mathcal{D}$, we have $\sigma \sim \sigma'$. Since σ and σ' both belong to $R(n)$, we conclude that $\sigma = \sigma'$.

To prove (3), we will show that $\mathcal{C}$ is a deformation retract of $\mathcal{D}$; that is, there exists a functor $H : \Delta^1 \times \mathcal{D} \to \mathcal{D}$ satisfying the following conditions:

(*i*) The restriction $H|_{\{0\}\times\mathcal{D}}$ is the identity functor $\mathrm{id}_{\mathcal{D}}$.

(*ii*) The restriction $H|_{\{1\}\times\mathcal{D}}$ factors through $\mathcal{C}$.

(*iii*) The restriction $H|_{\Delta^1\times\mathcal{C}}$ coincides with the projection map

$$\Delta^1 \times \mathcal{C} \to \mathcal{C} \subseteq \mathcal{D}.$$

(*iv*) For each object $D \in \mathcal{D}$, the restriction $H|_{\Delta^1\times\{D\}}$ is an isomorphism in $\mathcal{D}$.

Note that these conditions guarantee that the functor $H|_{\{1\}\times\mathcal{D}} : \mathcal{D} \to \mathcal{C}$ is a homotopy inverse to the inclusion map $\mathcal{C} \hookrightarrow \mathcal{D}$.

Let Q denote the set of pairs (S, H_S), where $S \subseteq \mathcal{D}$ is a simplicial subset which contains $\mathcal{C}$ and $H_S : \Delta^1 \times S \to \mathcal{D}$ is a morphism of simplicial sets which satisfies the analogues of conditions (*i*) through (*iv*). We regard Q as a partially ordered set, where $(S, H_S) \leq (S', H_{S'})$ if $S \subseteq S'$ and $H_S = H_{S'}|_{\Delta^1\times S}$. This partially ordered set satisfies the hypotheses of Zorn's lemma, and therefore contains a maximal element $(S_{\max}, H_{\max})$. To complete the proof, it will suffice to show that $S_{\max} = \mathcal{D}$. Assume otherwise. Then there is some n-simplex $\tau : \Delta^n \to \mathcal{D}$ which is not contained in $S_{\max}$. Choose n as small as possible, so that $\tau_0 = \tau|_{\partial\Delta^n}$ factors through $S_{\max}$. Then the composite map

$$\Delta^1 \times \partial\Delta^n \xrightarrow{\mathrm{id}\times\tau_0} \Delta^1 \times S_{\max} \xrightarrow{H_{\max}} \mathcal{D}$$

can be viewed as an isomorphism $\alpha_0 : \tau_0 \to \tau_0'$ in the ∞-category $\mathrm{Fun}(\partial\Delta^n, \mathcal{D})$, where τ_0' belongs to $\mathrm{Fun}(\partial\Delta^n, \mathcal{C})$. Using Proposition 4.4.5.8, we can lift α_0 to an isomorphism $\tau \to \tau'$ in the ∞-category $\mathrm{Fun}(\Delta^n, \mathcal{D})$. Let τ'' be the unique n-simplex of $\mathcal{D}$ which belongs to $R(n)$ and satisfies $\tau' \sim \tau''$. Then there exists an isomorphism $\beta : \tau' \to \tau''$ in the ∞-category $\mathrm{Fun}(\Delta^n, \mathcal{D})$ whose image in $\mathrm{Fun}(\partial\Delta^n, \mathcal{D})$ is an identity morphism. Using Theorem 1.5.6.1, we can lift the degenerate 2-simplex

$$\begin{tikzcd} & \tau_0' \arrow[dr, "\mathrm{id}"] & \\ \tau_0 \arrow[ur, "\alpha_0"] \arrow[rr, "\alpha_0"] & & \tau_0' \end{tikzcd}$$

of $\mathrm{Fun}(\partial\Delta^n, \mathcal{D})$ to a 2-simplex

$$\begin{tikzcd} & \tau' \arrow[dr, "\beta"] & \\ \tau \arrow[ur, "\alpha"] \arrow[rr, "\gamma"] & & \tau'' \end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\Delta^n, \mathcal{D})$. Let S denote the simplicial subset of $\mathcal{D}$ given by the union of $S_{\max}$ with the image of τ. Then $H_{\max}$ extends uniquely to a morphism $H_S : \Delta^1 \times S \to \mathcal{D}$ for which the composite map

$$\Delta^1 \times \Delta^n \xrightarrow{\mathrm{id} \times \tau} \Delta^1 \times S \xrightarrow{H_S} \mathcal{D}$$

coincides with γ. By construction, the pair (S, H_S) is an element of Q satisfying $(S, H_S) > (S_{\max}, H_{\max})$, contradicting the maximality of $(S_{\max}, H_{\max})$. □

4.7.7 Small Kan Complexes

03TJ In the setting of Kan complexes, essential κ-smallness can be tested at the level of homotopy groups.

03TK **Proposition 4.7.7.1.** *Let X be a Kan complex and let κ be an uncountable regular cardinal. Then X is essentially κ-small if and only if it satisfies the following pair of conditions:*

(1) *The set $\pi_0(X)$ is κ-small.*

(2) *For each vertex $x \in X$ and each integer $n > 0$, the homotopy group $\pi_n(X, x)$ is κ-small.*

Proof. By virtue of Proposition 4.7.6.15, we may assume without loss of generality that the Kan complex X is minimal. If X is essentially κ-small, then it is κ-small (Corollary 4.7.6.12), so that conditions (1) and (2) follow immediately from the definitions. Conversely, suppose that (1) and (2) are satisfied; we wish to show that X is κ-small. By virtue of Proposition 4.7.4.9, it will suffice to show that the collection of n-simplices of X is κ-small, for each $n \geq 0$. Our proof proceeds by induction on n. Using our inductive hypothesis (together with Remark 4.7.3.4 and Proposition 4.7.3.5), we see that the set $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^n, X)$ is κ-small. Since κ is regular, it will suffice to show that each fiber of the restriction map $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, X) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^n, X)$ is κ-small.

Set $E = \mathrm{Fun}(\Delta^n, X)$ and $B = \mathrm{Fun}(\partial \Delta^n, X)$, so that the inclusion map $\partial \Delta^n \hookrightarrow \Delta^n$ induces a Kan fibration $q : E \to B$ (Corollary 3.1.3.3) For each vertex $b \in B$, let E_b denote the fiber $\{b\} \times_B E$; we wish to show that the set of vertices of E_b is κ-small. Since the Kan complex X is minimal, each vertex of E_b belongs to a different connected component. It will therefore suffice to show that the set $\pi_0(E_b)$ is κ-small. If $n = 0$, this follows from condition (1). Let us therefore assume that $n > 0$, and identify b with a morphism of simplicial sets $\partial \Delta^n \to X$. If this morphism is not nullhomotopic, then the Kan complex E_b is empty and there is nothing to prove. We may therefore assume that there is a homotopy from b to a constant map $b' : \partial \Delta^n \to \{x\} \hookrightarrow X$. In this case, Proposition 5.2.2.19 supplies a homotopy equivalence of E_b with $E_{b'}$. We are therefore reduced to proving that the set $\pi_0(E_{b'}) \simeq \pi_n(X, x)$ is κ-small, which follows from condition (2). □

Corollary 4.7.7.2. *Let κ be an uncountable regular cardinal and let $f : X \to Y$ be a Kan fibration between Kan complexes, where Y is essentially κ-small. The following conditions are equivalent:* 03TL

(*a*) *The Kan complex X is essentially κ-small.*

(*b*) *For each vertex $y \in Y$, the fiber $X_y = \{y\} \times_Y X$ is essentially κ-small.*

Proof. The implication $(a) \Rightarrow (b)$ follows from Corollary 4.7.5.16 (and does not require the regularity of κ). Assume that condition (b) is satisfied; we will show that X satisfies the criteria of Proposition 4.7.7.1:

(1) Let y be a vertex of Y and let $[y]$ denote its image in $\pi_0(Y)$. Since f is a Kan fibration, the tautological map $\pi_0(X_y) \to \{[y]\} \times_{\pi_0(Y)} \pi_0(X)$ is a surjection. Assumption (b) guarantees that $\pi_0(X_y)$ is κ-small, so that the fiber $\{[y]\} \times_{\pi_0(Y)} \pi_0(X)$ is also κ-small. Since $\pi_0(Y)$ is κ-small, the regularity of κ guarantees that $\pi_0(X)$ is also κ-small.

(2) Fix a vertex $x \in X$ having image $y = f(x)$, and let $n > 0$ be a positive integer. For each integer $n > 0$, Proposition 3.2.6.2 supplies an exact sequence of groups

$$\pi_n(X_y, x) \to \pi_n(X, x) \xrightarrow{\pi_n(f)} \pi_n(Y, y).$$

Consequently, every nonempty fiber of the group homomorphism $\pi_n(f)$ carries a transitive action of the κ-small group $\pi_n(X_y, x)$, and is therefore κ-small. Since the group $\pi_n(Y, y)$ is κ-small, the regularity of κ guarantees that $\pi_n(X, x)$ is κ-small.

□

03TM **Exercise 4.7.7.3.** Let κ be an uncountable regular cardinal and let $f : X \to Y$ be Kan fibration between Kan complexes. Suppose that X is essentially κ-small, that each fiber $X_y = \{y\} \times_Y X$ is essentially κ-small, and that the morphism $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective. Show that Y is also essentially κ-small.

4.7.8 Local Smallness

03TN In mathematical practice, it is very common to encounter categories $\mathcal{C}$ which are not small but are nonetheless *locally small*: that is, for every pair of objects $X, Y \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is small. We now consider a quantitative counterpart of this condition in the ∞-categorical setting.

03TP **Definition 4.7.8.1.** Let κ be an uncountable cardinal. We say that an ∞-category $\mathcal{C}$ is *locally κ-small* if, for every pair of objects $X, Y \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is essentially κ-small.

03TQ **Example 4.7.8.2.** Let κ be an uncountable cardinal and let $\mathcal{C}$ be a category. Then the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ is locally κ-small if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is κ-small.

03TR **Example 4.7.8.3.** Let κ be an uncountable regular cardinal and let X be a Kan complex. Then X is locally κ-small if and only if, for every vertex $x \in X$ and every integer $n > 0$, the homotopy group $\pi_n(X, x)$ is κ-small.

03TS **Example 4.7.8.4.** Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is essentially κ-small. Then $\mathcal{C}$ is locally κ-small: that is, for every pair of objects $X, Y \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is essentially κ-small. This is a special case of Proposition 4.7.5.14, since $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ can be identified with the oriented fiber product $\{X\} \tilde{\times}_{\mathcal{C}} \{Y\}$.

03TT **Remark 4.7.8.5** (Homotopy Invariance)**.** Let κ be an uncountable cardinal and let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories. Then $\mathcal{C}$ is locally κ-small if and only if $\mathcal{D}$ is locally κ-small.

03TU **Variant 4.7.8.6.** Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *locally small* if, for every pair of objects $X, Y \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is essentially small (that is, it is homotopy equivalent to to a small Kan complex: see Variant 4.7.5.4).

03TW **Proposition 4.7.8.7.** *Let κ be an uncountable regular cardinal and let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is essentially κ-small.*

(2) *The ∞-category $\mathcal{C}$ is locally κ-small and the set of isomorphism classes $\pi_0(\mathcal{C}^{\simeq})$ is κ-small.*

(3) *The Kan complex* $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}$ *is essentially κ-small.*

(4) *For every finite simplicial set K, the Kan complex* $\mathrm{Fun}(K, \mathcal{C})^{\simeq}$ *is essentially κ-small.*

(5) *For every integer $n \geq 0$, the set $\pi_0(\mathrm{Fun}(\Delta^n, \mathcal{C})^{\simeq})$ is κ-small. Moreover, for every map $b : \partial \Delta^n \to \mathcal{C}$, the fundamental group $\pi_1(\mathrm{Fun}(\partial \Delta^n, \mathcal{C})^{\simeq}, b)$ is κ-small.*

Proof. The implication (1) ⇒ (2) follows from Example 4.7.8.4. We next show that (2) ⇒ (3). Assume that condition (2) is satisfied; we wish to show that the Kan complex $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}$ is essentially κ-small. Corollary 4.4.5.4 implies that the restriction map

$$\theta : \mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq} \to \mathrm{Fun}(\partial \Delta^1, \mathcal{C})^{\simeq} \simeq \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$$

is a Kan fibration. Moreover, for each vertex $(X, Y) \in \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$, the fiber $\theta^{-1}\{(X, Y)\}$ can be identified with the morphism space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, which is essentially κ-small by virtue of (2). Using Corollary 4.7.7.2 (and Remark 4.7.5.8), we are reduced to proving that the Kan complex $\mathcal{C}^{\simeq}$ is essentially κ-small. Fix a vertex $X \in \mathcal{C}^{\simeq}$. For $n \geq 2$, Example 4.6.1.13 supplies an isomorphism $\pi_n(\mathcal{C}^{\simeq}, X) \simeq \pi_{n-1}(\mathrm{Hom}_{\mathcal{C}}(X, X), \mathrm{id}_X)$, so that the homotopy group $\pi_n(\mathcal{C}^{\simeq}, X)$ is essentially small by virtue of assumption (2). Similarly, the fundamental group $\pi_1(\mathcal{C}^{\simeq}, X)$ can be identified with the subset of $\pi_0(\mathrm{Hom}_{\mathcal{C}}(X, X))$ spanned by the homotopy classes of isomorphisms, which is also κ-small. Since $\pi_0(\mathcal{C}^{\simeq})$ is κ-small by virtue of assumption (2), Proposition 4.7.7.1 implies that the Kan complex $\mathcal{C}^{\simeq}$ is essentially κ-small.

We now show that (3) implies (4). Assume that the Kan complex $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}$ is essentially κ-small and let K be a finite simplicial set; we wish to show that $\mathrm{Fun}(K, \mathcal{C})^{\simeq}$ is also essentially κ-small. We proceed by induction on the dimension n of K and the number of nondegenerate n-simplices of K. If K is empty, there is nothing to prove. Otherwise, there exists a pushout square of simplicial sets

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\
K' \arrow[r] & K.
\end{tikzcd}$$

Since the horizontal maps are monomorphisms, this diagram is also a categorical pushout square (Example 4.5.4.12) and therefore induces a homotopy pullback diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(\partial \Delta^n, \mathcal{C})^{\simeq} & \operatorname{Fun}(\Delta^n, \mathcal{C})^{\simeq} \arrow[l] \\
\operatorname{Fun}(K', \mathcal{C})^{\simeq} \arrow[u] & \operatorname{Fun}(K, \mathcal{C})^{\simeq} \arrow[l] \arrow[u]
\end{tikzcd}$$

Our inductive hypothesis guarantees that $\operatorname{Fun}(\partial \Delta^n, \mathcal{C})^{\simeq}$ and $\operatorname{Fun}(K', \mathcal{C})^{\simeq}$ are essentially κ-small. It will therefore suffice to show that the Kan complex $\operatorname{Fun}(\Delta^n, \mathcal{C})^{\simeq}$ is essentially κ-small (Corollary 4.7.5.16). If $n = 1$, this follows from assumption (3). If $n \geq 2$, then the inclusion map $\Lambda^n_1 \hookrightarrow \Delta^n$ induces a homotopy equivalence $\operatorname{Fun}(\Delta^n, \mathcal{C})^{\simeq} \to \operatorname{Fun}(\Lambda^n_1, \mathcal{C})^{\simeq}$, so that the desired result again follows from our inductive hypothesis. It will therefore suffice to treat the case $n = 0$: that is, to show that the Kan complex $\mathcal{C}^{\simeq}$ is essentially κ-small. This follows from Corollary 4.7.5.13, since $\mathcal{C}^{\simeq}$ is homotopy equivalent to the summand $\operatorname{Isom}(\mathcal{C})^{\simeq} \subseteq \operatorname{Fun}(\Delta^1, \mathcal{C})^{\simeq}$ (see Corollary 4.4.5.10).

The implication (4) $\Rightarrow$ (5) follows from Proposition 4.7.7.1. We will complete the proof by showing that (5) implies (1). Assume that condition (5) is satisfied; we will show that $\mathcal{C}$ is essentially κ-small. We now proceed as in the proof of Proposition 4.7.7.1. Using Proposition 4.7.6.15, we can reduce to the case where $\mathcal{C}$ is minimal. In this case, we wish to show that $\mathcal{C}$ is κ-small. By virtue of Proposition 4.7.4.9, it will suffice to show that the collection of n-simplices of $\mathcal{C}$ is κ-small, for each $n \geq 0$. Our proof proceeds by induction on n. Using our inductive hypothesis (together with Remark 4.7.3.4 and Proposition 4.7.3.5), we see that the set $\operatorname{Hom}_{\operatorname{Set}_{\Delta}}(\partial \Delta^n, \mathcal{C})$ is κ-small. Since κ is regular, it will suffice to show that each fiber of the restriction map $\operatorname{Hom}_{\operatorname{Set}_{\Delta}}(\Delta^n, \mathcal{C}) \to \operatorname{Hom}_{\operatorname{Set}_{\Delta}}(\partial \Delta^n, \mathcal{C})$ is κ-small.

Set $E = \operatorname{Fun}(\Delta^n, \mathcal{C})^{\simeq}$ and $B = \operatorname{Fun}(\partial \Delta^n, \mathcal{C})^{\simeq}$, so that the inclusion map $\partial \Delta^n \hookrightarrow \Delta^n$ induces a Kan fibration $q : E \to B$ (Corollary 4.4.5.4). Fix a vertex $b \in B$ and set $E_b = \{b\} \times_B E$; we wish to show that the set of vertices of E_b is κ-small. Since $\mathcal{C}$ is minimal, each vertex of E_b belongs to a different connected component. It will therefore suffice to show that the set of connected components $\pi_0(E_b)$ is κ-small. Assumption (5) guarantees that the set $\pi_0(E)$ is κ-small. Moreover Corollary 3.2.6.5 shows that every nonempty fiber of the map $\pi_0(E_b) \to \pi_0(E)$ is equipped with a transitive action of the fundamental group $\pi_1(B, b)$, which is also κ-small. Since κ is regular, it follows that the set $\pi_0(E_b)$ is also κ-small, as desired. □

03UR **Corollary 4.7.8.8.** *Let κ be an infinite cardinal, let λ be an uncountable cardinal of exponential cofinality $\geq \kappa$ (Definition 4.7.3.16), and let $\mathcal{C}$ be an ∞-category which is locally λ-small. Then, for every κ-small simplicial set K, the ∞-category* $\operatorname{Fun}(K, \mathcal{C})$ *is locally λ-small. Moreover, if κ is uncountable, then it suffices to assume that K is essentially κ-small.*

Proof. Let $F, F' : K \to \mathcal{C}$ be diagrams; we wish to show that the morphism space $\mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(F, F')$ is essentially λ-small. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by the essential images of F and F'. Proposition 4.7.8.7 guarantees that $\mathcal{C}_0$ is essentially λ-small. It will therefore suffice to show that $\mathrm{Fun}(K, \mathcal{C}_0)$ is locally λ-small, which follows immediately from Remark 4.7.5.10. □

Corollary 4.7.8.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. If $\mathcal{C}$ is essentially small and $\mathcal{D}$ is locally small, then the ∞-category* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *is locally small.* 03US

Variant 4.7.8.10. Let λ be an uncountable cardinal, let $\mathcal{C}$ be an ∞-category which is locally λ-small, and let K be a simplicial set. Suppose that K is κ-small, where $\kappa = \mathrm{ecf}(\lambda)$ is the exponential cofinality of λ. Then, for any diagram $f : K \to \mathcal{C}$, the ∞-categories $\mathcal{C}_{f/}$ and $\mathcal{C}_{/f}$ are locally λ-small. Moreover, if κ is uncountable, then it suffices to assume that K is essentially κ-small. 04JA

Proof. We will show that the slice ∞-category $\mathcal{C}_{/f}$ is locally λ-small; the analogous assertion for $\mathcal{C}_{f/}$ follows by a similar argument. Fix a pair of objects $X, Y \in \mathcal{C}_{/f}$; we wish to show that the morphism space $\mathrm{Hom}_{\mathcal{C}_{/f}}(X, Y)$ is essentially λ-small. Let $\mathcal{C}'$ be the smallest full subcategory of $\mathcal{C}$ which contains $f(K)$ together with the images of X and Y. Replacing $\mathcal{C}$ by $\mathcal{C}'$, we can reduce to the case where $\mathcal{C}$ is essentially κ-small. In this case, the ∞-category $\mathcal{C}_{/f}$ is essentially λ-small (Corollary 4.7.5.17), so the desired result follows from Example 4.7.8.4. □

Example 4.7.8.11. Let λ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is locally λ-small. Then, for every object $X \in \mathcal{C}$, the ∞-categories $\mathcal{C}_{/X}$ and $\mathcal{C}_{X/}$ are locally λ-small. 04JB

4.7.9 Small Fibrations

It will sometimes be convenient to work with a relative version of Definition 4.7.5.1. 047A

Definition 4.7.9.1. Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets and let κ be an uncountable regular cardinal. We say that U is *essentially κ-small* if, for every simplex $\sigma : \Delta^n \to \mathcal{C}$, the ∞-category $\Delta^n \times_{\mathcal{C}} \mathcal{E}$ is essentially κ-small. We say that U is *locally κ-small* if, for every simplex $\sigma : \Delta^n \to \mathcal{C}$, the ∞-category $\Delta^n \times_{\mathcal{C}} \mathcal{E}$ is locally κ-small. 047B

Variant 4.7.9.2. Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. We say that U is *essentially small* if, for every simplex $\sigma : \Delta^n \to \mathcal{C}$, the ∞-category $\Delta^n \times_{\mathcal{C}} \mathcal{E}$ is essentially small. We say that U is *locally small* if, for every n-simplex $\sigma : \Delta^n \to \mathcal{C}$, the ∞-category $\Delta^n \times_{\mathcal{C}} \mathcal{E}$ is locally small. 047C

047D **Remark 4.7.9.3.** Let κ be an uncountable regular cardinal and suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r] & \mathcal{C},
\end{tikzcd}$$

where U and U' are inner fibrations. If U is essentially κ-small, then U' is essentially κ-small. If U is locally κ-small, then U' is locally κ-small.

047E **Remark 4.7.9.4.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets and let κ be an uncountable regular cardinal. Then U is essentially κ-small if and only if it is locally κ-small and, for each vertex $C \in \mathcal{C}$, the set of isomorphism classes $\pi_0(\mathcal{E}_C^{\simeq})$ is κ-small. See Proposition 4.7.8.7.

047F **Proposition 4.7.9.5.** *Let κ be an uncountable regular cardinal, let $\mathcal{C}$ be an ∞-category which is locally κ-small, and let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration. The following conditions are equivalent:*

(1) *The inner fibration U is locally κ-small.*

(2) *For every edge $\Delta^1 \to \mathcal{C}$, the ∞-category $\Delta^1 \times_{\mathcal{C}} \mathcal{E}$ is locally κ-small.*

(3) *The ∞-category $\mathcal{E}$ is locally κ-small.*

Proof. The implication (1) ⇒ (2) is immediate from the definitions. Assume next that (2) is satisfied; we will prove (3). Let X and Y be objects of $\mathcal{E}$, and set $\overline{X} = U(X)$ and $\overline{Y} = U(Y)$. We wish to show that the Kan complex $\mathrm{Hom}_{\mathcal{E}}(X,Y)$ is essentially κ-small. By virtue of Proposition 4.6.1.21, the functor U induces a Kan fibration $\theta : \mathrm{Hom}_{\mathcal{E}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(\overline{X},\overline{Y})$. Our assumption that $\mathcal{C}$ is locally κ-small guarantees that the Kan complex $\mathrm{Hom}_{\mathcal{C}}(\overline{X},\overline{Y})$ is essentially κ-small. By virtue of Corollary 4.7.7.2, it will suffice to show that for every morphism $\overline{e} : \overline{X} \to \overline{Y}$ in $\mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{E}}(X,Y)_{\overline{e}} = \{\overline{e}\} \times_{\mathrm{Hom}_{\mathcal{C}}(\overline{X},\overline{Y})} \mathrm{Hom}_{\mathcal{E}}(X,Y)$ is essentially κ-small. This follows immediately from assumption (2).

We now complete the proof by showing that (3) implies (1). Assume that $\mathcal{E}$ is locally κ-small and choose a simplex $\sigma : \Delta^n \to \mathcal{C}$; we will show that the ∞-category $\mathcal{E}_\sigma = \Delta^n \times_{\mathcal{C}} \mathcal{E}$ is locally κ-small. Fix a pair of objects $\widetilde{X}, \widetilde{Y} \in \mathcal{E}_\sigma$; we wish to show that the Kan complex $\mathrm{Hom}_{\mathcal{E}_\sigma}(\widetilde{X},\widetilde{Y})$ is essentially κ-small. Let X and Y denote the images of $\widetilde{X}$ and $\widetilde{Y}$ in the ∞-category $\mathcal{E}$, and set $\overline{X} = U(X)$ and $\overline{Y} = U(Y)$ as above. If the Kan complex $\mathrm{Hom}_{\mathcal{E}_\sigma}(\widetilde{X},\widetilde{Y})$ is nonempty, then it can be identified with a fiber of the Kan fibration $\theta : \mathrm{Hom}_{\mathcal{E}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(\overline{X},\overline{Y})$, which is essentially κ-small by virtue of Corollary 4.7.7.2. □

Corollary 4.7.9.6. *Let κ be an uncountable regular cardinal and let $\mathcal{C}$ be an ∞-category which is essentially κ-small, and let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration. If the ∞-category $\mathcal{E}$ is essentially κ-small, then then U is essentially κ-small. The converse holds if U is an isofibration.* 047G

Proof. Assume first that $\mathcal{E}$ is essentially κ-small. Applying Proposition 4.7.9.5, we deduce that U is locally κ-small. It will therefore suffice to show that, for each object $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is essentially κ-small (Remark 4.7.9.4). Using Corollary 4.5.2.23, we can factor U as a composition $\mathcal{E} \xrightarrow{\iota} \mathcal{E}' \xrightarrow{U'} \mathcal{C}$, where U' is an isofibration and ι is an equivalence of ∞-categories. Then $\mathcal{E}'$ is essentially κ-small, so Corollary 4.7.5.16 guarantees that the fiber $\mathcal{E}'_C$ is essentially κ-small. Remark 4.5.2.24 guarantees that the map of fibers $\mathcal{E}_C \to \mathcal{E}'_C$ is fully faithful, so $\mathcal{E}_C$ is also essentially κ-small (Corollary 4.7.5.13).

Now suppose that U is an isofibration which is essentially κ-small; we wish to show that the ∞-category $\mathcal{E}$ is essentially κ-small. Proposition 4.7.9.5 guarantees that U is locally κ-small. It will therefore suffice to show that the set of isomorphism classes $\pi_0(\mathcal{E}^{\simeq})$ is κ-small. In fact, we will show that the core $\mathcal{E}^{\simeq}$ is an essentially κ-small Kan complex. This is a special case of Corollary 4.7.7.2, since U induces a Kan fibration $U^{\simeq} : \mathcal{E}^{\simeq} \to \mathcal{C}^{\simeq}$ whose fibers are essentially κ-small (see Proposition 4.4.3.7). □

Warning 4.7.9.7. If $U : \mathcal{E} \to \mathcal{C}$ is not assumed to be an isofibration, then the converse assertion of Corollary 4.7.9.6 does not necessarily hold. For example, suppose that $\mathcal{C} = \mathrm{N}_\bullet(\mathcal{C}_0)$ is the nerve of an essentially κ-small category $\mathcal{C}_0$, and let $\mathcal{E} = \mathrm{sk}_0(\mathcal{C})$ be the constant simplicial set associated to the collection of objects of $\mathcal{C}_0$. Then the inclusion map $\mathcal{E} \hookrightarrow \mathcal{C}$ is an essentially κ-small inner fibration (which is usually not an isofibration). However, the ∞-category $\mathcal{E}$ is essentially κ-small if and only if the set of objects of $\mathcal{C}_0$ is κ-small. 047H

Corollary 4.7.9.8. *Let κ be an uncountable regular cardinal. Then an ∞-category $\mathcal{C}$ is locally κ-small (in the sense of Definition 4.7.8.1) if and only if the inner fibration $U : \mathcal{C} \to \Delta^0$ is locally κ-small (in the sense of Definition 4.7.9.1). Similarly, $\mathcal{C}$ is essentially κ-small (in the sense of Definition 4.7.5.1) if and only if U is essentially κ-small.* 047J

Corollary 4.7.9.9 (Transitivity of Local Smallness)**.** *Let $V : \mathcal{E} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ be inner fibrations of simplicial sets, let κ be an uncountable regular cardinal, and suppose that U is locally κ-small. Then V is locally κ-small if and only if $U \circ V$ is locally κ-small.* 047K

Proof. Suppose first that V is locally κ-small. Choose an n-simplex $\sigma : \Delta^n \to \mathcal{C}$; we wish to show that the ∞-category $\Delta^n \times_{\mathcal{C}} \mathcal{E}$ is locally κ-small. This follows by applying Proposition 4.7.9.5 to the inner fibration of ∞-categories $(\mathrm{id} \times V) : \Delta^n \times_{\mathcal{C}} \mathcal{E} \to \Delta^n \times_{\mathcal{C}} \mathcal{D}$, which is locally κ-small by virtue of Remark 4.7.9.3.

Now suppose that $U \circ V$ is locally κ-small, and choose an n-simplex $\widetilde{\sigma} : \Delta^n \to \mathcal{D}$; we wish to show that the fiber product $\Delta^n \times_{\mathcal{D}} \mathcal{E}$ is locally κ-small. To prove this, we are free to replace $\mathcal{D}$ and $\mathcal{E}$ by $\Delta^n \times_{\mathcal{C}} \mathcal{D}$ and $\Delta^n \times_{\mathcal{C}} \mathcal{E}$, respectively, and thereby reduce to the case where $\mathcal{C} = \Delta^n$ is a locally κ-small ∞-category. In this case, our assumptions on U and $U \circ V$ guarantee that the ∞-categories $\mathcal{D}$ and $\mathcal{E}$ are also locally κ-small (Proposition 4.7.9.5), so that V is automatically locally κ-small (by Proposition 4.7.9.5 again). □

047L **Variant 4.7.9.10** (Transitivity of Essential Smallness)**.** Let $V : \mathcal{E} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ be inner fibrations of simplicial sets, let κ be an uncountable regular cardinal, and suppose that U is essentially κ-small. Then:

- If V is an essentially κ-small isofibration, then $U \circ V$ is essentially κ-small.
- If $U \circ V$ is essentially κ-small, then V is essentially κ-small.

Proof. We proceed as in the proof of Corollary 4.7.9.9. Assume first that V is an essentially κ-small isofibration, and choose an n-simplex $\sigma : \Delta^n \to \mathcal{C}$; we wish to show that the ∞-category $\Delta^n \times_{\mathcal{C}} \mathcal{E}$ is essentially κ-small. This follows by applying Corollary 4.7.9.6 to the isofibration of ∞-categories $(\mathrm{id} \times V) : \Delta^n \times_{\mathcal{C}} \mathcal{E} \to \Delta^n \times_{\mathcal{C}} \mathcal{D}$, which is essentially κ-small by virtue of Remark 4.7.9.3.

Now suppose that $U \circ V$ is essentially κ-small, and choose an n-simplex $\widetilde{\sigma} : \Delta^n \to \mathcal{D}$; we wish to show that the fiber product $\Delta^n \times_{\mathcal{D}} \mathcal{E}$ is essentially κ-small. To prove this, we are free to replace $\mathcal{D}$ and $\mathcal{E}$ by $\Delta^n \times_{\mathcal{C}} \mathcal{D}$ and $\Delta^n \times_{\mathcal{C}} \mathcal{E}$, respectively, and thereby reduce to the case where $\mathcal{C} = \Delta^n$. In particular, we may assume that $\mathcal{C}$ is an essentially κ-small ∞-category and that the functors U and $U \circ V$ are isofibrations (Example 4.4.1.6). Applying Corollary 4.7.9.6, we deduce that the ∞-categories $\mathcal{D}$ and $\mathcal{E}$ are essentially κ-small, so that V is also automatically essentially κ-small (by Corollary 4.7.9.6 again). □

047M **Corollary 4.7.9.11.** *Let κ be an uncountable regular cardinal and let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories. Then U is locally κ-small if and only if, for every edge $\Delta^1 \to \mathcal{C}$, the ∞-category $\Delta^1 \times_{\mathcal{C}} \mathcal{E}$ is locally κ-small.*

Proof. The "only if" direction is immediate from the definitions. To prove the converse, we may assume without loss of generality that $\mathcal{C} = \Delta^n$ is a standard simplex, in which case the desired result follows from Proposition 4.7.9.5. □

047N **Corollary 4.7.9.12.** *Let κ be an uncountable regular cardinal and let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories. Then U is essentially κ-small if and only if, for every edge $\Delta^1 \to \mathcal{C}$, the ∞-category $\Delta^1 \times_{\mathcal{C}} \mathcal{E}$ is essentially κ-small.*

Proof. Combine Corollary 4.7.9.11 with Remark 4.7.9.4. □

03TY **Warning 4.7.9.13.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories. If U is essentially κ-small, then the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is essentially κ-small for each object $C \in \mathcal{C}$.

Beware that the converse is false in general. For example, let S be a set and let $\mathcal{E}$ be the category containing a pair of objects X and Y, with morphisms given by

$$\mathrm{Hom}_{\mathcal{E}}(X,X) = \{\mathrm{id}_X\} \qquad \mathrm{Hom}_{\mathcal{E}}(Y,Y) = \{\mathrm{id}_Y\}$$
$$\mathrm{Hom}_{\mathcal{E}}(X,Y) = S \qquad \mathrm{Hom}_{\mathcal{E}}(Y,X) = \emptyset.$$

Then there is a unique isofibration $U : \mathrm{N}_{\bullet}(\mathcal{E}) \to \Delta^1$ satisfying $U(X) = 0$ and $U(Y) = 1$. The fibers $U^{-1}\{0\}$ and $U^{-1}\{1\}$ are isomorphic to Δ^0 (and are therefore essentially κ-small for every uncountable cardinal κ). However, the ∞-category $\mathrm{N}_{\bullet}(\mathcal{E})$ is essentially κ-small if and only if the set S is κ-small.

4.8 Truncations in Higher Category Theory

Recall that a simplicial set $\mathcal{C}$ is an ∞-*category* if, for every pair of integers $0 < i < m$, every inner horn $\sigma_0 : \Lambda^m_i \to \mathcal{C}$ can be extended to an m-simplex of $\mathcal{C}$ (Definition 1.4.0.1). In this case, $\mathcal{C}$ is (isomorphic to the nerve of) an ordinary category if and only if the extension σ is always unique (Proposition 1.3.4.1). More generally, we say that $\mathcal{C}$ is an $(n,1)$-*category* if the extension σ is unique whenever $m > n$ (Definition 4.8.1.1). In §4.8.1, we summarize the formal properties of this definition and give some basic examples. 0578

Beware that the notion of $(n,1)$-category is not homotopy-invariant: that is, if $\mathcal{C}$ and $\mathcal{D}$ are equivalent ∞-categories and $\mathcal{D}$ is an $(n,1)$-category, then $\mathcal{C}$ need not be an $(n,1)$-category. We can therefore ask the following:

Question 4.8.0.1. Let $\mathcal{C}$ be an ∞-category. Under what conditions does there exists an equivalence of ∞-categories $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is an $(n,1)$-category? 0579

We partially address Question 4.8.0.1 in §4.8.2 by introducing the notion of a *locally truncated* ∞-category. If m is an integer, we say that an ∞-category $\mathcal{C}$ is *locally m-truncated* if, for every pair of objects $X, Y \in \mathcal{C}$, the morphism space $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is m-truncated (Definition 4.8.2.1). It is easy to see that every $(n,1)$-category is locally $(n-1)$-truncated (Example 4.8.2.2). Conversely, if $\mathcal{C}$ is a locally $(n-1)$-truncated ∞ category, then there exists an equivalence $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is an $(n,1)$-category. We will give two proofs of this result:

- In §4.8.3, we show that a locally $(n-1)$-truncated ∞-category $\mathcal{C}$ is an $(n,1)$-category if and only if it is minimal in dimensions $\geq n$ (Proposition 4.8.3.1). In particular, if $\mathcal{D}$ is a minimal model of $\mathcal{C}$, then $\mathcal{D}$ is an $(n,1)$-category.
- In §4.8.4, we associate to every ∞-category $\mathcal{C}$ an $(n,1)$-category $\mathrm{h}_{\leq n}(\mathcal{C})$, which we call the *homotopy n-category of* $\mathcal{C}$ (Construction 4.8.4.9). The homotopy n-category $\mathrm{h}_{\leq n}(\mathcal{C})$ is equipped with a comparison functor $\mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C})$, which is an equivalence if and only if $\mathcal{C}$ is locally $(n-1)$-truncated (Corollary 4.8.4.16).

Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it satisfies the following three conditions:

- The functor F is essentially surjective: that is, every object of $\mathcal{D}$ is isomorphic to $F(X)$, for some objects $X \in \mathcal{C}$.
- The functor F is full: that is, for every pair of objects $X, Y \in \mathcal{C}$, the function $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective.
- The functor F is faithful: that is, for every pair of objects $X, Y \in \mathcal{C}$, the function $F_{X,Y}$ is injective.

057A **Exercise 4.8.0.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Show that F is faithful if and only if, for every diagram σ :

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

in the category $\mathcal{C}$, if $F(\sigma)$ is a commutative diagram in $\mathcal{D}$, then σ is also commutative.

To emphasize the parallels between the preceding conditions, it is convenient to reformulate them using the language of simplicial sets. To simplify the discussion, let us assume that the functor $F : \mathcal{C} \to \mathcal{D}$ is an isofibration (Definition 4.4.1.1). In this case:

(0) The functor F is essentially surjective if and only if it is surjective on objects: that is, every lifting problem

$$\begin{tikzcd}
\partial\Delta^0 \arrow[r] \arrow[d] & \mathrm{N}_\bullet(\mathcal{C}) \arrow[d, "\mathrm{N}_\bullet(F)"] \\
\Delta^0 \arrow[r] \arrow[ur, dashed] & \mathrm{N}_\bullet(\mathcal{D})
\end{tikzcd}$$

admits a solution.

(1) The functor F is full if and only if every lifting problem

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r] \arrow[d] & \mathrm{N}_\bullet(\mathcal{C}) \arrow[d, "\mathrm{N}_\bullet(F)"] \\
\Delta^1 \arrow[r] \arrow[ur, dashed] & \mathrm{N}_\bullet(\mathcal{D})
\end{tikzcd}$$

admits a solution.

(2) The functor F is faithful if and only if every lifting problem

$$\begin{tikzcd}
\partial\Delta^2 \arrow[r] \arrow[d] & \mathrm{N}_{\bullet}(\mathcal{C}) \arrow[d, "\mathrm{N}_{\bullet}(F)"] \\
\Delta^2 \arrow[r] \arrow[ur, dashed] & \mathrm{N}_{\bullet}(\mathcal{D}).
\end{tikzcd}$$

These conditions have a counterpart in the setting of ∞-categories:

Definition 4.8.0.3 (Preliminary)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration ∞-categories and let $m \geq 0$ be an integer. We say that F is *m-full* if every lifting problem `057B`

$$\begin{tikzcd}
\partial\Delta^m \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution.

In §4.8.5, we extend Definition 4.8.0.3 to the case where F is an arbitrary functor of ∞-categories (see Definition 4.8.5.10 for a homotopy-invariant formulation, and Proposition 4.8.5.30 for a comparison with Definition 4.8.0.3).

Example 4.8.0.4. Let $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ be a functor of ordinary categories and let $F : \mathrm{N}_{\bullet}(\mathcal{C}_0) \to \mathrm{N}_{\bullet}(\mathcal{D}_0)$ denote the induced of ∞-categories. Then: `057C`

- The functor F is 0-full if and only if F_0 is essentially surjective.
- The functor F is 1-full if and only if F_0 is full.
- The functor F is 2-full if and only if F_0 is faithful.
- For $m \geq 3$, the functor F is automatically m-full (see Exercise 1.3.1.5).

Fix an integer n. We will say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is *essentially n-categorical* if it is m-full for every nonnegative integer $m \geq n+2$ (Definition 4.8.6.1). In §4.8.6, we will see that this condition has many familiar specializations:

- A functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is essentially (-1)-categorical if and only if it is fully faithful, and essentially (-2)-categorical if and only if it is an equivalence of ∞-categories. See Remark 4.8.5.11.
- Let $\mathcal{C}$ be an ∞-category and let $n \geq -1$. Then the projection map $\mathcal{C} \to \Delta^0$ is essentially n-categorical if and only if $\mathcal{C}$ is locally $(n-1)$-truncated: that is, if and only if $\mathcal{C}$ is equivalent to an $(n,1)$-category. See Example 4.8.6.4.
- If $\mathcal{C}$ and $\mathcal{D}$ are Kan complexes, then a functor $F : \mathcal{C} \to \mathcal{D}$ is essentially n-categorical if and only if it is n-truncated, in the sense of Definition 3.5.9.1. See Example 4.8.6.3.

In §4.8.7, we introduce a dual version of this condition. We will say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is *categorically n-connective* if it is m-full for every integer $0 \leq m \leq n$ (Definition 4.8.7.1). Roughly speaking, this condition asserts that, up to equivalence, the ∞-category $\mathcal{D}$ can be be built from $\mathcal{C}$ using only simplices of dimension strictly larger than n (see Corollary 4.8.7.16 for a precise formulation). In the special case where $\mathcal{C}$ and $\mathcal{D}$ are Kan complexes, this recovers the theory of relative connectivity developed in §3.5.1 (Example 4.8.7.3). As with the usual notion of connectivity, it can sometimes be useful to extend the notion of categorical connectivity to morphisms of between simplicial sets which are not ∞-categories; we consider this extension in §4.8.9 (Definition 4.8.9.2).

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. In §4.8.8, we show that F admits a factorization

$$\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D},$$

where F' is categorically $(n+1)$-connective and G is essentially n-categorical (Theorem 4.8.8.3). Moreover, this factorization is unique up to equivalence (Remark 4.8.8.8). In the special case where $\mathcal{D} = \Delta^0$, this can be achieved by taking $\mathcal{D}'$ to be the homotopy n-category $\mathrm{h}_{\leq n}(\mathcal{C})$ constructed in §4.8.4 (Example 4.8.8.7). More generally, if F is an inner fibration of ∞-categories, we will show that the system of ∞-categories $\{\mathrm{h}_{\leq n}(\mathcal{C}_D)\}_{D \in \mathcal{D}}$ can be realized as the fibers of an inner fibration $G : \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \to \mathcal{D}$ which realizes the desired factorization (Construction 4.8.8.10).

4.8.1 $(n,1)$-Categories

057D Recall that a simplicial set $\mathcal{C}$ is (isomorphic to) the nerve of a category if and only if, for every pair of integers $0 < i < m$, the restriction map

$$\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^m, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Lambda^m_i, \mathcal{C})$$

is a bijection. In this section, we study a hierarchy of weaker filling conditions.

057E **Definition 4.8.1.1.** Let n be a positive integer. We say that a simplicial set $\mathcal{C}$ is an $(n,1)$-*category* if it satisfies the following condition for every pair of integers $0 < i < m$:

$(*)$ Every morphism of simplicial sets $\sigma_0 : \Lambda^m_i \to \mathcal{C}$ can be extended to an m-simplex σ of $\mathcal{C}$. Moreover, if $m > n$, then σ is unique.

Remark 4.8.1.2. Let n be a positive integer and let $\mathcal{C}$ be a simplicial set. If $\mathcal{C}$ is an $(n, 1)$-category, then it is an ∞-category. Conversely, if $\mathcal{C}$ is an ∞-category, then it is an $(n, 1)$-category if and only if, for every pair of integers $0 < i < m$ with $m > n$, the restriction map $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^m_i, \mathcal{C})$ is injective. 057F

Example 4.8.1.3. An ∞-category $\mathcal{C}$ is a $(1, 1)$-category if and only if it is isomorphic to $\mathrm{N}_\bullet(\mathcal{C}_0)$, for some category $\mathcal{C}_0$. By virtue of Proposition 4.8.1.7, this is a restatement of Proposition 1.3.4.1. Note that in this case, the category $\mathcal{C}_0$ is well-defined up to unique isomorphism (Proposition 1.3.3.1). 057G

Example 4.8.1.4. Let $\mathcal{C}$ be a 2-category, and suppose that every 2-morphism in $\mathcal{C}$ is an isomorphism. Then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ is a $(2, 1)$-category, in the sense of Definition 4.8.1.8. This follows by combining Propositions 4.8.1.7 and 2.3.3.1. 057H

Warning 4.8.1.5. We have now given several *a priori* different definitions for the notion of $(2, 1)$-category: 057J

(1) According to Definition 2.2.8.5, a $(2, 1)$-category is a 2-category $\mathcal{C}$ in which every 2-morphism is an isomorphism.

(2) According to Definition 4.8.1.8 (or Definition 4.8.1.1), a $(2, 1)$-category is a simplicial set which satisfies some additional conditions.

However, these definitions are compatible with one another. If $\mathcal{C}$ is a $(2, 1)$-category in the sense of (1), then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ is a $(2, 1)$-category in the sense of (2) (Example 4.8.1.4). We will later see that the converse is also true: every $(2, 1)$-category in the sense (2) is isomorphic to the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$, where $\mathcal{C}$ is a 2-category in which every 2-morphism is an isomorphism (in this case, Theorem 2.3.4.1 guarantees that $\mathcal{C}$ is unique up to non-strict isomorphism). See Proposition [?].

Exercise 4.8.1.6. Let n be a positive integer and let $\mathcal{C}$ be a differential graded category which satisfies the following condition: 057K

- For every pair of objects $X, Y \in \mathcal{C}$, the chain complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)_*$ is concentrated in degrees $< n$: that is, the abelian groups $\mathrm{Hom}_{\mathcal{C}}(X, Y)_m$ vanish for $m \geq n$.

Show that the differential graded nerve $\mathrm{N}^{\mathrm{dg}}_\bullet(\mathcal{C})$ is an $(n, 1)$-category (see the proof of Theorem 2.5.3.10).

It will be useful to work with a reformulation of Definition 4.8.1.1. Recall that:

- A simplicial set $\mathcal{C}$ is *weakly n-coskeletal* if the restriction map

  $$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^m, \mathcal{C})$$

  is a bijection for $m \geq n+2$ and an injection for $m = n+1$ (Definition 3.5.4.1).

- An ∞-category $\mathcal{C}$ is *minimal in dimension n* if, whenever σ and τ are n-simplices of $\mathcal{C}$ which are isomorphic relative to $\partial \Delta^n$, then $\sigma = \tau$ (Definition 4.7.6.4).

057L **Proposition 4.8.1.7.** *Let $\mathcal{C}$ be an ∞-category and let n be a positive integer. Then $\mathcal{C}$ is an $(n,1)$-category (in the sense of Definition 4.8.1.1) if and only if it is weakly n-coskeletal and minimal in dimension n.*

Proof. We proceed as in the proof of Proposition 3.5.5.12. Assume first that $\mathcal{C}$ is an $(n,1)$-category. Then, for any integer $m > n$, the composition of the restriction maps

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \mathcal{C}) \xrightarrow{\theta_m} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^m, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Lambda^m_1, \mathcal{C})$$

is a bijection. In particular, θ_m is an injection. To show that $\mathcal{C}$ is weakly n-coskeletal, it will suffice to show that θ_m is surjective for $m \geq n+2$. Fix a morphism $\sigma_0 : \partial \Delta^m \to \mathcal{C}$; we wish to show that σ_0 can be extended to an m-simplex of $\mathcal{C}$. Since $\mathcal{C}$ is an $(n,1)$-category, there is a unique m-simplex σ of $\mathcal{C}$ satisfying $\sigma|_{\Lambda^m_1} = \sigma_0|_{\Lambda^m_1}$. We complete the argument by observing that $\sigma|_{\partial \Delta^m} = \sigma_0$, by virtue of the injectivity of the map θ_{m-1}.

We next show that if $\mathcal{C}$ is an $(n,1)$-category, then it is minimal in dimension n. Let $\sigma_0, \sigma_1 : \Delta^n \to \mathcal{C}$ be n-simplices of $\mathcal{C}$ and let $h : \Delta^1 \times \Delta^n \to \mathcal{C}$ be a natural isomorphism from $\sigma_0 = h|_{\{0\} \times \Delta^n}$ to $\sigma_1 = h|_{\{1\} \times \Delta^n}$ whose restriction to $\Delta^1 \times \partial \Delta^n$ factors through $\partial \Delta^n$; we wish to show that $\sigma_0 = \sigma_1$. For $0 \leq i \leq n$, let $\alpha_i : [n+1] \to [1] \times [n]$ denote the nondecreasing function given by the formula

$$\alpha_i(j) = \begin{cases} (0, j) & \text{if } j \leq i \\ (1, j-1) & \text{if } j > i, \end{cases}$$

and let τ_i denote the $(n+1)$-simplex of $\mathcal{C}$ given by the composition

$$\Delta^{n+1} \xrightarrow{\alpha_i} \Delta^1 \times \Delta^n \xrightarrow{h} \mathcal{C}.$$

Let $\rho_i, \rho'_i : \Delta^n \to \mathcal{C}$ be the n-simplices of X given by $\rho_i = d^{n+1}_i(\tau_i)$ and $\rho'_i = d^{n+1}_{i+1}(\tau_i)$; by construction, we have

$$\sigma_0 = \rho'_n \qquad \rho_n = \rho'_{n-1} \qquad \cdots \qquad \rho_1 = \rho'_0 \qquad \rho_0 = \sigma_1.$$

We will complete the proof by showing that $\rho_i = \rho'_i$ for $0 \leq i \leq n$. We will treat the case $i > 0$ (the case $i < n$ follows by a similar argument). Using our assumption that h is constant along the boundary $\partial \Delta^n$, we see that the degenerate $(n+1)$-simplex $s_i^n(\rho_i)$ coincides with τ_i on the horn $\Lambda_i^{n+1} \subset \Delta^{n+1}$. Since $\mathcal{C}$ is an $(n,1)$-category, it follows that $\tau_i = s_i^n(\rho_i)$. Applying the face operator d_{i+1}^{n+1}, we obtain $\rho_i = \rho_i'$.

We now prove the converse. Assume that $\mathcal{C}$ is a weakly n-coskeletal ∞-category which is minimal in dimension n; we will show that it is an $(n,1)$-category. Fix a pair of integers $0 < i < m$ with $m > n$ and a pair of m-simplices $\tau_0, \tau_1 : \Delta^m \to X$ which coincide on the horn $\Lambda_i^m \subset \Delta^m$; we wish to show that $\tau_0 = \tau_1$. Since $\mathcal{C}$ is weakly n-coskeletal, it will suffice to prove that τ_0 and τ_1 coincide on the boundary $\partial \Delta^m$: that is, to show that the $(m-1)$-simplices $\sigma_0 = d_i^m(\tau_0)$ and $\sigma_1 = d_i^m(\tau_1)$ coincide. Note that σ_0 and σ_1 have the same restriction to the boundary $\partial \Delta^{m-1}$. Consequently, if $m \geq n+2$, the desired result follows from our assumption that $\mathcal{C}$ is weakly n-coskeletal. We may therefore assume that $m = n+1$. Since $\mathcal{C}$ is minimal in dimension n, it will suffice to show that there is an isomorphism from σ_0 to σ_1 (in the ∞-category $\operatorname{Fun}(\Delta^n, \mathcal{C})$) whose image in $\operatorname{Fun}(\partial \Delta^n, \mathcal{C})$ is an identity morphism. In fact, we will prove a stronger claim: there is an isomorphism from τ_0 to τ_1 in the ∞-category $\operatorname{Fun}(\Delta^m, \mathcal{C})$ whose image in $\operatorname{Fun}(\Lambda_i^m, \mathcal{C})$ is an identity morphism. This follows from the observation that the restriction map $\operatorname{Fun}(\Delta^m, \mathcal{C}) \to \operatorname{Fun}(\Lambda_i^m, \mathcal{C})$ is a trivial Kan fibration; see Proposition 1.5.7.6. □

Motivated by Proposition 4.8.1.7, we introduce a generalization of Definition 4.8.1.1.

Definition 4.8.1.8. Let n be an integer. We say that a simplicial set $\mathcal{C}$ is an $(n,1)$-*category* if it is an ∞-category which is weakly n-coskeletal and (if $n \geq 0$) minimal in dimension n. `057M`

For $n > 0$, Definitions 4.8.1.8 and 4.8.1.1 are equivalent: this is the content of Proposition 4.8.1.7. The advantage of Definition 4.8.1.8 is that it also makes sense for $n \leq 0$. However, for $n < 0$ it is rather trivial:

Example 4.8.1.9. A simplicial set $\mathcal{C}$ is a $(-1,1)$-category if and only if it is either empty or isomorphic to Δ^0. See Example 3.5.4.4. `057N`

Example 4.8.1.10. For $n \leq -2$, a simplicial set $\mathcal{C}$ is an $(n,1)$-category if and only it is isomorphic to Δ^0. See Example 3.5.4.3. `057P`

Example 4.8.1.11. Let X be a Kan complex and let $n \geq 0$ be an integer. Then X is an n-groupoid (in the sense of Definition 3.5.5.1) if and only if it is an $(n,1)$-category (in the sense of Definition 4.8.1.8). This is a reformulation of Proposition 3.5.5.12. `057Q`

Remark 4.8.1.12 (Monotonicity)**.** Let $\mathcal{C}$ be an $(m,1)$-category for some integer m. Then $\mathcal{C}$ is an $(n,1)$-category for every integer $n \geq m$. `057R`

Remark 4.8.1.13 (Symmetry)**.** Let n be an integer and let $\mathcal{C}$ be an $(n,1)$-category. Then the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is also an $(n,1)$-category. `057S`

057T **Remark 4.8.1.14.** Let n be an integer and let $\{\mathcal{C}_j\}_{j \in J}$ be a collection of $(n,1)$-categories. Then the product $\mathcal{C} = \prod_{j \in J} \mathcal{C}_j$ is also an $(n,1)$-category.

057U **Proposition 4.8.1.15.** *A simplicial set* $\mathcal{C}$ *is a* $(0,1)$*-category if and only if there exists an isomorphism* $\mathcal{C} \simeq \mathrm{N}_{\bullet}(Q)$*, for some partially ordered set* $(Q, \leq)$*.*

Proof. By virtue of Remark 4.8.1.12, we may assume without loss of generality that $\mathcal{C}$ is a $(1,1)$-category: that is, it is isomorphic to $\mathrm{N}_{\bullet}(\mathcal{C}_0)$, for some category $\mathcal{C}_0$ (see Example 4.8.1.3). In this case, $\mathcal{C}$ is a $(0,1)$-category if and only if it satisfies the following additional conditions:

(0) The ∞-category $\mathcal{C}$ is minimal in dimension 0: that is, if X and Y are isomorphic objects of $\mathcal{C}_0$, then $X = Y$.

(1) The restriction map $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^1, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^1, \mathcal{C})$ is injective: that is, for every pair of objects $X, Y \in \mathcal{C}_0$, there is at most one morphism from X to Y.

(2) The restriction map $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^2, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^2, \mathcal{C})$ is bijective: that is, every diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in the category $\mathcal{C}_0$ is automatically commutative.

Note that conditions (1) and (2) are equivalent to one another: they both assert that $\mathcal{C}_0$ can be recovered from the set of objects $Q = \mathrm{Ob}(\mathcal{C})$, endowed with the reflexive and transitive relation $\leq_Q$ defined by

$$(X \leq_Q Y) \Leftrightarrow \mathrm{Hom}_{\mathcal{C}_0}(X, Y) \neq \emptyset.$$

In this case, condition (0) is satisfied if and only if the relation $\leq_Q$ is also antisymmetric: that is, the relation $\leq_Q$ is a partial ordering of Q. □

057V **Remark 4.8.1.16.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a simplicial subset which is also an ∞-category. If $\mathcal{C}$ is an $(n,1)$-category for some integer $n \geq -1$, then $\mathcal{C}_0$ is also an $(n,1)$-category. For $n \geq 1$, this follows from Remark 4.8.1.2. The case $n = 0$ follows from Proposition 4.8.1.15 (since any subcategory of a partially ordered set is also a partially ordered set), and the case $n = -1$ is trivial (see Example 4.8.1.9).

057W **Example 4.8.1.17.** Let $n \geq 0$ and let $\mathcal{C}$ be an $(n,1)$-category. Then the core $\mathcal{C}^{\simeq}$ is an n-groupoid. This follows by combining Remark 4.8.1.16 with Example 4.8.1.11.

Remark 4.8.1.18. Let $\{\mathcal{C}_j\}_{j \in \mathcal{J}}$ be a diagram of simplicial sets having limit $\mathcal{C} = \varprojlim_{j \in \mathcal{J}} \mathcal{C}_j$ and let n be an integer. If each $\mathcal{C}_j$ is an $(n,1)$-category and $\mathcal{C}$ is an ∞-category, then $\mathcal{C}$ is also an $(n,1)$-category. For $n \leq -2$, this is trivial (Example 4.8.1.10). The case $n \geq -1$ follows from Remarks 4.8.1.14 and 4.8.1.16, since $\mathcal{C}$ can be identified with a simplicial subset of the product $\prod_{j \in J} \mathcal{C}_j$. 057X

Proposition 4.8.1.19. *Let n be a positive integer and let $\mathcal{C}$ be an $(n,1)$-category. Then, for every pair of objects $X, Y \in \mathcal{C}$, the pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ are $(n-1)$-groupoids.* 057Y

Proof. We will show that the right-pinched morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ is an $(n-1)$-groupoid; the analogous statement for the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ follows by a similar argument. If $n = 1$, then $\mathcal{C}$ can be identified with the nerve of an ordinary category $\mathcal{C}_0$ (Example 4.8.1.3) and the desired result follows from Example 4.6.5.12. We may therefore assume that $n > 1$. Since $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ is a Kan complex, it will suffice to show that it is an $(n-1,1)$-category (Example 4.8.1.11). Let $m \geq n$ and let σ_0 and σ_1 be m-simplices of $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ which satisfy $\sigma_0|_{\Lambda^m_i} = \sigma_1|_{\Lambda^m_i}$ for some $0 < i < m$; we wish to show that $\sigma_0 = \sigma_1$. Let us identify σ_0 and σ_1 with morphisms $\tau_0, \tau_1 : \Delta^{m+1} \to \mathcal{C}$ which carry the simplicial subset $\Delta^m \subset \Delta^{m+1}$ to the object X and the final vertex of Δ^{m+1} to the object Y. Our assumptions then guarantee that τ_0 and τ_1 have the same restriction to Λ^{m+1}_i. Since $\mathcal{C}$ is an $(n,1)$-category, it follows that $\tau_0 = \tau_1$. □

Corollary 4.8.1.20. *Let n be an integer and let $\mathcal{C}$ be an $(n,1)$-category. Then, for every pair of objects $X, Y \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is $(n-1)$-truncated.* 057Z

Proof. For $n \leq -1$, there is nothing to prove (see Examples 4.8.1.10 and 4.8.1.9). The case $n = 0$ follows from Proposition 4.8.1.15. We may therefore assume $n > 0$. By virtue of Proposition 4.6.5.10, it will suffice to show that the pinched morphism space $K = \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ is $(n-1)$-truncated. This follows from Example 3.5.7.2, since K is an $(n-1)$-groupoid (Proposition 4.8.1.19). □

Warning 4.8.1.21. Let $n \geq 1$ be an integer and let $\mathcal{C}$ be an $(n,1)$-category. For every pair of objects $X, Y \in \mathcal{C}$, Corollary 4.8.1.20 guarantees that the morphism space $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is $(n-1)$-truncated, and is therefore homotopy equivalent to an $(n-1)$-groupoid (for example, it is homotopy equivalent to the pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$). Beware that, for $n \geq 2$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ itself is generally not an $(n-1)$-groupoid. For example, this usually fails in the case where $\mathcal{C} = \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}_0)$ arises as the Duskin nerve of a 2-category $\mathcal{C}_0$: see Remark 8.1.8.8. However, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is always weakly $(n-1)$-coskeletal: see Corollary 4.8.3.5. 0580

0581 **Proposition 4.8.1.22** (Exponentation for $(n,1)$-Categories)**.** *Let n be an integer and let $\mathcal{C}$ be an $(n,1)$-category. Then, for any simplicial set K, the simplicial set $\operatorname{Fun}(K,\mathcal{C})$ is also an $(n,1)$-category.*

Proof. It follows from Theorem 1.5.3.7 that $\operatorname{Fun}(K,\mathcal{C})$ is an ∞-category. Since $\mathcal{C}$ is weakly n-coskeletal, the ∞-category $\operatorname{Fun}(K,\mathcal{C})$ is also weakly n-coskeletal (Corollary 3.5.4.13). To complete the proof, it will suffice to show that if $n \geq 0$, then $\operatorname{Fun}(K,\mathcal{C})$ is minimal in dimension n (Proposition 4.8.1.7). Suppose we are given a pair of n-simplices $\sigma_0, \sigma_1 : \Delta^n \to \operatorname{Fun}(K,\mathcal{C})$ and an isomorphism $\sigma_0 \xrightarrow{\sim} \sigma_1$ whose restriction to $\partial\Delta^n$ is an identity morphism; we wish to show that $\sigma_0 = \sigma_1$. Let us identify σ_0 and σ_1 with diagrams $f_0, f_1 : \Delta^n \times K \to \mathcal{C}$. Since $\mathcal{C}$ is weakly n-coskeletal, it will suffice to show that f_0 and f_1 coincide on m-simplices $\tau = (\tau', \tau'')$ of $\Delta^n \times K$ for $m \leq n$. If τ' factors through the boundary $\partial\Delta^n$, this follows immediately from the equality $\sigma_0|_{\partial\Delta^n} = \sigma_1|_{\partial\Delta^n}$. We may therefore assume without loss of generality that $m = n$ and that $\tau' : \Delta^m \to \Delta^n$ is the identity map. In this case, our assumption guarantees that there is an isomorphism of $f_0(\tau)$ with $f_1(\tau)$ whose image in $\operatorname{Fun}(\partial\Delta^n,\mathcal{C})$ is an identity morphism. The equality $f_0(\tau) = f_1(\tau)$ now follows from the fact that $\mathcal{C}$ is minimal in dimension n (Proposition 4.8.1.7). □

0582 **Corollary 4.8.1.23.** *Let n be an integer and let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of $(n,1)$-categories. Then the oriented fiber product $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ and the homotopy fiber product $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ are also $(n,1)$-categories.*

Proof. By definition, the oriented fiber product $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ can be realized as an iterated fiber product

$$\mathcal{C}_0 \times_{\operatorname{Fun}(\{0\},\mathcal{C})} \operatorname{Fun}(\Delta^1,\mathcal{C}) \times_{\operatorname{Fun}(\{1\},\mathcal{C})} \mathcal{C}_1,$$

which is an $(n,1)$-category by virtue of Propositions 4.8.1.22 and Remark 4.8.1.18. The homotopy fiber product $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is a full subcategory of $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$, which coincides with $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ for $n \leq -1$. Applying Remark 4.8.1.16, we see that it is also an $(n,1)$-category. □

4.8.2 Locally Truncated ∞-Categories

0583 We now formulate a homotopy-invariant counterpart of Definition 4.8.1.8.

0584 **Definition 4.8.2.1.** Let $\mathcal{C}$ be an ∞-category and let n be an integer. We say that $\mathcal{C}$ is *locally n-truncated* if, for every pair of objects $X, Y \in \mathcal{C}$, the Kan complex $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is n-truncated (see Definition 3.5.7.1).

0585 **Example 4.8.2.2.** Let n be an integer. Then every $(n,1)$-category is locally $(n-1)$-truncated (Corollary 4.8.1.20). In particular:

- If Q is a partially ordered set, then the nerve $\mathrm{N}_{\bullet}(Q)$ is locally (-1)-truncated ∞-category (Proposition 4.8.1.15).

- If $\mathcal{C}$ is an ordinary category, then the nerve $\mathrm{N}_\bullet(\mathcal{C})$ is a locally 0-truncated ∞-category (Example 4.8.1.3).
- If $\mathcal{C}$ is a 2-category in which every 2-morphism is an isomorphism, then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ is a locally 1-truncated ∞-category (Example 4.8.1.4).

Remark 4.8.2.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories. If $\mathcal{D}$ is locally n-truncated, then $\mathcal{C}$ is locally n-truncated. The converse holds if F is an equivalence of ∞-categories. In particular, if ∞-categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent, then $\mathcal{C}$ is locally n-truncated if and only if $\mathcal{D}$ is locally n-truncated. 0586

Let $\mathcal{C}$ be an ∞-category. Combining Example 4.8.2.2 with Remark 4.8.2.3, we see for $\mathcal{C}$ to be equivalent to an $(n,1)$-category, it is necessary for $\mathcal{C}$ to be locally $(n-1)$-truncated. In §4.8.3, we will prove that this condition is also sufficient, provided that $n \geq -1$ (Corollary 4.8.3.3).

Example 4.8.2.4. Let $n \geq -1$ be an integer and let X be a Kan complex. Then X is n-truncated (in the sense of Definition 3.5.7.1) if and only if it is locally $(n-1)$-truncated when regarded as an ∞-category (in the sense of Definition 4.8.2.1). This is reformulation of Example 3.5.9.18. See Corollary 4.8.3.11 for a more general statement. 0587

Remark 4.8.2.5. Let $\mathcal{C}$ be an ∞-category containing a pair of objects $X, Y \in \mathcal{C}$. For every integer n, the following conditions are equivalent: 0588

- The morphism space $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is n-truncated.
- The left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ is n-truncated.
- The right-pinched morphism space $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ is n-truncated.

This follows from Corollary 3.5.7.12, since the pinch inclusion maps

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y) \hookleftarrow \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$$

are homotopy equivalences (Proposition 4.6.5.10).

Proposition 4.8.2.6. *Let $\mathcal{C}$ be a locally Kan simplicial category. For every integer n, the following conditions are equivalent:* 0589

- *The homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is locally n-truncated.*
- *For every pair of objects $X, Y \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ is n-truncated.*

Proof. For every pair of objects $X, Y \in \mathcal{C}$, Theorem 4.6.8.5 supplies a homotopy equivalence from $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ to the pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(X,Y)$. The desired result now follows from Remark 4.8.2.5. □

058A **Variant 4.8.2.7.** Let $\mathcal{C}$ be a differential graded category. For every integer $n \geq -1$, the following conditions are equivalent:

- The differential graded nerve $\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})$ is locally n-truncated.
- For every pair of objects $X, Y \in \mathcal{C}$, the chain complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)_*$ is homologically n-truncated: that is, the homology groups $\mathrm{H}_m(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*)$ vanish for $m > n$.

Proof. For every pair of objects $X, Y \in \mathcal{C}$, Example 4.6.5.15 supplies an isomorphism from of the Eilenberg-MacLane space $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*)$ with the pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}^{\mathrm{dg}}(\mathcal{C})}(X,Y)$. The result now follows by combining Remark 4.8.2.5 with the criterion of Example 3.5.7.10. □

Example 4.8.2.2 admits a slight generalization:

058B **Proposition 4.8.2.8.** *Let $\mathcal{C}$ be an ∞-category and let n be an integer. If $\mathcal{C}$ is an n-coskeletal simplicial set (Definition 3.5.3.1), then it is locally $(n-2)$-truncated.*

Proof. For every pair of objects $X, Y \in \mathcal{C}$, our assumption that $\mathcal{C}$ is n-coskeletal guarantees that the pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ is $(n-1)$-coskeletal (Remark 4.6.5.4). In particular, it is $(n-2)$-truncated (Example 3.5.7.2). The desired result now follows from Remark 4.8.2.5. □

Our next goal is to show that every ∞-category $\mathcal{C}$ admits an optimal approximation by a locally n-truncated ∞-category.

058C **Definition 4.8.2.9.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. We say that F *exhibits $\mathcal{D}$ as a local n-truncation of $\mathcal{C}$* if the following conditions are satisfied:

(1) The functor F is essentially surjective (Definition 4.6.2.11).

(2) For every pair of objects $X, Y \in \mathcal{C}$, the induced map of Kan complexes

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

exhibits $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ as an n-truncation of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, in the sense of Definition 3.5.7.19.

058D **Remark 4.8.2.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. Suppose that F exhibits $\mathcal{D}$ as a local n-truncation of $\mathcal{C}$. Then $\mathcal{D}$ is locally n-truncated: that is, for every pair of objects $\overline{X}, \overline{Y} \in \mathcal{D}$, the morphism space $\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y})$ is n-truncated. To prove this, we can use the essential surjectivity of F to reduce to the case where $\overline{X} = F(X)$ and $\overline{Y} = F(Y)$ for some objects $X, Y \in \mathcal{C}$. In this case, the assertion follows from the observation that $\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y})$ is an n-truncation of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Conversely, if $\mathcal{D}$ is locally n-truncated, then F exhibits $\mathcal{D}$ as a local n-truncation of $\mathcal{C}$ if and only if it is essentially surjective and satisfies the following weaker version of condition (2):

($2'$) For every pair of objects $X, Y \in \mathcal{C}$, the map of Kan complexes

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is $(n + 1)$-connective.

Example 4.8.2.11. Let $\mathcal{C}$ be an ∞-category and let $\mathrm{h}\mathcal{C}$ be its homotopy category. Then the tautological map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ exhibits $\mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ as a local 0-truncation of $\mathcal{C}$. `058E`

Remark 4.8.2.12. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, where F exhibits $\mathcal{D}$ as a local n-truncation of $\mathcal{C}$. Then $(G \circ F) : \mathcal{C} \to \mathcal{E}$ exhibits $\mathcal{E}$ as an n-truncation of $\mathcal{C}$ if and only if G is an equivalence of ∞-categories. `058F`

Proposition 4.8.2.13. *Let $\mathcal{C}$ be an ∞-category, let $n \geq 0$ be an integer, and let $\mathrm{cosk}_n(\mathcal{C})$ denote the n-coskeleton of $\mathcal{C}$ (Notation 3.5.3.18). Then:* `058G`

(1) *The simplicial set $\mathrm{cosk}_n(\mathcal{C})$ is an ∞-category.*

(2) *The tautological map $\mathcal{C} \to \mathrm{cosk}_n(\mathcal{C})$ exhibits $\mathrm{cosk}_n(\mathcal{C})$ as a local $(n - 2)$-truncation of $\mathcal{C}$.*

Proof. We first prove (1). We proceed as in the proof of Proposition 3.5.3.23. Fix integers $0 < i < m$ and a morphism of simplicial sets $\sigma_0 : \Lambda^m_i \to \mathrm{cosk}_n(\mathcal{C})$; we wish to show that σ_0 can be extended to an m-simplex of $\mathrm{cosk}_n(\mathcal{C})$. Using Remark 3.5.3.21, we can identify σ_0 with a morphism of simplicial sets $f_0 : \mathrm{sk}_n(\Lambda^m_i) \to \mathcal{C}$; we wish to show that f_0 can be extended to the n-skeleton of Δ^m. If $n < m - 1$, then $\mathrm{sk}_n(\Lambda^m_i) = \mathrm{sk}_n(\Delta^m)$ and there is nothing to prove. We may therefore assume that $n \geq m - 1$, so that $\mathrm{sk}_n(\Lambda^m_i) = \Lambda^m_i$. In this case, our assumption that $\mathcal{C}$ is an ∞-category guarantees that f_0 can be extended to an n-simplex of $\mathcal{C}$.

We now prove (2). By construction, the tautological map $F : \mathcal{C} \to \mathrm{cosk}_n(\mathcal{C})$ is bijective on objects, and therefore essentially surjective. By virtue of Proposition 4.6.5.10, it will suffice to show that for every pair of objects $X, Y \in \mathcal{C}$, the induced map of pinched morphism spaces

$$\theta : \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{cosk}_n(\mathcal{C})}(X, Y)$$

exhibits $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{cosk}_n(\mathcal{C})}(X, Y)$ as an $(n-2)$-truncation of $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$. This is a special case of Example 3.5.7.23, since θ exhibits $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{cosk}_n(\mathcal{C})}(X, Y)$ as an $(n-1)$-coskeleton of $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ (Remark 4.6.5.4). □

058H **Proposition 4.8.2.14.** *Let n be an integer and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which exhibits $\mathcal{D}$ as a local n-truncation of $\mathcal{C}$. Then $\mathcal{C}$ is locally n-truncated if and only if F is an equivalence of ∞-categories.*

Proof. By assumption, F is essentially surjective. It follows from Theorem 4.6.2.20 that F is an equivalence of ∞-categories if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map of Kan complexes

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homotopy equivalence. Since $F_{X,Y}$ exhibits $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ as an n-truncation of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, this is equivalent to the requirement that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is n-truncated. □

058J **Corollary 4.8.2.15.** *Let $\mathcal{C}$ be an ∞-category and let $\mathrm{h}\mathcal{C}$ denote its homotopy category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is locally 0-truncated.*

(2) *The comparison map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ is an equivalence of ∞-categories.*

(3) *The comparison map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ is a trivial Kan fibration.*

(4) *The ∞-category $\mathcal{C}$ is equivalent to (the nerve of) an ordinary category.*

Proof. The implication (1) ⇒ (2) follows from Example 4.8.2.11 and Proposition 4.8.2.14. Since the comparison map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathcal{C})$ is an isofibration (Corollary 4.4.1.9), the equivalence (2) ⇔ (3) follows from Proposition 4.5.5.20. The implication (2) ⇒ (4) is clear, and the implication (4) ⇒ (1) follows from Example 4.8.2.2. □

058K **Exercise 4.8.2.16.** Show that an ∞-category $\mathcal{C}$ is locally (-1)-truncated if and only if there is an equivalence of ∞-categories $u : \mathcal{C} \to \mathrm{N}_{\bullet}(Q)$, for some partially ordered set Q. In this case, the morphism u is automatically a trivial Kan fibration (see Example 4.4.1.6 and Proposition 4.5.5.20).

058L **Corollary 4.8.2.17.** *Let $\mathcal{C}$ be an ∞-category and let $n \geq 0$ be an integer. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is locally $(n-2)$-truncated.*

(2) *The tautological map $\mathcal{C} \to \mathrm{cosk}_n(\mathcal{C})$ is an equivalence of ∞-categories.*

(3) *There exists an n-coskeletal ∞-category $\mathcal{D}$ which is equivalent to $\mathcal{C}$.*

Proof. The implication (1) ⇒ (2) follows from Propositions 4.8.2.13 and 4.8.2.14. The implication (2) ⇒ (3) is clear (since $\mathrm{cosk}_n(\mathcal{C})$ is an ∞-category), and the implication (3) ⇒ (1) follows from Proposition 4.8.2.8 (together with Remark 4.8.2.3). □

Local n-truncations can be characterized by a universal mapping property:

Proposition 4.8.2.18. *Let n be an integer and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, where $\mathcal{D}$ is locally n-truncated. The following conditions are equivalent:* 058M

(1) *The functor F exhibits $\mathcal{D}$ as a local n-truncation of $\mathcal{C}$, in the sense of Definition 4.8.2.9.*

(2) *For every locally n-truncated ∞-category $\mathcal{E}$, precomposition with F induces an equivalence of ∞-categories $\operatorname{Fun}(\mathcal{D}, \mathcal{C}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$.*

(3) *For every locally n-truncated ∞-category $\mathcal{E}$, precomposition with F induces a bijection*

$$\pi_0(\operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}) \to \pi_0(\operatorname{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}).$$

Proof. Without loss of generality we may assume that $n \geq -2$. We first show that (1) implies (2). By virtue of Corollary 4.8.2.17, we can assume without loss of generality that $\mathcal{D}$ and $\mathcal{E}$ are $(n+2)$-coskeletal. In this case, F factors (uniquely) as a composition $\mathcal{C} \xrightarrow{F'} \operatorname{cosk}_{n+2}(\mathcal{C}) \xrightarrow{F''} \mathcal{D}$, where F' exhibits $\operatorname{cosk}_{n+2}(\mathcal{C})$ as a local n-truncation of $\mathcal{C}$ (Proposition 4.8.2.13). Applying Remark 4.8.2.12, we see that F'' is an equivalence of ∞-categories. We may therefore replace $\mathcal{D}$ by $\operatorname{cosk}_{n+2}(\mathcal{C})$ and thereby reduce to the case where F exhibits $\mathcal{D}$ as an $(n+2)$-coskeleton of $\mathcal{C}$. In this case, Proposition 3.5.3.17 guarantees that the precomposition functor

$$\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \operatorname{Fun}(\mathcal{C}, \mathcal{E})$$

is an isomorphism of simplicial sets (and therefore an equivalence of ∞-categories).

The implication (2) $\Rightarrow$ (3) follows immediately from the definitions. We will complete the proof by showing that (3) implies (1). As before, we may assume that $\mathcal{D} = \operatorname{cosk}_{n+2}(\mathcal{D})$ is $(n+2)$-coskeletal from Proposition 4.5.1.22, so that F factors (uniquely) as a composition $\mathcal{C} \xrightarrow{F'} \operatorname{cosk}_{n+2}(\mathcal{C}) \xrightarrow{F''} \mathcal{D}$; we wish to show that the homotopy class $[F'']$ is an isomorphism in the homotopy category hQCat. Since $\operatorname{cosk}_{n+2}(\mathcal{C})$ and $\mathcal{D}$ are locally n-truncated, it will suffice to show that for every locally n-truncated ∞-category $\mathcal{E}$, the horizontal map in the diagram

$$\begin{tikzcd}
\pi_0(\operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}) \arrow[rr] \arrow[dr] & & \pi_0(\operatorname{Fun}(\operatorname{cosk}_{n+2}(\mathcal{C}), \mathcal{E})^{\simeq}) \arrow[dl] \\
& \pi_0(\operatorname{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}) &
\end{tikzcd}$$

is a bijection. This is clear, since the vertical maps are bijections. $\square$

Let hQCat denote the homotopy category of ∞-categories (Construction 4.5.1.1). For every integer $n \geq -1$, we let $\mathrm{hQCat}^{\leq n}$ denote the full subcategory of hQCat spanned by the those ∞-categories $\mathcal{C}$ which are locally $(n-1)$-truncated. We then have the following:

058N **Corollary 4.8.2.19.** *Let $n \geq -1$ be an integer. Then the inclusion functor* $\mathrm{hQCat}^{\leq n} \hookrightarrow \mathrm{hQCat}$ *admits a left adjoint, given on objects by the construction* $\mathcal{C} \mapsto \mathrm{cosk}_{n+1}(\mathcal{C})$.

4.8.3 Minimality Conditions

058P Let $\mathcal{C}$ be an ∞-category and let n be an integer. We then have the following implications (see Proposition 4.8.2.8):

$$\begin{tikzcd}
\mathcal{C} \text{ is an } (n,1)\text{-category} \arrow[d, Rightarrow] \\
\mathcal{C} \text{ is weakly } n\text{-coskeletal} \arrow[d, Rightarrow] \\
\mathcal{C} \text{ is } (n+1)\text{-coskeletal} \arrow[d, Rightarrow] \\
\mathcal{C} \text{ is locally } (n-1)\text{-truncated.}
\end{tikzcd}$$

Beware that, in general, none of these implications is reversible. However, the failure of reversibility can be measured using the minimality conditions introduced in §4.7.6.

058Q **Proposition 4.8.3.1.** *Let n be an integer and let $\mathcal{C}$ be an ∞-category. Assume that, if $n \leq -2$, then $\mathcal{C}$ is nonempty. Then:*

(1) *The ∞-category $\mathcal{C}$ is $(n+1)$-coskeletal if and only if it is locally $(n-1)$-truncated and minimal in dimensions $\geq n+2$ (see Definition 4.7.6.4).*

(2) *The ∞-category $\mathcal{C}$ is weakly n-coskeletal if and only if it is locally $(n-1)$-truncated and minimal in dimensions $\geq n+1$.*

(3) *The ∞-category $\mathcal{C}$ is an $(n,1)$-category if and only if it is locally $(n-1)$-truncated and minimal in dimensions $\geq n$.*

We will give the proof of Proposition 4.8.3.1 at the end of this section. First, let us collect some consequences.

Corollary 4.8.3.2. *Let $\mathcal{C}$ be a minimal ∞-category and let $n \geq -1$ be an integer. Then $\mathcal{C}$ is an $(n,1)$-category if and only if it is locally $(n-1)$-truncated.* 058R

Corollary 4.8.3.3. *Let $\mathcal{C}$ be an ∞-category and let $n \geq -1$ be an integer. Then $\mathcal{C}$ is locally $(n-1)$-truncated if and only if it is equivalent to an $(n,1)$-category.* 058S

Proof. Combine Proposition 4.7.6.15 with Corollary 4.8.3.2. □

Corollary 4.8.3.4. *Let n be an integer and let $\mathcal{C}$ be an ∞-category which is weakly $(n-1)$-coskeletal. Then $\mathcal{C}$ is an $(n,1)$-category.* 058T

Corollary 4.8.3.5. *Let n be an integer and let $\mathcal{C}$ be an $(n,1)$-category. Then, for every pair of objects $X, Y \in \mathcal{C}$, the morphism space $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is weakly $(n-1)$-coskeletal.* 058U

Proof. For $n < 0$, the result is trivial (see Example 4.8.1.9). We will therefore assume that $n \geq 0$. It follows from Corollary 4.8.1.23 that the morphism space $\mathrm{Hom}_{\mathcal{C}}(X,Y) = \{X\} \tilde{\times}_{\mathcal{C}} \{Y\}$ is also $(n,1)$-category; in particular, it is minimal in dimensions $\geq n$. Since $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is $(n-1)$-truncated (Corollary 4.8.1.20), it is locally $(n-2)$-truncated (Example 4.8.2.4). Applying Proposition 4.8.3.1, we conclude that $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is weakly $(n-1)$-coskeletal. □

Our proof of Proposition 4.8.3.1 will make use of some auxiliary results of independent interest. Recall that, if $\mathcal{C}$ is a simplicial set, then the *weak n-coskeleton* $\mathrm{cosk}^{\circ}_n(\mathcal{C})$ is the simplicial subset of $\mathrm{cosk}_n(\mathcal{C})$ given by the image of the tautological map $\mathrm{cosk}_{n+1}(\mathcal{C}) \to \mathrm{cosk}_n(\mathcal{C})$ (see Notation 3.5.4.19).

Proposition 4.8.3.6. *Let $\mathcal{C}$ be an ∞-category, let n be an integer, and let $\mathrm{cosk}^{\circ}_n(\mathcal{C})$ denote the weak n-coskeleton of $\mathcal{C}$. Then:* 058V

(1) *The simplicial set $\mathrm{cosk}^{\circ}_n(\mathcal{C})$ is an ∞-category.*

(2) *The tautological map $F : \mathcal{C} \to \mathrm{cosk}^{\circ}_n(\mathcal{C})$ is an inner fibration of ∞-categories.*

(3) *If $n \geq -1$, the functor F exhibits exhibits $\mathrm{cosk}^{\circ}_n(\mathcal{C})$ as a local $(n-1)$-truncation of $\mathcal{C}$.*

(4) *If $n \neq 0$, then F is an isofibration of ∞-categories.*

Proof. For $n < 0$, the weak coskeleton $\mathrm{cosk}^{\circ}_n(\mathcal{C})$ is either empty (if $n = -1$ and $\mathcal{C}$ is empty) or isomorphic to Δ^0; in either case, assertions (1) through (4) are clear. We may therefore assume that $n \geq 0$. The map F factors as a composition

$$\mathcal{C} \xrightarrow{F'} \mathrm{cosk}_{n+1}(\mathcal{C}) \xrightarrow{F''} \mathrm{cosk}^{\circ}_n(\mathcal{C}),$$

where F'' is a trivial Kan fibration (Proposition 3.5.4.22). Since $\mathrm{cosk}_{n+1}(\mathcal{C})$ is an ∞-category (Proposition 4.8.2.13), assertion (1) follows from Proposition 1.5.5.11.

To prove (2), we proceed as in Proposition 3.5.4.26. Suppose we are given a pair of integers $0 < i < m$; we wish to show that every lifting problem

058W
$$\begin{tikzcd}
\Lambda^m_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \operatorname{cosk}^{\circ}_n(\mathcal{C})
\end{tikzcd} \tag{4.48}$$

admits a solution. We consider two cases:

- If $m \leq n+1$, then we can choose an m-simplex σ of $\mathcal{C}$ satisfying $F(\sigma) = \overline{\sigma}$. Since F is bijective on simplices of dimension $\leq n$, the commutativity of the diagram (4.48) guarantees that $\sigma|_{\Lambda^m_i} = \sigma_0$.
- If $m \geq n+2$, then our assumption that $\mathcal{C}$ is an ∞-category guarantees that σ_0 can be extended to an m-simplex σ of X. The commutativity of the diagram (4.48) then guarantees that $F(\sigma)$ and $\overline{\sigma}$ have the same restriction to the horn $\Lambda^m_i \subset \Delta^m$. In particular, they have the same restriction to the n-skeleton of Δ^m, so $F(\sigma) = \overline{\sigma}$.

Since F'' is an equivalence of ∞-categories, assertion (3) follows by combining Proposition 4.8.2.13 with Remark 4.8.2.12. It remains to prove (4). Let Y be an object of $\mathcal{C}$ and suppose we are given an isomorphism morphism $\overline{u} : \overline{X} \to Y$ in the ∞-category $\operatorname{cosk}^{\circ}_n(\mathcal{C})$. If $n \geq 1$, then F is bijective on vertices and edges; it follows that we can write $\overline{u} = F(u)$ for a unique morphism $u : X \to Y$ in $\mathcal{C}$. To complete the proof, it will suffice to show that u is an isomorphism. Equivalently, we wish to show that the homotopy class $[u]$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$. This is clear: the nerve $\mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ is weakly 1-coskeletal (Example 3.5.4.6), so the tautological map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ factors (uniquely) through $\operatorname{cosk}^{\circ}_n(\mathcal{C})$ (Proposition 3.5.4.18). □

058X **Warning 4.8.3.7.** . The functor $F : \mathcal{C} \to \operatorname{cosk}^{\circ}_n(\mathcal{C})$ is generally not an isofibration in the case $n = 0$.

058Y **Warning 4.8.3.8.** In the situation of Proposition 4.8.3.6, the map $\mathcal{C} \to \operatorname{cosk}_{n+1}(\mathcal{C})$ is generally not an inner fibration.

058Z **Corollary 4.8.3.9.** *Let $\mathcal{C}$ be an ∞-category and let $n \geq -2$ be an integer. Then $\mathcal{C}$ is locally n-truncated if and only if the tautological map $F : \mathcal{C} \to \operatorname{cosk}^{\circ}_{n+1}(\mathcal{C})$ is a trivial Kan fibration.*

Proof. It follows from Proposition 4.8.3.6 that F exhibits $\operatorname{cosk}^{\circ}_{n+1}(\mathcal{C})$ as a local n-truncation of $\mathcal{C}$. Applying Proposition 4.8.2.14, we see that $\mathcal{C}$ is locally n-truncated if and only if F is an equivalence of ∞-categories. We wish to show that if this condition is satisfied, then F

is a trivial Kan fibration. By virtue of Proposition 4.5.5.20, it will suffice to show that F is an isofibration. For $n \neq -1$, this is automatic (Proposition 4.8.3.6). We will therefore assume that $n = -1$. Using Proposition 4.8.3.6, we see that F is an inner fibration. Fix a morphism $\overline{u} : \overline{X} \to \overline{Y}$ in the ∞-category $\operatorname{cosk}_0^{\circ}(\mathcal{C})$. Then there are unique objects $X, Y \in \mathcal{C}$ satisfying $\overline{X} = F(X)$ and $\overline{Y} = F(Y)$. Choose a morphism $u : X \to Y$ satisfying $F(u) = \overline{u}$. To complete the proof, it will suffice to show that if $\overline{u}$ is an isomorphism in $\operatorname{cosk}_0^{\circ}(\mathcal{C})$, then u is an isomorphism in $\mathcal{C}$. Let $\overline{v} : \overline{Y} \to \overline{X}$ be a homotopy inverse to $\overline{u}$. Then we can write $\overline{v} = F(v)$ for some morphism $v : Y \to X$ of $\mathcal{C}$. Since the mapping space $\operatorname{Hom}_{\mathcal{C}}(X, X)$ is either empty or contractible, the composition $v \circ u$ is automatically homotopic to id_X: that is v is a left homotopy inverse to u. A similar argument shows that v is right homotopy inverse to u, so that u is an isomorphism as desired. □

Corollary 4.8.3.10. *Let $\mathcal{C}$ be an ∞-category and let $n \geq 0$ be an integer. The following conditions are equivalent:* 0590

(1) *For every morphism $f : X \to Y$ of $\mathcal{C}$, the set $\pi_n(\operatorname{Hom}_{\mathcal{C}}(X, Y), f)$ consists of a single element.*

(2) *Every diagram $\partial \Delta^{n+2} \to \mathcal{C}$ can be extended to an $(n+2)$-simplex of $\mathcal{C}$.*

Proof. By virtue of Proposition 4.8.2.13, we can replace $\mathcal{C}$ by $\operatorname{cosk}_{n+2}(\mathcal{C})$ and thereby reduce to the case where the ∞-category $\mathcal{C}$ is $(n+2)$-coskeletal. In this case, the ∞-category $\mathcal{C}$ is locally n-truncated (Proposition 4.8.2.8), and satisfies condition (1) if and only if it is locally $(n-1)$-truncated. Applying Corollary 4.8.3.9, we see that (1) is equivalent to the following:

(1′) The tautological map $\mathcal{C} \to \operatorname{cosk}_n^{\circ}(\mathcal{C})$ is a trivial Kan fibration.

The equivalence of (1′) and (2) now follows from Corollary 3.5.4.24. □

Corollary 4.8.3.11. *Let $\mathcal{C}$ be an ∞-category and let $n \geq -2$ be an integer. Then $\mathcal{C}$ is locally n-truncated if and only if the restriction map* 0591

$$\operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^m, \mathcal{C}) \to \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^m, \mathcal{C})$$

is surjective for every integer $m \geq n+3$.

Proposition 4.8.3.12. *Let $\mathcal{C}$ be an ∞-category and let $m > 0$ be an integer. Then the restriction map* 0592

$$\theta_m : \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^m, \mathcal{C}) \to \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\partial \Delta^m, \mathcal{C})$$

is injective if and only if the following conditions are satisfied:

(1) *The ∞-category $\mathcal{C}$ is minimal in dimension m.*

(2) *The restriction map θ_{m+1} is surjective.*

0593 **Remark 4.8.3.13.** In the case $m = 0$, the formulation of Proposition 4.8.3.12 requires a slight modification. The restriction map θ_0 is injective if and only if $\mathcal{C}$ satisfies the following pair of conditions:

(1) The ∞-category $\mathcal{C}$ is minimal in dimension 0: that is, if X and Y are isomorphic objects of $\mathcal{C}$, then $X = Y$.

(2′) For every pair of objects $X, Y \in \mathcal{C}$, there exists an isomorphism from X to Y.

Note that condition (2′) is stronger than condition (2) of Proposition 4.8.3.12, which demands only that there exists a morphism from X to Y.

Proof of Proposition 4.8.3.12. Let $\mathcal{C}$ be an integer and let $m > 0$ be an integer. It follows immediately from the definitions that, if the restriction map

$$\theta_m : \operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \mathcal{C}) \to \operatorname{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^m, \mathcal{C})$$

is injective, then $\mathcal{C}$ is minimal in dimension m. We claim that, if this condition is satisfied, then θ_{m+1} is surjective: that is, every morphism $\tau_0 : \partial\Delta^{m+1} \to \mathcal{C}$ can be extended to an $(m+1)$-simplex of $\mathcal{C}$. Fix an integer $0 < i < m+1$. Our assumption that $\mathcal{C}$ is an ∞-category guarantees that we can choose an $(m+1)$-simplex τ of $\mathcal{C}$ satisfying $\tau|_{\Lambda^{m+1}_i} = \tau_0|_{\Lambda^{m+1}_i}$. In particular, τ and τ_0 have the same restriction to the $(m-1)$-skeleton of Δ^{m+1}. Invoking the injectivity of θ_m, we conclude that $\tau|_{\partial\Delta^{m+1}} = \tau_0$.

We now prove the converse. Assume that $\mathcal{C}$ is minimal in dimension m and that θ_{m+1} is surjective; we wish to show that θ_m is injective. Let σ_0 and σ_1 be m-simplices of $\mathcal{C}$ which have the same restriction to $\partial\Delta^m$; we wish to show that $\sigma_0 = \sigma_1$. Let

$$X(0) \subset X(1) \subset \cdots \subset X(m) \subset X(m+1) = \Delta^1 \times \Delta^m$$

be the filtration of Lemma 3.1.2.12, so that $X(0) = (\Delta^1 \times \partial\Delta^m) \cup (\{1\} \times \Delta^m)$ and the inclusion map $X(i) \hookrightarrow X(i+1)$ is inner anodyne for $0 \leq i < m$. There is a unique morphism of simplicial sets $h_0 : X(0) \to \mathcal{C}$ such that $h_0|_{\{1\}\times\Delta^m}$ coincides with σ_1, and $h_0|_{\Delta^1\times\partial\Delta^m}$ factors through the projection map $\Delta^1 \times \partial\Delta^m \twoheadrightarrow \partial\Delta^m$. Since $\mathcal{C}$ is an ∞-category, we can extend h_0 to a diagram $h_m : X(m) \to \mathcal{C}$. Invoking the surjectivity of θ_{m+1}, we see that h_m can be extended to a morphism $h : \Delta^1 \times \Delta^m \to \mathcal{C}$ satisfying $h|_{\{0\}\times\Delta^m} = \sigma_0$. By construction, h is an isomorphism from σ_0 to σ_1 in the ∞-category $\operatorname{Fun}(\Delta^m, \mathcal{C})$ whose image in $\operatorname{Fun}(\partial\Delta^m, \mathcal{C})$ is an identity morphism. Since $\mathcal{C}$ is minimal in dimension m, it follows that $\sigma_0 = \sigma_1$. □

Proof of Proposition 4.8.3.1. Let $\mathcal{C}$ be an ∞-category and let n be an integer. For every integer $m \geq 0$, we let $\theta_m : \operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \mathcal{C}) \to \operatorname{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^m, \mathcal{C})$ denote the restriction map. By virtue of Corollary 4.8.3.11, the map θ_m is surjective for $m \geq n+2$ if and only if $\mathcal{C}$ is locally $(n-1)$-truncated (and nonempty if $n \leq -2$). Assume that these equivalent conditions are satisfied. Then:

(1) The ∞-category $\mathcal{C}$ is $(n+1)$-coskeletal if and only if θ_m is injective for $m \geq n+2$. By virtue of Proposition 4.8.3.12 (and Remark 4.8.3.13), this is equivalent to the requirement that $\mathcal{C}$ is minimal in dimensions $\geq n+2$.

(2) The ∞-category $\mathcal{C}$ is weakly n-coskeletal if and only if θ_m is injective for $m \geq n+1$. By virtue of Proposition 4.8.3.12 (and Remark 4.8.3.13), this is equivalent to the requirement that $\mathcal{C}$ is minimal in dimensions $\geq n+1$.

(3) The ∞-category $\mathcal{C}$ is an $(n,1)$-category if and only if $\mathcal{C}$ is minimal in dimensions $\geq n$. This follows immediately from (2) (see Definition 4.8.1.8).

□

We conclude this section by recording another consequence of Proposition 4.8.3.12.

Corollary 4.8.3.14. *Let X be a Kan complex and let n be a nonnegative integer. The following conditions are equivalent:* 0594

(*a*) *The Kan complex X is n-reduced: that is, it has a single m-simplex for each $0 \leq m \leq n$ (Definition 3.5.2.8).*

(*b*) *The Kan complex X is $(n+1)$-connective and minimal in dimensions $\leq n$.*

Proof. Without loss of generality, we may assume that X is nonempty (otherwise, neither (*a*) nor (*b*) is satisfied). In this case, X is n-reduced if and only if the restriction map $\theta_m : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\partial \Delta^m, \mathcal{C})$ is injective for each $m \leq n$. Corollary 4.8.3.14 now follows by combining Proposition 4.8.3.12 (and Remark 4.8.3.13) with the criterion of Proposition 3.5.1.12. □

Corollary 4.8.3.15. *Let X be a minimal Kan complex and let $n \geq 0$ be an integer. Then X is n-reduced if and only if it is $(n+1)$-connective.* 0595

4.8.4 Higher Homotopy Categories

Let $\mathcal{C}$ be an ∞-category. In §1.4.5, we constructed the *homotopy category* $\mathrm{h}\mathcal{C}$, and showed that it is characterized (up to isomorphism) by the following universal property: for any 0596 category $\mathcal{D}$, there is a bijection

$$\begin{tikzcd}
\{\text{Functors of ordinary categories } \mathrm{h}\mathcal{C} \to \mathcal{D}\} \arrow[d, "\sim"] \\
\{\text{Functors of } \infty\text{-categories } \mathcal{C} \to \mathrm{N}_{\bullet}(\mathcal{D})\}.
\end{tikzcd}$$

This motivates the following:

0597 **Definition 4.8.4.1.** Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of ∞-categories and let n be an integer. We say that F *exhibits* $\mathcal{C}'$ *as a homotopy* n*-category of* $\mathcal{C}$ if the following conditions are satisfied:

(1) The ∞-category $\mathcal{C}'$ is an $(n, 1)$-category (Definition 4.8.1.8).

(2) For every $(n, 1)$-category $\mathcal{D}$, precomposition with F induces a bijection

$$\begin{tikzcd}
\{\text{Functors of } (n,1)\text{-categories } \mathcal{C}' \to \mathcal{D}\} \arrow[d, "\sim"] \\
\{\text{Functors of } \infty\text{-categories } \mathcal{C} \to \mathcal{D}\}.
\end{tikzcd}$$

0598 **Notation 4.8.4.2.** Let n be a nonnegative integer. We will see in a moment that for every ∞-category $\mathcal{C}$, there exists a functor $F : \mathcal{C} \to \mathcal{C}'$ which exhibits $\mathcal{C}'$ as a homotopy n-category of $\mathcal{C}$ (Corollary 4.8.4.16). It follows immediately from the definition that the simplicial set $\mathcal{C}'$ is unique up to (canonical) isomorphism and depends functorially on $\mathcal{C}$. To emphasize this dependence, we will often denote $\mathcal{C}'$ by $\mathrm{h}_{\leq n}(\mathcal{C})$ and refer to it as *the* homotopy n-category of $\mathcal{C}$. For a more explicit description of $\mathrm{h}_{\leq n}(\mathcal{C})$ (at least for $n > 0$), see Construction 4.8.4.9 (and Proposition 4.8.4.15).

0599 **Example 4.8.4.3.** Let $\mathcal{C}$ be an ∞-category and let $\mathrm{h}\mathcal{C}$ denote its homotopy category (Definition 1.4.5.3). Then the comparison map $F : \mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ of Construction 1.4.5.6 exhibits $\mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ as a homotopy 1-category of $\mathcal{C}$, in the sense of Definition 4.8.4.1. This is a reformulation of Proposition 1.4.5.7 (see Example 4.8.1.3). Stated more informally, there is a canonical isomorphism of simplicial sets $\mathrm{h}_{\leq 1}(\mathcal{C}) \simeq \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$. We will sometimes abuse notation by identifying the homotopy 1-category $\mathrm{h}_{\leq 1}(\mathcal{C})$ with the ordinary category $\mathrm{h}\mathcal{C}$.

059A **Exercise 4.8.4.4.** Let $\mathcal{C}$ be an ∞-category, and let $Q = \pi_0(\mathcal{C}^{\simeq})$ denote the collection of isomorphism classes of objects of $\mathcal{C}$. For each object $X \in \mathcal{C}$, let $[X] \in Q$ denote its isomorphism class. Show that:

- There is a partial ordering $\leq_Q$ on the set Q, where $[X] \leq_Q [Y]$ if and only if there exists a morphism from X to Y in the ∞-category $\mathcal{C}$.
- There is a unique functor $F : \mathcal{C} \to \mathrm{N}_\bullet(Q)$ which carries each object $X \in \mathcal{C}$ to the isomorphism class $[X] \in Q$.
- The functor F exhibits $\mathrm{N}_\bullet(Q)$ as a homotopy 0-category of $\mathcal{C}$, in the sense of Definition 4.8.4.1.

Example 4.8.4.5. Let $\mathcal{C}$ be an ∞-category. Then: `059B`

- For every integer $n \leq -2$, the unique functor $F : \mathcal{C} \to \Delta^0$ exhibits Δ^0 as a homotopy n-category of $\mathcal{C}$.
- If $\mathcal{C}$ is nonempty, then F also exhibits Δ^0 as a homotopy (-1)-category of $\mathcal{C}$.
- If $\mathcal{C}$ is empty, then the identity map $\mathrm{id} : \mathcal{C} \to \emptyset$ exhibits the empty simplicial set as a homotopy (-1)-category of $\mathcal{C}$.

Example 4.8.4.6. Let X be a Kan complex, let n be a nonnegative integer, and let $\pi_{\leq n}(X)$ denote the fundamental n-groupoid of X (Notation 3.5.6.6). Then the tautological map $u : X \to \pi_{\leq n}(X)$ exhibits $\pi_{\leq n}(X)$ as a homotopy n-category of X, in the sense of Definition 4.8.4.1. Since $\pi_{\leq n}(X)$ is an $(n,1)$-category (Example 4.8.1.11), it will suffice that for every $(n,1)$-category $\mathcal{D}$, precomposition with u induces a bijection $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\pi_{\leq n}(X), \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(X, \mathcal{D})$. By virtue of Proposition 4.4.3.17, we can replace $\mathcal{D}$ by its core $\mathcal{D}^{\simeq}$, and thereby reduce to the case where $\mathcal{D}$ is an n-groupoid (Example 4.8.1.17). In this case, the desired result follows from the universal property of Proposition 3.5.6.5. `059C`

In the situation of Notation 4.8.4.2, the homotopy n-category $\mathrm{h}_{\leq n}(\mathcal{C})$ automatically satisfies a stronger universal property:

Proposition 4.8.4.7. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of ∞-categories and let n be an integer. The following conditions are equivalent:* `059D`

(1) *For every $(n,1)$-category $\mathcal{D}$, precomposition with F induces a bijection of sets*

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}', \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathcal{D}).$$

(2) *For every $(n,1)$-category $\mathcal{D}$, precomposition with F induces an isomorphism of ∞-categories*

$$\mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Proof. Assume that (1) is satisfied; we will prove (2) (the reverse implication follows immediately from the definitions). Let $\mathcal{D}$ be an $(n,1)$-category; we wish to show that precomposition with F induces an isomorphism of simplicial sets from $\operatorname{Fun}(\mathcal{C}', \mathcal{D})$ to $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$. Equivalently, we wish to show that for every simplicial set K, the induced map

$$\operatorname{Hom}_{\mathrm{Set}_\Delta}(K, \operatorname{Fun}(\mathcal{C}', \mathcal{D})) \to \operatorname{Hom}_{\mathrm{Set}_\Delta}(K, \operatorname{Fun}(\mathcal{C}, \mathcal{D})).$$

This follows by applying condition (1) to the simplicial set $\operatorname{Fun}(K, \mathcal{D})$, which is an $(n,1)$-category by virtue of Proposition 4.8.1.22. □

059E **Corollary 4.8.4.8.** *Let $n \geq -1$ be an integer and let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of ∞-categories which exhibits $\mathcal{C}'$ as a homotopy n-category of $\mathcal{C}$ (Definition 4.8.4.1). Then F exhibits $\mathcal{C}'$ as a local $(n-1)$-truncation of $\mathcal{C}$ (Definition 4.8.2.9).*

Proof. Since $\mathcal{C}'$ is an $(n,1)$-category, it is locally $(n-1)$-truncated (Example 4.8.2.2). It will therefore suffice to show that, for every locally $(n-1)$-truncated ∞-category $\mathcal{D}$, precomposition with F induces an equivalence of ∞-categories $\theta : \operatorname{Fun}(\mathcal{C}', \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ (Proposition 4.8.2.18). By virtue of Corollary 4.8.3.3, we may assume that $\mathcal{D}$ is an $(n,1)$-category. In this case, Proposition 4.8.4.7 guarantees that θ is an isomorphism of simplicial sets. □

Our next goal is to show that every ∞-category $\mathcal{C}$ admits a homotopy n-category, for every integer n. For $n \leq 0$, this follows from Exercise 4.8.4.4 and Example 4.8.4.5. To handle the case $n > 0$, we will use a generalization of Construction 3.5.6.10.

059F **Construction 4.8.4.9.** Let $\mathcal{C}$ be an ∞-category, let n be a positive integer, and let $\operatorname{cosk}_n^\circ(\mathcal{C})$ denote the weak n-coskeleton of $\mathcal{C}$ (Notation 3.5.4.19). For every integer $m \geq 0$, we will identify m-simplices of $\operatorname{cosk}_n^\circ(\mathcal{C})$ with diagrams $\sigma : \operatorname{sk}_n(\Delta^m) \to \mathcal{C}$ which can be extended to the $(n+1)$-skeleton of Δ^m (Remark 3.5.4.21). Given two such morphisms $\sigma, \sigma' : \operatorname{sk}_n(\Delta^m) \to \mathcal{C}$, we write $\sigma \sim_m \sigma'$ if σ and σ' are isomorphic relative to $\operatorname{sk}_{n-1}(\Delta^m)$ (Definition 4.7.6.1). The construction

$$([m] \in \mathbf{\Delta}^{\mathrm{op}}) \mapsto \operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, \operatorname{cosk}_n^\circ(\mathcal{C})) / \sim_m$$

determines a simplicial set, which we will denote by $\mathrm{h}_{\leq n}(\mathcal{C})$. By construction, it is equipped with an epimorphism of simplicial sets $\operatorname{cosk}_n^\circ(\mathcal{C}) \twoheadrightarrow \mathrm{h}_{\leq n}(\mathcal{C})$, which determines a comparison map $\mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C})$.

059G **Remark 4.8.4.10.** In the situation of Construction 4.8.4.9, the relation $\sigma \sim_m \sigma'$ implies that $\sigma = \sigma'$ whenever $m < n$. It follows that the tautological map $\mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C})$ is bijective on simplices of dimension $< n$, and surjective on simplices of dimension n.

Proposition 4.8.4.11. *Let $\mathcal{C}$ be an ∞-category, let n be a positive integer, and let $\mathrm{h}_{\leq n}(\mathcal{C})$ be the simplicial set of Construction 4.8.4.9. Then, for every simplicial set A, the comparison map* 059H

$$\theta : \mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{cosk}^\circ_n(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}))$$

is surjective. Moreover, if $f_0, f_1 : A \to \mathrm{cosk}^\circ_n(\mathcal{C})$ are morphisms of simplicial sets which correspond to diagrams $u_0, u_1 : \mathrm{sk}_n(A) \to \mathcal{C}$, then $\theta(f_0) = \theta(f_1)$ if and only if u_0 and u_1 are isomorphic relative to $\mathrm{sk}_{n-1}(A)$.

Proof. We proceed as in the proof of Proposition 3.5.6.12. Fix a morphism of simplicial sets $g : A \to \mathrm{h}_{\leq n}(\mathcal{C})$. Using Remark 4.8.4.10 (and Proposition 1.1.4.12), we see that $g|_{\mathrm{sk}_n(A)}$ can be lifted to a morphism of simplicial sets $u : \mathrm{sk}_n(A) \to \mathcal{C}$. We will show that u can be extended to the $(n+1)$-skeleton of A (and is therefore classified by a morphism $f : A \to \mathrm{cosk}^\circ_n(\mathcal{C})$ satisfying $\theta(f) = g$; see Remark 3.5.4.21). By virtue of Proposition 1.1.4.12, this is equivalent to the assertion that for every $(n+1)$-simplex σ of A having restriction $\sigma_0 = \sigma|_{\partial\Delta^{n+1}}$, the composition $(g \circ \sigma_0) : \partial\Delta^{n+1} \to \mathcal{C}$ can be extended to an $(n+1)$-simplex of $\mathcal{C}$. Choose a lift of $g(\sigma)$ to an $(n+1)$-simplex of $\mathrm{cosk}^\circ_n(\mathcal{C})$, which we can identify with a diagram $\tau_0 : \partial\Delta^{n+1} \to \mathcal{C}$ which admits an extension $\tau : \Delta^{n+1} \to \mathcal{C}$. By construction, $g \circ \sigma_0$ and τ_0 coincide after composing with the comparison map $\mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C})$. Using Proposition 1.1.4.12 again, we see that $g \circ \sigma_0$ and τ_0 are isomorphic relative to $\mathrm{sk}_{n-1}(\Delta^{n+1})$. The desired result now follows from Corollary 4.4.5.3. This completes the proof that θ is surjective.

Now suppose that we are given a pair of morphisms $f_0, f_1 : A \to \mathrm{cosk}^\circ_n(\mathcal{C})$ satisfying $\theta(f_0) = \theta(f_1)$. We wish to show that the associated maps $u_0, u_1 : \mathrm{sk}_n(A) \to \mathcal{C}$ are isomorphic relative to $\mathrm{sk}_{n-1}(A)$ (the converse is immediate from the definitions). Using Remark 4.8.4.10, we deduce that u_0 and u_1 coincide on $\mathrm{sk}_{n-1}(A)$. By virtue of Proposition 1.1.4.12, we are reduced to showing that for every nondegenerate n-simplex σ of A, the compositions $u_0 \circ \sigma$ and $u_1 \circ \sigma$ are isomorphic relative to $\partial\Delta^n$. This follows from our assumption that the maps $\theta(f_0), \theta(f_1) : A \to \mathrm{h}_{\leq n}(\mathcal{C})$ coincide on the simplex σ. □

Remark 4.8.4.12. Let $\mathcal{C}$ be an ∞-category, let n be a positive integer, and let A be a simplicial set. Stated more informally, Proposition 4.8.4.11 asserts that $\mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}))$ can be viewed as a subquotient of the set $\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{sk}_n(A), \mathcal{C})$: 059J

- A diagram $u : \mathrm{sk}_n(A) \to \mathcal{C}$ determines a morphism from A to $\mathrm{h}_{\leq n}(\mathcal{C})$ if and only if u can be extended to the $(n+1)$-skeleton of A.
- Two diagrams $u_0, u_1 : \mathrm{sk}_n(A) \to \mathcal{C}$ determine the same morphism A to $\mathrm{h}_{\leq n}(\mathcal{C})$ if and only if they are isomorphic relative to the $(n-1)$-skeleton of A.

Compare with Remark 3.5.6.13.

Corollary 4.8.4.13. *Let $\mathcal{C}$ be an ∞-category and let n be a positive integer. Then the comparison map $\mathrm{cosk}^\circ_n(\mathcal{C}) \twoheadrightarrow \mathrm{h}_{\leq n}(\mathcal{C})$ of Construction 4.8.4.9 is a trivial Kan fibration.* 059K

Proof. Fix an integer $m \geq 0$; we wish to show that every lifting problem

059L
$$\begin{tikzcd}
\partial \Delta^m \arrow[r, "\sigma_0"] \arrow[d] & \operatorname{cosk}_n^{\circ}(\mathcal{C}) \arrow[d, "q"] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & \mathrm{h}_{\leq n}(\mathcal{C})
\end{tikzcd} \tag{4.49}$$

admits a solution.

Let σ be any m-simplex of $\operatorname{cosk}_n^{\circ}(\mathcal{C})$ satisfying $q(\sigma) = \overline{\sigma}$. By virtue of Remark 4.8.4.10, the commutativity of the diagram (4.49) guarantees that σ_0 and σ coincide on the $(n-1)$-skeleton of $\partial \Delta^m$. Consequently, if $m \leq n$, then σ is a solution to the the lifting problem (4.49). We will therefore assume that $m > n$. In this case, the boundary $\partial \Delta^m$ contains the n-skeleton of Δ^m. It will therefore suffice to show that σ_0 can be extended to an m-simplex σ' of $\operatorname{cosk}_n^{\circ}(\mathcal{C})$: the commutativity of the diagram (4.49) guarantees that any such extension satisfies the identity $q(\sigma') = \overline{\sigma}$ (Proposition 4.8.4.11). If $m \geq n+2$, then the existence of σ' is automatic (since $\operatorname{cosk}_n^{\circ}(\mathcal{C})$ is $(n+1)$-coskeletal). It will therefore suffice to treat the case $m = n+1$. In this case, we can identify σ_0 with a diagram $\tau_0 : \partial \Delta^{n+1} \to \mathcal{C}$, and we wish to show that τ_0 can be extended to an $(n+1)$-simplex of $\mathcal{C}$. Note that $\sigma|_{\partial \Delta^{n+1}}$ determines another diagram $\tau_1 : \partial \Delta^{n+1} \to \mathcal{C}$. Moreover, the commutativity of the diagram (4.49) guarantees that τ_0 and τ_1 are isomorphic relative to $\operatorname{sk}_{n-1}(\Delta^{n+1})$ (Proposition 4.8.4.11). Using Corollary 4.4.5.3, we are reduced to showing that τ_1 can be extended to an $(n+1)$-simplex of $\mathcal{C}$, which follows from the existence of σ. □

059M **Corollary 4.8.4.14.** *Let $\mathcal{C}$ be an ∞-category and let n be a positive integer. Then the simplicial set $\mathrm{h}_{\leq n}(\mathcal{C})$ of Construction 4.8.4.9 is an $(n,1)$-category.*

Proof. By virtue of Proposition 4.8.3.6, the weak n-coskeleton $\operatorname{cosk}_n^{\circ}(\mathcal{C})$ is an ∞-category. Combining Corollary 4.8.4.13 with Proposition 1.5.5.11, we conclude that $\mathrm{h}_{\leq n}(\mathcal{C})$ is also an ∞-category. To complete the proof, it will suffice to show that if σ and τ are m-simplices of $\mathrm{h}_{\leq n}(\mathcal{C})$ for some $m > n$ which satisfy $\sigma|_{\Lambda^m_i} = \tau|_{\Lambda^m_i}$ for some $0 < i < m$, then $\sigma = \tau$. Choose maps $\widetilde{\sigma}, \widetilde{\tau} : \operatorname{sk}_n(\Delta^m) \to \mathcal{C}$ representing σ and τ. Using Proposition 4.8.4.11, we can choose an isomorphism α from $\widetilde{\sigma}|_{\operatorname{sk}_n(\Lambda^m_i)}$ to $\widetilde{\tau}|_{\operatorname{sk}_n(\Lambda^m_i)}$ whose image in $\operatorname{Fun}(\operatorname{sk}_{n-1}(\Lambda^m_i), \mathcal{C})$ is an identity morphism. If $m \geq n+2$, then α is also an isomorphism from $\widetilde{\sigma}|_{\operatorname{sk}_n(\Delta^m)}$ to $\widetilde{\tau}|_{\operatorname{sk}_n(\Delta^m)}$, so that $\sigma = \tau$ as desired. In the case $m = n+1$, the morphisms $\widetilde{\sigma}$ and $\widetilde{\tau}$ can be extended to diagrams $\overline{\sigma}, \overline{\tau} : \Delta^{n+1} \to \mathcal{C}$. Using Proposition 1.5.7.6, we can extend α to an isomorphism of $\overline{\sigma}$ with $\overline{\tau}$. Restricting to the n-skeleton of Δ^m, we again conclude that $\sigma = \tau$. □

059N **Proposition 4.8.4.15.** *Let $\mathcal{C}$ be an ∞-category and let n be a positive integer. Then* *the comparison map $\mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C})$ of Construction 4.8.4.9 exhibits $\mathrm{h}_{\leq n}(\mathcal{C})$ as a homotopy n-category of $\mathcal{C}$.*

Proof. By virtue of Corollary 4.8.4.14, it will suffice to show that for every $(n,1)$-category $\mathcal{D}$, the composite map

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{h}_{\leq n}(\mathcal{C}), \mathcal{D}) \xrightarrow{\theta} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{cosk}_n^\circ(\mathcal{C}), \mathcal{D}) \xrightarrow{\theta'} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathcal{D})$$

is a bijection. Since $\mathcal{D}$ is weakly n-coskeletal, the map θ' is a bijection. By construction, $\mathrm{h}_{\leq n}(\mathcal{C})$ is a quotient of the weak coskeleton $\mathrm{cosk}_n^\circ(\mathcal{C})$, so θ is an injection. We will complete the proof by showing that θ is also a surjection: that is, every diagram $F : \mathrm{cosk}_n^\circ(\mathcal{C}) \to \mathcal{D}$ factors through $\mathrm{h}_{\leq n}(\mathcal{C})$. Let σ and σ' be m-simplices of $\mathrm{cosk}_n^\circ(\mathcal{C})$ satisfying $\sigma \sim_m \sigma'$ (see Construction 4.8.4.9); we wish to show that $F(\sigma) = F(\sigma')$. This follows from the observation that $\mathcal{D}$ is minimal in dimension n (Proposition 4.8.1.7). □

Corollary 4.8.4.16. *Let $\mathcal{C}$ be an ∞-category. For every integer n, there exists a functor $F : \mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C})$ which exhibits $\mathrm{h}_{\leq n}(\mathcal{C})$ as a homotopy n-category of $\mathcal{C}$. Moreover:* 059P

(1) *The functor F is bijective on m-simplices for $m < n$.*

(2) *The functor F factors (uniquely) as a composition $\mathcal{C} \xrightarrow{F'} \mathrm{cosk}_n^\circ(\mathcal{C}) \xrightarrow{F''} \mathrm{h}_{\leq n}(\mathcal{C})$, where F' is the inner fibration of Proposition 4.8.3.6.*

(3) *The functor F'' is a trivial Kan fibration.*

(4) *If $n \geq -1$, the functor F exhibits $\mathrm{h}_{\leq n}(\mathcal{C})$ as a local $(n-1)$-truncation of $\mathcal{C}$. In particular, $\mathcal{C}$ is locally $(n-1)$-truncated if and only if F is an equivalence of ∞-categories.*

(5) *The functor F is an isofibration.*

Proof. The existence of F follows from Example 4.8.4.5 (in the case $n < 0$), Exercise 4.8.4.4 (in the case $n = 0$), and Proposition 4.8.4.15 (in the case $n > 0$). Assertion (1) is vacuous for $n \leq 0$, and follows from Construction 4.8.4.9 for $n > 0$. Since $\mathrm{h}_{\leq n}(\mathcal{C})$ is an $(n,1)$-category, it is weakly n-coskeletal, so that assertion (2) follows from Proposition 3.5.4.18.

We next prove (3). For $n < 0$, the morphism F'' is an isomorphism (see Example 4.8.4.5) and there is nothing to prove. For $n > 0$, the desired result follows from Corollary 4.8.4.13. We may therefore assume that $n = 0$. We wish to show that every lifting problem

$$\begin{tikzcd}
\partial\Delta^m \arrow[r] \arrow[d] & \mathrm{cosk}_n^\circ(\mathcal{C}) \arrow[d, "F''"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathcal{C}'
\end{tikzcd}$$

admits a solution. For $m \geq 2$, this is automatic (since $\operatorname{cosk}_n^{\circ}(\mathcal{C})$ and $\mathcal{C}'$ are both 1-coskeletal). The cases $m = 0$ and $m = 1$ follow immediately from the construction of $\mathrm{h}_{\leq n}(\mathcal{C})$ given in Exercise 4.8.4.4.

Assertion (4) follows by combining (3) with Proposition 4.8.3.6. We now prove (5). For $n \neq 0$, the morphism F' is an isofibration (Proposition 4.8.3.6), so the desired result follows from (3). In the case $n = 0$, $\mathrm{h}_{\leq n}(\mathcal{C})$ is isomorphic to the nerve of a partially ordered set, so the result is automatic (Example 4.4.1.6). □

059Q **Corollary 4.8.4.17.** *Let $\mathcal{C}$ be an ∞-category, let n be an integer, and let $A \subseteq B$ be simplicial sets. If B has dimension $\leq n + 1$, then every lifting problem*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & \mathrm{h}_{\leq n}(\mathcal{C})
\end{tikzcd}$$

has a solution. If B has dimension $\leq n - 1$, then the solution is unique.

059R **Remark 4.8.4.18.** Let n be an integer. Then, for every collection of ∞-categories $\{\mathcal{C}_i\}_{i \in I}$, the canonical map

$$\mathrm{h}_{\leq n}(\prod_{i \in I} \mathcal{C}_i) \to \prod_{i \in I} \mathrm{h}_{\leq n}(\mathcal{C}_i)$$

is an isomorphism. This follows by inspecting the explicit descriptions supplied by Construction 4.8.4.9 (for the case $n > 0$), Exercise 4.8.4.4 (for the case $n = 0$) and Example 4.8.4.5 (for the case $n < 0$).

059S **Remark 4.8.4.19.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Then, for every integer $n \geq -1$, the homotopy category $\mathrm{h}_{\leq n}(\mathcal{C}_0)$ can be identified with the full subcategory of $\mathrm{h}_{\leq n}(\mathcal{C})$ spanned by the images of objects which belong $\mathcal{C}_0$.

059T **Proposition 4.8.4.20.** *Let n be an integer and suppose we are given a pullback diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r, "G_0"] \arrow[d, "G_1"] & \mathcal{C}_0 \arrow[d, "F_0"] \\
\mathcal{C}_1 \arrow[r, "F_1"] & \mathcal{C}\,.
\end{tikzcd}$$

If $\mathcal{C}$ is an $(n,1)$-category, then the diagram

$$\begin{tikzcd}
\mathrm{h}_{\leq n}(\mathcal{C}_{01}) \arrow[r] \arrow[d] & \mathrm{h}_{\leq n}(\mathcal{C}_{0}) \arrow[d] \\
\mathrm{h}_{\leq n}(\mathcal{C}_{1}) \arrow[r] & \mathrm{h}_{\leq n}(\mathcal{C})
\end{tikzcd}$$

is also a pullback square.

Proof. If $n \leq 0$, then we can identify $\mathcal{C}_{01}$ with the full subcategory of $\mathcal{C}_0 \times \mathcal{C}_1$ spanned by those objects (C_0, C_1) satisfying $F_0(C_0) = F_1(C_1)$. In this case, the desired result follows by combining Remarks 4.8.4.18 and 4.8.4.19. We may therefore assume without loss of generality that $n > 0$. Fix a simplicial set A; we wish to show that the tautological map

$$\theta : \mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}_{01})) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}_{0})) \times_{\mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}))} \mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}_{1})).$$

is a monomorphism. We first show that θ is injective. Suppose we are given a pair of maps $u, u' : A \to \mathrm{h}_{\leq n}(\mathcal{C}_{01})$ satisfying $\theta(u) = \theta(u')$; we wish to show that $u = u'$. Using Remark 4.8.4.12, we can choose representatives of u and u' by morphisms $\widetilde{u}, \widetilde{u}' : \mathrm{sk}_n(A) \to \mathcal{C}_{01}$. Our assumption that $\theta(u) = \theta(u')$ guarantees that there are natural isomorphisms

$$\alpha_0 : G_0 \circ \widetilde{u} \to G_0 \circ \widetilde{u}' \qquad \alpha_1 : G_1 \circ \widetilde{u} \to G_1 \circ \widetilde{u}'$$

which are the identity when restricted to $\mathrm{sk}_{n-1}(A)$. It follows from the proof of Proposition 4.8.1.7 shows that α_0 and α_1 have the same image in $\mathrm{Fun}(\mathrm{sk}_n(A), \mathcal{C})$. We can therefore identify the pair (α_0, α_1) with an isomorphism from $\widetilde{u}$ to $\widetilde{u}'$ (which is the identity when restricted to $\mathrm{sk}_{n-1}(A)$), which proves that $u = u'$.

We now prove that θ is surjective. Choose an element (u_0, u_1) of the fiber product $\mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}_{0})) \times_{\mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}))} \mathrm{Hom}_{\mathrm{Set}_\Delta}(A, \mathrm{h}_{\leq n}(\mathcal{C}_{1}))$. Using Remark 4.8.4.12, we can choose representatives of u_0 and u_1 by morphisms $\widetilde{u}_0 : \mathrm{sk}_{n+1}(A) \to \mathcal{C}_0$ and $\widetilde{u}_1 : \mathrm{sk}_{n+1}(A) \to \mathcal{C}_1$. Since $\mathcal{C}$ is an $(n,1)$-category, the tautological map $\mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C})$ is an isomorphism. It follows that $F_0 \circ \widetilde{u}_0$ coincides with $F_1 \circ \widetilde{u}_1$, so that the pair $(\widetilde{u}_0, \widetilde{u}_1)$ determines a morphism $\widetilde{u} : \mathrm{sk}_{n+1}(A) \to \mathcal{C}$. This represents a morphism $u : A \to \mathrm{h}_{\leq n}(\mathcal{C})$ satisfying $\theta(u) = (u_0, u_1)$. □

4.8.5 Full and Faithful Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Recall that F is *fully faithful* if, for every pair of objects $X, Y \in \mathcal{C}$, the map of morphism spaces 059U

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homotopy equivalence (Definition 4.6.2.1). It is sometimes convenient to break this into two separate conditions:

059V **Definition 4.8.5.1** (Full Functors)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that F is *full* if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is surjective on connected components.

059W **Definition 4.8.5.2** (Faithful Functors)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that F is *faithful* if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homotopy equivalence from $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to a summand of $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$.

059X **Remark 4.8.5.3.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is fully faithful (in the sense of Definition 4.6.2.1) if and only if it is both full and faithful (in the sense of Definitions 4.8.5.1 and 4.8.5.2).

059Y **Example 4.8.5.4.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then the functor of ∞-categories $\mathrm{N}_{\bullet}(F) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{D})$ is full (in the sense of Definition 4.8.5.1) if and only if the functor F is full (in the usual category-theoretic sense). Similarly, $\mathrm{N}_{\bullet}(F)$ is faithful (in the sense of Definition 4.8.5.2) if and only if F is faithful. Consequently, we can view Definitions 4.8.5.1 and Definition 4.8.5.2 as generalizations of their classical counterparts.

059Z **Remark 4.8.5.5.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is full (in the sense of Definition 4.8.5.1) if and only if the induced functor of homotopy categories $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is full (in the usual category-theoretic sense).

05A0 **Remark 4.8.5.6.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. If $G \circ F$ is full and F is essentially surjective, then G is also full.

05A1 **Exercise 4.8.5.7.** Let $f : X \to Y$ be a morphism of Kan complexes. Show that f is full (in the sense of Definition 4.8.5.1) if and only if it satisfies the following pair of conditions:

(a) The map of connected components $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is injective.

(b) For every vertex $x \in X$ having image $y = f(x)$, the map of fundamental groups $\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, y)$ is surjective.

The counterpart of Remark 4.8.5.5 for faithful functors is slightly more involved.

Proposition 4.8.5.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is faithful if and only if it satisfies the following pair of conditions:* `05A2`

(1) *The induced functor of homotopy categories $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is faithful.*

(2) *The diagram of ∞-categories*

`05A3`

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "F"] & \mathrm{N}_\bullet(\mathrm{h}\mathcal{C}) \arrow[d, "\mathrm{N}_\bullet(\mathrm{h}F)"] \\
\mathcal{D} \arrow[r] & \mathrm{N}_\bullet(\mathrm{h}\mathcal{D})
\end{tikzcd} \tag{4.50}$$

is a categorical pullback square.

Remark 4.8.5.9. In the situation of Proposition 4.8.5.8, the comparison map $\mathcal{D} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{D})$ is automatically an isofibration (Corollary 4.4.1.9). By virtue of Proposition 4.5.2.26, condition (2) is equivalent to the following: `05A4`

($2'$) The functor F induces an equivalence of ∞-categories $F' : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C}) \times_{\mathrm{N}_\bullet(\mathrm{h}\mathcal{D})} \mathcal{D}$.

Note that the functor F' is bijective on objects, and therefore essentially surjective. Using Theorem 4.6.2.20, we can reformulate ($2'$) as follows:

($2''$) For every pair of objects $X, Y \in \mathcal{C}$, the functor F induces a homotopy equivalence

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \pi_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \times_{\pi_0(\mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)))} \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)).$$

Proof of Proposition 4.8.5.8. By definition, a functor $F : \mathcal{C} \to \mathcal{D}$ is faithful if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ induces a homotopy equivalence from $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ to a summand of $\mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$. This is equivalent to the following pair of assertions:

($1_{X,Y}$) The map of sets $\pi_0(F_{X,Y}) : \pi_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \to \pi_0(\mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)))$ is injective.

($2_{X,Y}$) The map of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \pi_0(\mathrm{Hom}_{\mathcal{C}}(X,Y)) \times_{\pi_0(\mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)))} \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$

is a homotopy equivalence.

The desired result now follows by allowing the objects X and Y to vary (and applying Remark 4.8.5.9). □

Note the asymmetry between Remark 4.8.5.5 and Proposition 4.8.5.8: in the higher-categorical setting, fullness is a relatively weak condition which can be tested at the level of homotopy categories, but faithfulness is not. It will therefore be useful to further analyze Definition 4.8.5.2.

05A5 **Definition 4.8.5.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:

- We say that F is 0-*full* if it is essentially surjective: that is, every object of $\mathcal{D}$ is isomorphic to $F(X)$, for some object $X \in \mathcal{C}$ (Definition 4.6.2.11).
- We say that F is 1-*full* if it is full: that is, for objects $X, Y \in \mathcal{C}$ having images $\overline{X} = F(X)$ and $\overline{Y} = F(Y)$ in $\mathcal{D}$, the map
$$\pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y)) \to \pi_0(\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y}))$$
is surjective (Definition 4.8.5.1).
- For $n \geq 2$, we say that F is n-*full* if, for every morphism $u : X \to Y$ in the ∞-category $\mathcal{C}$ having image $\overline{u} : \overline{X} \to \overline{Y}$ in $\mathcal{D}$, the induced map
$$\pi_m(\mathrm{Hom}_{\mathcal{C}}(X, Y), u) \to \pi_m(\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y}), \overline{u})$$
is injective for $m = n - 2$ and surjective for $m = n - 1$.

05A6 **Remark 4.8.5.11.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:

- The functor F is faithful if and only if it is n-full for each $n \geq 2$ (see Example 3.5.9.3).
- The functor F is fully faithful if and only if it is n-full for each $n \geq 1$ (see Remark 4.8.5.3).
- The functor F is an equivalence of ∞-categories if and only if it is n-full for each $n \geq 0$ (see Theorem 4.6.2.20).

05A7 **Example 4.8.5.12.** Let $n \geq -2$ be an integer and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which exhibits $\mathcal{D}$ as a local n-truncation of $\mathcal{D}$ (see Definition 4.8.2.9). Then F is m-full for $m \leq n + 2$. In particular, for any ∞-category $\mathcal{C}$, the canonical maps

$$\mathcal{C} \to \mathrm{cosk}_{n+2}(\mathcal{C}) \qquad \mathcal{C} \to \mathrm{cosk}^{\circ}_{n+1}(\mathcal{C}) \qquad \mathcal{C} \to \mathrm{h}_{\leq n+1}(\mathcal{C})$$

are m-full for $m \leq n + 2$.

05A8 **Remark 4.8.5.13.** Let $\mathcal{C}$ be an ∞-category and let $n \geq -2$ be an integer. Then $\mathcal{C}$ is locally n-truncated if and only if the projection map $\mathcal{C} \to \Delta^0$ is m-full for all $m \geq n + 3$.

Remark 4.8.5.14 (Symmetry)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $n \geq 0$ be an integer. Then F is n-full if and only the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is n-full. `05A9`

Remark 4.8.5.15 (Composition)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let $n \geq 0$ be an integer. If F and G are n-full, then the composite functor $G \circ F$ is n-full. `05AA`

Remark 4.8.5.16 (Change of Target)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, where G is fully faithful. For $n \geq 1$, the functor F is n-full if and only if the composite functor $G \circ F$ is n-full. If G is an equivalence of ∞-categories, then this is also true when $n = 0$. `05AB`

Remark 4.8.5.17 (Isomorphism Invariance)**.** Let $F_0, F_1 : \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories which are isomorphic (as objects of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$). Then F_0 is n-full if and only if F_1 is n-full. To see this, let $\mathrm{Isom}(\mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{D})$ spanned by the isomorphisms (Example 4.4.1.14), so that the evaluation functors $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Isom}(\mathcal{D}) \to \mathcal{D}$ are equivalences of ∞-categories (Corollary 4.4.5.10). The assumption that F_0 and F_1 are isomorphic guarantees that there exists a functor $F : \mathcal{C} \to \mathrm{Isom}(\mathcal{D})$ satisfying $F_0 = \mathrm{ev}_0 \circ F$ and $F_1 = \mathrm{ev}_1 \circ F$. Using Remark 4.8.5.16, we see that F_0 is n-full if and only if F is n-full. Similarly, F_1 is n-full if and only if F is n-full. `05AC`

Remark 4.8.5.18 (Change of Source)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let $n \geq 0$ be an integer. If F is an equivalence of ∞-categories, then G is n-full if and only if $G \circ F$ is n-full. The "only if" direction follows immediately from Remark 4.8.5.15. For the converse, suppose that $G \circ F$ is n-full, and let $F^{-1} : \mathcal{D} \to \mathcal{C}$ be a homotopy inverse to F. Then F^{-1} is an equivalence of ∞-categories; in particular, it is n-full (Remark 4.8.5.11). Applying Remark 4.8.5.15, we conclude that the composition $G \circ F \circ F^{-1}$ is also n-full. This composition is isomorphic to G, so G is n-full as well (Remark 4.8.5.17). `05AD`

Remark 4.8.5.19. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $u : X \to Y$ be a morphism of $\mathcal{C}$ having image $\overline{u} : \overline{X} \to \overline{Y}$ in $\mathcal{D}$. For each integer n, the requirement that the map `05AE`

$$\pi_n(\mathrm{Hom}_{\mathcal{C}}(X, Y), u) \to \pi_n(\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y}), \overline{u})$$

is injective or surjective depends only on the isomorphism class of u (as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$).

In the setting of Kan complexes, Definition 4.8.5.10 can be simplified:

Proposition 4.8.5.20. *Let $f : X \to Y$ be a morphism of Kan complexes. Then:* `05AF`

(a) The morphism f is 0-full (in the sense of Definition 4.8.5.10) if and only if the induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is surjective.

(b) *For $n \geq 1$, the morphism f is n-full if and only if, for every vertex $x \in X$ having image $y = f(x)$, the induced map $\pi_m(f) : \pi_m(X, x) \to \pi_m(Y, y)$ is injective for $m = n - 1$ and surjective for $m = n$.*

Proof. Assertion (a) is immediate from the definitions. We will prove (b). The case $n = 1$ follows from Exercise 4.8.5.7. Let us therefore assume that $n \geq 2$. By definition, f is n-full if and only if, for every edge $u : x \to x'$ of X having image $v : y \to y'$ in Y, the induced map

$$\pi_{m-1}(\mathrm{Hom}_X(x, x'), u) \to \pi_{m-1}(\mathrm{Hom}_Y(y, y'), v)$$

is injective for $m = n - 1$ and surjective for $m = n$. By virtue of Remark 4.8.5.19, it suffices to check this in the special case where $u = \mathrm{id}_x$ is a degenerate edge of X. Assertion (b) now follows from isomorphisms

$$\pi_{m-1}(\mathrm{Hom}_X(x, x), \mathrm{id}_x) \simeq \pi_m(X, x) \qquad \pi_{m-1}(\mathrm{Hom}_Y(y, y), \mathrm{id}_y) \simeq \pi_m(Y, y)$$

of Example 4.6.1.13. □

05AG **Remark 4.8.5.21.** In the situation of Proposition 4.8.5.20, suppose that f is a Kan fibration. Then assertions (a) and (b) can be reformulated as follows:

(a') The Kan fibration f is 0-full if and only if, for each vertex $y \in Y$, the fiber $X_y = \{y\} \times_Y X$ is nonempty.

(b') For $n \geq 1$, the Kan fibration f is n-full if and only if, for every vertex $x \in X$ having image $y = f(x)$, the set $\pi_{n-1}(X_y, x)$ consists of a single element.

See Corollary 3.2.6.8 (and Variant 3.2.6.9 for the case $n = 1$).

05AH **Corollary 4.8.5.22.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $n \geq 1$. Then F is n-full if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map of morphism spaces*

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is $(n-1)$-full.

05AJ **Remark 4.8.5.23.** Stated more informally, Corollary 4.8.5.22 asserts that a functor $F : \mathcal{C} \to \mathcal{D}$ is n-full if it is surjective up to homotopy on n-morphisms (having fixed source and target). For an alternative formulation of this heuristic, see Proposition 4.8.5.30 below.

05AK **Corollary 4.8.5.24.** *Let $f : X \to Y$ be a morphism of Kan complexes and let n be an integer. Then:*

- *The morphism f is n-connective (in the sense of Definition 3.5.1.13) if and only if it is m-full for every nonnegative integer $m \leq n$.*

- *The morphism f is n-truncated (in the sense of Definition 3.5.9.1) if and only if it is m-full for every nonnegative integer $m \geq n+2$.*

Corollary 4.8.5.25. *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $n \geq 2$ be an integer. Then F is n-full if and only if, for every morphism $u : X \to Y$ of $\mathcal{C}$ having image $\overline{u} : \overline{X} \to \overline{Y}$ in $\mathcal{D}$, the set $\pi_{n-2}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\overline{u}}, u)$ consists of a single element. Here $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\overline{u}}$ denotes the fiber $\{\overline{u}\} \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y})} \mathrm{Hom}_{\mathcal{C}}(X,Y)$.* `05AL`

Proof. By virtue of Corollary 4.8.5.22, the functor F is n-full if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the map of Kan complexes

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is $(n-1)$-full. Since F is an inner fibration, $F_{X,Y}$ is a Kan fibration (Proposition 4.6.1.21). The desired result now follows from Remark 4.8.5.21. □

Variant 4.8.5.26. Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. Then F is full if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map `05AM`

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is surjective on vertices.

Proposition 4.8.5.27. *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $n \geq 1$ be an integer. The following conditions are equivalent:* `05AN`

(1) *The functor F is n-full.*

(2) *For every pullback diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r] & \mathcal{D}, \end{tikzcd}$$

the functor F' is n-full.

(3) *For every pullback diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\ \Delta^1 \arrow[r] & \mathcal{D}, \end{tikzcd}$$

the functor F' is n-full.

Proof. For $n \geq 2$, this follows from the criterion of Corollary 4.8.5.25. For $n = 1$, it follows from the criterion of Variant 4.8.5.26. □

05J2 **Corollary 4.8.5.28.** *Let $n > 0$ be an integer and suppose we are given a commutative diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D} \arrow[dl] \\ & \mathcal{E}, & \end{tikzcd}$$

where the vertical maps are inner fibrations. Then F is n-full if and only if, for every morphism u of $\mathcal{E}$, the induced map

$$F_u : \Delta^1 \times_{\mathcal{E}} \mathcal{C} \to \Delta^1 \times_{\mathcal{E}} \mathcal{D}$$

is n-full.

Proof. Using Proposition 4.1.3.2, we can reduce to the case where F is an inner fibration, in which case it follows from the criterion of Proposition 4.8.5.27. □

05AP **Corollary 4.8.5.29.** *Let $n \geq 0$ be an integer, and suppose we are given a categorical pullback diagram of ∞-categories*

05AQ
$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r, "G"] & \mathcal{D}. \end{tikzcd} \tag{4.51}$$

If F is n-full, then F' is n-full. The converse holds if G is full and essentially surjective.

Proof. The case $n = 0$ follows from Remark 4.6.2.18. We will therefore assume that $n \geq 1$. Using Corollary 4.5.2.23, we can reduce to the case where F and G are isofibrations. In this case, our assumption that (4.51) is a categorical pullback square guarantees that the induced map $\mathcal{C}' \to \mathcal{D}' \times_{\mathcal{D}} \mathcal{C}$ is an equivalence of ∞-categories (Proposition 4.5.2.26). Using Remark 4.8.5.18, $\mathcal{C}'$ by the fiber product $\mathcal{D}' \times_{\mathcal{D}} \mathcal{C}$ and thereby reduce to the case where the diagram (4.51) is a pullback square. Note that, if the functor G is full and essentially surjective, then every morphism of $\mathcal{D}$ can be lifted to a morphism of $\mathcal{D}'$. The desired result now follows from the criterion of Proposition 4.8.5.27. □

05AR **Proposition 4.8.5.30.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $n \geq 1$*

be an integer. Then F is n-full if and only if every lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution. If F is an isofibration, then this is also true in the case $n = 0$.

Proof. The case $n = 0$ reduces to the observation that an isofibration is essentially surjective if and only if it is surjective on objects. The case $n = 1$ is a reformulation of Variant 4.8.5.26. We may therefore assume without loss of generality that $n \geq 2$. Using Proposition 4.8.5.27, we can reduce to the case where $\mathcal{D} = \Delta^m$ is a standard simplex. In this case, the functor F is n-full if and only if, for every morphism $u : X \to Y$ of $\mathcal{C}$, the set $\pi_{n-2}(\mathrm{Hom}_{\mathcal{C}}(X, Y), u)$ consists of a single element (Corollary 4.8.5.25). The desired result now follows from Corollary 4.8.3.10. □

For later use, we record a few variants of Remark 4.8.5.15.

Proposition 4.8.5.31. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let $n \geq 1$. If $G \circ F$ is n-full and G is $(n+1)$-full, then F is n-full.* 05AS

Proof. We first treat the case $n = 1$. Fix a pair of objects $X, Y \in \mathcal{C}$ having images $\overline{X}$ and $\overline{Y}$ in $\mathcal{D}$. We wish to show that every morphism $\overline{u} : \overline{X} \to \overline{Y}$ is homotopic to $F(v)$, for some morphism $v : X \to Y$ in $\mathcal{C}$. Our assumption that $(G \circ F)$ is 1-full guarantees that we can choose v such that $(G \circ F)(v)$ is homotopic to $G(\overline{u})$. Since G is 2-full, the map $\pi_0(\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y})) \to \pi_0(\mathrm{Hom}_{\mathcal{E}}(G(\overline{X}), G(\overline{Y}))$ is injective, so that $F(v)$ is homotopic to $\overline{u}$ as desired.

We now treat the case $n \geq 2$. Without loss of generality, we may assume that F and G are inner fibrations. Using Proposition 4.8.5.27, we can further reduce to the case where $\mathcal{E}$ is a standard simplex. Fix a morphism $u : X \to Y$ of $\mathcal{C}$ having image $\overline{u} : \overline{X} \to \overline{Y}$ in $\mathcal{D}$. We wish to show that the map

$$\theta_m : \pi_m(\mathrm{Hom}_{\mathcal{C}}(X, Y), u) \to \pi_m(\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y}), \overline{u})$$

is injective for $m = n - 2$ and surjective for $m = n - 1$. This is clear: our assumption that $G \circ F$ is n-full guarantees that the set $\pi_{n-2}(\mathrm{Hom}_{\mathcal{C}}(X, Y), u)$ consists of a single element (so θ_{n-2} is automatically injective), and our assumption that G is $(n+1)$-full guarantees that the set $\pi_{n-1}(\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y}), \overline{u})$ consists of a single element (so that θ_{n-1} is automatically surjective). □

05AT **Exercise 4.8.5.32.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories where G is full and conservative. Show that if $G \circ F$ is essentially surjective, then F is also essentially surjective. Beware that the hypothesis that G is conservative cannot be omitted.

05AU **Proposition 4.8.5.33.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let $n \geq 2$. Assume that $G \circ F$ is n-full and that F is essentially surjective, full, and $(n-1)$-full. Then G is n-full.*

05AV **Remark 4.8.5.34.** For $n = 0$ and $n = 1$, we have stronger versions of Proposition 4.8.5.33:

- If $G \circ F$ is 0-full, then G is 0-full (this is a restatement of Remark 4.6.2.17).
- If $G \circ F$ is 1-full and F is 0-full, then G is 1-full (this is a restatement of Remark 4.8.5.6).

Proof of Proposition 4.8.5.33. Without loss of generality, we may assume that F and G are inner fibrations. Using Proposition 4.8.5.27, we can reduce to the case where $\mathcal{E}$ is a standard simplex. In this case, we must show that for every morphism $\overline{u} : \overline{X} \to \overline{Y}$ of $\mathcal{D}$, the set $\pi_{n-2}(\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y}), \overline{u})$ consists of a single element. Since F is full and essentially surjective, we can assume without loss of generality that $\overline{u} = F(u)$ for some morphism $u : X \to Y$ in the ∞-category $\mathcal{C}$. In this case, our assumption that F is $(n-1)$-full guarantees that the map

$$\pi_{n-2}(\mathrm{Hom}_{\mathcal{C}}(X, Y), u) \to \pi_{n-2}(\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Y}), \overline{u})$$

is surjective. It will therefore suffice to show that the set $\pi_{n-2}(\mathrm{Hom}_{\mathcal{C}}(X, Y), u)$ consists of a single element, which follows from our assumption that $G \circ F$ is n-full. □

4.8.6 Essentially Categorical Functors

05AW Let $\mathcal{C}$ be an ∞-category and let n be an integer. Combining Corollary 4.8.3.3 with Remark 4.8.5.13, we see that the following conditions are equivalent:

- The ∞-category $\mathcal{C}$ is equivalent to an $(n, 1)$-category.
- The projection map $\mathcal{C} \to \Delta^0$ is m-full for $m \geq n + 2$.

This motivates the following:

05AX **Definition 4.8.6.1.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. We say that F is *essentially n-categorical* if it is m-full for $m \geq n + 2$.

05AY **Example 4.8.6.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:

- The functor F is essentially 0-categorical if and only if it is faithful.

- The functor F is essentially (-1)-categorical if and only if it is fully faithful.
- The functor F is essentially (-2)-categorical if and only if it is an equivalence of ∞-categories. In this case, F is also essentially n-categorical for any $n \leq -2$.

This is a restatement of Remark 4.8.5.11.

Example 4.8.6.3. Let $f : X \to Y$ be a morphism of Kan complexes and let n be an integer. Then f is essentially n-categorical (in the sense of Definition 4.8.6.1) if and only if it is n-truncated (in the sense of Definition 3.5.9.1). See Corollary 4.8.5.24. 05AZ

Example 4.8.6.4. Let $\mathcal{C}$ be an ∞-category and let n be an integer. The following conditions are equivalent: 05B0

(1) The projection map $\mathcal{C} \to \Delta^0$ is essentially n-categorical.

(2) The ∞-category $\mathcal{C}$ is locally $(n-1)$-truncated. Moreover, if $n \leq -2$, then $\mathcal{C}$ is nonempty.

(3) The ∞-category $\mathcal{C}$ is equivalent to an $(n, 1)$-category.

(4) For $m \geq n + 2$, every morphism $\partial \Delta^m \to \mathcal{C}$ can be extended to an m-simplex of $\mathcal{E}$.

The equivalence (1) $\Leftrightarrow$ (2) follows from Remark 4.8.5.13, the equivalence (2) $\Leftrightarrow$ (3) from Corollary 4.8.3.3, and the equivalence (2) $\Leftrightarrow$ (4) from Corollary 4.8.3.11.

Remark 4.8.6.5 (Symmetry)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. Then F is essentially n-categorical if and only if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is essentially n-categorical. See Remark 4.8.5.14. 05B1

Remark 4.8.6.6 (Monotonicity)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $m \leq n$ be integers. If F is essentially m-categorical, then it is essentially n-categorical. 05B2

Remark 4.8.6.7 (Transitivity)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let n be an integer. Then: 05B3

(a) If F and G are essentially n-categorical, then the composite functor $G \circ F$ is essentially n-categorical.

(b) If $G \circ F$ is essentially n-categorical and G is essentially $(n + 1)$-categorical, then F is essentially n-categorical.

(c) If $G \circ F$ is essentially n-categorical and F is essentially $(n - 1)$-categorical, full, and essentially surjective, then G is essentially n-categorical.

Assertion (a) follows from Remark 4.8.5.15, assertion (b) from Proposition 4.8.5.31 (together with Exercise 4.8.5.32 in the case $n \leq -2$), and assertion (c) follows from Proposition 4.8.5.33 (together with Remark 4.5.1.18 in the case $n \leq -2$).

05B4 **Remark 4.8.6.8.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let n be an integer. Suppose that G is essentially n-categorical. Then F is essentially n-categorical if and only if $G \circ F$ is essentially n-categorical. This follows by combining Remarks 4.8.6.6 and 4.8.6.7.

05B5 **Example 4.8.6.9.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $n \geq -1$ be an integer. Suppose that $\mathcal{D}$ is locally $(n-1)$-truncated. Then F is essentially n-categorical if and only if $\mathcal{C}$ is also locally $(n-1)$-truncated. This follows by applying Remark 4.8.6.8 in the special case $\mathcal{E} = \Delta^0$ (see Example 4.8.6.4).

05B6 **Remark 4.8.6.10** (Homotopy Invariance)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let n be an integer. If F is an equivalence of ∞-categories, then $G \circ F$ is essentially n-categorical if and only if G is essentially n-categorical. If G is an equivalence of ∞-categories, then $G \circ F$ is essentially n-categorical if and only if F is essentially n-categorical. Both assertions are special cases of Remark 4.8.6.7.

05B7 **Remark 4.8.6.11** (Isomorphism Invariance)**.** Let $F_0, F_1 : \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories which are isomorphic (when regarded as objects of the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$). Then F_0 is essentially n-categorical if and only if F_1 is essentially n-categorical. See Remark 4.8.5.17.

05B8 **Remark 4.8.6.12.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $n \geq -1$. Then F is essentially n-categorical if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the map of Kan complexes

$$F_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is $(n-1)$-truncated. This follows by combining Example 4.8.6.3 with Corollary 4.8.5.22.

05B9 **Remark 4.8.6.13.** Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. For $n \geq 0$, F is essentially n-categorical if and only if the diagonal map $\delta : \mathcal{C} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ is essentially $(n-1)$-categorical. This follows by combining Remark 4.8.6.12 with Corollary 3.5.9.17, since F induces a Kan fibration $\operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$ for every pair of objects $X, Y \in \mathcal{C}$ (Proposition 4.6.1.21).

05BA **Warning 4.8.6.14.** Remark 4.8.6.13 is generally false in the case $n = -1$, even if we assume that F is an isofibration. For example, let $\mathcal{D}$ be an ∞-category and let $\mathcal{C} \subseteq \mathcal{D}$ be a subcategory. Then the inclusion map $F : \mathcal{C} \to \mathcal{D}$ is an inner fibration (which is even an isofibration, if $\mathcal{C}$ is a replete subcategory of $\mathcal{D}$). The diagonal $\delta : \mathcal{C} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ is an isomorphism of simplicial sets, and therefore an equivalence of ∞-categories. However, F need not be fully faithful.

05BB **Variant 4.8.6.15.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. For $n \geq 0$, the functor F is essentially n-categorical if and only if the composite map

$$\mathcal{C} \hookrightarrow \mathcal{C} \times_{\mathcal{D}} \mathcal{C} \hookrightarrow \mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}$$

is essentially $(n-1)$-categorical. To prove this, we can use Corollaries 4.5.2.23 and 4.5.2.20 to reduce to the situation where F is an isofibration. In this case, the desired result is a reformulation of Remark 4.8.6.13 (see Corollary 4.5.2.28).

Remark 4.8.6.16. Let n be an integer and suppose we are given a categorical pullback diagram of ∞-categories `05BC`

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r, "G"] & \mathcal{D}. \end{tikzcd}$$

If F is essentially n-categorical, then F' is essentially n-categorical. The converse holds if G is full and essentially surjective. See Corollary 4.8.5.29.

Proposition 4.8.6.17. *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $n \geq 0$ be an integer. The following conditions are equivalent:* `05BD`

(1) *The functor F is essentially n-categorical.*

(2) *For every pullback diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r] & \mathcal{D}, \end{tikzcd}$$

the functor F' is essentially n-categorical.

(3) *For every pullback diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r] & \mathcal{D}, \end{tikzcd}$$

where $\mathcal{D}'$ is locally $(n-1)$-truncated, the ∞-category $\mathcal{C}'$ is also locally $(n-1)$-truncated.

(4) *For every pullback diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^1 \arrow[r] & \mathcal{D},
\end{tikzcd}$$

the ∞-category $\mathcal{C}'$ is locally $(n-1)$-truncated.

Proof. Combine Proposition 4.8.5.27 with the criterion of Example 4.8.6.9. □

05BE **Warning 4.8.6.18.** Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $n \geq -1$ be an integer. If F is essentially n-categorical, then each fiber $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ of F is a locally $(n-1)$-truncated ∞-category. Beware that the converse is false in general, even if F is an isofibration. However, it holds under additional assumptions: see Variant 5.1.5.17.

Using Corollary 4.8.5.28, we immediately obtain the following:

05J3 **Variant 4.8.6.19.** Let $n \geq -1$ be an integer and suppose we are given a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D} \arrow[dl] \\
& \mathcal{E}, &
\end{tikzcd}$$

where the vertical maps are inner fibrations. Then F is essentially n-categorical if and only if, for every morphism u of $\mathcal{E}$, the induced functor

$$F_u : \Delta^1 \times_{\mathcal{E}} \mathcal{C} \to \Delta^1 \times_{\mathcal{E}} \mathcal{D}$$

is essentially n-categorical. In particular, F is fully faithful if and only if each F_u is fully faithful.

05BF **Proposition 4.8.6.20.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $n \geq -1$ be an integer. The following conditions are equivalent:*

(1) *The functor F is essentially n-categorical.*

(2) *For every integer $m \geq n+2$, every lifting problem*

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution.

(3) *For every simplicial set B and every simplicial subset $A \subseteq B$ which contains the $(n+1)$-skeleton of B, every lifting problem*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ B \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

admits a solution.

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 4.8.5.30. The implication (3) $\Rightarrow$ (2) is immediate, and the reverse implication follows from Proposition 1.1.4.12. ☐

Corollary 4.8.6.21. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let B be a simplicial set, and let $A \subseteq B$ be a simplicial subset. If F is essentially n-categorical, then the induced functor $F' : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B, \mathcal{D})$ is also essentially n-categorical.* 05BG

Proof. If $n \leq -2$, then F is an equivalence of ∞-categories; it then follows from Corollary 4.5.2.30 that F' is also an equivalence of ∞-categories. We may therefore assume that $n \geq -1$. Using Proposition 3.1.7.1, we can reduce to the case where F is an inner fibration, so that F' is also an inner fibration (Proposition 4.1.4.1). By virtue of Proposition 4.8.6.20, it will suffice to show that for every simplicial set B' and every simplicial subset $A' \subseteq B'$ which contains the $(n+1)$-skeleton of B, every lifting problem

05BH
$$\begin{tikzcd} A' \arrow[r] \arrow[d] & \mathrm{Fun}(B, \mathcal{C}) \arrow[d, "F'"] \\ B' \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B, \mathcal{D}) \end{tikzcd} \tag{4.52}$$

Unwinding the definitions, we can rewrite (4.52) as a lifting problem

$$\begin{tikzcd} (A \times B') \coprod_{(A \times A')} (B \times A') \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ B \times B' \arrow[r] \arrow[ur, dashed] & \mathcal{D}. \end{tikzcd}$$

The existence of a solution now follows from Proposition 4.8.6.20, since F is essentially n-categorical and $(A \times B') \coprod_{(A \times A')} (B \times A')$ contains the $(n+1)$-skeleton of $B \times B'$. ☐

05BJ **Corollary 4.8.6.22.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let B be a simplicial set, and let n be an integer. If F is essentially n-categorical, then the induced functor $\operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(B, \mathcal{D})$ is also essentially n-categorical.*

Proof. Apply Corollary 4.8.6.21 in the special case $A = \emptyset$. □

05BK **Corollary 4.8.6.23.** *Let $n \geq -1$ be an integer and let $F : \mathcal{C} \to \mathcal{D}$ be an essentially n-categorical inner fibration of ∞-categories. Then, for every diagram $B \to \mathcal{D}$, the ∞-category $\operatorname{Fun}_{/\mathcal{D}}(B, \mathcal{C})$ is locally $(n-1)$-truncated.*

Proof. It follows from Corollary 4.1.4.2 that F induces an inner fibration $F' : \operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(B, \mathcal{D})$, and from Corollary 4.8.6.22 that F' is essentially n-categorical. In particular, every fiber of F is locally $(n-1)$-truncated. □

We now study a special class of essentially n-categorical functors.

05BL **Definition 4.8.6.24.** Let n be a positive integer. We say that a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ is an *n-categorical inner fibration* if it satisfies the following condition:

$(*)$ For every pair of integers $0 < i < m$, every lifting problem

$$\begin{tikzcd}
\Lambda^m_i \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution. Moreover, if $m > n$, then the solution is unique.

It will sometimes be useful to extend Definition 4.8.6.24 to allow n to be an arbitrary integer.

05BM **Variant 4.8.6.25.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets.

- We say that U is a 0-*categorical inner fibration* if, for every morphism $\Delta^m \to \mathcal{D}$, the fiber product $\Delta^m \times_{\mathcal{D}} \mathcal{C}$ is isomorphic to the nerve of a partially ordered set.
- We say that F is a (-1)-*categorical inner fibration* if it induces an isomorphism from $\mathcal{C}$ to a full simplicial subset of $\mathcal{D}$ (Definition 4.1.2.15).
- For $n \leq -2$, we say that F is an *n-categorical inner fibration* if it is an isomorphsm of simplicial sets.

05BN **Example 4.8.6.26.** Let $\mathcal{C}$ be a simplicial set and let $F : \mathcal{C} \to \Delta^0$ be the projection map. Then F is an n-categorical inner fibration if and only if $\mathcal{C}$ is an $(n, 1)$-category.

Example 4.8.6.27. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then F is a 1-categorical inner fibration (Definition 4.8.6.24) if and only if it is an inner covering map (Definition 4.1.5.1). 05BP

Remark 4.8.6.28. Let $m \leq n$ be integers. If $F : \mathcal{C} \to \mathcal{D}$ is an m-categorical inner fibration, then it is also an n-categorical inner fibration (see Remark 4.8.1.12). In particular, F is an inner fibration. 05BQ

Remark 4.8.6.29. Suppose we are given a pullback diagram of simplicial sets 05BR

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D}' \arrow[r] & \mathcal{D}.
\end{tikzcd}$$

If F is an n-categorical inner fibration, then F' is also an n-categorical inner fibration.

Remark 4.8.6.30. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. It follows from Example 4.8.6.26 and Remark 4.8.6.29 that if F is an n-categorical inner fibration, then the fiber $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ is an $(n,1)$-category for each vertex $D \in \mathcal{D}$. Beware that the converse is generally false. 05BS

Remark 4.8.6.31 (Symmetry)**.** Let n be an integer and let $F : \mathcal{C} \to \mathcal{D}$ be an n-categorical inner fibration of simplicial sets. Then the opposite map $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is also an n-categorical inner fibration. 05BT

Proposition 4.8.6.32. *Let n be an integer, let $\mathcal{D}$ be an $(n,1)$-category, and let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. Then F is n-categorical if and only if $\mathcal{C}$ is an $(n,1)$-category.* 05BU

Proof. For $n \neq 0$, the desired result follows from immediately from the definitions. Let us therefore assume that $n = 0$, so that $\mathcal{D}$ is isomorphic to the nerve of a partially ordered set. If $\mathcal{C}$ is also isomorphic to the nerve of a partially ordered set, then any fiber product $\Delta^m \times_{\mathcal{D}} \mathcal{C}$ has the same property (since the formation of nerves commutes with fiber products). Conversely, suppose that F is a 0-categorical inner fibration. In this case, we claim that $\mathcal{C}$ satisfies the criteria of *** snip ***

(a) The simplicial set $\mathcal{C}$ is a $(1,1)$-category: this follows by applying Proposition 4.8.6.32 in the case $n = 1$.

(b) Let $u, u' : X \to Y$ be morphisms in $\mathcal{C}$ having the same source and target; we wish to show that $f = f'$. Our assumption that $\mathcal{D}$ is a $(0,1)$-category guarantees that $F(u) = F(u')$.

The desired result now follows from the observation that the fiber product $\Delta^1 \times_{\mathcal{D}} \mathcal{C}$ is a $(0,1)$-category.

(c) Let X and Y be isomorphic objects of $\mathcal{C}$; we wish to show that $X = Y$. Fix morphisms $u : X \to Y$ and $v : Y \to X$. Since $\mathcal{D}$ is a $(1,0)$-category, we have $F(u) = \mathrm{id}_D = F(v)$ for some object $D \in \mathcal{D}$. In this case, we can regard u and v as morphisms of the ∞-category $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$. Our assumption that F is a 0-categorical inner fibration guarantees that $\mathcal{C}_D$ is a $(0,1)$-category, so that $X = Y$.

□

05BV **Remark 4.8.6.33.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let n be an integer. Then F is an n-categorical inner fibration if and only if, for every pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\
\Delta^m \arrow[r] & \mathcal{D},
\end{tikzcd}$$

the projection map F' is an n-categorical inner fibration. For $n \geq 0$, this is equivalent to the requirement that $\mathcal{C}'$ is an $(n,1)$-category (Proposition 4.8.6.32).

05BW **Corollary 4.8.6.34** (Transitivity)**.** *Let n be an integer and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be inner fibrations of simplicial sets, where G is n-categorical. Then F is n-categorical if and only if $G \circ F$ is n-categorical.*

Proof. For $n < 0$, this follows immediately from the definitions. We may therefore assume that $n \geq 0$. Using Remark 4.8.6.33, we can reduce to the case where $\mathcal{E} = \Delta^m$ is a standard simplex. In this case, our assumption on G guarantees that $\mathcal{D}$ is an $(n,1)$-category. We wish to show that $\mathcal{C}$ is an $(n,1)$-category if and only if the inner fibration F is n-categorical, which follows from Proposition 4.8.6.32. □

05BX **Proposition 4.8.6.35.** *Let n be an integer and let $F : \mathcal{C} \to \mathcal{D}$ be an n-categorical inner fibration of ∞-categories. Then F is essentially n-categorical.*

For a partial converse, see Corollary 4.8.8.23.

Proof of Proposition 4.8.6.35. If $n = -2$, then F is an isomorphism of simplicial sets and therefore an equivalence of ∞-categories (Example 4.5.1.11). If $n = -1$, then F is an isomorphism from $\mathcal{C}$ onto a full subcategory of $\mathcal{D}$, and therefore fully faithful (Example 4.6.2.2). We may therefore assume without loss of generality that $n \geq 0$. By virtue of

Proposition 4.8.6.17, we may assume without loss of generality that $\mathcal{D}$ is a standard simplex; in this case, we wish to show that $\mathcal{C}$ is locally $(n-1)$-truncated. This follows from Example 4.8.2.2, since $\mathcal{C}$ is an $(n,1)$-category (Proposition 4.8.6.32). □

4.8.7 Categorically Connective Functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. Recall that F is *essentially* $(n-1)$-*categorical* if it is m-full for every nonnegative integer $m > n$. In this section, we study a dual version of this condition. 05BY

Definition 4.8.7.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. We say that F is *categorically* n-*connective* if it is m-full for every nonnegative integer $m \leq n$ (see Definition 4.8.5.10). 05BZ

Example 4.8.7.2. For small values of n, we can make Definition 4.8.7.1 more concrete: 05C0

- A functor $F : \mathcal{C} \to \mathcal{D}$ is categorically 1-connective if and only if it is full and essentially surjective.
- A functor $F : \mathcal{C} \to \mathcal{D}$ is categorically 0-connective if and only if it is essentially surjective.
- For $n < 0$, every functor $F : \mathcal{C} \to \mathcal{D}$ is categorically n-connective.

Example 4.8.7.3. Let $f : X \to Y$ be a morphism of Kan complexes and let n be an integer. Then f is categorically n-connective (in the sense of Definition 4.8.7.1) if and only if it is n-connective (in the sense of Definition 3.5.1.13). See Corollary 4.8.5.24. 05C1

Warning 4.8.7.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. If F is categorically n-connective, then it is an n-connective morphism of simplicial sets (Corollary 4.8.7.17). Beware that the converse is false in general. For example, the projection map $\Delta^1 \twoheadrightarrow \Delta^0$ is a homotopy equivalence (and therefore n-connective for every integer n) which is not categorically 2-connective. 05C2

Remark 4.8.7.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. Then F is categorically n-connective if and only if it satisfies the following pair of conditions: 05C3

- The functor F is locally $(n-1)$-connective. That is, for every pair of objects $X, Y \in \mathcal{C}$, the map of Kan complexes

  $$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

  is $(n-1)$-connective.

- If $n \geq 0$, then F is essentially surjective.

See Corollary 4.8.5.22.

05C4 **Remark 4.8.7.6** (Symmetry)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. Then F is categorically n-connective if and only if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is categorically n-connective.

05C5 **Remark 4.8.7.7** (Monotonicity)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $m \leq n$ be integers. If F is categorically n-connective, then it is categorically m-connective.

05C6 **Remark 4.8.7.8.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is an equivalence of ∞-categories if and only if it is categorically n-connective for every integer n. See Remark 4.8.5.11.

05C7 **Remark 4.8.7.9.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. It follows from Remark 4.8.7.5 that if F is categorically $(n+1)$-connective, then the induced map of homotopy n-categories $\mathrm{h}_{\leq n}(\mathcal{C}) \to \mathrm{h}_{\leq n}(\mathcal{D})$ is an equivalence. In particular, if F is categorically 2-connective, then it induces an equivalence of homotopy categories $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$.

05C8 **Remark 4.8.7.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:

- If F is categorically 1-connective and $\mathcal{C}$ is a Kan complex, then $\mathcal{D}$ is also a Kan complex.
- If F is categorically 2-connective, then $\mathcal{C}$ is a Kan complex if and only if $\mathcal{D}$ is a Kan complex.

05C9 **Remark 4.8.7.11.** Let n be an integer, and suppose we are given a categorical pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "F'"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D}' \arrow[r, "G"] & \mathcal{D}.
\end{tikzcd}$$

If F is categorically n-connective, then F' is categorically n-connective. The converse holds if G is full and essentially surjective. See Corollary 4.8.5.29.

05CA **Proposition 4.8.7.12** (Transitivity)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let n be an integer. Then:*

(1) *If F and G are categorically n-connective, then the composite functor $G \circ F$ is categorically n-connective.*

(2) *If $G \circ F$ is categorically n-connective, G is categorically $(n+1)$-connective, and $n \geq 1$, then F is categorically n-connective.*

(3) *If $G \circ F$ is categorically n-connective and F is categorically $(n-1)$-connective, then G is categorically n-connective.*

Proof. Assertions (1) and (3) follow by combining Remark 4.8.5.15 with Proposition 4.8.5.33 (supplemented by Remark 4.8.5.34), respectively. It will therefore suffice to prove (2). Assume that $n \geq 1$, that $G \circ F$ is categorically n-connective, and that G is categorically $(n+1)$-connective; we wish to prove that F is categorically n-connective: that is, that F is m-full for $m \leq n$. If $m > 0$, this follows from Proposition 4.8.5.31. It will therefore suffice to treat the case $m = 0$: that is, to show that F is essentially surjective. This follows from the essential surjectivity of $G \circ F$, since G induces an equivalence of homotopy categories (Remark 4.8.7.9). ☐

Proposition 4.8.7.13. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $n \geq 0$ be an integer. Suppose that F is bijective on simplices of dimension $< n$ and surjective on simplices of dimension n. Then F is categorically n-connective.* 05CB

Proof. Note that F is automatically essentially surjective (since it is surjective on objects). By virtue of Remark 4.8.7.5, it will suffice to show that for every pair of objects $X, Y \in \mathcal{C}$, the map of Kan complexes

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is $(n-1)$-connective. This follows from Corollary 3.5.2.2, since $F_{X,Y}$ is bijective on simplices of dimension $< n-1$ and surjective on simplices of dimension n. ☐

In the case where F is an isofibration, Definition 4.8.7.1 can be reformulated as a lifting property.

Proposition 4.8.7.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories and let n be an integer. The following conditions are equivalent:* 05CC

(1) *The functor F is categorically n-connective.*

(2) *For every integer $0 \leq m \leq n$, every lifting problem*

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution.

(3) *For every simplicial set B of dimension $\leq n$ and every simplicial subset $A \subseteq B$, every lifting problem*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
B \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution.

Proof. The equivalence (1) ⇔ (2) follows from Proposition 4.8.5.30. The implication (3) ⇒ (2) is immediate, and the reverse implication follows from Proposition 1.1.4.12. □

We now prove a partial converse to Proposition 4.8.7.13:

05CD **Proposition 4.8.7.15.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories and let n be an integer. The following conditions are equivalent:*

(1) *The functor F is categorically n-connective.*

(2) *The functor F factors as a composition*

$$\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{U} \mathcal{D},$$

where F' is a monomorphism which is bijective on m-simplices for $m \leq n$, and U is a trivial Kan fibration.

(3) *The functor U factors as a composition*

$$\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{U} \mathcal{D},$$

where F' is bijective on m-simplices for $m < n$, surjective on n-simplices, and U is categorically n-connective.

Proof. We proceed as in the proof of Corollary 3.5.2.4. The implication (2) ⇒ (3) is clear, and the implication (3) ⇒ (1) follows from Propositions 4.8.7.12 and 4.8.7.13. We will complete the proof by showing that (1) implies (2). Assume that F is categorically n-connective. Using a variant of Exercise 3.1.7.11, we can choose a factorization of F as a composition $\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{U} \mathcal{D}$ with the following properties;

(a) For every integer $m > n$, every lifting problem

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & \mathcal{D}' \arrow[d, "U"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution.

(*b*) The morphism F' can be realized as a transfinite pushout of inclusion maps $\partial \Delta^m \hookrightarrow \Delta^m$ for $m > n$.

It follows immediately from (*b*) that F is a monomorphism which is bijective on simplices of dimension $\leq n$. We will complete the proof by showing that U is a trivial Kan fibration: that is, every lifting problem

05CE

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & \mathcal{D}' \arrow[d, "U"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd} \tag{4.53}$$

admits a solution. For $m > n$, this follows from (*b*). For $m \leq n$, we can identify (4.53) with a lifting problem

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^m \arrow[r] \arrow[ur, dashed] & \mathcal{D},
\end{tikzcd}$$

which admits a solution by virtue of our assumption that F is a categorically n-connective isofibration (Proposition 4.8.7.14). □

Corollary 4.8.7.16. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. Then F is categorically n-connective if and only if it factors as a composition* 05CF

$$\mathcal{C} \xrightarrow{E} \mathcal{C}' \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D}$$

where E and G are equivalences of ∞-categories and F' is bijective on simplices of dimension $\leq n$. Moreover, we can arrange that E and F' are monomorphisms and that G is a trivial Kan fibration.

Proof. Combine Proposition 4.8.7.15 with Corollary 4.5.2.23. □

Corollary 4.8.7.17. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. If F is categorically n-connective, then it is n-connective.* 05CG

Proof. Using Corollary 4.8.7.16 (and Remark 4.5.3.4), we can reduce to the case where F is bijective on simplices of dimension $\leq n$. In this case, the desired result follows from Corollary 3.5.2.2. □

05CH **Corollary 4.8.7.18.** *Let n be an integer and let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories. Suppose that F is categorically n-connective and that G is essentially $(n-1)$-categorical. Then the diagram of ∞-categories*

05CJ
$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{B}, \mathcal{C}) \arrow[r, "\circ F"] \arrow[d, "G\circ"] & \operatorname{Fun}(\mathcal{A}, \mathcal{C}) \arrow[d, "G\circ"] \\
\operatorname{Fun}(\mathcal{B}, \mathcal{D}) \arrow[r, "\circ F"] & \operatorname{Fun}(\mathcal{A}, \mathcal{D})
\end{tikzcd} \tag{4.54}$$

is a categorical pullback square.

For a partial converse, see Proposition 4.8.9.1.

05CK **Remark 4.8.7.19.** In the situation of Corollary 4.8.7.18, suppose that one of the following additional conditions is satisfied:

(a) The functor F is a monomorphism of simplicial sets.

(b) The functor G is an isofibration.

Condition (a) guarantees that the horizontal maps in the diagram (4.54) are isofibrations (Corollary 4.4.5.3) and condition (b) guarantees that the vertical maps are isofibrations (Corollary 4.4.5.6). In either case, the conclusion of Corollary 4.8.7.18 is equivalent to the requirement that the functor

$$V : \operatorname{Fun}(\mathcal{B}, \mathcal{C}) \to \operatorname{Fun}(\mathcal{A}, \mathcal{C}) \times_{\operatorname{Fun}(\mathcal{A}, \mathcal{D})} \operatorname{Fun}(\mathcal{B}, \mathcal{D})$$

is an equivalence of ∞-categories (Proposition 4.5.2.26). If conditions (a) and (b) are both satisfied, then G is an isofibration of ∞-categories (Proposition 4.4.5.1). In this case, the conclusion of Corollary 4.8.7.18 is equivalent to the requirement that G is a trivial Kan fibration (Proposition 4.5.5.20).

Proof of Corollary 4.8.7.18. Using Corollary 4.8.7.16 we can reduce to the case where F is a monomorphism which is bijective on simplices of dimension $\leq n$. The desired result now follows from Corollary 4.8.6.21 (and Remark 4.8.7.19). □

05CL **Proposition 4.8.7.20.** *Let m and n be nonnegative integers and let $F : \mathcal{C} \to \mathcal{D}$ be a categorically $(m+n)$-connective functor of ∞-categories. Let B be a simplicial set of dimension $\leq m$, and let $A \subseteq B$ be a simplicial subset. Then the induced functor*

$$G : \operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(A, \mathcal{C}) \times_{\operatorname{Fun}(A, \mathcal{D})} \operatorname{Fun}(B, \mathcal{D})$$

is categorically n-connective.

Proof. We proceed as in the proof of Proposition 3.5.2.11. Using Corollary 4.5.2.23 (and Corollary 4.5.2.30), we can reduce to the case where F is an isofibration. In this case, G is also an isofibration (Proposition 4.4.5.1). By virtue of Proposition 4.8.7.14, it will suffice to show that if B' is a simplicial set of dimension $\leq n$ and $A' \subseteq B'$ is a simplicial subset, then every lifting problem

05CM

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & \mathrm{Fun}(B, \mathcal{C}) \arrow[d, "G"] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A, \mathcal{D})} \mathrm{Fun}(B, \mathcal{D})
\end{tikzcd} \tag{4.55}$$

admits a solution. Unwinding the definitions, we can rewrite (4.55) as a lifting problem

$$\begin{tikzcd}
(A \times B') \coprod_{(A \times A')} (B \times A') \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
B \times B' \arrow[r] \arrow[ur, dashed] & \mathcal{D}.
\end{tikzcd}$$

Since the simplicial set $B \times B'$ has dimension $\leq m+n$ (Proposition 1.1.3.6), the existence of a solution follows from our assumption that F is categorically $(m+n)$-connective (Proposition 4.8.7.14). ☐

Corollary 4.8.7.21. *Let m and n be nonnegative integers, let B be a simplicial set of dimension $\leq m$, and let $F : \mathcal{C} \to \mathcal{D}$ be functor of ∞-categories which is categorically $(m+n)$-connective. Then the induced map $\mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(B, \mathcal{D})$ is categorically n-connective.* 05CN

Proof. Applying Proposition 4.8.7.20 in the special case $A = \emptyset$. ☐

4.8.8 Relative Higher Homotopy Categories

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Recall that the *essential image* of F is the full subcategory $\mathcal{D}' \subseteq \mathcal{D}$ spanned by those objects which are isomorphic to $F(X)$, for some object $X \in \mathcal{C}$. The functor F then factors as a composition 05CP

$$\mathcal{C} \to \mathcal{D}_0 \hookrightarrow \mathcal{D},$$

where the functor on the left is essentially surjective and the functor on the right is fully faithful. It is sometimes useful to consider a different factorization.

05CQ **Proposition 4.8.8.1.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Then F factors as a composition*

$$\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D}$$

where G is faithful and F' is both full and essentially surjective.

Proof. We construct the category $\mathcal{D}'$ as follows:

- The objects of $\mathcal{D}'$ are the objects of $\mathcal{C}$. To avoid confusion, for each object $X \in \mathcal{C}$, we write $\overline{X}$ for the corresponding object of $\mathcal{D}'$.
- For every pair of objects $X, Y \in \mathcal{C}$, we take $\mathrm{Hom}_{\mathcal{D}'}(\overline{X}, \overline{Y})$ to be image of the map $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$. To avoid confusion, if $u : F(X) \to F(Y)$ is a morphism of $\mathcal{D}$ which belongs to the image of $F_{X,Y}$, we write $\overline{u} : \overline{X} \to \overline{Y}$ for the corresponding morphism of $\mathcal{D}'$.
- For every pair of objects $X, Y, Z \in \mathcal{C}$, the composition law

$$\circ : \mathrm{Hom}_{\mathcal{D}'}(\overline{Y}, \overline{Z}) \times \mathrm{Hom}_{\mathcal{D}'}(\overline{X}, \overline{Y}) \to \mathrm{Hom}_{\mathcal{D}'}(\overline{X}, \overline{Z})$$

  is the restriction of the composition law $\mathrm{Hom}_{\mathcal{D}}(F(Y), F(Z)) \times \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Z))$ for the category $\mathcal{D}$: that is, it satisfies the formula $\overline{v} \circ \overline{u} = \overline{v \circ u}$.

Let $F' : \mathcal{C} \to \mathcal{D}'$ be the functor which carries each object $X \in \mathcal{C}$ to the object $\overline{X} \in \mathcal{D}'$, and each morphism $u : X \to Y$ of $\mathcal{C}$ to the morphism $\overline{F(u)} : \overline{X} \to \overline{Y}$ of $\mathcal{D}'$. Let $G : \mathcal{D}' \to \mathcal{D}$ be the functor which carries each object $\overline{X} \in \mathcal{D}'$ to the object $F(X) \in \mathcal{D}$, and each morphism $\overline{u} : \overline{X} \to \overline{Y}$ of $\mathcal{D}'$ to the morphism $u : F(X) \to F(Y)$ of $\mathcal{D}$. Then the functor G is faithful, the functor F' is full and essentially surjective, and the composition $G \circ F'$ is equal to F. □

05CR **Exercise 4.8.8.2** (Uniqueness)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. The proof of Proposition 4.8.8.1 constructs a factorization

$$\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D}$$

where G is faithful and F' is both full and bijective on objects. Show that these properties characterize the category $\mathcal{D}'$ up to (unique) isomorphism.

Our goal in this section is to prove the following ∞-categorical generalization of Proposition 4.8.8.1:

05CS **Theorem 4.8.8.3.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. Then F admits a factorization $\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D}$ with the following properties:*

- *The functor G is essentially n-categorical: that is, it is m-full for $m \geq n + 2$.*

- *The functor F' is categorically $(n+1)$-connective: that is, it is m-full for $m \leq n+1$.*

Example 4.8.8.4. For $n \leq -2$, Theorem 4.8.8.3 asserts that every functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ admits a factorization $\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D}$, where the functor G is an equivalence of ∞-categories. This is trivial: we can take $\mathcal{D}' = \mathcal{D}$ and G to be the identity functor. `05CT`

Example 4.8.8.5. When $n = -1$, Theorem 4.8.8.3 asserts that every functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ admits a factorization $\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D}$, where the functor G is fully faithful and the functor F' is essentially surjective. For example, we can take $\mathcal{D}' \subseteq \mathcal{D}$ to be the essential image of the functor F, and $G : \mathcal{D}' \hookrightarrow \mathcal{D}$ to be the inclusion map. See Remark 4.6.2.12. `05CU`

Example 4.8.8.6. When $n = 0$, Theorem 4.8.8.3 asserts that every functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ admits a factorization $\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D}$, where the functor G is faithful and the functor F' is both full and essentially surjective. When $\mathcal{C}$ and $\mathcal{D}$ are (nerves of) ordinary categories, this follows from Proposition 4.8.8.1. To handle the general case, we can use (the proof of) Proposition 4.8.8.1 to factor the functor $\mathrm{h}F$ as a composition $\mathrm{h}\mathcal{C} \xrightarrow{F'_0} \mathcal{D}'_0 \xrightarrow{G_0} \mathrm{h}\mathcal{D}$ where G_0 is a faithful functor and F'_0 is a full functor which is essentially surjective (or even bijective on objects). To prove Theorem 4.8.8.3, we can take $\mathcal{D}'$ to be the fiber product $\mathrm{N}_{\bullet}(\mathcal{D}'_0) \times_{\mathrm{N}_{\bullet}(\mathrm{h}\mathcal{D})} \mathcal{D}$, and $G : \mathcal{D}' \to \mathcal{D}$ to be the functor given by projection onto the second factor (which is faithful by virtue of Proposition 4.8.5.8). `05CV`

Example 4.8.8.7. Let $\mathcal{C}$ be an ∞-category and let n be an integer. Then the projection map $\mathcal{C} \to \Delta^0$ factors as a composition `05CW`

$$\mathcal{C} \xrightarrow{F'} \mathrm{h}_{\leq n}(\mathcal{C}) \xrightarrow{G} \Delta^0,$$

where $\mathrm{h}_{\leq n}(\mathcal{C})$ is the homotopy n-category constructed in §4.8.4. This factorization satisfies the requirements of Theorem 4.8.8.3: the functor G is essentially n-categorical because $\mathrm{h}_{\leq n}(\mathcal{C})$ is an $(n,1)$-category (Example 4.8.6.4), and the functor F' is categorically $(n+1)$-connective by Example 4.8.5.12.

Remark 4.8.8.8 (Uniqueness)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then, for every integer n, the factorization of Theorem 4.8.8.3 is well-defined up to equivalence. More precisely, if the functor F admits two factorizations `05CX`

$$\mathcal{C} \xrightarrow{F'_0} \mathcal{D}'_0 \xrightarrow{G_0} \mathcal{D} \qquad \mathcal{C} \xrightarrow{F'_1} \mathcal{D}'_1 \xrightarrow{G_1} \mathcal{D}$$

where the functors F'_0 and F'_1 are essentially n-categorical, and the functors G_0 are G_1 are categorically $(n+1)$-connective, then we can find a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F'_0"] \arrow[d, equal] & \mathcal{D}'_0 \arrow[r, "G_0"] & \mathcal{D} \arrow[d, equal] \\
\mathcal{C} \arrow[r] \arrow[d, equal] & \mathcal{D}'_{01} \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"] & \mathcal{D} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "F'_1"] & \mathcal{D}'_1 \arrow[r, "G_1"] & \mathcal{D}
\end{tikzcd}$$

where the vertical maps are equivalences of ∞-categories. To prove this, we can use Corollary 4.5.2.23 to reduce to the case where F'_0 is a monomorphism of simplicial sets and G_1 is an isofibration. In this case, Corollary 4.8.7.18 (and Remark 4.8.7.19) guarantee that the functors F'_0 and G_1 induce a trivial Kan fibration

$$\operatorname{Fun}(\mathcal{D}'_0, \mathcal{D}'_1) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D}'_1) \times_{\operatorname{Fun}(\mathcal{C},\mathcal{D})} \operatorname{Fun}(\mathcal{D}'_0, \mathcal{D}).$$

In particular, this map is surjective on vertices, so the lifting problem

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F'_1"] \arrow[d, "F'_0"] & \mathcal{D}'_1 \arrow[d, "G_1"] \\
\mathcal{D}'_0 \arrow[r, "G_0"] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

has a solution. A choice of solution determines a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F'_0"] \arrow[d, equal] & \mathcal{D}'_0 \arrow[r, "G_0"] \arrow[d, "H"] & \mathcal{D} \arrow[d, equal] \\
\mathcal{C} \arrow[r, "F'_1"] & \mathcal{D}'_1 \arrow[r, "G_1"] & \mathcal{D}.
\end{tikzcd}$$

It follows from Proposition 4.8.7.12 that the functor H is categorically $(n+1)$-connective, and from Remark 4.8.6.7 that H is essentially n-categorical. Applying Remark 4.8.5.11, we conclude that H is an equivalence of ∞-categories.

Corollary 4.8.8.9. *Let $f : X \to Z$ be a morphism of Kan complexes and let n be an integer. Then f factors as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$, where f'' is n-truncated and f' is $(n+1)$-connective.* 05CY

Proof. Using Theorem 4.8.8.3, we can factor f as a composition $f'' \circ f'$, where $f'' : \mathcal{C} \to Z$ is an essentially n-categorical functor of ∞-categories and $f' : X \to \mathcal{C}$ is categorically $(n+1)$-connective. If $n \leq -1$, then f'' induces an equivalence from $\mathcal{C}$ to a summand of Z, so that $\mathcal{C}$ is a Kan complex. If $n \geq 0$, Remark 4.8.7.10 guarantees that $\mathcal{C}$ is a Kan complex. Setting $Y = \mathcal{C}$, we observe that f'' is n-truncated (Example 4.8.6.3) and f' is $(n+1)$-connective (Example 4.8.7.3). □

We will prove Theorem 4.8.8.3 in general by reducing to the special case studied in Example 4.8.8.7. For this, we will need a relative version of the construction $\mathcal{C} \mapsto \mathrm{h}_{\leq n}(\mathcal{C})$ introduced in §4.8.4.

Construction 4.8.8.10 (Relative Homotopy n-Categories)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets and let $n \geq 0$ be an integer. For every m-simplex σ of $\mathcal{D}$, let $\mathcal{C}_\sigma$ denote the fiber product $\Delta^m \times_{\mathcal{D}} \mathcal{C}$. We let $\mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D})_m$ denote the collection of pairs (σ, τ), where σ is an m-simplex of $\mathcal{D}$ and τ is a section of the projection map 05CZ

$$\mathrm{h}_{\leq n}(\mathcal{C}_\sigma) \to \mathrm{h}_{\leq n}(\Delta^m) \simeq \Delta^m.$$

If $f : [m'] \to [m]$ is a nondecreasing function, we let $f^* : \mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D})_m \to \mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D})_{m'}$ denote the map given by $f^*(\sigma, \tau) = (\sigma', \tau')$, where σ' is the composite map $\Delta^{m'} \xrightarrow{f} \Delta^m \xrightarrow{\sigma} \mathcal{D}$ and τ' is given by the composition

$$\begin{aligned} \Delta^{m'} & \simeq \Delta^{m'} \times_{\Delta^m} \Delta^m \\ & \xrightarrow{(\mathrm{id}, \tau)} \Delta^{m'} \times_{\Delta^m} \mathrm{h}_{\leq n}(\mathcal{C}_\sigma) \\ & \simeq \mathrm{h}_{\leq n}(\Delta^{m'} \times_{\Delta^m} \mathcal{C}_\sigma) \\ & \simeq \mathrm{h}_{\leq n}(\mathcal{C}_{\sigma'}), \end{aligned}$$

where the second isomorphism is provided by Proposition 4.8.4.20. By means of this construction, we can view the assignment $[m] \mapsto \mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D})_m$ as a simplicial set, which we will denote by $\mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D})$. Note that the construction $(\sigma, \tau) \mapsto \sigma$ determines a comparison map of simplicial sets $G : \mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D}) \to \mathcal{D}$.

It will be useful to extend this construction to the case where $n < 0$. If $n = -1$, we define $\mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D})$ to be the full simplicial subset of $\mathcal{D}$ whose vertices belong to the image of F, and we take $G : \mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D}) \hookrightarrow \mathcal{D}$ to be the inclusion map. If $n \leq -2$, we define $\mathrm{h}_{\leq n}(\mathcal{C} / \mathcal{D})$ to be the simplicial set $\mathcal{D}$, and G to be the identity morphism $\mathrm{id}_{\mathcal{D}}$.

05D0 **Example 4.8.8.11.** Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let n be an integer. Then there is a comparison map from the simplicial set $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ to the homotopy n-category $\mathrm{h}_{\leq n}(\mathcal{C})$. For $n \geq 0$, this map carries an m-simplex (σ, τ) of $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ to the m-simplex of $\mathrm{h}_{\leq n}(\mathcal{C})$ given by the composite map

$$\Delta^m \xrightarrow{\tau} \mathrm{h}_{\leq n}(\mathcal{C}_\sigma) \to \mathrm{h}_{\leq n}(\mathcal{C}).$$

If $\mathcal{D}$ is an $(n,1)$-category, then this comparison map is an isomorphism (Proposition 4.8.4.20).

05D1 **Example 4.8.8.12.** Let $\mathcal{C}$ be an ∞-category, so that the projection map $F : \mathcal{C} \to \Delta^0$ is an inner fibration. Since Δ^0 is an $(n,1)$-category, Example 4.8.8.11 supplies an isomorphism of simplicial sets $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \Delta^0) \simeq \mathrm{h}_{\leq n}(\mathcal{C})$.

05D2 **Remark 4.8.8.13** (Base Change)**.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathcal{D}' \arrow[r] & \mathcal{D},
\end{tikzcd}$$

where the vertical maps are inner fibrations. Then, for every integer n, the simplicial set $\mathrm{h}_{\leq n}(\mathcal{C}' \,/\, \mathcal{D}')$ can be identified with the fiber product $\mathcal{D}' \times_{\mathcal{D}} \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$. In particular, for every vertex $D \in \mathcal{D}$, we have a canonical isomorphism

$$\{D\} \times_{\mathcal{D}} \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \simeq \mathrm{h}_{\leq n}(\{D\} \times_{\mathcal{D}} \mathcal{C}).$$

05D3 **Proposition 4.8.8.14.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets and let n be an integer. Then the comparison map $G : \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \to \mathcal{D}$ of Construction 4.8.8.10 is an n-categorical inner fibration (see Definition 4.8.6.24).*

Proof. For $n < 0$, this is immediate from the construction. We may therefore assume without loss of generality that $n \geq 0$. Using Remarks 4.8.6.33 and 4.8.8.13, we can reduce to the case where $\mathcal{D} = \Delta^m$ is a standard simplex. In particular, $\mathcal{D}$ is an $(n,1)$-category. In this case, Example 4.8.8.11 guarantees that the simplicial set $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \simeq \mathrm{h}_{\leq n}(\mathcal{D})$ is an $(n,1)$-category. The desired result now follows from Proposition 4.8.6.32. □

05D4 **Remark 4.8.8.15.** Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets and let n be an integer. Then the comparison map $G : \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \to \mathcal{D}$ of Construction 4.8.8.10 fits into a

commutative diagram

$$\begin{tikzcd} & \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \arrow[dr, "G"] & \\ \mathcal{C} \arrow[ur, "F'"] \arrow[rr, "F"] & & \mathcal{D}. \end{tikzcd}$$

For $n \geq 0$, the morphism F' carries each m-simplex of $\mathcal{C}$ to the m-simplex $(F(\sigma), \tau)$ of $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$, where τ is the composite map

$$\Delta^m \xrightarrow{(\mathrm{id}, \sigma)} \Delta^m \times_{\mathcal{D}} \mathcal{C} = \mathcal{C}_\sigma \to \mathrm{h}_{\leq n}(\mathcal{C}_\sigma).$$

The simplicial set $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ of Construction 4.8.8.10 can be characterized by a universal mapping property:

Proposition 4.8.8.16. *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets and let n be an integer. Then, for every n-categorical inner fibration $\mathcal{D}' \to \mathcal{D}$, the comparison map of Remark 4.8.8.15 induces an isomorphism of simplicial sets* 05D5

$$\theta : \mathrm{Fun}_{/\mathcal{D}}(\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}), \mathcal{D}') \to \mathrm{Fun}_{/\mathcal{D}}(\mathcal{C}, \mathcal{D}').$$

Proof. We may assume without loss of generality that $n \geq 0$ (otherwise, the result follows immediately from the construction). For every morphism of simplicial sets $K \to \mathcal{D}$, Remark 4.8.8.15 determines a comparison map

$$\theta_K : \mathrm{Fun}_{/\mathcal{D}}(K \times_{\mathcal{D}} \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}), \mathcal{D}') \to \mathrm{Fun}_{/\mathcal{D}}(K \times_{\mathcal{D}} \mathcal{C}, \mathcal{D}').$$

We will prove that each θ_K is an isomorphism of simplicial sets; Proposition 4.8.8.16 then follows by taking $K = \mathcal{D}$. Note that the construction $K \mapsto \theta_K$ carries colimits (in the category of simplicial sets with a morphism to $\mathcal{D}$) to limits (in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_\Delta)$). By virtue of Remark 1.1.3.13, we can assume without loss of generality that K is a standard simplex. Replacing F by the projection map $K \times_{\mathcal{D}} \mathcal{C} \to K$ and $\mathcal{D}'$ by the fiber product $K \times_{\mathcal{C}} \mathcal{D}'$, we are reduced to proving Proposition 4.8.8.16 in the special case where $\mathcal{D}$ is a standard simplex: in particular, it is an $(n,1)$-category. In this case, $\mathcal{D}'$ is also an $(n,1)$-category (Proposition 4.8.6.32), and we can identify $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ with the homotopy n-category of $\mathcal{C}$ (Example 4.8.8.11). Applying Proposition 4.8.4.7, we see that the horizontal maps in the commutative diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}), \mathcal{D}') \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}') \arrow[d] \\ \mathrm{Fun}(\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}), \mathcal{D}) \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \end{tikzcd}$$

are isomorphisms. The desired result now follows by passing to fibers of the vertical maps. □

05D6 **Remark 4.8.8.17.** Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets, let n be an integer, and let $A \subseteq B$ be simplicial sets. If B has dimension $\leq n+1$, then every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F'"] \\
B \arrow[r] \arrow[ur, dashed] & \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})
\end{tikzcd}$$

has a solution. Moreover, if B has dimension $\leq n-1$, then the solution is unique. To prove this, we can assume without loss of generality that $B = \Delta^m$ is a standard simplex for some $m \leq n+1$, and that $A = \partial \Delta^m$ is its boundary (see Proposition 1.1.4.12). The case $n \leq -2$ is vacuous, and the case $n = -1$ is immediate from the definition. We may therefore assume that $n \geq 0$. Replacing F by the projection map $\Delta^m \times_{\mathcal{D}} \mathcal{C} \to \Delta^m$, we can reduce to the case where $\mathcal{D}$ is a standard simplex, so that U' identifies $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ with the homotopy n-category $\mathrm{h}_{\leq n}(\mathcal{C})$ (Example 4.8.8.11). In this case, the desired result follows from Corollary 4.8.4.17.

05D7 **Proposition 4.8.8.18.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets and let n be an integer. Then the comparison map $F' : \mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ of Remark 4.8.8.15 is an inner fibration.*

Proof. Without loss of generality, we may assume that $n \geq 0$. Using Remarks 4.1.1.13 and 4.8.8.13, we can reduce to the case where $\mathcal{D} = \Delta^m$ is a standard simplex. In this case, F' identifies with the tautological map $\mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C})$ (Example 4.8.8.11), so the desired result follows from Corollary 4.8.4.16. □

05D8 **Corollary 4.8.8.19.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let n be an integer. Then the simplicial set $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ is an ∞-category. Moreover, the functor $F' : \mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ of Remark 4.8.8.15 is categorically $(n+1)$-connective.*

Proof. Since $\mathcal{D}$ is an ∞-category and the comparison map $G : \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \to \mathcal{D}$ is an inner fibration (Proposition 4.8.8.14), the simplicial set $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ is also an ∞-category (Remark 4.1.1.9). Fix an integer $m \leq n+1$; we wish to show that the functor F' is m-full. For $n = -2$, there is nothing to prove. If $n = -1$, then U' is surjective on objects (by construction) and therefore essentially surjective. We may therefore assume without loss of generality that $n \geq 0$. Since U' is an inner fibration (Proposition 4.8.8.18), it will suffice to show that for every morphism $\Delta^1 \to \mathrm{h}_{\leq n}(\mathcal{E} \,/\, \mathcal{C})$, the projection map $\Delta^1 \times_{\mathrm{h}_{\leq n}(\mathcal{E} \,/\, \mathcal{C})} \mathcal{E} \to \Delta^1$ is m-full (Proposition 4.8.5.27. Using Remark 4.8.8.13, we can replace F by the projection map $\Delta^1 \times_{\mathcal{D}} \mathcal{C} \to \Delta^1$, and thereby reduce to the situation where $\mathcal{D} = \Delta^1$ is an $(n,1)$-category. In this case, the functor F' exhibits $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ as a homotopy n-category of $\mathcal{C}$ (Example 4.8.8.11), so the desired result follows from Example 4.8.5.12. □

Proof of Theorem 4.8.8.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let n be an integer. We wish to show that F factors as a composition $G \circ F'$, where G is essentially n-categorical and F' is categorically $(n+1)$-connective. Using Proposition 4.1.3.2, we can reduce to the case where F is an inner fibration. In this case, the factorization

$$\mathcal{C} \xrightarrow{F'} \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \xrightarrow{G} \mathcal{C}$$

of Remark 4.8.8.15 has the desired properties: Proposition 4.8.8.14 guarantees that G is an n-categorical inner fibration (and is therefore essentially n-categorical, by virtue of Proposition 4.8.6.35), and Corollary 4.8.8.19 guarantees that F' is categorically $(n+1)$-connective. □

Warning 4.8.8.20. Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. In the case $n = 0$, our proof of Theorem 4.8.8.3 shows that F factors as a composition 05D9

$$\mathcal{E} \xrightarrow{F'} \mathrm{h}_{\leq 0}(\mathcal{C} \,/\, \mathcal{D}) \xrightarrow{G} \mathcal{D},$$

where F' is fully faithful and essentially surjective, and G is a 0-categorical inner fibration (in particular, G is faithful). Beware that generally does not coincide with the factorization constructed in Example 4.8.8.6. If $u : X \to Y$ is an isomorphism in the ∞-category $\mathcal{C}$ having the property that $F(u)$ is an identity morphism in $\mathcal{D}$, then the functor F' carries X and Y to the same object of $\mathrm{h}_{\leq 0}(\mathcal{C} \,/\, \mathcal{D})$. Consequently, the functor F' is generally not bijective on objects.

A related phenomenon occurs in the case $n = -1$. By construction, $\mathrm{h}_{\leq -1}(\mathcal{C} \,/\, \mathcal{D})$ is the full subcategory of $\mathcal{D}$ spanned by objects of the form $F(X)$, where X is an object of $\mathcal{C}$. If the inner fibration U is not an isofibration, this subcategory might be smaller than the essential image of F.

We close this section with a few additional observations about Construction 4.8.8.10.

Proposition 4.8.8.21. *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets, let n be an integer, and let $G : \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \to \mathcal{D}$ be the comparison map of Construction 4.8.8.10. Then:* 05DA

(1) *If F is a left fibration, then G is a left fibration.*

(2) *If F is a right fibration, then G is a right fibration.*

(3) *If F is a Kan fibration, then G is a Kan fibration.*

(4) *If F is an isofibration of ∞-categories, then G is an isofibration of ∞-categories.*

Proof. We first prove (1). Assume that F is a left fibration, and suppose we are given integers $0 \leq i < n$; we wish to show that every lifting problem

05DB
$$\begin{tikzcd}
\Lambda^m_i \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \arrow[d, "G"] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{D}
\end{tikzcd} \tag{4.56}$$

admits a solution. If $m \leq n+2$, then σ_0 can be lifted to a morphism $\Lambda^m_i \to \mathcal{C}$ (Remark 4.8.8.17), so the desired result follows from our assumption that F is a left fibration. We may therefore assume that $m \geq n+3$. If $n = -2$, then G is an isomorphism and there is nothing to prove. If $n = -1$, then G identifies $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ with a full simplicial subset of $\mathcal{D}$, and the desired result follows from the observation that Λ^m_i contains every vertex of Δ^m. We may therefore assume that $n \geq 0$. Replacing F by the projection map $\Delta^m \times_{\mathcal{D}} \mathcal{C} \to \Delta^m$, we can reduce to the case where $\mathcal{D} = \Delta^m$ is a standard simplex. In this case, $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ is an $(n,1)$-category (Example 4.8.8.11). In particular, it is an $(n+1)$-coskeletal simplicial set, so the lifting problem (4.56) has a unique solution (since Λ^m_i contains the $(n+1)$-skeleton of Δ^m).

Assertion (2) follows by applying (1) to the opposite inner fibration $U^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. Assertion (3) follows by combining (1) and (2) with Example 4.2.1.5. It remains to prove (4). Fix an object $Y \in \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ and an isomorphism $\overline{e} : \overline{X} \to V(Y)$ in the ∞-category $\mathcal{D}$; we wish to show that $\overline{e}$ can be lifted to an isomorphism $e : X \to Y$ of $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$. If $n \leq -2$, then G is an isomorphism and the result is obvious. Otherwise, the comparison map $F' : \mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ is surjective on vertices, so we can choose an object $\widetilde{Y} \in \mathcal{C}$ satisfying $F'(\widetilde{Y}) = Y$. If F is an isofibration, then there exists an isomorphism $\widetilde{e} : \widetilde{X} \to \widetilde{Y}$ of $\mathcal{C}$ satisfying $F(\widetilde{e}) = \overline{e}$. It follows that $e = F'(\widetilde{e})$ is an isomorphism in $\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ satisfying $G(e) = \overline{e}$. □

05DC **Proposition 4.8.8.22.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let n be an integer. The following conditions are equivalent:*

(1) *The functor F is essentially n-categorical.*

(2) *The comparison map $F' : \mathcal{C} \to \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D})$ of Remark 4.8.8.15 is an equivalence of ∞-categories.*

Proof. It follows from Proposition 4.8.8.14 (and Proposition 4.8.6.35) that the comparison map $G : \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \to \mathcal{D}$ is essentially n-categorical. By virtue of Remark 4.8.6.8, we can replace (1) by the following condition:

(1′) The functor F' is essentially n-categorical: that is, it is m-full for $m \geq n+2$.

Since F' is also m-full for $m \leq n+1$ (Corollary 4.8.8.19), the equivalence (1′) $\Leftrightarrow$ (2) follows from Remark 4.8.5.11. □

Corollary 4.8.8.23. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. For every integer n, the following conditions are equivalent:* `05DD`

(1) *The functor F is essentially n-categorical.*

(2) *The functor F factors as a composition $\mathcal{C} \xrightarrow{F'} \mathcal{D}' \xrightarrow{G} \mathcal{D}$, where F' is an equivalence of ∞-categories and G is an n-categorical isofibration.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Proposition 4.8.6.35 (together with Remark 4.8.5.18). To prove the converse, we may assume without loss of generality that F is an isofibration (Corollary 4.5.2.23). In this case, the factorization $\mathcal{C} \xrightarrow{F'} \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \xrightarrow{G} \mathcal{D}$ of Remark 4.8.8.15 has the desired properties: Proposition 4.8.8.22 guarantees that F' is an equivalence of ∞-categories, Proposition 4.8.8.21 guarantees that G is an isofibration, and Proposition 4.8.8.14 guarantees that G is n-categorical. □

Proposition 4.8.8.24. *Let n be an integer, and suppose we are given a commutative diagram of ∞-categories* `05DE`

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D} \arrow[dl] \\
& \mathcal{E} &
\end{tikzcd}$$

where the vertical maps are inner fibrations. If F is categorically $(n+1)$-connective, then it induces an equivalence of ∞-categories $F' : \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{E}) \to \mathrm{h}_{\leq n}(\mathcal{D} \,/\, \mathcal{E})$.

Proof. We have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{E}) \arrow[r, "F'"] & \mathrm{h}_{\leq n}(\mathcal{D} \,/\, \mathcal{E}).
\end{tikzcd}$$

Here F is categorically $(n+1)$-connective by assumption, and the vertical maps are categorically $(n+1)$-connective by virtue of Corollary 4.8.8.19. Applying Proposition 4.8.7.12, we see that the functor F' is also categorically $(n+1)$-connective. We also have a commutative diagram

$$\begin{tikzcd}
\mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{E}) \arrow[rr, "F'"] \arrow[dr] & & \mathrm{h}_{\leq n}(\mathcal{D} \,/\, \mathcal{E}) \arrow[dl] \\
& \mathcal{E}, &
\end{tikzcd}$$

where the vertical maps are essentially n-categorical (Proposition 4.8.8.14). Using Remark 4.8.6.8, we see that F' is also essentially n-categorical. Using Remark 4.8.5.11, we see that F' is an equivalence of ∞-categories. □

05DF **Corollary 4.8.8.25.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let n be an integer. The following conditions are equivalent:*

(1) *The comparison map $G : \mathrm{h}_{\leq n}(\mathcal{C} \,/\, \mathcal{D}) \to \mathcal{D}$ is an equivalence of ∞-categories.*

(2) *The functor F is categorically $(n+1)$-connective.*

Proof. The implication (1) ⇒ (2) follows from Proposition 4.8.8.14 and Remark 4.8.5.16. The reverse implication follows by applying Proposition 4.8.8.24 in the special case $\mathcal{E} = \mathcal{D}$. □

4.8.9 Categorically Connective Morphisms of Simplicial Sets

05DG Using Theorem 4.8.8.3, we can give an alternative characterization of categorical connectivity.

05DH **Proposition 4.8.9.1.** *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor of ∞-categories and let n be an integer. The following conditions are equivalent:*

(1) *The functor F is categorically n-connective (Definition 4.8.7.1).*

(2) *For every essentially $(n-1)$-categorical functor of ∞-categories $U : \mathcal{C} \to \mathcal{D}$, the diagram*

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{B}, \mathcal{C}) \arrow[r, "\circ F"] \arrow[d, "U\circ"] & \mathrm{Fun}(\mathcal{A}, \mathcal{C}) \arrow[d, "U\circ"] \\
\mathrm{Fun}(\mathcal{B}, \mathcal{D}) \arrow[r, "\circ F"] & \mathrm{Fun}(\mathcal{A}, \mathcal{D})
\end{tikzcd}$$

is a categorical pullback square.

(3) *For every* $(n-1)$*-categorical isofibration* $U : \mathcal{C} \to \mathcal{B}$*, precomposition with* F *induces an equivalence of* ∞*-categories*

$$\theta_{\mathcal{C}} : \operatorname{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{C}) \to \operatorname{Fun}_{/\mathcal{B}}(\mathcal{A}, \mathcal{C}).$$

Proof. The implication (1) ⇒ (2) is a restatement of Corollary 4.8.7.18, and the implication (2) ⇒ (3) follows from Corollary 4.5.2.32. To show that (3) implies (1), we may assume without loss of generality that F is an isofibration. Then the comparison map $G : \mathrm{h}_{\leq n-1}(()\,\mathcal{A}\,/\,\mathcal{B}) \to \mathcal{B}$ $(n-1)$-categorical isofibration (Propositions 4.8.8.14 and 4.8.8.21). If $U : \mathcal{C} \to \mathcal{B}$ is another $(n-1)$-categorical isofibration, then we can use Proposition 4.8.8.16 to identify $\theta_{\mathcal{C}}$ with the map with the functor $\operatorname{Fun}_{/\mathcal{B}}(\mathcal{B}, \mathcal{B}') \to \operatorname{Fun}_{/\mathcal{B}}(\mathrm{h}_{\leq n-1}(\mathcal{A}\,/\,\mathcal{B}), \mathcal{C})$ given by precomposition with G. If condition (3) is satisfied, then G is an equivalence of ∞-categories, so that F is categorically n-connective by virtue of Corollary 4.8.8.25. □

Motivated by Proposition 4.8.9.1, we introduce a generalization of Definition 4.8.7.1.

Definition 4.8.9.2. Let $f : A \to B$ be a morphism of simplicial sets and let n be an integer. We say that f is *categorically* n*-connective* if, for every essentially $(n-1)$-categorical functor of ∞-categories $U : \mathcal{C} \to \mathcal{D}$, the diagram `05DJ`

$$\begin{tikzcd}
\operatorname{Fun}(B, \mathcal{C}) \arrow[r, "\circ f"] \arrow[d, "U\circ"] & \operatorname{Fun}(A, \mathcal{C}) \arrow[d, "U\circ"] \\
\operatorname{Fun}(B, \mathcal{D}) \arrow[r, "\circ f"] & \operatorname{Fun}(A, \mathcal{D})
\end{tikzcd}$$

is a categorical pullback square.

Remark 4.8.9.3. In the situation of Definition 4.8.9.2, we can assume without loss of generality that the functor $U : \mathcal{C} \to \mathcal{D}$ is an isofibration (see Corollary 4.5.2.23). Replacing $\mathcal{C}$ by the simplicial set $\mathrm{h}_{\leq n-1}(\mathcal{C}\,/\,\mathcal{D})$, we can further arrange that the isofibration U is $(n-1)$-categorical (Proposition 4.8.8.22). `05DK`

Remark 4.8.9.4. Let n be an integer. The notion of categorical n-connectivity is completely determined by the following two properties: `05DL`

(1) If $F : \mathcal{A} \to \mathcal{B}$ is a functor of ∞-categories, then it is categorically n-connective in the sense of Definition 4.8.9.2 if and only if it is categorically n-connective in the sense of Definition 4.8.7.1: that is, F is m-full for every nonnegative integer $m \leq n$ (see Proposition 4.8.9.1).

(2) Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r] & B', \end{tikzcd}$$

where the horizontal maps are categorical equivalences. Then f is categorically n-connective if and only if f' is categorically n-connective. See Proposition 4.5.2.19.

If $f : A \to B$ is any morphism of simplicial sets, then we can use Proposition 4.1.3.2 to choose a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & \mathcal{A} \arrow[d, "F"] \\ B \arrow[r] & \mathcal{B} \end{tikzcd}$$

where the horizontal maps are categorical equivalences and F is a functor of ∞-categories. Combining (1) and (2), we see that f is categorically n-connective if and only if the functor F is m-full for $m \leq n$.

05DM **Remark 4.8.9.5.** Let $f : A \to B$ be a morphism of simplicial sets. If f is categorically n-connective, then it is n-connective. This follows from Remark 4.8.9.4 and Corollary 4.8.7.17. Beware that the converse is false in general (Warning 4.8.7.4).

05DN **Remark 4.8.9.6** (Transitivity)**.** Let $f : A \to B$ and $g : B \to C$ be morphisms of simplicial sets and let n be an integer.

(1) Suppose that f and g are categorically n-connective. Then $g \circ f$ is categorically n-connective.

(2) Suppose that $g \circ f$ is categorically n-connective, g is categorically $(n+1)$-connective, and $n \geq 1$. Then f is categorically n-connective.

(3) Suppose that $g \circ f$ is categorically n-connective and that f is categorically $(n-1)$-connective. Then g is categorically n-connective.

To prove these assertions, we can use Remark 4.8.9.4 to reduce to the case where A, B, and C are ∞-categories, in which case the result follows from Proposition 4.8.7.12

05DP **Proposition 4.8.9.7.** *Suppose we are given a categorical pushout square of simplicial sets*

05DQ

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r] & B', \end{tikzcd} \tag{4.57}$$

where f is categorically n-connective. Then f' is also categorically n-connective.

Proof. Let $U : \mathcal{C} \to \mathcal{D}$ be an essentialy $(n-1)$-categorical functor of ∞-categories, and consider the cubical diagram

$$\begin{tikzcd}
\operatorname{Fun}(B', \mathcal{C}) \arrow[rr] \arrow[dr] \arrow[dd] & & \operatorname{Fun}(B, \mathcal{C}) \arrow[dr] \arrow[dd] & \\
& \operatorname{Fun}(B', \mathcal{D}) \arrow[rr] \arrow[dd] & & \operatorname{Fun}(B, \mathcal{D}) \arrow[dd] \\
\operatorname{Fun}(A', \mathcal{C}) \arrow[rr] \arrow[dr] & & \operatorname{Fun}(A, \mathcal{C}) \arrow[dr] & \\
& \operatorname{Fun}(A', \mathcal{D}) \arrow[rr] & & \operatorname{Fun}(A, \mathcal{D}).
\end{tikzcd}$$

Our assumption that f is categorically n-connective guarantees that the right face is a categorical pullback square, and our assumption on (4.57) guarantees that the front and back faces are categorical pullback squares. Applying Proposition 4.5.2.18, we conclude that the left face is also a categorical pullback square. □

Proposition 4.8.9.8. *Let $f : A \hookrightarrow B$ be a monomorphism of simplicial sets and let n be an integer. The following conditions are equivalent:* 05DR

(1) *The morphism f is categorically n-connective.*

(2) *For every essentially $(n-1)$-categorical functor of ∞-categories $U : \mathcal{C} \to \mathcal{D}$, the restriction map*

$$V : \operatorname{Fun}(B, \mathcal{C}) \to \operatorname{Fun}(A, \mathcal{C}) \times_{\operatorname{Fun}(A, \mathcal{D})} \operatorname{Fun}(B, \mathcal{D})$$

is an equivalence of ∞-categories.

(3) *For every essentially $(n-1)$-categorical isofibration of ∞-categories $U : \mathcal{C} \to \mathcal{D}$, the functor V is a trivial Kan fibration.*

(3) *Every lifting problem*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\ B \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

admits a solution, provided that U is an essentially $(n-1)$-categorical isofibration of ∞-categories.

Proof. The equivalences $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ follow from Remarks 4.8.7.19 and 4.8.9.3, and the implication $(3) \Rightarrow (4)$. is immediate. We will complete the proof by showing that (4) implies (3). Assume that condition (4) is satisfied, and let $U : \mathcal{C} \to \mathcal{D}$ be an essentially $(n-1)$-categorical isofibration of ∞-categories. We wish to show that, for every simplicial set B' and every simplicial subset $A' \subseteq B'$, every lifting problem

05DS

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & \operatorname{Fun}(B, \mathcal{C}) \arrow[d, "V"] \\ B' \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(A, \mathcal{C}) \times_{\operatorname{Fun}(A, \mathcal{D})} \operatorname{Fun}(B, \mathcal{D}) \end{tikzcd} \tag{4.58}$$

admits a solution. Unwinding the definitions, we can rewrite (4.58) as a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & \operatorname{Fun}(B', \mathcal{C}) \arrow[d, "V'"] \\ B \arrow[r] \arrow[ur, dashed] & \operatorname{Fun}(A', \mathcal{C}) \times_{\operatorname{Fun}(A', \mathcal{D})} \operatorname{Fun}(B', \mathcal{D}). \end{tikzcd}$$

The existence of a solution follows from (4), since V' is also an essentially $(n-1)$-categorical isofibration of ∞-categories (Corollary 4.8.6.21 and Proposition 4.4.5.1). □

05DT **Example 4.8.9.9.** Let B be a simplicial set and let $A \subseteq B$ be a simplicial subset which contains the n-skeleton of B. Then the inclusion map $A \hookrightarrow B$ is categorically n-connective. In particular, for every simplicial set B, the inclusion map $\operatorname{sk}_n(B) \hookrightarrow B$ is categorically n-connective.

05DU **Proposition 4.8.9.10.** *Let $n \geq 0$ be an integer and let $f : A \to B$ be a morphism of simplicial sets which is bijective on simplices of dimension $< n$ and surjective on n-simplices. Then f is categorically n-connective.*

Proof. Using Proposition 1.1.4.12, we can choose a simplicial subset $A' \subseteq \mathrm{sk}_n(A)$ which contains the $(n-1)$-skeleton of A, such that f restricts to an isomorphism of A' with the n-skeleton of B. It follows from Example 4.8.9.9 $f|_{A'}$ is categorically n-connective, and that the inclusion map $A' \hookrightarrow A$ is categorically $(n-1)$-connective. Applying Remark 4.8.9.6, we deduce that f is categorically n-connective. $\square$

Chapter 5

Fibrations of ∞-Categories

01J2 Let Ab denote the category of abelian groups. For every commutative ring A, we let $\mathrm{Mod}_A(\mathrm{Ab})$ denote the category of A-modules. Every homomorphism of commutative rings $u : A \to B$ determines a functor

$$T_u : \mathrm{Mod}_A(\mathrm{Ab}) \to \mathrm{Mod}_B(\mathrm{Ab}) \qquad T_u(M) = B \otimes_A M,$$

which we will refer to as *extension of scalars along u*. One can summarize the situation informally by saying that there is a functor from commutative rings to (large) categories, which carries each commutative ring A to the category $\mathrm{Mod}_A(\mathrm{Ab})$ and each ring homomorphism $u : A \to B$ to the functor T_u. However, we encounter the following subtleties:

(1) Let $u : A \to B$ and $v : B \to C$ be homomorphisms of commutative rings. Then the diagram of categories

$$\begin{tikzcd}
& \mathrm{Mod}_B(\mathrm{Ab}) \arrow[dr, "T_v"] & \\
\mathrm{Mod}_A(\mathrm{Ab}) \arrow[ur, "T_u"] \arrow[rr, "T_{vu}"] & & \mathrm{Mod}_C(\mathrm{Ab})
\end{tikzcd}$$

might not be *strictly* commutative. If M is an A-module, one cannot reasonably expect $C \otimes_A M$ to be *identical* to the iterated tensor product $C \otimes_B (B \otimes_A M)$. Instead, there is a canonical isomorphism

$$\mu_{v,u}(M) : C \otimes_B (B \otimes_A M) \simeq C \otimes_A M,$$

which depends functorially on M, so that the collection $\{\mu_{v,u}(M)\}_{M \in \mathrm{Mod}_A(\mathrm{Ab})}$ can be viewed as an isomorphism of functors $\mu_{v,u} : T_v \circ T_u \simeq T_{vu}$.

(2) Let A be a commutative ring, and let $\mathrm{id}_A : A \to A$ be the identity map. Then the extension of scalars functor $T_{\mathrm{id}_A} : \mathrm{Mod}_A(\mathrm{Ab}) \to \mathrm{Mod}_A(\mathrm{Ab})$ might not be *equal* to the identity functor $\mathrm{id}_{\mathrm{Mod}_A(\mathrm{Ab})}$. However, there is a natural isomorphism $\epsilon_A : \mathrm{id}_{\mathrm{Mod}_A(\mathrm{Ab})} \simeq T_{\mathrm{id}_A}$, which carries each A-module M to the A-module isomorphism

$$M \simeq A \otimes M \qquad x \mapsto 1 \otimes x.$$

Let Cat denote the ordinary category whose objects are categories (which, for the moment, we do not require to be small) and whose morphisms are functors. Because of the technical issues outlined above, the construction $A \mapsto \mathrm{Mod}_A(\mathrm{Ab})$ cannot be viewed as a functor from the category of commutative rings to the category Cat. However, this can be remedied using the language of 2-categories. Recall that Cat can be realized as the underlying category of a (strict) 2-category **Cat** (Example 2.2.0.4). The construction $A \mapsto \mathrm{Mod}_A(\mathrm{Ab})$ *can* be promoted to a functor of 2-categories

$$\mathrm{Mod}_\bullet : \{\text{Commutative rings}\} \to \mathbf{Cat},$$

whose composition and identity constraints are given by the natural isomorphisms $\mu_{v,u} : T_v \circ T_u \simeq T_{vu}$ and $\epsilon_A : \mathrm{id}_{\mathrm{Mod}_A(\mathrm{Ab})} \simeq T_{\mathrm{id}_A}$ described in (1) and (2) (see Definition 2.2.4.5).

It is often more convenient to encode the functoriality of the construction $A \mapsto \mathrm{Mod}_A(\mathrm{Ab})$ in a different way. Let $\mathcal{C}$ be an ordinary category. To every functor of 2-categories $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$, one can associate a new category $\int_{\mathcal{C}} \mathscr{F}$, called the *category of elements of* $\mathscr{F}$ (Definition 5.6.1.1). By definition, objects of the category $\int_{\mathcal{C}} \mathscr{F}$ are given by pairs (C, X), where C is an object of the category $\mathcal{C}$ and X is an object of the category $\mathscr{F}(C)$. The construction $(C, X) \mapsto C$ determines a forgetful functor $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$, whose fiber over an object $C \in \mathcal{C}$ can be identified with the category $\mathscr{F}(C)$. Moreover, the functor $\mathscr{F}$ can be recovered (up to isomorphism) from the category $\int_{\mathcal{C}} \mathscr{F}$ together with the functor U.

Passage from the data of the functor $\mathscr{F}$ to its category of elements $\int_{\mathcal{C}} \mathscr{F}$ has several advantages. It can be somewhat cumbersome to specify a functor of 2-categories $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ *explicitly*: one must give not only the values of $\mathscr{F}$ on objects and morphisms of $\mathcal{C}$, but also the composition and identity constraints of the functor $\mathscr{F}$ (see Definition 2.2.4.5). The same information is encoded *implicitly* in the composition law for morphisms in the category of elements $\int_{\mathcal{C}} \mathscr{F}$, in a way that is often easier to access in practice. For example, suppose that $\mathcal{C}$ is the category of commutative rings and that $\mathscr{F}$ is the functor $A \mapsto \mathrm{Mod}_A(\mathrm{Ab})$ described above. By definition, the functor $\mathscr{F}$ carries each ring homomorphism $u : A \to B$ to the extension of scalars functor

$$T_u : \mathrm{Mod}_A(\mathrm{Ab}) \to \mathrm{Mod}_B(\mathrm{Ab}) \qquad T_u(M) = B \otimes_A M.$$

Note that the construction of this functor requires certain choices, since the tensor product $B \otimes_A M$ is well-defined only up to (canonical) isomorphism. However, the category $\mathrm{Mod}(\mathrm{Ab}) = \int_{\mathcal{C}} \mathscr{F}$ has a more direct description which does not depend on these choices:

- The objects of Mod(Ab) are pairs (A, M), where A is a commutative ring and M is an A-module.
- A morphism from (A, M) to (B, N) in the category Mod(Ab) is a pair (u, f), where $u : A \to B$ is a homomorphism of commutative rings and $f : M \to N$ is a homomorphism of A-modules.

To characterize those categories which can be obtained as a category of elements $\int_{\mathcal{C}} \mathscr{F}$, it will be convenient to introduce some terminology.

`01RF` **Definition 5.0.0.1.** Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories and let $f : X \to Y$ be a morphism in the category $\mathcal{E}$.

- We say that f is *U-cartesian* if, for every object $W \in \mathcal{E}$, the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(W, X) \arrow[r, "f\circ"] \arrow[d, "U"] & \mathrm{Hom}_{\mathcal{E}}(W, Y) \arrow[d, "U"] \\
\mathrm{Hom}_{\mathcal{C}}(U(W), U(X)) \arrow[r, "U(f)\circ"] & \mathrm{Hom}_{\mathcal{C}}(U(W), U(Y))
\end{tikzcd}$$

  is a pullback square.

- We say that f is *U-cocartesian* if, for every object $Z \in \mathcal{E}$, the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(Y, Z) \arrow[r, "\circ f"] \arrow[d, "U"] & \mathrm{Hom}_{\mathcal{E}}(X, Z) \arrow[d, "U"] \\
\mathrm{Hom}_{\mathcal{C}}(U(Y), U(Z)) \arrow[r, "\circ U(f)"] & \mathrm{Hom}_{\mathcal{C}}(U(X), U(Z))
\end{tikzcd}$$

  is a pullback square.

`01RK` **Example 5.0.0.2.** Let Mod(Ab) be the category defined above and let CAlg(Ab) denote the category of commutative rings, so that the construction $(A, M) \mapsto A$ determines a forgetful functor $U : \mathrm{Mod}(\mathrm{Ab}) \to \mathrm{CAlg}(\mathrm{Ab})$. Then:

- A morphism $(u, f) : (A, M) \to (B, N)$ in the category Mod(Ab) is U-cartesian if and only if the underlying A-module homomorphism $f : M \to N$ is an isomorphism (so that the A-module M is obtained from the B-module N by *restriction of scalars* along the ring homomorphism u).

- A morphism $(u, f) : (A, M) \to (B, N)$ in the category Mod(Ab) is U-cocartesian if and only if the underlying A-module homomorphism $f : M \to N$ induces a B-module isomorphism $B \otimes_A M \simeq N$ (so that the B-module N is obtained from the A-module M by *extension of scalars* along the ring homomorphism u).

Definition 5.0.0.3. Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. We say that U is a *cartesian fibration* if it satisfies the following condition: 01RN

- For every object Y of the category $\mathcal{E}$ and every morphism $\overline{f} : \overline{X} \to U(Y)$ in the category $\mathcal{C}$, there exists a pair (X, f) where X is an object of $\mathcal{E}$ satisfying $U(X) = \overline{X}$ and $f : X \to Y$ is a U-cartesian morphism of $\mathcal{E}$ satisfying $U(f) = \overline{f}$.

We say that U is a *cocartesian fibration* if it satisfies the following dual condition:

- For every object X of the category $\mathcal{E}$ and every morphism $\overline{f} : U(X) \to \overline{Y}$ in the category $\mathcal{C}$, there exists a pair (Y, f) where Y is an object of $\mathcal{E}$ satisfying $U(Y) = \overline{Y}$ and $f : X \to Y$ is a U-cocartesian morphism of $\mathcal{E}$ satisfying $U(f) = \overline{f}$.

Warning 5.0.0.4. The terminology of Definition 5.0.0.3 is not standard. Many authors use the term *fibration* or *Grothendieck fibration* for what we refer to as a *cartesian fibration* of categories, and use the term *opfibration* or *Grothendieck opfibration* for what we refer to as a *cocartesian fibration* of categories. Our motivation is to be consistent with the terminology we will use for the analogous definitions in the ∞-categorical setting (see §5.1), where it is important to distinguish between several different notions of fibration. 01Q6

Example 5.0.0.5. Let Mod(Ab) be the category described in Example 5.0.0.2. Then the forgetful functor U : Mod(Ab) $\to$ CAlg(Ab) is both a cartesian fibration and a cocartesian fibration. 01RQ

Exercise 5.0.0.6. Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. Show that the following conditions are equivalent: 023F

- The functor U is a fibration in groupoids (Definition 4.2.2.1).
- The functor U is a cartesian fibration and every morphism of $\mathcal{E}$ is U-cartesian.
- The functor U is a cartesian fibration and, for every object $C \in \mathcal{C}$, the fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is a groupoid.

For a more general statement, see Proposition 5.1.4.14.

Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. A classical theorem of Grothendieck ([27]) asserts that U is a cocartesian fibration if $\mathcal{E}$ can be realized as the category of elements of **Cat**-valued functor on $\mathcal{C}$: that is, if and only if there exists a functor of 2-categories

$\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ and an isomorphism of categories $\mathcal{E} \simeq \int_{\mathcal{C}} \mathscr{F}$ which carries U to the forgetful functor $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ (Corollary 5.6.5.19). Moreover, the functor $\mathscr{F}$ is uniquely determined up to isomorphism. Fixing the category $\mathcal{C}$, the category of elements construction supplies a dictionary

023G
$$\{\text{Functors } \mathscr{F} : \mathcal{C} \to \mathbf{Cat}\} \simeq \{\text{Cocartesian fibrations } U : \mathcal{E} \to \mathcal{C}\}, \tag{5.1}$$

which is the starting point for the theory of *fibered categories.*

The goal of chapter is to introduce an ∞-categorical generalization of the correspondence (5.1). We begin in §5.1 by developing an ∞-categorical counterpart of the theory of (co)cartesian fibrations. Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets. We say that an edge e of $\mathcal{E}$ is *U-cocartesian* if every lifting problem

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C}
\end{tikzcd}$$

admits a solution, provided that $n \geq 2$ and the restriction $\sigma_0|_{\Delta^1}$ is equal to e (Definition 5.1.1.1). We will be primarily interested in the situation where U is an inner fibration of ∞-categories; in this case, we show that an edge $e \in \mathcal{E}$ is U-cocartesian if and only if it satisfies a homotopy-theoretic counterpart of Definition 5.0.0.1 (Proposition 5.1.2.1). We say that a morphism of simplicial sets $U : \mathcal{E} \to \mathcal{C}$ is a *cocartesian fibration* if it is an inner fibration having the property that, for every vertex $X \in \mathcal{E}$ and every edge $\overline{e} : U(X) \to \overline{Y}$, there exists a U-cocartesian edge $e : X \to Y$ satisfying $U(e) = \overline{e}$ (Definition 5.1.4.1). This can be regarded as a generalization of Definition 5.0.0.3: a functor of ordinary categories $U : \mathcal{E} \to \mathcal{C}$ is a cocartesian fibration if and only if the induced map $\mathrm{N}_\bullet(U) : \mathrm{N}_\bullet(\mathcal{E}) \to \mathrm{N}_\bullet(\mathcal{C})$ is a cocartesian fibration of simplicial sets (Example 5.1.4.2). It also generalizes the notion of left fibration introduced in §4.2: a morphism of simplicial sets $U : \mathcal{E} \to \mathcal{C}$ is a left fibration if and only if it is a cocartesian fibration and *every* edge of $\mathcal{E}$ is U-cocartesian (Proposition 5.1.4.14).

The remainder of this section is devoted to the problem of *classifying* cocartesian fibrations $U : \mathcal{E} \to \mathcal{C}$, where $\mathcal{C}$ is a fixed ∞-category. For each object $C \in \mathcal{C}$, let $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ denote the corresponding fiber of U. We can then ask the following:

0350 **Question 5.0.0.7.** What additional data is needed to reconstruct the ∞-category $\mathcal{E}$ from the collection of ∞-categories $\{\mathcal{E}_C\}_{C \in \mathcal{C}}$?

In §5.2, we give a partial answer to Question 5.0.0.7. Let $f : C \to D$ be a morphism in the ∞-category $\mathcal{C}$. For each object $X \in \mathcal{E}_C$, our assumption that U is a cocartesian fibration guarantees that we can lift f to a U-cocartesian morphism $\widetilde{f} : X \to Y$ of $\mathcal{E}$. We will see that the construction $X \mapsto Y$ can be upgraded to a functor of ∞-categories $f_! : \mathcal{E}_C \to \mathcal{E}_D$, which we will refer to as the *functor of covariant transport along* f (Definition 5.2.2.4). The construction of the functor $f_!$ requires some auxiliary choices, but its isomorphism class $[f_!]$ is uniquely determined (Proposition 5.2.2.8). Moreover, the construction $f \mapsto f_!$ is compatible with composition (Proposition 5.2.5.1), and therefore determines a functor of ordinary categories

$$\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat} \qquad C \mapsto \mathcal{E}_C;$$

here hQCat denotes the homotopy category of ∞-categories (Construction 4.5.1.1). We will refer to $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ as the *homotopy transport representation* of the cocartesian fibration U (Construction 5.2.5.2).

In some cases, the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ provides an answer to Question 5.0.0.7:

- If $U : \mathcal{E} \to \mathcal{C}$ is a left covering map of simplicial sets, then we can regard $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ as a functor from the homotopy category $\mathrm{h}\mathcal{C}$ to the category of sets. In this case, we can reconstruct $\mathcal{E}$ (up to isomorphism) as the fiber product

  $$\mathcal{C} \times_{\mathrm{N}_\bullet(\mathrm{Set})} \mathrm{N}_\bullet(\mathrm{Set}_*),$$

  where Set_* denotes the category of pointed sets (Proposition 5.2.7.2). It follows that the the construction $\mathcal{E} \mapsto \mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ defines an equivalence of categories

  $$\{\text{Left covering maps } U : \mathcal{E} \to \mathcal{C}\} \simeq \mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{Set}),$$

  which we regard as a generalization of the classical theory of covering spaces (Corollary 5.2.7.3).

- Suppose that $\mathcal{C} = \Delta^1$ is the standard 1-simplex. In this case, the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ records the data of the ∞-categories $\mathcal{E}_0$ and $\mathcal{E}_1$, together with (the isomorphism class of) the covariant transport functor $F : \mathcal{E}_0 \to \mathcal{E}_1$ associated to the nondegenerate edge of $\mathcal{C}$. From this data, one can reconstruct the ∞-category $\mathcal{E}$ up to equivalence. More precisely, we show that $\mathcal{E}$ is categorically equivalent to the mapping cylinder $(\Delta^1 \times \mathcal{E}_0) \coprod_{(\{1\} \times \mathcal{E}_0)} \mathcal{E}_1$; see Corollary 5.2.4.2.

In general, the homotopy transport representation of a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ does not contain enough information to reconstruct the ∞-category $\mathcal{E}$, even up to equivalence. The essence of the problem is that the functor $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ encodes only the *isomorphism classes* of the covariant transport functors associated to the morphisms of $\mathcal{C}$. To address Question 5.0.0.7, it is necessary to consider a refinement of $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ which witnesses the functoriality of the construction $C \mapsto \mathcal{E}_C$ *before* passing to the homotopy category hQCat. In §5.3, we specialize to the situation where $\mathcal{C} = \mathrm{N}_{\bullet}(\mathcal{C}_0)$ is (the nerve of) an ordinary category $\mathcal{C}_0$. In this case, we associate to each cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ a functor of ordinary categories $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}_0} : \mathcal{C}_0 \to \mathrm{QCat}$, which we refer to as the *strict transport representation* of $\mathcal{C}$ (Construction 5.3.1.5). The strict transport representation is a refinement of the homotopy transport representation: more precisely, there is a canonical isomorphism of $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ with the composite functor

$$\mathrm{h}\mathcal{C} \simeq \mathcal{C}_0 \xrightarrow{\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}_0}} \mathrm{QCat} \twoheadrightarrow \mathrm{hQCat}$$

(Corollary 5.3.1.8). Moreover, we show that this refinement provides an answer to Question 5.0.0.7: according to Theorem 5.3.5.6, the construction $\mathcal{E} \mapsto \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}_0}$ induces a bijection

$$\begin{tikzcd}
\{\text{Cocartesian Fibrations } \mathcal{E} \to \mathcal{C}\}/\text{Equivalence} \arrow[d] \\
\{\text{Functors } \mathcal{C}_0 \to \mathrm{QCat}\}/\text{Levelwise Equivalence}.
\end{tikzcd}$$

Moreover, the inverse bijection admits an explicit description: it carries (the equivalence class of) a functor $\mathscr{F} : \mathcal{C}_0 \to \mathrm{QCat}$ to (the equivalence class of) a cocartesian fibration $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}_0) \to \mathrm{N}_{\bullet}(\mathcal{C}_0)$. Here $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}_0)$ is an ∞-category which we refer to as the *$\mathscr{F}$-weighted nerve* of $\mathcal{C}_0$ (Definition 5.3.3.1).

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. In general, it is not reasonable to expect that the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ can be promoted to a *strictly commutative* diagram in the category of simplicial sets. In other words, $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ generally cannot be lifted to a morphism from $\mathcal{C}$ to the nerve $\mathrm{N}_{\bullet}(\mathrm{QCat})$. To address Question 5.0.0.7 in complete generality, we will instead contemplate *homotopy coherent* refinements of $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$, given by morphisms from $\mathcal{C}$ to the homotopy coherent nerve $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{QCat})$. Here we regard QCat as a locally Kan simplicial category, with morphism spaces given by $\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{D}, \mathcal{D}') = \mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq}$. The homotopy coherent nerve $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{QCat})$ is then an ∞-category which we will denote by $\mathcal{QC}$ and refer to as the *∞-category of small ∞-categories* (Construction 5.5.4.1). In §5.5, we study several variants of this construction. In particular, we introduce an ∞-category $\mathcal{QC}_{\mathrm{Obj}}$ whose objects are pairs $(\mathcal{D}, X)$, where $\mathcal{D}$ is a small ∞-category and X is an object of $\mathcal{D}$, and whose morphisms are pairs $(F, u) : (\mathcal{D}, X) \to (\mathcal{D}', X')$ where $F : \mathcal{D} \to \mathcal{D}'$ is a functor of ∞-categories and $u : F(X) \to X'$ is a morphism in $\mathcal{D}'$ (Definition 5.5.6.10).

The construction $(\mathcal{D}, X) \mapsto \mathcal{D}$ determines a forgetful functor $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$, which is a cocartesian fibration of ∞-categories (Proposition 5.5.6.11). In §5.0.0.7, we address Question 5.0.0.7 in general by showing that V is a *universal* cocartesian fibration. For any functor of ∞-categories $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$, we let $\int_{\mathcal{C}} \mathscr{F}$ denote the fiber product $\mathcal{C} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}$. We will refer to $\int_{\mathcal{C}} \mathscr{F}$ as the ∞-*category of elements of* $\mathscr{F}$ (Definition 5.6.2.4); by construction, its objects are pairs (C, X) where C is an object of $\mathcal{C}$ and X is an object of the ∞-category $\mathscr{F}(C)$. Note that projection onto the first factor determines a forgetful functor $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$, which is a cocartesian fibration of ∞-categories (since it is a pullback of the cocartesian fibration V). Our main result is that the construction $\mathscr{F} \mapsto \int_{\mathcal{C}} \mathscr{F}$ induces a bijection from the set of isomorphism classes in $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$ to the set of equivalence classes of ∞-categories equipped with a cocartesian fibration to $\mathcal{C}$ (Theorem 5.6.0.2). In particular, every cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ fits into a categorical pullback square

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}"] & \mathcal{QC},
\end{tikzcd}$$

where the functor $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathcal{QC}$ is uniquely determined up to isomorphism. The functor $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ is an ∞-categorical refinement of the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ (Remark 5.6.5.15), which we will refer to as the *covariant transport representation of* U (Definition 5.6.5.1).

Remark 5.0.0.8. The classical theory of fibered categories was introduced by Grothendieck in [27] (Exposé 6). 0158

5.1 Cartesian Fibrations

The goal in this section is to extend the theory of (co)cartesian fibrations to the setting of ∞-categories. The first step is to introduce an ∞-categorical analogue of Definition 5.0.0.1. Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories, and let $f : X \to Y$ be a morphism in $\mathcal{E}$. By definition, f is U-cartesian if and only if, for every morphism $h : W \to Y$ in $\mathcal{E}$, every commutative diagram 01T1

$$\begin{tikzcd}
& U(X) \arrow[dr, "U(f)"] & \\
U(W) \arrow[ur, "\overline{g}"] \arrow[rr, "U(h)"] & & U(Y)
\end{tikzcd}$$

in the category $\mathcal{C}$ can be lifted *uniquely* to a commutative diagram

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
W \arrow[ur, dashed, "g"] \arrow[rr, "h"] & & Y
\end{tikzcd}$$

in the category $\mathcal{E}$. Equivalently, the morphism f is U-cartesian if and only if every lifting problem

01T2
$$\begin{tikzcd}
\Lambda^2_2 \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{N}_\bullet(\mathcal{E}) \arrow[d, "\mathrm{N}_\bullet(U)"] \\
\Delta^2 \arrow[r] \arrow[ur, dashed, "\sigma"] & \mathrm{N}_\bullet(\mathcal{C})
\end{tikzcd} \tag{5.2}$$

has a unique solution, assuming that σ_0 carries the "final edge" $\mathrm{N}_\bullet(\{1 < 2\}) \subseteq \Lambda^2_2$ to the morphism f.

In the ∞-categorical setting, it is unreasonable to ask for the lifting problem (5.2) to admit a *unique* solution. Instead, we should require that the collection of possible choices for σ are, in some sense, parametrized by a contractible space. In §5.1.1, we formalize this idea by considering analogues of (5.2) for higher-dimensional simplices. If $U : \mathcal{E} \to \mathcal{C}$ is an arbitrary morphism of simplicial sets, we will say that an edge f of $\mathcal{E}$ is *U-cartesian* if every lifting problem

$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\sigma"] & \mathcal{C}
\end{tikzcd}$$

admits a solution, provided that $n \geq 2$ and σ_0 carries the "final edge" $\mathrm{N}_\bullet(\{n-1 < n\}) \subseteq \Lambda^n_n$ to f (Definition 5.1.1.1). In the special case where $\mathcal{E}$ and $\mathcal{C}$ are the nerves of ordinary categories, this reduces to the classical definition of cartesian morphism (Corollary 5.1.2.2).

The definition of U-cartesian edge makes sense for any morphism of simplicial sets $U : \mathcal{E} \to \mathcal{C}$. However, it has poor formal properties in general. We will be primarily interested in the case where $\mathcal{E}$ and $\mathcal{C}$ are ∞-categories and U is an inner fibration. Assume that these conditions are satisfied and let $f : X \to Y$ be a morphism of $\mathcal{E}$, having image $\overline{f} : \overline{X} \to \overline{Y}$ in $\mathcal{D}$. For every object $W \in \mathcal{E}$ having image $\overline{W} = U(W) \in \mathcal{C}$, composition with the homotopy class $[f]$ determines a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(W, X) \arrow[r, "{[f]\circ}"] \arrow[d] & \mathrm{Hom}_{\mathcal{E}}(W, Y) \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}}(\overline{W}, \overline{X}) \arrow[r, "{[\overline{f}]\circ}"] & \mathrm{Hom}_{\mathcal{C}}(\overline{W}, \overline{Y})
\end{tikzcd}$$

in the homotopy category hKan, which (after suitable modifications on the left hand side) can be lifted to a commutative diagram in the category of simplicial sets. In §5.1.2, we show that f is U-cartesian if and only if, for every object $W \in \mathcal{E}$, the resulting lift is a homotopy pullback diagram of Kan complexes (Proposition 5.1.2.1). This has a number of pleasant consequences: for example, it implies that the collection of U-cartesian morphisms is closed under composition (for a stronger statement, see Corollary 5.1.2.4).

Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E}' \arrow[r, "F"] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r] & \mathcal{C},
\end{tikzcd}$$

and let f be an edge of $\mathcal{E}'$. It follows immediately from the definitions that if $F(f)$ is U-cartesian, then f is U'-cartesian (Remark 5.1.1.11). The converse holds when U is a cartesian fibration (Remark 5.1.4.6), but is false in general. In §5.1.3, we address this point by introducing the more general notion of a *locally U-cartesian edge* of a simplicial set $\mathcal{E}$ equipped with a map $U : \mathcal{E} \to \mathcal{C}$ (Definition 5.1.3.1).

Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. In §5.1.4 we study the situation where $\mathcal{E}$ has "sufficiently many" U-cartesian edges in the following sense: for every vertex $Y \in \mathcal{E}$, every edge $\overline{f} : \overline{X} \to U(Y)$ of $\mathcal{C}$ can be lifted to a U-cartesian edge $f : X \to Y$ of $\mathcal{C}$. If this condition is satisfied, we say that U is a *cartesian fibration* of simplicial sets. This definition has the following features:

- A functor of ordinary categories $U : \mathcal{E} \to \mathcal{C}$ is a cartesian fibration (in the sense of Definition 5.0.0.3) if and only if the induced functor of ∞-categories $\mathrm{N}_{\bullet}(U) : \mathrm{N}_{\bullet}(\mathcal{E}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a cartesian fibration (Example 5.1.4.2).

- Every right fibration of simplicial sets $U : \mathcal{E} \to \mathcal{C}$ is a cartesian fibration. Conversely, a cartesian fibration $U : \mathcal{E} \to \mathcal{C}$ is a right fibration if and only if every fiber of U is a Kan complex (Proposition 5.1.4.14).

- The collection of cartesian fibrations is closed under the formation of pullbacks (Remark 5.1.4.6) and composition (Proposition 5.1.4.13).

- Let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of simplicial sets and let $f : K \to \mathcal{E}$ be any morphism of simplicial sets. Then the induced maps $\mathcal{E}_{/f} \to \mathcal{C}_{/(U \circ f)}$ and $\mathcal{E}_{f/} \to \mathcal{C}_{(U \circ f)/}$ are cartesian fibrations (Propositions 5.1.4.17 and 5.1.4.19).

Suppose we are given a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\
& \mathcal{E}, &
\end{tikzcd}$$

where U and V are isofibrations. Recall that, if F is an equivalence of ∞-categories, then the induced map of fibers $F_E : \mathcal{C}_E \to \mathcal{D}_E$ is also an equivalence of ∞-categories for every object $E \in \mathcal{E}$ (Corollary 4.5.2.32). The converse is false in general (Warning 4.5.2.33). Nevertheless, in §5.1.6 we show that the converse is true if we assume that U is a cartesian fibration and that F carries U-cartesian morphisms of $\mathcal{C}$ to V-cartesian morphisms of $\mathcal{D}$ (Theorem 5.1.6.1). In §5.1.7, we prove a counterpart of this result in the case where $\mathcal{E}$ is not assumed to be an ∞-category (Proposition 5.1.7.15): in this case, $\mathcal{C}$ and $\mathcal{D}$ need not be ∞-categories, but it is still possible to show that F is an equivalence of inner fibrations over $\mathcal{E}$ (see Definition 5.1.7.1).

01T3 **Remark 5.1.0.1.** The entirety of the preceding discussion can be dualized. If $U : \mathcal{E} \to \mathcal{C}$ is a morphism of simplicial sets, we will say that an edge f of $\mathcal{E}$ is *U-cocartesian* if it is U^{op}-cartesian when viewed as an edge of the opposite simplicial set $\mathcal{E}^{\mathrm{op}}$. We say that U is a *cocartesian fibration* if the opposite functor $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a cartesian fibration. For the sake of brevity, we will sometimes state our results only for cartesian fibrations (in which case there is always a counterpart for cocartesian fibrations, which can be obtained by passing to opposite simplicial sets).

5.1.1 Cartesian Edges of Simplicial Sets

01T4 Our first goal is to adapt Definition 5.0.0.1 to the setting of ∞-categories.

Definition 5.1.1.1. Let $q : X \to S$ be a morphism of simplicial sets, and let e be an edge of X. We say that e is *q-cartesian* if every lifting problem 01T5

$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution, provided that $n \geq 2$ and the composite map

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{n-1 < n\}) \hookrightarrow \Lambda^n_n \xrightarrow{\sigma_0} X$$

corresponds to the edge e.

We say that e is *q-cocartesian* if every lifting problem

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution, provided that $n \geq 2$ and the composite map

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{0 < 1\}) \hookrightarrow \Lambda^n_0 \xrightarrow{\sigma_0} X$$

corresponds to the edge e.

Remark 5.1.1.2. Let $q : X \to S$ be a morphism of simplicial sets and let $q^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ be the opposite morphism. Then an edge e of X is q-cartesian if and only if it is q^{op}-cocartesian (where we identify e with an edge of the opposite simplicial set X^{op}). 01T6

Example 5.1.1.3. Let $q : X \to S$ be a right fibration of simplicial sets. Then every edge of X is q-cartesian. Similarly, if $q : X \to S$ is a left fibration of simplicial sets, then every edge of X is q-cocartesian. 01T7

Example 5.1.1.4. Let $\mathcal{C}$ be an ∞-category, let $q : \mathcal{C} \to \Delta^0$ be the projection map, and let $e : X \to Y$ be a morphism in $\mathcal{C}$. The following conditions are equivalent: 01T8

- The morphism e is an isomorphism.
- The morphism e is q-cartesian.
- The morphism e is q-cocartesian.

This is a restatement of Theorem 4.4.2.6.

01T9 **Example 5.1.1.5.** Let $q : X \to S$ be a morphism of simplicial sets which restricts to an isomorphism from X to a full simplicial subset of S (see Definition 4.1.2.15). Then every edge of X is both q-cartesian and q-cocartesian.

01TA **Remark 5.1.1.6.** Let $p : X \to Y$ and $q : Y \to Z$ be morphisms of simplicial sets, and let e be an edge of the simplicial set X. If e is p-cartesian and $p(e)$ is a q-cartesian edge of Y, then e is $(q \circ p)$-cartesian. For a partial converse, see Corollary 5.1.2.6.

02R4 **Remark 5.1.1.7.** Let $q : X \to S$ be a morphism of simplicial sets, let $X' \subseteq X$ be a full simplicial subset, and let $q' = q|_{X'}$. If e is an edge of X' which is q-cartesian when viewed as an edge of X, then it it is q'-cartesian. This follows by combining Remark 5.1.1.6 with Example 5.1.1.5.

01TB **Proposition 5.1.1.8.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $e : X \to Y$ be a morphism in $\mathcal{C}$. The following conditions are equivalent:*

(1) *The morphism e is an isomorphism in $\mathcal{C}$.*

(2) *The morphism e is q-cartesian and $q(e)$ is an isomorphism in $\mathcal{D}$.*

(3) *The morphism e is q-cocartesian and $q(e)$ is an isomorphism in $\mathcal{D}$.*

Proof. We will prove the equivalence (1) ⇔ (2); the proof of the equivalence (1) ⇔ (3) is similar. The implication (1) ⇒ (2) follows from Proposition 4.4.2.13 and Remark 1.5.1.6. To prove the converse, let $p : \mathcal{D} \to \Delta^0$ denote the projection map. If $q(e)$ is an isomorphism in $\mathcal{C}$, then it is p-cartesian (Example 5.1.1.4). If, in addition, the morphism e is q-cartesian, then it is also $(p \circ q)$-cartesian (Remark 5.1.1.6) and is therefore an isomorphism in the ∞-category $\mathcal{C}$ (Example 5.1.1.4). □

01TC **Corollary 5.1.1.9.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. For every object $X \in \mathcal{C}$, the identity morphism $\mathrm{id}_X : X \to X$ is q-cartesian and q-cocartesian.*

023H **Corollary 5.1.1.10.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, where $\mathcal{D}$ is a Kan complex, and let $e : X \to Y$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:*

(1) *The morphism e is an isomorphism in $\mathcal{C}$.*

(2) *The morphism e is q-cartesian.*

(3) *The morphism e is q-cocartesian.*

Proof. Combine Propositions 5.1.1.8 and 1.4.6.10. □

Remark 5.1.1.11. Suppose we are given a pullback diagram of simplicial sets 01TD

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "q'"] & X \arrow[d, "q"] \\ S' \arrow[r] & S. \end{tikzcd}$$

Let e' be an edge of the simplicial set X', having image $e = f(e')$ in X. If e is q-cartesian, then e' is q'-cartesian. Similarly, if e is q-cocartesian, then e' is q'-cocartesian.

Remark 5.1.1.12. Let $q : X \to S$ be a morphism of simplicial sets and let e be an edge of the simplicial set X. The following conditions are equivalent: 01TE

- The edge e is q-cartesian.
- For every pullback diagram of simplicial sets

  $$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "q'"] & X \arrow[d, "q"] \\ S' \arrow[r] & S, \end{tikzcd}$$

  and every edge e' of X' satisfying $f(e') = e$, the edge e' is q'-cartesian.
- For every pullback diagram of simplicial sets

  $$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "q'"] & X \arrow[d, "q"] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

  and every edge e' of X' satisfying $f(e') = e$, the edge e' is q'-cartesian.

Proposition 5.1.1.13. *Let $q : X \to S$ be a morphism simplicial sets and let $e : x \to y$ be an edge of X. Then:* 01TF

- *The edge e is q-cartesian if and only if the natural map*

  $$X_{/e} \to X_{/y} \times_{S_{/q(y)}} S_{/q(e)}$$

  is a trivial Kan fibration of simplicial sets.

- *The edge e is q-cocartesian if and only if the natural map*

$$X_{e/} \to X_{x/} \times_{S_{q(x)/}} S_{q(e)/}$$

is a trivial Kan fibration of simplicial sets.

Proof. We will prove the first assertion; the proof of the second is similar. By definition, the natural map $X_{/e} \to X_{/y} \times_{S_{/q(y)}} S_{/q(e)}$ is a trivial Kan fibration if and only if, for every integer $n \geq 0$, every lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & X_{/e} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & X_{/y} \times_{S_{/q(y)}} S_{/q(e)}
\end{tikzcd}$$

admits a solution. By virtue of Lemma 4.3.6.15, this is equivalent to the datum of a lifting problem

$$\begin{tikzcd}
\Lambda^{n+2}_{n+2} \arrow[r, "\sigma_0"] \arrow[d] & X \arrow[d] \\
\Delta^{n+2} \arrow[r] \arrow[ur, dashed] & S,
\end{tikzcd}$$

where σ_0 carries the final edge $\mathrm{N}_\bullet(\{n+1 < n+2\}) \subseteq \Lambda^{n+2}_{n+2}$ to e. □

01TG **Corollary 5.1.1.14.** *Let $q : X \to S$ and $f : K \to X$ be morphisms of simplicial sets, and let $q' : X_{f/} \to S_{(q\circ f)/}$ be the morphism induced by q. Let $\overline{e} : \overline{x} \to \overline{y}$ be an edge of the simplicial set $X_{f/}$, and let $e : x \to y$ be its image in X. If e is q-cartesian, then $\overline{e}$ is q'-cartesian.*

Proof. Since e is q-cartesian, the restriction map

$$\theta : X_{/e} \to X_{/y} \times_{S_{/q(y)}} S_{/q(e)}$$

is a trivial Kan fibration (Proposition 5.1.1.13). We wish to show that the restriction map

$$\overline{\theta} : (X_{f/})_{/\overline{e}} \to (X_{f/})_{/\overline{y}} \times_{(S_{(q\circ f)/})_{/q'(\overline{y})}} (S_{(q\circ f)/})_{/q'(\overline{e})}.$$

is also a trivial Kan fibration. We can identify $\overline{e}$ with a morphism of simplicial sets $\overline{f} : K \to X_{/e}$, and $\overline{\theta}$ with the induced map

$$(X_{/e})_{\overline{f}/} \to (X_{/y} \times_{S_{/q(y)}} S_{/q(e)})_{(\theta\circ\overline{f})/}.$$

The desired result now follows from Corollary 4.3.7.17. □

Corollary 5.1.1.15. *Let $q : X \to S$ be a morphism of simplicial sets and let $e : x \to y$ be an edge of X. The following conditions are equivalent:* 01TH

(1) *The edge e is q-cartesian.*

(2) *Let $f : B \to X$ be a morphism of simplicial sets, let A be a simplicial subset of B, and let Y denote the fiber product $X_{f|_A/} \times_{S_{(q \circ f|_A)/}} S_{(q \circ f)/}$, so that the restriction map $X_{f/} \to X$ factors as a composition $X_{f/} \xrightarrow{\theta} Y \xrightarrow{\rho} X$. Then every lifting problem*

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d] & X_{f/} \arrow[d, "\theta"] \\
\Delta^1 \arrow[r, "e'"] \arrow[ur, dashed] & Y
\end{tikzcd}$$

admits a solution, provided that $\rho(e') = e$.

Proof. For a fixed simplicial set B with a simplicial subset $A \subseteq B$, condition (2) is equivalent to the requirement that every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X_{/e} \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & X_{/y} \times_{S_{/q(y)}} S_{/q(e)}
\end{tikzcd}$$

admits a solution. This condition is satisfied for *every* inclusion of simplicial sets $A \subseteq B$ if and only if the map $X_{/e} \to X_{/y} \times_{S_{/q(y)}} S_{/q(e)}$ is a trivial Kan fibration: that is, if and only if e is q-cartesian (Proposition 5.1.1.13). □

Remark 5.1.1.16. In the situation of Corollary 5.1.1.15, it is sufficient to verify condition (2) in the special case where $B = \Delta^n$ is a standard simplex and $A = \partial \Delta^n$ is its boundary. 01TJ

5.1.2 Cartesian Morphisms of ∞-Categories

Let $q : \mathcal{C} \to \mathcal{D}$ be a functor between ordinary categories and let $g : Y \to Z$ be a morphism in $\mathcal{C}$ having image $\overline{g} : \overline{Y} \to \overline{Z}$ in $\mathcal{D}$. Recall that g is *q-cartesian* if, for every object $X \in \mathcal{C}$ 01TK having image $\overline{X} = q(X)$ in $\mathcal{D}$, the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "g\circ"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y}) \arrow[r, "\overline{g}\circ"] & \mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Z})
\end{tikzcd}$$

is a pullback square (Definition 5.0.0.1). Our goal in this section is to give an analogous characterization of cartesian morphisms in the setting of ∞-categories.

We now encounter a slight complication: if X, Y, and Z are objects of an ∞-category $\mathcal{C}$ and $g : Y \to Z$ is a morphism, then the composition map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{g\circ} \mathrm{Hom}_{\mathcal{C}}(X,Z)$ is only well-defined up to homotopy. We can circumvent this difficulty using the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X,Y,Z)$ of Notation 4.6.9.1. By virtue of Corollary 4.6.9.5, the restriction map $\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ is a trivial Kan fibration of simplicial sets, and therefore induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Moreover, the "long edge" of Δ^2 determines a map of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z),$$

which we can regard as a surrogate for the composition map $\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{g\circ} \mathrm{Hom}_{\mathcal{C}}(X,Z)$. This construction depends functorially on $\mathcal{C}$ in the following sense: if $q : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories carrying X to $\overline{X} \in \mathcal{D}$ and g to $\overline{g} : \overline{Y} \to \overline{Z}$, then it induces a commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{Y},\overline{Z})} \{\overline{g}\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Z}),
\end{tikzcd}$$

where the vertical maps are determined by q and the horizontal maps are given by restriction. We can now state our main result, which we will prove at the end of this section:

01TL **Proposition 5.1.2.1.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $g : Y \to Z$ be a morphism in $\mathcal{C}$ having image $\overline{g} : \overline{Y} \to \overline{Z}$ in $\mathcal{D}$. Then g is q-cartesian if and only if, for* *every object $X \in \mathcal{C}$ having image $\overline{X} = q(X)$ in $\mathcal{D}$, the diagram of Kan complexes*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{Y},\overline{Z})} \{\overline{g}\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Z})
\end{tikzcd} \tag{5.3}$$ 01TM

is a homotopy pullback square.

Corollary 5.1.2.2. *Let $q : \mathcal{C} \to \mathcal{D}$ be a functor between categories, and let $\mathrm{N}_\bullet(q) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})$ be the induced morphism of simplicial sets. Let $g : Y \to Z$ be a morphism in the category $\mathcal{C}$. Then g is q-cartesian (in the sense of Definition 5.0.0.1) if and only if it is $\mathrm{N}_\bullet(q)$-cartesian (when regarded as an edge of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$).* 01TN

Proof. Combine Proposition 5.1.2.1 with Example 4.6.9.6. □

Corollary 5.1.2.3. *Let Q be a partially ordered set, let $q : \mathcal{C} \to \mathrm{N}_\bullet(Q)$ be an inner fibration of ∞-categories, and let $g : Y \to Z$ be a morphism in $\mathcal{C}$. Then g is q-cartesian if and only if, for every object $X \in \mathcal{C}$ satisfying $q(X) \leq q(Y)$, the map* 01TQ

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{[g]\circ} \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

of Notation 4.6.9.15 is an isomorphism in the homotopy category hKan*.*

Proof. By virtue of Proposition 5.1.2.1, the morphism g is q-cartesian if and only if, for each object $X \in \mathcal{C}$, the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[r, "\theta_X"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d] \\
\mathrm{Hom}_{\mathrm{N}_\bullet(Q)}(q(X),q(Y),q(Z)) \times_{\mathrm{Hom}_{\mathrm{N}_\bullet(Q)}(q(Y),q(Z))} \{q(g)\} \arrow[r] & \mathrm{Hom}_{\mathrm{N}_\bullet(Q)}(q(X),q(Z))
\end{tikzcd} \tag{5.4}$$ 01TR

is a homotopy pullback square. If $q(X) \nleq q(Y)$, then the Kan complexes on the left side of the diagram (5.4) are empty, so this condition is vacuous. If $q(X) \leq q(Y)$, then the Kan complexes on the lower half of the diagram are isomorphic to Δ^0, so that (5.4) is a homotopy pullback square if and only if θ_X is a homotopy equivalence (Corollary 3.4.1.5). We conclude by observing that, in the homotopy category hKan, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[rr] \arrow[dr, "\theta_X"'] & & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[dl, "{[g]\circ}"] \\
& \mathrm{Hom}_{\mathcal{C}}(X,Z), &
\end{tikzcd}$$

where the horizontal map is an isomorphism (Corollary 4.6.9.5). □

01TS **Corollary 5.1.2.4.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $\sigma : \Delta^2 \to \mathcal{C}$ be a 2-simplex of $\mathcal{C}$, which we will depict as a diagram*

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z.
\end{tikzcd}$$

- *Suppose that g is q-cartesian. Then f is q-cartesian if and only if h is q-cartesian.*
- *Suppose that f is q-cocartesian. Then g is q-cocartesian if and only if h is q-cocartesian.*

Proof. We will prove the first assertion; the second follows by a similar argument. For every simplex τ of the ∞-category $\mathcal{C}$, let $\overline{\tau}$ denote its image $q(\tau)$ in the ∞-category $\mathcal{D}$. By virtue of Proposition 5.1.2.1, it will suffice to show that for every object $W \in \mathcal{C}$, the following conditions are equivalent:

(a) The commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(W,X,Y) \times_{\mathrm{Hom}_{\mathcal{C}}(X,Y)} \{f\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(W,Y) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{X},\overline{Y}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y})} \{\overline{f}\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{Y})
\end{tikzcd}$$

is a homotopy pullback square.

(b) The commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(W,X,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(X,Z)} \{h\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(W,Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{X},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Z})} \{\overline{h}\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{Z})
\end{tikzcd}$$

is a homotopy pullback square.

By virtue of Corollaries 4.6.9.5 and 3.4.1.12, these conditions can be reformulated as follows:

(a') The commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(W,X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(X,Y,Z)} \{\sigma\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(W,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{X},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y},\overline{Z})} \{\overline{\sigma}\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{Y},\overline{Z})} \{\overline{g}\}
\end{tikzcd}$$

is a homotopy pullback square.

(b') The commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(W,X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(X,Y,Z)} \{\sigma\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(W,Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{X},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y},\overline{Z})} \{\overline{\sigma}\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{Z})
\end{tikzcd}$$

is a homotopy pullback square.

The equivalence of (a') and (b') follows by applying Proposition 3.4.1.11 to the commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(W,X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(X,Y,Z)} \{\sigma\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{X},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y},\overline{Z}))} \{\overline{\sigma}\} \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}}(W,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{Y},\overline{Z})} \{\overline{g}\} \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}}(W,Z) \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(\overline{W},\overline{Z}),
\end{tikzcd}$$

noting that the lower half of the diagram is a homotopy pullback square by virtue of our assumption that g is q-cartesian (Proposition 5.1.2.1). □

01TT **Corollary 5.1.2.5.** *Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, and let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms of $\mathcal{C}$ which are isomorphic as objects of the ∞-category* $\mathrm{Fun}(\Delta^1, \mathcal{C})$*. Then f is q-cartesian if and only if f' is q-cartesian. Similarly, f is q-cocartesian if and only if f' is q-cocartesian.*

Proof. Our assumption that f is isomorphic to f' in $\mathrm{Fun}(\Delta^1, \mathcal{C})$ guarantees that there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e"] & Y \arrow[d, "e'"] \\
X' \arrow[r, "f'"] & Y',
\end{tikzcd}$$

where e and e' are isomorphisms (and therefore q-cartesian by virtue of Proposition 5.1.1.8). The desired result now follows from Corollary 5.1.2.4. □

Using Proposition 5.1.2.1, we deduce the following stronger version of Remark 5.1.1.6:

01TU **Corollary 5.1.2.6** (Transitivity)**.** *Let $p : \mathcal{C} \to \mathcal{D}$ and $q : \mathcal{D} \to \mathcal{E}$ be inner fibrations of simplicial sets, and let $e : Y \to Z$ be an edge of the simplicial set $\mathcal{C}$.*

- *Assume that $p(e)$ is a q-cartesian edge of $\mathcal{D}$. Then e is p-cartesian if and only if it is $(q \circ p)$-cartesian.*
- *Assume that $p(e)$ is a q-cocartesian edge of $\mathcal{D}$. Then e is p-cocartesian if and only if it is $(q \circ p)$-cocartesian.*

Proof. We will prove the first assertion; the second follows by a similar argument. Using Remark 5.1.1.12, we can reduce to the case where $\mathcal{E}$ is an ∞-category (or even a simplex), so that $\mathcal{C}$ and $\mathcal{D}$ are also ∞-categories (Remark 4.1.1.9). Fix an object $X \in \mathcal{C}$, and set $r = q \circ p$. We have a commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, Y, Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{e\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X, Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(p(X), p(Y), p(Z)) \times_{\mathrm{Hom}_{\mathcal{C}}(p(Y),p(Z))} \{q(e)\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(p(X), p(Z)) \arrow[d] \\
\mathrm{Hom}_{\mathcal{E}}(r(X), r(Y), r(Z)) \times_{\mathrm{Hom}_{\mathcal{E}}(r(Y),r(Z))} \{r(e)\} \arrow[r] & \mathrm{Hom}_{\mathcal{E}}(r(X), r(Z)).
\end{tikzcd}$$

If $p(e)$ is a q-cartesian morphism of $\mathcal{D}$, then the bottom square is a homotopy pullback (Proposition 5.1.2.1). Invoking Proposition 3.4.1.11, we deduce that the upper square is a homotopy pullback if and only if the outer rectangle is a homotopy pullback. Allowing X to vary and invoking Proposition 5.1.2.1, we conclude that e is p-cartesian if and only if is r-cartesian. □

Proof of Proposition 5.1.2.1. Let $q : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, and let $g : Y \to Z$ be a morphism in the ∞-category $\mathcal{C}$ having image $\overline{g} : \overline{Y} \to \overline{Z}$ in the ∞-category $\mathcal{D}$. By virtue of Proposition 5.1.1.13, the morphism g is q-cartesian if and only if the restriction map

$$\theta : \mathcal{C}_{/g} \to \mathcal{C}_{/Z} \times_{\mathcal{D}_{/\overline{Z}}} \mathcal{D}_{/\overline{g}}$$

is a trivial Kan fibration of simplicial sets. Since q is an inner fibration, the morphism θ is a right fibration (Proposition 4.3.6.8). For each object $X \in \mathcal{C}$, θ restricts to a right fibration of simplicial sets

$$\theta_X : \{X\} \times_{\mathcal{C}} \mathcal{C}_{/g} \to \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Z} \times_{\mathcal{D}_{/\overline{Z}}} \mathcal{D}_{/\overline{g}} .$$

Note that if θ is a trivial Kan fibration, then so is θ_X. Conversely, if each θ_X is a trivial Kan fibration, then every fiber of θ is a contractible Kan complex, so that θ is a trivial Kan fibration by virtue of Proposition 4.4.2.14. To complete the proof, it will suffice to show that θ_X is a trivial Kan fibration if and only if the diagram (5.3) appearing in the statement of Proposition 5.1.2.1 is a homotopy pullback square.

For the remainder of the proof, let us regard the object $X \in \mathcal{C}$ as fixed, and set $\overline{X} = q(X)$.

We then have a commutative diagram of simplicial sets

01TV
$$\begin{tikzcd}
\{X\} \times_{\mathcal{C}} \mathcal{C}_{/g} \arrow[r] \arrow[d] & \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Z} \arrow[d, "\rho"] \\
\{\overline{X}\} \times_{\mathcal{D}} \mathcal{D}_{/\overline{g}} \arrow[r] & \{\overline{X}\} \times_{\mathcal{D}} \mathcal{D}_{/\overline{Z}} .
\end{tikzcd} \tag{5.5}$$

Corollary 4.3.6.11 guarantees that the restriction maps

$$\mathcal{C}_{/g} \to \mathcal{C}_{/Z} \to \mathcal{C} \qquad \mathcal{D}_{/\overline{g}} \to \mathcal{D}_{/\overline{Z}} \to \mathcal{D}$$

are right fibrations, so that each of the simplicial sets appearing in the diagram (5.5) is a Kan complex. The morphism ρ is a pullback of the restriction map $\mathcal{C}_{/Z} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/\overline{Z}}$, and is therefore a right fibration by virtue of Proposition 4.3.6.8. Applying Corollary 4.4.3.8, we deduce that ρ is a Kan fibration. The projection map

$$\{X\} \times_{\mathcal{C}} \mathcal{C}_{/Z} \times_{\mathcal{D}_{/\overline{Z}}} \mathcal{D}_{/\overline{g}} \to \{\overline{X}\} \times_{\mathcal{D}} \mathcal{D}_{/\overline{g}}$$

is a pullback of ρ, and therefore also a Kan fibration. In particular, the target of the right fibration θ_X is a Kan complex, so that θ_X is a Kan fibration (Corollary 4.4.3.8). It follows that θ_X is a trivial Kan fibration if and only if is a homotopy equivalence (Proposition 3.3.7.6): that is, if and only if the diagram (5.5) is a homotopy pullback square.

Let σ be an n-simplex of the simplicial set $\{X\} \times_{\mathcal{C}} \mathcal{C}_{/g}$. Then we can identify σ with a morphism of simplicial sets $u_\sigma : \Delta^n \star \Delta^1 \to \mathcal{C}$ such that $u_\sigma|_{\Delta^n}$ is the constant map taking the value X and $u_\sigma|_{\Delta^1} = g$. Let $\pi : \Delta^n \times \Delta^2 \to \Delta^n \star \Delta^1 \simeq \Delta^{n+2}$ be the map given on vertices by the formula

$$\pi(i,j) = \begin{cases} i & \text{if } j = 0 \\ n+1 & \text{if } j = 1 \\ n+2 & \text{if } j = 2. \end{cases}$$

The composition $u_\sigma \circ \pi : \Delta^n \times \Delta^2 \to \mathcal{C}$ can then be regarded as an n-simplex σ' of the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\operatorname{Hom}_{\mathcal{C}}(Y,Z)} \{g\}$. The construction $\sigma \mapsto \sigma'$ depends functorially on $[n] \in \mathbf{\Delta}$, and therefore determines a morphism of Kan complexes

$$\iota^{\mathrm{R}}_{X,g} : \{X\} \times_{\mathcal{C}} \mathcal{C}_{/g} \to \operatorname{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\operatorname{Hom}_{\mathcal{C}}(Y,Z)} \{g\}.$$

Note that the morphism $\iota_{X,g}$ fits into a commutative diagram

$$\begin{tikzcd}
\{X\} \times_{\mathcal{C}} \mathcal{C}_{/g} \arrow[r, "\iota^{\mathrm{R}}_{X,g}"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[d] \\
\{X\} \times_{\mathcal{C}} \mathcal{C}_{/Y} \arrow[r, "\iota^{\mathrm{R}}_{X,Y}"] & \mathrm{Hom}_{\mathcal{C}}(X,Y),
\end{tikzcd}$$

where the left vertical map is a pullback of the restriction morphism $\mathcal{C}_{/g} \to \mathcal{C}_{/Y}$ (and therefore a trivial Kan fibration by virtue of Corollary 4.3.6.13), the right vertical map is a pullback of the restriction morphism $\mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ (and therefore a trivial Kan fibration by virtue of Corollary 4.6.9.5), and $\iota^{\mathrm{R}}_{X,Y} : \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is the right-pinch inclusion map of Construction 4.6.5.7 (which is a homotopy equivalence of Kan complexes by virtue of Proposition 4.6.5.10). It follows that $\iota^{\mathrm{R}}_{X,g}$ is also a homotopy equivalence of Kan complexes. Applying the same construction to the ∞-category $\mathcal{D}$, we obtain a homotopy equivalence

$$\iota^{\mathrm{R}}_{\overline{X},\overline{g}} : \{\overline{X}\} \times_{\mathcal{D}} \mathcal{D}_{/\overline{g}} \to \mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{Y},\overline{Z})} \{\overline{g}\}.$$

We have a commutative diagram of Kan complexes

$$\begin{tikzcd}
\{X\} \times_{\mathcal{C}} \mathcal{C}_{/g} \arrow[rr] \arrow[dr, "\iota^{\mathrm{R}}_{X,g}"'] \arrow[dd] & & \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Z} \arrow[dr, "\iota^{\mathrm{R}}_{X,Z}"] \arrow[dd] & \\
& \mathrm{Hom}_{\mathcal{C}}(X,Y,Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[rr] \arrow[dd] & & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[dd] \\
\{\overline{X}\} \times_{\mathcal{D}} \mathcal{D}_{/\overline{g}} \arrow[rr] \arrow[dr, "\iota^{\mathrm{R}}_{\overline{X},\overline{g}}"'] & & \{\overline{X}\} \times_{\mathcal{D}} \mathcal{D}_{/\overline{Z}} \arrow[dr, "\iota^{\mathrm{R}}_{\overline{X},\overline{Z}}"] & \\
& \mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Y},\overline{Z}) \times_{\mathrm{Hom}_{\mathcal{D}}(\overline{Y},\overline{Z})} \{\overline{g}\} \arrow[rr] & & \mathrm{Hom}_{\mathcal{D}}(\overline{X},\overline{Z}),
\end{tikzcd}$$

where the right-pinch inclusion maps $\iota^{\mathrm{R}}_{X,Z}$ and $\iota^{\mathrm{R}}_{\overline{X},\overline{Z}}$ are homotopy equivalences (Proposition 4.6.5.10). Applying Corollary 3.4.1.12, we conclude that the front face (5.3) is a homotopy pullback square if and only if the back face (5.5) is a homotopy pullback square: that is, if and only if θ_X is a trivial Kan fibration. □

5.1.3 Locally Cartesian Edges

01TW It will often be convenient to consider a variant of Definition 5.1.1.1.

01TX **Definition 5.1.3.1.** Let $q : X \to S$ be a morphism of simplicial sets and let e be an edge of X having image $\overline{e} = q(e)$ in S. Form a pullback diagram of simplicial sets

$$\begin{tikzcd}
X_e \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\
\Delta^1 \arrow[r, "\overline{e}"] & S,
\end{tikzcd}$$

so that e lifts uniquely to an edge $\widetilde{e}$ of X_e having nondegenerate image in Δ^1. We say that e is *locally q-cartesian* if $\widetilde{e}$ is a q'-cartesian edge of the simplicial set X_e. We say that e is *locally q-cocartesian* if $\widetilde{e}$ is a q'-cocartesian edge of the simplicial set X_e.

01TY **Remark 5.1.3.2.** Let $q : X \to S$ be a morphism of simplicial sets and $q^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ be the opposite morphism. Then an edge e of X is locally q-cartesian if and only if it is locally q^{op}-cocartesian.

01U0 **Remark 5.1.3.3.** Let $q : X \to S$ be a morphism of simplicial sets. Then every q-cartesian edge of X is locally q-cartesian, and every q-cocartesian edge of X is locally q-cocartesian (see Remark 5.1.1.11).

01U1 **Remark 5.1.3.4.** Let $q : X \to S$ be a morphism of simplicial sets and let $e : x \to y$ be an edge of X. Suppose that S is isomorphic to a left cone $K^{\triangleleft}$ and that q carries the vertex $x \in X$ to the cone point of $K^{\triangleleft}$. Then e is q-cartesian if and only if it is locally q-cartesian. Similarly, if S is isomorphic to a right cone $L^{\triangleright}$ and q carries the vertex $y \in X$ to the cone point of $L^{\triangleright}$, then e is q-cocartesian if and only if it is locally q-cocartesian.

01U2 **Remark 5.1.3.5.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S' \arrow[r] & S.
\end{tikzcd}$$

Let e' be an edge of the simplicial set X', having image $e = f(e')$ in X. Then e is locally q-cartesian if and only if e' is locally q'-cartesian. Similarly, e is locally q-cocartesian if and only if e' is locally q'-cocartesian.

Example 5.1.3.6. Let $q : X \to S$ be a morphism of simplicial sets and let e be an edge of X such that $q(e) = \mathrm{id}_s$ is a degenerate edge of S. Suppose that the fiber $X_s = \{s\} \times_S X$ is an ∞-category (this condition is satisfied, for example, if q is an inner fibration). The following conditions are equivalent: 01U3

- The edge e is locally q-cartesian.
- The edge e is locally q-cocartesian.
- The edge e is an isomorphism in the ∞-category X_s.

To prove this, we can use Remark 5.1.3.5 to reduce to the situation where $S = \{s\}$ consists of a single vertex. In this case, the edge e is locally q-cartesian if and only if it is q-cartesian, and locally q-cocartesian if and only if it is q-cocartesian (Remark 5.1.3.4). The desired result now follows from Example 5.1.1.4.

Proposition 5.1.3.7. *Let $q : X \to S$ be an inner fibration of simplicial sets and let $\sigma : \Delta^2 \to X$ be a 2-simplex of X, which we will depict as a diagram* 01U4

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z. \end{tikzcd}$$

- *Suppose that g is q-cartesian. Then f is locally q-cartesian if and only if h is locally q-cartesian.*
- *Suppose that f is q-cocartesian. Then g is locally q-cocartesian if and only if h is locally q-cocartesian.*

Proof. We will prove the first assertion; the proof of the second is similar. Using Remarks 5.1.1.11 and 5.1.3.5, we can replace q by the projection map $\Delta^2 \times_S X \to \Delta^2$, and thereby reduce to the case where $S = \Delta^2$ and $q(\sigma)$ is the identity morphism id_{Δ^2}. In this case, both X and S are ∞-categories and the morphisms f and h are locally q-cartesian if and only if they are q-cartesian (Remark 5.1.3.4). The desired result now follows from Corollary 5.1.2.4. □

Remark 5.1.3.8 (Uniqueness of Locally Cartesian Lifts)**.** Let $q : X \to S$ be an inner fibration of simplicial sets and let $g : y \to z$ be a locally q-cartesian edge of X. Suppose that $h : x \to z$ is another edge of X satisfying $q(h) = q(g)$. Set $s = q(x) = q(y)$, and let 01U5

$X_s = \{s\} \times_S X$ denote the fiber of q over the vertex s. Our assumption that g is locally q-cartesian then guarantees that we can choose a 2-simplex σ of X satisfying

$$d^2_0(\sigma) = g \qquad d^2_1(\sigma) = h \qquad q(\sigma) = s^1_0(q(g)),$$

which we display informally as a diagram

$$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"] & & z;
\end{tikzcd}$$

here $f = d^2_2(\sigma)$ is a morphism in the ∞-category X_s. In this case, the following conditions are equivalent:

(1) The morphism f is an isomorphism in the ∞-category X_s.

(2) The morphism h is locally q-cartesian.

To see this, we can replace q by the projection map $\Delta^1 \times_S X \to \Delta^1$, and thereby reduce to the case where g and h are both lifts of the unique nondegenerate edge of $S = \Delta^1$. In this case, the morphism g is q-cartesian, and (1) is equivalent to the assertion that f is locally q-cartesian (Example 5.1.3.6). The equivalence of (1) and (2) is now a special case of Proposition 5.1.3.7.

01U6 **Corollary 5.1.3.9.** *Let $q : X \to S$ be an inner fibration of simplicial sets, let z be a vertex of X, and let $e : s \to q(z)$ be an edge of S. Suppose that there exists a q-cartesian edge $g : y \to z$ of X satisfying $q(g) = e$. Then any locally q-cartesian edge $h : x \to z$ satisfying $q(h) = e$ is q-cartesian.*

Proof. By virtue of Remark 5.1.1.12, we may assume without loss of generality that S is an ∞-category (or even a simplex). Applying Remark 5.1.3.8, we deduce that there is a 2-simplex of X as depicted in the diagram

$$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"] & & z,
\end{tikzcd}$$

where f is an isomorphism in the ∞-category X. Then f is also q-cartesian (Proposition 5.1.1.8), so Corollary 5.1.2.4 guarantees that h is q-cartesian. $\square$

We now record an analogue of Proposition 5.1.2.1 for detecting locally cartesian edges.

Notation 5.1.3.10. Let $q : X \to S$ be an inner fibration of simplicial sets, let y and z be vertices of X having images $s = q(y)$ and $t = q(z)$, and let $\overline{e} : s \to t$ be an edge of S. Recall that the relative morphism space $\mathrm{Hom}_X(y, z)_{\overline{e}}$ is defined to be the fiber product $\mathrm{Hom}_X(y, z) \times_{\mathrm{Hom}_S(s,t)} \{\overline{e}\}$ (Construction 4.6.1.15). 01U7

Let x be another vertex of X satisfying $q(x) = s$, and let σ denote the image of $\overline{e}$ under the degeneracy operator $\mathrm{Hom}_S(s, t) \to \mathrm{Hom}_S(s, s, t)$ (see Notation 4.6.9.1). It follows from Proposition 4.6.9.4 (and Example 4.6.1.17) that restriction along the inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a trivial Kan fibration of simplicial sets

$$\theta : \mathrm{Hom}_X(x, y, z) \times_{\mathrm{Hom}_S(s,s,t)} \{\sigma\} \to \mathrm{Hom}_X(y, z)_{\overline{e}} \times \mathrm{Hom}_{X_s}(x, y),$$

where $X_{\overline{y}}$ denotes the ∞-category given by the fiber $\{\overline{y}\} \times_S X$. In particular, the homotopy class $[\theta]$ is an isomorphism in the homotopy category hKan. Combining the inverse isomorphism $[\theta]^{-1}$ with the restriction map $\mathrm{Hom}_X(x, y, z) \times_{\mathrm{Hom}_S(s,s,t)} \{\sigma\} \to \mathrm{Hom}_X(x, z)_{\overline{e}}$, we obtain a composition law

$$\circ : \mathrm{Hom}_X(y, z)_{\overline{e}} \times \mathrm{Hom}_{X_s}(x, y) \to \mathrm{Hom}_X(x, z)_{\overline{e}}.$$

If $e : y \to z$ is an edge of X satisfying $q(e) = \overline{e}$, then the restriction of this composition law to $\{e\} \times \mathrm{Hom}_{X_s}(x, y)$ determines a morphism of Kan complexes $\mathrm{Hom}_{X_s}(x, y) \xrightarrow{[e]\circ} \mathrm{Hom}_X(x, z)_{\overline{e}}$, which is well-defined up to homotopy.

Proposition 5.1.3.11. *Let $q : X \to S$ be an inner fibration of simplicial sets, and let $e : y \to z$ be an edge of the simplicial set X having image $\overline{e} : s \to t$ in S. Then e is locally q-cartesian if and only if, for every object x of the ∞-category X_s, the composition map* 01U8

$$\mathrm{Hom}_{X_s}(x, y) \xrightarrow{[e]\circ} \mathrm{Hom}_X(x, z)_{\overline{e}}$$

of Notation 5.1.3.10 is an isomorphism in the homotopy category hKan.

Proof. Without loss of generality, we can replace $q : X \to S$ by the projection map $X \times_S \Delta^1 \to \Delta^1$ and thereby reduce to the case where $S = \Delta^1$ and $\overline{e}$ is the unique nondegenerate edge of Δ^1. In this case, the edge e is locally q-cartesian if and only if it is q-cartesian, and the desired result is a special case of Corollary 5.1.2.3. □

5.1.4 Cartesian Fibrations

We now introduce an ∞-categorical counterpart of Definition 5.0.0.3. 01U9

Definition 5.1.4.1. Let $q : X \to S$ be a morphism of simplicial sets. We say that q is a *cartesian fibration* if the following conditions are satisfied: 01UA

(1) The morphism q is an inner fibration.

(2) For every edge $\overline{e} : s \rightarrow t$ of the simplicial set S and every vertex $z \in X$ satisfying $q(z) = t$, there exists a q-cartesian edge $e : y \rightarrow z$ of X satisfying $q(e) = \overline{e}$.

We say that q is a *cocartesian fibration* if it satisfies condition (1) together with the following dual version of (2):

(2′) For every edge $\overline{e} : s \rightarrow t$ of the simplicial set S and every vertex $y \in X$ satisfying $q(y) = s$, there exists a q-cocartesian edge $e : y \rightarrow z$ of X satisfying $q(e) = \overline{e}$.

`01UB` **Example 5.1.4.2.** Let $q : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between ordinary categories. Then q is a cartesian fibration (in the sense of Definition 5.0.0.3) if and only if the induced morphism of simplicial sets $\mathrm{N}_\bullet(q) : \mathrm{N}_\bullet(\mathcal{C}) \rightarrow \mathrm{N}_\bullet(\mathcal{D})$ is a cartesian fibration (in the sense of Definition 5.1.4.1). Similarly, q is a cocartesian fibration if and only if $\mathrm{N}_\bullet(q)$ is a cocartesian fibration of simplicial sets. See Corollary 5.1.2.2.

`01UC` **Example 5.1.4.3.** Let X be a simplicial set and let $q : X \rightarrow \Delta^0$ denote the projection map. The following conditions are equivalent:

- The simplicial set X is an ∞-category.
- The morphism q is a cartesian fibration.
- The morphism q is a cocartesian fibration.

`01UD` **Remark 5.1.4.4.** Let $q : X \rightarrow S$ be a morphism of simplicial sets. Then q is a cartesian fibration if and only if the opposite morphism $q^{\mathrm{op}} : X^{\mathrm{op}} \rightarrow S^{\mathrm{op}}$ is a cocartesian fibration.

`01UE` **Remark 5.1.4.5.** Let $q : X \rightarrow S$ be an inner fibration of simplicial sets and let e be an edge of X. If q is a cartesian fibration, then e is q-cartesian if and only if it is locally q-cartesian (see Corollary 5.1.3.9). Similarly, if q is a cocartesian fibration, then e is q-cocartesian if and only if it is locally q-cocartesian.

`01UF` **Remark 5.1.4.6.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "q'"] & X \arrow[d, "q"] \\
S' \arrow[r] & S.
\end{tikzcd}$$

If q is a cartesian fibration, then q' is also a cartesian fibration. Moreover, an edge e' of X' is q'-cartesian if and only if $e = f(e')$ is a q-cartesian edge of X (this follows from Remarks

5.1.4.5 and 5.1.3.5). Similarly, if q is a cocartesian fibration, then q' is also a cocartesian fibration (and an edge e' of X' is q'-cocartesian if and only if $e = f(e')$ is a q-cocartesian edge of X).

Proposition 5.1.4.7. *Let $q : X \to S$ be a morphism of simplicial sets. Then q is a cartesian fibration if and only if, for every simplex $\sigma : \Delta^n \to S$, the projection map $q_\sigma : \Delta^n \times_S X \to \Delta^n$ is a cartesian fibration.* 023J

Proof. If q is a cartesian fibration, then Remark 5.1.4.6 guarantees that every pullback of q is a cartesian fibration. Conversely, suppose that for every n-simplex $\sigma : \Delta^n \to S$, the projection map $q_\sigma : \Delta^n \times_S X \to \Delta^n$ is a cartesian fibration. Applying this assumption in the case $n = 1$, we conclude that for every vertex $y \in X$ and every edge $\overline{e} : \overline{x} \to q(y)$ of S, there exists a locally q-cartesian edge $e : x \to y$ satisfying $q(e) = \overline{e}$. Moreover, Remark 4.1.1.13 guarantees that q is an inner fibration. It will therefore suffice to show that every locally q-cartesian edge e of X is q-cartesian. By virtue of Remark 5.1.1.12, it suffices to verify the analogous assertion for each of the projection maps $q_\sigma : \Delta^n \times_S X \to \Delta^n$, which follows from Remark 5.1.4.5 (since q_σ is assumed to be a cartesian fibration). □

Proposition 5.1.4.8. *Let $q : \mathcal{C} \to \mathcal{D}$ be a cartesian fibration of ∞-categories. Then q is an isofibration.* 01UG

Proof. Suppose we are given an object $Y \in \mathcal{C}$ and an isomorphism $\overline{e} : \overline{X} \to q(Y)$ in the ∞-category $\mathcal{D}$. We wish to show that there exists an isomorphism $e : X \to Y$ in the ∞-category $\mathcal{C}$ satisfying $q(e) = \overline{e}$. Our assumption that q is a cartesian fibration guarantees that we can write $\overline{e} = q(e)$, where $e : X \to Y$ is a q-cartesian morphism of $\mathcal{C}$. Since $\overline{e} = q(e)$ is an isomorphism, Proposition 5.1.1.8 guarantees that e is an isomorphism. □

Remark 5.1.4.9. In the statement of Proposition 5.1.4.8, the hypothesis that $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories is superfluous: we will later show that every cartesian fibration of simplicial sets is an isofibration (Corollary 5.6.7.5). 01UH

Corollary 5.1.4.10. *Let $q : X \to S$ be a morphism of simplicial sets, where S is a Kan complex. The following conditions are equivalent:* 023K

(1) *The morphism q is an isofibration.*

(2) *The morphism q is a cartesian fibration.*

(3) *The morphism q is a cocartesian fibration.*

Proof. We will prove the equivalence (1) ⇔ (2); the equivalence (1) ⇔ (3) follows by a similar argument. The implication (2) ⇒ (1) is a special case of Proposition 5.1.4.8. For the converse, suppose that q is an isofibration. Then q is an inner fibration. To complete the proof, we must show that for every vertex $y \in X$ and every edge $\overline{e} : \overline{x} \to q(y)$ of S, we can write $\overline{e} = q(e)$ for some q-cartesian edge e of X. Since S is a Kan complex, $\overline{e}$ is an isomorphism (Proposition 1.4.6.10). Our assumption that q is an isofibration then guarantees that we can write $\overline{e} = q(e)$ for some isomorphism $e : x \to y$ of X. The edge e is automatically q-cartesian by virtue of Corollary 5.1.1.10. □

01UJ **Proposition 5.1.4.11.** *Let $q : X \to S$ be a cartesian fibration of simplicial sets and let e be an edge of X such that $q(e) = \mathrm{id}_s$ is a degenerate edge of S. Then e is q-cartesian if and only if it is an isomorphism in the ∞-category $X_s = \{s\} \times_S X$.*

Proof. Combine Example 5.1.3.6 with Remark 5.1.4.5. □

01UK **Proposition 5.1.4.12.** *Let $q : X \to S$ be a cartesian fibration of simplicial sets and let $\sigma : \Delta^2 \to X$ be a 2-simplex of X, which we will depict as a diagram*

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z. \end{tikzcd}$$

Suppose that g is q-cartesian. Then f is q-cartesian if and only if h is q-cartesian.

Proof. Combine Proposition 5.1.3.7 with Remark 5.1.4.5. □

01UL **Proposition 5.1.4.13.** *Let $p : X \to Y$ and $q : Y \to Z$ be cartesian fibrations of simplicial sets. Then:*

- *The composite morphism $(q \circ p) : X \to Z$ is a cartesian fibration of simplicial sets.*
- *An edge e of X is $(q \circ p)$-cartesian if and only if e is p-cartesian and $p(e)$ is q-cartesian.*

Proof. It follows from Remark 4.1.1.8 that $q \circ p$ is an inner fibration. Let us say that an edge e of X is *special* if e is p-cartesian and $p(e)$ is q-cartesian. Remark 5.1.1.6 guarantees that every special edge of X is $(q \circ p)$-cartesian. Consequently, to prove the first assertion, it will suffice to verify the following:

(∗) For every edge $\overline{e} : z' \to z$ of Z and every vertex $x \in X$ satisfying $z = (q \circ p)(x)$, there exists a special edge $e : x' \to x$ of X satisfying $\overline{e} = (q \circ p)(e)$.

To prove (∗), set $y = p(x)$. Using our assumption that q is a cartesian fibration, we can choose a q-cartesian edge $\widetilde{e} : y' \to y$ of the simplicial set Y satisfying $q(\widetilde{e}) = \overline{e}$. Using our assumption that p is a cartesian fibration, we can choose a p-cartesian edge $e : x' \to x$ of X satisfying $p(e) = \widetilde{e}$. Then e is a special edge of X satisfying $(q \circ p)(e) = q(\widetilde{e}) = \overline{e}$.

To complete the proof, it will suffice to show that every $(q \circ p)$-cartesian edge $f : x'' \to x$ of X is special. Let $\overline{f} : z'' \to z$ be the image of f under $(q \circ p) : X \to Z$. Using $(*)$, we can choose a special edge $e : x' \to x$ satisfying $(q \circ p)(e) = \overline{f}$. Since e is $(q \circ p)$-cartesian, we can choose a 2-simplex σ of X satisfying

$$d_0^2(\sigma) = e \qquad d_1^2(\sigma) = f \qquad (q \circ p)(\sigma) = s_0^1(\overline{e}).$$

Set $g = d_2^2(\sigma)$, so that we can view σ informally as a diagram

$$\begin{tikzcd} & x' \arrow[dr, "e"] & \\ x'' \arrow[ur, "g"] \arrow[rr, "f"] & & x. \end{tikzcd}$$

Set $y' = p(x') \in Y$ and $z' = q(y') \in Z$. Since f is $(q \circ p)$-cartesian, the edge g is an isomorphism in the ∞-category $X_{z'}$ (Remark 5.1.3.8). Then g is p'-cartesian, where $p' : X_{z'} \to Y_{z'}$ is the projection map (Proposition 5.1.1.8). Applying Remark 5.1.3.5, we conclude that g is locally p-cartesian. Since p is a cartesian fibration, it follows that g is p-cartesian (Remark 5.1.4.5). Invoking Proposition 5.1.4.12, we deduce that f is also p-cartesian. Since $p(g)$ is an isomorphism in the ∞-category $Y_{z'}$, it is q-cartesian (Proposition 5.1.4.11). Applying Proposition 5.1.4.12, we conclude that $p(f)$ is also q-cartesian. □

Recall that an ∞-category $\mathcal{C}$ is a Kan complex if and only if every morphism in $\mathcal{C}$ is an isomorphism (Proposition 4.4.2.1). We now establish a relative version of this assertion:

Proposition 5.1.4.14. *Let $q : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:* 01UM

(1) *The morphism q is a right fibration.*

(2) *The morphism q is a cartesian fibration and every edge of X is q-cartesian.*

(3) *The morphism q is a cartesian fibration and, for every vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a Kan complex.*

Proof. The equivalence (1) ⇔ (2) is immediate from the definitions. The implication (2) ⇒ (3) follows from Propositions 5.1.4.11 and 4.4.2.1. We will complete the proof by showing that (3) implies (2). Assume that q is a cartesian fibration and that each fiber of q is a Kan complex. Let $h : x \to z$ be an edge of X; we wish to show that h is q-cartesian. Since q is a cartesian fibration, we can choose a q-cartesian edge $g : y \to z$ of X satisfying

$q(g) = q(h)$. The assumption that g is q-cartesian then guarantees the existence of a 2-simplex σ of X satisfying

$$d_0^2(\sigma) = g \qquad d_1^2(\sigma) = h \qquad q(\sigma) = s_0^1(q(h)),$$

as depicted in the diagram

$$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"] & & z.
\end{tikzcd}$$

Set $s = q(x)$, so that f is a morphism in the ∞-category $X_s = \{s\} \times_S X$. Since X_s is a Kan complex, f is an isomorphism (Proposition 1.4.6.10). Applying Remark 5.1.3.8 and Corollary 5.1.3.9, we deduce that h is q-cartesian. ☐

Recall that every ∞-category $\mathcal{C}$ has an underlying Kan complex $\mathcal{C}^{\simeq}$, obtained by discarding the noninvertible morphisms of $\mathcal{C}$ (Construction 4.4.3.1). Using Proposition 5.1.4.14, we can establish a relative version of this result.

`01UN` **Corollary 5.1.4.15.** *Let $q : X \to S$ be a cartesian fibration of simplicial sets, and let $X' \subseteq X$ be the simplicial subset spanned by those simplices $\sigma : \Delta^n \to X$ which carry each edge of Δ^n to a q-cartesian edge of X. Then the morphism $q|_{X'} : X' \to S$ is a right fibration of simplicial sets.*

Proof. Choose integers $0 < i \leq n$; we wish to show that every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & X' \arrow[d, "q|_{X'}"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & S
\end{tikzcd}$$

admits a solution. In the special case $i = n = 1$, this follows immediately from our assumption that q is a cartesian fibration. Assume therefore that $n \geq 2$. We first show that σ_0 can be extended to an n-simplex $\sigma : \Delta^n \to X$ satisfying $q \circ \sigma = \overline{\sigma}$. For $i < n$, this follows from the assumption that q is an inner fibration. For $i = n$, it follows from the assumption that the edge

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{n-1 < n\}) \hookrightarrow \Lambda^n_i \xrightarrow{\sigma_0} X' \hookrightarrow X$$

is q-cartesian. We now complete the proof by showing that σ factors through the simplicial subset X'; that is, it carries each edge of Δ^n to a q-cartesian edge of X. For $n \geq 3$, this is immediate (since every edge of Δ^n is contained in Λ^n_i). The case $n = 2$ follows from Proposition 5.1.4.12. □

Proposition 5.1.4.16. *Let $q : X \to S$ be a cartesian fibration of simplicial sets, and let $X' \subseteq X$ be a full simplicial subset with the following property:* 02R5

(∗) *For every vertex $y \in X'$ and every edge $\overline{e} : \overline{x} \to q(y)$ in S, there exists a vertex $x \in X'$ and a q-cartesian edge $e : x \to y$ of X satisfying $q(e) = \overline{e}$.*

Then $q' = q|_{X'}$ is a cartesian fibration from X' to S. Moreover, an edge e of X' is q'-cartesian if and only if it is q-cartesian.

Proof. Since the inclusion map $X' \hookrightarrow X$ is an inner fibration (see Definition 4.1.2.15), the restriction $q|_{X'}$ is also an inner fibration. Remark 5.1.1.7 guarantees that every edge of X' which is q-cartesian is also q'-cartesian, so that (∗) immediately guarantees that q' is a cartesian fibration. To complete the proof, we must show that if $e : x \to z$ is a q'-cartesian edge of X', then e is q-cartesian when viewed as an edge of X. Applying (∗), we can choose a q-cartesian edge $e' : y \to z$ satisfying $q(e') = q(e)$, where y belongs to X'. Then e' is also q'-cartesian, so Remark 5.1.3.8 guarantees that there exists a 2-simplex

$$\begin{tikzcd}
 & y \arrow[dr, "e'"] & \\
x \arrow[ur, "u"] \arrow[rr] & & z
\end{tikzcd}$$

of X', where u is an isomorphism in the ∞-category $\{q(x)\} \times_S X'$. It follows that u is also an isomorphism in the ∞-category $\{q(x)\} \times_S X$, and is therefore q-cartesian (Proposition 5.1.4.11). Applying Proposition 5.1.4.12, we see that the edge e is also q-cartesian. □

We now study the behavior of cartesian fibrations with respect to the slice and coslice constructions of §4.3.

Proposition 5.1.4.17. *Let $q : X \to S$ be a cartesian fibration of simplicial sets and let $f : K \to X$ be any morphism of simplicial sets. Then:* 01UP

(1) *The induced map $q' : X_{/f} \to S_{/(q \circ f)}$ is a cartesian fibration of simplicial sets.*

(2) *An edge e of $X_{/f}$ is q'-cartesian if and only if its image in X is q-cartesian.*

Proof. The morphism q' factors as a composition

$$X_{/f} \xrightarrow{u} X \times_S S_{/(q \circ f)} \xrightarrow{v} S_{/(q \circ f)}.$$

Since q is an inner fibration, the morphism u is a right fibration (Proposition 4.3.6.8). In particular, u is a cartesian fibration and every edge of $X_{/f}$ is u-cartesian (Proposition 5.1.4.14). The morphism v is a pullback of q, and is therefore a cartesian fibration (Remark 5.1.4.6). Moreover, an edge of $X \times_S S_{/(q \circ f)}$ is v-cartesian if and only if its image in X is q-cartesian. Assertions (1) and (2) now follow from Proposition 5.1.4.13. □

01UQ **Lemma 5.1.4.18.** *Let $q : X \to S$ be an inner fibration of simplicial sets, let $f : B \to X$ be a morphism of simplicial sets, let A be a simplicial subset of B, and let*

$$q' : X_{f/} \to X_{f|_A/} \times_{S_{(q \circ f|_A)/}} S_{(q \circ f)/}$$

be the induced map. Let $\overline{e}$ be an edge of the simplicial set $X_{f/}$, and let e be its image in X. If e is q-cartesian, then $\overline{e}$ is q'-cartesian.

Proof. Let $q'' : X_{f|_A/} \times_{S_{(q \circ f|_A)/}} S_{(q \circ f)/} \to S_{(q \circ f)/}$ be the projection map onto the second factor. Then q'' is a pullback of the map $u : X_{f|_A/} \to S_{(q \circ f|_A)/}$, and is therefore an inner fibration (Corollary 4.3.6.10). By virtue of Corollary 5.1.1.14, the edge $\overline{e}$ is $(q'' \circ q')$-cartesian and the image of $\overline{e}$ in $X_{f|_A/}$ is u-cartesian. It follows from Remark 5.1.1.11 that $q'(\overline{e})$ is q''-cartesian, so that $\overline{e}$ is q'-cartesian by virtue of Corollary 5.1.2.6. □

01UR **Proposition 5.1.4.19.** *Let $q : X \to S$ be a cartesian fibration of simplicial sets and let $f : K \to X$ be any morphism of simplicial sets. Then:*

(1) *The induced map $q' : X_{f/} \to S_{(q \circ f)/}$ is a cartesian fibration of simplicial sets.*

(2) *An edge e of $X_{f/}$ is q'-cartesian if and only if its image in X is q-cartesian.*

Proof. The morphism q' factors as a composition

$$X_{f/} \xrightarrow{u} X \times_S S_{(q \circ f)/} \xrightarrow{v} S_{(q \circ f)/},$$

where v is a pullback of q and is therefore a cartesian fibration by virtue of Remark 5.1.4.6. The morphism u is a left fibration (Proposition 4.3.6.8), and therefore an inner fibration. It follows that q' is an inner fibration (Remark 4.1.1.8).

Let us say that an edge e of $X_{f/}$ is *special* if its image in X is q-cartesian. If this condition is satisfied, then $u(e)$ is v-cartesian (Remark 5.1.4.6) and e is u-cartesian (Lemma 5.1.4.18), so that e is q'-cartesian by virtue of Remark 5.1.1.6. This proves the “if” direction of assertion (2).

To prove (1), it will suffice to show that for every vertex y of the simplicial set $X_{f/}$ and every edge $\overline{e} : \overline{x} \to q'(y)$ of the simplicial set $S_{(q \circ f)/}$, there exists a special edge $e : x \to y$ of $X_{f/}$ satisfying $q'(e) = \overline{e}$. Since v is a cartesian fibration, we can choose a v-cartesian edge $\widetilde{e} : \widetilde{x} \to u(y)$ of the simplicial set $X \times_S S_{(q \circ f)/}$. In this case, the image $\widetilde{e}$ in X is

q-cartesian (Remark 5.1.4.6). Corollary 5.1.1.15 then guarantees that there exists an edge $e : x \to y$ of $X_{f/}$ satisfying $u(e) = \widetilde{e}$. The edge e is automatically special and satisfies $q'(e) = (v \circ u)(e) = v(\widetilde{e}) = \overline{e}$, as desired.

To complete the proof of (2), it will suffice to show that every q'-cartesian edge $e : x \to z$ of $X_{f/}$ is special. It follows from the preceding argument that there exists a special edge $e' : y \to z$ satisfying $q'(e') = q'(e)$, which is also q'-cartesian. Applying Remark 5.1.3.8, we can choose a 2-simplex σ of $X_{f/}$ as indicated in the diagram

$$\begin{tikzcd}
 & y \arrow[dr, "e'"] & \\
x \arrow[ur, "e''"] \arrow[rr, "e"] & & z,
\end{tikzcd}$$

where e'' is an isomorphism in the ∞-category $\{q'(x)\} \times_{S_{(q \circ f)/}} X_{f/}$. Using Proposition 5.1.4.11, we deduce that the image of e'' in X is q-cartesian. Applying Proposition 5.1.4.12, we conclude the image of e in X is also q-cartesian, as desired. □

Proposition 5.1.4.20. *Suppose we are given a commutative diagram of simplicial sets* 02VV

$$\begin{tikzcd}
X_0 \arrow[r, "F_0"] \arrow[dr, "U_0"'] & X \arrow[d, "U"] & X_1 \arrow[l, "F_1"'] \arrow[dl, "U_1"] \\
 & S &
\end{tikzcd}$$

satisfying the following conditions:

- *The morphisms U_0 and U_1 are cartesian fibrations.*
- *The morphism F_0 carries U_0-cartesian edges of X_0 to U-cartesian edges of X.*
- *The morphism F_1 carries U_1-cartesian edges of X_1 to U-cartesian edges of X.*
- *The morphism F_1 is an isofibration.*

Then the induced map $U_{01} : X_0 \times_X X_1 \to S$ is also a cartesian fibration. Moreover, an edge $e = (e_0, e_1)$ of $X_0 \times_X X_1$ is U_{01}-cartesian if and only if e_0 is U_0-cartesian and e_1 is U_1-cartesian.

Proof. Let $\pi : X_0 \times_X X_1 \to X_0$ and $\pi' : X_0 \times_X X_1 \to X_1$ be the projection maps. Since F_1 is an isofibration, π is also an isofibration. In particular, π is an inner fibration, so the composition $U_{01} = U_0 \circ \pi$ is also an inner fibration. Let us say that an edge $e = (e_0, e_1)$ of $X_0 \times_X X_1$ is *special* if e_0 is U_0-cartesian and e_1 is U_1-cartesian. The second assumption guarantees that e is π-cartesian (Remark 5.1.1.11) and the first guarantees that $\pi(e)$ is U_0-cartesian. Applying Corollary 5.1.2.4, we deduce that every special edge of $X_0 \times_X X_1$ is U_{01}-cartesian.

To prove that U_{01} is a cartesian fibration, it will suffice to show that for every vertex $x = (x_0, x_1)$ of $X_0 \times_X X_1$ and every edge $\overline{e} : s \to U_{01}(x)$ in S, there exists a special edge $e : y \to x$ of $X_0 \times_X X_1$ satisfying $U_{01}(e) = \overline{e}$. Since U_0 is a cartesian fibration, we can choose a U_0-cartesian edge $e_0 : y_0 \to x_0$ of X_0 satisfying $U_0(e_0) = \overline{e}$. Similarly, we can choose a U_1-cartesian edge $e_1 : y_1 \to x_1$ of X_1 satisfying $U_1(e_1) = \overline{e}$. We now observe that $F_0(e_0)$ and $F_1(e_1)$ are U-cartesian edges of X having the same target and the same image in the simplicial set S. Applying Remark 5.1.3.8, we can choose a 2-simplex σ of X as indicated in the diagram

$$\begin{tikzcd}
 & F_1(y_1) \arrow[dr, "F_1(e_1)"] & \\
F_0(y_0) \arrow[ur, "v"] \arrow[rr, "F_0(e_0)"] & & F_1(x_1),
\end{tikzcd}$$

where v is an isomorphism in the ∞-category $X_s = \{s\} \times_S X$. Our assumption that F_1 is an isofibration guarantees that we can lift v to an isomorphism $\widetilde{v} : y_1' \to y_1$ in the ∞-category $\{s\} \times_S X_1$. Since F_1 is an inner fibration, we can lift σ to a 2-simplex $\widetilde{\sigma}$ of X_1 with boundary indicated in the diagram

$$\begin{tikzcd}
 & y_1 \arrow[dr, "e_1"] & \\
y_1' \arrow[ur, "\widetilde{v}"] \arrow[rr, "e_1'"] & & x_1.
\end{tikzcd}$$

It follows from Propositions 5.1.4.11 and 5.1.4.12 that e_1' is a U_1-cartesian edge of X_1, so that $e = (e_0, e_1')$ is a special edge of $X_0 \times_X X_1$ with target $x = (x_0, x_1)$ which satisfies $U_{01}(e) = \overline{e}$.

To complete the proof of Proposition 5.1.4.20, it will suffice to show that if $f : z \to x$ is a U_{01}-cartesian edge of the fiber product $X_0 \times_X X_1$, then f is special. Set $s = U_{01}(z)$. Applying the above argument, we can choose a special edge $e : y \to x$ satisfying $U_{01}(e) = U_{01}(f)$. Using Remark 5.1.3.8, we can choose a 2-simplex τ of $X_0 \times_X X_1$ with boundary indicated in the diagram

$$\begin{tikzcd} & y \arrow[dr, "e"] & \\ z \arrow[ur, "v"] \arrow[rr, "f"] & & x, \end{tikzcd}$$

where v is an isomorphism in the ∞-category $\{s\} \times_S (X_0 \times_X X_1)$. Applying Propositions 5.1.4.11 and 5.1.4.12 to the 2-simplices $\pi(\tau)$ and $\pi'(\tau)$, we conclude that the edges $\pi(f)$ and $\pi'(f)$ are U_0-cartesian and U_1-cartesian, as desired. □

As an application of Proposition 5.1.4.20, we record a generalization of Proposition 5.1.4.19 which will be useful later.

Corollary 5.1.4.21. *Suppose we are given a commutative diagram of simplicial sets* 03LL

$$\begin{tikzcd} X' \arrow[rr, "u"] \arrow[dr, "q'"'] & & X \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

where q and q' are cartesian fibrations and the morphism u carries q'-cartesian edges of X' to q-cartesian edges of X. Let $f : K \to X$ be a morphism of simplicial sets. Then q' induces a cartesian fibration $\widetilde{q}' : X' \times_X X_{f/} \to S_{(q \circ f)/}$. Moreover, an edge of $X' \times_X X_{f/}$ is $\widetilde{q}'$-cartesian if and only if its image in X' is q'-cartesian.

Proof. Let $\widetilde{u} : X' \times_S S_{(q \circ f)/} \to X \times_S S_{(q \circ f)/}$ denote the pullback of u, let $\widetilde{q} : X \times_S S_{(q \circ f)/} \to S_{(q \circ f)/}$ be given by projection onto the second factor, and let $v : X_{f/} \to X \times_S S_{(q \circ f)/}$ be the left fibration of Proposition 4.3.6.8. Note that $\widetilde{q}$ is a pullback of q, and therefore a cartesian fibration (Remark 5.1.4.6). Moreover, an edge of $X \times_S S_{(q \circ f)/}$ is $\widetilde{q}$-cartesian if and only if its image in X is q-cartesian. Similarly, the composite map $\widetilde{q} \circ \widetilde{u}$ is a pullback of q'. It follows that $\widetilde{q} \circ \widetilde{u}$ is a cartesian fibration, and that an edge of $X' \times_S S_{(q \circ f)/}$ is $(\widetilde{q} \circ \widetilde{u})$-cartesian if and only if its image in X' is q'-cartesian. Applying Proposition 5.1.4.19, we deduce that the composition $\widetilde{q} \circ v$ is a cartesian fibration, and that an edge of $X_{f/}$ is $(\widetilde{q} \circ v)$-cartesian if and only if its image in X is q-cartesian. The desired result now follows by applying Proposition

5.1.4.20 to the diagram

$$\begin{tikzcd}
X' \times_S S_{(q\circ f)/} \arrow[r, "\widetilde{u}"] \arrow[dr] & X \times_S S_{(q\circ f)/} \arrow[d, "\pi"] & X_{f/} \arrow[l, "v"'] \arrow[dl] \\
 & S_{(q\circ f)/}. &
\end{tikzcd}$$

□

5.1.5 Locally Cartesian Fibrations

01UW It will sometimes be convenient to consider a generalization of Definition 5.1.4.1

01UX **Definition 5.1.5.1.** Let $q : X \to S$ be a morphism of simplicial sets. We say that q is a *locally cartesian fibration* if the following conditions are satisfied:

(1) The morphism q is an inner fibration.

(2) For every edge $\overline{e} : s \to t$ of the simplicial set S and every vertex $z \in X$ satisfying $q(z) = t$, there exists a locally q-cartesian edge $e : y \to z$ of X satisfying $q(e) = \overline{e}$.

We say that q is a *locally cocartesian fibration* if it satisfies condition (1) together with the following dual version of (2):

(2′) For every edge $e : s \to t$ of the simplicial set S and every vertex $y \in X$ satisfying $q(y) = s$, there exists a locally q-cocartesian edge $e : y \to z$ of X satisfying $q(e) = \overline{e}$.

01UY **Example 5.1.5.2.** Let $q : X \to S$ be a morphism of simplicial sets. If q is a cartesian fibration, then it is a locally cartesian fibration. If q is a cocartesian fibration, then it is a locally cocartesian fibration.

01S3 **Exercise 5.1.5.3.** Let Q be a partially ordered set, let $\mathrm{Chain}[Q]$ denote the collection of all finite nonempty linearly ordered subsets of Q (Notation 3.3.2.1), and let $\mathrm{Max} : \mathrm{Chain}[Q] \to Q$ be the map which carries each element $S \in \mathrm{Chain}[Q]$ to the largest element of S.

- Show that the induced map of nerves $\mathrm{N}_\bullet(\mathrm{Max}) : \mathrm{N}_\bullet(\mathrm{Chain}[Q]) \to \mathrm{N}_\bullet(Q)$ is a locally cocartesian fibration.

- Show that, if $Q = [n]$ for $n \geq 2$, the functor $\mathrm{N}_\bullet(\mathrm{Max}) : \mathrm{N}_\bullet(\mathrm{Chain}[Q]) \to \mathrm{N}_\bullet(Q)$ is not a cocartesian fibration.

Remark 5.1.5.4. Let $q : X \to S$ be a morphism of simplicial sets. Then q is a locally cartesian fibration if and only if the opposite morphism $q^{\mathrm{op}} : X^{\mathrm{op}} \to S^{\mathrm{op}}$ is a locally cocartesian fibration. 01V0

Remark 5.1.5.5. Suppose we are given a pullback diagram of simplicial sets 01V1

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "q'"] & X \arrow[d, "q"] \\ S' \arrow[r] & S. \end{tikzcd}$$

If q is a locally cartesian fibration, then q' is also a locally cartesian fibration (see Remark 5.1.1.11). If q is a locally cocartesian fibration, then q' is also a locally cocartesian fibration.

Remark 5.1.5.6. Let $q : X \to S$ be an inner fibration of simplicial sets. The following conditions are equivalent: 01V2

- The morphism q is a locally cartesian fibration.
- For every pullback diagram

  $$\begin{tikzcd} X' \arrow[r] \arrow[d, "q'"] & X \arrow[d] \\ \Delta^1 \arrow[r] & S, \end{tikzcd}$$

  the morphism q' is a locally cartesian fibration.
- For every pullback diagram

  $$\begin{tikzcd} X' \arrow[r] \arrow[d, "q'"] & X \arrow[d] \\ \Delta^1 \arrow[r] & S, \end{tikzcd}$$

  the morphism q' is a cartesian fibration.

Remark 5.1.5.7. Let $p : X \to Y$ and $q : Y \to Z$ be morphisms of simplicial sets. If p is a cartesian fibration and q is a locally cartesian fibration, then the composition $q \circ p$ is a locally cartesian fibration. Moreover, an edge e of X is locally $(q \circ p)$-cartesian if and only if it is p-cartesian and $p(e)$ is locally q-cartesian of Y. To prove this, we can assume without loss of generality that $S = \Delta^1$. In this case, q is a cartesian fibration (Remark 5.1.5.6), so the desired result follows from Proposition 5.1.4.13. 01V3

01V4 **Warning 5.1.5.8.** The collection of locally (co)cartesian fibrations is not closed under composition.

01V5 **Proposition 5.1.5.9.** *Let $q : \mathcal{C} \to \mathcal{D}$ be a locally cartesian fibration of simplicial sets and let $g : Y \to Z$ be an edge of $\mathcal{C}$. The following conditions are equivalent:*

(1) *The edge g is q-cartesian.*

(2) *For every 2-simplex σ*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

of $\mathcal{C}$, the edge f is locally q-cartesian if and only if the edge h is locally q-cartesian.

(3) *For every 2-simplex σ*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

of $\mathcal{C}$, if f is locally q-cartesian, then h is locally q-cartesian.

Proof. The implication (1) ⇒ (2) follows from Corollary 5.1.2.6 (and does not require the assumption that q is a locally cartesian fibration), and the implication (2) ⇒ (3) is immediate. We will complete the proof by showing that (3) ⇒ (1). Using Remarks 5.1.1.12 and 5.1.3.5, we can reduce to the case where $\mathcal{D} = \Delta^n$ is a simplex. By virtue of Corollary 5.1.2.3, it will suffice to show that for each object $X \in \mathcal{C}$ satisfying $q(X) \leq q(Y)$, the composition map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{[g]\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

of Notation 4.6.9.15 is an isomorphism in the homotopy category hKan. Since q is a locally cartesian fibration, we can choose a locally q-cartesian morphism $f : X \to Y$ satisfying $q(W) = q(X)$. Using the fact that $\mathcal{C}$ is an ∞-category, we can choose a 2-simplex σ of $\mathcal{C}$ satisfying $d^2_0(\sigma) = g$ and $d^2_2(\sigma) = f$. Set $h = d^2_1(\sigma)$, so that we have a commutative diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in the ∞-category $\mathcal{C}$. If assumption (3) is satisfied, then h is also a locally q-cartesian morphism of $\mathcal{C}$. Invoking Proposition 4.6.9.12, we conclude that the diagram

$$\begin{tikzcd} & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[dr, "{[g]\circ}"] & \\ \mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[ur, "{[f]\circ}"] \arrow[rr, "{[h]\circ}"] & & \mathrm{Hom}_{\mathcal{C}}(X,Z) \end{tikzcd}$$

commutes (in the homotopy category hKan). Since f and h are locally q-cartesian, the horizontal and left diagonal map in this diagram are isomorphisms (in the homotopy category hKan), so the right diagonal map is an isomorphism as well. □

Corollary 5.1.5.10. *Let $q : X \to S$ be a locally cartesian fibration of simplicial sets. The following conditions are equivalent:* 01V6

1. *The morphism q is a cartesian fibration.*
2. *Every locally q-cartesian edge of X is q-cartesian.*
3. *For every 2-simplex σ:*

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z \end{tikzcd}$$

of the simplicial set X, if f and g are locally q-cartesian, then h is locally q-cartesian.

Proof. The implication (1) ⇒ (2) follows from Remark 5.1.4.5, the implication (2) ⇒ (1) is immediate, and the equivalence (2) ⇔ (3) follows from Proposition 5.1.5.9. □

Corollary 5.1.5.11. *Let $p : X \to S$ be an inner fibration of simplicial sets. Then p is a cartesian fibration if and only if every pullback $X \times_S \Delta^n \to \Delta^n$ is a cartesian fibration for $n \leq 2$.* 01V7

Corollary 5.1.5.12. *Let $q : X \to S$ be a locally cartesian fibration of simplicial sets. The following conditions are equivalent:* 046V

1. *The morphism q is a right fibration.*
2. *For every vertex $s \in S$, the fiber $X_s = \{s\} \times_S X$ is a Kan complex.*
3. *Every edge of X is locally q-cartesian.*

(4) *Every edge of X is q-cartesian.*

Proof. The implication (1) ⇒ (2) follows from Corollary 4.4.2.3. To show that (2) ⇒ (3), we may assume without loss of generality that $S = \Delta^1$. In this case, q is a cartesian fibration (Remark 5.1.5.6), so the desired result follows from Proposition 5.1.4.14. The implication (3) ⇒ (4) follows from Corollary 5.1.5.10. If condition (4) is satisfied, then q is a cartesian fibration (Corollary 5.1.5.10), so that (1) follows from Proposition 5.1.4.14. □

`01V8` **Proposition 5.1.5.13.** *Let $q : X \to S$ be a locally cartesian fibration of simplicial sets and let $f : K \to X$ be any morphism of simplicial sets. Then:*

(1) *The induced map $q' : X_{/f} \to S_{/(q \circ f)}$ is a locally cartesian fibration of simplicial sets.*

(2) *An edge e of $X_{/f}$ is locally q'-cartesian if and only if its image in X is locally q-cartesian.*

Proof. As in the proof of Proposition 5.1.4.17, we factor q' as a composition

$$X_{/f} \xrightarrow{u} X \times_S S_{/(q \circ f)} \xrightarrow{v} S_{/(q \circ f)},$$

where u is a cartesian fibration and every edge of $X_{/f}$ is u-cartesian (Proposition 5.1.4.14). The morphism v is a pullback of q, and is therefore a locally cartesian fibration (Remark 5.1.5.5). Moreover, an edge of $X \times_S S_{/(q \circ f)}$ is locally v-cartesian if and only if its image in X is locally q-cartesian (Remark 5.1.3.5). Assertions (1) and (2) now follow from Remark 5.1.5.7. □

`047P` **Proposition 5.1.5.14.** *Let κ be an uncountable regular cardinal and let $q : X \to S$ be a locally cartesian fibration of simplicial sets. Then q is locally κ-small (in the sense of Definition 4.7.9.1) if and only if, for every vertex $s \in S$, the ∞-category $X_s = \{s\} \times_S X$ is locally κ-small.*

Proof. Assume that, for every vertex $s \in S$, the ∞-category X_s is locally κ-small; we will show that q is locally κ-small (the reverse implication is immediate from the definitions). By virtue of Corollary 4.7.9.11, we may assume without loss of generality that $S = \Delta^1$. In this case, q is a cartesian fibration and we wish to show that X is locally κ-small. Fix a pair of objects $x, z \in X$; we wish to show that the Kan complex $\mathrm{Hom}_X(x, z)$ is essentially κ-small. We may assume without loss of generality that $q(x) \leq q(z)$ (otherwise, the Kan complex $\mathrm{Hom}_X(x, z)$ is empty). If $q(x) = q(z)$, then the desired result follows from our hypothesis. It will therefore suffice to treat the case where $q(x) = 0$ and $q(z) = 1$. Since q is a locally cartesian fibration, we can choose a q-cartesian morphism $f : y \to z$ of X, where $q(y) = 0$. In this case, composition with the homotopy class $[f]$ induces a homotopy equivalence of Kan complexes $\mathrm{Hom}_X(x, y) \to \mathrm{Hom}_X(x, z)$ (Corollary 5.1.2.3), so the desired result follows from the local κ-smallness of the ∞-category X_0. □

Corollary 5.1.5.15. *Let κ be an uncountable regular cardinal and let $q : X \to S$ be a locally cartesian fibration of simplicial sets. Then q is essentially κ-small if and only if, for every vertex $s \in S$, the ∞-category $X_s = \{s\} \times_S X$ is essentially κ-small.* 047Q

Proof. Combine Proposition 5.1.5.14 with Remark 4.7.9.4. □

Corollary 5.1.5.16. *Let κ be an uncountable regular cardinal and let $U : \mathcal{E} \to \mathcal{C}$ be a locally cartesian fibration of ∞-categories. Suppose that $\mathcal{C}$ is essentially κ-small. The following conditions are equivalent:* 05DV

(1) *The ∞-category $\mathcal{E}$ is essentially κ-small.*

(2) *For every vertex $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is essentially κ-small.*

Proof. Combine Corollaries 5.1.5.15 and 4.7.9.6. □

Variant 5.1.5.17. Let $U : \mathcal{E} \to \mathcal{C}$ be a locally cartesian fibration of ∞-categories and let $n \geq -1$ be an integer. Then U is essentially n-categorical (in the sense of Definition 4.8.6.1) if and only if, for each object $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is locally $(n-1)$-truncated (in the sense of Definition 4.8.2.1). 05DW

Proof. We proceed as in the proof of Proposition 5.1.5.14. Assume that, for each object $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C$ is locally $(n-1)$-truncated; we will show that the functor U is essentially n-categorical (the reverse implication follows from Proposition 4.8.6.17, and does not require the assumption that U is locally cartesian). By virtue of Proposition 4.8.5.27, we may assume without loss of generality that $\mathcal{C} = \Delta^1$. Fix a pair of objects $X, Z \in \mathcal{E}$; we wish to show that the map of Kan complexes $\operatorname{Hom}_{\mathcal{E}}(X, Z) \to \operatorname{Hom}_{\mathcal{C}}(U(X), U(Z))$ is n-truncated. We may assume that $U(X) \leq U(Z)$ (otherwise, both Kan complexes are empty and there is nothing to prove); in this case, we wish to show that the morphism space $\operatorname{Hom}_{\mathcal{E}}(X, Z)$ is n-truncated (Example 3.5.9.4). If $U(X) = U(Z)$, then the desired result follows from our hypothesis on the fibers of U. It will therefore suffice to treat the case where $U(X) = 0$ and $U(Z) = 1$. Since U is a locally cartesian fibration, we can choose a U-cartesian morphism $f : Y \to Z$ of $\mathcal{E}$ satisfying $U(Y) = 0$. In this case, composition with the homotopy calss $[f]$ induces a homotopy equivalence of Kan complexes $\operatorname{Hom}_{\mathcal{E}}(X, Y) \to \operatorname{Hom}_{\mathcal{E}}(X, Z)$ (Corollary 5.1.2.3). It will therefore suffice to show that the Kan complex $\operatorname{Hom}_{\mathcal{E}}(X, Y)$ is n-truncated, which follows from our assumption that the fiber $\mathcal{E}_0 = U^{-1}\{0\}$ is locally n-truncated. □

Corollary 5.1.5.18. *Let $U : \mathcal{E} \to \mathcal{C}$ be a right fibration of ∞-categories and let $n \geq -2$ be an integer. Then U is locally n-truncated if and only if, for every object $C \in \mathcal{C}$, the Kan complex $\mathcal{E}_C$ is $(n+1)$-truncated.* 05DX

Proof. Combine Variant 5.1.5.17 with Example 4.8.2.4. □

One advantage the theory of locally cartesian fibrations holds over the theory of cartesian fibrations is the following "fiberwise" existence criterion:

01V9 **Proposition 5.1.5.19.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd} X \arrow[rr, "r"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

satisfying the following conditions:

(1) *The maps p and q are locally cartesian fibrations, and r is an inner fibration.*

(2) *The map r carries locally p-cartesian edges of X to locally q-cartesian edges of Y.*

(3) *For every vertex s of S, the induced map $r_s : X_s \to Y_s$ is a locally cartesian fibration.*

Then r is a locally cartesian fibration.

01VA **Warning 5.1.5.20.** The analogue of Proposition 5.1.5.19 for cartesian fibrations is false.

Proof of Proposition 5.1.5.19. Choose a vertex $z \in X$ and an edge $\overline{h} : \overline{x} \to r(z)$ of the simplicial set Y. We wish to prove that there exists a locally r-cartesian edge $h : x \to z$ satisfying $r(h) = \overline{h}$. Since p is a locally cartesian fibration, we can choose a locally p-cartesian edge $g : y \to z$ satisfying $p(g) = q(\overline{h})$. Assumption (2) guarantees that $r(g)$ is locally q-cartesian, so we can choose a 2-simplex $\overline{\sigma}$ of Y satisfying

$$d_0^2(\overline{\sigma}) = r(g) \qquad d_1^2(\overline{\sigma}) = \overline{h} \qquad q(\overline{\sigma}) = s_0^1(q(\overline{h})),$$

as indicated in the diagram

$$\begin{tikzcd} & r(y) \arrow[dr, "r(g)"] & \\ \overline{x} \arrow[ur, "\overline{f}"] \arrow[rr, "\overline{h}"] & & r(z). \end{tikzcd}$$

Set $s = q(\overline{x})$, so that $\overline{f}$ can be regarded as an edge of the simplicial set Y_s. Invoking assumption (3), we conclude that there exists a locally r_s-cartesian edge $f : x \to y$ of X_s satisfying $r(f) = \overline{f}$. Since r is an inner fibration, we can choose a 2-simplex σ of X satisfying

$$d_0^2(\sigma) = g \qquad d_2^2(\sigma) = f \qquad r(\sigma) = \overline{\sigma},$$

as depicted in the diagram

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"] & & z. \end{tikzcd}$$

We will complete the proof by showing that h is locally r-cartesian. To prove this, we can replace X and Y by their pullbacks along the edge $\Delta^1 \xrightarrow{q(\overline{h})} S$, and thereby reduce to the case $S = \Delta^1$. In this case, the morphisms p and q are cartesian fibration (Remark 5.1.5.6), so that g is p-cartesian and $r(g)$ is q-cartesian (Remark 5.1.4.5). Applying Corollary 5.1.2.6, we conclude that g is r-cartesian. It follows from Remark 5.1.3.5 that f is locally r-cartesian, so that h is locally r-cartesian by virtue of Proposition 5.1.3.7. □

5.1.6 Fiberwise Equivalence

Let $\mathcal{D}$ be an ∞-category. Our primary goal in this section is to show that, when studying ∞-categories $\mathcal{C}$ equipped with a cartesian fibration $\mathcal{C} \to \mathcal{D}$, equivalence can be detected fiberwise. More precisely, we have the following result: 023L

Theorem 5.1.6.1. *Suppose we are given a commutative diagram of ∞-categories* 023M

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\ \mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}' \end{tikzcd}$$

where U is a cartesian fibration of ∞-categories, U' is an isofibration of ∞-categories, and $\overline{F}$ is an equivalence of ∞-categories. Then the functor F is an equivalence of ∞-categories if and only if it satisfies the following conditions:

(1) *For every object $D \in \mathcal{D}$ having image $D' = \overline{F}(D)$ in $\mathcal{D}'$, the induced functor*

$$F_D : \mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C} \to \{D'\} \times_{\mathcal{D}'} \mathcal{C}' = \mathcal{C}'_{D'}$$

is an equivalence of ∞-categories.

(2) *The functor F carries U-cartesian morphisms of $\mathcal{C}$ to U'-cartesian morphisms of $\mathcal{C}'$.*

Moreover, if these conditions are satisfied, then U' is also a cartesian fibration of ∞-categories.

We will give the proof of Theorem 5.1.6.1 at the end of this section. First, let us collect some of its consequences.

023N **Corollary 5.1.6.2.** *Suppose we are given a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}' .
\end{tikzcd}$$

Assume that U and U' are isofibrations of ∞-categories and that F and $\overline{F}$ are equivalences of ∞-categories. Then:

- *The functor U is a cartesian fibration if and only if U' is a cartesian fibration.*
- *The functor U is a cocartesian fibration if and only if U' is a cocartesian fibration.*

Proof. We will prove the first assertion; the second follows by a similar argument. It follows from Theorem 5.1.6.1 that if U is a cartesian fibration, then U' is also a cartesian fibration. To prove the converse, choose functors $G' : \mathcal{C}' \to \mathcal{C}$ and $\overline{G} : \mathcal{D}' \to \mathcal{D}$ which are homotopy inverse to the equivalences F and $\overline{F}$, respectively. We then have isomorphisms

$$U \circ G' \circ F \simeq U \simeq \overline{G} \circ \overline{F} \circ U = \overline{G} \circ U' \circ F$$

in the functor ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$. Since F is an equivalence of ∞-categories, it follows that there exists an isomorphism $\overline{\alpha} : U \circ G' \to \overline{G} \circ U'$ in the functor ∞-category $\operatorname{Fun}(\mathcal{C}', \mathcal{D})$. Using our assumption that U is an isofibration, we can lift $\overline{\alpha}$ to an isomorphism of functors $\alpha : G' \to G$ (Proposition 4.4.5.8). Applying Theorem 5.1.6.1 to the commutative diagram

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "G"] \arrow[d, "U'"] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{D}' \arrow[r, "\overline{G}"] & \mathcal{D},
\end{tikzcd}$$

we conclude that if U' is a cartesian fibration, then U is also a cartesian fibration. □

023P **Corollary 5.1.6.3.** *Suppose we are given a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}' .
\end{tikzcd}$$

Assume that U and U' are isofibrations of ∞-categories and that F and $\overline{F}$ are equivalences of ∞-categories. Then:

- *The functor U is a right fibration if and only if U' is a right fibration.*
- *The functor U is a left fibration if and only if U' is a left fibration.*

Proof. We will prove the first assertion; the second follows by a similar argument. Assume first that U' is a right fibration of ∞-categories. Then U' is a cartesian fibration (Proposition 5.1.4.14), so Corollary 5.1.6.2 implies that U is a cartesian fibration. To prove that U is a right fibration, it will suffice to show that for every object $D \in \mathcal{D}$, the fiber $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ is a Kan complex (Proposition 5.1.4.14). This follows from Remark 4.5.1.21, since the functor F induces an equivalence of ∞-categories $F_D : \mathcal{C}_D \to \mathcal{C}'_{\overline{F}(D)}$ (Corollary 4.5.2.32).

We now prove the reverse implication. Arguing as in the proof of Corollary 5.1.6.2, we can construct a commutative diagram

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "G"] \arrow[d, "U'"] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{D}' \arrow[r, "\overline{G}"] & \mathcal{D},
\end{tikzcd}$$

where G and $\overline{G}$ are homotopy inverses of the equivalences F and $\overline{F}$, respectively. It then follows from the preceding argument that if U is a right fibration of ∞-categories, then U' is also a right fibration of ∞-categories. □

Corollary 5.1.6.4. *Suppose we are given a commutative diagram of ∞-categories* 01VE

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}'.
\end{tikzcd}$$

where U and U' are right fibrations and the functor $\overline{F}$ is an equivalence of ∞-categories. Then F is an equivalence of ∞-categories if and only if, for every object $D \in \mathcal{D}$ having image $D' = \overline{F}(D) \in \mathcal{D}'$, the induced map of fibers $F_D : \mathcal{C}_D \to \mathcal{C}'_{D'}$ is a homotopy equivalence of Kan complexes.

The proof of Theorem 5.1.6.1 will require some preliminaries. Our first step is to show that if $U : \mathcal{C} \to \mathcal{D}$ is an isofibration of ∞-categories, then the collection of U-cartesian morphisms of $\mathcal{C}$ is invariant under categorical equivalence.

023Q **Lemma 5.1.6.5.** *Suppose we are given a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}',
\end{tikzcd}$$

where the functors U and U' are inner fibrations and the functors F and $\overline{F}$ are fully faithful. Let $g : Y \to Z$ be a morphism in $\mathcal{C}$. If $F(g)$ is a U'-cartesian morphism of $\mathcal{C}'$, then g is a U-cartesian morphism of $\mathcal{C}$.

Proof. By virtue of Proposition 5.1.2.1, it will suffice to show that for every object $X \in \mathcal{C}$, the diagram of Kan complexes

02R6
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, Y, Z) \times_{\mathrm{Hom}_{\mathcal{C}}(Y,Z)} \{g\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X, Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(U(X), U(Y), U(Z)) \times_{\mathrm{Hom}_{\mathcal{D}}(U(Y),U(Z))} \{U(g)\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}}(U(X), U(Z))
\end{tikzcd} \tag{5.6}$$

is a homotopy pullback square. Set $X' = F(X)$, $Y' = F(Y)$, $Z' = F(Z)$, and $g' = F(g)$. Since the functors F and $\overline{F}$ are fully faithful, (5.6) is homotopy equivalent to the diagram

02R7
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}'}(X', Y', Z') \times_{\mathrm{Hom}_{\mathcal{C}}(Y',Z')} \{g'\} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}'}(X', Z') \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}'}(U'(X'), U'(Y'), U'(Z')) \times_{\mathrm{Hom}_{\mathcal{D}'}(U'(Y),U'(Z))} \{U'(g')\} \arrow[r] & \mathrm{Hom}_{\mathcal{D}'}(U'(X'), U'(Z')).
\end{tikzcd} \tag{5.7}$$

Our assumption that g' is U'-cartesian guarantees that (5.7) is a homotopy pullback square of Kan complexes (Proposition 5.1.2.1), so that (5.6) is also a homotopy pullback square (Corollary 3.4.1.12). □

Proposition 5.1.6.6. *Suppose we are given a commutative diagram of ∞-categories* 023R

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}',
\end{tikzcd}$$

where the functors U and U' are isofibrations and the functors F and $\overline{F}$ are equivalences of ∞-categories. Let $g : Y \to Z$ be a morphism in $\mathcal{C}$. Then g is U-cartesian if and only if $F(g)$ is U'-cartesian.

Proof. It follows from Lemma 5.1.6.5 that if $F(g)$ is U'-cartesian, then g is U-cartesian. For the converse, suppose that g is U-cartesian. Arguing as in the proof of Corollary 5.1.6.2, we can construct a commutative diagram

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "G"] \arrow[d, "U'"] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{D}' \arrow[r, "\overline{G}"] & \mathcal{D},
\end{tikzcd}$$

where G and $\overline{G}$ are homotopy inverses of the equivalences F and $\overline{F}$, respectively. Then $G(F(g))$ is isomorphic to g as an object of the arrow ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Invoking Corollary 5.1.2.5, we conclude that $G(F(g))$ is U-cartesian, so that $F(g)$ is U'-cartesian by virtue of Lemma 5.1.6.5. □

Proposition 5.1.6.7. *Suppose we are given a commutative diagram of ∞-categories* 01VB

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "q"] & \mathcal{C}' \arrow[d, "q'"] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}'.
\end{tikzcd}$$

Assume that:

(1) *The functors q and q' are inner fibrations.*

(2) *The inner fibration q is a cartesian fibration and the functor F carries q-cartesian morphisms of $\mathcal{C}$ to locally q'-cartesian morphisms of $\mathcal{C}'$.*

(3) *The functor* $\overline{F} : \mathcal{D} \to \mathcal{D}'$ *is fully faithful.*

Then F *is fully faithful if and only if, for every object* $D \in \mathcal{D}$ *having image* $D' = \overline{F}(D) \in \mathcal{D}'$, *the induced map of fibers* $F_D : \mathcal{C}_D \to \mathcal{C}'_{D'}$ *is fully faithful.*

Proof. The "only if" direction follows from Proposition 4.6.2.8. For the converse, assume that each of the functors F_D is fully faithful; we will show that F is fully faithful. Let X and Z be objects of $\mathcal{C}$ having images $\overline{X}, \overline{Z} \in \mathcal{D}$; we wish to show that the upper horizontal map in the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, Z) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Z)) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(\overline{X}, \overline{Z}) \arrow[r] & \mathrm{Hom}_{\mathcal{D}'}(\overline{F}(\overline{X}), \overline{F}(\overline{Z}))
\end{tikzcd}$$

is a homotopy equivalence. Since q and q' are inner fibrations, the vertical maps are Kan fibrations (Proposition 4.6.1.21). Assumption (3) guarantees that the lower horizontal map is a homotopy equivalence. By virtue of Proposition 3.2.8.1, it will suffice to show that for every morphism $\overline{e} : \overline{X} \to \overline{Z}$ in $\mathcal{D}$, the induced map of fibers

$$\theta : \mathrm{Hom}_{\mathcal{C}}(X, Z)_{\overline{e}} \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Z))_{\overline{F}(\overline{e})}$$

is a homotopy equivalence.

Let $[\theta]$ denote the homotopy class of θ, regarded as a morphism in the homotopy category hKan. Since q is a cartesian fibration, there exists a q-cartesian morphism $g : Y \to Z$ of $\mathcal{C}$ satisfying $q(g) = \overline{e}$. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}_{\overline{X}}}(X, Y) \arrow[r] \arrow[d, "{[g]\circ}"] & \mathrm{Hom}_{\mathcal{C}'_{\overline{F}(\overline{X})}}(F(X), F(Y)) \arrow[d, "{[F(g)]\circ}"] \\
\mathrm{Hom}_{\mathcal{C}}(X, Z)_{\overline{e}} \arrow[r, "{[\theta]}"] & \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Z))_{\overline{F}(\overline{e})}
\end{tikzcd}$$

in hKan, where the vertical maps are given by the composition law of Notation 5.1.3.10. Assumption (2) guarantees that $F(g)$ is locally q'-cartesian, so that the vertical maps in this diagram are isomorphisms in hKan (Proposition 5.1.3.11). It will therefore suffice to show that the functor $F_{\overline{X}}$ induces a homotopy equivalence of mapping spaces $\mathrm{Hom}_{\mathcal{C}_{\overline{X}}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'_{\overline{F}(\overline{X})}}(F(X), F(Y))$, which follows from our assumption that $F_{\overline{X}}$ is fully faithful. □

Remark 5.1.6.8. In the situation of Proposition 5.1.6.7, we can replace (2) with the following *a priori* weaker assumption: 023S

($2'$) For every object $Z \in \mathcal{C}$ and every morphism $\overline{u} : \overline{Y} \to q(Z)$ in $\mathcal{D}$, there exists a q-cartesian $u : Y \to Z$ of $\mathcal{C}$ satisfying $q(u) = \overline{u}$ and for which $F(u)$ is locally q'-cartesian.

Assume that (2) is satisfied and let $v : X \to Z$ be any q-cartesian morphism in $\mathcal{C}$; we wish to show that $F(v)$ is locally q'-cartesian. To prove this, we can assume without loss of generality that $\mathcal{D} = \Delta^1 = \mathcal{D}'$ and that $\overline{F}$ is the identity map. Using ($2'$), we can choose another q-cartesian morphism $u : Y \to Z$ satisfying $q(u) = q(v)$ for which $F(u)$ is q'-cartesian. Applying Remark 5.1.3.8, we see that v can be obtained as a composition of u with an isomorphism in the ∞-category $\mathcal{C}$. Then $F(v)$ can be obtained as the composition of $F(u)$ with an isomorphism in the ∞-category $\mathcal{C}'$, and is therefore q-cartesian by virtue of Corollary 5.1.2.4 (and Proposition 5.1.1.8).

Proof of Theorem 5.1.6.1. Suppose we are given a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\
\mathcal{D} \arrow[r, "\overline{F}"] & \mathcal{D}'
\end{tikzcd}$$

where U is a cartesian fibration of ∞-categories, U' is an isofibration of ∞-categories, and $\overline{F}$ is an equivalence of ∞-categories. If F satisfies conditions (1) and (2) of Theorem 5.1.6.1, then it is fully faithful (Proposition 5.1.6.7) and essentially surjective (Remark 4.6.2.19), hence an equivalence of ∞-categories by virtue of Theorem 4.6.2.20. Conversely, if F is an equivalence of ∞-categories, then it satisfies conditions (1) and (2) by virtue of Corollary 4.5.2.32 and Proposition 5.1.6.7, respectively. To complete the proof, we must show that if these conditions are satisfied, then U' is also a cartesian fibration of ∞-categories.

Let Z' be an object of $\mathcal{C}'$ and let $\overline{g}' : \overline{Y}' \to U'(Z')$ be a morphism in $\mathcal{D}'$; we wish to show that $\overline{g}'$ can be lifted to a U'-cartesian morphism $Y' \to Z'$ in $\mathcal{C}'$. Since F is essentially surjective, we can choose an object $Z \in \mathcal{C}$ and an isomorphism $v : F(Z) \to Z'$ in the ∞-category $\mathcal{C}'$. Since $\overline{F}$ is essentially surjective, we can choose an object $\overline{Y} \in \mathcal{D}$ and an isomorphism $\overline{u} : \overline{F}(\overline{Y}) \to \overline{Y}'$ in the ∞-category $\mathcal{D}'$. Since $\overline{F}$ is fully faithful at the level of homotopy categories, we can choose a morphism $\overline{g} : \overline{Y} \to U(Z)$ in $\mathcal{D}$ for which the diagram

$$\begin{tikzcd}
\overline{F}(\overline{Y}) \arrow[r, "\overline{F}(\overline{g})"] \arrow[d, "\overline{u}"] & \overline{F}(U(Z)) \arrow[d, "U'(v)"] \\
\overline{Y}' \arrow[r, "\overline{g}'"] & \overline{Z}',
\end{tikzcd}$$

commutes in the homotopy category $\mathrm{h}\mathcal{D}'$, and can therefore be lifted to a commutative diagram $\overline{\sigma}$ in ∞-category $\mathcal{D}'$ (see Exercise 1.5.2.10). Using our assumption that U is a cartesian fibration, we can lift $\overline{g}$ to a U-cartesian morphism $g : Y \to Z$ of $\mathcal{C}$. Since U' is an isofibration, Corollary 4.4.5.9 guarantees that we can lift $\overline{\sigma}$ to a commutative diagram σ :

$$\begin{tikzcd}
F(Y) \arrow[r, "F(g)"] \arrow[d] & F(Z) \arrow[d, "v"] \\
Y' \arrow[r, "g'"] & Z'
\end{tikzcd}$$

in the ∞-category $\mathcal{C}'$, where the vertical maps are isomorphisms. To complete the proof, it will suffice to show that the morphism g' is U'-cartesian. This follows from Corollary 5.1.2.5, since the morphism $F(g)$ is U'-cartesian (Proposition 5.1.6.6). □

5.1.7 Equivalence of Inner Fibrations

0280 Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is an *equivalence of ∞-categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ and $F \circ G$ are isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$ as objects of the ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{D})$, respectively (Definition 4.5.1.10). In this section, we study a relative version of this notion, where $\mathcal{C}$ and $\mathcal{D}$ are simplicial sets equipped with inner fibrations $U : \mathcal{C} \to \mathcal{E}$ and $V : \mathcal{D} \to \mathcal{E}$ over the same base simplicial set $\mathcal{E}$ (which need not be an ∞-category). Recall that, in this case, the simplicial set

$$\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D}) = \{U\} \times_{\mathrm{Fun}(\mathcal{C}, \mathcal{E})} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is also an ∞-category (Corollary 4.1.4.8).

Definition 5.1.7.1. Suppose we are given a commutative diagram of simplicial sets 0281

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\
& \mathcal{E}, &
\end{tikzcd}$$

where U and V are inner fibrations. Let $G : \mathcal{D} \to \mathcal{C}$ be a morphism of simplicial sets. We say that G is a *homotopy inverse of* F *relative to* $\mathcal{E}$ if the following conditions are satisfied:

- The composition $U \circ G$ is equal to V: that is, the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "U"'] & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, "V"] \\
& \mathcal{E} &
\end{tikzcd}$$

  is commutative.

- The composite morphisms $G \circ F$and $F \circ G$ are isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{D}}$ as objects of the ∞-categories $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{C})$ and $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{D})$, respectively.

We say that F is an *equivalence of inner fibrations over* $\mathcal{E}$ if there exists a morphism of simplicial sets $G : \mathcal{D} \to \mathcal{C}$ which is a homotopy inverse of F relative to $\mathcal{E}$. We say that inner fibrations $U : \mathcal{C} \to \mathcal{E}$ and $V : \mathcal{D} \to \mathcal{E}$ are *equivalent* if there exists a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ which is an equivalence of inner fibrations over $\mathcal{E}$ (so that, in particular, we have $U = V \circ F$).

Example 5.1.7.2. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, so that the projection maps $U : \mathcal{C} \to \Delta^0$ and $V : \mathcal{D} \to \Delta^0$ are inner fibrations. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories if and only if it is an equivalence of inner fibrations over Δ^0. In particular, the inner fibrations U and V are equivalent (in the sense of Definition 5.1.7.1) if and only if the ∞-categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent (in the sense of Definition 4.5.1.10). 0282

Remark 5.1.7.3 (Two-out-of-Three)**.** Suppose we are given a commutative diagram of simplicial sets 0283

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[dr] & \mathcal{C}' \arrow[r, "F'"] \arrow[d] & \mathcal{C}'' \arrow[dl] \\
& \mathcal{E}, &
\end{tikzcd}$$

where the vertical maps are inner fibrations. If any two of the morphisms F, F', and $F' \circ F$ are equivalences of inner fibrations over $\mathcal{E}$, then so is the third. In particular, the collection of equivalences of inner fibrations over $\mathcal{E}$ is closed under composition.

0284 **Remark 5.1.7.4** (Functoriality)**.** Let $U : \mathcal{C} \to \mathcal{E}$ and $V : \mathcal{D} \to \mathcal{E}$ be inner fibrations of simplicial sets, and let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of inner fibrations over $\mathcal{E}$. For every morphism of simplicial sets $\mathcal{E}' \to \mathcal{E}$, the induced map $F' : \mathcal{E}' \times_{\mathcal{E}} \mathcal{C} \to \mathcal{E}' \times_{\mathcal{E}} \mathcal{D}$ is an equivalence of inner fibrations over $\mathcal{E}'$. In particular, for every object $E \in \mathcal{E}$, the induced map $F_E : \{E\} \times_{\mathcal{E}} \mathcal{C} \to \{E\} \times_{\mathcal{E}} \mathcal{D}$ is an equivalence of ∞-categories.

0285 **Proposition 5.1.7.5.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\
& \mathcal{E}. &
\end{tikzcd}$$

Then:

(1) *If U and V are inner fibrations and F is an equivalence of inner fibrations over $\mathcal{E}$, then F is a categorical equivalence of simplicial sets.*

(2) *If U and V are isofibrations and F is a categorical equivalence of simplicial sets, then it is an equivalence of inner fibrations over $\mathcal{E}$.*

Proof. We first prove (1). Assume that U and V are inner fibration and that F is an equivalence of inner fibrations over $\mathcal{E}$. We wish to show that F is a categorical equivalence of simplicial sets. Fix an ∞-category $\mathcal{K}$, and let $\theta_F : \pi_0(\mathrm{Fun}(\mathcal{D}, \mathcal{K})^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{K})^{\simeq})$ be the map given by precomposition with F. We wish to show that θ_F is a bijection. Let $G : \mathcal{D} \to \mathcal{C}$ be a homotopy inverse of F relative to $\mathcal{E}$, so that precomposition with G determines a map $\theta_G : \pi_0(\mathrm{Fun}(\mathcal{D}, \mathcal{K})^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{K})^{\simeq})$. We claim that θ_G is an inverse of θ_F. We will show that θ_G is a left inverse of θ_F; a similar argument will show that θ_G is a right inverse of θ_F. Fix a morphism $H : \mathcal{C} \to \mathcal{K}$; we wish to show that H is isomorphic to $H \circ G \circ F$ as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{K})$. This is clear, since postcomposition with H determines a functor of ∞-categories $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{K})$.

We now prove (2). Let Q be a contractible Kan complex containing a pair of distinct vertices x and y, and form a pushout diagram of simplicial sets

$$\begin{tikzcd}
\{x\} \times \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d] \\
Q \times \mathcal{C} \arrow[r] & \mathcal{M}.
\end{tikzcd}$$

Since the vertical maps are monomorphisms, this diagram is also a categorical pushout square (Proposition 4.5.4.11). In particular, if F is a categorical equivalence, then the map $Q \times \mathcal{C} \to \mathcal{M}$ is also a categorical equivalence (Proposition 4.5.4.10). Since Q is contractible, the inclusion $\{y\} \times \mathcal{C} \hookrightarrow Q \times \mathcal{C}$ is a categorical equivalence (Remark 4.5.3.7), so the inclusion $\{y\} \times \mathcal{C} \hookrightarrow \mathcal{M}$ is also a categorical equivalence. If U is an isofibration, then the lifting problem

$$\begin{tikzcd}
\{y\} \times \mathcal{C} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\mathcal{M} \arrow[r] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd}$$

admits a solution, which we can identify with a pair of morphisms $G : \mathcal{D} \to \mathcal{C}$ and $u : Q \to \mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{C})$ satisfying $u(x) = G \circ F$ and $u(y) = \mathrm{id}_{\mathcal{C}}$. It follows that $G \circ F$ is isomorphic to $\mathrm{id}_{\mathcal{C}}$ as an object of the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{C})$.

Repeating the above argument with F replaced by G, we conclude that there exists a morphism $H : \mathcal{C} \to \mathcal{D}$ in $(\mathrm{Set}_{\Delta})_{/S}$ such that $H \circ G$ is isomorphic to $\mathrm{id}_{\mathcal{D}}$ as an object of the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{D})$. Then F and H are both isomorphic to $H \circ G \circ F$ as objects of the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$, and are therefore isomorphic to each other. We may therefore assume without loss of generality that $H = F$, so that G is a homotopy inverse of F relative to $\mathcal{E}$. In particular, F is an equivalence of inner fibrations over $\mathcal{E}$. □

Warning 5.1.7.6. Assertion (2) of Proposition 5.1.7.5 need not be true if U and V are only assumed to be inner fibrations. For example, let $\mathcal{E}$ be an ∞-category and let $\mathcal{E}' \subseteq \mathcal{E}$ be a full subcategory for which the inclusion map $\iota : \mathcal{E}' \hookrightarrow \mathcal{E}$ is an equivalence. Then we have a commutative diagram of ∞-categories 0286

$$\begin{tikzcd}
\mathcal{E}' \arrow[rr, "\iota"] \arrow[dr, "\iota"'] & & \mathcal{E} \arrow[dl, "\mathrm{id}"] \\
& \mathcal{E}, &
\end{tikzcd}$$

where the vertical maps are inner fibrations. However, ι is not an equivalence of inner fibrations over $\mathcal{E}$ unless $\mathcal{E}' = \mathcal{E}$.

02LQ **Example 5.1.7.7.** Let $\mathcal{C}$ be an ∞-category and let $F : K \to \mathcal{C}$ be a diagram. It follows from Theorem 4.6.4.17 and Proposition 5.1.7.5 that the slice and coslice diagonal morphisms

$$\delta_{/F} : \mathcal{C}_{/F} \hookrightarrow \mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{F\} \qquad \delta_{F/} : \mathcal{C}_{F/} \hookrightarrow \{F\} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \mathcal{C}$$

are equivalences of right and left fibrations over $\mathcal{C}$, respectively. In particular, for every morphism of simplicial sets $\mathcal{D} \to \mathcal{C}$, the induced maps

$$\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/F} \hookrightarrow \mathcal{D} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{F\} \qquad \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{F/} \hookrightarrow \{F\} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \mathcal{D}$$

are equivalences of inner fibrations over $\mathcal{D}$ (Remark 5.1.7.4); in particular, they are categorical equivalences of simplicial sets (Proposition 5.1.7.5).

0287 **Corollary 5.1.7.8.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\ & \mathcal{E}, & \end{tikzcd}$$

where U and V are inner fibrations and $\mathcal{E} = \Delta^n$ is a standard simplex. Then F is an equivalence of inner fibrations over $\mathcal{E}$ if and only if it an equivalence of ∞-categories.

Proof. Our assumption that $\mathcal{E} = \Delta^n$ is a standard simplex guarantees that the inner fibrations U and V are isofibrations (Example 4.4.1.6), so the desired result follows from Proposition 5.1.7.5. □

0288 **Proposition 5.1.7.9.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\ & \mathcal{E}, & \end{tikzcd}$$

where U and V are inner fibrations. The following conditions are equivalent:

(1) *For every morphism of simplicial sets $B \to \mathcal{E}$, postcomposition with F induces a homotopy equivalence of Kan complexes* $\operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{C})^{\simeq} \to \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{D})^{\simeq}$.

(2) *For every morphism of simplicial sets $B \to \mathcal{E}$, postcomposition with F induces an equivalence of ∞-categories* $\operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{C}) \to \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{D})$.

(3) *The morphism F is an equivalence of inner fibrations over $\mathcal{E}$.*

(4) *For every simplex $\sigma : \Delta^n \to \mathcal{E}$, the induced map $F_\sigma : \Delta^n \times_{\mathcal{E}} \mathcal{C} \to \Delta^n \times_{\mathcal{E}} \mathcal{D}$ is an equivalence of ∞-categories.*

Proof. We first show that (1) implies (2). Assume that (1) is satisfied and let $B \to \mathcal{E}$ be a morphism of simplicial sets; we wish to show that the induced map $\operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{C}) \to \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{D})$ is an equivalence of ∞-categories. By virtue of Theorem 4.5.7.1, it will suffice to show that for every simplicial set B', the induced map

$$\operatorname{Fun}(B', \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{C}))^{\simeq} \to \operatorname{Fun}(B', \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{D}))^{\simeq}$$

is a homotopy equivalence of Kan complexes. This follows by applying (1) to the composite map $B' \times B \to B \to \mathcal{E}$.

We now prove that (2) implies (3). Assume that condition (2) is satisfied. Setting $B = \mathcal{D}$, we deduce that composition with F induces an equivalence of ∞-categories $\operatorname{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{C}) \to \operatorname{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{D})$. In particular, there exists a morphism $G : \mathcal{D} \to \mathcal{C}$ in $(\mathrm{Set}_\Delta)_{/\mathcal{E}}$ such that $F \circ G$ is isomorphic to $\mathrm{id}_{\mathcal{D}}$ as an object of the ∞-category $\operatorname{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{D})$. It follows that $F \circ G \circ F$ is isomorphic to F as an object of the ∞-category $\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$. Applying condition (2) in the case $B = \mathcal{C}$, we see that postcomposition with F induces an equivalence of ∞-categories $\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{C}) \to \operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$, so that $G \circ F$ is isomorphic to $\mathrm{id}_{\mathcal{C}}$ as an object of $\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{C})$. It follows that G is a homotopy inverse of F relative to $\mathcal{E}$. In particular, F is an equivalence of inner fibrations over $\mathcal{E}$.

The implication (3) ⇒ (4) follows by combining Remark 5.1.7.4 with Corollary 5.1.7.8. We will complete the proof by showing that (4) implies (1). Assume that condition (4) is satisfied, and let B be a simplicial set equipped with a morphism $B \to \mathcal{E}$. We wish to show that composition with F induces a homotopy equivalence of Kan complexes $\theta_B : \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{C})^{\simeq} \to \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{D})^{\simeq}$. Assume first that the simplicial set B has dimension $\leq n$, for some integer $n \geq -1$. Our proof proceeds by induction on n. If $n = -1$, then B is empty and there is nothing to prove. We may therefore assume without loss of generality that $n \geq 0$. Let A be the $(n-1)$-skeleton of B. Our inductive hypothesis guarantees that θ_A is a homotopy equivalence. By virtue of Proposition 3.2.8.1, it will suffice to verify the following:

($*$) The restriction maps

$$\operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{C}) \to \operatorname{Fun}_{/\mathcal{E}}(A, \mathcal{C}) \qquad \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{D}) \to \operatorname{Fun}_{/\mathcal{E}}(A, \mathcal{D})$$

are isofibrations of ∞-categories, and therefore induce Kan fibrations

$$\operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{C})^{\simeq} \to \operatorname{Fun}_{/\mathcal{E}}(A, \mathcal{C})^{\simeq} \qquad \operatorname{Fun}_{/\mathcal{E}}(B, \mathcal{D})^{\simeq} \to \operatorname{Fun}_{/\mathcal{E}}(A, \mathcal{D})^{\simeq};$$

see Proposition 4.4.3.7.

$(*')$ For every object $T \in \mathrm{Fun}_{/\mathcal{E}}(A, \mathcal{C})$, the induced map of fibers

$$\{T\} \times_{\mathrm{Fun}_{/\mathcal{E}}(A,\mathcal{C})} \mathrm{Fun}_{/\mathcal{E}}(B, \mathcal{C}) \to \{F \circ T\} \times_{\mathrm{Fun}_{/\mathcal{E}}(A,\mathcal{D})} \mathrm{Fun}_{/\mathcal{E}}(B, \mathcal{D})$$

is an equivalence of ∞-categories, and therefore induces a homotopy equivalence of Kan complexes

$$\{T\} \times_{\mathrm{Fun}_{/\mathcal{E}}(A,\mathcal{C})^{\simeq}} \mathrm{Fun}_{/\mathcal{E}}(B, \mathcal{C})^{\simeq} \to \{F \circ T\} \times_{\mathrm{Fun}_{/\mathcal{E}}(A,\mathcal{D})^{\simeq}} \mathrm{Fun}_{/\mathcal{E}}(B, \mathcal{D})^{\simeq}$$

(see Remark 4.5.1.19).

Let J denote the set of all nondegenerate n-simplices of B. Proposition 1.1.4.12 supplies a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{\sigma \in J} \partial \Delta^n \arrow[r] \arrow[d] & \coprod_{\sigma \in J} \Delta^n \arrow[d] \\
A \arrow[r] & B.
\end{tikzcd}$$

Consequently, to verify $(*)$ and $(*')$, we can assume without loss of generality that $B = \Delta^n$ is a standard simplex and that $A = \partial \Delta^n$ is its boundary. Replacing $\mathcal{C}$ and $\mathcal{D}$ by the fiber products $\Delta^n \times_{\mathcal{E}} \mathcal{C}$ and $\Delta^n \times_{\mathcal{E}} \mathcal{D}$, we can reduce further to the case where $\mathcal{E} = \Delta^n$ is a standard simplex. Applying Example 4.4.1.6, we deduce that U and V are isofibrations, so that assertion $(*)$ follows from Proposition 4.4.5.1. Invoking assumption (4), we deduce that F is an equivalence of ∞-categories, and therefore induces equivalences

$$\mathrm{Fun}(A, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{D}) \qquad \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(B, \mathcal{D}).$$

Assertion $(*')$ now follows from Corollary 4.5.2.32.

We now treat the case where B is a general simplicial set. For each $n \geq 0$, let $\mathrm{sk}_n(B)$ denote the n-skeleton of B (Construction 1.1.4.1). Using $(*)$ and Corollary 4.5.6.22, we see that θ_B can be realized as the inverse limit of a tower

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Fun}_{/S}(\mathrm{sk}_2(B), X)^{\simeq} \arrow[r] \arrow[d, "\theta_{\mathrm{sk}_2(B)}"] & \mathrm{Fun}_{/S}(\mathrm{sk}_1(B), X)^{\simeq} \arrow[r] \arrow[d, "\theta_{\mathrm{sk}_1(B)}"] & \mathrm{Fun}_{/S}(\mathrm{sk}_0(B), X)^{\simeq} \arrow[d, "\theta_{\mathrm{sk}_0(B)}"] \\
\cdots \arrow[r] & \mathrm{Fun}_{/S}(\mathrm{sk}_2(B), X')^{\simeq} \arrow[r] & \mathrm{Fun}_{/S}(\mathrm{sk}_1(B), X')^{\simeq} \arrow[r] & \mathrm{Fun}_{/S}(\mathrm{sk}_0(B), X')^{\simeq},
\end{tikzcd}$$

where each of the transition maps is a Kan fibration. The preceding arguments show that each of the vertical maps $\theta_{\mathrm{sk}_n(B)}$ is a homotopy equivalence of Kan complexes. Invoking Example 4.5.6.18, we deduce that θ_B is a homotopy equivalence of Kan complexes. $\square$

Corollary 5.1.7.10. *Suppose we are given a commutative diagram of ∞-categories* 05J4

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\
& \mathcal{E}, &
\end{tikzcd}$$

where U and V are inner fibrations. Then F is an equivalence of inner fibrations over $\mathcal{E}$ if and only if it satisfies the following pair of conditions:

(1) *The functor F is fully faithful.*

(2) *For every object $E \in \mathcal{E}$, the functor $F_E : \mathcal{C}_E \to \mathcal{D}_E$ is an equivalence of ∞-categories.*

Proof. If F is an equivalence of inner fibrations over $\mathcal{E}$, then it is an equivalence of ∞-categories (Proposition 5.1.7.5) and each of the functors F_E has the same property (Remark 5.1.7.4); this proves the necessity of conditions (1) and (2). Conversely, suppose that F satisfies conditions (1) and (2); we will show that F is an equivalence of inner fibrations over $\mathcal{E}$. By virtue of Proposition 5.1.7.9, it will suffice to show that for every morphism $\sigma : \Delta^n \to \mathcal{E}$, the induced map $F_\sigma : \Delta^n \times_{\mathcal{E}} \mathcal{C} \to \Delta^n \times_{\mathcal{E}} \mathcal{D}$ is an equivalence of ∞-categories. It follows from assumption (1) and Variant 4.8.6.19 that the functor F_σ is fully faithful, and from assumption (2) that the functor F_σ is essentially surjective. The desired result now follows from Theorem 4.6.2.20. □

Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then F determines a pullback functor $F^* : (\mathrm{Set}_\Delta)_{/\mathcal{D}} \to (\mathrm{Set}_\Delta)_{/\mathcal{C}}$, given on objects by the formula $F^*(\widetilde{\mathcal{D}}) = \mathcal{C} \times_{\mathcal{D}} \widetilde{\mathcal{D}}$.

Proposition 5.1.7.11. *Let $V : \widetilde{\mathcal{D}} \to \mathcal{D}$ be an isofibration of ∞-categories, let $\mathcal{C}$ be a simplicial set, and let $F, G : \mathcal{C} \to \mathcal{D}$ be morphisms of simplicial sets which are isomorphic when viewed as objects of the ∞-category* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. *Then the isofibrations $F^*(\widetilde{\mathcal{D}}) \to \mathcal{C}$ and $G^*(\widetilde{\mathcal{D}}) \to \mathcal{C}$ are equivalent (in the sense of Definition 5.1.7.1).* 028E

Warning 5.1.7.12. The conclusion of Proposition 5.1.7.11 does not necessarily hold if $V : \widetilde{\mathcal{D}} \to \mathcal{D}$ is assumed only to be an inner fibration of simplicial sets. See Warning 5.1.7.6. 028F

Proof of Proposition 5.1.7.11. Since F and G are isomorphic as objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, there exists a contractible Kan complex X containing vertices f and g and a functor $H : X \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ satisfying $H(f) = F$ and $H(g) = G$. Let us identify H with a morphism of simplicial sets $X \times \mathcal{C} \to \mathcal{D}$, and let $\widetilde{\mathcal{C}}$ denote the fiber product $(X \times \mathcal{C}) \times_{\mathcal{D}} \widetilde{\mathcal{D}}$. We will show that the inclusion maps

$$F^*(\widetilde{\mathcal{D}}) = \{f\} \times_X \widetilde{\mathcal{C}} \hookrightarrow \widetilde{\mathcal{C}} \hookleftarrow \{g\} \times_X \widetilde{\mathcal{C}} = G^*(\widetilde{\mathcal{D}})$$

are equivalences of inner fibrations over $\mathcal{C}$. To prove this, we may assume without loss of generality that $\mathcal{C} = \Delta^n$ is a standard simplex (Proposition 5.1.7.9); in this case, we wish to show that both inclusion maps are equivalences of ∞-categories (Corollary 5.1.7.8). This follows by applying Corollary 4.5.2.29 to the diagram of pullback squares

$$\begin{tikzcd}
F^*(\widetilde{\mathcal{D}}) \arrow[r] \arrow[d] & \widetilde{\mathcal{C}} \arrow[d] & G^*(\widetilde{\mathcal{D}}) \arrow[l] \arrow[d] \\
\{f\} \times \mathcal{C} \arrow[r] & X \times \mathcal{C} & \{g\} \times \mathcal{C}; \arrow[l]
\end{tikzcd}$$

here the vertical maps are isofibrations (since they are pullbacks of V) and the lower horizontal maps are equivalences of ∞-categories (since X is a contractible Kan complex). □

We now study properties of inner fibrations that are invariant under equivalence.

0289 **Lemma 5.1.7.13.** *Let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of simplicial sets and $F : \mathcal{C} \hookrightarrow \mathcal{D}$ be a monomorphism of simplicial sets. The following conditions are equivalent:*

(1) *The restriction $(U \circ F) : \mathcal{C} \to \mathcal{E}$ is an inner fibration and F is an equivalence of inner fibrations over $\mathcal{E}$.*

(2) *There exists a morphism $G : \mathcal{D} \to \mathcal{C}$ in $(\mathrm{Set}_\Delta)_{/\mathcal{E}}$ satisfying $G \circ F = \mathrm{id}_\mathcal{C}$ and an isomorphism $u : \mathrm{id}_\mathcal{D} \to F \circ G$ in the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{D})$ whose image in $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$ is the identity morphism $\mathrm{id}_F : F \to F \circ G \circ F = F$.*

Proof. We first show that (2) implies (1). Suppose that there exists a morphism $G : \mathcal{D} \to \mathcal{C}$ satisfying $G \circ F = \mathrm{id}_\mathcal{C}$. Then F and G exhibit $\mathcal{C}$ as a retract of $\mathcal{D}$ in the category $(\mathrm{Set}_\Delta)_{/\mathcal{E}}$. Since $U : \mathcal{D} \to \mathcal{E}$ is an isofibration, it follows that $(U \circ F) : \mathcal{C} \to \mathcal{E}$ is an isofibration (Remark 4.5.5.10). In particular, $U \circ F$ is an inner fibration (Remark 4.5.5.7). If there exists an isomorphism $u : \mathrm{id}_\mathcal{D} \to F \circ G$ in the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{D})$, then G is a homotopy inverse of F relative to $\mathcal{E}$, so that F is an equivalence of inner fibrations over $\mathcal{E}$.

We now show that (1) implies (2). Assume that $U \circ F$ is an inner fibration and that F is an equivalence of inner fibrations over $\mathcal{E}$. Let $G' : \mathcal{D} \to \mathcal{C}$ be a homotopy inverse of F relative to $\mathcal{E}$, so that there exists an isomorphism $e : \mathrm{id}_\mathcal{C} \to G' \circ F$ in the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{C})$. Applying Proposition 4.4.5.8, we can lift e to an isomorphism $\widetilde{e} : G \to G'$ in the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{C})$, where $G : \mathcal{D} \to \mathcal{C}$ satisfies $G \circ F = \mathrm{id}_\mathcal{C}$. Note that F is a categorical equivalence of simplicial sets (Proposition 5.1.7.5), and therefore induces a categorical equivalence

$$(\Delta^1 \times \mathcal{C}) \coprod_{(\partial \Delta^1 \times \mathcal{C})} (\partial \Delta^1 \times \mathcal{D}) \hookrightarrow \Delta^1 \times \mathcal{D}.$$

Since U is an isofibration, every lifting problem

$$\begin{tikzcd}
(\Delta^1 \times \mathcal{C}) \coprod_{(\partial \Delta^1 \times \mathcal{C})} (\partial \Delta^1 \times \mathcal{D}) \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\Delta^1 \times \mathcal{D} \arrow[r] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd}$$

admits a solution. In particular, there exists a morphism $u : \mathrm{id}_{\mathcal{D}} \to F \circ G$ in the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{D})$ whose image in $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$ is the identity map id_F. We will complete the proof by showing that u is an isomorphism in the ∞-category $\mathrm{Fun}_{/\mathcal{E}}(\mathcal{D}, \mathcal{D})$. Using the criterion of Proposition 4.4.4.9, we are reduced to checking that, for each vertex $D \in \mathcal{D}$ having image $E = U(D) \in \mathcal{E}$, the induced map $u_D : D \to (F \circ G)(D)$ is an isomorphism in the ∞-category $\mathcal{D}_E = \{E\} \times_{\mathcal{E}} \mathcal{D}$. This is clear, since D is isomorphic (in the ∞-category $\mathcal{D}_E$) to an object of the form $F(C)$ for $C \in \mathcal{C}_E$, and the morphism $u_{F(C)}$ is equal to the identity $\mathrm{id}_{F(C)}$. □

Proposition 5.1.7.14. *Let $U : \mathcal{C} \to \mathcal{E}$ and $V : \mathcal{D} \to \mathcal{E}$ be inner fibrations of simplicial sets which are equivalent to one another. Then:* 028A

(1) *The morphism U is an isofibration if and only if V is an isofibration.*

(2) *The morphism U is a cartesian fibration if and only if V is a cartesian fibration.*

(3) *The morphism U is a right fibration if and only if V is a right fibration.*

(4) *The morphism U is a cocartesian fibration if and only if V is a cocartesian fibration.*

(5) *The morphism U is a left fibration if and only if V is a left fibration.*

(6) *The morphism U is a Kan fibration if and only if V is a Kan fibration.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of inner fibrations over $\mathcal{E}$. We first prove (1). Assume that V is an isofibration; we will show that U is also an isofibration. Choose a monomorphism of simplicial sets $\mathcal{C} \hookrightarrow \mathcal{Q}$, where $\mathcal{Q}$ is a contractible Kan complex (Exercise 3.1.7.11). Replacing $\mathcal{D}$ by the product $\mathcal{D} \times \mathcal{Q}$, we can assume that F is a monomorphism of simplicial sets. In this case, Lemma 5.1.7.13 guarantees that F exhibits $\mathcal{C}$ as a retract of $\mathcal{D}$ in the category $(\mathrm{Set}_{\Delta})_{/\mathcal{E}}$, so that U is an isofibration by virtue of Remark 4.5.5.10.

To prove (2), we may assume without loss of generality that $\mathcal{E} = \Delta^n$ is a standard simplex (Proposition 5.1.4.7). In this case, U and V are isofibrations (Example 4.4.1.6) and F is an equivalence of ∞-categories (Corollary 5.1.7.8). It follows from Corollary 5.1.6.2 that U is a cartesian fibration if and only if V is a cartesian fibration.

To prove (3), suppose that U is a right fibration; we will show that V is a right fibration. It follows from (2) that V is a cartesian fibration. It will therefore suffice to show that, for each vertex $E \in \mathcal{E}$, the ∞-category $\{E\} \times_{\mathcal{E}} \mathcal{D}$ is a Kan complex (Proposition 5.1.4.14). By virtue of Remark 5.1.7.4, the morphism F induces an equivalence of ∞-categories $F_E : \{E\} \times_{\mathcal{E}} \mathcal{C} \to \{E\} \times_{\mathcal{E}} \mathcal{D}$. It will therefore suffice to show that $\{E\} \times_{\mathcal{E}} \mathcal{C}$ is a Kan complex (Remark 4.5.1.21), which follows from our assumption that U is a right fibration.

Assertions (4) and (5) follow by similar arguments. Assertion (6) follows by combining (3) and (5) (see Example 4.2.1.5). □

028B **Proposition 5.1.7.15.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\
& \mathcal{E}, &
\end{tikzcd}$$

where U and V are cartesian fibrations. Then F is an equivalence of inner fibrations over $\mathcal{E}$ if and only if the following conditions are satisfied:

(1) *For every vertex $E \in \mathcal{E}$, the induced map $F_E : \{E\} \times_{\mathcal{E}} \mathcal{C} \to \{E\} \times_{\mathcal{E}} \mathcal{D}$ is an equivalence of ∞-categories.*

(2) *The morphism F carries U-cartesian edges of $\mathcal{C}$ to V-cartesian edges of $\mathcal{D}$.*

Proof. By virtue of Proposition 5.1.7.9, we may assume without loss of generality that $\mathcal{E} = \Delta^n$ is a standard simplex, so that F is an equivalence of inner fibrations over $\mathcal{E}$ if and only if it is an equivalence of ∞-categories (Corollary 5.1.7.8). Since U and V are isofibrations (Example 4.4.1.6), the desired result follows from Theorem 5.1.6.1. □

028C **Corollary 5.1.7.16.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\
& \mathcal{E}, &
\end{tikzcd}$$

where U and V are right fibrations. Then F is an equivalence of inner fibrations if and only if, for every vertex $E \in \mathcal{E}$, the induced map $F_E : \{E\} \times_{\mathcal{E}} \mathcal{C} \to \{E\} \times_{\mathcal{E}} \mathcal{D}$ is a homotopy equivalence of Kan complexes.

Proof. Combine Proposition 5.1.7.15 with Proposition 5.1.4.14. □

5.2 Covariant Transport

Let $X \to S$ be a covering map of topological spaces. For every point $s \in S$, the fiber $X_s = \{s\} \times_S X$ is equipped with an action of the fundamental group $\pi_1(S, s)$. More generally, the construction $s \mapsto X_s$ determines a functor from the fundamental groupoid $\pi_{\leq 1}(S)$ to the category of sets, which will refer to as the *monodromy representation* of the covering map $X \to S$ (see Example 5.2.0.5 below). 019A

It will be convenient to place monodromy in a more general context. Recall that if $X \to S$ is a covering map of topological spaces, then the induced map $\mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(S)$ is a covering map of simplicial sets (Proposition 3.1.4.9). In particular, it is a left covering map of simplicial sets (Definition 4.2.3.8).

Construction 5.2.0.1 (Covariant Transport for Left Covering Maps)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a left covering map of simplicial sets. For each vertex $C \in \mathcal{C}$, the fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is a discrete simplicial set, which we will identify with its underlying set of vertices (Remark 4.2.3.17). If $\widetilde{C}$ is a vertex of $\mathcal{E}_C$ and $f : C \to D$ is an edge of $\mathcal{C}$, then our assumption that U is a left covering map guarantees that there is a unique edge $\widetilde{f} : \widetilde{C} \to f_!(\widetilde{C})$ of $\mathcal{E}$ satisfying $U(\widetilde{f}) = f$. The construction $\widetilde{C} \mapsto f_!(\widetilde{C})$ then determines a function $f_! : \mathcal{E}_C \to \mathcal{E}_D$, which we will refer to as *covariant transport along f*. 027S

Example 5.2.0.2. In the situation of Construction 5.2.0.1, suppose that $e = \mathrm{id}_C$ is a degenerate edge of $\mathcal{C}$. Then the covariant transport function $e_! : \mathcal{E}_C \to \mathcal{E}_C$ is the identity function. 0351

Proposition 5.2.0.3. *Let $U : \mathcal{E} \to \mathcal{C}$ be a left covering map of simplicial sets. Then there is a unique functor* 0352

$$\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{Set}$$

with the following properties:

- *For each vertex $C \in \mathcal{C}$, we have $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}(C) = \mathcal{E}_C$.*
- *Let $f : C \to D$ be an edge of $\mathcal{C}$, and let $[f]$ denote the corresponding morphism in the homotopy category $\mathrm{h}\mathcal{C}$. Then $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}([f])$ is the covariant transport function $f_! : \mathcal{E}_C \to \mathcal{E}_D$ of Construction 5.2.0.1.*

Proof. By virtue of Example 5.2.0.2 (and the proof of Proposition 1.3.6.4), it will suffice to show that if σ is a 2-simplex σ of $\mathcal{C}$ as indicated in the diagram

$$\begin{tikzcd}
& D \arrow[dr, "g"] & \\
C \arrow[ur, "f"] \arrow[rr, "h"] & & E,
\end{tikzcd}$$

then the covariant transport function $h_! : \mathcal{E}_C \to \mathcal{E}_E$ is equal to the composition $g_! \circ f_!$. Fix a vertex $X \in \mathcal{E}_C$. By construction, there is an edge $\widetilde{f} : X \to f_!(X)$ satisfying $U(\widetilde{f}) = f$ and an edge $\widetilde{h} : X \to h_!(X)$ satisfying $U(\widetilde{h}) = h$. Since U is a left covering map, we can lift σ (uniquely) to a 2-simplex of $\mathcal{E}$ with boundary indicated in the diagram

$$\begin{tikzcd}
 & f_!(X) \arrow[dr, "\widetilde{g}"] & \\
X \arrow[ur, "\widetilde{f}"] \arrow[rr, "\widetilde{h}"] & & h_!(X).
\end{tikzcd}$$

The edge $\widetilde{g}$ then satisfies $U(\widetilde{g}) = g$, and therefore witnesses the identity $g_!(f_!(X)) = h_!(X)$. □

0353 **Definition 5.2.0.4.** Let $U : \mathcal{E} \to \mathcal{C}$ be a left covering morphism of simplicial sets and let $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{Set}$ be the functor of Proposition 5.2.0.3. We will refer to $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ as the *homotopy transport representation* of U.

027T **Example 5.2.0.5** (The Monodromy Representation)**.** Let $f : X \to S$ be a covering map of topological spaces. Applying Proposition 5.2.0.3 to the induced map $\mathrm{Sing}_\bullet(X) \to \mathrm{Sing}_\bullet(S)$, we obtain a functor from the fundamental groupoid $\pi_{\leq 1}(S)$ to the category of sets, which we will denote by $\mathrm{hTr}_{X/S} : \pi_{\leq 1}(S) \to \mathrm{Set}$ and refer to as the *monodromy representation of* f. Concretely, it is given on objects by the formula $\mathrm{hTr}_{X/S}(s) = \{s\} \times_S X$.

0354 **Example 5.2.0.6.** Let Set_* denote the category of pointed sets, so that the forgetful functor $\mathrm{Set}_* \to \mathrm{Set}$ induces a left covering morphism of simplicial sets $\mathrm{N}_\bullet(\mathrm{Set}_*) \to \mathrm{N}_\bullet(\mathrm{Set})$ (Example 4.2.3.3). Then the homotopy transport functor $\mathrm{hTr}_{\mathrm{N}_\bullet(\mathrm{Set}_*)/\mathrm{N}_\bullet(\mathrm{Set})}$ is isomorphic to the identity functor $\mathrm{id}_{\mathrm{Set}} : \mathrm{Set} \to \mathrm{Set}$.

Our first goal in this section is to generalize the definition of the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ to the case where $U : \mathcal{E} \to \mathcal{C}$ is a cocartesian fibration of simplicial sets. In §5.2.2, we associate to each edge $f : C \to D$ of the simplicial set $\mathcal{C}$ a functor of ∞-categories $f_! : \mathcal{E}_C \to \mathcal{E}_D$, which we refer to as the *covariant transport functor* associated to f (Definition 5.2.2.4). Unlike the covariant transport *function* of Construction 5.2.0.1, the functor $f_!$ is not uniquely determined: it is well-defined only up to isomorphism (Proposition 5.2.2.8). To construct it (and to establish its uniqueness up to isomorphism), we will exploit the fact that postcomposition with U induces a cocartesian fibration $\mathrm{Fun}(\mathcal{E}_C, \mathcal{E}) \to \mathrm{Fun}(\mathcal{E}_C, \mathcal{C})$, which we prove in §5.2.1 (see Theorem 5.2.1.1).

In §5.2.5, we study the behavior of covariant transport with respect to composition. Suppose we are given a 2-simplex σ of the simplicial set $\mathcal{C}$, which we view as a commutative

diagram

$$\begin{tikzcd}
 & D \arrow[dr, "g"] & \\
C \arrow[ur, "f"] \arrow[rr, "h"] & & E.
\end{tikzcd}$$

In this case, we will show that there is an isomorphism of covariant transport functors $h_! \simeq g_! \circ f_!$ (Proposition 5.2.5.1). As a consequence, we can regard the construction $C \mapsto \mathcal{E}_C$ as a functor from the homotopy category $\mathrm{h}\mathcal{C}$ to the homotopy category hQCat of Construction 4.5.1.1, which we denote by $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ and refer to as the *homotopy transport representation* of the cocartesian fibration U (Construction 5.2.5.2).

The remainder of this section is devoted to the following:

Question 5.2.0.7. Let $\mathcal{C}$ be a simplicial set and let $\mathscr{F} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ be a functor. Can $\mathscr{F}$ be obtained as the homotopy transport representation of a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$? 023V

The answer to Question 5.2.0.7 is "no" in general. However, there are two important special cases where the answer is "yes":

- In §5.2.7, we show that any set-valued functor $\mathrm{h}\mathcal{C} \to \mathrm{Set}$ can be realized as the homotopy transport representation of a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$. Moreover, we can arrange that U is a left covering map. In this case, the simplicial set $\mathcal{E}$ is uniquely determined up to isomorphism (Corollary 5.2.7.3) and can be described explicitly using the classical *category of elements* construction, which we review in §5.2.6.

- Every functor of ∞-categories $\mathcal{E}_0 \to \mathcal{E}_1$ can be realized as the covariant transport functor associated to a cocartesian fibration $U : \mathcal{E} \to \Delta^1$: that is, Question 5.2.0.7 has an affirmative answer in the case $\mathcal{C} = \Delta^1$ (see Proposition 5.2.3.15). We prove this in §5.2.3 using an explicit construction which generalizes the join operation on simplicial sets (Construction 5.2.3.1). In §5.2.4, we show that the ∞-category $\mathcal{E}$ is determined uniquely up to equivalence (see Remark 5.2.4.3).

We will eventually give a complete answer to Question 5.2.0.7: a functor between ordinary categories $\mathscr{F} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ is (isomorphic to) the homotopy transport representation of a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ if and only if it can be promoted to a diagram $\widetilde{\mathscr{F}} : \mathcal{C} \to \mathcal{QC}$ (Remark 5.6.5.15), where $\mathcal{QC}$ denotes the ∞-category of small ∞-categories (Construction 5.5.4.1). In §5.2.8, we prove a preliminary result in this direction by showing that if $\mathcal{C}$ is an ∞-category, then the homotopy transport representation of any cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ can always be promoted to an *enriched* functor, where we regard $\mathrm{h}\mathcal{C}$ and QCat as enriched over the homotopy category of Kan complexes hKan (Construction 5.2.8.9).

023X **Remark 5.2.0.8.** In the preceding discussion, we have confined our attention to the case of cocartesian fibrations $U : \mathcal{E} \to \mathcal{C}$. Of course, all of our results have counterparts for cartesian fibrations, which can be obtained from passing to opposite ∞-categories.

5.2.1 Exponentiation for Cartesian Fibrations

01VF In this section, we study the behavior of (co)cartesian fibrations with respect to the formation of functor ∞-categories. Our main result can be stated as follows:

01VG **Theorem 5.2.1.1.** *Let $q : X \to S$ be a morphism of simplicial sets, let B be a simplicial set, and let $q' : \mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S)$ be the morphism given by postcomposition with q. Then:*

(1) *If q is a cartesian fibration of simplicial sets, then q' is also a cartesian fibration of simplicial sets.*

(2) *Assume that q is a cartesian fibration, and let e be an edge of the simplicial set $\mathrm{Fun}(B, X)$. Then e is q'-cartesian if and only if, for every vertex $b \in B$, the evaluation map $\mathrm{ev}_b : \mathrm{Fun}(B, X) \to \mathrm{Fun}(\{b\}, X) \simeq X$ carries e to a q-cartesian edge of X.*

(1′) *If q is a cocartesian fibration of simplicial sets, then q' is also a cocartesian fibration of simplicial sets.*

(2′) *Assume that q is a cocartesian fibration, and let e be an edge of the simplicial set $\mathrm{Fun}(B, X)$. Then e is q'-cocartesian if and only if, for every vertex $b \in B$, the evaluation map $\mathrm{ev}_b : \mathrm{Fun}(B, X) \to \mathrm{Fun}(\{b\}, X) \simeq X$ carries e to a q-cocartesian edge of X.*

01VH **Remark 5.2.1.2.** Let $\mathcal{C}$ be an ∞-category, so that the projection map $q : \mathcal{C} \to \Delta^0$ is a cartesian fibration (Example 5.1.4.3). In this case, part (1) of Theorem 5.2.1.1 is equivalent to the assertion that for every simplicial set B, the simplicial set $\mathrm{Fun}(B, \mathcal{C})$ is also an ∞-category (Theorem 1.5.3.7). By virtue of Proposition 5.1.4.11, part (2) is equivalent to the assertion that a morphism of $\mathrm{Fun}(B, \mathcal{C})$ is an isomorphism if and only if, for every vertex $b \in B$, its image under the evaluation functor $\mathrm{ev}_b : \mathrm{Fun}(B, \mathcal{C}) \to \mathcal{C}$ is an isomorphism in $\mathcal{C}$ (Theorem 4.4.4.4).

The proof of Theorem 5.2.1.1 will require some preliminaries. Let $q : X \to S$ be an inner fibration of simplicial sets. By definition, q is a cartesian fibration if and only if for every vertex $z \in X$ and every edge $\overline{e} : s \to q(z)$ of S, there exists a q-cartesian edge $e : y \to z$ in X satisfying $q(e) = \overline{e}$. To prove Theorem 5.2.1.1, we need to show that the edge e can be chosen to depend functorially on z.

Proposition 5.2.1.3. *Let $q : X \to S$ be an inner fibration of simplicial sets and let $Y \subseteq \mathrm{Fun}(\Delta^1, X)$ be the full simplicial subset of $\mathrm{Fun}(\Delta^1, X)$ spanned by those edges $e : \Delta^1 \to X$ which are q-cartesian (see Definition 4.1.2.17). Let $\theta : Y \to \mathrm{Fun}(\Delta^1, S) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\{1\}, X)$ denote the restriction map, and let $Z \subseteq \mathrm{Fun}(\Delta^1, S) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\{1\}, X)$ be the full simplicial subset spanned by those vertices which belong to the image of θ. Then $\theta : Y \to Z$ is a trivial Kan fibration of simplicial sets.* 01VK

Remark 5.2.1.4. In the situation of Proposition 5.2.1.3, the simplicial set Z coincides with $\mathrm{Fun}(\Delta^1, S) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\{1\}, X)$ if and only if q is a cartesian fibration. If this condition is satisfied, then Proposition 5.2.1.3 asserts that $\theta : Y \to \mathrm{Fun}(\Delta^1, S) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\{1\}, X)$ is a trivial Kan fibration. 043C

Proof of Proposition 5.2.1.3. Let $n > 0$ be an integer; we wish to show that every lifting problem of the form

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & Y \arrow[d, "\theta"] \\
\Delta^n \arrow[r] & \mathrm{Fun}(\Delta^1, S) \times_{\mathrm{Fun}(\{1\},S)} \mathrm{Fun}(\{1\}, X)
\end{tikzcd} \tag{5.8}$$ 01VL

admits a solution. Unwinding the definitions, we can rephrase (5.8) as a lifting problem

$$\begin{tikzcd}
(\Delta^1 \times \partial \Delta^n) \coprod_{(\{1\} \times \partial \Delta^n)} (\{1\} \times \Delta^n) \arrow[r, "h_0"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^1 \times \Delta^n \arrow[r, "\overline{h}"] \arrow[ur, dashed, "h"] & S,
\end{tikzcd}$$

where the morphism h_0 has the property that $h_0|_{\Delta^1 \times \{i\}}$ is a q-cartesian edge of X for $0 \leq i \leq n$. Let

$$(\Delta^1 \times \partial \Delta^n) \cup (\{1\} \times \Delta^n) = Y(0) \subset Y(1) \subset X(2) \subset \cdots \subset Y(n+1) = \Delta^1 \times \Delta^n$$

be the sequence of simplicial subsets appearing in the proof of Lemma 3.1.2.12, so that h_0 can be identified with a morphism of simplicial sets from $Y(0)$ to X. We will show that, for $0 \leq j \leq n+1$, there exists a morphism of simplicial sets $h_j : Y(j) \to X$ satisfying $h_j|_{Y(0)} = h_0$ and $q \circ h_j = \overline{h}|_{Y(j)}$ (taking $j = n+1$, this will complete the proof of Proposition 5.2.1.3). We proceed by induction on j, the case $j = 0$ being vacuous. Assume that $j > 0$ and that we have already constructed a morphism $h_{j-1} : Y(j-1) \to X$ satisfying $h_{j-1}|_{Y(0)} = h_0$

and $q \circ h_{j-1} = \overline{h}|_{Y(j-1)}$. By virtue of Lemma 3.1.2.12, we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^{n+1}_{j} \arrow[r, "\sigma_0"] \arrow[d] & Y(j-1) \arrow[d] \\
\Delta^{n+1} \arrow[r, "\sigma"] & Y(j).
\end{tikzcd}$$

Consequently, to prove the existence of h_i, it suffices to solve the lifting problem

$$\begin{tikzcd}
\Lambda^{n+1}_{j} \arrow[r, "h_{j-1} \circ \sigma_0"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^{n+1} \arrow[r, "\overline{h} \circ \sigma"] \arrow[ur, dashed] & S.
\end{tikzcd}$$

For $0 < j < n+1$, the existence of the desired solution follows from our assumption that q is an inner fibration. In the case $j = n+1$, the existence follows from the fact that the composite map

$$\Delta^1 \simeq \mathrm{N}_{\bullet}(\{n < n+1\}) \hookrightarrow \Lambda^{n+1}_{j} \xrightarrow{\sigma_0} Y(n) \xrightarrow{h_n} X$$

is the edge of X given by the restriction $h_0|_{\Delta^1 \times \{n\}}$, and is therefore q-cartesian. □

`01VM` **Lemma 5.2.1.5.** *Let $q : X \to S$ be an inner fibration of simplicial sets, let B be a simplicial set, and let $q' : \mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S)$ be the map given by postcomposition with q (so that q' is also an inner fibration; see Corollary 4.1.4.3). Let e be an edge of the simplicial set* $\mathrm{Fun}(B, X)$.

(1) *Suppose that, for every vertex $b \in B$, the evaluation map*

$$\mathrm{ev}_b : \mathrm{Fun}(B, X) \to \mathrm{Fun}(\{b\}, X) \simeq X$$

carries e to a q-cartesian edge of X. Then e is q'-cartesian.

(2) *Suppose that, for every vertex $b \in B$, the evaluation map*

$$\mathrm{ev}_b : \mathrm{Fun}(B, X) \to \mathrm{Fun}(\{b\}, X) \simeq X$$

carries e to a q-cocartesian edge of X. Then e is q'-cocartesian.

Proof. We will give a proof of (2); assertion (1) follows by a similar argument. We proceed as in the proof of Lemma 4.4.4.8. Suppose we are given an integer $n \geq 2$; we wish to show that every lifting problem

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{Fun}(B, X) \arrow[d, "q'"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathrm{Fun}(B, S)
\end{tikzcd}$$

admits a solution, provided that the composite map

$$\Delta^1 \simeq \mathrm{N}_{\bullet}(\{0 < 1\}) \hookrightarrow \Lambda^n_0 \xrightarrow{\sigma_0} \mathrm{Fun}(B, X)$$

is the edge e. Unwinding the definitions, we can rewrite this as a lifting problem

$$\begin{tikzcd}
B \times \Lambda^n_0 \arrow[r, "F_0"] \arrow[d] & X \arrow[d, "q"] \\
B \times \Delta^n \arrow[r, "\overline{F}"] \arrow[ur, dashed, "F"] & S.
\end{tikzcd}$$

Let P denote the collection of all pairs (A, F_A), where $A \subseteq B$ is a simplicial subset and $F_A : A \times \Delta^n \to X$ is a morphism of simplicial sets satisfying

$$F_A|_{A \times \Lambda^n_0} = F_0|_{A \times \Lambda^n_0} \qquad q \circ F_A = \overline{F}|_{A \times \Delta^n}$$

We regard P as partially ordered set, where $(A, F_A) \leq (A', F_{A'})$ if $A \subseteq A'$ and $F_A = F_{A'}|_{A \times \Delta^n}$. The partially ordered set P satisfies the hypotheses of Zorn's lemma, and therefore has a maximal element $(A_{\max}, F_{A_{\max}})$. We will complete the proof by showing that $A_{\max} = B$. Assume otherwise. Then there exists some nondegenerate m-simplex $\tau : \Delta^m \to B$ whose image is not contained in $A_{\max}$. Choosing m as small as possible, we can assume that τ carries the boundary $\partial \Delta^m$ into $A_{\max}$. Let $A' \subseteq B$ be the union of $A_{\max}$ with the image of τ, so that we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & A_{\max} \arrow[d] \\
\Delta^m \arrow[r] & A'.
\end{tikzcd}$$

We will complete the proof by showing that the lifting problem

$$\begin{tikzcd}
(A_{\max} \times \Delta^n) \coprod_{(A_{\max} \times \Lambda^n_0)} (A' \times \Lambda^n_0) \arrow[r, "(F_{A_{\max}}, F_0|_{A' \times \Lambda^n_0})"] \arrow[d] & X \arrow[d, "q"] \\
A' \times \Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution (contradicting the maximality of the pair $(A_{\max}, F_{A_{\max}})$).

Choose a sequence of simplicial subsets

$$Y(0) \subset Y(1) \subset Y(2) \subset \cdots \subset Y(t) = \Delta^m \times \Delta^n$$

satisfying the requirements of Lemma 4.4.4.7, so that $F_{A_{\max}}$ determines a map of simplicial sets $G_0 : Y(0) \to X$. We will show that, for $0 \leq s \leq t$, there exists a morphism of simplicial sets $G_s : Y(s) \to X$ satisfying $G_s|_{Y(0)} = G_0$ and $q \circ G_s = \overline{F}|_{Y(s)}$ (in the case $s = t$, this will complete the proof of Lemma 5.2.1.5). We proceed by induction on s, the case $s = 0$ being vacuous. Assume that $s > 0$ and that we have already constructed a morphism $G_{s-1} : Y(s-1) \to X$ satisfying $G_{s-1}|_{Y(0)} = F_0$ and $q \circ G_{s-1} = \overline{F}|_{Y(s-1)}$. By construction, there exist integers $\ell \geq 2$, $0 \leq k < \ell$ and a pushout diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^\ell_k \arrow[r, "\tau_0"] \arrow[d] & Y(s-1) \arrow[d] \\
\Delta^\ell \arrow[r, "\tau"] & Y(s).
\end{tikzcd}$$

Moreover, in the special case $k = 0$, we can assume that $\tau(0) = (0,0)$ and $\tau(1) = (0,1)$, so that the composite map

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{0 < 1\}) \hookrightarrow \Lambda^\ell_k \xrightarrow{\sigma_0} Y(s-1) \xrightarrow{G_{s-1}} X$$

corresponds to a q-cocartesian edge e' of X. To construct the desired extension F_s, it suffices to solve the lifting problem

$$\begin{tikzcd}
\Lambda^\ell_k \arrow[r, "G_{s-1} \circ \tau_0"] \arrow[d] & X \arrow[d, "q"] \\
\Delta^\ell \arrow[r, "\overline{F} \circ \tau"] \arrow[ur, dashed] & S.
\end{tikzcd}$$

For $0 < k < \ell$, the existence of the desired solution follows from our assumption that q is an inner fibration; when $k = 0$, it follows from the fact that e' is q-cocartesian. $\square$

Proof of Theorem 5.2.1.1. Assume that $q : X \to S$ is a cartesian fibration of simplicial sets (the case where q is a cocartesian fibration can be handled by a similar argument). Let B be any simplicial set and let $q' : \mathrm{Fun}(B, X) \to \mathrm{Fun}(B, S)$ be the map given by postcomposition with q. Then q' is an inner fibration (Corollary 4.1.4.3). Let us say that an edge e of the simplicial set $\mathrm{Fun}(B, X)$ is *special* if, for every vertex $b \in B$, the evaluation map $\mathrm{ev}_b : \mathrm{Fun}(B, X) \to \mathrm{Fun}(\{b\}, X) \simeq X$ carries e to a q-cartesian edge of X. By virtue of Lemma 5.2.1.5, every special edge of $\mathrm{Fun}(B, X)$ is q'-cartesian. Moreover, Proposition 5.2.1.3 guarantees that for every vertex $z \in \mathrm{Fun}(B, X)$ and every edge $\overline{e} : \overline{y} \to q'(z)$ of $\mathrm{Fun}(B, S)$, there exists a special edge $e : y \to z$ of $\mathrm{Fun}(B, X)$ satisfying $q'(e) = \overline{e}$. It follows that q' is a cartesian fibration.

To complete the proof, it will suffice to show that every q'-cartesian edge $e : x \to z$ of the simplicial set $\mathrm{Fun}(B, X)$ is special. By virtue of the preceding argument, there exists a special edge $e' : y \to z$ of $\mathrm{Fun}(B, X)$ satisfying $q'(e') = q'(e)$, which is also q'-cartesian. Applying Remark 5.1.3.8, we can choose a 2-simplex σ of $\mathrm{Fun}(B, X)$ as indicated in the diagram

$$\begin{tikzcd}
 & y \arrow[dr, "e'"] & \\
x \arrow[ur, "e''"] \arrow[rr, "e"] & & z,
\end{tikzcd}$$

where e'' is an isomorphism in the ∞-category $\{q'(x)\} \times_{\mathrm{Fun}(B,S)} \mathrm{Fun}(B, X)$. For each vertex $b \in B$, the evaluation functor ev_b carries σ to a 2-simplex

$$\begin{tikzcd}
 & \mathrm{ev}_b(y) \arrow[dr, "\mathrm{ev}_b(e')"] & \\
\mathrm{ev}_b(x) \arrow[ur, "\mathrm{ev}_b(x'')"] \arrow[rr, "\mathrm{ev}_b(e)"] & & \mathrm{ev}_b(z)
\end{tikzcd}$$

in the simplicial set X. Since e' is special, the edge $\mathrm{ev}_b(e')$ is q-cartesian. The edge $\mathrm{ev}_b(e'')$ is an isomorphism in a fiber of q, and is therfefore also q-cartesian (Proposition 5.1.4.11). Applying Proposition 5.1.4.12, we deduce that $\mathrm{ev}_b(e)$ is q-cartesian. Allowing the vertex b to vary, we conclude that e is a special edge of $\mathrm{Fun}(B, X)$, as desired. □

5.2.2 Covariant Transport Functors

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration between categories (Definition 5.0.0.3) and let 019M
$f : C \to D$ be a morphism in the category $\mathcal{C}$. If X is an object of the fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$, then our assumption that U is a cocartesian fibration guarantees that we can choose an object

$f_!(X)$ of the fiber $\mathcal{E}_D = \{D\} \times_{\mathcal{C}} \mathcal{E}$ together with a U-cocartesian morphism $\widetilde{f}_X : X \to f_!(X)$ satisfying $U(\widetilde{f}_X) = f$. In this case, we can view the construction $X \mapsto f_!(X)$ as a functor from the category $\mathcal{E}_C$ to the category $\mathcal{E}_D$:

01SA **Proposition 5.2.2.1.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of categories and let $f : C \to D$ be a morphism of $\mathcal{C}$. For each object $X \in \mathcal{E}_C$, let $\widetilde{f}_X$ be a U-cocartesian morphism of $\mathcal{E}$ having source X and satisfying $U(\widetilde{f}_X) = f$. Then there is a unique functor $f_! : \mathcal{E}_C \to \mathcal{E}_D$ with the following properties:*

- *For each object $X \in \mathcal{E}_C$, the object $f_!(X) \in \mathcal{E}_D$ is the target of the morphism $\widetilde{f}_X$.*
- *The construction $X \mapsto \widetilde{f}_X$ determines a natural transformation from the inclusion functor $\mathcal{E}_C \to \mathcal{E}$ to the functor $f_! : \mathcal{E}_C \to \mathcal{E}_D \subseteq \mathcal{E}$.*

Proof. For each object $X \in \mathcal{E}_C$, let $f_!(X)$ denote the target of the morphism $\widetilde{f}_X$. Let $u : X \to Y$ be a morphism in the category $\mathcal{E}_C$. Invoking our assumption that $\widetilde{f}_X$ is U-cocartesian, we see that there is a unique morphism $f_!(u) : f_!(X) \to f_!(Y)$ for which the diagram

01SB
$$\begin{tikzcd} X \arrow[r, "\widetilde{f}_X"] \arrow[d, "u"] & f_!(X) \arrow[d, "f_!(u)"] \\ Y \arrow[r, "\widetilde{f}_Y"] & f_!(Y) \end{tikzcd} \tag{5.9}$$

is commutative (in the category $\mathcal{E}$). Note that if $v : Y \to Z$ is another morphism in the category $\mathcal{E}_C$, then the calculation

$$f_!(v) \circ f_!(u) \circ \widetilde{f}_X = f_!(v) \circ \widetilde{f}_Y \circ u = \widetilde{f}_Z \circ v \circ u$$

shows that $f_!(v \circ u) = f_!(v) \circ f_!(u)$. Similarly, for each object $X \in \mathcal{E}_C$, the calculation $\widetilde{f}_X \circ \mathrm{id}_{f_!(X)} = \widetilde{f}_X = \mathrm{id}_X \circ \widetilde{f}_X$ shows that $f_!(\mathrm{id}_X) = \mathrm{id}_{f_!(X)}$. We can therefore regard $f_!$ as a functor from the category $\mathcal{E}_C$ to $\mathcal{E}_D$, and the commutativity of (5.9) guarantees that the construction $X \mapsto \widetilde{f}_X$ determines a natural transformation from the inclusion $\mathcal{E}_C \hookrightarrow \mathcal{E}$ to the functor $f_!$. □

023T **Construction 5.2.2.2.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of categories, let $f : C \to D$ be a morphism of the category $\mathcal{C}$, and let $f_! : \mathcal{E}_C \to \mathcal{E}_D$ be the functor of Proposition 5.2.2.1. We will refer to $f_!$ as the *functor of covariant transport along f*.

01SC **Warning 5.2.2.3.** In the situation of Construction 5.2.2.2, the covariant transport functor $f_! : \mathcal{E}_C \to \mathcal{E}_D$ depends not only on the cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ and the morphism

$f : C \to D$, but also on the system of U-cocartesian lifts $\{\widetilde{f}_X : X \to f_!(X)\}_{X \in \mathcal{E}_C}$. A different system of cocartesian lifts $\{\widetilde{f}'_X : X \to f'_!(X)\}_{X \in \mathcal{E}_C}$ will give rise to a different covariant transport functor $f'_! : \mathcal{E}_C \to \mathcal{E}_D$. However, there is a canonical isomorphism of functors $\alpha : f_! \simeq f'_!$, which is uniquely determined by the requirement that for every object $X \in \mathcal{E}_C$, the diagram

$$\begin{tikzcd} & f_!(X) \arrow[dr, "\alpha_X", "\sim"'] & \\ X \arrow[ur, "\widetilde{f}_X"] \arrow[rr, "\widetilde{f}'_X"] & & f'_!(X) \end{tikzcd}$$

is commutative.

We now apply the results of §5.2.1 to extend Construction 5.2.2.2 to the ∞-categorical setting.

Definition 5.2.2.4. Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets, let $f : C \to D$ be an edge of $\mathcal{C}$, and let $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ and $\mathcal{E}_D = \{D\} \times_{\mathcal{C}} \mathcal{E}$ denote the corresponding fibers of U. We will say that a functor $F : \mathcal{E}_C \to \mathcal{E}_D$ is *given by covariant transport along* f if there exists a morphism of simplicial sets $\widetilde{F} : \Delta^1 \times \mathcal{E}_C \to \mathcal{E}$ satisfying the following conditions: 019N

(1) The diagram of simplicial sets

$$\begin{tikzcd} \Delta^1 \times \mathcal{E}_C \arrow[r, "\widetilde{F}"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\ \Delta^1 \arrow[r, "f"] & \mathcal{C} \end{tikzcd}$$

commutes.

(2) The restriction $\widetilde{F}|_{\{0\} \times \mathcal{E}_C}$ is the identity map $\mathrm{id}_{\mathcal{E}_C}$, and the restriction $\widetilde{F}|_{\{1\} \times \mathcal{E}_C}$ is equal to F.

(3) For every object X of the ∞-category $\mathcal{E}_C$, the composite map

$$\Delta^1 \times \{X\} \hookrightarrow \Delta^1 \times \mathcal{E}_C \xrightarrow{\widetilde{F}} \mathcal{E}$$

is a locally U-cocartesian edge of the simplicial set $\mathcal{E}$.

If these conditions are satisfied, we say that the morphism $\widetilde{F}$ *witnesses F as given by covariant transport along f*.

019P **Example 5.2.2.5.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets and let C be a vertex of $\mathcal{C}$. Then the projection map

$$\Delta^1 \times \mathcal{E}_C \twoheadrightarrow \mathcal{E}_C \hookrightarrow \mathcal{E}$$

exhibits the identity functor $\mathrm{id}_{\mathcal{E}_C}$ as given by covariant transport along the degenerate edge id_C. See Example 5.1.3.6.

0355 **Example 5.2.2.6.** Let $U : \mathcal{E} \to \mathcal{C}$ be a left covering map of simplicial sets. Then, for every edge $f : C \to D$ in $\mathcal{C}$, there is a *unique* functor $\mathcal{E}_C \to \mathcal{E}_D$ given by covariant transport along f, which can be identified with the covariant transport *function* given by Construction 5.2.0.1.

01VR **Example 5.2.2.7.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration between ordinary categories, let $f : C \to D$ be a morphism in $\mathcal{C}$, and choose a collection of U-cocartesian morphisms $\{\widetilde{f}_X : X \to f_!(X)\}_{X \in \mathcal{E}_C}$ satisfying $U(\widetilde{f}_X) = f$. According to Proposition 5.2.2.1, there is a unique functor $f_! : \mathcal{E}_C \to \mathcal{E}_D$ for which the construction $X \mapsto \widetilde{f}_X$ determines a natural transformation of functors $\widetilde{f} : \mathrm{id}_{\mathcal{E}_C} \to f_!$. Passing to nerves, we obtain a natural transformation $\mathrm{id}_{\mathrm{N}_\bullet(\mathcal{E}_C)} \to \mathrm{N}_\bullet(f_!)$, which exhibits the functor

$$\mathrm{N}_\bullet(f_!) : \mathrm{N}_\bullet(\mathcal{E}_C) \to \mathrm{N}_\bullet(\mathcal{E}_D)$$

as given by covariant transport along f (regarded as an edge of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$).

Stated more informally, the covariant transport construction for cocartesian fibrations of ordinary categories (see Construction 5.2.2.2) can be regarded as a special case Definition 5.2.2.4.

01VS **Proposition 5.2.2.8.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let $f : C \to D$ be an edge of $\mathcal{C}$. Then:*

- *There exists a functor $F : \mathcal{E}_C \to \mathcal{E}_D$ which is given by covariant transport along f.*
- *An arbitrary functor $F' : \mathcal{E}_C \to \mathcal{E}_D$ is given by covariant transport along f if and only if it is isomorphic to F (as an object of the ∞-category $\mathrm{Fun}(\mathcal{E}_C, \mathcal{E}_D)$).*

023Y **Notation 5.2.2.9.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let $f : C \to D$ be an edge of the simplicial set $\mathcal{C}$. Applying Proposition 5.2.2.8, we conclude that the collection of functors $\mathcal{E}_C \to \mathcal{E}_D$ which are given by covariant transport along f comprise a single isomorphism class in the ∞-category $\mathrm{Fun}(\mathcal{E}_C, \mathcal{E}_D)$. We will denote this isomorphism class by $[f_!]$, which we regard as an element of the set $\pi_0(\mathrm{Fun}(\mathcal{E}_C, \mathcal{E}_D)^{\simeq})$. We will often use the notation $f_!$ to denote a particular choice of representative of this isomorphism class: that is, a particular choice of functor $\mathcal{E}_C \to \mathcal{E}_D$ which is given by covariant transport along f.

We now explain how to deduce Proposition 5.2.2.8 from Theorem 5.2.1.1. For this purpose, it will be convenient to introduce a bit more terminology.

Definition 5.2.2.10. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Let K be another simplicial set, let $H : \Delta^1 \times K \to \mathcal{E}$ be a morphism. We will say that H is a *U-cocartesian lift of $\overline{H} = U \circ H$* if, for every vertex $x \in K$, the restriction $H|_{\Delta^1 \times \{x\}}$ is a U-cocartesian edge of $\mathcal{E}$. 02R8

Remark 5.2.2.11. In the situation of Definition 5.2.2.10, we can identify H and $\overline{H}$ with edges of the simplicial sets $\mathrm{Fun}(K, \mathcal{E})$ and $\mathrm{Fun}(K, \mathcal{C})$, respectively. Then H is a U-cocartesian lift of $\overline{H}$ if and only if it is U'-cocartesian, where $U' : \mathrm{Fun}(K, \mathcal{E}) \to \mathrm{Fun}(K, \mathcal{C})$ is given by postcomposition with U. (see Theorem 5.2.1.1). 02R9

Example 5.2.2.12. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let $f : C \to D$ be an edge of $\mathcal{C}$, let $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ and $\mathcal{E}_D = \{D\} \times_{\mathcal{C}} \mathcal{E}$ denote the corresponding fibers of U. Suppose we are given a commutative diagram of simplicial sets 02RA

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E}_C \arrow[r, "\widetilde{F}"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^1 \arrow[r, "f"] & \mathcal{C},
\end{tikzcd}$$

where the restriction $\widetilde{F}|_{\{0\} \times \mathcal{E}_C}$ is the identity map from $\mathcal{E}_C$ to itself, and set $F = \widetilde{F}|_{\{1\} \times \mathcal{E}_C} \in \mathrm{Fun}(\mathcal{E}_C, \mathcal{E}_D)$. Then $\widetilde{F}$ witnesses F as given by covariant transport along f (in the sense of Definition 5.2.2.4) if and only if it is a U-cocartesian lift of $U \circ \widetilde{F}$ (in the sense of Definition 5.2.2.10).

Lemma 5.2.2.13. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let K be a simplicial set, and suppose we are given a lifting problem* 02RB

02RC

$$\begin{tikzcd}
\{0\} \times K \arrow[r, "H_0"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^1 \times K \arrow[r, "\overline{H}"] \arrow[ur, dashed, "H"] & \mathcal{C}.
\end{tikzcd} \tag{5.10}$$

Then:

(1) *The lifting problem (5.10) admits a solution $H : \Delta^1 \times K \to \mathcal{E}$ which is a U-cocartesian lift of $\overline{H}$.*

(2) *Let F be any object of the ∞-category* $\operatorname{Fun}_{/\mathcal{C}}(\{1\} \times K, \mathcal{E})$*. Then F is isomorphic to* $H|_{\{1\}\times K}$ *(as an object of* $\operatorname{Fun}_{/\mathcal{C}}(\{1\} \times K, \mathcal{E})$*) if and only if $F = H'|_{\{1\}\times K}$, where H' is another U-cocartesian lift of $\overline{H}$ which solves the lifting problem (5.10).*

Proof. By virtue of Remark 5.2.2.11 (and Theorem 5.2.1.1), we can replace U by the induced map $\operatorname{Fun}(K, \mathcal{E}) \to \operatorname{Fun}(K, \mathcal{C})$ and thereby reduce to the case where $K = \Delta^0$. In this case, assertion (1) follows immediately from our assumption that U is a cocartesian fibration, and assertion (2) follows from Remark 5.1.3.8. □

Proof of Proposition 5.2.2.8. Apply Lemma 5.2.2.13 in the special case where K is the ∞-category $\mathcal{E}_C$, $H_0 : K \to \mathcal{E}$ is the inclusion map, and $\overline{H}$ is the composite map $\Delta^1 \times K \to \Delta^1 \xrightarrow{f} \mathcal{C}$ (see Example 5.2.2.12). □

05J5 **Remark 5.2.2.14** (Functoriality)**.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "T"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where U and U' are cocartesian fibrations and T carries U-cocartesian edges of $\mathcal{E}$ to U'-cocartesian edges of $\mathcal{E}'$. Let $f : C \to D$ be an edge of , and let $F : \mathcal{E}_C \to \mathcal{E}_D$ and $F' : \mathcal{E}'_C \to \mathcal{E}'_D$ be a given by covariant transport along f. Then the diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E}_C \arrow[r, "F"] \arrow[d, "T_C"] & \mathcal{E}_D \arrow[d, "T_D"] \\
\mathcal{E}'_C \arrow[r, "F'"] & \mathcal{E}'_D
\end{tikzcd}$$

commutes up to isomorphism. To prove this, choose diagrams $\widetilde{F} : \Delta^1 \times \mathcal{E}_C \to \mathcal{E}$ and $\widetilde{F}' : \Delta^1 \times \mathcal{E}'_C \to \mathcal{E}'$ which exhibit F and F' as given by covariant transport along F. Then the morphisms

$$T \circ \widetilde{F}, \widetilde{F}' \circ (\mathrm{id}_{\Delta^1} \times T_C) : \Delta^1 \times \mathcal{E}_C \to \mathcal{E}'$$

are U'-cocartesian lifts of the map $\Delta^1 \times \mathcal{E}_C \to \Delta^1 \xrightarrow{f} \mathcal{C}$ which coincide when restricted to $\{0\} \times \mathcal{E}_C$, and are therefore isomorphic when restricted to $\{1\} \times \mathcal{E}_C$ (Lemma 5.2.2.13).

We also have a dual version of Definition 5.2.2.4:

01VN **Definition 5.2.2.15.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets, let C and D be vertices of $\mathcal{C}$, and let $f : C \to D$ be an edge of $\mathcal{C}$. We say that a functor $F : \mathcal{E}_D \to \mathcal{E}_C$ is *given by contravariant transport along* f if there exists a morphism of simplicial sets $\widetilde{F} : \Delta^1 \times \mathcal{E}_D \to \mathcal{E}$ satisfying the following conditions:

(1) The diagram of simplicial sets

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E}_D \arrow[r, "\widetilde{F}"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^1 \arrow[r, "f"] & \mathcal{C}
\end{tikzcd}$$

commutes.

(2) The restriction $\widetilde{F}|_{\{1\}\times\mathcal{E}_D}$ is equal to the identity map $\mathrm{id}_{\mathcal{E}_D}$, and the restriction $\widetilde{F}|_{\{0\}\times\mathcal{E}_D}$ is equal to F.

(3) For every object Y of the ∞-category $\mathcal{E}_D$, the composite map

$$\Delta^1 \times \{Y\} \hookrightarrow \Delta^1 \times \mathcal{E}_D \xrightarrow{\widetilde{F}} \mathcal{E}$$

is a locally U-cartesian edge of the simplicial set $\mathcal{E}$.

If these conditions are satisfied, we say that the morphism $\widetilde{F}$ *witnesses F as given by contravariant transport along f.*

Remark 5.2.2.16. Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets and let $f : C \to D$ be an edge of $\mathcal{C}$. Then a functor $F : \mathcal{E}_C \to \mathcal{E}_D$ is given by covariant transport along f if and only if the opposite functor $F^{\mathrm{op}} : \mathcal{E}_C^{\mathrm{op}} \to \mathcal{E}_D^{\mathrm{op}}$ is given by contravariant transport along f with respect to the cartesian fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. 019W

Proposition 5.2.2.8 has a counterpart for cartesian fibrations:

Proposition 5.2.2.17. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of simplicial sets and let $f : C \to D$ be an edge of $\mathcal{C}$. Then:* 019X

- *There exists a functor $F : \mathcal{E}_D \to \mathcal{E}_C$ which is given by contravariant transport along f.*
- *An arbitrary functor $F' : \mathcal{E}_D \to \mathcal{E}_C$ is given by contravariant transport along f if and only if it is isomorphic to F (as an object of the ∞-category $\mathrm{Fun}(\mathcal{E}_D, \mathcal{E}_C)$).*

Notation 5.2.2.18. Let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of simplicial sets and let $f : C \to D$ be an edge of the simplicial set $\mathcal{C}$. It follows from Proposition 5.2.2.17 that the collection of functors $\mathcal{E}_D \to \mathcal{E}_C$ which are given by contravariant transport along f comprise a single isomorphism class in the ∞-category $\mathrm{Fun}(\mathcal{E}_D, \mathcal{E}_C)$. We will denote this isomorphism class by $[f^*]$, which we regard as an element of the set $\pi_0(\mathrm{Fun}(\mathcal{E}_D, \mathcal{E}_C)^{\simeq})$. We will often use the notation f^* to denote a particular choice of representative of this isomorphism class: that is, a particular choice of functor $\mathcal{E}_D \to \mathcal{E}_C$ which is given by contravariant transport along f. 019Y

For Kan fibrations, there is a close relationship between covariant and contravariant transport:

01A1 **Proposition 5.2.2.19.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a Kan fibration of simplicial sets and let $f : C \to D$ be an edge of $\mathcal{C}$. Then the covariant and contravariant transport morphisms $[f_!] \in \mathrm{Hom}_{\mathrm{hKan}}(\mathcal{E}_C, \mathcal{E}_D)$ and $[f^*] \in \mathrm{Hom}_{\mathrm{hKan}}(\mathcal{E}_D, \mathcal{E}_C)$ are inverse to one another (as morphisms in the homotopy category* hKan*).*

Proof. Choose morphisms of Kan complexes $f_! : \mathcal{E}_C \to \mathcal{E}_D$ and $f^* : \mathcal{E}_D \to \mathcal{E}_C$ representing the homotopy classes $[f_!]$ and $[f^*]$, respectively. We will show that $f^* \circ f_!$ is homotopic to the identity morphism $\mathrm{id}_{\mathcal{E}_C}$; a similar argument will show that $f_! \circ f^*$ is homotopic to $\mathrm{id}_{\mathcal{E}_D}$. Let $\mathcal{D}$ denote the fiber product $\mathrm{Fun}(\mathcal{E}_C, \mathcal{E}) \times_{\mathrm{Fun}(\mathcal{E}_C, \mathcal{C})} \mathcal{C}$, and let $\pi : \mathcal{D} \to \mathcal{C}$ be the projection map onto the second factor. Since U is a Kan fibration, it follows from Corollary 3.1.3.2 that π is also a Kan fibration. Let $\widetilde{f} : \Delta^1 \times \mathcal{E}_C \to \mathcal{E}$ be a morphism witnessing $f_!$ as given by covariant transport along f. Then $\widetilde{f}$ determines an edge h of the simplicial set $\mathcal{D}$ satisfying $\pi(h) = f$. Let $\widetilde{f}' : \Delta^1 \times \mathcal{E}_D \to \mathcal{E}$ be a morphism which witnesses f^* as given by contravariant transport along f, so that the composite morphism

$$\Delta^1 \times \mathcal{E}_C \xrightarrow{\mathrm{id} \times f_!} \Delta^1 \times \mathcal{E}_D \xrightarrow{\widetilde{f}'} \mathcal{E}$$

determines an edge h' of the simplicial set $\mathcal{D}$ satisfying $\pi(h') = f$. The edges h and h' have the same target (the vertex of $\mathcal{D}$ corresponding to the morphism $f_!$). Invoking our assumption that π is a Kan fibration, we deduce that there exists a 2-simplex σ of $\mathcal{D}$ satisfying $d^2_0(\sigma) = h'$, $d^2_1(\sigma) = h$, and $\pi(\sigma) = s^1_0(f)$; we can represent σ as a diagram

$$\begin{tikzcd}
 & f^* \circ f_! \arrow[dr, "h'"] & \\
\mathrm{id}_{\mathcal{E}_C} \arrow[ur, dashed, "v"] \arrow[rr, "h"] & & f_!.
\end{tikzcd}$$

We now observe that the edge $v = d^2_2(\sigma)$ of $\mathcal{D}$ can be identified with a map of simplicial sets $V : \Delta^1 \times \mathcal{E}_C \to \mathcal{E}_C$ which is a homotopy from $\mathrm{id}_{\mathcal{E}_C} = V|_{\{0\} \times \mathcal{E}_C}$ to $f^* \circ f_! = V|_{\{1\} \times \mathcal{E}_C}$. □

We close this section by establishing a converse to Proposition 5.2.2.19:

01A2 **Theorem 5.2.2.20.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism U is a Kan fibration.*

(2) *The morphism U is a left fibration and, for every edge $f : C \to D$ of the simplicial set $\mathcal{C}$, the covariant transport morphism $[f_!] : \mathcal{E}_C \to \mathcal{E}_D$ is an isomorphism in the homotopy category* hKan.

(3) *The morphism U is a right fibration and, for every edge $f : C \to D$ of the simplicial set $\mathcal{C}$, the contravariant transport morphism $[f^*] : \mathcal{E}_D \to \mathcal{E}_C$ is an isomorphism in the homotopy category* hKan.

Proof. We will show that (1) ⇔ (2); the proof of the equivalence (1) ⇔ (3) is similar. The implication (1) ⇒ (2) is immediate from Proposition 5.2.2.19. For the converse, assume that $U : \mathcal{E} \to \mathcal{C}$ is a left fibration of simplicial sets and that, for every edge $f : C \to D$ of $\mathcal{C}$, the covariant transport morphism $[f_!]$ is an isomorphism in the homotopy category hKan. We wish to show that U is a Kan fibration. By virtue of Example 4.2.1.5, it will suffice to show that U is a right fibration. By Proposition 4.2.6.1, this is equivalent to the assertion that the induced map

$$\theta : \mathrm{Fun}(\Delta^1, \mathcal{E}) \to \mathrm{Fun}(\{1\}, \mathcal{E}) \times_{\mathrm{Fun}(\{1\}, \mathcal{C})} \mathrm{Fun}(\Delta^1, \mathcal{C})$$

is a trivial Kan fibration. Note that our assumption that U is a left fibration guarantees that θ is also a left fibration (Proposition 4.2.5.1).

Fix an edge $f : C \to D$ of the simplicial set $\mathcal{C}$ and let $\mathrm{Fun}(\Delta^1, \mathcal{E})_f$ denote the fiber $\mathrm{Fun}(\Delta^1, \mathcal{E}) \times_{\mathrm{Fun}(\Delta^1, \mathcal{C})} \{f\}$. Then evaluation at the vertex $1 \in \Delta^1$ induces a morphism $\theta_f : \mathrm{Fun}(\Delta^1, \mathcal{E})_f \to \mathcal{E}_D$. Note that θ_f is a pullback of θ, and is therefore also a left fibration. Since $\mathcal{E}_D$ is a Kan complex (Corollary 4.4.2.3), Corollary 4.4.3.8 guarantees that θ_f is a Kan fibration (so $\mathrm{Fun}(\Delta^1, \mathcal{C})_f$ is also a Kan complex). Evaluation at the vertex $0 \in \Delta^1$ induces another morphism of simplicial sets $u : \mathrm{Fun}(\Delta^1, \mathcal{E})_f \to \mathcal{E}_C$. Since U is a left fibration, the morphism u is a trivial Kan fibration. By construction, the homotopy class $[f_!]$ can be represented by the morphism of Kan complexes given by the composition

$$\mathcal{E}_C \xrightarrow{v} \mathrm{Fun}(\Delta^1, \mathcal{E})_f \xrightarrow{\theta_f} \mathcal{E}_D,$$

where v is a section of u (and therefore a homotopy equivalence). Consequently, our assumption that $[f_!]$ is an isomorphism in hKan guarantees that θ_f is a homotopy equivalence of Kan complexes (Remark 3.1.6.16). Applying Proposition 3.2.7.2, we deduce that the fibers of θ_f are contractible Kan complexes. Since every fiber of θ can also be viewed as a fiber of θ_f for some edge f of the simplicial set $\mathcal{C}$, it follows that the fibers of θ are also contractible Kan complexes. Invoking Proposition 4.4.2.14, we conclude that θ is a trivial Kan fibration, as desired. □

Corollary 5.2.2.21. *Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets. The following conditions are equivalent:* 0240

(1) *The morphism U is a covering map (Definition 3.1.4.1).*

(2) *The morphism U is a left covering map (Definition 4.2.3.8) and, for every edge $f : C \to D$ of the simplicial set $\mathcal{C}$, the covariant transport functor $f_! : \mathcal{E}_C \to \mathcal{E}_D$ is a bijection.*

(3) *The morphism U is a right covering map (Definition 4.2.3.8) and, for every edge $f : C \to D$ of the simplicial set $\mathcal{C}$, the contravariant transport morphism $f^* : \mathcal{E}_D \to \mathcal{E}_C$ is a bijection.*

Proof. Combine Theorem 5.2.2.20 with Corollary 4.2.3.20. □

5.2.3 Example: The Relative Join

0241 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Our goal in this section is to show that F is given by covariant transport, in the sense of Definition 5.2.2.4. More precisely, we will show that there exists a cocartesian fibration of ∞-categories $\mathcal{M} \to \Delta^1$ equipped with isomorphisms $\mathcal{C} \simeq \{0\} \times_{\Delta^1} \mathcal{M}$ and $\mathcal{D} = \{1\} \times_{\Delta^1} \mathcal{M}$ carrying F to a functor

$$\{0\} \times_{\Delta^1} \mathcal{M} \to \{1\} \times_{\Delta^1} \mathcal{M}$$

which is given by covariant transport along the nondegenerate edge of Δ^1 (Proposition 5.2.3.15). We will prove this by an explicit construction, using a generalization of the join operation studied in §4.3. (in §5.2.4, we will show that the ∞-category $\mathcal{M}$ is determined up to equivalence by the functor $F : \mathcal{C} \to \mathcal{D}$ (see Corollary 5.2.4.2 and Remark 5.2.4.3).

0242 **Construction 5.2.3.1** (The Relative Join)**.** Let $\mathcal{E}$ be a simplicial set. By virtue of Remark 4.3.3.23, there is a unique morphism of simplicial sets $\rho : \Delta^1 \times \mathcal{E} \to \mathcal{E} \star \mathcal{E}$ for which the diagram

$$\begin{tikzcd}
\{0\} \times \mathcal{E} \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{E}}"] & \Delta^1 \times \mathcal{E} \arrow[d, "\rho"] & \{1\} \times \mathcal{E} \arrow[l] \arrow[d, "\mathrm{id}_{\mathcal{E}}"] \\
\mathcal{E} \star \emptyset \arrow[r] & \mathcal{E} \star \mathcal{E} & \emptyset \star \mathcal{E} \arrow[l]
\end{tikzcd}$$

is commutative.

Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets. We let $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ denote the fiber product $(\mathcal{C} \star \mathcal{D}) \times_{(\mathcal{E} \star \mathcal{E})} (\Delta^1 \times \mathcal{E})$, so that we have a pullback diagram

$$\begin{tikzcd}
\mathcal{C} \star_{\mathcal{E}} \mathcal{D} \arrow[r] \arrow[d] & \mathcal{C} \star \mathcal{D} \arrow[d] \\
\Delta^1 \times \mathcal{E} \arrow[r, "\rho"] & \mathcal{E} \star \mathcal{E} .
\end{tikzcd}$$

We will refer to $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ as the *join of $\mathcal{C}$ and $\mathcal{D}$ relative to $\mathcal{E}$.*

Remark 5.2.3.2. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets, and let K be a simplicial set. By virtue of Remark 4.3.3.23, morphisms from K to the relative join $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ are given by maps $\pi : K \to \Delta^1$ together with commutative diagrams 0243

$$\begin{tikzcd}
\{0\} \times_{\Delta^1} K \arrow[r] \arrow[d] & K \arrow[d] & \{1\} \times_{\Delta^1} K \arrow[l] \arrow[d] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{E} & \mathcal{D}. \arrow[l, "G"']
\end{tikzcd}$$

Remark 5.2.3.3. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets. Then the inclusion maps $\mathcal{C} \hookrightarrow \mathcal{C} \star \mathcal{D} \hookleftarrow \mathcal{D}$ lift uniquely to monomorphisms 0244

$$\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{C} \star_{\mathcal{E}} \mathcal{D} \qquad \iota_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{C} \star_{\mathcal{E}} \mathcal{D},$$

which fit into a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\iota_{\mathcal{C}}"] \arrow[d] & \mathcal{C} \star_{\mathcal{E}} \mathcal{D} \arrow[d] & \mathcal{D} \arrow[l, "\iota_{\mathcal{D}}"'] \arrow[d] \\
\{0\} \arrow[r] & \Delta^1 & \{1\} \arrow[l]
\end{tikzcd}$$

in which both squares are pullbacks. In the future, we will often abuse notation by identifying $\mathcal{C}$ and $\mathcal{D}$ with their images under the monomorphisms $\iota_{\mathcal{C}}$ and $\iota_{\mathcal{D}}$, respectively (which are full simplicial subsets of the relative join $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$).

Example 5.2.3.4. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets. If $\mathcal{D}$ is empty, then the inclusion map $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ is an isomorphism of simplicial sets. If $\mathcal{C}$ is empty, then the inclusion map $\iota_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ is an isomorphism of simplicial sets. 0245

Example 5.2.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets, so that we have unique morphisms $F : \mathcal{C} \to \Delta^0$ and $G : \mathcal{D} \to \Delta^0$. Then the relative join $\mathcal{C} \star_{\Delta^0} \mathcal{D}$ agrees with the join $\mathcal{C} \star \mathcal{D}$ introduced in Construction 4.3.3.13. 0246

Example 5.2.3.6. Let $\mathcal{E}$ be a simplicial set. Then the relative join $\mathcal{E} \star_{\mathcal{E}} \mathcal{E}$ is isomorphic to $\Delta^1 \times \mathcal{E}$. 0247

Example 5.2.3.7. Let $\mathcal{E}$ be a simplicial set equipped with a morphism $\pi : \mathcal{E} \to \Delta^1$, and set $\mathcal{C} = \{0\} \times_{\Delta^1} \mathcal{E}$ and $\mathcal{D} = \{1\} \times_{\Delta^1} \mathcal{E}$. Then the relative join $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ is isomorphic to $\mathcal{E}$. 0248

0249 **Example 5.2.3.8.** Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors between categories. Then the relative join $\mathrm{N}_\bullet(\mathcal{C}) \star_{\mathrm{N}_\bullet(\mathcal{E})} \mathrm{N}_\bullet(\mathcal{D})$ can be identified with the nerve of the category

$$\mathcal{C} \star_{\mathcal{E}} \mathcal{D} = (\mathcal{C} \star \mathcal{D}) \times_{(\mathcal{E} \star \mathcal{E})} ([1] \times \mathcal{E}),$$

which can be described more concretely as follows:

- The set of objects $\mathrm{Ob}(\mathcal{C} \star_{\mathcal{E}} \mathcal{D})$ is the disjoint union of $\mathrm{Ob}(\mathcal{C})$ with $\mathrm{Ob}(\mathcal{D})$.
- For every pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C} \star_{\mathcal{E}} \mathcal{D})$, we have

$$\mathrm{Hom}_{\mathcal{C} \star_{\mathcal{E}} \mathcal{D}}(X, Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \text{if } X, Y \in \mathrm{Ob}(\mathcal{C}) \\ \mathrm{Hom}_{\mathcal{D}}(X, Y) & \text{if } X, Y \in \mathrm{Ob}(\mathcal{D}) \\ \mathrm{Hom}_{\mathcal{E}}(F(X), G(Y)) & \text{if } X \in \mathrm{Ob}(\mathcal{C}),\ Y \in \mathrm{Ob}(\mathcal{D}) \\ \emptyset & \text{if } X \in \mathrm{Ob}(\mathcal{D}),\ Y \in \mathrm{Ob}(\mathcal{C}). \end{cases}$$

024A **Remark 5.2.3.9** (Base Change)**.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d] & \mathcal{D}' \arrow[l] \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{E} & \mathcal{D}, \arrow[l]
\end{tikzcd}$$

where both squares are pullbacks. Then the induced diagram

$$\begin{tikzcd}
\mathcal{C}' \star_{\mathcal{E}'} \mathcal{D}' \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d] \\
\mathcal{C} \star_{\mathcal{E}} \mathcal{D} \arrow[r] & \mathcal{E}
\end{tikzcd}$$

is also a pullback square.

024B **Remark 5.2.3.10.** Let $G : \mathcal{D} \to \mathcal{E}$ be a fixed morphism of simplicial sets. Then the construction

$$(F : \mathcal{C} \to \mathcal{E}) \mapsto \mathcal{C} \star_{\mathcal{E}} \mathcal{D}$$

carries colimits in the category $(\mathrm{Set}_\Delta)_{/\mathcal{E}}$ to colimits in the category $(\mathrm{Set}_\Delta)_{\mathcal{D}/}$. In particular, the construction $\mathcal{C} \mapsto (\mathcal{C} \star_{\mathcal{E}} \mathcal{D})$ commutes with filtered colimits and carries pushout diagrams to pushout diagrams.

The relative join $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ of Construction 5.2.3.1 is defined for arbitrary diagrams of simplicial sets $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$. However, as our notation suggests, we will be primarily interested in the special case where $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are ∞-categories. In this case, we have the following generalization of Corollary 4.3.3.25:

Proposition 5.2.3.11. *Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Then the relative join $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ is an ∞-category.* `024C`

Lemma 5.2.3.12. *Let $U : \mathcal{E} \to \mathcal{E}'$ be an inner fibration of simplicial sets. Then the induced map* `024D`

$$\Delta^1 \times \mathcal{E} = \mathcal{E} \star_{\mathcal{E}} \mathcal{E} \to \mathcal{E} \star_{\mathcal{E}'} \mathcal{E}$$

is also an inner fibration of simplicial sets.

Proof. Suppose we are given integers $0 < i < n$; we wish to show that every lifting problem

`02RE`

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \Delta^1 \times \mathcal{E} \arrow[d] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{E} \star_{\mathcal{E}'} \mathcal{E}
\end{tikzcd} \tag{5.11}$$

admits a solution. Let α denote the composite map

$$\Delta^n \xrightarrow{\overline{\sigma}} \mathcal{E} \star_{\mathcal{E}'} \mathcal{E} \to \Delta^0 \star_{\Delta^0} \Delta^0 \simeq \Delta^1.$$

If α is a constant morphism, then the existence of σ is immediate. We may therefore assume without loss of generality that α is not constant. Write $\sigma_0 = (\alpha_0, \tau_0)$, where $\alpha_0 = \alpha|_{\Lambda^n_i}$ and $\tau_0 : \Lambda^n_i \to \mathcal{E}$ is a morphism of simplicial sets, and let $\overline{\tau}$ denote the composite map $\Delta^n \xrightarrow{\overline{\sigma}} \mathcal{E} \star_{\mathcal{E}'} \mathcal{E} \to \mathcal{E}'$. Since U is an inner fibration, the lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\tau_0"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^n \arrow[r, "\overline{\tau}"] \arrow[ur, dashed, "\tau"] & \mathcal{E}'
\end{tikzcd}$$

admits a solution. We now observe that the pair $\sigma = (\alpha, \tau)$ can be regarded as an n-simplex of $\Delta^1 \times \mathcal{E}$ which solves the lifting problem (5.11). □

02RF **Lemma 5.2.3.13.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "U"] & \mathcal{E} \arrow[d, "W"] & \mathcal{D} \arrow[l] \arrow[d, "V"] \\
\mathcal{C}' \arrow[r] & \mathcal{E}' & \mathcal{D}' \arrow[l]
\end{tikzcd}$$

in which the vertical morphisms are inner fibrations. Then the induced map

$$F : \mathcal{C} \star_{\mathcal{E}} \mathcal{D} \to \mathcal{C}' \star_{\mathcal{E}'} \mathcal{D}'$$

is also an inner fibration.

Proof. Unwinding the definitions, we see that F factors as a composition

$$\mathcal{C} \star_{\mathcal{E}} \mathcal{D} \xrightarrow{G} \mathcal{C} \star_{\mathcal{E}'} \mathcal{D} \xrightarrow{H} \mathcal{C}' \star_{\mathcal{E}'} \mathcal{D}',$$

where G is a pullback of the inner fibration $\mathcal{E} \star_{\mathcal{E}} \mathcal{E} \to \mathcal{E} \star_{\mathcal{E}'} \mathcal{E}$ of Lemma 5.2.3.12 and H is a pullback of the inner fibration $\mathcal{C} \star \mathcal{D} \to \mathcal{C}' \star \mathcal{D}'$ of Proposition 4.3.3.24. □

Proof of Proposition 5.2.3.11. Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Applying Lemma 5.2.3.13, we see that the natural map

$$\mathcal{C} \star_{\mathcal{E}} \mathcal{D} \to \Delta^0 \star_{\Delta^0} \Delta^0 \simeq \Delta^1$$

is an inner fibration of simplicial sets. Since Δ^1 is an ∞-category, it follows that $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$ is also an ∞-category (Remark 4.1.1.9). □

024F **Remark 5.2.3.14** (Morphism Spaces in the Relative Join)**.** Let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets. If X and Y are vertices of the relative join $\mathcal{C} \star_{\mathcal{E}} \mathcal{D}$, then we have canonical isomorphisms of simplicial sets

$$\mathrm{Hom}_{\mathcal{C} \star_{\mathcal{E}} \mathcal{D}}(X, Y) \simeq \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \text{if } X, Y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{D}}(X, Y) & \text{if } X, Y \in \mathcal{D} \\ \mathrm{Hom}_{\mathcal{E}}(F(X), G(Y)) & \text{if } X \in \mathcal{C},\ Y \in \mathcal{D} \\ \emptyset & \text{if } X \in \mathcal{D},\ Y \in \mathcal{C}\,. \end{cases}$$

The pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C} \star_{\mathcal{E}} \mathcal{D}}(X, Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C} \star_{\mathcal{E}} \mathcal{D}}(X, Y)$ admit similar descriptions.

We now specialize Construction 5.2.3.1 to the case where $\mathcal{D} = \mathcal{E}$ and the morphism $G : \mathcal{D} \to \mathcal{E}$ is the identity. Our goal is to prove the following:

Proposition 5.2.3.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:* 024J

(1) *The projection map $\pi : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \Delta^1$ is a cocartesian fibration of ∞-categories.*

(2) *The map*

$$\widetilde{F} : \Delta^1 \times \mathcal{C} \simeq (\mathcal{C} \star_{\mathcal{C}} \mathcal{C}) \to \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$$

witnesses the functor F as given by covariant transport along the nondegenerate edge of Δ^1.

The proof of Proposition 5.2.3.15 will require some preliminaries.

Lemma 5.2.3.16. *Suppose we are given a commutative diagram of simplicial sets* 02RG

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{D} \arrow[d, "V"] \\
\mathcal{C}' \arrow[r, "F'"] & \mathcal{D}',
\end{tikzcd}$$

so that U and V induce a morphism $W : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \mathcal{C}' \star_{\mathcal{D}'} \mathcal{D}'$. Let e be an edge of the simplicial set $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ satisfying the following conditions:

(1) *If e is contained in $\mathcal{C}$, then it is U-cocartesian when viewed as an edge of $\mathcal{C}$.*

(2) *The image of e under the map*

$$\rho : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \mathcal{D} \star_{\mathcal{D}} \mathcal{D} \simeq \Delta^1 \times \mathcal{D} \to \mathcal{D}$$

is V-cocartesian.

Then e is W-cocartesian.

Proof. Let $n \geq 2$ be an integer and suppose we are given a lifting problem

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \arrow[d, "W"] \\
\Delta^n \arrow[r, "\sigma'"] \arrow[ur, dashed, "\sigma"] & \mathcal{C}' \star_{\mathcal{D}'} \mathcal{D}',
\end{tikzcd} \qquad (5.12)$$

024H

where σ_0 carries $\mathrm{N}_\bullet(\{0 < 1\}) \subseteq \Lambda^n_0$ to the edge e. If σ' is contained in the simplicial subset $\mathcal{C}' \subseteq \mathcal{C}' \star_{\mathcal{D}'} \mathcal{D}'$, then the lifting problem (5.12) admits a solution by virtue of assumption (1).

Let us therefore assume that σ' is not contained in $\mathcal{C}'$. Let $\rho : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \mathcal{D}$ be as in (2), and define $\rho' : \mathcal{C}' \star_{\mathcal{D}'} \mathcal{D}' \to \mathcal{D}'$ similarly. Unwinding the definitions, we can rewrite (5.12) as a lifting problem

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\rho \circ \sigma_0"] \arrow[d] & \mathcal{D} \arrow[d, "V"] \\
\Delta^n \arrow[r, "\rho' \circ \sigma'"] \arrow[ur, dashed] & \mathcal{D}',
\end{tikzcd}$$

which admits a solution by virtue of assumption (2). □

`02RH` **Lemma 5.2.3.17.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{D} \arrow[d, "V"] \\
\mathcal{C}' \arrow[r, "F'"] & \mathcal{D}'.
\end{tikzcd}$$

Suppose that U and V are cocartesian fibrations, and that the morphism F carries U-cocartesian edges of $\mathcal{C}$ to V-cocartesian edges of $\mathcal{D}$. Then the induced map $W : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \mathcal{C}' \star_{\mathcal{D}'} \mathcal{D}'$ is also a cocartesian fibration. Moreover, an edge e of $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ is W-cocartesian if and only if it satisfies conditions (1) *and* (2) *of Lemma 5.2.3.16.*

Proof. It follows from Lemma 5.2.3.12 that W is an inner fibration of simplicial sets. Let us say that an edge of $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ is *special* if it satisfies conditions (1) and (2) of Lemma 5.2.3.16, so that every special edge of $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ is W-cocartesian. We consider three cases:

- Suppose that X belongs to $\mathcal{C}$ and $\overline{Y}$ belongs to $\mathcal{C}'$. In this case, our assumption that U is a cocartesian fibration guarantees that we can lift $\overline{e}$ to U-cocartesian edge $e : X \to Y$ of $\mathcal{C} \subseteq \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$. Since $F(e)$ is a V-cocartesian edge of $\mathcal{D}$, the edge e is special.

- Suppose that X belongs to $\mathcal{C}$ and $\overline{Y}$ belongs to $\mathcal{D}'$. In this case, we can identify $\overline{e}$ with an edge $\overline{e}_0 : V(F(X)) \to \overline{Y}$ of the simplicial set $\mathcal{D}'$. Since V is a cocartesian fibration, we can lift $\overline{e}_0$ to a V-cocartesian morphism $e_0 : F(X) \to Y$ of $\mathcal{D}$, which we can identify with a special edge $e : X \to Y$ of the simplicial set $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ satisfying $W(e) = \overline{e}$.

- Suppose that X belongs to $\mathcal{D}$ and $\overline{Y}$ belongs to $\mathcal{D}'$. In this case, our assumption that V is a cocartesian fibration guarantees that we can lift $\overline{e}$ to V-cocartesian edge $e : X \to Y$ of $\mathcal{D} \subseteq \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$, which is then special when regarded as an edge of $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$.

To complete the proof, it will suffice to show that every W-cocartesian edge $e : X \to Y$ of $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ is special. Applying the preceding argument, we can choose a special edge $e' : X \to Y'$ satisfying $W(e') = W(e)$. Set $\overline{Y} = W(Y) = W(Y')$. Since e and e' are both W-cocartesian, Remark 5.1.3.8 supplies a 2-simplex σ of the simplicial set $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ with boundary given by

$$\begin{tikzcd}
& Y \arrow[dr, "u"] & \\
X \arrow[ur, "e"] \arrow[rr, "e'"] & & Y',
\end{tikzcd}$$

where u is an isomorphism in the ∞-category $\{\overline{Y}\} \times_{(\mathcal{C}' \star_{\mathcal{D}'} \mathcal{D}')} (\mathcal{C} \star_{\mathcal{D}} \mathcal{D})$. Applying Remark 5.1.3.8 to the cocartesian fibrations U and V, we deduce that the edge e is also special. □

Example 5.2.3.18. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Applying Lemma 5.2.3.17 to the diagram 02RJ

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d] \\
\Delta^0 \arrow[r, equal] & \Delta^0,
\end{tikzcd}$$

we deduce that the projection map

$$\pi : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \Delta^0 \star_{\Delta^0} \Delta^0 \simeq \Delta^1$$

is a cocartesian fibration. Moreover, a morphism $e : X \to Y$ of the ∞-category $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$ is π-cocartesian if and only if it satisfies one of the following three conditions:

- The objects X and Y belong to $\mathcal{C}$ and e is an isomorphism in the ∞-category $\mathcal{C}$.
- The objects X and Y belong to $\mathcal{D}$ and e is an isomorphism in the ∞-category $\mathcal{D}$.
- The object X belongs to $\mathcal{C}$, the object Y belongs to $\mathcal{D}$, and e corresponds to an isomorphism $e_0 : F(X) \to Y$ in the ∞-category $\mathcal{D}$ (under the identification of Remark 5.2.3.14).

Proof of Proposition 5.2.3.15. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, so that the projection map $\pi : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \Delta^1$ of Example 5.2.3.18 is a cocartesian fibration. Note that the morphism

$$H : \Delta^1 \times \mathcal{C} \simeq \mathcal{C} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$$

satisfies $H|_{\{0\}\times\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$ and $H|_{\{1\}\times\mathcal{C}} = F$. To complete the proof, it will suffice to show that for every object $C \in \mathcal{C}$, the restriction $H|_{\Delta^1\times\{C\}}$ is a π-cocartesian morphism $e : X \to F(X)$ in the ∞-category $\mathcal{C} \star_{\mathcal{D}} \mathcal{D}$. This follows from the criterion of Example 5.2.3.18, since e corresponds to the identity morphism $\mathrm{id}_{F(X)} : F(X) \to F(X)$ under the identification of Remark 5.2.3.14. □

Passing to opposite ∞-categories, we obtain a dual version of Proposition 5.2.3.15:

024K **Variant 5.2.3.19.** Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. Then:

(1) The projection map $\pi : \mathcal{C} \star_{\mathcal{C}} \mathcal{D} \to \Delta^1$ is a cartesian fibration of ∞-categories.

(2) The map
$$h : \Delta^1 \times \mathcal{D} \simeq (\mathcal{D} \star_{\mathcal{D}} \mathcal{D}) \to \mathcal{C} \star_{\mathcal{C}} \mathcal{D}$$
witnesses the functor G as given by contravariant transport along the nondegenerate edge of Δ^1.

5.2.4 Fibrations over the 1-Simplex

024M Let $\mathcal{M}$ be an ∞-category equipped with a cocartesian fibration $\pi : \mathcal{M} \to \Delta^1$. Our goal in this section is to show that $\mathcal{M}$ is determined (up to equivalence) by the ∞-categories $\mathcal{C} = \{0\} \times_{\Delta^1} \mathcal{M}$, $\mathcal{D} = \{1\} \times_{\Delta^1} \mathcal{M}$, and the functor $F : \mathcal{C} \to \mathcal{D}$ given by covariant transport along the nondegenerate edge of Δ^1. This is a consequence of the following:

024N **Theorem 5.2.4.1.** *Let $U : \mathcal{M} \to \Delta^1$ be a functor of ∞-categories, and suppose we are given a commutative diagram σ :*

$$\begin{tikzcd}
\{1\} \times \mathcal{C} \arrow[r] \arrow[d] & \{1\} \times \mathcal{D} \arrow[d, "g"] \\
\Delta^1 \times \mathcal{C} \arrow[r, "h"] & \mathcal{M}
\end{tikzcd}$$

in the category $(\mathrm{Set}_{\Delta})_{/\Delta^1}$. Then σ is a categorical pushout diagram of simplicial sets (Definition 4.5.4.1) if and only if the following conditions are satisfied:

(1) *The restriction $h|_{\{0\}\times\mathcal{C}} : \mathcal{C} \to \{0\} \times_{\Delta^1} \mathcal{M}$ is a categorical equivalence of simplicial sets.*

(2) *The morphism $g : \mathcal{D} \to \{1\} \times_{\Delta^1} \mathcal{M}$ is a categorical equivalence of simplicial sets.*

(3) *For every vertex $C \in \mathcal{C}$, the restriction $h|_{\Delta^1\times\{C\}}$ is a U-cocartesian morphism of $\mathcal{M}$.*

Moreover, if these conditions are satisfied, then U is a cocartesian fibration.

Corollary 5.2.4.2. *Let $U : \mathcal{M} \to \Delta^1$ be a cocartesian fibration of ∞-categories with fibers $\mathcal{C} = \{0\} \times_{\Delta^1} \mathcal{M}$ and $\mathcal{D} = \{1\} \times_{\Delta^1} \mathcal{M}$. Let $h : \Delta^1 \times \mathcal{C} \to \mathcal{M}$ be a functor which witnesses the functor $F = h|_{\{1\} \times \mathcal{C}}$ as given by covariant transport along the nondegenerate edge of Δ^1 (Definition 5.2.2.4). Then h induces a categorical equivalence of simplicial sets* 024P

$$(\Delta^1 \times \mathcal{C}) \coprod_{(\{1\} \times \mathcal{C})} \mathcal{D} \to \mathcal{M} .$$

Proof. Combine Theorem 5.2.4.1 with Proposition 4.5.4.11. □

Remark 5.2.4.3. Let $U : \mathcal{M} \to \Delta^1$ be a cocartesian fibration of ∞-categories. It follows from Corollary 5.2.4.2 that the ∞-category $\mathcal{M}$ can be recovered (up to equivalence) from the ∞-categories $\mathcal{C} = \{0\} \times_{\Delta^1} \mathcal{M}$, $\mathcal{D} = \{1\} \times_{\Delta^1} \mathcal{M}$, and the covariant transport functor $F : \mathcal{C} \to \mathcal{D}$. Similarly, if $U : \mathcal{M} \to \Delta^1$ is a cartesian fibration, then the ∞-category $\mathcal{M}$ can be recovered from $\mathcal{C}$, $\mathcal{D}$, and the contravariant transport functor $G : \mathcal{D} \to \mathcal{C}$. 024Q

As an application of Theorem 5.2.4.1, we give an alternative characterization of the covariant transport functors introduced in §5.2.2.

Corollary 5.2.4.4. *Let $U : \mathcal{M} \to \Delta^1$ be a cocartesian fibration of ∞-categories and let $F : \mathcal{M}_0 \to \mathcal{M}_1$ be a functor. The following conditions are equivalent:* 046W

(1) *The functor F is given by covariant transport along the nondegenerate edge of Δ^1 (in the sense of Definition 5.2.2.4).*

(2) *There exists a functor $R : \mathcal{M} \to \mathcal{M}_1$ such that $R|_{\mathcal{M}_0} = F$, $R|_{\mathcal{M}_1} = \mathrm{id}$, and R carries U-cocartesian morphisms of $\mathcal{M}$ to isomorphisms in $\mathcal{M}_1$.*

Proof. Let e denote the nondegenerate edge of Δ^1. By virtue of Proposition 5.2.2.8, we can choose a functor $F' : \mathcal{M}_0 \to \mathcal{M}_1$ and a natural transformation $H : \Delta^1 \times \mathcal{M}_0 \to \mathcal{M}$ which exhibits F' as given by covariant transport along e. Let $R : \mathcal{M} \to \mathcal{M}_1$ be a functor satisfying condition (2). Then the composition

$$\Delta^1 \times \mathcal{M}_0 \xrightarrow{H} \mathcal{M} \xrightarrow{R} \mathcal{M}_1$$

can be regarded as a natural transformation from $R \circ H|_{\{0\} \times \mathcal{M}_0} = F$ to $R \circ H|_{\{1\} \times \mathcal{M}_1} = F'$. By assumption, this natural transformation carries each object of $\mathcal{M}_0$ to an isomorphism in the ∞-category $\mathcal{M}_1$, and is therefore an isomorphism of functors (Theorem 4.4.4.4). It follows that the functor F is also given by covariant transport along e (see Proposition 5.2.2.8). This proves the implication (2) ⇒ (1).

Now suppose that condition (1) is satisfied. Then we can assume that $F' = F$, so that we have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{M}_0 \arrow[r, "F"] \arrow[d] & \mathcal{M}_1 \arrow[d] \\
\mathcal{M}_0 \coprod \mathcal{M}_0 \arrow[r, "\mathrm{id} \coprod F"] \arrow[d] & \mathcal{M}_0 \coprod \mathcal{M}_1 \arrow[d] \\
\Delta^1 \times \mathcal{M}_0 \arrow[r, "H"] & \mathcal{M} .
\end{tikzcd}$$

The upper half of the diagram is a pushout square in which the vertical maps are monomorphisms, and therefore a categorical pushout square (Example 4.5.4.12). Theorem 5.2.4.1 guarantees that the outer rectangle is a categorical pushout square, so the lower half of the diagram is also a categorical pushout square (Proposition 4.5.4.8). It follows that the diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{M}, \mathcal{M}_1) \arrow[r, "\circ H"] \arrow[d] & \mathrm{Fun}(\Delta^1 \times \mathcal{M}_0, \mathcal{M}_1) \arrow[d] \\
\mathrm{Fun}(\mathcal{M}_0 \coprod \mathcal{M}_1, \mathcal{M}_1) \arrow[r] & \mathrm{Fun}(\mathcal{M}_0 \coprod \mathcal{M}_0, \mathcal{M}_1)
\end{tikzcd}$$

is a categorical pullback square (Proposition 4.5.4.4). Since the vertical maps are isofibrations (Corollary 4.4.5.3), Corollary 4.5.2.32 implies that composition with H induces an equivalence of ∞-categories

$$\{(F, \mathrm{id})\} \times_{\mathrm{Fun}(\mathcal{M}_0 \coprod \mathcal{M}_1, \mathcal{M}_1)} \mathrm{Fun}(\mathcal{M}, \mathcal{M}_1) \xrightarrow{\circ H} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{M}_0, \mathcal{M}_1)}(F, F).$$

It follows that we can choose a functor $R : \mathcal{M} \to \mathcal{M}_1$ such that $R|_{\mathcal{M}_0} = F$, $R|_{\mathcal{M}_1} = \mathrm{id}$, and the composition $R \circ H$ is homotopic to the identity (when regarded as a morphism from F to itself in the ∞-category $\mathrm{Fun}(\mathcal{M}_0, \mathcal{M}_1)$). To complete the proof, it will suffice to show that if $f : X \to Y$ is a U-cocartesian morphism of $\mathcal{M}$, then $U(f)$ is an isomorphism. We may assume without loss of generality that X belongs to $\mathcal{M}_0$ and Y belongs to $\mathcal{M}_1$ (otherwise, f is already an isomorphism and there is nothing to prove). In this case, Remark 5.1.3.8 guarantees that f is isomorphic (as an object of $\mathrm{Fun}(\Delta^1, \mathcal{M})$) to the edge $H|_{\Delta^1 \times \{X\}}$. It will therefore suffice to show that $(R \circ H)|_{\Delta^1 \times \{X\}}$ is an isomorphism in $\mathcal{M}_1$, which is clear (since it is homotopic to the identity morphism from $F(X)$ to itself). □

For any functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$, the projection map

$$\mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \Delta^0 \star_{\Delta^0} \Delta^0 \simeq \Delta^1$$

is a cocartesian fibration (Proposition 5.2.3.15). The commutative diagram of simplicial sets

$$\begin{tikzcd}
\emptyset \star_{\mathcal{C}} \mathcal{C} \arrow[r] \arrow[d] & \emptyset \star_{\mathcal{D}} \mathcal{D} \arrow[d] \\
\mathcal{C} \star_{\mathcal{C}} \mathcal{C} \arrow[r] & \mathcal{C} \star_{\mathcal{D}} \mathcal{D}
\end{tikzcd}$$

satisfies the hypotheses of Theorem 5.2.4.1, and is therefore a categorical pushout square. This is a special case of the following more general assertion, which does not require $\mathcal{C}$ and $\mathcal{D}$ to be ∞-categories:

Proposition 5.2.4.5. *Let $f : X \to Y$ be a morphism of simplicial sets. Then the diagram* 024R

0356
$$\begin{tikzcd}
\{1\} \times X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
\Delta^1 \times X \arrow[r] & X \star_Y Y
\end{tikzcd} \tag{5.13}$$

is a categorical pushout square of simplicial sets. Here the lower horizontal map is given by the composition

$$\Delta^1 \times X \simeq X \star_X X \xrightarrow{\mathrm{id} \star_f f} X \star_Y Y.$$

Example 5.2.4.6. In the special case $Y = \Delta^0$, Proposition 5.2.4.5 asserts that the diagram 024S

$$\begin{tikzcd}
\{1\} \times X \arrow[r] \arrow[d] & \Delta^0 \arrow[d] \\
\Delta^1 \times X \arrow[r] & X^{\triangleright}
\end{tikzcd}$$

is a categorical pushout square: that is, that the comparison map $X \diamond \Delta^0 \to X \star \Delta^0$ of Notation 4.5.8.3 is a categorical equivalence. This is the content of Proposition 4.5.8.12 (which is a special case of Theorem 4.5.8.8).

Corollary 5.2.4.7. *Suppose we are given a commutative diagram of simplicial sets* 0357

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X' \arrow[r] & Y',
\end{tikzcd}$$

where the vertical maps are categorical equivalences. Then the induced map $X \star_Y Y \to X' \star_{Y'} Y'$ *is also a categorical equivalence of simplicial sets.*

Proof. Combine Propositions 5.2.4.5 and 4.5.4.9. □

Proof of Proposition 5.2.4.5. The diagram (5.13) determines a morphism of simplicial sets

$$\lambda_X : (\Delta^1 \times X) \coprod_{(\{1\} \times X)} Y \to X \star_Y Y,$$

and we wish to show that λ_X is a categorical equivalence of simplicial sets (obtained by applying Construction 5.3.4.7 to the diagram $[1] \to \mathrm{Set}_{\Delta}$ determined by the morphism f). We wish to show that λ_X is a categorical equivalence of simplicial sets (Proposition 4.5.4.11). By virtue of Corollary 4.5.7.3, it will suffice to prove that for every map $\Delta^n \to Y$, the induced map

$$\Delta^n \times_Y ((\Delta^1 \times X) \coprod_{(\{1\} \times X)} Y) \to \Delta^n \times_Y (X \star_Y Y)$$

is a categorical equivalence. Using Remark 5.2.3.9, we can replace Y by Δ^n and X by the fiber product $\Delta^n \times_Y X$, and thereby reduce the proof of Proposition 5.2.4.5 to the special case where $Y = \Delta^n$ is a standard simplex.

Since the collection of categorical equivalences is closed under the formation of filtered colimits (Corollary 4.5.7.2), we may assume without loss of generality that the simplicial set X is finite (see Remark 3.6.1.8). In particular, X has dimension $\leq m$ for some integer $m \geq -1$. We proceed by induction on m. If $m = -1$, then X is empty and the morphism λ_X is an isomorphism (see Example 5.2.3.4). Assume that $m \geq 0$; we now proceed by induction on the number of nondegenerate m-simplices of X. If X does not have dimension $\leq m - 1$, then a choice of nondegenerate m-simplex of X determines a pushout diagram

$$\begin{tikzcd}
\partial \Delta^m \arrow[r] \arrow[d] & \Delta^m \arrow[d] \\
X' \arrow[r] & X,
\end{tikzcd}$$

where the horizontal maps are monomorphisms (Proposition 1.1.4.12). We then obtain a cubical diagram

$$\begin{tikzcd}
(\Delta^1 \times \partial \Delta^m)\coprod_{(\{1\} \times \partial \Delta^m)} Y \arrow[rr] \arrow[dr, "\lambda_{\partial \Delta^m}"'] \arrow[dd] & & (\Delta^1 \times \Delta^m)\coprod_{(\{1\} \times \Delta^m)} Y \arrow[dr, "\lambda_{\Delta^m}"'] \arrow[dd] & \\
& \partial \Delta^m \star_Y Y \arrow[rr] \arrow[dd] & & \Delta^m \star_Y Y \arrow[dd] \\
(\Delta^1 \times X')\coprod_{(\{1\} \times X')} Y \arrow[rr] \arrow[dr, "\lambda_{X'}"'] & & (\Delta^1 \times X)\coprod_{(\{1\} \times X)} Y \arrow[dr, "\lambda_X"'] & \\
& X' \star_Y Y \arrow[rr] & & X \star_Y Y,
\end{tikzcd}$$

where the front and back faces are categorical pushout squares (Proposition 4.5.4.11). Our inductive hypothesis guarantees that the morphisms $\lambda_{X'}$ and $\lambda_{\partial \Delta^m}$ are categorical equivalences. Consequently, to show that λ_X is a categorical equivalence, it will suffice to show that λ_{Δ^m} is a categorical equivalence. We can therefore replace X by Δ^m, and thereby reduce the proof of Proposition 5.2.4.5 to the special case where $f : \Delta^m \to \Delta^n$ is a morphism between standard simplices.

Suppose that $f(m) < n$. In this case, we can identify f with a morphism from $X = \Delta^m$ to the simplex Δ^{n-1} (regarded as a simplicial subset of Δ^n), and we can identify $X \star_Y Y$ with the right cone $(X \star_{\Delta^{n-1}} \Delta^{n-1})^{\triangleright}$. Under this identification, λ_X corresponds to the composition

$$\begin{aligned}
(\Delta^1 \times X)\coprod_{(\{1\} \times X)} (\Delta^{n-1})^{\triangleright} &\xrightarrow{\lambda'} (\Delta^1 \times X)^{\triangleright}\coprod_{(\{1\} \times X)^{\triangleright}} (\Delta^{n-1})^{\triangleright} \\
&\simeq (\Delta^1 \times X)\coprod_{(\{1\} \times X)} (\Delta^{n-1})^{\triangleright} \\
&\xrightarrow{\lambda''} (X \star_{\Delta^{n-1}} \Delta^{n-1})^{\triangleright},
\end{aligned}$$

where λ' is a pushout of the map

$$(\Delta^1 \times X)\coprod_{(\{1\} \times X)} (\{1\} \times X)^{\triangleright} \to (\Delta^1 \times X)^{\triangleright}$$

and is therefore inner anodyne by virtue of Example 4.3.6.5 (since the inclusion $\{1\} \times X \hookrightarrow \Delta^1 \times X$ is right anodyne; see Proposition 4.2.5.3). Consequently, to show that λ_X is a categorical equivalence, it will suffice to show that λ'' is a categorical equivalence. By virtue of Corollary 4.5.8.9, we are reduced to proving Proposition 5.2.4.5 for the map $f : X \to \Delta^{n-1}$. Applying this argument repeatedly, we can reduce to the case where $f(m) = n$.

Let $Z(0)$ denote the simplicial subset of $\Delta^1 \times \Delta^m$ given by the union of $\Delta^1 \times \partial \Delta^m$ with $\{1\} \times \Delta^m$, and let

$$Z(0) \subset Z(1) \subset Z(2) \subset \cdots \subset Z(m) \subset Z(m+1) = \Delta^1 \times \Delta^m$$

be the sequence of simplicial subsets appearing in Lemma 3.1.2.12. Note that λ_X carries $Z(m)$ into the simplicial subset $\partial \Delta^m \star_Y Y \subseteq X \star_Y Y$. We therefore obtain a cubical diagram of simplicial sets

$$\begin{tikzcd}
Z(0) \arrow[rr] \arrow[dr] \arrow[dd] & & (\Delta^1 \times \partial \Delta^m) \coprod_{(\{1\} \times \partial \Delta^m)} Y \arrow[dr, "\lambda_{\partial \Delta^m}"] \arrow[dd] & \\
& Z(m) \arrow[rr] \arrow[dd] & & \partial \Delta^m \star_Y Y \arrow[dd] \\
\Delta^1 \times \Delta^m \arrow[rr] \arrow[dr, "\mathrm{id}"] & & (\Delta^1 \times \Delta^m) \coprod_{(\{1\} \times \Delta^m)} Y \arrow[dr, "\lambda_{\Delta^m}"] & \\
& \Delta^1 \times \Delta^m \arrow[rr] & & \Delta^m \star_Y Y
\end{tikzcd}$$

where the front and back faces are pushout squares and the vertical maps are monomorphisms. It follows that the front and back faces are categorical pushout squares (Example 4.5.4.12). Our inductive hypothesis guarantees that $\lambda_{\partial \Delta^m}$ is a categorical equivalence, and the inclusion $Z(0) \hookrightarrow Z(m)$ is inner anodyne by construction (see Lemma 3.1.2.12). Applying Proposition 4.5.4.9, we conclude that λ_{Δ^m} is also a categorical equivalence. □

Proof of Theorem 5.2.4.1. Let $U : \mathcal{M} \to \Delta^1$ be a functor of ∞-categories and suppose we are given a commutative diagram σ :

$$\begin{tikzcd}
\{1\} \times \mathcal{C} \arrow[r, "F"] \arrow[d] & \{1\} \times \mathcal{D} \arrow[d, "g"] \\
\Delta^1 \times \mathcal{C} \arrow[r, "h"] & \mathcal{M}
\end{tikzcd}$$

in the category $(\mathrm{Set}_{\Delta})_{/\Delta^1}$. We wish to show that σ is a categorical pushout square if and only if conditions (1) through (3) of Theorem 5.2.4.1 are satisfied.

We first reduce to the case where $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories. Choose inner anodyne morphisms $\mathcal{C} \hookrightarrow \mathcal{C}'$ and $\mathcal{D} \hookrightarrow \mathcal{D}'$, where $\mathcal{C}'$ and $\mathcal{D}'$ are ∞-categories (Corollary 4.1.3.3). Since the fiber $\{1\} \times_{\Delta^1} \mathcal{M}$ is an ∞-category, we can extend g to a functor $g' : \mathcal{D}' \to \{1\} \times_{\Delta^1} \mathcal{M}$. Similarly, the composition $\mathcal{C} \xrightarrow{F} \mathcal{D} \hookrightarrow \mathcal{D}'$ extends to a functor of ∞-categories $F' : \mathcal{C}' \to \mathcal{D}'$. Using Exercise 3.1.7.11, we can factor F' as a composition $\mathcal{C}' \xrightarrow{F''} \mathcal{D}'' \xrightarrow{v} \mathcal{D}'$, where F'' is a monomorphism and v is a trivial Kan fibration. It follows from Lemma 1.5.7.5 that the inclusion map

$$(\Delta^1 \times \mathcal{C}) \coprod_{(\{1\} \times \mathcal{C})} (\{1\} \times \mathcal{C}') \hookrightarrow \Delta^1 \times \mathcal{C}'$$

is inner anodyne, so that we can extend h to a functor $h' : \Delta^1 \times \mathcal{C}' \to \mathcal{M}$ satisfying $h'|_{\{1\} \times \mathcal{C}'} = g' \circ F'$. By virtue of Proposition 4.5.4.9, σ is a categorical pushout square if and only if the diagram σ :

$$\begin{tikzcd}
\{1\} \times \mathcal{C}' \arrow[r, "F''"] \arrow[d] & \{1\} \times \mathcal{D}'' \arrow[d, "g' \circ v"] \\
\Delta^1 \times \mathcal{C}' \arrow[r, "h'"] & \mathcal{M}
\end{tikzcd}$$

is a categorical pushout square. We may therefore replace $\mathcal{C}$ and $\mathcal{D}$ by $\mathcal{C}'$ and $\mathcal{D}''$, and thereby reduce to the case where $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories and F is a monomorphism.

The assumption that F is a monomorphism guarantees that the natural map

$$\iota : (\Delta^1 \times \mathcal{C}) \coprod_{(\{1\} \times \mathcal{C})} \mathcal{D} \to \mathcal{C} \star_{\mathcal{D}} \mathcal{D}$$

is also a monomorphism, and Proposition 5.2.4.5 guarantees that ι is a categorical equivalence of simplicial sets. Since $\mathcal{M}$ is an ∞-category, Lemma 4.5.5.2 guarantees the existence of a functor $G : \mathcal{C} \star_{\mathcal{D}} \mathcal{D} \to \mathcal{M}$ satisfying $G|_{\Delta^1 \times \mathcal{C}} = h$ and $G|_{\mathcal{D}} = g$. By virtue of Proposition 4.5.4.9, the diagram σ is a categorical pushout square if and only if the functor G is an equivalence of ∞-categories.

Note that the functor G fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \star_{\mathcal{D}} \mathcal{D} \arrow[rr, "G"] \arrow[dr, "U'"'] & & \mathcal{M} \arrow[dl, "U"] \\
& \Delta^1, &
\end{tikzcd}$$

where U' is the cocartesian fibration of Proposition 5.2.3.15, and the functor U is an isofibration (Example 4.4.1.6). The desired result now follows by applying the criterion of Theorem 5.1.6.1 (and invoking Remark 5.1.6.8). □

5.2.5 The Homotopy Transport Representation

01VT We now study the behavior of the transport functors of §5.2.2 with respect to composition.

01VU **Proposition 5.2.5.1** (Transitivity)**.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let σ be a 2-simplex of $\mathcal{C}$, which we display as a diagram*

$$\begin{tikzcd} & D \arrow[dr, "g"] & \\ C \arrow[ur, "f"] \arrow[rr, "h"] & & E. \end{tikzcd}$$

Let $f_! : \mathcal{E}_C \to \mathcal{E}_D$ and $g_! : \mathcal{E}_D \to \mathcal{E}_E$ be functors which are given by covariant transport along f and g, respectively. Then the composite functor $(g_! \circ f_!) : \mathcal{E}_C \to \mathcal{E}_E$ is given by covariant transport along h.

Proof. Without loss of generality, we may replace U by the projection map $\Delta^2 \times_{\mathcal{C}} \mathcal{E} \to \Delta^2$, and thereby reduce to the case where $\mathcal{C} = \Delta^2$ and σ is the unique nondegenerate 2-simplex of $\mathcal{C}$. In this case, $\mathcal{E}$ is an ∞-category. Let $u : \mathrm{id}_{\mathcal{E}_C} \to f_!$ be a morphism in the ∞-category $\mathrm{Fun}(\mathcal{E}_C, \mathcal{E})$ which witnesses $f_!$ as given by covariant transport along f, and let $v : \mathrm{id}_{\mathcal{E}_D} \to g_!$ be a morphism in the ∞-category $\mathrm{Fun}(\mathcal{E}_D, \mathcal{E})$ which witnesses $g_!$ as given by covariant transport along g. Let $v' : f_! \to g_! \circ f_!$ denote the image of v under the functor $\mathrm{Fun}(\mathcal{E}_D, \mathcal{E}) \to \mathrm{Fun}(\mathcal{E}_C, \mathcal{E})$ given by precomposition with $f_!$. Let $w : \mathrm{id}_{\mathcal{E}_C} \to g_! \circ f_!$ be a composition of u with v' in the ∞-category $\mathrm{Fun}(\mathcal{E}_C, \mathcal{E})$. We will complete the proof by showing that w witnesses $g_! \circ f_!$ as given by covariant transport along h. To prove this, we must show that for every object $X \in \mathcal{E}_C$, the morphism $w_X : X \to (g_! \circ f_!)(X)$ is U-cocartesian. This follows from Corollary 5.1.2.4, since w_X is a composition of the U-cocartesian morphisms $u_X : X \to f_!(X)$ and $v_{f_!(X)} : f_!(X) \to (g_! \circ f_!)(X)$. □

019U **Construction 5.2.5.2** (The Homotopy Transport Representation: Covariant Case)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let hQCat denote the homotopy category of ∞-categories. It follows from Proposition 5.2.5.1 and Example 5.2.2.5 that there is a unique functor $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ with the following properties:

- For each vertex C of the simplicial set $\mathcal{C}$, $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}(C)$ is the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ (regarded as an object of hQCat).

- For each edge $f : C \to D$ of the simplicial set $\mathcal{C}$ representing a morphism $[f] \in \operatorname{Hom}_{\mathrm{h}\mathcal{C}}(C, D)$, we have $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}([f]) = [f_!]$. Here $[f_!]$ denotes the isomorphism class of the covariant transport functor of Notation 5.2.2.9, which we regarded as an element of the set
$$\operatorname{Hom}_{\mathrm{hQCat}}(\mathcal{E}_C, \mathcal{E}_D) = \pi_0(\operatorname{Fun}(\mathcal{E}_C, \mathcal{E}_D)^{\simeq}).$$

We will refer to $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ as the *homotopy transport representation* of the cocartesian fibration U.

Example 5.2.5.3. Let $U : \mathcal{E} \to \mathcal{C}$ be a left covering map of simplicial sets. Then the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ of Construction 5.2.5.2 coincides with the functor $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{Set}$ of Proposition 5.2.0.3 (here we abuse notation by identifying the category of sets with the full subcategory of hKan spanned by the discrete simplicial sets). 0358

Remark 5.2.5.4. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, and let $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ be the homotopy transport representation of Construction 5.2.5.2. It follows from Proposition 5.1.4.14 that U is a left fibration if and only if the functor $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ factors through the full subcategory $\mathrm{hKan} \subseteq \mathrm{hQCat}$. In particular, if U is a left fibration, then Construction 5.2.5.2 determines a functor $\mathrm{h}\mathcal{C} \to \mathrm{hKan}$ which we will also refer to as the *homotopy transport representation* of the left fibration U. 01VY

Remark 5.2.5.5. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let $f : C \to D$ be a morphism of $\mathcal{C}$, and let $f_! : \mathcal{E}_C \to \mathcal{E}_D$ be given by covariant transport along f. If f is an isomorphism in the ∞-category $\mathcal{C}$, then $f_!$ is an equivalence of ∞-categories. This follows from the observation that the homotopy transport functor $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ carries isomorphisms to isomorphisms. 02RD

Remark 5.2.5.6 (Base Change)**.** Suppose we are given a pullback diagram of simplicial sets 0359

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r] & \mathcal{C},
\end{tikzcd}$$

where U and U' are cocartesian fibrations. Then the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}'/\mathcal{C}'}$ is isomorphic to the composite functor

$$\mathrm{h}\mathcal{C}' \to \mathrm{h}\mathcal{C} \xrightarrow{\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}} \mathrm{hQCat}.$$

Construction 5.2.5.2 has an analogue for cartesian fibrations:

019Z **Construction 5.2.5.7** (The Homotopy Transport Representation: Contravariant Case)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of simplicial sets and let hQCat denote the homotopy category of ∞-categories (Construction 4.5.1.1). It follows from Proposition 5.2.5.1 and Example 5.2.2.5 that there is a unique functor $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathrm{h}\mathcal{C}^{\mathrm{op}} \to \mathrm{hQCat}$ satisfying the following conditions:

- For each vertex C of the simplicial set $\mathcal{C}$, $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ is the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ (regarded as an object of hQCat).
- For each edge $f : C \to D$ of the simplicial set $\mathcal{C}$ representing a morphism $[f] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, D)$, we have $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}([f]) = [f^*]$, where $[f^*]$ denotes the isomorphism class of the contravariant transport functor of Notation 5.2.2.18.

We will refer to $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ as the *homotopy transport representation* of the cartesian fibration U.

01VZ **Warning 5.2.5.8.** Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets which is both a cartesian fibration and a cocartesian fibration. Then Constructions 5.2.5.2 and 5.2.5.7 supply functors $\mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ and $\mathrm{h}\mathcal{C}^{\mathrm{op}} \to \mathrm{hQCat}$ respectively, which are both referred to as the homotopy transport representation of U and both denoted by $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$. We will see later that these two functors are interchangeable data: either can be recovered from the other (see Proposition 6.2.3.5).

01W0 **Example 5.2.5.9.** Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets. Combining Remark 5.2.5.4 with Theorem 5.2.2.20, we deduce that the following conditions are equivalent:

- The morphism U is a Kan fibration.
- The morphism U is a cocartesian fibration and the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ of Construction 5.2.5.2 factors through the subcategory $\mathrm{hKan}^{\simeq} \subseteq \mathrm{hQCat}$.
- The morphism U is a cartesian fibration and the homotopy transport representation $\mathrm{hTr}'_{\mathcal{E}\,/\,\mathcal{C}} : \mathrm{h}\mathcal{C}^{\mathrm{op}} \to \mathrm{hQCat}$ of Construction 5.2.5.7 factors through the subcategory $\mathrm{hKan}^{\simeq} \subseteq \mathrm{hQCat}$.

If these conditions are satisfied, then $\mathrm{hTr}'_{\mathcal{E}\,/\,\mathcal{C}}$ is given by the composition

$$\mathrm{h}\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{hTr}^{\mathrm{op}}_{\mathcal{E}\,/\,\mathcal{C}}} (\mathrm{hKan}^{\simeq})^{\mathrm{op}} \xrightarrow{\iota} \mathrm{hKan}^{\simeq},$$

where ι is the isomorphism which carries each morphism in $\mathrm{hKan}^{\simeq}$ to its inverse.

5.2.6 Elements of Set-Valued Functors

Throughout this section, we let Set denote the category of sets. 01Q7

Construction 5.2.6.1 (The Category of Elements)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a functor. We define a category $\int_{\mathcal{C}} \mathscr{F}$ as follows: 01Q8

- The objects of $\int_{\mathcal{C}} \mathscr{F}$ are pairs (C, x), where C is an object of $\mathcal{C}$ and x is an element of the set $\mathscr{F}(C)$.
- If (C, x) and (C', x') are objects of $\int_{\mathcal{C}} \mathscr{F}$, then a morphism from (C, x) to (C', x') in the category $\int_{\mathcal{C}} \mathscr{F}$ is a morphism $f : C \to C'$ in the category $\mathcal{C}$ for which the induced map $\mathscr{F}(f) : \mathscr{F}(C) \to \mathscr{F}(C')$ carries x to x'.
- Composition of morphisms in $\int_{\mathcal{C}} \mathscr{F}$ is given by composition of morphisms in $\mathcal{C}$.

We will refer to $\int_{\mathcal{C}} \mathscr{F}$ as the *category of elements* of the functor $\mathscr{F}$. Note that the construction $(C, x) \mapsto C$ determines a functor $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$, which we will refer to as the *forgetful functor*.

Variant 5.2.6.2. Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a functor. We define a category $\int^{\mathcal{C}} \mathscr{F}$ as follows: 01Q9

- The objects of $\int^{\mathcal{C}} \mathscr{F}$ are pairs (C, x), where C is an object of $\mathcal{C}$ and x is an element of the set $\mathscr{F}(C)$.
- If (C, x) and (C', x') are objects of $\int^{\mathcal{C}} \mathscr{F}$, then a morphism from (C, x) to (C', x') in the category $\int^{\mathcal{C}} \mathscr{F}$ is a morphism $f : C \to C'$ in the category $\mathcal{C}$ for which the induced map $\mathscr{F}(f) : \mathscr{F}(C') \to \mathscr{F}(C)$ carries x' to x.
- Composition of morphisms in $\int^{\mathcal{C}} \mathscr{F}$ is given by composition of morphisms in $\mathcal{C}$.

We will refer to $\int^{\mathcal{C}} \mathscr{F}$ as the *category of elements* of the functor $\mathscr{F}$. Note that the construction $(C, x) \mapsto C$ determines a functor $U : \int^{\mathcal{C}} \mathscr{F} \to \mathcal{C}$, which we will refer to as the *forgetful functor*.

Remark 5.2.6.3. Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a functor. Then we have a canonical isomorphism of categories 01QA

$$(\int_{\mathcal{C}} \mathscr{F})^{\mathrm{op}} \simeq (\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}),$$

where $\int_{\mathcal{C}} \mathscr{F}$ is the category of elements introduced in Construction 5.2.6.1 and $\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}$ is the category of elements introduced in Variant 5.2.6.2.

Example 5.2.6.4. Let $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. Then $\int^{\mathbf{\Delta}} X$ is the *category of simplices* $\mathbf{\Delta}_X$ introduced in Construction 1.1.3.9. 01QB

01QC **Example 5.2.6.5.** Let $\mathcal{C}$ be a category, let X be an object of $\mathcal{C}$, and let $h^X : \mathcal{C} \to \mathrm{Set}$ denote the functor corepresented by X (given on objects by the formula $h^X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$). Then the category of elements $\int_{\mathcal{C}} h^X$ can be identified with the coslice category $\mathcal{C}_{X/}$ of Variant 4.3.1.4. Similarly, if $h_X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is the functor represented by X (given on objects by $h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$), then the category of elements $\int^{\mathcal{C}} h_X$ can be identified with the slice category $\mathcal{C}_{/X}$.

01QD **Remark 5.2.6.6.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a functor. Then the category of elements $\int_{\mathcal{C}} \mathscr{F}$ fits into a pullback diagram

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r] \arrow[d] & \mathrm{Set}_* \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathrm{Set}.
\end{tikzcd}$$

Here Set_* denotes the category of pointed sets (see Example 4.2.3.3).

01QE **Remark 5.2.6.7.** Let $\mathcal{C}$ be a small category, let $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ be the category of set-valued functors on $\mathcal{C}^{\mathrm{op}}$, and let $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ be the Yoneda embedding (so that h carries each object $C \in \mathcal{C}$ to the representable functor $h_C = \mathrm{Hom}_{\mathcal{C}}(\bullet, C)$). For any object $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$, the category of elements $\int^{\mathcal{C}} \mathscr{F}$ fits into a pullback diagram

$$\begin{tikzcd}
\int^{\mathcal{C}} \mathscr{F} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})_{/\mathscr{F}} \arrow[d] \\
\mathcal{C} \arrow[r, "h"] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}).
\end{tikzcd}$$

This is essentially a reformulation of Yoneda's lemma (see Corollary 8.4.2.7 for an ∞-categorical counterpart).

We now show that, up to isomorphism, every functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ can be recovered from the category of elements $\int_{\mathcal{C}} \mathscr{F}$ (together with the forgetful functor $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$). Let Cat denote the category of (small) categories.

01QF **Proposition 5.2.6.8.** *Let $\mathcal{C}$ be a small category. Then:*

- *Construction 5.2.6.1 determines a fully faithful functor*

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \to \mathrm{Cat}_{/\mathcal{C}} \qquad \mathscr{F} \mapsto \int_{\mathcal{C}} \mathscr{F}.$$

- *Variant 5.2.6.2 determines a fully faithful functor*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Cat}_{/\mathcal{C}} \qquad \mathscr{F} \mapsto \int^{\mathcal{C}} \mathscr{F}.$$

Proof. We will prove the first assertion; the second follows by a similar argument. Let $\mathscr{F}$ and $\mathscr{G}$ be functors from $\mathcal{C}$ to the category of sets, and let $T : (\int_{\mathcal{C}} \mathscr{F}) \to (\int_{\mathcal{C}} \mathscr{G})$ be a functor for which the diagram

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[rr, "T"] \arrow[dr] & & \int_{\mathcal{C}} \mathscr{G} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

is *strictly* commutative, where the vertical maps are the forgetful functors. We wish to show that there is a unique natural transformation of functors

$$f : \mathscr{F} \to \mathscr{G} \qquad \{f_C : \mathscr{F}(C) \to \mathscr{G}(C)\}_{C \in \mathcal{C}}$$

for which the functor T is given on objects by the construction $T(C, x) = (C, f_C(x))$. Note that this requirement uniquely determines the function $f_C : \mathscr{F}(C) \to \mathscr{G}(C)$ for each object $C \in \mathcal{C}$. We must show that the resulting collection $\{f_C\}_{C \in \mathcal{C}}$ is a natural transformation: that is, for every morphism $u : C \to D$ in the category $\mathcal{C}$, the diagram of sets

$$\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "f_C"] \arrow[d, "\mathscr{F}(u)"] & \mathscr{G}(C) \arrow[d, "\mathscr{G}(u)"] \\
\mathscr{F}(D) \arrow[r, "f_D)"] & \mathscr{G}(D)
\end{tikzcd}$$

is commutative. Fix an element $x \in \mathscr{F}(C)$, so that u can be regarded as a morphism from (C, x) to $(D, \mathscr{F}(u)(x))$ in the category $\int_{\mathcal{C}} \mathscr{F}$. Applying the functor T, we deduce that u can also be regarded as a morphism from $(C, f_C(x))$ to $(D, f_D(\mathscr{F}(u)(x)))$ in the category $\int_{\mathcal{C}} \mathscr{G}$. It follows that $\mathscr{G}(u)(f_C(x)) = f_D(\mathscr{F}(u)(x))$, as desired. □

Remark 5.2.6.9. Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a functor, and let $\int_{\mathcal{C}} \mathscr{F}$ denote the category of elements of $\mathscr{F}$ (Construction 5.2.6.1). Then the forgetful functor $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is a left covering functor, in the sense of Definition 4.2.3.1. This follows from the pullback 01QP

diagram

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r] \arrow[d] & \mathrm{Set}_* \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathrm{Set}
\end{tikzcd}$$

of Remark 5.2.6.6, together with Remark 4.2.3.6 and Example 4.2.3.3. We will see in §5.2.7 that the converse is also true: for every left covering functor $U : \mathcal{E} \to \mathcal{C}$, there exists a functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ and isomorphism $\mathcal{E} \simeq \int_{\mathcal{C}} \mathscr{F}$ which is compatible with the functor U (Corollary 5.2.7.5). By virtue of Proposition 5.2.6.8, the functor $\mathscr{F}$ is unique up to canonical isomorphism.

5.2.7 Covering Space Theory

`01QG` Let S be a topological space. Every covering map $f : X \to S$ determines a functor from the fundamental groupoid $\pi_{\leq 1}(S)$ to the category of sets, given by the monodromy representation of Example 5.2.0.5. Under some mild assumptions on the topological space S, the converse is also true: every functor $\pi_{\leq 1}(S) \to \mathrm{Set}$ can be obtained as the monodromy representation of an essentially unique covering map $f : X \to S$. More precisely, we have the following:

`027R` **Theorem 5.2.7.1** (The Fundamental Theorem of Covering Space Theory)**.** *Let S be a topological space which is semilocally simply connected. Then the construction $X \mapsto \mathrm{hTr}_{X/S}$ determines an equivalence of categories*

$$\{\text{Covering maps } f : X \to S\} \to \mathrm{Fun}(\pi_{\leq 1}(S), \mathrm{Set}).$$

The proof of Theorem 5.2.7.1 can be broken into two parts:

(a) If S is a topological space which is semilocally simply connected, then the construction $X \mapsto \mathrm{Sing}_\bullet(X)$ induces an equivalence of categories

$$\begin{tikzcd}
\{\text{Covering maps of topological spaces } f : X \to S\} \arrow[d] \\
\{\text{Covering maps of simplicial sets } \mathcal{E} \to \mathrm{Sing}_\bullet(S)\}.
\end{tikzcd}$$

(b) For every Kan complex $\mathcal{C}$, the formation of monodromy representations determines an equivalence of categories

$$\{\text{Covering maps } \mathcal{E} \to \mathcal{C}\} \to \mathrm{Fun}(\pi_{\leq 1}(\mathcal{C}), \mathrm{Set}) \qquad \mathcal{E} \mapsto \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}.$$

The proof of (a) requires some point-set topology; we defer a discussion to §[?]. Our goal in this section is to give a proof of (b) (see Corollary 5.2.7.6). We will deduce (b) from a more general statement, which classifies left coverings of an arbitrary simplicial set $\mathcal{C}$ (Corollary 5.2.7.3).

Proposition 5.2.7.2. *Let $U : \mathcal{E} \to \mathcal{C}$ be a left covering map of simplicial sets, and let $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{Set}$ be the homotopy transport representation of Proposition 5.2.0.3. Then there is a canonical isomorphism of simplicial sets* 027V

$$\mathcal{E} \simeq \mathcal{C} \times_{\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})} \mathrm{N}_\bullet(\int_{\mathrm{h}\mathcal{C}} \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}).$$

Proof. Every vertex $X \in \mathcal{E}$ can be regarded as an element of the set $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}(U(X))$, and the construction $(X \in \mathcal{E}) \mapsto (\mathcal{E}_{U(X)}, X)$ determines a functor $\widetilde{\mathrm{hTr}}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{E} \to \mathrm{Set}_*$. Let us identify $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ with a morphism of simplicial sets from $\mathcal{C}$ to $\mathrm{N}_\bullet(\mathrm{Set})$ and $\widetilde{\mathrm{hTr}}_{\mathcal{E}/\mathcal{C}}$ with a morphism of simplicial sets from $\mathcal{E}$ to $\mathrm{N}_\bullet(\mathrm{Set}_*)$, so that we have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{\mathrm{hTr}}_{\mathcal{E}/\mathcal{C}}"] \arrow[d, "U"] & \mathrm{N}_\bullet(\mathrm{Set}_*) \arrow[d] \\
\mathcal{C} \arrow[r, "\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}"] & \mathrm{N}_\bullet(\mathrm{Set})
\end{tikzcd}$$

which we can identify with a morphism of simplicial sets $V : \mathcal{E} \to \mathcal{C} \times_{\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})} \mathrm{N}_\bullet(\int_{\mathrm{h}\mathcal{C}} \mathrm{hTr}_{\mathcal{E}/\mathcal{C}})$. Since U and the projection map $\int_{\mathcal{C}} \mathrm{hTr}_{\mathcal{E}/\mathcal{C}} \to \mathrm{h}\mathcal{C}$ are both left covering maps (Remark 5.2.6.9), it follows that V is a left covering map (Remark 4.2.3.14). By construction, V is bijective at the level of vertices, and is therefore an isomorphism of simplicial sets (Proposition 4.2.3.19). □

Corollary 5.2.7.3. *Let $\mathcal{C}$ be a simplicial set, and let $\mathrm{LCov}_{\mathcal{C}}$ denote the full subcategory of $(\mathrm{Set}_\Delta)_{/\mathcal{C}}$ spanned by the left covering maps $U : \mathcal{E} \to \mathcal{C}$. Then the formation of homotopy transport representations supplies an equivalence of categories* 027W

$$\mathrm{LCov}_{\mathcal{C}} \to \mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{Set}) \qquad (U : \mathcal{E} \to \mathcal{C}) \mapsto \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}.$$

Proof. Proposition 5.2.7.2 shows that the functor

$$(\mathscr{F} \in \mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{Set})) \mapsto \mathcal{C} \times_{\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})} \mathrm{N}_\bullet(\int_{\mathrm{h}\mathcal{C}} \mathscr{F}) \in \mathrm{LCov}_{\mathcal{C}}$$

is a left homotopy inverse to the functor $\mathcal{E} \mapsto \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$. By virtue of Example 5.2.0.6 and Remark 5.2.5.6, it is also a right homotopy inverse. □

027X **Corollary 5.2.7.4.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *There exists a pullback diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \mathrm{N}_{\bullet}(\mathcal{D}) \arrow[d, "\mathrm{N}_{\bullet}(V)"] \\
\mathcal{C} \arrow[r] & \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C}),
\end{tikzcd}$$

where $V : \mathcal{D} \to \mathrm{h}\mathcal{C}$ is a left covering functor (in the sense of Definition 4.2.3.1).

(2) *For every category $\mathcal{C}'$ and every morphism of simplicial sets $\mathrm{N}_{\bullet}(\mathcal{C}') \to \mathcal{C}$, the fiber product $\mathrm{N}_{\bullet}(\mathcal{C}') \times_{\mathcal{C}} \mathcal{E}$ is isomorphic to the nerve of a category $\mathcal{E}'$ and the projection $\mathcal{E}' \to \mathcal{C}'$ is left covering functor (in the sense of Definition 4.2.3.1).*

(3) *For every n-simplex $\sigma : \Delta^n \to \mathcal{C}$, the fiber product $\Delta^n \times_{\mathcal{C}} \mathcal{E}$ is isomorphic to the nerve of a category $\mathcal{E}'$ and the projection $\mathcal{E}' \to [n]$ is a left covering functor (in the sense of Definition 4.2.3.1).*

(4) *The morphism U is a left covering map of simplicial sets (in the sense of Definition 4.2.3.8).*

Proof. The implication (1) ⇒ (2) follows from Remark 4.2.3.6, the implication (2) ⇒ (3) is trivial, and the implication (3) ⇒ (4) follows by combining Remark 4.2.3.15 with Proposition 4.2.3.16. The implication (4) ⇒ (1) follows from Proposition 5.2.7.2. □

01R0 **Corollary 5.2.7.5.** *Let $\mathcal{C}$ be a category. Then:*

- *Construction 5.2.6.1 determines a fully faithful functor*

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \to \mathrm{Cat}_{/\mathcal{C}} \qquad \mathscr{F} \mapsto \int_{\mathcal{C}} \mathscr{F},$$

whose essential image consists of the left covering functors $U : \mathcal{E} \to \mathcal{C}$.

- *Variant 5.2.6.2 determines a fully faithful functor*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Cat}_{/\mathcal{C}} \qquad \mathscr{F} \mapsto \int^{\mathcal{C}} \mathscr{F},$$

whose essential image consists of the right covering functors $U : \mathcal{E} \to \mathcal{C}$.

027Z **Corollary 5.2.7.6.** *Let $\mathcal{C}$ be a Kan complex. Then the construction $(U : \mathcal{E} \to \mathcal{C}) \mapsto \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ induces an equivalence of categories*

$$\{\text{Covering maps } \mathcal{E} \to \mathcal{C}\} \to \mathrm{Fun}(\pi_{\leq 1}(\mathcal{C}), \mathrm{Set}).$$

Proof. Combine Corollaries 5.2.7.3 and 4.4.3.9. □

5.2.8 Parametrized Covariant Transport

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. To every morphism $f : C \to D$ in the ∞-category $\mathcal{C}$, Definition 5.2.2.4 associates a covariant transport functor $f_! : \mathcal{E}_C \to \mathcal{E}_D$, which is uniquely determined up to isomorphism (see Proposition 5.2.2.8). Our goal in this section is to show that the functor $f_!$ can be chosen to depend functorially on the morphism f: that is, the construction $f \mapsto f_!$ can be promoted to a functor from the Kan complex $\operatorname{Hom}_{\mathcal{C}}(C, D)$ to the ∞-category $\operatorname{Fun}(\mathcal{E}_C, \mathcal{E}_D)$. We begin by introducing a more elaborate version of Definition 5.2.2.4. 02RL

Definition 5.2.8.1 (Parametrized Covariant Transport)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let C and D be vertices of $\mathcal{C}$. We will say that a morphism $F : \operatorname{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}_D$ is *given by parametrized covariant transport* if there exists a morphism of simplicial sets $\widetilde{F} : \Delta^1 \times \operatorname{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}$ satisfying the following conditions: 02RM

(1) The diagram of simplicial sets

$$\begin{tikzcd}
\Delta^1 \times \operatorname{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \arrow[r, "\widetilde{F}"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^1 \times \operatorname{Hom}_{\mathcal{C}}(C, D) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

commutes (where the lower horizontal map is induced by the inclusion $\operatorname{Hom}_{\mathcal{C}}(C, D) \hookrightarrow \operatorname{Fun}(\Delta^1, \mathcal{C})$).

(2) The restriction $\widetilde{F}|_{\{0\} \times \operatorname{Hom}_{\mathcal{C}}(C,D) \times \mathcal{E}_C}$ is given by projection onto $\mathcal{E}_C$, and the restriction $\widetilde{F}|_{\{1\} \times \operatorname{Hom}_{\mathcal{C}}(C,D) \times \mathcal{E}_C}$ is equal to F.

(3) For every edge $f : C \to D$ of $\mathcal{C}$ and every object $X \in \mathcal{E}_C$, the composite map

$$\Delta^1 \times \{f\} \times \{X\} \hookrightarrow \Delta^1 \times \operatorname{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \xrightarrow{\widetilde{F}} \mathcal{E}$$

is a U-cocartesian edge of $\mathcal{E}$.

If these conditions are satisfied, we say that the morphism $\widetilde{F}$ *witnesses F as given by parametrized covariant transport.*

Remark 5.2.8.2. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let C and D be vertices of $\mathcal{C}$, and let $F : \operatorname{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}_D$ be given by parametrized covariant transport. Then, for every edge $f : C \to D$, the composite map 02RN

$$\{f\} \times \mathcal{E}_C \hookrightarrow \operatorname{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \xrightarrow{F} \mathcal{E}_D$$

is given by covariant transport along f, in the sense of Definition 5.2.2.4. In other words, we can identify F with a diagram $\mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Fun}(\mathcal{E}_C, \mathcal{E}_D)$, which carries each edge $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ to the covariant transport functor $f_!$ of Notation 5.2.2.9.

02RP **Example 5.2.8.3.** Let Set_* denote the category of pointed sets (Example 4.2.3.3), and let $V : \mathrm{Set}_* \to \mathrm{Set}$ denote the forgetful functor $(X, x) \mapsto X$. Then the induced map $\mathrm{N}_\bullet(V) : \mathrm{N}_\bullet(\mathrm{Set}_*) \to \mathrm{N}_\bullet(\mathrm{Set})$ is a cocartesian fibration (in fact, it is a left covering map), whose fiber over an object $X \in \mathrm{N}_\bullet(\mathrm{Set})$ can be identified with the set X. For every pair of sets X and Y, the evaluation map

$$\mathrm{ev} : \mathrm{Hom}_{\mathrm{Set}}(X, Y) \times X \to Y \qquad (f, x) \mapsto f(x)$$

is given by parametrized covariant transport (in the sense of Definition 5.2.8.1).

02RQ **Proposition 5.2.8.4.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, and let C and D be vertices of $\mathcal{C}$. Then:*

- *There exists a morphism $F : \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}_D$ which is given by parametrized covariant transport.*
- *An arbitrary diagram $F' : \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}_D$ is given by parametrized covariant transport if and only if it is isomorphic to F (as an object of the ∞-category $\mathrm{Fun}(\mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C, \mathcal{E}_D)$).*

Proof. Apply Lemma 5.2.2.13 to the simplicial set $K = \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C$. □

02RR **Remark 5.2.8.5** (Functoriality)**.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "G"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r, "\overline{G}"] & \mathcal{C}',
\end{tikzcd}$$

where U and U' are cocartesian fibrations. Let C and D be vertices of $\mathcal{C}$ having images $C' = \overline{G}(C)$ and $D' = \overline{G}(D)$, respectively, so that G induces functors $G_C : \mathcal{E}_C \to \mathcal{E}'_{C'}$ and $G_D : \mathcal{E}_D \to \mathcal{E}'_{D'}$. Let $\varphi : \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{C}'}(C', D')$ be the morphism induced by $\overline{G}$, and let

$$F : \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}_D \qquad F' : \mathrm{Hom}_{\mathcal{C}'}(C', D') \times \mathcal{E}'_{C'} \to \mathcal{E}'_{D'}$$

be given by parametrized covariant transport with respect to U and U'. Suppose that the morphism G carries U-cocartesian edges of $\mathcal{E}$ to U'-cocartesian edges of $\mathcal{E}'$. Then the

diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathcal{E}_C \arrow[r, "F"] \arrow[d, "\varphi \times G_C"] & \mathcal{E}_D \arrow[d, "G_D"] \\
\mathrm{Hom}_{\mathcal{C}'}(C',D') \times \mathcal{E}'_{C'} \arrow[r, "F'"] & \mathcal{E}'_{D'}
\end{tikzcd}$$

commutes up to isomorphism: that is, $G_D \circ F$ and $F' \circ (\varphi \times G_C)$ are isomorphic as objects of the ∞-category $\mathrm{Fun}(\mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathcal{E}_C, \mathcal{E}'_{D'})$. This follows by applying the uniqueness assertion of Lemma 5.2.2.13 to the lifting problem

$$\begin{tikzcd}
\{0\} \times \mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathcal{E}_C \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d, "U'"] \\
\Delta^1 \times \mathrm{Hom}_{\mathcal{C}}(C,D) \times \mathcal{E}_C \arrow[r] \arrow[ur, dashed] & \mathcal{C}' .
\end{tikzcd}$$

Variant 5.2.8.6 (Parametrized Contravariant Transport)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of simplicial sets and let C and D be vertices of $\mathcal{C}$. Applying Proposition 5.2.8.4 to the opposite cocartesian fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$, we obtain a diagram $\mathrm{Hom}_{\mathcal{C}}(C,D) \to \mathrm{Fun}(\mathcal{E}_D, \mathcal{E}_C)$, carrying each edge $f : C \to D$ to a functor $f^* : \mathcal{E}_D \to \mathcal{E}_C$ given by contravariant transport along f. 02RS

Let $\mathcal{C}$ be an ∞-category. Recall that, for every pair of objects $X, Y \in \mathcal{C}$, the morphism space $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ can be identified with the fiber over Y of the left fibration $\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \to \mathcal{C}$ of Proposition 4.6.4.11, or with the fiber over X of the right fibration $\mathcal{C} \tilde{\times}_{\mathcal{C}} \{Y\}$. In either case, parametrized transport recovers the composition law of $\mathcal{C}$:

Proposition 5.2.8.7. *Let $\mathcal{C}$ be an ∞-category containing objects C, D, and E. Then the composition law* 02GT

$$\circ : \mathrm{Hom}_{\mathcal{C}}(D,E) \times \mathrm{Hom}_{\mathcal{C}}(C,D) \to \mathrm{Hom}_{\mathcal{C}}(C,E)$$

of Construction 4.6.9.9 is given by parametrized covariant transport for the left fibration $U : \{C\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \to \mathcal{C}$ (in the sense of Definition 5.2.8.1), and also by parametrized contravariant transport for the right fibration $V : \mathcal{C} \tilde{\times}_{\mathcal{C}} \{E\} \to \mathcal{C}$.

Proof. We will prove the first assertion; the second follows by a similar argument. Let $S : \Delta^1 \times \Delta^1 \to \Delta^2$ be the morphism given on vertices by the formula $T(i,j) = i(j+1)$, and let T be a section of the trivial Kan fibration $\mathrm{Hom}_{\mathcal{C}}(C,D,E) \to \mathrm{Hom}_{\mathcal{C}}(D,E) \times \mathrm{Hom}_{\mathcal{C}}(C,D)$ (see Corollary 4.6.9.5). Then the composite map

$$\Delta^1 \times \Delta^1 \times \mathrm{Hom}_{\mathcal{C}}(D,E) \times \mathrm{Hom}_{\mathcal{C}}(C,D) \xrightarrow{S \times T} \Delta^2 \times \mathrm{Hom}_{\mathcal{C}}(C,D,E) \to \mathcal{C}$$

carries $\{0\} \times \Delta^1 \times \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D)$ to the vertex C, and can therefore be identified with a functor

$$\widetilde{F} : \Delta^1 \times \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \to \{C\} \tilde{\times}_{\mathcal{C}} \mathcal{C}.$$

which exhibits the composition law as given by parametrized covariant transport for the left fibration U. □

Proposition 5.2.5.1 has a counterpart for parametrized covariant transport:

02GU **Proposition 5.2.8.8.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. Let C, D, and E be objects of $\mathcal{C}$, and let*

$$F : \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}_D \qquad G : \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathcal{E}_D \to \mathcal{E}_E$$

$$H : \mathrm{Hom}_{\mathcal{C}}(C, E) \times \mathcal{E}_C \to \mathcal{E}_E$$

be given by parametrized covariant transport. Then the diagram

02RT
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \arrow[r, "\mathrm{id} \times F"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathcal{E}_D \arrow[d, "G"] \\
\mathrm{Hom}_{\mathcal{C}}(C, E) \times \mathcal{E}_C \arrow[r, "H"] & \mathcal{E}_E
\end{tikzcd} \tag{5.14}$$

commutes in the homotopy category hQCat*; here the left vertical map is given by the composition law of Construction 4.6.9.9.*

Proof. Let $\mathrm{Hom}_{\mathcal{C}}(C, D, E)$ be the Kan complex defined in Notation 4.6.9.1, let H' denote the composite map

$$\mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C \to \mathrm{Hom}_{\mathcal{C}}(C, E) \times \mathcal{E}_C \xrightarrow{H} \mathcal{E}_E,$$

and let H'' denote the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C & \to \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \\
& \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathcal{E}_D \\
& \xrightarrow{G} \mathcal{E}_E.
\end{aligned}$$

We will show that H' and H'' are isomorphic when regarded as objects of the ∞-category $\mathrm{Fun}(\mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C, \mathcal{E}_E)$. The homotopy commutativity of the diagram (5.14) will then follow by precomposing with any section of the trivial Kan fibration $\mathrm{Hom}_{\mathcal{C}}(C, D, E) \to \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D)$.

Choose morphisms

$$\widetilde{F} : \mathrm{N}_\bullet(\{0 < 1\}) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}$$

$$\widetilde{G} : \mathrm{N}_\bullet(\{1 < 2\}) \times \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathcal{E}_D \to \mathcal{E}$$

$$\widetilde{H} : \mathrm{N}_\bullet(\{0 < 2\}) \times \mathrm{Hom}_{\mathcal{C}}(C, E) \times \mathcal{E}_C \to \mathcal{E}$$

which witness F, G, and H as given by parametrized covariant transport, respectively. Composing with the projection maps

$$\mathrm{Hom}_{\mathcal{C}}(C, D) \leftarrow \mathrm{Hom}_{\mathcal{C}}(C, D, E) \to \mathrm{Hom}_{\mathcal{C}}(C, E),$$

we obtain morphisms

$$\widetilde{F}' : \mathrm{N}_\bullet(\{0 < 1\}) \times \mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C \to \mathcal{E}$$

$$\widetilde{H}' : \mathrm{N}_\bullet(\{0 < 2\}) \times \mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C \to \mathcal{E}\,.$$

Let $\widetilde{G}'$ denote the composite map

$$\begin{aligned} \mathrm{N}_\bullet(\{1 < 2\}) \times \mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C &\to \mathrm{N}_\bullet(\{1 < 2\}) \times \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \\ &\xrightarrow{F} \mathrm{N}_\bullet(\{1 < 2\}) \times \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathcal{E}_D \\ &\xrightarrow{\widetilde{G}} \mathcal{E}\,. \end{aligned}$$

Since U is an inner fibration, the lifting problem

$$\begin{tikzcd} \Lambda^2_1 \times \mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C \arrow[r, "(\widetilde{G}', \bullet, \widetilde{F}')"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\ \Delta^2 \times \mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C \arrow[r] \arrow[ur, dashed, "\Phi"] & \mathcal{C} \end{tikzcd}$$

admits a solution $\Phi : \Delta^2 \times \mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C \to \mathcal{E}$. Let $\widetilde{H}''$ denote the restriction of Φ to the product $\mathrm{N}_\bullet(\{0, 2\}) \times \mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C$. Using Proposition 5.1.4.12, we see that $\widetilde{H}''$ is a U-cocartesian lift of $U \circ \widetilde{H}'' = U \circ \widetilde{H}'$, in the sense of Definition 5.2.2.10. Applying the uniqueness assertion of Lemma 5.2.2.13, we conclude that the restrictions $H' = \widetilde{H}'|_{\{2\} \times \mathrm{Hom}_{\mathcal{C}}(C,D,E) \times \mathcal{E}_C}$ and $H'' = \widetilde{H}''|_{\{2\} \times \mathrm{Hom}_{\mathcal{C}}(C,D,E) \times \mathcal{E}_C}$ are isomorphic when regarded as objects of the ∞-category $\mathrm{Fun}(\mathrm{Hom}_{\mathcal{C}}(C, D, E) \times \mathcal{E}_C, \mathcal{E}_E)$, as desired. □

Using Proposition 5.2.8.8, we obtain the following refinement of Construction 5.2.5.2:

02GW **Construction 5.2.8.9** (Enriched Homotopy Transport: Covariant Case)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let us regard the homotopy category $\mathrm{h}\mathcal{C}$ as enriched over the homotopy category hKan of Kan complexes (Construction 4.6.9.13). It follows from Proposition 5.2.8.8 (and Example 5.2.2.5) that there is a unique hKan-enriched functor $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ with the following properties:

- For each object C of the ∞-category $\mathcal{C}$, $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ is the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ (regarded as an object of hQCat).

- For every pair of objects $C, D \in \mathcal{C}$, the induced map

$$\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Fun}(\mathcal{E}_C, \mathcal{E}_D)^{\simeq}$$

  in hKan corresponds to the parametrized covariant transport functor $\mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathcal{E}_C \to \mathcal{E}_D$ of supplied by Proposition 5.2.8.4 (which is well-defined up to isomorphism).

We will refer to $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}$ as the *enriched homotopy transport representation* of the cocartesian fibration U. Note that the underlying functor of ordinary categories $\mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ coincides with homotopy transport representation of Construction 5.2.5.2.

02RU **Remark 5.2.8.10** (Functoriality)**.** Suppose we are given a commutative diagram of ∞-categories

02RV

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "G"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r, "\overline{G}"] & \mathcal{C}',
\end{tikzcd} \tag{5.15}$$

where U and U' are cocartesian fibrations and the functor G carries U-cocartesian morphisms of $\mathcal{E}$ to U'-cocartesian morphisms of $\mathcal{E}'$. For each object $C \in \mathcal{C}$ having image $C' = \overline{G}(C)$, G restricts to a functor $G_C : \mathcal{E}_C \to \mathcal{E}'_{C'}$. It follows from Remark 5.2.8.5 that the construction $C \mapsto G_C$ determines a natural transformation of hKan-enriched functors $\alpha : \mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}} \to \mathrm{hTr}_{\mathcal{E}'\,/\,\mathcal{C}'} \circ \mathrm{h}\overline{G}$ from $\mathrm{h}\mathcal{C}$ to hQCat. Moreover, if (5.15) is a pullback square, then α is an isomorphism of hKan-enriched functors.

02RW **Variant 5.2.8.11** (Enriched Homotopy Transport: Left Fibrations)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a left fibration of ∞-categories. Applying Construction 5.2.8.9, we obtain an hKan-enriched functor

$$\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hKan},$$

given on objects by the formula $\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) = \{C\} \times_{\mathcal{C}} \mathcal{E}$.

Variant 5.2.8.12 (Enriched Homotopy Transport: Contravariant Case)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of ∞-categories. Applying Construction 5.2.8.9 to the opposite functor U^{op}, we deduce that there is a unique hKan-enriched functor $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C}^{\mathrm{op}} \to \mathrm{hQCat}$ with the following properties: 02GX

- For each object C of the ∞-category $\mathcal{C}$, $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}(C)$ is the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ (regarded as an object of hQCat).
- For every pair of objects $C, D \in \mathcal{C}$, the induced map

  $$\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Fun}(\mathcal{E}_D, \mathcal{E}_C)^{\simeq}$$

  is given by the parametrized contravariant transport functor $\mathcal{E}_D \times \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathcal{E}_C$ of Variant 5.2.8.6.

We will refer to $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ as the *enriched homotopy transport representation* of the cartesian fibration U. If U is a right fibration, then $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ takes values in the full subcategory $\mathrm{hKan} \subseteq \mathrm{hQCat}$.

Example 5.2.8.13. Let $\mathcal{C}$ be an ∞-category and let $\mathrm{h}\mathcal{C}$ denote its homotopy category, which we regard as enriched over the homotopy category hKan of Kan complexes. Applying Proposition 5.2.8.7, we obtain the following: 02GY

- For every object $C \in \mathcal{C}$, the corepresentable hKan-enriched functor

  $$\mathrm{h}\mathcal{C} \to \mathrm{hKan} \qquad D \mapsto \mathrm{Hom}_{\mathcal{C}}(C, D)$$

  is the enriched homotopy transport representation for the left fibration $\{C\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \to \mathcal{C}$.
- For every object $D \in \mathcal{C}$, the representable hKan-enriched functor

  $$\mathrm{h}\mathcal{C}^{\mathrm{op}} \to \mathrm{hKan} \qquad C \mapsto \mathrm{Hom}_{\mathcal{C}}(C, D)$$

  is the enriched homotopy transport representation for the right fibration $\mathcal{C} \tilde{\times}_{\mathcal{C}} \{D\} \to \mathcal{C}$.

5.3 Fibrations over Ordinary Categories

Let Set_{Δ} denote the category of simplicial sets, let $\mathrm{QCat} \subset \mathrm{Set}_{\Delta}$ denote the full subcategory spanned by the ∞-categories, and let hQCat denote its homotopy category (Construction 4.5.1.1). In §5.2.5, we associated to every cocartesian fibration of simplicial sets $U : \mathcal{E} \to S$ a functor $\mathrm{hTr}_{\mathcal{E}/S} : \mathrm{h}S \to \mathrm{hQCat}$ called the *homotopy transport representation of* U, given on objects by the formula $\mathrm{hTr}_{\mathcal{E}/S}(s) = \{s\} \times_S \mathcal{E}$ (Construction 5.2.5.2). In §5.3.1, we specialize to the situation where $S = \mathrm{N}_{\bullet}(\mathcal{C})$ is the nerve of an ordinary category $\mathcal{C}$. In this case, we 035A show that $\mathrm{hTr}_{\mathcal{E} / \mathrm{N}_\bullet(\mathcal{C})}$ can be lifted to a functor taking values in the category QCat. More precisely, we introduce a functor $\mathrm{sTr}_{\mathcal{E} / \mathcal{C}} : \mathcal{C} \to \mathrm{QCat}$ which we refer to as the *strict transport representation* of U (Construction 5.3.1.5), and show that the diagram

$$\begin{tikzcd} & \mathrm{QCat} \arrow[dr] & \\ \mathcal{C} \arrow[ur, "\mathrm{sTr}_{\mathcal{E} / \mathcal{C}}"] \arrow[rr, "\mathrm{hTr}_{\mathcal{E} / \mathrm{N}_\bullet(\mathcal{C})}"'] & & \mathrm{hQCat} \end{tikzcd}$$

commutes up to canonical isomorphism (Corollary 5.3.1.8).

Our primary goal in this section is to show that a cocartesian fibration $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ can be recovered, up to equivalence, from its strict transport representation $\mathrm{sTr}_{\mathcal{E} / \mathcal{C}}$. To formulate this precisely, we need another construction. In §5.3.3, we associate to every diagram $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ a simplicial set $\mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$, which we will refer to as the *$\mathscr{F}$-weighted nerve of $\mathcal{C}$* (Definition 5.3.3.1). The weighted nerve is equipped with a projection map $V : \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$, whose fiber over an object $C \in \mathcal{C}$ can be identified with the simplicial set $\mathscr{F}(C)$ (Example 5.3.3.8). If each of these simplicial sets is an ∞-category, then V is a cocartesian fibration of ∞-categories (Corollary 5.3.3.16). Our main results can be summarized as follows:

(1) Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories having weighted nerve $\mathcal{E} = \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$. Then there is a natural transformation from $\mathscr{F}$ to the strict transport representation $\mathrm{sTr}_{\mathcal{E} / \mathcal{C}}$, which carries each object $C \in \mathcal{C}$ to an equivalence of ∞-categories $\mathscr{F}(C) \to \mathrm{sTr}_{\mathcal{E} / \mathcal{C}}(C)$ (Corollary 5.3.4.19).

(2) Let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories having strict transport representation $\mathscr{F} = \mathrm{sTr}_{\mathcal{E} / \mathcal{C}}$. Then U is equivalent (in the sense of Definition 5.1.7.1) to the cocartesian fibration $\mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ (Theorem 5.3.5.6).

The proof of (1) is relatively straightforward. However, the proof of (2) is somewhat more difficult: given a cocartesian fibration $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ there is no obvious comparison map between the simplicial sets $\mathcal{E}$ and $\mathrm{N}^{\mathrm{sTr}_{\mathcal{E} / \mathcal{C}}}_\bullet(\mathcal{C})$. To relate them, we need an auxiliary construction. In §5.3.2, we associate to every diagram $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ a simplicial set $\underrightarrow{\mathrm{holim}}(\mathscr{F})$, which we refer to as the *homotopy colimit* of $\mathscr{F}$ (Construction 5.3.2.1). The formation of homotopy colimits plays an important role in the classical homotopy theory of simplicial sets: it can be regarded as a replacement for the usual notion of colimit (see Remark 5.3.2.9) which is compatible with weak homotopy equivalence (Proposition 5.3.2.18). Beware that the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ is generally not an ∞-category (even in the special case where $\mathscr{F}$ is a diagram of ∞-categories). Nevertheless, it is equipped with a projection map $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}(\mathcal{C})$, whose fiber over each object $C \in \mathcal{C}$ can be identified with the simplicial set $\mathscr{F}(C)$, and which behaves in certain respects like a cocartesian fibration. In §5.3.4, we make this heuristic precise by introducing the notion of a *scaffold*. If $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a cocartesian fibration of ∞-categories, we define a *scaffold of* U to be a commutative diagram

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[rr, "\lambda"] \arrow[dr] & & \mathcal{E} \arrow[dl, "U"'] \\
& \mathrm{N}_{\bullet}(\mathcal{C}), &
\end{tikzcd}$$

where λ restricts to a categorical equivalence $\mathscr{F}(C) \to \mathcal{E}_C$ for each $C \in \mathcal{C}$ and behaves well with respect to the collection of U-cocartesian morphisms of $\mathcal{E}$ (Definition 5.3.4.2). We are primarily interested in two examples:

- To any cocartesian fibration $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$, we associate a *universal scaffold* $\lambda_u : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$, where $\mathscr{F} = \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}$ is the strict transport representation of U (see Construction 5.3.4.7 and Proposition 5.3.4.8).
- To any diagram of ∞-categories $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$, we associate a *taut scaffold* $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$, where $\mathcal{E} = \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ is the $\mathscr{F}$-weighted nerve of $\mathcal{C}$ (see Construction 5.3.4.11 and Proposition 5.3.4.17).

In §5.3.5, we show that every scaffold $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ is a categorical equivalence of simplicial sets (Theorem 5.3.5.7). In particular, if $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a cocartesian fibration with strict transport representation $\mathscr{F} = \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}$, then we can exploit the taut and universal scaffolds

$$\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \xleftarrow{\lambda} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\lambda_u} \mathcal{E},$$

to deduce the existence of an equivalence of ∞-categories $\mathcal{E} \simeq \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ (compatible with the projection $\mathrm{N}_{\bullet}(\mathcal{C})$), thereby obtaining a proof of (2) (see Theorem 5.3.5.6).

We close this section by describing some other applications of our theory of scaffolds. Let $U : \mathcal{C} \to \mathcal{B}$ be a morphism of simplicial sets, let $\mathcal{D}$ be an ∞-category, and let $\mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B}, \mathcal{D})$ be the relative exponential of Construction 4.5.9.1. In §5.3.6, we show that if U is a cocartesian fibration, then the projection map $\mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B}, \mathcal{D}) \to \mathcal{B}$ is a cartesian fibration. More generally, for every cartesian fibration of simplicial sets $V : \mathcal{D} \to \mathcal{E}$, the induced map

$$\mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B}, \mathcal{D}) \xrightarrow{V \circ} \mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B}, \mathcal{E})$$

is also a cartesian fibration (Proposition 5.3.6.6). In §5.3.7, we apply this result to study the oriented fiber product of Definition 4.6.4.1. For any functor of ∞-categories $F : \mathcal{A} \to \mathcal{B}$, projection onto the second factor determines a cocartesian fibration $\mathcal{A} \tilde{\times}_{\mathcal{B}} \mathcal{B} \to \mathcal{B}$ (Corollary 5.3.7.3) which is, in some sense, freely generated by the ∞-category $\mathcal{A}$ (Theorem 5.3.7.6).

035B **Remark 5.3.0.1.** There is a close analogy between the homotopy colimit construction (studied in §5.3.2) and the weighted nerve construction (studied in §5.3.3).

- The formation of homotopy colimits determines a functor

  $$\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to (\mathrm{Set}_{\Delta})_{/ \mathrm{N}_{\bullet}(\mathcal{C})} \qquad \mathscr{F} \mapsto \underrightarrow{\mathrm{holim}}(\mathscr{F}).$$

  This functor has a right adjoint, which carries an object $\mathcal{E} \in (\mathrm{Set}_{\Delta})_{/ \mathrm{N}_{\bullet}(\mathcal{C})}$ to the diagram

  $$\mathcal{C} \to \mathrm{Set}_{\Delta} \qquad C \mapsto \operatorname{Fun}_{/ \mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \mathcal{E}).$$

  See Corollary 5.3.2.24.

- The formation of weighted nerves determines a functor

  $$\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to (\mathrm{Set}_{\Delta})_{/ \mathrm{N}_{\bullet}(\mathcal{C})} \qquad \mathscr{F} \mapsto \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}).$$

  This functor has a left adjoint, which carries an object $\mathcal{E} \in (\mathrm{Set}_{\Delta})_{/ \mathrm{N}_{\bullet}(\mathcal{C})}$ to the diagram

  $$\mathcal{C} \to \mathrm{Set}_{\Delta} \qquad C \mapsto \mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathcal{E}.$$

  See Corollary 5.3.3.25.

035C **Remark 5.3.0.2.** After restricting to diagrams of Kan complexes, the results of this section supply a dictionary

$$\begin{tikzcd}
\{\text{Left fibrations } \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})\} \arrow[d, shift left, "\mathrm{sTr}_{(-)/\mathcal{C}}"] \\
\{\text{Functors } \mathcal{C} \to \mathrm{Kan}\} \arrow[u, shift left, "\mathrm{N}_{\bullet}^{(-)}(\mathcal{C})"]
\end{tikzcd}$$

This dictionary was formulated in work of Heuts and Moerdijk (using the language of model categories) which is closely related to the contents of this section. For more details, we refer the reader to [28].

5.3.1 The Strict Transport Representation

Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a cocartesian fibration of ∞-categories. 035D To each morphism $f : C \to D$ of $\mathcal{C}$, the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}\,/\,\mathrm{N}_{\bullet}(\mathcal{C})}$ associates the homotopy class $[f_!]$, where $f_! : \mathcal{E}_C \to \mathcal{E}_D$ is given by covariant transport along f. Beware that the functor $f_!$ is only well-defined up to isomorphism. For example, the value of $f_!$ on an object $X \in \mathcal{E}_C$ depends on an auxiliary choice: namely, the choice of a U-cocartesian morphism $\widetilde{f} : X \to Y$ satisfying $U(\widetilde{f}) = f$ (once we have made this choice, we can take $f_!(X)$ to be the object $Y \in \mathcal{E}_D$). Our goal in this section is to show that, by replacing each fiber $\mathcal{E}_C$ by an equivalent ∞-category, the ambiguity in the definition of the transport functors can be eliminated. More precisely, we will associate to each object $C \in \mathcal{C}$ a simplicial set $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ with the following properties:

- There is a trivial Kan fibration of simplicial sets $\mathrm{ev}_C : \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \to \mathcal{E}_C$ (Proposition 5.3.1.7). In particular, $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ is an ∞-category which is equivalent to $\mathcal{E}_C$.
- Every morphism $f : C \to D$ in the category $\mathcal{C}$ determines a functor of ∞-categories $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(f) : \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \to \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(D)$, which does not depend on any auxiliary choices. Moreover, the assignment $f \mapsto \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(f)$ is compatible with composition, and therefore determines a functor $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathcal{C} \to \mathrm{QCat}$ which we will refer to as the *strict transport representation* of U (Construction 5.3.1.5).
- For every morphism $f : C \to D$ in $\mathcal{C}$, the diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \arrow[r, "\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(f)"] \arrow[d, "\mathrm{ev}_C"] & \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(D) \arrow[d, "\mathrm{ev}_D"] \\
\mathcal{E}_C \arrow[r, "f_!"] & \mathcal{E}_D
\end{tikzcd}$$

  commutes up to isomorphism. Consequently, the strict transport representation $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}$ can be regarded as a refinement of the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}\,/\,\mathrm{N}_{\bullet}(\mathcal{C})}$ of Construction 5.2.5.2.

We begin by considering a closely related construction.

Construction 5.3.1.1. Let Cat denote the ordinary category whose objects are (small) 035E categories and whose morphisms are functors. If $\mathcal{C}$ is a category, then the construction $C \mapsto \mathcal{C}_{C/}$ determines a functor $\mathcal{C} \to (\mathrm{Cat}_{/\mathcal{C}})^{\mathrm{op}}$, carrying each morphism $f : C \to D$ in $\mathcal{C}$ to the functor

$$\mathcal{C}_{D/} \to \mathcal{C}_{C/} \qquad (g : D \to E) \mapsto ((g \circ f) : C \to E).$$

For any morphism of simplicial sets $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$, we let $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathcal{C} \to \mathrm{Set}_\Delta$ denote the functor given on objects by the formula

$$\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) = \mathrm{Fun}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E}).$$

We will refer to $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}$ as the *weak transport representation* of U.

035F **Remark 5.3.1.2.** Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be an inner fibration of ∞-categories. Then, for every object $C \in \mathcal{C}$, the simplicial set

$$\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) = \mathrm{Fun}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E})$$

is an ∞-category (Corollary 4.1.4.8).

035G **Remark 5.3.1.3.** Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a morphism of simplicial sets. For each object $C \in \mathcal{C}$, we can regard the identity morphism id_C as an object of the coslice ∞-category $\mathcal{C}_{C/}$. Evaluation on id_C determines a morphism of simplicial sets

$$\mathrm{ev}_C : \mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \to \mathcal{E}_C\,.$$

Note that id_C is an initial object of the category $\mathcal{C}_{C/}$, so the inclusion map $\{\mathrm{id}_C\} \hookrightarrow \mathrm{N}_\bullet(\mathcal{C}_{C/})$ is left anodyne (Corollary 4.6.7.24). If U is a left fibration of ∞-categories, then ev_C is a trivial Kan fibration of simplicial sets. It follows that the simplicial set $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ is a Kan complex, and that ev_C is a homotopy equivalence of Kan complexes.

035H **Example 5.3.1.4.** Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a left covering map of simplicial sets. Then, for every object $C \in \mathcal{C}$, the evaluation map $\mathrm{ev}_C : \mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \to \mathcal{E}_C$ is an isomorphism of simplicial sets (Exercise 4.2.5.5). It follows that the simplicial set $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ is discrete (see Remark 4.2.3.17). We can therefore identify $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}$ with a functor from $\mathcal{C}$ to the category of sets, which is isomorphic to the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}\,/\,\mathrm{N}_\bullet(\mathcal{C})} : \mathcal{C} \to \mathrm{Set}$ of Definition 5.2.0.4.

Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories. For an object $C \in \mathcal{C}$, the evaluation map $\mathrm{ev}_C : \mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \to \mathcal{E}_C$ of Remark 5.3.1.3 is generally not an equivalence of ∞-categories. By definition, an object of $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ can be identified with a functor of ∞-categories $F : \mathrm{N}_\bullet(\mathcal{C}_{C/}) \to \mathcal{E}$ for which the diagram

$$\begin{tikzcd}
\mathrm{N}_\bullet(\mathcal{C}_{C/}) \arrow[rr, "F"] \arrow[dr] & & \mathcal{E} \arrow[dl, "U"] \\
& \mathrm{N}_\bullet(\mathcal{C}) &
\end{tikzcd}$$

is commutative. This functor carries id_C to an object $X = \mathrm{ev}_C(F) \in \mathcal{E}_C$, and carries each morphism $f : C \to D$ of $\mathcal{C}$ to an object $Y \in \mathcal{E}_D$ equipped with a morphism $\widetilde{f} : X \to Y$ satisfying $U(\widetilde{f}) = f$. To guarantee that this data can be recovered from X (at least up to isomorphism), we need to impose an additional condition which guarantees that $\widetilde{f}$ is U-cocartesian.

Construction 5.3.1.5 (The Strict Transport Representation)**.** Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories. For every object $C \in \mathcal{C}$, we let $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ denote the full subcategory of $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) = \mathrm{Fun}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E})$ spanned by those commutative diagrams 035J

$$\begin{tikzcd}
\mathrm{N}_\bullet(\mathcal{C}_{C/}) \arrow[rr, "F"] \arrow[dr] & & \mathcal{E} \arrow[dl, "U"] \\
& \mathrm{N}_\bullet(\mathcal{C}) &
\end{tikzcd}$$

where F carries each morphism of $\mathrm{N}_\bullet(\mathcal{C}_{C/})$ to a U-cocartesian morphism of $\mathcal{E}$. The construction $C \mapsto \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ determines a functor $\mathrm{sTr} : \mathcal{C} \to \mathrm{QCat}$, which we will refer to as the *strict transport representation* of the cocartesian fibration U.

Remark 5.3.1.6. In the situation of Construction 5.3.1.5, suppose that $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ is a left fibration of ∞-categories. It follows that every morphism of $\mathcal{E}$ is U-cocartesian (Proposition 5.1.4.14), so the strict transport representation $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathcal{C} \to \mathrm{QCat}$ coincides with the weak transport representation $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}$. 035K

We now wish to show that Construction 5.3.1.5 is a refinement of the homotopy transport representation introduced in §5.2.5. This is a consequence of the following generalization of Remark 5.3.1.3:

Proposition 5.3.1.7. *Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories. Then, for every object $C \in \mathcal{C}$, the evaluation map of Remark 5.3.1.3 induces a trivial Kan fibration of ∞-categories $\mathrm{ev}_C : \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \to \mathcal{E}_C$.* 035L

Corollary 5.3.1.8. *Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories. Then the diagram of functors* 035M

$$\begin{tikzcd}
& \mathrm{QCat} \arrow[dr] & \\
\mathcal{C} \arrow[ur, "\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}"] \arrow[rr, "\mathrm{hTr}_{\mathcal{E}\,/\,\mathrm{N}_\bullet(\mathcal{C})}"'] & & \mathrm{hQCat}
\end{tikzcd}$$

commutes up to natural isomorphism, given by the construction

$$(C \in \mathcal{C}) \mapsto (\mathrm{ev}_C : \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \to \mathrm{hTr}_{\mathcal{E}\,/\,\mathrm{N}_\bullet(\mathcal{C})}(C) = \mathcal{E}_C).$$

Proof. It follows from Proposition 5.3.1.7 that for each object $C \in \mathcal{C}$, the evaluation functor ev_C is a trivial Kan fibration, and therefore an isomorphism in the homotopy category hQCat. To complete the proof, it will suffice to show that the construction $C \mapsto \mathrm{ev}_C$ is a natural transformation: that is, for every morphism $f : C \to D$ of $\mathcal{C}$, the diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \arrow[r, "\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(f)"] \arrow[d, "\mathrm{ev}_C"] & \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(D) \arrow[d, "\mathrm{ev}_D"] \\
\mathcal{E}_C \arrow[r, "f_!"] & \mathcal{E}_D
\end{tikzcd}$$

commutes up to natural isomorphism. Let $s : \mathcal{E}_C \to \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C)$ be a section of the trivial Kan fibration ev_C. Then the homotopy class $[s]$ is an inverse of $[\mathrm{ev}_C]$ in the homotopy category hQCat. It will therefore suffice to show that the diagram

$$\begin{tikzcd}
\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \arrow[r, "\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(f)"] & \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(D) \arrow[d, "\mathrm{ev}_D"] \\
\mathcal{E}_C \arrow[r, "f_!"] \arrow[u, "s"] & \mathcal{E}_D
\end{tikzcd}$$

commutes up to isomorphism: that is, that the composite functor

$$\mathcal{E}_C \xrightarrow{s} \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \xrightarrow{\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(f)} \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(D) \xrightarrow{\mathrm{ev}_D} \mathcal{E}_D$$

is given by covariant transport along f.

Unwinding the definitions, we can identify the composition

$$\mathcal{E}_C \xrightarrow{s} \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \subseteq \mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) = \mathrm{Fun}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E})$$

with a functor $H : \mathrm{N}_\bullet(\mathcal{C}_{C/}) \times \mathcal{E}_C \to \mathcal{E}$. Let us regard id_C and f as objects of the category $\mathcal{C}_{C/}$, so that f lifts to a morphism $\widetilde{f} : \mathrm{id}_C \to f$ corresponding to an edge $e : \Delta^1 \to \mathrm{N}_\bullet(\mathcal{C}_{C/})$. Let H_e denote the composition

$$\Delta^1 \times \mathcal{E}_C \xrightarrow{e \times \mathrm{id}} \mathrm{N}_\bullet(\mathcal{C}_{C/}) \times \mathcal{E}_C \xrightarrow{H} \mathcal{E}\,.$$

Unwinding the definitions, we see that the commutative diagram

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E}_C \arrow[r, "H_e"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^1 \arrow[r, "f"] & \mathrm{N}_{\bullet}(\mathcal{C})
\end{tikzcd}$$

witnesses the composite functor $\mathrm{ev}_D \circ \mathrm{sTr}_{\mathcal{E}/\mathcal{C}}(f) \circ s$ as given by covariant transport along f, in the sense of Definition 5.2.2.4. □

Corollary 5.3.1.9 (Functoriality)**.** *Suppose we are given a commutative diagram of ∞-categories* 035N

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathrm{N}_{\bullet}(\mathcal{C}), &
\end{tikzcd}$$

where U and U' are cocartesian fibrations. The following conditions are equivalent:

(1) *The functor F carries U-cocartesian morphisms of $\mathcal{E}$ to U'-cocartesian morphisms of $\mathcal{E}'$.*

(2) *The induced map of weak transport representations $\mathrm{wTr}_{\mathcal{E}/\mathcal{C}} \to \mathrm{wTr}_{\mathcal{E}'/\mathcal{C}}$ carries $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$ into $\mathrm{sTr}_{\mathcal{E}'/\mathcal{C}}$.*

Proof. The implication (1) ⇒ (2) is immediate from the definitions. Conversely, suppose that condition (2) is satisfied, and let $f : X \to Y$ be a U-cocartesian morphism of $\mathcal{E}$; we wish to show that $F(f)$ is U'-cocartesian. Set $C = U(X)$. Using Proposition 5.3.1.7, we can choose an object $\widetilde{X} \in \mathrm{sTr}_{\mathcal{E}/\mathcal{C}}(C)$ satisfying $\mathrm{ev}_C(\widetilde{X}) = X$. Let us identify $\widetilde{X}$ with a functor of ∞-categories $G : \mathrm{N}_{\bullet}(\mathcal{C}_{C/}) \to \mathcal{E}$. Write $\overline{f}$ for the image $U(f)$, which we regard as a morphism in the coslice category $\mathcal{C}_{C/}$. Assumption (2) guarantees that F carries $\widetilde{X}$ to an object of $\mathrm{sTr}_{\mathcal{E}'/\mathcal{C}}(C)$, so that $(F \circ G)(\overline{f})$ is a U'-cocartesian morphism of $\mathcal{E}'$. Since f is isomorphic to $G(\overline{f})$ (as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{E})$), it follows that $F(f)$ is also U'-cocartesian. □

The remainder of this section is devoted to the proof of Proposition 5.3.1.7. With an eye toward future applications, we will formulate a more general result, which can be applied to cocartesian fibrations $U : \mathcal{E} \to \mathcal{C}$ where $\mathcal{C}$ is not given by (the nerve of) an ordinary category.

02T1 **Notation 5.3.1.10** (Cocartesian Sections)**.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. Then the simplicial set

$$\operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E}) = \{U'\} \times_{\operatorname{Fun}(\mathcal{E}', \mathcal{C})} \operatorname{Fun}(\mathcal{E}', \mathcal{E})$$

is an ∞-category (see Corollary 4.1.4.8). We let $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$ denote the full subcategory of $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$ whose objects are morphisms $F : \mathcal{E}' \to \mathcal{E}$ which satisfy the identity $U \circ F = U'$ and carry U'-cocartesian edges of $\mathcal{E}'$ to U-cocartesian edges of $\mathcal{E}$.

02T2 **Variant 5.3.1.11** (Cartesian Sections)**.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ be cartesian fibrations of simplicial sets. We let $\operatorname{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$ denote the full subcategory of $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$ whose objects are morphisms $F : \mathcal{E}' \to \mathcal{E}$ which satisfy the identity $U \circ F = U'$ and carry U'-cartesian edges of $\mathcal{E}'$ to U-cartesian edges of $\mathcal{E}$. Note that we have a canonical isomorphism of simplicial sets

$$\operatorname{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})^{\mathrm{op}} = \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\mathrm{op}}}(\mathcal{E}'^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}}).$$

In the special case $\mathcal{E}' = \mathcal{C}$, we will refer to $\operatorname{Fun}^{\mathrm{Cart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ as the *∞-category of cartesian sections of U*.

03LM **Remark 5.3.1.12.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E}' \arrow[rr, "F"] \arrow[dr, "U'"'] & & \mathcal{E} \arrow[dl, "U"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where U and U' are cocartesian fibrations. Let $e : X \to Y$ be an edge of $\mathcal{C}$. The following conditions are equivalent:

(1) For every U'-cocartesian edge $\widetilde{e} : \widetilde{X} \to \widetilde{Y}$ of $\mathcal{E}'$ satisfying $U'(\widetilde{e}) = e$, the image $F(\widetilde{e})$ is a U-cocartesian edge of $\mathcal{E}$.

(2) For every vertex $\widetilde{X}$ of $\mathcal{E}'$ satisfying $U'(\widetilde{X}) = X$, there exists a U'-cocartesian edge $\widetilde{e} : \widetilde{X} \to \widetilde{Y}$ of $\mathcal{E}'$ such that $F(\widetilde{e})$ is U-cocartesian and $U'(\widetilde{e}) = e$.

The implication (1) ⇒ (2) is immediate from the definitions, and the implication (2) ⇒ (1) follows from Remark 5.1.3.8.

Let W be the collection of edges of $\mathcal{C}$ which satisfy these conditions. Then W contains all degenerate edges of $\mathcal{C}$ and is closed under composition: that is, for every 2-simplex

$$\begin{tikzcd}
& Y \arrow[dr, "e'"] & \\
X \arrow[ur, "e"] \arrow[rr, "e''"] & & Z
\end{tikzcd}$$

of $\mathcal{C}$, if e and e' belong to W, then e'' also belongs to W (see Proposition 5.1.4.12).

Remark 5.3.1.13. We will be primarily interested in the special case of Notation 5.3.1.10 where $U' : \mathcal{E}' \to \mathcal{C}$ is a left fibration of simplicial sets. In this case, an object $F \in \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$ belongs to the full subcategory $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$ if and only if it carries *every* edge of $\mathcal{E}'$ to a U-cocartesian edge of $\mathcal{E}$ (Proposition 5.1.4.14). 035P

Example 5.3.1.14. Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories. Then the strict transport representation $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$ of Construction 5.3.1.5 is given on objects by the formula 035Q

$$\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}(C) = \operatorname{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E}).$$

Remark 5.3.1.15. Let $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. Then the full subcategory $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E}) \subseteq \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$ is replete (Example 4.4.1.12). That is, if F and G are isomorphic objects of $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$, then F carries U'-cocartesian edges of $\mathcal{E}'$ to U-cocartesian edges of $\mathcal{E}$ if and only if G has the same property. In fact, we can be more precise: for every particular edge e of $\mathcal{E}'$, the image $F(e)$ is U-cocartesian if and only if $G(e)$ is U-cocartesian. To prove this, we can assume without loss of generality that $\mathcal{C} = \Delta^1$, in which case it follows from Corollary 5.1.2.5. 02T3

Remark 5.3.1.16 (Detecting Isomorphisms)**.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ be cocartesian fibrations of ∞-categories, and let $\alpha : F \to G$ be a morphism in the ∞-category $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$. The following conditions are equivalent: 035R

(1) The morphism α is an isomorphism in the ∞-category $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$.

(2) The image of α is an isomorphism in the ∞-category $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$.

(3) The image of α is an isomorphism in the ∞-category $\operatorname{Fun}(\mathcal{E}', \mathcal{E})$.

(4) For each object $X \in \mathcal{C}$, the induced map $\alpha_X : F(X) \to G(X)$ is an isomorphism in the ∞-category $\mathcal{E}_X$.

(5) For each object $X \in \mathcal{C}$, the induced map $\alpha_X : F(X) \to G(X)$ is an isomorphism in the ∞-category $\mathcal{E}$.

The implications (1) ⇔ (2) is immediate, the equivalences (2) ⇔ (3) and (4) ⇔ (5) follow from Corollary 4.4.3.19, and the equivalence (3) ⇔ (5) follows from Theorem 4.4.4.4.

Remark 5.3.1.17 (Functoriality)**.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. Suppose that we are given a morphism of simplicial sets $F : \mathcal{C}_0 \to \mathcal{C}$, 02T4 and set $\mathcal{E}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$ and $\mathcal{E}'_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}'$. Then pullback along F determines a morphism of simplicial sets

$$F^* : \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E}) \to \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_0}(\mathcal{E}'_0, \mathcal{E}_0),$$

which we will refer to as the *restriction map*.

02T5 **Remark 5.3.1.18.** In the situation of Remark 5.3.1.17, suppose that $F : \mathcal{C}_0 \to \mathcal{C}$ is a monomorphism of simplicial sets. Then the restriction map $F^* : \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E}) \to \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_0}(\mathcal{E}'_0, \mathcal{E}_0)$ is an isofibration. To see this, we first observe that $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$ can be regarded as a replete subcategory of the fiber product

$$\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_0}(\mathcal{E}'_0, \mathcal{E}_0) \times_{\operatorname{Fun}_{/\mathcal{C}_0}(\mathcal{E}'_0, \mathcal{E}_0)} \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})$$

(Remark 5.3.1.15). It will therefore suffice to show that the restriction map

$$\operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E}) \to \operatorname{Fun}_{/\mathcal{C}_0}(\mathcal{E}'_0, \mathcal{E}_0) \simeq \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}'_0, \mathcal{E})$$

is an isofibration, which follows from Proposition 4.5.5.14.

02T6 **Remark 5.3.1.19.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ be cocartesian fibrations of simplicial sets, and let K be an arbitrary simplicial set. Then:

- The projection map $\mathcal{C} \times_{\operatorname{Fun}(K, \mathcal{C})} \operatorname{Fun}(K, \mathcal{E}) \to \mathcal{C}$ is also a cocartesian fibration.
- The canonical isomorphism

  $$\operatorname{Fun}(K, \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})) \simeq \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{C} \times_{\operatorname{Fun}(K,\mathcal{C})} \operatorname{Fun}(K, \mathcal{E}))$$

  restricts to an isomorphism of full subcategories

  $$\operatorname{Fun}(K, \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E})) \simeq \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{C} \times_{\operatorname{Fun}(K,\mathcal{C})} \operatorname{Fun}(K, \mathcal{E})).$$

Both assertions follow immediately from Theorem 5.2.1.1.

02T7 **Remark 5.3.1.20.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Let $\mathcal{E}^{\circ} \subseteq \mathcal{E}$ be the simplicial subset whose n-simplices are maps $\Delta^n \to \mathcal{E}$ which carry each edge of Δ^n to a U-cocartesian edge of $\mathcal{E}$, so that U restricts to a left fibration $U^{\circ} : \mathcal{E}^{\circ} \to \mathcal{C}$ (see Corollary 5.1.4.15). Then $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}^{\circ})$ can be identified with the core of the ∞-category $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$.

035S **Proposition 5.3.1.21.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let $F : \mathcal{C}_0 \to \mathcal{C}$ be a left anodyne morphism of simplicial sets, and set $\mathcal{E}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$. Then the restriction map*

$$F^* : \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \to \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_0}(\mathcal{C}_0, \mathcal{E}_0)$$

of Remark 5.3.1.17 is a trivial Kan fibration.

Proof. Since F is a monomorphism of simplicial sets, the functor F^* is an isofibration of ∞-categories (Remark 5.3.1.18). It will therefore suffice to show that F^* is an equivalence of ∞-categories (see Proposition 4.5.5.20). By virtue of Proposition 4.5.1.22, this is equivalent to the assertion that for simplicial set X, the induced map

$$\mathrm{Fun}(X, \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}))^{\simeq} \to \mathrm{Fun}(X, \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_0}(\mathcal{C}_0, \mathcal{E}_0))^{\simeq}$$

is a homotopy equivalence of Kan complexes (in fact, it suffices to verify this for $X = \Delta^1$; see Theorem 4.5.7.1). Replacing $\mathcal{E}$ by the fiber product $\mathcal{C} \times_{\mathrm{Fun}(X,\mathcal{C})} \mathrm{Fun}(X, \mathcal{E})$ and using Remark 5.3.1.19, we are reduced to proving that F^* restricts to a homotopy equivalence $F^* : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})^{\simeq} \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_0}(\mathcal{C}_0, \mathcal{E}_0)^{\simeq}$. Let $U^{\circ} : \mathcal{E}^{\circ} \to \mathcal{E}$ denote the underlying left fibration of U. Using Remark 5.3.1.20, we can identify θ with the map

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}^{\circ}) \to \mathrm{Fun}_{/\mathcal{C}_0}(\mathcal{C}_0, \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}^{\circ}) \simeq \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}_0, \mathcal{E}^{\circ}),$$

given by precomposition with F. Since F is left anodyne, this map is a trivial Kan fibration (Proposition 4.2.5.4). □

Corollary 5.3.1.22. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let $U' : \mathcal{E}' \to \mathcal{C}$ be a left fibration of ∞-categories, and let X be an initial object of $\mathcal{E}'$. Then evaluation at X induces a trivial Kan fibration of ∞-categories* 035T

$$\mathrm{ev}_X : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E}) \to \{X\} \times_{\mathcal{C}} \mathcal{E}\,.$$

Proof. By virtue of Remark 5.3.1.13, we can replace U by the projection map $\mathcal{E}' \times_{\mathcal{C}} \mathcal{E} \to \mathcal{E}'$ and thereby reduce to the case where U' is the identity map. In this case, the desired result follows from Proposition 5.3.1.21, since the inclusion map $\{X\} \hookrightarrow \mathcal{E}'$ is left anodyne (Corollary 4.6.7.24). □

Proof of Proposition 5.3.1.7. Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a cocartesian fibration of ∞-categories. By virtue of Example 5.3.1.14, it will suffice to show that the evaluation functor

$$\mathrm{ev}_C : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \mathcal{E}) \to \mathcal{E}_C$$

is a trivial Kan fibration. This is a special case of Corollary 5.3.1.22, since the identity morphism id_C is initial when viewed as an object of the coslice category $\mathcal{C}_{C/}$. □

We conclude by recording another special case of Corollary 5.3.1.22 which will be useful later:

Corollary 5.3.1.23. *Let $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleleft}$ be a cocartesian fibration of simplicial sets. Then evaluation at $\mathbf{0}$ induces a trivial Kan fibration of simplicial sets* 02TC

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \{\mathbf{0}\} \times_{\mathcal{C}^{\triangleleft}} \overline{\mathcal{E}}.$$

Proof. Combine Corollary 5.3.1.22 with Example 4.3.7.11. □

5.3.2 Homotopy Colimits of Simplicial Sets

035U Let $f_0 : A \to A_0$ and $f_1 : A \to A_1$ be morphisms of simplicial sets. Recall that the *homotopy pushout of* A_0 *with* A_1 *along* A is defined to be the simplicial set

$$A_0 \coprod\nolimits_A^{\mathrm{h}} A_1 = A_0 \coprod\nolimits_{(\{0\} \times A)} (\Delta^1 \times A) \coprod\nolimits_{(\{1\} \times A)} A_1$$

(see Construction 3.4.2.2). This construction has two essential properties:

(1) The formation of homotopy pushouts is compatible with weak homotopy equivalence. That is, if we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
A_0 \arrow[d] & A \arrow[l, "f_0"'] \arrow[r, "f_1"] \arrow[d] & A_1 \arrow[d] \\
B_0 & B \arrow[l, "g_0"'] \arrow[r, "g_1"] & B_1,
\end{tikzcd}$$

in which the vertical maps are weak homotopy equivalences, then the induced map $A_0 \coprod_A^{\mathrm{h}} A_1 \to B_0 \coprod_B^{\mathrm{h}} B_1$ is also a weak homotopy equivalence (Corollary 3.4.2.15).

(2) The homotopy pushout is equipped with a comparison map $A_0 \coprod_A^{\mathrm{h}} A_1 \twoheadrightarrow A_0 \coprod_A A_1$, which is a weak homotopy equivalence if either $f_0 : A_0 \to A$ or $f_1 : A_1 \to A$ is a monomorphism (Corollary 3.4.2.13).

Our goal in this section is to introduce a variant of the homotopy pushout construction which can be applied to more general diagrams of simplicial sets. To every category $\mathcal{C}$ and every functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$, we introduce a simplicial set $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ which we refer to as the *homotopy colimit of* $\mathscr{F}$ (Construction 5.3.2.1). The homotopy colimit satisfies an analogue of property (1): it is compatible both with weak homotopy equivalence (Proposition 5.3.2.18) and with categorical equivalence (Variant 5.3.2.19). Moreover, there is a natural epimorphism from the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ to the usual colimit $\varinjlim(\mathscr{F})$ (Remark 5.3.2.9). We will see later that this map is often a weak homotopy equivalence (Corollary 7.5.6.14).

035V **Construction 5.3.2.1.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor. For every integer $n \geq 0$, we let $\underrightarrow{\mathrm{holim}}(\mathscr{F})_n$ denote the set of all ordered pairs (σ, τ), where $\sigma : [n] \to \mathcal{C}$ is an n-simplex of the nerve $\mathrm{N}_{\bullet}(\mathcal{C})$ and τ is an n-simplex of the simplicial set $\mathscr{F}(\sigma(0))$.

If (σ, τ) is an element of $\underrightarrow{\mathrm{holim}}(\mathscr{F})_n$ and $\alpha : [m] \to [n]$ is a nondecreasing function of linearly ordered sets, we set $\alpha^*(\sigma, \tau) = (\sigma \circ \alpha, \tau') \in \underrightarrow{\mathrm{holim}}(\mathscr{F})_m$, where τ' is given by the composite map

$$\Delta^m \xrightarrow{\alpha} \Delta^n \xrightarrow{\tau} \mathscr{F}(\sigma(0)) \to \mathscr{F}((\sigma \circ \alpha)(0)).$$

By means of this construction, the assignment $[n] \mapsto \underrightarrow{\mathrm{holim}}(\mathscr{F})_n$ determines a simplicial set $\underrightarrow{\mathrm{holim}}(\mathscr{F}) = \underrightarrow{\mathrm{holim}}(\mathscr{F})_\bullet$ which we will refer to as the *homotopy colimit* of the diagram $\mathscr{F}$. Note that the construction $(\sigma, \tau) \mapsto \sigma$ determines a morphism of simplicial sets $U : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_\bullet(\mathcal{C})$, which we will refer to as the *projection map*.

Example 5.3.2.2 (Discrete Diagrams)**.** Let $\mathcal{C}$ be a category having only identity morphisms, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets. Then the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ can be identified with the disjoint union $\coprod_{C \in \mathcal{C}} \mathscr{F}(C)$. `035W`

Remark 5.3.2.3. Let $T : \mathcal{C}' \to \mathcal{C}$ be a functor between categories, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets indexed by $\mathcal{C}$, and let $\mathscr{F}'$ denote the composition $\mathscr{F} \circ T$. Then we have a pullback diagram of simplicial sets `035X`

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}') \arrow[r] \arrow[d, "U'"] & \underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[d, "U"] \\
\mathrm{N}_\bullet(\mathcal{C}') \arrow[r, "\mathrm{N}_\bullet(T)"] & \mathrm{N}_\bullet(\mathcal{C}),
\end{tikzcd}$$

where U and U' denote the projection maps of Construction 5.3.2.1. In particular, for every object $C \in \mathcal{C}$, we have a canonical isomorphism of simplicial sets

$$\mathscr{F}(C) \simeq \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}).$$

Example 5.3.2.4 (Constant Diagrams)**.** Let $\mathcal{C}$ be a category, let X be a simplicial set, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be the constant diagram taking the value X. Combining Remark 5.3.2.3 with Example 5.3.2.2, we obtain a canonical isomorphism of simplicial sets $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \simeq \mathrm{N}_\bullet(\mathcal{C}) \times X$. In particular, if $X = \Delta^0$, then the projection map $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_\bullet(\mathcal{C})$ is an isomorphism. `035Y`

Example 5.3.2.5 (Set-Valued Functors)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a diagram of sets indexed by $\mathcal{C}$. Let us abuse notation by identifying $\mathscr{F}$ with a diagram of simplicial sets (by identifying each of the sets $\mathscr{F}(C)$ as a discrete simplicial set). Then there is a canonical isomorphism of simplicial sets `035Z`

$$\underrightarrow{\mathrm{holim}}(\mathscr{F}) \simeq \mathrm{N}_\bullet(\int_{\mathcal{C}} \mathscr{F}).$$

Here $\int_{\mathcal{C}} \mathscr{F}$ denotes the category of elements of of the functor $\mathscr{F}$ (Construction 5.2.6.1).

Example 5.3.2.6 (Corepresentable Functors)**.** Let $\mathcal{C}$ be a category and let $h^C : \mathcal{C} \to \mathrm{Set}$ be the functor corepresented by an object $C \in \mathcal{C}$, given by $h^C(D) = \mathrm{Hom}_{\mathcal{C}}(C, D)$. Let us `0360` abuse notation by regarding h^C as a functor from $\mathcal{C}$ to the category of simplicial sets (by identifying each morphism set $\operatorname{Hom}_{\mathcal{C}}(C, D)$ with the corresponding discrete simplicial set). Combining Examples 5.3.2.5 and 5.2.6.5, we obtain a canonical isomorphism of simplicial sets $\underrightarrow{\mathrm{holim}}(h^C) \simeq \mathrm{N}_\bullet(\mathcal{C}_{C/})$.

0361 **Remark 5.3.2.7.** Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets indexed by $\mathcal{C}$, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that, for every object $C \in \mathcal{C}$ which does not belong to $\mathcal{C}_0$, the simplicial set $\mathscr{F}(C)$ is empty. Then the image of the projection map $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_\bullet(\mathcal{C})$ is contained in $\mathrm{N}_\bullet(\mathcal{C}_0)$. Setting $\mathscr{F}_0 = \mathscr{F}|_{\mathcal{C}_0}$, we deduce that the canonical map

$$\underrightarrow{\mathrm{holim}}(\mathscr{F}_0) \simeq \mathrm{N}_\bullet(\mathcal{C}_0) \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \hookrightarrow \underrightarrow{\mathrm{holim}}(\mathscr{F})$$

is an isomorphism.

0362 **Remark 5.3.2.8** (Functoriality)**.** Let $\mathcal{C}$ be a category. Then the formation of homotopy colimits determines a functor

$$\underrightarrow{\mathrm{holim}} : \operatorname{Fun}(\mathcal{C}, \mathrm{Set}_\Delta) \to (\mathrm{Set}_\Delta)_{/\mathrm{N}_\bullet(\mathcal{C})} \qquad \mathscr{F} \mapsto \underrightarrow{\mathrm{holim}}(\mathscr{F}).$$

Moreover, this functor preserves small limits and colimits.

0363 **Remark 5.3.2.9** (Comparison with the Colimit)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets and let $\{t_C : \mathscr{F}(C) \to X\}_{C \in \mathcal{C}}$ be a collection of morphisms which exhibit X as a colimit of the diagram $\mathscr{F}$. The morphisms t_C then determine a natural transformation $t_\bullet : \mathscr{F} \to \underline{X}$, where $\underline{X} : \mathcal{C} \to \mathrm{Set}_\Delta$ denotes the constant functor taking the value X. Using Example 5.3.2.4, we obtain a morphism of simplicial sets

$$\theta : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\underrightarrow{\mathrm{holim}}(t_\bullet)} \underrightarrow{\mathrm{holim}}(\underline{X}) \simeq \mathrm{N}_\bullet(\mathcal{C}) \times X \to X,$$

which we will refer to as the *comparison map*. Note that, for every vertex $C \in \mathcal{C}$, the restriction of θ to the fiber $\{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F})$ can be identified with the morphism t_C. Since X is the union of the images of the morphisms t_C, it follows that the comparison map $\theta : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \twoheadrightarrow \underrightarrow{\lim}(\mathscr{F})$ is an epimorphism of simplicial sets.

0364 **Example 5.3.2.10** (Disjoint Unions)**.** Let I be a set, which we regard as a category having only identity morphisms. Let $\mathscr{F} : I \to \mathrm{Set}_\Delta$ be a functor, which we identify with a collection of simplicial sets $\{X_i\}_{i \in I}$. Then the comparison map

$$\underrightarrow{\mathrm{holim}}(\mathscr{F}) \twoheadrightarrow \underrightarrow{\lim}(\mathscr{F}) = \coprod_{i \in I} X_i$$

is an isomorphism of simplicial sets.

Notation 5.3.2.11 (The Mapping Simplex)**.** Suppose we are given a diagram of simplicial sets 0365

$$X(0) \xrightarrow{f(1)} X(1) \xrightarrow{f(1)} X(2) \xrightarrow{f(3)} \cdots \xrightarrow{f(n)} X(n),$$

which we will identify with a functor $\mathscr{F} : [n] \to \mathrm{Set}_{\Delta}$. We denote the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ by $\underrightarrow{\mathrm{holim}}(X(0) \to \cdots \to X(n))$, and refer to it as the *mapping simplex* of the diagram $\mathscr{F}$.

Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be any diagram of simplicial sets and suppose we are given an n-simplex of $\mathrm{N}_{\bullet}(\mathcal{C})$, corresponding to a diagram $C_0 \to \cdots \to C_n$ in the category $\mathcal{C}$. By virtue of Remark 5.3.2.3, the fiber product $\Delta^n \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F})$ can be identified with the mapping simplex of the diagram $\mathscr{F}(C_0) \to \cdots \to \mathscr{F}(C_n)$. When $n = 0$, this mapping simplex can be identified with the simplicial set $\mathscr{F}(C_0)$ (Example 5.3.2.4). For larger values of n, the mapping simplex can be computed recursively:

Remark 5.3.2.12. Let $n \geq 1$ and let $\mathscr{F} : [n] \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets which we denote by 0252

$$X(0) \to X(1) \to X(2) \to \cdots \to X(n).$$

Let $\mathscr{F}' : [n] \to \mathrm{Set}_{\Delta}$ denote the constant diagram taking the value $X(0)$. Let $\mathscr{F}_0 \subseteq \mathscr{F}$ be the subfunctor given by the diagram

$$\emptyset \to X(1) \to X(2) \to \cdots \to X(n),$$

and define $\mathscr{F}'_0 \subseteq \mathscr{F}'$ similarly, so that we have a pushout diagram

$$\begin{tikzcd}
\mathscr{F}'_0 \arrow[r] \arrow[d] & \mathscr{F}_0 \arrow[d] \\
\mathscr{F}' \arrow[r] & \mathscr{F}
\end{tikzcd}$$

in the category $\mathrm{Fun}([n], \mathrm{Set}_{\Delta})$. Applying Remark 5.3.2.8, we deduce that the induced diagram of simplicial sets

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}'_0) \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}(\mathscr{F}_0) \arrow[d] \\
\underrightarrow{\mathrm{holim}}(\mathscr{F}') \arrow[r] & \underrightarrow{\mathrm{holim}}(\mathscr{F})
\end{tikzcd}$$

is also a pushout square. Using Example 5.3.2.4 and Remark 5.3.2.7, we can rewrite this diagram as

$$\begin{tikzcd}
\mathrm{N}_{\bullet}(\{1 < 2 < \cdots < n\}) \times X(0) \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}(X(1) \to \cdots \to X(n)) \arrow[d] \\
\Delta^n \times X(0) \arrow[r] & \underrightarrow{\mathrm{holim}}(X(0) \to \cdots \to X(n)).
\end{tikzcd}$$

0366 **Example 5.3.2.13** (The Mapping Cylinder)**.** Let $f : X \to Y$ be a morphism of simplicial sets, which we identify with a diagram $\mathscr{F} : [1] \to \mathrm{Set}_{\Delta}$. We will denote the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ by $\underrightarrow{\mathrm{holim}}(f : X \to Y)$ and refer to it as the *mapping cylinder* of the morphism f. Applying Remark 5.3.2.12, we obtain an isomorphism of simplicial sets

$$\underrightarrow{\mathrm{holim}}(f : X \to Y) \simeq (\Delta^1 \times X) \coprod_{(\{1\} \times X)} Y;$$

that is, the mapping cylinder $\underrightarrow{\mathrm{holim}}(f : X \to Y)$ can be identified with the homotopy pushout $X \coprod_X^{\mathrm{h}} Y$ of Construction 3.4.2.2.

0367 **Remark 5.3.2.14.** Let n be a nonnegative integer, and suppose we are given a diagram of simplicial sets

$$X(0) \to X(1) \to X(2) \to \cdots \to X(n).$$

For each integer $0 \leq i \leq n$, let $\Delta^n_{\geq i}$ denote the nerve of the linearly ordered set $\{i < i+1 < \cdots < n\}$, which we regard as a simplicial subset of Δ^n. Applying Remark 5.3.2.12 repeatedly, we can identify the mapping simplex $\underrightarrow{\mathrm{holim}}(X(0) \to \cdots \to X(n))$ with the iterated pushout

$$(\Delta^n \times X(0)) \coprod_{(\Delta^n_{\geq 1} \times X(0))} (\Delta^n_{\geq 1} \times X(1)) \coprod_{(\Delta^n_{\geq 2} \times X(1))} \cdots \coprod_{(\{n\} \times X(n-1))} (\{n\} \times X(n)).$$

0368 **Example 5.3.2.15** (Homotopy Quotients)**.** Let G be a group and let BG denote the associated groupoid (consisting of a single object with automorphism group G). Let X be a simplicial set equipped with an action of G, which we identify with a functor $\mathscr{F} : BG \to \mathrm{Set}_{\Delta}$. We will denote the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ by $X_{\mathrm{h}G}$, and refer to it as the *homotopy quotient of X by the action of G.*

0369 **Example 5.3.2.16.** Let $\mathcal{C}$ be the partially ordered set depicted in the diagram

$$\bullet \leftarrow \bullet \rightarrow \bullet$$

and suppose we are given a functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$, which we identify with a diagram of simplicial sets

$$A_0 \xleftarrow{f_0} A \xrightarrow{f_1} A_1.$$

The homotopy colimit $\varinjlim(\mathscr{F})$ can be identified with the iterated homotopy pushout

$$(A{\coprod}^{\mathrm{h}}_{A}A_0){\coprod}^{\mathrm{h}}_{A}A_1.$$

In particular, the comparison map $q_0 : A\coprod^{\mathrm{h}}_A A_0 \twoheadrightarrow A\coprod_A A_0 \simeq A_0$ induces an epimorphism of simplicial sets

$$q : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to A_0{\coprod}^{\mathrm{h}}_{A}A_1.$$

Note that q_0 is always a weak homotopy equivalence of simplicial sets (Corollary 3.4.2.13), so that q is also a weak homotopy equivalence (Corollary 3.4.2.14). Beware that q is never an isomorphism, except in the trivial case where the simplicial set A is empty (in which case the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ and the homotopy pushout $A_0\coprod^{\mathrm{h}}_A A_1$ can both be identified with the disjoint union $A_0 \coprod A_1$).

Exercise 5.3.2.17. Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets with the following properties: 036A

- For every object $C \in \mathcal{C}$, the simplicial set $\mathscr{F}(C)$ is a Kan complex.
- For every morphism $u : C \to C'$ in $\mathcal{C}$, the induced map $\mathscr{F}(u) : \mathscr{F}(C) \to \mathscr{F}(C')$ is a Kan fibration.

Show that the projection map $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a left fibration of simplicial sets.

We now apply the preceding analysis to study the homotopy invariance properties of Construction 5.3.2.1.

Proposition 5.3.2.18. *Let $\mathcal{C}$ be a category and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a levelwise weak homotopy equivalence between diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. Then the induced map $\underrightarrow{\mathrm{holim}}(\alpha) : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \underrightarrow{\mathrm{holim}}(\mathscr{G})$ is a weak homotopy equivalence of simplicial sets.* 036B

Proof. By virtue of Proposition 3.4.2.16, it will suffice to show that for every n-simplex $\Delta^n \to \mathrm{N}_{\bullet}(\mathcal{C})$, the induced map $\Delta^n \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \Delta^n \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{G})$ is a weak homotopy equivalence. Using Remark 5.3.2.3, we are reduced to proving Proposition 5.3.2.18 in the special case where $\mathcal{C}$ is the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$. We now proceed by induction on n. If $n = 0$, the desired result follows immediately from Example 5.3.2.2. Let us therefore assume that $n > 0$. Let $\mathscr{F}'$ denote the restriction of $\mathscr{F}$ to the full subcategory $\{1 < 2 < \cdots < n\}$ and define $\mathscr{G}'$ similarly. The natural transformation α

determines a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Delta^n \times \mathscr{F}(0) \arrow[d] & \mathrm{N}_\bullet(\{1 < \cdots < n\}) \times \mathscr{F}(0) \arrow[l] \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}(\mathscr{F}') \arrow[d] \\
\Delta^n \times \mathscr{G}(0) & \mathrm{N}_\bullet(\{1 < \cdots < n\}) \times \mathscr{G}(0) \arrow[l] \arrow[r] & \underrightarrow{\mathrm{holim}}(\mathscr{G}'),
\end{tikzcd}$$

where the left horizontal maps are monomorphisms, the right vertical map is a weak homotopy equivalence by virtue of our inductive hypothesis, and the other vertical maps are weak homotopy equivalences by virtue of our assumption on α. The desired result now follows by combining Corollary 3.4.2.14 with Remark 5.3.2.12. □

Using exactly the same argument, we see that the formation of homotopy colimits is compatible with categorical equivalence:

036C **Variant 5.3.2.19.** Let $\mathcal{C}$ be a category and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a levelwise categorical equivalence between diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_\Delta$. Then the induced map $\underrightarrow{\mathrm{holim}}(\alpha) : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \underrightarrow{\mathrm{holim}}(\mathscr{G})$ is a categorical equivalence of simplicial sets.

Proof. By virtue of Corollary 4.5.7.3, it will suffice to show that for every n-simplex $\Delta^n \to \mathrm{N}_\bullet(\mathcal{C})$, the induced map $\Delta^n \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \Delta^n \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{G})$ is a categorical equivalence of simplicial sets. Using Remark 5.3.2.3, we are reduced to proving Variant 5.3.2.19 in the special case where $\mathcal{C}$ is the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$. We now proceed by induction on n. If $n = 0$, the desired result follows immediately from Example 5.3.2.2. Let us therefore assume that $n > 0$. Let $\mathscr{F}'$ denote the restriction of $\mathscr{F}$ to the full subcategory $\{1 < 2 < \cdots < n\}$ and define $\mathscr{G}'$ similarly. The natural transformation α determines a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Delta^n \times \mathscr{F}(0) \arrow[d] & \mathrm{N}_\bullet(\{1 < \cdots < n\}) \times \mathscr{F}(0) \arrow[l] \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}(\mathscr{F}') \arrow[d] \\
\Delta^n \times \mathscr{G}(0) & \mathrm{N}_\bullet(\{1 < \cdots < n\}) \times \mathscr{G}(0) \arrow[l] \arrow[r] & \underrightarrow{\mathrm{holim}}(\mathscr{G}'),
\end{tikzcd}$$

where the left horizontal maps are monomorphisms, the right vertical map is a categorical equivalence by virtue of our inductive hypothesis, and the other vertical maps are categorical equivalences by virtue of our assumption on α. The desired result now follows by combining Corollary 4.5.4.14 with Remark 5.3.2.12. □

The homotopy colimit of Construction 5.3.2.1 can be characterized by a universal mapping property.

Construction 5.3.2.20. Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets indexed by $\mathcal{C}$. For each object $C \in \mathcal{C}$, we let 036D

$$f_C : \mathrm{N}_{\bullet}(\mathcal{C}_{C/}) \times \mathscr{F}(C) \to \underrightarrow{\mathrm{holim}}(\mathscr{F})$$

denote the morphism of simplicial sets given on n-simplices by the formula $f_C(\sigma, \tau) = (\overline{\sigma}, \overline{\tau})$, where $\overline{\sigma}$ denotes the image of σ in $\mathrm{N}_{\bullet}(\mathcal{C})$ and $\overline{\tau}$ denote the image of τ under the map $\mathscr{F}(C) \to \mathscr{F}(\overline{\sigma}(0))$. Note that we can identify f_C with a morphism of simplicial sets

$$u_{\mathscr{F},C} : \mathscr{F}(C) \to \mathrm{Fun}_{/\,\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \underrightarrow{\mathrm{holim}}(\mathscr{F})) = \mathrm{wTr}_{\underrightarrow{\mathrm{holim}}(\mathscr{F})/\mathcal{C}}(C).$$

This morphism depends functorially on C: that is, the collection $u_{\mathscr{F}} = \{u_{\mathscr{F},C}\}_{C \in \mathcal{C}}$ is a natural transformation from $\mathscr{F}$ to the weak transport representation $\mathrm{wTr}_{\underrightarrow{\mathrm{holim}}(\mathscr{F})/\mathcal{C}}$.

For every pair of functors $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$, let $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G})_{\bullet}$ denote the simplicial set parametrizing natural transformations from $\mathscr{F}$ to $\mathscr{G}$ (Example 2.4.2.2), described concretely by the formula

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G})_n = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G}^{\Delta^n}).$$

Here $\mathscr{G}^{\Delta^n} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ denotes the functor given by $\mathscr{G}^{\Delta^n}(C) = \mathrm{Fun}(\Delta^n, \mathscr{G}(C))$.

Proposition 5.3.2.21. *Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, let $\mathcal{E}$ be a simplicial set, and define $\mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ by the formula $\mathscr{G}(C) = \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \mathcal{E})$. Then composition with the natural transformation $u_{\mathscr{F}}$ of Construction 5.3.2.20 induces an isomorphism of simplicial sets* 036E

$$\Phi_{\mathscr{F}} : \mathrm{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G})_{\bullet}.$$

Proof. For every object $C \in \mathcal{C}$, let $h^C : \mathcal{C} \to \mathrm{Set}_{\Delta}$ denote the functor corepresented by C (given by $h^C(D) = \mathrm{Hom}_{\mathcal{C}}(C, D)$, regarded as a discrete simplicial set). For every simplicial set K, let $\underline{K} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ denote the constant functor taking the value K. For every functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ we have a coequalizer diagram

$$\coprod_{C \to D} h^D \times \underline{\mathscr{F}(C)} \rightrightarrows \coprod_{C} h^C \times \underline{\mathscr{F}(C)} \to \mathscr{F}$$

in the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$. Note that, if we regard the simplicial set $\mathcal{E}$ as fixed, then the construction $\mathscr{F} \mapsto \Phi_{\mathscr{F}}$ carries colimits in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$ to limits in the arrow category $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$. We can therefore assume without loss of generality that the functor $\mathscr{F}$ factors as a product $h^C \times \underline{K}$, for some object $C \in \mathcal{C}$ and some simplicial set K.

Fix an integer $n \geq 0$; we wish to show that $\Phi_{\mathscr{F}}$ induces a bijection from n-simplices of $\operatorname{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E})$ to n-simplices of $\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G})_{\bullet}$. Replacing $\mathcal{E}$ by the simplicial set $\operatorname{Fun}(K \times \Delta^n, \mathcal{E})$, we are reduced to proving that Construction 5.3.2.20 induces a bijection

$$\Phi_0 : \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\underrightarrow{\mathrm{holim}}(h^C), \mathcal{E}) \to \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(h^C, \mathscr{G}).$$

Let $\mathscr{G}_0 : \mathcal{C} \to \mathrm{Set}$ denote the functor given on objects by the formula

$$\mathscr{G}_0(C) = \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^0, \mathscr{G}(C)) = \operatorname{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \mathcal{E}).$$

Under the identification of $\underrightarrow{\mathrm{holim}}(h^C) \simeq \mathrm{N}_{\bullet}(\mathcal{C}_{C/})$ of Example 5.3.2.6, the function Φ_0 corresponds to the bijection $\mathscr{G}_0(C) \simeq \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathrm{Set})}(h^C, \mathscr{G}_0)$ supplied by Yoneda's lemma. □

036F **Corollary 5.3.2.22.** *Let $\mathcal{C}$ be a small category. Then the homotopy colimit functor*

$$\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to \mathrm{Set}_{\Delta} \qquad \mathscr{F} \mapsto \underrightarrow{\mathrm{holim}}(\mathscr{F})$$

admits a right adjoint, given by the construction

$$\mathrm{Set}_{\Delta} \to \operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \qquad \mathcal{E} \mapsto (C \mapsto \operatorname{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \mathcal{E})).$$

036G **Corollary 5.3.2.23.** *Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a morphism of simplicial sets, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor. Then composition with the natural transformation $u_{\mathscr{F}}$ of Construction 5.3.2.20 induces an isomorphism of simplicial sets*

$$\operatorname{Fun}_{/\,\mathrm{N}_{\bullet}(\mathcal{C})}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E}) \to \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{F}, \mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}})_{\bullet},$$

where $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}$ is the weak transport representation of Construction 5.3.1.1.

Proof. Define $\mathscr{G}, \mathscr{H} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ by the formulae $\mathscr{G}(C) = \operatorname{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \mathcal{E})$ and $\mathscr{H}(C) = \operatorname{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \mathrm{N}_{\bullet}(\mathcal{C}))$. We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\operatorname{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E}) \arrow[r] \arrow[d, "U \circ"] & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G})_{\bullet} \arrow[d, "U \circ"] \\
\operatorname{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathrm{N}_{\bullet}(\mathcal{C})) \arrow[r] & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{H})_{\bullet},
\end{tikzcd}$$

where the horizontal maps are isomorphisms by virtue of Proposition 5.3.2.21. Corollary 5.3.2.23 follows by restricting to fibers of the vertical maps. □

Corollary 5.3.2.24. *Let $\mathcal{C}$ be a small category. Then the homotopy colimit functor* 036H

$$\underrightarrow{\mathrm{holim}} : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to (\mathrm{Set}_{\Delta})_{/\,\mathrm{N}_{\bullet}(\mathcal{C})}$$

admits a right adjoint, given by the functor

$$(\mathrm{Set}_{\Delta})_{/\,\mathrm{N}_{\bullet}(\mathcal{C})} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \qquad (U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})) \mapsto (\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathcal{C} \to \mathrm{Set}_{\Delta})$$

of Construction 5.3.1.1.

5.3.3 The Weighted Nerve

Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes indexed by $\mathcal{C}$. 025W In §5.3.2, we introduced the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$, which is a simplicial set equipped with a projection map $U : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}(\mathcal{C})$. If $\mathscr{F}$ carries each morphism of $\mathcal{C}$ to a Kan fibration, then the projection map U is a left fibration of simplicial sets (Exercise 5.3.2.17). Beware that U is not a left fibration in general. In this section, we introduce a variant of the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ which we will refer to as the *$\mathscr{F}$-weighted nerve of $\mathcal{C}$* and denote by $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ (Definition 5.3.3.1). The weighted nerve is equipped with a projection map $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$, which is a left fibration provided that $\mathscr{F}$ is a diagram of Kan complexes (Corollary 5.3.3.19). In §5.3.5, we will construct a comparison map $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ (Construction 5.3.4.11) which is a categorical equivalence of simplicial sets (Corollary 5.3.5.9); in particular, it is a weak homotopy equivalence.

Definition 5.3.3.1 (The Weighted Nerve)**.** Let $\mathcal{C}$ be a category equipped with a functor 025X $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. For every integer $n \geq 0$, we let $\mathrm{N}^{\mathscr{F}}_{n}(\mathcal{C})$ denote the collection of all pairs (σ, τ), where $\sigma : [n] \to \mathcal{C}$ is an n-simplex of $\mathrm{N}_{\bullet}(\mathcal{C})$ which we identify with a diagram

$$C_0 \to C_1 \to C_2 \to \cdots \to C_{n-1} \to C_n$$

and τ is a collection of simplices $\{\tau_j : \Delta^j \to \mathscr{F}(C_j)\}_{0 \leq j \leq n}$ which fit into a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook] \arrow[d, "\tau_0"] & \Delta^1 \arrow[r, hook] \arrow[d, "\tau_1"] & \Delta^2 \arrow[r, hook] \arrow[d, "\tau_2"] & \cdots \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, "\tau_n"] \\
\mathscr{F}(C_0) \arrow[r] & \mathscr{F}(C_1) \arrow[r] & \mathscr{F}(C_2) \arrow[r] & \cdots \arrow[r] & \mathscr{F}(C_n).
\end{tikzcd}$$

For every nondecreasing function $\alpha : [m] \to [n]$, we define a map $\alpha^* : \mathrm{N}^{\mathscr{F}}_{n}(\mathcal{C}) \to \mathrm{N}^{\mathscr{F}}_{m}(\mathcal{C})$ by the formula $\alpha^*(\sigma, \tau) = (\sigma \circ \alpha, \tau')$, where $\tau' = \{\tau'_i : \Delta^i \to \mathscr{F}(\alpha(i))\}_{0 \leq i \leq m}$ is determined by the requirement that each τ'_i is equal to the composition

$$\Delta^i \xrightarrow{\alpha|_{\{0<1<\cdots<i\}}} \Delta^{\alpha(i)} \xrightarrow{\tau_{\alpha(i)}} \mathscr{F}(\alpha(i)).$$

By means of these restriction maps, we regard the construction $[n] \mapsto \mathrm{N}_n^{\mathscr{F}}(\mathcal{C})$ as a simplicial set. We will denote this simplicial set by $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ and refer to it as the *$\mathscr{F}$-weighted nerve of $\mathcal{C}$*. Note that there is an evident projection map $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$, given on simplices by the construction $(\sigma, \tau) \mapsto \sigma$.

025Y **Example 5.3.3.2.** Let X be a simplicial set, which we identify with the constant functor $\mathscr{F} : [0] \to \mathrm{Set}_\Delta$ taking the value X. Then the weighted nerve $\mathrm{N}_\bullet^{\mathscr{F}}([0])$ can be identified with the simplicial set X.

0267 **Remark 5.3.3.3** (Vertices of the Weighted Nerve)**.** Let $\mathcal{C}$ be a category equipped with a functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$. Then vertices of the weighted nerve $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ can be identified with pairs (C, x), where C is an object of $\mathcal{C}$ and x is a vertex of the simplicial set $\mathscr{F}(C)$.

0268 **Remark 5.3.3.4** (Edges of the Weighted Nerve)**.** Let $\mathcal{C}$ be a category equipped with a functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$, and let (C, x) and (D, y) be vertices of the weighted nerve $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ (see Remark 5.3.3.3). Edges of the weighted nerve $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ with source (C, x) and target (D, y) can be identified with pairs (f, e), where $f : C \to D$ is a morphism of the category $\mathcal{C}$ and $e : \mathscr{F}(f)(x) \to y$ is an edge of the simplicial set $\mathscr{F}(D)$.

036J **Remark 5.3.3.5.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a functor. Let K be an auxiliary simplicial set, and define $\mathscr{F}^K : \mathcal{C} \to \mathrm{Set}_\Delta$ by the formula $\mathscr{F}^K(C) = \mathrm{Fun}(K, \mathscr{F}(C))$. Then the weighted nerves of $\mathscr{F}$ and $\mathscr{F}^K$ are related by a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{N}_\bullet^{\mathscr{F}^K}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(K, \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})) \arrow[d] \\
\mathrm{N}_\bullet(\mathcal{C}) \arrow[r] & \mathrm{Fun}(K, \mathrm{N}_\bullet(\mathcal{C})).
\end{tikzcd}$$

036K **Example 5.3.3.6.** Let $\mathcal{C}$ be a category, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be the functor given on objects by the formula $\mathscr{F}(C) = \mathrm{N}_\bullet(\mathcal{C}_{/C})$. Then there is a canonical isomorphism of simplicial sets

$$\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \simeq \mathrm{N}_\bullet(\mathrm{Fun}([1], \mathcal{C})) = \mathrm{Fun}(\Delta^1, \mathrm{N}_\bullet(\mathcal{C})).$$

0261 **Remark 5.3.3.7** (Functoriality in $\mathcal{C}$)**.** Let $\mathcal{C}$ be a category equipped with a functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$, let $U : \mathcal{C}' \to \mathcal{C}$ be a functor between categories, and let $\mathscr{F}' : \mathcal{C}' \to \mathrm{Set}_\Delta$ denote the composition $\mathscr{F} \circ U$. Then there is a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{N}_{\bullet}^{\mathscr{F}'}(\mathcal{C}') \arrow[r] \arrow[d] & \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \arrow[d] \\
\mathrm{N}_{\bullet}(\mathcal{C}') \arrow[r, "\mathrm{N}_{\bullet}(U)"] & \mathrm{N}_{\bullet}(\mathcal{C}).
\end{tikzcd}$$

Example 5.3.3.8 (Fibers of the Weighted Nerve)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be functor. For each object $C \in \mathcal{C}$, Remark 5.3.3.7 and Example 5.3.3.2 supply an isomorphism of simplicial sets 0263

$$\mathscr{F}(C) \simeq \{C\} \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}).$$

Example 5.3.3.9 (The Weighted Nerve of a Constant Diagram)**.** Let $\mathcal{C}$ be a category, let X be a simplicial set, and let $\underline{X} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be the constant functor taking the value X. Then Remark 5.3.3.7 and Example 5.3.3.2 supply an isomorphism of simplicial sets $\mathrm{N}_{\bullet}^{\underline{X}}(\mathcal{C}) \simeq X \times \mathrm{N}_{\bullet}(\mathcal{C})$. 0262

Remark 5.3.3.10 (Functoriality in $\mathscr{F}$)**.** Let $\mathcal{C}$ be a category. Then the construction $\mathscr{F} \mapsto \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ determines a functor from the diagram category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$ to the category $(\mathrm{Set}_{\Delta})_{/\mathrm{N}_{\bullet}(\mathcal{C})}$ of simplicial sets over the nerve $\mathrm{N}_{\bullet}(\mathcal{C})$. This functor commutes with all limits and with filtered colimits. 0266

Exercise 5.3.3.11. Let $\mathcal{C}$ be a category and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a natural transformation between functors $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. Show that, if α is a levelwise trivial Kan fibration, then the induced map of weighted nerves $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathscr{G}}(\mathcal{C})$ is a trivial Kan fibration of simplicial sets. 036L

Example 5.3.3.12 (The Weighted Nerve of a Cone)**.** Let $\mathcal{C}$ be a category and let $\mathcal{C}^{\triangleright}$ denote the right cone on $\mathcal{C}$ (Example 4.3.2.5), and let $\mathbf{1} \in \mathcal{C}^{\triangleright}$ denote the final object. Suppose we are given a diagram of simplicial sets $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$. Set $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$ and $Y = \overline{\mathscr{F}}(\mathbf{1})$, so that $\overline{\mathscr{F}}$ determines a natural transformation $\alpha : \mathscr{F} \to \underline{Y}$ (where $\underline{Y} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ denotes the constant functor taking the value Y). Combining Remark 5.3.3.10 with Example 5.3.3.9, we obtain morphisms of simplicial sets 036M

$$\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \xrightarrow{\alpha} \mathrm{N}_{\bullet}^{\underline{Y}}(\mathcal{C}) \simeq Y \times \mathrm{N}_{\bullet}(\mathcal{C}) \to Y.$$

Unwinding the definitions, there is a canonical isomorphism of simplicial sets

$$\mathrm{N}_{\bullet}^{\overline{\mathscr{F}}}(\mathcal{C}^{\triangleright}) \simeq \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \star_{Y} Y,$$

where the right hand side denotes the relative join of Construction 5.2.3.1.

025Z **Example 5.3.3.13.** Let $f : X \to Y$ be a morphism of simplicial sets, which we identify with a functor $\mathscr{F} : [1] \to \mathrm{Set}_{\Delta}$ (so that $X = \mathscr{F}(0)$ and $Y = \mathscr{F}(1)$). Then Example 5.3.3.12 supplies an isomorphism of simplicial sets $\mathrm{N}_{\bullet}^{\mathscr{F}}([1]) \simeq X \star_{Y} Y$.

0264 **Example 5.3.3.14.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor. For every morphism $f : C \to D$ in $\mathcal{C}$, Remark 5.3.3.7 and Example 5.3.3.13 supply an isomorphism of simplicial sets

$$\Delta^1 \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \simeq \mathscr{F}(C) \star_{\mathscr{F}(D)} \mathscr{F}(D).$$

046X **Proposition 5.3.3.15.** *Let $\mathcal{C}$ be a category and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a natural transformation between functors $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. Assume that:*

- *For each object $C \in \mathcal{C}$, the morphism $\alpha_C : \mathscr{F}(C) \to \mathscr{G}(C)$ is a cocartesian fibration of simplicial sets.*
- *For each morphism $u : C \to D$ of $\mathcal{C}$, the morphism $\mathscr{F}(u) : \mathscr{F}(C) \to \mathscr{F}(D)$ carries α_C-cocartesian edges of $\mathscr{F}(C)$ to α_D-cocartesian edges of $\mathscr{F}(D)$.*

Then:

(1) *The induced map $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathscr{G}}(\mathcal{C})$ is a cocartesian fibration of simplicial sets.*

(2) *Let $(f, e) : (C, x) \to (D, y)$ be an edge of the simplicial set $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ (see Remark 5.3.3.4). Then (f, e) is U-cocartesian if and only if $e : \mathscr{F}(f)(x) \to y$ is an α_D-cocartesian edge of the simplicial set $\mathscr{F}(D)$.*

Proof. By virtue of Proposition 5.1.4.7 and Remark 5.3.3.7, we may assume without loss of generality that $\mathcal{C}$ is the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$ for some nonnegative integer n. We proceed by induction on n. If $n = 0$, then U can be identified with the cocartesian fibration $\alpha_0 : \mathscr{F}(0) \to \mathscr{G}(0)$ (Example 5.3.3.2), so that assertions (1) and (2) are immediate. Let us therefore assume that $n > 0$, so that $\mathcal{C}$ can be identified with the cone $\mathcal{C}_0^{\triangleright}$ for $\mathcal{C}_0 = [n-1]$. Set $\mathscr{F}_0 = \mathscr{F}|_{\mathcal{C}_0}$ and $\mathscr{G}_0 = \mathscr{G}|_{\mathcal{C}_0}$. It follows from our inductive hypothesis that U restricts to a cocartesian fibration $U_0 : \mathrm{N}_{\bullet}^{\mathscr{F}_0}(\mathcal{C}_0) \to \mathrm{N}_{\bullet}^{\mathscr{G}_0}(\mathcal{C}_0)$ is a cocartesian fibration of ∞-categories, and that an edge of $\mathrm{N}_{\bullet}^{\mathscr{F}_0}(\mathcal{C}_0)$ is U_0-cocartesian if and only if it satisfies the criterion described in (2). It follows that the functor $\mathrm{N}_{\bullet}^{\mathscr{F}_0}(\mathcal{C}_0) \to \mathscr{F}(n)$ described in Example 5.3.3.12 carries U_0-cocartesian morphisms to α_n-cocartesian morphisms of the ∞-category $\mathscr{F}(n)$. Unwinding the definitions, we can identify U with the map of relative joins

$$\mathrm{N}_{\bullet}^{\mathscr{F}_0}(\mathcal{C}_0) \star_{\mathscr{F}(n)} \mathscr{F}(n) \to \mathrm{N}_{\bullet}^{\mathscr{G}_0}(\mathcal{C}_0) \star_{\mathscr{G}(n)} \mathscr{G}(n).$$

Assertions (1) and (2) now follow from Lemma 5.2.3.17. □

046Y **Corollary 5.3.3.16.** *Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories indexed by $\mathcal{C}$. Then:*

(1) *The projection map* $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ *is a cocartesian fibration of simplicial sets.*

(2) *Let* $(f, e) : (C, x) \to (D, y)$ *be an edge of the simplicial set* $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ *(see Remark 5.3.3.4). Then* (f, e) *is* U*-cocartesian if and only if* $e : \mathscr{F}(f)(x) \to y$ *is an isomorphism in the* ∞*-category* $\mathscr{F}(D)$.

In particular, $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ *is an* ∞*-category.*

Proof. Apply Proposition 5.3.3.15 in the special case where $\mathscr{G}$ is the constant diagram taking the value Δ^0. □

Exercise 5.3.3.17. Let $\mathcal{C}$ be a category, let n be an integer, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets indexed by $\mathcal{C}$. Assume that, for each object $C \in \mathcal{C}$, the simplicial set $\mathscr{F}(C)$ is an $(n, 1)$-category (Definition 4.8.1.8). Show that the cocartesian fibration $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is n-categorical, in the sense of Definition 4.8.6.24. 05DY

In particular, if each of the simplicial sets $\mathscr{F}(C)$ is (isomorphic to) the nerve of an ordinary category, then the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ is also isomorphic to the nerve of an ordinary category. For a more precise statement, see Example 5.6.1.8.

Corollary 5.3.3.18. *Let* $\mathcal{C}$ *be a category and let* $\alpha : \mathscr{F} \to \mathscr{G}$ *be a natural transformation between functors* $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$*. Suppose that, for every object* $C \in \mathcal{C}$*, the morphism* $\alpha_C : \mathscr{F}(C) \to \mathscr{G}(C)$ *is a left fibration of simplicial sets. Then the induced map* $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathscr{G}}(\mathcal{C})$ *is also a left fibration of simplicial sets.* 046Z

Proof. Combine Propositions 5.1.4.14 and 5.3.3.15. □

Corollary 5.3.3.19. *Let* $\mathcal{C}$ *be a category and let* $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ *be a functor. Suppose that, for every object* $C \in \mathcal{C}$*, the simplicial set* $\mathscr{F}(C)$ *is a Kan complex. Then the projection map* $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ *is a left fibration.* 036N

Proof. Apply Corollary 5.3.3.18 in the special case where $\mathscr{G}$ is the constant diagram taking the value Δ^0. □

Corollary 5.3.3.20. *Let* $\mathcal{C}$ *be a category and let* $\alpha : \mathscr{F} \to \mathscr{F}'$ *be a natural transformation between functors* $\mathscr{F}, \mathscr{F}' : \mathcal{C} \to \mathrm{Set}_{\Delta}$*. Then* α *is a levelwise categorical equivalence if and only if the induced map* $T : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathscr{F}'}(\mathcal{C})$ *is a categorical equivalence of simplicial sets.* 036P

Proof. Assume first that α is a levelwise categorical equivalence. To prove that T is a categorical equivalence of simplicial sets, it will suffice to show that for every simplex $\sigma : \Delta^n \to \mathrm{N}_{\bullet}(\mathcal{C})$, the induced map $T_{\sigma} : \Delta^n \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \Delta^n \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathscr{F}'}(\mathcal{C})$ is a categorical equivalence of simplicial sets (Corollary 4.5.7.3). Using Remark 5.3.3.7, we can reduce to the special case where $\mathcal{C}$ is the linearly ordered $[n] = \{0 < 1 < \cdots < n\}$ for some $n \geq 0$. We now proceed by induction on n. If $n = 0$, the result is immediate from Example 5.3.3.2. The inductive step follows by combining Example 5.3.3.12 with Corollary 5.2.4.7.

We now prove the converse. Using Proposition 4.1.3.2, we can choose a commutative diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\alpha"] \arrow[d] & \mathscr{F}' \arrow[d] \\
\mathscr{G} \arrow[r, "\beta"] & \mathscr{G}'
\end{tikzcd}$$

in the category $\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$, where the vertical maps are levelwise categorical equivalences and the simplicial sets $\mathscr{G}(C)$ and $\mathscr{G}'(C)$ are ∞-categories for each $C \in \mathcal{C}$. Using the first part of the proof, we can replace α by β and thereby reduce to the special case where $\mathscr{F}$ and $\mathscr{F}'$ are diagrams of ∞-categories. In this case, the projection maps $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C}) \leftarrow \mathrm{N}_{\bullet}^{\mathscr{F}'}(\mathcal{C})$ are cocartesian fibrations of ∞-categories (Corollary 5.3.3.16). It then follows from Theorem 5.1.6.1 (together with Example 5.3.3.8) that if T is an equivalence of ∞-categories, then α is a levelwise categorical equivalence. □

036Q **Example 5.3.3.21.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. Suppose that, for every object $C \in \mathcal{C}$, the simplicial set $\mathscr{F}(C)$ is an ∞-category, so that the projection map $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a cocartesian fibration (Corollary 5.3.3.16). Define $\mathscr{F}^{\simeq} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ by the formula $\mathscr{F}^{\simeq}(C) = \mathscr{F}(C)^{\simeq}$. Then $\mathrm{N}_{\bullet}^{\mathscr{F}^{\simeq}}(\mathcal{C})$ can be identified with with simplicial subset of $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ spanned by those n-simplices which carry each edge of Δ^n to a U-cocartesian edge of $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$. That is, the projection map $U^{\simeq} : \mathrm{N}_{\bullet}^{\mathscr{F}^{\simeq}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is the underlying left fibration of the cocartesian fibration U (see Corollary 5.1.4.15).

026C **Remark 5.3.3.22** (The Homotopy Transport Representation)**.** Let $\mathcal{C}$ be a category equipped with a functor $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ and let $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ be the cocartesian fibration of Corollary 5.3.3.16. Then the homotopy transport representation

$$\mathrm{hTr}_{\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})/\mathrm{N}_{\bullet}(\mathcal{C})} : \mathcal{C} \to \mathrm{hQCat}$$

of Construction 5.2.5.2 is canonically isomorphic to the composition $\mathcal{C} \xrightarrow{\mathscr{F}} \mathrm{QCat} \twoheadrightarrow \mathrm{hQCat}$. To prove this, it suffices to observe that for every morphism $f : C \to D$ in $\mathcal{C}$, the functor

$$\mathscr{F}(f) : \mathscr{F}(C) \simeq \{C\} \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \{D\} \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \simeq \mathscr{F}(D)$$

is given by covariant transport along f, which follows immediately from Proposition 5.2.3.15 and Example 5.3.3.14.

We conclude this section by showing that the weighted nerve can be characterized by a universal mapping property.

Notation 5.3.3.23. Let $\mathcal{C}$ be a category and suppose we are given a morphism of simplicial sets $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$. For every object $C \in \mathcal{C}$, let $\mathscr{G}_{\mathcal{E}}(C)$ denote the fiber product $\mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathcal{E}$. The construction $C \mapsto \mathscr{G}_{\mathcal{E}}(C)$ then determines a functor $\mathscr{G}_{\mathcal{E}} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. 036R

Suppose we are given an n-simplex σ of $\mathcal{E}$. Then $U(\sigma)$ is an n-simplex of the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C})$, which we can identify with a diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n$$

in the category $\mathcal{C}$. For $0 \leq m \leq n$, we can view the diagram

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_m} C_m \xrightarrow{\mathrm{id}} C_m$$

as an m-simplex $\overline{\tau}_m$ of the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C}_{/C_m})$. The pair $(\overline{\tau}_m, U(\sigma)|_{\Delta^m})$ can then be viewed as an m-simplex τ_m of $\mathscr{G}_{\mathcal{E}}(C_m)$. Setting $\tau = (\tau_0, \tau_1, \cdots, \tau_n)$, we observe that the pair $(U(\sigma), \tau)$ can be regarded as an n-simplex $u_{\mathcal{E}}(\sigma)$ of the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{G}_{\mathcal{E}}}(\mathcal{C})$. Allowing n to vary, the construction $\sigma \mapsto u_{\mathcal{E}}(\sigma)$ determines a morphism of simplicial sets $u_{\mathcal{E}} : \mathcal{E} \to \mathrm{N}_{\bullet}^{\mathscr{G}_{\mathcal{E}}}(\mathcal{C})$ for which the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "u_{\mathcal{E}}"] \arrow[dr, "U"'] & & \mathrm{N}_{\bullet}^{\mathscr{G}_{\mathcal{E}}}(\mathcal{C}) \arrow[dl] \\
& \mathrm{N}_{\bullet}(\mathcal{C}) &
\end{tikzcd}$$

is commutative.

Proposition 5.3.3.24. *Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a morphism of simplicial sets, and let $u_{\mathcal{E}} : \mathcal{E} \to \mathrm{N}_{\bullet}^{\mathscr{G}_{\mathcal{E}}}(\mathcal{C})$ be the morphism of Notation 5.3.3.23. For every functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$, precomposition with $u_{\mathcal{E}}$ induces a bijection* 036S

$$T_{\mathcal{E}} : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{G}_{\mathcal{E}}, \mathscr{F}) \to \mathrm{Hom}_{(\mathrm{Set}_{\Delta})_{/\mathrm{N}_{\bullet}(\mathcal{C})}}(\mathcal{E}, \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})).$$

Corollary 5.3.3.25. *Let $\mathcal{C}$ be a category. Then the weighted nerve functor* 036T

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to (\mathrm{Set}_{\Delta})_{/\mathrm{N}_{\bullet}(\mathcal{C})} \qquad \mathscr{F} \mapsto \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$$

has a left adjoint, given by the construction $\mathcal{E} \mapsto \mathscr{G}_{\mathcal{E}}$ of Notation 5.3.3.23.

Proof of Proposition 5.3.3.24. The construction $\mathcal{E} \mapsto T_{\mathcal{E}}$ carries colimits in the category $(\mathrm{Set}_{\Delta})_{/\mathrm{N}_{\bullet}(\mathcal{C})}$ to limits in the arrow category $\mathrm{Fun}([1], \mathrm{Set})$. We can therefore assume without loss of generality that $\mathcal{E} = \Delta^n$ is a standard simplex, so that the morphism U determines a diagram $C_0 \to C_1 \to \cdots \to C_n$ in the category $\mathcal{C}$. Unwinding the definitions, we see that the codomain of $T_{\mathcal{E}}$ can be identified with the set of tuples $\tau = (\tau_0, \tau_1, \cdots, \tau_n)$, where $\tau_i : \Delta^i \to \mathscr{F}(C_i)$ are simplices for which the diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook] \arrow[d, "\tau_0"'] & \Delta^1 \arrow[r, hook] \arrow[d, "\tau_1"] & \Delta^2 \arrow[r, hook] \arrow[d, "\tau_2"'] & \cdots \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, "\tau_n"'] \\
\mathscr{F}(C_0) \arrow[r] & \mathscr{F}(C_1) \arrow[r] & \mathscr{F}(C_2) \arrow[r] & \cdots \arrow[r] & \mathscr{F}(C_n)
\end{tikzcd}$$

is commutative. Let us regard τ as fixed; we wish to prove that there is a unique natural transformation $\alpha : \mathscr{G}_{\mathcal{E}} \to \mathscr{F}$ satisfying $T_{\mathcal{E}}(\alpha) = \tau$.

Let D be an object of $\mathcal{C}$ and let $m \geq 0$ be an integer. Then m-simplices of the simplicial set $\mathscr{G}_{\mathcal{E}}(D) = \mathrm{N}_\bullet(\mathcal{C}_{/D}) \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathcal{E}$ can be identified with pairs (f, g), where $g : [m] \to [n]$ is a nondecreasing function and $f : C_{g(m)} \to D$ is a morphism in the category $\mathcal{C}$. Let $\alpha_D(f, g)$ denote the m-simplex of $\mathscr{F}(D)$ given by the composition

$$\Delta^m \xrightarrow{g} \Delta^{g(m)} \xrightarrow{\tau_{g(m)}} \mathscr{F}(C_{g(m)}) \xrightarrow{\mathscr{F}(f)} \mathscr{F}(D).$$

The construction $(f, g) \mapsto \alpha_D(f, g)$ determines a morphism of simplicial sets $\alpha_D : \mathscr{G}_{\mathcal{E}}(D) \to \mathscr{F}(D)$. The assignment $D \mapsto \alpha_D$ determines a natural transformation of functors $\alpha : \mathscr{G}_{\mathcal{E}} \to \mathscr{F}$ satisfying $T_{\mathcal{E}}(\alpha) = \tau$. This proves existence.

We now prove uniqueness. Suppose we are given another natural transformation $\alpha' : \mathscr{G}_{\mathcal{E}} \to \mathscr{F}$ satisfying $T_{\mathcal{E}}(\alpha') = \tau$; we wish to show that $\alpha = \alpha'$. Fix an object $D \in \mathcal{C}$ and an m-simplex of the simplicial set $\mathscr{G}_{\mathcal{E}}(D)$, which we identify with a pair (f, g) as above. We wish to verify that $\alpha_D(f, g)$ and $\alpha'_D(f, g)$ coincide (as m-simplices of the simplicial set $\mathscr{F}(D)$). Set $n' = g(m)$, so that the function g factors as a composition $[m] \xrightarrow{g} [n'] \xrightarrow{\iota} [n]$, where $\iota : [n'] \hookrightarrow [n]$ is the inclusion map. Since α_D and α'_D are morphisms of simplicial sets, it will suffice to prove that $\alpha_D(f, \iota)$ and $\alpha'_D(f, \iota)$ coincide (as n'-simplices of the simplicial set $\mathscr{F}(D)$). Since both α_D and α'_D are natural in D, we may assume without loss of generality that $D = C_{n'}$ and that f is the identity morphism. In this case, the identities $T_{\mathcal{E}}(\alpha) = \tau = T_{\mathcal{E}}(\alpha')$ give $\alpha_D(f, \iota) = \tau_{n'} = \alpha'_D(f, \iota)$. □

036U **Variant 5.3.3.26.** Let $\mathcal{C}$ be a category, and let us regard $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)$ as equipped with the simplicial enrichment described in Example 2.4.2.2. For every morphism of simplicial sets $\mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ and every functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$, precomposition with the morphism $u_{\mathcal{E}} : \mathcal{E} \to \mathrm{N}^{\mathscr{G}_{\mathcal{E}}}_\bullet(\mathcal{C})$ of Notation 5.3.3.23 induces an isomorphism of simplicial sets

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)}(\mathscr{G}_{\mathcal{E}}, \mathscr{F})_\bullet \to \mathrm{Fun}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathcal{E}, \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})).$$

To see that this map is bijective on m-simplices, we can replace $\mathcal{E}$ by the product $\Delta^m \times \mathcal{E}$ to reduce to the case $m = 0$, in which case it follows from Proposition 5.3.3.24.

5.3.4 Scaffolds of Cocartesian Fibrations

Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a (strictly commutative) diagram of ∞-categories indexed by $\mathcal{C}$. Our goal in this section is to show that the diagram $\mathscr{F}$ can be recovered, up to equivalence, from the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ of Definition 5.3.3.1. More precisely, we will show that there exists a levelwise categorical equivalence from $\mathscr{F}$ to the strict transport representation $\mathrm{sTr}_{\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})/\mathcal{C}}$ of Construction 5.3.1.5 (Corollary 5.3.4.19). 036V

We begin with some general remarks. Let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be any cocartesian fibration of simplicial sets and let $\mathrm{wTr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathrm{QCat}$ be the weak transport representation of U (Construction 5.3.1.1). Every levelwise categorical equivalence $\alpha : \mathscr{F} \to \mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$ can be viewed as a natural transformation from $\mathscr{F}$ to the weak transport representation $\mathrm{wTr}_{\mathcal{E}/\mathcal{C}}$, which we can identify (using Corollary 5.3.2.23) with a morphism from the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ into $\mathcal{E}$. Our first goal is to give an explicit characterization of the collection of morphisms $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ which arise in this way, which we will refer to as *scaffolds* of the cocartesian fibration U (Definition 5.3.4.2 and Remark 5.3.4.10).

Definition 5.3.4.1. Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets indexed by $\mathcal{C}$, and let e be an edge of the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$. Let us identify e with a pair $(f, \overline{e})$, where $f : C \to D$ is a morphism in the category $\mathcal{C}$ and $\overline{e}$ is an edge of the simplicial set $\mathscr{F}(C)$. We will say that the edge $e = (f, \overline{e})$ is *horizontal* if $\overline{e}$ is a degenerate edge of $\mathscr{F}(C)$. 036W

Definition 5.3.4.2. Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a cocartesian fibration of ∞-categories, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets. We will say that a morphism of simplicial sets $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ is a *scaffold* if it satisfies the following conditions: 0255

(0) The diagram of simplicial sets

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[rr, "\lambda"] \arrow[dr] & & \mathcal{E} \arrow[dl, "U"] \\
& \mathrm{N}_{\bullet}(\mathcal{C}) &
\end{tikzcd}$$

is commutative (where the left vertical map is the projection map of Construction 5.3.2.1).

(1) The morphism λ carries horizontal edges of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ to U-cocartesian morphisms of $\mathcal{E}$.

(2) For every object $C \in \mathcal{C}$, the induced map

$$\mathscr{F}(C) \simeq \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\lambda} \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathcal{E}$$

is a categorical equivalence of simplicial sets.

036X **Example 5.3.4.3.** Let n be a nonnegative integer and let $\mathcal{E}$ denote the nerve of the partially ordered set $Q = \{(i,j) \in [n] \times [n] : j \leq i\}$. Then there is a cocartesian fibration of ∞-categories $U : \mathcal{E} \to \Delta^n$, given on vertices by the formula $U(i,j) = i$. Let $\mathscr{F} : [n] \to \mathrm{Set}_\Delta$ denote the functor given by $\mathscr{F}(i) = \Delta^i$, so that vertices of the homotopy colimit can be identified with elements of Q. There is a unique morphism of simplicial sets $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ which is the identity at the level of vertices, which is a scaffold of the cocartesian fibration U. Moreover, λ is a monomorphism, and an n-simplex $(i_0, j_0) \leq (i_1, j_1) \leq \cdots \leq (i_n, j_n)$ belongs to the image of λ if and only if $j_n \leq i_0$. The case $n = 3$ is depicted in the following diagram, where the image of λ is indicated with solid arrows:

$$\begin{tikzcd}
 & & & (3,3) \\
 & & (2,2) \arrow[r] \arrow[ur, dashed] & (3,2) \arrow[u] \\
 & (1,1) \arrow[r] \arrow[ur, dashed] & (2,1) \arrow[r] \arrow[u] & (3,1) \arrow[u] \\
(0,0) \arrow[r] \arrow[ur, dashed] & (1,0) \arrow[r] \arrow[u] & (2,0) \arrow[r] \arrow[u] & (3,0).
\end{tikzcd}$$

0256 **Example 5.3.4.4.** Let $\mathcal{E}$ be an ∞-category equipped with a cocartesian fibration $U : \mathcal{E} \to \Delta^1$ having fibers $\mathcal{E}_0 = \{0\} \times_{\Delta^1} \mathcal{E}$ and $\mathcal{E}_1 = \{1\} \times_{\Delta^1} \mathcal{E}$. Choose a functor $F : \mathcal{E}_0 \to \mathcal{E}_1$ and a morphism $h : \Delta^1 \times \mathcal{E}_0 \to \mathcal{E}$ which witnesses F as given by covariant transport along the nondegenerate edge of Δ^1, in the sense of Definition 5.2.2.4. Then F can be identified with a diagram $\mathscr{F} : [1] \to \mathrm{QCat}$, and the map

$$\underrightarrow{\mathrm{holim}}(\mathscr{F}) = (\Delta^1 \times \mathcal{E}_0) \coprod_{(\{1\} \times \mathcal{E}_0)} \mathcal{E}_1 \xrightarrow{(h, \mathrm{id})} \mathcal{E}$$

is a scaffold.

036Y **Remark 5.3.4.5** (Isomorphism Invariance)**.** In the situation of Definition 5.3.4.2, suppose that we are given a pair of morphisms $\lambda, \lambda' : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ which are isomorphic when viewed as objects of the ∞-category $\mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E})$. Then λ is a scaffold if and only if λ' is a scaffold (see Corollary 5.1.2.5 and Remark 4.5.1.15).

Remark 5.3.4.6 (Change of $\mathcal{E}$)**.** Suppose we are given a commutative diagram 036Z

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "T"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathrm{N}_{\bullet}(\mathcal{C}), &
\end{tikzcd}$$

where the vertical maps are cocartesian fibrations and T is an equivalence of cocartesian fibrations over $\mathrm{N}_{\bullet}(\mathcal{C})$. Then a morphism $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ is a scaffold of the cocartesian fibration U if and only if $T \circ \lambda$ is a scaffold of the cocartesian fibration U'.

We now describe two important examples of scaffolds, both of which can be regarded as generalizations of Example 5.3.4.3.

Construction 5.3.4.7 (The Universal Scaffold)**.** Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ 026D be a cocartesian fibration of ∞-categories, and let $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$ denote the strict transport representation of U (Construction 5.3.1.5). For each $n \geq 0$, we can identify n-simplices of the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}})$ with pairs (σ, τ), where σ is an n-simplex of $\mathrm{N}_{\bullet}(\mathcal{C})$ (given by a diagram $C_0 \to C_1 \to \cdots \to C_n$ in the category $\mathcal{C}$) and τ is an n-simplex of the ∞-category $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}(C_0) = \mathrm{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}_{C_0/}), \mathcal{E})$, which we identify with a morphism of simplicial sets $\Delta^n \times \mathrm{N}_{\bullet}(\mathcal{C}_{C_0/}) \to \mathcal{E}$. Let us identify the diagram $C_0 \xrightarrow{\mathrm{id}} C_0 \to C_1 \to \cdots \to C_n$ with an n-simplex $\widetilde{\sigma}$ of the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C}_{C_0/})$, and let $\lambda_u(\sigma, \tau)$ denote the n-simplex of $\mathcal{E}$ given by the composite map

$$\Delta^n \xrightarrow{(\mathrm{id}, \widetilde{\sigma})} \Delta^n \times \mathrm{N}_{\bullet}(\mathcal{C}_{C_0/}) \xrightarrow{\tau} \mathcal{E}\,.$$

The construction $(\sigma, \tau) \mapsto \lambda_u(\sigma, \tau)$ determines a morphism of simplicial sets

$$\lambda_u : \underrightarrow{\mathrm{holim}}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}) \to \mathcal{E},$$

which we will refer to as the *universal scaffold* of the cocartesian fibration U.

Proposition 5.3.4.8. *Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a cocartesian fibration* 0370 *of ∞-categories. Then the morphism $\lambda_u : \underrightarrow{\mathrm{holim}}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}) \to \mathcal{E}$ of Construction 5.3.4.7 is a scaffold, in the sense of Definition 5.3.4.2.*

Proof. It is clear that the composition $U \circ \lambda_u$ coincides with the projection map $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}(\mathcal{C})$. Let e be a horizontal edge of the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}})$, determined by a morphism $\overline{e} : C \to D$ in the category $\mathcal{C}$ together with a degenerate edge id_T of the simplicial set $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}(C)$. Identifying T with an object of the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}_{C/}), \mathcal{E})$, we see that $\lambda_u(e)$ coincides with the morphism $T(\overline{e})$ and is therefore a U-cocartesian morphism of $\mathcal{E}$. To complete the proof, we observe that for every object $C \in \mathcal{C}$, the induced map

$$\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) \simeq \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}) \xrightarrow{\lambda} \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathcal{E}$$

agrees with the map $\mathrm{ev}_C : \mathrm{Fun}^{\mathrm{CCart}}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E}) \to \mathcal{E}_C$ given by evaluation on the initial object $\mathrm{id}_C \in \mathcal{C}_{C/}$, and is therefore a trivial Kan fibration of simplicial sets (Proposition 5.3.1.7). □

0371 **Corollary 5.3.4.9.** *Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories. Then there exists a diagram $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ and a scaffold $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$.*

0372 **Remark 5.3.4.10** (Universality)**.** Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets. Applying Corollary 5.3.2.23, we obtain a bijection from the set of morphisms $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ in the category $(\mathrm{Set}_\Delta)_{/\,\mathrm{N}_\bullet(\mathcal{C})}$ to the set of natural transformations $\alpha : \mathscr{F} \to \mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}$. Unwinding the definitions, we see that α factors through the subfunctor $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}} \subseteq \mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}$ if and only if λ satisfies condition (1) of Definition 5.3.4.2. If this condition is satisfied, then $\alpha : \mathscr{F} \to \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}$ is a levelwise categorical equivalence if and only if λ satisfies condition (2) of Definition 5.3.4.2. We therefore obtain a bijection

$$\begin{tikzcd}
\{\text{Levelwise categorical equivalences } \alpha : \mathscr{F} \to \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}\} \arrow[d, "\Phi"] \\
\{\text{Scaffolds } \lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}\}.
\end{tikzcd}$$

Concretely, this bijection carries a levelwise categorical equivalence $\alpha : \mathscr{F} \to \mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}$ to the composite map

$$\underrightarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\alpha} \underrightarrow{\mathrm{holim}}(\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}) \xrightarrow{\lambda_u} \mathcal{E},$$

where λ_u is the universal scaffold of Construction 5.3.4.7.

0373 **Construction 5.3.4.11** (The Taut Scaffold)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets. By definition, an n-simplex of the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ is a pair (σ, τ), where σ is an n-simplex of $\mathrm{N}_\bullet(\mathcal{C})$ (given by a diagram $C_0 \to \cdots \to C_n$ in the category $\mathcal{C}$) and τ is an n-simplex of the simplicial set $\mathscr{F}(C_0)$. For $0 \leq i \leq n$, let τ_i denote the composite map

$$\Delta^i \hookrightarrow \Delta^n \xrightarrow{\tau} \mathscr{F}(C_0) \to \mathscr{F}(C_i).$$

We then have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook] \arrow[d, "\tau_0"'] & \Delta^1 \arrow[r, hook] \arrow[d, "\tau_1"] & \Delta^2 \arrow[r, hook] \arrow[d, "\tau_2"'] & \cdots \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, "\tau_n"'] \\
\mathscr{F}(C_0) \arrow[r] & \mathscr{F}(C_1) \arrow[r] & \mathscr{F}(C_2) \arrow[r] & \cdots \arrow[r] & \mathscr{F}(C_n).
\end{tikzcd}$$

Consequently, we can view the pair $(\sigma, \{\tau_i\}_{0 \leq i \leq n})$ as an n-simplex of the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$. The construction $(\sigma, \tau) \mapsto (\sigma, \{\tau_i\}_{0 \leq i \leq n})$ determines a morphism of simplicial sets $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$. In the special case where $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ is a diagram of ∞-categories, we will refer to λ_t as the *taut scaffold* of the cocartesian fibration $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$.

Remark 5.3.4.12. Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor. Then the diagram of simplicial sets 0374

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[rr, "\lambda_t"] \arrow[dr] & & \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \arrow[dl] \\
& \mathrm{N}_{\bullet}(\mathcal{C}) &
\end{tikzcd}$$

commutes, where λ_t is the morphism of Construction 5.3.4.11 and the vertical morphism are the projection maps of Construction 5.3.2.1 and Definition 5.3.3.1.

Example 5.3.4.13. Let X be a simplicial set, which we identify with a diagram $\mathscr{F} : [0] \to \mathrm{Set}_{\Delta}$. Then the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ and the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{F}}([0])$ can both be identified with X (see Examples 5.3.2.2 and 5.3.3.2). Under these identifications, the taut scaffold $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}([0])$ of Construction 5.3.4.11 corresponds to the identity map id_X. 0375

Remark 5.3.4.14 (Functoriality)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets and let $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ be the morphism of Construction 5.3.4.11. If $T : \mathcal{C}' \to \mathcal{C}$ is any functor between categories, then λ induces a morphism 0376

$$\lambda_t' : \mathrm{N}_{\bullet}(\mathcal{C}') \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}(\mathcal{C}') \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}).$$

Setting $\mathscr{F}' = \mathscr{F} \circ T$, we can use Remarks 5.3.2.3 and 5.3.3.7 to identify λ_t' with a morphism from the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F}')$ to the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{F}'}(\mathcal{C}')$. This morphism coincides with the map obtained by applying Construction 5.3.4.11 to the diagram $\mathscr{F}'$.

026E **Example 5.3.4.15** (Comparison of Fibers)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets and let $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ be the morphism of Construction 5.3.4.11. Combining Example 5.3.4.13 with Remark 5.3.4.14, we see that for every object $C \in \mathcal{C}$, the induced map of fibers

$$\{C\} \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \{C\} \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$$

is an isomorphism of simplicial sets (under the identifications provided by Remark 5.3.2.3 and Example 5.3.3.8, it corresponds to the identity morphism $\mathrm{id} : \mathscr{F}(C) \to \mathscr{F}(C)$).

0377 **Example 5.3.4.16.** Let $f : X \to Y$ be a morphism of simplicial sets, which we identify with a diagram $\mathscr{F} : [1] \to \mathrm{Set}_{\Delta}$. Then the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ can be identified with the mapping cylinder $(\Delta^1 \times X)\coprod_{(\{1\}\times X)} Y$ (Example 5.3.2.13), and the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{F}}([1])$ can be identified with the relative join $X \star_Y Y$ (Example 5.3.3.13). Under these identifications, Construction 5.3.4.11 corresponds to a morphism of simplicial sets

$$\lambda_t : (\Delta^1 \times X)\coprod_{(\{1\}\times X)} Y \to X \star_Y Y.$$

Unwinding the definitions, we see that this map classifies the commutative diagram

0378
$$\begin{tikzcd}
\emptyset \star_X X \arrow[r] \arrow[d] & \emptyset \star_Y Y \arrow[d] \\
X \star_X X \arrow[r] & X \star_Y Y.
\end{tikzcd} \tag{5.16}$$

In particular, the morphism λ_t is an isomorphism if and only if (5.16) is a pushout square of simplicial sets.

0379 **Proposition 5.3.4.17.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories indexed by a category $\mathcal{C}$. Then the morphism $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ of Construction 5.3.4.11 is a scaffold of the cocartesian fibration $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$.*

Proof. Condition (0) of Definition 5.3.4.2 follows from Remark 5.3.4.12, condition (2) from Example 5.3.4.15, and condition (1) from the characterization of U-cocartesian morphisms supplied by Corollary 5.3.3.16. □

037A **Corollary 5.3.4.18.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets. Then there exists a cocartesian fibration of ∞-categories $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ and a scaffold $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$.*

Proof. Using Proposition 4.1.3.2, we can choose a diagram of ∞-categories $\mathscr{F}' : \mathcal{C} \to \mathrm{QCat}$ and a levelwise categorical equivalence $\alpha : \mathscr{F} \to \mathscr{F}'$. We can then take λ to be the composition $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\alpha} \underrightarrow{\mathrm{holim}}(\mathscr{F}') \xrightarrow{\lambda_t} \mathrm{N}_{\bullet}^{\mathscr{F}'}(\mathcal{C})$, where λ_t is the taut scaffold of Proposition 5.3.4.17. □

Corollary 5.3.4.19. *Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories indexed by $\mathcal{C}$, and let $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ be the cocartesian fibration of Corollary 5.3.3.16. Then there exists a levelwise categorical equivalence from $\mathscr{F}$ to the strict transport representation $\mathrm{sTr}_{\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})/\mathcal{C}}$.* 037B

Proof. Combine Proposition 5.3.4.17 with Remark 5.3.4.10 (for a more precise statement, see Construction 7.5.3.3). □

In certain cases, one can improve on Example 5.3.4.15.

Proposition 5.3.4.20. *Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor. Suppose that, for every morphism $u : C \to D$ in the category $\mathcal{C}$, the image $\mathscr{F}(u) : \mathscr{F}(C) \to \mathscr{F}(D)$ is a left covering map (Definition 4.2.3.8). Then the morphism $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ of Construction 5.3.4.11 is an isomorphism.* 037C

Proof. Let $(\sigma, \{\tau_i\}_{0 \le i \le n})$ be an n-simplex of the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$. We identify σ with a diagram $C_0 \to \cdots \to C_n$ in the category $\mathcal{C}$, and each τ_i with an i-simplex of the simplicial set $\mathscr{F}(C_i)$. We wish to show that there is a unique n-simplex τ of $\mathscr{F}(C_0)$ satisfying $\lambda_t(\sigma, \tau) = (\sigma, \{\tau_i\}_{0 \le i \le n})$. Note that, for this condition to be satisfied, the simplex τ must be a solution to the lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d] & \mathscr{F}(C_0) \arrow[d] \\
\Delta^n \arrow[r, "\tau_n"] \arrow[ur, dashed] & \mathscr{F}(C_n).
\end{tikzcd}$$

Since the inclusion $\{0\} \hookrightarrow \Delta^n$ is left anodyne (Example 4.3.7.11), our assumption that the right vertical map is a left covering guarantees that this lifting problem has a unique solution $\tau : \Delta^n \to \mathscr{F}(C_0)$ (Corollary 4.2.4.12). This proves uniqueness. To prove existence, write $\lambda_t(\sigma, \tau) = (\sigma, \{\tau'_i\}_{0 \le i \le n})$. We wish to prove that $\tau_i = \tau'_i$ for $0 \le i \le n$. For this, we observe that both τ_i and τ'_i can be viewed as solutions to a common lifting problem

$$\begin{tikzcd}
\{0\} \arrow[r, "\tau_i(0)"] \arrow[d] & \mathscr{F}(C_i) \arrow[d] \\
\Delta^i \arrow[r] \arrow[ur, dashed] & \mathscr{F}(C_n).
\end{tikzcd}$$

Since the inclusion $\{0\} \hookrightarrow \Delta^i$ is left anodyne (Example 4.3.7.11) and the right vertical map is a left covering, the solution to this lifting problem is uniquely determined (Corollary 4.2.4.12). □

037D **Example 5.3.4.21** (Set-Valued Functors)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a diagram of sets, and let us abuse notation by identifying $\mathscr{F}$ with a diagram of discrete simplicial sets. Then the taut scaffold $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ is an isomorphism. It follows that $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ can be identified with the nerve of the category of elements $\int_{\mathcal{C}} \mathscr{F}$ (see Example 5.3.2.5).

037E **Corollary 5.3.4.22.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a functor which carries each morphism of $\mathcal{C}$ to an isomorphism of simplicial sets. Then the morphism $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ of Remark 5.3.4.12 is an isomorphism.*

037F **Corollary 5.3.4.23.** *Let $\mathcal{C}$ be a groupoid and let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes. Then the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ is a Kan complex.*

Proof. Using Corollaries 5.3.4.22 and 5.3.3.19, we see that the map $U : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_\bullet(\mathcal{C})$ is a left fibration. Since $\mathrm{N}_\bullet(\mathcal{C})$ is a Kan complex (Proposition 1.3.5.2), it follows that U is a Kan fibration (Corollary 4.4.3.8), so that $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ is also a Kan complex (Remark 3.1.1.11). □

037G **Example 5.3.4.24** (Homotopy Quotients)**.** Let G be a group and let BG denote the associated groupoid (consisting of a single object with automorphism group G). Let X be a simplicial set equipped with an action of G, which we identify with a functor $\mathscr{F} : BG \to \mathrm{Set}_\Delta$. Applying Corollary 5.3.4.22, we obtain an isomorphism of simplicial sets $X_{\mathrm{h}G} \xrightarrow{\sim} \mathrm{N}_\bullet^{\mathscr{F}}(BG)$, where $X_{\mathrm{h}G}$ is the homotopy quotient of X by the action of G (Example 5.3.2.15). If X is a Kan complex, then Corollary 5.3.4.23 guarantees that $X_{\mathrm{h}G}$ is also a Kan complex.

5.3.5 Application: Classification of Cocartesian Fibrations

037H Let $\mathcal{C}$ be a category. In this section, we apply the results of §5.3.4 to classify cocartesian fibrations $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ up to equivalence. First, we need to introduce a bit of terminology.

037J **Definition 5.3.5.1.** Let $\mathcal{C}$ be a category and let $\mathscr{F}_0, \mathscr{F}_1 : \mathcal{C} \to \mathrm{QCat}$ be diagrams of ∞-categories indexed by $\mathcal{C}$. We will say that $\mathscr{F}_0$ and $\mathscr{F}_1$ are *levelwise equivalent* if there exists another diagram $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ equipped with levelwise categorical equivalences $\mathscr{F}_0 \to \mathscr{F} \leftarrow \mathscr{F}_1$ (see Definition 4.5.6.1).

037K **Proposition 5.3.5.2.** *Let $\mathcal{C}$ be a category and suppose we are given a pair of functors $\mathscr{F}_0, \mathscr{F}_1 \to \mathrm{QCat}$. Then $\mathscr{F}_0$ is levelwise equivalent to $\mathscr{F}_1$ (in the sense of Definition 5.3.5.1) if and only if the cocartesian fibrations $U_0 : \mathrm{N}_\bullet^{\mathscr{F}_0}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ and $U_1 : \mathrm{N}_\bullet^{\mathscr{F}_1}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ are equivalent (in the sense of Definition 5.1.7.1).*

037L **Corollary 5.3.5.3.** *For every category $\mathcal{C}$, levelwise equivalence determines an equivalence relation on the set of functors from $\mathcal{C}$ to QCat.*

Exercise 5.3.5.4. Give a direct proof of Corollary 5.3.5.3 (which does not use the characterization of Proposition 5.3.5.2). 037M

Proof of Proposition 5.3.5.2. Assume first that the functors $\mathscr{F}_0, \mathscr{F}_1 : \mathcal{C} \to \mathrm{QCat}$ are levelwise equivalent. Then there exists a functor $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ together with levelwise categorical equivalences $\mathscr{F}_0 \to \mathscr{F} \leftarrow \mathscr{F}_1$. Applying Corollary 5.3.3.20, we see that the induced maps $\mathrm{N}_\bullet^{\mathscr{F}_0}(\mathcal{C}) \to \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \leftarrow \mathrm{N}_\bullet^{\mathscr{F}_1}(\mathcal{C})$ are equivalences of cocartesian fibrations over $\mathrm{N}_\bullet(\mathcal{C})$.

We now prove the converse. Suppose that there exists a functor $T : \mathrm{N}_\bullet^{\mathscr{F}_0}(\mathcal{C}) \to \mathrm{N}_\bullet^{\mathscr{F}_1}(\mathcal{C})$ which is an equivalence of cocartesian fibrations over $\mathrm{N}_\bullet(\mathcal{C})$. Let $\lambda_0 : \underrightarrow{\mathrm{holim}}(\mathscr{F}_0) \to \mathrm{N}_\bullet^{\mathscr{F}_0}(\mathcal{C})$ and $\lambda_1 : \underrightarrow{\mathrm{holim}}(\mathscr{F}_1) \to \mathrm{N}_\bullet^{\mathscr{F}_1}(\mathcal{C})$ be the taut scaffolds of Construction 5.3.4.11. Then $T \circ \lambda_0$ is a scaffold of the cocartesian fibration U_1 (Remark 5.3.4.6). Applying Remark 5.3.4.10, we obtain levelwise categorical equivalences $\mathscr{F}_0 \to \mathrm{sTr}_{\mathrm{N}_\bullet^{\mathscr{F}_1}(\mathcal{C})/\mathcal{C}} \leftarrow \mathscr{F}_1$. $\square$

Warning 5.3.5.5. Let $\mathcal{C}$ be a category and let $\mathscr{F}_0, \mathscr{F}_1 : \mathcal{C} \to \mathrm{QCat}$ be diagrams. The assumption that $\mathscr{F}_0$ is levelwise equivalent to $\mathscr{F}_1$ (in the sense of Definition 5.3.5.1) does not guarantee the existence of a levelwise categorical equivalence directly from $\mathscr{F}_0$ to $\mathscr{F}_1$ (or in the opposite direction). 037N

Theorem 5.3.5.6. *Let $\mathcal{C}$ be a category. Then the weighted nerve functor $\mathscr{F} \mapsto \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ induces a bijection* 037P

$$\begin{tikzcd}
\{\text{Diagrams } \mathcal{C} \to \mathrm{QCat}\}/\text{Levelwise Equivalence} \arrow[d] \\
\{\text{Cocartesian Fibrations } \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})\}/\text{Equivalence}.
\end{tikzcd}$$

The inverse bijection carries (the equivalence class of) a cocartesian fibration $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ to (the equivalence class of) the strict transport representation $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$.

We will deduce Theorem 5.3.5.6 from the following result, which we prove at the end of this section:

Theorem 5.3.5.7. *Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a functor, and suppose we are given a commutative diagram of simplicial sets* 037Q

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[rr, "\lambda"] \arrow[dr] & & \mathcal{E} \arrow[dl, "U"] \\
& \mathrm{N}_\bullet(\mathcal{C}), &
\end{tikzcd}$$

where $\mathcal{E}$ is an ∞-category. The following conditions are equivalent:

(1) *The functor U is a cocartesian fibration and λ is a scaffold.*

(2) *The morphism λ is a categorical equivalence of simplicial sets.*

037R **Corollary 5.3.5.8.** *Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a cocartesian fibration of ∞-categories, and let $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$ denote the strict transport representation of Construction 5.3.1.5. Then the universal scaffold $\lambda_u : \underrightarrow{\mathrm{holim}}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}) \to \mathcal{E}$ of Construction 5.3.4.7 is a categorical equivalence of simplicial sets.*

Proof. Combine Theorem 5.3.5.7 with Proposition 5.3.4.8. □

037S **Corollary 5.3.5.9.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets. Then the morphism $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ of Construction 5.3.4.11 is a categorical equivalence of simplicial sets.*

Proof. Using Proposition 4.1.3.2, we can choose a diagram of ∞-categories $\mathscr{F}' : \mathcal{C} \to \mathrm{QCat}$ and a levelwise categorical equivalence $\alpha : \mathscr{F} \to \mathscr{F}'$. We then have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[r, "\lambda_t"] \arrow[d] & \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \arrow[d] \\
\underrightarrow{\mathrm{holim}}(\mathscr{F}') \arrow[r, "\lambda_t'"] & \mathrm{N}_{\bullet}^{\mathscr{F}'}(\mathcal{C})
\end{tikzcd}$$

where the horizontal maps are given by Construction 5.3.4.11 and the vertical maps are induced by the natural transformation α. Since α is a levelwise categorical equivalence, Variant 5.3.2.19 and Corollary 5.3.3.20 guarantee that the vertical maps are categorical equivalences of simplicial sets. Consequently, to show that λ_t is a categorical equivalence, it will suffice to show that λ_t' is a categorical equivalence. This is a special case of Theorem 5.3.5.7, since λ_t' is a scaffold of the cocartesian fibration $\mathrm{N}_{\bullet}^{\mathscr{F}'}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ (Proposition 5.3.4.17). □

037T **Example 5.3.5.10.** In the special case $\mathcal{C} = [1]$, Theorem 5.3.5.7 is a restatement of Theorem 5.2.4.1 and Corollary 5.3.5.9 is a restatement of Proposition 5.2.4.5.

Proof of Theorem 5.3.5.6. Let $\mathcal{C}$ be a category. It follows from Proposition 5.3.5.2 that the construction $\mathscr{F} \mapsto \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ determines an injective function

$$\begin{tikzcd}
\{\text{Diagrams } \mathcal{C} \to \mathrm{QCat}\}/\text{Levelwise Equivalence} \arrow[d, "\Phi"] \\
\{\text{Cocartesian Fibrations } \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})\}/\text{Equivalence}.
\end{tikzcd}$$

Moreover, the construction $(U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})) \mapsto \mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$ carries equivalences of cocartesian fibrations over $\mathrm{N}_{\bullet}(\mathcal{C})$ to levelwise categorical equivalences, and therefore induces a function

$$\begin{tikzcd}
\{\text{Cocartesian Fibrations } \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})\}/\text{Equivalence} \arrow[d, "\Psi"] \\
\{\text{Diagrams } \mathcal{C} \to \mathrm{QCat}\}/\text{Levelwise Equivalence}
\end{tikzcd}$$

in the opposite direction. We will show that $\Phi \circ \Psi$ is equal to the identity; it will then follow that Φ is a bijection and that $\Psi = \Phi^{-1}$ is the inverse bijection.

Fix a cocartesian fibration $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$, let $\mathscr{F} = \mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$ denote its strict transport representation, and let $U' : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ be the projection map. We wish to show that U and U' are equivalent as cocartesian fibrations over $\mathrm{N}_{\bullet}(\mathcal{C})$. Let $\lambda_u : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ denote the universal scaffold (Construction 5.3.4.7) and let $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ denote the taut scaffold (Construction 5.3.4.11). Then λ_t is a categorical equivalence of simplicial sets (Corollary 5.3.5.9). Applying Corollary 4.5.2.34, we see that precomposition with λ_t induces an equivalence of ∞-categories

$$\mathrm{Fun}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E}).$$

In particular, there exists a morphism $T : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathcal{E}$ such that $U \circ T = U'$ and $T \circ \lambda_t$ is isomorphic to λ_u (as an object of the ∞-category $\mathrm{Fun}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E})$). Since λ_u is a categorical equivalence of simplicial sets (Corollary 5.3.5.8), it follows that $T \circ \lambda_t$ is also a categorical equivalence of simplicial sets (Corollary 4.5.3.9). Applying the two-out-of-three property, we see that T is an equivalence of ∞-categories (Remark 4.5.3.5) and therefore an equivalence of cocartesian fibrations over $\mathrm{N}_{\bullet}(\mathcal{C})$ (Proposition 5.1.7.5). □

Proof of Theorem 5.3.5.7. We first show that (1) implies (2). Assume that $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a cocartesian fibration of simplicial sets and let $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ be a scaffold of U; we wish to show that λ is a categorical equivalence of simplicial sets. By virtue of Corollary 4.5.7.3, it will suffice to show that for every n-simplex $\sigma : \Delta^n \to \mathrm{N}_\bullet(\mathcal{C})$, the induced map

$$\Delta^n \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \Delta^n \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathcal{E}$$

is a categorical equivalence of simplicial sets. We may therefore assume without loss of generality that the category $\mathcal{C}$ is a linearly ordered set of the form $[n] = \{0 < 1 < \cdots < n\}$ for some $n \geq 0$.

We proceed by induction on n. If $n = 0$, the result is clear. Let us therefore assume that $n > 0$. Let $S = \mathrm{N}_\bullet(\{1 < \cdots < n\})$ be the 0th face of the simplex Δ^n and set $\mathcal{E}_+ = S \times_{\Delta^n} \mathcal{E}$. Let $\mathscr{F}_+$ denote the restriction of $\mathscr{F}$ to the subcategory $\{1 < \cdots < n\} \subset [n]$, so that our inductive hypothesis guarantees that λ restricts to a categorical equivalence $\lambda_+ : \underrightarrow{\mathrm{holim}}(\mathscr{F}_+) \to \mathcal{E}_+$. Note that Remark 5.3.2.12 supplies an isomorphism of simplicial sets

$$(\Delta^n \times \mathscr{F}(0)) \coprod_{(S \times \mathscr{F}(0))} \underrightarrow{\mathrm{holim}}(\mathscr{F}_+) \to \underrightarrow{\mathrm{holim}}(\mathscr{F}).$$

Let $V : \Delta^n \to \Delta^1$ be the morphism given on vertices by the formula $V(i) = \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{if } i > 0. \end{cases}$ Then V is a cocartesian fibration of simplicial sets, and the edge $\mathrm{N}_\bullet(\{0 < 1\}) \subseteq \Delta^n$ is V-cocartesian. It follows that, for every vertex x of the simplicial set $\mathscr{F}(0)$, the composite map

$$\Delta^1 \times \{x\} \hookrightarrow \Delta^n \times \mathscr{F}(0) \to \underrightarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\lambda} \mathcal{E}$$

is a $(V \circ U)$-cocartesian edge of $\mathcal{E}$. Applying Theorem 5.2.4.1 to the cocartesian fibration $V \circ U$, we deduce that the composition

$$\begin{aligned} (\Delta^1 \times \mathscr{F}(0)) \coprod_{(\{1\} \times \mathscr{F}(0))} \underrightarrow{\mathrm{holim}}(\mathscr{F}_+) & \xrightarrow{\iota} (\Delta^n \times \mathscr{F}(0)) \coprod_{(S \times \mathscr{F}(0))} \underrightarrow{\mathrm{holim}}(\mathscr{F}_+) \\ & \simeq \underrightarrow{\mathrm{holim}}(\mathscr{F}) \\ & \xrightarrow{\lambda} \mathcal{E}. \end{aligned}$$

is a categorical equivalence of simplicial sets. Consequently, to show that λ is a categorical equivalence of simplicial sets, it will suffice to show that ι is inner anodyne. By construction, ι is a pushout of the inclusion map

$$(\Delta^1 \coprod_{\{1\}} S) \times \mathscr{F}(0) \to \Delta^n \times \mathscr{F}(0).$$

By virtue of Lemma 1.5.7.5, it will suffice to show that the inclusion map $\Delta^1 \coprod_{\{1\}} S \hookrightarrow \Delta^n$ is inner anodyne. This is a special case of Example 4.3.6.5, since the inclusion $\{1\} \hookrightarrow S$ is left anodyne (Lemma 4.3.7.8).

We now show that (2) implies (1). Let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a functor of ∞-categories, and suppose that $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ is a categorical equivalence of simplicial sets such that $U \circ \lambda$ is equal to the projection map $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_\bullet(\mathcal{C})$. We first claim that U is an isofibration of ∞-categories. Since $\mathcal{E}$ is an ∞-category, the morphism U is an inner fibration (Proposition 4.1.1.10). It will therefore suffice to show that, for each object $\widetilde{C} \in \mathcal{E}$ having image $C = U(\widetilde{C}) \in \mathcal{C}$ and every isomorphism $e : C \to D$ of $\mathcal{C}$, there exists an isomorphism $\widetilde{e} : \widetilde{C} \to \widetilde{D}$ in $\mathcal{E}$ satisfying $U(\widetilde{e}) = e$. Since λ is a categorical equivalence, we can choose a vertex v of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ and an isomorphism $\widetilde{f} : \widetilde{C} \to \lambda(v)$ in $\mathcal{E}$. Let us identify v with a pair (C', X), where C' is an object of $\mathcal{C}$ and X is a vertex of the simplicial set $\mathscr{F}(C')$. Then $f = U(\widetilde{f})$ is an isomorphism from C to C' in the category $\mathcal{C}$. Replacing v by the pair $(C, \mathscr{F}(f^{-1})(X))$, we can reduce to the case where $C' = C$ and $f = \mathrm{id}_C$ so that $\widetilde{f}$ is an isomorphism in the ∞-category $\mathcal{E}_C$. In this case, we can take $\widetilde{e}$ to be any composition of $\widetilde{f}$ with the morphism $\lambda(e, \mathrm{id}_X) : \lambda(C, X) \to \lambda(D, \mathscr{F}(e)(X))$ of $\mathcal{E}$. This completes the proof that $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ is an isofibration.

Using Corollary 5.3.4.18, we can choose a cocartesian fibration $U' : \mathcal{E}' \to \mathrm{N}_\bullet(\mathcal{C})$ and a scaffold $\lambda' : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}'$. Then U' is an isofibration, so composition with λ induces a categorical equivalence $\mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathcal{E}, \mathcal{E}') \to \mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E}')$ (Corollary 4.5.2.34). It follows that there exists a functor $F : \mathcal{E} \to \mathcal{E}'$ satisfying $U' \circ F = U$ such that $F \circ \lambda$ is isomorphic to λ' as an object of the ∞-category $\mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{E}')$. Since λ' is a categorical equivalence of simplicial sets, the morphism $F \circ \lambda$ is also a categorical equivalence of simplicial sets (Corollary 4.5.3.9). Applying the two-out-of-three property (Remark 4.5.3.5), we deduce that F is an equivalence of ∞-categories. It follows that U is also a cocartesian fibration (Corollary 5.1.6.2) and that λ is a scaffold of U (Remark 5.3.4.6). □

We close this section by recording another consequence of Theorem 5.3.5.7.

Corollary 5.3.5.11. *Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories, and let $U' : \mathcal{E}' \to \mathrm{N}_\bullet(\mathcal{C})$ be an isofibration of ∞-categories. Then the composite map* 037U

$$\begin{aligned} \mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathcal{E}, \mathcal{E}') &\xrightarrow{\theta} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)}(\mathrm{wTr}_{\mathcal{E}/\mathcal{C}}, \mathrm{wTr}_{\mathcal{E}'/\mathcal{C}})_\bullet \\ &\to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}, \mathrm{wTr}_{\mathcal{E}'/\mathcal{C}})_\bullet \end{aligned}$$

is an equivalence of ∞-categories.

Proof. Using Corollary 5.3.2.23, we can identify θ with the functor

$$\mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathcal{E}, \mathcal{E}') \to \mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\underrightarrow{\mathrm{holim}}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}), \mathcal{E}')$$

given by precomposition with the universal scaffold λ_u. The desired result now follows by combining Corollaries 5.3.5.8 and 4.5.2.34. □

037V **Corollary 5.3.5.12.** *Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ and $U' : \mathcal{E}' \to \mathrm{N}_\bullet(\mathcal{C})$ be cocartesian fibrations of ∞-categories, having strict transport representations $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$ and $\mathrm{sTr}_{\mathcal{E}'/\mathcal{C}}$, respectively. Then the tautological map*

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathcal{E}, \mathcal{E}') \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}, \mathrm{sTr}_{\mathcal{E}'/\mathcal{C}})_\bullet$$

is an equivalence of ∞-categories.

Proof. By virtue of Remark 5.3.4.10, we have a pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathcal{E}, \mathcal{E}') \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}, \mathrm{sTr}_{\mathcal{E}'/\mathcal{C}})_\bullet \arrow[d] \\
\mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathcal{E}, \mathcal{E}') \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}, \mathrm{wTr}_{\mathcal{E}'/\mathcal{C}})_\bullet,
\end{tikzcd}$$

where the vertical maps are inclusions of replete full subcategories (and are therefore isofibrations; see Example 4.4.1.12). Since the bottom horizontal map is an equivalence of ∞-categories (Corollary 5.3.5.11), it follows that the upper horizontal map is also an equivalence of ∞-categories (Corollary 4.5.2.29). □

5.3.6 Application: Relative Exponentials

037W Let $U : \mathcal{C} \to \mathcal{B}$ be a cocartesian fibration of simplicial sets. Applying Construction 5.2.5.2, we obtain a homotopy transport representation

$$\mathrm{hTr}_{\mathcal{C}/\mathcal{B}} : \mathrm{h}\mathcal{B} \to \mathrm{hQCat} \qquad B \mapsto \mathcal{C}_B .$$

Let $\mathcal{D}$ be an ∞-category. In this section, we will show that the composite functor

$$\mathrm{h}\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathrm{hTr}^{\mathrm{op}}_{\mathcal{C}/\mathcal{B}}} \mathrm{hQCat}^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-,\mathcal{D})} \mathrm{hQCat}$$

can be realized as the homotopy transport representation of a *cartesian* fibration $\mathcal{C}' \to \mathcal{B}$. Moreover, we can take $\mathcal{C}'$ to be the relative exponential $\mathrm{Fun}(\mathcal{C}/\mathcal{B}, \mathcal{D})$ introduced in Construction 4.5.9.1 (Corollary 5.3.6.10). Our starting point is the following:

025H **Proposition 5.3.6.1.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a functor of ∞-categories which is either a cartesian fibration or a cocartesian fibration. Then U is exponentiable (in the sense of Definition 4.5.9.10). That is, if we are given any diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E}'' \arrow[r, "F"] \arrow[d] & \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}'' \arrow[r, "\overline{F}"] & \mathcal{C}' \arrow[r] & \mathcal{C}
\end{tikzcd}$$

where both squares are pullbacks and $\overline{F}$ *is a categorical equivalence, then* F *is also a categorical equivalence.*

Remark 5.3.6.2. In the statement of Proposition 5.3.6.1, the hypothesis that $\mathcal{C}$ is an ∞-category is not necessary: see Corollary 5.6.7.6. 025J

Our proof of Proposition 5.3.6.1 will require some preliminaries.

Lemma 5.3.6.3. *Let* $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ *be a diagram of simplicial sets and suppose we are given morphisms of simplicial sets* $A \xrightarrow{f} B \xrightarrow{g} \mathrm{N}_{\bullet}(\mathcal{C})$*, where* f *is inner anodyne. Then the induced map* 025A

$$\theta_g : A \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to B \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F})$$

is inner anodyne.

Proof. Let S be the collection of all morphisms of simplicial sets $f : A \to B$ having the property that, for every morphism $g : B \to \mathrm{N}_{\bullet}(\mathcal{C})$, the map θ_g is inner anodyne. It follows immediately from the definitions that S is weakly saturated (in the sense of Definition 1.5.4.12). Consequently, to show that every inner anodyne morphism belongs to S, it will suffice to prove that S contained every inner horn inclusion $f : \Lambda^n_i \hookrightarrow \Delta^n$, $0 < i < n$. Using Remark 5.3.2.3, we can reduce to the case where $\mathcal{C} = [n]$ and $g : \Delta^n \to \mathrm{N}_{\bullet}(\mathcal{C})$ is the identity map. In this case, Remark 5.3.2.14 shows that θ_g is a pushout of the inclusion map $\Lambda^n_i \times \mathscr{F}(0) \hookrightarrow \Delta^n \times \mathscr{F}(0)$, which is inner anodyne by virtue of Lemma 1.5.7.5. □

Lemma 5.3.6.4. *Let* $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ *be a diagram of simplicial sets, let* $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ *be a cocartesian fibration of ∞-categories, and let* $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{E}$ *be a scaffold. Then, for every morphism of simplicial sets* $S \to \mathrm{N}_{\bullet}(\mathcal{C})$*, the induced map* 037X

$$\lambda_S : S \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to S \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathcal{E}$$

is a categorical equivalence of simplicial sets.

Proof. By virtue of Corollary 4.5.7.3, we may assume without loss of generality that $S = \Delta^n$ is a standard simplex. Replacing $\mathcal{C}$ by the category $[n] = \{0 < 1 < \cdots < n\}$, we are reduced to proving that λ is a categorical equivalence, which follows from Theorem 5.3.5.7. □

Lemma 5.3.6.5. *Suppose we are given a pullback diagram of simplicial sets* 02RK

$$\begin{tikzcd}
\mathcal{E}' \arrow[r, "F"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r, "\overline{F}"] & \mathcal{C},
\end{tikzcd}$$

where $\overline{F}$ is inner anodyne. If U is either a cartesian fibration or a cocartesian fibration, then F is a categorical equivalence of simplicial sets.

Proof. We will give the proof under the assumption that U is a cocartesian fibration; the proof when U is a cartesian fibration is similar. Let S be the collection of all monomorphisms of simplicial sets $f : A \hookrightarrow B$ with the following property: for every morphism of simplicial sets $B \to \mathcal{C}$, the induced map $A \times_{\mathcal{C}} \mathcal{E} \hookrightarrow B \times_{\mathcal{C}} \mathcal{E}$ is a categorical equivalence. To complete the proof, it will suffice to show that the morphism $\overline{F} : \mathcal{C}' \hookrightarrow \mathcal{C}$ belongs to S. In fact, we claim that every inner anodyne morphism of simplicial sets belongs to S. Using Remark 4.5.3.6, Remark 4.5.3.5, Corollary 4.5.7.2, and Remark 4.5.4.13, we see that S is weakly saturated (see Definition 1.5.4.12). It will therefore suffice to show that S contains every inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$, $0 < i < n$. In particular, we are reduced to proving Lemma 5.3.6.5 in the special case where $\mathcal{C} = \mathrm{N}_\bullet(\mathcal{C}_0)$ is the nerve of a category $\mathcal{C}_0$. Applying Corollary 5.3.4.9, we deduce that there exists a diagram of ∞-categories $\mathscr{G} : \mathcal{C}_0 \to \mathrm{QCat}$ and a scaffold $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{G}) \to \mathcal{E}$. We then have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{C}' \times_{\mathcal{C}} \underrightarrow{\mathrm{holim}}(\mathscr{G}) \arrow[r, "\widetilde{F}"] \arrow[d] & \underrightarrow{\mathrm{holim}}(\mathscr{G}) \arrow[d, "\lambda"] \\
\mathcal{E}' \arrow[r, "F"] & \mathcal{E},
\end{tikzcd}$$

where the vertical maps are categorical equivalences (Lemma 5.3.6.4). Consequently, to show that F is a categorical equivalence, it will suffice to show that $\widetilde{F}$ is a categorical equivalence, which follows from Lemma 5.3.6.3. □

Proof of Proposition 5.3.6.1. Without loss of generality we may assume that $U : \mathcal{E} \to \mathcal{C}$ is a cocartesian fibration of ∞-categories. Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E}'' \arrow[r, "F"] \arrow[d] & \mathcal{E}' \arrow[r, "G"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}'' \arrow[r, "\overline{F}"] & \mathcal{C}' \arrow[r, "\overline{G}"] & \mathcal{C}
\end{tikzcd}$$

where both squares are pullbacks and $\overline{F}$ is a categorical equivalence. We wish to show that F is also a categorical equivalence. By virtue of Proposition 4.1.3.2, the morphism $\overline{G}$ factors as a composition $\mathcal{C}' \xrightarrow{\overline{G}'} \mathcal{B} \xrightarrow{\overline{G}''} \mathcal{C}$, where $\overline{G}'$ is inner anodyne and $\overline{G}''$ is an inner fibration. Note that the projection map $V : \mathcal{B} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{B}$ is a cocartesian fibration of ∞-categories.

We may therefore replace $\mathcal{C}$ by $\mathcal{B}$ and thereby reduce to the special case where $\overline{G}$ is inner anodyne. In this case, the morphism $G : \mathcal{E}' \to \mathcal{E}$ is a categorical equivalence of simplicial sets (Lemma 5.3.6.5). Consequently, to show that F is a categorical equivalence of simplicial sets, it will suffice to show that the composite map $(G \circ F) : \mathcal{E}'' \to \mathcal{E}$ is a categorical equivalence of simplicial sets (Remark 4.5.3.5).

Since $\overline{F}$ is a categorical equivalence and $\overline{G}$ is inner anodyne, it follows that the composite map $\overline{G} \circ \overline{F} : \mathcal{C}'' \to \mathcal{C}$ is also a categorical equivalence. Applying Proposition 4.1.3.2, we can factor $\overline{G} \circ \overline{F}$ as a composition $\mathcal{C}'' \xrightarrow{\overline{F}_0} \mathcal{C}'_0 \xrightarrow{\overline{G}_0} \mathcal{C}$, where $\overline{F}_0$ is inner anodyne and $\overline{G}_0$ is an inner fibration. Since $\mathcal{C}$ is an ∞-category, it follows that $\mathcal{C}'_0$ is also an ∞-category (Remark 4.1.1.9). Set $\overline{\mathcal{E}}'_0 = \mathcal{C}'_0 \times_{\mathcal{C}} \mathcal{E}$, so that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}'' \arrow[r, "F_0"] \arrow[d] & \mathcal{E}'_0 \arrow[r, "G_0"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}'' \arrow[r, "\overline{F}_0"] & \mathcal{C}'_0 \arrow[r, "\overline{G}_0"] & \mathcal{C}
\end{tikzcd}$$

satisfying $G \circ F = G_0 \circ F_0$. Since U is an isofibration (Proposition 5.1.4.8) and $\overline{G}_0$ is an equivalence of ∞-categories, it follows that G_0 is an equivalence ∞-categories (Corollary 4.5.2.29). Applying Lemma 5.3.6.5 to the square on the left, we see that F_0 is a categorical equivalence of simplicial sets. Invoking Remark 4.5.3.5, we deduce that $G \circ F = G_0 \circ F_0$ is also a categorical equivalence, as desired. □

We now formulate the main result of this section. In what follows, we assume that the reader is familiar with the relative exponential construction introduced in §4.5.9.

Proposition 5.3.6.6. *Let $U : \mathcal{C} \to \mathcal{B}$ be a cocartesian fibration of simplicial sets and let $V : \mathcal{D} \to \mathcal{E}$ be a cartesian fibration of simplicial sets. Then postcomposition with V induces a cartesian fibration of simplicial sets* 0470

$$V' : \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E}).$$

Moreover, an edge e of $\mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ is V'-cartesian if and only if it satisfies the following condition:

(∗) *Write $e = (\overline{e}, f)$, where $\overline{e}$ is an edge of $\mathcal{B}$ and $f : \Delta^1 \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ is a morphism of simplicial sets. Let $U_{\overline{e}} : \Delta^1 \times_{\mathcal{B}} \mathcal{C} \to \Delta^1$ be given by projection onto the first factor. Then f carries $U_{\overline{e}}$-cocartesian morphisms of $\Delta^1 \times_{\mathcal{B}} \mathcal{C}$ to V-cartesian morphisms of $\mathcal{D}$.*

Remark 5.3.6.7. For a more general version of Proposition 5.3.6.6 (which looses the requirement that U is a cocartesian fibration), see Corollary 7.3.7.6 (and Example 7.3.7.8). 0471

Before giving the proof of Proposition 5.3.6.6, let us note some of its consequences.

0472 **Corollary 5.3.6.8.** *Let $U : \mathcal{C} \to \mathcal{B}$ be a cocartesian fibration of simplicial sets and let $\mathcal{D}$ be an ∞-category. Then:*

(1) *The projection map $\pi : \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathcal{B}$ is a cartesian fibration of simplicial sets.*

(2) *Let e be an edge of the simplicial set $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$, corresponding to a pair $(\overline{e}, f)$, where $\overline{e}$ is an edge of the simplicial set $\mathcal{B}$ and $f : \Delta^1 \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories. Let $U_{\overline{e}} : \Delta^1 \times_{\mathcal{B}} \mathcal{C} \to \Delta^1$ be given by projection onto the first factor. Then e is π-cartesian if and only if the functor f carries $U_{\overline{e}}$-cocartesian morphisms to isomorphisms in the ∞-category $\mathcal{D}$.*

Proof. Apply Proposition 5.3.6.6 in the special case $\mathcal{E} = \Delta^0$ (and use Example 5.1.1.4). □

In the situation of Corollary 5.3.6.8, contravariant transport for the cartesian fibration π has a simple explicit description.

0473 **Proposition 5.3.6.9.** *Let $U : \mathcal{C} \to \mathcal{B}$ be a cocartesian fibration of simplicial sets, let $f : B \to B'$ be an edge of $\mathcal{B}$, and let $f_! : \mathcal{C}_B \to \mathcal{C}_{B'}$ be given by covariant transport along f for the cocartesian fibration U (see Definition 5.2.2.4). For every ∞-category $\mathcal{D}$, the functor*

$$\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})_{B'} \simeq \operatorname{Fun}(\mathcal{C}_{B'}, \mathcal{D}) \xrightarrow{\circ f_!} \operatorname{Fun}(\mathcal{C}_B, \mathcal{D}) \simeq \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})_B$$

is given by contravariant transport along f (for the cartesian fibration $\pi : \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathcal{B}$).

Proof. Replacing $\mathcal{C}$ by the fiber product $\Delta^1 \times_{\mathcal{B}} \mathcal{C}$, we can assume without loss of generality that $\mathcal{B} = \Delta^1$ and f is the nondegenerate edge of $\mathcal{B}$. By virtue of Corollary 5.2.4.4, we can choose a functor $R : \mathcal{C} \to \mathcal{C}_{B'}$ such that $R|_{\mathcal{C}_B} = f_!$, $R|_{\mathcal{C}_{B'}} = \mathrm{id}$, and R carries U-cocartesian morphisms of $\mathcal{C}$ to isomorphisms in $\mathcal{C}_{B'}$. Precomposition with functor $(U, R) : \mathcal{C} \to \Delta^1 \times \mathcal{C}_{B'}$ then determines a functor

$$H : \Delta^1 \times \operatorname{Fun}(\mathcal{C}_{B'}, \mathcal{D}) \simeq \operatorname{Fun}((\Delta^1 \times \mathcal{C}_{B'}) / \mathcal{B}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}).$$

Unwinding the definitions, we see that $H|_{\{0\} \times \operatorname{Fun}(\mathcal{C}_{B'}, \mathcal{D})}$ is the functor given by precomposition with $f_!$, and that $H|_{\{1\} \times \operatorname{Fun}(\mathcal{C}_{B'}, \mathcal{D})}$ is the identity functor from $\operatorname{Fun}(\mathcal{C}_{B'}, \mathcal{D})$ to itself. It will therefore suffice to show that, for each object $F \in \operatorname{Fun}(\mathcal{C}_{B'}, \mathcal{D})$, the restriction $H|_{\Delta^1 \times \{F\}}$ is a π-cartesian morphism of the ∞-category $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$. By virtue of Corollary 5.3.6.8, this is equivalent to the assertion that the composite functor $\mathcal{C} \xrightarrow{R} \mathcal{C}_{B'} \xrightarrow{F} \mathcal{D}$ carries U-cocartesian morphisms of $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. This follows from our assumption on R, since the functor F carries isomorphisms of $\mathcal{C}_{B'}$ to isomorphisms of $\mathcal{D}$. □

Corollary 5.3.6.10. *Let $U : \mathcal{C} \to \mathcal{B}$ be a cocartesian fibration of simplicial sets and let* 0474

$$\mathrm{hTr}_{\mathcal{C} / \mathcal{B}} : \mathrm{h}\mathcal{B} \to \mathrm{hQCat}$$

be the homotopy transport representation for U (Construction 5.2.5.2). Then, for any ∞-category $\mathcal{D}$, the composition

$$\mathrm{h}\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathrm{hTr}^{\mathrm{op}}_{\mathcal{C} / \mathcal{B}}} \mathrm{hQCat}^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-, \mathcal{D})} \mathrm{hQCat}$$

is the homotopy transport representation for the cartesian fibration $\mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathcal{B}$ of Corollary 5.3.6.8.

We will carry out the proof of Proposition 5.3.6.6 in several steps.

Lemma 5.3.6.11. *Let $U : \mathcal{C} \to \mathcal{B}$ be a cocartesian fibration of simplicial sets, let $V : \mathcal{D} \to \mathcal{E}$ be an isofibration of simplicial sets, and let e be an edge of the simplicial set $\mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$ which satisfies condition $(*)$ of Proposition 5.3.6.6. Then e is V'-cartesian, where $V' : \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})$ is given by postcomposition with V.* 02UA

Proof. Let $n \geq 2$ and suppose we are given a lifting problem

02UB
$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \arrow[d, "V'"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E}),
\end{tikzcd} \tag{5.17}$$

where σ_0 carries the final edge $\mathrm{N}_{\bullet}(\{n-1 < n\}) \subseteq \Lambda^n_n$ to e; we wish to show that this lifting problem admits a solution. Replacing U with the projection map $\Delta^n \times_{\mathcal{B}} \mathcal{C} \to \Delta^n$, we can assume without loss of generality that $\mathcal{B} = \Delta^n$ is a standard simplex, so that $\overline{\sigma}$ corresponds to a morphism of simplicial sets $\overline{F} : \mathcal{C} \to \mathcal{E}$. Invoking the universal property of the simplicial sets $\mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})$ (Proposition 4.5.9.5), we can rewrite (5.17) as a lifting problem

02UC
$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "\overline{F}"] \arrow[ur, dashed, "F"] & \mathcal{E}.
\end{tikzcd} \tag{5.18}$$

Note that since the edge e satisfies condition $(*)$, the morphism F_0 satisfies the following condition:

($*'$) If u is a U-cocartesian edge of $\mathcal{C}$ lying over the final edge $\mathrm{N}_\bullet(\{n-1<n\}) \subseteq \Delta^n$, then $F_0(u)$ is a V-cartesian edge of $\mathcal{D}$.

Using Corollary 5.3.4.9, we can choose a diagram of ∞-categories $\mathscr{F} : [n] \to \mathrm{QCat}$ and a scaffold $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathcal{C}$. Set $\mathcal{C}' = \underrightarrow{\mathrm{holim}}(\mathscr{F})$ and $\mathcal{C}'_0 = \Lambda^n_n \times_{\Delta^n} \mathcal{C}'$, so that λ restricts to a map $\lambda_0 : \mathcal{C}'_0 \to \mathcal{C}_0$. We then have a commutative diagram of ∞-categories

02UD

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C},\mathcal{D}) \arrow[r, "\circ\lambda"] \arrow[d] & \mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}',\mathcal{D}) \arrow[d] \\
\mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}_0,\mathcal{D}) \arrow[r, "\circ\lambda"] & \mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}'_0,\mathcal{D}).
\end{tikzcd} \tag{5.19}$$

Since V is an isofibration (Proposition 5.1.4.8), the vertical maps in this diagram are isofibrations (Proposition 4.5.5.14). Since λ and λ_0 are categorical equivalences of simplicial sets (Lemma 5.3.6.4), the horizontal maps are equivalences of ∞-categories. Applying Corollary 4.5.2.32, we deduce that the upper horizontal map in the diagram (5.19) restricts to an equivalence from each fiber of the left vertical map to the corresponding fiber of the right vertical map. Consequently, we can replace (5.18) with the lifting problem

02UE

$$\begin{tikzcd}
\mathcal{C}'_0 \arrow[r, "F_0\circ\lambda_0"] \arrow[d] & \mathcal{D} \arrow[d, "V"] \\
\mathcal{C}' \arrow[r, "\overline{F}\circ\lambda"] \arrow[ur, dashed] & \mathcal{E}.
\end{tikzcd} \tag{5.20}$$

Using Remark 5.3.2.12, we obtain a pushout square

$$\begin{tikzcd}
\Lambda^n_n \times \mathscr{F}(0) \arrow[r, "G_0"] \arrow[d] & \mathcal{C}'_0 \arrow[d] \\
\Delta^n \times \mathscr{F}(0) \arrow[r, "G"] & \mathcal{C}'.
\end{tikzcd}$$

Let us identify $F_0 \circ \lambda_0 \circ G_0$ with a morphism of simplicial sets $\tau_0 : \Lambda^n_n \to \mathrm{Fun}(\mathscr{F}(0),\mathcal{D})$, and $\overline{F} \circ \lambda \circ G$ with an n-simplex $\overline{\tau}$ of $\mathrm{Fun}(\mathscr{F}(0),\mathcal{E})$, so that we can rewrite (5.20) again as a lifting problem

$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "\tau_0"] \arrow[d] & \mathrm{Fun}(\mathscr{F}(0), \mathcal{E}) \arrow[d, "V''"] \\
\Delta^n \arrow[r, "\overline{\tau}"] \arrow[ur, dashed, "\tau"] & \mathrm{Fun}(\mathscr{F}(0), \mathcal{D}).
\end{tikzcd}$$

To show that this lifting problem admits a solution, it will suffice to show that τ_0 carries the final edge $\mathrm{N}_\bullet(\{n-1 < n\})$ of Λ^n_n to a V''-cocartesian edge of the simplicial set $\mathrm{Fun}(\mathscr{F}(0), \mathcal{E})$. Since λ is a scaffold, this follows by combining $(*')$ with the criterion of Theorem 5.2.1.1. □

Lemma 5.3.6.12. *Let $U : \mathcal{C} \to \mathcal{B}$ be a cocartesian fibration of simplicial sets, let $V : \mathcal{D} \to \mathcal{E}$ be a cartesian fibration of simplicial sets, and let $V' : \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})$ be the morphism given by postcomposition with V. Suppose we are given a vertex Y of the simplicial set $\mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ having image $\overline{Y} = V'(Y)$, and an edge $\overline{e} : \overline{X} \to \overline{Y}$ of the simplicial set $\mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})$. Then we can write $\overline{e} = V'(e)$ for some edge $e : X \to Y$ of $\mathrm{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ which satisfies condition $(*)$ of Proposition 5.3.6.6.* `02UF`

Proof. As in the proof of Lemma 5.3.6.11, we may assume without loss of generality that $\mathcal{B} = \Delta^1$, so that $\mathcal{C}$ is an ∞-category and $\overline{e}$ corresponds to a morphism $T : \mathcal{C} \to \mathcal{E}$. Replacing $\mathcal{D}$ by the fiber product $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, we can further reduce to the case where $\mathcal{C} = \mathcal{E}$ and T is the identity functor (so that $V : \mathcal{D} \to \mathcal{C}$ is a cartesian fibration of ∞-categories). Let $\mathcal{C}(0)$ and $\mathcal{C}(1)$ denote the fibers of $\mathcal{C}$ over the the vertices $0, 1 \in \Delta^1$, so that we can identify Y with a functor $\mathcal{C}(1) \to \mathcal{D}$ such that $V \circ Y$ is the inclusion map $\mathcal{C}(1) \hookrightarrow \mathcal{C}$. Applying Proposition 5.2.2.8, we can choose a functor $F : \mathcal{C}(0) \to \mathcal{C}(1)$ and a diagram

$$\begin{tikzcd}
\Delta^1 \times \mathcal{C}(0) \arrow[r, "H"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\Delta^1 \arrow[r, equal] & \mathcal{B}
\end{tikzcd}$$

which exhibits $F = H|_{\{1\} \times \mathcal{C}(0)}$ as given by covariant transport along the nondegenerate edge of $\mathcal{B} = \Delta^1$. Since V is a cartesian fibration, Proposition 5.2.1.3 guarantees that the lifting problem

$$\begin{tikzcd}
\{1\} \times \mathcal{C}(0) \arrow[r, "Y \circ F"] \arrow[d] & \mathcal{D} \arrow[d, "V"] \\
\Delta^1 \times \mathcal{C}(0) \arrow[r, "H"] \arrow[ur, dashed, "G"] & \mathcal{C}
\end{tikzcd}$$

admits a solution with the property that, for every object C of the ∞-category $\mathcal{C}(0)$, the restriction $G|_{\Delta^1 \times \{C\}}$ is a V-cartesian morphism of $\mathcal{D}$.

Let $\mathcal{C}' = (\Delta^1 \times \mathcal{C}(0)) \coprod_{(\{1\} \times \mathcal{C}(0))} \mathcal{C}(1)$ denote the mapping cylinder of the functor F. Amalgamating H with the inclusion map $\mathcal{C}(1) \hookrightarrow \mathcal{C}$, we obtain a morphism of simplicial sets $\overline{H} : \mathcal{C}' \to \mathcal{C}$ which is a categorical equivalence by virtue of Corollary 5.2.4.2. Amalgamating G with Y, we obtain a diagram $\overline{G} : \mathcal{C}' \to \mathcal{D}$ satisfying $V \circ \overline{G} = \overline{H}$. We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{D}) \arrow[r, "\circ G"] \arrow[d] & \operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}', \mathcal{D}) \arrow[d] \\
\operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}(1), \mathcal{D}) \arrow[r, equal] & \operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}(1), \mathcal{D}),
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories. Since V is an isofibration, the vertical maps in this diagram are isofibrations (Proposition 4.5.5.14). Applying Corollary 4.5.2.32, we deduce that the upper horizontal map in the diagram (5.19) restricts to an equivalence of the fibers of the vertical maps over the object $Y \in \operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}(1), \mathcal{D})$. It follows that there there exists a functor $E : \mathcal{C} \to \mathcal{D}$ such that $V \circ E = \mathrm{id}_{\mathcal{C}}$, $E|_{\mathcal{C}(1)} = Y$, and $E \circ \overline{H}$ is isomorphic to $\overline{G}$ as an object of the ∞-category $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}', \mathcal{D})$. By construction, we can identify E with an edge $e : X \to Y$ of $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$ satisfying $V'(e) = \overline{e}$. To complete the proof, it will suffice to show that e satisfies condition $(*)$ of Proposition 5.3.6.6. Let $f : C \to C'$ be a U-cocartesian morphism of $\mathcal{C}$; we wish to show that $E(f)$ is a V-cartesian morphism of $\mathcal{D}$. Without loss of generality, we may assume that $U(f)$ is the nondegenerate edge of $\mathcal{B} = \Delta^1$ (otherwise, f is an isomorphism and there is nothing to prove). By virtue of Remark 5.1.3.8, we can assume without loss of generality that $f : C \to F(C)$ is the U-cocartesian morphism given by the restriction $H|_{\Delta^1 \times \{C\}}$. In this case, $E(f)$ is isomorphic (as an object of the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{D})$) to the V-cartesian morphism $G|_{\Delta^1 \times \{C\}}$, and is therefore also V-cartesian (Corollary 5.1.2.5). □

Proof of Proposition 5.3.6.6. Let $U : \mathcal{C} \to \mathcal{B}$ be a cocartesian fibration of simplicial sets, let $V : \mathcal{D} \to \mathcal{E}$ be a cartesian fibration of simplicial sets, and let $V' : \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})$ be given by postcomposition with V. We first claim that V' is an inner fibration. To prove this, we may assume without loss of generality that $\mathcal{B}$ is a standard simplex, so that $U : \mathcal{C} \to \mathcal{B}$ is a cocartesian fibration of ∞-categories. Proposition 5.3.6.1 then guarantees that U is exponentiable, so that V' is an isofibration (Proposition 4.5.9.18) and therefore an inner fibration by virtue of Remark 4.5.5.7.

Let us say that an edge of $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$ is *special* if it satisfies condition $(*)$ appearing in the statement of Proposition 5.3.6.6. Lemma 5.3.6.11 guarantees that every special edge of

$\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ is V'-cartesian. Moreover, if Y is a vertex of $\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ and $\overline{e} : \overline{X} \to V'(Y)$ is an edge of $\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})$, then Lemma 5.3.6.12 guarantees that there exists a special edge $e : X \to Y$ of $\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ satisfying $V'(e) = \overline{e}$. It follows that V' is a cartesian fibration of simplicial sets.

To complete the proof of Proposition 5.3.6.6, we must show that every V'-cartesian edge $e : X \to Y$ of $\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ is special. Without loss of generality we may assume that $\mathcal{B} = \Delta^1$ and that e lies over the nondegenerate edge of $\mathcal{B}$, so that e corresponds to a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$. Replacing $\mathcal{D}$ by the fiber product $\mathcal{C} \times_{\mathcal{E}} \mathcal{D}$, we can assume that $\mathcal{E} = \mathcal{C}$ and that $V \circ F : \mathcal{C} \to \mathcal{E}$ is the identity functor, so that V is a cartesian fibration of ∞-categories. Using Lemma 5.3.6.12, we can choose a special edge $e' : X' \to Y$ of $\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ satisfying $V'(e') = V'(e)$, corresponding to another functor $F' : \mathcal{C} \to \mathcal{D}$ satisfying $V \circ F' = \mathrm{id}_{\mathcal{C}}$. Since e' is also V'-cartesian, it is isomorphic to e as an object of the ∞-category $\operatorname{Fun}_{/\mathcal{B}}(\Delta^1, \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}))$. It follows that F and F' are isomorphic as objects of the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$. If u is a U-cocartesian edge of $\mathcal{C}$, then $F(u)$ is isomorphic to the V-cartesian morphism $E'(u)$ (as an object of the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{D})$), and is therefore also V-cartesian (Corollary 5.1.2.5). □

5.3.7 Application: Path Fibrations

Recall that every morphism of Kan complexes $f : X \to Y$ admits a canonical factorization 01US

$$X \xrightarrow{\delta} P(f) \xrightarrow{\pi} Y,$$

where δ is a homotopy equivalence and π is the *path fibration*

$$P(f) = X \times_{\operatorname{Fun}(\{0\}, Y)} \operatorname{Fun}(\Delta^1, Y) \to \operatorname{Fun}(\{1\}, Y) \simeq Y$$

of Example 3.1.7.10. Note that the simplicial set $P(f) = X \tilde{\times}_Y Y$ is an example of an oriented fiber product (Definition 4.6.4.1), which is defined for *any* morphism of simplicial sets $f : X \to Y$. Beware that if X and Y are not Kan complexes, then δ need not be a homotopy equivalence and π need not be a Kan fibration. However, if $X = \mathcal{C}$ and $Y = \mathcal{D}$ are ∞-categories, then we have the following weaker statements:

(*a*) The functor $\delta : \mathcal{C} \to \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ is fully faithful, and its essential image is the homotopy fiber product $\mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D}$ of Construction 4.5.2.1 (Corollary 4.5.2.22).

(*b*) The functor $\pi : \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D} \to \mathcal{D}$ is a cocartesian fibration of ∞-categories (Corollary 5.3.7.3).

Moreover, the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ can be characterized by a universal mapping property: roughly speaking, the diagonal map δ exhibits the cocartesian fibration π as freely generated by the functor f (Theorem 5.3.7.6).

Our starting point is the following observation:

0475 **Lemma 5.3.7.1.** *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets, let e be an edge of $\operatorname{Fun}(\Delta^1, \mathcal{C})$, and let*

$$V : \operatorname{Fun}(\Delta^1, \mathcal{C}) = \mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{C} \to \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{C}$$

denote the morphism induced by U. Let $\operatorname{ev}_0, \operatorname{ev}_1 : \operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ be the evaluation maps. If $\operatorname{ev}_0(e)$ is U-cocartesian, then e is V-cocartesian. If $\operatorname{ev}_1(e)$ is U-cartesian, then e is V-cartesian.

Proof. Assume that $\operatorname{ev}_0(e)$ is U-cocartesian; we will show that e is V-cocartesian (the second assertion follows by a similar argument). Let $n \geq 2$; we wish to show that every lifting problem

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$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\sigma_0"] \arrow[d] & \operatorname{Fun}(\Delta^1, \mathcal{C}) \arrow[d, "V"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{C}
\end{tikzcd} \tag{5.21}$$

admits a solution, provided that the composite map

$$\Delta^1 \simeq \operatorname{N}_{\bullet}(\{0 < 1\}) \hookrightarrow \Lambda^n_0 \xrightarrow{\tau_0} \operatorname{Fun}(\Delta^1, \mathcal{C})$$

coincides with e. Let $X(0)$ denote the simplicial subset of $\Delta^1 \times \Delta^n$ given by the union of $\partial \Delta^1 \times \Delta^n$ with $\Delta^1 \times \Lambda^n_0$. Unwinding the definitions, we can rewrite (5.21) as a lifting problem

$$\begin{tikzcd}
X(0) \arrow[r, "\tau_0"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\Delta^1 \times \Delta^n \arrow[r, "\overline{\tau}"] \arrow[ur, dashed, "\tau"] & \mathcal{D}.
\end{tikzcd}$$

Choose a filtration

$$X(0) \subset X(1) \subset X(2) \subset \cdots \subset X(t) = \Delta^1 \times \Delta^n$$

satisfying the requirements of Lemma 4.4.4.7. We will complete the proof by showing that, for each $s \leq t$, the morphism τ_0 admits an extension $\tau_s : X(s) \to \mathcal{C}$ satisfying $U \circ \tau_s = \overline{\tau}|_{X(s)}$. The proof proceeds by induction on s, the case $s = 0$ being vacuous. Let us therefore assume that $0 < s \leq t$ and that τ_0 has already been extended to a morphism of simplicial sets

$\tau_{s-1} : X(s-1) \to \mathcal{C}$. By construction, we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^q_p \arrow[r, "\varphi_0"] \arrow[d] & X(s-1) \arrow[d] \\
\Delta^q \arrow[r, "\varphi"] & X(s)
\end{tikzcd}$$

for some $q \geq 2$ and $0 \leq p < q$. Consequently, to prove the existence of τ_s, it will suffice to show that $\tau_{s-1} \circ \varphi_0$ can be extended to a q-simplex of $\mathcal{C}$ lying over the simplex $\overline{\tau} \circ \varphi : \Delta^q \to \mathcal{D}$. For $p \neq 0$, the existence of this extension follows from our assumption that U is an inner fibration. To handle the case $p = 0$, we observe that the morphism φ carries the initial edge of Δ^q to the edge $(0,0) \to (0,1)$ of $\Delta^1 \times \Delta^n$, so that $\tau_{s-1} \circ \varphi_0$ carries the initial edge of Δ^q to the edge $\mathrm{ev}_0(e)$ of $\mathcal{C}$, which is U-cocartesian by assumption. □

Proposition 5.3.7.2. *Suppose we are given a commutative diagram of simplicial sets* 0477

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "F_0"] \arrow[d, "U_0"] & \mathcal{C} \arrow[d, "U"] & \mathcal{C}_1 \arrow[l, "F_1"'] \arrow[d, "U_1"'] \\
\mathcal{D}_0 \arrow[r, "G_0"] & \mathcal{D} & \mathcal{D}_1 \arrow[l, "G_1"']
\end{tikzcd}$$

where U_0, U_1, and U are cocartesian fibrations, and F_0 carries U_0-cocartesian edges of $\mathcal{C}_0$ to U-cocartesian edges of $\mathcal{C}$. Then the induced map

$$V : \mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1 \to \mathcal{D}_0 \tilde{\times}_{\mathcal{D}} \mathcal{D}_1$$

is a cocartesian fibration of simplicial sets. Moreover, an edge e of $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ is V-cocartesian if and only if it satisfies the following condition:

($*$) *Let e_0 and e_1 denote the images of e in $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively. Then e_0 is U_0-cocartesian and e_1 is U_1-cocartesian.*

Proof. Let us say that an edge e of $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ is *special* if it satisfies condition ($*$). We first show that if e is a special edge of $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$, then e is V-cocartesian. Let e_0 and e_1 denote the images of e in $\mathcal{C}_0$ and $\mathcal{C}_1$, respectively. Note that V factors as a composition

$$\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1 \xrightarrow{V'} \mathcal{C}_0 \tilde{\times}_{\mathcal{D}} \mathcal{C}_1 \xrightarrow{V''} \mathcal{D}_0 \tilde{\times}_{\mathcal{D}} \mathcal{D}_1 .$$

Here V' is a pullback of the projection map $\overline{V}' : \mathrm{Fun}(\Delta^1, \mathcal{C}) = \mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{C} \to \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{C}$. Since $F(e_0)$ is U-cocartesian, Lemma 5.3.7.1 implies that e is V'-cocartesian. Moreover, V'' is a pullback of the product map $(U_0 \times U_1) : \mathcal{C}_0 \times \mathcal{C}_1 \to \mathcal{D}_0 \times \mathcal{D}_1$. By assumption, e_0 is U_0-cocartesian and e_1 is U_1-cocartesian. It follows that $V'(e)$ is V''-cocartesian, so that e is V-cocartesian by virtue of Remark 5.1.1.6.

Since U_0, U_1, and U are inner fibrations, the morphisms $\overline{V}'$ and $(U_0 \times U_1)$ are also inner fibrations (see Proposition 4.1.4.1). It follows that V' and V'' are inner fibrations (Remark 4.1.1.5), so that V is an inner fibration (Remark 4.1.1.8). To show that V is a cocartesian fibration, it will suffice to show that if C is an object of $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ and $\overline{e} : V(C) \to \overline{C}'$ is an edge of $\mathcal{D}_0 \tilde{\times}_{\mathcal{D}} \mathcal{D}_1$, then there exists a special edge $e : C \to C'$ satisfying $V(e) = \overline{e}$. Let us identify C with a triple (C_0, C_1, u) where C_0 is a vertex of $\mathcal{C}_0$, C_1 is a vertex of $\mathcal{C}_1$, and $u : F_0(C_0) \to F_1(C_1)$ is an edge of $\mathcal{C}$. Similarly, we can identify $\overline{C}'$ with a triple $(\overline{C}'_0, \overline{C}'_1, \overline{u}')$ where $\overline{C}_0$ is a vertex of $\mathcal{D}_0$, $\overline{C}'_1$ is a vertex of $\mathcal{D}_1$, and $\overline{u}' : G_0(\overline{C}'_0) \to G_1(\overline{C}'_1)$ is an edge of $\mathcal{D}$. The edge $\overline{e}$ has images $\overline{e}_0 : U_0(C_0) \to \overline{C}'_0$ an $\overline{e}_1 : U_1(C_1) \to \overline{C}'_1$ in $\mathcal{D}_0$ and $\mathcal{D}_1$-respectively. Since U_0 is a cocartesian fibration, we can lift $\overline{e}_0$ to a U_0-cocartesian edge $e_0 : C_0 \to C'_0$ of $\mathcal{C}_0$. Similarly, we can lift $\overline{e}_1$ to a U_1-cocartesian edge $e_1 : C_1 \to C'_1$ of $\mathcal{C}_1$. The edge $\overline{e}$ also determines a map $\Delta^1 \times \Delta^1 \to \mathcal{D}$, which we depict informally in the diagram

$$\begin{tikzcd}
(G_0 \circ U_0)(C_0) \arrow[r, "U(u)"] \arrow[d, "G_0(\overline{e}_0)"] & (G_1 \circ U_1)(C_1) \arrow[d, "G_1(\overline{e}_1)"] \\
G_0(\overline{C}'_0) \arrow[r, "\overline{u}'"] & G_1(\overline{C}'_1).
\end{tikzcd}$$

Using our assumption that U is an inner fibration, we can lift the upper right triangle to a 2-simplex σ:

$$\begin{tikzcd}
F_0(C_0) \arrow[r, "u"] \arrow[dr, dashed, "v"] & F_1(C_1) \arrow[d, "F_1(e_1)"] \\
& F_1(C'_1)
\end{tikzcd}$$

of the simplicial set $\mathcal{C}$. Using the fact that $F_0(e_0)$ is U-cocartesian, we can lift the lower triangle to a 2-simplex τ

$$\begin{tikzcd}
F_0(C_0) \arrow[d, "F_0(e_0)"] \arrow[dr, "v"] & \\
F_0(C'_0) \arrow[r, dashed, "w"] & F_1(C'_1)
\end{tikzcd}$$

of $\mathcal{C}$. Setting $C' = (C'_0, C'_1, w) \in \mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$, we observe that the tuple (e_0, e_1, σ, τ) determines a special edge $e : C \to C'$ satisfying $V(e) = \overline{e}$.

We now complete the proof by showing that every V-cocartesian edge $f : C \to C''$ in $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ is special. Using the preceding argument, we can choose a special edge $e : C \to C'$ satisfying $V(e) = V(f)$. Set $\overline{C}' = V(C') = V(C'')$. Applying Remark 5.1.3.8, we deduce that there is a 2-simplex ρ :

$$\begin{tikzcd} & C' \arrow[dr, "s"] & \\ C \arrow[ur, "e"] \arrow[rr, "f"] & & C'' \end{tikzcd}$$

of the simplicial set $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$, where s is an isomorphism in the ∞-category $V^{-1}(\{\overline{C}'\})$. Applying Example 5.1.3.6, we deduce that the images of s in $\mathcal{C}_0$ is U_0-cocartesian, and the image of s in $\mathcal{C}_1$ is U_1-cocartesian. Since the collections of U_0-cocartesian and U_1-cocartesian edges are closed under composition (Corollary 5.1.2.4), we conclude that f is also special. □

Corollary 5.3.7.3. *Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be morphisms of simplicial sets and let* 0478

$$\mathcal{C}_0 \xleftarrow{\pi} \mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1 \xrightarrow{\pi'} \mathcal{C}_1$$

denote the projection maps. Then:

(1) *If $\mathcal{C}_0$ and $\mathcal{C}$ are ∞-categories, then π' is a cocartesian fibration of simplicial sets. Moreover, an edge e of $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ is π'-cocartesian if and only if $\pi(e)$ is an isomorphism in the ∞-category $\mathcal{C}_0$.*

(2) *If $\mathcal{C}_1$ and $\mathcal{C}$ are ∞-categories, then π is a cartesian fibration of simplicial sets. Moreover, an edge e of $\mathcal{C}_0 \tilde{\times}_{\mathcal{C}} \mathcal{C}_1$ is π-cartesian if and only if $\pi'(e)$ is an isomorphism in the ∞-category $\mathcal{C}_1$.*

Proof. Assertion (1) follows by applying Proposition 5.3.7.2 in the special case $\mathcal{D}_0 = \mathcal{D} = \Delta^0$ and $\mathcal{D}_1 = \mathcal{C}_1$. Assertion (2) follows by a similar argument. □

Example 5.3.7.4. Let $\mathcal{C}$ be an ∞-category. Applying Corollary 5.3.7.3 in the case where both F and G are the identity functor $\mathrm{id} : \mathcal{C} \to \mathcal{C}$, we deduce that the evaluation functor 01UU

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{0\}, \mathcal{C}) \simeq \mathcal{C}$$

is a cartesian fibration of ∞-categories, and the evaluation functor

$$\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\{1\}, \mathcal{C}) \simeq \mathcal{C}$$

is a cocartesian fibration of ∞-categories.

Corollary 5.3.7.5. *Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. Then:* 02VW

(1) *The restriction map $U : \mathrm{Fun}(K^{\triangleleft}, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$ is a cocartesian fibration. Moreover, a morphism e of $\mathrm{Fun}(K^{\triangleleft}, \mathcal{C})$ is U-cocartesian if and only if it carries the cone point $\mathbf{0} \in K^{\triangleleft}$ to an isomorphism in $\mathcal{C}$.*

(2) *The restriction map* $V : \mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$ *is a cartesian fibration. Moreover, a morphism* e *of* $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ *is* U*-cartesian if and only if it carries the cone point* $\mathbf{1} \in K^{\triangleright}$ *to an isomorphism in* $\mathcal{C}$*.*

Proof. We will prove (1); the proof of (2) is similar. Let $\Delta^0 \diamond K$ denote the blunt join of Notation 4.5.8.3, and let $c : \Delta^0 \diamond K \to \Delta^0 \star K = K^{\triangleleft}$ be the categorical equivalence of Theorem 4.5.8.8. We have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}(K^{\triangleleft}, \mathcal{C}) \arrow[rr, "\circ c"] \arrow[dr, "U"'] & & \mathrm{Fun}(\Delta^0 \diamond K, \mathcal{C}) \arrow[dl, "U'"] \\
& \mathrm{Fun}(K, \mathcal{C}) &
\end{tikzcd}$$

where the horizontal map is an equivalence of ∞-categories (Proposition 4.5.3.8) and the vertical maps are isofibrations (Corollary 4.4.5.3). Unwinding the definitions, we can identify $\mathrm{Fun}(\Delta^0 \diamond K, \mathcal{C})$ with the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K, \mathcal{C})$. Under this identification, the functor U' is given by projection onto the second factor, and is therefore a cocartesian fibration (Corollaryt 5.3.7.3). Applying Corollary 5.1.6.2, we deduce that U is also a cocartesian fibration. Moreover, a morphism e of $\mathrm{Fun}(K^{\triangleleft}, \mathcal{C})$ is U-cocartesian if and only if its image in $\mathrm{Fun}(\Delta^0 \diamond K, \mathcal{C})$ is U'-cocartesian (Proposition 5.1.6.6). Using the criterion of Corollary 5.3.7.3, we see that this is equivalent to the requirement that e carries the cone point $\mathbf{0} \in K^{\triangleleft}$ to an isomorphism in $\mathcal{C}$. □

037Z **Theorem 5.3.7.6.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor of* ∞*-categories, let* $\pi : \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D} \to \mathcal{D}$ *be given by projection onto the second factor, let* $\delta : \mathcal{C} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ *be the diagonal map. For every cocartesian fibration* $U : \mathcal{E} \to \mathcal{D}$*, precomposition with* δ *induces a trivial Kan fibration of* ∞*-categories*

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{D}}(\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{D}}(\mathcal{C}, \mathcal{E}).$$

Our proof of Theorem 5.3.7.6 will make use of an auxiliary construction.

02UH **Notation 5.3.7.7** (Cocartesian Direct Images)**.** Let $U : \mathcal{D} \to \mathcal{C}$ be a morphism of simplicial sets $\mathrm{id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ determines a section of the projection map $\pi : \mathrm{Fun}(\mathcal{D} / \mathcal{C}, \mathcal{D}) \to \mathcal{C}$. For every morphism of simplicial sets $V : \mathcal{E} \to \mathcal{D}$, we let $\mathrm{Res}_{\mathcal{D} / \mathcal{C}}(\mathcal{E})$ denote the fiber product $\mathcal{C} \times_{\mathrm{Fun}(\mathcal{D} / \mathcal{C}, \mathcal{D})} \mathrm{Fun}(\mathcal{D} / \mathcal{C}, \mathcal{E})$. Unwinding the definitions, we see that vertices of $\mathrm{Res}_{\mathcal{D} / \mathcal{C}}(\mathcal{E})$ can be identified with pairs (C, F), where C is a vertex of $\mathcal{C}$ and

$$F : \mathcal{D}_C = \{C\} \times_{\mathcal{C}} \mathcal{D} \to \{C\} \times_{\mathcal{C}} \mathcal{E} = \mathcal{E}_C$$

is a section of the map $V|_{\mathcal{E}_C} : \mathcal{E}_C \to \mathcal{D}_C$. If V is a cocartesian fibration, we let $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D} / \mathcal{C}}(\mathcal{E})$ denote the full simplicial subset of $\mathrm{Res}_{\mathcal{D} / \mathcal{C}}(\mathcal{E})$ spanned by those vertices (C, F) where F

carries each edge of $\mathcal{D}_C$ to V_C-cocartesian edge of $\mathcal{E}_C$. We will refer to $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$ as the *cocartesian direct image of $\mathcal{E}$ along U.*

Remark 5.3.7.8. Let $U : \mathcal{D} \to \mathcal{C}$ be a morphism of simplicial sets and let $V : \mathcal{E} \to \mathcal{D}$ be a cocartesian fibration of simplicial sets. Then the projection map $\pi : \mathrm{Res}_{\mathcal{D}/\mathcal{C}}(\mathcal{E}) \to \mathcal{C}$ restricts to a projection map $\pi^{\mathrm{CCart}} : \mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E}) \to \mathcal{C}$. Moreover, for each vertex $C \in \mathcal{C}$, the canonical isomorphism $\{C\} \times_{\mathcal{C}} \mathrm{Res}_{\mathcal{D}/\mathcal{C}}(\mathcal{E}) \simeq \mathrm{Fun}_{/\mathcal{D}_C}(\mathcal{D}_C, \mathcal{E}_C)$ restricts to an isomorphism of full subcategories $\{C\} \times_{\mathcal{C}} \mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E}) \simeq \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{D}_C}(\mathcal{D}_C, \mathcal{E}_C)$. `02UJ`

Proposition 5.3.7.9. *Let $V : \mathcal{E} \to \mathcal{D}$ be a cocartesian fibration of simplicial sets, let $U : \mathcal{D} \to \mathcal{C}$ be a cartesian fibration of simplicial sets. Then:* `02UK`

(1) *The projection map $\pi : \mathrm{Res}_{\mathcal{D}/\mathcal{C}}(\mathcal{E}) \to \mathcal{C}$ is a cocartesian fibration of simplicial sets.*

(2) *Let $e : X \to Y$ be a π-cocartesian edge of the simplicial set $\mathrm{Res}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$. If X belongs to the simplicial subset $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$, then Y also belongs to the simplicial subset $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$.*

(3) *The morphism π restricts to a cocartesian fibration $\pi^{\mathrm{CCart}} : \mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E}) \to \mathcal{C}$.*

(4) *An edge of the simplicial set $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$ is π^{CCart}-cocartesian if and only if it is π-cocartesian.*

Proof. Assertion (1) follows from Proposition 5.3.6.6 (after passing to opposite simplicial sets). To prove (2), we may assume without loss of generality that $\mathcal{C} = \Delta^1$ and $\pi(e)$ is the nondegenerate edge of $\mathcal{C}$. In this case, the simplicial sets $\mathcal{D}$ and $\mathcal{E}$ are ∞-categories, and we can identify the edge e with a morphism of simplicial sets $E : \mathcal{D} \to \mathcal{E}$ satisfying $V \circ E = \mathrm{id}_{\mathcal{D}}$. Let $u : D \to D'$ be a morphism in the ∞-category $\mathcal{D}_1 = \{1\} \times_{\mathcal{C}} \mathcal{D}$; we wish to show that $E(u)$ is a V-cocartesian morphism of $\mathcal{E}$. To prove this, let $G : \mathcal{D}_1 \to \mathcal{D}_0 = \{0\} \times_{\mathcal{C}} \mathcal{D}$ be given by contravariant transport along the nondegenerate edge of $\mathcal{C}$, so that we have a commutative diagram

$$\begin{tikzcd}
G(D) \arrow[r] \arrow[d, "G(u)"] & D \arrow[d, "u"] \\
G(D') \arrow[r] & D',
\end{tikzcd}$$

in the ∞-category where the horizontal maps are U-cartesian. Our assumption that e is π-cocartesian guarantees that the functor E carries U-cartesian morphisms of $\mathcal{D}$ to V-cocartesian morphisms of $\mathcal{E}$ (Proposition 5.3.6.6). We therefore obtain a commutative

diagram

$$\begin{tikzcd}
(E \circ G)(D) \arrow[r] \arrow[d, "(E \circ G)(u)"] & E(D) \arrow[d, "E(u)"] \\
(E \circ G)(D') \arrow[r] & E(D'),
\end{tikzcd}$$

where the horizontal maps are V-cocartesian. By virtue of Corollary 5.1.2.4, it will suffice to show that the morphism $(E \circ G)(u)$ is V-cocartesian, which follows from our assumption that X belongs to $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$. This completes the proof of (2); assertions (3) and (4) then follow by applying Proposition 5.1.4.16. □

In the situation of Proposition 5.3.7.9, the cocartesian direct image $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$ can be characterized by a universal property:

02UL **Proposition 5.3.7.10.** *Let $V : \mathcal{E} \to \mathcal{D}$ be a cocartesian fibration of simplicial sets and let $U : \mathcal{D} \to \mathcal{C}$ be a cartesian fibration of simplicial sets. For every cocartesian fibration of simplicial sets $W : \mathcal{C}' \to \mathcal{C}$, the canonical isomorphism*

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}', \mathrm{Res}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})) \xrightarrow{\sim} \mathrm{Fun}_{/\mathcal{D}}(\mathcal{C}' \times_{\mathcal{C}} \mathcal{D}, \mathcal{E})$$

restricts to an isomorphism of full simplicial subsets

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}', \mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{D}}(\mathcal{C}' \times_{\mathcal{C}} \mathcal{D}, \mathcal{E}).$$

Proof. Let $\pi : \mathrm{Res}_{\mathcal{D}/\mathcal{C}}(\mathcal{E}) \to \mathcal{C}$ denote the projection map and let $f : \mathcal{C}' \to \mathrm{Res}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$ be a morphism satisfying $\pi \circ f = W$, corresponding to a morphism of simplicial sets $F : \mathcal{C}' \times_{\mathcal{C}} \mathcal{D} \to \mathcal{E}$ for which $V \circ F$ is given by projection to the second factor. Note that we can regard F as a vertex of the simplicial subset $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{D}}(\mathcal{C}' \times_{\mathcal{C}} \mathcal{D}, \mathcal{E})$ if and only if it satisfies the following condition:

(a) For every edge (e', e) of the fiber product $\mathcal{C}' \times_{\mathcal{C}} \mathcal{D}$ for which e' is a W-cocartesian edge of $\mathcal{C}'$, the image $F(e', e)$ is a V-cocartesian edge of $\mathcal{E}$.

We wish to show that (a) is equivalent to the following pair of conditions:

(b) The morphism f factors through the full simplicial subset $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E}) \subseteq \mathrm{Res}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$. In other words, for every edge (e', e) of the fiber product $\mathcal{C}' \times_{\mathcal{C}} \mathcal{D}$ for which e' is a degenerate edge of $\mathcal{C}'$, the image $F(e', e)$ is a V-cocartesian edge of $\mathcal{E}$.

(c) For every W-cocartesian edge e' of $\mathcal{C}'$, the image $f(e')$ is a $\pi|_{\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})}$-cocartesian edge of $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{D}/\mathcal{C}}(\mathcal{E})$. By virtue of Propositions 5.3.7.9 and 5.3.6.6, this is equivalent to the assertion that for every edge (e', e) of the fiber product $\mathcal{C}' \times_{\mathcal{C}} \mathcal{D}$ where e' is W-cocartesian and e is U-cartesian, the image $F(e', e)$ is a V-cocartesian edge of $\mathcal{E}$.

The implications $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ are clear. For the converse, suppose that (b) and (c) are satisfied; we wish to prove (a). Let $(e', e) : (X', X) \to (Z', Z)$ be an edge of the fiber product $\mathcal{C}' \times_{\mathcal{C}} \mathcal{D}$, where $e' : X' \to Z'$ is W-cocartesian. Let $\overline{e} = U(e) = W(e')$ denote the corresponding edge of $\mathcal{C}$. Since U is a cartesian fibration, there exists a U-cartesian morphism $f : Y \to Z$ satisfying $U(f) = \overline{e}$. Let $\overline{\sigma}$ denote the left-degenerate 2-simplex $s_0^1(\overline{e})$. Since f is U-cartesian, we can lift $\overline{\sigma}$ to a 2-simplex of $\mathcal{D}$ as indicated in the diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "f"] & \\
X \arrow[ur] \arrow[rr, "e"] & & Z.
\end{tikzcd}$$

Writing σ' for the left-degenerate 2-simplex $s_0^1(e')$ of $\mathcal{C}'$, we obtain a 2-simplex $\tau = F(\sigma', \sigma)$ of $\mathcal{E}$. It follows from assumption (b) that the restriction $\tau|_{\mathrm{N}_\bullet(\{0<1\})}$ is a V-cocartesian edge of $\mathcal{E}$, and from assumption (c) that the restriction $\tau|_{\mathrm{N}_\bullet(\{1<2\})}$ is a V-cocartesian edge of $\mathcal{E}$. Applying Proposition 5.1.4.12, we conclude that $F(e', e) = \tau|_{\mathrm{N}_\bullet(\{0<2\})}$ is also a V-cocartesian edge of $\mathcal{E}$. $\square$

Proof of Theorem 5.3.7.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $U : \mathcal{E} \to \mathcal{D}$ be a cocartesian fibration of ∞-categories, and let $\delta : \mathcal{C} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ be the diagonal embedding. Since U is an isofibration (Proposition 5.1.4.8), the restriction map $\overline{\theta} : \mathrm{Fun}_{/\mathcal{D}}(\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{D}}(\mathcal{C}, \mathcal{E})$ is also an isofibration (Corollary 4.5.5.16). Because $\mathrm{Fun}_{/\mathcal{D}}^{\mathrm{CCart}}(\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}, \mathcal{E})$ is a replete full subcategory of $\mathrm{Fun}_{/\mathcal{D}}(\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}, \mathcal{E})$, it follows that $\overline{\theta}$ restricts to an isofibration $\theta : \mathrm{Fun}_{/\mathcal{D}}^{\mathrm{CCart}}(\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{D}}(\mathcal{C}, \mathcal{E})$. To prove Theorem 5.3.7.6, we will show that θ is an equivalence of ∞-categories (it is then automatically a trivial Kan fibration of simplicial sets: see Proposition 4.5.5.20).

Note that the functor $U : \mathcal{E} \to \mathcal{D}$ induces cocartesian fibrations $U' : \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{E} \to \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ and $U'' : \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \to \mathcal{C}$. Let $\pi' : \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D} \to \mathcal{C}$ be given by projection onto the first factor, so that π' is a cartesian fibration (Corollary 5.3.7.3). Let $\mathcal{M}$ denote the cocartesian direct image $\mathrm{Res}^{\mathrm{CCart}}_{\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D} / \mathcal{C}}(\mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{E})$ and let $T : \mathcal{M} \to \mathcal{C}$ be the projection map. Precomposition with the diagonal embedding $\delta : \mathcal{C} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ induces a restriction functor

$$\delta^* : \mathcal{M} \to \mathrm{Res}_{\mathcal{C} / \mathcal{C}}(\mathcal{C} \times_{\mathcal{D}} \mathcal{E}) = \mathcal{C} \times_{\mathcal{D}} \mathcal{E}$$

which fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "\delta^*"] \arrow[dr, "T"'] & & \mathcal{C} \times_{\mathcal{D}} \mathcal{E} \arrow[dl, "U''"] \\
 & \mathcal{C} &
\end{tikzcd}$$

It follows from Proposition 5.3.7.9 that T is a cocartesian fibration and that δ^* carries T-cocartesian morphisms of $\mathcal{M}$ to U''-cocartesian morphisms of $\mathcal{C} \times_{\mathcal{D}} \mathcal{E}$. Using Proposition 5.3.7.10, we can identify θ with the map

$$\operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{M}) \rightarrow \operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{C} \times_{\mathcal{D}}, \mathcal{E}) \simeq \operatorname{Fun}_{/\mathcal{D}}(\mathcal{C}, \mathcal{E})$$

given by postcomposition with δ^*. Consequently, to show that θ is an equivalence of ∞-categories, it will suffice to show that δ^* is an equivalence of cocartesian fibrations over $\mathcal{C}$. By virtue of Proposition 5.1.7.15), this can be checked fiberwise: that is, it suffices to show that for each object $C \in \mathcal{C}$, the induced map of fibers

$$\delta^*_C : \{C\} \times_{\mathcal{C}} \mathcal{M} \simeq \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{D}}(\{C\} \tilde{\times}_{\mathcal{D}} \mathcal{D}, \mathcal{E}) \rightarrow \{C\} \times_{\mathcal{D}} \mathcal{E}$$

is an equivalence of ∞-categories. This is a special case of Corollary 5.3.1.22, since $\delta(C)$ is an initial object of the ∞-category $\{C\} \tilde{\times}_{\mathcal{D}} \mathcal{D}$ (Proposition 4.6.7.22). □

5.4 $(\infty, 2)$-Categories

01W4 In §1.4, we defined an ∞-*category* to be a simplicial set $\mathcal{C}$ which satisfies the weak Kan extension condition: for $0 < i < n$, every morphism of simplicial sets $\Lambda^n_i \hookrightarrow \mathcal{C}$ can be extended to an n-simplex of $\mathcal{C}$ (Definition 1.4.0.1). Beware that this terminology is potentially confusing, because the theory of ∞-categories does not generalize the classical theory of 2-categories. For every 2-category $\mathcal{E}$, the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{E})$ is a simplicial set which determines $\mathcal{E}$ up to isomorphism (Theorem 2.3.4.1). However, the simplicial set $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{E})$ is an ∞-category if and only if $\mathcal{E}$ is a $(2, 1)$-category: that is, every 2-morphism in $\mathcal{E}$ is invertible (Theorem 2.3.2.1). Consequently, one can view the notions of 2-category and ∞-category as mutually incomparable extensions of the notion of $(2, 1)$-category. Our goal in this section is to show that these extensions admit a common generalization: a class of simplicial sets which we will refer to as $(\infty, 2)$-*categories*.

Our starting point is the notion of a *thin* 2-*simplex*, which was introduced in §2.3.2. Recall that if $\mathcal{C}$ is a simplicial set, then a 2-simplex σ of $\mathcal{C}$ is *thin* if every morphism of simplicial sets $\tau_0 : \Lambda^n_i \rightarrow \mathcal{C}$ can be extended to an n-simplex of $\mathcal{C}$, provided that $0 < i < n$, $n \geq 3$, and the 2-simplex $\tau_0|_{\mathrm{N}_{\bullet}(\{i-1<i<i+1\})}$ is equal to σ (Definition 2.3.2.3). By virtue of Example 2.3.2.4, $\mathcal{C}$ is an ∞-category if and only if it satisfies the following pair of conditions:

(1) Every morphism of simplicial sets $\Lambda^2_1 \rightarrow \mathcal{C}$ can be extended to a 2-simplex of $\mathcal{C}$.

(2) Every 2-simplex of $\mathcal{C}$ is thin.

We will obtain the notion of $(\infty, 2)$-category by weakening (2) to the requirement that degenerate 2-simplices of $\mathcal{C}$ are thin, but strengthening (1) to require that every map $\Lambda^2_1 \rightarrow \mathcal{C}$

can be extended to a thin 2-simplex of $\mathcal{C}$. We will also add additional axioms that guarantee the ability to fill outer horns of $\mathcal{C}$ in certain special circumstances (see Definition 5.4.1.1).

Every ∞-category is an $(\infty,2)$-category (Proposition 5.4.1.2), and every 2-category can be regarded as an $(\infty,2)$-category by passing to its Duskin nerve (Proposition 5.4.1.5). The situation is summarized in the following diagram

$$\begin{tikzcd}
\{\text{Groupoids}\} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \{\text{Categories}\} \arrow[d, phantom, "\cap"] & \\
\{\text{2-Groupoids}\} \arrow[r, phantom, "\subset"] \arrow[d, hook, "\mathrm{N}^{\mathrm{D}}_{\bullet}"] & \{(2,1)\text{-Categories}\} \arrow[r, phantom, "\subset"] \arrow[d, hook, "\mathrm{N}^{\mathrm{D}}_{\bullet}"] & \{\text{2-Categories}\} \arrow[d, hook, "\mathrm{N}^{\mathrm{D}}_{\bullet}"] \\
\{\text{Kan Complexes}\} \arrow[r, phantom, "\subset"] & \{\infty\text{-Categories}\} \arrow[r, phantom, "\subset"] & \{(\infty,2)\text{-Categories}\},
\end{tikzcd}$$

where none of the inclusions is reversible.

Let $\mathcal{C}$ be a simplicial set containing a pair of objects X and Y, and let $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ denote the pinched morphism spaces of Construction 4.6.5.1. If $\mathcal{C}$ is an ∞-category, then the simplicial sets $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ are Kan complexes (Proposition 4.6.5.5). In §5.4.3, we prove an analogous result: if $\mathcal{C}$ is an $(\infty,2)$-category, then the simplicial sets $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ are ∞-categories (Corollary 5.4.3.5). Recall that $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y)$ is defined as the fiber over Y of the projection map $q : \mathcal{C}_{X/} \to \mathcal{C}$, and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y)$ is defined as the fiber over X of the projection map $q' : \mathcal{C}_{/Y} \to \mathcal{C}$. When $\mathcal{C}$ is an ∞-category, the morphism q is a left fibration of simplicial sets and the morphism q' is a right fibration of simplicial sets (Corollary 4.3.6.11). Beware that, in the case where $\mathcal{C}$ is an $(\infty,2)$-category, the morphisms q and q' are generally not inner fibrations. Nevertheless, we will deduce that the fibers of q and q' are ∞-categories by showing that q and q' are *interior fibrations* (Definition 5.4.2.1), a class of morphisms which we introduce and study in §5.4.2. From this we deduce also that the simplicial sets $\mathcal{C}_{X/}$ and $\mathcal{C}_{/Y}$ are $(\infty,2)$-categories; moreover, an analogous result holds more generally for the slice and coslice constructions associated to any diagram $f : K \to \mathcal{C}$ (Corollary 5.4.3.4).

Suppose that we are given a 2-simplex σ of a simplicial set $\mathcal{C}$, whose 1-skeleton we indicate in the diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z.
\end{tikzcd}$$

Writing $q : \mathcal{C}_{X/} \to \mathcal{C}$ for the projection map, we can identify σ with an edge $\widetilde{g}$ of the simplicial set $\mathcal{C}_{X/}$ satisfying $q(\widetilde{g}) = g$. It follows immediately from the definition that if the 2-simplex σ is thin, then the edge $\widetilde{g}$ is q-cocartesian (in the sense of Definition 5.1.1.1); in particular, it is locally q-cocartesian. In §5.4.4, we prove that if $\mathcal{C}$ is an $(\infty, 2)$-category, then the converse holds: every locally q-cocartesian edge of $\mathcal{C}_{X/}$ is thin when viewed as a 2-simplex of $\mathcal{C}$ (Theorem 5.4.4.1). Roughly speaking, one can think of $\widetilde{g}$ as encoding the datum of a morphism γ from $g \circ f$ to h in the ∞-category $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Z)$; Theorem 5.4.4.1 confirms the heuristic that γ is an isomorphism if and only if σ is thin (in the case where $\mathcal{C}$ is the Duskin nerve of a 2-category, this is also the content of Theorem 2.3.2.5).

Let $\mathcal{C}$ be an $(\infty, 2)$-category. We define the *pith* of $\mathcal{C}$ to be the simplicial subset $\mathrm{Pith}(\mathcal{C}) \subseteq \mathcal{C}$ consisting of those simplices $\Delta^m \to \mathcal{C}$ which carry each 2-simplex of Δ^m to a thin 2-simplex of $\mathcal{C}$ (Construction 5.4.5.1). In §5.4.5, we show that $\mathrm{Pith}(\mathcal{C})$ is an ∞-category (Proposition 5.4.5.6) whose pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{Pith}(\mathcal{C})}(X, Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{Pith}(\mathcal{C})}(X, Y)$ can be identified with the cores of the ∞-categories $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$, respectively (Proposition 5.4.5.13). Roughly speaking, one can think of the ∞-category $\mathrm{Pith}(\mathcal{C})$ as obtained from the $(\infty, 2)$-category by "discarding" its noninvertible 2-morphisms. In particular, when $\mathcal{C}$ is the Duskin nerve of a 2-category $\mathcal{E}$, we can identify $\mathrm{Pith}(\mathcal{C})$ with the Duskin nerve of the $(2, 1)$-category $\mathrm{Pith}(\mathcal{E})$ introduced in Construction 2.2.8.9 (Example 5.4.5.4).

Let $\mathcal{C}$ and $\mathcal{D}$ be $(\infty, 2)$-categories. We define a *functor from $\mathcal{C}$ to $\mathcal{D}$* to be a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ which carries thin 2-simplices of $\mathcal{C}$ to thin 2-simplices of $\mathcal{D}$ (Definition 5.4.7.1). This definition can be somewhat cumbersome to work with in practice, because it requires us to check a condition for *every* thin 2-simplex of $\mathcal{C}$. In §5.4.7, we show that this is unnecessary: to verify that a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ is a functor, it suffices to show that every morphism $\sigma_0 : \Lambda^2_1 \to \mathcal{C}$ can be extended to a thin 2-simplex σ of $\mathcal{C}$ for which $F(\sigma)$ is a thin 2-simplex of $\mathcal{D}$ (Proposition 5.4.7.9). Here we can think of σ_0 as given by a pair of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, and the thinness assumption on $F(\sigma)$ corresponds heuristically to the requirement that F "preserves" the composition of f and g (up to isomorphism). Our proof will make use of a certain closure property enjoyed by the thin 2-simplices of an $(\infty, 2)$-category which we refer to as the *four-out-of-five* property, which we formulate and study in §5.4.6 (see Definition 5.4.6.8 and Proposition 5.4.6.11).

Recall that a 2-category $\mathcal{E}$ is *strict* if its unit and associativity constraints are identity morphisms (Example 2.2.1.4); in this case, we can view $\mathcal{E}$ as an ordinary category which is enriched over Cat (see Definition 2.2.0.1). This notion has a counterpart in the setting of $(\infty, 2)$-categories. Let Set_{Δ} denote the ordinary category of simplicial sets, and let **QCat** denote the full subcategory of Set_{Δ} whose objects are ∞-categories. Let $\mathcal{E}$ be a **QCat**-enriched category: that is, a simplicial category with the property that, for every pair of objects $X, Y \in \mathcal{C}$, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet}$ is an ∞-category. In §5.4.8, we show that the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{E})$ is an $(\infty, 2)$-category (Theorem 5.4.8.1). The construction $\mathcal{E} \mapsto \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{E})$ can be regarded as a generalization of the inclusion from strict 2-categories into general 2-categories (recall that if $\mathcal{E}$ is a strict 2-category, then its Duskin nerve can be identified with the homotopy coherent nerve of the associated simplicial category; see Example 2.4.3.11). Beware that not every $(\infty, 2)$-category $\mathcal{C}$ is *isomorphic* to the homotopy coherent nerve of a **QCat**-enriched category. Nevertheless, we will later prove a coherence theorem which guarantees that $\mathcal{C}$ is *equivalent* to the homotopy coherent nerve of a **QCat**-enriched category: see Theorem [?].

Remark 5.4.0.1. The ideas presented in this section are closely related to the work of Verity, who has proposed a simplicial framework for studying higher categories with noninvertible morphisms at all levels. We refer the reader to [57], [58], and [56] for Verity's work, and to [24] for a discussion of its relationship to the theory of $(\infty, 2)$-categories presented here. `01W5`

5.4.1 Definitions

We begin by introducing some terminology. `01W6`

Definition 5.4.1.1. Let $\mathcal{C}$ be a simplicial set. We will say that $\mathcal{C}$ is an *$(\infty, 2)$-category* if it satisfies the following axioms: `01W9`

(1) Every morphism of simplicial sets $\Lambda^2_1 \to \mathcal{C}$ can be extended to a thin 2-simplex of $\mathcal{C}$.

(2) Every degenerate 2-simplex of $\mathcal{C}$ is thin.

(3) Let $n \geq 3$ and let $\sigma_0 : \Lambda^n_0 \to \mathcal{C}$ be a morphism of simplicial sets with the property that the 2-simplex $\sigma_0|_{\mathrm{N}_{\bullet}(\{0<1<n\})}$ is left-degenerate (see Example 1.1.2.8) Then σ_0 can be extended to an n-simplex of $\mathcal{C}$.

(4) Let $n \geq 3$ and let $\sigma_0 : \Lambda^n_n \to \mathcal{C}$ be a morphism of simplicial sets with the property that the 2-simplex $\sigma_0|_{\mathrm{N}_{\bullet}(\{0<n-1<n\})}$ is right-degenerate. Then σ_0 can be extended to an n-simplex of $\mathcal{C}$.

Proposition 5.4.1.2. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is an $(\infty, 2)$-category.* `01WA`

Proof. Our assumption that $\mathcal{C}$ is an ∞-category guarantees that every 2-simplex of $\mathcal{C}$ is thin (Example 2.3.2.4). Consequently, condition (2) of Definition 5.4.1.1 is automatic, and condition (1) follows immediately from the definition. Conditions (3) and (4) follow from Theorem 4.4.2.6 (since every degenerate edge of $\mathcal{C}$ is an isomorphism). □

Remark 5.4.1.3. Let $\mathcal{C}$ be an $(\infty, 2)$-category. We will refer to vertices of $\mathcal{C}$ as *objects*, and to the edges of $\mathcal{C}$ as *morphisms*. If f is an edge of $\mathcal{C}$ satisfying $d^1_1(f) = X$ and $d^1_0(f) = Y$, then we say that f is a *morphism from X to Y* and write $f : X \to Y$. `01WB`

Suppose we are given morphisms $f : X \to Y$, $g : Y \to Z$, and $h : X \to Z$ of $\mathcal{C}$. We will say that a 2-simplex σ *witnesses h as a composition of f and g* if it is thin and satisfies $d^2_0(\sigma) = g$, $d^2_1(\sigma) = h$, and $d^2_2(\sigma) = f$, as indicated in the diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z.
\end{tikzcd}$$

Note that:

- When $\mathcal{C}$ is an ∞-category, this recovers the terminology of Definition 1.4.4.1 (since the 2-simplex σ is automatically thin).

- If $\mathcal{C}$ is the Duskin nerve of a 2-category $\mathcal{E}$, the 2-simplex σ can be identified with a 2-morphism $\gamma : g \circ f \Rightarrow h$ of $\mathcal{E}$, which is invertible if and only if σ is thin. In other words, σ witnesses h as a composition of f and g if and only if it encodes the datum of an isomorphism $g \circ f \overset{\sim}{\Longrightarrow} h$ in the category $\underline{\mathrm{Hom}}_{\mathcal{E}}(X, Z)$.

- Axiom (1) of Definition 5.4.1.1 asserts that the composition of 1-morphisms in $\mathcal{C}$ is defined (albeit not uniquely). More precisely, it asserts that for every pair of morphisms $f : X \to Y$ and $g : Y \to Z$, there exists a morphism $h : X \to Z$ and a 2-simplex which witnesses h as a composition of f and g.

01WC **Remark 5.4.1.4.** Let $\mathcal{C}$ be a simplicial set. Then $\mathcal{C}$ is an $(\infty, 2)$-category if and only if the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is an $(\infty, 2)$-category.

01WD **Proposition 5.4.1.5.** *Let $\mathcal{C}$ be a 2-category. Then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is an $(\infty, 2)$-category.*

Proof. Condition (1) of Definition 5.4.1.1 follows immediately from Theorem 2.3.2.5, and condition (2) from Corollary 2.3.2.7. We will verify (4); the proof of (3) is similar. Suppose we are given an integer $n \geq 3$ and a map $\sigma_0 : \Lambda^n_n \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$. for which the restriction $\sigma_0|_{\mathrm{N}_{\bullet}(\{0 < n-1 < n\})}$ is right-degenerate. We wish to show that σ_0 can be extended to an n-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$. We now consider three cases:

- Suppose that $n = 3$. Then σ_0 can be identified with a collection of objects $\{X_i\}_{0 \leq i \leq 3}$, 1-morphisms $\{f_{ji} : X_i \to X_j\}_{0 \leq i < j \leq 3}$, and 2-morphisms

$$\mu_{321} : f_{32} \circ f_{21} \Rightarrow f_{31} \qquad \mu_{320} : f_{32} \circ f_{20} \Rightarrow f_{30} \qquad \mu_{310} : f_{31} \circ f_{10} \Rightarrow f_{30}$$

in the 2-category $\mathcal{C}$. The assumption that $\sigma_0|_{\mathrm{N}_{\bullet}(\{0 < n-1 < n\})}$ is right-degenerate guarantees that $X_2 = X_3$, that $f_{20} = f_{30}$, that the 1-morphism f_{32} is the identity id_{X_2}, and that μ_{320} is the left unit constraint $\lambda_{f_{20}}$. To extend σ_0 to a 3-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}$, we must show that there exists a 2-morphism $\mu_{210} : f_{21} \circ f_{10} \Rightarrow f_{20}$ for which the diagram

$$\begin{tikzcd}
f_{32} \circ (f_{21} \circ f_{10}) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[d, Rightarrow, "\mathrm{id}_{f_{32}} \circ \mu_{210}"'] & & (f_{32} \circ f_{21}) \circ f_{10} \arrow[d, Rightarrow, "\mu_{321} \circ \mathrm{id}_{f_{10}}"] \\
f_{32} \circ f_{20} \arrow[dr, Rightarrow, "\mu_{320}"'] & & f_{31} \circ f_{10} \arrow[dl, Rightarrow, "\mu_{310}"] \\
& f_{30} &
\end{tikzcd} \tag{5.22}$$

01WE

is commutative, where $\alpha = \alpha_{f_{32}, f_{21}, f_{10}}$ is the associativity constraint for the composition of 1-morphisms in $\mathcal{C}$ (Proposition 2.3.1.9). This commutativity can be rewritten as an equation

$$\mu_{320}(\mathrm{id}_{f_{32}} \circ \mu_{210}) = \mu_{310}(\mu_{321} \circ \mathrm{id}_{f_{10}})\alpha.$$

This equation has a unique solution, because μ_{320} is invertible and horizontal composition with $\mathrm{id}_{f_{32}}$ induces an equivalence of categories $\underline{\mathrm{Hom}}_{\mathcal{C}}(X_0, X_2) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X_0, X_3)$.

- Suppose that $n = 4$. The restriction of σ_0 to the 2-skeleton of Δ^4 can be identified with a collection of objects $\{X_i\}_{0 \leq i \leq 4}$, 1-morphisms $\{f_{ji} : X_i \to X_j\}_{0 \leq i < j \leq 4}$, and 2-morphisms $\{\mu_{kji} : f_{kj} \circ f_{ji} \Rightarrow f_{ki}\}_{0 \leq i < j < k \leq 4}$ in the 2-category $\mathcal{C}$. The assumption that $\sigma_0|_{\mathrm{N}_{\bullet}(\{0 < n-1 < n\})}$ is right-degenerate guarantees that $X_3 = X_4$, that $f_{30} = f_{40}$, that the 1-morphism f_{43} is the identity id_{X_3}, and that μ_{430} is the left unit constraint

$\lambda_{f_{30}}$. Consider the diagram

$$\begin{tikzcd}
f_{43}(f_{31}f_{10}) \arrow[rrrrrr, Rightarrow, "\sim"] \arrow[ddddd, Rightarrow, "\mu_{310}"'] & & & & & & (f_{43}f_{31})f_{10} \arrow[ddddd, Rightarrow, "\mu_{431}"] \\
& f_{43}((f_{32}f_{21})f_{10}) \arrow[ul, Rightarrow, "\mu_{321}"'] \arrow[rrrr, Rightarrow, "\sim"] & & & & (f_{43}(f_{32}f_{21}))f_{10} \arrow[ur, Rightarrow, "\mu_{321}"] \arrow[d, Rightarrow, "\sim"] & \\
& f_{43}(f_{32}(f_{21}f_{10})) \arrow[u, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\mu_{210}"] \arrow[drr, Rightarrow, "\sim"] & & & & ((f_{43}f_{32})f_{21})f_{10} \arrow[d, Rightarrow, "\mu_{432}"] & \\
& f_{43}(f_{32}f_{20}) \arrow[d, Rightarrow, "\sim"] \arrow[ddl, Rightarrow, "\mu_{320}"'] & & (f_{43}f_{32})(f_{21}f_{10}) \arrow[urr, Rightarrow, "\sim"] \arrow[dll, Rightarrow, "\mu_{210}"'] \arrow[drr, Rightarrow, "\mu_{432}"] & & (f_{42}f_{21})f_{10} \arrow[d, Rightarrow, "\sim"'] \arrow[ddr, Rightarrow, "\mu_{421}"] & \\
& (f_{43}f_{32})f_{20} \arrow[rr, Rightarrow, "\mu_{432}"'] & & f_{42}f_{20} \arrow[d, Rightarrow, "\mu_{420}"] & & f_{42}(f_{21}f_{10}) \arrow[ll, Rightarrow, "\mu_{210}"] & \\
f_{43}f_{30} \arrow[rrr, Rightarrow, "\mu_{430}", "\sim"'] & & & f_{04} & & & f_{41}f_{10}; \arrow[lll, Rightarrow, "\mu_{410}"']
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X_0, X_4)$, where the unlabeled 2-morphisms are given by the associativity constraints. Note that the 4-cycles in this diagram commute by functoriality, and the central 5-cycle commutes by the pentagon identity of $\mathcal{C}$. Our assumption that σ_0 is defined on the horn Λ^4_4 guarantees that pentagonal cycles on the right and bottom of the diagram are commutative and that the outer cycle commutes. Since the 2-morphism μ_{430} is invertible, a diagram chase shows that the pentagonal cycle on the left of the diagram also commutes. Since f_{43} is an identity 1-morphism, horizontal composition with f_{43} is isomorphic to the identity (via the left unit constraint of Construction 2.2.1.11) and is therefore faithful. It follows that the diagram (5.22) is commutative, so that σ_0 extends (uniquely) to a 4-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$.

- If $n \geq 5$, then the horn Λ^n_n contains the 3-skeleton of Δ^n. In this case, the morphism $\sigma_0 : \Lambda^n_n \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ extends uniquely to an n-simplex of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ by virtue of Corollary 2.3.1.10.

□

5.4.2 Interior Fibrations

01WF Recall that a morphism of simplicial sets $q : \mathcal{C} \to \mathcal{D}$ is an *inner fibration* if it is weakly right orthogonal to the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for every pair of integers $0 < i < n$. In the setting of $(\infty,2)$-categories, it will be convenient to consider a variant of this condition.

Definition 5.4.2.1. Let $\mathcal{D}$ be an $(\infty, 2)$-category and let $q : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. We will say that q is an *interior fibration* if it satisfies the following conditions: 01WG

- Every lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

  admits a solution, provided that $0 < i < n$ and the restriction $\overline{\sigma}|_{\mathrm{N}_\bullet(\{i-1<i<i+1\})}$ is a thin 2-simplex of $\mathcal{D}$.

- For every vertex $X \in \mathcal{C}$, the degenerate edge id_X is q-cartesian and q-cocartesian.

Example 5.4.2.2. Let $\mathcal{D}$ be an ∞-category and let $q : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. The following conditions are equivalent: 01WH

(1) The morphism q is an interior fibration (in the sense of Definition 5.4.2.1).

(2) The morphism q is an inner fibration (in the sense of Definition 4.1.1.1).

The implication (1) ⇒ (2) follows from the observation that every 2-simplex of $\mathcal{D}$ is thin, and the implication (2) ⇒ (1) follows from Corollary 5.1.1.9. In particular, if either of these conditions is satisfied, then $\mathcal{C}$ is an ∞-category.

Remark 5.4.2.3. Let $\mathcal{D}$ be an $(\infty, 2)$-category and let $q : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then q is an interior fibration if and only if the opposite morphism $q^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is an interior fibration. 01WJ

Remark 5.4.2.4. Suppose we are given a pullback diagram of simplicial sets 01WK

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "q'"] & \mathcal{C} \arrow[d, "q"] \\
\mathcal{D}' \arrow[r, "F"] & \mathcal{D}.
\end{tikzcd}$$

Assume that $\mathcal{D}$ and $\mathcal{D}'$ are $(\infty, 2)$-categories and that the morphism F carries thin 2-simplices of $\mathcal{D}'$ to thin 2-simplices of $\mathcal{D}$ (that is, that F is a *functor of* $(\infty, 2)$*-categories*; see Definition 5.4.7.1). If q is an interior fibration, then q' is an interior fibration.

`01WL` **Remark 5.4.2.5.** Let $\mathcal{D}$ be an $(\infty, 2)$-category and let $q : \mathcal{C} \to \mathcal{D}$ be an interior fibration. Then, for every object $X \in \mathcal{D}$, the fiber $\mathcal{C}_X = \{X\} \times_{\mathcal{D}} \mathcal{C}$ is an ∞-category (this follows by combining Example 5.4.2.2 with Remark 5.4.2.4).

Our goal in this section is to show that, if $\mathcal{D}$ is an $(\infty, 2)$-category and $q : \mathcal{C} \to \mathcal{D}$ is an interior fibration of simplicial sets, then $\mathcal{C}$ is also an $(\infty, 2)$-category (Proposition 5.4.2.8). To prove this, we must exhibit a sufficiently large collection of thin 2-simplices of $\mathcal{C}$.

`01WM` **Lemma 5.4.2.6.** *Let $\mathcal{D}$ be an $(\infty, 2)$-category, let $q : \mathcal{C} \to \mathcal{D}$ be an interior fibration of simplicial sets, and let σ be a 2-simplex of $\mathcal{C}$. If $q(\sigma)$ is a thin 2-simplex of $\mathcal{D}$, then σ is a thin 2-simplex of $\mathcal{C}$.*

Proof. Suppose we are given a morphism of simplicial sets $\tau_0 : \Lambda^n_i \to \mathcal{C}$, where $n \geq 3$, $0 < i < n$, and τ_0 carries $\mathrm{N}_\bullet(\{i-1 < i < i+1\})$ to the 2-simplex σ. We wish to show that τ_0 can be extended to an n-simplex τ of $\mathcal{C}$. Let $\overline{\tau}_0 : \Lambda^n_i \to \mathcal{D}$ be the composition $q \circ \tau_0$. Since $q(\sigma)$ is a thin 2-simplex of $\mathcal{D}$, we can extend $\overline{\tau}_0$ to an n-simplex $\overline{\tau} : \Delta^n \to \mathcal{D}$. To complete the proof, it suffices to find a solution to the lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\tau_0"] \arrow[d] & \mathcal{C} \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\tau}"] \arrow[ur, dashed, "\tau"] & \mathcal{D},
\end{tikzcd}$$

which exists by virtue of our assumption that q is an interior fibration. □

`01WN` **Remark 5.4.2.7.** In the situation of Lemma 5.4.2.6, we will see later that the converse assertion is also true: if σ is a thin 2-simplex of $\mathcal{C}$, then $q(\sigma)$ is a thin 2-simplex of $\mathcal{D}$ (Proposition 5.4.7.10).

`01WP` **Proposition 5.4.2.8.** *Let $\mathcal{D}$ be an $(\infty, 2)$-category and let $q : \mathcal{C} \to \mathcal{D}$ be an interior fibration of simplicial sets. Then $\mathcal{C}$ is also an $(\infty, 2)$-category.*

Proof. We must verify that the simplicial set $\mathcal{C}$ satisfies each of the axioms of Definition 5.4.1.1:

(1) Let $f : X \to Y$ and $g : Y \to Z$ be edges of the simplicial set $\mathcal{C}$; we wish to show that there exists a thin 2-simplex $\Delta^2 \to \mathcal{C}$ satisfying $d^2_2(\sigma) = f$ and $d^2_0(\sigma) = g$, as indicated in the diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr] & & Z.
\end{tikzcd}$$

We first invoke our assumption that $\mathcal{D}$ is an $(\infty, 2)$-category to choose a thin 2-simplex $\overline{\sigma}$ of $\mathcal{D}$ satisfying $d^2_2(\overline{\sigma}) = q(f)$ and $d^2_0(\overline{\sigma}) = q(g)$. Since $\overline{\sigma}$ is thin, our assumption that q is an interior fibration guarantees that the lifting problem

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "(g,\bullet,f)"] \arrow[d] & \mathcal{C} \arrow[d, "q"] \\
\Delta^2 \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{D}
\end{tikzcd}$$

admits a solution. It follows from Lemma 5.4.2.6 that σ is a thin 2-simplex of $\mathcal{C}$.

(2) Let σ be a degenerate 2-simplex of $\mathcal{C}$. Then $q(\sigma)$ is a degenerate 2-simplex of $\mathcal{D}$. Since $\mathcal{D}$ is an $(\infty, 2)$-category $q(\sigma)$ is a thin 2-simplex of $\mathcal{D}$. Applying Lemma 5.4.2.6, we conclude that σ is a thin 2-simplex of $\mathcal{C}$.

(3) Let $n \geq 3$ and let $\tau_0 : \Lambda^n_0 \to \mathcal{C}$ be a morphism of simplicial sets with the property that the 2-simplex $\tau_0|_{\mathrm{N}_\bullet(\{0<1<n\})}$ is left-degenerate; we wish to show that τ_0 can be extended to an n-simplex τ of $\mathcal{C}$. Let $\overline{\tau}_0 : \Lambda^n_0 \to \mathcal{D}$ denote the composition $q \circ \tau_0$. Since $\mathcal{D}$ is an $(\infty, 2)$-category, we can extend $\overline{\tau}_0$ to an n-simplex $\overline{\tau} : \Delta^n \to \mathcal{D}$. To complete the proof, it will suffice to show that the lifting problem

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\tau_0"] \arrow[d] & \mathcal{C} \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\tau}"] \arrow[ur, dashed, "\tau"] & \mathcal{D}
\end{tikzcd}$$

admits a solution. We conclude by observing that the edge $\tau_0|_{\mathrm{N}_\bullet(\{0<1\})}$ is degenerate and is therefore q-cocartesian by virtue of our assumption that q is an interior fibration.

(4) Let $n \geq 3$ and let $\tau_0 : \Lambda^n_n \to \mathcal{C}$ be a morphism of simplicial sets with the property that the 2-simplex $\tau_0|_{\mathrm{N}_\bullet(\{0<n-1<n\})}$ is right-degenerate; we wish to show that τ_0 can be extended to an n-simplex τ of $\mathcal{C}$. This follows by the argument given above, applied to the opposite interior fibration $q^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

□

Proposition 5.4.2.9. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be interior fibrations of $(\infty, 2)$-categories. Then the composition $(G \circ F) : \mathcal{C} \to \mathcal{E}$ is also an interior fibration.* 01WQ

Proof. Suppose we are given an integer $n \geq 2$ and a lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "G \circ F"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{E} .
\end{tikzcd}$$

We wish to show that this lifting problem admits a solution if one of the following conditions is satisfied:

(a) The integer i is equal to 0 and $\sigma_0|_{\mathrm{N}_\bullet(\{0<1\})}$ is a degenerate edge of $\mathcal{C}$.

(b) The integer i satisfies $0 < i < n$ and the restriction $\overline{\sigma}|_{\mathrm{N}_\bullet(\{i-1<i<i+1\})}$ is a thin 2-simplex of $\mathcal{E}$.

(c) The integer i is equal to n and $\sigma_0|_{\mathrm{N}_\bullet(\{n-1<n\})}$ is a degenerate edge of $\mathcal{C}$.

Since G is an interior fibration, any of these hypotheses guarantee the existence of a solution to the associated lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "F \circ \sigma_0"] \arrow[d] & \mathcal{D} \arrow[d, "G"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\tau"] & \mathcal{E} .
\end{tikzcd}$$

It will therefore suffice to construct a solution to the lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^n \arrow[r, "\tau"] \arrow[ur, dashed, "\sigma"] & \mathcal{D} .
\end{tikzcd}$$

In cases (a) and (c), our assumption that F is an interior fibration immediately guarantees the existence of σ. In case (b), it suffices to verify that the restriction $\tau|_{\mathrm{N}_\bullet(\{i-1<i<i+1\})}$ is a thin 2-simplex of $\mathcal{D}$, which follows from Lemma 5.4.2.6. □

01WR **Proposition 5.4.2.10.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an interior fibration between $(\infty, 2)$-categories, and let X and Y be objects of $\mathcal{C}$. Then:*

(1) *The induced map of left-pinched morphism spaces* $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}^{\mathrm{L}}_{\mathcal{D}}(F(X),F(Y))$ *is a right fibration of simplicial sets.*

(2) *The induced map of right-pinched morphism spaces* $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}^{\mathrm{R}}_{\mathcal{D}}(F(X),F(Y))$ *is a left fibration of simplicial sets.*

Proof. We will prove (2); assertion (1) follows from a similar argument. We wish to show that, for every pair of integers $0 \leq i < n$, every lifting problem

01WS

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X,Y) \arrow[d] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathrm{Hom}^{\mathrm{R}}_{\mathcal{D}}(F(X),F(Y))
\end{tikzcd} \tag{5.23}$$

admits a solution. Unwinding the definitions, we can rewrite (5.23) as a lifting problem

$$\begin{tikzcd}
\Lambda^{n+1}_i \arrow[r, "\tau_0"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^{n+1} \arrow[r, "\overline{\tau}"] \arrow[ur, dashed, "\tau"] & \mathcal{D},
\end{tikzcd}$$

where the restriction $\tau_0|_{\mathrm{N}_\bullet(\{0<1<\cdots<n\})}$ is the constant map taking the value X. If $i = 0$, then this lifting problem admits a solution because the edge $\tau_0|_{\mathrm{N}_\bullet(\{0<1\})}$ is degenerate (and therefore F-cocartesian, by virtue of our assumption that F is an interior fibration). If $0 < i < n$, the solution exists by virtue of the fact that F is an interior fibration and $\overline{\tau}|_{\mathrm{N}_\bullet(\{i-1<i<i+1\})}$ is a degenerate 2-simplex of $\mathcal{D}$ (and therefore thin). □

5.4.3 Slices of $(\infty,2)$-Categories

The slice and coslice constructions of §4.3 provide many examples of interior fibrations of $(\infty,2)$-categories. 01WT

Proposition 5.4.3.1. *Let $\mathcal{C}$ be an $(\infty,2)$-category and let $f : K \to \mathcal{C}$ be a morphism of simplicial sets. Then the projection maps* 01WU

$$\mathcal{C}_{f/} \to \mathcal{C} \qquad \mathcal{C}_{/f} \to \mathcal{C}$$

are interior fibrations.

`01WV` **Warning 5.4.3.2.** In the situation of Proposition 5.4.3.1, the projection maps

$$\mathcal{C}_{f/} \to \mathcal{C} \qquad \mathcal{C}_{/f} \to \mathcal{C}$$

are generally not inner fibrations of simplicial sets.

`01WW` **Remark 5.4.3.3.** Let $\mathcal{C}$ be a simplicial set. Then axioms (3) and (4) of Definition 5.4.1.1 can be stated as follows:

($3'$) Let X be any vertex of $\mathcal{C}$ and let $q : \mathcal{C}_{/X} \to \mathcal{C}$ be the projection map. Then every degenerate edge of $\mathcal{C}_{/X}$ is q-cocartesian.

($4'$) Let X be any vertex of $\mathcal{C}$ and let $q' : \mathcal{C}_{X/} \to \mathcal{C}$ be the projection map. Then every degenerate edge of $\mathcal{C}_{X/}$ is q'-cartesian.

Note that ($3'$) and ($4'$) appear as special cases of the conclusion of Proposition 5.4.3.1.

`01WX` **Corollary 5.4.3.4.** *Let $\mathcal{C}$ be an $(\infty, 2)$-category and let $f : K \to \mathcal{C}$ be a morphism of simplicial sets. Then the simplicial sets $\mathcal{C}_{f/}$ and $\mathcal{C}_{/f}$ are $(\infty, 2)$-categories.*

Proof. Combine Proposition 5.4.3.1 with Proposition 5.4.2.8. □

`01WY` **Corollary 5.4.3.5.** *Let $\mathcal{C}$ be an $(\infty, 2)$-category. For every pair of objects X and Y, the pinched morphism spaces $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ of Construction 4.6.5.1 are ∞-categories.*

Proof. By definition, the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ is the fiber over Y of the projection map $\pi : \mathcal{C}_{X/} \to \mathcal{C}$. Since π is an interior fibration (Proposition 5.4.3.1), each of its fibers is an ∞-category (Remark 5.4.2.5). A similar argument shows that $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ is an ∞-category. □

`01WZ` **Warning 5.4.3.6.** Let $\mathcal{C}$ be an $(\infty, 2)$-category containing objects X and Y. Then the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of Construction 4.6.1.1 is generally not an ∞-category (see Warning 8.1.8.1).

`01X0` **Remark 5.4.3.7.** Let $\mathcal{C}$ be an $(\infty, 2)$-category containing X and Y. We will see later that the ∞-category $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ is naturally equivalent to the *opposite* of the ∞-category $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ (Proposition [?]). When $\mathcal{C}$ is the Duskin nerve of a 2-category, we can do better: the ∞-category $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)$ is *isomorphic* to the opposite of $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$; see Example 4.6.5.13.

We will deduce Proposition 5.4.3.1 from the following more precise result:

Proposition 5.4.3.8. *Let $f : K \to \mathcal{C}$ be a morphism of simplicial sets and let $f_0 : K_0 \to \mathcal{C}$ be the restriction of f to a simplicial subset $K_0 \subseteq K$. Then every lifting problem* 01X1

$$\begin{tikzcd}
\Lambda^m_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C}_{/f} \arrow[d] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & \mathcal{C}_{/f_0}
\end{tikzcd} \tag{5.24}$$

01X2

admits a solution provided that $m \geq 2$ and one of the following additional conditions is satisfied:

(*a*) *The simplicial set $\mathcal{C}$ is an $(\infty, 2)$-category, $i = 0$, and the composition*

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{0 < 1\}) \subseteq \Lambda^m_i \xrightarrow{\sigma_0} \mathcal{C}_{/f}$$

is a degenerate edge of $\mathcal{C}_{/f}$.

(*b*) *The integer i satisfies $0 < i < m$ and the composite map*

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{i-1 < i < i+1\}) \subseteq \Delta^m \xrightarrow{\overline{\sigma}} \mathcal{C}_{/f_0} \to \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

(*c*) *The integer i is equal to m and, for every vertex $x \in K$, the composite map*

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{m-1 < m\}) \star \{x\} \hookrightarrow \Lambda^m_i \star K \xrightarrow{\sigma} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

Proof. Unwinding the definitions, we can identify the diagram (5.24) with a morphism of simplicial sets

$$\overline{f} : (\Lambda^m_i \star K) \coprod_{(\Lambda^m_i \star K_0)} (\Delta^m \star K_0) \to \mathcal{C},$$

and we wish to show that $\overline{f}$ can be extended to a morphism $\Delta^m \star K \to \mathcal{C}$. Let P be the collection of all pairs (L, g), where L is a simplicial subset of K containing K_0 and $g : \Delta^m \star L \to \mathcal{C}$ is a morphism satisfying

$$g|_{\Delta^m \star K_0} = \overline{f}|_{\Delta^m \star K_0} \qquad g|_{\Lambda^m_i \star L} = \overline{f}|_{\Lambda^m_i \star L}.$$

We regard P as a partially ordered set, with $(L, g) \leq (L', g')$ if L is contained in L' and $g = g'|_{\Delta^m \star L}$. The partially ordered set P satisfies the hypotheses of Zorn's lemma and therefore admits a maximal element $(L_{\max}, g_{\max})$. We will complete the proof by showing that $L_{\mathrm{max}} = K$ (so that g_{max} is the desired extension of $\overline{f}$). Suppose otherwise. Then there is some nondegenerate simplex $\rho : \Delta^n \to K$ which is not contained in L_{max}. Choosing ρ so that n is as small as possible, we may assume without loss of generality that ρ carries the boundary $\partial \Delta^n$ into L_{max}. Let $L' \subseteq K$ be the simplicial subset given by the union of L_{max} together with the image of ρ, so that ρ determines a pushout diagram

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & L_{\mathrm{max}} \arrow[d] \\
\Delta^n \arrow[r] & L'.
\end{tikzcd}$$

We will show that g_{max} can be extended to a morphism of simplicial sets $g' : \Delta^m \star L' \to \mathcal{C}$ satisfying $g'|_{\Lambda^m_i \star L'} = \overline{f}|_{\Lambda^m_i \star L'}$; thereby contradicting the maximality of $(L_{\mathrm{max}}, g_{\mathrm{max}})$ and completing the proof of Proposition 5.4.3.8. Note that the composite maps

$$\Lambda^m_i \star \Delta^n \xrightarrow{\mathrm{id} \star \rho} \Lambda^m_i \star K \xrightarrow{\overline{f}} \mathcal{C}$$

$$\Delta^m \star \partial \Delta^n \xrightarrow{\mathrm{id} \star \rho} \Delta^m \star L_{\mathrm{max}} \xrightarrow{g_{\mathrm{max}}} \mathcal{C}$$

can be amalgamated to a morphism of simplicial sets

$$\tau_0 : (\Lambda^m_i \star \Delta^n) \coprod_{(\Lambda^m_i \star \partial \Delta^n)} (\Delta^m \star \partial \Delta^n) \to \mathcal{C},$$

whose source can be identified with the horn $\Lambda^{m+1+n}_i \subseteq \Delta^{m+1+n}$ (Lemma 4.3.6.15). We wish to show that τ_0 can be extended to a map

$$\tau : \Delta^m \star \Delta^n \simeq \Delta^{m+1+n} \to \mathcal{C}.$$

If $0 < i \leq m$, the desired extension exists because the composite map

$$\Delta^2 \simeq \mathrm{N}_{\bullet}(\{i-1 < i < i+1\}) \subseteq \Lambda^{m+1+n}_i \xrightarrow{\tau_0} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$ (by virtue of assumption (b) when $i < m$ or (c) in the case $i = m$). If $i = 0$, then the desired extension exists because assumption (a) guarantees that $\mathcal{C}$ is an $(\infty, 2)$-category and the 2-simplex

$$\Delta^2 \simeq \mathrm{N}_{\bullet}(\{0 < 1 < m+1+n\}) \subseteq \Lambda^{m+1+n}_i \xrightarrow{\tau_0} \mathcal{C}$$

is left-degenerate. □

Proof of Proposition 5.4.3.1. Let $\mathcal{C}$ be an $(\infty,2)$-category and let $f : K \to \mathcal{C}$ be a morphism of simplicial sets. We will show that the projection map $q : \mathcal{C}_{/f} \to \mathcal{C}$ is an interior fibration; the analogous assertion for the coslice simplicial set $\mathcal{C}_{f/}$ follows by a similar argument. Let $m \geq 2$ and suppose that we are given a lifting problem

$$\begin{tikzcd}
\Lambda^m_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C}_{/f} \arrow[d] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & \mathcal{C} .
\end{tikzcd}$$

We wish to show that a solution exists under any of the following additional assumptions:

(a) The integer i is equal to zero and the restriction $\sigma_0|_{\mathrm{N}_\bullet(\{0<1\})}$ is a degenerate edge of $\mathcal{C}_{/f}$.

(b) The integer i satisfies $0 < i < m$ and the composite map

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{i-1 < i < i+1\}) \subseteq \Delta^m \xrightarrow{\overline{\sigma}} \mathcal{C}_{/f_0} \to \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

(c) The integer i is equal to m and the restriction $\sigma_0|_{\mathrm{N}_\bullet(\{m-1<m\})}$ is a degenerate edge of $\mathcal{C}_{/f}$.

In cases (a) and (b), this follows immediately from Proposition 5.4.3.8. In case (c), we observe that for every vertex $x \in K$, the composite map

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{m-1 < m\}) \star \{x\} \hookrightarrow \Lambda^m_i \star K \xrightarrow{\sigma_0} \mathcal{C}$$

is a left-degenerate 2-simplex of $\mathcal{C}$. Since $\mathcal{C}$ is an $(\infty,2)$-category, this degenerate 2-simplex is thin, so that existence of the desired extension again follows from Proposition 5.4.3.8. □

In the situation of Proposition 5.4.3.1, the interior fibration $\mathcal{C}_{/f} \to \mathcal{C}$ behaves like a cartesian fibration (with the caveat that it need not be an inner fibration).

Proposition 5.4.3.9. *Let $\mathcal{C}$ be an $(\infty,2)$-category, let $f : K \to \mathcal{C}$ be a morphism of simplicial sets, and let $q : \mathcal{C}_{/f} \to \mathcal{C}$ be the projection map. Let Y be an object of the $(\infty,2)$-category $\mathcal{C}_{/f}$, and let $\overline{u} : \overline{X} \to q(Y)$ be a morphism in the $(\infty,2)$-category $\mathcal{C}$. Then $\overline{u}$ can be lifted to a morphism $u : X \to Y$ of $\mathcal{C}_{/f}$ with the following property:* 01X3

($*$) *For every vertex $z \in K$, the image of u in $\mathcal{C}_{/f(z)}$ is a thin 2-simplex of $\mathcal{C}$.*

`01X4` **Remark 5.4.3.10.** In the situation of Proposition 5.4.3.9, condition $(*)$ guarantees that u is a q-cartesian morphism of $\mathcal{C}_{/f}$ (this follows immediately from Proposition 5.4.3.8). In §5.4.4, we will prove the converse: every q-cartesian morphism of $\mathcal{C}_{/f}$ satisfies condition $(*)$ (Corollary 5.4.4.2).

Proposition 5.4.3.9 is a special case of the following more general assertion:

`01X5` **Proposition 5.4.3.11.** *Let $\mathcal{C}$ be an $(\infty, 2)$-category, let $f : K \to \mathcal{C}$ be a morphism of simplicial sets, and let $f_0 : K_0 \to \mathcal{C}$ be the restriction of f to a simplicial subset $K_0 \subseteq K$. Let $q : \mathcal{C}_{/f} \to \mathcal{C}_{/f_0}$ denote the projection map, and suppose we are given a lifting problem*

`01X6`
$$\begin{tikzcd}
\{1\} \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C}_{/f} \arrow[d, "q"] \\
\Delta^1 \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{C}_{/f_0}
\end{tikzcd} \qquad (5.25)$$

with the following property:

$(*_0)$ *For every vertex $x \in K_0$, the composition*

$$\Delta^2 \simeq \Delta^1 \star \{x\} \hookrightarrow \Delta^1 \star K_0 \xrightarrow{\overline{\sigma}} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

Then there exists an edge $\sigma : \Delta^1 \to \mathcal{C}_{/f}$ which solves the lifting problem problem (5.25) and which satisfies the following stronger version of $(*_0)$:

$(*)$ *For every vertex $x \in K$, the composition*

$$\Delta^2 \simeq \Delta^1 \star \{x\} \hookrightarrow \Delta^1 \star K \xrightarrow{\sigma} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

Proof. Arguing as in the proof of Proposition 5.4.3.8, we can reduce to the case where $K = \Delta^n$ is a standard simplex and $K_0 = \partial \Delta^n$ is its boundary. In this case, the lifting problem (5.25) determines a morphism of simplicial sets

$$\tau_0 : (\{1\} \star \Delta^n) \coprod_{(\{1\} \star \partial \Delta^n)} (\Delta^1 \star \partial \Delta^n) \to \mathcal{C},$$

whose source can be identified with the horn $\Lambda^{n+2}_1 \subseteq \Delta^{n+2}$ (Lemma 4.3.6.15), and we wish to extend τ to an $(n+2)$-simplex of $\mathcal{C}$. If $n > 0$, then the desired extension exists because τ_0 carries $\mathrm{N}_\bullet(\{0 < 1 < 2\})$ to a thin 2-simplex of $\mathcal{C}$ (by virtue of assumption $(*_0)$). If $n = 0$, then our assumption that $\mathcal{C}$ is an $(\infty, 2)$-category allows us to extend τ_0 to a thin 2-simplex of $\mathcal{C}$. $\square$

5.4.4 The Local Thinness Criterion

Let $\mathcal{C}$ be an $(\infty, 2)$-category and let σ be a 2-simplex of $\mathcal{C}$, whose restriction to the 1-skeleton of Δ^2 we indicate in the diagram 01X7

$$\begin{tikzcd} & Y \arrow[dr, "v"] & \\ X \arrow[ur, "u"] \arrow[rr, "w"] & & Z \end{tikzcd}$$

Roughly speaking, we can think of σ as encoding a 2-morphism $\gamma : v \circ u \Rightarrow w$, and we can think of the condition that σ is thin as corresponding to the requirement that γ is invertible. In the case where $\mathcal{C}$ is the Duskin nerve of a 2-category, this is the content of Theorem 2.3.2.5. For a general $(\infty, 2)$-category, we can formulate this heuristic more precisely as follows:

Theorem 5.4.4.1 (Local Thinness Criterion). *Let $\mathcal{C}$ be an $(\infty, 2)$-category and let σ be a 2-simplex of $\mathcal{C}$, which we represent by the diagram* 01X8

$$\begin{tikzcd} & Y \arrow[dr, "v"] & \\ X \arrow[ur, "u"] \arrow[rr, "w"] & & Z. \end{tikzcd}$$

The following conditions are equivalent:

(1) *The 2-simplex σ is thin.*

(2) *Let $q : \mathcal{C}_{/Z} \to \mathcal{C}$ denote the projection map. Then σ is q-cartesian when viewed as an edge of the simplicial set $\mathcal{C}_{/Z}$.*

(3) *The 2-simplex σ is locally q-cartesian when viewed as an edge of the simplicial set $\mathcal{C}_{/Z}$.*

(4) *Let $q' : \mathcal{C}_{X/} \to \mathcal{C}$ denote the projection map. Then σ is q'-cocartesian when viewed as an edge of the simplicial set $\mathcal{C}_{X/}$.*

(5) *The 2-simplex σ is locally q'-cocartesian when viewed as an edge of the simplicial set $\mathcal{C}_{X/}$.*

Proof. We will prove that (1) ⇔ (2) ⇔ (3); the proof that (1) ⇔ (4) ⇔ (5) follows by applying the same argument to the opposite $(\infty, 2)$-category $\mathcal{C}^{\mathrm{op}}$. The implication (1) ⇒ (2) follows from Proposition 5.4.3.8, and the implication (2) ⇒ (3) is immediate (see Remark 5.1.3.3). For each integer $n \geq 3$, consider the following weaker version of condition (1):

(1_n) For every integer $0 < i < n$ and every morphism of simplicial sets $\mu_0 : \Lambda^n_i \to \mathcal{C}$ for which the composition

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{i-1 < i < i+1\}) \hookrightarrow \Lambda^n_i \xrightarrow{\mu_0} \mathcal{C}$$

is equal to σ, there exists a map $\mu : \Delta^n \to \mathcal{C}$ extending μ_0.

Note that σ satisfies condition (1) if and only if it satisfies condition (1_n) for each $n \geq 3$. We will complete the proof by showing that $(3) \Rightarrow (1_n)$, using a fairly elaborate induction on n.

Assume that σ is locally q-cartesian when viewed as a morphism in the $(\infty, 2)$-category $\mathcal{C}_{/Z}$. Since $\mathcal{C}$ is an $(\infty, 2)$-category, we can choose a thin 2-simplex σ' satisfying $d^2_0(\sigma') = d^2_0(\sigma)$ and $d^2_2(\sigma') = d^2_2(\sigma)$, which we represent as a diagram

$$\begin{tikzcd}
& Y \arrow[dr, "v"] & \\
X \arrow[ur, "u"] \arrow[rr, "w'"] & & Z.
\end{tikzcd}$$

The implication $(1) \Rightarrow (3)$ shows that σ' is also locally q-cartesian when viewed as an edge of the simplicial set $\mathcal{C}_{/Z}$. Let us regard the edge u as a morphism of simplicial sets $\Delta^1 \to \mathcal{C}$, and let $\mathcal{E}$ denote the fiber product $\Delta^1 \times_{\mathcal{C}} \mathcal{C}_{/Z}$. Since q is an interior fibration, it follows from Remark 5.4.2.4 and Example 5.4.2.2 that the projection map $\pi : \mathcal{E} \to \Delta^1$ is an inner fibration. Moreover, we can identify σ and σ' with π-cartesian edges of $\mathcal{E}$ having nondegenerate images under π. Applying Remark 5.1.3.8, we see that there exists a 2-simplex of $\mathcal{E}$ which exhibits σ' as a composition of σ with an isomorphism in $\mathcal{E}$. The image of this 2-simplex under the projection map $\mathcal{E} \to \mathcal{C}_{/Z}$ can be identified with a 3-simplex ρ of $\mathcal{C}$ such that $d^3_0(\rho) = \sigma$, $d^3_1(\rho) = \sigma'$, and $d^3_3(\rho) = s^1_0(u)$ is left-degenerate; the restriction of ρ to the 1-skeleton of Δ^3 we can represent by the diagram

$$\begin{tikzcd}
& X \arrow[r, "u"] \arrow[drr, "w"] & Y \arrow[dr, "v"] & \\
X \arrow[ur, "\mathrm{id}_X"] \arrow[urr, "u"] \arrow[rrr, "w'"] & & & Z.
\end{tikzcd}$$

By construction, the remaining face $\sigma'' = d^3_2(\rho)$ is an isomorphism when viewed as a morphism in the ∞-category $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Z) = \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Z}$, and is therefore locally q-cartesian (Example 5.1.3.6). In particular, our inductive hypothesis guarantees that the simplex σ'' satisfies condition (1_m) for $3 \leq m < n$.

Fix a morphism of simplicial sets $\mu_0 : \Lambda^n_i \to \mathcal{C}$ as in condition (1_n); we wish to show that μ_0 can be extended to an n-simplex μ of $\mathcal{C}$. Let $\delta^{i-1}_{n+1} : \Delta^n \hookrightarrow \Delta^{n+1}$ denote the inclusion of the $(i-1)$st face, given on vertices by the formula

$$\delta^{i-1}_{n+1}(j) = \begin{cases} j & \text{if } j < i-1 \\ j+1 & \text{if } j \geq i-1. \end{cases}$$

We will construct an $(n+1)$-simplex $\nu : \Delta^{n+1} \to \mathcal{C}$ which satisfies the following conditions:

(a) The composite map

$$\Lambda^n_i \hookrightarrow \Delta^n \xrightarrow{\delta^{i-1}_{n+1}} \Delta^{n+1} \xrightarrow{\nu} \mathcal{C}$$

is equal to μ_0.

(b) The composite map

$$\Delta^3 \simeq \mathrm{N}_\bullet(\{i-1 < i < i+1 < i+2\}) \hookrightarrow \Delta^{n+1} \xrightarrow{\nu} \mathcal{C}$$

is equal to the 3-simplex ρ.

(c) For every integer $0 \leq j < i-1$, the 2-simplex

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{j < i-1 < i\}) \hookrightarrow \Delta^{n+1} \xrightarrow{\nu} \mathcal{C}$$

is right-degenerate (in particular, it is thin).

(d) For every integer $i+2 < j \leq n+1$, the 2-simplex

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{i-1 < i < j\}) \hookrightarrow \Delta^{n+1} \xrightarrow{\nu} \mathcal{C}$$

is left-degenerate (in particular, it is thin).

Assuming that this construction is possible, we complete the proof by observing that $\mu = \nu \circ \delta^{i-1}_{n+1}$ provides the desired extension of μ_0 (by virtue of assumption (a)).

The construction of the $(n+1)$-simplex ν will take place in several steps. We define simplicial subsets

$$K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq K_3 \subsetneq K_4 \subsetneq \Delta^{n+1}$$

and maps $\nu_j : K_j \to \mathcal{C}$ as follows:

- Let $K_0 \subseteq \Delta^{n+1}$ be the image of the horn Λ^n_i under δ^{i-1}_{n+1}, so that δ^{i-1}_{n+1} induces an isomorphism $\Lambda^n_i \xrightarrow{\sim} K_0$. It follows that there is a unique morphism of simplicial sets $\nu_0 : K_0 \to \mathcal{C}$ satisfying $\mu_0 = \nu_0 \circ \delta^{i-1}_{n+1}|_{\Lambda^n_i}$. By construction, the map ν_0 satisfies condition (a).

- Let $K_1 \subseteq \Delta^{n+1}$ be the union of K_0 with the 3-simplex $\mathrm{N}_\bullet(\{i-1 < i < i+1 < i+2\})$. It follows from the identity $d^3_0(\rho) = \sigma$ that ν_0 extends uniquely to a map $\nu_1 : K_1 \to \mathcal{C}$ satisfying condition (b).

- Let K_2 be the simplicial subset of Δ^{n+1} obtained by removing those nondegenerate simplices which contain *all* of the vertices $\{0 < 1 < \cdots < i-2 < i+2 < i+3 < \cdots < n+1\}$ and *at least one* of the vertices $\{i-1, i\}$. We will prove below that ν_1 can be extended to a map $\nu_2 : K_2 \to \mathcal{C}$ which satisfies conditions (c) and (d).

- Let $\delta^i_{n+1} : \Delta^n \hookrightarrow \Delta^{n+1}$ denote the inclusion of the ith face, given on vertices by the formula
$$\delta^i_{n+1}(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$
Let K_3 be the union of K_2 with the image of δ^i_{n+1}. Note that δ^i_{n+1} determines a pushout diagram of simplicial sets
$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K_2 \arrow[d] \\ \Delta^n \arrow[r, "\delta^i_{n+1}"] & K_3. \end{tikzcd}$$
Let α_0 denote the composite map $\Lambda^n_i \xrightarrow{\delta^i_{n+1}} K_2 \xrightarrow{\nu_2} \mathcal{C}$. Since ν_1 satisfies condition (b), α_0 carries $\mathrm{N}_\bullet(\{i-1 < i < i+1\})$ to the thin 2-simplex σ' of $\mathcal{C}$, and can therefore be extended to an n-simplex α of $\mathcal{C}$. It follows that ν_2 extends uniquely to a morphism of simplicial sets $\nu_3 : K_3 \to \mathcal{C}$ satisfying $\nu_3 \circ \delta^i_{n+1} = \alpha$.

- Let K_4 denote the horn $\Lambda^{n+1}_{i-1} \subsetneq \Delta^n$. Note that K_4 can be written as the union of K_3 with the image of the face inclusion $\delta^{i+1}_{n+1} : \Delta^n \hookrightarrow \Delta^{n+1}$, given on vertices by the formula
$$\delta^{i+1}_{n+1}(j) = \begin{cases} j & \text{if } j \leq i \\ j+1 & \text{if } j > i. \end{cases}$$
Moreover, we have a pushout diagram
$$\begin{tikzcd} \Lambda^n_{i-1} \arrow[r] \arrow[d] & K_3 \arrow[d] \\ \Delta^n \arrow[r, "\delta^{i+1}_{n+1}"] & K_4. \end{tikzcd}$$

Let β_0 denote the composite map

$$\Lambda^n_{i-1} \xrightarrow{\delta^{i+1}_{n+1}} K_3 \xrightarrow{\nu_3} \mathcal{C}\,.$$

If $i > 1$, then condition (c) guarantees that the restriction $\beta_0|_{\mathrm{N}_\bullet(\{i-2<i-1<i\})}$ is a right-degenerate 2-simplex of $\mathcal{C}$. If $i = 1$, then condition (d) guarantees that the restriction $\beta_0|_{\mathrm{N}_\bullet(\{0<1<n\})}$ is a left-degenerate 2-simplex of $\mathcal{C}$. In either case, our assumption that $\mathcal{C}$ is an $(\infty, 2)$-category guarantees that β_0 can be extended to an n-simplex β of $\mathcal{C}$, so that ν_3 can be extended uniquely to a map $\nu_4 : K_4 \to \mathcal{C}$ satisfying $\nu_4 \circ \delta^{i+1}_{n+1} = \beta$.

- If $i > 1$, then condition (c) guarantees that the map $\nu_4 : \Lambda^{n+1}_{i-1} \to \mathcal{C}$ carries $\mathrm{N}_\bullet(\{i-2 < i-1 < i\})$ to a right-degenerate 2-simplex of $\mathcal{C}$. If $i = 1$, then condition (d) guarantees that ν_4 carries $\mathrm{N}_\bullet(\{0 < 1 < n+1\})$ to a left-degenerate 2-simplex of $\mathcal{C}$. In either case, our assumption that $\mathcal{C}$ is an $(\infty, 2)$-category guarantees that we can extend ν_4 to an $(n+1)$-simplex $\nu : \Delta^{n+1} \to \mathcal{C}$, thereby completing the proof of Theorem 5.4.4.1.

It remains to show that ν_1 admits an extension $\nu_2 : K_2 \to \mathcal{C}$ which satisfies conditions (c) and (d). Let us say that a simplex $\tau : \Delta^m \to K_2$ is *free* if it is nondegenerate, not contained in K_1, and there exists an integer $0 \le j \le m$ satisfying $\tau(j) = i$. Note that in this case, we automatically have $j > 0$ and $\tau(j-1) = i-1$ (otherwise, τ would be contained in K_1). Moreover, if τ is any nondegenerate m-simplex of K_2 which is not contained in K_1, then τ is either free or can be realized uniquely as a face of a free $(m+1)$-simplex $\tau' : \Delta^{m+1} \to K_2$ (obtained by adjoining i to the image of τ).

Let $\{\tau_1, \tau_2, \cdots, \tau_t\}$ be an enumeration of the collection of all free simplices of K_2, chosen so $\dim(\tau_1) \le \dim(\tau_2) \le \cdots \le \dim(\tau_t)$. For $0 \le s \le t$, let $K_2(s)$ denote the union of K_1 with the images of the maps $\{\tau_1, \tau_2, \cdots, \tau_s\}$, so that we have inclusions of simplicial sets

$$K_1 = K_2(0) \subset K_2(1) \subset K_2(2) \subset \cdots \subset K_2(t) = K_2.$$

We will complete the proof by inductively constructing a compatible sequence of maps $\nu_2(s) : K_2(s) \to \mathcal{C}$ satisfying $\nu_2(0) = \nu_1$ together with the following translation of conditions (c) and (d):

$(*_s)$ If the simplex τ_s has dimension 2, then the 2-simplex $\nu_2 \circ \tau_s$ of $\mathcal{C}$ is left-degenerate if $\tau_s(1) = i$ and right-degenerate if $\tau_s(2) = i$.

Assume that $s > 0$ and that the map $\nu_2(s-1)$ has already been constructed. Set $\tau = \tau_s : \Delta^m \to K_2$, so that there is a unique integer $1 \le j \le m$ satisfying $\tau(j) = i$. Note that for $0 \le k \le m$ with $k \ne j$, the face $d^m_k(\tau)$ is either free or belongs to K_1; in either case, it belongs to $K_2(s-1)$. Moreover, the face $d^m_j(\tau)$ is neither free, nor contained in K_1, nor contained as a face of any other free m-simplex of K_2. It follows that τ determines a pushout diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^m_j \arrow[r] \arrow[d] & K_2(s-1) \arrow[d] \\
\Delta^m \arrow[r, "\tau"] & K_2(s).
\end{tikzcd}$$

Let $\xi_0 : \Lambda^m_j \to \mathcal{C}$ denote the composite map $\Lambda^m_j \xrightarrow{\tau} K_2(s-1) \xrightarrow{\nu_2(s-1)} \mathcal{C}$; we wish to show that ξ_0 can be extended to an m-simplex of $\mathcal{C}$. If $m = 2$, then there is a unique such extension which satisfies condition $(*_s)$ (since, by construction, the morphism ν_1 carries $\mathrm{N}_\bullet(\{i-1 < i\})$ to the degenerate edge id_X of $\mathcal{C}$). We may therefore assume that $m \geq 3$. We consider several cases:

- If $j = m$, then it follows from assumption $(*_{s'})$ for $s' < s$ that ξ_0 carries $\mathrm{N}_\bullet(\{0 < m-1 < m\})$ to a right-degenerate 2-simplex of $\mathcal{C}$, so the desired extension exists by virtue of our assumption that $\mathcal{C}$ is an $(\infty, 2)$-category.

- If $j < m$ and $\tau(j+1) = i+1$, then it follows from (b) that ξ_0 carries $\mathrm{N}_\bullet(\{j-1 < j < j+1\})$ to the left-degenerate 2-simplex $d^3_3(\rho)$. Since $\mathcal{C}$ is an $(\infty, 2)$-category, this 2-simplex is thin so that ξ_0 can be extended to an m-simplex of $\mathcal{C}$.

- If $j < m$ and $\tau(j+1) > i+2$, then it follows from assumption $(*_{s'})$ for $s' < s$ that ξ_0 carries $\mathrm{N}_\bullet(\{j-1 < j < j+1\})$ to a left-degenerate 2-simplex of $\mathcal{C}$. Since $\mathcal{C}$ is an $(\infty, 2)$-category, this 2-simplex is thin so that ξ_0 can be extended to an m-simplex of $\mathcal{C}$.

- If $j < m$ and $\tau(j+1) = i+2$, then it follows from (b) that ξ_0 carries $\mathrm{N}_\bullet(\{j-1 < j < j+1\})$ to the 2-simplex σ'' of $\mathcal{C}$. In this case, our assumption that τ belongs to K_2 guarantees that $m < n$, so the existence of the desired extension follows the fact that σ'' satisfies condition (1_m) (by virtue of our inductive hypothesis).

□

Theorem 5.4.4.1 immediately generalizes to other slice constructions:

01X9 **Corollary 5.4.4.2.** *Let $\mathcal{C}$ be an $(\infty, 2)$-category, let $f : K \to \mathcal{C}$ be a morphism of simplicial sets, and let $q : \mathcal{C}_{/f} \to \mathcal{C}$ denote the projection map. Let $u : X \to Y$ be a morphism in the $(\infty, 2)$-category $\mathcal{C}_{/f}$. The following conditions are equivalent:*

(1) *For every vertex $z \in K$, the composite map*

$$\Delta^2 \simeq \Delta^1 \star \{z\} \hookrightarrow \Delta^1 \star K \xrightarrow{u} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

(2) *The morphism u is q-cartesian.*

(3) *The morphism u is locally q-cartesian.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 5.4.3.8, and the implication (2) $\Rightarrow$ (3) is immediate (see Remark 5.1.3.3). We will show that (3) $\Rightarrow$ (1). Fix a vertex $z \in K$; we wish to show that the composite map

$$\Delta^2 \simeq \Delta^1 \star \{z\} \hookrightarrow \Delta^1 \star K \xrightarrow{u} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$. Set $Z = f(z) \in \mathcal{C}$, so that q factors as a composition

$$\mathcal{C}_{/f} \xrightarrow{q'} \mathcal{C}_{/Z} \xrightarrow{q''} \mathcal{C}\,.$$

By virtue of Theorem 5.4.4.1, it will suffice to show that the $q'(u)$ is a locally q''-cartesian morphism of the $(\infty,2)$-category $\mathcal{C}_{/Z}$.

Set $\overline{u} = q(u)$, which we regard as a morphism $\overline{q} : \overline{X} \to \overline{Y}$ in the $(\infty,2)$-category $\mathcal{C}$. By virtue of Proposition 5.4.3.9, we can lift $\overline{u}$ to a morphism $u' : X' \to Y$ in $\mathcal{C}_{/f}$ which satisfies condition (1) (and therefore also satisfies (3)). Regard $\overline{u}$ as a 1-simplex of $\mathcal{C}$ and let $\mathcal{E}$ denote the fiber product $\Delta^1 \times_{\mathcal{C}} \mathcal{C}_{/f}$. Since q is an interior fibration (Proposition 5.4.3.1), the projection map $\pi : \mathcal{E} \to \Delta^1$ is also an interior fibration (Remark 5.4.2.4) and therefore an inner fibration (Example 5.4.2.2). Let us abuse notation by identifying u and u' with morphisms in the ∞-category $\mathcal{E}$ lying over the unique nondegenerate edge of Δ^1. Assumption (3) then guarantees that u and u' are π-cartesian. Invoking Remark 5.1.3.8, we deduce that there exists a 2-simplex $\rho : \Delta^2 \to \mathcal{E}$, which we display as a diagram

$$\begin{tikzcd}
 & X' \arrow[dr, "u'"] & \\
X \arrow[ur, "v"] \arrow[rr, "u"] & & Y,
\end{tikzcd}$$

where v is an isomorphism in the ∞-category $\{0\} \times_{\Delta^1} \mathcal{E} \simeq \{\overline{X}\} \times_{\mathcal{C}} \mathcal{C}_{/f}$. It follows that $q'(v)$ is an isomorphism in the ∞-category $\{\overline{X}\} \times_{\mathcal{C}} \mathcal{C}_{/Z}$. Since u' satisfies condition (1), Theorem 5.4.4.1 guarantees that $q'(u')$ is locally q''-cartesian. Invoking Remark 5.1.3.8 again, we deduce that $q'(u)$ is locally q''-cartesian, as desired. □

5.4.5 The Pith of an $(\infty,2)$-Category

Let $\mathcal{C}$ be a 2-category. Recall that the *pith of* $\mathcal{C}$ is the subcategory $\mathrm{Pith}(\mathcal{C}) \subseteq \mathcal{C}$ obtained by removing the non-invertible 2-morphisms of $\mathcal{C}$ (Construction 2.2.8.9). In this section, we generalize this definition to the setting of $(\infty,2)$-categories. 01XA

01XB **Construction 5.4.5.1.** Let $\mathcal{C}$ be an $(\infty, 2)$-category. We let $\mathrm{Pith}(\mathcal{C}) \subseteq \mathcal{C}$ denote the simplicial subset consisting of those simplices $\sigma : \Delta^n \to \mathcal{C}$ which carry every 2-simplex of Δ^n to a thin 2-simplex of $\mathcal{C}$. We will refer to $\mathrm{Pith}(\mathcal{C})$ as the *pith of* $\mathcal{C}$.

01XC **Remark 5.4.5.2.** Let $\mathcal{C}$ be an $(\infty, 2)$-category. Then every degenerate 2-simplex of $\mathcal{C}$ is thin. Consequently, to check that a simplex $\sigma : \Delta^n \to \mathcal{C}$ belongs to the pith $\mathrm{Pith}(\mathcal{C})$, it suffices to check that σ carries every *nondegenerate* 2-simplex of Δ^n to a thin 2-simplex of $\mathcal{C}$. In particular:

- Every object of $\mathcal{C}$ belongs to $\mathrm{Pith}(\mathcal{C})$.
- Every morphism of $\mathcal{C}$ belongs to $\mathrm{Pith}(\mathcal{C})$.
- A 2-simplex σ of $\mathcal{C}$ belongs to $\mathrm{Pith}(\mathcal{C})$ if and only if it is thin.

01XD **Remark 5.4.5.3.** Let $\mathcal{C}$ be an $(\infty, 2)$-category. Then $\mathrm{Pith}(\mathcal{C})$ is the largest simplicial subset of $\mathcal{C}$ which does not contain any non-thin 2-simplices of $\mathcal{C}$.

01XE **Example 5.4.5.4.** Let $\mathcal{C}$ be a 2-category and let $\mathrm{Pith}(\mathcal{C})$ denote its pith (Construction 2.2.8.9). Then the inclusion $\mathrm{Pith}(\mathcal{C}) \hookrightarrow \mathcal{C}$ induces an isomorphism of simplicial sets $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathcal{C})) \simeq \mathrm{Pith}(\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}))$. This is an immediate consequence of Theorem 2.3.2.5.

01XF **Example 5.4.5.5.** Let $\mathcal{C}$ be an ∞-category. Then $\mathrm{Pith}(\mathcal{C}) = \mathcal{C}$ (see Example 2.3.2.4).

01XG **Proposition 5.4.5.6.** *Let $\mathcal{C}$ be an $(\infty, 2)$-category. Then $\mathrm{Pith}(\mathcal{C})$ is an ∞-category.*

Our proof of Proposition 5.4.5.6 will make use of a closure property of the collection of thin 2-simplices of an $(\infty, 2)$-category $\mathcal{C}$.

01XH **Definition 5.4.5.7.** Let $\mathcal{C}$ be a simplicial set and let T be a collection of 2-simplices of $\mathcal{C}$. We will say that T *has the inner exchange property* if the following condition is satisfied:

$(*)$ Let $\sigma : \Delta^3 \to \mathcal{C}$ be a 3-simplex of $\mathcal{C}$. For every triple of integers $0 \leq i < j < k \leq 3$, let σ_{kji} be the face of σ given by the restriction $\sigma|_{\mathrm{N}_{\bullet}(\{i<j<k\})}$. Assume that the outer faces σ_{210} and σ_{321} belong to T. Then σ_{310} belongs to T if and only if σ_{320} belongs to T.

01XJ **Remark 5.4.5.8.** Let $\mathcal{C}$ be a simplicial set, let T be a collection of 2-simplices of $\mathcal{C}$, and let T^{op} denote the set T, regarded as a collection of simplices of the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$. Then T has the inner exchange property if and only if T^{op} has the inner exchange property.

01XK **Remark 5.4.5.9.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let T be a collection of 2-simplices of $\mathcal{D}$. If T has the inner exchange property, then the inverse image $F^{-1}(T)$ has the inner exchange property.

Proposition 5.4.5.10 (Inner Exchange)**.** *Let $\mathcal{C}$ be an $(\infty, 2)$-category. Then the collection of thin 2-simplices of $\mathcal{C}$ has the inner exchange property (Definition 5.4.5.7).* 01XL

Remark 5.4.5.11. To get a feeling for the content of Proposition 5.4.5.10, let us specialize to the case where $\mathcal{C} = \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})$ is the Duskin nerve of a 2-category $\mathcal{D}$. In this case, we can identify a 3-simplex $\sigma : \Delta^3 \to \mathcal{C}$ with a collection of objects $\{X_i\}_{0 \leq i \leq 3}$ of $\mathcal{D}$, a collection of 1-morphisms $\{f_{ji} : X_i \to X_j\}_{0 \leq i < j \leq 3}$, and a collection of 2-morphisms $\{\mu_{kji} : f_{kj} \circ f_{ji} \Rightarrow f_{ki}\}$ for which the diagram 01XM

$$\begin{tikzcd}
f_{32} \circ (f_{21} \circ f_{10}) \arrow[rr, Rightarrow, "\alpha", "\sim"'] \arrow[d, Rightarrow, "\mathrm{id}_{f_{32}} \circ \mu_{210}"'] & & (f_{32} \circ f_{21}) \circ f_{10} \arrow[d, Rightarrow, "\mu_{321} \circ \mathrm{id}_{f_{10}}"] \\
f_{32} \circ f_{20} \arrow[dr, Rightarrow, "\mu_{320}"'] & & f_{31} \circ f_{10} \arrow[dl, Rightarrow, "\mu_{310}"] \\
& f_{30} &
\end{tikzcd}$$

is commutative, where $\alpha = \alpha_{f_{32}, f_{21}, f_{10}}$ is the associativity constraint for the composition of 1-morphisms in $\mathcal{C}$ (Proposition 2.3.1.9). The assumption that the outer faces of σ are thin guarantees that the 2-morphisms μ_{321} and μ_{210} are isomorphisms. In this case, Proposition 5.4.5.10 asserts that μ_{320} is an isomorphism if and only if μ_{310} is an isomorphism, which follows by inspection.

Proof of Proposition 5.4.5.10. Let $\mathcal{C}$ be an $(\infty, 2)$-category, let $\sigma : \Delta^3 \to \mathcal{C}$ be a 3-simplex of $\mathcal{C}$ and let $C = \sigma(3) \in \mathcal{C}$ be the image of the final vertex. Let us regard the face $\sigma_{210} = \sigma|_{\mathrm{N}_{\bullet}(\{0<1<2\})}$ as a morphism of simplicial sets from Δ^2 to $\mathcal{C}$, and let $\mathcal{E}$ denote the pullback $\Delta^2 \times_{\mathcal{C}} \mathcal{C}_{/C}$. Note that the projection map $\mathcal{C}_{/C} \to \mathcal{C}$ is an interior fibration (Proposition 5.4.3.1). If σ_{210} is thin, then the projection map $\pi : \mathcal{E} \to \Delta^2$ is also an interior fibration (Remark 5.4.2.4); since Δ^2 is an ∞-category, it is an inner fibration (Example 5.4.2.2). Unwinding the definitions, we can identify σ with a 2-simplex of $\mathcal{E}$ lying over the unique nondegenerate 2-simplex of Δ^2, which we display as a diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z.
\end{tikzcd}$$

If $\sigma_{321} = \sigma|_{\mathrm{N}_\bullet(\{1<2<3\})}$ is a thin 2-simplex of $\mathcal{C}$, then the "easy direction" of Theorem 5.4.4.1 guarantees that g is π-cartesian. It follows that f is π-cartesian if and only if h is π-cartesian (Corollary 5.1.2.4). Equivalently, f is locally π-cartesian if and only if h is locally π-cartesian (see Remark 5.1.3.4). Applying the "hard direction" of Theorem 5.4.4.1, we conclude that the 2-simplex $\sigma_{310} = \sigma|_{\mathrm{N}_\bullet(\{0<1<3\})}$ is thin if and only if the 2-simplex $\sigma_{320} = \sigma|_{\mathrm{N}_\bullet(\{0<2<3\})}$ is thin. □

Proof of Proposition 5.4.5.6. Let $\mathcal{C}$ be an $(\infty, 2)$-category. Suppose we are given integers $0 < i < n$ and a morphism of simplicial sets $\sigma_0 : \Lambda^n_i \to \mathrm{Pith}(\mathcal{C})$; we wish to show that σ_0 can be extended to an n-simplex $\sigma : \Delta^n \to \mathrm{Pith}(\mathcal{C})$. If $n = 2$, then condition (1) of Definition 5.4.1.1 guarantees that we can extend σ_0 to a thin 2-simplex of $\mathcal{C}$, which then belongs to $\mathrm{Pith}(\mathcal{C})$ by virtue of Remark 5.4.5.2. We may therefore assume that $n \geq 3$. In this case, we observe that the composite map

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{i-1 < i < i+1\}) \hookrightarrow \Lambda^n_i \xrightarrow{\sigma_0} \mathrm{Pith}(\mathcal{C}) \to \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$, so that we can extend σ_0 to an n-simplex $\sigma : \Delta^n \to \mathcal{C}$. To complete the proof, it will suffice to show that σ carries each 2-simplex of Δ^n to a thin 2-simplex of $\mathcal{C}$. If $n \geq 4$, this is automatic (since every 2-simplex of Δ^n is contained in the horn Λ^n_i). In the case $n = 3$, it follows from our assumption that the collection of thin 2-simplices of $\mathcal{C}$ has the inner exchange property (Proposition 5.4.5.10). □

047R **Definition 5.4.5.12.** Let $\mathcal{C}$ be an $(\infty, 2)$-category. We say that a morphism $f : X \to Y$ of $\mathcal{C}$ is an *isomorphism* if it is an isomorphism when viewed as a morphism in the ∞-category $\mathrm{Pith}(\mathcal{C})$. We say that objects $X, Y \in \mathcal{C}$ are *isomorphic* if there is an isomorphism from X to Y (that is, if X and Y are isomorphic when viewed as objects of the ∞-category $\mathrm{Pith}(\mathcal{C})$).

Let $\mathcal{C}$ be an $(\infty, 2)$-category. Heuristically, one can think of the ∞-category $\mathrm{Pith}(\mathcal{C})$ as obtained from $\mathcal{C}$ by removing its noninvertible 2-morphisms, just as the core $\mathcal{E}^{\simeq}$ of an ∞-category $\mathcal{E}$ is obtained by removing its noninvertible morphisms (see Construction 4.4.3.1). We now make this heuristic more precise (see Corollary 5.4.7.12 for a relative version):

01XN **Proposition 5.4.5.13.** *Let $\mathcal{C}$ be an $(\infty, 2)$-category containing objects X and Y. Then the inclusion $\mathrm{Pith}(\mathcal{C}) \hookrightarrow \mathcal{C}$ induces isomorphisms of simplicial sets*

$$\mathrm{Hom}^{\mathrm{L}}_{\mathrm{Pith}(\mathcal{C})}(X, Y) \simeq \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y)^{\simeq} \qquad \mathrm{Hom}^{\mathrm{R}}_{\mathrm{Pith}(\mathcal{C})}(X, Y) \simeq \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)^{\simeq}.$$

Proof. Let σ be an n-simplex of the simplicial set $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$, which we view as a morphism of simplicial sets $\tau : \Delta^{n+1} \to \mathcal{C}$ whose restriction to the face $\Delta^n \subseteq \Delta^{n+1}$ equal to the constant map $\Delta^n \to \{X\} \hookrightarrow \mathcal{C}$. Then σ belongs to the simplicial subset $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{Pith}(\mathcal{C})}(X, Y) \subseteq \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ if and only if, for every 2-simplex $\rho : \Delta^2 \to \Delta^{n+1}$, the composition $\tau \circ \rho$ is a thin 2-simplex of $\mathcal{C}$. Note that this condition is automatically satisfied if ρ is degenerate, or takes values in the subset $\Delta^n \subseteq \Delta^{n+1}$ (since every degenerate 2-simplex of $\mathcal{C}$ is thin). Consequently, it suffices to verify this condition in the case where ρ is the right cone of a map $\rho_0 : \Delta^1 \to \Delta^n$. In this case, $\tau \circ \rho$ is thin if and only if the edge $\Delta^1 \xrightarrow{\rho_0} \Delta^n \xrightarrow{\sigma} \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ is an isomorphism in the ∞-category $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$ (Theorem 5.4.4.1). Allowing τ_0 to vary, we obtain the identification $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{Pith}(\mathcal{C})}(X, Y) \simeq \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)^{\simeq}$; the proof of the analogous statement for left-pinched morphism spaces is similar. □

Proposition 5.4.5.14. *Let $\mathcal{C}$ be an $(\infty, 2)$-category and let $f : K \to \mathcal{C}$ be a morphism of simplicial sets. Then:* `01XP`

(1) *The projection map $\pi : \mathcal{C}_{/f} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C}) \to \mathrm{Pith}(\mathcal{C})$ is a cartesian fibration of ∞-categories. Moreover, a morphism u of $\mathcal{C}_{/f} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C})$ is π-cartesian if and only if, for every vertex $z \in K$, the composite map*

$$\Delta^2 \simeq \Delta^1 \star \{z\} \hookrightarrow \Delta^1 \star K \xrightarrow{u} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

(2) *The projection map $\pi' : \mathcal{C}_{f/} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C}) \to \mathrm{Pith}(\mathcal{C})$ is a cocartesian fibration of ∞-categories. Moreover, a morphism v of $\mathcal{C}_{f/} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C})$ is π'-cocartesian if and only if, for every vertex $x \in K$, the composite map*

$$\Delta^2 \simeq \{x\} \star \Delta^1 \hookrightarrow K \star \Delta^1 \xrightarrow{v} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

Proof. We will prove (1); the proof of (2) is similar. It follows from Remark 5.4.2.4 that π is an interior fibration. Since $\mathrm{Pith}(\mathcal{C})$ is an ∞-category (Proposition 5.4.5.6), it is an inner fibration of ∞-categories (Example 5.4.2.2). Let us say that a morphism u of $\mathcal{C}_{/f} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C})$ is *special* if, for every vertex $z \in K$, the composite map

$$\Delta^2 \simeq \Delta^1 \star \{z\} \hookrightarrow \Delta^1 \star K \xrightarrow{u} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$. Let $\overline{\pi} : \mathcal{C}_{/f} \to \mathcal{C}$ be the projection map. It follows from Corollary 5.4.4.2 that every special morphism of $\mathcal{C}_{/f} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C})$ is $\overline{\pi}$-cartesian when viewed as a morphism of $\mathcal{C}_{/f}$, and therefore also π-cartesian (Remark 5.1.1.11). Conversely, any π-cartesian morphism of $\mathcal{C}_{/f} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C})$ is locally $\overline{\pi}$-cartesian when viewed as a morphism of $\mathcal{C}_{/f}$, and therefore special (again by Corollary 5.4.4.2). To complete the proof, it will suffice to show that if Y is an object of $\mathcal{C}_{/f}$, then any morphism $\overline{u} : \overline{X} \to q(\overline{Y})$ in $\mathrm{Pith}(\mathcal{C})$ can be lifted to a special morphism $u : X \to Y$ of $\mathcal{C}_{/f} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C})$, which follows from Proposition 5.4.3.9. □

5.4.6 The Four-out-of-Five Property

01XQ Let $\mathcal{C}$ be an ∞-category. Recall that the collection of isomorphisms in $\mathcal{C}$ has the "two-out-of-three" property: if $f : X \to Y$ and $g : Y \to Z$ are composable morphisms of $\mathcal{C}$ and any two of the morphisms f, g, and $g \circ f$ is an isomorphism, then so is the third (Remark 1.4.6.3). This can be regarded as a special case of a more general closure property.

01XR **Definition 5.4.6.1.** Let $\mathcal{C}$ be a simplicial set and let W be a collection of edges of $\mathcal{C}$. We will say that W has the *two-out-of-six property* if it satisfies the following condition:

$(*)$ Let σ be a 3-simplex of $\mathcal{C}$ and, for every pair of integers $0 \leq i < j \leq 3$, let σ_{ji} denote the edge of $\mathcal{C}$ given by $\sigma|_{\mathrm{N}_\bullet(\{i<j\})}$. If the edges σ_{20} and σ_{31} belong to W, then the edges σ_{10}, σ_{21}, σ_{32}, and σ_{30} also belong to W.

01XS **Exercise 5.4.6.2.** Let $\mathcal{C}$ be a simplicial set and let W be a collection of edges of $\mathcal{C}$ which has the two-out-of-six property. Show that W has the two-out-of-three property. That is, for any 2-simplex σ of $\mathcal{C}$, if any two of the faces $d^2_0(\sigma)$, $d^2_1(\sigma)$, and $d^2_2(\sigma)$ belong to W, then so does the third.

01XT **Remark 5.4.6.3.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of edges of $\mathcal{C}$. We can informally summarize Definition 5.4.6.1 as follows: a collection of morphisms W of $\mathcal{C}$ has the two-out-of-six property if, for every triple of composable morphisms $f : A \to B$, $g : B \to C$, and $h : C \to D$, if the compositions $g \circ f$ and $h \circ g$ belong to W, then the morphisms f, g, h, and $h \circ g \circ f$ belong to W. Beware that this summary is somewhat imprecise, since the compositions $g \circ f$, $h \circ g$, and $h \circ g \circ f$ are *a priori* only well-defined up to homotopy.

01XU **Remark 5.4.6.4.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let W be a collection of edges of $\mathcal{D}$. If W has the two-out-of-six property, then the inverse image $F^{-1}(W)$ also has the two-out-of-six property.

0051 **Proposition 5.4.6.5** (Two-out-of-Six)**.** *Let $\mathcal{C}$ be an ∞-category and let W be the collection of isomorphisms in $\mathcal{C}$. Then W has the two-out-of-six property.*

Proof. By definition, a morphism f of $\mathcal{C}$ is an isomorphism if and only if its homotopy class $[f]$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}$ (Definition 1.4.6.1). By virtue of Remark 5.4.6.4, we can replace $\mathcal{C}$ by the nerve $\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ and thereby reduce to the case where $\mathcal{C} = \mathrm{N}_\bullet(\mathcal{C}')$ for some category $\mathcal{C}'$. Let σ be a 3-simplex of $\mathcal{C}$, corresponding to a triple of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

in $\mathcal{C}'$, and suppose that $g \circ f$ and $h \circ g$ are isomorphisms. Then $g \circ f$ admits an inverse $u : C \to A$. It follows that $g \circ (f \circ u) = (g \circ f) \circ u = \mathrm{id}_C$, so that g admits a right inverse. A similar argument shows that g also admits a left inverse, and is therefore an isomorphism

(Remark 1.4.6.7). Applying the two-out-three property, we deduce that f and h are also isomorphisms. Since the collection of isomorphisms is closed under composition, it also follows that $h \circ g \circ f$ is an isomorphism. □

Proposition 5.4.6.5 admits a converse:

Proposition 5.4.6.6. *Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$ which has the two-out-of-six property. If W contains every identity morphism of $\mathcal{C}$, then it contains every isomorphism of $\mathcal{C}$.* 01XV

In other words, the collection of isomorphisms in an ∞-category $\mathcal{C}$ is the smallest collection of morphisms which contains all identity morphisms and has the two-out-of-six property.

Warning 5.4.6.7. The analogue of Proposition 5.4.6.6 for the two-out-of-three property is false in general. For example, if $\mathcal{C}$ is the nerve of a category, then the collection of identity morphisms of $\mathcal{C}$ has the two-out-of-three property, but usually does not contain all the isomorphisms of $\mathcal{C}$. 01XW

Proof of Proposition 5.4.6.6. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be an isomorphism in $\mathcal{C}$. Then f admits a homotopy inverse $g : Y \to X$. Let σ be a 2-simplex of $\mathcal{C}$ which witnesses id_X as a composition f and g, and let σ' be a 2-simplex of $\mathcal{C}$ which witnesses id_Y as a composition of g and f. Then the triple $(\sigma', s_0^1(f), \bullet, \sigma)$ can be regarded as a morphism of simplicial sets $\tau_0 : \Lambda^3_2 \to \mathcal{C}$. Since $\mathcal{C}$ is an ∞-category, we can extend τ_0 to a 3-simplex $\tau : \Delta^3 \to \mathcal{C}$, whose restriction to the 1-skeleton of Δ^3 is indicated in the diagram

$$\begin{tikzcd}
& Y \arrow[r, "g"] \arrow[drr, "\mathrm{id}_Y"] & X \arrow[dr, "f"] & \\
X \arrow[ur, "f"] \arrow[urr, "\mathrm{id}_X"] \arrow[rrr, "f"] & & & Y.
\end{tikzcd}$$

It follows that if W is a collection of morphisms of $\mathcal{C}$ which contains id_X, id_Y, and has the two-out-of-six property, then W also contains the isomorphism f. □

Our goal in this section is to prove analogues of Propositions 5.4.6.5 and Proposition 5.4.6.6 in the setting of $(\infty, 2)$-categories, where we replace the set $W \subseteq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^1, \mathcal{C})$ of isomorphisms with the set $T \subseteq \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^2, \mathcal{C})$ of thin 2-simplices.

Definition 5.4.6.8. Let $\mathcal{C}$ be a simplicial set and let T be a collection of 2-simplices of $\mathcal{C}$. We say that T has the *four-out-of-five property* if it satisfies the following condition: 01XX

$(*)$ Let $\sigma : \Delta^4 \to \mathcal{C}$ be a 4-simplex of $\mathcal{C}$. For every triple of integers $0 \leq i < j < k \leq 4$, let σ_{kji} denote the 2-simplex of $\mathcal{C}$ given by the restriction of σ to $\mathrm{N}_{\bullet}(\{i < j < k\})$. If the 2-simplices σ_{310}, σ_{420}, σ_{321}, and σ_{432} belong to T, then the 2-simplex σ_{430} also belongs to T.

`01XY` **Warning 5.4.6.9.** Definition 5.4.6.8 is not self-dual. Let T be a collection of 2-simplices of $\mathcal{C}$ which satisfies the four-out-of-five property and let T^{op} denote the same set, regarded as a collection of 2-simplices of the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$. Then T^{op} need not satisfy the four-out-of-five property.

`01XZ` **Remark 5.4.6.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let T be a collection of 2-simplices of $\mathcal{D}$. If T has the four-out-of-five-property, then the inverse image $F^{-1}(T)$ also has the four-out-of-five property.

`01Y0` **Proposition 5.4.6.11** (Four-out-of-Five)**.** *Let $\mathcal{C}$ be an $(\infty,2)$-category and let T be the collection of all thin 2-simplices of $\mathcal{C}$. Then T has the four-out-of-five property.*

`01Y1` **Warning 5.4.6.12.** Let $\mathcal{C}$ be an $(\infty,2)$-category and let $\sigma : \Delta^4 \to \mathcal{C}$ be a 4-simplex of $\mathcal{C}$. For $0 \leq i < j < k \leq 4$, let σ_{kji} denote the restriction $\sigma|_{\mathrm{N}_{\bullet}(\{i<j<k\})}$. Proposition 5.4.6.11 asserts that, if the 2-simplices σ_{310}, σ_{420}, σ_{321}, and σ_{432} are thin, then σ_{430} is also thin. Beware that the remaining 2-simplices σ_{210}, σ_{410}, σ_{320}, σ_{421}, and σ_{431} need not be thin.

`01Y2` **Example 5.4.6.13.** To get a feeling for the content of Proposition 5.4.6.11, let us consider the special case where $\mathcal{C} = \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}')$ is the Duskin nerve of a strict 2-category $\mathcal{C}'$. Let σ be a 4-simplex of $\mathcal{C}$, which we identify with a collection of objects $\{X_i\}_{0 \leq i \leq 4}$, 1-morphisms $\{f_{ji} : X_j \to X_i\}_{0 \leq i < j \leq 4}$, and 2-morphisms $\{\mu_{kji} : f_{kj} \circ f_{ji} \Rightarrow f_{ki}\}_{0 \leq i < j < k \leq 4}$ of $\mathcal{C}$ satisfying the condition described in Proposition 2.3.1.9. Proposition 5.4.6.11 asserts that if the 2-morphisms μ_{310}, μ_{420}, μ_{321}, and μ_{432} are invertible, then the 2-morphism μ_{430} is also invertible. This follows by inspecting the cubical diagram

$$\begin{tikzcd}
f_{43} \circ f_{32} \circ f_{21} \circ f_{10} \arrow[rrrr, Rightarrow, "\mu_{210}"] \arrow[drr, Rightarrow, "\mu_{321}", "\sim"'] \arrow[dd, Rightarrow, "\mu_{432}"] & & & & f_{43} \circ f_{32} \circ f_{20} \arrow[drr, Rightarrow, "\mu_{320}"] \arrow[dd, Rightarrow, "\mu_{432}", "\sim"'] & & \\
& & f_{43} \circ f_{31} \circ f_{10} \arrow[rrrr, Rightarrow, "\mu_{310}", "\sim"'] \arrow[dd, Rightarrow, "\mu_{431}"] & & & & f_{43} \circ f_{30} \arrow[dd, Rightarrow, "\mu_{430}"] \\
f_{42} \circ f_{21} \circ f_{10} \arrow[rrrr, Rightarrow, "\mu_{210}"] \arrow[drr, Rightarrow, "\mu_{421}"] & & & & f_{42} \circ f_{20} \arrow[drr, Rightarrow, "\mu_{420}", "\sim"'] & & \\
& & f_{41} \circ f_{10} \arrow[rrrr, Rightarrow, "\mu_{410}"] & & & & f_{40}
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}'}(X_0, X_4)$ and applying the two-out-of-six property to the chain of 2-morphisms

$$f_{43} \circ f_{32} \circ f_{21} \circ f_{10} \xRightarrow{\mu_{210}} f_{43} \circ f_{32} \circ f_{20} \xRightarrow{\mu_{320}} f_{43} \circ f_{30} \xRightarrow{\mu_{430}} f_{40}.$$

Proof of Proposition 5.4.6.11. Let $\mathcal{C}$ be an $(\infty, 2)$-category and let $\sigma : \Delta^4 \to \mathcal{C}$ be a 4-simplex. For every triple of integers $0 \leq i < j < k \leq 4$, let σ_{kji} denote the 2-simplex of $\mathcal{C}$ given by the restriction of σ to $\mathrm{N}_\bullet(\{i < j < k\})$. Assume that the 2-simplices σ_{310}, σ_{420}, σ_{321}, and σ_{432} are thin. We wish to show that σ_{430} is also thin.

Set $X = \sigma(0) \in \mathcal{C}$. Let $\mathcal{E}$ denote the fiber product $\mathcal{C}_{X/} \times_{\mathcal{C}} \mathrm{Pith}(\mathcal{C})$ and let $\pi : \mathcal{E} \to \mathrm{Pith}(\mathcal{C})$ be the projection map, so that π is a cocartesian fibration of ∞-categories (Proposition 5.4.5.14). For $1 \leq i \leq 4$, let $\mathcal{E}_i$ denote the ∞-category $\{\sigma(i)\} \times_{\mathrm{Pith}(\mathcal{C})} \mathcal{E}$, so that the edge $\sigma|_{\mathrm{N}_\bullet(\{0<i\})}$ of $\mathcal{C}$ can be identified with an object $Y_i \in \mathcal{E}_i$. For $1 \leq i < j \leq 4$, let us identify the 2-simplex $\sigma|_{\mathrm{N}_\bullet(\{0<i<j\})}$ with a morphism $f_{j,i} : Y_i \to Y_j$ in $\mathcal{E}$. By virtue of Proposition 5.4.5.14), it will suffice to show that the morphism $f_{4,3} : Y_3 \to Y_4$ is π-cocartesian.

For $2 \leq i \leq 4$, let $F_i : \mathcal{E}_{i-1} \to \mathcal{E}_i$ be given by covariant transport along the edge $\sigma|_{\mathrm{N}_\bullet(\{i-1<i\})}$ of $\mathrm{Pith}(\mathcal{C})$ (see Definition 5.2.2.4) so that we have a sequence of functors

$$\mathcal{E}_1 \xrightarrow{F_2} \mathcal{E}_2 \xrightarrow{F_3} \mathcal{E}_3 \xrightarrow{F_4} \mathcal{E}_4 .$$

Let $H_i : \Delta^1 \times \mathcal{E}_{i-1} \to \mathcal{E}$ be a functor which witnesses that F_i is given by covariant transport along $\sigma|_{\mathrm{N}_\bullet(\{i-1<i\})}$, so that $h_i = H_i|_{\Delta^1 \times \{Y_{i-1}\}}$ is a π-cocartesian morphism of $\mathcal{E}$. It follows that the morphism $f_{i,i-1}$ can be written as a composition

$$Y_{i-1} \xrightarrow{h_i} F_i(Y_{i-1}) \xrightarrow{g_i} Y_i,$$

where g_i is a morphism in the ∞-category $\mathcal{E}_i$. To complete the proof, it will suffice to show that the morphism $g_4 : F_4(Y_3) \to Y_4$ is an isomorphism in the ∞-category $\mathcal{E}_4$ (see Remark 5.1.3.8).

Note that we have a chain of 1-morphisms

$$(F_4 \circ F_3 \circ F_2)(Y_1) \xrightarrow{(F_4 \circ F_3)(g_2)} (F_4 \circ F_3)(Y_2) \xrightarrow{F_4(g_3)} F_4(Y_3) \xrightarrow{g_4} Y_4$$

in the ∞-category $\mathcal{E}_4$. Since the collection of isomorphisms in the homotopy category $\mathrm{h}\mathcal{E}_4$ satisfies the two-out-of-six property, it will suffice to prove the following:

(a) The composition

$$(F_4 \circ F_3 \circ F_2)(Y_1) \xrightarrow{[(F_4 \circ F_3)(g_2)]} (F_4 \circ F_3)(Y_2) \xrightarrow{[F_4(g_3)]} F_4(Y_3)$$

is an isomorphism in the homotopy category $\mathrm{h}\mathcal{E}_4$.

(b) The composition

$$(F_4 \circ F_3)(Y_2) \xrightarrow{[F_4(g_3)]} F_4(Y_3) \xrightarrow{[g_4]} Y_4$$

is an isomorphism in the homotopy category $\mathrm{h}\mathcal{E}_4$.

We will deduce (a) from the following slightly stronger assertion:

(a') The composition

$$(F_3 \circ F_2)(Y_1) \xrightarrow{[F_3(g_2)]} F_3(Y_2) \xrightarrow{[g_3]} Y_3$$

is an isomorphism in the homotopy category $\mathrm{h}\mathcal{E}_3$.

To prove (a'), we first note that the 2-simplex σ_{321} is thin, and can therefore be regarded as a 2-simplex of $\mathrm{Pith}(\mathcal{C})$. Let $\mathcal{E}'$ denote the fiber product $\mathrm{N}_\bullet(\{1 < 2 < 3\}) \times_{\mathrm{Pith}(\mathcal{C})} \mathcal{E}$, and let $\pi' : \mathcal{E}' \to \mathrm{N}_\bullet(\{1 < 2 < 3\})$ be the projection map. In the homotopy category $\mathrm{h}\mathcal{E}'$, we have a commutative diagram

$$\begin{tikzcd}
Y_1 \arrow[r, "{[h_2]}"] \arrow[dr, "{[f_{2,1}]}"'] & F_2(Y_1) \arrow[r] \arrow[d, "{[g_2]}"] & (F_3 \circ F_2)(Y_1) \arrow[d, "{[F_3(g_2)]}"] \\
 & Y_2 \arrow[r, "{[h_3]}"] \arrow[dr, "{[f_{3,2}]}"'] & F_3(Y_2) \arrow[d, "{[g_3]}"] \\
 & & Y_3,
\end{tikzcd}$$

where the upper horizontal composition is the homotopy class of a π'-cocartesian morphism (Corollary 5.1.2.4). It follows that the vertical composition on the right is an isomorphism if and only if the diagonal composition is also the homotopy class of a π'-cocartesian morphism (Remark 5.1.3.8). We now observe that the 3-simplex $\sigma|_{\mathrm{N}_{\bullet}(\{0<1<2<3\})}$ witnesses the identity $[f_{3,2}] \circ [f_{2,1}] = [f_{3,1}]$ in the homotopy category $\mathrm{h}\mathcal{E}'$. It will therefore suffice to show that $f_{3,1}$ is a π'-cocartesian morphism of the ∞-category $\mathcal{E}'$, which follows from Proposition 5.4.5.14 and our assumption that σ_{310} is thin. This completes the proof of (a). The proof of (b) follows by the same argument, using the thinness of the 2-simplices σ_{432} and σ_{420} in place of σ_{321} and σ_{310}. □

We now prove a partial converse to Proposition 5.4.6.11, which can be regarded as an $(\infty, 2)$-categorical analogue of Proposition 5.4.6.6.

Proposition 5.4.6.14. *Let $\mathcal{C}$ be an $(\infty, 2)$-category and let T be a collection of 2-simplices of $\mathcal{C}$. Assume that:* 01Y3

(1) *Every degenerate 2-simplex of $\mathcal{C}$ belongs to T.*

(2) *For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, there exists a thin 2-simplex σ of $\mathcal{C}$ which belongs to T and satisfies $d^2_2(\sigma) = f$ and $d^2_0(\sigma) = g$, as indicated in the diagram*

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr] & & Z.
\end{tikzcd}$$

(3) *The collection T has the inner exchange property (Definition 5.4.5.7).*

(4) *The collection T has the four-out-of-five property (Definition 5.4.6.8).*

Then every thin 2-simplex of $\mathcal{C}$ belongs to T.

Proof. Let σ be a thin 2-simplex of $\mathcal{C}$, whose 1-skeleton we represent by the diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z.
\end{tikzcd}$$

Applying assumption (2), we can choose a thin 2-simplex σ' of $\mathcal{C}$ which belongs to T whose restriction to the 1-skeleton of Δ^2 is represented by the diagram

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h'"] & & Z.
\end{tikzcd}$$

The edge g determines a morphism of simplicial sets $\Delta^1 \to \mathcal{C}$. Let $\mathcal{E}$ denote the fiber product $\Delta^1 \times_{\mathcal{C}} \mathcal{C}_{X/}$. Since the projection map $\mathcal{C}_{X/} \to \mathcal{C}$ is an interior fibration (Proposition 5.4.3.1), it follows from Remark 5.4.2.4 and Example 5.4.2.2 that the projection map $\pi : \mathcal{E} \to \Delta^1$ is an inner fibration; in particular, $\mathcal{E}$ is an ∞-category. Moreover, we can identify the edges f, h, and h' of $\mathcal{C}$ with objects $\widetilde{Y}$, $\widetilde{Z}$, and $\widetilde{Z}'$ of $\mathcal{E}$, and the 2-simplices σ and σ' with morphisms $\widetilde{h} : \widetilde{Y} \to \widetilde{Z}$ and $\widetilde{h}' : \widetilde{Y} \to \widetilde{Z}'$. Since σ and σ' are both thin, the morphisms $\widetilde{h}$ and $\widetilde{h}'$ are both π-cocartesian (Theorem 5.4.4.1). It follows that $\widetilde{h}$ and $\widetilde{h}'$ are isomorphic when viewed as objects of the ∞-category $\mathcal{E}_{\widetilde{Y}/}$ (see Remark 5.1.3.8). We can therefore choose a 2-simplex ρ of $\mathcal{E}_{\widetilde{Y}/}$ whose 1-skeleton is given by the diagram

$$\begin{tikzcd}
& \widetilde{h}' \arrow[dr] & \\
\widetilde{h} \arrow[ur] \arrow[rr, "\mathrm{id}_{\widetilde{h}}"] & & \widetilde{h},
\end{tikzcd}$$

which we can identify with a 4-simplex $\tau : \Delta^4 \to \mathcal{C}$. For $0 \leq i < j < k \leq 4$, let τ_{kji} denote the 2-simplex of $\mathcal{C}$ given by $\tau|_{\mathrm{N}_\bullet(\{i<j<k\})}$. By construction, the 2-simplex τ_{310} is equal to σ', and therefore belongs to T. Moreover, the 2-simplices τ_{420}, τ_{321}, τ_{431}, and τ_{432} are right-degenerate, and therefore belong to T by virtue of assumption (1). Since T has the four-out-of-five-property, it follows that τ_{430} belongs to T. Applying the inner exchange property to the 3-simplex $\tau|_{\mathrm{N}_\bullet(\{0<1<3<4\})}$, we deduce that the 2-simplex $\sigma = \tau_{410}$ also belongs to T, as desired. □

5.4.7 Functors of $(\infty, 2)$-Categories

01Y4 Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Recall that a *functor* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ (Definition 1.5.0.1). In this case, it is automatic that F carries isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$ (Remark 1.5.1.6). Beware that the $(\infty, 2)$-categorical analogue of this statement is false: if $\mathcal{C}$ and $\mathcal{D}$ are $(\infty, 2)$-categories, then a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ will generally not carry thin 2-simplices of $\mathcal{C}$ to thin 2-simplices of $\mathcal{D}$. This motivates the following:

Definition 5.4.7.1. Let $\mathcal{C}$ and $\mathcal{D}$ be $(\infty, 2)$-categories. A *functor from* $\mathcal{C}$ *to* $\mathcal{D}$ is a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ which carries thin 2-simplices of $\mathcal{C}$ to thin 2-simplices of $\mathcal{D}$. 01Y5

Example 5.4.7.2. Let $\mathcal{C}$ be an $(\infty, 2)$-category and let $\mathcal{D}$ be an ∞-category. Then every 2-simplex of $\mathcal{D}$ is thin, so every morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ is a functor. In particular, when $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, Definition 5.4.7.1 reduces to Definition 1.5.0.1. 01Y6

Example 5.4.7.3. Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. By virtue of Theorem 2.3.4.1 and Corollary 2.3.4.5, passage to the Duskin nerve induces a bijection 01Y7

$$\begin{tikzcd}
\{\text{Strictly unitary functors of 2-categories } \mathcal{C} \to \mathcal{D}\} \arrow[d] \\
\{\text{Functors of } (\infty, 2)\text{-categories } \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})\}.
\end{tikzcd}$$

Remark 5.4.7.4 (Functoriality)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be $(\infty, 2)$-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then F is a functor (Definition 5.4.7.1) if and only it carries $\mathrm{Pith}(\mathcal{C})$ into $\mathrm{Pith}(\mathcal{D})$. If this condition is satisfied, then $\mathrm{Pith}(F) = F|_{\mathrm{Pith}(\mathcal{C})}$ can be regarded as a functor from the ∞-category $\mathrm{Pith}(\mathcal{C})$ to the ∞-category $\mathrm{Pith}(\mathcal{D})$. 01Y8

Remark 5.4.7.5. Let $\mathcal{C}$ and $\mathcal{D}$ be $(\infty, 2)$-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. If F is a functor of $(\infty, 2)$-categories and $u : X \to Y$ is an isomorphism in the $(\infty, 2)$-category $\mathcal{C}$, then $F(u) : F(X) \to F(Y)$ is an isomorphism in the $(\infty, 2)$-category $\mathcal{D}$ (see Definition 5.4.5.12). This follows by applying Remark 1.5.1.6 to the functor $\mathrm{Pith}(F) : \mathrm{Pith}(\mathcal{C}) \to \mathrm{Pith}(\mathcal{D})$ of Remark 5.4.7.4. Beware that, if F is not assumed to be a functor, then $F(u)$ need not be an isomorphism. 047S

Remark 5.4.7.6. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{D}$ be an $(\infty, 2)$-category. Then every functor $F : \mathcal{C} \to \mathcal{D}$ takes values in the pith $\mathrm{Pith}(\mathcal{D}) \subseteq \mathcal{D}$. Consequently, the inclusion $\mathrm{Pith}(\mathcal{D}) \hookrightarrow \mathcal{D}$ induces a bijection 01Y9

$$\begin{tikzcd}
\{\text{Functors of } \infty\text{-categories from } \mathcal{C} \text{ to } \mathrm{Pith}(\mathcal{D})\} \arrow[d] \\
\{\text{Functors of } (\infty, 2)\text{-categories from } \mathcal{C} \text{ to } \mathcal{D}\}.
\end{tikzcd}$$

Note that this property (together with Proposition 5.4.5.6) characterize the simplicial set $\mathrm{Pith}(\mathcal{D})$ up to unique isomorphism.

01YA **Remark 5.4.7.7.** The existence of morphisms between $(\infty, 2)$-categories which do not preserve thin 2-simplices should be viewed as a feature of our formalism, rather than a bug. Recall that, if $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, then Theorem 2.3.4.1 supplies a bijection

$$\begin{tikzcd}
\{\text{Strictly unitary lax functors } \mathcal{C} \to \mathcal{D}\} \arrow[d, "\sim"] \\
\{\text{Morphisms of simplicial sets } \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{D})\}.
\end{tikzcd}$$

Consequently, we can think of general morphisms of simplicial sets as providing a generalization of the notion of (strictly) unitary lax functors to the setting of $(\infty, 2)$-categories.

01YB **Warning 5.4.7.8.** For every pair of simplicial sets $\mathcal{C}$ and $\mathcal{D}$, we let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the simplicial set introduced in Construction 1.5.3.1. When working with $(\infty, 2)$-categories, this notation is potentially confusing. By construction, vertices of the simplicial set $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ can be identified with morphisms of simplicial sets $F : \mathcal{C} \to \mathcal{D}$. If $\mathcal{C}$ and $\mathcal{D}$ are $(\infty, 2)$-categories, then such morphisms need not carry thin 2-simplices of $\mathcal{C}$ to thin 2-simplices of $\mathcal{D}$, and therefore need not correspond to *functors* from $\mathcal{C}$ to $\mathcal{D}$ in the sense of Definition 5.4.7.1. We will return to this point in §[?].

The following criterion is often useful for checking that a morphism of $(\infty, 2)$-categories $F : \mathcal{C} \to \mathcal{D}$ is a functor:

01YC **Proposition 5.4.7.9.** *Let $\mathcal{C}$ and $\mathcal{D}$ be $(\infty, 2)$-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism F is a functor: that is, it carries thin 2-simplices of $\mathcal{C}$ to thin 2-simplices of $\mathcal{D}$.*

(2) *For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$ of $\mathcal{C}$, there exists a thin 2-simplex σ of $\mathcal{C}$ with $d^2_0(\sigma) = g$ and $d^2_2(\sigma) = f$, as indicated in the diagram*

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr] & & Z,
\end{tikzcd}$$

such that $F(\sigma)$ is a thin 2-simplex of $\mathcal{D}$.

Proof. The implication (1) $\Rightarrow$ (2) is immediate. To prove the converse, let T be the collection of all 2-simplices of $\mathcal{C}$ for which $F(\sigma)$ is a thin 2-simplex of $\mathcal{D}$. Since the collection of thin 2-simplices of $\mathcal{D}$ has the four-out-of-five property (Proposition 5.4.6.11), it follows that T also has the four-out-of-five property (Remark 5.4.6.10). Since the collection of thin 2-simplices of $\mathcal{D}$ has the inner exchange property (Proposition 5.4.5.10), T has the inner exchange property (Remark 5.4.5.9). Since $\mathcal{D}$ is an $(\infty, 2)$-category, every degenerate 2-simplex of $\mathcal{D}$ is thin, so every degenerate 2-simplex of $\mathcal{C}$ belongs to T. If condition (2) is satisfied, then Proposition 5.4.6.14 guarantees that every thin 2-simplex of $\mathcal{C}$ belongs to T, so that F is a functor. □

Proposition 5.4.7.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be an interior fibration of $(\infty, 2)$-categories (Definition 5.4.2.1). Then:* 01YD

(1) *The morphism F is a functor of $(\infty, 2)$-categories: that is, it carries thin 2-simplices of $\mathcal{C}$ to thin 2-simplices of $\mathcal{D}$, and therefore induces a functor $\operatorname{Pith}(F) : \operatorname{Pith}(\mathcal{C}) \to \operatorname{Pith}(\mathcal{D})$.*

(2) *The diagram of simplicial sets*

$$\begin{tikzcd}
\operatorname{Pith}(\mathcal{C}) \arrow[r] \arrow[d, "\operatorname{Pith}(F)"] & \mathcal{C} \arrow[d, "F"] \\
\operatorname{Pith}(\mathcal{D}) \arrow[r] & \mathcal{D}
\end{tikzcd}$$

is a pullback square.

(3) *The functor $\operatorname{Pith}(F) : \operatorname{Pith}(\mathcal{C}) \to \operatorname{Pith}(\mathcal{D})$ is an inner fibration of ∞-categories.*

Proof. We will prove assertion (1) by showing that F satisfies the criterion of Proposition 5.4.7.9. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of $\mathcal{C}$. Since $\mathcal{D}$ is an $(\infty, 2)$-category, we can choose a thin 2-simplex $\overline{\sigma}$ of $\mathcal{D}$ satisfying $d^2_0(\overline{\sigma}) = F(g)$ and $d^2_2(\overline{\sigma}) = F(f)$, which we depict as a diagram

$$\begin{tikzcd}
 & F(Y) \arrow[dr, "F(g)"] & \\
F(X) \arrow[ur, "F(f)"] \arrow[rr] & & F(Z).
\end{tikzcd}$$

Since F is an interior fibration, the lifting problem

$$\begin{tikzcd}
\Lambda^2_1 \arrow[r, "(g, \bullet, f)"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\Delta^2 \arrow[r, "\overline{\sigma}"] \arrow[ur, "\sigma"] & \mathcal{D}
\end{tikzcd}$$

admits a solution. Then σ is a thin 2-simplex of $\mathcal{C}$ (Lemma 5.4.2.6) for which the image $\overline{\sigma} = F(\sigma)$ is a thin 2-simplex of $\mathcal{D}$.

We now prove (2). Let τ be an m-simplex of the simplicial set $\mathcal{C}$, and suppose that $F(\tau)$ belongs to the pith $\operatorname{Pith}(\mathcal{D})$. We wish to show that τ belongs to $\operatorname{Pith}(\mathcal{C})$: that is, that it carries each 2-simplex of Δ^m to a thin 2-simplex of $\mathcal{C}$. This follows immediately from Lemma 5.4.2.6, since the composite map

$$\Delta^2 \to \Delta^m \xrightarrow{\tau} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a thin 2-simplex of $\mathcal{D}$.

Combining (2) with Remark 5.4.2.4, we conclude that the functor $\operatorname{Pith}(F) : \operatorname{Pith}(\mathcal{C}) \to \operatorname{Pith}(\mathcal{D})$ is an interior fibration. Since $\operatorname{Pith}(\mathcal{D})$ is an ∞-category (Proposition 5.4.5.6), it follows that $\operatorname{Pith}(F)$ is an inner fibration (Example 5.4.2.2). □

`01YE` **Corollary 5.4.7.11.** *Let $\mathcal{C}$ be an $(\infty, 2)$-category, let $f : K \to \mathcal{C}$ be a morphism of simplicial sets, and let*

$$q' : \mathcal{C}_{f/} \to \mathcal{C} \qquad q : \mathcal{C}_{/f} \to \mathcal{C}$$

be the projection maps. Then:

(1) *The functor $\operatorname{Pith}(q) : \operatorname{Pith}(\mathcal{C}_{/f}) \to \operatorname{Pith}(\mathcal{C})$ is a cartesian fibration of ∞-categories. Moreover, a morphism u of $\operatorname{Pith}(\mathcal{C}_{/f})$ is $\operatorname{Pith}(q)$-cartesian if and only if, for every vertex $z \in K$, the composite map*

$$\Delta^2 \simeq \Delta^1 \star \{z\} \hookrightarrow \Delta^1 \star K \xrightarrow{u} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

(2) *The functor $\operatorname{Pith}(q') : \operatorname{Pith}(\mathcal{C}_{f/}) \to \operatorname{Pith}(\mathcal{C})$ is a cocartesian fibration of ∞-categories. Moreover, a morphism v of $\operatorname{Pith}(\mathcal{C}_{f/})$ is $\operatorname{Pith}(q')$-cocartesian if and only if, for every vertex $x \in K$, the composite map*

$$\Delta^2 \simeq \{x\} \star \Delta^1 \hookrightarrow K \star \Delta^1 \xrightarrow{v} \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$.

Proof. Combine Propositions 5.4.7.10 and 5.4.5.14. □

Specializing Corollary 5.4.7.11 to the case $K = \Delta^0$, we obtain the following:

Corollary 5.4.7.12. *Let $\mathcal{C}$ be an $(\infty,2)$-category and let Z be an object of $\mathcal{C}$. Then:* 01YF

(1) *The projection map $\pi : \mathcal{C}_{/Z} \to \mathcal{C}$ induces a cartesian fibration of ∞-categories* $\mathrm{Pith}(\pi) : \mathrm{Pith}(\mathcal{C}_{/Z}) \to \mathrm{Pith}(\mathcal{C})$.

(2) *A morphism u of* $\mathrm{Pith}(\mathcal{C}_{/Z})$ *is* $\mathrm{Pith}(\pi)$*-cartesian if and only if it corresponds to a thin 2-simplex of $\mathcal{C}$ (in this case, it is also π-cartesian when viewed as a morphism of $\mathcal{C}_{/Z}$).*

(3) *The inclusion* $\mathrm{Pith}(\mathcal{C}) \hookrightarrow \mathcal{C}$ *induces an isomorphism from* $\mathrm{Pith}(\mathcal{C})_{/Z}$ *to the (non-full) subcategory of* $\mathrm{Pith}(\mathcal{C}_{/Z})$ *spanned by the π-cartesian morphisms.*

Proof. Assertions (1) and (2) follow from Corollary 5.4.7.11, and assertion (3) is an immediate consequence of (2). □

Remark 5.4.7.13. Recall that every cartesian fibration of simplicial sets $\pi : \mathcal{E} \to \mathcal{D}$ has an underlying right fibration $\pi' : \mathcal{E}' \to \mathcal{D}$, given by restricting π to the simplicial subset $\mathcal{E}' \subseteq \mathcal{E}$ spanned by those simplices $\sigma : \Delta^n \to \mathcal{E}$ which carry each edge of Δ^n to π-cartesian edge of $\mathcal{E}$. Corollary 5.4.7.12 asserts that, when π is the cartesian fibration $\mathrm{Pith}(\mathcal{C}_{/Z}) \to \mathrm{Pith}(\mathcal{C})$ associated to a choice of object Z of an $(\infty,2)$-category $\mathcal{C}$, then π' can be identified with the right fibration $\mathrm{Pith}(\mathcal{C})_{/Z} \to \mathrm{Pith}(\mathcal{C})$ supplied by Corollary 4.3.6.11; compare with Proposition 5.4.5.13. 01YG

We can also use Proposition 5.4.7.10 to deduce a relative version of Proposition 5.4.3.1:

Corollary 5.4.7.14. *Let $\mathcal{C}$ be an $(\infty,2)$-category, let $f : K \to \mathcal{C}$ be a morphism of simplicial sets, and let $f_0 = f|_{K_0}$ denote the restriction of f to a simplicial subset $K_0 \subseteq K$. Then the projection maps* 01YH

$$\mathcal{C}_{f/} \to \mathcal{C}_{f_0/} \qquad \mathcal{C}_{/f} \to \mathcal{C}_{/f_0}$$

are interior fibrations of $(\infty,2)$-categories.

Warning 5.4.7.15. In the situation of Corollary 5.4.7.14, the induced map $\mathrm{Pith}(\mathcal{C}_{/f}) \to \mathrm{Pith}(\mathcal{C}_{/f_0})$ is generally not a cartesian fibration, and the induced map $\mathrm{Pith}(\mathcal{C}_{f/}) \to \mathrm{Pith}(\mathcal{C}_{f_0/})$ is generally not a cocartesian fibration. 01YJ

Proof of Corollary 5.4.7.14. We will show that the map of slice simplicial sets $q : \mathcal{C}_{/f} \to \mathcal{C}_{/f_0}$ is an interior fibration; the analogous statement for coslice simplicial sets follows by a similar argument. We first observe that $\mathcal{C}_{/f_0}$ is an $(\infty, 2)$-category (Corollary 5.4.3.4). Suppose we are given an integer $n \geq 2$ and a lifting problem

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C}_{/f} \arrow[d, "q"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{C}_{/f_0} .
\end{tikzcd}$$

We wish to show that this lifting problem admits a solution provided that one of the following conditions is satisfied:

(a) The integer i is equal to 0 and $\sigma_0|_{\mathrm{N}_\bullet(\{0<1\})}$ is a degenerate edge of $\mathcal{C}_{/f}$.

(b) The integer i satisfies $0 < i < n$ and the restriction $\overline{\sigma}|_{\mathrm{N}_\bullet(\{i-1<i<i+1\})}$ is a thin 2-simplex of $\mathcal{C}_{/f_0}$.

(c) The integer i is equal to n and $\sigma_0|_{\mathrm{N}_\bullet(\{n-1<n\})}$ is a degenerate edge of $\mathcal{C}_{/f}$.

In cases (a) and (c), this follows immediately from Proposition 5.4.3.8. In case (b), it suffices (by virtue of Proposition 5.4.3.8) to verify that the composite map

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{i-1 < i < i+1\}) \subseteq \Delta^n \xrightarrow{\overline{\sigma}} \mathcal{C}_{/f_0} \to \mathcal{C}$$

is a thin 2-simplex of $\mathcal{C}$. This follows from our hypothesis, since the projection map $\mathcal{C}_{/f_0} \to \mathcal{C}$ preserves thin 2-simplices (Proposition 5.4.7.10). □

5.4.8 Strict $(\infty, 2)$-Categories

01YK Let $\mathcal{C}$ be a simplicial category. If $\mathcal{C}$ is locally Kan, then Theorem 2.4.5.1 guarantees that the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is an ∞-category. Our goal in this section is to establish an $(\infty, 2)$-categorical variant of this result:

01YL **Theorem 5.4.8.1.** *Let $\mathcal{C}$ be a simplicial category. Suppose that, for every pair of objects X and Y, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet$ is an ∞-category. Then the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is an $(\infty, 2)$-category.*

We will deduce Theorem 5.4.8.1 from the following thinness criterion for 2-simplices of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$.

01YM **Proposition 5.4.8.2.** *Let $\mathcal{C}$ be a simplicial category. Suppose that, for every pair of objects X and Y, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet$ is an ∞-category. Let σ be a 2-simplex of the* *homotopy coherent nerve* $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$*, which we identify with a (not necessarily commutative) diagram*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

in $\mathcal{C}$ *together with an edge* $\mu : g \circ f \to h$ *in the simplicial set* $\mathrm{Hom}_{\mathcal{C}}(X, Z)_{\bullet}$*. If* μ *is an isomorphism, then* σ *is thin.*

Proof. Suppose we are given integers $n \geq 3$, $0 < i < n$, and a morphism of simplicial sets $\tau_0 : \Lambda^n_i \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ for which the restriction $\tau_0|_{\mathrm{N}_{\bullet}(\{i-1<i<i+1\})}$ is the 2-simplex σ. We wish to show that τ_0 can be extended to an n-simplex of $\mathcal{C}$. Let $\mathrm{Path}[n]_{\bullet}$ be the simplicial category described in Notation 2.4.3.1, and let us identify $\mathrm{Path}[\Lambda^n_i]_{\bullet}$ with the simplicial subcategory of $\mathrm{Path}[n]_{\bullet}$ described in Proposition 2.4.5.8. Then τ_0 can be identified with a simplicial functor $F_0 : \mathrm{Path}[\Lambda^n_i]_{\bullet} \to \mathcal{C}$, and we wish to show that τ_0 can be extended to a simplicial functor $F : \mathrm{Path}[n]_{\bullet} \to \mathcal{C}$.

For $0 \leq j \leq n$, let C_j denote the object of $\mathcal{C}$ given by $F_0(j)$. For $1 \leq j \leq n$, let $u_j : C_{j-1} \to C_j$ be the morphism in $\mathcal{C}$ obtained by applying F_0 to the unique vertex of $\mathrm{Hom}_{\mathrm{Path}[\Lambda^n_i]}(j-1, j)$, so that we have a chain of composable morphisms

$$C_0 \xrightarrow{u_1} C_1 \xrightarrow{u_2} \cdots \xrightarrow{u_n} C_n$$

in the simplicial category $\mathcal{C}$. Let $\square^{n-1}$ denote the simplicial cube of dimension $(n-1)$ and let $\sqcap^{n-1}_i \subseteq \square^{n-1}$ denote the hollow cube of Notation 2.4.5.5, so that Remark 2.4.5.4 and Proposition 2.4.5.8 supply isomorphisms

$$\mathrm{Hom}_{\mathrm{Path}[n]}(0, n)_{\bullet} \simeq \square^{n-1} \qquad \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_i]}(0, n)_{\bullet} \simeq \sqcap^{n-1}_i.$$

Let λ_0 denote the composite map

$$\sqcap^{n-1}_i \simeq \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_i]}(0, n)_{\bullet} \xrightarrow{F_0} \mathrm{Hom}_{\mathcal{C}}(C_0, C_n)_{\bullet}.$$

By virtue of Corollary 2.4.5.10, it will suffice to show that λ_0 can be extended to a morphism of simplicial sets $\lambda : \square^{n-1} \to \mathrm{Hom}_{\mathcal{C}}(C_0, C_n)_{\bullet}$.

Let I denote the set $\{1, 2, \ldots, i-1, i+1, \cdots, n-1\}$, so that we can identify $\square^{n-1}$ with the product $\Delta^1 \times \square^I$. Under this identification, $\sqcap^{n-1}_i$ corresponds to the pushout

$$(\Delta^1 \times \partial \square^I) \coprod_{(\{0\} \times \partial \square^I)} (\{0\} \times \square^I).$$

Let $v \in \square^I$ be the initial vertex (corresponding to the empty subset of I), and let e be the edge of $\operatorname{Hom}_{\mathcal{C}}(C_0, C_n)_\bullet$ given by the composite map

$$\Delta^1 \times \{v\} \hookrightarrow \Delta^1 \times \partial \square^I \hookrightarrow \sqcap_i^{n-1} \xrightarrow{\lambda_0} \operatorname{Hom}_{\mathcal{C}}(C_0, C_n)_\bullet.$$

Unwinding the definitions, we see that e is the image of μ under the morphism of simplicial sets

$$\operatorname{Hom}_{\mathcal{C}}(C_{i-1}, C_{i+1})_\bullet \to \operatorname{Hom}_{\mathcal{C}}(C_0, C_n)_\bullet \qquad \rho \mapsto u_n \circ u_{n-1} \circ \cdots u_{i+2} \circ \rho \circ u_{i-1} \circ \cdots \circ u_1,$$

and is therefore an isomorphism in the ∞-category $\operatorname{Hom}_{\mathcal{C}}(C_0, C_n)_\bullet$. Note that every simplex of $\square^I$ which is not contained in the boundary $\partial \square^I$ has initial vertex v. The existence of the desired extension λ now follows from Proposition 4.4.5.8. $\square$

01YN **Example 5.4.8.3.** Let $\mathcal{C}$ be a simplicial category. Suppose that, for every pair of objects X and Y, the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X, Y)_\bullet$ is an ∞-category. Then the inclusion $\mathrm{N}_\bullet(\mathcal{C}) \hookrightarrow \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ of Remark 2.4.3.8 carries every 2-simplex of the ordinary nerve $\mathrm{N}_\bullet(\mathcal{C})$ to a thin 2-simplex of the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$.

To verify the outer horn-filling conditions which appear in Definition 5.4.1.1, we will need a variant of Proposition 2.4.5.8.

01YP **Proposition 5.4.8.4.** *Let $n \geq 2$ be an integer and let $F : \operatorname{Path}[\Lambda_n^n]_\bullet \to \operatorname{Path}[\Delta^n]_\bullet$ be the simplicial functor induced by the horn inclusion $\Lambda_n^n \hookrightarrow \Delta^n$. Then:*

(a) *The functor F is bijective on objects; in particular, we can identify the objects of $\operatorname{Path}[\Lambda_n^n]_\bullet$ with elements of the set $[n] = \{0 < 1 < \cdots < n\}$.*

(b) *For $(0, n-1) \neq (i, j) \neq (0, n)$, the functor F induces an isomorphism of simplicial sets*

$$\operatorname{Hom}_{\operatorname{Path}[\Lambda_n^n]}(i, j)_\bullet \simeq \operatorname{Hom}_{\operatorname{Path}[\Delta^n]}(i, j)_\bullet.$$

(c) *The functor F induces a monomorphism of simplicial sets*

$$\operatorname{Hom}_{\operatorname{Path}[\Lambda_n^n]}(0, n-1)_\bullet \hookrightarrow \operatorname{Hom}_{\operatorname{Path}[\Delta^n]}(0, n-1)_\bullet,$$

whose image can be identified with the boundary

$$\partial \square^{n-2} \subseteq \square^{n-2} \simeq \operatorname{Hom}_{\operatorname{Path}[\Delta^n]}(0, n-1)_\bullet$$

introduced in Notation 2.4.5.5.

(*d*) *The functor F induces a monomorphism of simplicial sets*

$$\mathrm{Hom}_{\mathrm{Path}[\Lambda^n_n]}(0,n)_{\bullet} \hookrightarrow \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n)_{\bullet},$$

whose image can be identified with the hollow cube

$$\sqcup^{n-1}_{n-1} \subseteq \square^{n-1} \simeq \mathrm{Hom}_{\mathrm{Path}[\Delta^n]}(0,n)_{\bullet}$$

introduced in Notation 2.4.5.5.

Proof. Assertion (a) is immediate from Theorem 2.4.4.10. To prove the remaining assertions, fix an integer $m \geq 0$. Using Lemma 2.4.4.16, we see that $\mathrm{Path}[\Delta^n]_m$ can be identified with the path category $\mathrm{Path}[G]$ of a directed graph G which can be described concretely as follows:

- The vertices of G are the elements of the set $[n] = \{0 < 1 < \cdots < n\}$.
- For $0 \leq i < j \leq n$, an edge of G with source j and target k is a chain of subsets

$$\{i, i+1, \ldots, j-1, j\} \supseteq I_0 \supseteq \cdots \supseteq I_m = \{i, j\}$$

Using Theorem 2.4.4.10, we see that $\mathrm{Path}[\Lambda^n_n]_m$ can be identified with the path category of the directed subgraph $G' \subseteq G$ having the same vertices, where an edge $\overrightarrow{I} = (I_0 \supseteq \cdots \supseteq I_m)$ of G belongs to G' if and only if the subset $I_0 \subseteq [n]$ corresponds to a simplex of Δ^n which belongs to the horn Λ^n_n: that is, if and only if $[n-1] \nsubseteq I_0$. We now argue as follows:

- For $(0, n-1) \neq (i, j) \neq (0, n)$, every path from i to j in the graph G is also a path in the graph G'. This proves (b).
- Let τ be a morphism from 0 to $n-1$ in the category $\mathrm{Path}[n]_m$, which we identify with a chain of subsets

$$[n-1] \supseteq I_0 \supseteq I_1 \supseteq \cdots \supseteq I_m \supseteq \{0, n-1\}.$$

  Then τ belongs to $\mathrm{Path}[\Lambda^n_n]_m$ if and only if $I_0 \neq [n-1]$ or $I_m \neq \{0, n-1\}$: that is, if and only if τ corresponds to an m-simplex of the cube $\partial\square^{n-2} \subseteq \square^{n-2}$. This proves ($c$).

- Let τ be a morphism from 0 to n in the category $\mathrm{Path}[n]_m$, which we identify with a chain of subsets

$$[n] \supseteq I_0 \supseteq I_1 \supseteq \cdots \supseteq I_m \supseteq \{0, n\}.$$

  Then τ belongs to $\mathrm{Path}[\Lambda^n_n]_m$ if and only if $I_0 \neq [n]$ or $\{0, n\} \neq I_m \neq \{0, n-1, n\}$: that is, if and only if τ corresponds to an m-simplex of the hollow cube $\sqcup^{n-1}_{n-1} \subseteq \square^{n-1}$. This proves ($d$).

□

01YQ **Corollary 5.4.8.5.** *Let $\mathcal{C}$ be a simplicial category, let $n \geq 2$ be an integer, and let $\sigma_0 : \Lambda^n_n \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ be a morphism of simplicial sets, which we identify with a simplicial functor $F : \mathrm{Path}[\Lambda^n_n]_\bullet \to \mathcal{C}$ inducing a map of simplicial sets*

$$\lambda_0 : \sqcup^{n-1}_{n-1} \simeq \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_n]}(0, n)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet.$$

Suppose that F carries the edge $\mathrm{N}_\bullet(\{n-1 < n\}) \subseteq \Lambda^n_n$ to an isomorphism in $\mathcal{C}$. Then the restriction map

$$\begin{tikzcd}
\{\text{Maps } \sigma : \Delta^n \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \text{ with } \sigma_0 = \sigma|_{\Lambda^n_n}\} \arrow[d, "\theta"] \\
\{\text{Maps } \lambda : \square^{n-1} \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet \text{ with } \lambda_0 = \lambda|_{\sqcup^{n-1}_{n-1}}\}
\end{tikzcd}$$

is bijective.

Proof. By virtue of Corollary 2.4.6.13, we can identify θ with a pullback of the restriction map

$$\begin{tikzcd}
\{\text{Maps } \sigma_1 : \partial\Delta^n \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \text{ with } \sigma_0 = \sigma_1|_{\Lambda^n_n}\} \arrow[d, "\theta'"] \\
\{\text{Maps } \lambda_1 : \partial\square^{n-1} \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet \text{ with } \lambda_0 = \lambda_1|_{\sqcup^{n-1}_{n-1}}\}.
\end{tikzcd}$$

It will therefore suffice to show that θ' is bijective. Let us identify Δ^{n-1} with a simplicial subset of Δ^n (via the map which is the identity on vertices), so that the boundary $\partial\Delta^{n-1}$ is contained in the horn Λ^n_n. Let τ_0 denote the restriction of σ_0 to $\partial\Delta^{n-1}$, let μ_0 denote the λ_0 to the simplicial subset $\partial\square^{n-2} \times \{0\} \subseteq \sqcup^{n-1}_{n-1}$. Note that μ_0 can be written as a composition

$$\partial\square^{n-2} \simeq \mathrm{Hom}_{\mathrm{Path}[\partial\Delta^{n-1}]}(0, n-1)_\bullet \xrightarrow{\nu_0} \mathrm{Hom}_{\mathcal{C}}(F(0), F(n-1))_\bullet \xrightarrow{e\circ} \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_\bullet,$$

where ν_0 is determined by τ_0. Using the identifications

$$\partial\Delta^n \simeq \Delta^{n-1} \coprod_{\partial\Delta^{n-1}} \Lambda^n_n \qquad \partial\square^{n-1} \simeq (\square^{n-2} \times \{0\}) \coprod_{(\partial\square^{n-2} \times \{0\})} \sqcup^{n-1}_{n-1},$$

we can identify θ' with composition

$$\begin{tikzcd}
\{\text{Maps } \tau : \Delta^{n-1} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}) \text{ with } \tau_0 = \tau|_{\partial \Delta^n}\} \arrow[d] \\
\{\text{Maps } \nu : \square^{n-2} \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n-1))_{\bullet} \text{ with } \nu = \nu_0|_{\partial \square^{n-2}}\} \arrow[d, "e\circ"] \\
\{\text{Maps } \mu : \square^{n-2} \to \mathrm{Hom}_{\mathcal{C}}(F(0), F(n))_{\bullet} \text{ with } \mu = \mu_0|_{\partial \square^{n-2}}\}.
\end{tikzcd}$$

Here the first map is bijective by virtue of Corollary 2.4.6.13, and the second by virtue of our assumption that e is an isomorphism in the simplicial category $\mathcal{C}$. $\square$

Proof of Theorem 5.4.8.1. Let $\mathcal{C}$ be a simplicial category with the property that, for every pair of objects $X, Y \in \mathcal{C}$, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ is an ∞-category. Using Example 5.4.8.3, we immediately deduce that every degenerate 2-simplex of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is thin, and that every morphism $\Lambda^2_1 \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ can be extended to a thin 2-simplex of $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$. We will complete the proof that $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is an $(\infty,2)$-category by showing that, if $n \geq 3$ and $\sigma_0 : \Lambda^n_n \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is a morphism of simplicial sets for which the 2-simplex $\sigma_0|_{\mathrm{N}_{\bullet}(\{0<n-1<n\})}$ is right-degenerate, then σ_0 can be extended to an n-simplex σ of $\mathcal{C}$ (the dual assertion regarding extension of maps $\Lambda^n_0 \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ follows by the same argument, applied to the opposite simplicial category $\mathcal{C}^{\mathrm{op}}$). Let us identify σ_0 with a simplicial functor $F : \mathrm{Path}[\Lambda^n_n]_{\bullet} \to \mathcal{C}$, carrying each element $i \in [n]$ to an object $C_i \in \mathcal{C}$.

Let $\square^{n-1}$ denote the simplicial cube of dimension $(n-1)$ and let $\sqcup^{n-1}_{n-1} \subseteq \square^{n-1}$ denote the hollow cube of Notation 2.4.5.5, so that Remark 2.4.5.4 and Proposition 5.4.8.4 supply isomorphisms

$$\mathrm{Hom}_{\mathrm{Path}[n]}(0,n)_{\bullet} \simeq \square^{n-1} \qquad \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_n]}(0,n) \simeq \sqcap^{n-1}_{n-1}.$$

Let λ_0 denote the composite map

$$\sqcup^{n-1}_{n-1} \simeq \mathrm{Hom}_{\mathrm{Path}[\Lambda^n_n]}(0,n)_{\bullet} \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}}(C_0, C_n)_{\bullet}.$$

Note that our degeneracy assumption on $\sigma_0|_{\mathrm{N}_{\bullet}(\{0<n-1<n\})}$ guarantees that the functor F induces an isomorphism $C_{n-1} \simeq C_n$ in the category $\mathcal{C}$. By virtue of Corollary 5.4.8.5, it will suffice to show that λ_0 can be extended to a morphism of simplicial sets $\lambda : \square^{n-1} \to \mathrm{Hom}_{\mathcal{C}}\, C_0, C_n)_{\bullet}$.

Let us identify $\sqcup_{n-1}^{n-1}$ with the pushout

$$(\partial\square^{n-2}\times\Delta^1)\coprod_{(\partial\square^{n-2}\times\{1\})}(\square^{n-2}\times\{1\}).$$

Let v be the final vertex of the cube $\partial\square^{n-2}$ (corresponding to the set $\{1, 2, \ldots, n-2\}$, regarded as a subset of itself). Our assumption that the 2-simplex $\sigma_0|_{\mathrm{N}_\bullet(\{0<n-1<n\})}$ is right-degenerate guarantees that the composite map

$$\{v\}\times\Delta^1 \hookrightarrow \sqcup_{n-1}^{n-1} \xrightarrow{\lambda_0} \mathrm{Hom}_{\mathcal{C}}(C_0, C_n)_\bullet.$$

is a degenerate edge of the ∞-category $\mathrm{Hom}_{\mathcal{C}}(C_0, C_n)_\bullet$; in particular, it is an isomorphism of $\mathrm{Hom}_{\mathcal{C}}(C_0, C_n)_\bullet$. Note that every simplex of $\square^{n-2}$ which is not contained in the boundary $\partial\square^{n-2}$ has final vertex v. The existence of the desired extension λ now follows by applying Proposition 4.4.5.8. $\square$

01YR **Proposition 5.4.8.6** (Functoriality). *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of simplicial categories. Assume that:*

- *For every pair of objects $C, C' \in \mathcal{C}$, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(C, C')_\bullet$ is an ∞-category.*
- *For every pair of objects $D, D' \in \mathcal{D}$, the simplicial set $\mathrm{Hom}_{\mathcal{D}}(D, D')_\bullet$ is an ∞-category.*

Then the induced map $\mathrm{N}^{\mathrm{hc}}_\bullet(F) : \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{D})$ is a functor of $(\infty, 2)$-categories: that is, it carries thin 2-simplices of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ to thin 2-simplices of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{D})$.

Proof. It follows from Theorem 5.4.8.1 that the simplicial sets $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ and $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{D})$ are $(\infty, 2)$-categories. We will show that the morphism $\mathrm{N}^{\mathrm{hc}}_\bullet(F)$ is a functor by verifying the criterion of Proposition 5.4.7.9. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in the category $\mathcal{C}$ (or, equivalently, in the $(\infty, 2)$-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$). Then f and g determine a 2-simplex of the nerve $\mathrm{N}_\bullet(\mathcal{C})$, which we identify with 2-simplex σ of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ (see Remark 2.4.3.8). By virtue of Example 5.4.8.3, σ is a thin 2-simplex of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ and its image $\mathrm{N}^{\mathrm{hc}}_\bullet(F)(\sigma)$ is a thin 2-simplex of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{D})$. $\square$

We are now equipped to establish the converse of Proposition 5.4.8.2:

01YS **Proposition 5.4.8.7.** *Let $\mathcal{C}$ be a simplicial category. Suppose that, for every pair of objects X and Y, the simplicial set $\mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet$ is an ∞-category. Let σ be a 2-simplex of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$, which we identify with a (not necessarily commutative) diagram*

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z
\end{tikzcd}$$

in $\mathcal{C}$ together with an edge $\mu : g \circ f \to h$ in the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X,Z)_{\bullet}$*. Then σ is thin if and only if μ is an isomorphism in the ∞-category* $\operatorname{Hom}_{\mathcal{C}}(X,Z)_{\bullet}$*.*

Proof. It follows from Proposition 5.4.8.2 that if μ is an isomorphism, then σ is thin. Conversely, assume that σ is thin; we wish to show that μ is an isomorphism. Define a strict 2-category $\mathcal{E}$ as follows:

- The objects of $\mathcal{E}$ are the objects of $\mathcal{C}$.
- For every pair of objects $A, B \in \mathcal{C}$, we define $\underline{\operatorname{Hom}}_{\mathcal{E}}(A,B)$ to be the homotopy category of the ∞-category $\operatorname{Hom}_{\mathcal{C}}(A,B)_{\bullet}$.
- For every triple of objects $A, B, C \in \mathcal{C}$, we define the composition law

  $$\circ : \underline{\operatorname{Hom}}_{\mathcal{E}}(B,C) \times \underline{\operatorname{Hom}}_{\mathcal{E}}(A,B) \to \underline{\operatorname{Hom}}_{\mathcal{E}}(A,C)$$

  to be the functor of homotopy categories induced by the composition law

  $$\operatorname{Hom}_{\mathcal{C}}(B,C)_{\bullet} \times \operatorname{Hom}_{\mathcal{C}}(A,B)_{\bullet} \to \operatorname{Hom}_{\mathcal{C}}(A,C)$$

  of the simplicial category $\mathcal{C}$.

Let $\mathcal{D}$ denote the simplicial category obtained by applying the construction of Example 2.4.2.8 to the strict 2-category $\mathcal{E}$: the simplicial category $\mathcal{D}$ has the same objects as $\mathcal{C}$, with simplicial morphism spaces given by

$$\operatorname{Hom}_{\mathcal{D}}(A,B)_{\bullet} = \mathrm{N}_{\bullet}(\underline{\operatorname{Hom}}_{\mathcal{E}}(A,B)) = \mathrm{N}_{\bullet}(\mathrm{h}\operatorname{Hom}_{\mathcal{C}}(A,B)_{\bullet}).$$

There is an evident functor of simplicial categories $F : \mathcal{C} \to \mathcal{D}$, which is the identity on objects and which induces the unit map $\operatorname{Hom}_{\mathcal{C}}(A,B)_{\bullet} \to \mathrm{N}_{\bullet}(\mathrm{h}\operatorname{Hom}_{\mathcal{C}}(A,B)_{\bullet})$ on simplicial morphism spaces. Invoking Proposition 5.4.8.6, we see that the induced map $\mathrm{N}^{\mathrm{hc}}_{\bullet}(F)$ carries σ to a thin 2-simplex of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{D})$, which we can identify with the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{E})$ of the 2-category $\mathcal{E}$ (Example 2.4.3.11). Using the description of the thin simplices of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{E})$ supplied by Theorem 2.3.2.5, we conclude that the homotopy class $[\mu]$ is an isomorphism in the category $\underline{\operatorname{Hom}}_{\mathcal{E}}(X,Z) = \mathrm{h}\operatorname{Hom}_{\mathcal{C}}(X,Z)_{\bullet}$, so that μ is an isomorphism in the ∞-category $\operatorname{Hom}_{\mathcal{C}}(X,Z)_{\bullet}$. □

Corollary 5.4.8.8. *Let $\mathcal{C}$ be a simplicial category having the property that, for every pair of objects $X, Y \in \mathcal{C}$, the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ is an ∞-category. Let $\mathcal{C}'$ denote the simplicial subcategory of $\mathcal{C}$ having the same objects, with morphism simplicial sets given by $\operatorname{Hom}_{\mathcal{C}'}(X,Y)_{\bullet} = \operatorname{Hom}_{\mathcal{C}}(X,Y)^{\simeq}_{\bullet}$. Then the inclusion of simplicial categories $\mathcal{C}' \hookrightarrow \mathcal{C}$ induces an isomorphism of ∞-categories $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}') \simeq \operatorname{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}))$.* 01YT

Proof. Let σ be an n-simplex of the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$, which we identify with a simplicial functor $F : \mathrm{Path}[n]_\bullet \to \mathcal{C}$ carrying each $i \in [n]$ to an object $C_i \in \mathcal{C}$. If $T \subseteq [n]$ is a nonempty subset having smallest element i and largest element k, let us write $F(T)$ for the corresponding vertex of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(C_i, C_k)_\bullet$. If $S \subseteq T$ is a subset containing i and k, let us write $F(S \subseteq T) : F(T) \to F(S)$ for the corresponding edge of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(C_i, C_k)_\bullet$. Let us abuse notation by identifying $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C}')$ with a simplicial subset of $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$. Unwinding the definitions, we see that σ is contained in $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C}')$ if and only if the following condition is satisfied:

(1) For every inclusion $S \subseteq T$ of nonempty subsets of $[n]$ having the same smallest element i and largest element k, the edge $F(S \subseteq T) : F(T) \to F(S)$ is an isomorphism in the ∞-category $\mathrm{Hom}_{\mathcal{C}}(C_i, C_k)_\bullet$.

Using the thinness criterion of Proposition 5.4.8.7, we see that σ belongs to the pith $\mathrm{Pith}(\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C}))$ if and only if the following *a priori* weaker condition is satisfied:

(2) For every triple of elements $0 \leq i \leq j \leq k \leq n$, the edge

$$F(\{i,k\} \subseteq \{i,j,k\}) : F(\{i,j,k\}) \to F(\{i,k\})$$

is an isomorphism in the ∞-category $\mathrm{Hom}_{\mathcal{C}}(C_i, C_k)_\bullet$).

To complete the proof, it will suffice to show that (2) ⇒ (1). Assume that (2) is satisfied, and suppose that we are given nonempty subsets $S \subseteq T$ of $[n]$ having the same smallest element i and largest element k. We wish to show that $F(S \subseteq T)$ is an isomorphism in the ∞-category $\mathrm{Hom}_{\mathcal{C}}(C_i, C_k)_\bullet$. Since the collection of isomorphisms contains all identity morphisms and is closed under composition (Remark 1.4.6.3), we may assume without loss of generality that the difference $T \setminus S$ contains exactly one element j. Set $S_- = \{s \in S : s < j\}$ and $S_+ = \{s \in S : s > j\}$. Let i' be the largest element of S_-, and let k' denote the smallest element of S_+. Unwinding the definitions, we see that the edge $F(S \subseteq T)$ is the image of $F(\{i', k'\} \subseteq \{i', j, k'\})$ under the functor

$$\mathrm{Hom}_{\mathcal{C}}(C_{i'}, C_{k'})_\bullet \xrightarrow{F(S_+) \circ \bullet \circ F(S_-)} \mathrm{Hom}_{\mathcal{C}}(C_i, C_k)_\bullet,$$

and is therefore an isomorphism by virtue of assumption (2). □

5.4.9 Comparison of Homotopy Transport Representations

02RX Let $\mathcal{C}$ be a locally Kan simplicial category containing an object X. Since the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ is an ∞-category (Theorem 2.4.5.1), the projection map $U : \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{X/} \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ is a left fibration (Proposition 4.3.6.1). Let $\mathrm{hTr}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{X/} / \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})} : \mathrm{hN}_\bullet^{\mathrm{hc}}(\mathcal{C}) \to \mathrm{hKan}$ denote the homotopy transport representation of U. Combining Example 5.2.8.13 with Corollary 4.6.9.20, we obtain the following concrete description of the functor $\mathrm{hTr}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{X/} / \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})}$:

Proposition 5.4.9.1. *Let $\mathcal{C}$ be locally Kan simplicial category containing an object X, and let $\Phi : \mathrm{h}\mathcal{C} \xrightarrow{\sim} \mathrm{hN}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ be the isomorphism of Proposition 2.4.6.9. Then the diagram of functors* 02RY

$$\begin{tikzcd}
& \mathrm{h}\mathcal{C} \arrow[dl, "\Phi"', "\sim"] \arrow[dr, "\mathrm{Hom}_{\mathcal{C}}(X{,}\bullet)_{\bullet}"] & \\
\mathrm{hN}^{\mathrm{hc}}_{\bullet}(\mathcal{C}) \arrow[rr, "\mathrm{hTr}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/} / \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}"] & & \mathrm{hKan}
\end{tikzcd}$$

commutes up to isomorphism.

Our goal in this section is to formulate and prove a stronger version of Proposition 5.4.9.1, which differs in three respects:

- We drop the assumption that the simplicial category $\mathcal{C}$ is locally Kan, and assume instead that the simplicial set $\mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet}$ is an ∞-category for every pair of objects $Y, Z \in \mathcal{C}$. In this case, the nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ need not be an ∞-category, so the projection map $U : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ need not be a left fibration. However, Theorem 5.4.8.1 guarantees that $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is an $(\infty, 2)$-category, so that U restricts to a cocartesian fibration of ∞-categories $\mathrm{Pith}(U) : \mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}) \to \mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}))$ (Corollary 5.4.7.11). Note that Proposition 2.4.6.9 and Corollary 5.4.8.8 supply an isomorphism of homotopy categories $\Phi : \mathrm{h}\mathcal{C}' \xrightarrow{\sim} \mathrm{hPith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}))$, where $\mathcal{C}' \subseteq \mathcal{C}$ is the locally Kan simplicial subcategory with morphism spaces given by $\mathrm{Hom}_{\mathcal{C}'}(Y, Z)_{\bullet} = \mathrm{Hom}_{\mathcal{C}}(Y, Z)^{\simeq}_{\bullet}$.

- Proposition 5.4.9.1 asserts that a certain diagram commutes up to isomorphism. However, it is possible to be more precise. For every pair of objects $X, Y \in \mathcal{C}$, Theorem 4.6.8.9 supplies an equivalence of ∞-categories

  $$\theta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Y) = \mathrm{hTr}_{\mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}) / \mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}))}(Y),$$

  so that the homotopy class $[\theta_{X,Y}]$ can be viewed as an isomorphism in the category hQCat. We will show that $[\theta_{X,Y}]$ depends functorially on Y, so that the construction $Y \mapsto [\theta_{X,Y}]$ furnishes a natural isomorphism of functors

  $$\mathrm{Hom}_{\mathcal{C}}(X, \bullet)_{\bullet} \to \mathrm{hTr}_{\mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}) / \mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}))} \circ \Phi$$

- Since $\mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}))$ is an ∞-category, we can regard the homotopy transport representation

  $$\mathrm{hTr}_{\mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}) / \mathrm{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}))} : \mathrm{hPith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})) \to \mathrm{hQCat}$$

  as an hKan-enriched functor (Construction 5.2.8.9). Similarly, we can regard Φ as an isomorphism of hKan-enriched categories (Corollary 4.6.9.20), and the construction

$Y \mapsto \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ determines an hKan-enriched functor from $\mathrm{h}\mathcal{C}'$ to hQCat. We will show that the natural isomorphism $Y \mapsto [\theta_Y]$ is compatible with these hKan-enrichments.

Our main result is the following:

`02RZ` **Theorem 5.4.9.2.** *Let $\mathcal{C}$ be a simplicial category having the property that, for every pair of objects $Y, Z \in \mathcal{C}$, the simplicial set $\operatorname{Hom}_{\mathcal{C}}(Y,Z)_{\bullet}$ is an ∞-category. Let X be an object of $\mathcal{C}$, let hTr denote the (enriched) homotopy transport representation associated to the cocartesian fibration $\operatorname{Pith}(U) : \operatorname{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}) \to \operatorname{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}))$, and let $\mathcal{C}' \subseteq \mathcal{C}$ be the locally Kan simplicial subcategory defined above. Then the diagram of hKan-enriched functors*

$$\begin{tikzcd}
 & \mathrm{h}\mathcal{C}' \arrow[dl, "\Phi"', "\sim"] \arrow[dr, "\operatorname{Hom}_{\mathcal{C}}(X, \bullet)_{\bullet}"] & \\
\mathrm{hPith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})) \arrow[rr, "\mathrm{hTr}"] & & \mathrm{hQCat}
\end{tikzcd}$$

commutes up to natural isomorphism, given explicitly by the map

$$Y \mapsto ([\theta_{X,Y}] : \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \to \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)).$$

of Construction 4.6.8.3.

Proof. For every object $Y \in \mathcal{C}$, the comparison functor

$$\theta_{X,Y} : \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \to \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y)$$

is an equivalence of ∞-categories (Theorem 4.6.8.9), so its homotopy class $[\theta_{X,Y}]$ is an isomorphism when regarded as a morphism in the homotopy category hQCat. To complete the proof, it will suffice to show that the construction $Y \mapsto [\theta_{X,Y}]$ determines a natural transformation of hKan-enriched functors. Let Y and Z be objects of $\mathcal{C}$, so that the map $\theta_{Y,Z}$ restricts to a homotopy equivalence of Kan complexes $\theta^{\simeq}_{Y,Z} : \operatorname{Hom}_{\mathcal{C}'}(Y,Z)_{\bullet} \to$

$\operatorname{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}')}(Y,Z)$. We wish to show that the diagram of Kan complexes

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$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}'}(Y,Z)_{\bullet} \arrow[r] \arrow[d, "\theta^{\simeq}_{Y,Z}"] & \operatorname{Fun}(\operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}, \operatorname{Hom}_{\mathcal{C}}(X,Z)_{\bullet})^{\simeq} \arrow[dd, "\theta_{X,Z}\circ"] \\
\operatorname{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}')}(Y,Z) \arrow[d, "\rho"] & \\
\operatorname{Fun}(\operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}}(X,Y), \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}}(X,Z))^{\simeq} \arrow[r, "\circ\theta_{X,Y}"] & \operatorname{Fun}(\operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet}, \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}}(X,Z))^{\simeq}
\end{tikzcd} \tag{5.26}$$

commutes up to homotopy, where ρ is given by parametrized covariant transport for the cocartesian fibration $\operatorname{Pith}(U) : \operatorname{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}) \to \operatorname{Pith}(\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})) \simeq \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}')$.

We will show that there exists a functor of ∞-categories

$$H : \Delta^1 \times \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \times \operatorname{Hom}_{\mathcal{C}}(Y,Z)_{\bullet} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}$$

satisfying the following requirements:

(*a*) The diagram of simplicial sets

$$\begin{tikzcd}
\Delta^1 \times \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \times \operatorname{Hom}_{\mathcal{C}}(Y,Z)_{\bullet} \arrow[rr, "H"] \arrow[d] & & \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/} \arrow[d, "U"] \\
\Delta^1 \times \operatorname{Hom}_{\mathcal{C}}(Y,Z)^{\simeq}_{\bullet} \arrow[r, "\theta_{Y,Z}"] & \Delta^1 \times \operatorname{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(Y,Z) \arrow[r] & \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})
\end{tikzcd}$$

commutes.

(*b*) The restriction $H_0 = H|_{\{0\}\times \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \times \operatorname{Hom}_{\mathcal{C}}(Y,Z)_{\bullet}}$ is given by the composition

$$\operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \times \operatorname{Hom}_{\mathcal{C}}(Y,Z)_{\bullet} \to \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \xrightarrow{\theta_{X,Y}} \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Y).$$

(*c*) The restriction $H_1 = H|_{\{1\}\times \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \times \operatorname{Hom}_{\mathcal{C}}(Y,Z)_{\bullet}}$ is given by the composition

$$\operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet} \times \operatorname{Hom}_{\mathcal{C}}(Y,Z)_{\bullet} \xrightarrow{\circ} \operatorname{Hom}_{\mathcal{C}}(X,Z)_{\bullet} \xrightarrow{\theta_{X,Z}} \operatorname{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X,Z).$$

(*d*) For every pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, the composite map

$$\Delta^1 \times \{f\} \times \{g\} \hookrightarrow \Delta^1 \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \xrightarrow{H} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})_{X/}$$

is a U-cocartesian morphism of the $(\infty,2)$-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})_{X/}$ (that is, it corresponds to a thin 2-simplex of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$; see Theorem 5.4.4.1.

Assume for the moment that there exists a morphism H satisfying these requirements. Note that the restriction $H|_{\{1\}\times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \times \mathrm{Hom}_{\mathcal{C}'}(Y,Z)_\bullet}$ can be identified with a map of Kan complexes

$$\lambda : \mathrm{Hom}_{\mathcal{C}'}(Y,Z)_\bullet \to \mathrm{Fun}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet, \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_\bullet}(X,Z))^{\simeq}.$$

It follows from requirement (*c*) that λ is given by clockwise composition around the diagram (5.26), and from requirements (*a*), (*b*), and (*d*) that λ is also given (up to homotopy) by counterclockwise composition around the diagram (5.26). It follows that the diagram (5.26) commutes up to homotopy, as desired.

It remains to construct the morphism H. Fix an auxiliary symbol e, let $n \geq 0$, and let σ be an n-simplex of the simplicial set $\Delta^1 \times \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet$. We will identify σ with a triple $(\alpha, f_\sigma, g_\sigma)$, where $\alpha : [n] \to [1]$ is a nondecreasing function, f_σ is an n-simplex of $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$, and Fg_σ is an n-simplex of $\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet$. Let $\mathrm{Path}[\{e\} \star [n]]_\bullet$ denote the simplicial path category of the linearly ordered set $\{e\} \star [n] = \{e < 0 < \cdots < n\}$ (see Notation 2.4.3.1). To the n-simplex σ, we associate a simplicial functor $h_\sigma : \mathrm{Path}[\{e\} \star [n]]_\bullet \to \mathcal{C}$ as follows:

- On objects, the functor h_σ is given by the formula

$$h_\sigma(i) = \begin{cases} X & \text{if } i = e \\ Y & \text{if } 0 \leq i \leq n \text{ and } \alpha(i) = 0 \\ Z & \text{if } 0 \leq i \leq n \text{ and } \alpha(i) = 1. \end{cases}$$

- Let $i < j$ be elements of the linearly ordered set $\{e\} \star [n]$, so that $\mathrm{Hom}_{\mathrm{Path}[\{e\}\star[n]]}(i,j)_\bullet$ can be identified with the nerve $\mathrm{N}_\bullet(Q)$, where Q is the collection of all subsets $K \subseteq \{e\} \star [n]$ having smallest element i and largest element j (and we regard Q as ordered by reverse inclusion). The simplicial functor h_σ is given on morphisms by a map of simplicial sets $u_{i,j} : \mathrm{N}_\bullet(Q) \to \mathrm{Hom}_{\mathcal{C}}(h_\sigma(i), h_\sigma(j))$. If $0 \leq i < j \leq n$ with $\alpha(i) = \alpha(j)$, we take $u_{i,j}$ to be the constant map taking the value id_Y (if $\alpha(i) = 0$) or id_Z (if $\alpha(i) = 1$). The remaining cases can be described as follows:

  (a') If $0 \leq i < j \leq n$ satisfy $\alpha(i) = 0$ and $\alpha(j) = 1$, then $u_{i,j}$ is given by the composition

  $$\mathrm{N}_\bullet(Q) \xrightarrow{r_+} \Delta^n \xrightarrow{g_\sigma} \mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet,$$

  where r_+ is given on vertices by the formula $r_+(K) = \min\{k \in K : \alpha(k) = 1\}$.

(b') If $i = e$ and $\alpha(j) = 0$, then $u_{i,j}$ is given by the composition

$$\mathrm{N}_\bullet(Q) \xrightarrow{r_-} \Delta^n \xrightarrow{f_\sigma} \mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet,$$

where r_- is given on vertices by the formula $r_-(K) = \min\{k \in K : k > e\}$.

(c') If $i = e$ and $\alpha(j) = 1$, then $u_{i,j}$ is given by the composition

$$\mathrm{N}_\bullet(Q) \xrightarrow{(r_+, r_-)} \Delta^n \times \Delta^n \xrightarrow{g_\sigma \times f_\sigma} \mathrm{Hom}_{\mathcal{C}}(Y, Z)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z)_\bullet,$$

where r_- and r_+ are defined as above.

Note that we can identify h_σ with a morphism of simplicial sets $\{e\} \star \Delta^n \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ carrying $\{e\}$ to the vertex X, which we can view as an n-simplex $H(\sigma)$ of the $(\infty, 2)$-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})_{X/}$. The construction $\sigma \mapsto H(\sigma)$ determines a morphism of simplicial sets

$$H : \Delta^1 \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(Y, Z)_\bullet \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})_{X/}.$$

Requirements (a), (b), and (c) follow immediately from (a'), (b'), and (c') (together with the definitions of the maps $\theta_{Y,Z}$, $\theta_{X,Y}$, and $\theta_{X,Z}$, respectively). Requirement (d) follows from the description of the thin 2-simplices of $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ supplied by Proposition 5.4.8.7. □

5.5 The ∞-Categories $\mathcal{S}$ and $\mathcal{QC}$

Let Kan denote the category of Kan complexes and let hKan denote its homotopy category (Construction 3.1.5.10). There is an evident forgetful functor $U : \mathrm{Kan} \to \mathrm{hKan}$, which carries each Kan complex X to itself and each morphism of Kan complexes $f : X \to Y$ to its homotopy class $[f] \in \pi_0(\mathrm{Fun}(X, Y))$. Broadly speaking, homotopy theory is concerned with questions about Kan complexes which are invariant under homotopy equivalence. Since a morphism of Kan complexes f is a homotopy equivalence if and only if its homotopy class $[f]$ is an isomorphism, it is tempting to characterize homotopy theory as the study of the category hKan. Beware that this characterization is somewhat misleading: many questions belonging to the purview of homotopy theory cannot be formulated at the level of the homotopy category. For example, suppose we are given a commutative diagram of Kan complexes σ : 01YV

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y. \end{tikzcd}$$

One can then ask if σ is a homotopy pullback square (Definition 3.4.1.1). Though the answer to this question depends only on the homotopy type of the diagram σ (Corollary 3.4.1.12), it does *not* depend only on the associated diagram $U(\sigma)$ in the homotopy category hKan (see Example 3.4.1.13).

Roughly speaking, the problem is that passage from the category of Kan complexes Kan to its homotopy category hKan destroys too much information. To remedy the situation, it is convenient to consider a refinement of the homotopy category hKan. Note that Kan has the structure of a simplicial category (see Example 2.4.2.1). In §5.5.1, we show that the homotopy coherent nerve $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan})$ is an ∞-category (Proposition 5.5.1.2), which we will denote by $\mathcal{S}$ and refer to as *the ∞-category of spaces* (Construction 5.5.1.1). After passing to nerves, the forgetful functor $U : \mathrm{Kan} \to \mathrm{hKan}$ factors as a composition

$$\mathrm{N}_{\bullet}(\mathrm{Kan}) \xhookrightarrow{U'} \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S} \xrightarrow{U''} \mathrm{N}_{\bullet}(\mathrm{hKan})$$

with the following features:

- The functor U' is a monomorphism of simplicial sets which is bijective on vertices and edges. In particular, we can identify objects of the ∞-category $\mathcal{S}$ with Kan complexes, and morphisms in the ∞-category $\mathcal{S}$ with morphisms of Kan complexes (Remark 5.5.1.3).
- The functor U'' exhibits hKan as a homotopy category of the ∞-category $\mathcal{S}$ (Remark 5.5.1.6). In particular, a map of Kan complexes $f : X \to Y$ is a homotopy equivalence if and only if it is an isomorphism when regarded as a morphism of the ∞-category $\mathcal{S}$ (Remark 5.5.1.4).

In §5.5.3, we introduce a variant of the ∞-category $\mathcal{S}$ whose objects are *pointed* Kan complexes (X, x). Here there are (at least) two different ways we might proceed:

- Let Kan_* denote the category of pointed Kan complexes (Definition 3.2.1.5). Note that Kan_* can be identified with the coslice category $\mathrm{Kan}_{\Delta^0/}$, where we regard the standard simplex Δ^0 as an object of the category Kan. This identification determines a simplicial enrichment of the category Kan_*, and we can obtain an ∞-category $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}_*)$ by passing to the homotopy coherent nerve.
- If we regard Δ^0 as an object of the ∞-category $\mathcal{S}$, then we can instead form the coslice ∞-category $\mathcal{S}_{\Delta^0/}$. We will denote this ∞-category by $\mathcal{S}_*$ and refer to it as the *∞-category of pointed spaces* (Construction 5.5.3.1).

Beware that the ∞-categories $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}_*)$ and $\mathcal{S}_*$ are not isomorphic as simplicial sets. However, there is a natural comparison functor $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}_*) \hookrightarrow \mathcal{S}_*$ which is an equivalence of ∞-categories (Proposition 5.5.3.8). This is a special case of a general assertion concerning the compatibility of the homotopy coherent nerve with (co)slice constructions (Theorem 5.5.2.21), which we formulate and prove in §5.5.2.

In §5.5.5, we consider an enlargement of the ∞-category $\mathcal{S}$. Let Set_{Δ} denote the category of simplicial sets and let $\mathbf{QCat} \subseteq \mathrm{Set}_{\Delta}$ denote the full subcategory spanned by the ∞-categories, which we again regard as a simplicial category (see Example 2.4.2.1). The homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathbf{QCat})$ is an $(\infty,2)$-category (Proposition 5.5.5.2), which we will denote by $\boldsymbol{\mathcal{QC}}$ and refer to as *the* $(\infty,2)$*-category of* ∞*-categories* (Construction 5.5.5.1). For many applications, it is convenient to work instead with the underlying ∞-category $\mathcal{QC} = \mathrm{Pith}(\boldsymbol{\mathcal{QC}})$, which we study in §5.5.4. Both of these constructions have pointed analogues, which we introduce and compare in §5.5.6.

Warning 5.5.0.1. The constructions of this section depend on a choice of dichotomy between "small" and "large" mathematical objects, and we implicitly assume that the categories $\mathrm{Set}_{\Delta} \supseteq \mathbf{QCat} \supseteq \mathrm{Kan}$ consist only of small simplicial sets. In particular, the objects of $\mathcal{S}$ are small Kan complexes, and the objects of $\mathcal{QC}$ are small ∞-categories. By contrast, the ∞-categories $\mathcal{S}$ and $\mathcal{QC}$ are not themselves small. In particular, one cannot regard $\mathcal{QC}$ as an object of itself, or the Kan complex $\mathcal{S}^{\simeq}$ as an object of $\mathcal{S}$. 01YW

5.5.1 The ∞-Category of Spaces

We begin by introducing a refinement of Construction 3.1.5.10. 01YX

Construction 5.5.1.1 (The ∞-Category of Spaces)**.** Let Kan denote the category of Kan complexes. We view Kan as a simplicial category, with simplicial morphism sets given by the construction 00TZ

$$\mathrm{Hom}_{\mathrm{Kan}}(X,Y)_{\bullet} = \mathrm{Fun}(X,Y).$$

We let $\mathcal{S}$ denote the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan})$ (Definition 2.4.3.5). We will refer to $\mathcal{S}$ as *the* ∞*-category of spaces.*

Proposition 5.5.1.2. *The simplicial set $\mathcal{S}$ is an ∞-category.* 01YY

Proof. By virtue of Theorem 2.4.5.1, it suffices to show that the simplicial category Kan is locally Kan: that is, for every pair of Kan complexes X and Y, the simplicial set $\mathrm{Fun}(X,Y)$ is also a Kan complex. This is a special case of Corollary 3.1.3.4. □

Remark 5.5.1.3. Let $\mathrm{N}_{\bullet}(\mathrm{Kan})$ denote the nerve of the category of Kan complexes, where we view Kan as an ordinary category. There is an evident monomorphism of simplicial sets 01YZ

$$\iota : \mathrm{N}_{\bullet}(\mathrm{Kan}) \hookrightarrow \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan}) = \mathcal{S},$$

which is bijective on simplices of dimension ≤ 1 (Example 2.4.3.9). In other words:

- The objects of the ∞-category $\mathcal{S}$ are Kan complexes.
- If X and Y are Kan complexes, then morphisms $f : X \to Y$ in the ∞-category $\mathcal{S}$ can be identified with morphisms of simplicial sets from X to Y.

However, ι is not bijective on simplices of dimension ≥ 2. For example, 2-simplices of $\mathcal{S}$ can be identified with diagrams of Kan complexes

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[d, Rightarrow, "\mu"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z
\end{tikzcd}$$

which commute up to a *specified* homotopy $\mu : (g \circ f) \to h$.

`01Z0` **Remark 5.5.1.4.** Let $f : X \to Y$ be a morphism of Kan complexes. Then f is a homotopy equivalence (in the sense of Definition 3.1.6.1) if and only if it is an isomorphism when viewed as a morphism of the ∞-category $\mathcal{S}$.

`01Z1` **Remark 5.5.1.5.** Let X and Y be Kan complexes. Then Remark 4.6.8.6 supplies a canonical homotopy equivalence of Kan complexes $\mathrm{Fun}(X, Y) \to \mathrm{Hom}_{\mathcal{S}}(X, Y)$. Beware that this homotopy equivalence is generally not an isomorphism.

`00U0` **Remark 5.5.1.6.** Let X and Y be Kan complexes, and let $f, g : X \to Y$ be morphisms. Then f and g are homotopic as morphisms of simplicial sets (that is, they belong to the same connected component of the Kan complex $\mathrm{Fun}(X, Y)$) if and only if they are homotopic as morphisms in the ∞-category $\mathcal{S}$ (Definition 1.4.3.1). Consequently, the category hKan of Construction 3.1.5.10 can be identified with the homotopy category of the ∞-category $\mathcal{S}$ (this is a special case of Proposition 2.4.6.9). Moreover, this identification is compatible (via the homotopy equivalences of Remark 5.5.1.5) with the hKan-enrichments supplied by Remark 3.1.5.12 and Construction 4.6.9.13 (see Corollary 4.6.9.20).

`01Z2` **Remark 5.5.1.7** (Comparison with Sets)**.** For every set S, let $\underline{S}$ denote the associated constant simplicial set (Construction 1.1.5.2). The construction $S \mapsto \underline{S}$ determines a fully faithful embedding from the category of sets to the category of Kan complexes. Passing to homotopy coherent nerves, we obtain a functor of ∞-categories $\mathrm{N}_{\bullet}(\mathrm{Set}) \to \mathcal{S}$. This functor is fully faithful: in fact, it is an isomorphism from $\mathrm{N}_{\bullet}(\mathrm{Set})$ to the full subcategory of $\mathcal{S}$ spanned by Kan complexes of the form $\underline{S}$. We will generally abuse notation by identifying (the nerve of) the category Set with its image in $\mathcal{S}$: in particular, we will not distinguish between a set S and the associated constant simplicial set $\underline{S}$, viewed as an object of $\mathcal{S}$. We can summarize the situation informally by saying that the ∞-category $\mathcal{S}$ is an enlargement of the ordinary category Set.

Remark 5.5.1.8 (Comparison with Groupoids)**.** Let **Cat** denote the (strict) 2-category of small categories, let $\mathbf{Gpd} \subseteq \mathbf{Cat}$ denote the full subcategory spanned by the groupoids, and let $\mathbf{Gpd}_{\bullet}$ denote the associated simplicial category (Example 2.4.2.8), which we can describe concretely as follows: `01Z3`

- The objects of the simplicial category $\mathbf{Gpd}_{\bullet}$ are small groupoids.
- If $\mathcal{C}$ and $\mathcal{D}$ are groupoids, then the simplicial set $\mathrm{Hom}_{\mathbf{Gpd}}(\mathcal{C}, \mathcal{D})_{\bullet}$ is the nerve of the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Note that if $\mathcal{C}$ is a groupoid, then the nerve $\mathrm{N}_{\bullet}(\mathcal{C})$ is a Kan complex (Proposition 1.3.5.2). By virtue of Proposition 1.5.3.3, the construction $\mathcal{C} \mapsto \mathrm{N}_{\bullet}(\mathcal{C})$ determines a fully faithful embedding of simplicial categories $\mathbf{Gpd}_{\bullet} \hookrightarrow \mathrm{Kan}$. Passing to homotopy coherent nerves and invoking Example 2.4.3.11, we obtain a functor of ∞-categories

$$\mathrm{N}_{\bullet}^{\mathrm{D}}(\mathbf{Gpd}) \simeq \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathbf{Gpd}_{\bullet}) \hookrightarrow \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S},$$

where $\mathrm{N}_{\bullet}^{\mathrm{D}}(\mathbf{Gpd})$ is the Duskin nerve of the 2-category **Gpd** (Construction 2.3.1.1). This functor restricts to an *isomorphism* of $\mathrm{N}_{\bullet}^{\mathrm{D}}(\mathbf{Gpd})$ with the full subcategory of $\mathcal{S}$ spanned by those Kan complexes of the form $\mathrm{N}_{\bullet}(\mathcal{C})$, where $\mathcal{C}$ is a small groupoid. We can informally summarize the situation informally by saying that the ∞-category $\mathcal{S}$ is an enlargement of the 2-category of groupoids **Gpd**.

Remark 5.5.1.9 (Comparison with Topological Spaces)**.** Let Top denote the category of topological spaces and continuous functions, endowed with the simplicial enrichment described in Example 2.4.1.5. The geometric realization construction $X \mapsto |X|$ determines a functor of simplicial categories $| \bullet | : \mathrm{Kan} \to \mathrm{Top}$ (see Construction 3.6.5.1). Moreover, if X and Y are Kan complexes, then Proposition 3.6.5.2 guarantees that the induced map `01Z4`

$$\mathrm{Fun}(X, Y) = \mathrm{Hom}_{\mathrm{Kan}}(X, Y)_{\bullet} \to \mathrm{Hom}_{\mathrm{Top}}(|X|, |Y|)_{\bullet}$$

is a homotopy equivalence of Kan complexes. Applying Corollary 4.6.8.8, we deduce that the induced map

$$\mathcal{S} = \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) \xrightarrow{|\bullet|} \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Top})$$

is a fully faithful functor of ∞-categories. The essential image of this functor is the full subcategory $\mathcal{T}_0 \subseteq \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Top})$ spanned by those topological spaces which have the homotopy type of a CW complex (Proposition 3.6.5.3). We therefore obtain an equivalence of ∞-category $\mathcal{S} \xrightarrow{|\bullet|} \mathcal{T}_0$ (Theorem 4.6.2.20), which has a homotopy inverse induced by the simplicial functor $X \mapsto \mathrm{Sing}_{\bullet}(X)$.

5.5.2 Digression: Slicing and the Homotopy Coherent Nerve

01Z5 Let $\mathcal{C}$ be a category and let $\mathrm{N}_\bullet(\mathcal{C})$ denote its nerve. For every object $X \in \mathcal{C}$, Example 4.3.5.8 supplies canonical isomorphisms

$$\mathrm{N}_\bullet(\mathcal{C}_{/X}) \simeq \mathrm{N}_\bullet(\mathcal{C})_{/X} \qquad \mathrm{N}_\bullet(\mathcal{C}_{X/}) \simeq \mathrm{N}_\bullet(\mathcal{C})_{X/}.$$

Our goal in this section is to establish a counterpart of this result in the case where $\mathcal{C}$ is a (locally Kan) simplicial category. In this case, the slice and coslice categories $\mathcal{C}_{/X}$ and $\mathcal{C}_{X/}$ inherit simplicial enrichments (Construction 5.5.2.1), and there are natural comparison maps

$$\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}_{/X}) \hookrightarrow \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})_{/X} \qquad \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}_{X/}) \hookrightarrow \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})_{X/}.$$

Beware that these maps are generally not isomorphisms at the level of simplicial sets (Warning 5.5.2.19). However, we will show that, under some mild assumptions, they are *equivalences* of ∞-categories (Theorem 5.5.2.21).

01Z6 **Construction 5.5.2.1** (Slices of Simplicial-Categories)**.** Let $\mathcal{C}$ be a simplicial category and let X be an object of $\mathcal{C}$. We define a simplicial category $\mathcal{C}_{/X}$ as follows:

- The objects of $\mathcal{C}_{/X}$ are pairs (C, f), where C is an object of $\mathcal{C}$ and $f : C \to X$ is a vertex of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(C, X)_\bullet$.
- Let (C, f) and (D, g) be objects of $\mathcal{C}_{/X}$. We let $\mathrm{Hom}_{\mathcal{C}_{/X}}((C, f), (D, g))_\bullet$ denote the simplicial set given by the fiber product

  $$\mathrm{Hom}_{\mathcal{C}}(C, D)_\bullet \times_{\mathrm{Hom}_{\mathcal{C}}(C,X)_\bullet} \{f\},$$

  which we regard as a simplicial subset of $\mathrm{Hom}_{\mathcal{C}}(C, D)_\bullet$. More precisely, we let $\mathrm{Hom}_{\mathcal{C}_{/X}}((C, f), (D, g))_\bullet$ denote the simplicial subset of $\mathrm{Hom}_{\mathcal{C}}(C, D)_\bullet$ consisting of those n-simplices σ for which the composite map is equal to the constant map $\Delta^n \twoheadrightarrow \{f\}$.
- Let (C, f), (D, g), and (E, h) be objects of $\mathcal{C}_{/X}$. Then the composition law

  $$\circ : \mathrm{Hom}_{\mathcal{C}_{X/}}((D, g), (E, h))_\bullet \times \mathrm{Hom}_{\mathcal{C}_{/X}}((C, f), (D, g))_\bullet \to \mathrm{Hom}_{\mathcal{C}_{/X}}((C, f), (E, h))_\bullet$$

  for the simplicial category $\mathcal{C}_{/X}$ is given by the restriction of the composition law

  $$\circ : \mathrm{Hom}_{\mathcal{C}}(D, E)_\bullet \times \mathrm{Hom}_{\mathcal{C}}(C, D)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(C, E)_\bullet$$

  for the simplicial category $\mathcal{C}$.

01Z7 **Exercise 5.5.2.2.** Let $\mathcal{C}$ be a simplicial category containing an object X. Show that the simplicial categories $\mathcal{C}_{/X}$ and $\mathcal{C}_{X/}$ of Construction 5.5.2.1 are well-defined (that is, the composition law of Construction 5.5.2.1 is unital and associative).

Variant 5.5.2.3 (Coslices of Simplicial-Categories)**.** Let $\mathcal{C}$ be a simplicial category and let X be an object of $\mathcal{C}$. We define a simplicial category $\mathcal{C}_{X/}$ as follows: 01Z8

- The objects of $\mathcal{C}_{X/}$ are pairs (C, f), where C is an object of $\mathcal{C}$ and f is a vertex of the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X, C)_{\bullet}$.
- Let (C, f) and (D, g) be objects of $\mathcal{C}_{X/}$. We let $\operatorname{Hom}_{\mathcal{C}_{X/}}((C, f), (D, g))_{\bullet}$ denote the simplicial set given by the fiber product

  $$\operatorname{Hom}_{\mathcal{C}}(C, D)_{\bullet} \times_{\operatorname{Hom}_{\mathcal{C}}(X,D)_{\bullet}} \{g\},$$

  which we regard as a simplicial subset of $\operatorname{Hom}_{\mathcal{C}}(C, D)_{\bullet}$.
- Let (C, f), (D, g), and (E, h) be objects of $\mathcal{C}_{X/}$. Then the composition law

  $$\circ : \operatorname{Hom}_{\mathcal{C}_{/X}}((D, g), (E, h))_{\bullet} \times \operatorname{Hom}_{\mathcal{C}_{X/}}((C, f), (D, g))_{\bullet} \rightarrow \operatorname{Hom}_{\mathcal{C}_{X/}}((C, f), (E, h))_{\bullet}$$

  for the simplicial category $\mathcal{C}_{X/}$ is given by the restriction of the composition law

  $$\circ : \operatorname{Hom}_{\mathcal{C}}(D, E)_{\bullet} \times \operatorname{Hom}_{\mathcal{C}}(C, D)_{\bullet} \rightarrow \operatorname{Hom}_{\mathcal{C}}(C, E)_{\bullet}$$

  for the simplicial category $\mathcal{C}$.

Remark 5.5.2.4. Let $\mathcal{C}$ be a simplicial category containing an object X, which we also regard as an object of the opposite simplicial category $\mathcal{C}^{\mathrm{op}}$. Then there is a canonical isomorphism of simplicial categories $(\mathcal{C}_{X/})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{/X}$. 01Z9

Remark 5.5.2.5. For every simplicial category $\mathcal{C}$, let $\mathcal{C}^{\circ}$ denote the underlying ordinary category of $\mathcal{C}$ (Example 2.4.1.4). If X is an object of $\mathcal{C}$, then we have canonical isomorphisms 01ZA

$$(\mathcal{C}_{/X})^{\circ} \simeq (\mathcal{C}^{\circ})_{/X} \qquad (\mathcal{C}_{X/})^{\circ} \simeq (\mathcal{C}^{\circ})_{X/},$$

where the left hand sides are defined using the slice and coslice operations on simplicial categories (Construction 5.5.2.1 and Variant 5.5.2.3) and the right hand sides are defined using the slice and coslice operations on ordinary categories (Construction 4.3.1.1 and Variant 4.3.1.4). In other words, the slice and coslice constructions are compatible with the forgetful functor from simplicial categories to ordinary categories. We can summarize the situation more informally as follows: if $\mathcal{C}$ is a category and X is an object of $\mathcal{C}$, then any simplicial enrichment of $\mathcal{C}$ determines a simplicial enrichment on the slice and coslice categories $\mathcal{C}_{/X}$ and $\mathcal{C}_{X/}$.

Remark 5.5.2.6. Let $\mathcal{C}$ be an ordinary category and let $\underline{\mathcal{C}}$ denote the associated constant simplicial category (Example 2.4.2.4). Then the simplicial categories $\underline{\mathcal{C}}_{/X}$ and $\underline{\mathcal{C}}_{X/}$ of Construction 5.5.2.1 and Variant 5.5.2.3 are also constant, associated to the ordinary categories $\mathcal{C}_{/X}$ and $\mathcal{C}_{X/}$ of Construction 4.3.1.1 and Variant 4.3.1.4, respectively. In other words, the slice and coslice constructions are compatible with the operation of regarding an ordinary category as a (constant) simplicial category. 01ZB

01ZC **Warning 5.5.2.7.** Let $\mathcal{C}$ be a simplicial category and let $\mathrm{h}\mathcal{C}$ denote the homotopy category of $\mathcal{C}$ (Construction 2.4.6.1). For every object $X \in \mathcal{C}$, there is a natural comparison map $\mathrm{h}(\mathcal{C}_{/X}) \to (\mathrm{h}\mathcal{C})_{/X}$, which carries an object (C, f) of the slice simplicial category $\mathcal{C}_{/X}$ to the object $(C, [f])$ of the slice category $(\mathrm{h}\mathcal{C})_{/X}$, where $[f] \in \pi_0(\mathrm{Hom}_{\mathcal{C}}(C, X)_\bullet)$ denotes the homotopy class of f. Beware that this functor is generally *not* an equivalence of categories (see Warning 3.2.1.11).

We now characterize the simplicial category $\mathcal{C}_{/X}$ of Construction 5.5.2.1 by a universal property.

01ZD **Notation 5.5.2.8.** Let $\mathcal{C}$ be a simplicial category. We define a simplicial category $\mathcal{C}^{\triangleleft}$ as follows:

- The set of objects $\mathrm{Ob}(\mathcal{C}^{\triangleleft})$ is the (disjoint) union $\mathrm{Ob}(\mathcal{C}) \cup \{X_0\}$, where X_0 is an auxiliary symbol.
- The simplicial morphism sets in $\mathcal{C}^{\triangleleft}$ are given by
$$\mathrm{Hom}_{\mathcal{C}^{\triangleleft}}(C, D)_\bullet = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(C, D)_\bullet & \text{if } C, D \in \mathrm{Ob}(\mathcal{C}) \\ \Delta^0 & \text{if } C = X_0 \\ \emptyset & \text{otherwise.} \end{cases}$$
- For objects $C, D, E \in \mathrm{Ob}(\mathcal{C}^{\triangleleft})$, the composition law
$$\circ : \mathrm{Hom}_{\mathcal{C}^{\triangleleft}}(D, E)_\bullet \times \mathrm{Hom}_{\mathcal{C}^{\triangleleft}}(C, D)_\bullet \to \mathrm{Hom}_{\mathcal{C}^{\triangleleft}}(C, E)_\bullet$$
is given by the composition law on $\mathcal{C}$ in the case where $C, D, E \in \mathrm{Ob}(\mathcal{C})$, and is otherwise uniquely determined (since either the left hand side is empty or the right hand side is Δ^0).

More informally, the simplicial category $\mathcal{C}^{\triangleleft}$ is obtained from $\mathcal{C}$ by adjoining a (new) initial object X_0. We will refer to $\mathcal{C}^{\triangleleft}$ as the *left cone on* $\mathcal{C}$, and to the object $X_0 \in \mathcal{C}^{\triangleleft}$ as the *cone point*.

01ZE **Variant 5.5.2.9.** Let $\mathcal{C}$ be a simplicial category. We define a simplicial category $\mathcal{C}^{\triangleright}$ as follows:

- The set of objects $\mathrm{Ob}(\mathcal{C}^{\triangleright})$ is given by the (disjoint) union $\mathrm{Ob}(\mathcal{C}) \cup \{Y_0\}$, where Y_0 is an auxiliary symbol.
- The simplicial morphism sets in $\mathcal{C}^{\triangleright}$ are given by
$$\mathrm{Hom}_{\mathcal{C}^{\triangleright}}(C, D)_\bullet = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(C, D)_\bullet & \text{if } C, D \in \mathrm{Ob}(\mathcal{C}) \\ \Delta^0 & \text{if } D = Y_0 \\ \emptyset & \text{otherwise.} \end{cases}$$

- For objects $C, D, E \in \mathrm{Ob}(\mathcal{C}^{\triangleright})$, the composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}^{\triangleright}}(D, E)_{\bullet} \times \mathrm{Hom}_{\mathcal{C}^{\triangleright}}(C, D)_{\bullet} \to \mathrm{Hom}_{\mathcal{C}^{\triangleright}}(C, E)_{\bullet}$$

  is given by the composition law on $\mathcal{C}$ in the case where $C, D, E \in \mathrm{Ob}(\mathcal{C})$, and is otherwise uniquely determined.

More informally, the simplicial category $\mathcal{C}^{\triangleright}$ is obtained from $\mathcal{C}$ by adjoining a (new) final object Y_0. We will refer to $\mathcal{C}^{\triangleright}$ as the *right cone on* $\mathcal{C}$, and to the object $Y_0 \in \mathcal{C}^{\triangleright}$ as the *cone point*.

Remark 5.5.2.10. Let $\mathcal{C}$ be a simplicial category. Then there is a canonical isomorphism of simplicial categories $(\mathcal{C}^{\triangleleft})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{\triangleright}$. `01ZF`

Remark 5.5.2.11. For every simplicial category $\mathcal{C}$, let $\mathcal{C}^{\circ}$ denote the underlying ordinary category of $\mathcal{C}$ (Example 2.4.1.4). Then we have canonical isomorphisms `01ZG`

$$(\mathcal{C}^{\triangleleft})^{\circ} \simeq (\mathcal{C}^{\circ})^{\triangleleft} \qquad (\mathcal{C}^{\triangleright})^{\circ} \simeq (\mathcal{C}^{\circ})^{\triangleright},$$

where the left hand sides are defined using Notation 5.5.2.8 and Variant 5.5.2.9, and the right hand sides are defined in Example 4.3.2.5. In other words, the formation of cones is compatible with the forgetful functor from simplicial categories to ordinary categories.

Remark 5.5.2.12. Let $\mathcal{C}$ be an ordinary category and let $\underline{\mathcal{C}}$ denote the associated constant simplicial category (Example 2.4.2.4). Then the simplicial categories $\underline{\mathcal{C}}^{\triangleleft}$ and $\underline{\mathcal{C}}^{\triangleright}$ of Notation 5.5.2.8 and Variant 5.5.2.9 are also constant, associated to the ordinary categories $\mathcal{C}^{\triangleleft}$ and $\mathcal{C}^{\triangleright}$ of Example 4.3.2.5. In other words, the formation of cones is compatible with the operation of regarding an ordinary category as a (constant) simplicial category. `01ZH`

Remark 5.5.2.13. For every simplicial category $\mathcal{C}$, let $\mathrm{h}\mathcal{C}$ denote its homotopy category. Then there are canonical isomorphisms of categories `01ZJ`

$$\mathrm{h}(\mathcal{C}^{\triangleleft}) \simeq (\mathrm{h}\mathcal{C})^{\triangleleft} \qquad \mathrm{h}(\mathcal{C}^{\triangleright}) \simeq (\mathrm{h}\mathcal{C})^{\triangleright}.$$

In other words, the formation of cones is compatible with the passage from a simplicial category to its homotopy category.

Remark 5.5.2.14. For every simplicial category $\mathcal{C}$, let $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ denote the homotopy coherent nerve of $\mathcal{C}$. Then there are canonical isomorphisms of simplicial sets `01ZK`

$$\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\triangleleft}) \simeq \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})^{\triangleleft} \qquad \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}^{\triangleright}) \simeq \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})^{\triangleright},$$

which are uniquely determined by the requirements that they restrict to the identity on $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ and preserve the cone points. In other words, the formation of cones is compatible with the homotopy coherent nerve.

01ZL **Construction 5.5.2.15.** Let $\mathcal{C}$ be a simplicial category and let Y be an object of $\mathcal{C}$. We define a simplicial functor $V : (\mathcal{C}_{/Y})^{\triangleright} \to \mathcal{C}$ as follows:

- The functor V carries each object $(C, f) \in \mathcal{C}_{/Y}$ to the object $C \in \mathcal{C}$, and carries the cone point $Y_0 \in (\mathcal{C}_{/Y})^{\triangleright}$ to the object $Y \in \mathcal{C}$.
- If (C, f) and (D, g) are objects of $\mathcal{C}_{/Y}$, then the induced map of simplicial sets
$$\mathrm{Hom}_{(\mathcal{C}_{/Y})^{\triangleright}}((C, f), (D, g))_{\bullet} \to \mathrm{Hom}_{\mathcal{C}}(V(C, f), V(D, g))_{\bullet}$$
is equal to the inclusion map $\mathrm{Hom}_{\mathcal{C}_{/Y}}((C, f), (D, g))_{\bullet} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(C, D)_{\bullet}$.
- If (C, f) is an object of $\mathcal{C}_{/Y}$, then the induced map
$$\Delta^0 = \mathrm{Hom}_{(\mathcal{C}_{/Y})^{\triangleright}}((C, f), Y_0)_{\bullet} \to \mathrm{Hom}_{\mathcal{C}}(V(C, f), V(Y_0))_{\bullet} = \mathrm{Hom}_{\mathcal{C}}(C, Y)_{\bullet}$$
is equal to the vertex f.

We will refer to V as the *right cone contraction functor*. Similarly, to every object $X \in \mathcal{C}$ we can associate a simplicial functor $V' : (\mathcal{C}_{X/})^{\triangleleft} \to \mathcal{C}$ carrying the cone point of $(\mathcal{C}_{X/})^{\triangleleft}$ to the object X, which we will refer to as the *left cone contraction functor*.

01ZM **Proposition 5.5.2.16.** *Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial categories. Let X_0 and Y_0 denote the cone points of $\mathcal{D}^{\triangleleft}$ and $\mathcal{D}^{\triangleright}$, respectively. Then:*

- *For every object $Y \in \mathcal{C}$, postcomposition with the right cone contraction functor $V : (\mathcal{C}_{/Y})^{\triangleright} \to \mathcal{C}$ of Construction 5.5.2.15 induces a bijection*
$$\begin{tikzcd}
\{\text{Simplicial functors } F : \mathcal{D} \to \mathcal{C}_{/Y}\} \arrow[d, "\sim"] \\
\{\text{Simplicial functors } G : \mathcal{D}^{\triangleright} \to \mathcal{C} \text{ with } G(Y_0) = Y\}
\end{tikzcd}$$
- *For every object $X \in \mathcal{C}$, postcomposition with the left cone contraction functor $V' : (\mathcal{C}_{X/})^{\triangleleft} \to \mathcal{C}$ of Construction 5.5.2.15 induces a bijection*
$$\begin{tikzcd}
\{\text{Simplicial functors } F : \mathcal{D} \to \mathcal{C}_{X/}\} \arrow[d, "\sim"] \\
\{\text{Simplicial functors } G : \mathcal{D}^{\triangleleft} \to \mathcal{C} \text{ with } G(X_0) = X\}
\end{tikzcd}$$

Proof. We will prove the first assertion; the proof of the second is similar. Fix a simplicial functor $G : \mathcal{D}^{\triangleright} \to \mathcal{C}$ and set $Y = G(Y_0)$. We wish to show that there is a unique simplicial functor $F : \mathcal{D} \to \mathcal{C}_{/Y}$ for which the composition

$$\mathcal{D}^{\triangleright} \xrightarrow{F^{\triangleright}} (\mathcal{C}_{/Y})^{\triangleright} \xrightarrow{V} \mathcal{C}$$

is equal to G. For each object $D \in \mathcal{D}$, the simplicial functor G induces a morphism of simplicial sets

$$\Delta^0 = \operatorname{Hom}_{\mathcal{D}^{\triangleright}}(D, Y_0)_{\bullet} \xrightarrow{G} \operatorname{Hom}_{\mathcal{C}}(G(D), G(Y_0))_{\bullet},$$

which we can identify with a vertex f of the simplicial set $\operatorname{Hom}_{\mathcal{C}}(G(D), Y)_{\bullet}$. The simplicial functor F is then given on objects by the formula $F(D) = (G(D), f)$, and is determined on morphisms by the requirement that the composition

$$\operatorname{Hom}_{\mathcal{D}}(D, E)_{\bullet} \xrightarrow{F} \operatorname{Hom}_{\mathcal{C}_{/Y}}(F(D), F(E))_{\bullet} \subseteq \operatorname{Hom}_{\mathcal{C}}(G(D), G(E))_{\bullet}$$

coincides with the map of simplicial sets determined by the simplicial functor G. □

Construction 5.5.2.17. Let $\mathcal{C}$ be a simplicial category, let X be an object of $\mathcal{C}$, and let $V : (\mathcal{C}_{/X})^{\triangleright} \to \mathcal{C}$ be the right cone contraction functor of Construction 5.5.2.15. Passing to homotopy coherent nerves (and invoking Remark 5.5.2.14), we obtain a map 01ZN

$$\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}_{/X})^{\triangleright} \simeq \mathrm{N}^{\mathrm{hc}}_{\bullet}((\mathcal{C}_{/X})^{\triangleright}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$$

carrying the cone point to the vertex X, which we can further identify with a morphism of simplicial sets $c : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}_{/X}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{/X}$. We will refer to c as the *slice comparison morphism.* Similarly, the left cone contraction functor $V' : (\mathcal{C}_{X/})^{\triangleleft} \to \mathcal{C}$ induces a morphism of simplicial sets $c' : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}_{X/}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}$, which we will refer to as the *coslice comparison morphism.*

Example 5.5.2.18. Let $\mathcal{C}$ be an ordinary category, which we identify with the associated constant simplicial category $\underline{\mathcal{C}}$ of Example 2.4.2.4. For every object $X \in \mathcal{C}$, the slice and coslice comparison morphisms 01ZP

$$c : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}_{/X}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{/X} \qquad c' : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}_{X/}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}$$

of Construction 5.5.2.17 can be identified with the isomorphisms $\mathrm{N}_{\bullet}(\mathcal{C}_{/X}) \simeq \mathrm{N}_{\bullet}(\mathcal{C})_{/X}$ and $\mathrm{N}_{\bullet}(\mathcal{C}_{X'}) \simeq \mathrm{N}_{\bullet}(\mathcal{C})_{X/}$ described in Example 4.3.5.8.

Warning 5.5.2.19. Let $\mathcal{C}$ be a simplicial category containing an object X. Then the slice and coslice comparison morphisms 01ZQ

$$c : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}_{/X}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{/X} \qquad c' : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}_{X/}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})_{X/}$$

of Construction 5.5.2.17 are always bijective at the level of vertices (on the left side, vertices of either of the simplicial sets $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}_{/X})$ and $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})_{/X}$ can be identified with pairs (C, f), where C is an object of $\mathcal{C}$ and f is a morphism from C to X). Beware that c and c' are generally *not* bijective on simplices of dimension ≥ 1. Unwinding the definitions, we see that edges of the simplicial set $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}_{/X})$ can be identified with diagrams

$$\begin{tikzcd}
C \arrow[rr, "h"] \arrow[dr, "f"'] & & D \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

in the category $\mathcal{C}$ which are strictly commutative, while edges of $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})_{/X}$ can be identified with diagrams which commute up to a *specified* homotopy $\mu : g \circ h \to f$ in $\mathrm{Hom}_{\mathcal{C}}(C, X)_{\bullet}$.

01ZR **Exercise 5.5.2.20.** Let $\mathcal{C}$ be a simplicial category and let X be an object of $\mathcal{C}$. Show that the slice and coslice comparison morphisms

$$c : \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}_{/X}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})_{/X} \qquad c' : \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}_{X/}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})_{X/}$$

are monomorphisms of simplicial sets.

We are now ready to state the main result of this section. For the sake of brevity, we will formulate the statement only for coslice categories (one can deduce a dual statement for slice categories by replacing $\mathcal{C}$ by its opposite).

01ZS **Theorem 5.5.2.21.** *Let $\mathcal{C}$ be a locally Kan simplicial category and let X be an object of $\mathcal{C}$ with the following property:*

($*$) *For every morphism $f : X \to Y$ and every object $Z \in \mathcal{C}$, the morphism of simplicial sets $\mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X, Z)_{\bullet}$ is a Kan fibration.*

Then the coslice comparison morphism $c' : \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}_{X/}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})_{X/}$ of Construction 5.5.2.17 is an equivalence of ∞-categories.

For many applications, hypothesis ($*$) of Theorem 5.5.2.21 is too strong: it is often satisfied only for morphisms $f : X \to Y$ which are sufficiently well-behaved. We therefore consider a somewhat more general situation:

01ZT **Proposition 5.5.2.22.** *Let $\mathcal{C}$ be a locally Kan simplicial category, let X be an object of $\mathcal{C}$, and let $\mathcal{E}$ be a full simplicial subcategory of $\mathcal{C}_{X/}$ with the following property:*

($*$) *For every pair of objects (Y, f) and (Z, g) of the simplicial category $\mathcal{E} \subseteq \mathcal{C}_{X/}$, the morphism of simplicial sets $\mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X, Z)_{\bullet}$ is a Kan fibration.*

Then the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{E})$ *is an* ∞*-category, and the coslice comparison morphism* $c' : \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C}_{X/}) \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{X/}$ *of Construction 5.5.2.17 restricts to a fully faithful functor of* ∞*-categories* $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{E}) \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{X/}$.

Proof of Theorem 5.5.2.21 from Proposition 5.5.2.22. Let $\mathcal{C}$ be a locally Kan simplicial category and let X be an object of $\mathcal{C}$ which satisfies hypothesis $(*)$ of Theorem 5.5.2.21. Applying Proposition 5.5.2.22 in the case $\mathcal{E} = \mathcal{C}_{X/}$, we conclude that the coslice comparison morphism $c' : \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C}_{X/}) \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{X/}$ is fully faithful. Since c' is bijective on vertices, it is also essentially surjective, and is therefore an equivalence of ∞-categories by virtue of Theorem 4.6.2.20. □

Proof of Proposition 5.5.2.22. Let $\mathcal{C}$ be a locally Kan simplicial category containing an object X, and let $\mathcal{E} \subseteq \mathcal{C}_{X/}$ be a full simplicial subcategory satisfying hypothesis $(*)$ of Proposition 5.5.2.22. For every pair of objects $(Y, f), (Z, g) \in \mathcal{E}$, the simplicial set $\mathrm{Hom}_{\mathcal{E}}((Y, f), (Z, g))_\bullet$ is the fiber of the Kan fibration

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z)_\bullet \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X, Z)_\bullet$$

over the vertex g, and is therefore a Kan complex (Remark 3.1.1.9). Applying Theorem 2.4.5.1, we conclude that the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{E})$ is an ∞-category. We wish to show that, for every pair of objects $(Y, f), (Z, g) \in \mathcal{E}$ as above, the coslice comparison morphism c' induces a homotopy equivalence of morphism spaces

$$\mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{E})}((Y, f), (Z, g)) \to \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{X/}}((Y, f), (Z, g)).$$

By virtue of Proposition 4.6.5.10, this is equivalent to the requirement that c' induces a homotopy equivalence $\rho : \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{E})}((Y, f), (Z, g)) \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet(\mathcal{C})_{X/}}((Y, f), (Z, g))$ of left-pinched morphism spaces.

Construction 4.6.8.3 supplies comparison maps

$$\overline{\theta} : \mathrm{Hom}_{\mathcal{E}}((Y, f), (Z, g))_\bullet \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{E})}((Y, f), (Z, g))$$

$$\theta_{Y,Z} : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_\bullet \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})}(Y, Z) \qquad \theta_{X,Z} : \mathrm{Hom}_{\mathcal{C}}(X, Z)_\bullet \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})}(X, Z),$$

which are homotopy equivalences of Kan complexes by virtue of Theorem 4.6.8.5. Let us regard $f : X \to Y$ as an edge of the simplicial set $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$, and let Q denote the fiber $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{f/} \times_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})} \{Z\}$. Since the inclusion $\{1\} \hookrightarrow \Delta^1$ is right anodyne, the restriction map $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{f/} \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{Y/}$ is a trivial Kan fibration (Proposition 4.3.6.12), and therefore restricts to a trivial Kan fibration

$$\pi : Q \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{Y/} \times_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})} \{Z\} = \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})}(Y, Z).$$

In particular, Q is a Kan complex and π is a homotopy equivalence. Let π' denote the restriction map

$$Q \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})_{X/} \times_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})} \{Z\} = \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})}(X, Z).$$

Note that π' is a pullback of the left fibration $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})_{f/} \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})_{X/}$ (Corollary 4.3.6.11), and is therefore also a left fibration (Remark 4.2.1.8). Since the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})}(X, Z)$ is a Kan complex (Proposition 4.6.5.5), the morphism π' is a Kan fibration (Corollary 4.4.3.8). We will construct an auxiliary map of Kan complexes $\lambda : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \to Q$ with the following properties:

(a) The composition $\mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \xrightarrow{\lambda} Q \xrightarrow{\pi} \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})}(Y, Z)$ is equal to $\theta_{Y,Z}$.

(b) The cubical diagram of Kan complexes

01ZU

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}((Y,f),(Z,g))_{\bullet} \arrow[rr] \arrow[dr, "\rho\circ\overline{\theta}"] \arrow[dd] & & \mathrm{Hom}_{\mathcal{C}}(Y,Z)_{\bullet} \arrow[dr, "\lambda"] \arrow[dd, "\circ f"] & \\
& \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}(\mathcal{C})_{X/}}((Y,f),(Z,g)) \arrow[rr] \arrow[dd] & & Q \arrow[dd, "\pi'"] \\
\{g\} \arrow[rr] \arrow[dr, equal] & & \mathrm{Hom}_{\mathcal{C}}(X,Z)_{\bullet} \arrow[dr, "\theta_{X,Z}"] & \\
& \{g\} \arrow[rr] & & \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})}(X,Z)
\end{tikzcd}$$ (5.27)

is commutative.

Suppose that such a map has been constructed. It follows from (a) that λ is a homotopy equivalence. Moreover, the front and back faces of the diagram (5.27) are pullback squares of simplicial sets. Since the vertical maps

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \to \mathrm{Hom}_{\mathcal{C}}(X, Z)_{\bullet} \qquad \pi' : Q \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})}(X, Z)$$

are Kan fibrations, these faces are also homotopy pullback squares (Example 3.4.1.3). Since λ, $\theta_{X,Z}$, and the identity map $\mathrm{id} : \{g\} \to \{g\}$ are homotopy equivalences of Kan complexes, it follows from Corollary 3.4.1.12 that the map $\rho \circ \overline{\theta}$ is also a homotopy equivalence of Kan complexes. Since $\overline{\theta}$ is a homotopy equivalence, we conclude that ρ is a homotopy equivalence as desired.

We now complete the proof by constructing the morphism $\lambda : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet} \to Q$. Let σ be an n-simplex of the simplicial set $\mathrm{Hom}_{\mathcal{C}}(Y, Z)_{\bullet}$, so that $\theta_{Y,Z}(\sigma)$ is an n-simplex of the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})}(Y, Z)$ which we can identify with a simplicial functor $F_{\sigma} : \mathrm{Path}[\{y\} \star [n]]_{\bullet} \to \mathcal{C}$ such that $F_{\sigma}(y) = Y$ and $F_{\sigma}|_{\mathrm{Path}[n]_{\bullet}}$ is the constant functor taking the value Z (see Construction 4.6.8.3). We extend F_σ to a simplicial functor $F_\sigma^+ : \mathrm{Path}[\{x\} \star \{y\} \star [n]]_\bullet \to \mathcal{C}$ as follows:

- The functor F_σ^+ carries x to the object $X \in \mathcal{C}$.
- For every element $i \in \{y\} \star [n]$, the induced map of simplicial sets

  $$\mathrm{Hom}_{\mathrm{Path}[\{x\}\star\{y\}\star[n]]}(x,i)_\bullet \to \mathrm{Hom}_{\mathcal{C}}(X, F_\sigma(i))_\bullet$$

  is given by the composition

  $$\begin{aligned} \mathrm{Hom}_{\mathrm{Path}[\{x\}\star\{y\}\star[n]]}(x,i)_\bullet &\xrightarrow{u} \mathrm{Hom}_{\mathrm{Path}[\{y\}\star[n]]}(y,i)_\bullet \\ &\xrightarrow{F_\sigma} \mathrm{Hom}_{\mathcal{C}}(Y, F_\sigma(i))_\bullet \\ &\xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X, F_\sigma(i))_\bullet, \end{aligned}$$

  where u is induced by the map of partially ordered sets $\{x\} \star \{y\} \star [n] \to \{y\} \star [n]$ which is the identity on $\{y\} \star [n]$ and carries x to y.

Then F_σ^+ determines a morphism of simplicial sets $\{x\}\star\{y\}\star\Delta^n \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ carrying $\{x\}\star\{y\}$ to the edge f and Δ^n to the vertex Z, which we can identify with an n-simplex $\lambda(\sigma)$ of the Kan complex Q. The construction $\sigma \mapsto \lambda(\sigma)$ depends functorially on $[n] \in \mathbf{\Delta}$, and therefore induces a morphism of simplicial sets $\lambda : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_\bullet \to Q$ which is easily verified to satisfy conditions (a) and (b). □

5.5.3 The ∞-Category of Pointed Spaces

We now study a variant of Construction 5.5.1.1. 01ZV

Construction 5.5.3.1 (The ∞-Category of Pointed Spaces)**.** Let $\mathcal{S} = \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan})$ denote the ∞-category of spaces, and regard the Kan complex Δ^0 as an object of $\mathcal{S}$. We let $\mathcal{S}_*$ denote the coslice ∞-category $\mathcal{S}_{\Delta^0/}$. We will refer to $\mathcal{S}_*$ as *the ∞-category of pointed spaces.* 01ZW

Proposition 5.5.3.2. *The simplicial set $\mathcal{S}_*$ is an ∞-category, and the projection map $\mathcal{S}_* \to \mathcal{S}$ is a left fibration of ∞-categories.* 01ZX

Proof. By virtue of Proposition 5.5.1.2, the simplicial set $\mathcal{S}$ is an ∞-category. It follows that for every object $X \in \mathcal{S}$, the projection map $\mathcal{S}_{X/} \to \mathcal{S}$ is a left fibration (Corollary 4.3.6.11). Taking $X = \Delta^0$, we conclude that the projection map $\mathcal{S}_* \to \mathcal{S}$ is a left fibration, so that $\mathcal{S}_*$ is an ∞-category (Remark 4.2.1.4). □

Example 5.5.3.3 (Objects of $\mathcal{S}_*$)**.** By definition, an object of the ∞-category $\mathcal{S}_*$ is an edge $e : \Delta^0 \to X$ of the simplicial set $\mathcal{S} = \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan})$ whose source is the Kan complex Δ^0. By virtue of Remark 5.5.1.3, this is the same data as a morphism $e : \Delta^0 \to X$ in the ordinary category of Kan complexes: that is, the data of a pointed Kan complex (X, x) (Definition 3.2.1.5). 01ZY

01ZZ **Example 5.5.3.4** (Morphisms of $\mathcal{S}_*$)**.** Let (X, x) and (Y, y) be pointed Kan complexes, regarded as objects of the ∞-category $\mathcal{S}_*$. By definition, a morphism from (X, x) to (Y, y) in the ∞-category $\mathcal{S}_*$ can be identified with a 2-simplex σ of the simplicial set $\mathcal{S} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan})$, which we can identify with a diagram of simplicial sets

$$\begin{tikzcd} & X \arrow[dr, "f"] \arrow[d, Rightarrow, "h"] & \\ \Delta^0 \arrow[ur, "x"] \arrow[rr, "y"'] & {} & Y \end{tikzcd}$$

which commutes up to a *specified* homotopy h. In other words, a morphism from (X, x) to (Y, y) in the ∞-category $\mathcal{S}_*$ can be identified with a pair (f, h), where $f : X \to Y$ is a morphism of Kan complexes and $h : f(x) \to y$ is an edge of the simplicial set Y.

02S1 **Remark 5.5.3.5.** Let X be a Kan complex, which we regard as an object of the ∞-category $\mathcal{S}$. Then Theorem 4.6.8.5 supplies a homotopy equivalence

$$\theta_X : X = \mathrm{Hom}_{\mathrm{Kan}}(\Delta^0, X)_{\bullet} \to \mathrm{Hom}^{\mathrm{L}}_{\mathcal{S}}(\Delta^0, X) = \{X\} \times_{\mathcal{S}} \mathcal{S}_* .$$

Beware that θ_X is generally not an isomorphism of simplicial sets.

02S2 **Proposition 5.5.3.6.** *Let $U : \mathcal{S}_* \to \mathcal{S}$ be the left fibration of Proposition 5.5.3.2, and let*

$$\mathrm{hTr}_{\mathcal{S}_* / \mathcal{S}} : \mathrm{h}\mathcal{S} \to \mathrm{hKan}$$

be the enriched homotopy transport representation of Variant 5.2.8.11. Then $\mathrm{hTr}_{\mathcal{S}_* / \mathcal{S}}$ *is homotopy inverse (as an* hKan*-enriched functor) to the isomorphism* $\mathrm{hKan} \simeq \mathrm{h}\mathcal{S}$ *of Remark 5.5.1.6. In particular,* $\mathrm{hTr}_{\mathcal{S}_* / \mathcal{S}}$ *is an equivalence of* hKan*-enriched categories.*

Proof. Apply Theorem 5.4.9.2 to the simplicial category Kan. □

02S3 **Remark 5.5.3.7.** The statement of Proposition 5.5.3.6 can be made more precise: Theorem 5.4.9.2 supplies an explicit hKan-enriched isomorphism from the identity functor $\mathrm{id}_{\mathrm{hKan}}$ to the composition

$$\mathrm{hKan} \xrightarrow{\sim} \mathrm{h}\mathcal{S} \xrightarrow{\mathrm{hTr}_{\mathcal{S}_* / \mathcal{S}}} \mathrm{hKan},$$

which carries each Kan complex X to the homotopy equivalence $\theta_X : X \to \{X\} \times_{\mathcal{S}} \mathcal{S}_* = \mathrm{hTr}_{\mathcal{S}_* / \mathcal{S}}(X)$ of Remark 5.5.3.5.

Let Kan_* denote the category of pointed Kan complexes (Definition 3.2.1.5). For every pair of pointed Kan complexes (X, x) and (Y, y), we let

$$\mathrm{Hom}_{\mathrm{Kan}_*}((X, x), (Y, y))_{\bullet} = \mathrm{Fun}(X, Y) \times_{\mathrm{Fun}(\{x\}, Y)} \{y\}$$

be the simplicial set parametrizing *pointed* morphisms from X to Y. If (Z, z) is another pointed Kan complex, we have an evident composition law

$$\circ : \mathrm{Hom}_{\mathrm{Kan}_*}((Y,y),(Z,z))_\bullet \times \mathrm{Hom}_{\mathrm{Kan}_*}((X,x),(Y,y))_\bullet \to \mathrm{Hom}_{\mathrm{Kan}_*}((X,x),(Z,z)),$$

which endows Kan_* with the structure of a simplicial category. Note that this construction is a special case of Variant 5.5.2.3, since Kan_* can be identified with the coslice category $\mathrm{Kan}_{\Delta^0/}$. Applying Construction 5.5.2.17, we obtain a coslice comparison functor

$$\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_*) = \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_{\Delta^0/}) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan})_{\Delta^0/} = \mathcal{S}_* .$$

Proposition 5.5.3.8. *The coslice comparison functor $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_*) \to \mathcal{S}_*$ is an equivalence of ∞-categories.* 0200

Proof. Note that, for every pair of pointed Kan complexes (X, x) and (Y, y), the evaluation map $\mathrm{Fun}(X, Y) \to \mathrm{Fun}(\{x\}, Y)$ is a Kan fibration (Corollary 3.1.3.3). Proposition 5.5.3.8 is therefore a special case of Theorem 5.5.2.21. □

Warning 5.5.3.9. The coslice comparison functor $F : \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_*) \to \mathcal{S}_*$ of Proposition 5.5.3.8 is bijective on vertices: objects of either $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_*)$ and $\mathcal{S}_*$ can be identified with pointed Kan complexes (X, x). However, it is not bijective on edges (and is therefore not an isomorphism of simplicial sets). If (X, x) and (Y, y) are pointed Kan complexes, then a morphism from (X, x) to (Y, y) in the ∞-category $\mathcal{S}_*$ can be identified with a pair (f, h), where $f : X \to Y$ is a morphism of Kan complexes and $h : f(x) \to y$ is an edge of the Kan complex Y. The pair (f, h) belongs to the image of F if and only if the edge h is degenerate (which guarantees in particular that $f(x) = y$, so that f is a morphism of pointed Kan complexes). 0201

Corollary 5.5.3.10. *The coslice comparison functor $\Phi : \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_*) \to \mathcal{S}_*$ induces an isomorphism of homotopy categories $\mathrm{h}\mathit{\Phi} : \mathrm{hKan}_* \xrightarrow{\sim} \mathrm{h}\mathcal{S}_*$, where hKan_* denotes the homotopy category of pointed Kan complexes (Construction 3.2.1.12).* 0202

Proof. It follows from Propositions 2.4.6.9 and 5.5.3.8 that the functor $\mathrm{h}\mathit{\Phi}$ is an equivalence of categories. Since it is bijective on objects, it is an isomorphism of categories. □

Note that the coslice comparison functor $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_*) \to \mathcal{S}_*$ is a monomorphism of simplicial sets (Exercise 5.5.2.20). Heuristically, we can think of $\mathcal{S}_*$ as an enlargement of the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_*)$ which is obtained by allowing morphisms between pointed Kan complexes which preserve base points only up to (specified) homotopy. By virtue of Proposition 5.5.3.8, this enlargement gives rise to an equivalent ∞-category. However, the ∞-category $\mathcal{S}_*$ is in some respects more convenient to work with, because the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ is a left fibration of ∞-categories. The composite functor $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}_*) \to \mathcal{S}_* \to \mathcal{S}$ does not share this property:

0203 **Warning 5.5.3.11.** There is an evident simplicial functor from the category Kan_* of pointed Kan complexes to the category Kan of Kan complexes, given on objects by the construction $(X, x) \mapsto X$. Passing to homotopy coherent nerves, we obtain a functor of ∞-categories $U : \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan}_*) \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$. Beware that the functor U is not a left fibration of simplicial sets. For example, suppose we are given a 2-simplex σ of $\mathcal{S}$, corresponding to a diagram of Kan complexes

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] \arrow[d, Rightarrow, "\mu"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"'] & {} & Z
\end{tikzcd}$$

which commutes up to a homotopy $\mu : (g \circ f) \to h$ (see Remark 5.5.1.3). Pick a vertex $x \in X$ and set $y = f(x)$ and $z = h(x)$, so that we have morphisms of pointed Kan complexes $f : (X, x) \to (Y, y)$ and $h : (X, x) \to (Z, z)$. This data determines a lifting problem

$$\begin{tikzcd}
\Lambda_0^2 \arrow[r, "(\bullet, h, f)"] \arrow[d] & \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan}_*) \arrow[d, "U"] \\
\Delta^2 \arrow[r, "\sigma"] \arrow[ur, dashed] & \mathcal{S},
\end{tikzcd}$$

which admits a solution if and only if $\mu(x) : g(y) \to z$ is a degenerate edge of the Kan complex Z (in which case $g(y) = z$, so that $g : (Y, y) \to (Z, z)$ is also a morphism of pointed Kan complexes).

0204 **Example 5.5.3.12** (Pointed Sets as Pointed Spaces)**.** Let Set_* denote the category of pointed sets (see Example 4.2.3.3). Every pointed set (X, x) can be regarded as a pointed Kan complex by identifying X with the corresponding constant simplicial set. This construction determines a fully faithful embedding $\mathrm{Set}_* \hookrightarrow \mathrm{Kan}_*$. Composing with the equivalence of Proposition 5.5.3.8, we obtain a functor of ∞-categories

$$\mathrm{N}_\bullet(\mathrm{Set}_*) \hookrightarrow \mathrm{N}_\bullet(\mathrm{Kan}_*) \hookrightarrow \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan}_*) \hookrightarrow \mathcal{S}_* .$$

It follows from Remark 5.5.1.7 that this functor is fully faithful: in fact, it is an *isomorphism* from $\mathrm{N}_\bullet(\mathrm{Set}_*)$ to the full subcategory of $\mathcal{S}_*$ spanned by those pointed Kan complexes (X, x) where the simplicial set X is constant.

For every group G, let $B_\bullet G$ denote its classifying simplicial set (Construction 1.3.2.5), which we regard as a Kan complex (Proposition 1.2.5.9) having a unique vertex. The construction $G \mapsto B_{\bullet}G$ determines a functor from the category **Group** of groups to the category Kan_* of pointed Kan complexes. Passing to nerves, we obtain a functor of ∞-categories

$$\mathrm{N}_{\bullet}(\mathbf{Group}) \to \mathrm{N}_{\bullet}(\mathrm{Kan}_*) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}_*) \to \mathcal{S}_* .$$

Proposition 5.5.3.13. *The functor* 0205

$$\mathrm{N}_{\bullet}(\mathbf{Group}) \to \mathcal{S}_* \qquad G \mapsto B_{\bullet}G$$

is fully faithful.

Proof. By virtue of Proposition 5.5.3.8 and Corollary 4.6.8.8, it will suffice to show that the construction $G \mapsto B_{\bullet}G$ determines a weakly fully faithful functor from **Group** (regarded as a constant simplicial category) to the simplicial category Kan_*. In other words, we must show that for every pair of groups G and H, the canonical map

$$\theta : \{\text{Group homomorphisms from } G \text{ to } H\} \to \mathrm{Hom}_{\mathrm{Kan}_*}(B_{\bullet}G, B_{\bullet}H)_{\bullet}$$

is a homotopy equivalence of Kan complexes. In fact, we claim that θ is an isomorphism of simplicial sets. Let BG denote the category having a single object X with automorphism group G, and let BH denote the category having a single object Y with automorphism group H. Proposition 1.5.3.3 then supplies an isomorphism

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Kan}_*}(B_{\bullet}G, B_{\bullet}H)_{\bullet} &= \mathrm{Fun}(\mathrm{N}_{\bullet}(BG), \mathrm{N}_{\bullet}(BH)) \times_{\mathrm{N}_{\bullet}(BH)} \mathrm{N}_{\bullet}(\{Y\}) \\
&\simeq \mathrm{N}_{\bullet}(\mathrm{Fun}(BG, BH)) \times_{\mathrm{N}_{\bullet}(BH)} \mathrm{N}_{\bullet}(\{Y\}) \\
&\simeq \mathrm{N}_{\bullet}(\mathrm{Fun}(BG, BH) \times_{BH} \{Y\}).
\end{aligned}$$

Note that if $F, F' : BG \to BH$ are functors and $\alpha : F \to F'$ is a natural transformation with the property that $\alpha_X : F(X) \to F'(X)$ is the identity morphism id_Y, then the functors F and F' are equal and α is the identity transformation (since X is the only object of the category BG). It follows that the fiber product category $\mathrm{Fun}(BG, BH) \times_{BH} \{Y\}$ is discrete: that is, it has only identity morphisms. We conclude by observing that the set of objects of the category $\mathrm{Fun}(BG, BH) \times_{BH} \{Y\}$ can be identified with the set of group homomorphisms from G to H. □

Remark 5.5.3.14 (Comparison with Pointed Topological Spaces)**.** Let Top_* denote the category whose objects are pointed topological spaces (X, x) and whose morphisms $f : (X, x) \to (Y, y)$ are continuous functions $f : X \to Y$ satisfying $f(x) = y$. We regard Top_* as a simplicial category, where the n-simplices of $\mathrm{Hom}_{\mathrm{Top}_*}((X, x), (Y, y))_{\bullet}$ are continuous maps $f : |\Delta^n| \times X \to Y$ satisfying $f(t, x) = y$ for every point $t \in |\Delta^n|$. 0206

The construction $(X, x) \mapsto (|X|, x)$ determines a simplicial functor from the category Kan_* of pointed Kan complexes to the category Top_* of pointed topological spaces. Moreover, if (X,x) and (Y,y) are pointed Kan complexes, then we have a commutative diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Hom}_{\mathrm{Kan}}(X,Y)_{\bullet} \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathrm{Top}}(|X|,|Y|)_{\bullet} \arrow[d] \\
Y \arrow[r] & \operatorname{Sing}_{\bullet}(|Y|),
\end{tikzcd}$$

where the vertical maps are Kan fibrations given by evaluation at x and the horizontal maps are homotopy equivalences (Proposition 3.6.5.2). Passing to the fiber over the vertex $y \in Y$, we deduce that the induced map

$$\operatorname{Hom}_{\mathrm{Kan}_*}((X,x),(Y,y))_{\bullet} \to \operatorname{Hom}_{\mathrm{Top}_*}((|X|,x),(|Y|,y))_{\bullet}$$

is also a homotopy equivalence of Kan complexes. Allowing (X,x) and (Y,y) to vary, we deduce that geometric realization $|\bullet| : \mathrm{Kan}_* \to \mathrm{Top}_*$ is a weakly fully faithful functor of simplicial categories (Definition 4.6.8.7), and therefore induces a fully faithful functor of ∞-categories $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan}_*) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Top}_*)$ (Corollary 4.6.8.8). Composing this functor with a homotopy inverse to the equivalence $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan}_*) \to \mathcal{S}_*$ of Proposition 5.5.3.8, we obtain a fully faithful functor $\mathcal{S}_* \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Top}_*)$.

0207 **Exercise 5.5.3.15.** Let (X,x) be a pointed topological space. Show that (X,x) belongs to the essential image of the functor $\mathcal{S}_* \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Top}_*)$ if and only if the topological space X has the homotopy type of a CW complex and the inclusion map $\{x\} \hookrightarrow X$ is a Hurewicz cofibration (that is, the union $(\{0\} \times X) \cup ([0,1] \times \{x\})$ is a retract of the product space $[0,1] \times X$).

5.5.4 The ∞-Category of ∞-Categories

0208 Let hQCat denote the homotopy category of (small) ∞-categories (Construction 4.5.1.1). Recall that the objects of hQCat are (small) ∞-categories, and a morphism from $\mathcal{C}$ to $\mathcal{D}$ in hQCat is an isomorphism class of functors from $\mathcal{C}$ to $\mathcal{D}$. In this section, we show that hQCat can be realized as the homotopy category of an ∞-category $\mathcal{QC}$, which we will refer to as *the ∞-category of ∞-categories*. Proceeding as in §5.5.1, we will realize $\mathcal{QC}$ as the homotopy coherent nerve of a simplicial category.

0209 **Construction 5.5.4.1** (The ∞-Category of ∞-Categories)**.** We define a simplicial category QCat as follows:

- The objects of QCat are (small) ∞-categories.

- If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then the simplicial set $\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet}$ is the core $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$ of the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.
- If $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are ∞-categories, then the composition law
$$\circ : \mathrm{Hom}_{\mathrm{QCat}}(\mathcal{D}, \mathcal{E})_{\bullet} \times \mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet} \rightarrow \mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{E})_{\bullet}$$
is induced by the composition map $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

We let $\mathcal{QC}$ denote the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat})$. We will refer to $\mathcal{QC}$ as the *∞-category of ∞-categories.*

Remark 5.5.4.2. Many authors use the term *quasicategory* for what we refer to as an ∞-category (see Remark 1.4.0.2); the notations of Construction 5.5.4.1 reflect this alternative terminology. `020A`

Proposition 5.5.4.3. *The simplicial set $\mathcal{QC}$ is an ∞-category.* `020B`

Proof. For every pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the core $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$ is a Kan complex (Corollary 4.4.3.11). It follows that the simplicial category QCat of Construction 5.5.4.1 is locally Kan, so its homotopy coherent nerve $\mathcal{QC} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat})$ is an ∞-category by virtue of Theorem 2.4.5.1. □

Remark 5.5.4.4. The low-dimensional simplices of $\mathcal{QC}$ are simple to describe: `020C`

- An object of $\mathcal{QC}$ is a (small) ∞-category $\mathcal{C}$.
- If $\mathcal{C}$ and $\mathcal{D}$ are objects of $\mathcal{QC}$, then a morphism from $\mathcal{C}$ to $\mathcal{D}$ in $\mathcal{QC}$ is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$.
- A 2-simplex of $\mathcal{QC}$ can be identified with a diagram
$$\begin{tikzcd} & \mathcal{D} \arrow[dr, "G"] \arrow[d, Rightarrow, "\mu", "\sim"'] & \\ \mathcal{C} \arrow[ur, "F"] \arrow[rr, "H"'] & {} & \mathcal{E} \end{tikzcd}$$
where $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are (small) ∞-categories, F, G, and H are functors, and $\mu : G \circ F \rightarrow H$ is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

Remark 5.5.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Then Remark 4.6.8.6 supplies a homotopy equivalence of Kan complexes $\phi : \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} \rightarrow \mathrm{Hom}_{\mathcal{QC}}(\mathcal{C}, \mathcal{D})$. Beware that this homotopy equivalence is generally not an isomorphism. `020F`

020D **Remark 5.5.4.6.** Let hQCat denote the homotopy category of ∞-categories (Construction 4.5.1.1), which we view as an hKan-enriched category (see Remark 3.1.5.12). Applying Proposition 2.4.6.9 and Corollary 4.6.9.20, we obtain a canonical isomorphism of hKan-enriched categories $\Phi : \mathrm{hQCat} \xrightarrow{\sim} \mathrm{h}\mathcal{QC}$, which is given on objects by the construction $\Phi(\mathcal{C}) = \mathcal{C}$ and on morphism spaces by the homotopy equivalences

$$\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_\bullet = \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} \to \mathrm{Hom}_{\mathcal{QC}}(\mathcal{C}, \mathcal{D})$$

of Remark 5.5.4.5.

020E **Remark 5.5.4.7.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Then F is an equivalence of ∞-categories (in the sense of Definition 4.5.1.10) if and only if it is an *isomorphism* in the ∞-category $\mathcal{QC}$.

020G **Remark 5.5.4.8** (Comparison with Kan Complexes)**.** Every Kan complex is an ∞-category (Example 1.4.0.3). Moreover, if X and Y are Kan complexes, then the simplicial set $\mathrm{Fun}(X, Y)$ is also a Kan complex (Corollary 3.1.3.4), and therefore coincides with its core $\mathrm{Fun}(X, Y)^{\simeq}$. It follows that we can regard the simplicial category Kan of Construction 5.5.1.1 as a full simplicial subcategory of QCat. Passing to homotopy coherent nerves, we deduce that the ∞-category $\mathcal{S} = \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan})$ is the full subcategory of $\mathcal{QC} = \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{QCat})$ spanned by the Kan complexes.

020H **Remark 5.5.4.9** (Comparison with Categories)**.** Let **Cat** denote the strict 2-category of small categories (Example 2.2.0.4), let Pith(**Cat**) denote its pith (Construction 2.2.8.9), and let us abuse notation by identifying Pith(**Cat**) with the simplicial category described in Example 2.4.2.8. Concretely, this simplicial category can be described as follows:

- The objects of Pith(**Cat**) are small categories.
- If $\mathcal{C}$ and $\mathcal{D}$ are objects of Pith(**Cat**), then the simplicial set $\mathrm{Hom}_{\mathrm{Pith}(\mathbf{Cat})}(\mathcal{C}, \mathcal{D})_\bullet$ is the nerve of the groupoid $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural isomorphisms.

By virtue of Proposition 1.5.3.3, the construction $\mathcal{C} \mapsto \mathrm{N}_\bullet(\mathcal{C})$ determines a fully faithful embedding of simplicial categories $\mathrm{Pith}(\mathbf{Cat}) \hookrightarrow \mathrm{QCat}$. Passing to homotopy coherent nerves (and invoking Example 2.4.3.11), we obtain a functor of ∞-categories $\mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{Pith}(\mathbf{Cat})) \to \mathcal{QC}$. Unwinding the definitions, we see that this functor induces an *isomorphism* from the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{Pith}(\mathbf{Cat}))$ to the full subcategory of $\mathcal{QC}$ spanned by those ∞-categories of the form $\mathrm{N}_\bullet(\mathcal{C})$, where $\mathcal{C}$ is an ordinary category.

03UT **Variant 5.5.4.10.** Let κ be an uncountable cardinal. We let $\mathcal{QC}^{<\kappa}$ denote the full subcategory of $\mathcal{QC}$ spanned by the ∞-categories which are κ-small. We will refer to $\mathcal{QC}^{<\kappa}$ as the *∞-category of essentially κ-small ∞-categories.*

Remark 5.5.4.11 (Set-Theoretic Conventions)**.** By definition, the objects of the ∞-category $\mathcal{QC} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat})$ are *small* ∞-categories. According to the convention of Remark 4.7.0.5, this means that we restrict our attention to essentially λ-small Kan complexes, where λ is some fixed uncountable strongly inaccessible cardinal. In this case, the definitions given in Variant 5.5.4.10 are appropriate only for uncountable cardinals $\kappa < \lambda$. More generally, if κ is an arbitrary uncountable cardinal, we can define $\mathcal{QC}^{<\kappa}$ to be the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}^{<\kappa})$, where $\mathrm{QCat}^{<\kappa}$ denotes the (simplicially enriched) category of κ-small ∞-categories. We then have three cases: 03UU

(*a*) If $\kappa < \lambda$, then $\mathcal{QC}^{<\kappa}$ is a full subcategory of $\mathcal{QC}$.

(*b*) If $\kappa = \lambda$, then $\mathcal{QC}^{<\kappa}$ coincides with $\mathcal{QC}$.

(*c*) If $\kappa > \lambda$, then $\mathcal{QC}$ is a full subcategory of $\mathcal{QC}^{<\kappa}$.

To simplify the exposition, we will often implicitly assume that we are in case (*a*), as suggested in Variant 5.5.4.10. However, it will be convenient to also allow case (*c*) when working with ∞-categories which are not necessarily small (such as $\mathcal{QC}$ itself).

Variant 5.5.4.12. Let κ be an uncountable cardinal. We let $\mathcal{S}^{<\kappa}$ denote the full subcategory of $\mathcal{S}$ spanned by the κ-small Kan complexes, which we also regard as a full subcategory of $\mathcal{QC}^{<\kappa}$. Similarly, we let $\mathcal{S}^{<\kappa}_{*}$ denote the full subcategory of $\mathcal{S}_{*}$ spanned by those pointed Kan complexes (X, x) where X is κ-small. 03UV

Remark 5.5.4.13. Let κ and λ be regular cardinals and suppose that κ is less than or equal to the exponential cofinality $\mathrm{ecf}(\lambda)$ (see Definition 4.7.3.16). Then the ∞-category $\mathcal{QC}^{<\kappa}$ is locally λ-small. This follows by combining Remarks 5.5.4.5 and 4.7.5.10. It follows that the full subcategory $\mathcal{S}^{<\kappa} \subseteq \mathcal{QC}^{<\kappa}$ is also locally λ-small. 03UW

5.5.5 The $(\infty, 2)$-Category of ∞-Categories

For some applications, it will be convenient to work with a variant of Construction 5.5.4.1, which retains information about non-invertible natural transformations of functors. 020J

Construction 5.5.5.1 (The $(\infty, 2)$-Category of ∞-Categories)**.** Let Set_{Δ} denote the category of simplicial sets, endowed with the simplicial enrichment of Example 2.4.2.1. We let $\mathbf{QCat}$ denote the full simplicial subcategory of Set_{Δ} spanned by the (small) ∞-categories, which we can describe concretely as follows: 020K

- The objects of $\mathbf{QCat}$ are (small) ∞-categories.
- If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, then the simplicial set $\mathrm{Hom}_{\mathbf{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet}$ is the ∞-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

We let $\boldsymbol{\mathcal{QC}}$ denote the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathbf{QCat})$. We will refer to $\boldsymbol{\mathcal{QC}}$ as the *$(\infty, 2)$-category of ∞-categories.*

020L **Proposition 5.5.5.2.** *The simplicial set $\boldsymbol{\mathcal{QC}}$ is an $(\infty, 2)$-category.*

Proof. For every pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$, Theorem 1.5.3.7 guarantees that the simplicial set $\mathrm{Hom}_{\mathbf{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is an ∞-category. The desired result is now a special case of Theorem 5.4.8.1. □

020M **Remark 5.5.5.3.** The low-dimensional simplices of $\boldsymbol{\mathcal{QC}}$ are simple to describe:

- An object of $\boldsymbol{\mathcal{QC}}$ is a (small) ∞-category $\mathcal{C}$.
- If $\mathcal{C}$ and $\mathcal{D}$ are objects of $\mathcal{QC}$, then a morphism from $\mathcal{C}$ to $\mathcal{D}$ in $\boldsymbol{\mathcal{QC}}$ is a functor $F : \mathcal{C} \to \mathcal{D}$.
- A 2-simplex σ of $\boldsymbol{\mathcal{QC}}$ can be identified with a diagram

$$\begin{tikzcd} & \mathcal{D} \arrow[dr, "G"] \arrow[d, Rightarrow, "\mu"] & \\ \mathcal{C} \arrow[ur, "F"] \arrow[rr, "H"'] & {} & \mathcal{E} \end{tikzcd}$$

  where $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are (small) ∞-categories, F, G, and H are functors, and $\mu : G \circ F \to H$ is a morphism in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$. Moreover, σ is thin if and only if μ is an isomorphism of functors (Proposition 5.4.8.7).

020N **Remark 5.5.5.4** (Comparison with $\mathcal{QC}$)**.** Let QCat and **QCat** be the simplicial categories defined in Constructions 5.5.4.1 and 5.5.5.1, respectively. There is an evident comparison map $\mathrm{QCat} \hookrightarrow \mathbf{QCat}$ which is the identity at the level of objects, and which is given on morphism spaces by the inclusion maps

$$\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D}) = \mathrm{Hom}_{\mathbf{QCat}}(\mathcal{C}, \mathcal{D}).$$

Passing to the homotopy coherent nerve, we obtain a functor of $(\infty, 2)$-categories $\mathcal{QC} \hookrightarrow \boldsymbol{\mathcal{QC}}$ which restricts to an isomorphism of ∞-categories $\mathcal{QC} \simeq \mathrm{Pith}(\boldsymbol{\mathcal{QC}})$ (Corollary 5.4.8.8).

020P **Remark 5.5.5.5.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Then Theorem 4.6.8.5 supplies an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Hom}^{\mathrm{L}}_{\boldsymbol{\mathcal{QC}}}(\mathcal{C}, \mathcal{D})$. Beware that this equivalence is generally not an isomorphism at the level of simplicial sets.

Remark 5.5.5.6 (Comparison with Kan Complexes)**.** Since every Kan complex is an ∞-category (Example 1.4.0.3), we can identify the simplicial category Kan of Construction 5.5.1.1 with a full simplicial subcategory of **QCat**. Passing to homotopy coherent nerves, we can identify ∞-category of spaces $\mathcal{S} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan})$ with the full subcategory of $\mathcal{QC} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathbf{QCat})$ spanned by the Kan complexes. 020Q

Remark 5.5.5.7 (Comparison with Categories)**.** Let **Cat** denote the strict 2-category of small categories (Example 2.2.0.4). By virtue of Proposition 1.5.3.3, the construction $\mathcal{C} \mapsto \mathrm{N}_{\bullet}(\mathcal{C})$ induces an isomorphism from the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathbf{Cat})$ to the full subcategory of $\mathcal{QC}$ spanned by those ∞-categories of the form $\mathrm{N}_{\bullet}(\mathcal{C})$, where $\mathcal{C}$ is an ordinary category. 020R

Remark 5.5.5.8 (Passage to the Homotopy Category)**.** Let $\mathrm{Cat}_{\bullet}$ denote the simplicial category associated to the strict 2-category **Cat** (see Example 2.4.2.8). For every pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$, Corollary 1.5.3.5 supplies a comparison map 025K

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{D})) \simeq \mathrm{N}_{\bullet}(\mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{h}\mathcal{D})).$$

This construction is compatible with composition, and therefore determines a functor of simplicial categories

$$\mathbf{QCat} \to \mathrm{Cat}_{\bullet} \qquad \mathcal{C} \mapsto \mathrm{h}\mathcal{C}.$$

Passing to homotopy coherent nerves (and invoking Example 2.4.3.11), we obtain a functor of $(\infty, 2)$-categories

$$\mathcal{QC} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathbf{QCat}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Cat}_{\bullet}) \simeq \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathbf{Cat}).$$

Stated more informally, the construction $\mathcal{C} \mapsto \mathrm{h}\mathcal{C}$ determines a functor from the $(\infty, 2)$-category $\mathcal{QC}$ to the ordinary 2-category **Cat**.

Variant 5.5.5.9. Let κ be an uncountable cardinal. We let $\mathcal{QC}^{<\kappa}$ denote the full simplicial subset of $\mathcal{QC}$ spanned by those ∞-categories $\mathcal{C}$ which are κ-small. Then $\mathcal{QC}^{<\kappa}$ is an $(\infty, 2)$-category, which we will refer to as *the* $(\infty, 2)$*-category of essentially* κ*-small* ∞*-categories.* 03UX

5.5.6 ∞-Categories with a Distinguished Object

In this section, we study pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is a (small) ∞-category and $C \in \mathcal{C}$ is a distinguished object. Our goal is to organize the collection of such pairs into an ∞-category. We consider several variants of this construction which are related by inclusion maps 020S

$$\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}_{*}) \hookrightarrow \mathcal{QC}_{*} \hookrightarrow \mathcal{QC}_{\mathrm{Obj}} \hookrightarrow \boldsymbol{\mathcal{QC}}_{\mathrm{Obj}};$$

their interrelationships can be described informally as follows:

- Morphisms from $(\mathcal{C}, C)$ to $(\mathcal{D}, D)$ in the ∞-category $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{QCat}_*)$ are given by functors $F : \mathcal{C} \to \mathcal{D}$ which satisfy $F(C) = D$ (that is, F is strictly compatible with the choice of distinguished objects).

- Morphisms from $(\mathcal{C}, C)$ to $(\mathcal{D}, D)$ in the ∞-category $\mathcal{QC}_*$ are given by pairs (F, α), where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\alpha : F(C) \to D$ is an isomorphism in the ∞-category $\mathcal{D}$ (that is, F is compatible with the choice of distinguished objects up to isomorphism). The inclusion $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{QCat}_*) \hookrightarrow \mathcal{QC}_*$ is an equivalence of ∞-categories (Proposition 5.5.6.6).

- Morphisms from $(\mathcal{C}, C)$ to $(\mathcal{D}, D)$ in the ∞-category $\mathcal{QC}_{\mathrm{Obj}}$ are given by pairs (F, α), where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $\alpha : F(C) \to D$ is an morphism in the ∞-category $\mathcal{D}$ which is not required to be an isomorphism; this ∞-category contains $\mathcal{QC}_*$ as a (non-full) subcategory (Remark 5.5.6.16).

- The simplicial set $\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ is an $(\infty, 2)$-category having the same objects and morphisms as $\mathcal{QC}_{\mathrm{Obj}}$, but which also contains information about non-invertible natural transformations between functors (see Example 5.5.6.17).

020T **Construction 5.5.6.1.** Let $\mathcal{QC}$ denote the ∞-category of ∞-categories (Construction 5.5.4.1), and regard the Kan complex Δ^0 as an object of $\mathcal{QC}$. We let $\mathcal{QC}_*$ denote the coslice simplicial set $\mathcal{QC}_{\Delta^0/}$.

020U **Proposition 5.5.6.2.** *The simplicial set $\mathcal{QC}_*$ is an ∞-category, and the projection map $\mathcal{QC}_* \to \mathcal{QC}$ is a left fibration of ∞-categories.*

Proof. By virtue of Proposition 5.5.4.3, the simplicial set $\mathcal{QC}$ is an ∞-category. It follows that for every object $\mathcal{C} \in \mathcal{QC}$, the projection map $\mathcal{QC}_{\mathcal{C}/} \to \mathcal{QC}$ is a left fibration (Corollary 4.3.6.11). Taking $\mathcal{C} = \Delta^0$, we conclude that the projection map $\mathcal{QC}_* \to \mathcal{QC}$ is a left fibration, so that $\mathcal{QC}_*$ is an ∞-category (Remark 4.2.1.4). □

020V **Example 5.5.6.3** (Objects and Morphisms of $\mathcal{QC}_*$)**.** The low-dimensional simplices of the ∞-category $\mathcal{QC}_*$ are easy to describe:

- The objects of $\mathcal{QC}_*$ can be identified with pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is a (small) ∞-category and $C \in \mathcal{C}$ is an object (which we identify with the morphism $\Delta^0 \to \mathcal{C}$ taking the value C).

- Let $(\mathcal{C}, C)$ and $(\mathcal{D}, D)$ be objects of $\mathcal{QC}_*$. A morphism from $(\mathcal{C}, C)$ to $(\mathcal{D}, D)$ in the ∞-category $\mathcal{QC}_*$ can be identified with a pair (F, α), where $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories and $\alpha : F(C) \to D$ is an isomorphism in the ∞-category $\mathcal{D}$.

Warning 5.5.6.4. By analogy with Definition 3.2.1.5, it would be natural to refer to the objects $(\mathcal{C}, C)$ of $\mathcal{QC}_*$ as *pointed* ∞-categories. We will avoid using this terminology, since it conflicts with another (related but distinct) notion of pointed ∞-category that we will consider later (Definition [?]). 020W

Remark 5.5.6.5 (Comparison with Pointed Spaces)**.** Let us regard the ∞-category of spaces $\mathcal{S}$ as a full subcategory of the ∞-category $\mathcal{QC}$ (Remark 5.5.4.8). The inclusion $\mathcal{S} \hookrightarrow \mathcal{QC}$ determines a functor of coslice ∞-categories $\mathcal{S}_* \to \mathcal{QC}_*$. This functor restricts to an isomorphism from $\mathcal{S}_*$ with the full subcategory of $\mathcal{QC}_*$ spanned by those pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is a Kan complex. 020X

Let QCat denote the ordinary category whose objects are (small) ∞-categories and whose morphisms are functors, and let QCat_* denote the coslice category $\mathrm{QCat}_{\Delta^0/}$. The simplicial enrichment of QCat (described in Construction 5.5.4.1) determines a simplicial enrichment of the coslice category QCat_* (see Variant 5.5.2.3), and Construction 5.5.2.17 yields a coslice comparison functor

$$\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}_*) = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}_{\Delta^0/}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat})_{\Delta^0/} = \mathcal{QC}_*.$$

Proposition 5.5.6.6. *The coslice comparison functor* $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}_*) \to \mathcal{QC}_*$ *is an equivalence of ∞-categories.* 020Y

Proof. By virtue of Theorem 5.5.2.21, it will suffice to show that for every pair of objects $(\mathcal{C}, C), (\mathcal{D}, D) \in \mathcal{QC}_*$, the restriction map

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} = \mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet} \to \mathrm{Hom}_{\mathrm{QCat}}(\{C\}, \mathcal{D}) = \mathrm{Fun}(\{C\}, \mathcal{D})^{\simeq}$$

is a Kan fibration. This follows from Proposition 4.4.3.7, since the restriction functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\{C\}, \mathcal{D})$ is an isofibration of ∞-categories (Corollary 4.4.5.3). □

Warning 5.5.6.7. The coslice comparison functor $U : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}_*) \to \mathcal{QC}_*$ of Proposition 5.5.6.6 is bijective on vertices: objects of either $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}_*)$ and $\mathcal{QC}_*$ can be identified with pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is an ∞-category and C is an object of $\mathcal{C}$. However, it is not bijective on edges (and is therefore not an isomorphism of simplicial sets). If $(\mathcal{C}, C)$ and $(\mathcal{D}, D)$ are objects of $\mathcal{QC}_*$, then a morphism from $(\mathcal{C}, C)$ to $(\mathcal{D}, D)$ in the ∞-category $\mathcal{QC}_*$ can be identified with a pair (F, α), where $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories and $\alpha : F(C) \to D$ is an isomorphism in the ∞-category $\mathcal{D}$. The pair (F, α) belongs to the image of U if and only if the isomorphism α is a degenerate edge of $\mathcal{D}$ (which guarantees in particular that $F(C) = D$). 020Z

We now introduce an enlargement of the ∞-category $\mathcal{QC}_*$.

0210 **Construction 5.5.6.8.** Let $\boldsymbol{\mathcal{QC}}$ denote the $(\infty,2)$-category of ∞-categories (Construction 5.5.5.1), and regard the Kan complex Δ^0 as an object of $\boldsymbol{\mathcal{QC}}$. We let $\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ denote the coslice simplicial set $\boldsymbol{\mathcal{QC}}_{\Delta^0/}$.

0211 **Proposition 5.5.6.9.** *The simplicial set $\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ is an $(\infty,2)$-category. Moreover, the projection map $\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}} \to \boldsymbol{\mathcal{QC}}$ is an interior fibration of $(\infty,2)$-categories.*

Proof. It follows from Proposition 5.5.5.2 that $\boldsymbol{\mathcal{QC}}$ is an $(\infty,2)$-category. The desired conclusion now follows from Corollary 5.4.3.4 and Proposition 5.4.3.1. □

0212 **Definition 5.5.6.10.** Let $\mathcal{QC}_{\mathrm{Obj}}$ denote the pith of the $(\infty,2)$-category $\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ (see Construction 5.4.5.1).

0213 **Proposition 5.5.6.11.**

(1) *The simplicial set $\mathcal{QC}_{\mathrm{Obj}}$ is an ∞-category.*

(2) *The projection map $\widetilde{V} : \boldsymbol{\mathcal{QC}}_* = \boldsymbol{\mathcal{QC}}_{\Delta^0/} \to \boldsymbol{\mathcal{QC}}$ restricts to a functor*

$$V : \mathcal{QC}_{\mathrm{Obj}} = \mathrm{Pith}(\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}) \to \mathrm{Pith}(\boldsymbol{\mathcal{QC}}) = \mathcal{QC}\,.$$

(3) *The diagram*

$$\begin{tikzcd} \mathcal{QC}_{\mathrm{Obj}} \arrow[r] \arrow[d, "V"] & \boldsymbol{\mathcal{QC}}_{\mathrm{Obj}} \arrow[d, "\widetilde{V}"] \\ \mathcal{QC} \arrow[r] & \boldsymbol{\mathcal{QC}} \end{tikzcd}$$

is a pullback square of simplicial sets.

(4) *The functor V is a cocartesian fibration of ∞-categories.*

Proof. Assertion (1) follows from Proposition 5.4.5.6. Since $\widetilde{V}$ is an interior fibration (Proposition 5.5.6.9), assertions (2) and (3) follow from Proposition 5.4.7.10. Assertion (4) is a special case of Corollary 5.4.7.11. □

0214 **Example 5.5.6.12** (Objects and Morphisms of $\mathcal{QC}_{\mathrm{Obj}}$)**.** The inclusion of simplicial sets $\mathcal{QC} \hookrightarrow \boldsymbol{\mathcal{QC}}$ induces a functor of ∞-categories $\iota : \mathcal{QC}_* \hookrightarrow \mathcal{QC}_{\mathrm{Obj}}$. The functor ι is bijective on vertices. In particular, we can identify the objects of $\mathcal{QC}_{\mathrm{Obj}}$ with pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is a (small) ∞-category and $C \in \mathcal{C}$ is an object. However, it is not bijective on edges. Unwinding the definitions, we see that a morphism $\widetilde{F}$ from $(\mathcal{C}, C)$ to $(\mathcal{D}, D)$ in the ∞-category $\mathcal{QC}_{\mathrm{Obj}}$ can be identified with a pair (F, α), where $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories and $\alpha : F(C) \to D$ is a morphism in the ∞-category $\mathcal{D}$. For every such pair (F, α), the following conditions are equivalent:

- The morphism $\widetilde{F} = (F, \alpha)$ belongs to the image of the inclusion map $\iota : \mathcal{QC}_{*} \hookrightarrow \mathcal{QC}_{\mathrm{Obj}}$.
- The morphism $\alpha : F(C) \to D$ is an isomorphism in the ∞-category $\mathcal{D}$.
- The morphism $\widetilde{F}$ is V-cocartesian, where $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ is the cocartesian fibration of Proposition 5.5.6.11.

Remark 5.5.6.13 (Fibers of V)**.** Let $\mathcal{C}$ be a small ∞-category, which we regard as an object of the ∞-category $\mathcal{QC}$. Then Construction 4.6.8.3 supplies a comparison map 02S4

$$\begin{aligned} \mathcal{C} &= \mathrm{Hom}_{\mathbf{QCat}}(\Delta^0, \mathcal{C}) \\ &\xrightarrow{\theta_{\mathcal{C}}} \mathrm{Hom}^{\mathrm{L}}_{\boldsymbol{\mathcal{QC}}}(\Delta^0, \mathcal{C}) \\ &= \{\mathcal{C}\} \times_{\boldsymbol{\mathcal{QC}}} \boldsymbol{\mathcal{QC}}_{\mathrm{Obj}} \\ &= \{\mathcal{C}\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}, \end{aligned}$$

which is an equivalence of ∞-categories (Theorem 4.6.8.9). Beware that $\theta_{\mathcal{C}}$ is generally not an isomorphism of simplicial sets (though it is bijective on n-simplices for $n \leq 1$; see Example 5.5.6.12).

We have the following generalization of Proposition 5.5.3.6:

Proposition 5.5.6.14. *Let $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ be the cocartesian fibration of Proposition 5.5.6.11 and let* 02S5

$$\mathrm{hTr}_{\mathcal{QC}_{\mathrm{Obj}} / \mathcal{QC}} : \mathrm{h}\mathcal{QC} \to \mathrm{hQCat}$$

denote the enriched homotopy transport representation of Construction 5.2.8.9. Then $\mathrm{hTr}_{\mathcal{QC}_{\mathrm{Obj}} / \mathcal{QC}}$ *is homotopy inverse (as an* hKan*-enriched functor) to the isomorphism* $\mathrm{hQCat} \simeq \mathrm{h}\mathcal{QC}$ *supplied by Remark 5.5.4.6. In particular,* $\mathrm{hTr}_{\mathcal{QC}_{\mathrm{Obj}} / \mathcal{QC}}$ *is an equivalence of* hKan*-enriched categories.*

Proof. Apply Theorem 5.4.9.2 to the simplicial category **QCat**. □

Remark 5.5.6.15. The statement of Proposition 5.5.6.14 can be made more precise: Theorem 5.4.9.2 supplies an explicit hKan-enriched isomorphism from the identity functor $\mathrm{id}_{\mathrm{hQCat}}$ to the composition 02S6

$$\mathrm{hQCat} \xrightarrow{\sim} \mathrm{h}\mathcal{QC} \xrightarrow{\mathrm{hTr}_{\mathcal{QC}_{\mathrm{Obj}} / \mathcal{QC}}} \mathrm{hQCat},$$

which carries each small ∞-category $\mathcal{C}$ to the equivalence

$$\theta_{\mathcal{C}} : \mathcal{C} \to \{\mathcal{C}\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}} = \mathrm{hTr}_{\mathcal{QC}_{\mathrm{Obj}} / \mathcal{QC}}(\mathcal{C})$$

described in Remark 5.5.6.13.

0215 **Remark 5.5.6.16.** The inclusion map $\iota : \mathcal{QC}_* \hookrightarrow \mathcal{QC}_{\mathrm{Obj}}$ is an isomorphism from $\mathcal{QC}_*$ to the (non-full) subcategory of $\mathcal{QC}_{\mathrm{Obj}}$ spanned by those morphisms which satisfy the conditions of Example 5.5.6.12. In other words, the projection map $\mathcal{QC}_* \to \mathcal{QC}$ is the underlying left fibration of the cocartesian fibration $\mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ (see Corollary 5.4.7.12).

Note that the inclusion map $\mathcal{QC}_{\mathrm{Obj}} = \mathrm{Pith}(\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}) \hookrightarrow \boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ is bijective on simplices of dimension ≤ 1 (Remark 5.4.5.2). However, it is not bijective at the level of 2-simplices.

0216 **Example 5.5.6.17** (2-Simplices of $\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$)**.** By virtue of Example 5.5.6.12, a morphism of simplicial sets $\sigma_0 : \partial\Delta^2 \to \boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ can be identified with the following data:

- A collection of ∞-categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ equipped with distinguished objects $C \in \mathcal{C}$, $D \in \mathcal{D}$, and $E \in \mathcal{E}$.
- A collection of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{E}$, and $H : \mathcal{C} \to \mathcal{E}$.
- A collection of morphisms $\alpha : F(C) \to D$, $\beta : G(D) \to E$, and $\gamma : H(C) \to E$ in the ∞-categories $\mathcal{D}$ and $\mathcal{E}$.

Unwinding the definitions, we see that extending σ_0 to a 2-simplex σ of $\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ is equivalent to choosing a natural transformation of functors $\mu : (G \circ F) \to H$ and a morphism of simplicial sets $\theta : \square^2 \to \mathcal{E}$ whose restriction to the boundary $\partial\square^2$ is indicated in the diagram

$$\begin{tikzcd}
(G \circ F)(C) \arrow[r, "\mu(C)"] \arrow[d, "G(\alpha)"] & H(C) \arrow[d, "\gamma"] \\
G(D) \arrow[r, "\beta"] & E.
\end{tikzcd}$$

Moreover:

- The 2-simplex σ belongs to the image of $\mathcal{QC}_{\mathrm{Obj}} \hookrightarrow \boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ if and only if $\mu : G \circ F \to H$ is an isomorphism in the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$.
- The 2-simplex σ belongs to the image of $\mathcal{QC}_* \hookrightarrow \boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ if and only if μ, α, β, and γ are all isomorphisms.
- The 2-simplex σ belongs to the image of $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}_*) \hookrightarrow \boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ if and only if μ, α, β, and γ are identity morphisms (so that $H = G \circ F$, $D = F(C)$, and $E = G(D)$) and the morphism $\theta : \square^2 \to \mathcal{E}$ is constant.

03UY **Variant 5.5.6.18.** Let κ be an uncountable cardinal. We let $\boldsymbol{\mathcal{QC}}^{<\kappa}_{\mathrm{Obj}}$ denote the full simplicial subset of $\boldsymbol{\mathcal{QC}}_{\mathrm{Obj}}$ spanned by those pairs $(\mathcal{C}, C)$ where the ∞-category $\mathcal{C}$ is κ-small, and we define $\mathcal{QC}^{<\kappa}_{\mathrm{Obj}} = \mathrm{Pith}(\boldsymbol{\mathcal{QC}}^{<\kappa}_{\mathrm{Obj}})$ similarly. The projection map $\mathcal{QC}^{<\kappa}_{\mathrm{Obj}} \to \mathcal{QC}^{<\kappa}$ is then a cocartesian fibration of ∞-categories, whose fibers are κ-small.

5.6 Classification of Cocartesian Fibrations

Our goal in this section is to address the following: 027M

Question 5.6.0.1. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. To what extent can U be recovered from the collection of ∞-categories $\{\mathcal{E}_C\}_{C \in \mathcal{C}}$? 0380

In §5.2.7, we gave an answer to Question 5.6.0.1 under the assumption that U is a left covering map. In this case, the construction $C \mapsto \mathcal{E}_C$ determines a functor $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{Set}$. Moreover, the ∞-category $\mathcal{E}$ can be recovered (up to isomorphism) as the fiber product $\mathcal{C} \times_{\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})} \int_{\mathrm{h}\mathcal{C}} \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ (Proposition 5.2.7.2), where the second factor denotes the *category of elements* of the set-valued functor $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ (Construction 5.2.6.1).

In the setting of classical category theory, Grothendieck gave a complete answer to Question 5.6.0.1. Let $\mathcal{C}$ be an ordinary category, and let **Cat** denote the (strict) 2-category of small categories (Example 2.2.0.4), and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories. In §5.6.1, we introduce a category $\int_{\mathcal{C}} \mathscr{F}$ whose objects are pairs (C, X) where C is an object of $\mathcal{C}$ and X is an object of the category $\mathscr{F}(C)$. We will refer to $\int_{\mathcal{C}} \mathscr{F}$ as the *category of elements* of the functor $\mathscr{F}$ (Definition 5.6.1.1). The category $\int_{\mathcal{C}} \mathscr{F}$ is equipped with a cocartesian fibration $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$, given on objects by the construction $(C, X) \mapsto C$. Moreover, the cocartesian fibration U is *essentially small*: that is, for each object $C \in \mathcal{C}$, the fiber $U^{-1}\{C\}$ is an essentially small category (since it is equivalent to the small category $\mathscr{F}(C)$). In [27], Grothendieck showed that, up to isomorphism, every essentially small cocartesian fibration can be obtained in this way (Corollary 5.6.5.19).

In §5.6.2, we introduce an ∞-categorical counterpart of the preceding construction. Let $\mathcal{QC}_{\mathrm{Obj}}$ denote the ∞-category of Construction 5.5.6.10, whose objects are pairs $(\mathcal{A}, X)$ where $\mathcal{A}$ is a (small) ∞-category and X is an object of $\mathcal{A}$. For every morphism of simplicial sets $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$, we let $\int_{\mathcal{C}} \mathscr{F}$ denote the fiber product $\mathcal{C} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}$. By construction, vertices of $\int_{\mathcal{C}} \mathscr{F}$ can be identified with pairs (C, X), where C is a vertex of $\mathcal{C}$ and X is an object of the ∞-category $\mathscr{F}(C)$. Projection onto the first factor determines a cocartesian fibration of simplicial sets $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$, given on objects by the construction $(C, X) \mapsto C$ (Proposition 5.6.2.2). In particular, if $\mathcal{C}$ is an ∞-category, then the simplicial set $\int_{\mathcal{C}} \mathscr{F}$ is also an ∞-category, which we refer to as the *∞-category of elements of $\mathscr{F}$* (Definition 5.6.2.4). This construction has the following features:

- Let $\mathcal{C}$ be an ordinary category and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories, so that the construction $C \mapsto \mathrm{N}_\bullet(\mathscr{F}(C))$ determines a functor of ∞-categories $\mathrm{N}_\bullet(\mathscr{F}) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathcal{QC}$. In §5.6.3, we construct a canonical isomorphism of simplicial sets

$$\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet(\mathscr{F}) \simeq \mathrm{N}_\bullet(\int_{\mathcal{C}} \mathscr{F})$$

where the left hand side is the ∞-category of elements of the functor $\mathrm{N}_{\bullet}(\mathscr{F})$ and the right hand side is the nerve of the ordinary category of elements of the functor $\mathscr{F}$ (Proposition 5.6.3.4). Consequently, we can view the ∞-category of elements construction as a generalization of the classical category of elements construction.

- Let $\mathcal{C}$ be an ordinary category and let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a functor of ordinary categories. Passing to the homotopy coherent nerve, we obtain a functor of ∞-categories $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{QC}$. In §5.6.4, we construct a comparison map
$$\theta : \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$$
and show that it is an equivalence of ∞-categories (Proposition 5.6.4.8). In other words, we can think of the ∞-category of elements as a variant of the weighted nerve construction, which can be applied to homotopy coherent diagrams which are not strictly commutative. Beware that θ is usually not an isomorphism of simplicial sets.

It is not difficult to show that if diagrams $\mathscr{F}, \mathscr{F}' : \mathcal{C} \to \mathcal{QC}$ are isomorphic (as objects of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$), then the cocartesian fibrations
$$\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C} \qquad \int_{\mathcal{C}} \mathscr{F}' \to \mathcal{C}$$
are equivalent (see Proposition 5.6.2.19). It follows that the construction $\mathscr{F} \mapsto \int_{\mathcal{C}} \mathscr{F}$ determines a function from the collection of isomorphism classes in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$ to the collection of equivalence classes of cocartesian fibrations over $\mathcal{C}$. We will show that, modulo set-theoretic technicalities, this function is a bijection.

028K **Theorem 5.6.0.2** (Universality Theorem)**.** *Let $\mathcal{C}$ be a simplicial set. Then the construction*
$$(\mathscr{F} : \mathcal{C} \to \mathcal{QC}) \mapsto (\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C})$$
induces a bijection from $\pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{QC})^{\simeq})$ to the set of equivalence classes of essentially small cocartesian fibrations $U : \mathcal{E} \to \mathcal{C}$.

027Q **Warning 5.6.0.3.** In the statement of Theorem 5.6.0.2, the essential smallness assumption cannot be omitted: if the cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ is equivalent to $\int_{\mathcal{C}} \mathscr{F}$ for some diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$, then each fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is equivalent to the small ∞-category $\mathscr{F}(C)$ (see Example 5.6.2.18).

028M **Remark 5.6.0.4.** We can summarize Theorem 5.6.0.2 more informally by saying that the projection map $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ is *universal* among essentially small cocartesian fibrations. Note that this property characterizes the ∞-category $\mathcal{QC}$ (and the cocartesian fibration V) up to equivalence.

Remark 5.6.0.5. We will later show that the bijection of Theorem 5.6.0.2 can be upgraded to an equivalence of ∞-categories; see Theorem [?]. 028N

Corollary 5.6.0.6. *Let $\mathcal{C}$ be a simplicial set. Then the construction* 028T

$$(\mathscr{F} : \mathcal{C} \to \mathcal{S}) \mapsto (\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C})$$

induces a bijection from $\pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\simeq})$ to the set of equivalence classes of essentially small left fibrations $U : \mathcal{E} \to \mathcal{C}$.

Example 5.6.0.7. Let $\mathcal{C}$ be a locally small ∞-category and let X be an object of $\mathcal{C}$. It follows from Corollary 5.6.0.6 that there is an essentially unique functor $h^X : \mathcal{C} \to \mathcal{S}$ for $\int_{\mathcal{C}} h^X$ is equivalent to $\mathcal{C}_{X/}$ as left fibrations over $\mathcal{C}$. We will refer to $h^X : \mathcal{C} \to \mathcal{S}$ as the *functor corepresented by X*. For every object $Y \in \mathcal{C}$, we have isomorphisms 0381

$$h^X(Y) \simeq \{Y\} \times_{\mathcal{C}} \int_{\mathcal{C}} h^X \simeq \{Y\} \times_{\mathcal{C}} \mathcal{C}_{X/} = \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

in the homotopy category hKan, depending functorially on Y. In §5.6.6, we will show that this property characterizes the functor h^X up to isomorphism (Theorem 5.6.6.13).

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. We will say that a diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ is a *covariant transport representation for U* if there exists an equivalence $\alpha : \mathcal{E} \to \int_{\mathcal{C}} \mathscr{F}$ of cocartesian fibrations over $\mathcal{C}$ (Definition 5.6.5.1). Theorem 5.6.0.2 asserts that if U is essentially small, then there exists a covariant transport representation for U which is uniquely determined up to isomorphism (as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$). In fact, we will prove something stronger: the covariant transport representation of U is unique up to a contractible space of choices. In §5.6.8, we formulate this statement more precisely by introducing a Kan complex $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ whose vertices are pairs $(\mathscr{F}, \alpha)$ as above (see Notation 5.6.8.1). We prove the contractibility of $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ in §5.6.9: as we will see, it is a formal consequence of the fact that the homotopy transport representation of the cocartesian fibration $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ determines an equivalence of hKan-enriched categories $\mathrm{hTr}_{\mathcal{QC}_{\mathrm{Obj}} \,/\, \mathcal{QC}} : \mathrm{h}\mathcal{QC} \to \mathrm{hQCat}$ (Proposition 5.5.6.14).

Remark 5.6.0.8. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration between (small) simplicial sets. We will denote the covariant transport representation of U by $\mathrm{Tr}_{\mathcal{E} \,/\, \mathcal{C}}$; it can be regarded as a homotopy coherent refinement of the homotopy transport representation $\mathrm{hTr}_{\mathcal{E} \,/\, \mathcal{C}}$ introduced in Construction 5.2.5.2 (see Remark 5.6.5.8 for a precise statement). We can summarize the situation with the following informal answer to Question 5.6.0.1: 0382

- For every essentially small cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$, the construction $C \mapsto \mathcal{E}_C$ determines a functor of ∞-categories $\mathrm{Tr}_{\mathcal{E} \,/\, \mathcal{C}} : \mathcal{C} \to \mathcal{QC}$. Moreover, we can recover $\mathcal{E}$ (up to equivalence) as the ∞-category of elements $\int_{\mathcal{C}} \mathrm{Tr}_{\mathcal{E} \,/\, \mathcal{C}}$.

0383 **Remark 5.6.0.9.** In the statement of Theorem 5.6.0.2, it is not necessary to assume that the simplicial set $\mathcal{C}$ is an ∞-category. This additional generality will play an essential role in our proof (which will require us to analyze the restriction of the cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ to simplicial subsets of $\mathcal{C}$). Moreover, it has a number of pleasant consequences: since $\mathcal{QC}$ is an ∞-category, it guarantees that every cocartesian fibration of simplicial sets is *equivalent* to the pullback of a cocartesian fibration between ∞-categories. In §5.6.7, we use this to prove a sharper statement: every cocartesian fibration of simplicial sets is *isomorphic* to the pullback of a cocartesian fibration between ∞-categories (Corollary 5.6.7.3). From this, we deduce that every cocartesian fibration of simplicial sets is an isofibration (Corollary 5.6.7.5), and that the collection of categorical equivalences of simplicial sets is stable under the formation of pullback by cocartesian fibrations (Corollary 5.6.7.6).

5.6.1 Elements of Category-Valued Functors

01R1 Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a functor. In §5.2.6 we introduced the *category of elements* $\int_{\mathcal{C}} \mathscr{F}$, whose objects are pairs (C, x) where C is an object of $\mathcal{C}$ and x is an element of the set $\mathscr{F}(C)$ (Construction 5.2.6.1). In this section, we study a generalization of this construction, where we allow $\mathscr{F}$ to be a $\mathcal{C}$-indexed diagram of categories (rather than a $\mathcal{C}$-indexed diagram of sets). In what follows, we let **Cat** denote the (strict) 2-category of small categories (Example 2.2.0.4).

025N **Definition 5.6.1.1** (The Category of Elements: Covariant Version)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories. We define a category $\int_{\mathcal{C}} \mathscr{F}$ as follows:

- The objects of $\int_{\mathcal{C}} \mathscr{F}$ are pairs (C, X), where C is an object of $\mathcal{C}$ and X is an object of the category $\mathscr{F}(C)$.
- Let (C, X) and (D, Y) be objects of $\int_{\mathcal{C}} \mathscr{F}$. Then a morphism from (C, X) to (D, Y) in the category $\int_{\mathcal{C}} \mathscr{F}$ is a pair (f, u), where $f : C \to D$ is a morphism in the category $\mathcal{C}$ and $u : \mathscr{F}(f)(X) \to Y$ is a morphism in the category $\mathscr{F}(D)$.
- Let $(f, u) : (C, X) \to (D, Y)$ and $(g, v) : (D, Y) \to (E, Z)$ be morphisms in the category $\int_{\mathcal{C}} \mathscr{F}$. Then the composition $(g, v) \circ (f, u)$ is the pair $(g \circ f, w)$, where $w : \mathscr{F}(g \circ f)(X) \to Z$ is the morphism of $\mathscr{F}(E)$ given by the composition
$$\mathscr{F}(g \circ f)(X) \xrightarrow{\mu_{g,f}^{-1}(X)} (\mathscr{F}(g) \circ \mathscr{F}(f))(X) \xrightarrow{\mathscr{F}(g)(u)} \mathscr{F}(g)(Y) \xrightarrow{v} Z,$$
where $\mu_{g,f} : \mathscr{F}(g) \circ \mathscr{F}(f) \simeq \mathscr{F}(g \circ f)$ denotes the composition constraint for the functor $\mathscr{F}$.

We will refer to $\int_{\mathcal{C}} \mathscr{F}$ as the *category of elements of* $\mathscr{F}$.

Remark 5.6.1.2. The category of elements $\int_{\mathcal{C}} \mathscr{F}$ was originally introduced by Grothendieck in [27]. For this reason, many authors refer to the category $\int_{\mathcal{C}} \mathscr{F}$ as the *Grothendieck construction* on the functor $\mathscr{F}$. 025M

Proposition 5.6.1.3. *Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to$ **Cat** be a functor of 2-categories. Then the category of elements $\int_{\mathcal{C}} \mathscr{F}$ is well-defined: that is, the composition law described in Definition 5.6.1.1 is unital and associative.* 01R3

Proof. Let (D, Y) be an object of $\int_{\mathcal{C}} \mathscr{F}$. We let $\mathrm{id}_{(D,Y)}$ denote the morphism from (D, Y) to itself given by the pair $(\mathrm{id}_D, \epsilon_D^{-1}(Y))$, where $\epsilon_D : \mathrm{id}_{\mathscr{F}(D)} \xrightarrow{\sim} \mathscr{F}(\mathrm{id}_D)$ is the identity constraint for the functor $\mathscr{F}$. We first show that $\mathrm{id}_{(D,Y)}$ is a (two-sided) unit for the composition law on $\int_{\mathcal{C}} \mathscr{F}$. We consider two cases:

- Let (C, X) be another object of $\int_{\mathcal{C}} \mathscr{F}$ and let $(f, u) : (C, X) \to (D, Y)$ be a morphism in $\int_{\mathcal{C}} \mathscr{F}$. We wish to show that the composition $\mathrm{id}_{(D,Y)} \circ (f, u)$ is equal to (f, u) (as a morphism from (C, X) to (D, Y)). Unwinding the definitions, this is equivalent to the assertion that the morphism $u : \mathscr{F}(f)(X) \to Y$ is equal to the composition

  $$\mathscr{F}(f)(X) \xrightarrow{\mu^{-1}_{\mathrm{id}_D, f}(X)} (\mathscr{F}(\mathrm{id}_D) \circ \mathscr{F}(f))(X) \xrightarrow{\mathscr{F}(\mathrm{id}_D)(u)} \mathscr{F}(\mathrm{id}_D)(Y) \xrightarrow{\epsilon_D^{-1}(Y)} Y.$$

  Using the commutativity of the diagram

  $$\begin{tikzcd}
  \mathscr{F}(f)(X) \arrow[r, "\epsilon_D(\mathscr{F}(f)(X))", "\sim"'] \arrow[d, "u"] & (\mathscr{F}(\mathrm{id}_D) \circ \mathscr{F}(f))(X) \arrow[d, "\mathscr{F}(\mathrm{id}_D)(u)"] \\
  Y \arrow[r, "\epsilon_D(Y)", "\sim"'] & \mathscr{F}(\mathrm{id}_D)(Y),
  \end{tikzcd}$$

  we are reduced to showing that the composition

  $$\mathscr{F}(f)(X) \xrightarrow{\mu^{-1}_{\mathrm{id}_D, f}(X)} (\mathscr{F}(\mathrm{id}_D) \circ \mathscr{F}(f))(X) \xrightarrow{\epsilon_D^{-1}(\mathscr{F}(f)(X))} \mathscr{F}(f)(X)$$

  is equal to the identity, which follows from axiom (a) of Definition 2.2.4.5.

- Let (E, Z) be another object of $\int_{\mathcal{C}} \mathscr{F}$, and let $(g, v) : (D, Y) \to (E, Z)$ be a morphism in $\int_{\mathcal{C}} \mathscr{F}$. We wish to show that the composition $(g, v) \circ \mathrm{id}_{(D,Y)}$ is equal to (g, v) (as a morphism from (D, Y) to (E, Z)). Unwinding the definitions, this is equivalent to the assertion that the morphism $v : \mathscr{F}(g)(Z) \to Y$ is equal to the composition

  $$\mathscr{F}(g)(Y) \xrightarrow{\mu^{-1}_{g, \mathrm{id}_D}(Y)} (\mathscr{F}(g) \circ \mathscr{F}(\mathrm{id}_D))(Y) \xrightarrow{\mathscr{F}(g)(\epsilon_D^{-1}(Y))} \mathscr{F}(g)(Y) \xrightarrow{v} Z,$$

  which follows from from axiom (b) of Definition 2.2.4.5.

We now show that composition of morphisms in $\int_{\mathcal{C}} \mathscr{F}$ is associative. Suppose we are given a composable sequence

$$(B, W) \xrightarrow{(e,t)} (C, X) \xrightarrow{(f,u)} (D, Y) \xrightarrow{(g,v)} (E, Z)$$

of morphisms of $\int_{\mathcal{C}} \mathscr{F}$. Unwinding the definitions, we obtain equalities

$$(g, v) \circ ((f, u) \circ (e, t)) = (g \circ f \circ e, v \circ \mathscr{F}(g)(u) \circ w)$$

$$((g, v) \circ (f, u)) \circ (e, t)) = (g \circ f \circ e, v \circ \mathscr{F}(g)(u) \circ w')$$

where $w, w' : \mathscr{F}(g \circ f \circ e)(W) \to (\mathscr{F}(e) \circ \mathscr{F}(f))(X)$ are the morphisms in the category $\mathscr{F}(E)$ given by clockwise and counterclockwise composition in the diagram

$$\begin{tikzcd}
\mathscr{F}(g \circ f \circ e)(W) \arrow[r, "\mu^{-1}_{g, f \circ e}(W)", "\sim"'] \arrow[d, "\mu^{-1}_{g \circ f, e}(W)", "\sim"'] & (\mathscr{F}(g) \circ \mathscr{F}(f \circ e))(W) \arrow[d, "\mathscr{F}(g)(\mu^{-1}_{f,e}(W))", "\sim"'] \\
(\mathscr{F}(g \circ f) \circ \mathscr{F}(e))(W) \arrow[r, "\mu^{-1}_{g,f}(\mathscr{F}(e)(W))", "\sim"'] \arrow[d, "\mathscr{F}(g \circ f)(t)"] & (\mathscr{F}(g) \circ \mathscr{F}(f) \circ \mathscr{F}(e))(W) \arrow[d, "(\mathscr{F}(g) \circ \mathscr{F}(f))(t)"] \\
\mathscr{F}(g \circ f)(X) \arrow[r, "\mu^{-1}_{g,f}(X)", "\sim"'] & (\mathscr{F}(g) \circ \mathscr{F}(f))(X).
\end{tikzcd}$$

It will therefore suffice to show that this diagram commutes. For the upper square, this follows from axiom (c) of Definition 2.2.4.5. For the lower square, it follows from the naturality of the composition constraint $\mu_{g,f}$. □

Definition 5.6.1.1 has a counterpart for contravariant functors:

025P **Definition 5.6.1.4** (The Category of Elements: Contravariant Version)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a functor of 2-categories (Definition 2.2.4.5). We define a category $\int^{\mathcal{C}} \mathscr{F}$ as follows:

- The objects of $\int^{\mathcal{C}} \mathscr{F}$ are pairs (C, X), where C is an object of $\mathcal{C}$ and X is an object of the category $\mathscr{F}(C)$.
- Let (C, X) and (D, Y) be objects of $\int^{\mathcal{C}} \mathscr{F}$. Then a morphism from (C, X) to (D, Y) in the category $\int^{\mathcal{C}} \mathscr{F}$ is a pair (f, u), where $f : C \to D$ is a morphism in the category $\mathcal{C}$ and $u : X \to \mathscr{F}(f)(Y)$ is a morphism in the category $\mathscr{F}(C)$.

- Let $(f,u) : (C,X) \to (D,Y)$ and $(g,v) : (D,Y) \to (E,Z)$ be morphisms in the category $\int^{\mathcal{C}} \mathscr{F}$. Then the composition $(g,v) \circ (f,u)$ is the pair $(g \circ f, w)$, where $w : X \to \mathscr{F}(g \circ f)(Z)$ is the morphism of $\mathscr{F}(C)$ given by the composition
$$X \xrightarrow{u} \mathscr{F}(f)(Y) \xrightarrow{\mathscr{F}(f)(v)} (\mathscr{F}(f) \circ \mathscr{F}(g))(Z) \xrightarrow{\mu_{f,g}(Z)} \mathscr{F}(g \circ f)(Z),$$
where $\mu_{f,g} : \mathscr{F}(f) \circ \mathscr{F}(g) \simeq \mathscr{F}(g \circ f)$ denotes the composition constraint for the lax functor $\mathscr{F}$.

We will refer to $\int^{\mathcal{C}} \mathscr{F}$ as the *category of elements* of the functor $\mathscr{F}$.

Remark 5.6.1.5. The category of elements $\int^{\mathcal{C}} \mathscr{F}$ can be defined more generally when $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a lax functor of 2-categories. We will return to this point in §[?] (see Definition [?]). `025Q`

Remark 5.6.1.6. Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories. Then the construction $(C \in \mathcal{C}) \mapsto \mathscr{F}(C)^{\mathrm{op}}$ determines a functor of 2-categories $\mathscr{F}^{\mathrm{op}} : \mathcal{C} = (\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathbf{Cat}$. In this case, we have a canonical isomorphism of categories `01R5`

$$\int^{\mathcal{C}^{\mathrm{op}}} (\mathscr{F}^{\mathrm{op}}) \simeq (\int_{\mathcal{C}} \mathscr{F})^{\mathrm{op}},$$

where the left hand side is given by Definition 5.6.1.4 and the right hand side is given by Definition 5.6.1.1.

Example 5.6.1.7 (Set-Valued Functors)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a functor from $\mathcal{C}$ to the category of sets. Then we can also regard $\mathscr{F}$ as a functor from $\mathcal{C}$ to the 2-category **Cat** (by composing with the fully faithful embedding $\mathrm{Set} \hookrightarrow \mathbf{Cat}$, carrying each set S to the associated discrete category). In this case, the category $\int_{\mathcal{C}} \mathscr{F}$ of Definition 5.6.1.1 agrees with the category of elements of $\mathscr{F}$ defined in Construction 5.2.6.1. Similarly, for every functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, the category $\int^{\mathcal{C}} \mathscr{F}$ can be identified with the category of elements of $\mathscr{F}$ defined in Variant 5.2.6.2. `01R6`

Example 5.6.1.8. Let Cat denote the category whose objects are (small) categories and whose morphisms are functors, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Cat}$ be a functor of ordinary categories. Composing with the nerve functor $\mathrm{N}_\bullet : \mathrm{Cat} \to \mathrm{Set}_\Delta$, we obtain a functor $\mathscr{F}' : \mathcal{C} \to \mathrm{Set}_\Delta$. There is a canonical isomorphism of simplicial sets $\mathrm{N}_\bullet^{\mathscr{F}'}(\mathcal{C}) \simeq \mathrm{N}_\bullet(\int_{\mathcal{C}} \mathscr{F})$, where the left hand side denotes the weighted nerve of Definition 5.3.3.1 and $\int_{\mathcal{C}} \mathscr{F}$ denotes the category of elements introduced in Definition 5.6.1.1. See Exercise 5.3.3.17. `0260`

Example 5.6.1.9. Let $\mathscr{I}$ denote the inclusion from the ordinary category Cat (regarded as a 2-category having only identity 2-morphisms) to the 2-category **Cat**, and let $\mathrm{Cat}_*^{\mathrm{lax}}$ denote the category of elements $\int_{\mathrm{Cat}} \mathscr{I}$. The category $\mathrm{Cat}_*^{\mathrm{lax}}$ can be described concretely as follows: `01R7`

- The objects of $\mathrm{Cat}_*^{\mathrm{lax}}$ are pairs $(\mathcal{C}, X)$, where $\mathcal{C}$ is a category and X is an object of $\mathcal{C}$.
- A morphism from $(\mathcal{C}, X)$ to $(\mathcal{D}, Y)$ is a pair (F, u), where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $u : F(X) \to Y$ is a morphism in the category $\mathcal{D}$.
- If $(F, u) : (\mathcal{C}, X) \to (\mathcal{D}, Y)$ and $(G, v) : (\mathcal{D}, Y) \to (\mathcal{E}, Z)$ are morphisms in $\mathrm{Cat}_*^{\mathrm{lax}}$, then their composition is the pair $(G \circ F, w)$, where w is the morphism of $\mathcal{E}$ given by the composition
$$(G \circ F)(X) \xrightarrow{G(u)} G(Y) \xrightarrow{v} Z.$$

0384 **Example 5.6.1.10.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ denote the (strict) functor given on objects by the formula $\mathscr{F}(C) = \mathcal{C}_{/C}$. Then the category of elements $\int_{\mathcal{C}} \mathscr{F}$ can be identified with the arrow category $\mathrm{Fun}([1], \mathcal{C})$.

01RA **Notation 5.6.1.11.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories. Then the category of elements $\int_{\mathcal{C}} \mathscr{F}$ is equipped with a forgetful functor $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$, given on objects by the construction $(C, X) \mapsto C$ and on morphisms by the construction $(f, u) \mapsto f$. Similarly, for every functor of 2-categories $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, the category of $\int^{\mathcal{C}} \mathscr{F}$ of Definition 5.6.1.4 is equipped with a forgetful functor $U : \int^{\mathcal{C}} \mathscr{F} \to \mathcal{C}$.

025R **Remark 5.6.1.12** (Fibers of the Forgetful Functor)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories. For every object $C \in \mathcal{C}$, there is a canonical isomorphism of categories
$$\mathscr{F}(C) \simeq \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F},$$
which carries each object $X \in \mathscr{F}(C)$ to the object $(C, X) \in \int_{\mathcal{C}} \mathscr{F}$ and each morphism $u : X \to Y$ in $\mathscr{F}$ to the morphism $(\mathrm{id}_C, u \circ \epsilon_C(X)) : (C, X) \to (C, Y)$ of $\int_{\mathcal{C}} \mathscr{F}$ (here $\epsilon_C : \mathscr{F}(\mathrm{id}_C) \simeq \mathrm{id}_{\mathscr{F}(C)}$ denotes the identity constraint on the functor $\mathscr{F}$). Similarly, for each functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, we have a canonical isomorphism
$$\mathscr{F}(C) \simeq \{C\} \times_{\mathcal{C}} \int^{\mathcal{C}} \mathscr{F}.$$

01RB **Remark 5.6.1.13.** Let $V : \mathcal{D} \to \mathcal{C}$ be a functor between categories. If $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ is a functor of 2-categories, then the composition $(\mathscr{F} \circ V) : \mathcal{D} \to \mathbf{Cat}$ is also a functor of 2-categories, and we have a pullback diagram of categories

$$\begin{tikzcd}
\int_{\mathcal{D}} (\mathscr{F} \circ V) \arrow[r] \arrow[d] & \int^{\mathcal{C}} \mathscr{F} \arrow[d] \\
\mathcal{D} \arrow[r, "V"] & \mathcal{C}
\end{tikzcd}$$

where the vertical maps are the forgetful functors of Notation 5.6.1.11. Similarly, for every functor of 2-categories $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, we have a pullback diagram

$$\begin{tikzcd}
\int^{\mathcal{D}}(\mathscr{F} \circ V^{\mathrm{op}}) \arrow[r] \arrow[d] & \int^{\mathcal{C}} \mathscr{F} \arrow[d] \\
\mathcal{D} \arrow[r, "V"] & \mathcal{C} .
\end{tikzcd}$$

Example 5.6.1.14. Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Cat}$ be a functor between ordinary categories, which we can identify with a *strict* functor from $\mathcal{C}$ to the 2-category **Cat**. Applying Remark 5.6.1.13, we deduce that the category of elements $\int_{\mathcal{C}} \mathscr{F}$ fits into a pullback diagram 01RC

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r] \arrow[d] & \mathrm{Cat}_{*}^{\mathrm{lax}} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathrm{Cat} .
\end{tikzcd}$$

Proposition 5.6.1.15. *Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories, and let $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ denote the forgetful functor. Then a morphism $(f, u) : (C, X) \to (D, Y)$ of $\int_{\mathcal{C}} \mathscr{F}$ is U-cocartesian if and only if $u : \mathscr{F}(f)(X) \to Y$ is an isomorphism in the category $\mathscr{F}(D)$.* 025S

Proof. Assume first that u is an isomorphism; we wish to show that (f, u) is a U-cocartesian morphism of the category $\int_{\mathcal{C}} \mathscr{F}$. Fix a morphism $g : D \to E$ of $\mathcal{C}$ and an object $Z \in \mathscr{F}(E)$; we wish to show that every morphism $(g \circ f, w) : (C, X) \to (E, Z)$ in the category $\int_{\mathcal{C}} \mathscr{F}$ can be written uniquely as a composition $(g, v) \circ (f, u)$ for some morphism $(g, v) : (D, Y) \to (E, Z)$. Unwinding the definitions, we wish to show that there is a unique morphism $v : \mathscr{F}(g)(Y) \to Z$ in the category $\mathscr{F}(E)$ for which the composition

$$\mathscr{F}(g \circ f)(X) \xrightarrow{\mu_{g,f}^{-1}(X)} (\mathscr{F}(g) \circ \mathscr{F}(f))(X) \xrightarrow{\mathscr{F}(g)(u)} \mathscr{F}(g)(Y) \xrightarrow{v} Z$$

is equal to w. This is clear, since $\mu_{g,f}^{-1}(X)$ and $\mathscr{F}(g)(u)$ are isomorphisms.

Now suppose that (f, u) is a U-cocartesian morphism of the category $\int_{\mathcal{C}} \mathscr{F}$; we wish to show that u is an isomorphism. Let $\iota : \mathscr{F}(D) \to \{D\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$ be the isomorphism of Remark 5.6.1.12. Then the morphism (f, u) factors as a composition

$$(C, X) \xrightarrow{(f, \mathrm{id})} (D, \mathscr{F}(f)(X)) \xrightarrow{\iota(u)} (D, Y).$$

The first half of the argument shows that the morphism (f, id) is also U-cocartesian, so that $\iota(u)$ is an isomorphism in the fiber $\{D\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$. Since ι is an isomorphism of categories, it follows that u is an isomorphism in the category $\mathscr{F}(D)$. □

025T **Corollary 5.6.1.16.** *Let $\mathcal{C}$ be a category. If $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ is a functor of 2-categories, then the forgetful functor $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is a cocartesian fibration of categories. If $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ is a functor of 2-categories, then the forgetful functor $\int^{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is a cartesian fibration of categories.*

Proof. We will prove the first assertion; the second follows by a similar argument. Let (C, X) be an object of the category $\int_{\mathcal{C}} \mathscr{F}$ and let $f : C \to D$ be a morphism in $\mathcal{C}$; we wish to show that f can be lifted to a U-cocartesian morphism $(f, u) : (C, X) \to (D, Y)$ of $\int_{\mathcal{C}} \mathscr{F}$. This follows immediately from the criterion of Proposition 5.6.1.15: for example, we can take $Y = \mathscr{F}(f)(X)$ and u to be the identity morphism. □

025U **Remark 5.6.1.17.** In §5.6.5, we will prove a converse to Corollary 5.6.1.16: for every cocartesian fibration of categories $U : \mathcal{E} \to \mathcal{C}$, there exists a functor of 2-categories $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ and an isomorphism of categories $\int_{\mathcal{C}} \mathscr{F} \simeq \mathcal{E}$ whose composition with U is the forgetful functor of Notation 5.6.1.11. See Corollary 5.6.5.19.

025V **Remark 5.6.1.18** (Covariant Transport)**.** Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories, and let $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ be the forgetful functor of Notation 5.6.1.11. For each object $C \in \mathcal{C}$, let

$$\iota_C : \mathscr{F}(C) \simeq \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F} \subseteq \int_{\mathcal{C}} \mathscr{F}$$

be the isomorphism of Remark 5.6.1.12. Note that every morphism $f : C \to D$ in $\mathcal{C}$ determines a natural transformation of functors $\widetilde{f} : \iota_C \to \iota_D \circ \mathscr{F}(f)$, which carries an object $X \in \mathscr{F}(C)$ to the U-cocartesian morphism $(f, \mathrm{id}) : (C, X) \to (D, \mathscr{F}(f)(X))$. It follows that $\widetilde{f}$ identifies $\mathscr{F}(f)$ with the covariant transport functor $f_!$ of Notation 5.2.2.2.

5.6.2 Elements of $\mathcal{QC}$-Valued Functors

026H Let QCat denote the ordinary category whose objects are ∞-categories and whose morphisms are functors (Construction 5.5.4.1). To every functor $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$, the weighted nerve construction of Definition 5.3.3.1 supplies a cocartesian fibration of ∞-categories $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ (Corollary 5.3.3.16), whose fiber over an object $C \in \mathcal{C}$ is isomorphic to the ∞-category $\mathscr{F}(C)$ (Example 5.3.3.8). The utility of this construction is limited by the fact that it applies only to *strictly commutative* diagrams in QCat: that is, Definition 5.3.3.1 requires $\mathcal{C}$ to be an ordinary category and $\mathscr{F}$ to be a functor of ordinary categories. Our goal in this section is to introduce a *homotopy coherent* variant of the weighted nerve which is associated to any functor of ∞-categories $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$; here $\mathcal{QC}$ denotes the ∞-category of ∞-categories introduced in Construction 5.5.4.1.

Definition 5.6.2.1. Let $\mathcal{C}$ be a simplicial set and let $\mathscr{F} : \mathcal{C} \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$ be a morphism of simplicial sets. We let $\int_{\mathcal{C}} \mathscr{F}$ denote the fiber product $\mathcal{C} \times_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})} \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})_{\Delta^0/}$, so that we have a pullback diagram of simplicial sets 026J

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r] \arrow[d, "U"] & \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})_{\Delta^0/} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta}).
\end{tikzcd}$$

We will refer to $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ as the *projection map.*

The simplicial set $\int_{\mathcal{C}} \mathscr{F}$ of Definition 5.6.2.1 is defined for an arbitrary morphism $\mathscr{F} : \mathcal{C} \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$. However, we will be primarily interested in the case where $\mathscr{F}$ takes values in the simplicial subset $\mathcal{QC} \subseteq \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$ introduced in Construction 5.5.4.1.

Proposition 5.6.2.2. *Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a morphism of simplicial sets. Then the projection map $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is a cocartesian fibration of simplicial sets.* 026K

Proof. By construction, the morphism U fits into a pullback diagram

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r] \arrow[d, "U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC},
\end{tikzcd}$$

where

$$\mathcal{QC}_{\mathrm{Obj}} = \mathcal{QC} \times_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})} \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})_{\Delta^0/}$$

is the ∞-category introduced in Construction 5.5.6.10. It will therefore suffice to show that the projection map $\mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ is a cocartesian fibration of simplicial sets, which follows from Proposition 5.5.6.11. □

Corollary 5.6.2.3. *Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a functor of ∞-categories. Then the simplicial set $\int_{\mathcal{C}} \mathscr{F}$ of Definition 5.6.2.1 is an ∞-category.* 026L

Definition 5.6.2.4. Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a functor of ∞-categories. We will refer to $\int_{\mathcal{C}} \mathscr{F}$ as the ∞-*category of elements of $\mathscr{F}$.* 026M

026N **Remark 5.6.2.5.** Let $\mathcal{C}$ be an ordinary category equipped with a strictly unitary functor of 2-categories $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$. Then the construction $C \mapsto \mathrm{N}_\bullet(\mathscr{F}(C))$ determines a functor of ∞-categories $\mathrm{N}_\bullet(\mathscr{F}) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathcal{QC}$ (see Remark 5.5.4.9). In §5.6.3, we will construct a canonical isomorphism

$$\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet(\mathscr{F}) \simeq \mathrm{N}_\bullet(\int_{\mathcal{C}} \mathscr{F}),$$

where the simplicial set on the left hand side is given by Definition 5.6.2.1 and $\int_{\mathcal{C}} \mathscr{F}$ is the category of elements introduced in Definition 5.6.1.1 (see Proposition 5.6.3.4). Stated more informally, we can regard the ∞-category of elements construction (Definition 5.6.2.4) as a generalization of the classical category of elements construction (Definition 5.6.1.1).

026P **Warning 5.6.2.6.** In §5.6.1, we introduced a variant of the category of elements construction for *contravariant* **Cat**-valued functors $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ (see Definition 5.6.1.4), which is characterized by the formula

$$\int^{\mathcal{C}} \mathscr{F} = (\int_{\mathcal{C}^{\mathrm{op}}} \mathscr{F}^{\mathrm{op}})^{\mathrm{op}}.$$

In the ∞-categorical setting, the situation is more subtle: the involution $\mathcal{E} \mapsto \mathcal{E}^{\mathrm{op}}$ does not preserve the simplicial structure on the category QCat and therefore does not induce an involution on the simplicial set $\mathcal{QC} = \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{QCat})$. We will return to this point in §[?].

026Q **Warning 5.6.2.7.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a functor of ordinary categories. Passing to the homotopy coherent nerve, we obtain a functor of ∞-categories $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathscr{F}) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathcal{QC}$. Beware that the simplicial set $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathrm{hc}}(\mathscr{F})$ is usually not *isomorphic* to the weighted nerve $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ of Definition 5.3.3.1, even in the special case $\mathcal{C} = \Delta^0$. However, in §5.6.4 we will construct a comparison map

$$\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \to \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathrm{hc}}(\mathscr{F})$$

which is an equivalence of ∞-categories (Proposition 5.6.4.8).

026R **Example 5.6.2.8** (Set-Valued Functors)**.** Let Set denote the category of sets, and let us regard the nerve $\mathrm{N}_\bullet(\mathrm{Set})$ as a simplicial subset of the homotopy coherent nerve $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Set}_\Delta)$. Let $\mathscr{F} : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{Set})$ be a morphism of simplicial sets, which we can identify with a functor of categories $\mathrm{h}\mathscr{F} : \mathrm{h}\mathcal{C} \to \mathrm{Set}$. Using Example 5.5.3.12 and Remark 5.2.6.6, we obtain a canonical isomorphism of simplicial sets

$$\int_{\mathcal{C}} \mathscr{F} \simeq \mathcal{C} \times_{\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})} \mathrm{N}_\bullet(\int_{\mathrm{h}\mathcal{C}} \mathrm{h}\mathscr{F}),$$

where $\int_{\mathcal{C}} \mathscr{F}$ is the simplicial set of Definition 5.6.2.1 and $\int_{\mathrm{h}\mathcal{C}} \mathrm{h}\mathscr{F}$ is the category of elements introduced in Construction 5.2.6.1. In particular, the projection map $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is a left covering map.

Example 5.6.2.9 ($\mathcal{S}$-Valued Functors)**.** Let $\mathcal{S}$ denote the ∞-category of spaces (Construction 5.5.1.1), which we view as a full simplicial subset of $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$, and let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a morphism of simplicial sets. Then the simplicial set $\int_{\mathcal{C}} \mathscr{F}$ fits into pullback diagram 026S

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r] \arrow[d, "\pi"] & \mathcal{S}_{*} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{S},
\end{tikzcd}$$

where $\mathcal{S}_{*}$ is the ∞-category of pointed spaces (Construction 5.5.3.1). In this case, Proposition 5.5.3.2 guarantees that the projection map $\pi : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is a left fibration of simplicial sets.

Example 5.6.2.10 ($\mathcal{QC}$-Valued Functors)**.** Let $\mathcal{QC}$ denote the $(\infty, 2)$-category of ∞-categories (Construction 5.5.5.1), which we view as a full simplicial subset of $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$, and let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a morphism of simplicial sets. We then have a pullback diagram of simplicial sets 026T

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r] \arrow[d, "\pi"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC},
\end{tikzcd}$$

where $\mathcal{QC}_{\mathrm{Obj}}$ is the $(\infty, 2)$-category of Construction 5.5.6.10 (Construction 5.5.6.8). If $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ is a functor of $(\infty, 2)$-categories, then Proposition 5.5.6.9 and Remark 5.4.2.4 guarantee that $\pi : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is an interior fibration; in particular, $\int_{\mathcal{C}} \mathscr{F}$ is also an $(\infty, 2)$-category.

Warning 5.6.2.11. Let $\mathcal{C}$ be an $(\infty, 2)$-category and let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a morphism of simplicial sets. If $\mathscr{F}$ is not a functor, then $\int_{\mathcal{C}} \mathscr{F}$ need not be an $(\infty, 2)$-category (this phenomenon arises already in the case $\mathcal{C} = \Delta^2$). 026U

Example 5.6.2.12 (Objects of the ∞-Category of Elements)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$ be a morphism of simplicial sets. Then vertices of the simplicial set $\int_{\mathcal{C}} \mathscr{F}$ can be identified with pairs (C, X), where C is a vertex of $\mathcal{C}$ and X is a vertex of the simplicial set $\mathscr{F}(C)$ (see Example 5.5.6.12). Moreover, the projection map $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is given on vertices by the construction $U(C, X) = C$. 026V

Example 5.6.2.13 (Morphisms of the ∞-Category of Elements)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$ be a morphism of simplicial sets. Let (C, X) and (D, Y) be vertices of the simplicial set $\int_{\mathcal{C}} \mathscr{F}$. Edges of $\int_{\mathcal{C}} \mathscr{F}$ from (C, X) to (D, Y) can be identified with pairs (f, u), where $f : C \to D$ is 026W an edge of the simplicial set $\mathcal{C}$ and $u : \mathscr{F}(f)(X) \to Y$ is an edge of the simplicial set $\mathscr{F}(D)$ (see Example 5.5.6.12). Moreover, the projection map $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is given on edges by the construction $U(f, u) = f$.

026X **Remark 5.6.2.14** (Cocartesian Morphisms of the ∞-Category of Elements)**.** Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a morphism of simplicial sets, so that the projection map $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is a cocartesian fibration of simplicial sets (Proposition 5.6.2.2). Then an edge $(f, u) : (C, X) \to (D, Y)$ of $\int_{\mathcal{C}} \mathscr{F}$ is U-cocartesian if and only if $u : \mathscr{F}(f)(X) \to Y$ is an isomorphism in the ∞-category $\mathscr{F}(D)$ (see Example 5.5.6.12).

026Z **Example 5.6.2.15** (2-Simplices of the ∞-Category of Elements)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$ be a morphism of simplicial sets and let $\sigma_0 : \partial \Delta^2 \to \int_{\mathcal{C}} \mathscr{F}$ be a morphism of simplicial sets, which we depict informally as a diagram

$$\begin{tikzcd}
& (D, Y) \arrow[dr, "{(g,v)}"] & \\
(C, X) \arrow[ur, "{(f,u)}"] \arrow[rr, "{(h,w)}"] & & (E, Z).
\end{tikzcd}$$

Extensions of σ_0 to a 2-simplex of $\int_{\mathcal{C}} \mathscr{F}$ can be identified with pairs (μ, θ), where $\mu : \mathscr{F}(g) \circ \mathscr{F}(f) \to \mathscr{F}(h)$ is an edge of the simplicial set $\mathrm{Fun}(\mathscr{F}(C), \mathscr{F}(E))$, and $\theta : \square^2 \to \mathscr{F}(E)$ is a morphism of simplicial sets whose restriction to the boundary $\partial \square^2$ is indicated in the diagram

$$\begin{tikzcd}
(\mathscr{F}(g) \circ \mathscr{F}(f))(X) \arrow[r, "\mu(X)"] \arrow[d, "\mathscr{F}(g)(u)"] & \mathscr{F}(h)(X) \arrow[d, "w"] \\
\mathscr{F}(g)(Y) \arrow[r, "v"] & Z
\end{tikzcd}$$

(see Example 5.5.6.17). Moreover, the projection map $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is given on 2-simplices by the construction $U(\mu, \theta) = \mu$.

0270 **Example 5.6.2.16.** Let $\mathcal{E}$ be a simplicial set, which we identify with the morphism of simplicial sets $\Delta^0 \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})$ taking the value $\mathcal{E}$. Then the simplicial set $\int_{\Delta^0} \mathcal{E}$ can be identified with the left-pinched morphism space $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})}(\Delta^0, \mathcal{E})$. In particular, Construction 4.6.8.3 supplies a comparison morphism

$$\theta_{\mathcal{E}} : \mathcal{E} = \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^0, \mathcal{E})_{\bullet} \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Set}_{\Delta})}(\Delta^0, \mathcal{E}) = \int_{\Delta^0} \mathcal{E}.$$

If $\mathcal{E}$ is an ∞-category, then $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Set}_{\Delta})}(\Delta^0, \mathcal{E})$ is also an ∞-category, and the comparison morphism ρ is an equivalence of ∞-categories (Theorem 4.6.8.9). Beware that $\theta_{\mathcal{E}}$ is generally not an isomorphism (though it is always a monomorphism which is bijective on simplices of dimension ≤ 1). For example, Example 5.6.2.15 implies that 2-simplices of $\int_{\Delta^0} \mathcal{E}$ can be identified with morphisms of simplicial sets $\rho : \Delta^1 \times \Delta^1 \to \mathcal{E}$ for which the restriction $\rho|_{\Delta^1 \times \{0\}}$ is a degenerate edge of $\mathcal{E}$, as indicated in the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "u"'] \arrow[dr] & X \arrow[d, "w"] \arrow[dl, phantom, "\sigma", near start] \\
Y \arrow[r, "v"'] \arrow[ur, phantom, "\tau", near start] & Z.
\end{tikzcd}$$

The corresponding 2-simplex of $\int_{\Delta^0} \mathcal{E}$ belongs to the image of $\theta_{\mathcal{E}}$ if and only if σ is a left-degenerate 2-simplex of $\mathcal{E}$ (in which case it is given by $\theta_{\mathcal{E}}(\tau)$).

Remark 5.6.2.17. Let $U : \mathcal{C}' \to \mathcal{C}$ and $\mathscr{F} : \mathcal{C} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Set}_{\Delta})$ be morphisms of simplicial sets, and let $\mathscr{F}'$ denote the composition $(\mathscr{F} \circ U) : \mathcal{C}' \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Set}_{\Delta})$. Then the simplicial set $\int_{\mathcal{C}'} \mathscr{F}'$ can be identified with the fiber product $\mathcal{C}' \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$. 0271

Example 5.6.2.18 (Fibers of the ∞-Category of Elements)**.** Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a morphism of simplicial sets. For each vertex $C \in \mathcal{C}$, Remark 5.6.2.17 and Example 5.6.2.16 supply a canonical isomorphism 0272

$$\{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F} \simeq \mathrm{Hom}^{\mathrm{L}}_{\mathcal{QC}}(\Delta^0, \mathscr{F}(C)).$$

In particular, Construction 4.6.8.3 supplies a comparison functor $\theta_C : \mathscr{F}(C) \to \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$ which is an equivalence of ∞-categories (Theorem 4.6.8.9), but generally not an isomorphism of simplicial sets.

Proposition 5.6.2.19. *Let $\mathcal{C}$ be a simplicial set, let $\mathscr{F}, \mathscr{F}' : \mathcal{C} \to \mathcal{QC}$ be diagrams, and let $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ and $U' : \int_{\mathcal{C}} \mathscr{F}' \to \mathcal{C}$ be the projection maps. If $\mathscr{F}$ and $\mathscr{F}'$ are isomorphic as objects of the diagram ∞-category* $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$*, then U and U' are equivalent as cocartesian fibrations over $\mathcal{C}$ (in the sense of Definition 5.1.7.1).* 02S7

Proof. Apply Proposition 5.1.7.11 to the cocartesian fibration $\mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ of Proposition 5.5.6.11. □

Proposition 5.6.2.20. *Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a functor of ∞-categories, let $\mathcal{E} = \int_{\mathcal{C}} \mathscr{F}$ denote the ∞-category of elements of $\mathscr{F}$, and let* 02S8

$$\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{QC}$$

denote the enriched homotopy transport representation associated to the cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ *(see Construction 5.2.8.9). Then there is a canonical isomorphism of* hKan*-enriched functors* $\theta : \mathrm{h}\mathscr{F} \to \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$*, which carries each object* $C \in \mathcal{C}$ *to the comparison map*

$$\theta_C : \mathscr{F}(C) \to \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}(C) = \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$$

of Example 5.6.2.18.

Proof. By virtue of Remarks 5.2.8.10 and 5.6.2.17, we may assume without loss of generality that $\mathcal{C} = \mathcal{QC}$ and that $\mathscr{F}$ is the identity functor. In this case, the desired result is a restatement of Proposition 5.5.6.14. $\square$

027K **Corollary 5.6.2.21.** *Let* $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ *be a morphism of simplicial sets, let* $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ *be the cocartesian fibration of Proposition 5.6.2.2, and let*

$$f_! : \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F} \to \{D\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$$

be a functor which is given by covariant transport along an edge $f : C \to D$ *of* $\mathcal{C}$ *(Definition 5.2.2.4). Then the diagram*

$$\begin{tikzcd}
\mathscr{F}(C) \arrow[r, "\sim"] \arrow[d, "{[\mathscr{F}(f)]}"] & \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F} \arrow[d, "{[f_!]}"] \\
\mathscr{F}(D) \arrow[r, "\sim"] & \{D\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}
\end{tikzcd}$$

commutes in the homotopy category hQCat *(where the horizontal maps are the equivalences described in Example 5.6.2.18).*

Proof. Without loss of generality, we may assume that $\mathcal{C} = \Delta^1$, in which case the desired result reduces to Proposition 5.6.2.20. $\square$

027L **Corollary 5.6.2.22.** *Let* $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ *be a functor of* ∞*-categories and set* $\mathcal{E} = \int_{\mathcal{C}} \mathscr{F}$*. Then there is a canonical isomorphism* $\mathrm{h}\mathscr{F} \xrightarrow{\sim} \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ *in the functor category* $\mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{h}\mathcal{QC})$*, which carries each vertex* $C \in \mathcal{C}$ *to the comparison map*

$$\theta_C : \mathscr{F}(C) \to \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}(C) = \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$$

of Example 5.6.2.18.

5.6.3 Comparison with the Category of Elements

Let **Cat** denote the 2-category of small categories (Example 2.2.0.4) and let $\mathcal{QC}$ denote the $(\infty, 2)$-category of small ∞-categories (Construction 5.5.5.1). Suppose we are given a category $\mathcal{C}$ equipped with a functor $\mathscr{F} : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{QC}$. Composing with the functor 0273

$$\mathcal{QC} \to \mathrm{N}_{\bullet}^{\mathrm{D}}(\mathbf{Cat}) \qquad \mathcal{C} \mapsto \mathrm{h}\mathcal{C}$$

of Remark 5.5.5.8 and invoking Corollary 2.3.4.5, we obtain a (strictly unitary) functor of 2-categories $\mathrm{h}\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$, which carries each object $C \in \mathcal{C}$ to the homotopy category of the ∞-category $\mathscr{F}(C)$. Our goal in this section is to compare the ∞-category $\int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F}$ of Definition 5.6.2.1 with the ordinary category $\int_{\mathcal{C}} \mathrm{h}\mathscr{F}$ of Definition 5.6.1.1. We begin with two simple observations:

- Objects of the ∞-category $\int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F}$ can be identified with pairs (C, X), where C is an object of $\mathcal{C}$ and X is an object of the ∞-category $\mathscr{F}(C)$ (Example 5.6.2.12). Since the ∞-category $\mathscr{F}(C)$ and its homotopy category $\mathrm{h}\mathscr{F}(\mathscr{C})$ have the same objects, we can also identify such pairs with objects of the ordinary category $\int_{\mathcal{C}} \mathrm{h}\mathscr{F}$.
- Let (C, X) and (D, Y) be objects of the ∞-category $\int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F}$. By definition, morphisms from (C, X) to (D, Y) in the ∞-category $\int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F}$ can be identified with pairs (f, u), where $f : C \to D$ is a morphism in the category $\mathcal{C}$ and $u : \mathscr{F}(f)(X) \to Y$ is a morphism in the ∞-category $\mathscr{F}(D)$ (Example 5.6.2.13). Every such pair determines a morphism $(f, [u])$ in the ordinary category $\int_{\mathcal{C}} \mathrm{h}\mathscr{F}$, where $[u]$ denotes the homotopy class of u (regarded as a morphism in the homotopy category $\mathrm{h}\mathscr{F}(D)$).

Proposition 5.6.3.1. *Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{QC}$ be a functor of ∞-categories. Then there is a unique functor of ∞-categories* 0274

$$T : \int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F} \to \mathrm{N}_{\bullet}(\int_{\mathcal{C}} \mathrm{h}\mathscr{F})$$

which is the identity on objects and which carries each morphism (f, u) of $\int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F}$ to the pair $(f, [u])$, regarded as a morphism in the ordinary category $\int_{\mathcal{C}} \mathrm{h}\mathscr{F}$. Moreover, the functor T exhibits the classical category of elements $\int_{\mathcal{C}} \mathrm{h}\mathscr{F}$ as the homotopy category of the ∞-category of elements $\int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F}$.

Stated more informally, Proposition 5.6.3.1 asserts that there is a canonical isomorphism of categories

$$\mathrm{h} \int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F} \xrightarrow{\sim} \int_{\mathcal{C}} \mathrm{h}\mathscr{F}.$$

In other words, passage to the homotopy category intertwines the classical category of elements construction (Definition 5.6.1.1) with the ∞-category of elements construction introduced in §5.6.2.

Proof of Proposition 5.6.3.1. We first prove the existence of the functor T appearing in the statement of Proposition 5.6.3.1 (the uniqueness is immediate). Since the induced functor of 2-categories $\mathrm{h}\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ is strictly unitary, the construction $(f, u) \mapsto (f, [u])$ carries degenerate edges of the ∞-category $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathscr{F}$ to identity morphisms in the category $\int_{\mathcal{C}} \mathrm{h}\mathscr{F}$. It will therefore suffice to show that, for every 2-simplex σ of the simplicial set $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathscr{F}$ whose boundary is indicated in the diagram

$$\begin{tikzcd}
& (D,Y) \arrow[dr, "{(g,v)}"] & \\
(C,X) \arrow[ur, "{(f,u)}"] \arrow[rr, "{(h,w)}"] & & (E,Z),
\end{tikzcd}$$

we have an identity $(h, [w]) = (g, [v]) \circ (f, [u])$ in the category $\int_{\mathcal{C}} \mathscr{F}$. Note that the functor $\mathscr{F}$ determines a natural isomorphism $\mu : \mathscr{F}(g) \circ \mathscr{F}(f) \xrightarrow{\sim} \mathscr{F}(h)$ in the ∞-category $\mathrm{Fun}(\mathscr{F}(C), \mathscr{F}(E))$. Unwinding the definitions, we see that the composition $(g, [v]) \circ (f, [u])$ is equal to $(h, [v] \circ [\mathscr{F}(f)(u)] \circ [\mu(X)]^{-1})$. We are therefore reduced to proving the commutativity of the diagram

$$\begin{tikzcd}
(\mathscr{F}(g) \circ \mathscr{F}(f))(X) \arrow[r, "{[\mu(X)]}"] \arrow[d, "{[\mathscr{F}(u)]}"] & (\mathscr{F}(h)(X) \arrow[d, "{[w]}"] \\
\mathscr{F}(g)(Y) \arrow[r, "{[v]}"] & Z
\end{tikzcd}$$

in the homotopy category $\mathrm{h}\mathscr{F}(Z)$. This commutativity is witnessed by the existence of a diagram

$$\begin{tikzcd}
(\mathscr{F}(g) \circ \mathscr{F}(f))(X) \arrow[r, "\mu(X)"] \arrow[d, "\mathscr{F}(u)"] & (\mathscr{F}(h)(X) \arrow[d, "w"] \\
\mathscr{F}(g)(Y) \arrow[r, "v"] & Z
\end{tikzcd}$$

in the ∞-category $\mathscr{F}(Z)$ itself, which is supplied by the datum of the 2-simplex σ (see Example 5.6.2.15). This completes the construction of the functor T.

It follows immediately from the definitions that the functor T is bijective at the level of objects and that, for every pair of objects (C, X) and (D, Y), the induced map

$$\theta : \pi_0(\mathrm{Hom}_{\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathscr{F}}((C,X),(D,Y)) \to \mathrm{Hom}_{\int_{\mathcal{C}} \mathrm{h}\mathscr{F}}((C,X),(D,Y))$$

is surjective. To complete the proof, we must show that θ is also injective. Fix a pair of morphisms $(f,u) : (C,X) \to (D,Y)$ and $(f',u') : (C,X) \to (D,Y)$ in the ∞-category $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathscr{F}$ having the same image under T, so that $f = f'$ as elements of $\mathrm{Hom}_{\mathcal{C}}(C,D)$ and the morphisms $u, u' : \mathscr{F}(f)(X) \to Y$ are homotopic in the ∞-category $\mathscr{F}(D)$. By virtue of Corollary 1.4.3.7, there exists a morphism of simplicial sets $\theta : \Box^2 \to \mathscr{F}(D)$ whose restriction to the boundary $\partial \Box^2$ is indicated in the diagram

$$\begin{tikzcd}
\mathscr{F}(f)(X) \arrow[r, "\mathrm{id}"] \arrow[d, "u"] & \mathscr{F}(f)(X) \arrow[d, "u'"] \\
Y \arrow[r, "\mathrm{id}"] & Y.
\end{tikzcd}$$

By virtue of Example 5.6.2.15, θ determines a 2-simplex of the ∞-category $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathscr{F}$ whose boundary is indicated in the diagram

$$\begin{tikzcd}
& (D,Y) \arrow[dr, "(\mathrm{id}_D{,}\mathrm{id}_Y)"] & \\
(C,X) \arrow[ur, "(f{,}u)"] \arrow[rr, "(f'{,}u')"] & & (D,Y),
\end{tikzcd}$$

which we can regard as a homotopy from (f,u) to (f',u'). □

In the statement of Proposition 5.6.3.1, it is essential that the source of the functor $\mathscr{F}$ is (the nerve of) an ordinary category. For a more general functor of ∞-categories $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$, one cannot expect to obtain the homotopy category of $\int_{\mathcal{C}} \mathscr{F}$ from the construction of Definition 5.6.1.1, because the forgetful functor $\mathrm{h}\int_{\mathcal{C}} \mathscr{F} \to \mathrm{h}\mathcal{C}$ need not be a cocartesian fibration. However, this difficulty does not arise in the case where $\mathscr{F}$ is a set-valued functor:

Proposition 5.6.3.2. *Let $\mathcal{C}$ be a simplicial set equipped with a morphism $\mathscr{F} : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{Set})$, which we can identify with a functor $\mathrm{h}\mathscr{F} : \mathrm{h}\mathcal{C} \to \mathrm{Set}$. Then the isomorphism of simplicial sets* 0275

$$\int_{\mathcal{C}} \mathscr{F} \simeq \mathcal{C} \times_{\mathrm{N}_\bullet(\mathrm{h}\mathcal{C})} \mathrm{N}_\bullet\Big(\int_{\mathrm{h}\mathcal{C}} \mathrm{h}\mathscr{F}\Big)$$

of Example 5.6.2.8 induces an isomorphism of categories

$$\mathrm{h}\int_{\mathcal{C}} \mathscr{F} \to \int_{\mathrm{h}\mathcal{C}} \mathrm{h}\mathscr{F}.$$

Proof. Using Proposition 4.1.3.2, we can factor the unit map $\mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ as a composition

$$\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$$

where F is inner anodyne and G is an inner fibration of simplicial sets (so that $\mathcal{C}'$ is an ∞-category). It follows that $\mathscr{F}$ extends uniquely to a morphism $\mathscr{F}' : \mathcal{C}' \to \mathrm{N}_{\bullet}(\mathrm{Set})$. Using Remark 5.6.2.17, we obtain a pullback diagram of simplicial sets

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r, "\widetilde{F}"] \arrow[d, "U"] & \int_{\mathcal{C}'} \mathscr{F}' \arrow[d] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{C}',
\end{tikzcd}$$

where the vertical maps are cocartesian fibrations (Proposition 5.6.2.2). Since F is inner anodyne, the map $\widetilde{F}$ is a categorical equivalence of simplicial sets (Proposition 5.3.6.1). Moreover, since F is bijective at the level of vertices, $\widetilde{F}$ is also bijective at the level of vertices. It follows that F and $\widetilde{F}$ induce isomorphisms of homotopy categories

$$\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{C}' \qquad \mathrm{h}\widetilde{F}' : \mathrm{h}\int_{\mathcal{C}} \mathscr{F} \to \mathrm{h}\int_{\mathcal{C}'} \mathscr{F}'.$$

Replacing $\mathcal{C}$ by $\mathcal{C}'$, we are reduced to proving Proposition 5.6.3.2 in the special case where $\mathcal{C}$ is an ∞-category.

Let $\mathcal{D}$ be the category of elements $\int_{\mathrm{h}\mathcal{C}} \mathrm{h}\mathscr{F}$, so that Example 5.6.2.8 supplies a pullback diagram of simplicial sets

$$\begin{tikzcd}
\int_{\mathcal{C}} \mathscr{F} \arrow[r, "G"] \arrow[d, "U"] & \mathrm{N}_{\bullet}(\mathcal{D}) \arrow[d] \\
\mathcal{C} \arrow[r, "\overline{G}"] & \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C}).
\end{tikzcd}$$

We wish to show that G exhibits $\mathcal{D}$ as a homotopy category of the ∞-category $\int_{\mathcal{C}} \mathscr{F}$. Note that, since $\overline{G}$ is bijective at the level of vertices, the functor G has the same property. It will therefore suffice to show that, for every pair of objects $X, Y \in \int_{\mathcal{C}} \mathscr{F}$, the induced map

$$\theta_{X,Y} : \mathrm{Hom}_{\int_{\mathcal{C}} \mathscr{F}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(G(X), G(Y))$$

exhibits $\mathrm{Hom}_{\mathcal{D}}(G(X), G(Y))$ as the set of connected components of the Kan complex $\mathrm{Hom}_{\int_{\mathcal{C}} \mathscr{F}}(X, Y)$. Equivalently, we wish to show that each fiber of the map $\theta_{X,Y}$ is a connected (and therefore nonempty) Kan complex. This is clear, since $\theta_{X,Y}$ is a pullback of the map

$$\overline{\theta}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(U(X), U(Y)) \to \pi_0(\mathrm{Hom}_{\mathcal{C}}(U(X), U(Y))) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(U(X), U(Y)),$$

whose fibers are the connected components of $\mathrm{Hom}_{\mathcal{C}}(U(X), U(Y))$. □

Corollary 5.6.3.3. *Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets. Then U is a left covering map (in the sense of Definition 4.2.3.8) if and only if the following pair of conditions is satisfied:* 027Y

(1) *The induced map $\mathrm{h}U : \mathrm{h}\mathcal{E} \to \mathrm{h}\mathcal{C}$ is a left covering functor (in the sense of Definition 4.2.3.1).*

(2) *The diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \mathrm{N}_\bullet(\mathrm{h}\mathcal{E}) \arrow[d, "\mathrm{N}_\bullet(\mathrm{h}U)"] \\
\mathcal{C} \arrow[r] & \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})
\end{tikzcd}$$

is a pullback square.

Proof. The sufficiency of conditions (1) and (2) follows from Proposition 4.2.3.16 and Remark 4.2.3.15. To prove the converse, assume that U is a left covering map. By virtue of Corollary 5.2.7.4, we may assume that $\mathcal{E} = \int_{\mathcal{C}} \mathscr{F}$ for some morphism of simplicial sets $\mathscr{F} : \mathcal{C} \to \mathrm{N}_\bullet(\mathrm{Set})$. Let us abuse notation by identifying $\mathscr{F}$ with a functor from the homotopy category $\mathrm{h}\mathcal{C}$ to the category of sets. Using Proposition 5.6.3.2, we can identify $\mathrm{h}\mathcal{E}$ with the category of elements $\int_{\mathrm{h}\mathcal{C}} \mathscr{F}$ of Construction 5.2.6.1. Condition (1) now follows from Remark 5.2.6.9, and condition (2) by combining Example 5.6.2.8 with Remark 5.6.2.17. □

We now consider a variant of the situation described in Proposition 5.6.3.1. Let $\mathcal{C}$ be an ordinary category and suppose we are given a strictly unitary functor of 2-categories $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$. Passing to the Duskin nerve (and using Remark 5.5.5.7 to identify $\mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{Cat})$ with a full subcategory of $\mathcal{QC}$), we obtain a functor of ∞-categories $\mathrm{N}^{\mathrm{D}}_\bullet(\mathscr{F}) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathcal{QC}$. Identifying $\mathrm{hN}^{\mathrm{D}}_\bullet(\mathscr{F})$ with the original functor $\mathscr{F}$, Proposition 5.6.3.1 yields a comparison functor

$$T : \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{D}}_\bullet(\mathscr{F}) \to \mathrm{N}_\bullet(\int_{\mathcal{C}} \mathscr{F}).$$

Proposition 5.6.3.4. *Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to$ **Cat** be a strictly unitary functor of 2-categories. Then the comparison map* 0276

$$T : \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{D}}_\bullet(\mathscr{F}) \to \mathrm{N}_\bullet(\int_{\mathcal{C}} \mathscr{F})$$

is an isomorphism of simplicial sets.

Stated more informally, Proposition 5.6.3.4 asserts that we can regard the classical category of elements construction (Definition 5.6.1.1) as a special case of Definition 5.6.2.4.

Proof of Proposition 5.6.3.4. By virtue of Proposition 5.6.3.1, it will suffice to show that the simplicial set $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathrm{D}}(\mathscr{F})$ is isomorphic to the nerve of a category. We will prove this by verifying the criterion of Proposition 1.3.4.1. Fix $0 < i < n$; we wish to show that every morphism of simplicial sets $\sigma_0 : \Lambda_i^n \to \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathrm{D}}(\mathscr{F})$ can be extended uniquely to an n-simplex σ of $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathrm{D}}(\mathscr{F})$. Let $\overline{\sigma}_0$ denote the composition of σ_0 with the projection map $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathrm{D}}(\mathscr{F}) \to \mathrm{N}_\bullet(\mathcal{C})$. Proposition 1.3.4.1 then guarantees that $\overline{\sigma}_0$ extends uniquely to a morphism of simplicial sets $\overline{\sigma} : \Delta^n \to \mathrm{N}_\bullet(\mathcal{C})$. It will therefore suffice to show that the lifting problem

0277
$$\begin{tikzcd}
\Lambda_i^n \arrow[r, "\sigma_0"] \arrow[d] & \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathrm{D}}(\mathscr{F}) \arrow[d, "\pi"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathrm{N}_\bullet(\mathcal{C})
\end{tikzcd} \tag{5.28}$$

has a unique solution.

We begin by treating the special case $n = 2$ (so that $i = 1$). In this case, we can identify σ_0 with a pair of composable morphisms

$$(C, X) \xrightarrow{(f,u)} (D, Y) \xrightarrow{(g,v)} (E, Z)$$

in the ∞-category $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathrm{D}}(\mathscr{F})$. Set $h = g \circ f \in \mathrm{Hom}_{\mathcal{C}}(C, E)$, so that the composition constraint of $\mathscr{F}$ determines an isomorphism of functors $\mu : \mathscr{F}(g) \circ \mathscr{F}(f) \xrightarrow{\sim} \mathscr{F}(h)$. Unwinding the definitions (using Example 5.6.2.15), we are reduced to proving that there is a unique morphism $w : \mathscr{F}(h)(X) \to Z$ in the category $\mathscr{F}(E)$ for which the diagram

$$\begin{tikzcd}
(\mathscr{F}(g) \circ \mathscr{F}(f))(X) \arrow[r, "\mu(X)"] \arrow[d, "\mathscr{F}(u)"] & \mathscr{F}(h)(X) \arrow[d, "w"] \\
\mathscr{F}(g)(Y) \arrow[r, "v"] & Z
\end{tikzcd}$$

commutes. This is clear, since $\mu(X)$ is an isomorphism in the category $\mathscr{F}(E)$.

We now treat the case $n \geq 3$. Note that the existence of a solution to the lifting problem (5.28) is automatic (since the projection map π is a cocartesian fibration; see Proposition 5.6.2.2). It will therefore suffice to show that σ is unique. Using Lemma 4.3.6.15 and Remark

5.5.5.7, we can rewrite (5.28) as a lifting problem

$$\begin{tikzcd}
\Lambda^{n+1}_{i+1} \arrow[r, "\tau_0"] \arrow[d] & \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathbf{Cat}) \arrow[d] \\
\Delta^{n+1} \arrow[r] \arrow[ur, dashed, "\tau"] & \Delta^0.
\end{tikzcd}$$

The uniqueness of its solution is now an immediate consequence of Proposition 2.3.1.9, since the horn Λ^{n+1}_{i+1} contains the 2-skeleton of Δ^{n+1}. □

Corollary 5.6.3.5. *Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a morphism of simplicial sets which factors through the full subcategory $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathbf{Cat})) \subset \mathcal{QC}$ of Remark 5.5.4.9. Then the projection map $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is an inner covering map of simplicial sets.* 0278

Proof. By virtue of Corollary 4.1.5.11, we may assume without loss of generality that $\mathcal{C} = \Delta^n$ is a standard simplex. In this case, we wish to show that the simplicial set $\int_{\mathcal{C}} \mathscr{F}$ is isomorphic to the nerve of an ordinary category (see Proposition 4.1.5.10), which is a special case of Proposition 5.6.3.4. □

Warning 5.6.3.6. Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a morphism of simplicial sets. In the language of §4.8, Corollary 5.6.3.5 asserts that if each of the ∞-categories $\mathscr{F}(C)$ is a $(1,1)$-category, then the cocartesian fibration $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is 1-categorical (see Example 4.8.6.27). Beware that for $n \geq 2$, the assumption that each $\mathscr{F}(C)$ is an $(n,1)$-category does not guarantee that the cocartesian fibration U is n-categorical. However, if $\mathcal{C}$ is an ∞-category, then it is essentially n-categorical: this is an immediate consequence of Variant 5.1.5.17. 05DZ

5.6.4 Comparison with the Weighted Nerve

Let $\mathcal{C}$ be a category which is equipped with a functor $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$. In §5.3.3 and §5.6.3, we introduced two different cocartesian fibrations associated to $\mathscr{F}$: 0279

- The projection map $U : \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$, where $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ denotes the $\mathscr{F}$-weighted nerve of $\mathcal{C}$ (Definition 5.3.3.1). For each object $C \in \mathcal{C}$, the fiber $U^{-1}\{C\}$ is isomorphic (as a simplicial set) to the ∞-category $\mathscr{F}(C)$ (Example 5.3.3.8).
- The projection map $U' : \int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F}) \to \mathrm{N}_{\bullet}(\mathcal{C})$, where $\int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$ denotes the ∞-category of elements of the functor $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}) = \mathcal{QC}$. For each object $C \in \mathcal{C}$, the fiber $U'^{-1}\{C\}$ is equivalent (but not necessarily isomorphic) to the ∞-category $\mathscr{F}(C)$ (Example 5.6.2.18).

Our goal in this section is to show that these constructions are equivalent (though not necessarily isomorphic).

027A **Construction 5.6.4.1** (The Comparison Map)**.** Let $\mathcal{C}$ be a category and let $\vec{C}$ be an n-simplex of the nerve $\mathrm{N}_{\bullet}(\mathcal{C})$, given by a diagram

$$C_0 \to C_1 \to C_2 \to \cdots \to C_{n-1} \to C_n$$

in the category $\mathcal{C}$. Let $\mathscr{F}$ be a functor from $\mathcal{C}$ to the category of simplicial sets and suppose that we are given a collection of simplices $\vec{\sigma} = \{\sigma_j : \Delta^j \to \mathscr{F}(C_j)\}_{0 \leq j \leq n}$ which fit into a commutative diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook] \arrow[d, "\sigma_0"] & \Delta^1 \arrow[r, hook] \arrow[d, "\sigma_1"] & \Delta^2 \arrow[r, hook] \arrow[d, "\sigma_2"] & \cdots \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, "\sigma_n"] \\
\mathscr{F}(C_0) \arrow[r] & \mathscr{F}(C_1) \arrow[r] & \mathscr{F}(C_2) \arrow[r] & \cdots \arrow[r] & \mathscr{F}(C_n).
\end{tikzcd}$$

To this data, we can associate a commutative diagram of simplicial sets

027B
$$\begin{tikzcd}
\Delta^n \arrow[r] \arrow[d, "\vec{C}"] & \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Set}_{\Delta})_{\Delta^0/} \arrow[d] \\
\mathrm{N}_{\bullet}(\mathcal{C}) \arrow[r, "\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})"] & \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Set}_{\Delta}),
\end{tikzcd} \qquad (5.29)$$

where the upper horizontal map is given by the simplicial functor

$$F : \mathrm{Path}[\{x\} \star [n]]_{\bullet} \to \mathrm{Set}_{\Delta}$$

described as follows:

- The functor F carries x to the simplicial set Δ^0 (so that F can be identified with an n-simplex of the coslice simplicial set $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Set}_{\Delta})_{\Delta^0/}$).
- The restriction of F to the simplicial path category $\mathrm{Path}[n]_{\bullet}$ is given by the composition

  $$\mathrm{Path}[n]_{\bullet} \to [n] \xrightarrow{\vec{C}} \mathcal{C} \xrightarrow{\mathscr{F}} \mathrm{Set}_{\Delta}$$

  (as required by the commutativity of the diagram (5.29)).

- For $0 \leq m \leq n$, the induced map of simplicial sets
$$\mathrm{Hom}_{\mathrm{Path}[\{x\}\star[n]]}(x,m)_\bullet \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(F(x),F(m))_\bullet = \mathscr{F}(C_m)$$
is given by the composition $\mathrm{Hom}_{\mathrm{Path}[\{x\}\star[n]]}(x,m)_\bullet \xrightarrow{\rho} \Delta^m \xrightarrow{\sigma_m} \mathscr{F}(C_m)$, where ρ is induced by the morphism of partially ordered sets
$$\mathrm{Hom}_{\mathrm{Path}[\{x\}\star[n]]}(x,m) \to [m] \qquad (S \subseteq \{x\}\star[n]) \mapsto \min(S \setminus \{x\}).$$

Note that we can identify the diagram (5.29) with an n-simplex $\theta(\overrightarrow{C}, \overrightarrow{\sigma})$ of the simplicial set $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$. The construction $(\overrightarrow{C}, \overrightarrow{\sigma}) \mapsto \theta(\overrightarrow{C}, \overrightarrow{\sigma})$ then determines a morphism of simplicial sets $\theta : \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$, which we will refer to as the *comparison map*.

Example 5.6.4.2 (The Comparison Map on Vertices)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F}$ be a functor from $\mathcal{C}$ to the category of simplicial sets. Let us identify vertices of the weighted nerve $\mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$ with pairs (C,X), where C is an object of $\mathcal{C}$ and X is a vertex of the simplicial set $\mathscr{F}(C)$ (Remark 5.3.3.3). Under this identification, the comparison map 027C
$$\theta : \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$$
of Construction 5.6.4.1 is given on vertices by the construction $(C,X) \mapsto (C,X)$, where we identify (C,X) with a vertex of $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$ using Example 5.6.2.12. In particular, the morphism θ is bijective at the level of vertices.

Example 5.6.4.3 (The Comparison Map on Edges)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F}$ be a functor from $\mathcal{C}$ to the category of simplicial sets. Let (C,X) and (D,Y) be vertices of the weighted nerve $\mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$. Using Remark 5.3.3.4, we can identify edges of $\mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$ having source (C,X) and target (D,Y) with pairs (f,u), where $f : C \to D$ is a morphism in the category $\mathcal{C}$ and $u : \mathscr{F}(f)(X) \to Y$ is an edge of the simplicial set $\mathscr{F}(D)$. Under this identification, the comparison map 027D
$$\theta : \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$$
of Construction 5.6.4.1 is given on edges by the construction $(f,u) \mapsto (f,u)$, where we identify (f,u) with an edge of the simplicial set $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$ using Example 5.6.2.13. In particular, the morphism θ is bijective at the level of edges.

Warning 5.6.4.4. Let $\mathcal{C}$ be a category and let $\mathscr{F}$ be a functor from $\mathcal{C}$ to the category of simplicial sets. The comparison map $\theta : \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$ of Construction 5.6.4.1 is generally not bijective on n-simplices for $n \geq 2$ (even in the special case $\mathcal{C} = [0]$). 027E

Exercise 5.6.4.5. Let $\mathcal{C}$ be a category and let $\mathscr{F}$ be a functor from $\mathcal{C}$ to the category of simplicial sets. Show that the comparison map $\theta : \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$ of Construction 5.6.4.1 is a monomorphism of simplicial sets. 027F

027G **Remark 5.6.4.6.** Let $\mathcal{C}$ be a category and let $\mathscr{F}$ be a functor from $\mathcal{C}$ to the category of simplicial sets. Then the diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \arrow[rr, "\theta"] \arrow[dr] & & \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F}) \arrow[dl] \\
& \mathrm{N}_\bullet(\mathcal{C}) &
\end{tikzcd}$$

is commutative, where the vertical morphisms are the projection maps of Definitions 5.3.3.1 and 5.6.2.1 and θ is the comparison morphism of Construction 5.6.4.1

027H **Example 5.6.4.7.** Let $\mathcal{C}$ be a category and let $\mathscr{F}$ be a functor from $\mathcal{C}$ to the category of simplicial sets. For every object $C \in \mathcal{C}$, the comparison morphism $\theta : \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$ of Construction 5.6.4.1 induces a morphism of simplicial sets

$$\theta_C : \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F}).$$

Under the isomorphisms

$$\mathscr{F}(C) \simeq \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \qquad \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Set}_\Delta)}(\Delta^0, \mathscr{F}(C)) \simeq \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$$

supplied by Examples 5.3.3.8 and 5.6.2.18, we can identify θ_C with the comparison map $\mathscr{F}(C) \to \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Set}_\Delta)}(\Delta^0, \mathscr{F}(C))$ of Construction 4.6.8.3.

027J **Proposition 5.6.4.8.** *Let $\mathcal{C}$ be a category equipped with a functor $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$, and let*

$$\begin{tikzcd}
\mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \arrow[rr, "\theta"] \arrow[dr, "U"'] & & \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F}) \arrow[dl, "U'"] \\
& \mathrm{N}_\bullet(\mathcal{C}) &
\end{tikzcd}$$

be the commutative diagram of Remark 5.6.4.6. Then:

(1) *For each object $C \in \mathcal{C}$, the morphism θ induces an equivalence of ∞-categories*

$$\theta_C : \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F}).$$

(2) *A morphism f of the weighted nerve $\mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$ is U-cocartesian if and only if $\theta(f)$ is a U'-cocartesian morphism of the ∞-category $\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$.*

(3) *The functor θ is an equivalence of ∞-categories.*

Proof. Assertion (1) follows from Example 5.6.4.7 and Theorem 4.6.8.9. Assertion (2) follows from Example 5.6.4.3 together with the descriptions of U-cocartesian and U'-cocartesian morphisms supplied by Corollary 5.3.3.16 and Remark 5.6.2.14. Assertion (3) follows by combining (1) and (2) with Theorem 5.1.6.1 (since U and U' are cocartesian fibrations, by virtue of Corollary 5.3.3.16 and Proposition 5.6.2.2). □

5.6.5 The Universality Theorem

Throughout this section, we let $\mathcal{QC}_{\mathrm{Obj}}$ denote the ∞-category of pairs $(\mathcal{C}, C)$, where $\mathcal{C}$ is a small ∞-category and C is an object of $\mathcal{C}$ (Definition 5.5.6.10), and we let $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ denote the forgetful functor (given on objects by the formula $V(\mathcal{C}, C) = \mathcal{C}$). 028U

Definition 5.6.5.1. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. We will say that a commutative diagram of simplicial sets 028V

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC}
\end{tikzcd}$$

witnesses $\mathscr{F}$ as a covariant transport representation of U if the induced map

$$\mathcal{E} \to \mathcal{C} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}} = \int_{\mathcal{C}} \mathscr{F}$$

is an equivalence of cocartesian fibrations over $\mathcal{C}$, in the sense of Definition 5.1.7.1. We say that $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ *is a covariant transport representation of U* if there exists a diagram which witnesses $\mathscr{F}$ as a covariant transport representation of U.

Remark 5.6.5.2. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a functor. By virtue of Proposition 5.1.7.5, a diagram 028W

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC}
\end{tikzcd}$$

witnesses $\mathscr{F}$ as a covariant transport representation for U if and only if the induced map $\mathcal{E} \to \int_{\mathcal{C}} \mathscr{F}$ is an equivalence of ∞-categories. We will later extend this observation to the case where $\mathcal{C}$ is a general simplicial set (Corollary 5.6.7.8).

028X **Remark 5.6.5.3.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. A commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC}
\end{tikzcd}$$

witnesses $\mathscr{F}$ as a covariant transport representation of U if and only if it satisfies the following pair of conditions:

(a) For every vertex $C \in \mathcal{C}$, the map of fibers

$$\widetilde{\mathscr{F}}_C : \mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E} \to \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$$

is an equivalence of ∞-categories.

(b) The morphism $\widetilde{\mathscr{F}}$ carries U-cocartesian edges of $\mathcal{E}$ to V-cocartesian edges of $\mathcal{QC}_{\mathrm{Obj}}$.

See Proposition 5.1.7.15. Moreover, we can replace (b) by the following *a priori* weaker condition (see Remark 5.1.6.8):

(b') For every vertex $X \in \mathcal{E}$ and every edge $\overline{e} : U(X) \to \overline{Y}$ in $\mathcal{C}$, there exists a U-cocartesian edge $e : X \to Y$ of $\mathcal{E}$ for which $U(e) = \overline{e}$ and and $\widetilde{\mathscr{F}}(e)$ is a V-cocartesian edge of $\mathcal{QC}_{\mathrm{Obj}}$.

028Z **Example 5.6.5.4** (Left Covering Maps)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a left covering map of simplicial sets and let $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{Set}$ be the homotopy transport representation of U (Example 5.2.5.3), so that $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ can be identified with a morphism of simplicial sets $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{Set})$. Combining Proposition 5.2.7.2 with Example 5.6.2.8, we obtain a canonical isomorphism of simplicial sets $\mathcal{E} \simeq \int_{\mathcal{C}} \mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$, which exhibits $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ as a covariant transport representation of U (in the sense of Definition 5.6.5.1).

0290 **Example 5.6.5.5** (Fibrations over a Point)**.** Let $\mathcal{E}$ be a small ∞-category, which we identify with a morphism $\mathscr{F} : \Delta^0 \to \mathcal{QC}$. Then $\mathscr{F}$ is a covariant transport representation of the projection map $U : \mathcal{E} \to \Delta^0$. More precisely, Example 5.6.2.16 supplies an equivalence of ∞-categories $\mathcal{E} \to \int_{\Delta^0} \mathscr{F}$ which witnesses $\mathscr{F}$ as a covariant transport representation of U. More generally, a functor $\Delta^0 \to \mathcal{QC}$ is a covariant transport representation of U if and only if it corresponds to an ∞-category which is equivalent to $\mathcal{E}$.

0385 **Example 5.6.5.6** (Weighted Nerves)**.** Let $\mathcal{C}$ be an ordinary category, let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a functor, and let $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ be the weighted nerve of Definition 5.3.3.1. Then the projection map $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a cocartesian fibration (Corollary 5.3.3.16). Moreover, the equivalence

$$\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F})$$

of Proposition 5.6.4.8 exhibits $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F})$ as a covariant transport representation for U.

Example 5.6.5.7 (Strict Transport)**.** Let $\mathcal{C}$ be an ordinary category, let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a cocartesian fibration of ∞-categories, and let $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathrm{QCat}$ be the strict transport representation of U (Construction 5.3.1.5). Then the functor 0386

$$\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{QCat}) = \mathcal{QC}$$

is a covariant transport representation for U (in the sense of Definition 5.6.5.1). In other words, U is equivalent to the cocartesian fibration $U' : \int_{\mathcal{C}} \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}) \to \mathrm{N}_{\bullet}(\mathcal{C})$. To see this, we observe that both U and U' are equivalent to the cocartesian fibration $\mathrm{N}_{\bullet}^{\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$: this follows from Theorem 5.3.5.6 and Proposition 5.6.4.8.

Remark 5.6.5.8. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, and let $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ be the homotopy transport representation of U (Construction 5.2.5.2). Let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a morphism of simplicial sets and let $\mathrm{h}\mathscr{F} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ be the induced functor between homotopy categories. Let $\alpha : \mathcal{E} \to \int_{\mathcal{C}} \mathscr{F}$ be an equivalence of cocartesian fibrations over $\mathcal{C}$. By virtue of Corollary 5.6.2.22, α induces an isomorphism from $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ to $\mathrm{h}\mathscr{F}$ in the functor category $\mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{hQCat})$. Stated more informally, any covariant transport representation of U provides a lifting of the homotopy transport representation $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ from the ordinary category $\mathrm{Fun}(\mathrm{h}\mathcal{C}, \mathrm{hQCat})$ to the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$. Moreover, if the simplicial set $\mathcal{C}$ is an ∞-category, then the identification $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} \simeq \mathrm{h}\mathscr{F}$ is an isomorphism of hKan-enriched functors (Proposition 5.6.2.20). 0291

Remark 5.6.5.9. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let $\mathscr{F}, \mathscr{F}' : \mathcal{C} \to \mathcal{QC}$ be morphisms which are isomorphic as objects of the diagram ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$. Then $\mathscr{F}$ is a covariant transport representation of U if and only if $\mathscr{F}'$ is a covariant transport representation of U. This follows immediately from Proposition 5.6.2.19. 0292

We now formulate a stronger version of Theorem 5.6.0.2:

Theorem 5.6.5.10 (Relative Universality Theorem)**.** *Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibration of simplicial sets, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a simplicial subset having inverse image $\mathcal{E}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E} \subseteq \mathcal{E}$, and let $U_0 : \mathcal{E}_0 \to \mathcal{C}_0$ be the restriction $U|_{\mathcal{E}_0}$. Suppose we are given a* 0293 *commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "\widetilde{\mathscr{F}}_0"] \arrow[d, "U_0"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\mathcal{C}_0 \arrow[r, "\mathscr{F}_0"] & \mathcal{QC}
\end{tikzcd}$$

which witnesses $\mathscr{F}_0$ as a covariant transport representation of U_0. Then there exists a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC}
\end{tikzcd}$$

which witnesses $\mathscr{F}$ as a covariant transport representation of U, where $\mathscr{F}_0 = \mathscr{F}|_{\mathcal{C}_0}$ and $\widetilde{\mathscr{F}_0} = \widetilde{\mathscr{F}}|_{\mathcal{E}_0}$.

We will give a reformulation of Theorem 5.6.5.10 in §5.6.8 (see Theorem 5.6.8.3), which we prove in §5.6.9.

0294 **Corollary 5.6.5.11.** *Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibration of simplicial sets, let $\mathcal{C}' \subseteq \mathcal{C}$ be a simplicial subset, and let $\mathscr{F}' : \mathcal{C}' \to \mathcal{QC}$ be a covariant transport representation for the projection map $\mathcal{C}' \times_{\mathcal{C}} \mathcal{E} \to \mathcal{C}'$. Then there exists a morphism $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ satisfying $\mathscr{F}' = \mathscr{F}|_{\mathcal{C}'}$ which is a covariant transport representation of U.*

0295 **Corollary 5.6.5.12.** *Let $\mathcal{Q}$ be a full subcategory of $\mathcal{QC}$ and let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets having the property that, for each vertex $C \in \mathcal{C}$, the fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is equivalent to an ∞-category which belongs to $\mathcal{Q}$. Then there exists a morphism $\mathscr{F} : \mathcal{C} \to \mathcal{Q} \subseteq \mathcal{QC}$ which is a covariant transport representation of U.*

Proof. For each vertex $C \in \mathcal{C}$, choose an ∞-category $\mathscr{F}'(C) \in \mathcal{Q}$ which is equivalent to the fiber $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$. The construction $C \mapsto \mathscr{F}'(C)$ determines a morphism of simplicial sets $\mathscr{F}' : \mathcal{C}' \to \mathcal{Q}$, where $\mathcal{C}' = \mathrm{sk}_0(\mathcal{C})$ is the 0-skeleton of $\mathcal{C}$, which is a covariant transport representation of the projection map $\mathcal{C}' \times_{\mathcal{C}} \mathcal{E} \to \mathcal{C}'$ (see Example 5.6.5.5). Applying Corollary 5.6.5.11, we can extend $\mathscr{F}'$ to a morphism $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ which is a covariant transport representation of U. By construction, the morphism $\mathscr{F}$ takes values in the full subcategory $\mathcal{Q} \subseteq \mathcal{QC}$. □

Corollary 5.6.5.13. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let $\mathscr{F}_0, \mathscr{F}_1 : \mathcal{C} \to \mathcal{QC}$ be covariant transport representations for U. Then $\mathscr{F}_0$ and $\mathscr{F}_1$ are isomorphic as objects of the ∞-category* $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$. 0296

Proof. Let $U_{\Delta^1} : \Delta^1 \times \mathcal{E} \to \Delta^1 \times \mathcal{C}$ be the product of U with the identity map id_{Δ^1}, and define $U_{\partial\Delta^1} : \partial\Delta^1 \times \mathcal{E} \to \partial\Delta^1 \times \mathcal{C}$ similarly. Note that the map $(\mathscr{F}_0, \mathscr{F}_1) : \partial\Delta^1 \times \mathcal{C} \to \mathcal{QC}$ is a covariant transport representation of $U_{\partial\Delta^1}$. Applying Corollary 5.6.5.11, we deduce that U_{Δ^1} admits a covariant transport representation $\mathscr{F} : \Delta^1 \times \mathcal{C} \to \mathcal{QC}$ which satisfies $\mathscr{F}|_{\{0\}\times\mathcal{C}} = \mathscr{F}_0$ and $\mathscr{F}|_{\{1\}\times\mathcal{C}} = \mathscr{F}_1$. Let us identify $\mathscr{F}$ with a morphism $u : \mathscr{F}_0 \to \mathscr{F}_1$ in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$. We will complete the proof by showing that u is an isomorphism. By virtue of Theorem 4.4.4.4, it will suffice to show that for each vertex $C \in \mathcal{C}$, the induced map $u_C : \mathscr{F}_0(C) \to \mathscr{F}_1(C)$ is an isomorphism in $\mathcal{QC}$. Using Remark 5.6.5.8 (and Remark 5.2.8.5), we see that the homotopy class $[u_C]$ is isomorphic (as an object of the arrow category $\mathrm{Fun}([1], \mathrm{hQCat})$) to the homotopy class of the functor $\mathcal{E}_C \to \mathcal{E}_C$ given by covariant transport along the degenerate edge id_C of $\mathcal{C}$: that is, the homotopy class of the identity functor $\mathrm{id}_{\mathcal{E}_C}$. □

Proof of Theorem 5.6.0.2. Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibrations of simplicial sets. We wish to show that U admits a covariant transport representation $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$, which is uniquely determined up to isomorphism (as an object of the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$). The existence statement follows by applying Theorem 5.6.5.10 in the special case $\mathcal{C}_0 = \emptyset$, and the uniqueness follows from Corollary 5.6.5.13. □

Notation 5.6.5.14 (The Covariant Transport Representation)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibration. We let $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ denote a covariant transport representation of U, regarded as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$ (which exists by virtue of Corollary 5.6.5.12). We write $[\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}]$ for the isomorphism class of the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$, regarded as an object of the set $\pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{QC})^{\simeq})$. By virtue of Corollary 5.6.5.13, the isomorphism class $[\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}]$ is well-defined: that is, it depends only on the cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$. Beware that $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ is not uniquely determined: in fact, any diagram isomorphic to $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ is also a covariant transport representation of U (Remark 5.6.5.9). Nevertheless, it will be convenient to abuse terminology and refer to $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ as *the* covariant transport representation of U, with the caveat that it is well-defined only up to isomorphism. 019S

Remark 5.6.5.15. Let $\mathcal{C}$ be a simplicial set equipped with a functor $\overline{\mathscr{F}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$. It follows from Corollary 5.6.5.12 that the functor $\overline{\mathscr{F}}$ is isomorphic to the homotopy transport representation of a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ if and only if it can be promoted to a diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$. 0297

Corollary 5.6.5.16. *Let $\mathcal{C}$ be a small category. Then passage to the homotopy coherent* 0387 *nerve induces a bijection*

$$\begin{tikzcd}
\{\text{Functors of ordinary categories } \mathcal{C} \to \mathrm{QCat}\}/\text{Levelwise equivalence} \arrow[d] \\
\{\text{Functors of } \infty\text{-categories } \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{QC}\}/\text{Isomorphism.}
\end{tikzcd}$$

Proof. Combine Example 5.6.5.7, Theorem 5.3.5.6, and Theorem 5.6.0.2. □

0388 **Remark 5.6.5.17** (Rectification)**.** Corollary 5.6.5.16 is a prototypical example of a *rectification* result. If $\mathcal{C}$ is an ordinary category, then a functor of ∞-categories $\mathscr{F} : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{QC}$ can be viewed as a *homotopy coherent* diagram in the simplicial category QCat:

- To every object X of the category $\mathcal{C}$, the functor $\mathscr{F}$ associates an ∞-category $\mathscr{F}(X)$.
- To every morphism $u : X \to Y$ of the category $\mathcal{C}$, the functor $\mathscr{F}$ associates a functor of ∞-categories $\mathscr{F}(u) : \mathscr{F}(X) \to \mathscr{F}(Y)$.
- To every pair of composable morphisms $u : X \to Y$ and $v : Y \to Z$ in the category $\mathcal{C}$, the functor $\mathscr{F}$ associates an isomorphism of functors $\alpha_{u,v} : \mathscr{F}(v) \circ \mathscr{F}(u) \to \mathscr{F}(v \circ u)$.
- When applied to higher-dimensional simplices of $\mathrm{N}_{\bullet}(\mathcal{C})$, the functor $\mathscr{F}$ provides additional data which encode coherence laws satisfied by the isomorphisms $\alpha_{u,v}$.

Corollary 5.6.5.16 asserts that we can always find a *strictly commutative* diagram $\mathscr{G} : \mathcal{C} \to \mathrm{QCat}$ which is isomorphic to $\mathscr{F}$ in the ∞-category $\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}), \mathcal{QC})$. In particular, the diagram $\mathscr{G}$ carries each object $X \in \mathcal{C}$ to an ∞-category $\mathscr{G}(X)$ which is equivalent to $\mathscr{F}(X)$ (beware that we generally cannot arrange that $\mathscr{G}(X)$ is *isomorphic* to $\mathscr{F}(X)$ as a simplicial set).

In §[?], we will prove a more refined version of this result, which allows us to describe the entire ∞-category $\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}), \mathcal{QC})$ in terms of strictly commutative diagrams indexed by $\mathcal{C}$ (Proposition [?]).

Using Theorem 5.6.5.10, we obtain the following converse of Corollary 5.6.3.5.

029A **Proposition 5.6.5.18.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism U is an inner covering map (Definition 4.1.5.1), a cocartesian fibration, and each fiber of U is small.*

(2) *There exists a morphism of simplicial sets $\mathscr{F} : \mathcal{C} \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathbf{Cat})) \subseteq \mathcal{QC}$ and an isomorphism $G : \mathcal{E} \simeq \int_{\mathcal{C}} \mathscr{F}$ in the category $(\mathrm{Set}_{\Delta})_{/\mathcal{C}}$.*

Proof. The implication (2) ⇒ (1) follows from Corollary 5.6.3.5 and Proposition 5.6.2.2. For each vertex $C \in \mathcal{C}$, our assumption that U is an inner covering map guarantees that the fiber $\{C\} \times_{\mathcal{C}} \mathcal{E}$ is isomorphic to the nerve of a (small) category $\mathscr{F}_0(C)$ (Example 4.1.5.3). Let $\mathcal{C}_0$ be the 0-skeleton of $\mathcal{C}$, so that the construction $C \mapsto \mathscr{F}_0(C)$ determines a morphism of simplicial sets $\mathscr{F}_0 : \mathcal{C}_0 \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathbf{Cat}))$. Let $\mathcal{E}_0$ denote the inverse image $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$, so that Proposition 5.6.3.4 supplies an isomorphism of simplicial sets $G_0 : \mathcal{E}_0 \simeq \int_{\mathcal{C}_0} \mathscr{F}_0$. In particular, G_0 is an equivalence of cocartesian fibrations over $\mathcal{C}_0$. Invoking Theorem 5.6.5.10, we can extend $\mathscr{F}_0$ to a diagram $\mathscr{F} : \mathcal{C} \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathbf{Cat}))$ and G_0 to a morphism of simplicial sets $G : \mathcal{E} \to \int_{\mathcal{C}} \mathscr{F}$ which is an equivalence of cocartesian fibrations over $\mathcal{C}$. We will complete the proof by showing that G is an isomorphism of simplicial sets. To prove this, it will suffice to show that for every simplex $\sigma : \Delta^n \to \mathcal{C}$, the induced map

$$G_\sigma : \Delta^n \times_{\mathcal{C}} \mathcal{E} \to \Delta^n \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$$

is an isomorphism of simplicial sets. Replacing U by the projection map $\Delta^n \times_{\mathcal{C}} \mathcal{E} \to \Delta^n$, we are reduced to proving that G is an isomorphism under the additional assumption that $\mathcal{C} = \Delta^n$ is a standard simplex. Since U and the projection map $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ are inner covering maps, the simplicial sets $\mathcal{E}$ and $\int_{\mathcal{C}} \mathscr{F}$ are isomorphic to the nerves of their homotopy categories $\mathrm{h}\mathcal{E}$ and $\mathrm{h}\int_{\mathcal{C}} \mathscr{F}$, respectively; it will therefore suffice to show that the functor of ordinary categories $\mathrm{h}G : \mathrm{h}\mathcal{E} \to \mathrm{h}\int_{\mathcal{C}} \mathscr{F}$ is an isomorphism. Our assumption that G is an equivalence of cocartesian fibrations over $\mathcal{C} = \Delta^n$ guarantees that it is an equivalence of ∞-categories (Corollary 5.1.7.8), so that $\mathrm{h}G$ is an equivalence of ordinary categories. It will therefore suffice to show that the functor $\mathrm{h}G$ is bijective on objects: that is, that the morphism G is bijective on vertices. This is clear, since the morphism $G_0 = G|_{\mathcal{E}_0}$ is an isomorphism. □

Corollary 5.6.5.19 (Grothendieck)**.** *Let $U : \mathcal{E} \to \mathcal{C}$ be functor between categories. The following conditions are equivalent:* 01SU

(1) *The functor U is a cocartesian fibration and each fiber of U is a small category.*

(2) *There exists a functor of 2-categories $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ and an isomorphism $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{E}$ whose composition with U coincides with the forgetful functor $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$.*

Proof. We will show that (1) ⇒ (2); the reverse implication follows from Corollary 5.6.1.16. Note that the map $\mathrm{N}_{\bullet}(U) : \mathrm{N}_{\bullet}(\mathcal{E}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a cocartesian fibration of simplicial sets (Example 5.1.4.2) and an inner covering map (Proposition 4.1.5.10). By virtue of Proposition 5.6.5.18, there exists a morphism of simplicial sets $\mathscr{F}' : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Pith}(\mathbf{Cat}))$ and an isomorphism of simplicial sets $V : \int_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathscr{F} \simeq \mathrm{N}_{\bullet}(\mathcal{E})$ which is compatible with $\mathrm{N}_{\bullet}(U)$. By virtue of Theorem 2.3.4.1 (and Corollary 2.3.4.5), we have $\mathscr{F}' = \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathscr{F})$ for a unique functor of 2-categories $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$. In this case, we can use Proposition 5.6.3.4 to identify

$\int_{\mathrm{N}_\bullet(\mathcal{C})} \mathscr{F}'$ with the nerve of the ordinary category of elements $\int_{\mathcal{C}} \mathscr{F}$. Under this identification, V corresponds to the nerve of an isomorphism $\int_{\mathcal{C}} \mathscr{F}' \simeq \mathcal{E}$ which is compatible with U. □

Let **Gpd** $\subseteq$ **Cat** denote the full subcategory spanned by the groupoids.

`01SZ` **Corollary 5.6.5.20.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a functor between categories. The following conditions are equivalent:*

- *The functor U is an opfibration in groupoids (Variant 4.2.2.4) and each fiber of U is a small groupoid.*
- *There exists a functor of 2-categories $\mathscr{F} : \mathcal{C} \to$ **Gpd** and an isomorphism of categories $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{E}$ which carries U to the forgetful functor $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$.*

Proof. Combine Corollary 5.6.5.19 with Exercise 5.0.0.6. □

5.6.6 Application: Corepresentable Functors

`02HU` Let $\mathcal{C}$ be a category. Every object $X \in \mathcal{C}$ determines a functor

$$h^X : \mathcal{C} \to \mathrm{Set} \qquad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

which we refer to as the *functor corepresented by* X. We say that a functor from $\mathcal{C}$ to Set is *corepresentable* if it is isomorphic to h^X for some object $X \in \mathcal{C}$. Our goal in this section is to develop an ∞-categorical counterpart of the notion of corepresentable functor (and the dual notion of representable functor), where we replace the ordinary category Set by the ∞-category $\mathcal{S}$ of Construction 5.5.1.1.

We begin with an elementary observation. Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a functor between ordinary categories. For each object $X \in \mathcal{C}$, Yoneda's lemma supplies a bijection

$$\mathscr{F}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set})}(h^X, \mathscr{F}).$$

Concretely, this bijection carries each element $x \in \mathscr{F}(X)$ to a natural transformation $\alpha_x : h^X \to \mathscr{F}$, characterized by the requirement that it carries each $Y \in \mathcal{C}$ to the composite map

`0389`
$$h^X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\mathscr{F}} \mathrm{Hom}_{\mathrm{Set}}(\mathscr{F}(X), \mathscr{F}(Y)) \xrightarrow{\mathrm{ev}_x} \mathscr{F}(Y). \qquad (5.30)$$

The functor $\mathscr{F}$ is corepresentable if it is possible to choose the object $X \in \mathcal{C}$ and the element $x \in \mathscr{F}(X)$ so that the map (5.30) is bijective, for each $Y \in \mathcal{C}$. This motivates the following:

`038A` **Definition 5.6.6.1** (Corepresentable Functors)**.** Let $\mathcal{C}$ be an ∞-category containing an object X, let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor, and let x be a vertex of the Kan complex $\mathscr{F}(X)$. We will say that x *exhibits* $\mathscr{F}$ *as corepresented by* X if, for every object $Y \in \mathcal{C}$, the composite map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \xrightarrow{\mathscr{F}} \mathrm{Hom}_{\mathcal{S}}(\mathscr{F}(X), \mathscr{F}(Y)) \\ & \simeq \mathrm{Fun}(\mathscr{F}(X), \mathscr{F}(Y)) \\ & \xrightarrow{\mathrm{ev}_x} \mathscr{F}(Y) \end{aligned}$$

is an isomorphism in the homotopy category hKan; here the second map is the inverse of the homotopy equivalence $\mathrm{Fun}(\mathscr{F}(X), \mathscr{F}(Y)) \to \mathrm{Hom}_{\mathcal{S}}(X, Y)$ supplied by Remark 5.5.1.5.

We say that the functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ is *corepresentable by* X if there exists a vertex $x \in \mathscr{F}(X)$ which exhibits $\mathscr{F}$ as corepresented by X. We say that the functor $\mathscr{F}$ is *corepresentable* if it is corepresentable by X, for some object $X \in \mathcal{C}$.

Variant 5.6.6.2 (Representable Functors)**.** Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and write X^{op} for the corresponding object of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. Given a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, we say that a vertex $x \in \mathscr{F}(X^{\mathrm{op}})$ *exhibits* $\mathscr{F}$ *as represented by* X if it exhibits $\mathscr{F}$ as corepresented by the object X^{op}, in the sense of Definition 5.6.6.1. We say that a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is *representable by* X if it is corepresentable by X^{op}, and that $\mathscr{F}$ is *representable* if it is representable by X for some object $X \in \mathcal{C}$. `038B`

Remark 5.6.6.3. Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor of ∞-categories and let $X \in \mathcal{C}$ be an object, and let $x \in \mathscr{F}(X)$ be a vertex. The condition that x exhibits $\mathscr{F}$ as corepresented by X depends only on the connected component $[x] \in \pi_0(\mathscr{F}(X))$. `038C`

Remark 5.6.6.4. Let $\mathcal{C}$ be an ∞-category containing an object X, let $\alpha : \mathscr{F} \to \mathscr{G}$ be a morphism in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$, and let x be a vertex of the Kan complex $\mathscr{F}(X)$. Then any two of the following conditions imply the third: `038D`

The vertex $x \in \mathscr{F}(X)$ exhibits the functor $\mathscr{F}$ as corepresented by X.

The vertex $\alpha(x) \in \mathscr{G}(X)$ exhibits the functor $\mathscr{G}$ as corepresented by X.

The natural transformation α is an isomorphism.

In particular, if $\mathscr{F}$ and $\mathscr{G}$ are isomorphic objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$, then $\mathscr{F}$ is corepresentable by X if and only if $\mathscr{G}$ is corepresentable by X.

Remark 5.6.6.5. Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ and $U : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. Suppose we are given an object $Y \in \mathcal{D}$ and a vertex $\eta \in \mathscr{F}(U(Y))$. Then: `03LN`

- If U is fully faithful and η exhibits the functor $\mathscr{F}$ as corepresented by $U(Y)$, then it also exhibits the functor $\mathscr{F} \circ U$ as Y.

- If U is an equivalence of ∞-categories and η exhibits the functor $\mathscr{F} \circ U$ as corepresented by η, then it also exhibits $\mathscr{F}$ as corepresented by $U(Y)$.
- If U is an equivalence of ∞-categories, then the functor $\mathscr{F}$ is corepresentable if and only if $\mathscr{F} \circ U$ is corepresentable.

`038E` **Remark 5.6.6.6.** Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor of ∞-categories, let $u : X \to Y$ be a morphism in $\mathcal{C}$, and let $x \in \mathscr{F}(X)$ be a vertex having image $y = \mathscr{F}(u)(x) \in \mathscr{F}(Y)$. Then any two of the following conditions imply the third:

- The vertex x exhibits the functor $\mathscr{F}$ as corepresented by X.
- The vertex y exhibits the functor $\mathscr{F}$ as corepresented by Y.
- The morphism u is an isomorphism.

`038F` **Remark 5.6.6.7** (Uniqueness of the Corepresenting Object)**.** Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor of ∞-categories which is corepresentable by an object $X \in \mathcal{C}$. Let Y be another object of $\mathcal{C}$. Then $\mathscr{F}$ is corepresentable by Y if and only if Y is isomorphic to X. The "if" direction follows immediately from Remark 5.6.6.6. Conversely, suppose that $\mathscr{F}$ is corepresentable by Y. Choose vertices $x \in \mathscr{F}(X)$ and $y \in \mathscr{F}(Y)$ which exhibit $\mathscr{F}$ as corepresented by X and Y, respectively. Since evaluation at x induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathscr{F}(Y)$, we can choose a morphism $u : X \to Y$ such that $\mathscr{F}(u)(x)$ and y belong to the same connected component of $\mathscr{F}(Y)$. Then $\mathscr{F}(u)(x)$ also exhibits $\mathscr{F}$ as corepresented by Y (Remark 5.6.6.3), so that u is an isomorphism in $\mathcal{C}$ by virtue of Remark 5.6.6.6.

`02FS` **Remark 5.6.6.8.** Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor of ∞-categories. Then the construction $Y \mapsto \pi_0(\mathscr{F}(Y))$ determines a functor from the homotopy category $\mathrm{h}\mathcal{C}$ to the category of sets, which we will denote by $\pi_0(\mathscr{F})$. Suppose that X is an object of $\mathcal{C}$ and $x \in \mathscr{F}(X)$ exhibits $\mathscr{F}$ as corepresented by X. Then, for every object $Y \in \mathcal{C}$, evaluation on the connected component $[x] \in \pi_0(\mathscr{F}(X))$ induces a bijection

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y)) \to \pi_0(\mathscr{F}(Y)).$$

It follows that the functor $\pi_0(\mathscr{F}) : \mathrm{h}\mathcal{C} \to \mathrm{Set}$ is corepresentable by X, in the sense of classical category theory.

`038G` **Warning 5.6.6.9.** The converse of Remark 5.6.6.8 is false in general. For example, let $\mathcal{C}$ be an ∞-category containing an object X, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set} \subset \mathcal{S}$ be the functor given on objects by the formula $\mathscr{F}(Y) = \pi_0(\mathrm{Hom}_{\mathcal{C}}(X, Y))$. Then $\pi_0(\mathscr{F})$ is corepresentable by the object X (when regarded as a functor from $\mathrm{h}\mathcal{C}$ to the category of sets), but the functor $\mathscr{F}$ is usually not corepresentable.

In spite of Warning 5.6.6.9, the corepresentability of a functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ *can* be tested at the level of the homotopy category $\mathrm{h}\mathcal{C}$. The caveat is that we must equip $\mathrm{h}\mathcal{C}$ with the enrichment described in Construction 4.6.9.13.

Definition 5.6.6.10. Let hKan denote the homotopy category of Kan complexes (Construction 3.1.5.10) and let $\mathcal{C}$ be an hKan-enriched category containing an object X. We will say that an hKan-enriched functor $\mathscr{F} : \mathcal{C} \to \mathrm{hKan}$ is *corepresentable by* X if there exists a vertex $x \in \mathscr{F}(X)$ such that, for every object $Y \in \mathcal{C}$, the induced map 02FR

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \times \{x\} \hookrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \times \mathscr{F}(X) \to \mathscr{F}(Y)$$

is an isomorphism in the homotopy category hKan. In this case, we also say that *x exhibits $\mathscr{F}$ as corepresented by the object X*. We say that the functor $\mathscr{F}$ is *corepresentable* if it is corepresentable by X for some object $X \in \mathcal{C}$.

We say that an hKan-enriched functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{hKan}$ is *representable by* X if it is corepresentable by the object $X^{\mathrm{op}} \in \mathcal{C}^{\mathrm{op}}$, and that $\mathscr{F}$ is *representable* if it is representable by some object of $\mathcal{C}$.

Remark 5.6.6.11. Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor of ∞-categories, and let $\mathrm{h}\mathscr{F} : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{S} = \mathrm{hKan}$ be the induced functor of hKan-enriched homotopy categories (see Construction 4.6.9.13). Then: 038H

- A vertex $x \in \mathscr{F}(X)$ exhibits $\mathscr{F}$ as corepresented by an object $X \in \mathcal{C}$ (in the sense of Definition 5.6.6.1) if and only if it exhibits $\mathrm{h}\mathscr{F}$ as corepresented by the object $X \in \mathrm{h}\mathcal{C}$ (in the sense of Definition 5.6.6.10).
- The functor $\mathscr{F}$ is corepresentable by an object $X \in \mathcal{C}$ if and only if $\mathrm{h}\mathscr{F}$ is corepresentable by $X \in \mathrm{h}\mathcal{C}$.
- The functor $\mathscr{F}$ is corepresentable if and only if $\mathrm{h}\mathscr{F}$ is corepresentable as an hKan-enriched functor.

Remark 5.6.6.12. Let $\mathcal{C}$ be an hKan-enriched category. Then an hKan-enriched functor $\mathscr{F} : \mathcal{C} \to \mathrm{hKan}$ is corepresentable by an object $X \in \mathcal{C}$ if and only if it is isomorphic (as an hKan-enriched functor) to the functor 02FT

$$\mathcal{C} \to \mathrm{hKan} \qquad Y \mapsto \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y).$$

Let $\mathcal{C}$ be an ∞-category. It follows from Remarks 5.6.6.4 and 5.6.6.7 that there is a unique function

$$\begin{tikzcd}
\{\text{Isomorphism classes of corepresentable functors } \mathcal{C} \to \mathcal{S}\} \arrow[d] \\
\{\text{Isomorphism classes of objects of } \mathcal{C}\},
\end{tikzcd}$$

which carries (the isomorphism class of) a corepresentable functor $\mathscr{F}$ to (the isomorphic class of) an object $X \in \mathcal{C}$ which corepresents $\mathscr{F}$. Our main goal in this section is to show that, modulo set-theoretic considerations, this map is bijective.

038J **Theorem 5.6.6.13.** *Let $\mathcal{C}$ be a locally small ∞-category. Then, for every object $X \in \mathcal{C}$, there exists a functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ which is corepresentable by X. Moreover, the functor $\mathscr{F}$ is uniquely determined up to isomorphism.*

038L **Notation 5.6.6.14** (Corepresentable Functors)**.** Let $\mathcal{C}$ be a locally small ∞-category. For every object $X \in \mathcal{C}$, Theorem 5.6.6.13 asserts that there exists a functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ which is corepresented by X, which is uniquely determined up to isomorphism. To emphasize this uniqueness, we will typically denote the functor $\mathscr{F}$ by h^X and refer to it as *the functor corepresented by X*. For every object $Y \in \mathcal{C}$, we can apply the same argument to the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ to obtain a functor represented by Y, which we will typically denote by $h_Y : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ and refer to as *the functor represented by Y*. Note that Remark 5.6.6.12 supplies isomorphisms $h^X(Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq h_Y(X)$ in the homotopy category hKan, depending functorially on the pair $(X, Y) \in \mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{h}\mathcal{C}$.

038M **Remark 5.6.6.15.** Let $\mathcal{C}$ be an ∞-category. Then every object $X \in \mathcal{C}$ determines an hKan-enriched functor

$$\mathrm{Hom}_{\mathcal{C}}(X, \bullet) : \mathrm{h}\mathcal{C} \to \mathrm{hKan} \qquad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

Theorem 5.6.6.13 asserts $\mathrm{Hom}_{\mathcal{C}}(X, \bullet)$ can be promoted, in an essentially unique way, to a functor of ∞-categories $h^X : \mathcal{C} \to \mathcal{S}$ (see Remark 5.6.6.12). Beware that this is a special feature of corepresentable functors. In general, an hKan-enriched functor $\mathscr{F} : \mathrm{h}\mathcal{C} \to \mathrm{hKan}$ cannot be promoted to a functor of ∞-categories. Moreover, when such a promotion exists, it need not be unique.

038N **Remark 5.6.6.16** (Functoriality)**.** Let $\mathcal{C}$ be a locally small ∞-category. We will see later that the corepresentable functor $h^X : \mathcal{C} \to \mathcal{S}$ and the representable functor $h_Y : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ of Notation 5.6.6.14 depend functorially on the objects X and Y, respectively. More precisely, the construction

$$\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) : \mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{h}\mathcal{C} \to \mathrm{hKan} \qquad (X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

can be promoted to a functor of ∞-categories $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ with the following properties:

- For each object $X \in \mathcal{C}$, the restriction $H|_{\{X\} \times \mathcal{C}}$ is corepresentable by X.
- For each object $Y \in \mathcal{C}$, the restriction $H|_{\mathcal{C}^{\mathrm{op}} \times \{Y\}}$ is representable by Y.

See Proposition 8.3.3.2.

Unlike its classical counterpart, Theorem 5.6.6.13 is nontrivial: given an object X of an ∞-category $\mathcal{C}$, there is no immediately obvious candidate for a functor $h^X : \mathcal{C} \to \mathcal{S}$ which is corepresented by X. However, the situation is better when $\mathcal{C}$ arises from a simplicially enriched category.

Proposition 5.6.6.17. *Let $\mathcal{C}$ be a locally Kan simplicial category, let X be an object of $\mathcal{C}$, and let* 038P

$$\mathscr{F} : \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan}) = \mathcal{S}.$$

denote the homotopy coherent nerve of the simplicial functor $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet}$. Then the identity morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)_{\bullet} = \mathscr{F}(X)$ exhibits the functor $\mathscr{F}$ as corepresented by X, in the sense of Definition 5.6.6.1.

Proof. Fix an object $Y \in \mathcal{C}$. We then have a commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \arrow[r, "U"] \arrow[d, "\theta", "\sim"'] & \mathrm{Hom}_{\mathrm{Kan}}(\mathscr{F}(X), \mathscr{F}(Y))_{\bullet} \arrow[r, "\mathrm{ev}"] \arrow[d, "\theta'", "\sim"'] & \mathscr{F}(Y) \\
\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Y) \arrow[r, "V"] & \mathrm{Hom}_{\mathcal{S}}(\mathscr{F}(X), \mathscr{F}(Y)), &
\end{tikzcd}$$

where the vertical maps are supplied by Construction 4.6.8.3 (applied in the simplicial categories $\mathcal{C}$ and Kan, respectively) and ev is given by evaluation at the vertex $\mathrm{id}_X \in \mathscr{F}(X)$. Let θ'^{-1} denote a homotopy inverse to θ' (which exists by virtue of Theorem 4.6.8.5). Proposition 5.6.6.17 asserts that the composition $\mathrm{ev} \circ \theta'^{-1} \circ V$ is a homotopy equivalence. Since θ is also a homotopy equivalence (Theorem 4.6.8.5), this is equivalent to the assertion that $\mathrm{ev} \circ U$ is a homotopy equivalence. This is clear: the composition $\mathrm{ev} \circ U$ is the identity map from the Kan complex $\mathscr{F}(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet}$ to itself. □

Remark 5.6.6.18. Let $\mathcal{C}$ be a locally Kan simplicial category. The preceding proof shows that if $\mathcal{C}$ satisfies the conclusion of Proposition 5.6.6.17, then it also satisfies the conclusion of Theorem 4.6.8.5: that is, the comparison map $\theta : \mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})}(X, Y)$ is a homotopy equivalence for every pair of objects $X, Y \in \mathcal{C}$. Note however that we have already used Theorem 4.6.8.5 (applied to the simplicial category Kan) implicitly to give the *definition* of a corepresentable functor in the ∞-categorical setting. 038Q

The rest of this section is devoted to the proof of Theorem 5.6.6.13. Fix a locally small ∞-category $\mathcal{C}$ and an object $X \in \mathcal{C}$. We can then use the dictionary of Corollary 5.6.0.6 to identify functors $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ with essentially small left fibrations $U : \mathcal{E} \to \mathcal{C}$. We will show that $\mathscr{F}$ is corepresentable by an object $X \in \mathcal{C}$ if and only if the ∞-category $\mathcal{E}$ has an initial object $\widetilde{X}$ satisfying $U(\widetilde{X}) = X$ (Proposition 5.6.6.21). We will then show that this condition guarantees that U is equivalent to the left fibration $U_0 : \mathcal{C}_{X/} \to \mathcal{C}$ (Proposition 5.6.6.21). Combining these assertions, we see that a functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ is corepresentable by X if and only if it is a covariant transport representation for U_0, so that the existence and uniqueness assertions of Theorem 5.6.6.13 follow from Theorem 5.6.0.2.

038R **Proposition 5.6.6.19.** *Let $U : \mathcal{D} \to \mathcal{C}$ be a left fibration of ∞-categories and let $\widetilde{X} \in \mathcal{D}$ be an object having image $X = U(\widetilde{X})$. The following conditions are equivalent:*

(1) *There exists an equivalence $F : \mathcal{C}_{X/} \to \mathcal{D}$ of left fibrations over $\mathcal{C}$ satisfying $F(\mathrm{id}_X) = \widetilde{X}$.*

(2) *The object $\widetilde{X} \in \mathcal{D}$ is initial (Definition 4.6.7.1).*

(3) *For every left fibration $V : \mathcal{E} \to \mathcal{C}$, evaluation on the object $\widetilde{X}$ induces a trivial Kan fibration $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E}) \to \{X\} \times_{\mathcal{C}} \mathcal{E}$.*

(4) *For every left fibration $V : \mathcal{E} \to \mathcal{C}$, evaluation on the object $\widetilde{X}$ induces a bijection*

$$\pi_0(\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E})) \to \pi_0(\{X\} \times_{\mathcal{C}} \mathcal{E}).$$

Proof. If $F : \mathcal{C}_{X/} \to \mathcal{D}$ is an equivalence of left fibrations over $\mathcal{C}$, then it is an equivalence of ∞-categories (Proposition 5.1.7.5). Since $\mathrm{id}_X : X \to X$ is initial when regarded as an object of the ∞-category $\mathcal{C}_{X/}$ (Proposition 4.6.7.22), Corollary 4.6.7.20 guarantees that $\widetilde{X}$ is an initial object of $\mathcal{D}$. This proves the implication (1) ⇒ (2). The implication (2) ⇒ (3) follows by combining Corollary 4.6.7.24 with Proposition 4.2.5.4, and the implication (3) ⇒ (4) is immediate.

We will complete the proof by showing that (4) implies (1). Note that the object $\mathrm{id}_X \in \mathcal{C}_{X/}$ satisfies condition (1) and therefore also satisfies condition (3). It follows that there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{X/} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D} \arrow[dl, "U"] \\
& \mathcal{C} &
\end{tikzcd}$$

satisfying $F(\mathrm{id}_X) = \widetilde{X}$. To complete the proof, it will suffice to show that if condition (4) is satisfied, then F is an equivalence of left fibrations over $\mathcal{C}$. For every left fibration $V : \mathcal{E} \to \mathcal{C}$, we have a commutative diagram of sets

$$\begin{tikzcd}
\pi_0(\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D},\mathcal{E})) \arrow[rr, "\circ[F]"] \arrow[dr] & & \pi_0(\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}_{X/},\mathcal{E})) \arrow[dl] \\
& \pi_0(\{X\}\times_{\mathcal{C}}\mathcal{E}) &
\end{tikzcd},$$

where the vertical maps are given by evaluation on the objects $\widetilde{X}\in\mathcal{D}$ and $\mathrm{id}_X\in\mathcal{C}_{X/}$, and are therefore bijective. It follows that the horizontal map is also bijective. □

Corollary 5.6.6.20. *Suppose we are given a commutative diagram of ∞-categories* 03J6

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E} \arrow[dl, "V"] \\
& \mathcal{C} &
\end{tikzcd}$$

where U and V are left fibrations. Let $\widetilde{X}\in\mathcal{D}$ be an initial object. Then F is an equivalence of ∞-categories if and only if $F(\widetilde{X})$ is an initial object of $\mathcal{E}$.

Proof. If F is an equivalence of ∞-categories, then it carries initial objects to initial objects by virtue of Corollary 4.6.7.20. Conversely, suppose that $F(\widetilde{X})$ is an initial object of $\mathcal{E}$; we wish to show that F is an equivalence of ∞-categories. Set $X = U(\widetilde{X})$. Applying Proposition 5.6.6.19, we deduce that there is a functor $G\in\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E},\mathcal{D})$ such that $(G\circ F)(\widetilde{X})$ is isomorphic to $\widetilde{X}$ as an object of the ∞-category $\mathcal{D}_X = \{X\}\times_{\mathcal{C}}\mathcal{D}$. Applying Proposition 5.6.6.19 again, we deduce that $G\circ F$ is isomorphic to $\mathrm{id}_{\mathcal{D}}$ as an object of the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D},\mathcal{D})$; in particular, F is a right homotopy inverse to G. Since G carries $F(\widetilde{X})$ to an initial object of $\mathcal{D}$, we can apply the same argument (with the roles of $\mathcal{D}$ and $\mathcal{E}$ reversed) to show that G has a left homotopy inverse. It follows that G is an equivalence of ∞-categories, so that F is also an equivalence of ∞-categories. □

Proposition 5.6.6.21. *Let $U:\mathcal{D}\to\mathcal{C}$ be an essentially small left fibration of ∞-categories and let $X\in\mathcal{C}$ be an object. Then:* 02J8

(1) *Let $\widetilde{X}\in\mathcal{D}$ be an object satisfying $U(\widetilde{X}) = X$. Then $\widetilde{X}$ is an initial object of $\mathcal{D}$ if and only if, for every object $Y\in\mathcal{C}$, the composition*

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)\xrightarrow{\theta}\mathrm{Fun}(\mathcal{D}_X,\mathcal{D}_Y)\xrightarrow{\mathrm{ev}_{\widetilde{X}}}\mathcal{D}_Y$$

is a homotopy equivalence, where θ is given by parametrized covariant transport (see Definition 5.2.8.1).

(2) *Let* $\mathrm{hTr}_{\mathcal{D}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hKan}$ *be the homotopy transport representation of* U*, which we regard as an* hKan*-enriched functor (Variant 5.2.8.12). Then* $\widetilde{X}$ *is an initial object of* $\mathcal{D}$ *if and only if it exhibits* $\mathrm{hTr}_{\mathcal{D}/\mathcal{C}}$ *as corepresented by* X*, in the sense of Definition 5.6.6.10.*

(3) *The homotopy transport representation* $\mathrm{hTr}_{\mathcal{D}/\mathcal{C}}$ *is corepresentable by the object* X *if and only if there exists an initial object* $\widetilde{X} \in \mathcal{D}$ *satisfying* $U(\widetilde{X}) = X$*.*

(4) *Let* $\mathrm{Tr}_{\mathcal{D}/\mathcal{C}} : \mathcal{C} \to \mathcal{S}$ *be a covariant transport representation for* U*. Then* $\mathrm{Tr}_{\mathcal{D}/\mathcal{C}}$ *is corepresentable by the object* X *if and only if there exists an initial object* $\widetilde{X} \in \mathcal{D}$ *satisfying* $U(\widetilde{X}) = X$*.*

Proof. Let $\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C}$ be the oriented fiber product of Definition 4.6.4.1, and let us regard id_X as an initial object of $\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C}$ (Proposition 4.6.7.22). Using Proposition 5.6.6.19, we can choose a functor of ∞-categories $F : \{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ satisfying $F(\mathrm{id}_X) = \widetilde{X}$ which fits into a commutative diagram

$$\begin{tikzcd}
\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \arrow[rr, "F"] \arrow[dr] & & \mathcal{D} \arrow[dl, "U"] \\
& \mathcal{C}. &
\end{tikzcd}$$

Using Proposition 5.6.6.19, we see that $\widetilde{X}$ is an initial object of $\mathcal{D}$ if and only if F is an equivalence of left fibrations over $\mathcal{C}$. By virtue of Corollary 5.1.7.16, this is equivalent to the requirement that for each object $Y \in \mathcal{C}$, the functor F restricts to a homotopy equivalence of Kan complexes

$$F_Y : \mathrm{Hom}_{\mathcal{C}}(X,Y) = \{X\} \tilde{\times}_{\mathcal{C}} \{Y\} \to \mathcal{D}_Y$$

Assertion (1) follows from the observation that F_Y is homotopic to the composition of the parametrized covariant transport morphism $\theta : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Fun}(\mathcal{D}_X, \mathcal{D}_Y)$ with the evaluation map $\mathrm{ev}_{\widetilde{X}} : \mathrm{Fun}(\mathcal{D}_X, \mathcal{D}_Y) \to \mathcal{D}_Y$ (see Remark 5.2.8.5 and Proposition 5.2.8.7). The implication (1) ⇒ (2) follows from Remark 5.6.5.8, the implication (2) ⇒ (3) is immediate, and the implication (3) ⇒ (4) follows from Remark 5.6.6.11. □

Proof of Theorem 5.6.6.13. Let $\mathcal{C}$ be a locally small ∞-category and let X be an object of $\mathcal{C}$. We wish to show that there exists a functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ which is corepresentable by X, and that $\mathscr{F}$ is uniquely determined up to isomorphism (as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$). By virtue of Proposition 5.6.6.21 and Corollary 5.6.0.6, this is equivalent to the assertion that there exists a left fibration $U : \mathcal{D} \to \mathcal{C}$ together with an initial object $\widetilde{X} \in \mathcal{D}$ satisfying $U(\widetilde{X}) = X$, and that the left fibration U is uniquely determined up to equivalence (in the sense of Definition 5.1.7.1). To prove existence, we can take $\mathcal{D} = \mathcal{C}_{X/}$ and $\widetilde{X}$ to be the identity morphism id_X (Proposition 4.6.7.22). The uniqueness assertion follows from Proposition 5.6.6.19. □

5.6.7 Application: Extending Cocartesian Fibrations

In §3.3.8, we showed that every Kan fibration of simplicial sets $f : X \to S$ can be obtained as the pullback of a Kan fibration between Kan complexes. Our goal in this section is to prove an analogous result for cocartesian fibrations of simplicial sets (Corollary 5.6.7.3). Our starting point is the following: `029B`

Lemma 5.6.7.1. *Suppose we are given a commutative diagram of simplicial sets* `029C`

`029D`

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "G_0"] \arrow[d, "U_0"] & \mathcal{E}' \arrow[d, "V"] \\
\mathcal{C}_0 \arrow[r] & \mathcal{C}
\end{tikzcd} \qquad (5.31)$$

where the vertical maps are inner fibrations, the bottom horizontal map exhibits $\mathcal{C}_0$ as a simplicial subset of $\mathcal{C}$, and G_0 induces an equivalence $\mathcal{E}_0 \to \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}'$ of inner fibrations over $\mathcal{C}_0$. Then (5.31) can be extended to a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d, "U_0"] & \mathcal{E} \arrow[r, "G"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "V"] \\
\mathcal{C}_0 \arrow[r] & \mathcal{C} \arrow[r, equal] & \mathcal{C},
\end{tikzcd}$$

where U is an inner fibration, G is an equivalence of inner fibrations over $\mathcal{C}$, and the square on the left induces an isomorphism of simplicial sets $\mathcal{E}_0 \simeq \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$.

Proof. Choose a monomorphism of simplicial sets $\mathcal{E}_0 \hookrightarrow \mathcal{Q}$, where $\mathcal{Q}$ is a contractible Kan complex (see Exercise 3.1.7.11). Replacing $\mathcal{E}'$ with the product $\mathcal{E}' \times \mathcal{Q}$, we can reduce to the case where G_0 is a monomorphism of simplicial sets. Let $\mathcal{E}$ denote the simplicial subset of $\mathcal{E}'$ consisting of those simplices $\sigma : \Delta^m \to \mathcal{E}'$ for which the induced map $\mathcal{C}_0 \times_{\mathcal{C}} \Delta^m \to \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}'$ factors through G_0. To complete the proof, it will suffice to verify the following:

(a) The morphism $U = V|_{\mathcal{E}}$ is an inner fibration from $\mathcal{E}$ to $\mathcal{C}$.

(b) The inclusion $\mathcal{E} \hookrightarrow \mathcal{E}'$ is an equivalence of inner fibrations over $\mathcal{C}$.

By virtue of Remark 4.1.1.13 and Proposition 5.1.7.9, it suffices to prove (a) and (b) in the special case where $\mathcal{C} = \Delta^n$ is a standard simplex. In this case, the morphism $V : \mathcal{E}' \to \mathcal{C}$ is an isofibration (Example 4.4.1.6).

Let $\mathcal{E}'_0$ denote the fiber product $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}'$. Applying Lemma 5.1.7.13 to the morphism $G_0 : \mathcal{E}_0 \to \mathcal{E}'_0$ (which is an equivalence of inner fibrations over $\mathcal{C}_0$), we deduce that there exists a morphism $R_0 : \mathcal{E}'_0 \to \mathcal{E}_0$ in the category $(\mathrm{Set}_{\Delta})_{/\mathcal{C}_0}$ such that $R_0 \circ G_0 = \mathrm{id}_{\mathcal{E}_0}$, and an isomorphism $\alpha_0 : \mathrm{id}_{\mathcal{E}'_0} \to G_0 \circ R_0$ in the ∞-category $\mathrm{Fun}_{/\mathcal{C}_0}(\mathcal{E}'_0, \mathcal{E}'_0)$ whose image in $\mathrm{Fun}_{/\mathcal{C}_0}(\mathcal{E}_0, \mathcal{E}'_0)$ is degenerate. Applying Proposition 4.4.5.8 (and the criterion of Proposition 4.4.4.9), we can choose a morphism $R : \mathcal{E}' \to \mathcal{E}'$ in $(\mathrm{Set}_{\Delta})_{/\mathcal{C}}$ such that $R|_{\mathcal{E}'_0} = G_0 \circ R_0$ and an isomorphism $\alpha : \mathrm{id}_{\mathcal{E}'} \to R$ in the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E}')$ whose image in $\mathrm{Fun}_{/\mathcal{C}_0}(\mathcal{E}'_0, \mathcal{E}'_0)$ is equal to α_0.

We now prove (a). Suppose we are given a lifting problem

$$\begin{tikzcd}
A \arrow[r, "f_0"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
B \arrow[r, "\overline{f}"] \arrow[ur, dashed, "f"] & \mathcal{C},
\end{tikzcd}$$

where the left vertical map is inner anodyne. Since $V : \mathcal{E}' \to \mathcal{C}$ is an inner fibration, we can extend f_0 to a morphism $f' : B \to \mathcal{E}$ satisfying $V \circ f' = \overline{f}$. Set $B_0 = \mathcal{C}_0 \times_{\mathcal{C}} B$ and $A_0 = \mathcal{C}_0 \times_{\mathcal{C}} A$, and define

$$f_1 : (A \coprod_{A_0} B_0) \to \mathcal{E}$$

by the formula $f_1|_A = f_0$ and $f_1|_{B_0} = R \circ f'|_{B_0}$. Note that there is an isomorphism

$$\beta : f'|_{A \coprod_{A_0} B_0} \to f_1$$

in the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(A \coprod_{A_0} B_0, \mathcal{E}')$, whose image in $\mathrm{Fun}_{/\mathcal{C}}(A, \mathcal{E}')$ is degenerate and whose image in $\mathrm{Fun}_{/\mathcal{C}}(B_0, \mathcal{E}')$ is the restriction of α. Applying Proposition 4.4.5.8, we deduce that f_1 admits an extension $f : B \to \mathcal{C}$ satisfying $U \circ f = \overline{f}$.

To prove (b), we observe that the morphism $R : \mathcal{E}' \to \mathcal{E}$ is a homotopy inverse of the inclusion $\iota : \mathcal{E} \hookrightarrow \mathcal{E}'$ relative to $\mathcal{C}$. By construction, α determines an isomorphism from $\mathrm{id}_{\mathcal{E}'}$ to the composition $\iota \circ R$ in the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}', \mathcal{E}')$, and the restriction of α determines an isomorphism from $\mathrm{id}_{\mathcal{E}}$ to $R \circ \iota$ in the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E})$. □

029E **Proposition 5.6.7.2** (Extending Cocartesian Fibrations)**.** *Let $\mathcal{C}$ be a simplicial set, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a simplicial subset, and let $U_0 : \mathcal{E}_0 \to \mathcal{C}_0$ be a cocartesian fibration of simplicial sets. Suppose that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a categorical equivalence of simplicial sets. Then* *there exists a pullback diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d, "U_0"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}_0 \arrow[r] & \mathcal{C},
\end{tikzcd}$$

where U is a cocartesian fibration.

Proof. By virtue of Theorem 5.6.0.2, there exists a morphism of simplicial sets $\mathscr{F}_0 : \mathcal{C}_0 \to \mathcal{QC}$ and an equivalence $G_0 : \mathcal{E}_0 \to \int_{\mathcal{C}_0} \mathscr{F}_0$ of cocartesian fibrations over $\mathcal{C}_0$. Since $\mathcal{QC}$ is an ∞-category (Proposition 5.5.4.3), our assumption that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a categorical equivalence guarantees that we can extend $\mathscr{F}_0$ to a morphism of simplicial sets $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$. We can then identify G_0 with an equivalence $\mathcal{E}_0 \to \mathcal{C}_0 \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}$ of cocartesian fibrations over $\mathcal{C}_0$. Applying Lemma 5.6.7.1, we can write U_0 as the pullback of an inner fibration $U : \mathcal{E} \to \mathcal{C}$ which is equivalent to the projection map $V : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ as an inner fibration over $\mathcal{C}$. Since V is a cocartesian fibration (Proposition 5.6.2.2), it follows that U is also a cocartesian fibration (Proposition 5.1.7.14). □

Corollary 5.6.7.3. *Let $U_0 : \mathcal{E}_0 \to \mathcal{C}_0$ be a cocartesian fibration of simplicial sets. Then there exists a pullback diagram* 029F

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d, "U_0"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}_0 \arrow[r, "F"] & \mathcal{C},
\end{tikzcd}$$

where U is a cocartesian fibration of ∞-categories and F is inner anodyne.

Proof. Using Corollary 4.1.3.3, we can choose an inner anodyne map $F : \mathcal{C}_0 \hookrightarrow \mathcal{C}$, where $\mathcal{C}$ is an ∞-category. Since F is a categorical equivalence of simplicial sets (Corollary 4.5.3.14), Proposition 5.6.7.2 guarantees that U_0 is the pullback of a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$. □

Remark 5.6.7.4. In the situation of Corollary 5.6.7.3, if U_0 is a left fibration, then U is also a left fibration. To see this, it suffices to show that the fibers of U are Kan complexes (Proposition 5.1.4.14). This is clear, since every fiber of U is also a fiber of U_0 (note that the inner anodyne morphism $F : \mathcal{C}_0 \to \mathcal{C}$ is bijective at the level of vertices; see Exercise 1.5.6.6). 02LR

Corollary 5.6.7.5. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then U is an isofibration.* 029G

Proof. By virtue of Corollary 5.6.7.3, we may assume without loss of generality that U is a cocartesian fibration of ∞-categories, in which case the desired result follows from Proposition 5.1.4.8. $\square$

029H **Corollary 5.6.7.6.** *Let $U : \mathcal{E} \rightarrow \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then U is exponentiable (Definition 4.5.9.10). In particular, for any pullback diagram of simplicial sets*

029J
$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r, "\overline{F}"] & \mathcal{C},
\end{tikzcd} \tag{5.32}$$

if $\overline{F}$ is a categorical equivalence, then F is also a categorical equivalence.

Proof. By virtue of Corollary 5.6.7.3 and Remark 4.5.9.14, we may assume that U is a cocartesian fibration of ∞-categories, in which case the desired result follows from Proposition 5.3.6.1. $\square$

03TX **Corollary 5.6.7.7.** *Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \rightarrow \mathcal{C}$ be a cocartesian fibration of simplicial sets, and suppose that $\mathcal{C}$ is essentially κ-small. The following conditions are equivalent:*

(1) *The simplicial set $\mathcal{E}$ is essentially κ-small.*

(2) *For every vertex $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is essentially κ-small.*

Proof. Using Corollaries 5.6.7.3 and 5.6.7.6, we can reduce to the situation where $\mathcal{C}$ is an ∞-category. In this case, the desired result is a special case of Corollary 5.1.5.16. $\square$

029K **Corollary 5.6.7.8.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E} \arrow[dl, "V"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where U and V are cocartesian fibrations. Then F is an equivalence of cocartesian fibrations over $\mathcal{C}$ (Definition 5.1.7.1) if and only if it is a categorical equivalence of simplicial sets.

Proof. Combine Proposition 5.1.7.5 with Corollary 5.6.7.5. $\square$

5.6.8 Transport Witnesses

Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibration of ∞-categories. Theorem 5.6.0.2 asserts that there exists a commutative diagram 02S9

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC}
\end{tikzcd}$$

which witnesses $\mathscr{F}$ as a covariant transport representation for U; here $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ is the cocartesian fibration of Proposition 5.5.6.11. In this section, we formulate a stronger statement, which asserts that the collection of all such diagrams is parametrized by a contractible Kan complex (Theorem 5.6.8.3).

Notation 5.6.8.1. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. We let $\mathrm{TW}(\mathcal{E} / \mathcal{C})$ denote the simplicial subset of the fiber product 02SA

$$\mathrm{Fun}(\mathcal{C}, \mathcal{QC}) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})$$

whose n-simplices are diagrams

$$\begin{tikzcd}
\Delta^n \times \mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "\mathrm{id}_{\Delta^n} \times U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\Delta^n \times \mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC}
\end{tikzcd}$$

which witness $\mathscr{F}$ as a covariant transport representation for the cocartesian fibration $(\mathrm{id}_{\Delta^n} \times U) : \Delta^n \times \mathcal{E} \to \Delta^n \times \mathcal{C}$.

Example 5.6.8.2. Let $\mathcal{E}$ be an ∞-category and let $U : \mathcal{E} \to \Delta^0$ denote the projection map. Note that projection onto the first factor determines a morphism of simplicial sets 02SB

$$\mathrm{TW}(\mathcal{E} / \Delta^0) \to \mathrm{Fun}(\Delta^0, \mathcal{QC}) = \mathcal{QC}.$$

Unwinding the definitions, we see that the fiber of this morphism over a small ∞-category $\mathcal{E}'$ can be identified with the full subcategory

$$\mathrm{Equiv}(\mathcal{E}, \{\mathcal{E}'\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}) \subseteq \mathrm{Fun}(\mathcal{E}, \{\mathcal{E}'\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}})^{\simeq}$$

spanned by the equivalences of ∞-categories $\mathcal{E} \to \{\mathcal{E}'\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}$.

We will prove the following result in §5.6.9:

02SC **Theorem 5.6.8.3.** *Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibration of simplicial sets. Then the simplicial set* $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ *is a contractible Kan complex.*

02SD **Remark 5.6.8.4.** Theorem 5.6.8.3 is an immediate consequence of Theorem 5.6.5.10. We will see at the end of this section that the converse is also true.

The remainder of this section is devoted to establishing some formal properties of the simplicial sets $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ which will be useful for the proof of Theorem 5.6.8.3.

02SE **Lemma 5.6.8.5.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC},
\end{tikzcd}$$

where U is a cocartesian fibration. Let $j : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be an inner anodyne morphism of simplicial sets, let $\mathcal{E}_0$ denote the fiber product $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$, and let $U_0 : \mathcal{E}_0 \to \mathcal{C}_0$ denote the projection map. If $\widetilde{\mathscr{F}}|_{\mathcal{E}_0}$ witnesses $\mathscr{F}|_{\mathcal{C}_0}$ as a covariant transport representation for U_0, then $\widetilde{\mathscr{F}}$ witnesses $\mathscr{F}$ as a covariant transport representation for U.

Proof. Let S denote the collection of all morphisms of simplicial sets $i : A \to B$ with the following property: for every morphism of simplicial sets $B \to \mathcal{C}$, if the restriction $\widetilde{\mathscr{F}}|_{A \times_{\mathcal{C}} \mathcal{E}}$ witnesses $\mathscr{F}|_A$ as a covariant transport representation for the projection map $A \times_{\mathcal{C}} \mathcal{E} \to A$, then $\widetilde{\mathscr{F}}|_{B \times_{\mathcal{C}} \mathcal{E}}$ witnesses $\mathscr{F}|_B$ as a covariant transport representation for the projection map $B \times_{\mathcal{C}} \mathcal{E} \to B$. To prove Lemma 5.6.8.5, it will suffice to show that every inner anodyne morphism of simplicial sets belongs to S. It is not difficult to see that the collection of morphisms S is weakly saturated, in the sense of Definition 1.5.4.12. It will therefore suffice to show that, for every pair of integers $0 < i < n$, the inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ belongs to S. We may therefore assume without loss of generality that $\mathcal{C} = \Delta^n$ and $\mathcal{C}_0 = \Lambda^n_i$ is an inner horn.

Since every vertex of Δ^n is contained in Λ^n_i, it follows immediately that the pair $(\mathscr{F}, \widetilde{\mathscr{F}})$ satisfies condition (a) of Remark 5.6.5.3. To verify (b), let $e : X \to Z$ be an U-cocartesian edge of $\mathcal{E}$ having image $\overline{e} = U(e)$ in Δ^n; we wish to show that $\widetilde{\mathscr{F}}(e)$ is a V-cocartesian edge of $\mathcal{E}'$. If $\overline{e}$ belongs to the horn Λ^n_i, then this follows from our assumption on $\widetilde{F}|_{\mathcal{E}_0}$. We may therefore assume without loss of generality that $\mathcal{C} = \Delta^2$ and that $\overline{e} : 0 \to 2$ is the "long" edge of the simplex Δ^2. Since U is a cocartesian fibration, there exists a U-cocartesian edge

$e' : X \to Y$ of $\mathcal{E}$, where $U(Y) = 1$. Our assumption that e' is U-cocartesian guarantees the existence of a 2-simplex

$$\begin{tikzcd}
 & Y \arrow[dr, "e''"] & \\
X \arrow[ur, "e'"] \arrow[rr, "e"] & & Z
\end{tikzcd}$$

of $\mathcal{E}$, and Proposition 5.1.4.12 implies that e'' is also U-cocartesian. Since $\widetilde{\mathscr{F}}|_{\mathcal{C}_0}$ carries U_0-cocartesian morphisms of $\mathcal{E}_0$ to V-cocartesian morphisms of $\mathcal{QC}_{\mathrm{Obj}}$, it follows that $\widetilde{\mathscr{F}}(e')$ and $\widetilde{\mathscr{F}}(e'')$ are V-cocartesian edges of $\mathcal{E}'$. Applying Proposition 5.1.4.12 again, we deduce that $\widetilde{\mathscr{F}}(e)$ is also V-cocartesian. □

Lemma 5.6.8.6. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then:* 02SF

(1) *The fiber product $\mathcal{M} = \mathrm{Fun}(\mathcal{C}, \mathcal{QC}) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})$ is an ∞-category.*

(2) *The simplicial set $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ is a replete subcategory of $\mathcal{M}$ (see Example 4.4.1.12).*

In particular, the simplicial set $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ is an ∞-category.

Proof. Since V is an inner fibration, the induced map $V' : \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{QC})$ is also an inner fibration (Corollary 4.1.4.3). The projection map $\mathcal{M} \to \mathrm{Fun}(\mathcal{C}, \mathcal{QC})$ is a pullback of V', and is therefore also an inner fibration. Since $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$ is an ∞-category (Theorem 1.5.3.7), assertion (1) follows from Remark 4.1.1.9.

We now prove (2). We first show that $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ is a subcategory of $\mathcal{M}$: that is, that the inclusion map $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C}) \hookrightarrow \mathcal{M}$ is an inner fibration. Fix integers $0 < i < n$ and let σ be an n-simplex of $\mathcal{M}$ for which the restriction $\sigma|_{\Lambda^n_i}$ belongs to $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$; we wish to show that σ is an n-simplex of $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$. Unwinding the definitions, we can identify σ with a commutative diagram

$$\begin{tikzcd}
\Delta^n \times \mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "\mathrm{id}_{\Delta^n} \times U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\Delta^n \times \mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC};
\end{tikzcd}$$

we wish to show that $\widetilde{\mathscr{F}}$ witnesses $\mathscr{F}$ as a covariant transport representation for the cocartesian fibration $\mathrm{id}_{\Delta^n} \times U$. This follows from Lemma 5.6.8.5, since the inclusion $\Lambda^n_i \times \mathcal{C} \hookrightarrow \Delta^n \times \mathcal{C}$ is inner anodyne (Lemma 1.5.7.5).

We now complete the proof by showing that the subcategory $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C}) \subseteq \mathcal{M}$ is replete. Let u be an isomorphism in the ∞-category $\mathcal{M}$, which we identify with a commutative diagram

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "\mathrm{id}_{\Delta^1} \times U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\Delta^1 \times \mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC}.
\end{tikzcd}$$

Set $\mathscr{F}_0 = \mathscr{F}|_{\{0\} \times \mathcal{C}}$ and $\widetilde{\mathscr{F}}_0 = \widetilde{\mathscr{F}}|_{\{0\} \times \mathcal{E}}$, and suppose that the pair $(\mathscr{F}_0, \widetilde{\mathscr{F}}_0)$ is an object of the ∞-category $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ (that is, $\widetilde{\mathscr{F}}_0$ witnesses $\mathscr{F}_0$ as a covariant transport representation for U. We wish to show that $\widetilde{\mathscr{F}}$ witnesses $\mathscr{F}$ as a covariant transport representation for $(\mathrm{id}_{\Delta^1} \times U) : \Delta^1 \times \mathcal{E} \to \Delta^1 \times \mathcal{C}$.

We first verify condition (b) of Remark 5.6.5.3. Let e be an $(\mathrm{id}_{\Delta^1} \times U)$-cocartesian edge of the simplicial set $\Delta^1 \times \mathcal{E}$; we wish to show that $\widetilde{\mathscr{F}}(e)$ is a V-cocartesian morphism of $\mathcal{QC}_{\mathrm{Obj}}$. Write $e = (\varphi_{ij}, \overline{e})$, where $\varphi_{ij} : i \to j$ is an edge of Δ^1 and $\overline{e} : X \to Y$ is a U-cocartesian edge of $\mathcal{E}$. We consider three cases:

(1) Suppose that $i = j = 0$. Then $\widetilde{\mathscr{F}}(e) = \widetilde{\mathscr{F}}_0(\overline{e})$ is V-cocartesian by virtue of our assumption that $\widetilde{\mathscr{F}}_0$ witnesses $\mathscr{F}_0$ as a covariant transport representation for U.

(2) Suppose that $i = 0$ and $j = 1$. In this case, there exists a 2-simplex of $\Delta^1 \times \mathcal{E}$ whose boundary is indicated in the diagram

$$\begin{tikzcd}
 & (0, Y) \arrow[dr, "{(\varphi_{01}, \mathrm{id}_Y)}"] & \\
(0, X) \arrow[ur, "{(\varphi_{00}, \overline{e})}"] \arrow[rr, "{(\varphi_{01}, \overline{e})}"] & & (1, Y).
\end{tikzcd}$$

Our assumption that u is an isomorphism in the ∞-category $\mathcal{M}$ guarantees that $\widetilde{\mathscr{F}}(\varphi_{01}, \mathrm{id}_Y)$ is an isomorphism in the ∞-category $\mathcal{QC}_{\mathrm{Obj}}$, and is therefore V-cocartesian (Proposition 5.1.1.8). It follows from case (1) that $\widetilde{\mathscr{F}}(\varphi_{00}, \overline{e})$ is also a V-cocartesian morphism of $\mathcal{QC}_{\mathrm{Obj}}$. Since the collection of V-cocartesian morphisms of $\mathcal{QC}_{\mathrm{Obj}}$ is closed under composition (Corollary 5.1.2.4), we conclude that $\widetilde{\mathscr{F}}(\varphi_{01}, \overline{e})$ is also V-cocartesian.

(3) Suppose that $i = j = 1$. In this case, there exists a 2-simplex of $\Delta^1 \times \mathcal{E}$ whose boundary

is indicated in the diagram

$$\begin{tikzcd}
 & (1,X) \arrow[dr, "{(\varphi_{11},\overline{e})}"] & \\
(0,X) \arrow[ur, "{(\varphi_{01},\mathrm{id}_X)}"] \arrow[rr, "{(\varphi_{01},\overline{e})}"] & & (1,Y).
\end{tikzcd}$$

Our assumption that u is an isomorphism in the ∞-category $\mathcal{M}$ guarantees that $\widetilde{\mathscr{F}}(\varphi_{01}, \mathrm{id}_X)$ is an isomorphism in the ∞-category $\mathcal{QC}_{\mathrm{Obj}}$, and is therefore V-cocartesian (Proposition 5.1.1.8). It follows from case (2) that $\widetilde{\mathscr{F}}(\varphi_{01}, \overline{e})$ is also a V-cocartesian morphism of $\mathcal{QC}_{\mathrm{Obj}}$, so that $\widetilde{\mathscr{F}}(\varphi_{11}, \overline{e})$ is V-cocartesian by virtue of Corollary 5.1.2.4.

We now complete the proof by showing that the pair $(\widetilde{\mathscr{F}}, \mathscr{F})$ satisfies condition (a) of Remark 5.6.5.3. Let (i, C) be a vertex of the product $\Delta^1 \times \mathcal{C}$, so that $\widetilde{\mathscr{F}}$ restricts to a functor of ∞-categories

$$\widetilde{\mathscr{F}}_{(i,C)} : \{i\} \times \mathcal{E}_C \to \{\mathscr{F}(i,C)\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}} .$$

We wish to show that the functor $\widetilde{\mathscr{F}}_{(i,C)}$ is an equivalence of ∞-categories. If $i = 0$, this follows from our assumption that $\widetilde{\mathscr{F}}_0$ witnesses $\mathscr{F}_0$ as a covariant transport representation for U. We may therefore assume without loss of generality that $i = 1$. Set $v = \mathscr{F}(\varphi_{01}, \mathrm{id}_C)$ and let

$$v_! : \{\mathscr{F}(0,C)\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}} \to \{\mathscr{F}(1,C)\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}$$

be the functor given by covariant transport along v. Since u is an isomorphism in the ∞-category $\mathcal{M}$, v is an isomorphism in the ∞-category $\mathcal{QC}$ so that $v_!$ is an equivalence of ∞-categories (Remark 5.2.5.5). Combining the first part of the proof with Remark 5.2.8.5, we deduce that the diagram of ∞-categories

$$\begin{tikzcd}
\{0\} \times \mathcal{E}_C \arrow[r, "\widetilde{\mathscr{F}}_{(0,C)}"] \arrow[d, "\sim"] & \{\mathscr{F}(0,C)\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "v_!"] \\
\{1\} \times \mathcal{E}_C \arrow[r, "\widetilde{\mathscr{F}}_{(1,C)}"] & \{\mathscr{F}(1,C)\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}
\end{tikzcd}$$

commutes up to isomorphism (that is, it determines a commutative diagram in the homotopy category hQCat). Since $v_!$ and $\widetilde{\mathscr{F}}_{(0,C)}$ are equivalences of ∞-categories, it follows that $\widetilde{\mathscr{F}}_{(1,C)}$ is also an equivalence of ∞-categories. □

Lemma 5.6.8.7. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then the simplicial set* $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ *is a Kan complex.* 02SG

Proof. Since $\mathrm{TW}(\mathcal{E} / \mathcal{C})$ is an ∞-category (Lemma 5.6.8.6), it will suffice to show that every morphism u of $\mathrm{TW}(\mathcal{E} / \mathcal{C})$ is an isomorphism (Proposition 4.4.2.1). Let us identify u with a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E} \arrow[r, "\widetilde{\mathscr{F}}"] \arrow[d, "\mathrm{id}_{\Delta^1} \times U"] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\Delta^1 \times \mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{QC}
\end{tikzcd}$$

satisfying conditions (a) and (b) of Remark 5.6.5.3.

Passing to homotopy categories, we see that $\mathscr{F}$ induces a functor $\mathrm{h}\mathscr{F} : [1] \times \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{QC} \simeq \mathrm{hQCat}$. Applying Remark 5.6.5.8, we see that $\mathrm{h}\mathscr{F}$ is isomorphic to the composite functor $[1] \times \mathrm{h}\mathcal{C} \twoheadrightarrow \mathrm{h}\mathcal{C} \xrightarrow{\mathrm{hTr}_{\mathcal{E} / \mathcal{C}}} \mathrm{hQCat}$, where $\mathrm{hTr}_{\mathcal{E} / \mathcal{C}}$ denotes the homotopy transport representation of Construction 5.2.5.2. It follows that, for every vertex $C \in \mathcal{C}$, the morphism $\mathscr{F}$ carries the edge $\Delta^1 \times \{C\}$ to an isomorphism $\overline{e}$ in $\mathcal{QC}$. If X is an object of $\mathcal{E}$ satisfying $U(X) = C$, then $\widetilde{\mathscr{F}}$ carries $\Delta^1 \times \{X\}$ to a V-cocartesian morphism e of $\mathcal{QC}_{\mathrm{Obj}}$ satisfying $V(e) = \overline{e}$, which is then also an isomorphism by virtue of Corollary 5.1.1.10. Allowing C and X to vary and applying Theorem 4.4.4.4, we deduce that $\mathscr{F}$ and $\widetilde{\mathscr{F}}$ are isomorphisms when regarded as morphisms in the ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{QC})$ and $\mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})$, respectively.

Set $\mathcal{M} = \mathrm{Fun}(\mathcal{C}, \mathcal{QC}) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})$. Applying Corollary 4.4.3.19 to the pullback diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}}) \arrow[d] \\
\mathrm{Fun}(\mathcal{C}, \mathcal{QC}) \arrow[r] & \mathrm{Fun}(\mathcal{E}, \mathcal{QC}),
\end{tikzcd}$$

we deduce that u is an isomorphism when regarded as a morphism of the ∞-category $\mathcal{M}$. Since $\mathrm{TW}(\mathcal{E} / \mathcal{C})$ is replete subcategory of $\mathcal{M}$ (Lemma 5.6.8.6), it follows that u is also an isomorphism when regarded as a morphism of $\mathrm{TW}(\mathcal{E} / \mathcal{C})$ (Example 4.4.2.9). □

02SH **Remark 5.6.8.8.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. It follows from Lemmas 5.6.8.6 and 5.6.8.7 that $\mathrm{TW}(\mathcal{E} / \mathcal{C})$ can be identified with the full subcategory of the Kan complex

$$\mathrm{Fun}(\mathcal{C}, \mathcal{QC})^{\simeq} \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{QC})^{\simeq}} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})^{\simeq}$$

spanned by those pairs $(\mathscr{F}, \widetilde{\mathscr{F}})$ which witness $\mathscr{F}$ as a covariant transport representation for U.

Notation 5.6.8.9 (Functoriality)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Suppose we are given an arbitrary morphism of simplicial sets $f : \mathcal{C}_0 \to \mathcal{C}$, and set $\mathcal{E}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$. Precomposition with f and with the projection map $\mathcal{E}_0 \to \mathcal{E}$ determines a morphism of simplicial sets 02SJ

$$f^* : \mathrm{TW}(\mathcal{E} \,/\, \mathcal{C}) \to \mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0),$$

which we will refer to as the *restriction map.* Note that the construction $\mathcal{C}_0 \mapsto \mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0)$ carries colimits in the category $(\mathrm{Set}_{\Delta})_{/\mathcal{C}}$ to limits in the category of simplicial sets.

Lemma 5.6.8.10. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Let $\mathcal{C}_0$ be a simplicial subset of $\mathcal{C}$ and set $\mathcal{E}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$. Then:* 02SK

(1) *The restriction map $\theta : \mathrm{TW}(\mathcal{E} \,/\, \mathcal{C}) \to \mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0)$ of Notation 5.6.8.9 is a Kan fibration between Kan complexes.*

(2) *If the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is inner anodyne, then θ is a trivial Kan fibration.*

Proof. We first prove (1). Since the simplicial set $\mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0)$ is a Kan complex (Lemma 5.6.8.7), it will suffice to show that θ is an isofibration. Define fiber products

$$\mathcal{M} = \mathrm{Fun}(\mathcal{C}, \mathcal{QC}) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})$$

$$\mathcal{M}_0 = \mathrm{Fun}(\mathcal{C}_0, \mathcal{QC}) \times_{\mathrm{Fun}(\mathcal{E}_0, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}}),$$

so that we have a commutative diagram

02SL

$$\begin{tikzcd}
\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C}) \arrow[r] \arrow[d, "\theta"] & \mathcal{M} \arrow[d, "\overline{\theta}"] \\
\mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0) \arrow[r] & \mathcal{M}_0 .
\end{tikzcd} \tag{5.33}$$

It follows from Lemma 5.6.8.6 that $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ is a replete subcategory of $\mathcal{M}$, and therefore also a replete subcategory of the fiber product $\mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0) \times_{\mathcal{M}_0} \mathcal{M}$. It will therefore suffice to show that the projection map $\mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0) \times_{\mathcal{M}_0} \mathcal{M} \to \mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0)$ is an isofibration of ∞-categories. Since the collection of isofibrations is stable under pullback, we are reduced to showing that the map $\overline{\theta} : \mathcal{M} \to \mathcal{M}_0$ is an isofibration. We now observe that $\overline{\theta}$ factors as a composition

$$\begin{aligned}
\mathcal{M} \quad &= \quad \mathrm{Fun}(\mathcal{C}, \mathcal{QC}) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}}) \\
&\xrightarrow{\overline{\theta}'} \quad \mathrm{Fun}(\mathcal{C}, \mathcal{QC}) \times_{\mathrm{Fun}(\mathcal{E}_0, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}}) \\
&\xrightarrow{\overline{\theta}''} \quad \mathrm{Fun}(\mathcal{C}_0, \mathcal{QC}) \times_{\mathrm{Fun}(\mathcal{E}_0, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}}) \\
&= \quad \mathcal{M}_0,
\end{aligned}$$

where $\overline{\theta}''$ is a pullback of the restriction map

$$\psi'' : \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}}) \to \mathrm{Fun}(\mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}}) \times_{\mathrm{Fun}(\mathcal{E}_0, \mathcal{QC})} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}).$$

Since the forgetful functor $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ is an isofibration, ψ'' is also an isofibration (Propositions 4.4.5.1). Similarly, $\overline{\theta}'$ is a pullback of the restriction map $\psi' : \mathrm{Fun}(\mathcal{C}, \mathcal{QC}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{QC})$, which is an isofibration by virtue of Corollary 4.4.5.3. It follows that $\overline{\theta} = \overline{\theta}'' \circ \overline{\theta}'$ is also an isofibration. This completes the proof of (1).

We now prove (2). Suppose that the inclusion map $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is inner anodyne; we wish to show that θ is a trivial Kan fibration. Applying Proposition 1.5.7.6, we deduce that ψ' is a trivial Kan fibration of simplicial sets. Since U is a cocartesian fibration, the inclusion map $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ is a categorical equivalence (Lemma 5.3.6.5). Applying Proposition 4.5.5.18, we deduce that ψ'' is a trivial Kan fibration. It follows that the morphisms $\overline{\theta}'$ and $\overline{\theta}''$ are also trivial Kan fibrations, so that $\overline{\theta} = \overline{\theta}'' \circ \overline{\theta}'$ is a trivial Kan fibration. Applying Lemma 5.6.8.5, we see that the diagram (5.33) is a pullback square, so that θ is also a trivial Kan fibration. □

Proof of Theorem 5.6.5.10 from Theorem 5.6.8.3. Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibration of simplicial. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a simplicial subset and set $\mathcal{E}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$. Applying Theorem 5.6.8.3, we see that the simplicial sets $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ and $\mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0)$ are contractible Kan complexes. It follows that the restriction map $\theta : \mathrm{TW}(\mathcal{E} \,/\, \mathcal{C}) \to \mathrm{TW}(\mathcal{E}_0 \,/\, \mathcal{C}_0)$ is a homotopy equivalence. Since θ is also Kan fibration (Lemma 5.6.8.10), it is a trivial Kan fibration (Proposition 3.2.7.2). In particular, θ is surjective on vertices, which is a restatement of Theorem 5.6.5.10. □

5.6.9 Proof of the Universality Theorem

02SM Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibration of simplicial sets. Our goal in this section is to prove Theorem 5.6.8.3, which that the space of transport witnesses $\mathrm{TW}(\mathcal{E} \,/\, \mathcal{C})$ of Notation 5.6.8.1 is a contractible Kan complex. The main step is to establish the following:

02SN **Lemma 5.6.9.1.** *Let $U : \mathcal{E} \to \Delta^1$ be a cocartesian fibration having fibers $\mathcal{E}_0 = \{0\} \times_{\Delta^1} \mathcal{E}$ and $\mathcal{E}_1 = \{1\} \times_{\Delta^1} \mathcal{E}$. Then the restriction map*

$$\theta : \mathrm{TW}(\mathcal{E} \,/\, \Delta^1) \to \mathrm{TW}(\mathcal{E}_0 \amalg \mathcal{E}_1 \,/\, \partial \Delta^1)$$

is a trivial Kan fibration of simplicial sets.

Proof. It follows from Lemma 5.6.8.10 that θ is a Kan fibration; we wish to show that it is a trivial Kan fibration. Fix a pair of small ∞-categories $\mathcal{D}_0$ and $\mathcal{D}_1$. Set $\mathcal{E}'_0 = \{\mathcal{D}_0\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}$

and $\mathcal{E}'_1 = \{\mathcal{D}_1\} \times_{\mathcal{QC}} \mathcal{QC}_{\mathrm{Obj}}$, and let $\mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0)$ and $\mathrm{Equiv}(\mathcal{E}_1, \mathcal{E}'_1)$ be the Kan complexes introduced in Example 5.6.8.2, so that the fiber

$$\{(\mathcal{D}_0, \mathcal{D}_1)\} \times_{\mathrm{Fun}(\partial \Delta^1, \mathcal{QC})} \mathrm{TW}(\mathcal{E}_0 \amalg \mathcal{E}_1 \,/\, \partial \Delta^1)$$

can be identified with the product $\mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Equiv}(\mathcal{E}_1, \mathcal{E}'_1)$. Let $\mathrm{TW}(\mathcal{E} \,/\, \Delta^1)_{\mathcal{D}_0, \mathcal{D}_1}$ denote the fiber product $\{(\mathcal{D}_0, \mathcal{D}_1)\} \times_{\mathrm{Fun}(\partial \Delta^1, \mathcal{QC})} \mathrm{TW}(\mathcal{E} \,/\, \Delta^1)$, so that θ restricts to a Kan fibration

$$\theta_{\mathcal{D}_0, \mathcal{D}_1} : \mathrm{TW}(\mathcal{E} \,/\, \Delta^1)_{\mathcal{D}_0, \mathcal{D}_1} \to \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Equiv}(\mathcal{E}_1, \mathcal{E}'_1).$$

Note that every fiber of θ can also be viewed as a fiber of $\theta_{\mathcal{D}_0, \mathcal{D}_1}$ for suitably chosen ∞-categories $\mathcal{D}_0$ and $\mathcal{D}_1$. Consequently, to show that θ is a trivial Kan fibration, it will suffice to show that each of the morphisms $\theta_{\mathcal{D}_0, \mathcal{D}_1}$ is a trivial Kan fibration, or alternatively that it is a homotopy equivalence (see Proposition 3.2.7.2).

For the remainder of the proof, we will regard the ∞-categories $\mathcal{D}_0$ and $\mathcal{D}_1$ as fixed. Let $\mathcal{B}^+$ denote the fiber product

$$\mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \times_{\mathrm{Fun}(\Delta^1 \times \mathcal{E}_0, \mathcal{QC})^{\simeq}} \mathrm{Fun}(\Delta^1 \times \mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}})^{\simeq}.$$

Let $\pi^+ : \mathcal{B}^+ \to \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1)$ be given by projection onto the first factor, and let

$$r_0^+ : \mathcal{B}^+ \to \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_0)^{\simeq} \qquad r_1^+ : \mathcal{B}^+ \to \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq}$$

be given by restriction to the simplicial subsets $\{0\} \times \mathcal{E}_0$ and $\{1\} \times \mathcal{E}_0$, respectively. Combining Propositions 4.4.5.1 and 4.4.3.7, we deduce that the map

$$(r_0^+, r_1^+, \pi^+) : \mathcal{B}^+ \to \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_0)^{\simeq} \times \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq} \times \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1)$$

is a Kan fibration. In particular, the simplicial set $\mathcal{B}^+$ is a Kan complex.

Let $\mathcal{B}$ denote the summand $\mathcal{B}^+$ spanned by those pairs $(e, \widetilde{e})$, where $e : \mathcal{D}_0 \to \mathcal{D}_1$ is a functor and $\widetilde{e} : \Delta^1 \times \mathcal{E}_0 \to \mathcal{QC}_{\mathrm{Obj}}$ is a morphism fitting into a commutative diagram

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E}_0 \arrow[r, "\widetilde{e}"] \arrow[d] & \mathcal{QC}_{\mathrm{Obj}} \arrow[d, "V"] \\
\Delta^1 \arrow[r, "e"] & \mathcal{QC}
\end{tikzcd}$$

which satisfies the following pair of conditions:

(i) The restriction $\widetilde{e}|_{\{0\} \times \mathcal{E}_0} : \mathcal{E}_0 \to \mathcal{E}'_0$ is an equivalence of ∞-categories.

(ii) For each object $Z \in \mathcal{E}_0$, the composite map

$$\Delta^1 \times \{Z\} \hookrightarrow \Delta^1 \times \mathcal{E}_0 \xrightarrow{\widetilde{e}} \mathcal{QC}_{\mathrm{Obj}}$$

is a V-cocartesian morphism of $\mathcal{QC}_{\mathrm{Obj}}$.

Condition (i) ensures that r_0^+ restricts to a morphism of Kan complexes $r_0 : \mathcal{B} \to \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0)$. Moreover, π^+ and r_1^+ restrict to morphisms $\pi : \mathcal{B} \to \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1)$ and $r_1 : \mathcal{B} \to \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq}$, respectively. Since $\mathcal{B}$ is a summand of $\mathcal{B}^+$, the map

$$(r_0, r_1, \pi) : \mathcal{B} \to \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq} \times \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1)$$

is also a Kan fibration.

It follows from Theorem 5.2.1.1 that composition with V induces a cocartesian fibration $V' : \mathrm{Fun}(\mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}}) \to \mathrm{Fun}(\mathcal{E}_0, \mathcal{QC})$. Moreover, a morphism of the ∞-category $\mathrm{Fun}(\mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}})$ is V'-cocartesian if and only if it corresponds to a morphism of simplicial sets $\widetilde{e} : \Delta^1 \times \mathcal{E}_0 \to \mathcal{QC}_{\mathrm{Obj}}$ satisfying condition (ii). Let $\mathrm{Fun}'(\Delta^1 \times \mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}})$ spanned by morphisms which satisfy this condition. Unwinding the definitions, we have a pullback square

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dl, "{(r_0,\pi)}"'] \arrow[dr] & \\
\mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \arrow[dr] & & \mathrm{Fun}'(\Delta^1 \times \mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}}) \arrow[dl] \\
 & \mathrm{Fun}(\{0\} \times \mathcal{E}_0, \mathcal{QC}_{\mathrm{Obj}}) \times_{\mathrm{Fun}(\{0\} \times \mathcal{E}_0, \mathcal{QC})} \mathrm{Fun}(\Delta^1 \times \mathcal{E}_0, \mathcal{QC}), &
\end{tikzcd}$$

where the bottom right map is a trivial Kan fibration (Proposition 5.2.1.3). It follows that the map $(r_0, \pi) : \mathcal{B} \to \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1)$ is a trivial Kan fibration of simplicial sets.

Let $s : \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \to \mathcal{B}$ be a section of the trivial Kan fibration (r_0, π), and let T denote the composite map

$$\mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \xrightarrow{s} \mathcal{B} \xrightarrow{(r_0, r_1)} \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq}.$$

For every equivalence of ∞-categories $F : \mathcal{E}_0 \to \mathcal{E}'_0$, we can regard $T|_{\{F\} \times \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1)}$ as a morphism of Kan complexes $T_F : \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \to \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq}$. Unwinding the definitions, we can identify T_F with the composition

$$\mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \xrightarrow{T'} \mathrm{Fun}(\mathcal{E}'_0, \mathcal{E}'_1)^{\simeq} \xrightarrow{\circ F} \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq},$$

where T' is given by parametrized covariant transport for the cocartesian fibration $V : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC}$ (Definition 5.2.8.1). It follows from Proposition 5.5.6.14 that T' is a homotopy equivalence. Our assumption that F is an equivalence of ∞-categories then guarantees that T_F is also a homotopy equivalence. Allowing $F \in \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0)$ to vary and applying Proposition 3.2.8.1, we conclude that T is a homotopy equivalence. Since s is homotopy inverse to the trivial Kan fibration (r_0, π), it is also a homotopy equivalence. Applying the two-out-of-three property (Remark 3.1.6.7), we conclude that the map

$$(r_0, r_1) : \mathcal{B} \to \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq}$$

is also a homotopy equivalence. Since (r_0, r_1) is also a Kan fibration, it is a trivial Kan fibration (Proposition 3.3.7.6).

Using Proposition 5.2.2.8, we can choose a functor $\lambda : \mathcal{E}_0 \to \mathcal{E}_1$ and a natural transformation $h : \Delta^1 \times \mathcal{E}_0 \to \mathcal{E}$ which witnesses λ as given by covariant transport along the nondegenerate edge of Δ^1 (in the sense of Definition 5.2.2.4). Form a pullback diagram

02SP

$$\begin{tikzcd}
\widetilde{\mathcal{B}} \arrow[r] \arrow[d, "(\widetilde{r}_0, \widetilde{r}_1)"] & \mathcal{B} \arrow[d, "(r_0, r_1)"] \\
\mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Equiv}(\mathcal{E}_1, \mathcal{E}'_1) \arrow[r, "\circ\lambda"] & \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq}.
\end{tikzcd} \quad (5.34)$$

Let $\mathcal{M}$ denote the pushout $(\Delta^1 \times \mathcal{E}_0) \coprod_{(\{1\} \times \mathcal{E}_0)} \mathcal{E}_1$, so that we can identify $\widetilde{\mathcal{B}}$ with a summand of the Kan complex

$$\mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \times_{\mathrm{Fun}(\mathcal{M}, \mathcal{QC})^{\simeq}} \mathrm{Fun}(\mathcal{M}, \mathcal{QC}_{\mathrm{Obj}})^{\simeq}.$$

Note that h induces a categorical equivalence of simplicial sets $h^+ : \mathcal{M} \to \mathcal{E}$ (Corollary

5.2.4.2). We have a commutative diagram of Kan complexes

02SQ
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{QC})^{\simeq}} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})^{\simeq} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \arrow[d] \\
\mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})^{\simeq} \arrow[r, "V\circ"] \arrow[d, "\circ h^{+}"] & \mathrm{Fun}(\mathcal{E}, \mathcal{QC})^{\simeq} \arrow[d, "\circ h^{+}"] \\
\mathrm{Fun}(\mathcal{M}, \mathcal{QC}_{\mathrm{Obj}})^{\simeq} \arrow[r, "V\circ"] & \mathrm{Fun}(\mathcal{M}, \mathcal{QC})^{\simeq},
\end{tikzcd} \tag{5.35}$$

where the upper vertical are homotopy equivalences (since h^{+} is a categorical equivalence) and the horizontal maps are Kan fibrations (Corollary 4.4.5.7). Note that the top and bottom squares of (5.35) are homotopy pullback squares (Example 3.4.1.3 and Corollary 3.4.1.5). It follows that the outer rectangle is also a homotopy pullback square (Proposition 3.4.1.11): that is, precomposition with h^{+} induces a homotopy equivalence of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \times_{\mathrm{Fun}(\mathcal{E}, \mathcal{QC})^{\simeq}} \mathrm{Fun}(\mathcal{E}, \mathcal{QC}_{\mathrm{Obj}})^{\simeq} \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\mathcal{QC}}(\mathcal{D}_0, \mathcal{D}_1) \times_{\mathrm{Fun}(\mathcal{M}, \mathcal{QC})^{\simeq}} \mathrm{Fun}(\mathcal{M}, \mathcal{QC}_{\mathrm{Obj}})^{\simeq}.
\end{tikzcd}$$

Applying Remark 5.6.5.3, we see that $\mathrm{TW}(\mathcal{E} / \Delta^1)_{\mathcal{D}_0, \mathcal{D}_1}$ can be identified with the inverse image of $\widetilde{\mathcal{B}}$ under the homotopy equivalence φ. In particular, φ restricts to a homotopy equivalence $\varphi_0 : \mathrm{TW}(\mathcal{E} / \Delta^1)_{\mathcal{D}_0, \mathcal{D}_1} \to \widetilde{\mathcal{B}}$. Unwinding the definitions, we see that the morphism

$$\theta_{\mathcal{D}_0, \mathcal{D}_1} : \mathrm{TW}(\mathcal{E} / \Delta^1)_{\mathcal{D}_0, \mathcal{D}_1} \to \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Equiv}(\mathcal{E}_1, \mathcal{E}'_1)$$

coincides with the the composition $(\widetilde{r}_0, \widetilde{r}_1) \circ \varphi_0$. Since $(\widetilde{r}_0, \widetilde{r}_1)$ is a pullback of the trivial Kan fibration $(r_0, r_1) : \mathcal{B} \to \mathrm{Equiv}(\mathcal{E}_0, \mathcal{E}'_0) \times \mathrm{Fun}(\mathcal{E}_0, \mathcal{E}'_1)^{\simeq}$, it is also a trivial Kan fibration. In a particular, $(\widetilde{r}_0, \widetilde{r}_1)$ is a homotopy equivalence, so that the composite map $\theta_{\mathcal{D}_0, \mathcal{D}_1} = (\widetilde{r}_0, \widetilde{r}_1) \circ \varphi_0$ is also a homotopy equivalence, as desired.

□

02SR **Lemma 5.6.9.2.** *Let $\mathcal{E}$ be an essentially small ∞-category. Then the simplicial set $\mathrm{TW}(\mathcal{E} / \Delta^0)$ is a contractible Kan complex.*

Proof. It follows from Lemma 5.6.8.7 that the simplicial set $\mathrm{TW}(\mathcal{E}\,/\,\Delta^0)$ is a Kan complex. Since $\mathcal{E}$ is essentially small, the Kan complex $\mathrm{TW}(\mathcal{E}\,/\,\Delta^0)$ is nonempty. It will therefore suffice to show that the diagonal map

$$\delta : \mathrm{TW}(\mathcal{E}\,/\,\Delta^0) \to \mathrm{TW}(\mathcal{E}\,/\,\Delta^0) \times \mathrm{TW}(\mathcal{E}\,/\,\Delta^0)$$

is a homotopy equivalence (Corollary 3.5.1.33). Unwinding the definitions, we see that δ factors as a composition

$$\begin{aligned} \mathrm{TW}(\mathcal{E}\,/\,\Delta^0) &\xrightarrow{\delta'} \mathrm{Fun}(\Delta^1, \mathrm{TW}(\mathcal{E}\,/\,\Delta^0)) \\ &\simeq \mathrm{TW}(\Delta^1 \times \mathcal{E}\,/\,\Delta^1) \\ &\xrightarrow{\delta''} \mathrm{TW}(\partial\Delta^1 \times \mathcal{E}\,/\,\partial\Delta^1) \\ &\simeq \mathrm{TW}(\mathcal{E}\,/\,\Delta^0) \times \mathrm{TW}(\mathcal{E}\,/\,\Delta^0). \end{aligned}$$

Since the 1-simplex Δ^1 is contractible (Example 3.2.4.2), the morphism δ' is a homotopy equivalence. It will therefore suffice to show that the restriction map δ'' is a homotopy equivalence, which follows from Lemma 5.6.9.1. □

Proof of Theorem 5.6.8.3. Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small cocartesian fibration of simplicial sets. We wish to show that the simplicial set $\mathrm{TW}(\mathcal{E}\,/\,\mathcal{C})$ is a contractible Kan complex.

For every simplicial set $\mathcal{C}_0$ equipped with a morphism $\mathcal{C}_0 \to \mathcal{C}$, let $X(\mathcal{C}_0)$ denote the simplicial set $\mathrm{TW}(\mathcal{E}_0\,/\,\mathcal{C}_0)$, where $\mathcal{E}_0$ is the fiber product $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$. Note that the simplicial set $X(\mathcal{C}) = \mathrm{TW}(\mathcal{E}\,/\,\mathcal{C})$ can be realized as the inverse limit of the tower

$$\cdots \to X(\mathrm{sk}_2(\mathcal{C})) \to X(\mathrm{sk}_1(\mathcal{C})) \to X(\mathrm{sk}_0(\mathcal{C})),$$

where each of the transition maps is a Kan fibration (Lemma 5.6.8.10). Consequently, to show that $X(\mathcal{C})$ is a contractible Kan complex, it will suffice to show that each of the simplicial sets $X(\mathrm{sk}_k(\mathcal{C}))$ is a contractible Kan complex. Replacing $\mathcal{C}$ by $\mathrm{sk}_k(\mathcal{C})$, we can assume that the simplicial set $\mathcal{C}$ has dimension $\leq k$, for some integer $k \geq -1$.

We now proceed by induction on k. In the case $k = -1$, the simplicial set $\mathcal{C}$ is empty and $\mathrm{TW}(\mathcal{E}\,/\,\mathcal{C})$ is isomorphic to Δ^0. We may therefore assume without loss of generality that $k \geq 0$. Let S be the collection of nondegenerate k-simplices of $\mathcal{C}$, so that Proposition 1.1.4.12 supplies a pushout diagram of simplicial sets

$$\begin{tikzcd} \coprod_{\sigma \in S} \partial\Delta^k \arrow[r] \arrow[d] & \coprod_{\sigma \in S} \Delta^k \arrow[d] \\ \mathcal{C}_0 \arrow[r] & \mathcal{C}, \end{tikzcd}$$

where $\mathcal{C}_0 = \mathrm{sk}_{k-1}(\mathcal{C})$ is the $(k-1)$-skeleton of $\mathcal{C}$. It follows from our inductive hypothesis that the simplicial set $X(\mathcal{C}_0)$ is a contractible Kan complex. Consequently, to show that $X(\mathcal{C})$ is a contractible Kan complex, it will suffice to show that the restriction map $\theta : X(\mathcal{C}) \to X(\mathcal{C}_0)$ is a trivial Kan fibration. Note that θ is a pullback of the restriction map

$$\theta_0 : X(\coprod_{\sigma \in S} \Delta^k) \to X(\coprod_{\sigma \in S} \partial \Delta^k).$$

We will complete the proof by showing that θ_0 is a trivial Kan fibration. Since θ_0 is a Kan fibration (Lemma 5.6.8.10), this is equivalent to the assertion that θ_0 is a homotopy equivalence (Proposition 3.2.7.2). Our inductive hypothesis guarantees that the Kan complex $X(\coprod_{\sigma \in S} \partial \Delta^k)$ is contractible. We are therefore reduced to showing that the Kan complex $X(\coprod_{\sigma \in S} \Delta^k)$ is also contractible. Since the collection of contractible Kan complexes is closed under products, we are reduced to verifying the contractibility of the simplicial set $X(\mathcal{C}_0)$ in the special case where $\mathcal{C}_0 = \Delta^k$ is a standard simplex of dimension k. We now consider several cases:

- In the case $k = 0$, the desired result follows from Lemma 5.6.9.2.
- In the case $k = 1$, Lemma 5.6.9.1 supplies a trivial Kan fibration $X(\Delta^1) \to X(\partial \Delta^1)$. Our inductive hypothesis guarantees that the Kan complex $X(\partial \Delta^1)$ is contractible, so that $X(\Delta^1)$ is also contractible.
- In the case $k \geq 2$, we can choose an integer $0 < i < k$. In this case, the inclusion $\Lambda^k_i \hookrightarrow \Delta^k$ is inner anodyne, so the restriction map $X(\Delta^k) \to X(\Lambda^k_i)$ is a trivial Kan fibration (Lemma 5.6.8.10). Our inductive hypothesis guarantees that the Kan complex $X(\Lambda^k_i)$ is contractible, so that $X(\Delta^k)$ is also contractible.

□

Part II

Higher Category Theory

Chapter 6

Adjoint Functors

02C9 6.1 Adjunctions in 2-Categories

02CA We begin by reviewing the theory of adjoint functors in the setting of classical category theory, originally introduced in [34].

02CB **Definition 6.1.0.1** (Kan)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. A Hom-*adjunction between F and G* is a collection of bijections

$$\rho_{C,D} : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \simeq \mathrm{Hom}_{\mathcal{C}}(C, G(D))$$

which depend functorially on $C \in \mathcal{C}$ and $D \in \mathcal{D}$ (that is, the construction $(C, D) \mapsto \rho_{C,D}$ is an isomorphism in the functor category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathrm{Set})$). In this case, we say that the construction $(C, D) \mapsto \rho_{C,D}$ *exhibits F as a left adjoint to G and G as a right adjoint to F*.

In the situation of Definition 6.1.0.1, functoriality imposes strong constraints on the construction $(C, D) \mapsto \rho_{C,D}$. For each object $C \in \mathcal{C}$, let $\eta_C : C \to (G \circ F)(C)$ be the morphism of $\mathcal{C}$ given by the image of the identity morphism $\mathrm{id}_{F(C)}$ under the bijection

$$\rho_{C,F(C)} : \mathrm{Hom}_{\mathcal{D}}(F(C), F(C)) \simeq \mathrm{Hom}_{\mathcal{C}}(C, (G \circ F)(C)).$$

For every morphism $f : F(C) \to D$ in $\mathcal{D}$, the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(C), F(C)) \arrow[r, "\rho_{C,F(C)}", "\sim"'] \arrow[d, "f\circ"] & \mathrm{Hom}_{\mathcal{D}}(C, (G \circ F)(C)) \arrow[d, "G(f)\circ"] \\
\mathrm{Hom}_{\mathcal{D}}(F(C), D) \arrow[r, "\rho_{C,D}", "\sim"'] & \mathrm{Hom}_{\mathcal{C}}(C, G(D))
\end{tikzcd}$$

supplies an equality

$$\rho_{C,D}(f) = \rho_{C,D}(f \circ \mathrm{id}_{F(C)}) = G(f) \circ \rho_{C,F(C)}(\mathrm{id}_{F(C)}) = G(f) \circ \eta_C.$$

In particular, the bijection $\rho_{C,D}$ is completely determined by the morphism η_C. Moreover, the functoriality of $\rho_{\bullet,\bullet}$ in the first variable guarantees that the construction $C \mapsto \eta_C$ is a natural transformation from the identity functor $\mathrm{id}_{\mathcal{C}}$ to the composition $G \circ F$. Similarly, the inverse bijections $\rho_{C,D}^{-1} : \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \simeq \mathrm{Hom}_{\mathcal{D}}(F(C), D)$ can be recovered from the collection of morphisms $\{\epsilon_D = \rho^{-1}_{G(D),D}(\mathrm{id}_{G(D)})\}_{D \in \mathcal{D}}$, which comprise a natural transformation of functors $\epsilon : (F \circ G) \to \mathrm{id}_{\mathcal{D}}$. This leads to a reformulation of Definition 6.1.0.1:

Definition 6.1.0.2. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between categories. An *adjunction between* F *and* G is a pair (η, ϵ), where $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ are natural transformations satisfying the following compatibility conditions: 02CC

$(Z1)$ For each object $C \in \mathcal{C}$, the composite morphism

$$F(C) \xrightarrow{F(\eta_C)} (F \circ G \circ F)(C) \xrightarrow{\epsilon_{F(C)}} F(C)$$

is equal to the identity $\mathrm{id}_{F(C)}$.

$(Z2)$ For each object $D \in \mathcal{D}$, the composite morphism

$$G(D) \xrightarrow{\eta_{G(D)}} (G \circ F \circ G)(D) \xrightarrow{G(\epsilon_D)} G(D)$$

is equal to the identity $\mathrm{id}_{G(D)}$.

If these conditions are satisfied, then we will refer to η as the *unit* of the adjunction (η, ϵ) and to ϵ as the *counit* of the adjunction (η, ϵ). In this case, we will say that (η, ϵ) *exhibits* F *as a left adjoint to* G and also that it *exhibits* G *as a right adjoint to* F.

Example 6.1.0.3. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between categories, and let $\{\rho_{C,D}\}_{C \in \mathcal{C}, D \in \mathcal{D}}$ be a Hom-adjunction between F and G (in the sense of Definition 6.1.0.1). Let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ be the natural transformations given by the formulae 02CD

$$\eta_C = \rho_{C,F(C)}(\mathrm{id}_{F(C)}) \in \mathrm{Hom}_{\mathcal{C}}(C, (G \circ F)(C))$$
$$\epsilon_D = \rho^{-1}_{G(D),D}(\mathrm{id}_{G(D)}) \in \mathrm{Hom}_{\mathcal{D}}((F \circ G)(D), D).$$

Then the pair (η, ϵ) is an adjunction between F and G (in the sense of Definition 6.1.0.2). Condition $(Z1)$ follows from the observation that for each object $C \in \mathcal{C}$, we have

$$\begin{aligned} \mathrm{id}_{F(C)} &= \rho^{-1}_{C,F(C)}(\rho_{C,F(C)}(\mathrm{id}_{F(C)})) \\ &= \rho^{-1}_{C,F(C)}(\eta_C) \\ &= \rho^{-1}_{C,F(C)}(\mathrm{id}_{(G\circ F)(C)} \circ \eta_C) \\ &= \rho^{-1}_{(G\circ F)(C),F(C)}(\mathrm{id}_{(G\circ F)(C)}) \circ F(\eta_C) \\ &= \epsilon_{F(C)} \circ F(\eta_C). \end{aligned}$$

The verification of $(Z2)$ is similar.

02CE **Exercise 6.1.0.4.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between categories. Show that every adjunction (η, ϵ) between F and G can be obtained by applying the construction of Example 6.1.0.3 to a unique Hom-adjunction $\{\rho_{C,D}\}_{C \in \mathcal{C}, D \in \mathcal{D}}$ between F and G (see Example 6.1.2.7).

It follows from Exercise 6.1.0.4 that Definitions 6.1.0.1 and 6.1.0.2 are essentially equivalent to one another. However, an advantage of Definition 6.1.0.2 is that it can be formulated entirely in the language of functors and natural transformations: that is, it uses only the structure of the 2-category **Cat** of Example 2.2.0.4. In §6.1.1, we exploit this observation to generalize the notion of adjunction to an arbitrary 2-category. Given a 2-category $\mathcal{C}$ containing 1-morphisms $f : C \to D$ and $g : D \to C$, we define an *adjunction between* f *and* g to be a pair of 2-morphisms

$$\eta : \mathrm{id}_C \Rightarrow g \circ f \qquad \epsilon : f \circ g \Rightarrow \mathrm{id}_D$$

satisfying analogues of the compatibility conditions $(Z1)$ and $(Z2)$ above (Definition 6.1.1.1).

Our first goal is to adapt Exercise 6.1.0.4 to the setting of a general 2-category $\mathcal{C}$. Suppose we are given 1-morphisms $f : C \to D$, $g : D \to C$, $c : T \to C$, and $d : T \to D$ in $\mathcal{C}$. In §6.1.2, we show that every adjunction (η, ϵ) between f and g determines a bijection

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d) \simeq \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d),$$

depending functorially on c and d (see Corollary 6.1.2.6 and Remark 6.1.2.4). Here the map from right to left is constructed using the unit map $\eta : \mathrm{id}_C \Rightarrow g \circ f$, and from left to right using the counit $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$. As an application, we show that an adjunction (η, ϵ) is completely determined by the unit η (or the counit ϵ), and give a criterion which can be used to test if an arbitrary 2-morphism $\eta : \mathrm{id}_C \Rightarrow g \circ f$ is the unit of an adjunction (see Proposition 6.1.2.9, Variant 6.1.2.12, and Proposition 6.1.2.13)

Let $\mathcal{C}$ be a 2-category and let $f : C \to D$ be a 1-morphism in $\mathcal{C}$. In §6.1.3, we show that if f admits a right adjoint g, then g is uniquely determined up to (canonical) isomorphism (Corollary 6.1.3.3). Moreover, the formation of right adjoints can be regarded as a (partially defined) functor from $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, D)^{\mathrm{op}}$ to $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, C)$ (Notation 6.1.3.5), with a (partially defined) inverse given by the formation of left adjoints (Notation 6.1.3.8). In §6.1.4, we consider the special case where $f : C \xrightarrow{\sim} D$ is an *isomorphism* in $\mathcal{C}$: in this case, f automatically admits a right adjoint (and a left adjoint), which can be identified with a homotopy inverse isomorphism $D \xrightarrow{\sim} C$ (Proposition 6.1.4.1).

In §6.1.5, we show that the formation of adjoints is compatible with composition. More precisely, if $f : C \to D$ and $f' : D \to E$ are 1-morphisms in a 2-category $\mathcal{C}$ which admit right adjoints $g : D \to C$ and $g' : E \to D$, respectively, then the composition $(f' \circ f) : C \to E$ also admits a right adjoint, which is canonically isomorphic to the composition $(g \circ g') : E \to C$ (Corollary 6.1.5.5).

The theory of adjunctions can be usefully applied to many 2-categories $\mathcal{C}$ other than **Cat** (for example, we will use it in §6.2 to generalize the theory of adjoint functors to the setting of ∞-categories). In §6.1.6, we consider the case where $\mathcal{C}$ has a single object X, and can therefore be identified with the monoidal category $\mathcal{E} = \underline{\mathrm{End}}_{\mathcal{C}}(X)$ (see Example 2.2.2.5). Specializing the theory of adjunctions to this situation, we recover the classical notion of a *duality datum* in $\mathcal{E}$ (Definition 6.1.6.1).

6.1.1 Adjunctions

Our goal in this section is to generalize the notion of an adjunction to an arbitrary 2-category $\mathcal{C}$. Here Definition 6.1.0.2 adapts without essential change; the only additional complications are the fact that the associativity and unit constraints of $\mathcal{C}$ need not be strict. 02CF

Definition 6.1.1.1. Let $\mathcal{C}$ be a 2-category, let C and D be objects of $\mathcal{C}$, and let $f : C \to D$ and $g : D \to C$ be 1-morphisms in $\mathcal{C}$. An *adjunction between f and g* is a pair of 2-morphisms (η, ϵ), where $\eta : \mathrm{id}_C \Rightarrow g \circ f$ is a morphism in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$ and $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ is a morphism in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, D)$, which satisfy the following compatibility conditions: 02CG

($Z1$) The composition

$$f \xRightarrow[\sim]{\rho_f^{-1}} f \circ \mathrm{id}_C \xRightarrow{\mathrm{id}_f \circ \eta} f \circ (g \circ f) \xRightarrow[\sim]{\alpha_{f,g,f}} (f \circ g) \circ f \xRightarrow{\epsilon \circ \mathrm{id}_f} \mathrm{id}_D \circ f \xRightarrow[\sim]{\lambda_f} f$$

is the identity 2-morphism from f to itself. Here λ_f and ρ_f are the left and right unit constraints of the 2-category $\mathcal{C}$ (Construction 2.2.1.11) and $\alpha_{f,g,f}$ is the associativity constraint for the 2-category $\mathcal{C}$.

($Z2$) The composition

$$g \xRightarrow[\sim]{\lambda_g^{-1}} \mathrm{id}_C \circ g \xRightarrow{\eta \circ \mathrm{id}_g} (g \circ f) \circ g \xRightarrow[\sim]{\alpha_{g,f,g}^{-1}} g \circ (f \circ g) \xRightarrow{\mathrm{id}_g \circ \epsilon} g \circ \mathrm{id}_D \xRightarrow[\sim]{\rho_g} g$$

is the identity 2-morphism from g to itself.

If these conditions are satisfied, then we will refer to η as the *unit* of the adjunction (η, ϵ) and to ϵ as the *counit* of the adjunction (η, ϵ). In this case, we say that (η, ϵ) *exhibits f as a left adjoint of g*, and also that it *exhibits g as a right adjoint of f*.

Example 6.1.1.2. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between categories, which we regard as 1-morphisms in the strict 2-category **Cat** of Example 2.2.0.4. An adjunction between F and G in the 2-category **Cat** is an adjunction between F and G in the usual category-theoretic sense: that is, a pair of natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ which satisfy the requirements of Definition 6.1.0.2. 02CH

02CJ **Remark 6.1.1.3.** Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ be 2-morphisms of $\mathcal{C}$. Then the pair (η, ϵ) is an adjunction between f and g in the 2-category $\mathcal{C}$ if and only if the pair $(\eta^{\mathrm{op}}, \epsilon^{\mathrm{op}})$ it is an adjunction between g^{op} and f^{op} in the opposite 2-category $\mathcal{C}^{\mathrm{op}}$ (Construction 2.2.3.1). Note that in this case, g^{op} is the left adjoint, while f^{op} is the right adjoint.

02CK **Remark 6.1.1.4.** Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ be 2-morphisms of $\mathcal{C}$. Then the pair (η, ϵ) is an adjunction between f and g in the 2-category $\mathcal{C}$ if and only if the pair $(\epsilon^{\mathrm{c}}, \eta^{\mathrm{c}})$ is an adjunction between g^{c} and f^{c} in the conjugate 2-category $\mathcal{C}^{\mathrm{c}}$ (Construction 2.2.3.4). Note that in this case, ϵ^{c} is the unit of the adjunction and η^{c} is the counit. Similarly, g^{c} is the left adjoint and f^{c} is the right adjoint.

02CL **Remark 6.1.1.5** (Isomorphism Invariance)**.** Let $\mathcal{C}$ be a 2-category, let $f, f' : C \to D$ and $g, g' : D \to C$ be 1-morphisms in $\mathcal{C}$, and let (η, ϵ) be an adjunction between f and g. Suppose we are given invertible 2-morphisms $\beta : g \xRightarrow{\sim} g'$ and $\gamma : f \xRightarrow{\sim} f'$. Let η' denote the composition $\mathrm{id}_C \xRightarrow{\eta} g \circ f \xRightarrow[\sim]{\beta \circ \gamma} g' \circ f'$ and let ϵ' denote the composition $f' \circ g' \xRightarrow[\sim]{\gamma^{-1} \circ \beta^{-1}} f \circ g \xRightarrow{\epsilon} \mathrm{id}_D$. Then the pair (η', ϵ') is an adjunction between f' and g'.

02CM **Exercise 6.1.1.6** (Functoriality)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of 2-categories. Suppose we are given 1-morphisms $f : C \to D$ and $g : D \to C$ in $\mathcal{C}$. Let (η, ϵ) be an adjunction between f and g in the 2-category $\mathcal{C}$, let η' denote the composition

$$\mathrm{id}_{F(C)} \xRightarrow[\sim]{} F(\mathrm{id}_C) \xRightarrow{F(\eta)} F(g \circ f) \xRightarrow[\sim]{\mu_{g,f}^{-1}} F(g) \circ F(f),$$

and let ϵ' denote the composition

$$F(f) \circ F(g) \xRightarrow[\sim]{\mu_{f,g}} F(f \circ g) \xRightarrow{F(\epsilon)} F(\mathrm{id}_D) \xRightarrow[\sim]{} \mathrm{id}_{F(D)},$$

where $\mu_{f,g}$ and $\mu_{g,f}$ are the composition constraints of the functor F and the unlabeled isomorphisms are the identity constraints of F. Show that the pair (η', ϵ') is an adjunction between $F(f)$ and $F(g)$ in the 2-category $\mathcal{D}$.

02CN **Example 6.1.1.7.** Let $\mathcal{C}$ be an ordinary category which admits fiber products, and let $\mathrm{Corr}(\mathcal{C})$ denote the 2-category of correspondences in $\mathcal{C}$ (Example 2.2.2.1). Every morphism $f : X \to Y$ in $\mathcal{C}$ determines diagrams

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "f"] & & & X \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}_X"] & \\
X & & Y & Y & & X
\end{tikzcd}$$

which we can regard as 1-morphisms $f_! : X \to Y$ and $f^! : Y \to X$ in the 2-category $\mathrm{Corr}(\mathcal{C})$. Unwinding the definitions, we see that the compositions $f^! \circ f_!$ and $f_! \circ f^!$ are given (up to isomorphism) by the diagrams

$$\begin{tikzcd}
& X \times_Y X \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & & & X \arrow[dl, "f"'] \arrow[dr, "f"] & \\
X & & X & Y & & Y,
\end{tikzcd}$$

where $\pi_0, \pi_1 : X \times_Y X \to X$ are the projection maps. We can therefore regard the diagonal map $\delta : X \to X \times_Y X$ as a 2-morphism from id_X to $f^! \circ f_!$ in $\mathrm{Corr}(\mathcal{C})$, and the morphism $f : X \to Y$ as a 2-morphism from $f_! \circ f^!$ to id_Y in $\mathrm{Corr}(\mathcal{C})$. The pair (δ, f) is an adjunction between $f_!$ and $f^!$.

6.1.2 Adjuncts

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between categories. By virtue of Exercise 6.1.0.4, every adjunction (η, ϵ) between F and G determines a collection of bijections 02CP

$$\rho_{C,D} : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \simeq \mathrm{Hom}_{\mathcal{C}}(C, G(D)),$$

depending functorially on $C \in \mathcal{C}$ and $D \in \mathcal{D}$. In this section, we establish an analogue of this statement for adjunctions in an arbitrary 2-category.

Construction 6.1.2.1. Let $\mathcal{C}$ be a 2-category containing objects T, C, and D, together with 1-morphisms $f : C \to D$, $g : D \to C$, $c : T \to C$, and $d : T \to D$. 02CQ

- Let $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ and $\beta : c \Rightarrow g \circ d$ be 2-morphisms of $\mathcal{C}$. We will refer to the composition

  $$f \circ c \xRightarrow{\mathrm{id}_f \circ \beta} f \circ (g \circ d) \xRightarrow[\sim]{\alpha_{f,g,d}} (f \circ g) \circ d \xRightarrow{\epsilon \circ \mathrm{id}_d} \mathrm{id}_D \circ d \xRightarrow[\sim]{\lambda_d} d$$

  as the *left adjunct of β with respect to ϵ*, or more simply as the *left adjunct of β* if the 2-morphism ϵ is clear from context. Here λ_d and $\alpha_{f,g,d}$ are the left unit and associativity constraints for the 2-category $\mathcal{C}$.

- Let $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\gamma : f \circ c \Rightarrow d$ be 2-morphisms of $\mathcal{C}$. We will refer to the composition

  $$c \xRightarrow[\sim]{\lambda_c^{-1}} \mathrm{id}_C \circ c \xRightarrow{\eta} (g \circ f) \circ c \xRightarrow[\sim]{\alpha_{g,f,c}^{-1}} g \circ (f \circ c) \xRightarrow{\mathrm{id}_g \circ \gamma} g \circ d$$

  as the *right adjunct of γ with respect to η*, or more simply as the *right adjunct of γ* if the 2-morphism η is clear from context. Here again λ_c and $\alpha_{g,f,c}$ are the left unit and associativity constraints for the 2-category $\mathcal{C}$.

02CR **Example 6.1.2.2.** Let $\mathcal{C}$ be a 2-category containing 1-morphisms $f : C \to D$ and $g : D \to C$. Then:

- Every 2-morphism $\eta : \mathrm{id}_C \Rightarrow g \circ f$ is equal to the right adjunct of the right unit constraint $\rho_f : f \circ \mathrm{id}_D \overset{\sim}{\Longrightarrow} f$ (with respect to η).

- Every 2-morphism $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ is equal to the left adjunct of $\rho_g^{-1} : g \overset{\sim}{\Longrightarrow} g \circ \mathrm{id}_D$ (with respect to ϵ).

02CS **Example 6.1.2.3.** Let $\mathcal{C}$ be a 2-category containing 1-morphisms $f : C \to D$ and $g : D \to C$, and suppose we are given 2-morphisms $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$. Then (η, ϵ) is an adjunction between f and g if and only if the following conditions are satisfied:

($Z1$) The left adjunct of η (with respect to ϵ) is equal to the right unit constraint $\rho_f : f \circ \mathrm{id}_C \overset{\sim}{\Longrightarrow} f$.

($Z2$) The right adjunct of ϵ (with respect to η) is the inverse $\rho_g^{-1} : g \overset{\sim}{\Longrightarrow} g \circ \mathrm{id}_D$ of the right unit constraint.

02CT **Remark 6.1.2.4** (Functoriality)**.** Let $\mathcal{C}$ be a 2-category containing objects T, C, and D, together with 1-morphisms $f : C \to D$, $g : D \to C$, $c, c' : T \to C$, and $d, d' : T \to D$. Then:

- If $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\varphi : c \Rightarrow c'$ are 2-morphisms of $\mathcal{C}$, then the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c', d) \arrow[r] \arrow[d, "\mathrm{id}_f \circ \varphi"] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c', g \circ d) \arrow[d, "\varphi"] \\
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d) \arrow[r] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d)
\end{tikzcd}$$

  is commutative, where the horizontal maps are given by the formation of right adjuncts with respect to η.

- If $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ and $\varphi : c \Rightarrow c'$ are 2-morphisms of $\mathcal{C}$, then the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c', g \circ d) \arrow[r] \arrow[d, "\varphi"] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c', d) \arrow[d, "\mathrm{id}_f \circ \varphi"] \\
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d) \arrow[r] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d)
\end{tikzcd}$$

  is commutative, where the horizontal maps are given by the formation of left adjuncts with respect to ϵ.

- If $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\psi : d \Rightarrow d'$ are 2-morphisms of $\mathcal{C}$, then the diagram of sets

  $$\begin{tikzcd}
  \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d) \arrow[r] \arrow[d, "\psi"] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d) \arrow[d, "\mathrm{id}_g \circ \psi"] \\
  \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d') \arrow[r] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d')
  \end{tikzcd}$$

  is commutative, where the horizontal maps are given by the formation of right adjuncts with respect to η.

- If $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ and $\psi : d \Rightarrow d'$ are 2-morphisms of $\mathcal{C}$, then the diagram of sets

  $$\begin{tikzcd}
  \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d) \arrow[r] \arrow[d, "\mathrm{id}_g \circ \psi"] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d) \arrow[d, "\psi"] \\
  \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d') \arrow[r] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d'),
  \end{tikzcd}$$

  is commutative, where the horizontal maps are given by the formation of left adjuncts with respect to ϵ.

Stated more informally, Construction 6.1.2.1 depends functorially on the 1-morphisms $c : T \to C$ and $d : T \to D$.

Proposition 6.1.2.5. *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ be 2-morphisms. Suppose we are given another object $T \in \mathcal{C}$ equipped with 1-morphisms $c : T \to C$ and $d : T \to D$, together with 2-morphisms $\beta : c \Rightarrow g \circ d$ and $\gamma : f \circ c \Rightarrow d$. Then:* 02CU

(1) *If the pair (η, ϵ) satisfies condition $(Z1)$ of Definition 6.1.1.1 and β is the right adjunct of γ, then γ is the left adjunct of β.*

(2) *If the pair (η, ϵ) satisfies condition $(Z2)$ of Definition 6.1.1.1 and γ is the left adjunct of β, then β is the right adjunct of γ.*

Proof. We will prove (1); the proof of (2) follows by applying the same argument in the conjugate 2-category $\mathcal{C}^{\mathrm{c}}$. Consider the diagram

$$\begin{tikzcd}
f \circ c \arrow[r, Rightarrow, "\lambda_c^{-1}"] \arrow[d, Rightarrow, "\rho_f^{-1}"] & f \circ (\mathrm{id}_C \circ c) \arrow[r, Rightarrow, "\eta"] \arrow[dl, Rightarrow, "\sim"] & f \circ ((g \circ f) \circ c) \arrow[r, Rightarrow, "\sim"] \arrow[dl, Rightarrow, "\sim"'] & f \circ (g \circ (f \circ c)) \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, "\sim"'] & f \circ (g \circ d) \arrow[d, Rightarrow, "\sim"'] \\
(f \circ \mathrm{id}_C) \circ c \arrow[r, Rightarrow, "\eta"] & (f \circ (g \circ f) \circ c \arrow[r, Rightarrow, "\sim"] & ((f \circ g) \circ f) \circ c \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\epsilon"] & (f \circ g) \circ (f \circ c) \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, "\epsilon"] & (f \circ g) \circ d \arrow[d, Rightarrow, "\epsilon"] \\
 & & (\mathrm{id}_D \circ f) \circ c \arrow[r, Rightarrow, "\sim"] \arrow[dr, Rightarrow, "\lambda_f"] & \mathrm{id}_D \circ (f \circ c) \arrow[r, Rightarrow, "\gamma"] \arrow[d, Rightarrow, "\lambda_{f \circ c}"] & \mathrm{id}_D \circ d \arrow[d, Rightarrow, "\lambda_d"] \\
 & & & f \circ c \arrow[r, Rightarrow, "\gamma"] & d
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(T, D)$, where the unlabeled morphisms are given by the associativity constraints of $\mathcal{C}$ (and their inverses). Our assumption that β is the right adjunct of γ guarantees that the composition along the top line coincides with $\mathrm{id}_f \circ \beta$. Consequently, the left adjunct of β is the 2-morphism of $\mathcal{C}$ given by clockwise composition around the outside of the diagram. On the other hand, axiom $(Z1)$ of Definition 6.1.1.1 guarantees counterclockwise composition around the outside of the diagram coincides with γ. To complete the proof, it will suffice to show that the diagram commutes. The commutativity of the triangular regions follows from Propositions 2.2.1.14 and 2.2.1.16. The commutativity of the bottom right square follows from the naturality of left unit constraints (Remark 2.2.1.13) and the commutativity of the middle right square from the functoriality of composition. The remaining squares commute by the naturality of the associativity constraints of $\mathcal{C}$, and the five-sided region commutes by virtue of the pentagon identity. □

02CV **Corollary 6.1.2.6.** *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and suppose we are given 2-morphisms $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$. The following conditions are equivalent:*

(1) *The pair (η, ϵ) is an adjunction between f and g (in the sense of Definition 6.1.1.1).*

(2) *For every object $T \in \mathcal{C}$ and every pair of 1-morphisms $c : T \to C$ and $d : T \to D$, the formation of left and right adjuncts (Construction 6.1.2.1) supplies mutually inverse bijections*

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d) \simeq \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d).$$

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 6.1.2.5. For the converse, we first observe that $\eta : \mathrm{id}_C \Rightarrow g \circ f$ is equal to the right adjunct of the right unit constraint $\rho_f : f \circ \mathrm{id}_D \overset{\sim}{\Longrightarrow} f$ with respect to η (Example 6.1.2.2). If assumption (2) is satisfied, then ρ_f is the left adjunct of η with respect to ϵ. Similarly, assumption (2) guarantees that $\rho_g^{-1} : g \overset{\sim}{\Longrightarrow} g \circ \mathrm{id}_D$ is the right adjunct of ϵ with respect to η, so that the pair (η, ϵ) is an adjunction by virtue of Example 6.1.2.3. □

Example 6.1.2.7. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between categories, and let (η, ϵ) be an adjunction between F and G. Suppose we are given objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, which we identify with functors $C : \{*\} \to \mathcal{C}$ and $D : \{*\} \to \mathcal{D}$, respectively. Applying Corollary 6.1.2.6 to the 2-category **Cat**, we obtain a bijection 02CW

$$\rho_{C,D} : \mathrm{Hom}_{\mathcal{D}}(F(C), D) \simeq \mathrm{Hom}_{\mathcal{D}}(C, G(D)).$$

This bijection depends functorially on C and D (Remark 6.1.2.4), and can therefore be regarded as a Hom-adjunction between F and G in the sense of Definition 6.1.0.1. Note that, for every morphism $f : F(C) \to D$ in $\mathcal{C}$, the image $\rho_{C,D}(f) \in \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ is given explicitly by the composition $C \xrightarrow{\eta_C} (G \circ F)(C) \xrightarrow{G(f)} G(D)$. In particular, the morphism $\eta_C : C \to (G \circ F)(C)$ can be recovered by applying $\rho_{C,F(C)}$ to the identity morphism $\mathrm{id}_{F(C)}$. Similarly, for each object $D \in \mathcal{D}$, the morphism $\epsilon_D : (F \circ G)(D) \to D$ can be recovered by applying $\rho^{-1}_{G(D),D}$ to the identity morphism $\mathrm{id}_{G(D)}$. In other words, the adjunction (η, ϵ) is obtained by applying the construction of Example 6.1.0.3 to the Hom-adjunction $\{\rho_{C,D}\}_{C \in \mathcal{C}, D \in \mathcal{D}}$.

Corollary 6.1.2.8. *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and suppose we are given 2-morphisms $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ satisfying condition $(Z1)$ of Definition 6.1.1.1. Let $\gamma : g \Rightarrow g$ denote the 2-morphism given by the composition* 02CX

$$g \underset{\sim}{\overset{\lambda_g^{-1}}{\Longrightarrow}} \mathrm{id}_C \circ g \overset{\eta \circ \mathrm{id}_g}{\Longrightarrow} (g \circ f) \circ g \underset{\sim}{\overset{\alpha^{-1}_{g,f,g}}{\Longrightarrow}} g \circ (f \circ g) \overset{\mathrm{id}_g \circ \epsilon}{\Longrightarrow} g \circ \mathrm{id}_D \underset{\sim}{\overset{\rho_g}{\Longrightarrow}} g.$$

Then γ is idempotent: that is, $\gamma^2 = \gamma$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, C)$. In particular, if γ has either a left or a right inverse, then $\gamma = \mathrm{id}_g$ (so that (η, ϵ) is an adjunction between f and g).

Proof. Let γ' denote the composition $g \overset{\gamma}{\Rightarrow} g \overset{\rho_g^{-1}}{\Longrightarrow} g \circ \mathrm{id}_D$. Then γ' is the right adjunct of ϵ with respect to η (see Example 6.1.2.3). Invoking Remark 6.1.2.4, we deduce that the horizontal composition $\gamma'\gamma$ is the right adjunct of ϵ' with respect to η, where ϵ' denotes the composite map $f \circ g \overset{\mathrm{id}_f \circ \gamma}{\Longrightarrow} f \circ g \overset{\epsilon}{\Rightarrow} \mathrm{id}_D$. Combining Example 6.1.2.2 with Remark 6.1.2.4, we see that ϵ' is the left adjunct of γ' with respect to ϵ. Since the pair (η, ϵ) satisfies $(Z1)$, it follows that $\gamma'\gamma = \gamma'$, Composing with the right unit constraint ρ_g, we conclude that $\gamma\gamma = \gamma$. □

02CY **Proposition 6.1.2.9.** *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let $\eta : \mathrm{id}_C \Rightarrow g \circ f$ be a 2-morphism of $\mathcal{C}$. The following conditions are equivalent:*

(1) *For every object $T \in \mathcal{C}$ and every pair of 1-morphisms $c : T \to C$ and $d : T \to D$, the formation of right adjuncts with respect to η (Construction 6.1.2.1) induces a bijection*

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d) \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d).$$

(2) *There exists a 2-morphism $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ for which (η, ϵ) is an adjunction between f and g.*

Moreover, if these conditions are satisfied, then the 2-morphism ϵ is uniquely determined.

Proof. The implication (2) ⇒ (1) follows from Corollary 6.1.2.6. Conversely, suppose that condition (1) is satisfied. Applying (1) in the case $T = D$, $c = g$, and $d = \mathrm{id}_D$, we conclude that there is a unique 2-morphism $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ whose right adjunct is equal to the inverse $\rho_g^{-1} : g \overset{\sim}{\Longrightarrow} g \circ \mathrm{id}_D$ of the right unit constraint ρ_g, so that the pair (η, ϵ) satisfies condition $(Z2)$ of Definition 6.1.1.1 (Example 6.1.2.3). We will complete the proof by showing that (η, ϵ) also satisfies condition $(Z1)$. Let $\gamma : f \circ \mathrm{id}_C \Rightarrow f$ be the left adjunct of η. It follows from Proposition 6.1.2.5 that the right adjunct of γ is equal to η, which is also the right adjunct of the unit constraint $\rho_f : f \circ \mathrm{id}_C \overset{\sim}{\Longrightarrow} f$. Invoking assumption (1), we conclude that $\gamma = \rho_f$, which is a restatement of $(Z1)$ (Example 6.1.2.3). □

02CZ **Definition 6.1.2.10.** Let $\mathcal{C}$ be a 2-category and let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$. We say that a 2-morphism $\eta : \mathrm{id}_C \Rightarrow g \circ f$ *is the unit of an adjunction* if it satisfies the equivalent conditions of Proposition 6.1.2.9: that is, if there exists a 2-morphism $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ for which the pair (η, ϵ) is an adjunction. If this condition is satisfied, we will say that η *exhibits f as a left adjoint of g* and also that η *exhibits g as a right adjoint of f*.

In the 2-category **Cat**, we can formulate a sharper version of Proposition 6.1.2.9:

02D0 **Variant 6.1.2.11.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between categories and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be a natural transformation. The following conditions are equivalent:

(1) For every pair of objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the formation of right adjuncts with respect to η induces a bijection $\mathrm{Hom}_{\mathcal{D}}(F(C), D) \to \mathrm{Hom}_{\mathcal{C}}(C, G(D))$.

(2) There exists a natural transformation $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ for which (η, ϵ) is an adjunction between F and G.

Moreover, if these conditions are satisfied, then the natural transformation ϵ is uniquely determined.

Proof. We will prove that (1) $\Rightarrow$ (2); the remaining assertions follow immediately from Proposition 6.1.2.9. Fix an object $D \in \mathcal{D}$. Applying assertion (1) in the case $C = G(D)$, we deduce that there is a unique morphism $\epsilon_D : (F \circ G)(D) \to D$ for which the composition

$$G(D) \xrightarrow{\eta_{G(D)}} (G \circ F \circ G)(D) \xrightarrow{G(\epsilon_D)} G(D)$$

is the identity morphism from $G(D)$ to itself.

We first claim that the construction $D \mapsto \epsilon_D$ is a natural transformation of functors from $F \circ G$ to $\mathrm{id}_{\mathcal{D}}$. Let $h : D \to D'$ be a morphism in the category $\mathcal{D}$; we wish to show that the diagram

$$\begin{tikzcd}
(F \circ G)(D) \arrow[r, "\epsilon_D"] \arrow[d, "(F \circ G)(h)"] & D \arrow[d, "h"] \\
(F \circ G)(D') \arrow[r, "\epsilon_{D'}"] & D'
\end{tikzcd}$$

commutes. Consider the diagram

$$\begin{tikzcd}
G(D) \arrow[r, "\eta_{G(D)}"] \arrow[d, "G(h)"] & (G \circ F \circ G)(D) \arrow[r, "G(\epsilon_D)"] \arrow[d, "G(F(G(h)))"] & G(D) \arrow[d, "G(h)"] \\
G(D') \arrow[r, "\eta_{G(D')}"] & (G \circ F \circ G)(D') \arrow[r, "G(\epsilon_{D'})"] & G(D')
\end{tikzcd}$$

in the category $\mathcal{C}$. It follows from the definitions of ϵ_D and $\epsilon_{D'}$ that both horizontal compositions are equal to the identity, so the outer rectangle commutes. Since η is a natural transformation, the left square commutes. It follows that the compositions $G(h) \circ G(\epsilon_D) \circ \eta_{G(D)}$ and $G(\epsilon_{D'}) \circ G(F(G(h))) \circ \eta_{G(D)}$ are the same: that is, the morphisms

$$h \circ \epsilon_D, \epsilon_{D'} \circ F(G(h)) \in \mathrm{Hom}_{\mathcal{D}}((F \circ G)(D), D')$$

have the same right adjunct. Invoking assumption (1), we deduce that $h \circ \epsilon_D = \epsilon_{D'} \circ F(G(h))$, as desired.

It follows immediately from the construction that the pair of natural transformations (η, ϵ) satisfies condition $(Z2)$ of Definition 6.1.0.2. To complete the proof, it will suffice to show that it also satisfies condition $(Z1)$. Let C be an object of $\mathcal{C}$; we wish to show that the composite map

$$F(C) \xrightarrow{F(\eta_C)} (F \circ G \circ F)(C) \xrightarrow{\epsilon_{F(C)}} F(C)$$

is equal to the identity map $\mathrm{id}_{F(C)}$. Note that the right adjunct of $\epsilon_{F(C)} \circ F(\eta_C)$ is the composite map

$$C \xrightarrow{\eta_C} (G \circ F)(C) \xrightarrow{(G\circ F)(\eta_C)} (G \circ F \circ G \circ F)(C) \xrightarrow{G(\epsilon_{F(C)})} (G \circ F)(C).$$

By virtue of the fact that (η, ϵ) satisfies $(Z2)$, this composition is equal to η_C, which is also the right adjunct of the identity map $\mathrm{id}_{F(C)}$. Invoking assumption (1), we conclude that $\epsilon_{F(C)} \circ F(\eta_C) = \mathrm{id}_{F(C)}$, as desired. □

We can give another characterization of the units of adjunctions by applying Proposition 6.1.2.9 in the opposite 2-category $\mathcal{C}^{\mathrm{op}}$:

02D1 **Variant 6.1.2.12.** Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : C \to D$ be 1-morphisms of $\mathcal{C}$, and let $\eta : \mathrm{id}_C \Rightarrow g \circ f$ be a 2-morphism of $\mathcal{C}$. Then η is the unit of an adjunction if and only if the following condition is satisfied:

- For every object $T \in \mathcal{C}$ and every pair of morphisms $c : C \to T$ and $d : D \to T$, the 2-morphism η determines a bijection

  $$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,T)}(c \circ g, d) \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,T)}(c, d \circ f),$$

  carrying each 2-morphism $\beta : c \circ g \Rightarrow d$ to the composition

  $$c \underset{\sim}{\overset{\rho_c^{-1}}{\Longrightarrow}} c \circ \mathrm{id}_C \overset{\mathrm{id}_c \circ \eta}{\Longrightarrow} c \circ (g \circ f) \underset{\sim}{\overset{\alpha_{c,g,f}}{\Longrightarrow}} (c \circ g) \circ f \overset{\beta \circ \mathrm{id}_f}{\Longrightarrow} d \circ f.$$

For the reader's convenience, let us also record a conjugate version of the preceding discussion:

02D2 **Proposition 6.1.2.13.** *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ be a 2-morphism of $\mathcal{C}$. The following conditions are equivalent:*

(1) *For every object $T \in \mathcal{C}$ and every pair of 1-morphisms $c : T \to C$ and $d : T \to D$, the formation of left adjuncts with respect to ϵ (Construction 6.1.2.1) induces a bijection*

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d) \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d)$$

(2) *For every object $T \in \mathcal{C}$ and every pair of 1-morphisms $c : C \to T$ and $d : D \to T$, the 2-morphism ϵ determines a bijection*

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,T)}(c, d \circ f) \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,T)}(c \circ g, d)$$

carrying each 2-morphism $\gamma : c \Rightarrow d \circ f$ to the composition

$$c \circ g \overset{\gamma \circ \mathrm{id}_g}{\Longrightarrow} (d \circ f) \circ g \underset{\sim}{\overset{\alpha_{d,f,g}^{-1}}{\Longrightarrow}} d \circ (f \circ g) \overset{\mathrm{id}_d \circ \epsilon}{\Longrightarrow} d \circ \mathrm{id}_D \underset{\sim}{\overset{\rho_d}{\Longrightarrow}} d.$$

(3) *There exists a* 2*-morphism* $\eta : \mathrm{id}_C \Rightarrow g \circ f$ *for which* (η, ϵ) *is an adjunction between* f *and* g*.*

Moreover, if these conditions are satisfied, then the 2*-morphism* η *is uniquely determined.*

Proof. Apply Proposition 6.1.2.9 and Variant 6.1.2.12 to the conjugate 2-category $\mathcal{C}^{\mathrm{c}}$. □

Definition 6.1.2.14. Let $\mathcal{C}$ be a 2-category and let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$. We say that a 2-morphism $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ *is the counit of an adjunction* if it satisfies the equivalent conditions of Proposition 6.1.2.13: that is, there exists a 2-morphism $\eta : \mathrm{id}_C \Rightarrow g \circ f$ for which the pair (η, ϵ) is an adjunction. If this condition is satisfied, we will say that ϵ *exhibits* f *as a left adjoint of* g and also that ϵ *exhibits* g *as a right adjoint of* f. 02D3

6.1.3 Uniqueness of Adjoints

Let $\mathcal{C}$ be a 2-category and let $f : C \to D$ be a 1-morphism of $\mathcal{C}$. We will say that a 1-morphism $g : D \to C$ *is a right adjoint of* f if there exists an adjunction (η, ϵ) between f and g, in the sense of Definition 6.1.1.1. Beware that the right adjoint of f is usually not unique: if g is a right adjoint of f, then any 1-morphism $g' : D \to C$ which is isomorphic to g can also be regarded as a right adjoint to f (see Remark 6.1.1.5). However, we will show in this section that this is the only source of ambiguity: the right adjoint of a 1-morphism f (if it exists) is well-defined up to canonical isomorphism. 02D4

Proposition 6.1.3.1. *Let* $\mathcal{C}$ *be a* 2*-category, let* $f : C \to D$ *and* $g : D \to C$ *be* 1*-morphisms of* $\mathcal{C}$*, and let* $\eta : \mathrm{id}_C \Rightarrow g \circ f$ *be the unit of an adjunction. Then:* 02D5

(1) *For every* 1*-morphism* $f' : C \to D$*, the function*

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,D)}(f, f') \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,C)}(\mathrm{id}_C, g \circ f') \qquad \gamma \mapsto (\mathrm{id}_g \circ \gamma)\eta$$

is a bijection.

(2) *For every* 1*-morphism* $g' : D \to C$*, the function*

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,C)}(g, g') \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,C)}(\mathrm{id}_C, g' \circ f) \qquad \beta \mapsto (\beta \circ \mathrm{id}_f)\eta$$

is a bijection.

Proof. Let $\rho_f : f \circ \mathrm{id}_C \xRightarrow{\sim} f$ be the right unit constraint. To prove (1), we observe that the composition

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,D)}(f \circ \mathrm{id}_C, f') \xrightarrow[\sim]{\rho_f^{-1}} \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,D)}(f, f') \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,C)}(\mathrm{id}_C, g \circ f')$$

is given by the formation of right adjuncts (see Example 6.1.2.2 and Remark 6.1.2.4), and is therefore bijective by (Proposition 6.1.2.5). Assertion (2) follows by a similar argument. □

02D6 **Variant 6.1.3.2.** Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ be the counit of an adjunction. Then:

(1) For every 1-morphism $f' : C \to D$, the function

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,D)}(f', f) \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,D)}(f' \circ g, \mathrm{id}_D) \qquad \gamma \mapsto \epsilon(\gamma \circ \mathrm{id}_g)$$

is a bijection.

(2) For every 1-morphism $g' : D \to C$, the function

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,C)}(g', g) \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,D)}(f \circ g', \mathrm{id}_D) \qquad \beta \mapsto \epsilon(\mathrm{id}_f \circ \beta)$$

is a bijection.

Proof. Apply Proposition 6.1.3.1 to the conjugate 2-category $\mathcal{C}^{\mathrm{c}}$. □

02D7 **Corollary 6.1.3.3.** *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let (η, ϵ) be an adjunction between f and g. Let $g' : D \to C$ be another 1-morphism of $\mathcal{C}$. Then:*

(1) *For every 2-morphism $\eta' : \mathrm{id}_C \Rightarrow g' \circ f$, there is a unique 2-morphism $\beta : g \Rightarrow g'$ for which η' is equal to the composition $\mathrm{id}_C \overset{\eta}{\Rightarrow} g \circ f \xRightarrow{\beta \circ \mathrm{id}_f} g' \circ f$. Moreover, β is an isomorphism if and only if η' is the unit of an adjunction.*

(2) *For every 2-morphism $\epsilon' : f \circ g' \Rightarrow \mathrm{id}_D$, there is a unique 2-morphism $\gamma : g' \Rightarrow g$ for which ϵ' factors as a composition $f \circ g' \xRightarrow{\mathrm{id}_f \circ \gamma} f \circ g \overset{\epsilon}{\Rightarrow} \mathrm{id}_D$. Moreover, γ is an isomorphism if and only ϵ' is the counit of an adjunction.*

Proof. We will prove (1); the proof of (2) similar. Let $\eta' : \mathrm{id}_C \Rightarrow g' \circ f$ be a 2-morphism of $\mathcal{C}$. It follows from Proposition 6.1.3.1 that there is a unique 2-morphism $\beta : g \Rightarrow g'$ satisfying $\eta' = (\beta \circ \mathrm{id}_f)\eta$. If β is an isomorphism, then η' is the unit of an adjunction by virtue of Remark 6.1.1.5. Conversely, suppose that η' is the unit of an adjunction. To prove that β is an isomorphism, it will suffice to show that for every 1-morphism $g'' : D \to C$, precomposition with β induces a bijection $\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,C)}(g', g'') \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,C)}(g, g'')$. This is clear: we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,C)}(g', g'') \arrow[rr, "\beta"] \arrow[dr, swap, "\eta'"] & & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,C)}(g, g'') \arrow[dl, "\eta"] \\
& \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(C,C)}(\mathrm{id}_C, g'' \circ f), &
\end{tikzcd}$$

where the vertical maps are bijective by virtue of Proposition 6.1.3.1. □

Proposition 6.1.3.4. *Let $\mathcal{C}$ be a 2-category containing 1-morphisms $f, f' : C \to D$ and $g, g' : D \to C$. Let (η, ϵ) be an adjunction between f and g, and let (η', ϵ') be an adjunction between f' and g'. Then every 2-morphism $\beta : f \Rightarrow f'$ determines a 2-morphism $\beta^R : g' \Rightarrow g$, which is uniquely determined by* either *of the following properties:* 02D8

(1) *The diagram*

$$\begin{tikzcd}
\mathrm{id}_C \arrow[r, Rightarrow, "\eta"] \arrow[d, Rightarrow, "\eta'"] & g \circ f \arrow[d, Rightarrow, "\mathrm{id}_g \circ \beta"] \\
g' \circ f' \arrow[r, Rightarrow, "\beta^R \circ \mathrm{id}_{f'}"] & g \circ f'
\end{tikzcd}$$

commutes (in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$).

(2) *The diagram*

$$\begin{tikzcd}
f \circ g' \arrow[r, Rightarrow, "\mathrm{id}_f \circ \beta^R"] \arrow[d, Rightarrow, "\beta \circ \mathrm{id}_{g'}"] & f \circ g \arrow[d, Rightarrow, "\epsilon"] \\
f' \circ g' \arrow[r, Rightarrow, "\epsilon'"] & \mathrm{id}_D
\end{tikzcd}$$

commutes (in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, D)$).

Proof. It follows from Corollary 6.1.3.3 that there is a unique morphism β^R satisfying condition (1). We will prove that β^R also satisfies condition (2) (it is also uniquely determined by condition (2), by virtue of Corollary 6.1.3.3). Note (2) is equivalent to the assertion that $\epsilon'(\beta \circ \mathrm{id}_{g'})$ is the left adjunct of $\rho_g^{-1}\beta^R$ with respect to ϵ (in the sense of Construction 6.1.2.1). By virtue of Proposition 6.1.2.5, this is equivalent to the assertion that $\rho_g^{-1}\beta^R$ is the right adjunct of $\epsilon'(\beta \circ \mathrm{id}_{g'})$ with respect to η. This follows from the commutativity of the outer rectangle in the diagram

$$\begin{tikzcd}
g' \arrow[r, Rightarrow, "\lambda_{g'}^{-1}", "\sim"'] \arrow[ddd, equal] & \mathrm{id}_C \circ g' \arrow[r, Rightarrow, "\eta \circ \mathrm{id}_{g'}"] \arrow[d, Rightarrow, "\eta' \circ \mathrm{id}_{g'}"'] & (g \circ f) \circ g' \arrow[r, Rightarrow, "\alpha_{g,f,g'}^{-1}", "\sim"'] \arrow[d, Rightarrow, "(\mathrm{id}_g \circ \beta) \circ \mathrm{id}_{g'}"'] & g \circ (f \circ g') \arrow[ddl, Rightarrow, "\mathrm{id}_g \circ (\beta \circ \mathrm{id}_{g'})"] \arrow[dddd, Rightarrow, "\mathrm{id}_g \circ \epsilon' (\beta \circ \mathrm{id}_{g'})"] \\
 & (g' \circ f') \circ g' \arrow[r, Rightarrow, "(\beta^R \circ \mathrm{id}_{f'}) \circ \mathrm{id}_{g'}"] \arrow[d, Rightarrow, "\alpha_{g',f',g'}^{-1}"', "\sim"] & (g \circ f') \circ g' \arrow[d, Rightarrow, "\alpha_{g,f',g'}^{-1}"', "\sim"] & \\
 & g' \circ (f' \circ g') \arrow[r, Rightarrow, "\beta^R \circ (\mathrm{id}_{f'} \circ \mathrm{id}_{g'})"] \arrow[d, Rightarrow, "\mathrm{id}_{g'} \circ \epsilon'"] & g \circ (f' \circ g') \arrow[ddr, Rightarrow, "\mathrm{id}_g \circ \epsilon'"] & \\
g' \arrow[r, Rightarrow, "\rho_{g'}^{-1}"] \arrow[d, Rightarrow, "\beta^R"] & g' \circ \mathrm{id}_D \arrow[drr, Rightarrow, "\beta^R \circ \mathrm{id}_{\mathrm{id}_D}"] & & \\
g \arrow[rrr, Rightarrow, "\rho_g^{-1}"] & & & g \circ \mathrm{id}_D
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, C)$. Here the upper middle square commutes by virtue of condition (1), the rectangle on the left commutes by virtue of the assumption that (η', ϵ') is an adjunction, and the commutativity of the rest of the diagram follows by the naturality properties of the associativity and unit constraints of the 2-category $\mathcal{C}$. □

02D9 **Notation 6.1.3.5.** Let $\mathcal{C}$ be a 2-category containing a pair of objects C and D, and let $\underline{\mathrm{LHom}}_{\mathcal{C}}(C, D)$ denote the full subcategory of $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, D)$ spanned by those 1-morphisms $f : C \to D$ which admit a right adjoint $g : D \to C$. In this case, Corollary 6.1.3.3 guarantees that the 1-morphism g is determined uniquely up to isomorphism. We will sometimes abuse terminology by referring to g as *the* right adjoint of f and denoting it by f^R. The construction $f \mapsto f^R$ extends to a functor of categories $\underline{\mathrm{LHom}}_{\mathcal{C}}(C, D)^{\mathrm{op}} \to \underline{\mathrm{Hom}}_{\mathcal{C}}(D, C)$, which carries each 2-morphism $\beta : f \Rightarrow f'$ to the 2-morphism $\beta^R : f'^R \Rightarrow f^R$ described in Proposition 6.1.3.4.

02DA **Warning 6.1.3.6.** Let $\mathcal{C}$ be a 2-category and let $f : C \to D$ be a 1-morphism of $\mathcal{C}$. It follows from Corollary 6.1.3.3 that if f admits a right adjoint f^R, then f^R is characterized (up to canonical isomorphism) by the requirement that it represents the functor

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(D, C)^{\mathrm{op}} \to \mathrm{Set} \qquad g \mapsto \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,D)}(f \circ g, \mathrm{id}_D).$$

Beware that it is possible for this functor to be representable by a 1-morphism $g : D \to C$ which is *not* a right adjoint to f (in which case f cannot admit any right adjoint); see Warning 6.1.6.16.

The preceding discussion has an obvious counterpart for left adjoints:

Corollary 6.1.3.7. *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 2-morphisms of $\mathcal{C}$, and let (η, ϵ) be an adjunction between f and g. Let $f' : C \to D$ be another 1-morphism of $\mathcal{C}$. Then:* 02DB

(1) *For every 2-morphism $\eta' : \mathrm{id}_C \Rightarrow g \circ f'$, there is a unique 2-morphism $\beta : f \Rightarrow f'$ for which η' is equal to the composition $\mathrm{id}_C \xRightarrow{\eta} g \circ f \xRightarrow{\mathrm{id}_g \circ \beta} g \circ f'$. Moreover, β is an isomorphism if and only if η' is the unit of an adjunction.*

(2) *For every 2-morphism $\epsilon' : f' \circ g \Rightarrow \mathrm{id}_D$, there is a unique 2-morphism $\gamma : f' \Rightarrow f$ for which ϵ' factors as a composition $f' \circ g \xRightarrow{\gamma \circ \mathrm{id}_g} f \circ g \xRightarrow{\epsilon} \mathrm{id}_D$. Moreover, γ is an isomorphism if and only ϵ' is the counit of an adjunction.*

Proof. Apply Corollary 6.1.3.7 to the opposite 2-category $\mathcal{C}^{\mathrm{op}}$. □

Notation 6.1.3.8. Let $\mathcal{C}$ be a 2-category containing a pair of objects C and D, and let $\underline{\mathrm{RHom}}_{\mathcal{C}}(D, C)$ denote the full subcategory of $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, C)$ spanned by those 1-morphisms $g : D \to C$ which admit a left adjoint $f : C \to D$. In this case, Corollary 6.1.3.3 guarantees that the 1-morphism f is uniquely determined up to isomorphism. We will sometimes abuse terminology by referring to f as *the* left adjoint of g and denoting it by g^L. The construction $g \mapsto g^L$ determines an equivalence of categories $\underline{\mathrm{RHom}}_{\mathcal{C}}(D, C) \to \underline{\mathrm{LHom}}_{\mathcal{C}}(C, D)^{\mathrm{op}}$, which is homotopy inverse to the functor $f \mapsto f^R$ described in Notation 6.1.3.5. 02DC

6.1.4 Adjoints of Isomorphisms

Let $\mathcal{C}$ be a 2-category and let $f : C \to D$ be an isomorphism in $\mathcal{C}$, so that f admits a homotopy inverse $g : D \to C$ (Definition 2.2.8.17). Then the 1-morphism g is both right adjoint and left adjoint to f. More precisely, we have the following: 02DD

Proposition 6.1.4.1. *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let $\eta : \mathrm{id}_C \Rightarrow g \circ f$ be a 2-morphism of $\mathcal{C}$. Assume that either f or g is an isomorphism in $\mathcal{C}$. Then η is the unit of an adjunction (in the sense of Definition 6.1.2.10) if and only if it is an isomorphism in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$.* 02DE

We will give the proof of Proposition 6.1.4.1 at the end of this section.

Corollary 6.1.4.2. *Let $\mathcal{C}$ be a 2-category and let $f : C \to D$ be an isomorphism in $\mathcal{C}$. Then any homotopy inverse to f is both a left adjoint and a right adjoint of f.* 02DF

Proof. Let $g : D \to C$ be a homotopy inverse to f, so that there exists an isomorphism $\eta : \mathrm{id}_C \overset{\sim}{\Longrightarrow} g \circ f$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$. It follows from Proposition 6.1.4.1 that η is the unit of an adjunction, and therefore exhibits g as a right adjoint to f. A similar argument shows that g is left adjoint to f. □

02DG **Remark 6.1.4.3.** Let $\mathcal{C}$ be a 2-category and let $f : C \to D$ be a 1-morphism of $\mathcal{C}$. By definition, f is an isomorphism if and only if there exists a 1-morphism $g : D \to C$ together with isomorphisms

$$\eta : \mathrm{id}_C \overset{\sim}{\Longrightarrow} g \circ f \qquad \epsilon : f \circ g \overset{\sim}{\Longrightarrow} \mathrm{id}_D$$

in the categories $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, D)$, respectively. The main content of Proposition 6.1.4.1 is that, if such isomorphisms exist, then we can choose η and ϵ to be *compatible* in the sense that they satisfy conditions $(Z1)$ and $(Z2)$ of Definition 6.1.1.1. Note that in this case, η is determined by ϵ and vice versa (Proposition 6.1.2.9).

02DH **Corollary 6.1.4.4.** *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let (η, ϵ) be an adjunction between f and g. The following conditions are equivalent:*

(1) *The 1-morphism f is an isomorphism in $\mathcal{C}$.*

(2) *The 1-morphism g is an isomorphism in $\mathcal{C}$.*

(3) *The 2-morphisms η and ϵ are isomorphisms in $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, D)$, respectively. In particular, f and g are homotopy inverse to one another.*

Proof. The implication (3) ⇒ (1) and (3) ⇒ (2) are immediate from the definitions, and the reverse implications follow by applying Proposition 6.1.4.1 to $\mathcal{C}$ and the conjugate 2-category $\mathcal{C}^{\mathrm{c}}$. □

02DJ **Warning 6.1.4.5.** In the situation of Corollary 6.1.4.4, it is possible for the unit $\eta : \mathrm{id}_C \Rightarrow g \circ f$ to be an isomorphism while the counit $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ is not, or vice versa (in which case, the 1-morphisms f and g cannot be isomorphisms).

We will deduce Proposition 6.1.4.1 from the following more general result:

02DK **Proposition 6.1.4.6.** *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let $\eta : \mathrm{id}_C \Rightarrow g \circ f$ be a 2-morphism of $\mathcal{C}$ which satisfies the following conditions:*

- *The 2-morphisms*

$$(\mathrm{id}_f \circ \eta) : f \circ \mathrm{id}_C \Rightarrow f \circ (g \circ f) \qquad (\eta \circ \mathrm{id}_g) : \mathrm{id}_C \circ g \Rightarrow (g \circ f) \circ g$$

are isomorphisms.

- *For every object $T \in \mathcal{C}$, the composition functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(T, D) \xrightarrow{g\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, C)$ is fully faithful.*

Then η is the unit of an adjunction (η, ϵ). Moreover, the counit map $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ is an isomorphism.

Proof. Since postcomposition with g induces a fully faithful functor

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(D, D) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(D, C),$$

there is a unique 2-morphism $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ for which the horizontal composition $\mathrm{id}_g \circ \epsilon$ is equal to the composite map

$$g \circ (f \circ g) \xRightarrow{\alpha_{g,f,g}} (g \circ f) \circ g \xRightarrow{(\eta \circ \mathrm{id}_g)^{-1}} \mathrm{id}_C \circ g \xRightarrow{\lambda_g} g \xRightarrow{\rho_g^{-1}} g \otimes \mathrm{id}_D .$$

Moreover, ϵ is an isomorphism and the pair (η, ϵ) automatically satisfies condition $(Z2)$ of Definition 6.1.1.1. Let β denote the composition

$$f \underset{\sim}{\xRightarrow{\rho_f^{-1}}} f \circ \mathrm{id}_C \xRightarrow{\mathrm{id}_f \circ \eta} f \circ (g \circ f) \underset{\sim}{\xRightarrow{\alpha_{f,g,f}}} (f \circ g) \circ f \xRightarrow{\epsilon \circ \mathrm{id}_f} \mathrm{id}_D \circ f \underset{\sim}{\xRightarrow{\lambda_f}} f$$

Since ϵ and $\mathrm{id}_f \circ \eta$ are isomorphisms, it follows that β is an isomorphism. Applying Corollary 6.1.2.8, we see that $\beta^2 = \beta$, so that $\beta = \mathrm{id}_f$. $\square$

Proof of Proposition 6.1.4.1. Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms in $\mathcal{C}$, and assume that g is an isomorphism (the case where f is an isomorphism can be treated by applying a similar argument in the opposite 2-category $\mathcal{C}^{\mathrm{op}}$). Suppose first that $\eta : \mathrm{id}_C \Rightarrow g \circ f$ is an isomorphism in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$. It follows that the horizontal compositions

$$(\mathrm{id}_f \circ \eta) : f \Rightarrow f \circ (g \circ f) \qquad (\eta \circ \mathrm{id}_g) : g \Rightarrow (g \circ f) \circ g$$

are isomorphisms in $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, D)$ and $\underline{\mathrm{Hom}}_{\mathcal{D}}(D, C)$, respectively. For each object $T \in \mathcal{C}$, our assumption that g is an isomorphism guarantees that the composition functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(T, D) \xrightarrow{g\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, C)$ is an equivalence of categories, and therefore fully faithful. Invoking the criterion of Proposition 6.1.4.7, we conclude that η is the unit of an adjunction.

We now prove the converse. Suppose that the 2-morphism $\eta : \mathrm{id}_C \Rightarrow g \circ f$ is the unit of an adjunction. Our assumption that g is an isomorphism guarantees that we can choose a 1-morphism $f' : C \to D$ and an isomorphism $\eta' : \mathrm{id}_C \xRightarrow{\sim} g \circ f'$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$. It follows from the first part of the proof that η' is the unit of an adjunction. Applying Corollary 6.1.3.7, we deduce that there is a unique isomorphism $\beta : f \xRightarrow{\sim} f'$ for which η' is equal to the composition

$$\mathrm{id}_C \xRightarrow{\eta} g \circ f \xRightarrow{\mathrm{id}_g \circ \beta} g \circ f'.$$

Since η' and $(\mathrm{id}_g \circ \beta)$ are isomorphisms in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$, it follows that η is also an isomorphism. □

We close this section by proving a converse of Proposition 6.1.4.6, which characterizes adjunctions (η, ϵ) for which the counit ϵ is an isomorphism.

02DL **Proposition 6.1.4.7.** *Let $\mathcal{C}$ be a 2-category, let $f : C \to D$ and $g : D \to C$ be 1-morphisms of $\mathcal{C}$, and let (η, ϵ) be an adjunction between f and g. The following conditions are equivalent:*

(1) *The 2-morphism $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ is an isomorphism.*

(1′) *The 1-morphism $f \circ g$ is an isomorphism.*

(2) *The 2-morphisms*

$$(\mathrm{id}_f \circ \eta) : f \circ \mathrm{id}_C \Rightarrow f \circ (g \circ f) \qquad (\eta \circ \mathrm{id}_g) : \mathrm{id}_C \circ g \Rightarrow (g \circ f) \circ g$$

are isomorphisms. Moreover, for every object $T \in \mathcal{C}$, the functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(T, C) \xrightarrow{f\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, D)$ is essentially surjective.

(2′) *The 2-morphism $\eta \circ \mathrm{id}_g : \mathrm{id}_C \circ g \Rightarrow (g \circ f) \circ g$ is an isomorphism. Moreover, for every object $T \in \mathcal{C}$, the composite functor*

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(T, D) \xrightarrow{g\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, C) \xrightarrow{f\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, D)$$

is essentially surjective.

(3) *For every object $T \in \mathcal{C}$, the functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(T, D) \xrightarrow{g\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, C)$ is fully faithful.*

(3′) *The functor $\underline{\mathrm{Hom}}_{\mathcal{D}}(D, D) \xrightarrow{g\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(D, C)$ is fully faithful.*

Proof. We first show that (1) and (1′) are equivalent. If ϵ is an isomorphism, then $f \circ g$ is isomorphic to id_D (as an object of the category $\underline{\mathrm{Hom}}_{\mathcal{D}}(D, D)$) and is therefore an isomorphism of $\mathcal{C}$ (Remark 2.2.8.23). Conversely, suppose that $f \circ g$ is an isomorphism. Then it is invertible when viewed as an object of the monoidal category $\underline{\mathrm{End}}_{\mathcal{C}}(D)$. Since (η, ϵ) is an adjunction, we can regard $f \circ g$ as a coalgebra object of $\underline{\mathrm{End}}_{\mathcal{C}}(D)$ with counit ϵ (see Remark [?]). Applying Proposition 2.1.5.23 (to the monoidal category $\underline{\mathrm{End}}_{\mathcal{C}}(D)^{\mathrm{op}}$), we deduce that ϵ is an isomorphism.

We now show that (1) implies (2). Assume that $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ is an isomorphism. Axiom $(Z1)$ of Definition 6.1.1.1 guarantees that the composition

$$f \overset{\sim}{\Longrightarrow} f \circ \mathrm{id}_C \xRightarrow{\mathrm{id}_f \circ \eta} f \circ (g \circ f) \overset{\sim}{\Longrightarrow} (f \circ g) \circ f \xRightarrow[\sim]{\epsilon \circ \mathrm{id}_f} \mathrm{id}_D \circ f \overset{\sim}{\Longrightarrow} f$$

is equal to the identity 2-morphism id_f, which proves that the horizontal composition $\mathrm{id}_f \circ \eta$ is an isomorphism in $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, D)$. Similarly, it follows from axiom $(Z2)$ of Definition

6.1.1.1 that the horizontal composition $\eta \circ \mathrm{id}_g$ is an isomorphism in $\underline{\mathrm{Hom}}_{\mathcal{C}}(D, C)$. For every 1-morphism $d : T \to D$ in $\mathcal{C}$, the map

$$f \circ (g \circ d) \xRightarrow[\sim]{\alpha_{f,g,d}} (f \circ g) \circ d \xRightarrow[\sim]{\epsilon \circ \mathrm{id}_d} \mathrm{id}_D \circ d \xRightarrow[\sim]{\lambda_d} d$$

is an isomorphism, so that d belongs to the essential image of the functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(T, C) \xrightarrow{f\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, D)$.

We now show that (2) implies (2′). Let $d : T \to D$ be a 1-morphism of $\mathcal{C}$. If the functor $\underline{\mathrm{Hom}}_{\mathcal{C}}(T, C) \xrightarrow{f\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, D)$ is essentially surjective, then d is isomorphic to $f \circ c$ for some 1-morphism $c : T \to C$ of $\mathcal{C}$. If $\mathrm{id}_f \circ \eta$ is an isomorphism, then the chain of isomorphisms

$$f \circ c \overset{\sim}{\Longrightarrow} (f \circ \mathrm{id}_C) \circ c \xRightarrow{(\mathrm{id}_f \circ \eta) \circ \mathrm{id}_c} (f \circ (g \circ f)) \circ c \overset{\sim}{\Longrightarrow} f \circ ((g \circ f) \circ c) \overset{\sim}{\Longrightarrow} f \circ (g \circ (f \circ c))$$

shows that d belongs to the essential image of the composite functor

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(T, D) \xrightarrow{g\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, C) \xrightarrow{f\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(T, D).$$

We next show that (2′) implies (3). Fix an object $T \in \mathcal{C}$ and a pair of 1-morphisms $d, d' : T \to D$; we wish to show that the composition map

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(d', d) \to \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(g \circ d', g \circ d)$$

is a bijection. By virtue of assumption (2′), we may assume that $d' = f \circ c$, where $c : T \to C$ is a 1-morphism of the form $g \circ d''$. By virtue of Proposition 6.1.2.9, the composition

$$\begin{array}{rcl} \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, d) & \to & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(g \circ (f \circ c), g \circ d) \\ & \simeq & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}((g \circ f) \circ c, g \circ d) \\ & \xrightarrow{\eta \circ \mathrm{id}_c} & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(\mathrm{id}_C \circ c, g \circ d) \\ & \simeq & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ d) \end{array}$$

is a bijection. It will therefore suffice to show that the 2-morphism $(\eta \circ \mathrm{id}_c) : \mathrm{id}_C \circ c \Rightarrow (g \circ f) \circ c$ is an isomorphism. This follows from assumption (2′), since $(\eta \circ \mathrm{id}_c)$ can be rewritten as a composition

$$\mathrm{id}_C \circ (g \circ d'') \overset{\sim}{\Longrightarrow} (\mathrm{id}_C \circ g) \circ d'' \xRightarrow{(\eta \circ \mathrm{id}_g) \circ \mathrm{id}_{d''}} ((g \circ f) \circ g) \circ d'' \simeq (g \circ f) \circ (g \circ d'').$$

The implication (3) ⇒ (3′) is clear. We will complete the proof by showing that (3′) implies (1). Assume that (3′) is satisfied; we wish to show that the 2-morphism $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ is an isomorphism. To prove this, it will suffice to show that for every 1-morphism $u : D \to D$, vertical precomposition with ϵ induces a bijection $\mathrm{Hom}_{\underline{\mathrm{End}}_{\mathcal{C}}(D)}(\mathrm{id}_D, u) \to \mathrm{Hom}_{\underline{\mathrm{End}}_{\mathcal{C}}(D)}(f \circ g, u)$. We now observe that this map fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\mathrm{End}}_{\mathcal{C}}(D)}(\mathrm{id}_D, u) \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}_g \circ"] & \mathrm{Hom}_{\underline{\mathrm{End}}_{\mathcal{C}}(D)}(f \circ g, u) \arrow[d] \\
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,C)}(g \circ \mathrm{id}_D, g \circ u) \arrow[r, "\sim"] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(D,C)}(g, g \circ u)
\end{tikzcd}$$

where the bottom horizontal map is induced by the right unit constraint $\rho_g : g \circ \mathrm{id}_D \overset{\sim}{\Longrightarrow} g$, the right vertical map is given by the formation of right adjuncts with respect to η (and is therefore bijective by virtue of Corollary 6.1.2.6), and the left vertical map is bijective by virtue of assumption (3′). □

6.1.5 Composition of Adjunctions

02DM We now show that the formation of right and left adjoints is compatible with composition of 1-morphisms.

02DN **Construction 6.1.5.1.** Let $\mathcal{C}$ be a 2-category containing objects C, D, and E, together with 1-morphisms

$$f : C \to D \qquad f' : D \to E \qquad g : D \to C \qquad g' : E \to D$$

and 2-morphisms $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\eta' : \mathrm{id}_D \Rightarrow g' \circ f'$. We let $c(\eta, \eta')$ denote the 2-morphism given by the composition

$$\mathrm{id}_C \overset{\eta}{\Longrightarrow} g \circ f \overset{\sim}{\Longrightarrow} g \circ (\mathrm{id}_D \circ f) \overset{\eta'}{\Longrightarrow} g \circ ((g' \circ f') \circ f) \overset{\sim}{\Longrightarrow} g \circ (g' \circ (f' \circ f)) \overset{\sim}{\Longrightarrow} (g \circ g') \circ (f' \circ f),$$

where the unlabeled isomorphisms are given by the unit and associativity constraints of $\mathcal{C}$. We will refer to $c(\eta, \eta')$ as the *contraction of* η *and* η'.

02DP **Remark 6.1.5.2.** In the situation of Construction 6.1.5.1, let $\mathcal{C}^{\mathrm{op}}$ be the opposite of the 2-category $\mathcal{C}$, so that we can identify η and η' with 2-morphisms

$$\eta^{\mathrm{op}} : \mathrm{id}_{C^{\mathrm{op}}} \Rightarrow f^{\mathrm{op}} \circ g^{\mathrm{op}} \qquad \eta'^{\mathrm{op}} : \mathrm{id}_{D^{\mathrm{op}}} \Rightarrow f'^{\mathrm{op}} \circ g'^{\mathrm{op}}.$$

Then the 2-morphism $c(\eta, \eta')^{\mathrm{op}}$ can be identified with the contraction $c(\eta'^{\mathrm{op}}, \eta^{\mathrm{op}})$, formed in the 2-category $\mathcal{C}^{\mathrm{op}}$. In other words, $c(\eta, \eta')$ can also be computed as the composition

$$\mathrm{id}_C \overset{\eta}{\Longrightarrow} g \circ f \overset{\sim}{\Longrightarrow} (g \circ \mathrm{id}_D) \circ f \overset{\eta'}{\Longrightarrow} (g \circ (g' \circ f'))) \circ f \overset{\sim}{\Longrightarrow} ((g \circ g') \circ f') \circ f \overset{\sim}{\Longrightarrow} (g \circ g') \circ (f' \circ f).$$

This follows from the commutativity of the diagram

$$\begin{tikzcd}
& g \circ f \arrow[dl, Rightarrow, "\mathrm{id}_g \circ \lambda_f^{-1}"', "\sim"] \arrow[dr, Rightarrow, "\rho_g^{-1} \circ \mathrm{id}_f", "\sim"'] & \\
g \circ (\mathrm{id}_D \circ f) \arrow[rr, Rightarrow, "\alpha_{g,\mathrm{id}_D,f}", "\sim"'] \arrow[d, Rightarrow, "\mathrm{id}_g \circ (\eta' \circ \mathrm{id}_f)"'] & & (g \circ \mathrm{id}_D) \circ f \arrow[d, Rightarrow, "(\mathrm{id}_g \circ \eta') \circ \mathrm{id}_F"] \\
g \circ ((g' \circ f') \circ f) \arrow[rr, Rightarrow, "\alpha_{g,g' \circ f',f}", "\sim"'] \arrow[d, Rightarrow, "\mathrm{id}_g \circ \alpha_{g',f',f}^{-1}"', "\sim"] & & (g \circ (g' \circ f')) \circ f \arrow[d, Rightarrow, "\alpha_{g,g',f'} \circ \mathrm{id}_{f'}", "\sim"'] \\
g \circ (g' \circ (f' \circ f)) \arrow[dr, Rightarrow, "\alpha_{g,g',f' \circ f}"', "\sim"] & & ((g \circ g') \circ f') \circ f \arrow[dl, Rightarrow, "\alpha_{g \circ g',f',f}^{-1}", "\sim"'] \\
& (g \circ g') \circ (f' \circ f) &
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(C, C)$. Here the upper triangle commutes by virtue of the triangle identity (Proposition 2.2.1.14), the middle square commutes by the naturality of the associativity constraints of $\mathcal{C}$, and the lower region commutes by virtue of the pentagon identity.

Proposition 6.1.5.3. *Let $\mathcal{C}$ be a 2-category containing objects C, D, and E, together with 1-morphisms* `02DQ`

$$f : C \to D \qquad g : D \to C \qquad f' : D \to E \qquad g' : E \to D$$

and 2*-morphisms* $\eta : \mathrm{id}_C \Rightarrow g \circ f$ *and* $\eta' : \mathrm{id}_D \Rightarrow g' \circ f'$*. Let* T *be another object of* $\mathcal{C}$ *equipped with* 1*-morphisms* $c : T \to C$ *and* $e : T \to E$*. Then the diagram*

02DR

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,E)}((f' \circ f) \circ c, e) \arrow[r, "\alpha_{f',f,c}", "\sim"'] \arrow[dd] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,E)}(f' \circ (f \circ c), e) \arrow[d] \\
 & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, g' \circ e) \arrow[d] \\
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, (g \circ g') \circ e) \arrow[r, "\alpha_{g,g',e}", "\sim"'] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ (g' \circ e))
\end{tikzcd} \tag{6.1}$$

is commutative. Here the right vertical maps are given by the formation of right adjuncts with respect to η *and* η' *(in the sense of Construction 6.1.2.1), while the left vertical map is given by the formation of right adjuncts with respect to the contraction* $c(\eta, \eta')$ *of Construction 6.1.5.1.*

Proof. Fix a 2-morphism $\beta : (f' \circ f) \circ c \Rightarrow e$ in $\mathcal{C}$. Clockwise and counterclockwise composition around the outside of the diagram (6.1) determines two elements of $\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, (g \circ g') \circ e)$, and we wish to prove that these two elements are the same. Unwinding the definitions, we see that these elements can be obtained as the vertical composition of $c \xRightarrow[\sim]{\lambda_c^{-1}} \mathrm{id}_C \circ c \xRightarrow{\eta \circ \mathrm{id}_c} (g \circ f) \circ c$ with 2-morphisms given by clockwise and counterclockwise composition around the outside of the diagram

$$\begin{tikzcd}
(gf)c \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\lambda_f^{-1}"', "\sim"] & g(fc) \arrow[d, Rightarrow, "\lambda_f^{-1}"', "\sim"] \arrow[dr, Rightarrow, "\lambda_{fc}^{-1}", "\sim"'] & & \\
(g(\mathrm{id}_D\, f))c \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\eta'"] & g((\mathrm{id}_D\, f)c) \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\eta'"] & g(\mathrm{id}_D(fc)) \arrow[d, Rightarrow, "\eta'"] & \\
(g((g'f')f))c \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\sim"] & g(((g'f')f)c) \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\sim"] & g((g'f')(fc)) \arrow[r, Rightarrow, "\sim"] & g(g'(f'(fc))) \arrow[dl, Rightarrow, "\sim"] \\
(g(g'(f'f)))c \arrow[r, Rightarrow, "\sim"] \arrow[d, Rightarrow, "\sim"] & g((g'(f'f))c) \arrow[r, Rightarrow, "\sim"] & g(g'((f'f)c)) \arrow[r, Rightarrow, "\beta"] \arrow[d, Rightarrow, "\sim"] & g(g'e) \arrow[d, Rightarrow] \\
((gg')(f'f))c \arrow[rr, Rightarrow, "\sim"] & & (gg')((f'f)c) \arrow[r, Rightarrow, "\beta"] & (gg')e
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(T, C)$; here denote the composition of 1-morphisms u and v in $\mathcal{C}$ by uv (rather than $u \circ v$) to simplify the notation, and the unlabeled isomorphisms are given by the associativity constraints of $\mathcal{C}$. It will therefore suffice to observe that this diagram is commutative. The commutativity of the pentagonal regions follows from the pentagon identity in $\mathcal{C}$, the commutativity of the triangle from Proposition 2.2.1.16, and the commutativity of each square from the naturality of the associativity constraints of $\mathcal{C}$. □

Corollary 6.1.5.4. *Let $\mathcal{C}$ be a 2-category containing objects C, D, and E, together with 1-morphisms* `02DS`

$$f : C \to D \qquad g : D \to C \qquad f' : D \to E \qquad g' : E \to D$$

and 2-morphisms $\eta : \mathrm{id}_C \Rightarrow g \circ f$ and $\eta' : \mathrm{id}_D \Rightarrow g' \circ f'$. If η and η' are units of adjunctions, then the contraction $c(\eta, \eta') : \mathrm{id}_C \Rightarrow (g \circ g') \circ (f' \circ f)$ is also the unit of an adjunction.

Proof. Combine Proposition 6.1.5.3 with the criterion of Proposition 6.1.2.9. □

From Corollary 6.1.5.4, we can extract the following slightly less precise consequence:

02DT **Corollary 6.1.5.5.** *Let $\mathcal{C}$ be a 2-category containing objects C, D, and E, together with 1-morphisms*

$$f : C \to D \qquad g : D \to C \qquad f' : D \to E \qquad g' : E \to D.$$

If f is left adjoint to g and f' is left adjoint to g', then $f' \circ f$ is left adjoint to $g \circ g'$.

02DU **Corollary 6.1.5.6.** *Let $\mathcal{C}$ be a 2-category containing 1-morphisms $u : C \to D$ and $v : D \to E$. If u and v admit left adjoints, then $v \circ u$ admits a left adjoint. If u and v admit right adjoints, then $v \circ u$ admits a right adjoint.*

We can also formulate a more precise version of Corollary 6.1.5.4, which explicitly describes the counit of a composite adjunction. For this, we need a variant of Construction 6.1.5.1:

02DV **Construction 6.1.5.7.** Let $\mathcal{C}$ be a 2-category containing objects C, D, and E, together with 1-morphisms

$$f : C \to D \qquad g : D \to C \qquad f' : D \to E \qquad g' : E \to D$$

and 2-morphisms $\epsilon : f \circ g \Rightarrow \mathrm{id}_D$ and $\epsilon' : f' \circ g' \Rightarrow \mathrm{id}_E$ be 2-morphisms of $\mathcal{C}$. We let $c(\epsilon, \epsilon')$ denote the 2-morphism given by the composition

$$(f' \circ f) \circ (g \circ g') \xRightarrow{\sim} f' \circ (f \circ (g \circ g')) \xRightarrow{\sim} f' \circ ((f \circ g) \circ g') \xRightarrow{\epsilon} f' \circ (\mathrm{id}_D \circ g') \xRightarrow{\sim} f' \circ g' \xRightarrow{\epsilon'} \mathrm{id}_E$$

We will refer to $c(\epsilon, \epsilon')$ as the *contraction of ϵ and ϵ'*.

02DW **Remark 6.1.5.8.** In the situation of Construction 6.1.5.7, we can identify ϵ and ϵ' with 2-morphisms

$$\epsilon^{\mathrm{c}} : \mathrm{id}_{D^{\mathrm{c}}} \Rightarrow f^{\mathrm{c}} \circ g^{\mathrm{c}} \qquad {\epsilon'}^{\mathrm{c}} : \mathrm{id}_{E^{\mathrm{c}}} \Rightarrow {f'}^{\mathrm{c}} \circ {g'}^{\mathrm{c}}$$

in the conjugate 2-category $\mathcal{C}^{\mathrm{c}}$ (Construction 2.2.3.4). The contraction $c(\epsilon, \epsilon')$ can then be described as the conjugate of the 2-morphism $c({\epsilon'}^{\mathrm{c}}, \epsilon^{\mathrm{c}})$ obtained by applying Construction 6.1.5.1 to the 2-category $\mathcal{C}^{\mathrm{c}}$.

02DX **Corollary 6.1.5.9.** *Let $\mathcal{C}$ be a 2-category containing objects C, D, and E, together with 1-morphisms*

$$f : C \to D \qquad g : D \to C \qquad f' : D \to E \qquad g' : E \to D.$$

Let (η, ϵ) be an adjunction between f and g, and let (η', ϵ') be an adjunction between f' and g'. Then the pair $(c(\eta, \eta'), c(\epsilon, \epsilon'))$ is an adjunction between $f' \circ f$ and $g \circ g'$. Here $c(\eta, \eta')$ is the contraction of η with η' (in the sense of Construction 6.1.5.1), and $c(\epsilon, \epsilon')$ is the contraction of ϵ with ϵ' (in the sense of Construction 6.1.5.7).

Proof. By virtue of Proposition 6.1.5.3 and Corollary 6.1.2.6, it will suffice to show that for every object $T \in \mathcal{C}$ equipped with 1-morphisms $c : T \to C$ and $e : T \to E$, the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,E)}((f' \circ f) \circ c, e) \arrow[r, "\alpha_{f',f,c}", "\sim"'] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,E)}(f' \circ (f \circ c), e) \\
& \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,D)}(f \circ c, g' \circ e) \arrow[u] \\
\mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, (g \circ g') \circ e) \arrow[uu] \arrow[r, "\alpha_{g,g',e}", "\sim"'] & \mathrm{Hom}_{\underline{\mathrm{Hom}}_{\mathcal{C}}(T,C)}(c, g \circ (g' \circ e)) \arrow[u]
\end{tikzcd}$$

commutes, where the right vertical maps are given by the formation of left adjuncts with respect to ϵ and ϵ', and the left vertical map is given by the formation of left adjuncts with respect to the contraction $c(\epsilon, \epsilon')$ of Construction 6.1.5.7. This follows by applying Proposition 6.1.5.3 to the conjugate 2-category $\mathcal{C}^{\mathrm{c}}$. ☐

6.1.6 Duality in Monoidal Categories

We now specialize the theory of adjunctions to the setting of 2-categories of the form 02DY $B\mathcal{C}$ (Example 2.2.2.5), where $\mathcal{C}$ is a monoidal category. Throughout this section, we write $\mathbf{1}$ for the unit object of a monoidal category $\mathcal{C}$.

Definition 6.1.6.1. Let $\mathcal{C}$ be a monoidal category containing objects X and Y. A *duality* 02DZ *datum* is a pair $(\mathrm{coev}, \mathrm{ev})$, where $\mathrm{coev} : \mathbf{1} \to Y \otimes X$ and $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ are morphisms of $\mathcal{C}$ satisfying the following compatibility conditions:

($Z1$) The composition

$$X \xrightarrow[\sim]{\rho_X^{-1}} X \otimes \mathbf{1} \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}} X \otimes (Y \otimes X) \xrightarrow[\sim]{\alpha_{X,Y,X}} (X \otimes Y) \otimes X \xrightarrow{\mathrm{ev} \otimes \mathrm{id}_X} \mathbf{1} \otimes X \xrightarrow[\sim]{\lambda_X} X$$

is equal to the identity morphism id_X. Here the isomorphism $\alpha_{X,Y,X}$ is the associativity constraint for the monoidal category $\mathcal{C}$, and the isomorphisms λ_X and ρ_X are the left and right unit constraints of Construction 2.1.2.17.

($Z2$) The composition

$$Y \xrightarrow[\sim]{\lambda_Y^{-1}} \mathbf{1} \otimes Y \xrightarrow{\mathrm{coev} \otimes \mathrm{id}_Y} (Y \otimes X) \otimes Y \xrightarrow[\sim]{\alpha_{Y,X,Y}^{-1}} Y \otimes (X \otimes Y) \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}} Y \otimes \mathbf{1} \xrightarrow[\sim]{\rho_Y} Y$$

is equal to the identity morphism id_Y.

If these conditions are satisfied, then we will refer to coev as the *coevaluation morphism* of the duality datum $(\mathrm{coev}, \mathrm{ev})$, and to ev as the *evaluation morphism* of the duality datum $(\mathrm{coev}, \mathrm{ev})$. In this case, we say that the pair $(\mathrm{coev}, \mathrm{ev})$ *exhibits X as a left dual of Y*, also that it *exhibits Y as a right dual of X*.

02E0 **Remark 6.1.6.2** (Duals as Adjoints)**.** Let $\mathcal{C}$ be a monoidal category containing objects X and Y, which we regard as 1-morphisms of the 2-category $B\mathcal{C}$ described in Example 2.2.2.5. Suppose we are given a pair of morphisms

$$\mathrm{coev} : \mathbf{1} \to Y \otimes X \qquad \mathrm{ev} : X \otimes Y \to \mathbf{1}$$

in $\mathcal{C}$, which we identify with 2-morphisms of $B\mathcal{C}$. Then the pair $(\mathrm{coev}, \mathrm{ev})$ is a duality datum in the monoidal category $\mathcal{C}$ (in the sense of Definition 6.1.6.1) if and only if it is an adjunction in the 2-category $B\mathcal{C}$ (in the sense of Definition 6.1.1.1).

02E1 **Remark 6.1.6.3** (Adjoints as Duals)**.** Let $\mathcal{C}$ be a 2-category, let X be an object of $\mathcal{C}$, let $f, g : X \to X$ be 1-morphisms of $\mathcal{C}$, and let $\eta : \mathrm{id}_X \Rightarrow g \circ f$ and $\epsilon : f \circ g \Rightarrow \mathrm{id}_X$ be 2-morphisms of $\mathcal{C}$. Then the pair (η, ϵ) is an adjunction in the 2-category $\mathcal{C}$ (in the sense of Definition 6.1.1.1) if and only if it is a duality datum in the monoidal category $\underline{\mathrm{End}}_{\mathcal{C}}(X)$ of Remark 2.2.1.7.

02E2 **Remark 6.1.6.4.** Let $\mathcal{C}$ be a monoidal category containing objects X and Y and morphisms

$$\mathrm{coev} : \mathbf{1} \to Y \otimes X \qquad \mathrm{ev} : X \otimes Y \to \mathbf{1}.$$

Then:

- The pair $(\mathrm{coev}, \mathrm{ev})$ is a duality datum in the monoidal category $\mathcal{C}$ if and only if it is a duality datum in the reverse monoidal category $\mathcal{C}^{\mathrm{rev}}$ of Example 2.1.3.5. Note that passage to the reverse monoidal category reverses the roles of X and Y: if X is the left dual of Y in the monoidal category $\mathcal{C}$, then it is the right dual of Y in the monoidal category $\mathcal{C}^{\mathrm{rev}}$ (and vice-versa).

- The pair $(\mathrm{coev}, \mathrm{ev})$ is a duality datum in $\mathcal{C}$ if and only if the pair $(\mathrm{ev}^{\mathrm{op}}, \mathrm{coev}^{\mathrm{op}})$ is a duality datum in the opposite monoidal category $\mathcal{C}^{\mathrm{op}}$ (see Example 2.1.3.4). Note that passage to the opposite monoidal category reverses the roles of evaluation and coevaluation: $\mathrm{ev}^{\mathrm{op}}$ is the coevaluation morphism for the duality datum $(\mathrm{ev}^{\mathrm{op}}, \mathrm{coev}^{\mathrm{op}})$, while $\mathrm{coev}^{\mathrm{op}}$ is the evaluation morphism. Similarly, if X is the left dual of Y in the monoidal category $\mathcal{C}$, then it is the right dual of Y in the opposite monoidal category $\mathcal{C}^{\mathrm{op}}$ (and vice-versa).

02E3 **Proposition 6.1.6.5.** *Let $\mathcal{C}$ be a monoidal category and let $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:*

(1) *For every pair of objects $C, D \in \mathcal{C}$, the composite map*

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(C, Y \otimes D) &\rightarrow \operatorname{Hom}_{\mathcal{C}}(X \otimes C, X \otimes (Y \otimes D)) \\ &\simeq \operatorname{Hom}_{\mathcal{C}}(X \otimes C, (X \otimes Y) \otimes D) \\ &\xrightarrow{\mathrm{ev}} \operatorname{Hom}_{\mathcal{C}}(X \otimes C, \mathbf{1} \otimes D) \\ &\simeq \operatorname{Hom}_{\mathcal{C}}(X \otimes C, D)\end{aligned}$$

is a bijection.

(2) *For every pair of objects $C, D \in \mathcal{C}$, the composite map*

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(C, D \otimes X) &\rightarrow \operatorname{Hom}_{\mathcal{C}}(C \otimes Y, (D \otimes X) \otimes Y) \\ &\simeq \operatorname{Hom}_{\mathcal{C}}(C \otimes Y, D \otimes (X \otimes Y)) \\ &\xrightarrow{\mathrm{ev}} \operatorname{Hom}_{\mathcal{C}}(C \otimes Y, D \otimes \mathbf{1}) \\ &\simeq \operatorname{Hom}_{\mathcal{C}}(C \otimes Y, D)\end{aligned}$$

is a bijection.

(3) *There exists a morphism* $\mathrm{coev} : \mathbf{1} \rightarrow Y \otimes X$ *for which the pair* $(\mathrm{coev}, \mathrm{ev})$ *is a duality datum, in the sense of Definition 6.1.6.1.*

Moreover, if these conditions are satisfied, then the morphism $\mathrm{coev} : \mathbf{1} \rightarrow Y \otimes X$ *is unique.*

Proof. Apply Proposition 6.1.2.13 to the 2-category $B\mathcal{C}$ of Example 2.2.2.5. □

Definition 6.1.6.6. Let $\mathcal{C}$ be a monoidal category. We will say that a morphism $\mathrm{ev} : X \otimes Y \rightarrow \mathbf{1}$ in $\mathcal{C}$ is a *duality datum* if it satisfies the equivalent conditions of Proposition 6.1.6.5: that is, if there exists a morphism $\mathrm{coev} : \mathbf{1} \rightarrow Y \otimes X$ for which the pair $(\mathrm{coev}, \mathrm{ev})$ is a duality datum in the sense of Definition 6.1.6.1. 02E4

Applying Proposition 6.1.6.5 to the opposite monoidal category $\mathcal{C}^{\mathrm{op}}$, we obtain the following:

Variant 6.1.6.7. Let $\mathcal{C}$ be a monoidal category and let $\mathrm{coev} : \mathbf{1} \rightarrow Y \otimes X$ be a morphism of $\mathcal{C}$. The following conditions are equivalent: 02E5

(1) For every pair of objects $C, D \in \mathcal{C}$, the composite map

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(X \otimes C, D) &\rightarrow \operatorname{Hom}_{\mathcal{C}}(Y \otimes (X \otimes C), Y \otimes D) \\ &\simeq \operatorname{Hom}_{\mathcal{C}}((Y \otimes X) \otimes C, Y \otimes D) \\ &\xrightarrow{\mathrm{coev}} \operatorname{Hom}_{\mathcal{C}}(\mathbf{1} \otimes C, Y \otimes D) \\ &\simeq \operatorname{Hom}_{\mathcal{C}}(C, Y \otimes D)\end{aligned}$$

is a bijection.

(2) For every pair of objects $C, D \in \mathcal{C}$, the composite map

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(C \otimes Y, D) &\to \mathrm{Hom}_{\mathcal{C}}((C \otimes Y) \otimes X, D \otimes X) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(C \otimes (Y \otimes X), D \otimes X) \\
&\xrightarrow{\mathrm{coev}} \mathrm{Hom}_{\mathcal{C}}(C \otimes \mathbf{1}, D \otimes X) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(C, D \otimes X)
\end{aligned}$$

is a bijection.

(3) There exists a morphism $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ for which the pair $(\mathrm{coev}, \mathrm{ev})$ is a duality datum, in the sense of Definition 6.1.6.1.

Moreover, if these conditions are satisfied, then the morphism $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ is unique.

02E6 **Definition 6.1.6.8.** Let $\mathcal{C}$ be a monoidal category. We will say that a morphism $\mathrm{coev} : \mathbf{1} \to Y \otimes X$ in $\mathcal{C}$ is a *duality datum* if it satisfies the equivalent conditions of Variant 6.1.6.7: that is, if there exists a morphism $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ for which the pair $(\mathrm{coev}, \mathrm{ev})$ is a duality datum in the sense of Definition 6.1.6.1.

02E7 **Definition 6.1.6.9.** Let $\mathcal{C}$ be a monoidal category. Then:

- We say that an object $X \in \mathcal{C}$ is *right dualizable* if there exists an object $Y \in \mathcal{C}$ and a duality datum $\mathrm{ev} : X \otimes Y \to \mathbf{1}$. In this case, we will also say that Y *is a right dual of* X, or that the morphism ev *exhibits* Y *as a right dual of* X.

- We say that an object $Y \in \mathcal{C}$ is *left dualizable* if there exists an object $X \in \mathcal{C}$ and a duality datum $\mathrm{ev} : X \otimes Y \to \mathbf{1}$. In this case, we will also say that X *is a left dual of* Y, or that the morphism ev *exhibits* X *as a left dual of* Y.

02E8 **Example 6.1.6.10.** Let $\mathcal{C}$ be a monoidal category. We say that an object $X \in \mathcal{C}$ is *invertible* if there exists an object $Y \in \mathcal{C}$ such that the tensor products $Y \otimes X$ and $X \otimes Y$ are isomorphic to the unit object $\mathbf{1}$. If this condition is satisfied, then any choice of isomorphism $\mathbf{1} \simeq Y \otimes X$ is a duality datum (this is a special case of Proposition 6.1.4.1). In particular, the object Y is a right dual of X. Similarly, Y is a left dual of X.

02E9 **Exercise 6.1.6.11.** Let $\mathcal{C}$ be a category which admits finite products, and regard $\mathcal{C}$ as equipped with the monoidal structure given by cartesian products (Example 2.1.3.2). Show that an object $X \in \mathcal{C}$ is left (or right) dualizable if and only if it is isomorphic to the final object $\mathbf{1}$.

02EA **Exercise 6.1.6.12.** Let k be a field and let Vect_k denote the category of vector spaces over k, equipped with the monoidal structure described in Example 2.1.3.1. Show that an object $V \in \mathrm{Vect}_k$ is left (or right) dualizable if and only if it is finite-dimensional as a vector space over k.

It is instructive to contrast Definition 6.1.6.6 with a slightly more general notion of duality.

Definition 6.1.6.13. Let $\mathcal{C}$ be a monoidal category containing objects X and Y. We will say that a morphism $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ *exhibits Y as a weak right dual of X* if, for every object $W \in \mathcal{C}$, the composite map 02EB

$$\mathrm{Hom}_{\mathcal{C}}(W, Y) \to \mathrm{Hom}_{\mathcal{C}}(X \otimes W, X \otimes Y) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathcal{C}}(X \otimes W, \mathbf{1})$$

is bijective. We say that ev *exhibits X as a weak left dual of Y* if, for every object $Z \in \mathcal{C}$, the composite map

$$\mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z \otimes Y, X \otimes Y) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathcal{C}}(Z \otimes Y, \mathbf{1})$$

is bijective.

Remark 6.1.6.14. Let $\mathcal{C}$ be a monoidal category and let X be an object of $\mathcal{C}$. It follows immediately from the definition that if there exists a morphism $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ which exhibits Y as a weak right dual of X, then the pair (Y, ev) is unique up to isomorphism and depends functorially on X. To emphasize this dependence we will sometimes denote the object Y by $X^{\vee}$ and abuse terminology by referring to it as *the* weak right dual of X. 02EC

Similarly, if Y is a fixed object of $\mathcal{C}$ and there exists a morphism $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ which exhibits X as a weak left dual of Y, then the pair (X, ev) is uniquely determined up to isomorphism and depends functorially on Y. We will emphasize this dependence by denoting the object X by ${}^{\vee}Y$ and referring to it as *the* weak left dual of Y.

Proposition 6.1.6.15. *Let $\mathcal{C}$ be a monoidal category and let* $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ *be a morphism of $\mathcal{C}$. Then:* 02ED

(1) *If the morphism* ev *exhibits Y as a right dual of X (Definition 6.1.6.6), then it exhibits Y as a weak right dual of X (Definition 6.1.6.13). The converse holds if X is right dualizable.*

(2) *If the morphism* ev *exhibits X as a left dual of Y, then it exhibits X as a weak left dual of Y. The converse holds if Y is left dualizable.*

Proof. We will prove (1); the proof of (2) is similar. If $\mathrm{ev} : X \otimes Y \to \mathbf{1}$ is a duality datum, then it exhibits Y as a weak right dual of X by virtue of Variant 6.1.3.2 (applied to the 2-category $B\mathcal{C}$). Conversely, suppose that ev exhibits Y as a weak right dual of X. If there exists another object $Y' \in \mathcal{C}$ and a duality datum $\mathrm{ev}' : X \otimes Y' \to \mathbf{1}$, then the universal property of Y guarantees that there is a unique morphism $u : Y' \to Y$ for which ev' is equal to the composite map $X \otimes Y' \xrightarrow{\mathrm{id}_X \otimes u} X \otimes Y \xrightarrow{\mathrm{ev}} \mathbf{1}$. Since ev' exhibits Y' as a weak right dual of X, the morphism u must be an isomorphism, so that the morphism ev is also a duality datum. □

02EE **Warning 6.1.6.16.** In the situation of Proposition 6.1.6.15, it is possible for an object $X \in \mathcal{C}$ to admit a weak right dual which is not a right dual. For example, let $\mathcal{C} = \mathrm{Vect}_k$ be the category of vector spaces over a field k, equipped with the monoidal structure of Example 2.1.3.1. Let V be a vector space over k and let $V^* = \mathrm{Hom}_k(V, k)$ be its dual space. Then the evaluation map

$$\mathrm{ev} : V \otimes_k V^* \to k \qquad v \otimes \lambda \mapsto \lambda(v)$$

exhibits V^* as a weak (right) dual of V (in the sense of Definition 6.1.6.13). However, it is a duality datum only when V is finite-dimensional over k (Exercise 6.1.6.12).

02EF **Remark 6.1.6.17.** Let $\mathcal{C}$ be a monoidal category containing objects X and Y. If both X and Y are right dualizable, then the tensor product $X \otimes Y$ is also right dualizable; moreover we have a canonical isomorphism $(X \otimes Y)^{\vee} \simeq Y^{\vee} \otimes X^{\vee}$ (see Corollary 6.1.5.4 for a more precise statement). Similarly, if both X and Y are left dualizable, then the tensor product $X \otimes Y$ is left dualizable, and there is a canonical isomorphism ${}^{\vee}(X \otimes Y) \simeq {}^{\vee}Y \otimes {}^{\vee}X$.

02EG **Exercise 6.1.6.18.** Let $\mathcal{C}$ be a monoidal category containing objects X and Y. Show that, if X is weakly right dualizable and Y is right dualizable, then the tensor product $X \otimes Y$ is weakly right dualizable (and that there is a canonical isomorphism $(X \otimes Y)^{\vee} \simeq Y^{\vee} \otimes X^{\vee}$).

6.2 Adjoint Functors Between ∞-Categories

02EH 6.2.1 Adjunctions of ∞-Categories

02EJ We now adapt Definition 6.1.0.2 to the setting of ∞-categories.

02EK **Definition 6.2.1.1.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. We will say that a pair of natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ are *compatible up to homotopy* if the following conditions are satisfied:

$(Z1)$ The identity isomorphism $\mathrm{id}_F : F \to F$ is a composition of the natural transformations

$$F = F \circ \mathrm{id}_{\mathcal{C}} \xrightarrow{\mathrm{id}_F \circ \eta} F \circ G \circ F \qquad F \circ G \circ F \xrightarrow{\epsilon \circ \mathrm{id}_F} \mathrm{id}_{\mathcal{D}} \circ F = F$$

in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, in the sense of Definition 1.4.4.1.

$(Z2)$ The identity isomorphism $\mathrm{id}_G : G \to G$ is a composition of the natural transformations

$$G = \mathrm{id}_{\mathcal{C}} \circ G \xrightarrow{\eta \circ \mathrm{id}_G} G \circ F \circ G \qquad G \circ F \circ G \xrightarrow{\mathrm{id}_G \circ \epsilon} G \circ \mathrm{id}_{\mathcal{D}} = G$$

in the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

We say that a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ *is the unit of an adjunction* if there exists a natural transformation $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ which is compatible with η up to homotopy. We say that a natural transformation $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ *is the counit of an adjunction* if there exists a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ which is compatible with ϵ up to homotopy.

Definition 6.2.1.2. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. We say that F *is a left adjoint of* G, or that G *is a right adjoint of* F, if there exists a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ which is the unit of an adjunction between F and G. In this case, we say that η *exhibits* F *as a left adjoint of* G and also that it *exhibits* G *as a right adjoint of* F. Equivalently, F is a left adjoint of G if there exists a natural transformation $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ which is the counit of an adjunction between F and G; in this case, we say that ϵ *exhibits* F *as a left adjoint of* G and also that it *exhibits* G *as a right adjoint of* F. `02EL`

Notation 6.2.1.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. We say that F *is a left adjoint*, or that F *admits a right adjoint*, if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ which is right adjoint to F. We let $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ which are left adjoints. `02EM`

Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories. We say that G *is a right adjoint*, or that G *admits a left adjoint*, if there exists a functor $F : \mathcal{C} \to \mathcal{D}$ which is left adjoint to G. We let $\mathrm{RFun}(\mathcal{D}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ spanned by those functors $G : \mathcal{D} \to \mathcal{C}$ which are right adjoints.

Remark 6.2.1.4. Let $\mathrm{h}_2\mathbf{QCat}$ be the homotopy 2-category of ∞-categories (see Construction 4.5.1.23). Suppose we are given functors of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, which we regard as 1-morphisms in the 2-category $\mathrm{h}_2\mathbf{QCat}$. Let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ be natural transformations and let $[\eta]$ and $[\epsilon]$ denote their homotopy classes, which we regard as 2-morphisms in $\mathrm{h}_2\mathbf{QCat}$. Then η and ϵ are compatible up to homotopy (in the sense of Definition 6.2.1.1) if and only if the pair $([\eta], [\epsilon])$ is an adjunction in the 2-category $\mathrm{h}_2\mathbf{QCat}$ (in the sense of Definition 6.1.1.1). `02EN`

Remark 6.2.1.5. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories, and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ be natural transformations. Axioms $(Z1)$ and $(Z2)$ of Definition 6.2.1.1 can be restated as follows: `02EP`

$(Z1)$ There exists a 2-simplex σ of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with boundary as indicated in the diagram

$$\begin{tikzcd}
 & F \circ G \circ F \arrow[dr, "\epsilon \circ \mathrm{id}_F"] & \\
F \circ \mathrm{id}_{\mathcal{C}} \arrow[ur, "\mathrm{id}_F \circ \eta"] \arrow[rr, "\mathrm{id}_F"] & & \mathrm{id}_{\mathcal{D}} \circ F.
\end{tikzcd}$$

($Z2$) There exists a 2-simplex τ of the ∞-category $\operatorname{Fun}(\mathcal{D}, \mathcal{C})$ with boundary as indicated in the diagram

$$\begin{tikzcd}
 & G \circ F \circ G \arrow[dr, "\mathrm{id}_G \circ \epsilon"] & \\
\mathrm{id}_{\mathcal{C}} \circ G \arrow[ur, "\eta \circ \mathrm{id}_G"] \arrow[rr, "\mathrm{id}_G"] & & G \circ \mathrm{id}_{\mathcal{D}} .
\end{tikzcd}$$

In this case, we will say that the 2-simplices σ and τ *witness* the axioms ($Z1$) and ($Z2$), respectively.

02EQ **Remark 6.2.1.6.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories, and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be a natural transformation. It follows from Remark 6.2.1.4 that the condition that η is the unit of an adjunction (in the sense of Definition 6.2.1.1) depends only on the homotopy class $[\eta]$, regarded as a morphism in the category $\mathrm{hFun}(\mathcal{C}, \mathcal{C})$. Moreover, if $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ is a counit which is compatible with η up to homotopy, then the homotopy class $[\epsilon]$ is uniquely determined (see Proposition 6.1.2.9). Beware that it is *only* the homotopy class of ϵ that is uniquely determined: if $\epsilon' : F \circ G \to \mathrm{id}_{\mathcal{D}}$ is homotopic to ϵ, then it is also compatible with η up to homotopy.

02ER **Remark 6.2.1.7.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be a natural tranformation. Then η is the unit of an adjunction between F and G if and only if the opposite natural transformation $\eta^{\mathrm{op}} : G^{\mathrm{op}} \circ F^{\mathrm{op}} \to \mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$ is the counit of an adjunction between the functors $G^{\mathrm{op}} : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ and $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. Note that in this case, η^{op} exhibits G^{op} as the *left* adjoint of F^{op}.

02ES **Remark 6.2.1.8** (Composition of Adjoints). Let $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories which admit right adjoints. Then the composite functor $(F' \circ F) : \mathcal{C} \to \mathcal{E}$ also admits a right adjoint. More precisely, if $G : \mathcal{D} \to \mathcal{C}$ and $G' : \mathcal{E} \to \mathcal{D}$ are right adjoints of F and F', respectively, then the composite functor $(G \circ G') : \mathcal{E} \to \mathcal{C}$ is right adjoint to $(F' \circ F) : \mathcal{C} \to \mathcal{E}$ (see Corollary 6.1.5.5).

02ET **Example 6.2.1.9.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between ordinary categories, and suppose we are given natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$. Then the pair (η, ϵ) is an adjunction between F and G (in the sense of Definition 6.1.0.2) if and only if the induced maps

$$\mathrm{N}_{\bullet}(\eta) : \mathrm{id}_{\mathrm{N}_{\bullet}(\mathcal{C})} \to \mathrm{N}_{\bullet}(G) \circ \mathrm{N}_{\bullet}(F) \qquad \mathrm{N}_{\bullet}(\epsilon) : \mathrm{N}_{\bullet}(F) \circ \mathrm{N}_{\bullet}(G) \to \mathrm{id}_{\mathrm{N}_{\bullet}(\mathcal{D})}$$

are compatible up to homotopy, in the sense of Definition 6.2.1.1. In particular:

- A natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ is the unit of an adjunction between functors of ordinary categories F and G if and only if $\mathrm{N}_\bullet(\eta) : \mathrm{id}_{\mathrm{N}_\bullet(\mathcal{C})} \to \mathrm{N}_\bullet(G) \circ \mathrm{N}_\bullet(F)$ is the unit of an adjunction between functors of ∞-categories $\mathrm{N}_\bullet(F)$ and $\mathrm{N}_\bullet(G)$.
- A natural transformation $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{C}}$ is the counit of an adjunction between functors of ordinary categories F and G if and only if $\mathrm{N}_\bullet(\epsilon) : \mathrm{N}_\bullet(F) \circ \mathrm{N}_\bullet(G) \to \mathrm{id}_{\mathrm{N}_\bullet(\mathcal{D})}$ is the unit of an adjunction between functors of ∞-categories $\mathrm{N}_\bullet(F)$ and $\mathrm{N}_\bullet(G)$.
- A functor of ordinary categories $F : \mathcal{C} \to \mathcal{D}$ admits a right adjoint G if and only if the induced functor of ∞-categories $\mathrm{N}_\bullet(F) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})$ admits a right adjoint (in which case $\mathrm{N}_\bullet(G)$ is a right adjoint of $\mathrm{N}_\bullet(F)$).
- A functor of ordinary categories $G : \mathcal{D} \to \mathcal{C}$ admits a left adjoint F if and only if the induced functor of ∞-categories $\mathrm{N}_\bullet(G) : \mathrm{N}_\bullet(\mathcal{D}) \to \mathrm{N}_\bullet(\mathcal{C})$ admits a left adjoint (in which case $\mathrm{N}_\bullet(F)$ is a left adjoint of $\mathrm{N}_\bullet(G)$).

Proposition 3.1.6.9 generalizes to the setting of ∞-categories:

Remark 6.2.1.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$. The existence of natural transformations 02EU

$$\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F \qquad \epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$$

guarantees that F and G are simplicial homotopy inverses of one another, in the sense of Definition 3.1.6.1. In particular, F and G are homotopy equivalences of simplicial sets.

Example 6.2.1.11. Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories, and let $G : \mathcal{D} \to \mathcal{C}$ be a homotopy inverse of F. Then G is also a right adjoint of F. More precisely, any isomorphism $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ in the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ is the unit of an adjunction between F and G (Proposition 6.1.4.1). Similarly, G is a left adjoint of F. 02EV

Remark 6.2.1.12. Let $F : X \to Y$ be a morphism of Kan complexes. Then F admits a right adjoint (in the sense of Notation 6.2.1.3) if and only if F is a homotopy equivalence. This follows by combining Remark 6.2.1.10 with Example 6.2.1.11. 02EW

Remark 6.2.1.12 can be regarded as a special case of the following more general assertion:

Proposition 6.2.1.13. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories and let* 02EX

$$\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F \qquad \epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$$

be natural transformations which are compatible up to homotopy. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by those objects $C \in \mathcal{C}$ for which the unit $\eta_C : C \to (G \circ F)(C)$ is an isomorphism, and let $\mathcal{D}' \subseteq \mathcal{D}$ be the full subcategory spanned by those objects $D \in \mathcal{D}$ for which the counit $\epsilon_D : (F \circ G)(D) \to D$ is an isomorphism. Then F and G restrict to functors $F' : \mathcal{C}' \to \mathcal{D}'$ and $G' : \mathcal{D}' \to \mathcal{C}'$ which are homotopy inverse to one another.

Proof. Let C be an object of $\mathcal{C}'$, so that $\eta_C : C \to (G \circ F)(C)$ is an isomorphism. Since η and ϵ are compatible up to homotopy, the identity morphism $\mathrm{id}_{F(C)}$ is a composition of $F(\eta_C) : F(C) \to (F \circ G \circ F)(C)$ with $\epsilon_{F(C)} : (F \circ G \circ F)(C) \to F(C)$ in the ∞-category $\mathcal{D}$. It follows that $\epsilon_{F(C)}$ is an isomorphism in $\mathcal{D}$ (Remark 1.4.6.3), so that $F(C)$ belongs to the full subcategory $\mathcal{D}' \subseteq \mathcal{D}$. Setting $F' = F|_{\mathcal{C}'}$, we obtain a functor $F' : \mathcal{C}' \to \mathcal{D}'$. A similar argument shows that we can regard $G' = G|_{\mathcal{D}'}$ as a functor from $\mathcal{D}'$ to $\mathcal{C}'$. The unit morphism η restricts to a natural transformation of functors $\eta' : \mathrm{id}_{\mathcal{C}'} \to G' \circ F'$. By construction, η' carries each object $C \in \mathcal{C}'$ to an isomorphism, and is therefore an isomorphism in the functor ∞-category $\mathrm{Fun}(\mathcal{C}', \mathcal{C}')$ (Theorem 4.4.4.4). Similarly, the counit ϵ restricts to a natural isomorphism $\epsilon' : F' \circ G' \to \mathrm{id}_{\mathcal{D}'}$, so that F' and G' are homotopy inverse to one another. □

02EY **Proposition 6.2.1.14.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which admits a right adjoint. Let $G : \mathcal{D} \to \mathcal{C}$ be another functor of ∞-categories and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be a natural transformation. The following conditions are equivalent:*

(1) *The natural transformation η is the unit of an adjunction between the ∞-categories $\mathcal{C}$ and $\mathcal{D}$.*

(2) *The induced map $\mathrm{id}_{\mathrm{h}\mathcal{C}} \to \mathrm{h}G \circ \mathrm{h}F$ is the unit of an adjunction between the homotopy categories $\mathrm{h}\mathcal{C}$ and $\mathrm{h}\mathcal{D}$.*

Proof. The implication (1) ⇒ (2) follows from the observation that the formation of homotopy categories defines a (strict) functor of 2-categories

$$\mathrm{h}_2\mathbf{QCat} \to \mathbf{Cat} \qquad \mathcal{C} \mapsto \mathrm{h}\mathcal{C},$$

and therefore carries adjunctions to adjunctions (see Exercise 6.1.1.6). We will show that (2) implies (1). By assumption, the functor F admits a right adjoint $G' : \mathcal{D} \to \mathcal{C}$. Let $\eta' : \mathrm{id}_{\mathcal{C}} \to F \circ G'$ be the unit of an adjunction. Applying Corollary 6.1.3.3, we deduce that there exists a natural transformation $\gamma : G' \to G$ such that η is a composition of the natural transformations

$$\eta' : \mathrm{id}_{\mathcal{C}} \to F \circ G' \qquad (\mathrm{id}_F \circ \gamma) : F \circ G' \to F \circ G$$

in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$. If assumption (2) is satisfied, then the image of γ in the functor category $\mathrm{Fun}(\mathrm{h}\mathcal{D}, \mathrm{h}\mathcal{C})$ is an isomorphism: that is, γ carries each object $D \in \mathcal{D}$ to an isomorphism $\gamma_D : G'(D) \to G(D)$ in the ∞-category $\mathcal{C}$. Applying Theorem 4.4.4.4, we conclude that γ is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$, so that the criterion of Corollary 6.1.3.3 guarantees that η is also the unit of an adjunction. □

02EZ **Corollary 6.2.1.15.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ be the induced functor of homotopy categories. If F admits a right adjoint G, then $\mathrm{h}F$ also admits a right adjoint, which can be identified with the functor $\mathrm{h}G$.*

Warning 6.2.1.16. The implication (2) ⇒ (1) of Proposition 6.2.1.14 generally fails if the functor $F : \mathcal{C} \to \mathcal{D}$ does not have a right adjoint. For example, let X be a simply connected Kan complex, let $F : \Delta^0 \to X$ be the map corresponding to a vertex $x \in X$, and let $G : X \to \Delta^0$ be the projection map. Since X is simply connected, the functors $\mathrm{h}F$ and $\mathrm{h}G$ are equivalences of ordinary categories. In particular, the identity transformation from $\mathrm{id}_{\Delta^0} = G \circ F$ to itself determines the unit of an adjunction between $\mathrm{h}F$ and $\mathrm{h}G$. However, the functors F and G cannot be adjoint unless the Kan complex X is contractible (see Remark 6.2.1.10) `02F0`

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between ∞-categories and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be a natural transformation. By virtue of Variant 6.1.2.11, the natural transformation η exhibits $\mathrm{h}G$ as a right adjoint to $\mathrm{h}F$ if and only if, for every pair of objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the composite map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(F(C), D) &= \pi_0(\mathrm{Hom}_{\mathcal{D}}(F(C), D)) \\ &\xrightarrow{G} \pi_0(\mathrm{Hom}_{\mathcal{C}}((G \circ F)(C), G(D))) \\ &\xrightarrow{\circ[\eta_C]} \pi_0(\mathrm{Hom}_{\mathcal{C}}(C, G(D)) \\ &= \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, G(D)) \end{aligned}$$

is a bijection. If η exhibits G as a right adjoint to F, then we can say more:

Proposition 6.2.1.17. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between ∞-categories and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be the unit of an adjunction. Then, for every pair of objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the composite map* `02F1`

$$\mathrm{Hom}_{\mathcal{D}}(F(C), D) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}((G \circ F)(C), G(D)) \xrightarrow{\circ[\eta_C]} \mathrm{Hom}_{\mathcal{C}}(C, G(D))$$

is an isomorphism in the homotopy category hKan*; here the second map is given by the composition law of Construction 4.6.9.9.*

Proof. It will suffice to show that, for every Kan complex T, the induced map

$$\begin{aligned} \pi_0(\mathrm{Fun}(T, \mathrm{Hom}_{\mathcal{D}}(F(C), D))) &= \mathrm{Hom}_{\mathrm{hKan}}(T, \mathrm{Hom}_{\mathcal{D}}(F(C), D)) \\ &\xrightarrow{\theta} \mathrm{Hom}_{\mathrm{hKan}}(T, \mathrm{Hom}_{\mathcal{C}}(C, G(D))) \\ &= \pi_0(\mathrm{Fun}(T, \mathrm{Hom}_{\mathcal{C}}(C, G(D)))) \end{aligned}$$

is bijective. Let $\underline{C} \in \mathrm{Fun}(T, \mathcal{C})$ and $\underline{D} \in \mathrm{Fun}(T, \mathcal{D})$ be the constant morphisms taking the values C and D, respectively. Unwinding the definitions, we see that θ can be identified with the map

$$\mathrm{Hom}_{\mathrm{hFun}(T,\mathcal{D})}(F \circ \underline{C}, \underline{D}) \to \mathrm{Hom}_{\mathrm{hFun}(T,\mathcal{C})}(\underline{C}, G \circ \underline{D})$$

given by the formation of right adjuncts with respect to the homotopy class $[\eta]$ (regarded as a 2-morphism in the category $\mathrm{h}_2\mathbf{QCat}$). The bijectivity of θ now follows from the criterion of Proposition 6.1.2.9. □

02F2 **Remark 6.2.1.18.** We will see later that the converse of Proposition 6.2.1.17 also holds: if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors of ∞-categories and $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ is a natural transformation which induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{D}}(F(C), D) \simeq \mathrm{Hom}_{\mathcal{C}}(C, G(D))$ for every pair of objects $(C, D) \in \mathcal{C} \times \mathcal{D}$, then η is the unit of an adjunction between F and G (Corollary 6.2.4.5).

02F3 **Remark 6.2.1.19.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. It follows from Proposition 6.1.3.4 that if F admits a right adjoint G, then G is well-defined up to isomorphism as an object of the functor ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$. We will sometimes emphasize this by referring to G as *the* right adjoint of F and denoting it by F^R. By virtue of Notation 6.1.3.8, the construction $F \mapsto F^R$ determines an equivalence of homotopy categories $\mathrm{hLFun}(\mathcal{C}, \mathcal{D}) \to \mathrm{hRFun}(\mathcal{D}, \mathcal{C})^{\mathrm{op}}$. We will see later that this construction can be upgraded to an equivalence of ∞-categories $\mathrm{LFun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{RFun}(\mathcal{D}, \mathcal{C})^{\mathrm{op}}$ (see Corollary 8.3.4.10).

02F4 **Warning 6.2.1.20.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. The following data are essentially equivalent to one another:

- The datum of a functor $F : \mathcal{C} \to \mathcal{D}$ which admits a right adjoint.
- The datum of a functor $G : \mathcal{D} \to \mathcal{C}$ which admits a left adjoint.
- The datum of a triple (F, G, η), where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors and $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ is the unit of an adjunction between F and G.
- The datum of a triple (F, G, ϵ), where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ is the counit of an adjunction between F and G.
- The datum of a quintuple $(F, G, \eta, \epsilon, \sigma)$, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ are natural transformations which are compatible up to homotopy, and $\sigma : \Delta^2 \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is a 2-simplex witnessing axiom $(Z1)$ of Definition 6.2.1.1 (see Remark 6.2.1.5).
- The datum of a quintuple $(F, G, \eta, \epsilon, \tau)$, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ are natural transformations which are compatible up to homotopy, and $\tau : \Delta^2 \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is a 2-simplex witnessing axiom $(Z2)$ of Definition 6.2.1.1.

The following data are *not* equivalent to the above (or to each other):

- The datum of a pair (F, G), where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors which are adjoint to one another.

- The datum of a quadruple (F, G, η, ϵ), where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ are natural transformations which are compatible up to homotopy,

- The datum of a sextuple $(F, G, \eta, \epsilon, \sigma, \tau)$, where $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors, $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ are natural transformations, and $\sigma : \Delta^2 \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\tau : \Delta^2 \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ are 2-simplices witnessing axioms $(Z1)$ and $(Z2)$ of Definition 6.2.1.1.

To say that a functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ is somewhat imprecise: one should really specify a *witness* to the adjointness of F and G, which can take the form of either a unit $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ or a counit $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$. Given *both* a unit η and a counit ϵ, one can further demand evidence of their compatibility, which can take the form of a 2-simplex $\sigma : \Delta^2 \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ witnessing axiom $(Z1)$ or a 2-simplex $\tau : \Delta^2 \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$ witnessing axiom $(Z2)$. If one specifies *both* of the witnesses σ and τ, then one can further demand a witness to the compatibility of σ with τ; we will return to this point in §[?].

6.2.2 Reflective Subcategories

Let $\mathcal{C}$ be an ∞-category. Our goal in this section is to characterize those full subcategories $\mathcal{C}' \subseteq \mathcal{C}$ for which the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ admits a left or right adjoint. 02F5

Definition 6.2.2.1. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory. We say that a morphism $u : X \to Y$ in $\mathcal{C}$ *exhibits Y as a $\mathcal{C}'$-reflection of X* if Y belongs to $\mathcal{C}'$ and, for every object $Z \in \mathcal{C}'$, the precomposition map $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ [u]} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is an isomorphism in the homotopy category hKan. We say that u *exhibits X as a $\mathcal{C}'$-coreflection of Y* if X belongs to $\mathcal{C}'$ and, for every object $W \in \mathcal{C}'$, the postcomposition map $\mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{[u] \circ} \mathrm{Hom}_{\mathcal{C}}(W, Y)$ is an isomorphism in the homotopy category hKan. 02F6

We say that a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is *reflective* if it is full and, for every object $X \in \mathcal{C}$, there exists a morphism $u : X \to Y$ which exhibits Y as a $\mathcal{C}'$-reflection of X. We say that the subcategory $\mathcal{C}'$ is *coreflective* if it is full and, for every object $Y \in \mathcal{C}$, there exists a morphism $u : X \to Y$ which exhibits X as a $\mathcal{C}'$-coreflection of Y.

Remark 6.2.2.2. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, so that we can identify $\mathcal{C}'^{\mathrm{op}}$ with a full subcategory of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. Then: 02F7

- A morphism $u : X \to Y$ in $\mathcal{C}$ exhibits Y as a $\mathcal{C}'$-reflection of X if and only if $u^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ exhibits Y^{op} as a $\mathcal{C}'^{\mathrm{op}}$-coreflection of X^{op}.

- The subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is reflective if and only if the subcategory $\mathcal{C}'^{\mathrm{op}} \subseteq \mathcal{C}^{\mathrm{op}}$ is coreflective.

02F8 **Remark 6.2.2.3.** Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, and suppose we are given a pair of morphisms $u : X \to Y$ and $w : X \to Z$ of $\mathcal{C}$, where Y and Z belong to the subcategory $\mathcal{C}'$. If u exhibits Y as a $\mathcal{C}'$-reflection of X, then we can realize w as a composition of u with another morphism $v : Y \to Z$ of $\mathcal{C}'$, which is uniquely determined up to homotopy. Moreover, v is an isomorphism if and only if w exhibits Z as a $\mathcal{C}'$-reflection of X. Stated more informally: a $\mathcal{C}'$-reflection of X, if it exists, is unique up to isomorphism.

02F9 **Example 6.2.2.4.** Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, and let $u : X \to Y$ be a morphism of $\mathcal{C}$. If X belongs to the subcategory $\mathcal{C}'$, then u exhibits Y as a $\mathcal{C}'$-reflection of X if and only if it is an isomorphism. Similarly, if Y belongs to $\mathcal{C}'$, then u exhibits X as a $\mathcal{C}'$-coreflection of Y if and only if it is an isomorphism.

02SS **Example 6.2.2.5.** Let $\mathcal{C}$ be an ∞-category which contains a final object, and let $\mathcal{C}^{\mathrm{fin}}$ denote the full subcategory of $\mathcal{C}$ spanned by its final objects (so that $\mathcal{C}^{\mathrm{fin}}$ is a contractible Kan complex: see Corollary 4.6.7.14). Then $\mathcal{C}^{\mathrm{fin}}$ is a reflective subcategory of $\mathcal{C}$.

02ST **Example 6.2.2.6.** Let $\mathcal{S}$ denote the ∞-category of spaces (Construction 5.5.1.1) and let $\mathcal{QC}$ denote the ∞-category of (small) ∞-categories (Construction 5.5.4.1). Then $\mathcal{S}$ is a reflective and coreflective subcategory of $\mathcal{QC}$. If $\mathcal{C}$ is a small ∞-category, then the inclusion map $\mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$ exhibits the core $\mathcal{C}^{\simeq}$ as a $\mathcal{S}$-coreflection of $\mathcal{C}$ (this follows by combining Proposition 4.4.3.17 with Remark 5.5.4.6), and the comparison map $\mathcal{C} \to \mathrm{Ex}^{\infty}(\mathcal{C})$ exhibits the Kan complex $\mathrm{Ex}^{\infty}(\mathcal{C})$ as a $\mathcal{S}$-reflection of $\mathcal{C}$ (this follows by combining Proposition 3.3.6.7 with Remark 5.5.4.6).

04JC **Example 6.2.2.7.** Let Top denote the category whose objects are topological spaces and whose morphisms are continuous functions Let us regard Top as a simplicial category (Example 2.4.1.5), and let $\mathcal{T} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Top})$ denote its homotopy coherent nerve. Let $\mathcal{T}_0 \subseteq \mathcal{T}$ be the full subcategory spanned by those topological spaces which have the homotopy type of a CW complex. Then:

- A continuous function between topological spaces $f : X \to Y$ is a weak homotopy equivalence (in the sense of Definition 3.6.3.1) if and only if it exhibits X as a $\mathcal{T}_0$-colocalization of Y. This is restatement of Corollary 3.6.5.4.
- The full subcategory $\mathcal{T}_0 \subseteq \mathcal{T}$ is coreflective. That is, for every topological space Y, there exists a weak homotopy equivalence $f : X \to Y$, where X has the homotopy type of a CW complex. For example, we can take f to be the counit map $|\mathrm{Sing}_{\bullet}(Y)| \to Y$ (see Corollary 3.6.4.2).

Definition 6.2.2.1 can be rephrased as a lifting property:

Proposition 6.2.2.8. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, and let $f : X \to Y$ be a morphism in $\mathcal{C}$, where $Y \in \mathcal{C}'$. The following conditions are equivalent:* 03XA

(1) *The morphism f exhibits Y as a $\mathcal{C}'$-reflection of X, in the sense of Definition 6.2.2.1.*

(2) *For every object $Z \in \mathcal{C}'$, the restriction map $\mathcal{C}_{f/} \times_{\mathcal{C}} \{Z\} \to \mathcal{C}_{X/} \times_{\mathcal{C}} \{Z\}$ is a homotopy equivalence of Kan complexes.*

(3) *The restriction map $u : \mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}' \to \mathcal{C}_{X/} \times_{\mathcal{C}} \mathcal{C}'$ is an equivalence of ∞-categories.*

(4) *The restriction map u is a trivial Kan fibration.*

(5) *For $n \geq 2$, every morphism of simplicial sets $\sigma_0 : \Lambda^n_0 \to \mathcal{C}$ can be extended to an n-simplex of $\mathcal{C}$, provided that σ_0 carries the initial edge $\Delta^1 = \mathrm{N}_\bullet(\{0 < 1\})$ to the morphism f and satisfies $\sigma_0(i) \in \mathcal{C}'$ for $i \geq 2$.*

Proof. The equivalence (1) ⇔ (2) follows from Proposition 4.6.9.16, the equivalence (2) ⇔ (3) from Corollary 5.1.7.16 (and Proposition 5.1.7.5). Corollary 4.3.6.11 guarantees that u is a left fibration. In particular, it is an isofibration, so the equivalence (3) ⇔ (4) is a special case of Proposition 4.5.5.20. The equivalence (4) ⇔ (5) follows by unwinding definitions. □

Corollary 6.2.2.9. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, and let $q : K \to \mathcal{C}'$ be a morphism of simplicial sets. Let $\overline{f} : \overline{X} \to \overline{Y}$ be a morphism in the ∞-category $\mathcal{C}_{/q}$, having image $f : X \to Y$ in $\mathcal{C}$. If f exhibits Y as a $\mathcal{C}'$-reflection of X, then $\overline{f}$ exhibits $\overline{Y}$ as a $\mathcal{C}'_{/q}$-reflection of $\overline{X}$.* 04JD

Proof. Set $\mathcal{D} = \mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}'$ and $\mathcal{E} = \mathcal{C}_{X/} \times_{\mathcal{C}} \mathcal{C}'$. Our assumption that f exhibits Y as a $\mathcal{C}'$-reflection of X guarantees that the restriction map $u : \mathcal{D} \to \mathcal{E}$ is a trivial Kan fibration (Proposition 6.2.2.8). We wish to show that the analogous restriction map

$$\overline{u} : (\mathcal{C}_{/q})_{\overline{f}/} \times_{\mathcal{C}_{/q}} \mathcal{C}'_{/q} \to (\mathcal{C}_{/q})_{\overline{X}/} \times_{\mathcal{C}_{/q}} \mathcal{C}'_{/q}$$

is also trivial Kan fibration. Let us regard $\overline{f}$ as a morphism of simplicial sets $\Delta^1 \star K \to \mathcal{C}$, which we can identify with a diagram $\overline{q} : K \to \mathcal{D}$. Under this identification, $\overline{u}$ corresponds to the map $\mathcal{D}_{/\overline{q}} \to \mathcal{E}_{/u \circ \overline{q}}$ induced by u, which is a trivial Kan fibration by virtue of Corollary 4.3.7.17. □

Corollary 6.2.2.10. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, and let $q : K \to \mathcal{C}'$ be a diagram. If $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$, then $\mathcal{C}'_{/q}$ is a reflective subcategory of $\mathcal{C}_{/q}$.* 04JE

Proof. Let $\overline{X}$ be an object of the slice ∞-category $\mathcal{C}_{/q}$; we wish to show that there exists a morphism $\overline{f} : \overline{X} \to \overline{Y}$ which exhibits $\overline{Y}$ as a $\mathcal{C}'_{/q}$-reflection of $\overline{X}$. Let X denote the image of

$\overline{X}$ in the ∞-category $\mathcal{C}$. Since $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$, we can choose a morphism $f : X \to Y$ in $\mathcal{C}$ which exhibits Y as a $\mathcal{C}'$-reflection of X. By virtue of Corollary 6.2.2.9, it will suffice to show that f can be lifted to a morphism $\overline{f} : \overline{X} \to \overline{Y}$ in $\mathcal{C}_{/q}$. Unwinding the definitions, we can rewrite this as a lifting problem

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & \mathcal{C}_{f/} \times_{\mathcal{C}} \mathcal{C}' \arrow[d] \\
K \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{X/} \times_{\mathcal{C}} \mathcal{C}',
\end{tikzcd}$$

which admits a solution by virtue of the fact that the right vertical map is a trivial Kan fibration (Proposition 6.2.2.8). □

Our next goal is to prove the following:

02FA **Proposition 6.2.2.11.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, and let $\iota : \mathcal{C}' \hookrightarrow \mathcal{C}$ be the inclusion map. Then ι admits a left adjoint if and only if $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$. Similarly, ι admits a right adjoint if and only if $\mathcal{C}'$ is a coreflective subcategory of $\mathcal{C}$.*

The first step toward proving Proposition 6.2.2.11 is to show that if $X \in \mathcal{C}$ is an object which admits a $\mathcal{C}'$-reflection $u : X \to Y$, then the pair (u, Y) can be chosen to depend functorially on X.

02FB **Definition 6.2.2.12.** Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, and let $L : \mathcal{C} \to \mathcal{C}$ be a functor. We will say that a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to L$ *exhibits L as a $\mathcal{C}'$-reflection functor* if, for every object $X \in \mathcal{C}$, the morphism $\eta_X : X \to L(X)$ exhibits $L(X)$ as a $\mathcal{C}'$-reflection of $\mathcal{C}$, in the sense of Definition 6.2.2.1. We say that a natural transformation $\epsilon : L \to \mathrm{id}_{\mathcal{C}}$ *exhibits L as a $\mathcal{C}'$-coreflection functor* if, for every object $Y \in \mathcal{C}$, the morphism $\epsilon_Y : L(Y) \to Y$ exhibits $L(Y)$ as a $\mathcal{C}'$-coreflection of Y.

02FC **Remark 6.2.2.13.** In the situation of Definition 6.2.2.12, the assumption that $\eta : \mathrm{id}_{\mathcal{C}} \to L$ exhibits L as a $\mathcal{C}'$-reflection functor guarantees in particular that for every object $X \in \mathcal{C}$, the image $L(X)$ belongs to the full subcategory $\mathcal{C}' \subseteq \mathcal{C}$. Consequently, we can also view L as a functor from $\mathcal{C}$ to $\mathcal{C}'$.

02FD **Lemma 6.2.2.14.** *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory. Then $\mathcal{C}'$ is reflective if and only if there exists a functor $L : \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to L$ which exhibits L as a $\mathcal{C}'$-reflection functor.*

Proof. Assume that $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$; we will show that there exists a functor $L : \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to L$ which exhibits L as a

$\mathcal{C}'$-reflection functor (the reverse implication is immediate from the definitions). Let $\mathcal{E}$ be the full subcategory of $\mathcal{C} \times \Delta^1$ spanned by those objects (X, i) having the property that if $i = 1$, then X belongs to the full subcategory $\mathcal{C}'$. Let $\pi : \mathcal{E} \to \Delta^1$ denote the projection map. Let $\widetilde{u} : (X, 0) \to (Y, 1)$ be a morphism in $\mathcal{E}$, corresponding to a morphism $u : X \to Y$ in $\mathcal{C}$ for which the target Y belongs to $\mathcal{C}'$. By virtue of Corollary 5.1.2.3, the morphism $\widetilde{u}$ is π-cocartesian if and only if u exhibits Y as a $\mathcal{C}'$-localization of X. Consequently, our assumption that $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$ guarantees that π is a cocartesian fibration of ∞-categories. Applying Proposition 5.2.2.8, we deduce that there exists a functor

$$L : \mathcal{C} \simeq \{0\} \times_{\Delta^1} \mathcal{E} \to \{1\} \times_{\Delta^1} \mathcal{E} \simeq \mathcal{C}'$$

and a morphism $\widetilde{\eta} : \mathrm{id}_{\mathcal{C}} \to L$ in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ which carries each object $X \in \mathcal{C}$ to a π-cocartesian morphism $(X, 0) \to (L(X), 1)$ in $\mathcal{E}$. Composing with the projection map $\pi : \mathcal{E} \to \Delta^1$, we obtain a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to L$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ which exhibits L as a $\mathcal{C}'$-reflection functor. □

Proposition 6.2.2.15. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory, and let $\iota : \mathcal{C}' \hookrightarrow \mathcal{C}$ be the inclusion map. Let $L : \mathcal{C} \to \mathcal{C}'$ be a functor of ∞-categories and let $\eta : \mathrm{id}_{\mathcal{C}} \to \iota \circ L$ be a natural transformation. The following conditions are equivalent:* 02FE

(1) *The natural transformation η is the unit of an adjunction: that is, it exhibits L as a left adjoint to the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$.*

(2) *The natural transformation η exhibits L as a $\mathcal{C}'$-reflection functor: that is, for every object $X \in \mathcal{C}$, the morphism $\eta_X : X \to L(X)$ exhibits $L(X)$ as a $\mathcal{C}'$-reflection of X.*

(3) *For every object $X \in \mathcal{C}$, the morphism $L(\eta_X) : L(X) \to L(L(X))$ is an isomorphism in $\mathcal{C}'$. Moreover, if X belongs to $\mathcal{C}'$, then $\eta_X : X \to L(X)$ is an isomorphism.*

Moreover, if these conditions are satisfied, then any natural transformation $\epsilon : L \circ \iota \to \mathrm{id}_{\mathcal{C}'}$ which is compatible with η up to homotopy (in the sense of Definition 6.2.1.1) is an isomorphism in the functor ∞-category $\mathrm{Fun}(\mathcal{C}', \mathcal{C}')$.

Proof. We first show that (1) implies (2). Let X be an object of $\mathcal{C}$, so that η determines a morphism $\eta_X : X \to L(X)$. For every object $Y \in \mathcal{C}'$, Proposition 6.2.1.17 guarantees that composition with the homotopy class $[\eta_X]$ induces an isomorphism

$$\mathrm{Hom}_{\mathcal{C}'}(L(X), Y) = \mathrm{Hom}_{\mathcal{C}}(L(X), Y) \xrightarrow{\circ[\eta_X]} \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

in the homotopy category hKan. It follows that η_X exhibits $L(X)$ as a $\mathcal{C}'$-reflection of X. Allowing X to vary, we conclude that η exhibits L as a $\mathcal{C}'$-reflection functor.

We now show that (2) implies (3). Assume that, for every object $X \in \mathcal{C}$, the morphism $\eta_X : X \to L(X)$ exhibits $L(X)$ as a $\mathcal{C}'$-reflection of X. Note that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "\eta_X"] & L(X) \arrow[d, "\eta_{L(X)}"] \\
L(X) \arrow[r, "L(\eta_X)"] & L(L(X))
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, obtained by applying the natural transformation η to the morphism $\eta_X : X \to L(X)$. For each object $Y \in \mathcal{C}$, we obtain a commutative diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(L(X), Y) \arrow[l, "\circ[\eta_X]"'] \\
\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(L(X), Y) \arrow[u, "\circ[\eta_X]"] & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(L(L(X)), Y). \arrow[l, "\circ[L(\eta_X)]"'] \arrow[u, "\circ[\eta_{L(X)}]"']
\end{tikzcd}$$

If Y belongs to the subcategory $\mathcal{C}' \subseteq \mathcal{C}$, then the vertical maps and the upper horizontal map in this diagram are bijective. It follows that the lower horizontal map is bijective as well. Allowing Y to vary, we deduce that the homotopy class $[L(\eta_X)]$ is an isomorphism in the homotopy category $\mathrm{h}\mathcal{C}'$, so that $L(\eta_X)$ is an isomorphism in the ∞-category $\mathcal{C}'$. In the special case where X belongs to $\mathcal{C}'$, Example 6.2.2.4 guarantees that η_X is already an isomorphism before applying the functor L.

We now show that (3) implies (1). Note that η determines natural transformations

$$\eta' : L \to L \circ \iota \circ L \qquad (X \in \mathcal{C}) \mapsto (L(\eta_X) \in \mathrm{Hom}_{\mathcal{C}'}(L(X), L(L(X))))$$

$$\eta'' : \iota \to \iota \circ L \circ \iota \qquad (Y \in \mathcal{C}') \mapsto (\eta_Y \in \mathrm{Hom}_{\mathcal{C}}(Y, L(Y))).$$

If condition (3) is satisfied, then Theorem 4.4.4.4 guarantees that η' and η'' are isomorphisms in the ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ and $\mathrm{Fun}(\mathcal{C}', \mathcal{C})$, respectively. Invoking the criterion of Proposition 6.1.4.6, we conclude that η is the unit of an adjunction. □

Proof of Proposition 6.2.2.11. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory. It follows from Proposition 6.2.2.15 that the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ admits a left adjoint if and only if there exists a functor $L : \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to L$ which exhibits L as a $\mathcal{C}'$-reflection functor. By virtue of Lemma 6.2.2.14, this is equivalent to the requirement that $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$. The analogous characterization of coreflective subcategories follows by a similar argument. □

02SU **Example 6.2.2.16.** Combining Example 6.2.2.6 with Proposition 6.2.2.11, we see that the inclusion functor $\mathcal{S} \hookrightarrow \mathcal{QC}$ admits both a right adjoint (given on objects by the construction $\mathcal{C} \mapsto \mathcal{C}^{\simeq}$) and a left adjoint (given on objects by the construction $\mathcal{C} \mapsto \mathrm{Ex}^{\infty}(\mathcal{C})$).

02FF **Corollary 6.2.2.17.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. The following conditions are equivalent:*

(1) *The functor G is fully faithful and the essential image of G is a reflective subcategory of $\mathcal{C}$.*

(2) *The functor G is fully faithful and admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$.*

(3) *There exist a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural isomorphism $\epsilon : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ which is the counit of an adjunction between F and G.*

(4) *The functor G admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$ for which the composition $(F \circ G) : \mathcal{D} \to \mathcal{D}$ is an equivalence of ∞-categories.*

Proof. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the essential image of G. If G is fully faithful, then it induces an equivalence $\mathcal{D} \to \mathcal{C}'$ (Corollary 4.6.2.22). The equivalence (1) $\Leftrightarrow$ (2) follows by applying Proposition 6.2.2.11 to the subcategory $\mathcal{C}' \subseteq \mathcal{C}$, and the implication (2) $\Rightarrow$ (3) follows by applying Proposition 6.2.2.15 to the subcategory $\mathcal{C}' \subseteq \mathcal{C}$. To show that (3) $\Rightarrow$ (2), we observe that if a natural isomorphism $\epsilon : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ is the counit of an adjunction, then G restricts to an equivalence of $\mathcal{D}$ with a full subcategory of $\mathcal{C}$ (Proposition 6.2.1.13), and is therefore fully faithful. The equivalence (3) $\Leftrightarrow$ (4) is a special case of Proposition 6.1.4.7. □

04JF **Remark 6.2.2.18.** In the situation of Corollary 6.2.2.17, suppose that $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ is the unit of an adjunction between F and G. Then an object $C \in \mathcal{C}$ belongs to the essential image of G if and only if the unit map $\eta_C : C \to (G \circ F)(C)$ is an isomorphism. The "if" direction is obvious. To prove the converse, we may assume without loss of generality that $C = G(D)$, for some object $D \in \mathcal{D}$. In this case, the morphism $\eta_C = \eta_{G(D)}$ fits into a commutative diagram

$$\begin{tikzcd}
 & (G \circ F \circ G)(D) \arrow[dr, "G(\epsilon_D)"] & \\
G(D) \arrow[ur, "\eta_{G(D)}"] \arrow[rr, "\mathrm{id}_{G(D)}"] & & G(D),
\end{tikzcd}$$

where $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ is compatible with ϵ up to homotopy. Since ϵ is an isomorphism, it follows that $\eta_C = \eta_{G(D)}$ is also an isomorphism.

03UZ **Corollary 6.2.2.19.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is an equivalence if and only if it satisfies the following pair of conditions:*

(1) *The functor F is conservative. That is, a morphism u of $\mathcal{C}$ is an isomorphism if and only if $F(u)$ is an isomorphism in $\mathcal{D}$.*

(2) *The functor F admits a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$.*

Proof. Suppose that conditions (1) and (2) are satisfied; we will show that F is an equivalence of ∞-categories (the converse is immediate from the definitions). Combining assumption (2) with Corollary 6.2.2.17, we can choose a functor $G : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism $\epsilon : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ which is the counit of an adjunction between F and G. Let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be a natural transformation which is compatible up to homotopy with ϵ, in the sense of Definition 6.2.1.1. For each object $C \in \mathcal{C}$, the morphism η_C fits into a commutative diagram

$$\begin{tikzcd}
& (F \circ G \circ F)(C) \arrow[dr, "\epsilon_{F(C)}"] & \\
F(C) \arrow[ur, "F(\eta_C)"] \arrow[rr, "\mathrm{id}_{F(C)}"] & & F(C)
\end{tikzcd}$$

in the ∞-category $\mathcal{D}$, where $\epsilon_{F(C)}$ and $\mathrm{id}_{F(C)}$ are isomorphisms. It follows that $F(\eta_C)$ is also an isomorphism in $\mathcal{D}$. Applying assumption (1), we deduce that η_C is an isomorphism in $\mathcal{C}$. Allowing the object C to vary (and invoking the criterion of Theorem 4.4.4.4), we deduce that η is also a natural isomorphism, so that F and G are homotopy inverse to one another. □

02FG **Corollary 6.2.2.20.** *Let $\mathcal{C}$ be an ∞-category, let L be a functor from $\mathcal{C}$ to itself, and let $\eta : \mathrm{id}_{\mathcal{C}} \to L$ be a natural transformation. The following conditions are equivalent:*

(1) *For every object $X \in \mathcal{C}$, the morphisms $L(\eta_X) : L(X) \to L(L(X))$ and $\eta_{L(X)} : L(X) \to L(L(X))$ are isomorphisms.*

(2) *There exists a full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ for which η exhibits L as a $\mathcal{C}'$-reflection functor, in the sense of Definition 6.2.2.12.*

Proof. The implication (2) ⇒ (1) follows from Proposition 6.2.2.15. Conversely, suppose that condition (1) is satisfied, and let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by those objects of the form $L(X)$ for $X \in \mathcal{C}$. Assumption (1) guarantees that η_Y is an isomorphism for each $Y \in \mathcal{C}'$, so that η exhibits L as a $\mathcal{C}'$-reflection functor by virtue of Proposition 6.2.2.15. □

02FH **Exercise 6.2.2.21.** Suppose that the conditions of Corollary 6.2.2.20 are satisfied and let $\mathcal{C}' \subseteq \mathcal{C}$ be a full subcategory of $\mathcal{C}$. Show that η exhibits L as a $\mathcal{C}'$-reflection functor if and only if the following conditions are satisfied:

- For each object $X \in \mathcal{C}$, the object $L(X)$ is contained in $\mathcal{C}'$.
- For each object $Y \in \mathcal{C}'$, there exists an isomorphism $Y \to L(X)$ for some object $X \in \mathcal{C}$.

If the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is replete (Example 4.4.1.12), then it is uniquely determined by these conditions.

Reflective subcategories are stable under pullback along cocartesian fibrations:

Proposition 6.2.2.22. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let $\mathcal{C}' \subseteq \mathcal{C}$ be a reflective subcategory. Then the pullback $\mathcal{E}' = \mathcal{C}' \times_{\mathcal{C}} \mathcal{E}$ is a reflective subcategory of $\mathcal{E}$. Moreover, a morphism $f : X \to Y$ in $\mathcal{E}$ exhibits Y as a $\mathcal{E}'$-reflection of X if and only if it satisfies the following pair of conditions:* `02SV`

(1) *The morphism f is U-cocartesian.*

(2) *The morphism $U(f) : U(X) \to U(Y)$ exhibits $U(Y)$ as a $\mathcal{C}'$-reflection of $U(X)$ in the ∞-category $\mathcal{C}$.*

Proof. We first show that, if $f : X \to Y$ is a morphism of $\mathcal{E}$ satisfying conditions (1) and (2), then f exhibits Y as a $\mathcal{E}'$-reflection of X. It follows from condition (2) that $U(Y)$ belongs to $\mathcal{C}'$, so that Y belongs to $\mathcal{E}'$. It will therefore suffice to show that for each object $Z \in \mathcal{E}$, precomposition with f induces a homotopy equivalence $\theta : \mathrm{Hom}_{\mathcal{E}}(Y, Z) \to \mathrm{Hom}_{\mathcal{E}}(X, Z)$. Let us abuse notation by identifying θ with the restriction map $\{f\} \times_{\mathrm{Hom}_{\mathcal{E}}(X,Y)} \mathrm{Hom}_{\mathcal{E}}(X, Y, Z) \to \mathrm{Hom}_{\mathcal{E}}(X, Z)$, so that we have a commutative diagram of Kan complexes

$$\begin{tikzcd}
\{f\} \times_{\mathrm{Hom}_{\mathcal{E}}(X,Y)} \mathrm{Hom}_{\mathcal{E}}(X, Y, Z) \arrow[r, "\theta"] \arrow[d] & \mathrm{Hom}_{\mathcal{E}}(X, Z) \arrow[d] \\
\{U(f)\} \times_{\mathrm{Hom}_{\mathcal{C}}(U(X),U(Y))} \mathrm{Hom}_{\mathcal{C}}(U(X), U(Y), U(Z)) \arrow[r, "\overline{\theta}"] & \mathrm{Hom}_{\mathcal{C}}(U(X), U(Z)).
\end{tikzcd}$$

Assumption (1) guarantees that this diagram is a homotopy pullback square (Proposition 5.1.2.1), and assumption (2) guarantees that $\overline{\theta}$ is a homotopy equivalence of Kan complexes. Applying Corollary 3.4.1.5, we conclude that θ is also a homotopy equivalence.

We now show that $\mathcal{E}'$ is a reflective subcategory of $\mathcal{E}$. Fix an object $X \in \mathcal{E}$. Since $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}$, there exists a morphism $\overline{f} : U(X) \to \overline{Y}$ in $\mathcal{C}$ which exhibits $\overline{Y}$ as a $\mathcal{C}'$-reflection of $U(X)$. Since U is a cocartesian fibration, we can write $\overline{f} = U(f)$ for some U-cocartesian morphism $f : X \to Y$ of $\mathcal{E}$. By construction, the morphism f satisfies conditions (1) and (2), and therefore exhibits Y as an $\mathcal{E}'$-reflection of X.

To complete the proof, it will suffice to show that if $h : X \to Z$ is another morphism which exhibits Z as a $\mathcal{E}'$-reflection of X, then h also satisfies conditions (1) and (2). By virtue of Remark 6.2.2.3, there exists a 2-simplex

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z,
\end{tikzcd}$$

of $\mathcal{E}$, where $g : Y \to Z$ is an isomorphism of $\mathcal{E}'$. In particular, g is U-cocartesian (Proposition 5.1.1.8), so that h satisfies (1) by virtue of Corollary 5.1.2.4. Since $U(g)$ is an isomorphism in $\mathcal{C}'$, condition (2) follows from Remark 6.2.2.3. □

6.2.3 Correspondences

02FJ Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. To every morphism $e : C \to D$ of $\mathcal{C}$, Proposition 5.2.2.8 supplies a covariant transport functor

$$e_! : \mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E} \to \{D\} \times_{\mathcal{C}} \mathcal{E} = \mathcal{E}_D,$$

which is well-defined up to isomorphism. Our goal in this section is to show that U is a cartesian fibration if and only if each of the functors $e_! : \mathcal{E}_C \to \mathcal{E}_D$ admits a right adjoint (Proposition 6.2.3.5). Moreover, if this condition is satisfied, then the right adjoint to $e_!$ is given by the contravariant transport functor $e^* : \mathcal{E}_D \to \mathcal{E}_C$ of Proposition 5.2.2.17. We begin by analyzing the special case $\mathcal{C} = \Delta^1$.

02FK **Lemma 6.2.3.1.** *Let $\mathcal{E}$ be an ∞-category equipped with a functor $U : \mathcal{E} \to \Delta^1$, having fibers $\mathcal{E}_0 = \{0\} \times_{\Delta^1} \mathcal{E}$ and $\mathcal{E}_1 = \{1\} \times_{\Delta^1} \mathcal{E}$. Let $f : X \to Y$ be a morphism of $\mathcal{E}$. Then:*

- *The morphism f exhibits X as a $\mathcal{E}_0$-coreflection of Y (in the sense of Definition 6.2.2.1) if and only if X belongs to $\mathcal{E}_0$ and f is π-cartesian.*
- *The morphism f exhibits Y as a $\mathcal{E}_1$-reflection of X if and only if Y belongs to $\mathcal{E}_1$ and f is π-cocartesian.*

Proof. This is a special case of Corollary 5.1.2.3. □

02FL **Corollary 6.2.3.2.** *Let $\mathcal{E}$ be an ∞-category equipped with a functor $U : \mathcal{E} \to \Delta^1$. Then:*

- *The functor U is a cartesian fibration if and only if the full subcategory $\{0\} \times_{\Delta^1} \mathcal{E} \subseteq \mathcal{E}$ is coreflective.*
- *The functor U is a cocartesian fibration if and only if the full subcategory $\{1\} \times_{\Delta^1} \mathcal{E} \subseteq \mathcal{E}$ is reflective.*

Remark 6.2.3.3. Let $U : \mathcal{E} \to \Delta^1$ be a functor of ∞-categories having fibers $\mathcal{E}_0 = \{0\} \times_{\Delta^1} \mathcal{E}$ and $\mathcal{E}_1 = \{1\} \times_{\Delta^1} \mathcal{E}$. Suppose that U is a cocartesian fibration, so that the full subcategory $\mathcal{E}_1 \subseteq \mathcal{E}$ is reflective (Corollary 6.2.3.2). By virtue of Lemma 6.2.2.14, there exists a $\mathcal{E}_1$-reflection functor $L : \mathcal{E} \to \mathcal{E}_1$. Then the restriction $L|_{\mathcal{E}_0} : \mathcal{E}_0 \to \mathcal{E}_1$ is given by covariant transport along the unique nondegenerate edge e of Δ^1 (in the sense of Definition 5.2.2.4). More precisely, if $\eta : \mathrm{id}_{\mathcal{E}} \to L$ is a natural transformation which exhibits L as a $\mathcal{E}_1$-reflection functor, then η carries each object $X \in \mathcal{E}$ to a U-cocartesian morphism $\eta_X : X \to L(X)$, so that η restricts to a natural transformation $\mathrm{id}_{\mathcal{E}_0} \to L|_{\mathcal{E}_0}$ which witnesses $L|_{\mathcal{E}_0}$ as given by covariant transport along e. `02FM`

Similarly, if U is a cartesian fibration, then the full subcategory $\mathcal{E}_0 \subseteq \mathcal{E}$ is coreflective; if $L' : \mathcal{E} \to \mathcal{E}_0$ is a $\mathcal{E}_0$-coreflection functor, then the restriction $L'|_{\mathcal{E}_1} : \mathcal{E}_1 \to \mathcal{E}_0$ is given by contravariant transport along e, in the sense of Definition 5.2.2.15.

Proposition 6.2.3.4. *Let $\mathcal{E}$ be an ∞-category equipped with a cocartesian fibration $U : \mathcal{E} \to \Delta^1$, having fibers $\mathcal{E}_0 = \{0\} \times_{\Delta^1} \mathcal{E}$ and $\mathcal{E}_1 = \{1\} \times_{\Delta^1} \mathcal{E}$. Let $F : \mathcal{E}_0 \to \mathcal{E}_1$ be a functor given by covariant transport along the nondegenerate edge e of Δ^1. Then the functor F admits a right adjoint if and only if U is a cartesian fibration. In this case, the right adjoint to F is given by contravariant transport along e.* `02FN`

Proof. Let $\iota_0 : \mathcal{E}_0 \hookrightarrow \mathcal{E}$ and $\iota_1 : \mathcal{E}_1 \hookrightarrow \mathcal{E}$ denote the inclusion maps. Since U is a cocartesian fibration, $\mathcal{E}_1$ is a reflective subcategory of $\mathcal{E}$ (Corollary 6.2.3.2). Let $L : \mathcal{E} \to \mathcal{E}_1$ be a $\mathcal{E}_1$-reflection functor (Lemma 6.2.2.14). Without loss of generality, we may assume that the functor $F : \mathcal{E}_0 \to \mathcal{E}_1$ factors as a composition $\mathcal{E}_0 \xrightarrow{\iota_0} \mathcal{E} \xrightarrow{L} \mathcal{E}_1$ (Remark 6.2.3.3). Note that L is a left adjoint to the inclusion $\iota_1 : \mathcal{E}_1 \hookrightarrow \mathcal{E}$ (Proposition 6.2.2.15).

Suppose that U is also a cartesian fibration, so that the subcategory $\mathcal{E}_0 \subseteq \mathcal{E}$ is coreflective (Corollary 6.2.3.2). Let $L' : \mathcal{E} \to \mathcal{E}_0$ be a $\mathcal{E}_0$-coreflection functor (Corollary 6.2.3.2), so that L' can be regarded as a right adjoint to ι_0 (Proposition 6.2.2.15). Invoking Remark 6.2.1.8, we conclude that the composite functor $F = L \circ \iota_0$ has a right adjoint G, given by the composition $L' \circ \iota_1 = L'|_{\mathcal{E}_1}$. Moreover, Remark 6.2.3.3 guarantees that $G : \mathcal{E}_1 \to \mathcal{E}_0$ is given by contravariant transport along e.

We now prove the converse. Suppose that the functor $F : \mathcal{E}_0 \to \mathcal{E}_1$ admits a right adjoint $G : \mathcal{E}_1 \to \mathcal{E}_0$. Fix an object $Z \in \mathcal{E}_1$; we wish to show that there exists an object $Y \in \mathcal{E}_0$ and a U-cartesian morphism $f : Y \to Z$. Let $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{E}_1}$ be the counit of an adjunction between F and G. Set $Y = G(Z)$, so that ϵ determines a morphism $\epsilon_Z : F(Y) \to Z$ in the ∞-category $\mathcal{E}_1$. Let $\eta : \mathrm{id}_{\mathcal{E}} \to L$ be a natural transformation which exhibits L as a $\mathcal{E}_1$-reflection functor, so that η determines a morphism $\eta_Y : Y \to F(Y)$. Let $f : Y \to Z$ be a composition of η_Y with ϵ_Z. We will complete the proof by showing that f is U-cartesian. To prove this, it will suffice to show that for every object $X \in \mathcal{E}_0$, the composite map

$$\mathrm{Hom}_{\mathcal{E}_0}(X, Y) \xrightarrow{[\eta_Y]\circ} \mathrm{Hom}_{\mathcal{E}}(X, F(Y)) = \mathrm{Hom}_{\mathcal{E}}(X, (F \circ G)(Z)) \xrightarrow{[\epsilon_Z]\circ} \mathrm{Map}_{\mathcal{E}}(X, Z)$$

is an isomorphism in the homotopy category hKan (see Corollary 5.1.2.3). Unwinding the definitions, we see that this map factors as a composition

$$\mathrm{Hom}_{\mathcal{E}_0}(X, G(Z)) \xrightarrow{F} \mathrm{Hom}_{\mathcal{E}_1}(F(X), (F \circ G)(Z)) \xrightarrow{[\epsilon_Z]\circ} \mathrm{Hom}_{\mathcal{E}_1}(F(X), Z) \xrightarrow{\circ[\eta_X]} \mathrm{Hom}_{\mathcal{E}}(X, Z),$$

where the composition of the first two maps is an isomorphism in hKan because ϵ is the counit of an adjunction (see Proposition 6.2.1.17), and third is an isomorphism because η_X exhibits $F(X)$ as a $\mathcal{E}_1$-reflection of X. □

02FP **Proposition 6.2.3.5.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. The following conditions are equivalent:*

(1) *The morphism U is a cartesian fibration of simplicial sets.*

(2) *For every edge $e : C \to D$ of the simplicial set $\mathcal{C}$, the covariant transport functor $e_! : \mathcal{E}_C \to \mathcal{E}_D$ of Notation 5.2.2.9 admits a right adjoint.*

Moreover, if these conditions are satisfied and $e : C \to D$ is an edge of $\mathcal{C}$, then the contravariant transport functor $e^* : \mathcal{E}_D \to \mathcal{E}_C$ of Notation 5.2.2.18 is right adjoint to $e_!$.

Proof. Assume first that condition (1) is satisfied and let $e : C \to D$ be an edge of the simplicial set $\mathcal{C}$, which we identify with a morphism $\Delta^1 \to \mathcal{C}$. Applying Proposition 6.2.3.4 to the projection map $\Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \Delta^1$, we deduce that the covariant transport functor $e_! : \mathcal{E}_C \to \mathcal{E}_D$ is right adjoint to the contravariant transport functor $e^* \mathcal{E}_D \to \mathcal{E}_C$, which proves (2).

We now show that (2) implies (1). By virtue of Proposition 5.1.4.7, we may assume without loss of generality that $\mathcal{C} = \Delta^n$ is a standard simplex. For $0 \leq i \leq n$, let $\mathcal{E}_i$ denote the fiber $\{i\} \times_{\Delta^n} \mathcal{E}$, which we regard as a full subcategory of $\mathcal{E}$. We wish to show that, for every pair of integers $0 \leq j < k \leq n$ and every object $Z \in \mathcal{E}_k$, there exists an object $Y \in \mathcal{E}_j$ and a U-cartesian morphism $g : Y \to Z$ in $\mathcal{E}$. Proposition 6.2.3.4 implies that the projection map $\mathrm{N}_\bullet(\{j < k\}) \times_{\Delta^n} \mathcal{E} \to \mathrm{N}_\bullet(\{j < k\})$ is a cartesian fibration, so we can choose an object $Y \in \mathcal{E}_j$ and a morphism $g : Y \to Z$ which is locally U-cartesian. We will complete the proof by showing that g is U-cartesian. To prove this, we must show that for each integer $0 \leq i \leq j$ and each object $W \in \mathcal{E}_i$, composition with the homotopy class $[g]$ induces an isomorphism $\mathrm{Hom}_{\mathcal{E}}(W, Y) \xrightarrow{[g]\circ} \mathrm{Hom}_{\mathcal{E}}(W, Z)$ in the homotopy category of Kan complexes hKan (see Corollary 5.1.2.3). Since U is a cocartesian fibration, we can choose a U-cocartesian morphism $f : W \to X$, where X belongs to $\mathcal{E}_j$. We conclude by observing that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(X,Y) \arrow[r, "{[g]\circ}", "\sim"'] \arrow[d, "\circ[f]", "\sim"'] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "\circ[f]", "\sim"'] \\
\mathrm{Hom}_{\mathcal{E}}(W,Y) \arrow[r, "{[g]\circ}"] & \mathrm{Hom}_{\mathcal{E}}(W,Z)
\end{tikzcd}$$

in the homotopy category hKan, where the upper horizontal map is an isomorphism by virtue of our assumption that g is locally U-cartesian, and the vertical maps are isomorphisms by virtue of our assumption that f is U-cocartesian (Corollary 5.1.2.3). $\square$

6.2.4 Local Existence Criterion

`02FQ` Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between categories. Suppose that G admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$. For each object $X \in \mathcal{C}$, the value $F(X) \in \mathcal{D}$ is determined, up to canonical isomorphism, by the property that it *corepresents* the functor $Z \mapsto \mathrm{Hom}_{\mathcal{D}}(X, G(Z))$: that is, there exists a bijection $\mathrm{Hom}_{\mathcal{D}}(F(X), Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G(Z))$ which depends functorially on Z. This observation has a converse: if, for every object $X \in \mathcal{C}$, the functor

$$\mathcal{D} \to \mathrm{Set} \qquad Z \mapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Z))$$

is corepresentable by an object of $\mathcal{D}$, then the functor G admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$ (Corollary 6.2.4.4). Our goal in this section is to establish a counterpart of this criterion in the ∞-categorical setting. We begin with a simple observation.

`02FV` **Proposition 6.2.4.1.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. Then G admits a left adjoint if and only if, for every object $X \in \mathcal{C}$, the following condition is satisfied:*

($*_X$) *There exists an object $Y \in \mathcal{D}$ and a morphism $u : X \to G(Y)$ in $\mathcal{C}$ such that, for every object $Z \in \mathcal{D}$, the composite map*

$$\mathrm{Hom}_{\mathcal{D}}(Y,Z) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Z)) \xrightarrow{\circ[u]} \mathrm{Hom}_{\mathcal{C}}(X, G(Z))$$

is a homotopy equivalence of Kan complexes.

Proof. We first prove necessity. Suppose that there exists a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ which exhibits F as a left adjoint of G. Fix an object $X \in \mathcal{C}$ and set $Y = F(X)$. Then η determines a morphism $\eta_X : X \to G(Y)$ which satisfies the requirement of condition ($*_X$) (Proposition 6.2.1.17).

We now prove sufficiency. Let $\mathcal{E}$ denote the relative join $\mathcal{C} \star_{\mathcal{C}} \mathcal{D}$ and let $U : \mathcal{E} \to \Delta^1$ be the cartesian fibration of Proposition 5.2.3.15. Let us abuse notation by identifying the fibers

$\{0\} \times_{\Delta^1} \mathcal{E}$ and $\{1\} \times_{\Delta^1} \mathcal{E}$ with $\mathcal{C}$ and $\mathcal{D}$, respectively. Fix an object $X \in \mathcal{C}$, and suppose that there exists an object $Y \in \mathcal{D}$ together with a morphism $u : X \to G(Y)$ satisfying the requirement of condition $(*_X)$. Then we can identify u with a morphism $f : X \to Y$ in the ∞-category $\mathcal{E}$. Our assumption on u guarantees that the morphism f is U-cocartesian (see Corollary 5.1.2.3). Consequently, if condition $(*_X)$ is satisfied for every object $X \in \mathcal{C}$, then U is a cocartesian fibration. Applying Proposition 6.2.3.4, we conclude that G admits a left adjoint. □

02J9 **Corollary 6.2.4.2.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. The following conditions are equivalent:*

(1) *The functor G admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$.*

(2) *For every left fibration $\widetilde{\mathcal{C}} \to \mathcal{C}$, if the ∞-category $\widetilde{\mathcal{C}}$ has an initial object, then the ∞-category $\mathcal{D} \times_{\mathcal{C}} \widetilde{\mathcal{C}}$ also has an initial object.*

(3) *For every object $X \in \mathcal{C}$, the ∞-category $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/}$ has an initial object.*

(4) *For every corepresentable* hKan-*enriched functor $\lambda : \mathrm{h}\mathcal{C} \to \mathrm{hKan}$, the composite functor*

$$\mathrm{h}\mathcal{D} \xrightarrow{\mathrm{h}G} \mathrm{h}\mathcal{C} \xrightarrow{\lambda} \mathrm{hKan}$$

is also corepresentable (in the sense of Definition 5.6.6.10).

(5) *For every corepresentable functor $\lambda : \mathcal{C} \to \mathcal{S}$ of ∞-categories, the composite functor*

$$\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{\lambda} \mathcal{S}$$

is also corepresentable (in the sense of Definition 5.6.6.1).

Proof. The equivalence (1) ⇔ (4) is a reformulation of Proposition 6.2.4.1. The implication (2) ⇒ (3) is immediate. To see that (3) implies (4), we observe that if $\lambda : \mathrm{h}\mathcal{C} \to \mathrm{hKan}$ is an hKan-enriched functor which is corepresentable by an object $X \in \mathcal{C}$, then $\lambda \circ \mathrm{h}G$ is isomorphic to the enriched homotopy transport representation of the left fibration $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/} \to \mathcal{D}$. If $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/}$ has an initial object, then this functor is corepresentable by virtue of Proposition 5.6.6.21.

The implication (4) ⇒ (5) follows from Remark 5.6.6.11. We will complete the proof by showing that (5) implies (2). Let $U : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a left fibration, and let $\mathrm{Tr}_{\widetilde{\mathcal{C}}/\mathcal{C}} : \mathcal{C} \to \mathcal{S}$ be a covariant transport representation for U (see Definition 5.6.5.1). If $\widetilde{\mathcal{C}}$ has an initial object, then the functor $\mathrm{Tr}_{\widetilde{\mathcal{C}}/\mathcal{C}}$ is corepresentable (Proposition 5.6.6.21). Assumption (5) then guarantees that the functor $\mathrm{Tr}_{\widetilde{\mathcal{C}}/\mathcal{C}} \circ G$ is also corepresentable. Identifying $\mathrm{Tr}_{\widetilde{\mathcal{C}}/\mathcal{C}} \circ G$ with the covariant transport representation of the left fibration $\mathcal{D} \times_{\mathcal{C}} \widetilde{\mathcal{C}} \to \mathcal{D}$, we see that the ∞-category $\mathcal{D} \times_{\mathcal{C}} \widetilde{\mathcal{C}}$ also has an initial object (Proposition 5.6.6.21). □

Remark 6.2.4.3. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories which satisfies the equivalent conditions of Corollary 6.2.4.2, so that G admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$. For each object $X \in \mathcal{C}$, the value $F(X) \in \mathcal{D}$ admits several characterizations: 038S

- The object $F(X)$ corepresents the hKan-enriched functor

$$\mathrm{h}\mathcal{D} \xrightarrow{\mathrm{h}G} \mathrm{h}\mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, \bullet)} \mathrm{hKan}.$$

- The object $F(X)$ corepresents the functor of ∞-categories

$$\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{h^X} \mathcal{S},$$

  where h^X is the functor corepresented by X.

- The object $F(X)$ is the image in $\mathcal{D}$ of an initial object of the ∞-category $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/}$.

Corollary 6.2.4.4. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor between ordinary categories. The following conditions are equivalent:* 02FW

(1) *The functor G admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$.*

(2) *For every object $X \in \mathcal{C}$, the set-valued functor*

$$\mathcal{D} \to \mathrm{Set} \qquad Z \mapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Z))$$

is corepresentable.

Corollary 6.2.4.5. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors between ∞-categories, and let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be a natural transformation. The following conditions are equivalent:* 02FX

(1) *The natural transformation η is the unit of an adjunction between F and G.*

(2) *For every pair of objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the composite map*

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}((G \circ F)(X), G(Y)) \xrightarrow{\circ[\eta_X]} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

is a homotopy equivalence of Kan complexes.

(3) *The functor F admits a right adjoint. Moreover, for every pair of objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, the composite map*

$$\mathrm{Hom}_{\mathrm{h}\mathcal{D}}(F(X), Y) \xrightarrow{G} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}((G \circ F)(X), G(Y)) \xrightarrow{\circ[\eta_X]} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, G(Y))$$

is a bijection of sets.

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 6.2.1.17, the implication (2) $\Rightarrow$ (3) follows from Proposition 6.2.4.1. We will complete the proof by showing that (3) $\Rightarrow$ (1). Note that, if condition (3) is satisfied, then the natural transformation η exhibits $\mathrm{h}G : \mathrm{h}\mathcal{D} \to \mathrm{h}\mathcal{C}$ as a right adjoint of the functor $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ (see Variant 6.1.2.11). Invoking Proposition 6.2.1.14, we deduce that η is the unit of an adjunction between F and G. $\square$

02KD **Corollary 6.2.4.6.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories and let $u : K \to \mathcal{D}$ be a morphism of simplicial sets, so that G induces a functor of coslice ∞-categories $G' : \mathcal{D}_{/u} \to \mathcal{C}_{/(G \circ u)}$. If the functor G admits a left adjoint, then the functor G' also admits a left adjoint.*

Proof. We will use the criterion of Corollary 6.2.4.2. Fix an object $\overline{X} \in \mathcal{C}_{/(G \circ u)}$; we wish to show that the ∞-category

$$\mathcal{E} = \mathcal{D}_{/u} \times_{\mathcal{C}_{/G \circ u}} (\mathcal{C}_{/(G \circ u)})_{\overline{X}/}$$

has an initial object. Let X denote the image of $\overline{X}$ in the ∞-category $\mathcal{C}$. Unwinding the definitions, we can identify $\overline{X}$ with a morphism of simplicial sets $\overline{u} : K \to (\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/})$, and $\mathcal{E}$ with the slice ∞-category $(\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/})_{/\overline{u}}$. Since G admits a left adjoint, the ∞-category $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/}$ has an initial object (Corollary 6.2.4.2). Applying Corollary 7.1.3.20, we conclude that $\mathcal{E}$ also has an initial object. $\square$

6.2.5 Digression: ∞-Categories with Short Morphisms

03XB Let $\mathcal{C}$ be a category. Recall that $\mathcal{C}$ is *free* if every morphism $f : X \to Y$ of $\mathcal{C}$ factors uniquely as a composition

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n = Y,$$

where each f_i is an indecomposable morphism of $\mathcal{C}$ (see Proposition 1.3.7.11). In this case, Proposition 1.5.7.3 asserts that the inclusion map $G \hookrightarrow \mathrm{N}_{\bullet}(\mathcal{C})$ is inner anodyne, where G is the 1-dimensional simplicial set whose vertices are the objects of $\mathcal{C}$ and whose nondegenerate edges are the indecomposable morphisms of $\mathcal{C}$. Our goal in this section is to prove a more general result, where we relax the assumption that $\mathcal{C}$ is free. First, we need a definition.

03XC **Definition 6.2.5.1.** Let $\mathcal{C}$ be an ∞-category and let S be a collection of morphisms of $\mathcal{C}$. An *S-optimal factorization of f* is a 2-simplex

03XD
$$\begin{tikzcd} & Y \arrow[dr, "s"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] & & Z \end{tikzcd} \qquad (6.2)$$

of $\mathcal{C}$, corresponding to a morphism $\widetilde{g} : \widetilde{X} \to \widetilde{Y}$ in the ∞-category $\mathcal{C}_{/Z}$ with the following properties:

- The morphism $s : Y \to Z$ belongs to S.
- Let $\widetilde{Y}'$ be an object of $\mathcal{C}_{/Z}$ corresponding to a morphism $s' : Y' \to Z$ which belongs to S. Then composition with $\widetilde{g}$ induces a homotopy equivalence of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}_{/Z}}(\widetilde{Y}, \widetilde{Y}') \to \mathrm{Hom}_{\mathcal{C}_{/Z}}(\widetilde{X}, \widetilde{Y}').$$

If these conditions are satisfied, we say that the diagram (6.2) is an *S-optimal factorization of f*.

Example 6.2.5.2. In the situation of Definition 6.2.5.1, assume that $\mathcal{C}$ is (the nerve of) an ordinary category. Then an S-optimal factorization of a morphism $f : X \to Z$ is a pair of morphisms $X \xrightarrow{g} Y \xrightarrow{s} Z$, where $s \in S$ and $s \circ g = f$, which has the following universal property: for every other pair of morphisms $X \xrightarrow{g'} Y' \xrightarrow{s'} Z$ with $s' \in S$ and $s' \circ g' = f$, there is a unique morphism $h : Y \to Y'$ satisfying $h \circ g = g'$ and $s' \circ h = s$, as indicated in the diagram 03XE

$$\begin{tikzcd}
 & Y \arrow[dr, "s"] \arrow[dd, dashed, "h"] & \\
X \arrow[ur, "g"] \arrow[dr, "g'"'] & & Z \\
 & Y'. \arrow[ur, "s'"'] &
\end{tikzcd}$$

Stated more informally, the pair (g, s) is *universal* among all factorizations of f through a morphism which belongs to S.

Example 6.2.5.3. Let G be a directed graph, let $\mathcal{C} = \mathrm{Path}[G]$ denote its path category (Construction 1.3.7.1), and let S be the collection of morphisms of $\mathcal{C}$ which are either identity morphisms or are indecomposable. Then every morphism $f : X \to Z$ in $\mathcal{C}$ admits a (unique) S-optimal factorization: 03XF

- If $f = \mathrm{id}_X$ is an identity morphism, then its S-optimal factorization is given by the diagram $X \xrightarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$.
- If f is not an identity morphism, then it admits a unique factorization $X \xrightarrow{g} Y \xrightarrow{s} Z$, where s is an indecomposable morphism of $\mathcal{C}$ (that is, a morphism which corresponds to an edge of the graph G); this factorization is S-optimal.

03XG **Definition 6.2.5.4.** Let $\mathcal{C}$ be an ∞-category. A *class of short morphisms for* $\mathcal{C}$ is a collection S of morphisms of $\mathcal{C}$ with the following properties:

(1) Every identity morphism of $\mathcal{C}$ belongs to S.

(2) For every 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "f'"] & \\ X \arrow[ur, "f''"] \arrow[rr, "f"] & & Z \end{tikzcd}$$

of the ∞-category $\mathcal{C}$, if f and f' belong to S, then f'' also belongs to S.

(3) Every morphism $f : X \to Z$ of $\mathcal{C}$ admits an S-optimal factorization (Definition 6.2.5.1).

(4) Every morphism of $\mathcal{C}$ can be obtained as a composition of morphisms which belong to S.

03XH **Remark 6.2.5.5.** Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms for $\mathcal{C}$. Let $f : X \to Y$ and $g : X \to Y$ be morphisms of $\mathcal{C}$ which are homotopic. If f belongs to S, then g also belongs to S. This follows by applying property (2) of Definition 6.2.5.4 to a 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "\mathrm{id}"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] & & Y. \end{tikzcd}$$

03XJ **Notation 6.2.5.6.** Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms for $\mathcal{C}$. We let $\mathcal{C}^{\mathrm{short}} \subseteq \mathcal{C}$ denote the simplicial subset consisting of those simplices $\sigma : \Delta^n \to \mathcal{C}$ having the following property:

$(*)$ For every pair of integers $0 \leq i \leq j \leq n$, the induced morphism $\sigma(i) \to \sigma(j)$ belongs to S.

Note that, since S contains all identity morphisms of $\mathcal{C}$, condition $(*)$ is automatically satisfied in the case $i = j$. In particular, every vertex of $\mathcal{C}$ is contained in $\mathcal{C}^{\mathrm{short}}$, and an edge of $\mathcal{C}$ is contained in $\mathcal{C}^{\mathrm{short}}$ if and only if belongs to S.

03XK **Remark 6.2.5.7.** Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms for $\mathcal{C}$. Then a simplex $\sigma : \Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}^{\mathrm{short}}$ if and only if, for every integer $0 \leq i < n$, the morphism $\sigma(i) \to \sigma(n)$ belongs to S. Condition $(*)$ of Notation 6.2.5.6 can be deduced from this *a priori* weaker assumption by applying assumption (2) of Definition 6.2.5.4 to the diagrams

$$\begin{tikzcd}
 & \sigma(j) \arrow[dr] & \\
\sigma(i) \arrow[ur] \arrow[rr] & & \sigma(n)
\end{tikzcd}$$

for $i \leq j \leq n$.

Remark 6.2.5.8. Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms of $\mathcal{C}$. Then the simplicial set $\mathcal{C}^{\mathrm{short}}$ is never an ∞-category, except in the trivial situation where S is the class of all morphisms of $\mathcal{C}$ (in which case we have $\mathcal{C}^{\mathrm{short}} = \mathcal{C}$). However, for every object $Z \in \mathcal{C}$, the simplicial set $\mathcal{C}^{\mathrm{short}}_{/Z} = (\mathcal{C}^{\mathrm{short}})_{/Z}$ is an ∞-category, since it can be identified with the full subcategory of $\mathcal{C}_{/Z}$ spanned by those morphisms $s : Y \to Z$ which belong to S. 03XL

Remark 6.2.5.9. Let $\mathcal{C}$ be an ∞-category, let S be a class of short morphisms of $\mathcal{C}$, and let $f : X \to Z$ be a morphism of $\mathcal{C}$, which we identify with an object $\widetilde{X}$ of the slice ∞-category $\mathcal{C}_{/Z}$. Then an S-optimal factorization of f can be viewed as a morphism $\widetilde{X} \to \widetilde{Y}$ in $\mathcal{C}_{/Z}$ which exhibits $\widetilde{Y}$ as a $\mathcal{C}^{\mathrm{short}}_{/Z}$-reflection of $\widetilde{X}$, in the sense of Definition 6.2.2.1. Consequently, condition (3) of Definition 6.2.5.4 is equivalent to the requirement that the full subcategory $\mathcal{C}^{\mathrm{short}}_{/Z} \subseteq \mathcal{C}_{/Z}$ is reflective, for each object $Z \in \mathcal{C}$. 03XM

We can now state our main result.

Theorem 6.2.5.10. *Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms for $\mathcal{C}$. Then the inclusion map $\mathcal{C}^{\mathrm{short}} \hookrightarrow \mathcal{C}$ is an inner anodyne morphism of simplicial sets.* 03XN

Example 6.2.5.11. Let G be a directed graph and let $\mathcal{C} = \mathrm{Path}[G]$ denote its path category (Construction 1.3.7.1). Let S be the collection of morphisms of $\mathcal{C}$ which are either identity morphisms or are indecomposable. Then S is a class of short morphisms for the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$ (the existence of S-optimal factorizations follows from Example 6.2.5.3, and the remaining requirements are immediate from the definitions). Moreover, the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C})^{\mathrm{short}}$ can be identified with the directed graph G (regarded as a 1-dimensional simplicial set; see §1.1.6). Applying Theorem 6.2.5.10 in this case, we recover the statement that the inclusion map $G \hookrightarrow \mathrm{N}_{\bullet}(\mathcal{C})$ is inner anodyne (Proposition 1.5.7.3). 03XP

Our proof of Theorem 6.2.5.10 will require some auxiliary constructions.

Notation 6.2.5.12. Let $\mathcal{C}$ be an ∞-category, let S be a class of short morphisms for $\mathcal{C}$, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. We let $\ell(f)$ denote the smallest integer n such that f 03XQ can be written as the composition of n morphisms of S: that is, there exists an n-simplex $\sigma : \Delta^n \to \mathcal{C}$ which carries the spine $\mathrm{Spine}[n]$ into $\mathcal{C}^{\mathrm{short}}$, for which the composition

$$\Delta^1 \to \mathrm{N}_{\bullet}(\{0, n\}) \hookrightarrow \Delta^n \xrightarrow{\sigma} \mathcal{C}$$

coincides with f. Note that condition (4) of Definition 6.2.5.4 guarantees that $\ell(f) < \infty$. We will refer to $\ell(f)$ as the *S-length* of f. Note that $\ell(f) = 0$ if and only if f is an identity morphism of $\mathcal{C}$, and $\ell(f) \leq 1$ if and only if f belongs to S.

03XR **Lemma 6.2.5.13.** *Let $\mathcal{C}$ be an ∞-category, let S be a class of short morphisms of $\mathcal{C}$, and suppose we are given a 2-simplex*

03XS
$$\begin{tikzcd} & Y \arrow[dr, "s"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"] & & Z, \end{tikzcd} \tag{6.3}$$

of $\mathcal{C}$, where s belongs to S. Then:

(a) If $\ell(f) \geq 1$, then $\ell(g) \leq \ell(f)$.

(b) If $\ell(f) \geq 2$ and the factorization (6.3) is S-optimal, then $\ell(g) = \ell(f) - 1$.

Proof. We prove (a) and (b) by simultaneous induction on the length $n = \ell(f)$. If $n = 1$, then assertion (a) follows from condition (2) of Definition 6.2.5.4 and assertion (b) is vacuous. We therefore assume that $n \geq 2$. We first prove (b). Choose a factorization

$$\begin{tikzcd} & Y' \arrow[dr, "s'"] & \\ X \arrow[ur, "g'"] \arrow[rr, "f"] & & Z, \end{tikzcd}$$

where $s' \in S$ and $\ell(g') = n - 1$. If the factorization (6.3) is S-optimal, then we can choose a

3-simplex

$$\begin{tikzcd}
 & Y \arrow[dd, "h"] \arrow[dr, "s"] & \\
X \arrow[ur, "g"] \arrow[dr, "g'"'] & & Z \\
 & Y'. \arrow[ur, "s'"'] &
\end{tikzcd}$$

Condition (2) of Definition 6.2.5.4 guarantees that h belongs to S. Our inductive hypothesis guarantees that the left half of the diagram satisfies assertion (a); so that $\ell(g) \leq \ell(g') = \ell(f) - 1$. The reverse inequality follows immediately from the definition.

We now prove (a). Choose an S-optimal factorization

$$\begin{tikzcd}
 & Y'' \arrow[dr, "s''"] & \\
X \arrow[ur, "g''"] \arrow[rr, "f"] & & Z.
\end{tikzcd}$$

It follows from the preceding argument that $\ell(g'') = n - 1$. We then have a commutative diagram

$$\begin{tikzcd}
 & Y'' \arrow[dd, "j"] \arrow[dr, "s''"] & \\
X \arrow[ur, "g''"] \arrow[dr, "g"'] & & Z \\
 & Y, \arrow[ur, "s"'] &
\end{tikzcd}$$

and condition (2) of Definition 6.2.5.4 guarantees that j belongs to S. We therefore obtain $\ell(g) \leq \ell(g'') + \ell(j) \leq (n-1) + 1 = n$, as desired. □

Notation 6.2.5.14. Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms of $\mathcal{C}$. For every n-simplex σ of $\mathcal{C}$, we let $\mathrm{pr}(\sigma)$ denote the smallest nonnegative integer p such that, for $p \leq q \leq n$, the morphism $\sigma(q) \to \sigma(n)$ belongs to S. We will refer to $\mathrm{pr}(\sigma)$ as the *priority* of σ. This definition has the following properties: 03XT

- The simplex σ has priority 0 if and only if it belongs to the simplicial subset $\mathcal{C}^{\mathrm{short}}$ of Notation 6.2.5.6 (see Remark 6.2.5.7).
- For each $0 \leq i \leq n$, the face $\tau = d_i^n(\sigma)$ satisfies $\mathrm{pr}(\tau) \leq \mathrm{pr}(\sigma)$. The inequality is strict if $i < \mathrm{pr}(\sigma)$, and equality holds if $\mathrm{pr}(\sigma) \leq i < n$.
- For each $0 \leq i \leq n$, the degenerate simplex $\tau = s_i^n(\sigma)$ satisfies

$$\mathrm{pr}(\tau) = \begin{cases} \mathrm{pr}(\sigma) + 1 & \text{if } 0 \leq i < \mathrm{pr}(\sigma) \\ \mathrm{pr}(\sigma) & \text{if } \mathrm{pr}(\sigma) \leq i \leq n \end{cases}$$

In the situation of Notation 6.2.5.14, suppose that $\mathcal{C}$ is (the nerve of) an ordinary category. An n-simplex of $\mathcal{C}$ can then be viewed as a diagram

$$X_0 \to X_1 \to \cdots \to X_n,$$

whose transition maps we denote by $f_{j,i} : X_i \to X_j$. If σ does not belong to $\mathcal{C}^{\mathrm{short}}$, then the priority of σ is the smallest integer p for which the morphism $f_{n,p-1} : X_{p-1} \to X_n$ does not belong to S. Then $f_{n,p-1}$ admits an S-optimal factorization $X_{p-1} \xrightarrow{g} Y \xrightarrow{s} X_n$. Since the morphism $f_{n,p} : X_p \to X_n$ belongs to S, there is a unique morphism $h : Y \to X_p$ for which the diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "s"] \arrow[dd, "h"] & \\
X_{p-1} \arrow[ur, "g"] \arrow[dr, "f_{p,p-1}"'] & & X_n \\
 & X_p \arrow[ur, "f_{n,p}"'] &
\end{tikzcd}$$

is commutative. The diagram

$$X_0 \to \cdots \to X_{p-1} \xrightarrow{g} Y \xrightarrow{h} X_p \to \cdots \to X_n$$

then determines an $(n+1)$-simplex σ^+ of $\mathcal{C}$ having priority p, which satisfies $d_p^{n+1}(\sigma^+) = \sigma$. To prove Theorem 6.2.5.10, we will extend the construction $\sigma \mapsto \sigma^+$ to the case where $\mathcal{C}$ is an ∞-category.

03XU **Lemma 6.2.5.15.** *Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms of $\mathcal{C}$. Then there is a function which associates to each n-simplex σ of $\mathcal{C}$ which does not belong to $\mathcal{C}^{\mathrm{short}}$ an $(n+1)$-simplex σ^+ of $\mathcal{C}$, which has the following properties:*

(1) *The face operators satisfy*

$$d_i^{n+1}(\sigma^+) = \begin{cases} \sigma & \text{if } i = \mathrm{pr}(\sigma) \\ d_{i-1}^n(\sigma)^+ & \text{if } \mathrm{pr}(\sigma) < i \leq n. \end{cases}$$

(2) *Let $\sigma = s_j^{n-1}(\tau)$ be a degenerate n-simplex of $\mathcal{C}$. Then*

$$\sigma^+ = \begin{cases} s_j^n(\tau^+) & \text{if } 0 \leq j < \mathrm{pr}(\tau) \\ s_{j+1}^n(\tau^+) & \text{if } \mathrm{pr}(\tau) \leq j < n. \end{cases}$$

(3) *If $\sigma = \tau^+$ for some $(n-1)$-simplex τ of $\mathcal{C}$ having priority $p > 0$, then $\sigma^+ = s_p^n(\sigma)$.*

(4) *If $\mathrm{pr}(\sigma) = n$, then the 2-simplex*

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{n-1 < n < n+1\}) \hookrightarrow \Delta^{n+1} \xrightarrow{\sigma^+} \mathcal{C}$$

is an S-optimal factorization.

Exercise 6.2.5.16. Prove Lemma 6.2.5.15 in the special case where $\mathcal{C}$ is (the nerve of) an ordinary category. 03XV

We defer the (somewhat tedious) proof of Lemma 6.2.5.15 until the end of this section.

Proof of Theorem 6.2.5.10. Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms for $\mathcal{C}$. For every nonnegative integer p, let $\mathcal{C}^{\leq p}$ denote the smallest simplicial subset of $\mathcal{C}$ which contains all simplices of priority $\leq p$ (so that a nondegenerate simplex of $\mathcal{C}$ belongs to $\mathcal{C}^{\leq p}$ if and only if it has priority $\leq p$). Then $\mathcal{C}$ is the colimit of the sequence of inclusion maps

$$\mathcal{C}^{\mathrm{short}} = \mathcal{C}^{\leq 0} \hookrightarrow \mathcal{C}^{\leq 1} \hookrightarrow \mathcal{C}^{\leq 2} \hookrightarrow \cdots$$

We will complete the proof by showing that each of these inclusion maps is inner anodyne. For the remainder of the proof, we fix a positive integer p; our goal is to show that the inclusion map $\mathcal{C}^{\leq p-1} \hookrightarrow \mathcal{C}^{\leq p}$ is inner anodyne.

Choose a function $\sigma \mapsto \sigma^+$ satisfying the requirements of Lemma 6.2.5.15. For each integer $n \geq 0$, let $\mathcal{C}^{\leq p}(n)$ denote the simplicial subset of $\mathcal{C}^{\leq p}$ generated by $\mathcal{C}^{\leq p-1}$ together with all simplices of the form σ^+, where σ is a simplex of $\mathcal{C}$ having priority p and dimension $\leq n$. By virtue of requirement (2) of Lemma 6.2.5.15, it suffices to allow σ to range over *nondegenerate* simplices which satisfy these conditions. Note that each $\mathcal{C}^{\leq p}(n)$ contains the n-skeleton of $\mathcal{C}^{\leq p}$, so that $\mathcal{C}^{\leq p}$ can be realized as the colimit of the sequence

$$\mathcal{C}^{\leq p-1} = \mathcal{C}^{\leq p}(0) \hookrightarrow \mathcal{C}^{\leq p}(1) \hookrightarrow \mathcal{C}^{\leq p}(2) \hookrightarrow \mathcal{C}^{\leq p}(3) \hookrightarrow \cdots$$

It will therefore suffice to show that each of these inclusions is inner anodyne. For the remainder of the proof, we fix an integer $n > 0$; our goal is to show that the inclusion map $\mathcal{C}^{\leq p}(n-1) \hookrightarrow \mathcal{C}^{\leq p}(n)$ is inner anodyne.

Let $\{\sigma_t\}_{t \in T}$ denote the collection of n-simplices of $\mathcal{C}$ which have priority p but are not contained in $\mathcal{C}^{\leq p}(n-1)$. We first claim that $\mathcal{C}^{\leq p}(n)$ is generated by $\mathcal{C}^{\leq p}(n-1)$ together with the collection of $(n+1)$-simplices $\{\sigma_t^+\}_{t \in T}$. To prove this, it suffices to show that if σ is an n-simplex of $\mathcal{C}$ which belongs to $\mathcal{C}^{\leq p}(n-1)$, then σ^+ also belongs to $\mathcal{C}^{\leq p}(n-1)$. If σ is degenerate, then we can write $\sigma^+ = s_i^n(\sigma_0^+)$ when σ_0 is an $(n-1)$-simplex of $\mathcal{C}$ having priority $\leq p$ (Lemma 6.2.5.15), and the desired conclusion follows from the observation that σ_0^+ is contained in $\mathcal{C}^{\leq p}(n-1)$. We may therefore assume that σ is a nondegenerate n-simplex of $\mathcal{C}$. If σ has priority $< p$, then σ^+ also has priority $< p$ and is therefore contained in $\mathcal{C}^{\leq p-1}$. We may therefore assume that σ has priority p, and must therefore be of the form τ^+ where τ is an $(n-1)$-simplex of $\mathcal{C}$ having priority p. Condition (3) of Lemma 6.2.5.15 then guarantees that σ^+ can be obtained from σ by applying a degeneracy operator, and is therefore contained in $\mathcal{C}^{\leq p}(n-1)$ as desired.

For each $t \in T$, we define the *complexity of* t to be the integer $c(t) = \sum_{q=p}^{n} \ell(\sigma_t(p-1) \to \sigma_t(q))$. Using Proposition 4.7.1.35, we can choose a well-ordering on T for which the complexity function

$$c : T \to \mathbf{Z}_{\geq 0} \qquad t \mapsto c(t)$$

is nondecreasing. For each $t \in T$, let $\mathcal{C}^{\leq p}_{\leq t}(n)$ denote the simplicial subset of $\mathcal{C}^{\leq p}(n)$ generated by $\mathcal{C}^{\leq p}(n-1)$ together with the simplices σ_s^+ for $s \leq t$, and define $\mathcal{C}^{\leq p}_{<t}(n)$ similarly. Then the inclusion map $\mathcal{C}^{\leq p}(n-1) \hookrightarrow \mathcal{C}^{\leq p}(n)$ can be realized as a transfinite composition of inclusion maps $\{\mathcal{C}^{\leq p}_{<t}(n) \hookrightarrow \mathcal{C}^{\leq p}_{\leq t}(n)\}_{t \in T}$. It will therefore suffice to show that each of these inclusion maps is inner anodyne.

Fix an element $t \in T$ and let $L_t \subseteq \Delta^{n+1}$ be the inverse image of $\mathcal{C}^{\leq p}_{<t}(n)$ under the map $\sigma_t^+ : \Delta^{n+1} \to \mathcal{C}$, so that we have a pullback diagram of simplicial sets

03XW
$$\begin{tikzcd}
L_t \arrow[r] \arrow[d] & \mathcal{C}^{\leq p}_{<t}(n) \arrow[d] \\
\Delta^{n+1} \arrow[r, "\sigma_t^+"] & \mathcal{C}^{\leq p}_{\leq t}(n).
\end{tikzcd} \tag{6.4}$$

We will complete the proof by showing that L_t coincides with the inner horn $\Lambda^{n+1}_p \subseteq \Delta^{n+1}$, so that the diagram (6.4) is also a pushout square (Lemma 3.1.2.11). This is equivalent to the following more concrete assertion:

$(*_t)$ For $0 \leq i \leq n+1$, the n-simplex $d_i^{n+1}(\sigma_t^+)$ belongs to $\mathcal{C}^{\leq p}_{<t}(n)$ if and only if $i \neq p$.

Our proof proceeds by induction on t. We consider several cases:

- For $0 \leq i < p$, the n-simplex $d_i^{n+1}(\sigma_t^+)$ has priority $< p$, and is therefore contained in $\mathcal{C}^{<p} \subseteq \mathcal{C}^{\leq p}(n-1) \subseteq \mathcal{C}^{\leq p}_{<t}(n)$.
- For $i = p$, the n-simplex $d_i^{n+1}(\sigma_t^+)$ coincides with σ_t (Lemma 6.2.5.15), which is not contained in $\mathcal{C}^{\leq p}(n-1)$. Consequently, if σ_t is contained in $\mathcal{C}^{\leq p}_{<t}(n-1)$, then there exists some $t' < t$ such that σ_t is contained in $\mathcal{C}^{\leq p}_{\leq t'}(n-1)$ but not in $\mathcal{C}^{\leq p}_{<t'}(n-1)$. Applying our inductive hypothesis, we deduce that $\sigma_t = \sigma_{t'}$, which contradicts the inequality $t' < t$.
- For $p < i \leq n$, condition (1) of Lemma 6.2.5.15 implies that $d_i^{n+1}(\sigma_t^+)$ coincides with $d_{i-1}^n(\sigma_t)^+$, and therefore belongs to $\mathcal{C}^{\leq p}(n-1) \subseteq \mathcal{C}^{\leq p}_{<t}(n)$.
- Suppose that $i = n+1$ and set $\tau = d_i^{n+1}(\sigma_t^+)$; we wish to show that τ is contained in $\mathcal{C}^{\leq p}_{<t}(n)$. Note that τ has priority $\leq p$. If τ is contained in $\mathcal{C}^{\leq p}(n-1)$, there is nothing to prove. We may therefore assume without loss of generality that τ is contained in T: that is, we have $\tau = \sigma_{t'}$ for some $t' \in T$. Set $X = \sigma_t(p-1) = \sigma_{t'}(p-1)$. For $p \leq q \leq n$, let $f_q : X \to \sigma_t(q)$ be the morphism determined by σ_t, and define $f'_q : X \to \sigma_{t'}(q)$ similarly. By construction, the morphism f_q coincides with f'_{q+1} for $p \leq q < n$. Moreover, the restriction of σ_t^+ to the 2-simplex $\mathrm{N}_\bullet(\{p-1 < p < n+1\}$ determines a diagram

$$\begin{tikzcd}
 & \sigma_{t'}(p) \arrow[dr] & \\
X \arrow[ur, "f'_p"] \arrow[rr, "f_n"] & & \sigma_t(n)
\end{tikzcd}$$

  which is an S-optimal factorization of f_n, so that $\ell(f'_p) = \ell(f_n) - 1$ by virtue of Lemma 6.2.5.13. It follows that the complexity $c(\sigma_{t'})$ is given by

$$\begin{aligned}
c(\sigma_{t'}) &= \sum_{q=p}^{n} \ell(f'_q) \\
&= \ell(f'_p) + \sum_{q=p+1}^{n} \ell(f'_q) \\
&= \ell(f_n) - 1 + \sum_{q=p}^{n-1} \ell(f_q) \\
&= (\sum_{q=p}^{n} \ell(f_q)) - 1 \\
&= c(\sigma_t) - 1.
\end{aligned}$$

  We therefore have $t' < t$, so that $\tau = \sigma_{t'} = d_p^{n+1}(\sigma_{t'}^+)$ is contained in $\mathcal{C}^{\leq p}_{\leq t'}(n) \subseteq \mathcal{C}^{\leq p}_{<t}(n)$.

$\square$

Proof of Lemma 6.2.5.15. Let $\mathcal{C}$ be an ∞-category and let S be a class of short morphisms for $\mathcal{C}$. Our construction proceeds by recursion. Fix an integer $n \geq 0$. Assume that we have constructed a function $\tau \mapsto \tau^{+}$ on simplices of $\mathcal{C}$ having dimension $< n$ and priority > 0, satisfying conditions (1) through (4) of Lemma 6.2.5.15. Let σ be an n-simplex of $\mathcal{C}$ having priority > 0; we wish to show that there is an $(n+1)$-simplex σ^{+} which also satisfies conditions (1) through (4). Let us say that σ of $\mathcal{C}$ is *free* if it is not of the form τ^{+}, where τ is an $(n-1)$-simplex of priority > 0. We divide the construction into three cases:

(a) The n-simplex σ is not free.

(b) The n-simplex σ is free and degenerate.

(c) The n-simplex σ is free and nondegenerate.

We begin with case (a). Assume that $\sigma = \tau^{+}$, where τ is an $(n-1)$-simplex of $\mathcal{C}$ having priority $p > 0$. It follows from our inductive hypothesis that σ has the same priority p, and that $\tau = d^{n}_{p}(\sigma)$. In particular, τ is uniquely determined by σ. In this case, we define $\sigma^{+} = s^{n}_{p}(\sigma)$, so that condition (3) is satisfied by construction. Since $p \leq n-1 < n$, condition (4) is vacuous. Note that the faces $d^{n+1}_{p}(\sigma^{+})$ and $d^{n+1}_{p+1}(\sigma^{+})$ coincide with $\sigma = \tau^{+} = d^{n}_{p}(\sigma)^{+}$, so that condition (1) is satisfied for $i = p$ and $i = p+1$. For $p+1 < i \leq n$, we compute

$$\begin{aligned} d^{n+1}_{i}(\sigma^{+}) &= d^{n+1}_{i}(s^{n}_{p}(\tau^{+})) \\ &= s^{n-1}_{p}(d^{n}_{i-1}(\tau^{+})) \\ &= s^{n-1}_{p}(d^{n-1}_{i-2}(\tau)^{+}) \\ &= (d^{n-1}_{i-2}(\tau)^{+})^{+} \\ &= d^{n}_{i-1}(\tau^{+})^{+} \\ &= d^{n}_{i-1}(\sigma)^{+}. \end{aligned}$$

It remains to verify condition (2). Suppose that $\sigma = s^{n-1}_{j}(\sigma')$ for some $(n-1)$-simplex σ' of $\mathcal{C}$. Note that, since σ has priority p, we must have $j \neq p-1$ (see Notation 6.2.5.14). We first consider the case $j < p-1$, so that σ' has priority $p-1$. In this case, we wish to show that $\sigma^{+} = s^{n}_{j}(\sigma'^{+})$. Set $\tau' = d^{n-1}_{p-1}(\sigma')$. We then have

$$\tau = d^{n}_{p}(\sigma) = d^{n}_{p}(s^{n-1}_{j}(\sigma')) = s^{n-2}_{j}(d^{n-1}_{p-1}(\sigma')) = s^{n-2}_{j}(\tau'),$$

so that $\sigma = \tau^{+} = s^{n-1}_{j}(\tau'^{+})$. Applying the face operator d^{n}_{j}, we obtain $\sigma' = \tau'^{+}$, so that $\sigma'^{+} = s^{n-1}_{p-1}(\sigma')$. The desired result now follows from the calculation

$$\sigma^{+} = s^{n}_{p}(\sigma) = s^{n}_{p}(s^{n-1}_{j}(\sigma')) = s^{n}_{j}(s^{n-1}_{p-1}(\sigma')) = s^{n}_{j}(\sigma'^{+}).$$

We now treat the case $j \geq p$, so that σ' has priority p. In this case, we wish to show that $\sigma^{+} = s^{n}_{j+1}(\sigma'^{+})$. If $j = p$, this follows from the calculation

$$\begin{aligned} \sigma^{+} &= s^{n}_{p}(\sigma) \\ &= s^{n}_{p}(s^{n-1}_{p}(\sigma')) \\ &= s^{n}_{p+1}(s^{n-1}_{p}(\sigma')) \\ &= s^{n}_{p+1}(\sigma) \\ &= s^{n}_{p+1}(\tau^{+}). \end{aligned}$$

Let us therefore assume that $j > p$, and set $\tau' = d^{n-1}_{p}(\sigma')$. We then have

$$\tau = d^{n}_{p}(\sigma) = d^{n}_{p}(s^{n-1}_{j}(\sigma')) = s^{n-2}_{j-1}(d^{n-1}_{p}(\sigma')) = s^{n-2}_{j-1}(\tau'),$$

so that $\sigma = \tau^{+} = s^{n-1}_{j}(\tau'^{+})$. Applying the face operator d^{n}_{j}, we deduce that $\sigma' = \tau'^{+}$, so that $\sigma'^{+} = s^{n-1}_{p}(\sigma')$. The desired result now follows from the calculation

$$\sigma^{+} = s^{n}_{p}(\sigma) = s^{n}_{p}(s^{n-1}_{j}(\sigma')) = s^{n}_{j+1}(s^{n-1}_{p}(\sigma')) = s^{n}_{j+1}(\sigma'^{+}).$$

This completes our treatment of case (a).

We now consider case (b). Assume that σ is a free simplex of $\mathcal{C}$ of the form $s^{n-1}_{j}(\tau)$. Choose j as small as possible and let p be the priority of τ. We first treat the case where $j < p$, so that σ has priority $p+1$ (see Notation 6.2.5.14). In this case, we define $\sigma^{+} = s^{n}_{j}(\tau^{+})$, so that

$$d^{n+1}_{p+1}(\sigma^{+}) = d^{n+1}_{p+1}(s^{n}_{j}(\tau^{+})) = s^{n-1}_{j}(d^{n}_{p}(\tau^{+})) = s^{n-1}_{j}(\tau) = \sigma.$$

For $p + 1 < i \leq n$, a similar calculation gives

$$\begin{aligned} d^{n+1}_{i}(\sigma^{+}) &= d^{n+1}_{i}(s^{n}_{j}(\tau^{+})) \\ &= s^{n-1}_{j}(d^{n}_{i-1}(\tau^{+})) \\ &= s^{n-1}_{j}(d^{n-1}_{i-2}(\tau)^{+}) \\ &= s^{n-1}_{j}(d^{n-1}_{i-2}(\tau))^{+} \\ &= d^{n}_{i-1}(s^{n-1}_{j}(\tau))^{+} \\ &= d^{n}_{i-1}(\sigma)^{+}, \end{aligned}$$

which proves (1).

To verify (2), suppose that $\sigma = s^{n-1}_{j'}(\tau')$, for some $(n-1)$-simplex τ' of $\mathcal{C}$. Note that we must have $j' \geq j$. Since σ has priority $p+1$, we also have $j' \neq p$. Assume first that $j' < p$, so that τ' has priority p. In this case, we wish to show that $\sigma^{+} = s^{n}_{j'}(\tau'^{+})$. If $j' = j$, this is immediate. We may therefore assume that $j' > j$, so that we can write $\tau' = s^{n-2}_{j}(\tau'')$

and $\tau = s^{n-2}_{j'-1}(\tau'')$ for some unique $(n-2)$-simplex τ'' of $\mathcal{C}$. In this case, the desired result follows from the calculation

$$\begin{aligned} \sigma^+ &= s^n_j(\tau^+) \\ &= s^n_j(s^{n-2}_{j'-1}(\tau'')^+) \\ &= s^n_j(s^{n-1}_{j'-1}(\tau''^+)) \\ &= s^n_{j'} s^{n-1}_j(\tau''^+) \\ &= s^n_{j'}(s^{n-2}_j(\tau'')^+) \\ &= s^n_{j'}(\tau'^+). \end{aligned}$$

If $j' > p$, then τ' instead has priority $p+1$, and the desired result follows instead from the calculation

$$\begin{aligned} \sigma^+ &= s^n_j(\tau^+) \\ &= s^n_j(s^{n-2}_{j'-1}(\tau'')^+) \\ &= s^n_j(s^{n-1}_{j'}(\tau''^+)) \\ &= s^n_{j'+1} s^{n-1}_j(\tau''^+) \\ &= s^n_{j'+1}(s^{n-2}_j(\tau'')^+) \\ &= s^n_{j'+1}(\tau'^+). \end{aligned}$$

Condition (3) is vacuous (since we have assumed that σ is free). To prove (4), we note that if σ has priority n, then τ has priority $(n-1)$; the desired result now follows from the observation that the restriction of σ^+ to $\mathrm{N}_\bullet(\{n-1 < n < n+1\})$ coincides with the restriction of τ^+ to $\mathrm{N}_\bullet(\{n-2 < n-1 < n\})$, and is therefore an S-optimal factorization. This completes the construction in the case $j < p$.

We now treat the case $j \geq p$, so that the simplex $\sigma = s^{n-1}_j(\tau)$ has priority p. In this case, we set $\sigma^+ = s^n_{j+1}(\tau^+)$. Condition (3) again vacuous (since σ is assumed to be free), and condition (4) is vacuous since $p < n$. We next prove (1). Note that we have

$$d^{n+1}_p(\sigma^+) = d^{n+1}_p(s^n_{j+1}(\tau^+)) = s^{n-1}_j(d^n_p(\tau^+)) = s^{n-1}_j(\tau) = \sigma.$$

To complete the proof of (1), we must show that $d^{n+1}_i(\sigma^+) = d^n_{i-1}(\sigma)^+$ for $p < i \leq n$. For $i \leq j$, this follows from the calculation

$$\begin{aligned} d^{n+1}_i(\sigma^+) &= d^{n+1}_i(s^n_{j+1}(\tau^+)) \\ &= s^{n-1}_j(d^n_i(\tau^+)) \\ &= s^{n-1}_j(d^{n-1}_{i-1}(\tau)^+) \\ &= s^{n-2}_{j-1}(d^{n-1}_{i-1}(\tau))^+ \\ &= d^n_{i-1}(s^{n-1}_j(\tau))^+ \\ &= d^n_{i-1}(\sigma)^+. \end{aligned}$$

For $j+2 < i \leq n$, it follows instead from the calculation

$$\begin{array}{rcl} d_i^{n+1}(\sigma^+) & = & d_i^{n+1}(s_{j+1}^n(\tau^+)) \\ & = & s_{j+1}^{n-1}(d_{i-1}^n(\tau^+)) \\ & = & s_{j+1}^{n-1}(d_{i-2}^{n-1}(\tau)^+) \\ & = & s_j^{n-2}(d_{i-2}^{n-1}(\tau))^+ \\ & = & d_{i-1}^n(s_j^{n-1}(\tau))^+ \\ & = & d_{i-1}^n(\sigma)^+. \end{array}$$

It will therefore suffice to treat the case $i \in \{j+1, j+2\}$, in which case we have

$$\begin{array}{rcl} d_i^{n+1}(\sigma^+) & = & d_i^{n+1}(s_{j+1}^n(\tau^+)) \\ & = & \tau^+ \\ & = & d_{i-1}^n(s_j^{n-1}(\tau))^+ \\ & = & d_{i-1}^n(\sigma)^+. \end{array}$$

To verify condition (2), suppose that $\sigma = s_{j'}^{n-1}(\tau')$. By construction, we then have $j' \geq j \geq p$, so that the simplex τ' has priority p. We wish to show that $\sigma^+ = s_{j'+1}^n(\tau'^+)$. If $j' = j$, this is immediate. We may therefore assume that $j' > j$, so that we can write $\tau' = s_j^{n-2}(\tau'')$ and $\tau = s_{j'-1}^{n-2}(\tau'')$ as above. In this case, the desired result follows from the calculation

$$\begin{array}{rcl} \sigma^+ & = & s_{j+1}^n(\tau^+) \\ & = & s_{j+1}^n(s_{j'-1}^{n-2}(\tau'')^+) \\ & = & s_{j+1}^n(s_{j'}^{n-1}(\tau''^+)) \\ & = & s_{j'+1}^n s_{j+1}^{n-1}(\tau''^+) \\ & = & s_{j'+1}^n(s_j^{n-2}(\tau'')^+) \\ & = & s_{j'+1}^n(\tau'^+). \end{array}$$

This completes the treatment of case (b).

We now consider case (c). For the remainder of the proof, we assume that the simplex σ is free and nondegenerate, of priority $p > 0$. Let us decompose Δ^{n+1} as a join $\Delta^{p-1} \star \Delta^{n-p} \star \{z\}$. In what follows, we write x for the final vertex of Δ^{p-1} (corresponding to the element $p-1 \in [n+1]$) and y for the initial vertex of Δ^{n-p} (corresponding to the element $p \in [n+1]$). Note that the n-simplices σ and $\{d_i^{n+1}(\sigma)^+\}_{p \leq i < n}$ determine a morphism of simplicial sets $\sigma^\dagger : \Delta^{p-1} \star \partial \Delta^{n-p} \star \{z\} \to \mathcal{C}$. Unwinding the definitions, we see that an $(n+1)$-simplex σ^+ of $\mathcal{C}$ satisfying condition (1) can be identified with an extension of $\sigma^\dagger$ to the join $\Delta^{p-1} \star \Delta^{n-p} \star \{z\} \simeq \Delta^{n+1}$. We wish to show that such an extension can always be found, which additionally satisfies condition (4) in the case $p = n$ (note that conditions (2) and (3) are vacuous, by virtue of our assumption that σ is free and nondegenerate).

Let $\overline{\sigma}^{\dagger}$ denote the restriction of $\sigma^{\dagger}$ to $\{x\} \star \partial\Delta^{n-p} \star \{z\}$. Since the inclusion $\{x\} \hookrightarrow \Delta^{p-1}$ is right anodyne (see Example 4.3.7.11), it will suffice to show that $\overline{\sigma}^{\dagger}$ can be extended to an $(n+2-p)$-simplex $\overline{\sigma}^{+}$ of $\mathcal{C}$, having the additional property that $\overline{\sigma}^{+}$ is an S-optimal factorization in the case $p = n$. If $p = n$, the existence of $\overline{\sigma}^{+}$ follows from our assumption that S is a class of short morphisms for $\mathcal{C}$. We therefore assume that $p < n$. Set $Z = \sigma^{\dagger}(z)$, so that we can identify $\overline{\sigma}^{\dagger}$ with a morphism of simplicial sets $\rho_0 : \Lambda^{n+1-p}_{0} \to \mathcal{C}_{/Z}$; we wish to extend ρ_0 to an $(n+1-p)$-simplex of $\mathcal{C}_{/Z}$. For $0 < i \leq n+1-p$, the image $\rho_0(i)$ belongs to the full subcategory $\mathcal{C}^{\mathrm{short}}_{/Z} \subseteq \mathcal{C}_{/Z}$. By virtue of Proposition 6.2.2.8, it will suffice to show that the restriction of ρ_0 to Δ^1 exhibits $\rho_0(1)$ as a $\mathcal{C}^{\mathrm{short}}_{/Z}$-reflection of $\rho_0(0)$. This is equivalent to the assertion that the 2-simplex

$$\begin{tikzcd}
 & \sigma^{\dagger}(y) \arrow[dr] & \\
\sigma^{\dagger}(x) \arrow[ur] \arrow[rr] & & \sigma^{\dagger}(z)
\end{tikzcd}$$

is an S-optimal factorization of the lower horizontal morphism (Remark 6.2.5.9), which follows from our inductive hypothesis. □

6.3 Localization

01M4 Let $\mathcal{C}$ be a category and let W be a collection of morphisms in $\mathcal{C}$. One can then construct a new category by formally adjoining an inverse to each morphism of W.

01M5 **Definition 6.3.0.1.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories and let W be a collection of morphisms of $\mathcal{C}$. We say that F *exhibits $\mathcal{D}$ as a strict localization of $\mathcal{C}$ with respect to W* if, for every category $\mathcal{E}$, precomposition with F induces a bijection

$$\begin{tikzcd}
\{\text{Functors } \mathcal{D} \to \mathcal{E}\} \arrow[d] \\
\{\text{Functors } \mathcal{C} \to \mathcal{E} \text{ carrying each } w \in W \text{ to an isomorphism in } \mathcal{E}\}.
\end{tikzcd}$$

01M6 **Remark 6.3.0.2** (Existence and Uniqueness)**.** Let $\mathcal{C}$ be a category and let W be a collection of morphisms in $\mathcal{C}$. Then there exists a category $W^{-1}\mathcal{C}$ and a functor $F : \mathcal{C} \to W^{-1}\mathcal{C}$ which exhibits $W^{-1}\mathcal{C}$ as a strict localization of $\mathcal{C}$ with respect to W. Moreover, the category $W^{-1}\mathcal{C}$ is determined uniquely up to isomorphism. In what follows, we will sometimes abuse terminology by referring to $W^{-1}\mathcal{C}$ as *the* strict localization of $\mathcal{C}$ with respect to W. Explicitly, the category $W^{-1}\mathcal{C}$ can be constructed from $\mathcal{C}$ by adjoining a new morphism $w^{-1} : Y \to X$ for each morphism $w : X \to Y$ of W, and imposing the relations $w^{-1} \circ w = \mathrm{id}_X$ and $w \circ w^{-1} = \mathrm{id}_Y$. From this description, we see that the functor F induces a bijection $\mathrm{Ob}(\mathcal{C}) \simeq \mathrm{Ob}(W^{-1}\mathcal{C})$.

Example 6.3.0.3. Let Kan denote the category of Kan complexes and let hKan denote the homotopy category of Kan complexes (Construction 3.1.5.10). Then the quotient functor Kan → hKan exhibits hKan as a strict localization of Kan with respect to the collection of all homotopy equivalences (see Corollary 3.1.7.7). `01Q2`

Warning 6.3.0.4. Let $\mathcal{C}$ be a category and let W be a collection of morphisms of $\mathcal{C}$. If $\mathcal{C}$ is small, then the strict localization $W^{-1}\mathcal{C}$ is also small. Beware that if $\mathcal{C}$ is only assumed to be locally small (Variant 4.7.8.6), then $W^{-1}\mathcal{C}$ need not be locally small. However, one can often ensure that $W^{-1}\mathcal{C}$ is locally small by imposing additional assumptions on the collection of morphisms W. `01M7`

Remark 6.3.0.5. Let $\mathcal{C}$ be a category, let W be a collection of morphisms of $\mathcal{C}$, and let $F : \mathcal{C} \to W^{-1}\mathcal{C}$ be a functor which exhibits $W^{-1}\mathcal{C}$ as a strict localization of $\mathcal{C}$ with respect to W. Then, for every category $\mathcal{E}$, the precomposition functor $\mathrm{Fun}(W^{-1}\mathcal{C}, \mathcal{E}) \xrightarrow{\circ F} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ induces an isomorphism from $\mathrm{Fun}(W^{-1}\mathcal{C}, \mathcal{E})$ to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ spanned by those functors $\mathcal{C} \to \mathcal{E}$ which carry each element $w \in W$ to an isomorphism in $\mathcal{E}$. Bijectivity at the level of objects follows immediately from the definition. At the level of morphisms, it follows from the bijectivity of the map `01M8`

$$\begin{tikzcd}
\{\text{Functors } W^{-1}\mathcal{C} \to \mathrm{Fun}([1], \mathcal{E})\} \arrow[d] \\
\{\text{Functors } \mathcal{C} \to \mathrm{Fun}([1], \mathcal{E}) \text{ carrying } W \text{ to isomorphisms}\}.
\end{tikzcd}$$

Beware that Definition 6.3.0.1 is not invariant under equivalence. If $\mathcal{C}$ is a category, W is a collection of morphisms in $\mathcal{C}$, and $\mathcal{D}$ is a category which is equivalent but not isomorphic to the strict localization $W^{-1}\mathcal{C}$, then $\mathcal{D}$ is *not* a strict localization of $\mathcal{C}$ with respect to W. We can remedy the situation by introducing a more liberal notion of localization.

Definition 6.3.0.6. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories and let W be a collection of morphisms of $\mathcal{C}$. We will say that F *exhibits* $\mathcal{D}$ *as a* 1*-categorical localization of* $\mathcal{C}$ *with respect to* W if, for every category $\mathcal{E}$, precomposition with F induces a fully faithful functor $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$, whose essential image consists of those functors $\mathcal{C} \to \mathcal{E}$ which carry each $w \in W$ to an isomorphism in $\mathcal{E}$. `01M9`

01MA **Example 6.3.0.7.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. If F exhibits $\mathcal{D}$ as a strict localization of $\mathcal{C}$ with respect to W, then F exhibits $\mathcal{D}$ as a 1-categorical localization of $\mathcal{C}$ with respect to W (see Remark 6.3.0.5). The converse is false (except in the trivial case where $\mathcal{C}$ is empty).

01Q3 **Example 6.3.0.8.** Let Set_{Δ} denote the category of simplicial sets, and let hKan denote the homotopy category of Kan complexes (Construction 3.1.5.10). Then the fibrant replacement functor $\mathrm{Ex}^{\infty} : \mathrm{Set}_{\Delta} \to \mathrm{hKan}$ exhibits hKan as a 1-categorical localization of Set_{Δ} with respect to the collection W of weak homotopy equivalences (see Variant 3.1.7.8). However, it does not exhibit hKan as a strict localization of Set_{Δ} with respect to W (since it is not bijective on objects).

01MB **Remark 6.3.0.9.** Let $\mathcal{C}$ be a category, let W be a collection of morphisms in $\mathcal{C}$, and let $F : \mathcal{C} \to W^{-1}\mathcal{C}$ be a functor which exhibits $W^{-1}\mathcal{C}$ as a strict localization of $\mathcal{C}$ with respect to W. Let $G : \mathcal{C} \to \mathcal{D}$ be another functor. Then G exhibits $\mathcal{D}$ as a 1-categorical localization of $\mathcal{C}$ with respect to W if and only if the following conditions are satisfied:

- The functor G carries each $w \in W$ to an isomorphism in $\mathcal{D}$, and therefore factors uniquely as a composition $\mathcal{C} \xrightarrow{F} W^{-1}\mathcal{C} \xrightarrow{G'} \mathcal{D}$.
- The functor $G' : W^{-1}\mathcal{C} \to \mathcal{D}$ is an equivalence of categories.

Our goal in this section is to adapt the notion of localization to the setting of ∞-categories. We begin in §6.3.1 by introducing an ∞-categorical counterpart of Definition 6.3.0.6. Given an ∞-category $\mathcal{C}$ and a collection W of morphisms of $\mathcal{C}$, we say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W* if, for every ∞-category $\mathcal{E}$, precomposition with F induces a fully faithful functor of ∞-categories $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$, whose essential image consists of those functors which carry each element of W to an isomorphism in $\mathcal{E}$ (Definition 6.3.1.9). In §6.3.2, we show that such a localization always exists (Proposition 6.3.2.1) and is uniquely determined up to equivalence (Remark 6.3.2.2); we will often emphasize this uniqueness by denoting the ∞-category $\mathcal{D}$ by $\mathcal{C}[W^{-1}]$.

Let $\mathcal{C}$ be an ordinary category, and let W be a collection of morphisms of $\mathcal{C}$. Then W can also be regarded as a collection of morphisms of the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$. By virtue of Proposition 6.3.2.1, there exists a functor of ∞-categories $F : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{D}$ which exhibits $\mathcal{D}$ as a localization of $\mathrm{N}_{\bullet}(\mathcal{C})$ with respect to W. In this case, it is not hard to see that the induced map $\mathcal{C} \simeq \mathrm{hN}_{\bullet}(\mathcal{C}) \xrightarrow{\mathrm{h}F} \mathrm{h}\mathcal{D}$ exhibits the homotopy category $\mathrm{h}\mathcal{D}$ as a 1-categorical localization of $\mathcal{C}$ with respect to W, in the sense of Definition 6.3.0.6 (Example 6.3.1.18). Beware that, in this situation, the unit map $\mathcal{D} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{D})$ is generally *not* an equivalence. In other words, the formation of localizations (in the ∞-categorical setting) generally does not carry ordinary categories to ordinary categories, even up to equivalence. In fact, we prove in §6.3.7 that *every* ∞-category $\mathcal{D}$ can be obtained by localizing (the nerve of) a partially ordered set (Theorem 6.3.7.1). The proof will make use of some basic stability properties for the class of localizations, which we establish in §6.3.4.

In general, it is very difficult to give an explicit description of the localization of an ∞-category $\mathcal{C}$ with respect to a class of morphisms W. In §6.3.3, we study a special case in which such a description is available. We will say that a localization functor $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is *reflective* if it admits a right adjoint. In this case, the right adjoint $G : \mathcal{C}[W^{-1}] \to \mathcal{C}$ is automatically fully faithful, and its essential image is a reflective subcategory $\mathcal{C}' \subseteq \mathcal{C}$ (Proposition 6.3.3.6). Reflective localizations are extremely common in practice, and will play a central role in the theory of locally presentable ∞-categories which we develop in §[?].

Warning 6.3.0.10. It also is possible to contemplate a version of Definition 6.3.0.1 in the ∞-categorical setting. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Let us say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *exhibits $\mathcal{D}$ as a strict localization of $\mathcal{C}$ with respect to W* if, for every ∞-category $\mathcal{E}$, precomposition with F induces a bijection 01MC

$$\begin{tikzcd}
\{\text{Functors } \mathcal{D} \to \mathcal{E}\} \arrow[d] & . \\
\{\text{Functors } \mathcal{C} \to \mathcal{E} \text{ carrying each } w \in W \text{ to an isomorphism in } \mathcal{E}\} &
\end{tikzcd}$$

However, this definition is useless. One can show that an ∞-category $\mathcal{C}$ admits a strict localization with respect to W only in the trivial case where every element of W is already an isomorphism in $\mathcal{C}$ (in which case we can take F to be the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$). Roughly speaking, the problem is that if $w : X \to Y$ is an isomorphism in an ∞-category $\mathcal{C}$, then the homotopy inverse isomorphism $w^{-1} : Y \to X$ is only well-defined up to homotopy (or up to a contractible space of choices), in contrast with classical category theory where the inverse isomorphism w^{-1} is unique.

6.3.1 Localizations of ∞-Categories

We begin by introducing some terminology. 01ME

Notation 6.3.1.1. Let $\mathcal{C}$ be a simplicial set, let W be a collection of edges of $\mathcal{C}$, and let $\mathcal{E}$ be an ∞-category. We let $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ spanned by those morphisms $F : \mathcal{C} \to \mathcal{E}$ that carry each edge of W to an isomorphism in $\mathcal{E}$. 01MF

Remark 6.3.1.2. In the context of Notation 6.3.1.1, we will usually be interested in the situation where the simplicial set $\mathcal{C}$ is an ∞-category (as suggested by the notation). However, it will be technically convenient to allow more general simplicial sets as well. 01MG

01MH **Example 6.3.1.3.** Let $\mathcal{C}$ be a simplicial set and let W be a collection of degenerate edges of $\mathcal{C}$. Then, for every ∞-category $\mathcal{E}$, we have $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) = \mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

01MJ **Example 6.3.1.4.** Let $\mathcal{C}$ be a simplicial set and let W be a collection of edges of $\mathcal{C}$. If $\mathcal{E}$ is a Kan complex, then $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) = \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ (see Proposition 1.4.6.10).

01MK **Example 6.3.1.5.** Let $W = \{\mathrm{id}_{\Delta^1}\}$ consist of the single nondegenerate edge of the standard 1-simplex Δ^1. For every ∞-category $\mathcal{E}$, $\mathrm{Fun}(\Delta^1[W^{-1}], \mathcal{E})$ is the full subcategory $\mathrm{Isom}(\mathcal{E}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{E})$ spanned by the isomorphisms in $\mathcal{E}$ (Example 4.4.1.14).

01ML **Example 6.3.1.6.** Let $\mathcal{C}$ be a simplicial set and let $\mathrm{h}\mathcal{C}$ denote its homotopy category (Definition 1.3.6.1). Let W be a collection of edges of $\mathcal{C}$, let $[W]$ denote the collection of morphisms in $\mathrm{h}\mathcal{C}$ which belong to the image of W, and let $F : \mathrm{h}\mathcal{C} \to \mathcal{D}$ be a functor of ordinary categories which exhibits $\mathcal{D}$ as a strict localization of $\mathrm{h}\mathcal{C}$ with respect to $[W]$ (Definition 6.3.0.1). If $\mathcal{E}$ is an ordinary category, then we have a canonical isomorphism of simplicial sets

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{N}_\bullet(\mathcal{E})) \simeq \mathrm{N}_\bullet(\mathrm{Fun}(\mathcal{D}, \mathcal{E})).$$

01MM **Remark 6.3.1.7.** Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets and let W be a collection of edges of $\mathcal{C}$. For every ∞-category $\mathcal{E}$, the canonical isomorphism $\mathrm{Fun}(\mathcal{C}, \mathrm{Fun}(\mathcal{D}, \mathcal{E})) \simeq \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}, \mathcal{E}))$ restricts to an isomorphism of full subcategories

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathrm{Fun}(\mathcal{D}, \mathcal{E})) \simeq \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})).$$

This follows immediately from the criterion of Theorem 4.4.4.4.

01MN **Remark 6.3.1.8.** Let $\mathcal{C}$ be a simplicial set, let W be a collection of edges of $\mathcal{C}$, and let $\mathcal{E}$ be an ∞-category. Then the full subcategory $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is replete. That is, if $F, F' : \mathcal{C} \to \mathcal{E}$ are isomorphic objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$, then F carries edges of W to isomorphisms in $\mathcal{E}$ if and only if F' carries edges of W to isomorphisms in $\mathcal{E}$ (see Example 4.4.1.14).

01MP **Definition 6.3.1.9.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let W be a collection of edges of $\mathcal{C}$. We say that F *exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W* if, for every ∞-category $\mathcal{E}$, the precomposition map $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ is fully faithful, and its essential image is the full subcategory $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{E})$.

01MQ **Remark 6.3.1.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. If F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to a collection of edges W, then, for every ∞-category $\mathcal{E}$ and every morphism $G : \mathcal{D} \to \mathcal{E}$, the composite map $(G \circ F) : \mathcal{C} \to \mathcal{E}$ carries each element of W to an isomorphism in $\mathcal{E}$. In particular, if $\mathcal{D}$ itself is an ∞-category, then F carries each element of W to an isomorphism in $\mathcal{D}$.

Exercise 6.3.1.11. Let $\mathcal{C}$ be a simplicial set, let W be a collection of edges of $\mathcal{C}$, and let $F, F' : \mathcal{C} \to \mathcal{D}$ be a pair of diagrams taking values in an ∞-category $\mathcal{D}$. Suppose that F and F' are isomorphic when viewed as objects of the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$. Show that F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W if and only if F' exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W. 02SW

Example 6.3.1.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let W be a collection of degenerate edges of $\mathcal{C}$. Then F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W if and only if it is a categorical equivalence of simplicial sets (see Proposition 4.5.3.8). 01MR

Proposition 6.3.1.13. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets and let W be a collection of edges of $\mathcal{C}$. The following conditions are equivalent:* 01MS

(1) *The morphism F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W (Definition 6.3.1.9).*

(2) *For every ∞-category $\mathcal{E}$, the functor $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ factors through the full subcategory $\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ and induces an equivalence of ∞-categories $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$.*

(3) *For every ∞-category $\mathcal{E}$, the functor $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ factors through the full subcategory $\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ and induces a homotopy equivalence of Kan complexes $\operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \to \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq}$.*

(4) *For every ∞-category $\mathcal{E}$, the functor $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ factors through the full subcategory $\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ and induces a bijection of sets $\pi_0(\operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}) \to \pi_0(\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq})$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Corollary 4.6.2.22 (and the repleteness of the full subcategory $\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \subseteq \operatorname{Fun}(\mathcal{C}, \mathcal{E})$). The implication (2) $\Rightarrow$ (3) follows from Remark 4.5.1.19 and the implication (3) $\Rightarrow$ (4) from Remark 3.1.6.5. We will complete the proof by showing that (4) $\Rightarrow$ (2). Assume that $F : \mathcal{C} \to \mathcal{D}$ satisfies condition (4), and let $\mathcal{E}$ be an ∞-category; we wish to show that the precomposition functor $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ is an equivalence of ∞-categories. For this, it will suffice to show that for every simplicial set $\mathcal{B}$, the induced map

$$\theta : \pi_0(\operatorname{Fun}(\mathcal{B}, \operatorname{Fun}(\mathcal{D}, \mathcal{E}))^{\simeq}) \to \pi_0(\operatorname{Fun}(\mathcal{B}, \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}))^{\simeq})$$

is a bijection. Using Remark 6.3.1.7, we can identify θ with the map

$$\pi_0(\operatorname{Fun}(\mathcal{D}, \operatorname{Fun}(\mathcal{B}, \mathcal{E}))^{\simeq}) \to \pi_0(\operatorname{Fun}(\mathcal{C}[W^{-1}], \operatorname{Fun}(\mathcal{B}, \mathcal{E}))^{\simeq}),$$

which is bijective by virtue of assumption (4). □

01MT **Example 6.3.1.14.** Let $W = \{\mathrm{id}_{\Delta^1}\}$ consist of the single nondegenerate edge of the standard 1-simplex Δ^1. Then the projection map $\Delta^1 \to \Delta^0$ exhibits Δ^0 as a localization of Δ^1 with respect to W. To prove this, it will suffice to show that for every ∞-category $\mathcal{E}$, the construction $X \mapsto \mathrm{id}_X$ induces an equivalence of ∞-categories $\mathcal{E} = \mathrm{Fun}(\Delta^0, \mathcal{E}) \to \mathrm{Fun}(\Delta^1[W^{-1}], \mathcal{E}) = \mathrm{Isom}(\mathcal{E})$, which follows from Corollary 4.5.3.13.

038T **Remark 6.3.1.15.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to a collection of edges W, and let $U : \overline{\mathcal{E}} \to \mathcal{E}$ be an isofibration of ∞-categories. Then, for every diagram $\mathcal{D} \to \mathcal{E}$, precomposition with F induces a fully faithful functor

$$\mathrm{Fun}_{/\mathcal{E}}(\mathcal{D}, \overline{\mathcal{E}}) \to \mathrm{Fun}_{/\mathcal{E}}(\mathcal{C}, \overline{\mathcal{E}}),$$

whose essential image is spanned by those functors $G : \mathcal{C} \to \overline{\mathcal{E}}$ which carry each edge of W to an isomorphism in the ∞-category $\overline{\mathcal{E}}$. This follows by applying Corollary 4.5.2.32 to the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{D}, \overline{\mathcal{E}}) \arrow[r, "\circ F"] \arrow[d, "U\circ"] & \mathrm{Fun}(\mathcal{C}[W^{-1}], \overline{\mathcal{E}}) \arrow[d] \\
\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \arrow[r] & \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}).
\end{tikzcd}$$

01MU **Remark 6.3.1.16.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to a collection of edges W. Then, for every Kan complex $\mathcal{E}$, precomposition with F induces a homotopy equivalence of Kan complexes

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) = \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

(see Example 6.3.1.4). It follows that F is a weak homotopy equivalence of simplicial sets.

01MV **Remark 6.3.1.17.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to a collection of edges W. Let $[W]$ denote the collection of morphisms in the homotopy category $\mathrm{h}\mathcal{C}$ which belong to the image of W. Then the induced functor $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ exhibits the homotopy category $\mathrm{h}\mathcal{D}$ as a 1-categorical localization of $\mathrm{h}\mathcal{C}$ with respect to $[W]$, in the sense of Definition 6.3.0.6. This follows immediately from Example 6.3.1.6.

01MW **Example 6.3.1.18.** Let $\mathcal{C}$ be an ordinary category and let W be a collection of morphisms of $\mathcal{C}$, which we identify with edges of the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C})$. Let $F : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{D}$ be a morphism of simplicial sets which exhibits $\mathcal{D}$ as a localization of $\mathrm{N}_{\bullet}(\mathcal{C})$ with respect to W. Then the induced functor $\mathcal{C} \simeq \mathrm{hN}_{\bullet}(\mathcal{C}) \xrightarrow{\mathrm{h}F} \mathrm{h}\mathcal{D}$ exhibits the homotopy category $\mathrm{h}\mathcal{D}$ as a 1-categorical localization of $\mathcal{C}$ with respect to W, in the sense of in the sense of Definition 6.3.0.6.

Remark 6.3.1.19. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets, and let W be a collection of edges of $\mathcal{C}$. If any two of the following three conditions is satisfied, then so is the third: 01MX

- The morphism F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W.
- The morphism $G \circ F$ exhibits $\mathcal{E}$ as a localization of $\mathcal{C}$ with respect to W.
- The morphism G is a categorical equivalence of simplicial sets.

Proposition 6.3.1.20. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, where $\mathcal{D}$ is an ∞-category, and let W be the collection of all edges of $\mathcal{C}$. The following conditions are equivalent:* 01MY

(1) *The morphism F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W.*

(2) *The ∞-category $\mathcal{D}$ is a Kan complex and F is a weak homotopy equivalence of simplicial sets.*

Proof. We first prove that (2) implies (1). Assume that $\mathcal{D}$ is a Kan complex and that $F : \mathcal{C} \to \mathcal{D}$ is a weak homotopy equivalence; we wish to show that F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W. By virtue of Proposition 6.3.1.13, it will suffice to show that for every ∞-category $\mathcal{E}$, composition with F induces a homotopy equivalence of Kan complexes $\theta : \mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \to \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq}$. Since $\mathcal{D}$ is a Kan complex, Proposition 4.4.3.22 allows us to identify θ with the canonical map

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}^{\simeq}) \xrightarrow{\circ F} \mathrm{Fun}(\mathcal{C}, \mathcal{E}^{\simeq}),$$

which is a homotopy equivalence by virtue of our assumption that F is a weak homotopy equivalence.

We now show that (1) implies (2). Assume that F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W. Invoking Remark 6.3.1.16, we deduce that F is a weak homotopy equivalence. We wish to show that $\mathcal{D}$ is a Kan complex. Choose a weak homotopy equivalence $G : \mathcal{D} \to \mathcal{E}$, where $\mathcal{E}$ is a Kan complex (Corollary 3.1.7.2). Then the composite map $(G \circ F) : \mathcal{C} \to \mathcal{E}$ is also a weak homotopy equivalence (Remark 3.1.6.16). Invoking the implication (2) ⇒ (1), we conclude that $G \circ F$ exhibits $\mathcal{E}$ as a localization of $\mathcal{C}$ with respect to W. It follows from Remark 6.3.1.19 that G is an equivalence of ∞-categories. Since $\mathcal{E}$ is a Kan complex, it follows that the ∞-category $\mathcal{D}$ is also a Kan complex (Remark 4.5.1.21). □

Proposition 6.3.1.21 (Transitivity)**.** *Let $F : \mathcal{C} \to \mathcal{C}'$ and $F' : \mathcal{C}' \to \mathcal{C}''$ be morphisms of simplicial sets. Let W and W' be collections of edges of $\mathcal{C}$ satisfying the following conditions:* 02LS

- *The morphism F exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to W.*

- *The morphism F' exhibits $\mathcal{C}''$ as a localization of $\mathcal{C}'$ with respect to $F(W')$.*

Then the composite morphism $(F' \circ F) : \mathcal{C} \to \mathcal{C}''$ exhibits $\mathcal{C}''$ as a localization of $\mathcal{C}$ with respect to $W \cup W'$.

Proof. Let $\mathcal{E}$ be an ∞-category; we wish to prove that precomposition with $F' \circ F$ induces an equivalence from $\operatorname{Fun}(\mathcal{C}'', \mathcal{E})$ to the full subcategory $\operatorname{Fun}(\mathcal{C}[(W \cup W')^{-1}], \mathcal{E}) \subseteq \operatorname{Fun}(\mathcal{C}, \mathcal{E})$. We have a commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{C}'', \mathcal{E}) \arrow[r, "\circ F'"] & \operatorname{Fun}(\mathcal{C}'[F(W')^{-1}], \mathcal{E}) \arrow[r, "\circ F"] \arrow[d] & \operatorname{Fun}(\mathcal{C}[(W \cup W')^{-1}], \mathcal{E}) \arrow[d] \\
& \operatorname{Fun}(\mathcal{C}', \mathcal{E}) \arrow[r, "\circ F"] & \operatorname{Fun}(\mathcal{C}[W^{-1}],
\end{tikzcd}$$

where the horizontal functors on the left and lower right are equivalences of ∞-categories. Since the square is a pullback and the vertical maps are isofibrations (Remark 6.3.1.8), it follows that the horizontal map on the upper right is also an equivalence of ∞-categories (Corollary 4.5.2.29). □

02LT **Corollary 6.3.1.22.** *Let $\mathcal{C}$ be a simplicial set, let W and W' be collections of edges of $\mathcal{C}$, and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W. Suppose that, for every edge $w \in W'$, the image $F(w)$ is a degenerate edge of $\mathcal{D}$. Then F also exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to $W \cup W'$.*

Proof. Combine Proposition 6.3.1.21 with Example 6.3.1.12. □

6.3.2 Existence of Localizations

01MZ Our goal in this section is to prove the following:

01N0 **Proposition 6.3.2.1** (Existence of Localizations)**.** *Let $\mathcal{C}$ be a simplicial set and let W be a collection of edges of $\mathcal{C}$. Then there exists an ∞-category $\mathcal{D}$ and a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W.*

01N1 **Remark 6.3.2.2** (Uniqueness of Localizations)**.** Let $\mathcal{C}$ be a simplicial set and let W be a collection of edges of $\mathcal{C}$. Proposition 6.3.2.1 asserts that there exists an ∞-category $\mathcal{D}$ and a morphism $F : \mathcal{C} \to \mathcal{D}$ which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W. In this case, for every ∞-category $\mathcal{E}$, composition with F induces a bijection

$$\operatorname{Hom}_{\mathrm{hCat}_\infty}(\mathcal{D}, \mathcal{E}) = \pi_0(\operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}) \to \pi_0(\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq})$$

(Proposition 6.3.1.13). In other words, the ∞-category $\mathcal{D}$ corepresents the functor

$$\mathrm{hCat}_{\infty} \to \mathrm{Set} \qquad \mathcal{E} \mapsto \pi_0(\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq}).$$

It follows that $\mathcal{D}$ is uniquely determined (up to canonical isomorphism) as an object of the homotopy category hCat_{∞}. We will sometimes emphasize this uniqueness by referring to $\mathcal{D}$ as *the* localization of $\mathcal{C}$ with respect to W, and denoting it by $\mathcal{C}[W^{-1}]$. Beware that the localization $\mathcal{C}[W^{-1}]$ is *not* well-defined up to isomorphism as a simplicial set: in fact, any equivalent ∞-category can also be regarded as a localization of $\mathcal{C}$ with respect to W (Remark 6.3.1.19).

Warning 6.3.2.3. Let $\mathcal{C}$ be a simplicial set, let W be a collection of edges of $\mathcal{C}$, and let $\mathcal{E}$ be an ∞-category. We have now given two different definitions for the ∞-category $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$: 01N2

(1) According to Notation 6.3.1.1, $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ spanned by those diagrams $F : \mathcal{C} \to \mathcal{E}$ which carry each edge of W to an isomorphism in $\mathcal{E}$.

(2) By the convention of Remark 6.3.2.2, $\mathcal{C}[W^{-1}]$ denotes an ∞-category equipped with a diagram $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ which exhibits $\mathcal{C}[W^{-1}]$ as a localization of $\mathcal{C}$ with respect to W. We can then consider the ∞-category of functors from $\mathcal{C}[W^{-1}]$ to $\mathcal{E}$, which we will temporarily denote by $\mathrm{Fun}'(\mathcal{C}[W^{-1}], \mathcal{E})$.

Beware that these ∞-categories are not identical. However, they are equivalent: if $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ exhibits $\mathcal{C}[W^{-1}]$ as a localization of $\mathcal{C}$ with respect to W, then composition with F induces an equivalence of ∞-categories $\mathrm{Fun}'(\mathcal{C}[W^{-1}], \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ (Proposition 6.3.1.13). Note that the ∞-category $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$ does not depend on any auxiliary choices: it is well-defined up to equality as a simplicial subset of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$. By contrast, the ∞-category $\mathrm{Fun}'(\mathcal{C}[W^{-1}], \mathcal{E})$ depends on the choice of the functor $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ (and is therefore well-defined up to equivalence, but not up to isomorphism).

Our proof of Proposition 6.3.2.1 will make use of the following:

Lemma 6.3.2.4. *Let Q be a contractible Kan complex, let $e : \Delta^1 \hookrightarrow Q$ be a monomorphism of simplicial sets, and let $W = \{\mathrm{id}_{\Delta^1}\}$ consist of the single nondegenerate edge of Δ^1. Then, for any ∞-category $\mathcal{E}$, precomposition with e induces a trivial Kan fibration of simplicial sets* 01N3

$$\theta : \mathrm{Fun}(Q, \mathcal{E}) \to \mathrm{Fun}(\Delta^1[W^{-1}], \mathcal{E}) = \mathrm{Isom}(\mathcal{E}).$$

Proof. Since e is a monomorphism, Corollary 4.4.5.3 immediately implies that θ is an isofibration when regarded as a functor from $\mathrm{Fun}(Q, \mathcal{E})$ to $\mathrm{Fun}(\Delta^1, \mathcal{E})$. Using the pullback

diagram

$$\begin{tikzcd}
\operatorname{Fun}(Q, \mathcal{E}) \arrow[r] \arrow[d, "\theta"] & \operatorname{Fun}(Q, \mathcal{E}) \arrow[d, "\theta"] \\
\operatorname{Isom}(\mathcal{E}) \arrow[r] & \operatorname{Fun}(\Delta^1, \mathcal{E}),
\end{tikzcd}$$

we deduce that θ is also an isofibration when regarded as a functor from $\operatorname{Fun}(Q, \mathcal{E})$ to $\operatorname{Isom}(\mathcal{E})$. Consequently, to show that θ is a trivial Kan fibration, it will suffice to show that it is an equivalence of ∞-categories (Proposition 4.5.5.20). In other words, we are reduced to proving that the morphism e exhibits Q as a localization of Δ^1 with respect to W. Let $q : Q \to \Delta^0$ denote the projection map. Since Q is contractible, the morphism q is an equivalence of ∞-categories. By virtue of Remark 6.3.1.19, we are reduced to proving that the composite map $\Delta^1 \xrightarrow{e} Q \xrightarrow{q} \Delta^0$ exhibits Δ^0 as a localization of Δ^1 with respect to W, which follows from Example 6.3.1.14. □

We will deduce Proposition 6.3.2.1 from the following more precise result:

01N4 **Proposition 6.3.2.5.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, where $\mathcal{D}$ is an ∞-category. Let W be a collection of edges of $\mathcal{C}$ such that, for each $w \in W$, the image $F(w)$ is an isomorphism in $\mathcal{D}$. Then F factors as a composition*

$$\mathcal{C} \xrightarrow{G} \mathcal{C}[W^{-1}] \xrightarrow{H} \mathcal{D},$$

where G exhibits $\mathcal{C}[W^{-1}]$ as a localization of $\mathcal{C}$ with respect to W and H is an inner fibration (so that $\mathcal{C}[W^{-1}]$ is also an ∞-category). Moreover, this factorization can be chosen to depend functorially on the diagram $F : \mathcal{C} \to \mathcal{D}$ and the collection of edges W, in such a way that the construction $(F : \mathcal{C} \to \mathcal{D}, W) \mapsto \mathcal{C}[W^{-1}]$ commutes with filtered colimits.

Proof. For each element $w \in W$, the image $F(w)$ can be regarded as a morphism from Δ^1 to the core $\mathcal{D}^{\simeq}$. By virtue of Proposition 3.1.7.1, we can (functorially) choose a factorization of this morphism as a composition

$$\Delta^1 \xrightarrow{i_w} Q_w \xrightarrow{q_w} \mathcal{D}^{\simeq},$$

where i_w is anodyne and q_w is a Kan fibration. Since $\mathcal{D}^{\simeq}$ is a Kan complex, Q_w is also a Kan complex, which is contractible by virtue of the fact that i_w is anodyne. Form a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{w \in W} \Delta^1 \arrow[r] \arrow[d, "\coprod_{w \in W} i_w"] & \mathcal{C} \arrow[d, "i"] \\
\coprod_{w \in W} Q_w \arrow[r] & \mathcal{C}'.
\end{tikzcd}$$

We first claim that $i : \mathcal{C} \to \mathcal{C}'$ exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to W. Let $\mathcal{E}$ be an ∞-category. Note that if $G : \mathcal{C} \to \mathcal{E}$ is a morphism of simplicial sets which factors through $\mathcal{C}'$, then for each $w \in W$ the morphism $G(w)$ belongs to the image of a functor $Q_w \to \mathcal{E}$, and is therefore an isomorphism in $\mathcal{E}$. It follows that composition with i induces a functor $\theta : \operatorname{Fun}(\mathcal{C}', \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$, and we wish to show that θ is an equivalence of ∞-categories. This follows by inspecting the commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{C}', \mathcal{E}) \arrow[r, "\theta"] \arrow[d] & \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \arrow[r] \arrow[d] & \operatorname{Fun}(\mathcal{C}, \mathcal{E}) \arrow[d] \\
\prod_{w \in W} \operatorname{Fun}(Q_w, \mathcal{E}) \arrow[r, "\theta'"] & \prod_{w \in W} \operatorname{Isom}(\mathcal{E}) \arrow[r] & \prod_{w \in W} \operatorname{Fun}(\Delta^1, \mathcal{E}).
\end{tikzcd}$$

The outer rectangle is a pullback square by the definition of $\mathcal{C}'$, and the right square is a pullback by the definition of $\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$. It follows that the left square is also a pullback. Lemma 6.3.2.4 implies that θ' is a trivial Kan fibration, so that θ is also a trivial Kan fibration (hence an equivalence of ∞-categories by Proposition 4.5.3.11).

Note that the morphism $F : \mathcal{C} \to \mathcal{D}$ and the collection of morphisms $\{q_w : Q_w \to \mathcal{D}^{\simeq} \subseteq \mathcal{D}\}_{w \in W}$ can be amalgamated to a single morphism of simplicial sets $F' : \mathcal{C}' \to \mathcal{D}$. Applying Proposition 4.1.3.2, we can (functorially) factor F' as a composition $\mathcal{C}' \xrightarrow{G'} \mathcal{C}[W^{-1}] \xrightarrow{H} \mathcal{D}$, where G' is inner anodyne and H is an inner fibration. We conclude by observing that the composite map $G = (G' \circ i) : \mathcal{C} \to \mathcal{C}[W^{-1}]$ exhibits $\mathcal{C}[W^{-1}]$ as a localization of $\mathcal{C}$ with respect to W, by virtue of Remark 6.3.1.19. □

Proof of Proposition 6.3.2.1. Apply Proposition 6.3.2.5 in the special case $\mathcal{D} = \Delta^0$. □

Variant 6.3.2.6. Let κ be an uncountable cardinal, and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to some collection of edges W (Definition 6.3.1.9). If $\mathcal{C}$ is essentially κ-small, then $\mathcal{D}$ is essentially κ-small. 03TZ

Proof. Without loss of generality, we may assume that F is a monomorphism of simplicial sets. Choose a categorical equivalence of simplicial sets $u : \mathcal{C} \to \overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is κ-small, and form a pushout diagram of simplicial sets

03U0

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "u"] \arrow[d, "F"] & \overline{\mathcal{C}} \arrow[d, "\overline{F}"] \\
\mathcal{D} \arrow[r, "v"] & \overline{\mathcal{D}}
\end{tikzcd} \tag{6.5}$$

Then (6.5) is a categorical pushout square (Example 4.5.4.12), so v is also a categorical equivalence (Proposition 4.5.4.10). Moreover, the morphism $\overline{F}$ exhibits $\overline{D}$ as a localization of

$\overline{\mathcal{C}}$ with respect to $u(W)$ (Corollary 6.3.4.3). We may therefore replace F by $\overline{F}$, and thereby reduce to proving Variant 6.3.2.6 in the special case where $\mathcal{C}$ is κ-small. In particular, the set of edges W is κ-small.

Let Q be a contractible Kan complex which is equipped with a monomorphism $\Delta^1 \hookrightarrow Q$ and has only countably many simplices. Form a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{w \in W} \Delta^1 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "G"] \\
\coprod_{w \in W} Q \arrow[r] & \mathcal{C}',
\end{tikzcd}$$

so that $\mathcal{C}'$ is κ-small (Remark 4.7.4.6). It follows from Corollary 6.3.4.3 that the morphism G exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to W. Using Proposition 4.7.5.5, we can choose an inner anodyne morphism $\mathcal{C}' \hookrightarrow \mathcal{C}''$, where $\mathcal{C}''$ is a κ-small ∞-category. Then $\mathcal{C}''$ is also a localization of $\mathcal{C}$ with respect to W, so Remark 6.3.2.2 supplies a categorical equivalence of simplicial sets $\mathcal{D} \to \mathcal{C}''$. It follows that $\mathcal{D}$ is essentially κ-small, as desired. □

6.3.3 Reflective Localizations

02FY It will often be convenient to work with the following variant of Definition 6.3.1.9.

04JG **Definition 6.3.3.1.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We will say that F is a *localization functor* if there exists a collection of morphisms W in $\mathcal{C}$ such that F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W.

In the situation of Definition 6.3.3.1, there is always a canonical choice for the collection of morphisms W:

04JH **Proposition 6.3.3.2.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let W be the collection of all morphisms w of $\mathcal{C}$ such that $F(w)$ is an isomorphism in $\mathcal{D}$. Then F is a localization functor if and only if it exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W.*

Proof. Assume that F is a localization functor; we will show that it exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W (the reverse implication follows immediately from the definitions). Let $\mathcal{E}$ be an ∞-category; we wish to show that composition with F induces an equivalence of ∞-categories $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})$. Choose a collection of morphisms W' of $\mathcal{C}$ such that F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W'. Note that W' is contained in W, so that we can regard $\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{D})$ as a full subcategory of $\operatorname{Fun}(\mathcal{C}[W'^{-1}], \mathcal{D})$. It will therefore suffice to show that the composite functor

$$\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E}) \hookrightarrow \operatorname{Fun}(\mathcal{C}[W'^{-1}], \mathcal{E})$$

is an equivalence of ∞-categories, which follows from our assumption on the functor F. □

We now use the ideas of §6.2.2 to describe a large class of localization functors.

Definition 6.3.3.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that F is a *reflective localization functor* if it admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$ which is fully faithful. `02G9`

Remark 6.3.3.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Using Corollary 6.2.2.17 to the right adjoint of F, we see that the following conditions are equivalent: `04JJ`

- The functor F is a reflective localization: that is, it admits a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$.
- There exists a functor $G : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism $\epsilon : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ which is the counit of an adjunction between F and G.
- The functor F admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$ for which the composition $(F \circ G) : \mathcal{D} \to \mathcal{D}$ is an equivalence of ∞-categories.

Moreover, if these conditions are satisfied, then the essential image of G is a reflective subcategory of $\mathcal{C}$.

Remark 6.3.3.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a reflective localization of ∞-categories. Then there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ which is the unit of an adjunction between F and G. The natural transformation η is generally not an isomorphism (unless F is an equivalence of ∞-categories). However, our assumption that F is a reflective localization guarantees that there exists a natural isomorphism $\epsilon : F \circ G \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ which is compatible with η up to homotopy. In particular, for every object $X \in \mathcal{C}$, the composition `04JK`

$$F(X) \xrightarrow{F(\eta_X)} (F \circ G \circ F)(X) \xrightarrow{\epsilon_{F(X)}} F(X)$$

is homotopic to the identity $\mathrm{id}_{F(X)}$, which guarantees that $F(\eta_X)$ is an isomorphism in $\mathcal{D}$.

Proposition 6.3.3.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:* `04JL`

(1) *The functor F is a reflective localization (in the sense of Definition 6.3.3.3): that is, it admits a fully faithful right adjoint.*

(2) *The functor F is a localization functor (in the sense of Definition 6.3.3.1) which admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$.*

Proof. We first show that (2) implies (1). Let $G : \mathcal{D} \to \mathcal{C}$ be a right adjoint to F and let $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ be the counit of an adjunction. We wish to show that, if F is a localization functor, then ϵ is an isomorphism (Remark 6.3.3.4). By virtue of Proposition 6.1.4.7 (applied to the *opposite* of the 2-category $\mathrm{h}_2\mathbf{QCat}$), it will suffice to show that for any ∞-category $\mathcal{E}$, precomposition with the isomorphism class $[F] \in \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq})$ induces a monomorphism

$$\pi_0(\mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}) \xrightarrow{\circ [F]} \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}),$$

which follows immediately from our assumption on F.

We now prove the converse. Assume that F is a reflective localization functor and let W be the collection of all morphisms w of $\mathcal{C}$ such that $F(w)$ is an isomorphism in $\mathcal{D}$; we will show that F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W. Fix an ∞-category $\mathcal{E}$, so that precomposition with F induces a function

$$\varphi : \pi_0(\mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq}).$$

Let $G : \mathcal{D} \to \mathcal{C}$ and $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ be as above, so that precomposition with G induces a function

$$\psi : \pi_0(\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq}) \subseteq \pi_0(\mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}).$$

We will show that ψ is inverse to φ. By assumption, the natural transformation ϵ is an isomorphism. It follows that, for every functor $E : \mathcal{D} \to \mathcal{E}$, ϵ induces an isomorphism $E \circ F \circ G \xrightarrow{\sim} E$. Passing to isomorphism classes, we obtain an equality $[E] = [E \circ F \circ G] = \psi([E \circ F]) = (\psi \circ \varphi)([E])$. Allowing E to vary, we conclude that $\psi \circ \varphi$ is the identity on $\pi_0(\mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq})$.

We now complete the proof by showing that $\varphi \circ \psi$ is the identity on $\pi_0(\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq})$. Let $H : \mathcal{C} \to \mathcal{E}$ be a functor of ∞-categories which carries each morphism of W to an isomorphism in $\mathcal{E}$; we wish to show that H is isomorphic to $H \circ G \circ F$. Choose a unit map $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ which is compatible with ϵ up to homotopy. For every object $C \in \mathcal{C}$, Remark 6.3.3.5 guarantees that the morphism $\eta_C : C \to (G \circ F)(C)$ belongs to W, so that $H(\eta_C)$ is an isomorphism in the ∞-category $\mathcal{E}$. Allowing the object C to vary (and invoking the criterion of Theorem 4.4.4.4), we conclude that η induces an isomorphism from H to $H \circ G \circ F$ in the functor ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$. □

04JM **Example 6.3.3.7.** Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a reflective subcategory, and let $L : \mathcal{C} \to \mathcal{C}'$ be a $\mathcal{C}'$-reflection functor (Definition 6.2.2.12). Then L is a reflective localization functor (since it is left adjoint to the inclusion map $\mathcal{C}' \hookrightarrow \mathcal{C}$; see Proposition 6.2.2.15). In particular, L exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to the collection of morphisms w such that $L(w)$ is an isomorphism.

04JN **Remark 6.3.3.8.** Let $\mathcal{C}$ be an ∞-category. Up to equivalence, every reflective localization functor $F : \mathcal{C} \to \mathcal{D}$ can be obtained from the construction of Example 6.3.3.7. More precisely, if G is a fully faithful right adjoint to F, then it induces an equivalence of $\mathcal{D}$ with a reflective subcategory $\mathcal{C}' \subseteq \mathcal{C}$ (carrying F to the $\mathcal{C}'$-reflection functor $(G \circ F) : \mathcal{C} \to \mathcal{C}'$); see Corollary 6.2.2.17.

Remark 6.3.3.9. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, where F is a reflective localization functor. Then G is a reflective localization functor if and only if $(G \circ F)$ is a reflective localization functor. In particular, the collection of reflective localization functors is closed under composition. See Remarks 6.2.1.8 and 4.6.2.5. 04JP

Warning 6.3.3.10. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. If F and G are localization functors, the composition $G \circ F$ need not be a localization functor. For example, let $\mathcal{C}$ be the 1-dimensional simplicial set corresponding to the directed graph depicted in the diagram 04JQ

$$\bullet \xrightarrow{e} \bullet \xleftarrow{w} \bullet \xrightarrow{e'} \bullet.$$

There is a functor $F : \mathcal{C} \to \Delta^2$ which carries e and e' to the edges $0 \to 1$ and $1 \to 2$, respectively. It is not difficult to show that F exhibits Δ^2 as a localization of $\mathcal{C}$ with respect to w. However, the localization of Δ^2 with respect to its "long edge" $0 \to 2$ cannot be realized directly as a localization of $\mathcal{C}$.

Variant 6.3.3.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that F is a *coreflective localization functor* if it admits a fully faithful left adjoint $\mathcal{D} \to \mathcal{C}$. Equivalently, F is a coreflective localization functor if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a reflective localization functor (see Remark 6.2.1.7). It follows from Proposition 6.3.3.6 that every coreflective localization functor is a localization functor (in the sense of Definition 6.3.3.1). 04JR

Warning 6.3.3.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a localization functor of ∞-categories which is both reflective and coreflective. Then F admits both a left adjoint $F^L : \mathcal{D} \to \mathcal{C}$ and a right adjoint $F^R : \mathcal{D} \to \mathcal{C}$, which are automatically fully faithful (Proposition 6.3.3.6). Let $\mathcal{C}^L \subseteq \mathcal{C}$ be the full subcategory spanned by the essential image of the functor F^L, and define $\mathcal{C}^R \subseteq \mathcal{C}$ similarly. When viewed as abstract ∞-categories, $\mathcal{C}^L$ and $\mathcal{C}^R$ are equivalent (since they are both equivalent to the ∞-category $\mathcal{D}$). Beware that they generally do not coincide *as subcategories of* $\mathcal{C}$. See Warning 9.1.1.23. 04JS

6.3.4 Stability Properties of Localizations

Our goal in this section is to record some basic formal properties of the localization construction $\mathcal{C} \mapsto \mathcal{C}[W^{-1}]$ introduced in §6.3.2. We first show that localization commutes with the formation of filtered colimits. More precisely, we have the following: 01N5

Proposition 6.3.4.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets which is given as the colimit (in the arrow category* $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$*) of a filtered diagram of morphisms $\{F_\alpha : \mathcal{C}_\alpha \to \mathcal{D}_\alpha\}$. Assume that:* 01N6

- *Each morphism F_α exhibits $\mathcal{D}_\alpha$ as a localization of $\mathcal{C}_\alpha$ with respect to some collection of edges W_α.*

- *Each of the transition maps $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ of the diagram carries W_α into W_β.*

Let us regard $W = \varinjlim W_\alpha$ as a collection of edges of the simplicial set $\mathcal{C}$. Then F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W.

Proof. Using Corollary 4.1.3.3, we can choose a compatible family of inner anodyne morphisms $G_\alpha : \mathcal{D}_\alpha \to \mathcal{E}_\alpha$, where each $\mathcal{E}_\alpha$ is an ∞-category. Set $\mathcal{E} = \varinjlim \mathcal{E}_\alpha$, so that the morphisms G_α determine a map of simplicial sets $G : \mathcal{D} \to \mathcal{E}$. Since each G_α is a categorical equivalence of simplicial sets, each of the composite maps $(G_\alpha \circ F_\alpha) : \mathcal{C}_\alpha \to \mathcal{E}_\alpha$ exhibits $\mathcal{E}_\alpha$ as a localization of $\mathcal{C}_\alpha$ with respect to W_α. In particular, each of the morphisms $G_\alpha \circ F_\alpha$ carries edges of W_α to isomorphisms in the ∞-category $\mathcal{E}_\alpha$ (Remark 6.3.1.10). Applying Proposition 6.3.2.5, we can (functorially) factor each of the morphisms $G_\alpha \circ F_\alpha$ as a composition

$$\mathcal{C}_\alpha \xrightarrow{G'_\alpha} \mathcal{C}_\alpha[W_\alpha^{-1}] \xrightarrow{F'_\alpha} \mathcal{E}_\alpha,$$

where each $\mathcal{C}_\alpha[W_\alpha^{-1}]$ is an ∞-category, each of the morphisms G'_α exhibits $\mathcal{C}_\alpha[W_\alpha^{-1}]$ as a localization of $\mathcal{C}_\alpha$ with respect to W_α, and the colimit map $G' : \mathcal{C} \to \varinjlim_\alpha \mathcal{C}_\alpha[W_\alpha^{-1}]$ exhibits $\mathcal{C}[W^{-1}] = \varinjlim_\alpha \mathcal{C}_\alpha[W_\alpha^{-1}]$ as a localization of $\mathcal{C}$ with respect to W. We then have a filtered diagram of commutative squares

$$\begin{tikzcd}
\mathcal{C}_\alpha \arrow[r, "F_\alpha"] \arrow[d, "G'_\alpha"] & \mathcal{D}_\alpha \arrow[d, "G_\alpha"] \\
\mathcal{C}_\alpha[W_\alpha^{-1}] \arrow[r, "F'_\alpha"] & \mathcal{E}_\alpha
\end{tikzcd}$$

having colimit

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "G'"] & \mathcal{D} \arrow[d, "G"] \\
\mathcal{C}[W^{-1}] \arrow[r, "F'"] & \mathcal{E}.
\end{tikzcd}$$

Applying Remark 6.3.1.19, we deduce that each of the morphisms F'_α is a categorical equivalence of simplicial sets. Since the collection of categorical equivalences is stable under filtered colimits (Corollary 4.5.7.2), the morphism F' is also a categorical equivalence of simplicial sets. Applying Remark 6.3.1.19 again, we deduce that $F' \circ G'$ exhibits $\mathcal{E}$ as a localization of $\mathcal{C}$ with respect to W. Since each G_α is a categorical equivalence, Corollary 4.5.7.2 also guarantees that G is a categorical equivalence. Using the equality $G \circ F = F' \circ G'$ and applying Remark 6.3.1.19 again, we conclude that F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to W, as desired. □

We now show that localization is compatible with the formation of categorical pushout squares.

Proposition 6.3.4.2. *Suppose we are given a commutative diagram of simplicial sets* 01N7

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[rr, "G"] \arrow[dd, "H"] \arrow[dr, "F_{01}"] & & \mathcal{C}_0 \arrow[dd, "H'" near start] \arrow[dr, "F_0"] & \\
& \mathcal{D}_{01} \arrow[rr] \arrow[dd] & & \mathcal{D}_0 \arrow[dd] \\
\mathcal{C}_1 \arrow[rr, "G'" near end] \arrow[dr, "F_1"] & & \mathcal{C} \arrow[dr, "F"] & \\
& \mathcal{D}_1 \arrow[rr] & & \mathcal{D}
\end{tikzcd}$$

with the following properties:

(*a*) *The back face*

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r, "G"] \arrow[d, "H"] & \mathcal{C}_0 \arrow[d, "H'"] \\
\mathcal{C}_1 \arrow[r, "G'"] & \mathcal{C}
\end{tikzcd}$$

is a categorical pushout square of simplicial sets.

(*b*) *The morphism of simplicial sets $F_{01} : \mathcal{C}_{01} \to \mathcal{D}_{01}$ exhibits $\mathcal{D}_{01}$ as a localization of $\mathcal{C}_{01}$ with respect to some collection of edges W_{01}.*

(*c*) *The morphism of simplicial sets $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ exhibits $\mathcal{D}_0$ as a localization of $\mathcal{C}_0$ with respect to some collection of edges W_0 containing $G(W_{01})$.*

(*d*) *The morphism of simplicial sets $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ exhibits $\mathcal{D}_1$ as a localization of $\mathcal{C}_1$ with respect to some collection of edges W_1 containing $H(W_{01})$.*

Then the following conditions are equivalent:

(1) *The front face*

$$\begin{tikzcd}
\mathcal{D}_{01} \arrow[r] \arrow[d] & \mathcal{D}_0 \arrow[d] \\
\mathcal{D}_1 \arrow[r] & \mathcal{D}
\end{tikzcd}$$

is a categorical pushout square of simplicial sets.

(2) *The morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to the collection of edges $W = H'(W_0) \cup G'(W_1)$.*

Proof. Let $\mathcal{E}$ be an ∞-category. Assumption (a) guarantees that the diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{C}, \mathcal{E})^{\simeq} \arrow[r] \arrow[d] & \operatorname{Fun}(\mathcal{C}_0, \mathcal{E})^{\simeq} \arrow[d] \\
\operatorname{Fun}(\mathcal{C}_1, \mathcal{E})^{\simeq} \arrow[r] & \operatorname{Fun}(\mathcal{C}_{01}, \mathcal{E})^{\simeq}
\end{tikzcd}$$

is a homotopy pullback square. Applying Proposition 3.4.1.14, we deduce that the diagram of summands

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq} \arrow[r] \arrow[d] & \operatorname{Fun}(\mathcal{C}_0[W_0^{-1}], \mathcal{E})^{\simeq} \arrow[d] \\
\operatorname{Fun}(\mathcal{C}_1[W_1^{-1}], \mathcal{E})^{\simeq} \arrow[r] & \operatorname{Fun}(\mathcal{C}_{01}[W_{01}^{-1}], \mathcal{E})^{\simeq}
\end{tikzcd}$$

is also a homotopy pullback square. Invoking Corollary 3.4.1.12, we conclude that the following conditions are equivalent:

$(1_{\mathcal{E}})$ The diagram of Kan complexes

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \arrow[r] \arrow[d] & \operatorname{Fun}(\mathcal{D}_0, \mathcal{E})^{\simeq} \arrow[d] \\
\operatorname{Fun}(\mathcal{D}_1, \mathcal{E})^{\simeq} \arrow[r] & \operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}
\end{tikzcd}$$

is a homotopy pullback square.

($2_{\mathcal{E}}$) Precomposition with F induces a homotopy equivalence of Kan complexes

$$\operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\circ F} \operatorname{Fun}(\mathcal{C}[W^{-1}], \mathcal{E})^{\simeq}.$$

We now observe that condition (1) is equivalent to the requirement that ($1_{\mathcal{E}}$) holds for every ∞-category $\mathcal{E}$ (by definition), and condition (2) is equivalent to the requirement that ($2_{\mathcal{E}}$) holds for every ∞-category $\mathcal{E}$ (Proposition 6.3.1.13). □

Corollary 6.3.4.3. *Suppose we are given a categorical pushout diagram of simplicial sets* 03E3

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "F"] & \mathcal{C}' \arrow[d, "F'"] \\
\mathcal{D} \arrow[r] & \mathcal{D}',
\end{tikzcd}$$

where F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to some collection of edges W. Then F' exhibits $\mathcal{D}'$ as a localization of $\mathcal{C}'$ with respect to $G(W)$.

Proof. Apply Proposition 6.3.4.2 to the cubical diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{C}' \arrow[dr] \arrow[dd] & \\
& \mathcal{C} \arrow[rr] \arrow[dd] & & \mathcal{C}' \arrow[dd] \\
\mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{C}' \arrow[dr] & \\
& \mathcal{D} \arrow[rr] & & \mathcal{D}' .
\end{tikzcd}$$

□

Example 6.3.4.4 (Contracting an Edge)**.** Let $\mathcal{C}$ be a simplicial set and let e be an edge 03E4 of $\mathcal{C}$ which corresponds to a monomorphism of simplicial sets $\Delta^1 \hookrightarrow \mathcal{C}$ (that is, the source and target of e are distinct when regarded as vertices of $\mathcal{C}$). Let $\mathcal{C}'$ denote the simplicial set obtained from $\mathcal{C}$ by collapsing the edge e, so that we have a pushout square of simplicial sets

$$\begin{tikzcd}
\Delta^1 \arrow[r, "e"] \arrow[d] & \mathcal{C} \arrow[d, "T"] \\
\Delta^0 \arrow[r] & \mathcal{C}' .
\end{tikzcd}$$

Since the horizontal maps in this diagram are monomorphisms, it is also a categorical pushout square (Example 4.5.4.12). Combining Corollary 6.3.4.3 with Example 6.3.1.14, we see that T exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to the singleton $W = \{e\}$.

6.3.5 Fiberwise Localization

02LU Suppose we are given a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r, "G"] & \mathcal{C}',
\end{tikzcd}$$

where U and U' are cocartesian fibrations and the functor F carries U-cocartesian morphisms of $\mathcal{E}$ to U'-cocartesian morphisms of $\mathcal{E}'$. For each object $C \in \mathcal{C}$, write $F_C : \mathcal{E}_C \to \mathcal{E}'_{G(C)}$ for the induced map of fibers. It follows from Theorem 5.1.6.1 that if the functors $\{F_C\}_{C \in \mathcal{C}}$ and G are equivalences of ∞-categories, then F is also an equivalence of ∞-categories. Our goal in this section is to prove a generalization of this result, which gives a sufficient condition for F to exhibit $\mathcal{E}'$ as a localization of $\mathcal{E}$.

038U **Theorem 6.3.5.1.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r, "\overline{F}"] & \mathcal{C}'
\end{tikzcd}$$

which satisfies the following conditions:

(1) *The morphisms U and U' are cocartesian fibrations.*

(2) *The morphism F carries U-cocartesian edges of $\mathcal{E}$ to U'-cocartesian edges of $\mathcal{E}'$.*

(3) *For every vertex $C \in \mathcal{C}$ having image $C' = \overline{F}(C) \in \mathcal{C}'$, the induced functor of ∞-categories $F_C : \mathcal{E}_C \to \mathcal{E}'_{C'}$ exhibits $\mathcal{E}'_{C'}$ as the localization of $\mathcal{E}_C$ with respect to some collection of morphisms W_C of $\mathcal{E}_C$.*

(4) *The morphism $\overline{F}$ exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to some collection of morphisms $\overline{W}$ of $\mathcal{C}$.*

Set $W_- = \bigcup_{C \in \mathcal{C}} W_C$ and let W_+ be the collection of all U-cocartesian edges e such that $U(e)$ belongs to $\overline{W}$. Then F exhibits $\mathcal{E}'$ as a localization of $\mathcal{E}$ with respect to $W_- \cup W_+$.

We begin by proving a special case of Theorem 6.3.5.1, where $\overline{F}$ is assumed to be an isomorphism.

Proposition 6.3.5.2. *Suppose we are given a commutative diagram of simplicial sets* 02LW

02LX
$$\begin{tikzcd} \mathcal{E} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\ & \mathcal{C} & \end{tikzcd} \tag{6.6}$$

with the following properties:

(1) *The morphisms U and U' are cocartesian fibrations.*

(2) *The morphism F carries U-cocartesian edges of $\mathcal{E}$ to U'-cocartesian edges of $\mathcal{E}'$.*

(3) *For every vertex $C \in \mathcal{C}$, the induced functor of ∞-categories $F_C : \mathcal{E}_C \to \mathcal{E}'_C$ exhibits $\mathcal{E}'_C$ as the localization of $\mathcal{E}_C$ with respect to some collection of morphisms W_C.*

Set $W = \bigcup_{C \in \mathcal{C}} W_C$, which we regard as a collection of edges of the simplicial set $\mathcal{E}$. Then F exhibits $\mathcal{E}'$ as a localization of $\mathcal{E}$ with respect to W.

Proof. Let $\mathcal{D}$ be an ∞-category, so that precomposition with F induces a functor $F^* : \mathrm{Fun}(\mathcal{E}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{D})$. We wish to show that the functor F^* is fully faithful, and that its essential image is the full subcategory $\mathrm{Fun}(\mathcal{E}[W^{-1}], \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{D})$. Let $\mathcal{B} = \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{D})$ and $\mathcal{B}' = \mathrm{Fun}(\mathcal{E}' / \mathcal{C}, \mathcal{D})$ be the relative exponentials of Construction 4.5.9.1, and let $\pi : \mathcal{B} \to \mathcal{C}$ and $\pi' : \mathcal{B}' \to \mathcal{C}$ denote the projection maps. Combining assumption (1) with Corollary 5.3.6.8, we see that π and π' are cartesian fibrations.

For each vertex $C \in \mathcal{C}$, let us identify the fibers $\mathcal{B}_C = \{C\} \times_{\mathcal{C}} \mathcal{B}$ and $\mathcal{B}'_C = \{C\} \times_{\mathcal{C}} \mathcal{B}'$ with the ∞-categories $\mathrm{Fun}(\mathcal{E}_C, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{E}'_C, \mathcal{D})$, respectively. Precomposition with F

induces a morphism of simplicial sets $G : \mathcal{B}' \to \mathcal{B}$ satisfying $\pi \circ G = \pi'$, given on each fiber by the functor

$$\mathcal{B}'_C = \mathrm{Fun}(\mathcal{E}'_C, \mathcal{D}) \xrightarrow{\circ F_C} \mathrm{Fun}(\mathcal{E}_C, \mathcal{D}) = \mathcal{B}_C .$$

Combining assumption (2) with Corollary 5.3.6.8, we see that G carries π'-cartesian edges of $\mathcal{B}'$ to π-cartesian edges of $\mathcal{B}$. In particular, for every edge $e : X \to Y$ of $\mathcal{C}$, the diagram of ∞-categories

038V

$$\begin{tikzcd} \mathcal{B}'_Y \arrow[r, "e^*"] \arrow[d, "G_Y"] & \mathcal{B}'_X \arrow[d, "G_X"] \\ \mathcal{B}_Y \arrow[r, "e^*"] & \mathcal{B}_X \end{tikzcd} \qquad (6.7)$$

commutes up to isomorphism, where the horizontal functors are given by contravariant transport along e (see Remark 5.2.8.5).

Let us identify the vertices of $\mathcal{B}$ with pairs (C, ρ), where C is a vertex of $\mathcal{C}$ and $\rho : \mathcal{E}_C \to \mathcal{D}$ is a functor of ∞-categories. Let $\mathcal{B}^0 \subseteq \mathcal{B}$ denote the full simplicial subset spanned by those vertices (C, ρ) for which the functor ρ carries each edge of W_C to an isomorphism in the ∞-category $\mathcal{D}$. It follows from assumption (3) that for every vertex C, the functor $G_C : \mathcal{B}'_C \to \mathcal{B}_C$ is fully faithful, and its essential image can be identified with the full subcategory $\mathcal{B}^0_C = \{C\} \times_{\mathcal{C}} \mathcal{B}^0 \subseteq \mathcal{B}_C$. Combining this observation with the homotopy commutativity of the diagram (6.7), we see that for every edge $e : X \to Y$ in $\mathcal{E}$, the contravariant transport functor $e^* : \mathcal{B}_Y \to \mathcal{B}_X$ carries $\mathcal{B}^0_Y$ into $\mathcal{B}^0_X$. It follows that π restricts to a cartesian fibration of simplicial sets $\pi^0 : \mathcal{B}^0 \to \mathcal{C}$, and that an edge of $\mathcal{B}^0$ is π^0-cartesian if and only if it is π-cartesian when viewed as an edge of $\mathcal{B}$ (Proposition 5.1.4.16). In particular, the morphism $G : \mathcal{B}' \to \mathcal{B}^0 \subseteq \mathcal{B}$ carries π'-cartesian edges of $\mathcal{B}'$ to π^0-cartesian edges of $\mathcal{B}^0$, and therefore induces an equivalence $\mathcal{B}' \to \mathcal{B}^0$ of cartesian fibrations over $\mathcal{C}$ (Proposition 5.1.7.15). We complete the proof by observing that $F^* : \mathrm{Fun}(\mathcal{E}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}[W^{-1}], \mathcal{D})$ can be identified with the functor

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{B}') \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{B}^0)$$

given by precomposition with G, and is therefore an equivalence of ∞-categories. □

02LY **Corollary 6.3.5.3.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Suppose that, for every vertex $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is weakly contractible. Let W be the collection of all edges e of $\mathcal{E}$ having the property that $U(e)$ is a degenerate edge of $\mathcal{C}$. Then U exhibits $\mathcal{C}$ as a localization of $\mathcal{E}$ with respect to W.*

Proof. For each vertex $C \in \mathcal{C}$, let W_C be the collection of all morphisms in the ∞-category $\mathcal{E}_C$. Since $\mathcal{E}_C$ is weakly contractible, the projection map $\mathcal{E}_C \to \{C\}$ exhibits $\{C\}$ as a localization of $\mathcal{E}_C$ with respect to W_C (Proposition 6.3.1.20). The desired result now follows by applying Proposition 6.3.5.2 to the commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "U"] \arrow[dr, "U"'] & & \mathcal{C} \arrow[dl, "\mathrm{id}_{\mathcal{C}}"] \\
& \mathcal{C}. &
\end{tikzcd}$$

□

We now consider another special case of Theorem 6.3.5.1.

Proposition 6.3.5.4. *Suppose we are given a pullback diagram of simplicial sets* 038W

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r, "\overline{F}"] & \mathcal{C}',
\end{tikzcd}$$

where U and U' are cocartesian fibrations. Suppose that $\overline{F}$ exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to some collection of edges $\overline{W}$, and let W denote the collection of U-cocartesian edges e of $\mathcal{E}$ which satisfy $U(e) \in \overline{W}$. Then F exhibits $\mathcal{E}'$ as a localization of $\mathcal{E}$ with respect to W.

Proof. Using Corollary 4.1.3.3, we can choose an inner anodyne map $\mathcal{C}' \hookrightarrow \mathcal{C}''$, where $\mathcal{C}''$ is an ∞-category. By virtue of Proposition 5.6.7.2, we can assume that U' is the pullback of a cocartesian fibration of simplicial sets $U'' : \mathcal{E}'' \to \mathcal{C}''$. Applying Proposition 5.3.6.1, we deduce that the inclusion map $\mathcal{E}' \hookrightarrow \mathcal{E}''$ is a categorical equivalence of simplicial sets. We may therefore replace U' by U'', and thereby reduce to proving Proposition 6.3.5.4 in the special case where $\mathcal{C}'$ is an ∞-category.

Fix an ∞-category $\mathcal{D}$. We wish to show that the functor $F^* : \mathrm{Fun}(\mathcal{E}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{D})$ is fully faithful and that its essential image is the full subcategory $\mathrm{Fun}(\mathcal{E}[W^{-1}], \mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{E}, \mathcal{D})$. Let $\mathcal{B}' = \mathrm{Fun}(\mathcal{E}' / \mathcal{C}, \mathcal{D})$ and $\pi' : \mathcal{B}' \to \mathcal{C}$ be as in the proof of Proposition 6.3.5.2, so that we have canonical isomorphisms

$$\mathrm{Fun}(\mathcal{E}', \mathcal{D}) \simeq \mathrm{Fun}_{/\mathcal{C}'}(\mathcal{C}', \mathcal{B}') \qquad \mathrm{Fun}(\mathcal{E}, \mathcal{D}) \simeq \mathrm{Fun}_{/\mathcal{C}'}(\mathcal{C}, \mathcal{B}')$$

Note that a morphism $G : \mathcal{E} \to \mathcal{D}$ carries each edge of W to an isomorphism in $\mathcal{D}$ if and only if the corresponding object $g \in \mathrm{Fun}_{/\mathcal{C}'}(\mathcal{C}, \mathcal{B}')$ carries each element $\overline{e} \in \overline{W}$ to a π'-cartesian edge of $\mathcal{B}'$ (see Corollary 5.3.6.8). Since $\overline{F}$ carries each edge $\overline{e} \in \overline{W}$ to an isomorphism in $\mathcal{C}'$, this is equivalent to the requirement that $g(\overline{e})$ is an isomorphism in $\mathcal{B}'$ (Proposition 5.1.1.8). We are therefore reduced to showing that composition with $\overline{F}$ induces a fully faithful functor $\mathrm{Fun}_{/\mathcal{C}'}(\mathcal{C}', \mathcal{B}') \to \mathrm{Fun}_{/\mathcal{C}'}(\mathcal{C}, \mathcal{B}')$, whose essential image is spanned by those functors $g \in \mathrm{Fun}_{/\mathcal{C}'}(\mathcal{C}, \mathcal{B}')$ which carry each edge of $\overline{W}$ to an isomorphism in $\mathcal{B}'$. This is a special case of Remark 6.3.1.15. □

038X **Corollary 6.3.5.5.** *Suppose we are given a pullback diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r, "\overline{F}"] & \mathcal{C}',
\end{tikzcd}$$

where U and U' are left fibrations. Suppose that $\overline{F}$ exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to some collection of edges $\overline{W}$. Then F exhibits $\mathcal{E}'$ as a localization of $\mathcal{E}$ with respect to $W = U^{-1}(\overline{W})$.

Proof. Combine Proposition 6.3.5.4 with Proposition 5.1.4.14. □

Proof of Theorem 6.3.5.1. Suppose we are given a commutative diagram of simplicial sets

038Y
$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r, "\overline{F}"] & \mathcal{C}'
\end{tikzcd} \tag{6.8}$$

which satisfies the hypotheses of Theorem 6.3.5.1. Fix an ∞-category $\mathcal{D}$. We wish to show that precomposition with F induces a fully faithful functor $F^* : \mathrm{Fun}(\mathcal{E}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}, \mathcal{D})$, whose essential image consists of those morphisms $G : \mathcal{E} \to \mathcal{D}$ which carry each edge of $W_- \cup W_+$ to an isomorphism in $\mathcal{D}$. Let $\pi : \mathcal{C} \times_{\mathcal{C}'} \mathcal{E}' \to \mathcal{E}'$ be given by projection onto the second factor. Note that the pair (U, F) determines a morphism of simplicial sets $\widetilde{F} : \mathcal{E} \to \mathcal{C} \times_{\mathcal{C}'} \mathcal{E}'$ satisfying $\pi \circ \widetilde{F} = F$, so that F^* factors as a composition

$$\mathrm{Fun}(\mathcal{E}', \mathcal{D}) \xrightarrow{\pi^*} \mathrm{Fun}(\mathcal{C} \times_{\mathcal{C}'} \mathcal{E}', \mathcal{D}) \xrightarrow{\widetilde{F}^*} \mathrm{Fun}(\mathcal{E}, \mathcal{D}).$$

Let W' be the collection of all edges of $\mathcal{C} \times_{\mathcal{C}'} \mathcal{E}'$ of the form $(\overline{e}, f)$, where $\overline{e}$ belongs to $\overline{W}$ and f is a U'-cocartesian edge of $\mathcal{E}'$. It follows from Proposition 6.3.5.4 that the functor π^*

is fully faithful, and that its essential image consists of those morphisms $G' : \mathcal{C} \times_{\mathcal{C}'} \mathcal{E}' \to \mathcal{D}$ which carry each edge of W' to an isomorphism in $\mathcal{D}$. Applying Proposition 6.3.5.2, we see that the functor $\widetilde{F}^*$ is also fully faithful, and that its essential image consists of those morphisms $G : \mathcal{E} \to \mathcal{D}$ which carry each edge of W_- to an isomorphism in $\mathcal{D}$. To complete the proof, it will suffice to show the following:

$(*)$ A morphism of simplicial sets $G' : \mathcal{C} \times_{\mathcal{C}'} \mathcal{E}' \to \mathcal{D}$ carries each edge of W' to an isomorphism in $\mathcal{D}$ if and only if $G' \circ \widetilde{F}$ carries each edge of W_+ to an isomorphism in $\mathcal{D}$.

The "only if" assertion is immediate (since $\widetilde{F}(W_+)$ is contained in W'). The converse follows from the observation that every edge $(\overline{e}, f)$ is isomorphic, when viewed as an object of the ∞-category $\operatorname{Fun}_{/\mathcal{C}}(\Delta^1, \mathcal{C} \times_{\mathcal{C}'} \mathcal{E}')$, to $\widetilde{F}(e)$, where $e : X \to Y$ is any U-cocartesian edge of $\mathcal{E}$ for which $U(e) = \overline{e}$ and $F(X)$ is isomorphic to the domain of f as an object of the ∞-category $\{X\} \times_{\mathcal{C}'} \mathcal{E}'$. □

We now record a few other thematically related results which will be useful later.

Proposition 6.3.5.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to a collection of edges W. Let K be any simplicial set, and let W_K denote the collection of edges $e = (e', e'')$ of the product $K \times \mathcal{C}$ for which e' is a degenerate edge of K and e'' belongs to W. Then the induced map $F_K : K \times \mathcal{C} \to K \times \mathcal{D}$ exhibits $K \times \mathcal{D}$ as the localization of $K \times \mathcal{C}$ with respect to W_K.* `02LV`

Proof. Let $\mathcal{E}$ be an ∞-category, and let

$$\theta : \operatorname{Fun}(K \times \mathcal{D}, \mathcal{E}) \to \operatorname{Fun}(K \times \mathcal{C}, \mathcal{E})$$

be the functor given by precomposition with F_K. We wish to show that F_K is fully faithful, and that its essential image is the full subcategory $\operatorname{Fun}((K \times \mathcal{C})[W_K^{-1}], \mathcal{E})$ of Notation 6.3.1.1. Unwinding the definitions, we can identify θ with the functor

$$\theta' : \operatorname{Fun}(\mathcal{D}, \operatorname{Fun}(K, \mathcal{E})) \to \operatorname{Fun}(\mathcal{C}, \operatorname{Fun}(K, \mathcal{E}))$$

given by precomposition with F. Under this identification $\operatorname{Fun}((K \times \mathcal{C})[W_K^{-1}], \mathcal{E})$ corresponds to the full subcategory $\operatorname{Fun}(\mathcal{C}[W^{-1}], \operatorname{Fun}(K, \mathcal{E})) \subseteq \operatorname{Fun}(\mathcal{C}, \operatorname{Fun}(K, \mathcal{E}))$ (see Theorem 4.4.4.4), so that the desired result follows from our assumption on the functor F. □

6.3.6 Universal Localizations

`02LZ`

The formation of localizations is generally not compatible with fiber products. If

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r] & S
\end{tikzcd}$$

is a pullback diagram of simplicial sets where the morphism f exhibits S as a localization of X (with respect to some collection of edges of X), then the morphism f' need not have the same property. To address this point, it will be convenient to introduce a more restrictive notion of localization.

02M0 **Definition 6.3.6.1.** Let $f : X \to S$ be a morphism of simplicial sets. We will say that f is *universally localizing* if, for every morphism of simplicial sets $S' \to S$, the projection map $S' \times_S X \to S'$ exhibits S' as a localization of $S' \times_S X$ with respect to some collection of edges W.

047T **Example 6.3.6.2.** Let $f : X \to S$ be a cocartesian fibration of simplicial sets. If each fiber $X_s = \{s\} \times_S X$ is weakly contractible, then f is universally localizing. See Corollary 6.3.5.3.

If $f : X \to S$ is a universally localizing morphism of simplicial sets, then it exhibits S as a localization of X with respect to *some* collection of edges W. It is possible to be more precise: we can take W to be the collection of edges of X having degenerate image in S.

02M1 **Proposition 6.3.6.3.** *Let $f : X \to S$ be a morphism of simplicial sets. For every morphism $T \to S$, let W_T denote the collection of all edges $w = (w_T, w_X)$ of the fiber product $T \times_S X$ for which w_T is a degenerate edge of T. The following conditions are equivalent:*

(1) *For every morphism of simplicial sets $T \to S$, the projection map $T \times_S X \to T$ exhibits T as a localization of $T \times_S X$ with respect to W_T.*

(2) *The morphism f is universally localizing, in the sense of Definition 6.3.6.1.*

(3) *For every simplex $\sigma : \Delta^n \to S$, the projection map $\Delta^n \times_S X \to \Delta^n$ exhibits Δ^n as a localization of $\Delta^n \times_S X$ with respect to some collection of edges of $\Delta^n \times_S X$.*

(4) *For every simplex $\sigma : \Delta^n \to S$, the projection map $\Delta^n \times_S X \to \Delta^n$ exhibits Δ^n as a localization of $\Delta^n \times_S X$ with respect to W_{Δ^n}.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are immediate. We next show that (3) implies (4). Let σ be an n-simplex of S, and suppose that the projection map $\pi : \Delta^n \times_S X \to \Delta^n$ exhibits Δ^n as a localization of $\Delta^n \times_S X$ with respect to some collection of edges W. Since

Δ^n is an ∞-category in which every isomorphism is an identity morphism, the diagram π must carry each edge of W to a degenerate edge of Δ^n: that is, we have $W \subseteq W_{\Delta^n}$. Applying Corollary 6.3.1.22, we deduce that π also exhibits Δ^n as a localization of $\Delta^n \times_S X$ with respect to W_{Δ^n}.

We now complete the proof by showing that (4) implies (1). Let us say that a simplicial set T is *good* if, for every morphism $T \to S$, the projection map $T \times_S X \to T$ exhibits T as a localization of $T \times_S X$ with respect to W_T. Assume that condition (4) is satisfied, so that every standard simplex Δ^m is good. We wish to show that every simplicial set T is good. Using Proposition 6.3.4.1, we see that the collection of good simplicial sets is closed under filtered colimits; we may therefore assume without loss of generality that T is finite. If $T = \emptyset$, the result is obvious. We may therefore assume that T has dimension n for some integer $n \geq 0$. We proceed by induction on n and on the number of nondegenerate n-simplices of T. Fix a nondegenerate n-simplex $\sigma : \Delta^n \to T$. Using Proposition 1.1.4.12, we see that there is a pushout square of simplicial sets

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d, "\sigma"] \\
T' \arrow[r] & T,
\end{tikzcd}$$

where T' is a simplicial set of dimension $\leq n$ having fewer nondegenerate n-simplices than T. By virtue of Proposition 6.3.4.2, to show that T is good, it will suffice to show that the simplicial sets Δ^n, $\partial \Delta^n$, and T' are good. In the first case this follows from assumption (4), and in the remaining cases it follows from our inductive hypothesis. □

Corollary 6.3.6.4. *Let $f : X \to S$ be a universally localizing morphism of simplicial sets, and let W be the collection of edges w of X for which $f(w)$ is a degenerate edge of S. Then f exhibits S as a localization of X with respect to W.* 02M2

Remark 6.3.6.5. Let $f : X \to S$ be a universally localizing morphism of simplicial sets. Then f is a weak homotopy equivalence (see Remark 6.3.1.16). 02M3

Remark 6.3.6.6. Let X be a simplicial set. Then the projection map $X \to \Delta^0$ is universally localizing if and only if X is weakly contractible. This follows by combining Propositions 6.3.1.20 and 6.3.5.6. 02M4

02M5 **Remark 6.3.6.7.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r] & S. \end{tikzcd}$$

If f is universally localizing, then f' is universally localizing.

04JT **Proposition 6.3.6.8.** *Let $f : X \to S$ be a morphism of simplicial sets which admits a section $u : S \hookrightarrow X$. Suppose that $u \circ f$ and id_X belong to the same connected component of the simplicial set $\mathrm{Fun}_{/S}(X, X)$. Then f is universally localizing.*

Proof. Let W be the collection of all edges w of X such that $f(w)$ is degenerate in S. Since our hypothesis is stable under the formation of pullbacks, it will suffice to show that f exhibits S as a localization of X with respect to W. Fix an ∞-category $\mathcal{C}$; we wish to show that composition with f induces a bijection

$$\alpha : \pi_0(\mathrm{Fun}(S, \mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(X[W^{-1}], \mathcal{C})^{\simeq}).$$

The injectivity of α follows immediately from the existence of the section u. To prove surjectivity, it will suffice to show that for every object $g \in \mathrm{Fun}(X[W^{-1}], \mathcal{C})$ is isomorphic to $g \circ u \circ f$. Since $u \circ f$ and id_X belong to the same connected component of $\mathrm{Fun}_{/S}(X, X)$, it suffices to observe that postcomposition with g carries every edge of $\mathrm{Fun}_{/S}(X, X)$ to an isomorphism in the ∞-category $\mathrm{Fun}(X, \mathcal{C})$. □

02M6 **Proposition 6.3.6.9.** *Let $f : X \to S$ be a universally localizing morphism of simplicial sets. Then f is surjective.*

Proof. Let $\sigma : \Delta^n \to S$ be an n-simplex of S; we wish to show that σ can be lifted to an n-simplex of X. Assume otherwise, so that the inclusion map $\partial \Delta^n \times_S X \hookrightarrow \Delta^n \times_S X$ is an isomorphism. We have a commutative diagram of simplicial sets

$$\begin{tikzcd} \partial \Delta^n \times_S X \arrow[r, "\sim"] \arrow[d] & \Delta^n \times_S X \arrow[d] \\ \partial \Delta^n \arrow[r] & \Delta^n, \end{tikzcd}$$

where the vertical maps are weak homotopy equivalences (see Remarks 6.3.6.7 and 6.3.6.5). It follows that the inclusion $\partial \Delta^n \hookrightarrow \Delta^n$ is also a weak homotopy equivalence, which is a contradiction (since the relative homology group $\mathrm{H}_n(\Delta^n, \partial \Delta^n; \mathbf{Z}) \simeq \mathbf{Z}$ is nonzero). □

Proposition 6.3.6.10. *Let $f : X \to Y$ and $g : Y \to Z$ be universally localizing morphisms of simplicial sets. Then the composition $(g \circ f) : X \to Z$ is universally localizing.* 02M7

Proof. Suppose we are given a morphism of simplicial sets $Z' \to Z$. Set $X' = Z' \times_Z X$, and let W be the collection of those edges w of X' having degenerate image in Z'. We will show that the projection map $\pi : X' \to Z'$ exhibits Z' as a localization of X' with respect to W. Set $Y' = Z' \times_Z Y$, so that π factors as a composition $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z'$. It follows from Proposition 6.3.6.9 (and Remark 6.3.6.7) that f' is a surjection of simplicial sets. In particular, the image $f'(W)$ is the collection of all edges u of Y' having the property that $g'(u)$ is a degenerate edge of Z'.

Let $W_0 \subseteq W$ be the collection of those edges w of X' for which $f'(w)$ is a degenerate edge of Y'. Applying Proposition 6.3.6.3, we conclude that f' exhibits Y' as a localization of X' with respect to W_0, and that g' exhibits Z' as a localization of Y' with respect to $f'(W)$. Applying Proposition 6.3.1.21, we conclude that $\pi = g' \circ f'$ exhibits Z' as the localization of X' with respect to $W_0 \cup W = W$, as desired. □

Corollary 6.3.6.11. *Let $f : X \to S$ be a universally localizing morphism of simplicial sets, and let K be a weakly contractible simplicial set. Then the composite map $X \times K \to X \xrightarrow{f} S$ is universally localizing.* 02M8

Proof. By virtue of Proposition 6.3.6.10, it will suffice to show that the projection map $X \times K \to X$ is universally localizing. Using Remark 6.3.6.7, we can reduce to the case $X = \Delta^0$, in which case the desired result follows from Remark 6.3.6.6. □

Proposition 6.3.6.12. *The collection of universally localizing morphisms is closed under the formation of filtered colimits (when regarded as a full subcategory of the arrow category* $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$*).* 02M9

Proof. Suppose that $f : X \to S$ is a morphism of simplicial sets which can be realized as the colimit of a filtered diagram $\{f_\alpha : X_\alpha \to S_\alpha\}$ in the category $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$, where each f_α is universally localizing. We wish to show that f is universally localizing. Fix a morphism of simplicial sets $T \to S$ and let W be the collection of all edges $w = (w_T, w_X)$ of $T \times_S X$ for which w_T is a degenerate edge of T. Note that the projection map $f_T : T \times_S X \to T$ can be realized as a filtered colimit of morphisms $f_{T,\alpha} : T \times_S X_\alpha \to T \times_S S_\alpha$. For each index α, let W_α denote the collection of edges of $T \times_S X_\alpha$ having degenerate image in T. Since f_α is universally localizing, Proposition 6.3.6.3 guarantees that $f_{T,\alpha}$ exhibits $T \times_S S_\alpha$ as a localization of $T \times_S X_\alpha$ with respect to W_α. Applying Proposition 6.3.4.1, we conclude that f_T exhibits T as a localization of $T \times_S X$ with respect to W. □

02MA **Proposition 6.3.6.13.** *Suppose we are given a commutative diagram of simplicial sets*

02MB
$$\begin{tikzcd}
X_{01} \arrow[rr] \arrow[dr, "f_{01}"] \arrow[dd] & & X_0 \arrow[dr, "f_0"] \arrow[dd] & \\
& S_{01} \arrow[rr] \arrow[dd] & & S_0 \arrow[dd] \\
X_1 \arrow[rr] \arrow[dr, "f_1"] & & X \arrow[dr, "f"] & \\
& S_1 \arrow[rr] & & S
\end{tikzcd} \tag{6.9}$$

with the following properties:

(*a*) *The front and back faces*

$$\begin{tikzcd}
S_{01} \arrow[r] \arrow[d] & S_0 \arrow[d] \\
S_1 \arrow[r] & S
\end{tikzcd} \qquad \begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

are pushout squares.

(*b*) *The morphisms $S_{01} \to S_0$ and $X_{01} \to X_0$ are monomorphisms.*

(*c*) *The morphisms f_{01}, f_0, and f_1 are universally localizing.*

Then the morphism f is universally localizing.

Proof. Fix a morphism of simplicial sets $T \to S$; we wish to show that the projection map $f_T : T \times_S X \to T$ exhibits T as a localization of $T \times_S X$ with respect to some collection of morphisms W. Since the hypotheses of Proposition 6.3.6.13 are stable with respect to pullback, we may assume without loss of generality that $T = S$. Let W_0 be the collection of edges w of X_0 having the property that $f_0(w)$ is a degenerate edge of S_0, and define W_1 and W_{01} similarly. Combining assumption (c) with Proposition 6.3.6.3, we conclude that the morphism f_0 (respectively f_1, f_{01}) exhibits the simplicial set S_0 (respectively S_1, S_{01}) as a localization of X_0 (respectively X_1, X_{01}) with respect to W_0 (respectively W_1, W_{01}). Let W be the collection of edges of X given by the union of the images of W_0 and W_1. Note that conditions (a) and (b) guarantee that the front and back faces of the diagram (6.9) are categorical pushout squares (Proposition 4.5.4.11). Applying Proposition 6.3.4.2, we conclude that f exhibits S as a localization of X with respect to W. □

6.3.7 Subdivision and Localization

Our goal in this section is to prove the following: 02MC

Theorem 6.3.7.1. *Let S be a simplicial set. Then there exists a partially ordered set $(A, \leq)$ and a universally localizing morphism $\mathrm{N}_\bullet(A) \to S$.* 02MD

We begin by proving a weaker version of Theorem 6.3.7.1, which asserts that every simplicial set S admits a universally localizing morphism $\mathrm{N}_\bullet(\mathcal{C}) \to S$, for some category $\mathcal{C}$. Here it is possible to be completely explicit: we can take $\mathcal{C}$ to be the category of simplices $\mathbf{\Delta}_S$ introduced in Construction 1.1.3.9 (Corollary 6.3.7.5).

Proposition 6.3.7.2. *Let S be a simplicial set, let $\mathrm{Sd}(S)$ denote the subdivision of S (Definition 3.3.3.1), and let $\lambda_S : \mathrm{Sd}(S) \to S$ denote the last vertex map (Construction 3.3.4.3). Then λ_S is universally localizing.* 02ME

Remark 6.3.7.3. Let S be a simplicial set. Combining Proposition 6.3.7.2 with Remark 6.3.6.5, we recover the assertion that the last vertex map $\lambda_S : \mathrm{Sd}(S) \to S$ is a weak homotopy equivalence. In other words, we can regard Proposition 6.3.7.2 as a refinement of Proposition 3.3.4.8. 02MF

Proof of Proposition 6.3.7.2. By virtue of Proposition 6.3.6.12, we may assume without loss of generality that the simplicial set S is finite. If S is empty, there is nothing to prove. We may therefore assume that S has dimension n for some integer $n \geq 0$. We proceed by induction on n and on the number of nondegenerate n-simplices of S. Fix a nondegenerate n-simplex $\sigma : \Delta^n \to S$. Using Proposition 1.1.4.12, we see that there is a pushout square of simplicial sets

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d, "\sigma"] \\
S' \arrow[r] & S,
\end{tikzcd}$$

where S' is a simplicial set of dimension $\leq n$ having fewer nondegenerate n-simplices than

S. Applying Proposition 6.3.6.13 to the commutative diagram

$$\begin{tikzcd}
\mathrm{Sd}(\partial \Delta^n) \arrow[rr] \arrow[dd] \arrow[dr, "\lambda_{\partial \Delta^n}"] & & \mathrm{Sd}(\Delta^n) \arrow[dd] \arrow[dr, "\lambda_{\Delta^n}"] & \\
& \partial \Delta^n \arrow[rr] \arrow[dd] & & \Delta^n \arrow[dd] \\
\mathrm{Sd}(S') \arrow[rr] \arrow[dr, "\lambda_{S'}"] & & \mathrm{Sd}(S) \arrow[dr, "\lambda_S"] & \\
& S' \arrow[rr] & & S,
\end{tikzcd}$$

we are reduced to showing that the morphisms $\lambda_{S'}$, $\lambda_{\partial \Delta^n}$, and λ_{Δ^n} are universally localizing. In the first two cases, this follows from our inductive hypothesis. We are therefore reduced to proving Proposition 6.3.7.2 in the special case where $S = \Delta^n$ is a standard simplex. Using Example 3.3.3.5, we can identify the subdivision $\mathrm{Sd}(S) = \mathrm{Sd}(\Delta^n)$ with the nerve of the partially ordered set $\mathrm{Chain}[n]$ of nonempty subsets $P \subseteq [n]$. Under this identification, λ_S is obtained from the map of partially ordered sets

$$\mathrm{Chain}[n] \to [n] \qquad (S \subseteq [n]) \mapsto \max(S).$$

We now observe that this map admits section $\mu : [n] \to \mathrm{Chain}[n]$, given by the construction $i \mapsto \{0 < 1 < \cdots < i\}$, and there is a (unique) natural transformation $\mathrm{id}_{\mathrm{Sd}(S)} \to \mu \circ \lambda_S$ which belongs to $\mathrm{Fun}_{/S}(\mathrm{Sd}(S), \mathrm{Sd}(S))$. The desired result now follows from the criterion of Proposition 6.3.6.8. □

04JU **Variant 6.3.7.4.** Let S be a simplicial set and let $\psi_S : \mathrm{N}_{\bullet}(\mathbf{\Delta}_S) \to \mathrm{Sd}(S)$ be the comparison map of Construction 3.3.3.9. Then ψ_S is universally localizing.

Proof. Note that the functor $S \mapsto \mathrm{N}_{\bullet}(\mathbf{\Delta}_S)$ preserves small colimits (Variant 3.3.3.19). Proceeding as in the proof of Proposition 6.3.7.2, we can reduce to the case where $S = \Delta^n$ is a standard simplex. In this case, we can identify ψ_S with (the nerve of) the functor

$$\mathbf{\Delta}_S \to \mathrm{Chain}[n] \qquad (\alpha : [m] \to [n] \mapsto \mathrm{im}(\alpha) \subseteq [n])$$

This functor admits a section ϕ, which identifies $\mathrm{Chain}[n]$ with the subcategory $\mathbf{\Delta}_S^{\mathrm{nd}} \subseteq \mathbf{\Delta}_S$ of *nondegenerate* simplices of S. Note that there is a (unique) natural transformation from the identity functor $\mathrm{id}_{\mathbf{\Delta}_S}$ to $\phi \circ \psi_S$ which belongs to $\mathrm{Fun}_{/\,\mathrm{Sd}(S)}(\mathrm{N}_{\bullet}(\mathbf{\Delta}_S), \mathrm{N}_{\bullet}(\mathbf{\Delta}_S))$, so the desired result follows from the criterion of Proposition 6.3.6.8. □

Corollary 6.3.7.5. *Let S be a simplicial set. Then the composite morphism* 04JV

$$\mathrm{N}_{\bullet}(\mathbf{\Delta}_S) \xrightarrow{\psi_S} \mathrm{Sd}(S) \xrightarrow{\lambda_S} S$$

is universally localizing.

Proof. By virtue of Proposition 6.3.6.10, this follows from the observation that the morphisms λ_S and ψ_S are universally localizing (Proposition 6.3.7.2 and Variant 6.3.7.4). □

We now study some additional assumptions on the simplicial set S which will allow us to replace the category $\mathbf{\Delta}_S$ of Corollary 6.3.7.5 by a partially ordered set.

Definition 6.3.7.6. Let S be a simplicial set. We say that S is *nonsingular* if, for every nondegenerate n-simplex σ of S, the corresponding map $\sigma : \Delta^n \to S$ is a monomorphism of simplicial sets. 02MG

Remark 6.3.7.7. Recall that a simplicial set S is *braced* if the collection of nondegenerate simplices of S is closed under the face operators (Definition 3.3.1.1). Every nonsingular simplicial set is braced. However, the converse is false. For example, the quotient $\Delta^1 / \partial \Delta^1$ is braced, but is not nonsingular. 02MH

Example 6.3.7.8. Let $(A, \leq)$ be a partially ordered set. Then the nerve $\mathrm{N}_{\bullet}(A)$ is a nonsingular simplicial set. In particular, for every integer $n \geq 0$, the standard simplex Δ^n is nonsingular. 02MJ

Remark 6.3.7.9. Let S be a nonsingular simplicial set. Then every simplicial subset $S' \subseteq S$ is also nonsingular. 02MK

Remark 6.3.7.10. Let S be a simplicial set which can be written as a union of a collection of simplicial subsets $\{S_\alpha \subseteq S\}$. If each S_α is nonsingular, then S is nonsingular. 02ML

Remark 6.3.7.11. Let S and T be nonsingular simplicial sets. Then the join $S \star T$ is nonsingular. In particular, if S is nonsingular, then the cone $S^{\triangleright}$ is also nonsingular. 02MM

Remark 6.3.7.12. Let S be a simplicial set and A denote the collection of simplicial subsets $S' \subseteq S$ which are isomorphic to a standard simplex. We regard $\mathrm{Sub}_{\mathbf{\Delta}}(S)$ as a partially ordered set with respect to inclusion. If S is nonsingular, the construction 02MN

$$(\sigma : \Delta^n \to S) \mapsto (\mathrm{im}(\sigma) \subseteq S)$$

determines an isomorphism of categories $\mathbf{\Delta}_S^{\mathrm{nd}} \simeq A$, where $\mathbf{\Delta}_S^{\mathrm{nd}}$ denotes category of nondegenerate simplices of S (Notation 3.3.3.11). Combining this observation with Proposition 3.3.3.16, we obtain an isomorphism of simplicial sets $\mathrm{N}_{\bullet}(A) \to \mathrm{Sd}(S)$.

02MP **Corollary 6.3.7.13.** *Let S be a nonsingular simplicial set. Then there exists a partially ordered set A and a universally localizing morphism $\mathrm{N}_{\bullet}(A) \to S$.*

Proof. Combine Proposition 6.3.7.2 with Remark 6.3.7.12. □

For our purposes, Corollary 6.3.7.13 is a poor replacement for Theorem 6.3.7.1: an ∞-category $\mathcal{C}$ is rarely nonsingular when regarded as a simplicial set (see Exercise 3.3.1.2). We will deduce the general form of Theorem 6.3.7.1 by combining Corollary 6.3.7.13 with the following result:

02MQ **Proposition 6.3.7.14.** *Let S be a simplicial set. Then there exists a universally localizing morphism $\varphi : \widetilde{S} \to S$, where $\widetilde{S}$ is nonsingular.*

The proof of Proposition 6.3.7.14 will make use of the following:

02MR **Lemma 6.3.7.15.** *Let $\{S_\alpha\}$ be a diagram of nonsingular simplicial sets. Then the limit $\varprojlim_{\alpha} S_\alpha$ is also nonsingular.*

Proof. By virtue of Remark 6.3.7.9, it will suffice to show that the product $S = \prod_{\alpha} S_\alpha$ is nonsingular. Let $\sigma : \Delta^n \to S$ be a nondegenerate simplex of S; we wish to show that σ is a monomorphism of simplicial sets. For each index α, Proposition 1.1.3.8 guarantees that there exists a commutative diagram

$$\begin{tikzcd}
\Delta^n \arrow[r, "\sigma"] \arrow[d, "\tau_\alpha"] & S \arrow[d] \\
\Delta^{n_\alpha} \arrow[r, "\sigma_\alpha"] & S_\alpha,
\end{tikzcd}$$

where σ_α is a nondegenerate simplex S_α. Our assumption that S_α is nondegenerate guarantees that σ_α is a monomorphism of simplicial sets, so that the product map

$$\prod_{\alpha} \Delta^{n_\alpha} \xrightarrow{\prod_\alpha \sigma_\alpha} \prod_{\alpha} S_\alpha = S$$

is also a monomorphism. It will therefore suffice to show that $\tau = \{\tau_\alpha\}$ determines a monomorphism of simplicial sets $\Delta^n \to \prod_\alpha \Delta^{n_\alpha}$. Since $\prod_\alpha \Delta^{n_\alpha}$ can be identified with the nerve of the partially ordered set $\prod_\alpha [n_\alpha]$, it is a nonsingular simplicial set (Example 6.3.7.8). It will therefore suffice to show that τ is nondegenerate, which follows immediately from our assumption that σ is nondegenerate. □

Proof of Proposition 6.3.7.14. Let S be a simplicial set. For each integer $k \geq 0$, let $\mathrm{sk}_k(S)$ denote the k-skeleton of S (Construction 1.1.4.1). We will construct a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{sk}}_0(S) \arrow[r, hook] \arrow[d, "\varphi_0"] & \widetilde{\mathrm{sk}}_1(S) \arrow[r, hook] \arrow[d, "\varphi_1"] & \widetilde{\mathrm{sk}}_2(S) \arrow[r, hook] \arrow[d, "\varphi_2"] & \cdots \\
\mathrm{sk}_0(S) \arrow[r, hook] & \mathrm{sk}_1(S) \arrow[r, hook] & \mathrm{sk}_2(S) \arrow[r, hook] & \cdots
\end{tikzcd}$$

where each of the horizontal maps is a monomorphism, each of the vertical maps is universally localizing, and each of the simplicial sets $\widetilde{\mathrm{sk}}_k(S)$ is nonsingular. It then follows from Remark 6.3.7.10 that the colimit $\widetilde{S} = \varinjlim_k \widetilde{\mathrm{sk}}_k(S)$ is nonsingular. Applying Proposition 6.3.6.12, we conclude that the morphisms φ_k determine a universally localizing morphism $\varphi : \widetilde{S} \to S$.

The construction of the morphisms $\varphi_k : \widetilde{\mathrm{sk}}_k(S) \to \mathrm{sk}_k(S)$ proceeds by induction. If $k = 0$, we can take $\widetilde{\mathrm{sk}}_k(S) = \mathrm{sk}_k(S)$ and φ_k to be the identity morphism. Let us therefore assume that $k > 0$, and that the morphism $\varphi_{k-1} : \widetilde{\mathrm{sk}}_{k-1}(S) \to \mathrm{sk}_{k-1}(S)$ has already been constructed. Let S_k^{nd} denote the set of nondegenerate k-simplices of S, let T denote the coproduct $\coprod_{\sigma \in S_k^{\mathrm{nd}}} \Delta^k$, and let $T_0 \subseteq T$ denote the coproduct $\coprod_{\sigma \in S_k^{\mathrm{nd}}} \partial \Delta^k$, so that Proposition 1.1.4.12 supplies a pushout diagram

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d] & T \arrow[d] \\
\mathrm{sk}_{k-1}(S) \arrow[r] & \mathrm{sk}_k(S).
\end{tikzcd}$$

Note that T is nonsingular (Example 6.3.7.8), so the simplicial subset $T_0 \subseteq T$ is also nonsingular (Remark 6.3.7.9). Let $\widetilde{T}_0$ denote the fiber product $T_0 \times_{\mathrm{sk}_{k-1}(S)} \widetilde{\mathrm{sk}}_{k-1}(S)$, and we define $\widetilde{\mathrm{sk}}_k(S)$ to be the pushout of the diagram

$$(\widetilde{\mathrm{sk}}_{k-1}(S) \times \widetilde{T}_0^{\triangleright}) \hookleftarrow \widetilde{T}_0 \hookrightarrow (T \times \widetilde{T}_0^{\triangleright}).$$

Note that the cone point of $\widetilde{T}_0^{\triangleright}$ determines an embedding $\widetilde{\mathrm{sk}}_{k-1}(S) \to \widetilde{\mathrm{sk}}_k(S)$. Moreover, we have a commutative diagram

02MS

$$\begin{tikzcd}
\widetilde{\mathrm{sk}}_{k-1}(S) \times \widetilde{T}_0^{\triangleright} \arrow[d] & \widetilde{T}_0 \arrow[l] \arrow[r] \arrow[d] & T \times \widetilde{T}_0^{\triangleright} \arrow[d] \\
\mathrm{sk}_{k-1}(S) & T_0 \arrow[l] \arrow[r] & T.
\end{tikzcd} \tag{6.10}$$

which determines an extension of φ_{k-1} to a map

$$\varphi_k : \widetilde{\mathrm{sk}}_k(S) \to \mathrm{sk}_{k-1}(S) \coprod_{T_0} T \simeq \mathrm{sk}_k(S).$$

Since the cone $\widetilde{T}_0^{\triangleright}$ is weakly contractible, it follows from Corollary 6.3.6.11 that the vertical maps in the diagram (6.10) are universally localizing. Applying Proposition 6.3.6.13, we deduce that φ_k is also universally localizing.

To complete the proof, it will suffice to show that the simplicial set $\widetilde{\mathrm{sk}}_k(S)$ is nonsingular. By virtue of Remark 6.3.7.10, it will suffice to show that the simplicial subsets $\widetilde{\mathrm{sk}}_{k-1}(S) \times \widetilde{T}_0^{\triangleright}$ and $T \times \widetilde{T}_0^{\triangleright}$ are nonsingular. Since $\widetilde{\mathrm{sk}}_{k-1}(S)$ is nonsingular (by our inductive hypothesis) and T is nonsingular (Example 6.3.7.8), we are reduced to proving that the cone $\widetilde{T}_0^{\triangleright}$ is nonsingular (Lemma 6.3.7.15). By virtue of Remark 6.3.7.11, we can reduce further to showing that $\widetilde{T}_0$ is nonsingular. This follows from Remark 6.3.7.9 and Lemma 6.3.7.15, since $\widetilde{T}_0$ can be identified with a simplicial subset of the product $T \times \widetilde{\mathrm{sk}}_{k-1}(S)$. □

02MT **Remark 6.3.7.16.** Let S be a finite simplicial set. In this case, each of the simplicial sets $\widetilde{\mathrm{sk}}_k(S)$ constructed in the proof of Proposition 6.3.7.14 will also be finite. Specializing to the case $k \geq \dim(S)$, we obtain a universally localizing morphism

$$\widetilde{\mathrm{sk}}_k(S) \to \mathrm{sk}_k(S) = S$$

where the simplicial set $\widetilde{\mathrm{sk}}_k(S)$ is both finite and nonsingular.

Proof of Theorem 6.3.7.1. Let S be a simplicial set. Applying Proposition 6.3.7.14, we can choose a universally localizing morphism $\varphi : \widetilde{S} \to S$, where $\widetilde{S}$ is a nonsingular simplicial set. Let $A = \mathrm{Sub}_{\mathbf{\Delta}}(\widetilde{S})$ denote the partially ordered set of simplicial subsets of $\widetilde{S}$ which are isomorphic to a standard simplex, so that Corollary 6.3.7.13 supplies a universally localizing morphism $\lambda_{\widetilde{S}} : \mathrm{N}_{\bullet}(A) \to \widetilde{S}$. Applying Proposition 6.3.6.10, we deduce that the composite morphism

$$\mathrm{N}_{\bullet}(A) \xrightarrow{\lambda_{\widetilde{S}}} \widetilde{S} \xrightarrow{\varphi} S$$

is also universally localizing. □

Combining the preceding argument with Remark 6.3.7.16, we also obtain the following:

02MU **Variant 6.3.7.17.** Let S be a finite simplicial set. Then there exists a finite partially ordered set $(A, \leq)$ and a universally localizing morphism $\mathrm{N}_{\bullet}(A) \to S$.

02MV **Exercise 6.3.7.18.** Let S be a simplicial set and let $\widetilde{S}$ be the smallest simplicial subset of $S \times \mathrm{N}_{\bullet}(\mathbf{Z}_{\geq 0})$ which contains all simplices of the form (σ, τ), where τ is a nondegenerate simplicial subset of $\mathrm{N}_{\bullet}(\mathbf{Z}_{\geq 0})$ (that is, it corresponds to a *strictly increasing* sequence of nonnegative integers). Show that $\widetilde{S}$ is nonsingular, and that projection onto the first factor determines a universally localizing morphism $\widetilde{S} \twoheadrightarrow S$.

Chapter 7

Limits and Colimits

02H0 In this chapter, we extend the classical theory of limits and colimits to the setting of higher category theory. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that an object $Y \in \mathcal{D}$ is a *limit of* F if there exists a natural transformation $\alpha : \underline{Y} \to F$ having the following universal property: for every object $X \in \mathcal{C}$, composition with α induces a homotopy equivalence of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\underline{X}, F);$$

here $\underline{X}, \underline{Y} \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the constant functors taking the values X and Y, respectively. In this case, the object Y is uniquely determined up to isomorphism; to emphasize this, we often denote Y by $\varprojlim(F)$, or by $\varprojlim_{C \in \mathcal{C}}(F(C))$. In §7.1, we summarize the formal properties of this notion (as well as the dual notion of *colimit*, which plays an equally essential role in the theory).

Throughout this book, we will often be faced with the problem of *computing* (or *describing*) the limit of a diagram $F : \mathcal{C} \to \mathcal{D}$. In such situations, it is useful to have some flexibility to modify the ∞-category $\mathcal{C}$. In §7.2, we introduce the notion of a *left cofinal* morphism of simplicial sets $e : \mathcal{C}' \to \mathcal{C}$ (Definition 7.2.1.1). If $e : \mathcal{C}' \to \mathcal{C}$ is left cofinal, then an object of $\mathcal{D}$ is a limit of F if and only if it is a limit of the composite map $F' = F \circ e$ (see Corollary 7.2.2.11, and Corollary 7.4.5.14 for a converse). When $\mathcal{C}$ is an ∞-category, cofinality admits a simple characterization: a morphism $e : \mathcal{C}' \to \mathcal{C}$ is left cofinal if and only if, for each object $C \in \mathcal{C}$, the simplicial set $\mathcal{C}' \times_{\mathcal{C}} \mathcal{C}_{/C}$ is weakly contractible (Theorem 7.2.3.1). We will encounter many situations where this criterion is easy to verify. In such cases, it is harmless to replace $\mathcal{C}$ by $\mathcal{C}'$ for the purpose of calculating the limit of a diagram $F : \mathcal{C} \to \mathcal{D}$.

In §7.3, we consider another important technique for computing limits. Suppose we are given a cartesian fibration of ∞-categories $U : \mathcal{E} \to \mathcal{C}$. Under some mild assumptions, one can show that the limit of a diagram $F : \mathcal{E} \to \mathcal{D}$ obeys a transitivity formula, which we can write informally as

$$\varprojlim_{X \in \mathcal{E}} (F(X)) \simeq \varprojlim_{C \in \mathcal{C}} (\varprojlim_{X \in \mathcal{E}_C} F(X)).$$

More precisely, suppose that for every object $C \in \mathcal{C}$, the diagram $F_C = F|_{\mathcal{E}_C}$ admits a limit in the ∞-category $\mathcal{D}$. Then one can construct a new functor $G : \mathcal{C} \to \mathcal{D}$, given on objects by the formula $G(C) = \varprojlim(F_C)$; we refer to G as a *right Kan extension of F along U* (see Definition 7.3.1.2 and Proposition 7.3.4.4). Moreover, an object of the ∞-category $\mathcal{D}$ is a limit of the functor F if and only if it is a limit of the functor G (Corollary 7.3.8.20).

The remainder of this chapter is devoted to studying limits and colimits in special situations. Let $\mathcal{S}$ denote the ∞-category of spaces (Construction 5.5.1.1). For any ∞-category $\mathcal{C}$, Corollary 5.6.0.6 supplies a bijection from the set of isomorphism classes of functors $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ and the set of equivalence classes of essentially small left fibrations $U : \mathcal{E} \to \mathcal{C}$. In §7.4, we use this identification to give an explicit description of limits and colimits in $\mathcal{S}$:

(1) The Kan complex $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ parametrizing sections of U is a limit of the diagram $\mathscr{F}$.

(2) A Kan complex X is a colimit of $\mathscr{F}$ if and only if there exists a weak homotopy equivalence $\mathcal{E} \to X$ (Corollary 7.4.5.4).

These assertions are special cases of more general results which apply to diagrams taking values in the ∞-category $\mathcal{QC} \supset \mathcal{S}$; see Theorems 7.4.1.1 and7.4.3.6.

Recall that the ∞-category $\mathcal{S}$ is defined as the homotopy coherent nerve of the ordinary category of Kan complexes Kan. In particular, if $\mathscr{F}_0 : \mathcal{C}_0 \to \mathrm{Kan}$ is a functor between ordinary categories, then passing to the homotopy coherent nerve gives a functor of ∞-categories $\mathscr{F} : \mathcal{C} \to \mathcal{S}$, where $\mathcal{C} = \mathrm{N}_{\bullet}(\mathcal{C}_0)$. In this case, there is a natural candidate for the corresponding left fibration $U : \mathcal{E} \to \mathcal{C}$, obtained by taking $\mathcal{E}$ to be the weighted nerve $\mathrm{N}_{\bullet}^{\mathscr{F}_0}(\mathcal{C}_0)$ of Definition 5.3.3.1. In §7.5, we combine this observation with assertions (1) and (2) to compare limits and colimits in the ∞-category $\mathcal{S}$ with the classical theory of homotopy limits and colimits introduced by Bousfield and Kan in [6].

In §7.6, we provide a detailed discussion of some special classes of limits which arise frequently in practice, such as products (Definition 7.6.1.3), powers (Definition 7.6.2.1), pullbacks (Definition 7.6.3.1), equalizers (Definition 7.6.5.4), and sequential limits (Definition 7.6.6.1). From these primitives, many other examples can be constructed: for example, arbitrary limits in an ∞-category $\mathcal{D}$ can be built by combining products and equalizers (see Corollary 7.6.5.25 and Proposition 7.6.7.9).

7.1 Limits and Colimits

02H1

Let $\mathcal{K}$ and $\mathcal{C}$ be categories. For every object $X \in \mathcal{C}$, let $\underline{X}$ denote the constant functor from $\mathcal{K}$ to $\mathcal{C}$, carrying each object of $\mathcal{K}$ to X and each morphism of $\mathcal{K}$ to the identity morphism id_X. If $U : \mathcal{K} \to \mathcal{C}$ is an arbitrary functor, then a *limit of* U is an object of $\mathcal{C}$ which represents the functor $X \mapsto \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})}(\underline{X}, U)$. This can be formulated more precisely as follows:

Definition 7.1.0.1. Let $F : \mathcal{K} \to \mathcal{C}$ be a functor between categories. Let Y be an object of $\mathcal{C}$ and let $\alpha : \underline{Y} \to F$ be a natural transformation of functors. We say that the natural transformation α *exhibits* Y *as a limit of* F if the following condition is satisfied: 02H2

$(*)$ For every object $X \in \mathcal{C}$, composition with α induces a bijection from $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to the set $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})}(\underline{X}, F)$ of natural transformations from $\underline{X}$ to F.

Our goal in this section is to introduce an ∞-categorical counterpart of Definition 7.1.0.1. Let $\mathcal{C}$ be an ∞-category, let $F : \mathcal{K} \to \mathcal{C}$ be a diagram, let $\alpha : \underline{Y} \to F$ be a natural transformation. For every object $X \in \mathcal{C}$, composition with α induces a map of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{X}, F),$$

which is well-defined up to homotopy. We will say that α *exhibits* Y *as a limit of* F if this map is a homotopy equivalence for each $X \in \mathcal{C}$ (Definition 7.1.1.1). In §7.1.1, we provide a detailed analysis of this notion and its formal properties (as well as the dual notion of *colimit*, which is defined in a similar way).

In §4.6.7, we introduced the notion of a final object of an ∞-category $\mathcal{C}$ (Definition 4.6.7.1). This can be regarded as a special case of the general theory of limits: an object $Y \in \mathcal{C}$ is final if and only if it is a limit of the empty diagram (Example 7.1.1.6). Conversely, if K is an arbitrary simplicial set equipped with a diagram $F : K \to \mathcal{C}$, we will see that a natural transformation $\alpha : \underline{Y} \to F$ exhibits Y as a limit of F if and only if it is final when viewed as an object of the ∞-category $\mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}$ (Proposition 7.1.2.1). Recall that $\mathcal{C} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \{F\}$ is equivalent (but not isomorphic) to the slice ∞-category $\mathcal{C}_{/F}$ (Theorem 4.6.4.17). In §7.1.2, we use this observation to reformulate the notion of limit: an object Y is a limit of a diagram $F : K \to \mathcal{C}$ if there exists a diagram $\overline{F} : K^{\triangleleft} \to \mathcal{C}$ which carries the cone point of $K^{\triangleleft}$ to the object Y and which is final when viewed as an object of the slice ∞-category $\mathcal{C}_{/F}$ (Corollary 7.1.2.2). In this situation, we will refer to $\overline{F}$ as a *limit diagram* in the ∞-category $\mathcal{C}$ (Definintion 7.1.2.4).

In §7.1.3, we study the dependence of K-indexed limits on the ambient ∞-category in which they are formed. We say that a functor of ∞-categories $G : \mathcal{C} \to \mathcal{D}$ *preserves* K*-indexed limits* if, for every diagram $F : K \to \mathcal{C}$, the induced functor $\mathcal{C}_{/F} \to \mathcal{D}_{/(G \circ F)}$ carries final objects of $\mathcal{C}_{/F}$ to final objects of $\mathcal{D}_{/(G \circ F)}$ (Definition 7.1.3.4). We illustrate the concept in this section with a few elementary examples (and will encounter many others later in this book):

- If $G : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then it preserves K-indexed limits for every simplicial set K (Proposition 7.1.3.9).
- Let $\mathcal{C}$ be an ∞-category which admits K-indexed limits, and let $f : A \to \mathcal{C}$ be any morphism of simplicial sets. Then the coslice ∞-category $\mathcal{C}_{f/}$ also admits K-indexed limits, and the projection map $\mathcal{C}_{f/} \to \mathcal{C}$ preserves K-indexed limits (Corollary 7.1.3.20).
- Let $G : \mathcal{C} \to \mathcal{D}$ be a right fibration of ∞-categories, and suppose that $\mathcal{D}$ admits K-indexed limits. If K is weakly contractible, then the ∞-category $\mathcal{C}$ also admits K-indexed limits, and the right fibration F preserves K-indexed limits (Corollary 7.1.5.18).

For many applications, it will be useful to consider a relative version of the theory of limit diagrams. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that an object $Y \in \mathcal{C}$ is *U-final* if, for every object $X \in \mathcal{C}$, the functor U induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(U(X), U(Y))$ (Definition 7.1.4.1). We say that a diagram $\overline{F} : K^{\triangleleft} \to \mathcal{C}$ with restriction $F = \overline{F}|_K$ is a *U-limit diagram* if it is $U_{/F}$-final when regarded as an object of the ∞-category $\mathcal{C}_{/F}$, where $U_{/F} : \mathcal{C}_{/F} \to \mathcal{D}_{/(U \circ F)}$ is the projection map (Definition 7.1.5.1). In the special case $\mathcal{D} = \Delta^0$, we recover the usual notions of final object and limit diagram, respectively (Examples 7.1.4.2 and 7.1.5.3). Moreover, most of the basic features of final objects and limit diagrams have counterparts in the relative setting, which we summarize in §7.1.4 and §7.1.5. Even if one is ultimately interested in the "absolute" theory, the language of relative limits is a useful tool: we illustrate this point in §7.1.6 by using the relative language to study limits in an ∞-category of the form $\mathrm{Fun}(B, \mathcal{C})$ (our main result is that, under mild assumptions, such limits can be computed pointwise: see Proposition 7.1.6.1)

02VX **Remark 7.1.0.2.** The preceding discussion has centered around the theory of limits. There is also a dual theory of *colimits* in the ∞-categorical setting, which can be obtained by passing to opposite ∞-categories. Every assertion concerning limits has a counterpart for colimits (and vice versa). We will often use this implicitly (for example, by stating a result only for colimits but later using the dual assertion for limits).

7.1.1 Limits and Colimits in ∞-Categories

02VY Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. For each object $X \in \mathcal{C}$, we let $\underline{X} \in \mathrm{Fun}(K, \mathcal{C})$ denote the constant diagram $K \to \{X\} \hookrightarrow \mathcal{C}$. Note that the construction $X \mapsto \underline{X}$ determines a functor of ∞-categories $\mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$, carrying each morphism $f : X \to Y$ to a natural transformation $\underline{f} : \underline{X} \to \underline{Y}$.

02VZ **Definition 7.1.1.1.** Let $\mathcal{C}$ be an ∞-category containing an object Y, let K be a simplicial set, and let $u : K \to \mathcal{C}$ be a diagram. We say that a natural transformation $\alpha : \underline{Y} \to u$ *exhibits Y as a limit of u* if the following condition is satisfied:

$(*)$ For each object $X \in \mathcal{C}$, the composition

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{X}, \underline{Y}) \xrightarrow{[\alpha]\circ} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{X}, u)$$

is an isomorphism in the homotopy category hKan, where the second map is described in Notation 4.6.9.15.

We will say that a natural transformation $\beta : u \to \underline{Y}$ *exhibits* Y *as a colimit of* u if the following dual condition is satisfied:

$(*')$ For each object $Z \in \mathcal{C}$, the composition

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{Y}, \underline{Z}) \xrightarrow{\circ[\beta]} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(u, \underline{Z})$$

is an isomorphism in the homotopy category hKan.

Remark 7.1.1.2. Stated more informally, a natural transformation $\alpha : \underline{Y} \to u$ exhibits Y as a limit of u if and only if postcomposition with α induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{X}, u)$ for each object $X \in \mathcal{C}$. Similarly, a natural transformation $\beta : u \to \underline{Y}$ exhibits Y as a colimit of u if and only if precomposition with β induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(u, \underline{Z})$ for each object $Z \in \mathcal{C}$. 02W0

Remark 7.1.1.3. Let $\mathcal{C}$ be an ∞-category containing an object Y and let $u : K \to \mathcal{C}$ be a diagram. Then a natural transformation $\alpha : \underline{Y} \to u$ exhibits Y as a limit of u if and only if it exhibits Y as a colimit of the induced diagram $u^{\mathrm{op}} : K^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$, when regarded as a morphism in the ∞-category $\mathrm{Fun}(K^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Fun}(K, \mathcal{C})^{\mathrm{op}}$. 02JC

Example 7.1.1.4. Let $\mathcal{C}$ be an ordinary category, let K be a simplicial set, and suppose we are given a diagram $u : K \to \mathrm{N}_{\bullet}(\mathcal{C})$, which we can identify with a functor of ordinary categories $U : \mathrm{h}K \to \mathcal{C}$ (see Proposition 1.4.5.7). If Y is an object of $\mathcal{C}$, then we can use Corollary 1.5.3.5 to identify natural transformations $\underline{Y} \to u$ (of diagrams in the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$) with natural transformations $\underline{Y} \to U$ (of diagrams in the ordinary category $\mathcal{C}$). Under this identification, a natural transformation $\underline{Y} \to u$ exhibits Y as a limit of u (in the ∞-categorical sense of Definition 7.1.1.1) if and only if it exhibits Y as a limit of U (in the classical sense of Definition 7.1.0.1). 02JD

Example 7.1.1.5. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. The following conditions are equivalent: 02JR

- The morphism f is an isomorphism from X to Y in the ∞-category $\mathcal{C}$ (Definition 1.4.6.1).
- The morphism f exhibits X as a limit of the diagram $\{Y\} \hookrightarrow \mathcal{C}$.

- The morphism f exhibits Y as a colimit of the diagram $\{X\} \hookrightarrow \mathcal{C}$.

038Z **Example 7.1.1.6.** Let $\mathcal{C}$ be an ∞-category. Then an object $Y \in \mathcal{C}$ is initial (in the sense of Definition 4.6.7.1) if and only if it is a colimit of the empty diagram $\emptyset \hookrightarrow \mathcal{C}$. Similarly, Y is final if and only if it is a limit of the empty diagram.

02W1 **Remark 7.1.1.7.** Let $\mathcal{C}$ be an ∞-category, let $u : K \to \mathcal{C}$ be a diagram, and let $Y \in \mathcal{C}$ be an object. If $\alpha : \underline{Y} \to u$ is a natural transformation, then the condition that α exhibits Y as a limit of u depends only on its homotopy class $[\alpha]$ (as a morphism in the ∞-category $\operatorname{Fun}(K, \mathcal{C})$). Similarly, if $\beta : u \to \underline{Y}$ is a natural transformation, then the condition that β exhibits Y as a colimit of u depends only on its homotopy class $[\beta]$.

02W2 **Remark 7.1.1.8.** Let $\mathcal{C}$ be an ∞-category containing an object Y, let K be a simplicial set, and let $\beta : u \to u'$ be an isomorphism in the ∞-category $\operatorname{Fun}(K, \mathcal{C})$. Suppose we are given a natural transformation $\alpha : \underline{Y} \to u$, and let $\alpha' : \underline{Y} \to u'$ be any composition of α with β. Then α exhibits Y as a limit of u if and only if α' exhibits Y as a limit of u'. Similarly, if $\gamma' : u' \to \underline{Y}$ is a natural transformation and $\gamma : u \to \underline{Y}$ is a composition of β with γ', then γ exhibits Y as a colimit of u if and only if γ' exhibits Y as a colimit of u'.

02W3 **Remark 7.1.1.9.** Let $\mathcal{C}$ be an ∞-category, let $u : K \to \mathcal{C}$ be a diagram, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Suppose we are given a natural transformation of diagrams $\beta : \underline{Y} \to u$, and let $\alpha : \underline{X} \to u$ be a composition of β with the constant natural transformation $\underline{f} : \underline{X} \to \underline{Y}$. Then any two of the following three properties imply the third:

- The natural transformation α exhibits X as a limit of the diagram u.
- The natural transformation β exhibits Y as a limit of the diagram u.
- The morphism $f : X \to Y$ is an isomorphism in the ∞-category $\mathcal{C}$.

02W4 **Remark 7.1.1.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories, let $u : K \to \mathcal{C}$ be a diagram, and let $Y \in \mathcal{C}$ be an object equipped with a natural transformation $\alpha : \underline{Y} \to u$. If $F(\alpha) : \underline{F(Y)} \to (F \circ u)$ exhibits $F(Y)$ as a limit of the diagram $(F \circ u) : K \to \mathcal{D}$, then α exhibits Y as a limit of u. The converse holds if F is an equivalence of ∞-categories.

02JB **Definition 7.1.1.11.** Let $\mathcal{C}$ be an ∞-category and let $u : K \to \mathcal{C}$ be a diagram. We say that an object $Y \in \mathcal{C}$ *is a limit of* u if there exists a natural transformation $\alpha : \underline{Y} \to u$ which exhibits Y as a limit of u, in the sense of Definition 7.1.1.1. We say that Y *is a colimit of* u if there exists a natural transformation $\beta : u \to \underline{Y}$ which exhibits Y as a colimit of u.

02JH **Proposition 7.1.1.12.** *Let $\mathcal{C}$ be an ∞-category and let $u : K \to \mathcal{C}$ be a diagram. Then:*

- *Suppose that the diagram u has limit $Y \in \mathcal{C}$. Then an object $X \in \mathcal{C}$ is a limit of u if and only if it is isomorphic to Y.*

- *Suppose that the diagram u has colimit $Y \in \mathcal{C}$. Then an object $X \in \mathcal{C}$ is a colimit of u if and only if it is isomorphic to Y.*

Proof. Let $\beta : \underline{Y} \to u$ be a natural transformation which exhibits Y as a limit of the diagram u. For any object X and any natural transformation $\alpha : \underline{X} \to u$, there exists a morphism $f : X \to Y$ such that α is a composition of β with the constant natural transformation $\underline{f} : \underline{X} \to \underline{Y}$. If α also exhibits X as a limit of the diagram u, then f is an isomorphism (Remark 7.1.1.9); in particular, X is isomorphic to Y. Conversely, if $f : X \to Y$ is an isomorphism, then any composition of $\underline{f}$ with β is a natural transformation $\underline{X} \to u$ which exhibits X as a limit of u (Remark 7.1.1.9). This proves the first assertion; the proof of the second follows by applying the same argument to the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. □

Notation 7.1.1.13. Let $\mathcal{C}$ be an ∞-category and let $u : K \to \mathcal{C}$ be a diagram. It follows from Proposition 7.1.1.12 that, if the diagram u admits a limit Y, then the isomorphism class of the object Y depends only on the diagram u. To emphasize this dependence, we will often denote Y by $\varprojlim(u)$ and refer to it as *the* limit of the diagram u. Similarly, if u admits a colimit $X \in \mathcal{C}$, we will often denote X by $\varinjlim(u)$ and refer to it as *the* colimit of the diagram u. Beware that this terminology is somewhat abusive, since the objects $\varprojlim(u)$ and $\varinjlim(u)$ are only well-defined up to isomorphism. 02JJ

In situations where the limit $\varprojlim(u)$ and colimit $\varinjlim(u)$ are defined, they depend functorially on the diagram $u : K \to \mathcal{C}$.

Definition 7.1.1.14. Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. We will say that $\mathcal{C}$ *admits K-indexed limits* if, for every diagram $u : K \to \mathcal{C}$, there exists an object $Y \in \mathcal{C}$ which is a limit of u. We will say that $\mathcal{C}$ *admits K-indexed colimits* if, for every diagram $u : K \to \mathcal{C}$, there exists an object $X \in \mathcal{C}$ which is a colimit of u. 02JK

Remark 7.1.1.15. Let $u : K \to K'$ be a categorical equivalence of simplicial sets. Then an ∞-category $\mathcal{C}$ admits K-indexed colimits if and only if it admits K'-indexed colimits. 03U1

Variant 7.1.1.16. It will often be useful to extend the terminology of Definition 7.1.1.14, replacing the individual simplicial set K by a collection of simplicial sets. For example: 0390

- We say that an ∞-category $\mathcal{C}$ *admits finite limits* if it admits K-indexed limits for every finite simplicial set K (Definition 3.6.1.1), every diagram $f : K \to \mathcal{C}$ admits a limit.

- We say that an ∞-category $\mathcal{C}$ *admits finite colimits* if it admits K-indexed colimits for every finiite simplicial set K.

- We say that an ∞-category $\mathcal{C}$ is *complete* if it admits K-indexed limits for every small simplicial set K.

- We say that an ∞-category $\mathcal{C}$ is *cocomplete* if it admits K-indexed colimits for every small simplicial set K.

Let $\mathcal{C}$ be an ∞-category. For every simplicial set K, precomposition with the projection map $K \to \Delta^0$ determines a functor

$$\delta : \mathcal{C} \simeq \operatorname{Fun}(\Delta^0, \mathcal{C}) \to \operatorname{Fun}(K, \mathcal{C}).$$

We will refer to δ as the *diagonal functor*: it carries each object $X \in \mathcal{C}$ to the constant diagram $\underline{X} : K \to \mathcal{C}$ taking the value X.

`02JL` **Proposition 7.1.1.17.** *Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. Then:*

- *The ∞-category $\mathcal{C}$ admits K-indexed limits if and only if the diagonal functor $\delta : \mathcal{C} \to \operatorname{Fun}(K, \mathcal{C})$ admits a right adjoint G. If this condition is satisfied, then the right adjoint $G : \operatorname{Fun}(K, \mathcal{C}) \to \mathcal{C}$ carries each diagram $u : K \to \mathcal{C}$ to a limit $\varprojlim(u) \in \mathcal{C}$.*

- *The ∞-category $\mathcal{C}$ admits K-indexed colimits if and only if the diagonal functor $\delta : \mathcal{C} \to \operatorname{Fun}(K, \mathcal{C})$ admits a left adjoint F. If this condition is satisfied, then the left adjoint $F : \operatorname{Fun}(K, \mathcal{C}) \to \mathcal{C}$ carries each diagram $u : K \to \mathcal{C}$ to a colimit $\varinjlim(u) \in \mathcal{C}$.*

Proof. Apply Proposition 6.2.4.1. □

7.1.2 Limit and Colimit Diagrams

`02JA` Let $\mathcal{C}$ be an ∞-category, let $u : K \to \mathcal{C}$ be a diagram, and let $\mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\}$ denote the oriented fiber product of Construction 4.6.4.1. By definition, we can identify objects of $\mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\}$ with pairs (Y, α), where Y is an object of $\mathcal{C}$ and $\alpha : \underline{Y} \to u$ is a natural transformation (here $\underline{Y}$ denotes the constant diagram $K \to \{Y\} \hookrightarrow \mathcal{C}$). Using Proposition 5.6.6.21, we can reformulate Definition 7.1.1.1 as follows:

`02W7` **Proposition 7.1.2.1.** *Let $\mathcal{C}$ be an ∞-category containing an object Y and let $u : K \to \mathcal{C}$ be a diagram. Then:*

- *A natural transformation $\alpha : \underline{Y} \to u$ exhibits Y as a limit of the diagram u if and only if it is final when regarded as an object of the oriented fiber product $\mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\}$.*

- *A natural transformation $\beta : u \to \underline{Y}$ exhibits Y as a colimit of the diagram u if and only if it is initial when regarded as an object of the oriented fiber product $\{u\} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \mathcal{C}$.*

Proof. We will prove the first assertion; the second follows by a similar argument. Projection onto the first factor determines a right fibration $\theta : \mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\} \to \mathcal{C}$. For each object $X \in \mathcal{C}$, we can identify $\theta^{-1}(X)$ with the morphism space $\operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(\underline{X}, u)$. Let

$$\rho_X : \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(\underline{Y}, u) \times \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(\underline{X}, u)$$

be the parametrized contravariant transport map of Variant 5.2.8.6. Using Remark 5.2.8.5 and Proposition 5.2.8.7, we see that ρ_X factors as a composition

$$\begin{aligned} \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(\underline{Y}, u) \times \operatorname{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(\underline{Y}, u) \times \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(\underline{X}, \underline{Y}) \\ &\xrightarrow{\circ} \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(\underline{X}, u), \end{aligned}$$

given on objects by the construction $(\alpha, f) \mapsto \alpha \circ \underline{f}$. It follows that a natural transformation $\alpha : \underline{Y} \rightarrow u$ exhibits Y as a limit of u if and only if, for every object $X \in \mathcal{C}$, the restriction $\rho_X|_{\{\alpha\} \times \operatorname{Hom}_{\mathcal{C}}(X,Y)}$ is a homotopy equivalence of Kan complexes. By virtue of Proposition 5.6.6.21, this is equivalent to the requirement that α is final when regarded as an object of the ∞-category $\mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\}$. □

Corollary 7.1.2.2. *Let $\mathcal{C}$ be an ∞-category, let $u : K \rightarrow \mathcal{C}$ be a diagram, and let $Y \in \mathcal{C}$ be an object. The following conditions are equivalent:* 02W8

(1) *The object Y is a limit of the diagram u.*

(2) *The object Y represents the right fibration $\mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\} \rightarrow \mathcal{C}$ given by projection onto the first factor.*

(3) *The object Y represents the right fibration $\mathcal{C}_{/u} \rightarrow \mathcal{C}$ of Proposition 4.3.6.1.*

Proof. The equivalence (1) ⇔ (2) follows immediately from Proposition 7.1.2.1, and the equivalence (2) ⇔ (3) follows from the observation that the slice diagonal $\mathcal{C}_{/u} \hookrightarrow \mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\}$ of Construction 4.6.4.13 is an equivalence of ∞-categories (Theorem 4.6.4.17). □

Corollary 7.1.2.3. *Let $\mathcal{C}$ be an ∞-category and let $u : K \rightarrow \mathcal{C}$ be a diagram. The following conditions are equivalent:* 02W9

(1) *The diagram u has a limit in $\mathcal{C}$.*

(2) *The oriented fiber product $\mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\} \rightarrow \mathcal{C}$ has a final object.*

(3) *The slice ∞-category $\mathcal{C}_{/u}$ has a final object.*

Let $u : K \rightarrow \mathcal{C}$ be a diagram in an ∞-category $\mathcal{C}$. If Y is an object of $\mathcal{C}$, then supplying a natural transformation of diagrams $\alpha : \underline{Y} \rightarrow u$ is equivalent to giving a morphism of simplicial sets $\overline{u} : \Delta^0 \diamond K \rightarrow \mathcal{C}$ satisfying $\overline{u}|_{\Delta^0} = Y$ and $\overline{u}|_K = u$, where

$$\Delta^0 \diamond K = \Delta^0 \coprod_{(\{0\} \times K)} (\Delta^1 \times K)$$

is the simplicial set introduced in Notation 4.5.8.3. In practice, a datum of this type can be somewhat cumbersome to work with. For example, if K is an ∞-category, then $\Delta^0 \diamond K$ need not be an ∞-category. It is therefore often convenient to work with the following variant of Definition 7.1.1.1:

02JM **Definition 7.1.2.4.** Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ be a morphism of simplicial sets carrying the cone point of $K^{\triangleleft}$ to an object $Y \in \mathcal{C}$. Set $u = \overline{u}|_K$, so that the diagram $\overline{u}$ can be identified with an object of the slice ∞-category $\mathcal{C}_{/u}$. We will say that $\overline{u}$ is a *limit diagram* if it is a final object of $\mathcal{C}_{/u}$. If this condition is satisfied, we say that $\overline{u}$ *exhibits* Y *as a limit of the diagram* u.

02JN **Variant 7.1.2.5.** Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let $\overline{u} : K^{\triangleright} \to \mathcal{C}$ be a morphism of simplicial sets carrying the cone point of $K^{\triangleright}$ to an object $Y \in \mathcal{C}$. Set $u = \overline{u}|_K$, so that the diagram $\overline{u}$ can be identified with an object of the coslice ∞-category $\mathcal{C}_{u/}$. We will say that $\overline{u}$ is a *colimit diagram* if it is an initial object of $\mathcal{C}_{u/}$. If this condition is satisfied, we say that $\overline{u}$ *exhibits* Y *as a colimit of the diagram* u.

02WA **Remark 7.1.2.6.** Let $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ be as in Definition 7.1.2.4. Then $\overline{u}$ is a limit diagram if and only if the composite map

$$\Delta^1 \times K \simeq K \star_K K \to \Delta^0 \star_{\Delta^0} K = K^{\triangleleft} \xrightarrow{\overline{u}} \mathcal{C}$$

corresponds to a natural transformation $\alpha : \underline{Y} \to u$ which exhibits Y as a limit of u, in the sense of Definition 7.1.1.1. This follows from the characterization of Proposition 7.1.2.1, together with the observation that the slice diagonal $\mathcal{C}_{/u} \hookrightarrow \mathcal{C} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \{u\}$ of Construction 4.6.4.13 is an equivalence of ∞-categories (Theorem 4.6.4.17).

02JP **Remark 7.1.2.7.** Let $\mathcal{C}$ be an ∞-category and let $u : K \to \mathcal{C}$ be a diagram. Then an object $Y \in \mathcal{C}$ is a limit of u (in the sense of Definition 7.1.1.11) if and only if there exists a diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ which exhibits Y as a limit of u. This is a reformulation of Corollary 7.1.2.2. Similarly, Y is a colimit of u if and only if there exists a diagram $\overline{u}' : K^{\triangleright} \to \mathcal{C}$ which exhibits Y as a colimit of u.

02JS **Remark 7.1.2.8.** Let $\mathcal{C}$ be an ∞-category and let $f : K \to \mathcal{C}$ be a morphism of simplicial sets. An extension $\overline{f} : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram in $\mathcal{C}$ if and only if the opposite map $\overline{f}^{\mathrm{op}} : (K^{\mathrm{op}})^{\triangleleft} \to \mathcal{C}^{\mathrm{op}}$ is a limit diagram in the ∞-category $\mathcal{C}^{\mathrm{op}}$.

02JQ **Example 7.1.2.9.** Let $\mathcal{C}$ be an ∞-category. Then an object $Y \in \mathcal{C}$ is final (in the sense of Definition 4.6.7.1) if and only if the map

$$(\emptyset)^{\triangleleft} \simeq \Delta^0 \xrightarrow{Y} \mathcal{C}$$

is a limit diagram in $\mathcal{C}$. Similarly, Y is initial if and only if the map

$$(\emptyset)^{\triangleright} \simeq \Delta^0 \xrightarrow{Y} \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$.

Example 7.1.2.10. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. The following conditions are equivalent: 02WB

- The morphism f is an isomorphism.
- When regarded as a morphism $(\Delta^0)^{\triangleleft} \to \mathcal{C}$, f is a limit diagram.
- When regarded as a morphism $(\Delta^0)^{\triangleright} \to \mathcal{C}$, f is a colimit diagram.

This is a restatement of Proposition 4.6.7.22 (and also of Example 7.1.1.5, by virtue of Remark 7.1.2.6).

Remark 7.1.2.11. Let $\mathcal{C}$ be an ∞-category, let $g : B \to \mathcal{C}$ be a morphism of simplicial sets, and suppose we are given a diagram $\overline{f} : A^{\triangleleft} \to \mathcal{C}_{/g}$, which we can identify with a morphism of simplicial sets 02JT

$$\overline{q} : (A \star B)^{\triangleleft} \simeq A^{\triangleleft} \star B \to \mathcal{C}\,.$$

Then $\overline{f}$ is a limit diagram in the slice ∞-category $\mathcal{C}_{/g}$ if and only if $\overline{q}$ is a limit diagram in the ∞-category $\mathcal{C}$.

Proposition 7.1.2.12. *Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a morphism with restriction $f = \overline{f}|_K$. The following conditions are equivalent:* 02JU

(1) *The morphism $\overline{f}$ is a limit diagram (Definition 7.1.2.4).*

(2) *The restriction map $\mathcal{C}_{/\overline{f}} \to \mathcal{C}_{/f}$ is a trivial Kan fibration.*

(3) *The restriction map $\mathcal{C}_{/\overline{f}} \to \mathcal{C}_{/f}$ is an equivalence of ∞-categories.*

(4) *For every object $X \in \mathcal{C}$, the restriction map $\{X\} \times_{\mathcal{C}} \mathcal{C}_{/\overline{f}} \to \{X\} \times \times_{\mathcal{C}} \mathcal{C}_{/f}$ is a homotopy equivalence of Kan complexes.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 4.6.7.10. Note that the restriction map $\mathcal{C}_{/\overline{f}} \to \mathcal{C}_{/f}$ is a right fibration of ∞-categories (Corollary 4.3.6.11), and therefore an isofibration (Example 4.4.1.11). The equivalence (2) $\Leftrightarrow$ (3) now follows from Proposition 4.5.5.20, and the equivalence (3) $\Leftrightarrow$ (4) follows from Corollary 5.1.6.4. $\square$

Proposition 7.1.2.13. *Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let $\overline{\rho} : \overline{F} \to \overline{G}$ be a natural transformation between diagrams $\overline{F}, \overline{G} : K^{\triangleleft} \to \mathcal{C}$. Assume that, for every vertex $x \in K$, the morphism $\overline{\rho}_x : \overline{F}(x) \to \overline{G}(x)$ is an isomorphism in $\mathcal{C}$. Then any two of the following conditions imply the third:* 0391

(1) *The morphism of simplicial sets $\overline{F}$ is a limit diagram in $\mathcal{C}$.*

(2) *The morphism of simplicial sets $\overline{G}$ is a limit diagram in $\mathcal{C}$.*

(3) *The natural transformation $\overline{\rho}$ carries the cone point $\mathbf{0} \in K^{\triangleleft}$ to an isomorphism $\overline{\rho}_{\mathbf{0}} : \overline{F}(\mathbf{0}) \to \overline{G}(\mathbf{0})$.*

Proof. Set $F = \overline{F}|_K$ and $G = \overline{G}|_K$, so that $\overline{\rho}$ restricts to an isomorphism $\rho : F \to G$ in the ∞-category $\operatorname{Fun}(K, \mathcal{C})$ (Theorem 4.4.4.4). Set $X = \overline{F}(\mathbf{0})$ and $Y = \overline{F}(\mathbf{0})$, and let $\underline{X}, \underline{Y} : K \to \mathcal{C}$ be the constant maps taking the values X and Y, respectively. Let c denote the composition $\Delta^1 \times K \simeq K \star_K K \to \Delta^0 \star_{\Delta^0} K = K^{\triangleleft}$. Then the composition

$$\Delta^1 \times \Delta^1 \times K \xrightarrow{\operatorname{id} \times c} \Delta^1 \times K^{\triangleleft} \xrightarrow{\overline{\rho}} \mathcal{C}$$

can be identified with a commutative diagram

$$\begin{tikzcd}
\underline{X} \arrow[r, "\underline{f}"] \arrow[d, "\alpha"'] \arrow[dr, "\gamma"] & \underline{Y} \arrow[d, "\beta"] \\
F \arrow[r, "\rho", "\sim"'] & G
\end{tikzcd}$$

in the ∞-category $\operatorname{Fun}(K, \mathcal{C})$. Using Remark 7.1.2.6, we can reformulate conditions (1) and (2) as follows:

(1′) The natural transformation α exhibits X as a limit of F.

(2′) The natural transformation β exhibits Y as a limit of G.

Since ρ is an isomorphism, we can use Remark 7.1.1.8 to reformulate (1′) as follows:

(1″) The natural transformation γ exhibits X as a limit of G.

It will therefore suffice to show that any two of the conditions (1″), (2′), and (3) imply the third, which is a special case of Remark 7.1.1.9. □

02K1 **Corollary 7.1.2.14.** *Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. Then:*

(1) *Let $\overline{u}, \overline{v} : K^{\triangleleft} \to \mathcal{C}$ be a pair of diagrams which are isomorphic when regarded as objects of the ∞-category $\operatorname{Fun}(K^{\triangleleft}, \mathcal{C})$. Then $\overline{u}$ is a limit diagram if and only if $\overline{v}$ is a limit diagram.*

(2) *Let $\overline{u}, \overline{v} : K^{\triangleright} \to \mathcal{C}$ be a pair of diagrams which are isomorphic when regarded as objects of the ∞-category $\operatorname{Fun}(K^{\triangleright}, \mathcal{C})$. Then $\overline{u}$ is a colimit diagram if and only if $\overline{v}$ is a colimit diagram.*

02K2 **Corollary 7.1.2.15.** *Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and suppose we are given a pair of morphisms $u, v : K \to \mathcal{C}$ which are isomorphic as objects of the ∞-category $\operatorname{Fun}(K, \mathcal{C})$. Then:*

(1) *The morphism u can be extended to a limit diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ if and only if v can be extended to a limit diagram $\overline{v} : K^{\triangleleft} \to \mathcal{C}$.*

(2) *The morphism u can be extended to a colimit diagram $\overline{u} : K^{\triangleright} \to \mathcal{C}$ if and only if v can be extended to a colimit diagram $\overline{v} : K^{\triangleright} \to \mathcal{C}$.*

Proof. We will prove (1); the proof of (2) is similar. Suppose that u can be extended to a limit diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$. Since the diagrams u and v are isomorphic, it follows from Corollary 4.4.5.3 that $\overline{u}$ is isomorphic to a diagram $\overline{v} : K^{\triangleleft} \to \mathcal{C}$ satisfying $\overline{v}|_K = v$. Applying Corollary 7.1.2.14, we conclude that $\overline{v}$ is also a limit diagram. □

7.1.3 Preservation of Limits and Colimits

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Beware that, in general, F need not carry (co)limit diagrams in $\mathcal{C}$ to (co)limit diagrams in $\mathcal{D}$. This motivates the following: 02JV

Definition 7.1.3.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $q : K \to \mathcal{C}$ be a diagram. Suppose that q can be extended to a limit diagram $\overline{q} : K^{\triangleleft} \to \mathcal{C}$. We say that *the limit of q is preserved by F* if the composition $F \circ \overline{q}$ is a limit diagram in the ∞-category $\mathcal{D}$. Similarly, if q can be extended to a colimit diagram $\overline{q} : K^{\triangleright} \to \mathcal{C}$, we say that *the colimit of q is preserved by F* if $F \circ \overline{q}$ is a colimit diagram in the ∞-category $\mathcal{D}$. 0392

Remark 7.1.3.2. In the situation of Definition 7.1.3.1, the condition that F preserves the (co)limit of a diagram $q : K \to \mathcal{C}$ depends only on the diagram q, and not on the extension $\overline{q}$ (see Corollary 7.1.2.14). 0393

Remark 7.1.3.3. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $q : K \to \mathcal{C}$ be a diagram which admits a limit in $\mathcal{C}$. Choose an object $X \in \mathcal{C}$ and a natural transformation $\alpha : \underline{X} \to q$ which exhibits X as a limit of q. Then F preserves the limit of q if and only if the natural transformation $F(\alpha)$ exhibits the object $F(X)$ as a limit of the diagram $F \circ q$. 0394

Definition 7.1.3.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let K be a simplicial set. We will say that F *preserves K-indexed limits* if, for every limit diagram $\overline{q} : K^{\triangleleft} \to \mathcal{C}$, the composite map $(F \circ \overline{q}) : K^{\triangleleft} \to \mathcal{D}$ is a limit diagram in $\mathcal{D}$. We will say that F *preserves K-indexed colimits* if, for every colimit diagram $\overline{q} : K^{\triangleright} \to \mathcal{C}$, the composite map $(F \circ \overline{q}) : K^{\triangleright} \to \mathcal{D}$ is a colimit diagram in $\mathcal{D}$. 02JW

Example 7.1.3.5. Let $F : \mathcal{C} \to \mathcal{D}$ be any functor of ∞-categories. Then F preserves Δ^0-indexed limits and colimits. By virtue of Example 7.1.2.10, this is equivalent to the observation that F carries isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$ (see Remark 1.5.1.6). 02JX

Warning 7.1.3.6. In the formulation of Definition 7.1.3.4, it is not necessary to assume that the ∞-category $\mathcal{C}$ admits K-indexed limits or colimits. For example, if $\mathcal{C}$ is an ∞-category 02JY which contains no limit diagrams $\overline{q} : K^{\triangleleft} \to \mathcal{C}$, then *every* functor $F : \mathcal{C} \to \mathcal{D}$ preserves K-indexed limits. In practice, we will usually (but not always) apply the terminology of Definition 7.1.3.4 in cases where the ∞-category admits K-indexed limits or colimits, so that the conclusion of Definition 7.1.3.4 is non-vacuous.

02WC **Exercise 7.1.3.7.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let K be a simplicial set. Show that F preserves K-indexed limits if and only if it satisfies the following condition:

- For every diagram $u : K \to \mathcal{C}$ and every natural transformation $\alpha : \underline{Y} \to u$ which exhibits an object $Y \in \mathcal{C}$ as a limit of u (in the sense of Definition 7.1.1.1), the image $F(\alpha) : \underline{F(Y)} \to (F \circ u)$ exhibits the object $F(Y) \in \mathcal{D}$ as a limit of the diagram $(F \circ u) : K \to \mathcal{D}$.

0395 **Variant 7.1.3.8.** It will often be useful to extend the terminology of Definition 7.1.3.4, replacing the individual simplicial set K by a collection of simplicial sets.

- We say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *preserves finite limits* if it preserves K-indexed limits, for every finite simplicial set K.

- We say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *preserves finite colimits* if it preserves K-indexed colimits, for every finite simplicial set K.

- We say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *preserves small limits* if it preserves K-indexed limits, for every small simplicial set K.

- We say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *preserves small colimits* if it preserves K-indexed colimits, for every small simplicial set K.

Let us begin with a trivial example.

02JZ **Proposition 7.1.3.9.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories and let K be a simplicial set. Then:*

(1) *A morphism $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ is a limit diagram if and only if the composition $F \circ \overline{u}$ is a limit diagram in $\mathcal{D}$.*

(2) *A morphism $\overline{u} : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram if and only if the composition $F \circ \overline{u}$ is a colimit diagram in $\mathcal{D}$.*

In particular, the equivalence F preserves K-indexed limits and colimits.

Proof. We will prove (1); the proof of (2) is similar. Let $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ be a diagram and set $u = \overline{u}|_K$. We then have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C}_{/\overline{u}} \arrow[r] \arrow[d] & \mathcal{D}_{/(F \circ \overline{u})} \arrow[d] \\
\mathcal{C}_{/u} \arrow[r] & \mathcal{D}_{/(F \circ u)} .
\end{tikzcd}$$

Since F is an equivalence of ∞-categories, the horizontal maps in this diagram are also equivalences of ∞-categories (Corollary 4.6.4.19). It follows that the left vertical map is an equivalence of ∞-categories if and only if the right vertical map is an equivalence of ∞-categories. The desired result now follows from the criterion of Proposition 7.1.2.12. □

Variant 7.1.3.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories and let $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ be a morphism of simplicial sets. If $F \circ \overline{u}$ is a limit diagram in the ∞-category $\mathcal{D}$, then $\overline{u}$ is a limit diagram in the ∞-category $\mathcal{C}$. 02WD

Proof. Combine Remark 7.1.1.10 with Exercise 7.1.3.7. □

Corollary 7.1.3.11. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories and let $u : K \to \mathcal{C}$ be a morphism of simplicial sets. Then:* 02K3

(1) *The morphism u can be extended to a limit diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ if and only if the composite map $(F \circ u) : K \to \mathcal{D}$ can be extended to a limit diagram $K^{\triangleleft} \to \mathcal{D}$.*

(2) *The morphism u can be extended to a colimit diagram $\overline{u} : K^{\triangleright} \to \mathcal{C}$ if and only if the composite map $(F \circ u) : K \to \mathcal{D}$ can be extended to a colimit diagram $K^{\triangleright} \to \mathcal{D}$.*

Proof. We will prove (1); the proof of (2) is similar. If u can be extended to a limit diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$, then Proposition 7.1.3.9 guarantees that $F \circ \overline{u}$ is a limit diagram in $\mathcal{D}$ extending $F \circ u$. Conversely, suppose that $F \circ u$ can be extended to a limit diagram $\overline{v} : K^{\triangleleft} \to \mathcal{D}$. Let $G : \mathcal{D} \to \mathcal{C}$ be an equivalence of ∞-categories which is homotopy inverse to F, so that $G \circ F$ is isomorphic to the identity functor $\mathrm{id}_{\mathcal{C}}$. Then $(G \circ \overline{v}) : K^{\triangleleft} \to \mathcal{C}$ is a limit diagram in $\mathcal{C}$ (Proposition 7.1.3.9), and the restriction $(G \circ \overline{v})|_K = (G \circ F \circ u)$ is isomorphic to u as an object of the ∞-category $\mathrm{Fun}(K, \mathcal{C})$. Applying Corollary 7.1.2.15, we deduce that u can be extended to a limit diagram $\overline{p} : K^{\triangleleft} \to \mathcal{C}$. □

Corollary 7.1.3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which are equivalent to one another, and let K be a simplicial set. Then $\mathcal{C}$ admits K-indexed (co)limits if and only if $\mathcal{D}$ admits K-indexed (co)limits.* 02K4

02K5 **Remark 7.1.3.13.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let K be a simplicial set, and let $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ be a limit diagram with restriction $u = \overline{u}|_K$. The following conditions are equivalent:

(1) The composition $(F \circ \overline{u}) : K^{\triangleleft} \to \mathcal{D}$ is a limit diagram.

(2) For every limit diagram $\overline{u}' : K^{\triangleleft} \to \mathcal{C}$ with $\overline{u}'|_K = u$, the composition $(F \circ \overline{u}') : K^{\triangleleft} \to \mathcal{D}$ is a limit diagram.

The implication (2) $\Rightarrow$ (1) is immediate. For the converse, we observe that if $\overline{u}' : K^{\triangleleft} \to \mathcal{C}$ is another limit diagram with $\overline{u}'|_K = u$, then $\overline{u}$ and $\overline{u}'$ are isomorphic when viewed as objects of the slice ∞-category $\mathcal{C}_{/u}$, so that $F \circ \overline{u}$ and $F \circ \overline{u}'$ are isomorphic when viewed as objects of the ∞-category $\mathcal{D}_{/(F \circ u)}$. Since $F \circ \overline{u}$ is a final object of $\mathcal{D}_{/(F \circ u)}$, it follows that $F \circ \overline{u}'$ is also a final object of $\mathcal{D}_{/(F \circ u)}$ (Corollary 4.6.7.15).

A conservative functor $F : \mathcal{C} \to \mathcal{D}$ which preserves K-indexed limits also reflects them:

02K6 **Proposition 7.1.3.14.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a conservative functor of ∞-categories and let K be a simplicial set.*

- *Suppose that $\mathcal{C}$ admits K-indexed limits and the functor F preserves K-indexed limits. Then a morphism $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ is a limit diagram in $\mathcal{C}$ if and only if $(F \circ \overline{u}) : K^{\triangleleft} \to \mathcal{D}$ is a limit diagram in $\mathcal{D}$.*

- *Suppose that $\mathcal{C}$ admits K-indexed colimits and the functor F preserves K-indexed colimits. Then a morphism $\overline{u} : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram in $\mathcal{C}$ if and only if $(F \circ \overline{u}) : K^{\triangleright} \to \mathcal{D}$ is a colimit diagram in $\mathcal{D}$.*

Proposition 7.1.3.14 is an immediate consequence of the following more precise assertion:

02K7 **Lemma 7.1.3.15.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a conservative functor of ∞-categories and let $u : K \to \mathcal{C}$ be a diagram. Suppose that u can be extended to a limit diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ for which the composition $(F \circ \overline{u}) : K^{\triangleleft} \to \mathcal{D}$ is also a limit diagram. Let $\overline{u}' : K^{\triangleleft} \to \mathcal{C}$ be an arbitrary extension of u. Then $\overline{u}'$ is a limit diagram in $\mathcal{C}$ if and only if $F \circ \overline{u}'$ is a limit diagram in $\mathcal{D}$.*

Proof. Let us identify $\overline{u}$ and $\overline{u}'$ with objects C and C' of the slice ∞-category $\mathcal{C}_{/u}$. Our assumption that $\overline{u}$ is a limit diagram guarantees that C is a final object of $\mathcal{C}_{/u}$, so there exists a morphism $f : C' \to C$ in $\mathcal{C}_{/u}$. Note that $\overline{u}'$ is a limit diagram if and only if the object C' is also final: that is, if and only if the morphism f is an isomorphism.

Let $g : D' \to D$ be the image of f under the functor $F_{/u} : \mathcal{C}_{/u} \to \mathcal{D}_{/(F \circ u)}$. Our assumption that $F \circ \overline{u}$ is a limit diagram guarantees that D is a final object of $\mathcal{D}_{/(F \circ u)}$. Consequently, g

is an isomorphism if and only if the object D' is also final: that is, if and only if $(F \circ \overline{u}')$ is a limit diagram in $\mathcal{D}$.

To complete the proof, it will suffice to show that f is an isomorphism in $\mathcal{C}_{/u}$ if and only if $g = F_{/u}(f)$ is an isomorphism in $\mathcal{D}_{/(F \circ u)}$. In fact, the functor $F_{/u}$ is conservative: this follows from our assumption that F is conservative, by virtue of Corollary 4.4.2.12. $\square$

Definition 7.1.3.16. Let $F : \mathcal{C} \to \mathcal{D}$ be a conservative functor of ∞-categories and let K be a simplicial set. We will say that the functor F *creates K-indexed limits* if the following condition is satisfied: 02K8

- Let $u : K \to \mathcal{C}$ be a diagram for which the induced map $(F \circ u) : K \to \mathcal{D}$ admits a limit in $\mathcal{D}$. Then u can be extended to a limit diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ for which the composition $(F \circ \overline{u}) : K^{\triangleleft} \to \mathcal{D}$ is a limit diagram in $\mathcal{D}$.

We say that the functor F *creates K-indexed colimits* if it satisfies the following dual condition:

- Let $u : K \to \mathcal{C}$ be a diagram for which the induced map $(F \circ u) : K \to \mathcal{D}$ admits a colimit in $\mathcal{D}$. Then u can be extended to a colimit diagram $\overline{q} : K^{\triangleright} \to \mathcal{C}$ for which the composition $(F \circ \overline{u}) : K^{\triangleright} \to \mathcal{D}$ is a colimit diagram in $\mathcal{D}$.

Remark 7.1.3.17. Let $F : \mathcal{C} \to \mathcal{D}$ be a conservative functor of ∞-categories and let $u : K \to \mathcal{C}$ be a diagram. Suppose that F creates K-indexed limits and that $F \circ u$ can be extended to a limit diagram $K^{\triangleleft} \to \mathcal{D}$. Then an extension $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ of u is a limit diagram if and only if $F \circ \overline{u}$ is a limit diagram in $\mathcal{D}$ (see Lemma 7.1.3.15). 02K9

Proposition 7.1.3.18. *Let K be a simplicial set, let $\mathcal{D}$ be an ∞-category which admits K-indexed limits, and let $F : \mathcal{C} \to \mathcal{D}$ be a conservative functor of ∞-categories. The following conditions are equivalent:* 02KA

(1) *The ∞-category $\mathcal{C}$ admits K-indexed limits and the functor F preserves K-indexed limits.*

(2) *The functor F creates K-indexed limits.*

Proof. The implication (1) ⇒ (2) is immediate. Conversely, suppose that (2) is satisfied and let $u : K \to \mathcal{C}$ be a diagram. Since $\mathcal{D}$ admits K-indexed limits, $F \circ u$ can be extended to a limit diagram in $\mathcal{D}$. Since F creates K-indexed limits, it follows that there exists a limit diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ with $\overline{u}|_K = u$ such that $F \circ \overline{u}$ is a limit diagram in $\mathcal{D}$. Applying Remark 7.1.3.13, we see that this holds for *every* limit diagram $\overline{u} : K^{\triangleleft} \to \mathcal{C}$ satisfying $\overline{u}|_K = u$, which proves (1). $\square$

The following is an important example of Definition 7.1.3.16:

`02KB` **Proposition 7.1.3.19.** *Let $\mathcal{C}$ be an ∞-category, let A be a simplicial set, and let $f : A \to \mathcal{C}$ be a diagram. Then:*

(1) *The projection map $\mathcal{C}_{f/} \to \mathcal{C}$ creates K-indexed limits, for every simplicial set K.*

(2) *The projection map $\mathcal{C}_{/f} \to \mathcal{C}$ creates K-indexed colimits, for every simplicial set K.*

Proof. We will prove (1); the proof of (2) is similar. Let K be a simplicial set and let $p : K \to \mathcal{C}_{f/}$ be a diagram, which we will identify with a morphism of simplicial sets $q : A \star K \to \mathcal{C}$ satisfying $q|_A = f$. Set $g = q|_K$, so that q can also be identified with a diagram $f' : A \to \mathcal{C}_{/g}$. Suppose that g can be extended to a limit diagram $\overline{g} : K^{\triangleleft} \to \mathcal{C}$. Then the projection map $\mathcal{C}_{/\overline{g}} \to \mathcal{C}_{/g}$ is a trivial Kan fibration (Proposition 7.1.2.12), so that f' can be lifted to a diagram $f'' : A \to \mathcal{C}_{/\overline{g}}$. We can then identify f'' with a morphism of simplicial sets $\overline{q} : A \star K^{\triangleleft} \to \mathcal{C}$ extending q, or equivalently with a morphism $\overline{p} : K^{\triangleleft} \to \mathcal{C}_{f/}$ extending p. We will complete the proof by showing that $\overline{p}$ is a limit diagram. To prove this, it will suffice to show that $\overline{p}$ is final when regarded as an object of the slice ∞-category $(\mathcal{C}_{f/})_{/p} \simeq (\mathcal{C}_{/g})_{f'/}$. This follows from Proposition 4.6.7.12, since $\overline{g}$ is a final object of $\mathcal{C}_{/g}$. □

`02KC` **Corollary 7.1.3.20.** *Let $\mathcal{C}$ be an ∞-category, let $f : A \to \mathcal{C}$ be a morphism of simplicial sets, and let K be an arbitrary simplicial set. Then:*

(1) *If $\mathcal{C}$ admits K-indexed limits, then the coslice ∞-category $\mathcal{C}_{f/}$ admits K-indexed limits. Moreover, a morphism $K^{\triangleleft} \to \mathcal{C}_{f/}$ is a limit diagram if and only if its image in $\mathcal{C}$ is a limit diagram.*

(2) *If $\mathcal{C}$ admits K-indexed colimits, then the slice ∞-category $\mathcal{C}_{/f}$ admits K-indexed colimits. Moreover, a morphism $K^{\triangleright} \to \mathcal{C}_{/f}$ is a colimit diagram if and only if its image in $\mathcal{C}$ is a colimit diagram.*

Proof. Combine Propositions 7.1.3.19 and 7.1.3.18 with Remark 7.1.3.17. □

`02KE` **Corollary 7.1.3.21.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$. For every simplicial set K, the functor F preserves K-indexed colimits and the functor G preserves K-indexed limits.*

Proof. We will show that F preserves K-indexed colimits; the assertion that G preserves K-indexed limits can be proved by a similar argument. Let $u : K \to \mathcal{C}$ be a morphism of simplicial sets, so that F induces a functor $F' : \mathcal{C}_{u/} \to \mathcal{D}_{(F \circ u)/}$. We wish to show that the functor F' carries initial objects of $\mathcal{C}_{u/}$ to initial objects of $\mathcal{D}_{(F \circ u)/}$. It follows from Corollary 6.2.4.6 that the functor F' also admits a right adjoint. We may therefore replace F by F' and thereby reduce to the case where $K = \emptyset$. In this case, we must show that if X is an initial object of $\mathcal{C}$, then $F(X)$ is an initial object of $\mathcal{D}$. Choose an object $Y \in \mathcal{D}$; we wish to show that the morphism space $\mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ is a contractible Kan complex. Proposition 6.2.1.17 supplies a homotopy equivalence of Kan complexes $\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$. We conclude by observing that the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ is contractible, by virtue of our assumption that the object $X \in \mathcal{C}$ is initial. □

Corollary 7.1.3.22 (Colimits in a Reflective Localization)**.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a reflective subcategory (Definition 6.2.2.1), and let $u : K \to \mathcal{C}'$ be a diagram. If u admits a colimit in $\mathcal{C}$, then it also admits a colimit in $\mathcal{C}'$.* `04JW`

Proof. By virtue of Proposition 6.2.2.11, the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}'$. If u admits a colimit in $\mathcal{C}$, then $L \circ u$ admits a colimit in $\mathcal{C}'$ (Corollary 7.1.3.21). Since u factors through $\mathcal{C}'$, it is isomorphic to $L \circ u$ and therefore also admits a colimit in $\mathcal{C}'$ (Remark 7.1.1.8). □

Warning 7.1.3.23. In the situation of Corollary 7.1.3.22, the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ generally does not preserve the colimit of the diagram u. If $C = \varinjlim(u)$ is a colimit of u in the ∞-category $\mathcal{C}$, then C usually does not belong to $\mathcal{C}'$. The colimit of u in the ∞-category $\mathcal{C}'$ is instead given by the localization $L(C)$. `03Y0`

Variant 7.1.3.24 (Limits in a Reflective Localization)**.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a reflective subcategory. Then a diagram $u : K \to \mathcal{C}'$ admits a limit in $\mathcal{C}'$ if and only if it admits a limit in $\mathcal{C}$. In this case, the limit of u is preserved by the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$. `04JX`

We will deduce Variant 7.1.3.24 from the following special case:

Lemma 7.1.3.25. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a reflective subcategory. If $\mathcal{C}$ contains a final object X, then $\mathcal{C}'$ contains an object which is isomorphic to X. In particular, if $\mathcal{C}'$ is replete, then it contains every final object of $\mathcal{C}$.* `04JY`

Proof. Choose a morphism $f : X \to Y$ in $\mathcal{C}$ which exhibits Y as a $\mathcal{C}'$-reflection of X (see Definition 6.2.2.1). Since X is a final object of $\mathcal{C}$, we can choose a morphism $g : Y \to X$. We will complete the proof by showing that g is a homotopy inverse to f: that is, the homotopy classes $[f]$ and $[g]$ are inverses of one another in the homotopy category $\mathrm{h}\mathcal{C}$. Since X is a final object of $\mathcal{C}$, the equality $[g] \circ [f] = [\mathrm{id}_X]$ is automatic. We wish to prove the equality $[f] \circ [g] = [\mathrm{id}_Y]$. Since f exhibits Y as a $\mathcal{C}'$-reflection of X, precomposition with $[f]$ induces a bijection $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$. The desired result now follows from the calculation

$$([f] \circ [g]) \circ [f] = [f] \circ ([g] \circ [f]) = [f] \circ [\mathrm{id}_X] = [f] = [\mathrm{id}_Y] \circ [f].$$

□

Proof of Variant 7.1.3.24. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a reflective subcategory, and let $u : K \to \mathcal{C}'$ be a diagram. Applying Corollary 6.2.2.10, we deduce that $\mathcal{C}'_{/u}$ is a reflective subcategory of $\mathcal{C}_{/u}$. If the diagram u admits a limit in $\mathcal{C}$, then the slice ∞-category $\mathcal{C}_{/u}$ has a final object X. Applying Lemma 7.1.3.25, we deduce that X is isomorphic to an object of $\mathcal{C}'_{/u}$, which is also a final object of $\mathcal{C}_{/u}$ and therefore also of $\mathcal{C}'_{/u}$. In particular, the diagram u has a limit in $\mathcal{C}'$ (which is preserved by the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$). □

04JZ **Proposition 7.1.3.26.** *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be a reflective subcategory. If $\mathcal{C}$ has a final object X, then $\mathcal{C}'$ contains an object which is isomorphic to X. In particular, if $\mathcal{C}'$ is replete, then it contains every final object of $\mathcal{C}$.*

Proof. Choose a morphism $f : X \to Y$ which exhibits Y as a $\mathcal{C}'$-reflection of X.

□

03Y1 **Corollary 7.1.3.27.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a reflective subcategory of $\mathcal{C}$, and let K be a simplicial set. If $\mathcal{C}$ admits K-indexed limits, then $\mathcal{C}_0$ also admits K-indexed limits. If $\mathcal{C}$ admits K-indexed colimits, then $\mathcal{C}_0$ also admits K-indexed colimits.*

Proof. Combine Variant 7.1.3.24 with Corollary 7.1.3.22. □

7.1.4 Relative Initial and Final Objects

02WE In §4.6.7, we introduced the notions of *initial* and *final* object of an ∞-category $\mathcal{C}$ (Definition 4.6.7.1). In this section, we study the more general notions of *U-initial* and *U-final* objects, where $U : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories.

02KU **Definition 7.1.4.1.** Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that an object $Y \in \mathcal{C}$ is *U-final* if, for every object $X \in \mathcal{C}$, the functor U induces a homotopy equivalence

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(U(X), U(Y)).$$

We say that Y is *U-initial* if, for every object $Z \in \mathcal{C}$, the functor U induces a homotopy equivalence

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{D}}(U(Y), U(Z)).$$

02WF **Example 7.1.4.2.** Let $\mathcal{C}$ be an ∞-category and let $U : \mathcal{C} \to \Delta^0$ be the projection map. Then an object $Y \in \mathcal{C}$ is U-initial if and only if it is initial, and U-final if and only if it is final.

0396 **Remark 7.1.4.3.** Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by the U-initial objects. Then the restriction $U|_{\mathcal{C}_0} : \mathcal{C}_0 \to \mathcal{D}$ is fully faithful. Similarly, U is fully faithful when restricted to the full subcategory of U-final objects of $\mathcal{C}$.

Example 7.1.4.4. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent: 02WG

- The functor U is fully faithful.
- Every object of $\mathcal{C}$ is U-initial.
- Every object of $\mathcal{C}$ is U-final.

Remark 7.1.4.5. Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. Then an object $Y \in \mathcal{C}$ is U-initial if and only if it is U^{op}-final, when regarded as an object of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. 02WH

Remark 7.1.4.6 (Transitivity)**.** Let $U : \mathcal{C} \to \mathcal{D}$ and $V : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, and let $Y \in \mathcal{C}$ be an object for which $U(Y)$ is V-final. Then Y is U-final if and only if it is $(V \circ U)$-final. 02WJ

Remark 7.1.4.7. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $Y \in \mathcal{C}$ be an object. Suppose that $U(Y)$ is a final object of $\mathcal{D}$. Then Y is a final object of $\mathcal{C}$ if and only if it is a U-final object of $\mathcal{C}$ (apply Remark 7.1.4.6 in the special case $\mathcal{E} = \Delta^0$). 02WK

Remark 7.1.4.8. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $V : \mathcal{C} \to \mathcal{D}$ be another functor which is isomorphic to U (as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$). Then an object $Y \in \mathcal{C}$ is U-initial if and only if it is V-initial. To prove this, let Z be an object of $\mathcal{C}$ and let $\alpha : U \to V$ be an isomorphism of functors, so that we have a commutative diagram 02WL

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y, Z) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{D}}(U(Y), U(Z)) \arrow[d, "{[\alpha_Z]\circ}"] \\
\mathrm{Hom}_{\mathcal{D}}(V(Y), V(Z)) \arrow[r, "{\circ[\alpha_Y]}"] & \mathrm{Hom}_{\mathcal{D}}(U(Y), V(Z))
\end{tikzcd}$$

in the homotopy category hKan, where the bottom horizontal and right vertical maps are homotopy equivalences. It follows that the upper horizontal map is a homotopy equivalence if and only if the left vertical map is a homotopy equivalence. Similarly, the object Y is U-final if and only if it is V-final.

Remark 7.1.4.9. Suppose we are given a commutative diagram of ∞-categories 02WM

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\
\mathcal{D} \arrow[r] & \mathcal{D}',
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories. Then an object $X \in \mathcal{C}$ is U-initial if and only if $F(X) \in \mathcal{C}'$ is U'-initial, and U-final if and only if $F(X)$ is U'-final.

02WN **Proposition 7.1.4.10.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $f : X \to Y$ be a morphism in $\mathcal{C}$ with the property that $U(f)$ is an isomorphism. Then any two of the following three conditions imply the third:*

(1) *The object X is U-initial.*

(2) *The object Y is U-initial.*

(3) *The morphism f is an isomorphism.*

Proof. Fix an object $Z \in \mathcal{C}$. We claim that any two of the following three conditions imply the third:

(1_Z) The functor U induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X, Z) \to \mathrm{Hom}_{\mathcal{C}}(U(X), U(Z))$.

(2_Z) The functor U induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(U(Y), U(Z))$.

(3_Z) Precomposition $[f]$ induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ (see Notation 4.6.9.15).

This follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y, Z) \arrow[r, "\circ[f]"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X, Z) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(U(Y), U(Z)) \arrow[r, "\circ[U(f)]"] & \mathrm{Hom}_{\mathcal{D}}(U(X), U(Z))
\end{tikzcd}$$

in the homotopy category hKan, since the bottom horizontal map is a homotopy equivalence (by virtue of our assumption that $U(f)$ is an isomorphism). Proposition 7.1.4.10 follows by allowing the object Z to vary. □

02WP **Corollary 7.1.4.11.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $f : X \to Y$ be an isomorphism in $\mathcal{C}$. Then the object X is U-initial if and only if Y is U-initial, and the object X is U-final if and only if Y is U-final.*

02WQ **Corollary 7.1.4.12** (Uniqueness)**.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let X and Y be U-initial objects of $\mathcal{C}$. Then X and Y are isomorphic if and only if $U(X)$ and $U(Y)$ are isomorphic as objects of $\mathcal{D}$.*

Proof. Assume that there exists an isomorphism $\overline{f} : U(X) \to U(Y)$ in the ∞-category $\mathcal{D}$. Since X is U-initial, the functor U induces a homotopy equivalence $\operatorname{Hom}_{\mathcal{C}}(X,Y) \to \operatorname{Hom}_{\mathcal{D}}(U(X),U(Y))$. It follows that there exists a morphism $f : X \to Y$ in $\mathcal{C}$ such that $U(f)$ is homotopic to $\overline{f}$. In particular, $U(f) : U(X) \to U(Y)$ is also an isomorphism in $\mathcal{D}$. Applying Proposition 7.1.4.10, we deduce that f is an isomorphism. In particular, the objects X and Y are isomorphic. □

Recall that a functor of ∞-categories $U : \mathcal{C} \to \mathcal{D}$ is a *coreflective localization* if it admits a fully faithful left adjoint $\mathcal{D} \to \mathcal{C}$ (Proposition 6.3.3.6). This condition has a simple formulation in terms of relatively final objects:

Proposition 7.1.4.13. *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then U is a coreflective localization functor if and only if, for every object $D \in \mathcal{D}$, there exists a U-initial object $C \in \mathcal{C}$ and an isomorphism $D \to U(C)$ in the ∞-category $\mathcal{D}$.* 0397

We will deduce Proposition 7.1.4.13 from a slightly more precise result.

Lemma 7.1.4.14. *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by the U-initial objects, and suppose that the restriction $U_0 = U|_{\mathcal{C}_0}$ is essentially surjective. Then:* 0398

(1) *The functor $U_0 : \mathcal{C}_0 \to \mathcal{D}$ is an equivalence of ∞-categories.*

(2) *Let $e : X \to Y$ be a morphism in $\mathcal{C}$, where X is U-initial. Then e exhibits X as a $\mathcal{C}_0$-coreflection of Y (in the sense of Definition 6.2.2.1) if and only if $U(e)$ is an isomorphism in the ∞-category $\mathcal{D}$.*

(3) *The full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ is coreflective.*

(4) *Let $F_0 : \mathcal{D} \to \mathcal{C}_0$ be a homotopy inverse of the functor U_0, and let $F : \mathcal{D} \to \mathcal{C}$ be a composition of F_0 with the inclusion map $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$. Then F is a left adjoint of U.*

(5) *The functor U is a coreflective localization.*

Proof. Note that the functor $U_0 : \mathcal{C}_0 \to \mathcal{D}$ is automatically fully faithful (Remark 7.1.4.3). Our assumption that U_0 is essentially surjective then guarantees that it is an equivalence of ∞-categories, which proves (1).

We next prove the following:

$(*)$ For every object $Y \in \mathcal{C}$, there exists a morphism $e : X \to Y$ in $\mathcal{C}$, where X is U-initial and $U(e)$ is an isomorphism in $\mathcal{D}$.

To prove $(*)$, we observe that the essential surjectivity of U_0 guarantees that there exists a U-initial object $X \in \mathcal{C}$ and an isomorphism $\overline{e} : U(X) \to U(Y)$ in the ∞-category $\mathcal{D}$. Since X is U-initial, the functor U induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(U(X),U(Y))$. Modifying $\overline{e}$ by a homotopy, we can assume without loss of generality that $\overline{e} = U(e)$ for some morphism $X \to Y$ of $\mathcal{C}$.

We now prove (2). Let $e : X \to Y$ be a morphism in $\mathcal{C}$, where the object X is U-initial. Assume first that $U(e)$ is an isomorphism in $\mathcal{D}$. We wish to show that, for every U-initial object $C \in \mathcal{C}$, postcomposition with e induces a homotopy equivalence of Kan complexes $\mathrm{Hom}_{\mathcal{C}}(C,X) \to \mathrm{Hom}_{\mathcal{C}}(C,Y)$. This follows by inspecting the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C,X) \arrow[r, "\circ[e]"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(C,Y) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(U(C),U(X)) \arrow[r, "\circ[U(e)]"] & \mathrm{Hom}_{\mathcal{D}}(U(C),U(Y))
\end{tikzcd}$$

in the homotopy category of Kan complexes hKan; here the vertical maps are homotopy equivalences by virtue of our assumption that C is U-initial, and the bottom horizontal map is a homotopy equivalence by virtue of our assumption that $U(e)$ is an isomorphism.

We now prove the converse. Assume that $e : X \to Y$ exhibits X as a $\mathcal{C}_0$-coreflection of Y; we wish to show that $U(e)$ is an isomorphism. Using $(*)$, we can choose a U-initial object $X' \in \mathcal{C}$ and a morphism $e' : X' \to Y$ such that $U(e')$ is an isomorphism in $\mathcal{D}$. It follows from the previous step that e' exhibits X' as a $\mathcal{C}_0$-coreflection of Y. It follows that e can be realized as the composition of e' with an isomorphism $v : X \to X'$ in the ∞-category $\mathcal{C}$ (Remark 6.2.2.3). Then $U(e)$ is a composition of the isomorphisms $U(v)$ and $U(e')$ in the ∞-category $\mathcal{D}$, and is therefore also an isomorphism.

Assertion (3) follows immediately from (2) and $(*)$. Combining (3) with Proposition 6.2.2.11, we see that there exists a functor $L : \mathcal{C} \to \mathcal{C}_0$ and a natural transformation $\eta : L \to \mathrm{id}_{\mathcal{C}}$ which exhibits L as a $\mathcal{C}_0$-colocalization functor: that is, it carries each object $Y \in \mathcal{C}$ to a morphism $\eta_Y : L(Y) \to Y$ where $L(Y)$ is U-initial and $U(\eta_Y)$ is an isomorphism. In particular, η induces an isomorphism $U_0 \circ L \to U$ in the ∞-category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ (Theorem 4.4.4.4). It follows from assumption (1) that the functor U_0 admits a homotopy inverse $F_0 : \mathcal{D} \to \mathcal{C}_0$, which is also a left adjoint of U_0 (Example 6.2.1.11). Moreover, the inclusion functor $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is left adjoint to L (Proposition 6.2.2.15). It follows that the composition $F = \iota \circ F_0$ is left adjoint to $U_0 \circ L$ (Remark 6.2.1.8), and therefore also to U. This proves (4). Moreover, the functor F is fully faithful (since F_0 is an equivalence of ∞-categories and ι is the inclusion of a full subcategory), so assertion (5) follows from Proposition 6.3.3.6. □

Proof of Proposition 7.1.4.13. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Assume that U

is a coreflective localization functor: we will show that, for every object $D \in \mathcal{D}$, there exists a U-initial object $C \in \mathcal{C}$ and an isomorphism $D \to U(C)$ in $\mathcal{D}$ (the converse follows from Lemma 7.1.4.14). Using Proposition 6.3.3.6, we see that there exists a functor $F : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism $\eta : \mathrm{id}_{\mathcal{D}} \to U \circ F$ which is the unit of an adjunction between F and U. In particular, for every object $D \in \mathcal{D}$, we have an isomorphism $\eta_D : D \to U(C)$ for $C = F(D)$. We will complete the proof by showing that the object C is U-initial. Fix an object $X \in \mathcal{C}$; we wish to show that the functor U induces a homotopy equivalence of Kan complexes $\rho : \mathrm{Hom}_{\mathcal{C}}(F(D), X) \to \mathrm{Hom}_{\mathcal{D}}((U \circ F)(D), U(X))$. Since $\eta_D : D \to (U \circ F)(D)$ is an isomorphism, this is equivalent to the requirement that the composite map

$$\mathrm{Hom}_{\mathcal{C}}(F(D), X) \to \mathrm{Hom}_{\mathcal{D}}((U \circ F)(D), U(X)) \xrightarrow{\circ[\eta_D]} \mathrm{Hom}_{\mathcal{D}}(D, U(X))$$

is a homotopy equivalence of Kan complexes, which follows from our assumption that η is the unit of an adjunction (Proposition 6.2.1.17). □

Corollary 7.1.4.15. *Let $U : \mathcal{C} \to \mathcal{D}$ be an isofibration of ∞-categories. Then U is a coreflective localization functor if and only if, for every object $Y \in \mathcal{D}$, the fiber $\mathcal{C}_Y = \{Y\} \times_{\mathcal{D}} \mathcal{C}$ contains a U-initial object of $\mathcal{C}$.* 0399

Proof. Assume that U is a coreflective localization functor. We will show that, for each object $Y \in \mathcal{D}$, the ∞-category $\mathcal{C}_Y$ contains a U-initial object of $\mathcal{C}$ (the converse follows immediately from Proposition 7.1.4.13). Using Proposition 7.1.4.13, we see that there exists a U-initial object $X \in \mathcal{C}$ and an isomorphism $e : Y \to U(X)$ in $\mathcal{D}$. Since U is an isofibration, we can lift e to an isomorphism $\widetilde{e} : \widetilde{Y} \to X$ in the ∞-category $\mathcal{C}$. Our assumption that X is U-initial then guarantees that $\widetilde{Y}$ is also U-initial (Corollary 7.1.4.11). □

Proposition 7.1.4.16. *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:* 02WR

(1) *An object $Y \in \mathcal{C}$ is U-initial if and only if U induces an equivalence of ∞-categories $U' : \mathcal{C}_{Y/} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{U(Y)/}$.*

(2) *An object $Y \in \mathcal{C}$ is U-final if and only if U induces an equivalence of ∞-categories $U'' : \mathcal{C}_{/Y} \to \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/U(Y)}$.*

Proof. We will prove (2); the proof of (1) is similar. Fix an object $Y \in \mathcal{C}$, so that the morphism U'' of (1) fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{/Y} \arrow[rr, "U''"] \arrow[dr] & & \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/U(Y)} \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

where the vertical maps are right fibrations (Proposition 4.3.6.1). Applying Corollary 5.1.7.16, we see that U'' is an equivalence of ∞-categories if and only if, for every object $X \in \mathcal{C}$, the induced map of fibers

$$U''_X : \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Y} \rightarrow \{X\} \times_{\mathcal{D}} \mathcal{D}_{/U(Y)}$$

is a homotopy equivalence of Kan complexes. By virtue of Proposition 4.6.5.10, this is equivalent to the requirement that U induces a homotopy equivalence $\operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{D}}(U(X), U(Y))$. □

`02WS` **Corollary 7.1.4.17.** *Let $U : \mathcal{C} \rightarrow \mathcal{D}$ be an inner fibration of ∞-categories and let Y be an object of $\mathcal{C}$. The following conditions are equivalent:*

(1) *The object Y is U-initial.*

(2) *The induced map $U' : \mathcal{C}_{Y/} \rightarrow \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{U(Y)/}$ is a trivial Kan fibration.*

(3) *Every lifting problem*

$$\begin{tikzcd}
\partial \Delta^n \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

has a solution, provided that $n > 0$ and $\sigma_0(0) = Y$.

Proof. Since U is an inner fibration, the morphism U' is a left fibration (Corollary 4.3.6.9). In particular, it is a trivial Kan fibration if and only if it is an equivalence of ∞-categories (Proposition 4.5.5.20). The equivalence (1) ⇔ (2) now follows from Proposition 7.1.4.16. The equivalence (2) ⇔ (3) is immediate from the definitions. □

`02WT` **Corollary 7.1.4.18.** *Let $U : \mathcal{C} \rightarrow \mathcal{D}$ be an inner fibration of ∞-categories. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory of $\mathcal{C}$ whose objects are U-initial, and let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the full subcategory of $\mathcal{D}$ spanned by objects of the form $U(C)$ for $C \in \mathcal{C}_0$. Then the functor $U|_{\mathcal{C}_0} : \mathcal{C}_0 \rightarrow \mathcal{D}_0$ is a trivial Kan fibration.*

Proof. Suppose we are given a lifting problem

$$\begin{tikzcd}
\partial \Delta^n \arrow[r, "\sigma_0"] \arrow[d] & \mathcal{C}_0 \arrow[d, "U_0"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & \mathcal{D}_0 .
\end{tikzcd}$$

If $n = 0$, this lifting problem admits a solution by the definition of the subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$. If $n > 0$, then $\sigma_0(0)$ is a U-initial object of $\mathcal{C}$, so Corollary 7.1.4.17 guarantees that σ_0 can be extended to an n-simplex $\sigma : \Delta^n \to \mathcal{C}$ satisfying $U(\sigma) = \overline{\sigma}$. We conclude by observing that σ automatically factors through the full subcategory $\mathcal{C}_0$ (since every vertex of Δ^n is contained in the boundary $\partial \Delta^n$). □

Proposition 7.1.4.19. *Suppose we are given a commutative diagram of ∞-categories* 02WU

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\ & \mathcal{E}, & \end{tikzcd}$$

where U and V are inner fibrations. Let $E \in \mathcal{E}$ be an object, and let $F_E : \mathcal{C}_E \to \mathcal{D}_E$ denote the corresponding restriction of F. Then:

(1) *If $X \in \mathcal{C}_E$ is F-initial when viewed as an object of the ∞-category $\mathcal{C}$, then X is F_E-initial.*

(2) *Assume that U and V are cartesian fibrations, and that the functor F carries U-cartesian morphisms of $\mathcal{C}$ to V-cartesian morphisms of $\mathcal{D}$. If X is F_E-initial, then it is F-initial when viewed as an object of $\mathcal{C}$.*

Proof. We first prove (1). Assume that X is F-initial. For every object $Y \in \mathcal{C}_E$, we have a commutative diagram of Kan complexes

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \arrow[dl] \\ & \mathrm{Hom}_{\mathcal{E}}(E,E). & \end{tikzcd}$$

Our assumption that X is F-initial guarantees that ρ is a homotopy equivalence. Since U and V are inner fibrations, the vertical maps are Kan fibrations (Proposition 4.6.1.21). Applying Corollary 3.3.7.5, we conclude that ρ restricts to a homotopy equivalence

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_E}(X,Y) &= \mathrm{Hom}_{\mathcal{C}}(X,Y) \times_{\mathrm{Hom}_{\mathcal{E}}(E,E)} \{\mathrm{id}_E\} \\ &\to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \times_{\mathrm{Hom}_{\mathcal{E}}(E,E)} \{\mathrm{id}_E\} \\ &= \mathrm{Hom}_{\mathcal{D}_E}(F(X),F(Y)). \end{aligned}$$

Allowing Y to vary over objects of $\mathcal{C}_E$, it follows that X is an F_E-initial object of $\mathcal{C}$.

We now prove (2). Assume that U and V are cartesian fibrations, that the functor F carries U-cartesian morphisms of $\mathcal{C}$ to V-cartesian morphisms of $\mathcal{D}$, and that X is F_E-initial. We wish to show that X is F-initial. Fix an object $Z \in \mathcal{C}$; we must show that the horizontal map in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[rr, "\theta"] \arrow[dr] & & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Z)) \arrow[dl] \\
& \mathrm{Hom}_{\mathcal{E}}(U(X),U(Z)) &
\end{tikzcd}$$

is a homotopy equivalence. Since the vertical maps are Kan fibrations (Proposition 4.6.1.21), it will suffice to show that the induced map

$$\theta_{\overline{f}} : \mathrm{Hom}_{\mathcal{C}}(X,Z) \times_{\mathrm{Hom}_{\mathcal{E}}(U(X),U(Z))} \{\overline{f}\} \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Z)) \times_{\mathrm{Hom}_{\mathcal{E}}(U(X),U(Z))} \{\overline{f}\}$$

is a homotopy equivalence, for each morphism $\overline{f} : U(X) \to U(Z)$ in the ∞-category $\mathcal{E}$ (Corollary 3.3.7.5). Since U is a cartesian fibration, we can write $\overline{f} = U(f)$, where $f : Y \to Z$ is a U-cartesian morphism in $\mathcal{C}$. By assumption, the image $F(f) : F(Y) \to F(Z)$ is a V-cartesian morphism in the ∞-category $\mathcal{D}$. Using Proposition 5.1.3.11, we can replace $\theta_{\overline{f}}$ with the morphism

$$\mathrm{Hom}_{\mathcal{C}_E}(X,Y) \to \mathrm{Hom}_{\mathcal{D}_E}(F(X),F(Y)),$$

which is a homotopy equivalence by virtue of our assumption that X is F_E-initial. □

02WW **Exercise 7.1.4.20.** Let $U : \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration of ∞-categories, and let $C \in \mathcal{C}$ be an object having image $D = U(C)$ in $\mathcal{D}$. Show that C is U-initial if and only if the following condition is satisfied:

(∗) For every morphism $f : D \to D'$ in $\mathcal{D}$, the covariant transport functor $f_! : \mathcal{C}_D \to \mathcal{C}_{D'}$ carries C to an initial object of the ∞-category $\mathcal{C}_{D'}$.

For a more general statement, see Proposition 7.3.9.2.

02WV **Corollary 7.1.4.21.** *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, and let $C \in \mathcal{C}$ be an object having image $D = U(C)$ in $\mathcal{D}$. If the object C is U-initial, then it is initial when regarded as an object of the ∞-category $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$. The converse holds if U is a cartesian fibration.*

Proof. Apply Proposition 7.1.4.19 in the special case where $\mathcal{E} = \mathcal{D}$ and $\mathcal{E}' = \{D\}$. □

039A **Corollary 7.1.4.22.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a cartesian fibration of ∞-categories. The following conditions are equivalent:*

(1) *For each object $D \in \mathcal{D}$, the ∞-category $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ has an initial object.*

(2) *The functor U is a coreflective localization: that is, it admits a fully faithful left adjoint $F : \mathcal{D} \to \mathcal{C}$.*

Proof. Combine Corollaries 7.1.4.15 and 7.1.4.21. □

7.1.5 Relative Limits and Colimits

We now introduce a relative version of Definition 7.1.2.4. 02KF

Definition 7.1.5.1. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a morphism of simplicial sets with restriction $f = \overline{f}|_K$, so that U induces a functor $U_{/f} : \mathcal{C}_{/f} \to \mathcal{D}_{/(U \circ f)}$. We will say that $\overline{f}$ is a *U-limit diagram* if it is $U_{/f}$-final when viewed as an object of the ∞-category $\mathcal{C}_{/f}$. Similarly, we say that a morphism $\overline{g} : K^{\triangleright} \to \mathcal{C}$ with restriction $g = \overline{g}|_K$ is a *U-colimit diagram* if $\overline{g}$ is $U_{g/}$-initial when viewed as an object of the ∞-category $\mathcal{C}_{g/}$, where $U_{/g} : \mathcal{C}_{g/} \to \mathcal{D}_{(U \circ g)/}$ denotes the functor induced by U. 02KG

Remark 7.1.5.2. Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. Then a morphism $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram if and only if the opposite map $\overline{f}^{\mathrm{op}} : (K^{\mathrm{op}})^{\triangleright} \to \mathcal{C}^{\mathrm{op}}$ is an U^{op}-colimit diagram. 02KH

Example 7.1.5.3. Let $\mathcal{C}$ be an ∞-category and $U : \mathcal{C} \to \Delta^0$ be the projection map. Then a morphism $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram (in the sense of Definition 7.1.5.1) if and only if it is a limit diagram (in the sense of Definition 7.1.2.4). Similarly, a morphism $\overline{g} : K^{\triangleright} \to \mathcal{C}$ is a U-colimit diagram if and only if it is a colimit diagram. 02KK

Example 7.1.5.4. Let $U : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories. Then every morphism $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram, and every morphism $\overline{g} : K^{\triangleright} \to \mathcal{C}$ is a U-colimit diagram. This follows by combining Example 7.1.4.4 with Corollary 4.6.4.20. 02KM

Example 7.1.5.5. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then an object $C \in \mathcal{C}$ is U-final if and only if it is a U-limit diagram when viewed as a morphism of simplicial sets $(\emptyset)^{\triangleleft} \simeq \Delta^0 \to \mathcal{C}$. Similarly, C is U-initial if and only if it is a U-colimit diagram when viewed as a morphism of simplicial sets $(\emptyset)^{\triangleright} \simeq \Delta^0 \to \mathcal{C}$. 02WX

Remark 7.1.5.6. Suppose we are given a commutative diagram of ∞-categories 02WY

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{C}' \arrow[d, "U'"] \\
\mathcal{D} \arrow[r] & \mathcal{D}',
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories. Then a morphism of simplicial sets $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram if and only if $F \circ \overline{f}$ is a U'-limit diagram. Similarly, a morphism of simplicial sets $\overline{g} : K^{\triangleright} \to \mathcal{C}$ is a U-colimit diagram if and only if $F \circ \overline{g}$ is a U'-colimit diagram. This follows by combining Remark 7.1.4.9 with Corollary 4.6.4.19.

02WZ **Remark 7.1.5.7.** Let $U_0 : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $U_1 : \mathcal{C} \to \mathcal{D}$ be a functor which is isomorphic to U_0 (as an object of the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$). Then a diagram $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a U_0-limit diagram if and only if it is a U_1-limit diagram (see Remark 7.1.4.8). This follows by applying Remark 7.1.5.6 to each square of the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[d, "U_0"] & \mathcal{C} \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[d, "U"] & \mathcal{C} \arrow[d, "U_1"] \\
\operatorname{Fun}(\{0\}, \mathcal{D}) & \operatorname{Isom}(\mathcal{D}) \arrow[l, "\mathrm{ev}_0"'] \arrow[r, "\mathrm{ev}_1"] & \operatorname{Fun}(\{1\}, \mathcal{D}),
\end{tikzcd}$$

where $U : \mathcal{C} \to \operatorname{Isom}(\mathcal{D})$ classifies an isomorphism between U_0 and U_1; note that ev_0 and ev_1 are trivial Kan fibrations by virtue of Corollary 4.4.5.10.

02KJ **Remark 7.1.5.8.** Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a morphism, and set $f = \overline{f}|_K$, so that U induces a functor

$$U' : \mathcal{C}_{/\overline{f}} \to \mathcal{C}_{/f} \times_{\mathcal{D}_{/(U \circ f)}} \mathcal{D}_{/(U \circ \overline{f})} .$$

By virtue of Proposition 7.1.4.16, the following conditions are equivalent:

(1) The morphism $\overline{f}$ is a U-limit diagram.

(2) The functor U' is an equivalence of ∞-categories.

If U is an inner fibration of ∞-categories, then the functor U' is automatically a right fibration (Proposition 4.3.6.8). In this case, we can replace (1) and (2) by either of the following conditions:

(3) The functor U' is a trivial Kan fibration.

(4) Each fiber of U' is a contractible Kan complex.

The equivalence of (2) $\Leftrightarrow$ (3) follows from Proposition 4.5.5.20, and the equivalence (3) $\Leftrightarrow$ (4) from Proposition 4.4.2.14.

02KL **Example 7.1.5.9.** Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories. Then:

- A morphism e of $\mathcal{C}$ is U-cartesian (in the sense of Definition 5.1.1.1) if and only if it is a U-limit diagram when viewed as a morphism of simplicial sets $(\Delta^0)^{\triangleleft} \to \mathcal{C}$.
- A morphism f of $\mathcal{C}$ is U-cocartesian (in the sense of Definition 5.1.1.1) if and only if it is a U-colimit diagram when viewed as a morphism of simplicial sets $(\Delta^0)^{\triangleright} \to \mathcal{C}$.

This follows by combining Remark 7.1.5.8 with Proposition 5.1.1.13.

Example 7.1.5.10. Let K be a weakly contractible simplicial set and let $U : \mathcal{C} \to \mathcal{D}$ be a right fibration of ∞-categories. Then every morphism $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram (see Proposition 4.3.7.6). Similarly, if U is a left fibration, then every morphism $\overline{g} : K^{\triangleright} \to \mathcal{C}$ is a U-colimit diagram. 02KN

Remark 7.1.5.11. Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let K be a simplicial set. Using Remark 7.1.5.8, we see that a morphism $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram if and only if every lifting problem 02X0

$$\begin{tikzcd}
\partial\Delta^n \star K \arrow[r, "\rho"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\Delta^n \star K \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

admits a solution, provided that $n \geq 1$ and the the restriction of ρ to $\{n\} \star K \simeq K^{\triangleleft}$ coincides with $\overline{f}$.

Proposition 7.1.5.12. *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$. Then $\overline{f}$ is a U-limit diagram if and only if, for every object $C \in \mathcal{C}$, the diagram of morphism spaces* 02X1

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleleft},\mathcal{C})}(\underline{C}, \overline{f}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{C}|_K, \overline{f}|_K) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleleft},\mathcal{D})}(U \circ \underline{C}, U \circ \overline{f}) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(U \circ \underline{C}|_K, U \circ \overline{f}|_K)
\end{tikzcd} \tag{7.1}$$ 02X2

is a homotopy pullback square; here we let $\underline{C} \in \mathrm{Fun}(K^{\triangleleft}, \mathcal{C})$ denote the constant diagram taking the value C.

Proof. Set $f = \overline{f}|_K$. Note that the restriction maps

$$\mathcal{C}_{/\overline{f}} \to \mathcal{C}_{/f} \qquad \mathcal{C}_{/f} \to \mathcal{C} \qquad \mathcal{D}_{/(U\circ\overline{f})} \to \mathcal{D}_{/(U\circ f)}$$

are right fibrations of simplicial sets (Corollary 4.3.6.11). It follows that we can regard the map

$$U' : \mathcal{C}_{/\overline{f}} \to \mathcal{C}_{/f} \times_{\mathcal{D}_{/(U \circ f)}} \mathcal{D}_{/(U \circ \overline{f})}$$

of Remark 7.1.5.8 as a functor between ∞-categories which are right-fibered over $\mathcal{C}$. Combining Remark 7.1.5.8 with the criterion of Corollary 5.1.6.4, we see that $\overline{f}$ is a U-colimit diagram if and only if, for every object $C \in \mathcal{C}$, the induced map

$$U'_C : \{C\} \times_{\mathcal{C}} \mathcal{C}_{/\overline{f}} \to \{C\} \times_{\mathcal{C}} \mathcal{C}_{/f} \times_{\mathcal{D}_{/(U \circ f)}} \mathcal{D}_{/(U \circ \overline{f})}$$

is a homotopy equivalence of Kan complexes.

To complete the proof, it will suffice to show that U'_C is a homotopy equivalence if and only if the diagram (7.1) is a homotopy pullback square. To see this, we note that Proposition 4.6.5.10 supplies a levelwise homotopy equivalence of (7.1) with the diagram

02X3
$$\begin{tikzcd}
\{C\} \times_{\mathcal{C}} \mathcal{C}_{/\overline{f}} \arrow[r] \arrow[d] & \{C\} \times_{\mathcal{C}} \mathcal{C}_{/f} \arrow[d] \\
\{U(C)\} \times_{\mathcal{D}} \mathcal{D}_{/(U \circ \overline{f})} \arrow[r] & \{U(C)\} \times_{\mathcal{D}} \mathcal{D}_{/(U \circ f)} .
\end{tikzcd} \tag{7.2}$$

It will therefore suffice to show that (7.2) is a homotopy pullback square if and only if U'_C is a homotopy equivalence (Corollary 3.4.1.12). This is a special case of Example 3.4.1.3, since the horizontal maps in the diagram (7.2) are Kan fibrations (combine Corollaries 4.3.6.11 and 4.4.3.8). □

02X4 **Proposition 7.1.5.13.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\overline{u}, \overline{v}' : K^{\triangleleft} \to \mathcal{C}$ be diagrams which are isomorphic when viewed as objects of the ∞-category* $\operatorname{Fun}(K^{\triangleleft}, \mathcal{C})$*. Then $\overline{u}$ is a U-limit diagram if and only if $\overline{v}$ is a U-limit diagram.*

Proof. We proceed as in the proof of Corollary 7.1.2.14. Let $\operatorname{Isom}(\mathcal{C})$ denote the full subcategory of $\operatorname{Fun}(\Delta^1, \mathcal{C})$ spanned by the isomorphisms in $\mathcal{C}$, and define $\operatorname{Isom}(\mathcal{D}) \subseteq \operatorname{Fun}(\Delta^1, \mathcal{D})$ similarly. For $i \in \{0, 1\}$, the evaluation functors

$$\operatorname{ev}_i : \operatorname{Isom}(\mathcal{C}) \to \mathcal{C} \qquad \operatorname{ev}_i : \operatorname{Isom}(\mathcal{D}) \to \mathcal{D}$$

are trivial Kan fibrations (Corollary 4.4.5.10), and therefore equivalences of ∞-categories (Proposition 4.5.3.11). Our assumption that $\overline{u}$ and $\overline{v}$ are isomorphic guarantees that we can choose a diagram $\overline{w} : K^{\triangleleft} \to \operatorname{Isom}(\mathcal{C})$ satisfying $\operatorname{ev}_0 \circ \overline{w} = \overline{u}$ and $\operatorname{ev}_1 \circ \overline{w} = \overline{v}$. Applying

Remark 7.1.5.6 to the commutative diagram

$$\begin{tikzcd}
\operatorname{Isom}(\mathcal{C}) \arrow[r, "\mathrm{ev}_0"] \arrow[d, "U'"] & \mathcal{C} \arrow[d, "U"] \\
\operatorname{Isom}(\mathcal{D}) \arrow[r, "\mathrm{ev}_0"] & \mathcal{D},
\end{tikzcd}$$

we see that $\overline{u}$ is a U-limit diagram if and only if $\overline{w}$ is a U'-limit diagram. A similar argument shows that this is equivalent to the requirement that $\overline{v}$ is a U-limit diagram. □

Proposition 7.1.5.14 (Transitivity)**.** *Let $U : \mathcal{C} \to \mathcal{D}$ and $V : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories.* 02KQ

(1) *Let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a morphism of simplicial sets such that $U \circ \overline{f}$ is a V-limit diagram. Then $\overline{f}$ is a U-limit diagram if and only if it is a $(V \circ U)$-limit diagram.*

(2) *Let $\overline{g} : K^{\triangleright} \to \mathcal{C}$ be a morphism of simplicial sets such that $U \circ \overline{g}$ is a V-colimit diagram. Then $\overline{g}$ is a U-colimit diagram if and only if it is a $(V \circ U)$-colimit diagram.*

Proof. Apply Remark 7.1.4.6. □

Corollary 7.1.5.15. *Let $U : \mathcal{C} \to \mathcal{D}$ and $V : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, where V is fully faithful. Then:* 02X5

(1) *A morphism $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram if and only if it is a $(V \circ U)$-limit diagram.*

(2) *A morphism $\overline{g} : K^{\triangleright} \to \mathcal{C}$ is a U-colimit diagram if and only if it is a $(V \circ U)$-colimit diagram.*

Proof. Combine Proposition 7.1.5.14 with Example 7.1.5.4. □

Corollary 7.1.5.16. *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:* 02KR

(1) *Let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a morphism of simplicial sets such that $U \circ \overline{f}$ is a limit diagram in $\mathcal{D}$. Then $\overline{f}$ is a limit diagram in $\mathcal{C}$ if and only if it is a U-limit diagram.*

(2) *Let $\overline{g} : K^{\triangleright} \to \mathcal{C}$ be a morphism of simplicial sets such that $U \circ \overline{g}$ is a colimit diagram in $\mathcal{D}$. Then $\overline{g}$ is a colimit diagram in $\mathcal{C}$ if and only if it is a U-colimit diagram.*

Proof. Apply Proposition 7.1.5.14 in the case $\mathcal{E} = \Delta^0$ (and use Example 7.1.5.3). □

Corollary 7.1.5.17. *Let K be a weakly contractible simplicial set and let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. If U is a left fibration, then it creates K-indexed colimits. If U is a right fibration, then it creates K-indexed limits.* 02KS

Proof. Assume U is a right fibration; we will show that it creates K-indexed limits (the analogous statement for left fibrations follows by a similar argument). Let $f : K \to \mathcal{C}$ be a diagram and suppose that $U \circ f$ can be extended to a limit diagram $g : K^{\triangleleft} \to \mathcal{D}$. Since the inclusion $K \hookrightarrow K^{\triangleleft}$ is right anodyne (Example 4.3.7.10), our assumption that U is a right fibration guarantees that the lifting problem

$$\begin{tikzcd}
K \arrow[r, "f"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
K^{\triangleleft} \arrow[r, "g"] \arrow[ur, dashed, "\overline{f}"] & \mathcal{D}
\end{tikzcd}$$

has a solution. Since K is weakly contractible, the morphism $\overline{f}$ is automatically a U-limit diagram (Example 7.1.5.10). Applying Corollary 7.1.5.16, we see that $\overline{f}$ is a limit diagram. □

02KT **Corollary 7.1.5.18.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let K be a weakly contractible simplicial set. Then:*

- *If U is a right fibration and the ∞-category $\mathcal{D}$ admits K-indexed limits, then $\mathcal{C}$ also admits K-indexed limits and U preserves K-indexed limits.*
- *If U is a left fibration and the ∞-category $\mathcal{D}$ admits K-indexed colimits, then $\mathcal{C}$ also admits K-indexed colimits and U preserves K-indexed colimits.*

Proof. Combine Corollary 7.1.5.17 with Proposition 7.1.3.18. □

Proposition 7.1.5.19 (Base Change)**.** *Suppose we are given a commutative diagram of ∞-categories* 02X6

$$\begin{tikzcd}
\mathcal{C}' \arrow[rr, "F'"] \arrow[dd, "G"] \arrow[dr, "U'"] & & \mathcal{D}' \arrow[dl] \arrow[dd] \\
& \mathcal{E}' \arrow[dd] & \\
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"] & & \mathcal{D} \arrow[dl, "V"] \\
& \mathcal{E} &
\end{tikzcd} \qquad (7.3)$$

02X7

where each square is a pullback and the diagonal maps are inner fibrations. Let $\overline{f} : K^{\triangleright} \to \mathcal{C}'$ be a morphism of simplicial sets. Then:

(1) *If $G \circ \overline{f}$ is an F-colimit diagram in the ∞-category $\mathcal{C}$, then $\overline{f}$ is an F'-colimit diagram in the ∞-category $\mathcal{C}'$.*

(2) *Assume that U and V are cartesian fibrations, and that the functor F carries U-cartesian morphisms of $\mathcal{C}$ to V-cartesian morphisms of $\mathcal{D}$. If $\overline{f}$ is an F'-colimit diagram in the ∞-category $\mathcal{C}'$, then $G \circ \overline{f}$ is an F-colimit diagram in the ∞-category $\mathcal{C}$.*

Proof. Set $f = \overline{f}|_K$. By virtue of Corollary 4.3.6.10 and Proposition 5.1.4.19, we can replace

(7.3) by the commutative diagram

$$\begin{tikzcd}
\mathcal{C}'_{f/} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{D}'_{(F'\circ f)/} \arrow[dl] \arrow[dd] \\
& \mathcal{E}'_{(U'\circ f)/} \arrow[dd] & \\
\mathcal{C}_{(G\circ f)/} \arrow[rr] \arrow[dr] & & \mathcal{D}_{(F\circ G\circ f)/} \arrow[dl] \\
& \mathcal{E}_{(U\circ G\circ f)/} &
\end{tikzcd}$$

and thereby reduce to the special case $K = \emptyset$. In this case, the desired result follows from Proposition 7.1.4.19. □

02KY **Corollary 7.1.5.20.** *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let $D \in \mathcal{D}$ be an object, and let*

$$\overline{f} : K^{\triangleright} \to \mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$$

be a diagram. If $\overline{f}$ is a U-colimit diagram in $\mathcal{C}$, then it is a colimit diagram in the ∞-category $\mathcal{C}_D$. The converse holds if U is a cartesian fibration.

Proof. Apply Proposition 7.1.5.19 in the special case $\mathcal{E} = \mathcal{D}$ and $\mathcal{E}' = \{D\}$. □

02KZ **Remark 7.1.5.21.** Corollary 7.1.5.20 has an obvious counterpart for U-limit diagrams under the assumption that $U : \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration, which can be proved in the same way. It also has a more subtle counterpart for U-colimit diagrams when U is a cocartesian fibration (or U-limit diagrams when U is a cartesian fibration), which we will discuss in §7.3.9 (see Proposition 7.3.9.2).

Beware that the conclusion of Corollary 7.1.5.20 does not necessarily hold if U is not a cartesian fibration. However, we have the following slightly weaker result:

05J6 **Proposition 7.1.5.22.** *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let $D \in \mathcal{D}$ be an object, and let $\overline{f} : K^{\triangleright} \to \mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ be a diagram. Then $\overline{f}$ is a U-colimit diagram in $\mathcal{C}$ if and only if it satisfies the following condition:*

$(*)$ *For every morphism $e : D \to D'$ in $\mathcal{D}$, the morphism*

$$K^{\triangleleft} \xrightarrow{\overline{f}} \mathcal{C}_D \hookrightarrow \Delta^1 \times_{\mathcal{D}} \mathcal{C}$$

is a U'-colimit diagram, where $U' : \Delta^1 \times_{\mathcal{D}} \mathcal{C} \to \Delta^1$ is given by projection onto the first factor.

Proof. Assume that condition $(*)$ is satisfied; we will show that $\overline{f}$ is a U-colimit diagram (the converse follows from Proposition 7.1.5.19). Set $f = \overline{f}|_K$. By virtue of Proposition 7.1.5.12, it will suffice to show that for each vertex $C \in \mathcal{C}$, the diagram of Kan complexes

05J7

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleleft},\mathcal{C})}(\overline{f}, \underline{C}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(f, \underline{C}|_K) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleright},\mathcal{D})}(U \circ \overline{f}, U \circ \underline{C}) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(U \circ f, U \circ \underline{C}|_K)
\end{tikzcd} \tag{7.4}$$

is a homotopy pullback square, where $\underline{C} \in \mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ is the constant diagram taking the value C. Since U is an inner fibration, the vertical maps in (7.4) are Kan fibrations (Proposition 4.6.1.21 and Corollary 4.1.4.3). Using the criterion of Example 3.4.1.4, it will suffice to show that for every vertex $u \in \mathrm{Hom}_{\mathrm{Fun}(K^{\triangleright},\mathcal{D})}(U \circ \overline{f}, U \circ \underline{C})$, the induced map

$$\begin{tikzcd}
\{u\} \times_{\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleright},\mathcal{D})}(U\circ\overline{f}, U\circ\underline{C})} \mathrm{Hom}_{\mathrm{Fun}(K^{\triangleright},\mathcal{C})}(\overline{f}, \underline{C}) \arrow[d, "\theta_u"] \\
\{u\} \times_{\mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(U\circ f, U\circ\underline{C}|_K)} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(f, \underline{C}|_K)
\end{tikzcd}$$

is a homotopy equivalence of Kan complexes. Set $D' = U(C)$, so that u can be identified with a morphism of simplicial sets $K^{\triangleright} \to \mathrm{Hom}_{\mathcal{D}}(D, D')$, and the condition that θ_u is a homotopy equivalence depends only on the homotopy class of u. Since the simplicial set $K^{\triangleright}$ is weakly contractible (Example 4.3.7.11), we may assume without loss of generality that $u : K^{\triangleright} \to \mathrm{Hom}_{\mathcal{D}}(D, D')$ is the constant map taking the value e, for some morphism $e : D \to D'$ in $\mathcal{D}$. The desired result now follows from $(*)$. □

Remark 7.1.5.23. In the situation of Proposition 7.1.5.22, we can replace condition $(*)$ by the following: 05J8

$(*')$ For every morphism $e : D \to D'$ in $\mathcal{D}$, the morphism

$$K^{\triangleleft} \xrightarrow{\overline{f}} \mathcal{C}_D \hookrightarrow \Delta^1 \times_{\mathcal{D}} \mathcal{C}$$

is a colimit diagram in the ∞-category $\Delta^1 \times_{\mathcal{D}} \mathcal{C}$.

See Corollary 7.1.5.16.

7.1.6 Limits and Colimits of Functors

02X8 Let $\mathcal{C}$ be an ∞-category and let B be a simplicial set. For every vertex $b \in B$, we let

$$\mathrm{ev}_b : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(\{b\}, \mathcal{C}) \simeq \mathcal{C}$$

denote the functor given by evaluation at b. Our goal in this section is to show that the collection of functors $\{\mathrm{ev}_b\}_{b \in B}$ creates colimits in the following sense:

02X9 **Proposition 7.1.6.1.** *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, and let $f : K \to \mathrm{Fun}(B, \mathcal{C})$ be a diagram. Assume that, for every vertex $b \in B$, the composite diagram*

$$K \xrightarrow{f} \mathrm{Fun}(B, \mathcal{C}) \xrightarrow{\mathrm{ev}_b} \mathcal{C}$$

admits a colimit in the ∞-category $\mathcal{C}$. Then:

(1) *The diagram f admits a colimit in* $\mathrm{Fun}(B, \mathcal{C})$.

(2) *Let $\overline{f} : K^{\triangleright} \to \mathrm{Fun}(B, \mathcal{C})$ be an extension of f. Then $\overline{f}$ is a colimit diagram if and only if, for every vertex $b \in B$, the morphism*

$$K^{\triangleright} \xrightarrow{\overline{f}} \mathrm{Fun}(B, \mathcal{C}) \xrightarrow{\mathrm{ev}_b} \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$.

02XA **Corollary 7.1.6.2.** *Let K be a simplicial set and let $\mathcal{C}$ be an ∞-category which admits K-indexed colimits. Then, for every simplicial set B, the ∞-category* $\mathrm{Fun}(B, \mathcal{C})$ *also admits K-indexed colimits. Moreover, a morphism of simplicial sets $\overline{f} : K^{\triangleright} \to \mathrm{Fun}(B, \mathcal{C})$ is a colimit diagram if and only if, for every vertex $b \in B$, the morphism*

$$K^{\triangleright} \xrightarrow{\overline{f}} \mathrm{Fun}(B, \mathcal{C}) \xrightarrow{\mathrm{ev}_b} \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$.

We will give a proof of Proposition 7.1.6.1 at the end of this section. Our strategy is to deduce Proposition 7.1.6.1 from a pair of more general results which apply to *relative* colimit diagrams (Corollaries 7.1.6.7 and 7.1.6.11). The increased flexibility of the relative setting will allow us to reduce to the case $K = \emptyset$, by virtue of the following:

02XB **Proposition 7.1.6.3.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let K be a simplicial set, and let*

$$U' : \mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C}) \times_{\mathrm{Fun}(K, \mathcal{D})} \mathrm{Fun}(K^{\triangleright}, \mathcal{D})$$

be the restriction map. Then a morphism of simplicial sets $\overline{f} : K^{\triangleright} \to \mathcal{C}$ is a U-colimit diagram if and only if it is U'-initial when viewed as an object of the ∞-category $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$.

Proof. Set $f = \overline{f}|_K$, so that U' restricts to a functor

$$U'' : \{f\} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \to \{U \circ f\} \times_{\mathrm{Fun}(K,\mathcal{D})} \mathrm{Fun}(K^{\triangleright}, \mathcal{D}).$$

We have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{f/} \arrow[r] \arrow[d, "F_{f/}"] & \{f\} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \arrow[d, "U''"] \\
\mathcal{D}_{(F \circ f)/} \arrow[r] & \{F \circ f\} \times_{\mathrm{Fun}(K,\mathcal{D})} \mathrm{Fun}(K^{\triangleright}, \mathcal{D}),
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories (see Example 4.6.6.8). Applying Remark 7.1.4.9, we see that $\overline{f}$ is an U-colimit diagram if and only if it is U''-initial when viewed as an object of the fiber $\{f\} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K^{\triangleright})$.

We have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \arrow[rr, "U'"] \arrow[dr, "V"'] & & \mathrm{Fun}(K, \mathcal{C}) \times_{\mathrm{Fun}(K,\mathcal{D})} \mathrm{Fun}(K^{\triangleright}, \mathcal{D}) \arrow[dl, "V'"] \\
& \mathrm{Fun}(K, \mathcal{C}). &
\end{tikzcd}$$

Applying Corollary 5.3.7.5, we see that V and V' are cartesian fibrations and that U' carries V-cartesian morphisms of $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ to V'-cartesian morphisms of $\mathrm{Fun}(K, \mathcal{C}) \times_{\mathrm{Fun}(K,\mathcal{D})} \mathrm{Fun}(K^{\triangleright}, \mathcal{D})$. It follows from Proposition 7.1.4.19, that $\overline{f}$ is U''-initial (when regarded as an object of $\{f\} \times_{\mathrm{Fun}(K,\mathcal{C})} \mathrm{Fun}(K^{\triangleright}, \mathcal{C})$) if and only if it is U'-initial (when viewed as an object of $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$). □

Remark 7.1.6.4. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\overline{f} : K^{\triangleright} \to \mathcal{C}$ be a morphism of simplicial sets having restriction $f = \overline{f}|_K$. Proposition 7.1.6.3 asserts that $\overline{f}$ is a U-colimit diagram if and only if, for every diagram $\overline{g} : K^{\triangleright} \to \mathcal{C}$ having restriction $g = \overline{g}|_K$, the diagram of Kan complexes 02XC

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleright}, \mathcal{C})}(\overline{f}, \overline{g}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(K, \mathcal{C})}(f, g) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleright}, \mathcal{D})}(U \circ \overline{f}, U \circ \overline{g}) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(K, \mathcal{D})}(U \circ f, U \circ g)
\end{tikzcd}$$

is a homotopy pullback square. However, it suffices to verify this condition in the special case where $\overline{g}$ is a *constant* diagram: that is the content of Proposition 7.1.5.12.

02XD **Corollary 7.1.6.5.** *Let $\mathcal{C}$ be an ∞-category, let K be a simplicial set, and let*

$$U : \mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$$

denote the restriction map. Then a morphism of simplicial sets $\overline{f} : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram if and only if it is U-initial when viewed as an object of the ∞-category $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$.

Proof. Apply Proposition 7.1.6.3 in the special case $\mathcal{D} = \Delta^0$. □

02XE **Corollary 7.1.6.6.** *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let B and K be simplicial sets, and let $A \subseteq B$ be a simplicial subset which contains every vertex of B. Suppose we are given a lifting problem*

02XF

$$\begin{tikzcd}
(B \times K) \coprod_{(A \times K)} (A \times K^{\triangleright}) \arrow[r, "f"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
B \times K^{\triangleright} \arrow[r] \arrow[ur, dashed, "\overline{f}"] & \mathcal{D}
\end{tikzcd} \tag{7.5}$$

which satisfies the following condition:

- (∗) *Let $\sigma : \Delta^n \to B$ be an n-simplex which does not belong to A, and let $a = \sigma(0)$ be the initial vertex. Then the restriction*

$$f_a = f|_{\{a\} \times K^{\triangleright}} : K^{\triangleright} \to \mathcal{C}$$

is a U-colimit diagram.

Then the lifting problem (7.5) admits a solution $\overline{f} : B \times K^{\triangleright} \to \mathcal{C}$.

Proof. Set $\mathcal{C}' = \mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ and $\mathcal{D}' = \mathrm{Fun}(K, \mathcal{C}) \times_{\mathrm{Fun}(K, \mathcal{D})} \mathrm{Fun}(K^{\triangleright}, \mathcal{D})$, so that U induces an inner fibration $U' : \mathcal{C}' \to \mathcal{D}'$ (Proposition 4.1.4.1). We can then rewrite (7.5) as a lifting problem

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d] & \mathcal{C}' \arrow[d, "U'"] \\
B \arrow[r, "g_0"] \arrow[ur, dashed] & \mathcal{D}'.
\end{tikzcd}$$

Let P be the partially ordered set of pairs (A', g'), where $A' \subseteq B$ is a simplicial subset containing A, and $g' : A' \to \mathcal{C}'$ is a morphism satisfying $g'|_A = g$ and $U' \circ g' = g_0|_{A'}$. The partially ordered set P satisfies the hypotheses of Zorn's lemma and therefore contains a maximal element $(A_{\max}, g_{\max})$. To complete the proof, it will suffice to show that $A_{\max} = B$. Assume otherwise: then there exists some n-simplex $\sigma : \Delta^n \to B$ which is not contained in $A_{\max}$. Choose n as small as possible, so that σ carries the boundary $\partial \Delta^n$ into $A_{\max}$. Since every vertex of A is contained in B, we must have $n > 0$. Moreover, it follows from $(*)$ together with Proposition 7.1.6.3 that the vertex $a = \sigma(0)$ is a U'-initial object of $\mathcal{C}'$. Applying Corollary 7.1.4.17, we deduce that the lifting problem

$$\begin{tikzcd}
\partial \Delta^n \arrow[r, "g_{\max} \circ \sigma"] \arrow[d] & \mathcal{C}' \arrow[d, "U'"] \\
\Delta^n \arrow[r, "g_0 \circ \sigma"] \arrow[ur, dashed] & \mathcal{D}'
\end{tikzcd}$$

has a solution, which contradicts the maximality of $(A_{\max}, g_{\max})$. □

Corollary 7.1.6.7. *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let B and K be simplicial sets, and suppose we are given a lifting problem* 02XG

$$\begin{tikzcd}
B \times K \arrow[r, "f"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
B \times K^{\triangleright} \arrow[r, "\overline{g}"] \arrow[ur, dashed, "\overline{f}"] & \mathcal{D}
\end{tikzcd} \tag{7.6}$$

02XH

Assume that, for each vertex $b \in B$, the restriction $f|_{\{b\} \times K}$ can be extended to a U-colimit diagram $\overline{f}_b : K^{\triangleright} \to \mathcal{C}$ satisfying $U \circ \overline{f}_b = \overline{g}|_{\{b\} \times K^{\triangleright}}$. Then the lifting problem (7.6) admits a solution $\overline{f} : B \times K^{\triangleright} \to \mathcal{C}$ satisfying $\overline{f}|_{\{b\} \times K^{\triangleright}} = \overline{f}_b$ for each $b \in B$.

Proof. Apply Corollary 7.1.6.6 in the special case where $A = \operatorname{sk}_0(B)$ is the 0-skeleton of B. □

We can now prove a weak form of Proposition 7.1.6.1:

Corollary 7.1.6.8. *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, and let $f : K \to \operatorname{Fun}(B, \mathcal{C})$ be a diagram. Assume that, for every vertex $b \in B$, the diagram* 02XJ

$$K \xrightarrow{f} \operatorname{Fun}(B, \mathcal{C}) \xrightarrow{\operatorname{ev}_b} \mathcal{C}$$

has a colimit in $\mathcal{C}$. Then f can be extended to a morphism $\overline{f} : K^{\triangleright} \to \operatorname{Fun}(B, \mathcal{C})$ having the property that each composition $K^{\triangleright} \xrightarrow{\overline{f}} \operatorname{Fun}(B, \mathcal{C}) \xrightarrow{\operatorname{ev}_b} \mathcal{C}$ is a colimit diagram in $\mathcal{C}$.

Proof. Apply Corollary 7.1.6.7 in the special case $\mathcal{D} = \Delta^0$. □

To complete the proof of Proposition 7.1.6.1, we must show that the morphism $\overline{f} : K^{\triangleright} \to \mathrm{Fun}(B, \mathcal{C})$ appearing in the statement of Corollary 7.1.6.8 is a colimit diagram. As above, it will be convenient to deduce this from a stronger assertion about relative colimit diagrams.

02XK **Proposition 7.1.6.9.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Let B be a simplicial set and let A be a simplicial subset, so that F induces a functor*

$$F' : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B, \mathcal{D}).$$

Suppose we are given a diagram $\overline{f} : K^{\triangleright} \to \mathrm{Fun}(B, \mathcal{C})$ satisfying the following condition:

($*$) *Let $\sigma : \Delta^n \to B$ be an n-simplex of B which is not contained in A and set $b = \sigma(0)$. Then the composite map $K^{\triangleright} \xrightarrow{\overline{f}} \mathrm{Fun}(B, \mathcal{C}) \xrightarrow{\mathrm{ev}_b} \mathcal{C}$ is an F-colimit diagram in the ∞-category $\mathcal{C}$.*

Then $\overline{f}$ is an F'-colimit diagram in the ∞-category $\mathrm{Fun}(B, \mathcal{C})$.

Proof. As in the proof of Corollary 7.1.6.6, we can replace F by the restriction functor

$$\mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C}) \times_{\mathrm{Fun}(K,\mathcal{D})} \mathrm{Fun}(K^{\triangleright}, \mathcal{D})$$

and thereby reduce to the special case $K = \emptyset$ (Proposition 7.1.6.3). In this case, we view $\overline{f}$ as an object of the ∞-category $\mathrm{Fun}(B, \mathcal{C})$, and we wish to show that this object is F'-initial.

Using Proposition 4.1.3.2, we can factor F as a composition $\mathcal{C} \xrightarrow{G} \mathcal{E} \xrightarrow{U} \mathcal{D}$, where U is an inner fibration (so that $\mathcal{E}$ is an ∞-category) and G is inner anodyne (and therefore an equivalence of ∞-categories). Note that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(B, \mathcal{C}) \arrow[r, "F'"] \arrow[d, "G\circ"] & \mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}(A, \mathcal{C}) \arrow[d, "G\circ"] \\
\mathrm{Fun}(B, \mathcal{E}) \arrow[r, "U'"] & \mathrm{Fun}(A, \mathcal{E}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B, \mathcal{D}) \arrow[r] & \mathrm{Fun}(A, \mathcal{E}),
\end{tikzcd}$$

where the vertical maps on the left and right are equivalences of ∞-categories (Remark 4.5.1.16). Since the square on the right is a pullback diagram and the right horizontal maps are isofibrations (Corollary 4.4.5.3), it follows that the vertical map in the middle is also an equivalence of ∞-categories (Corollary 4.5.2.29). Consequently, to show that $\overline{f}$ is F'-initial, it will suffice to show that $G \circ \overline{f}$ is U'-initial when viewed as an object of $\mathrm{Fun}(B, \mathcal{E})$ (Remark

7.1.4.9). Since U' is an inner fibration (Proposition 4.1.4.1), it will suffice to verify that $\overline{f}$ satisfies the criterion of Corollary 7.1.4.17: every lifting problem

02XL

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{Fun}(B,\mathcal{E}) \arrow[d, "U'"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(A,\mathcal{E}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B,\mathcal{D})
\end{tikzcd} \tag{7.7}$$

has a solution, provided that $n > 0$ and $\sigma_0(0) = \overline{f}$. Unwinding the definitions, we can rewrite (7.7) as a lifting problem

$$\begin{tikzcd}
(\partial\Delta^n \times B)\coprod_{(\partial\Delta^n \times A)}(\Delta^n \times B) \arrow[r, "g"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^n \times B \arrow[r] & \mathcal{D}.
\end{tikzcd}$$

Since $n > 0$, every vertex of the simplicial set $\Delta^n \times B$ is contained in $\partial\Delta^n \times B$. Moreover, if $\tau : \Delta^m \to \Delta^n \times B$ is an m-simplex which does not belong to $(\partial\Delta^n \times B)\coprod_{(\partial\Delta^n \times A)}(\Delta^n \times B)$, then condition $(*)$ (and Remark 7.1.4.9) guarantee that g carries $\tau(0)$ to a U'-initial vertex of $\mathcal{E}$. The existence of the desired solution now follows from Corollary 7.1.6.6 (applied in the special case $K = \emptyset$). □

Corollary 7.1.6.10. *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, let $A \subseteq B$ be a simplicial subset, and let $U : \mathrm{Fun}(B,\mathcal{C}) \to \mathrm{Fun}(A,\mathcal{C})$ be the restriction functor. Let $\overline{f} : K^{\triangleright} \to \mathrm{Fun}(B,\mathcal{C})$ be a diagram satisfying the following condition:* 02XM

- $(*)$ *Let $\sigma : \Delta^n \to B$ be an n-simplex of B which is not contained in A and set $b = \sigma(0)$. Then the composite map $K^{\triangleright} \xrightarrow{\overline{f}} \mathrm{Fun}(B,\mathcal{C}) \xrightarrow{\mathrm{ev}_b} \mathcal{C}$ is a colimit diagram in the ∞-category $\mathcal{C}$.*

Then $\overline{f}$ is a U-colimit diagram in the ∞-category $\mathrm{Fun}(B,\mathcal{C})$.

Proof. Apply Proposition 7.1.6.9 in the special case $\mathcal{D} = \Delta^0$. □

Corollary 7.1.6.11. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let B be a simplicial set, and let $F' : \mathrm{Fun}(B,\mathcal{C}) \to \mathrm{Fun}(B,\mathcal{D})$ be given by composition with F. Let $\overline{f} : K^{\triangleright} \to \mathrm{Fun}(B,\mathcal{C})$ be a diagram. Assume that, for every vertex $b \in B$, the composition* 02XN

$$K^{\triangleright} \to \mathrm{Fun}(B,\mathcal{C}) \xrightarrow{\mathrm{ev}_b} \mathcal{C}$$

is an F-colimit diagram in the ∞-category $\mathcal{C}$. Then $\overline{f}$ is an F'-colimit diagram in the ∞-category $\operatorname{Fun}(B, \mathcal{C})$*.*

Proof. Apply Proposition 7.1.6.9 in the special case $A = \emptyset$. □

02XP **Corollary 7.1.6.12.** *Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, and let $\overline{f} : K^{\triangleright} \to \operatorname{Fun}(B, \mathcal{C})$ be a diagram. Assume that, for each vertex $b \in B$, the composite map $K^{\triangleright} \xrightarrow{\overline{f}} \operatorname{Fun}(B, \mathcal{C}) \xrightarrow{\operatorname{ev}_b} \mathcal{C}$ is a colimit diagram in $\mathcal{C}$. Then $\overline{f}$ is a colimit diagram in* $\operatorname{Fun}(B, \mathcal{C})$*.*

Proof. Apply Corollary 7.1.6.11 in the special case $\mathcal{D} = \Delta^0$ (or Corollary 7.1.6.10) in the special case $A = \emptyset$. □

Proof of Proposition 7.1.6.1. Let $\mathcal{C}$ be an ∞-category, let B be a simplicial set, and let $f : K \to \operatorname{Fun}(B, \mathcal{C})$ be a diagram. Assume that, for every vertex $b \in B$, the composite diagram

$$K \xrightarrow{f} \operatorname{Fun}(B, \mathcal{C}) \xrightarrow{\operatorname{ev}_b} \mathcal{C}$$

admits a colimit in the ∞-category $\mathcal{C}$. Applying Corollary 7.1.6.8, we see that f admits an extension $\overline{f} : K^{\triangleright} \to \operatorname{Fun}(B, \mathcal{C})$ with the property that, for every vertex $b \in B$, the composition $\operatorname{ev}_b \circ \overline{f}$ is a colimit diagram in $\mathcal{C}$. Applying Corollary 7.1.6.12, we see any such extension is a colimit diagram in $\operatorname{Fun}(B, \mathcal{C})$. To complete the proof, it will suffice to show the converse: if $\overline{f}' : K^{\triangleright} \to \operatorname{Fun}(B, \mathcal{C})$ is any colimit diagram extending f and $b \in B$ is a vertex, then $\operatorname{ev}_b \circ \overline{f}'$ is also a colimit diagram in $\mathcal{C}$. In this case, the extension $\overline{f}'$ is isomorphic to $\overline{f}$ as an object of the ∞-category $\operatorname{Fun}(K^{\triangleright}, \operatorname{Fun}(B, \mathcal{C}))$. It follows that $\operatorname{ev}_b \circ \overline{f}'$ is isomorphic to $\operatorname{ev}_b \circ \overline{f}$ as an object of the ∞-category $\operatorname{Fun}(K^{\triangleright}, \mathcal{C})$ and therefore a colimit diagram by virtue of Corollary 7.1.2.14. □

7.2 Cofinality

02MZ Let $\mathcal{C}$ be an ∞-category and let $f : B \to \mathcal{C}$ be a diagram in $\mathcal{C}$ indexed by a simplicial set B. In §7.1, we introduced the definition of a *limit* $\varprojlim(f)$ and *colimit* $\varinjlim(f)$ of the diagram f (Definition 7.1.1.11). In practice, it is often convenient to replace f by a simpler diagram having the same limit (or colimit). The primary goal of this section is to introduce a general formalism which will allow us to make replacements of this sort.

We begin in §7.2.1 by introducing the notions of *left cofinal* and *right cofinal* morphisms of simplicial sets (Definition 7.2.1.1). Roughly speaking, one can regard left cofinality as a homotopy-invariant replacement for the notion of left anodyne morphism introduced in Definition 4.2.4.1. More precisely, the collection of left cofinal morphisms of simplicial sets is uniquely determined by the following assertions:

- A monomorphism of simplicial sets $f : A \hookrightarrow B$ is left cofinal if and only if it is left anodyne (Proposition 7.2.1.3).
- Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] & B', \end{tikzcd}$$

where the vertical maps are categorical equivalences. Then f is left cofinal if and only if f' is left cofinal (Corollary 7.2.1.22).

In §7.2.2, we connect the notion of cofinality with the theory of limits and colimits developed in §7.1. Let $\mathcal{C}$ be an ∞-category, and let $g : B \to \mathcal{C}$ be a diagram in $\mathcal{C}$. We will show that if $f : A \to B$ is a left cofinal morphism of simplicial sets, then the limit of the diagram g (if it exists) can be identified with with the limit of the composite diagram $(g \circ f) : A \to \mathcal{C}$ (Corollary 7.2.2.11). Similarly, if f is right cofinal, then the colimit of g can be identified with the colimit of $g \circ f$. Consequently, cofinality is a very useful tool for computing (or verifying the existence of) limits and colimits.

In §7.2.3, we specialize to the study of cofinal functors between ∞-categories. Our main result asserts that a functor $F : \mathcal{C} \to \mathcal{D}$ is right cofinal if and only if, for every object $D \in \mathcal{D}$, the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible (Theorem 7.2.3.1). In particular, the weak contractibility of each slice $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ guarantees that F is a weak homotopy equivalence of simplicial sets: this is an ∞-categorical generalization of Quillen's "Theorem A" (see Example 7.2.3.3). We will deduce Theorem 7.2.3.1 from a general fact about the stability of right cofinality with respect to pullback along cocartesian fibrations (Proposition 7.2.3.12), which is of independent interest.

We devote the second half of this section to studying properties of ∞-categories which are closely related to the notion of cofinality. We say that an ∞-category $\mathcal{C}$ is *filtered* if, for every finite simplicial set K and every diagram $f : K \to \mathcal{C}$, the coslice ∞-category $\mathcal{C}_{f/}$ is nonempty (Definition 7.2.4.3). In §7.2.4, we show that if this property is satisfied for *every* finite simplicial set K, then one can say more: every such coslice ∞-category $\mathcal{C}_{f/}$ is weakly contractible. It follows that $\mathcal{C}$ is filtered if and only if the diagonal map $\mathcal{C} \to \operatorname{Fun}(K, \mathcal{C})$ is right cofinal for every finite simplicial set K (Proposition 7.2.4.10).

To show that an ∞-category $\mathcal{C}$ is filtered, it is not necessary to show that the coslice ∞-category $\mathcal{C}_{f/}$ is nonempty for *every* finite diagram $f : K \to \mathcal{C}$. In §7.2.5, we show that it suffices to verify this condition in the case where $K = \partial \Delta^n$ is the boundary of a standard simplex, for each $n \geq 0$ (Lemma 7.2.5.13). Using this observation, we show that the condition that an ∞-category $\mathcal{C}$ is filtered can be formulated entirely at the level of the homotopy category $\mathrm{h}\mathcal{C}$, viewed as an hKan-enriched category (Theorem 7.2.5.5). As an application, we show that our notion of filtered ∞-category generalizes the classical notion of a filtered category: that is, an ordinary category $\mathcal{C}$ is filtered if and only if the nerve $\mathrm{N}_\bullet(\mathcal{C})$ is a filtered ∞-category (Corollary 7.2.5.8). We also formulate a counterpart of this result for the homotopy coherent nerve of a locally Kan simplicial category (Corollary 7.2.5.10).

Our primary interest in the notion of filtered ∞-category stems from the exactness properties enjoyed by filtered colimits. We will see later that a small ∞-category $\mathcal{C}$ is filtered if and only if the colimit functor $\varinjlim : \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \to \mathcal{S}$ preserves finite limits (Theorem [?]). In §7.2.6 we establish a version of this statement, which reformulates the condition that $\mathcal{C}$ is filtered in terms of fiber products of ∞-categories which are left-fibered over $\mathcal{C}$ (Corollary 7.2.6.3). As a consequence, we show that if $F : \mathcal{C}' \to \mathcal{C}$ is a right cofinal functor of ∞-categories where $\mathcal{C}'$ is filtered, then $\mathcal{C}$ is also filtered (Proposition 7.2.7.1). In §7.2.7, we establish a partial converse to this assertion: if $\mathcal{C}$ is a filtered ∞-category, then there exists a directed partially ordered set $(A, \leq)$ and a right cofinal functor $\mathrm{N}_\bullet(A) \to \mathcal{C}$ (Theorem 7.2.7.2).

For many applications, it will be useful to consider a generalization of the notion of filtered ∞-category. In §7.2.8, we introduce the larger class of *sifted* simplicial sets. We say that a simplicial set K is *sifted* if, for every finite set I, the diagonal map $\delta : K \to K^I$ is right cofinal (Definition 7.2.8.1). Equivalently, a simplicial set K is sifted if it is weakly contractible and the diagonal $K \hookrightarrow K \times K$ is right cofinal (Proposition 7.2.8.8). Every filtered ∞-category is sifted (Example 7.2.8.4), but the converse is false: for example, the ∞-category $\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}$ is sifted (Proposition 7.2.8.10), but is not filtered.

7.2.1 Cofinal Morphisms of Simplicial Sets

02N0 Recall that a morphism of simplicial sets $f : A \to B$ is *left anodyne* if, for every left fibration $q : X \to S$, every lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & X \arrow[d, "q"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

admits a solution (Proposition 4.2.4.5). Beware that this condition can only be satisfied if f is a monomorphism of simplicial sets, and is therefore not invariant under categorical equivalence. Our goal in this section is to introduce an enlargement of the collection of left anodyne morphisms which does not suffer from this defect.

Definition 7.2.1.1 (Joyal)**.** Let $f : A \to B$ be a morphism of simplicial sets. We say that f is *left cofinal* if, for every left fibration $q : \widetilde{B} \to B$, precomposition with f induces a homotopy equivalence of Kan complexes $\mathrm{Fun}_{/B}(B, \widetilde{B}) \to \mathrm{Fun}_{/B}(A, \widetilde{B})$ (see Corollary 4.4.2.5). We say that f is *right cofinal* if, for every right fibration $q : \widetilde{B} \to B$, precomposition with f induces a homotopy equivalence of Kan complexes $\mathrm{Fun}_{/B}(B, \widetilde{B}) \to \mathrm{Fun}_{/B}(A, \widetilde{B})$. `02N1`

Remark 7.2.1.2. Let $f : A \to B$ be a morphism of simplicial sets. Then f is left cofinal if and only if the opposite morphism $f^{\mathrm{op}} : A^{\mathrm{op}} \to B^{\mathrm{op}}$ is right cofinal. `02N2`

Proposition 7.2.1.3. *Let $f : A \to B$ be a morphism of simplicial sets. Then f is left anodyne if and only if it is a left cofinal monomorphism. Similarly, f is right anodyne if and only if it is a right cofinal monomorphism.* `02N3`

Proof. We will prove the first assertion; the second follows by a similar argument. Assume first that f is left anodyne. Then f is a monomorphism (Remark 4.2.4.4). For every left fibration of simplicial sets $\widetilde{B} \to B$, the restriction map $\theta : \mathrm{Fun}_{/B}(B, \widetilde{B}) \to \mathrm{Fun}_{/B}(A, \widetilde{B})$ is a pullback of the map

$$\mathrm{Fun}(B, \widetilde{B}) \to \mathrm{Fun}(B, B) \times_{\mathrm{Fun}(A,B)} \mathrm{Fun}(A, \widetilde{B}),$$

and is therefore a trivial Kan fibration (Proposition 4.2.5.4). In particular, u is a homotopy equivalence (Proposition 3.1.6.10). Allowing $\widetilde{B}$ to vary, we conclude that f is left cofinal.

We now prove the converse. Assume that f is a left cofinal monomorphism; we wish to show that f is left anodyne. By virtue of Proposition 4.2.4.5, it will suffice to show that every lifting problem

`02N4`

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & X \arrow[d, "q"] \\
B \arrow[r, "g"] \arrow[ur, dashed] & S
\end{tikzcd} \qquad (7.8)$$

admits a solution, provided that q is a left fibration of simplicial sets. Let us regard the morphism g as fixed, and consider the restriction map

$$\theta : \mathrm{Fun}_{/B}(B, X \times_S B) \to \mathrm{Fun}_{/B}(A, X \times_S B).$$

Since f is a monomorphism, the morphism θ is a left fibration (Proposition 4.2.5.1). Since the target simplicial set $\mathrm{Fun}_{/B}(A, X \times_S B)$ is a Kan complex (Corollary 4.4.2.5), it follows that θ is a Kan fibration (Corollary 4.4.3.8). Our assumption that f is left cofinal guarantees that θ is a homotopy equivalence, and therefore a trivial Kan fibration (Proposition 3.2.7.2). In particular, it is surjective at the level of vertices, which guarantees that (7.8) admits a solution. □

03LQ **Example 7.2.1.4.** Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. Then the inclusion map $\{X\} \hookrightarrow \mathcal{C}$ is right cofinal if and only if X is a final object of $\mathcal{C}$. This follows by combining Proposition 7.2.1.3 with Corollary 4.6.7.24. Similarly, the inclusion map $\{X\} \hookrightarrow \mathcal{C}$ is left cofinal if and only if X is an initial object of $\mathcal{C}$.

02N5 **Proposition 7.2.1.5.** *Let $f : A \to B$ be a morphism of simplicial sets. Then:*

(1) *If f is either left cofinal or right cofinal, then it is a weak homotopy equivalence.*

(2) *If f is a weak homotopy equivalence and B is a Kan complex, then f is left and right cofinal.*

Proof. We first prove (1). Let X be a Kan complex. Then the projection map $X \times B \to B$ is a Kan fibration (Remark 3.1.1.6), and therefore both a left and a right fibration (Example 4.2.1.5). Consequently, if f is either left cofinal or right cofinal, the induced map

$$\mathrm{Fun}(B, X) \simeq \mathrm{Fun}_{/B}(B, X \times B) \to \mathrm{Fun}_{/B}(A, X \times B) \simeq \mathrm{Fun}(A, X)$$

is a homotopy equivalence of Kan complexes. Allowing X to vary, we conclude that f is a weak homotopy equivalence.

We now prove (2). Assume that B is a Kan complex and that f is a weak homotopy equivalence; we will show that f is left cofinal (the proof that f is right cofinal is similar). Let $q : \widetilde{B} \to B$ be a left fibration. Since B is a Kan complex, q is a Kan fibration (Corollary 4.4.3.8); in particular, $\widetilde{B}$ is a Kan complex. Applying Corollary 3.1.3.4, we obtain a commutative diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Fun}(B, \widetilde{B}) \arrow[r, "\circ f"] \arrow[d, "q\circ"] & \mathrm{Fun}(A, \widetilde{B}) \arrow[d, "q\circ"] \\
\mathrm{Fun}(B, B) \arrow[r, "\circ f"] & \mathrm{Fun}(A, B),
\end{tikzcd}$$

where the vertical maps are Kan fibrations (Corollary 3.1.3.2). Our assumption that f is a weak homotopy equivalence guarantees that the horizontal maps are homotopy equivalences (Proposition 3.1.6.17). Applying Proposition 3.2.8.1, we deduce that the map $\mathrm{Fun}_{/B}(B, \widetilde{B}) \to \mathrm{Fun}_{/B}(A, \widetilde{B})$ is also a homotopy equivalence. □

02N6 **Proposition 7.2.1.6.** *Let $f : A \to B$ and $g : B \to C$ be morphisms of simplicial sets, and suppose that f is left cofinal. Then g is left cofinal if and only if the composite map $g \circ f$ is left cofinal. In particular, the collection of left cofinal morphisms is closed under composition.*

Proof. Let $q : \widetilde{C} \to C$ be a left fibration of simplicial sets, and let

$$\mathrm{Fun}_{/C}(C, \widetilde{C}) \xrightarrow{g^*} \mathrm{Fun}_{/C}(B, \widetilde{C}) \xrightarrow{f^*} \mathrm{Fun}_{/C}(A, \widetilde{C})$$

be the morphisms given by precomposition with g and f. Our assumption that f is left cofinal guarantees that f^* is a homotopy equivalence. It follows that g^* is a homotopy equivalence if and only if $f^* \circ g^*$ is a homotopy equivalence (Remark 3.1.6.7). □

Corollary 7.2.1.7. *Let $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$ be monomorphisms of simplicial sets. If both f and $g \circ f$ are left anodyne, then g is left anodyne. If f and $g \circ f$ are right anodyne, then g is right anodyne.* `02N7`

Proof. Combine Propositions 7.2.1.6 and 7.2.1.3. □

Warning 7.2.1.8. Let $g : \Delta^1 \to \Delta^0$ be the projection map and let $f : \{1\} \hookrightarrow \Delta^1$ be the inclusion. Then g and $g \circ f$ are left cofinal (Proposition 7.2.1.5). However, the morphism f is *not* left cofinal, since it is not left anodyne (see Example 4.2.4.6). Consequently, the collection of left cofinal morphisms does not satisfy the two-out-of-three property. `02N8`

Corollary 7.2.1.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let X be an initial object of $\mathcal{C}$. Then F is left cofinal if and only if $F(X)$ is an initial object of $\mathcal{D}$.* `043E`

Proof. Combine Proposition 7.2.1.6 with Example 7.2.1.4. □

Proposition 7.2.1.10. *Let A be a simplicial set, let W be a collection of edges of A, and let $f : A \to B$ be a morphism of simplicial sets which exhibits B as a localization of A with respect to W (see Definition 6.3.1.9). Then f is both left and right cofinal.* `02N9`

Proof. We will show that f is left cofinal; the proof that f is right cofinal is similar. Let $q : \widetilde{B} \to B$ be a left fibration; we wish to show that composition with f induces a homotopy equivalence $f^* : \mathrm{Fun}_{/B}(B, \widetilde{B}) \to \mathrm{Fun}_{/B}(A, \widetilde{B})$. Applying Corollary 5.6.7.3 (and Remark 5.6.7.4), we deduce that there exists a pullback diagram of simplicial sets

$$\begin{tikzcd}
\widetilde{B} \arrow[r] \arrow[d, "q"] & \widetilde{\mathcal{C}} \arrow[d, "Q"] \\
B \arrow[r, "g"] & \mathcal{C},
\end{tikzcd}$$

where Q is a left fibration of ∞-categories. Let $\mathrm{Fun}(A[W^{-1}], \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(A, \mathcal{C})$ spanned by those diagrams which carry each edge of W to an isomorphism in

$\mathcal{C}$ (Notation 6.3.1.1), and define $\operatorname{Fun}(A[W^{-1}], \widetilde{\mathcal{C}})$ similarly. We have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\operatorname{Fun}(B, \widetilde{\mathcal{C}}) \arrow[r, "\circ f"] \arrow[d, "Q\circ"] & \operatorname{Fun}(A[W^{-1}], \widetilde{\mathcal{C}}) \arrow[r] \arrow[d, "Q\circ"] & \operatorname{Fun}(A, \widetilde{\mathcal{C}}) \arrow[d, "Q\circ"] \\
\operatorname{Fun}(B, \mathcal{C}) \arrow[r, "\circ f"] & \operatorname{Fun}(A[W^{-1}], \mathcal{C}) \arrow[r] & \operatorname{Fun}(A, \mathcal{C}),
\end{tikzcd}$$

where the vertical maps on both sides are left fibrations (Corollary 4.2.5.2). Since Q is a left fibration of ∞-categories, it is conservative (Proposition 4.4.2.11), so the right side of the diagram is a pullback square. In particular, the vertical map in the middle is also a left fibration. Our assumption that f exhibits B as a localization of A with respect to W guarantees that the left horizontal maps are equivalences of ∞-categories. Applying Corollary 4.5.2.32, we conclude that the map of fibers

$$\begin{aligned}
\operatorname{Fun}_{/B}(B, \widetilde{B}) \simeq \{g\} \times_{\operatorname{Fun}(B,\mathcal{C})} \operatorname{Fun}(B, \widetilde{\mathcal{C}}) &\rightarrow \{g \circ f\} \times_{\operatorname{Fun}(A[W^{-1}],\mathcal{C})} \operatorname{Fun}(A[W^{-1}], \widetilde{\mathcal{C}}) \\
&= \{g \circ f\} \times_{\operatorname{Fun}(A,\mathcal{C})} \operatorname{Fun}(A, \widetilde{\mathcal{C}}) \\
&\simeq \operatorname{Fun}_{/B}(A, \widetilde{B})
\end{aligned}$$

is an equivalence of ∞-categories, and therefore a homotopy equivalence of Kan complexes (Example 4.5.1.13). □

02NA **Corollary 7.2.1.11.** *Let $f : A \twoheadrightarrow B$ be a universally localizing morphism of simplicial sets (see Definition 6.3.6.1). Then f is both left and right cofinal.*

02NB **Corollary 7.2.1.12.** *Let $\mathcal{C}$ be a simplicial set. Then there exists a partially ordered set $(A, \leq)$ and a morphism of simplicial sets $F : \mathrm{N}_{\bullet}(A) \to \mathcal{C}$ which is both left and right cofinal. Moreover, if the simplicial set $\mathcal{C}$ is finite, then we can arrange that the partially ordered set $(A, \leq)$ is finite.*

Proof. Combine Theorem 6.3.7.1 (and Variant 6.3.7.17) with Corollary 7.2.1.11. □

02NC **Corollary 7.2.1.13.** *Let $f : A \to B$ be a categorical equivalence of simplicial sets. Then f is left cofinal and right cofinal.*

Proof. Combine Proposition 7.2.1.10 with Example 6.3.1.12. □

02ND **Corollary 7.2.1.14.** *Let $q : X \to S$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism q is left cofinal and a left fibration.*

(2) *The morphism q is right cofinal and a right fibration.*

(3) *The morphism q is a trivial Kan fibration.*

Proof. If q is a trivial Kan fibration, then it is both a left fibration and a right fibration (Example 4.2.1.5). Moreover, q is also a categorical equivalence of simplicial sets (Proposition 4.5.3.11), hence left and right cofinal by virtue of Corollary 7.2.1.13. This proves the implications (3) ⇒ (1) and (3) ⇒ (2).

We will complete the proof by showing that (1) ⇒ (3) (the proof of the implication (2) ⇒ (3) is similar). Assume that q is a left cofinal left fibration. Then composition with q induces a homotopy equivalence of Kan complexes $\mathrm{Fun}_{/S}(S, X) \to \mathrm{Fun}_{/S}(X, X)$. In particular, the morphism q admits a section $f : S \to X$ such that id_X and $f \circ q$ belong to the same connected component of $\mathrm{Fun}_{/S}(X, X)$. For each vertex $s \in S$, let $X_s = \{s\} \times_S X$ be the fiber of q over s. Then the identity map $\mathrm{id} : X_s \to X_s$ is homotopic to the constant map $X_s \to \{f(s)\} \hookrightarrow X_s$. It follows that the Kan complex X_s is contractible. Allowing s to vary, we conclude that the left fibration q is a trivial Kan fibration (Proposition 4.4.2.14). □

Corollary 7.2.1.15. *Let $f : X \to Z$ be a morphism of simplicial sets. Then f is left cofinal if and only if it factors as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$, where f' is left anodyne and f'' is a trivial Kan fibration.* 02NE

Proof. Suppose first that we can write $f = f'' \circ f'$, where f' is left anodyne and f'' is a trivial Kan fibration. Proposition 7.2.1.3 guarantees that f' is left cofinal, and Proposition 7.2.1.5 guarantees that f'' is left cofinal. Applying Proposition 7.2.1.6, we conclude that f is also left cofinal.

We now prove the converse. Assume that $f : X \to Z$ is left cofinal. Applying Proposition 4.2.4.7, we can write f as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$, where f' is left anodyne and f'' is a left fibration. Then f' is also left cofinal (Proposition 7.2.1.3). Applying Proposition 7.2.1.6, we deduce that f'' is left cofinal. It then follows from Corollary 7.2.1.14 that f'' is a trivial Kan fibration. □

Corollary 7.2.1.16. *Suppose we are given a categorical pushout diagram of simplicial sets* 02NF

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] \\ X' \arrow[r, "f'"] & Z'. \end{tikzcd} \tag{7.9}$$ 02NG

If f is left cofinal, then f' is also left cofinal.

Proof. By virtue of Corollary 7.2.1.15, we may assume that f factors as a composition $X \xrightarrow{g} Y \xrightarrow{h} Z$, where g is left anodyne and h is a trivial Kan fibration. Setting $Y' = Y \coprod_X X'$, we can expand (7.9) to a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & Y \arrow[r, "h"] \arrow[d] & Z \arrow[d] \\
X' \arrow[r, "g'"] & Y' \arrow[r, "h'"] & Z'.
\end{tikzcd}$$

Note that the square on the left is a pushout diagram in which the horizontal maps are monomorphisms, and therefore a categorical pushout diagram (Example 4.5.4.12). Applying Proposition 4.5.4.8, we deduce that the square on the right is also a categorical pushout diagram. Since h is a categorical equivalence (Proposition 4.5.3.11), it follows that h' is also a categorical equivalence (Proposition 4.5.4.10). In particular, h' is left cofinal (Corollary 7.2.1.13). The morphism g' is left anodyne (since it is a pushout of g), and is therefore also left cofinal (Proposition 7.2.1.3). Applying Proposition 7.2.1.6, we deduce that $f' = h' \circ g'$ is also left cofinal. □

02NH **Corollary 7.2.1.17.** *The collection of left cofinal morphisms of simplicial sets is closed under the formation of filtered colimits (when regarded as a full subcategory of the arrow category* $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$*).*

Proof. For every morphism of simplicial sets $f : X \to Z$, let $X \xrightarrow{f'} Q(f) \xrightarrow{f''} Y$ be the factorization of Proposition 4.2.4.7, so that f' is left anodyne, f'' is a left fibration, and the construction $f \mapsto Q(f)$ is a functor which commutes with filtered colimits. Using Propositions 7.2.1.5, 7.2.1.6, and Corollary 7.2.1.14, we see that f is left cofinal if and only if the morphism $f'' : Q(f) \to Z$ is a trivial Kan fibration. Since the collection of trivial Kan fibrations is closed under filtered colimits (Remark 1.5.5.3), it follows that the collection of left cofinal morphisms is also closed under filtered colimits. □

02NJ **Corollary 7.2.1.18.** *The collection of left anodyne morphisms of simplicial sets is closed under the formation of filtered colimits (when regarded as a full subcategory of the arrow category* $\mathrm{Fun}([1], \mathrm{Set}_{\Delta})$*).*

Proof. Combine Corollary 7.2.1.17 with Proposition 7.2.1.3. □

02NK **Corollary 7.2.1.19.** *Let $f : X \to Z$ be a left cofinal morphism of simplicial sets. Then, for every simplicial set K, the product map $(f \times \mathrm{id}_K) : X \times K \to Z \times K$ is left cofinal.*

Proof. By virtue of Corollary 7.2.1.15, the morphism f factors as a composition $X \xrightarrow{f'} Y \xrightarrow{f''} Z$, where f' is left anodyne and f'' is a trivial Kan fibration. It follows that $f \times \mathrm{id}_K$ factors as a composition

$$X \times K \xrightarrow{f' \times \mathrm{id}_K} Y \times K \xrightarrow{f'' \times \mathrm{id}_K} Z \times K.$$

We now note that $f' \times \mathrm{id}_K$ is left anodyne (Proposition 4.2.5.3) and $f'' \times \mathrm{id}_K$ is a trivial Kan fibration (Remark 1.5.5.2). Applying Corollary 7.2.1.15, we deduce that $f \times \mathrm{id}_K$ is left cofinal. □

Corollary 7.2.1.20. *Let $f : X \to Y$ and $f' : X' \to Y'$ be left cofinal morphisms of simplicial sets. Then the product map $(f \times f') : X \times X' \to Y \times Y'$ is left cofinal.* 02NL

Proof. Factoring $f \times f'$ as a composition

$$X \times X' \xrightarrow{f \times \mathrm{id}_{X'}} Y \times X' \xrightarrow{\mathrm{id}_Y \times f'} Y \times Y',$$

the desired result follows by combining Corollary 7.2.1.19 with Proposition 7.2.1.6. □

We now prove that cofinality is invariant under categorical equivalence.

Proposition 7.2.1.21. *Let $f : A \to B$ and $g : B \to C$ be morphisms of simplicial sets, and suppose that g is a categorical equivalence. Then f is left cofinal if and only if $g \circ f$ is left cofinal.* 02NM

Proof. Since g is a categorical equivalence, the construction $\widetilde{C} \mapsto B \times_C \widetilde{C}$ induces a bijection from equivalence classes of left fibrations over C to equivalence classes of left fibrations over B (Corollary 5.6.0.6). It follows that f is left cofinal if and only if it satisfies the following condition:

$(*)$ For every left fibration $q : \widetilde{C} \to C$, the restriction map $f^* : \mathrm{Fun}_{/C}(B, \widetilde{C}) \to \mathrm{Fun}_{/C}(A, \widetilde{C})$ is a homotopy equivalence of Kan complexes.

It will therefore suffice to show that, for every left fibration $q : \widetilde{C} \to C$, the restriction map $f^* : \mathrm{Fun}_{/C}(B, \widetilde{C}) \to \mathrm{Fun}_{/C}(A, \widetilde{C})$ is a homotopy equivalence if and only if the restriction map $(g \circ f)^* : \mathrm{Fun}_{/C}(C, \widetilde{C}) \to \mathrm{Fun}_{/C}(A, \widetilde{C})$ is a homotopy equivalence. This is clear, since our assumption that g is a categorical equivalence guarantees that the restriction map $g^* : \mathrm{Fun}_{/C}(C, \widetilde{C}) \to \mathrm{Fun}_{/C}(B, \widetilde{C})$ is a homotopy equivalence (Corollary 7.2.1.13). □

Corollary 7.2.1.22. *Suppose we are given a commutative diagram of simplicial sets* 02NN

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "g'"] \\
A' \arrow[r, "f'"] & B',
\end{tikzcd}$$

where g and g' are categorical equivalences. Then f is left cofinal if and only if f' is left cofinal.

Proof. By virtue of Proposition 7.2.1.21, the morphism f is left cofinal if and only if the composite morphism $g' \circ f$ is left cofinal. Similarly, Proposition 7.2.1.6 guarantees that f' is left cofinal if and only if $f' \circ g$ is left cofinal. We conclude by observing that $g' \circ f = f' \circ g$. □

02NP **Corollary 7.2.1.23.** *Let $\mathcal{C}$ be an ∞-category and suppose we are given a pair of diagrams $f_0, f_1 : K \to \mathcal{C}$ indexed by a simplicial set K. Suppose that f_0 and f_1 are isomorphic as objects of the ∞-category $\mathrm{Fun}(K, \mathcal{C})$. Then f_0 is left cofinal if and only if f_1 is left cofinal.*

Proof. Let $\mathrm{Isom}(\mathcal{C}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ be the full subcategory spanned by the isomorphisms of $\mathcal{C}$ (see Example 4.4.1.14). Let $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Isom}(\mathcal{C}) \to \mathcal{C}$ be the morphisms given by evaluation at the vertices $0, 1 \in \Delta^1$, so that ev_0 and ev_1 are trivial Kan fibrations (Corollary 4.4.5.10). Fix an isomorphism of f_0 with f_1, which we identify with a diagram $F : K \to \mathrm{Isom}(\mathcal{C})$ satisfying $\mathrm{ev}_0 \circ F = f_0$ and $\mathrm{ev}_1 \circ F = f_1$. Applying Corollary 7.2.1.22 to the diagram

$$\begin{tikzcd}
K \arrow[r, "F"] \arrow[d, "\mathrm{id}"] & \mathrm{Isom}(\mathcal{C}) \arrow[d, "\mathrm{ev}_0"] \\
K \arrow[r, "f_0"] & \mathcal{C},
\end{tikzcd}$$

we deduce that f_0 is left cofinal if and only if F is left cofinal. By the same reasoning, this is equivalent to the condition that f_1 is left cofinal. □

7.2.2 Cofinality and Limits

02NQ Let $\mathcal{C}$ be an ∞-category. In §7.1.2, we introduced the notion of a *limit* $\varprojlim(G)$ and *colimit* $\varinjlim(G)$ for a diagram $G : B \to \mathcal{C}$ (Definition 7.1.1.11). Our goal in this section is to show that, if $F : A \to B$ is a left cofinal morphism of simplicial sets, then the limit $\varprojlim(G)$ (if it exists) can be identified with the limit $\varprojlim(G \circ F)$. Similarly, if $F : A \to B$ is right cofinal, then the colimit $\varinjlim(G)$ (if it exists) can be identified with the colimit $\varinjlim(G \circ F)$. Our proof is based on the following characterization of (left) cofinality:

02NR **Proposition 7.2.2.1.** *Let $F : A \to B$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism F is left cofinal (in the sense of Definition 7.2.1.1).*

(2) *The diagram*

04K0

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "F"] & A^{\triangleleft} \arrow[d, "F^{\triangleleft}"] \\
B \arrow[r] & B^{\triangleleft}
\end{tikzcd} \tag{7.10}$$

is a categorical pushout square of simplicial sets.

(3) *The diagram*

04K1

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "F"] & \Delta^0 \diamond A \arrow[d] \\
B \arrow[r] & \Delta^0 \diamond B
\end{tikzcd} \tag{7.11}$$

is a categorical pushout square (here $\diamond$ denotes the blunt join introduced in Notation 4.5.8.3).

(4) *For every ∞-category $\mathcal{C}$ and every diagram $G : B \to \mathcal{C}$, composition with F induces an equivalence of ∞-categories*

$$\mathcal{C} \tilde{\times}_{\mathrm{Fun}(B,\mathcal{C})} \{G\} \to \mathcal{C} \tilde{\times}_{\mathrm{Fun}(A,\mathcal{C})} \{G \circ F\}.$$

(5) *For every ∞-category $\mathcal{C}$ and every diagram $G : B \to \mathcal{C}$, the restriction map $\mathcal{C}_{/G} \to \mathcal{C}_{/(G \circ F)}$ is an equivalence of ∞-categories.*

(6) *For every ∞-category $\mathcal{C}$, every diagram $G : B \to \mathcal{C}$, and every object $X \in \mathcal{C}$, precomposition with F induces a homotopy equivalence of Kan complexes*

$$\mathrm{Hom}_{\mathrm{Fun}(B,\mathcal{C})}(\underline{X}, G) \to \mathrm{Hom}_{\mathrm{Fun}(A,\mathcal{C})}(\underline{X} \circ F, G \circ F);$$

here $\underline{X} : B \to \mathcal{C}$ denotes the constant diagram taking the value X.

(7) *For every ∞-category $\mathcal{C}$, every diagram $G : B \to \mathcal{C}$, and every object $X \in \mathcal{C}$, precomposition with F induces a homotopy equivalence of Kan complexes*

$$\mathrm{Fun}_{/\mathcal{C}}(B, \mathcal{C}_{X/}) \to \mathrm{Fun}_{/\mathcal{C}}(A, \mathcal{C}_{X/}).$$

Proof. We first show that (1) implies (2). Let $F : A \to B$ be a left cofinal morphism of simplicial sets; we wish to show that the diagram (7.10) is a categorical pushout square. By virtue of Corollary 7.2.1.15 (and Proposition 4.5.4.8), we may assume that F is either left anodyne or a trivial Kan fibration. In the second case, the vertical morphisms in the diagram (7.10) are categorical equivalences (see Corollary 4.5.8.9), so the desired result is a special case of Proposition 4.5.4.10. In the second case, Example 4.3.6.5 guarantees that the induced map $B \coprod_A A^{\triangleleft} \hookrightarrow B^{\triangleleft}$ is inner anodyne, so the desired result follows from Proposition 4.5.4.11.

Notation 4.5.8.3 supplies a comparison map from the diagram (7.11) to the diagram (7.10), whcih is a levelwise categorical equivalence by virtue of Theorem 4.5.8.8. The equivalence (2) ⇔ (3) now follows from Proposition 4.5.4.9.

We next show that (3) implies (4). Let $\mathcal{C}$ be an ∞-category and let $G : B \to \mathcal{C}$ be a diagram. If condition (3) is satisfied, then the diagram of ∞-categories

$$\begin{tikzcd}
\operatorname{Fun}(\Delta^0 \diamond B, \mathcal{C}) \arrow[r] \arrow[d] & \operatorname{Fun}(\Delta^0 \diamond A, \mathcal{C}) \arrow[d] \\
\operatorname{Fun}(B, \mathcal{C}) \arrow[r, "\circ F"] & \operatorname{Fun}(A, \mathcal{C})
\end{tikzcd}$$

is a categorical pullback square. Corollary 4.4.5.3 guaranteees that the vertical maps in this diagram are isofibrations. Invoking Corollary 4.5.2.31 (together with the definition of the blunt join), we deduce that the induced map

$$\begin{aligned}
\mathcal{C} \tilde{\times}_{\operatorname{Fun}(B,\mathcal{C})} \{G\} &\simeq \operatorname{Fun}(\Delta^0 \diamond B, \mathcal{C}) \times_{\operatorname{Fun}(B,\mathcal{C})} \{G\} \\
&\to \operatorname{Fun}(\Delta^0 \diamond A, \mathcal{C}) \times_{\operatorname{Fun}(A,\mathcal{C})} \{G \circ F\} \\
&\simeq \mathcal{C} \tilde{\times}_{\operatorname{Fun}(A,\mathcal{C})} \{G \circ F\}
\end{aligned}$$

is an equivalence of ∞-categories.

We next prove the equivalences (4) ⇔ (5) ⇔ (6) ⇔ (7). Let $G : B \to \mathcal{C}$ be as above. Applying Construction 4.6.4.13, we obtain a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C}_{/G} \arrow[r] \arrow[d, "\theta"] & \mathcal{C} \tilde{\times}_{\operatorname{Fun}(B,\mathcal{C})} \{G\} \arrow[d, "\theta'"] \\
\mathcal{C}_{/(G \circ F)} \arrow[r] & \mathcal{C} \tilde{\times}_{\operatorname{Fun}(A,\mathcal{C})} \{G \circ F\},
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories (Theorem 4.6.4.17). It follows that θ is an equivalence of ∞-categories if and only if θ' is an equivalence of ∞-categories. This proves the equivalence (4) ⇔ (5). Note that the functor θ' fits into a commutative

diagram

$$\begin{tikzcd}
\mathcal{C} \tilde{\times}_{\mathrm{Fun}(B,\mathcal{C})} \{G\} \arrow[rr, "\theta'"] \arrow[dr] & & \mathcal{C} \tilde{\times}_{\mathrm{Fun}(A,\mathcal{C})} \{G \circ F\} \arrow[dl] \\
& \mathcal{C}, &
\end{tikzcd}$$

where the vertical maps are right fibrations (Corollary 4.6.4.12). Applying Corollary 5.1.7.16 and Proposition 5.1.7.5, we see that θ' is an equivalence of ∞-categories if and only if it induces a homotopy equivalence

$$\theta'_X : \{\underline{X}\} \tilde{\times}_{\mathrm{Fun}(B,\mathcal{C})} \{G\} \to \{\underline{X} \circ F\} \tilde{\times}_{\mathrm{Fun}(A,\mathcal{C})} \{G \circ F\}$$

for each object $X \in \mathcal{C}$, which proves the equivalence (4) ⇔ (6). Unwinding the definitions, we can identify θ' with the lower horizontal map appearing in the diagram

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{C}}(B, \mathcal{C}_{X/}) \arrow[r, "\theta''_X"] \arrow[d] & \mathrm{Fun}_{/\mathcal{C}}(A, \mathcal{C}_{X/}) \arrow[d] \\
\mathrm{Fun}_{/\mathcal{C}}(B, \{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C}) \arrow[r, "\theta'_X"] & \mathrm{Fun}_{/\mathcal{C}}(A, \{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C}),
\end{tikzcd}$$

where the vertical maps are given by postcomposition with the coslice diagonal morphism $\rho : \mathcal{C}_{X/} \to \{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C}$. Theorem 4.6.4.17 guarantees that ρ is an equivalence of ∞-categories. It is therefore also an an equivalence of left fibrations over $\mathcal{C}$ (Proposition 5.1.7.5), so that the vertical maps are homotopy equivalences. It follows that θ'_X is a homotopy equivalence if and only if θ''_X is a homotopy equivalence, which proves the equivalence (6) ⇔ (7).

We now complete the proof by showing that (7) implies (1). Assume that condition (7) is satisfied; we wish to show that F is left cofinal. Let $q : \widetilde{B} \to B$ be a left fibration; we must show that composition with F induces a homotopy equivalence $\mathrm{Fun}_{/B}(B, \widetilde{B}) \to \mathrm{Fun}_{/B}(A, \widetilde{B})$. To prove this, we are free to replace $q : \widetilde{B} \to B$ by any other left fibration which is equivalent to it (in the sense of Definition 5.1.7.1). We may therefore assume without loss of generality that there exists a pullback diagram of simplicial sets

$$\begin{tikzcd}
\widetilde{B} \arrow[r] \arrow[d, "q"] & \mathcal{S}_* \arrow[d, "q_{\mathrm{univ}}"] \\
B \arrow[r, "G"] & \mathcal{S},
\end{tikzcd}$$

where $q_{\text{univ}} : \mathcal{S}_* \to \mathcal{S}$ is the universal left fibration of Corollary 5.6.0.6. We are then reduced to proving that F induces a homotopy equivalence $\operatorname{Fun}_{/\mathcal{S}}(B, \mathcal{S}_*) \to \operatorname{Fun}_{\mathcal{S}}(A, \mathcal{S}_*)$, which is a special case of (7) (applied to the ∞-category $\mathcal{C} = \mathcal{S}$ and the object $X = \Delta^0$). □

02XQ **Corollary 7.2.2.2.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $e : A \to B$ be a left cofinal morphism of simplicial sets. Then a morphism of simplicial sets $\overline{f} : B^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram if and only if the composite map*

$$A^{\triangleleft} \xrightarrow{e^{\triangleleft}} B^{\triangleleft} \xrightarrow{\overline{f}} \mathcal{C}$$

is a U-limit diagram.

Proof. Set $f = \overline{f}|_B$ and apply Remark 7.1.4.9 to the commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C}_{/f} \arrow[r] \arrow[d] & \mathcal{C}_{/(f \circ e)} \arrow[d] \\
\mathcal{D}_{/(U \circ f)} \arrow[r] & \mathcal{D}_{/(U \circ f \circ e)},
\end{tikzcd}$$

noting that the horizontal maps are equivalences by virtue of Proposition 7.2.2.1. □

02XR **Corollary 7.2.2.3.** *Let $\mathcal{C}$ be an ∞-category and let $e : A \to B$ be a left cofinal morphism of simplicial sets. Then a morphism of simplicial sets $\overline{f} : B^{\triangleleft} \to \mathcal{C}$ is a limit diagram if and only if the composite map*

$$A^{\triangleleft} \xrightarrow{e^{\triangleleft}} B^{\triangleleft} \xrightarrow{\overline{f}} \mathcal{C}$$

is a limit diagram.

Proof. Apply Corollary 7.2.2.2 in the special case $\mathcal{D} = \Delta^0$ (see Example 7.1.5.3). □

02NT **Remark 7.2.2.4.** The converse of Corollary 7.2.2.3 is also true: if $e : A \to B$ is a morphism of simplicial sets having the property that precomposition with the induced map $e^{\triangleleft} : A^{\triangleleft} \to B^{\triangleleft}$ carries limit diagrams to limit diagrams, then e is left cofinal. Moreover, it suffices check this condition for diagrams in the ∞-category $\mathcal{S}$ of spaces (see Corollary 7.4.5.14).

02XV **Corollary 7.2.2.5.** *Let $U : \mathcal{D} \to \mathcal{E}$ be an inner fibration of ∞-categories and let $\mathcal{C}$ be an ∞-category containing an object Y. Then:*

- *If Y is an initial object of $\mathcal{C}$, then a diagram $\mathcal{C}^{\triangleleft} \to \mathcal{D}$ is a U-limit diagram if and only if it carries $\{Y\}^{\triangleleft} \simeq \Delta^1$ to a U-cartesian morphism of $\mathcal{D}$.*

- *If Y is a final object of $\mathcal{K}$, then a diagram $\mathcal{C}^{\triangleleft} \to \mathcal{D}$ is a U-colimit diagram if and only if it carries $\{Y\}^{\triangleright} \simeq \Delta^1$ to a U-cocartesian morphism of $\mathcal{D}$.*

Proof. If Y is an initial object of $\mathcal{C}$, then the inclusion map $\{Y\} \hookrightarrow \mathcal{C}$ is left cofinal (Corollary 4.6.7.24). The first assertion now follows by combining Corollary 7.2.2.2 with Example 7.1.5.9. The second assertion follows by a similar argument. □

Corollary 7.2.2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Then:* `02XW`

- *If $\mathcal{C}$ has an initial object Y, then a functor $\mathcal{C}^{\triangleleft} \to \mathcal{D}$ is a limit diagram if and only if it carries $\{Y\}^{\triangleleft} \simeq \Delta^1$ to an isomorphism in the ∞-category $\mathcal{D}$.*
- *If $\mathcal{C}$ has a final object Y, then a functor $\mathcal{C}^{\triangleright} \to \mathcal{D}$ is a colimit diagram if and only if it carries $\{Y\}^{\triangleright} \simeq \Delta^1$ to an isomorphism in the ∞-category $\mathcal{D}$.*

Proof. Apply Corollary 7.2.2.5 in the special case $\mathcal{E} = \Delta^0$ (and use Example 5.1.1.4). □

Corollary 7.2.2.7. *Let $\mathcal{C}$ be an ∞-category containing an object $C \in \mathcal{C}$ and let $e : A \to B$ be a left cofinal morphism of simplicial sets. Suppose we are given a diagram $f : B \to \mathcal{C}$ and a natural transformation $\alpha : \underline{C} \to f$, where $\underline{C} \in \operatorname{Fun}(B, \mathcal{C})$ denotes the constant diagram taking the value C. Then α exhibits C as a limit of f (in the sense of Definition 7.1.1.1) if and only if the induced natural transformation $\alpha|_A : \underline{C}|_A \to f|_A$ exhibits C as a limit of the diagram $f|_A$.* `02XS`

Proof. By virtue of Remark 7.1.1.7, we are free to modify the natural transformation α by a homotopy and may therefore assume that it corresponds to a morphism of simplicial sets $\Delta^0 \diamond B \to \mathcal{C}$ which factors through the categorical equivalence $\Delta^0 \diamond B \twoheadrightarrow \Delta^0 \star B$ of Theorem 4.5.8.8. In this case, the desired result follows from Corollary 7.2.2.3 and Remark 7.1.2.6. □

Corollary 7.2.2.8. *Let $e : A \to B$ be a morphism of simplicial sets and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. If e is left cofinal and the functor F preserves A-indexed limits, then F preserves B-indexed limits. If e is right cofinal and the functor F preserves A-indexed colimits, then the functor F preserves B-indexed colimits.* `02NV`

Proposition 7.2.2.9. *Suppose we are given a commutative diagram of simplicial sets* `02XT`

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
B^{\triangleleft} \arrow[r, "\overline{g}"] & \mathcal{D},
\end{tikzcd}$$

where U is an inner fibration of ∞-categories. Let $e : A \to B$ be a left cofinal morphism of simplicial sets. The following conditions are equivalent:

(1) *There exists a U-limit diagram $\overline{f} : B^{\triangleleft} \to \mathcal{C}$ satisfying $\overline{f}|_B = f$ and $U \circ \overline{f} = \overline{g}$.*

(2) *There exists a U-limit diagram $\overline{f}_0 : A^{\triangleleft} \to \mathcal{C}$ satisfying $\overline{f}_0|_A = f \circ e$ and $U \circ \overline{f}_0 = \overline{g} \circ e^{\triangleleft}$.*

Proof. The implication (1) ⇒ (2) follows by observing that if $\overline{f} : B^{\triangleleft} \to \mathcal{C}$ is a U-limit diagram, then the left cofinality of e guarantees that $\overline{f} \circ e^{\triangleleft}$ is also a U-limit diagram (Corollary 7.2.2.2). We will complete the proof by showing that (2) implies (1). By virtue of Corollary 7.2.1.15, we can assume that the morphism e is either left anodyne or a trivial Kan fibration. We first treat the case where e is a trivial Kan fibration. Let $s : B \to A$ be a section of e, and let $\overline{f}_0 : A^{\triangleleft} \to \mathcal{C}$ satisfy the requirements of (2). Let $\overline{f}$ denote the composite map

$$B^{\triangleleft} \xrightarrow{s^{\triangleleft}} A^{\triangleleft} \xrightarrow{\overline{f}_0} \mathcal{C}.$$

It follows immediately from the construction that $\overline{f}|_B = f$ and $U \circ \overline{f} = \overline{g}$. Moreover, the composition $\overline{f} \circ e^{\triangleleft}$ is isomorphic to $\overline{f}_0$ (as an object of the ∞-category $\mathrm{Fun}(B^{\triangleleft}, \mathcal{C})$), and is therefore also a U-limit diagram (Proposition 7.1.5.13). Since e is left cofinal, it follows that $\overline{f}$ is also a U-limit diagram (Corollary 7.2.2.2).

We now treat the case where e is left anodyne. In this case, the induced map $A^{\triangleleft} \coprod_A B \hookrightarrow B^{\triangleleft}$ is inner anodyne. Since U is an inner fibration, we can extend f to a morphism $\overline{f} : B^{\triangleleft} \to \mathcal{C}$ satisfying $U \circ \overline{f} = \overline{g}$ and $\overline{f} \circ e^{\triangleleft} = \overline{f}_0$. Since e is left cofinal, the morphism $\overline{f}$ is automatically a U-limit diagram (Corollary 7.2.2.2). □

02NS **Corollary 7.2.2.10.** *Let $\mathcal{C}$ be an ∞-category and let $e : A \to B$ be a left cofinal morphism of simplicial sets. Then a diagram $f : B \to \mathcal{C}$ has a limit if and only if the composite diagram $(f \circ e) : A \to \mathcal{C}$ has a limit.*

Proof. If $\overline{f} : B^{\triangleleft} \to \mathcal{C}$ is a colimit diagram extending f, then Corollary 7.2.2.3 guarantees that $\overline{f} \circ e^{\triangleleft} : A^{\triangleleft} \to \mathcal{C}$ is a colimit diagram extending $f \circ e$. Conversely, if $f \circ e$ can be extended to a colimit diagram, then Proposition 7.2.2.9 (applied in the special case $\mathcal{D} = \Delta^0$) guarantees that f can also be extended to a colimit diagram. □

02XU **Corollary 7.2.2.11.** *Let $\mathcal{C}$ be an ∞-category, let $e : A \to B$ be a left cofinal morphism of simplicial sets, and let $f : B \to \mathcal{C}$ be a diagram. Then an object $X \in \mathcal{C}$ is a limit of f if and only if it is a limit of the diagram $(f \circ e) : A \to \mathcal{C}$.*

Proof. If an object $X \in \mathcal{C}$ is a limit of f, then we can choose a limit diagram $\overline{f} : B^{\triangleleft} \to \mathcal{C}$ carrying the cone point of $f^{\triangleleft}$ to the object X. Applying Corollary 7.2.2.10, we deduce that $\overline{f} \circ e^{\triangleleft}$ exhibits X as a limit of the diagram $f \circ e$. Conversely, if X is a limit of the diagram $f \circ e$, then Corollary 7.2.2.10 guarantees that the diagram f admits a limit $Y \in \mathcal{C}$. The preceding argument shows that Y is also a limit of the diagram $f \circ e$. Applying Proposition 7.1.1.12, we deduce that Y is isomorphic to X, so that X is also a limit of the diagram f. □

Corollary 7.2.2.12. *Let $e : A \to B$ be a morphism of simplicial sets and let $\mathcal{C}$ be an ∞-category. If e is left cofinal and $\mathcal{C}$ admits A-indexed limits, then $\mathcal{C}$ also admits B-indexed limits. If e is right cofinal and $\mathcal{C}$ admits A-indexed colimits, then $\mathcal{C}$ also admits B-indexed colimits.* 02NU

Corollary 7.2.2.13. *Let $e : A \to B$ be a morphism of simplicial sets and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. If e is left cofinal and the functor F creates A-indexed limits, then F creates B-indexed limits. If e is right cofinal and the functor F creates A-indexed colimits, then the functor F creates B-indexed colimits.* 02NW

Corollary 7.2.2.14. *Suppose we are given lifting problem* 039B

039C

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C}^{\triangleleft} \arrow[r] \arrow[ur, dashed, "\overline{f}"] & \mathcal{E},
\end{tikzcd} \tag{7.12}$$

where $\mathcal{C}$ is an ∞-category and U is a cartesian fibration of ∞-categories. If $\mathcal{C}$ has a final object C, then (7.12) admits a solution $\overline{f} : \mathcal{C}^{\triangleleft} \to \mathcal{D}$ which is a U-limit diagram.

Proof. Using Proposition 7.2.2.9 and Corollary 4.6.7.24, we can replace $\mathcal{C}$ by the simplicial set $\{C\} \simeq \Delta^0$, in which case the desired result follows from our assumption that U is a cartesian fibration (see Example 7.1.5.9). □

7.2.3 Quillen's Theorem A for ∞-Categories

The following result provides a concrete criterion for establishing the cofinality of a functor between ∞-categories. 02NX

Theorem 7.2.3.1 (Joyal)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, where $\mathcal{D}$ is an ∞-category. Then:* 02NY

(1) *The morphism F is left cofinal if and only if, for every object $X \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/X}$ is weakly contractible.*

(2) *The morphism F is right cofinal if and only if, for every object $X \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/}$ is weakly contractible.*

Remark 7.2.3.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, where $\mathcal{D}$ is an ∞-category. For every object $X \in \mathcal{D}$, the slice and coslice diagonal morphisms of Construction 4.6.4.13 induce categorical equivalences 02NZ

$$\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/X} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathcal{D}} \{X\} \qquad \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/} \hookrightarrow \{X\} \tilde{\times}_{\mathcal{D}} \mathcal{C}$$

(Example 5.1.7.7). We can therefore reformulate Theorem 7.2.3.1 as follows:

(1′) The morphism F is left cofinal if and only if, for every object $X \in \mathcal{D}$, the simplicial set $\mathcal{C} \tilde{\times}_{\mathcal{D}} \{X\}$ is weakly contractible.

(2′) The morphism F is right cofinal if and only if, for every object $X \in \mathcal{D}$, the simplicial set $\{X\} \tilde{\times}_{\mathcal{D}} \mathcal{C}$ is weakly contractible.

02P0 **Example 7.2.3.3** (Quillen's Theorem A)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Suppose that, for every object $X \in \mathcal{D}$, the category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/}$ has weakly contractible nerve. Applying Theorem 7.2.3.1, we deduce that the induced morphism of simplicial sets $\mathrm{N}_{\bullet}(F) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{D})$ is right cofinal. In particular, it is a weak homotopy equivalence (Proposition 7.2.1.5). This recovers a classical result of Quillen (see [45]).

03U2 **Corollary 7.2.3.4.** *Let $(S, \leq)$ and $(T, \leq)$ be linearly ordered sets, and let $f : S \to T$ be a nondecreasing function. The following conditions are equivalent:*

(1) *The function $f : S \to T$ is cofinal in the sense of Definition 4.7.1.26. That is, for every element $t \in T$, there exists an element $s \in S$ satisfying $t \leq f(s)$.*

(2) *The induced morphism of simplicial sets $\mathrm{N}_{\bullet}(S) \to \mathrm{N}_{\bullet}(T)$ is right cofinal, in the sense of Definition 7.2.1.1.*

Proof. For each $t \in T$, set $S_{\geq t} = \{s \in S : t \leq f(s)\}$, which we regard as a linearly ordered subset of S. Using Theorem 7.2.3.1, we can rewrite conditions (1) and (2) as follows:

(1′) For each element $t \in T$, the linearly ordered set $S_{\geq t}$ is nonempty.

(2′) For each element $t \in T$, the linearly ordered set $S_{\geq t}$ has weakly contractible nerve.

The implication (2′) ⇒ (1′) is immediate, and the reverse implication follows from Corollary 3.2.8.5. □

02XX **Corollary 7.2.3.5.** *Let $\mathcal{C}$ be an ∞-category and let $\overline{f} : A^{\triangleleft} \to \mathcal{C}$ be a diagram, where A is a weakly contractible simplicial set. The following conditions are equivalent:*

(1) *The diagram $\overline{f}$ carries each edge of $A^{\triangleleft}$ to an isomorphism in $\mathcal{C}$.*

(2) *The restriction $f = \overline{f}|_{A}$ carries each edge of A to an isomorphism in $\mathcal{C}$, and $\overline{f}$ is a limit diagram.*

Proof. Without loss of generality, we may assume that f carries each edge of A to an isomorphism in $\mathcal{C}$. Under this assumption, we can restate (1) and (2) as follows:

($1'$) For every vertex $a \in A$, the edge

$$\Delta^1 \simeq \{a\}^{\triangleleft} \hookrightarrow A^{\triangleleft} \xrightarrow{\overline{f}} \mathcal{C}$$

is an isomorphism in the ∞-category $\mathcal{C}$.

($2'$) The morphism $\overline{f}$ is a limit diaram.

Using Corollary 3.1.7.2, we can choose an anodyne morphism $i : A \hookrightarrow B$, where B is a Kan complex. Note that f can be regarded as a morphism from A to the core $\mathcal{C}^{\simeq}$, which is also a Kan complex (Corollary 4.4.3.11). We can therefore extend f to a morphism of Kan complexes $g : B \to \mathcal{C}^{\simeq}$. Moreover, the morphism i is left cofinal (Proposition 7.2.1.5) and therefore left anodyne (Proposition 7.2.1.3). It follows that the induced map $B \coprod_A A^{\triangleleft} \hookrightarrow B^{\triangleleft}$ is inner anodyne (Example 4.3.6.5), so that we can choose a functor $\overline{g} : B^{\triangleleft} \to \mathcal{C}$ satisfying $\overline{g}|_B = g$ and $\overline{g}|_{A^{\triangleleft}} = \overline{f}$.

It follows from Corollary 7.2.2.3 that $\overline{f}$ is a limit diagram if and only if $\overline{g}$ is a limit diagram. Since A is weakly contractible, the Kan complex B is contractible. In particular, every vertex $a \in A$ can be regarded as a final object of B. The equivalence of ($1'$) and ($2'$) now follows from Corollary 7.2.2.6. □

Corollary 7.2.3.6. *Let $\mathcal{C}$ be an ∞-category and let $U : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories. Then:* 02XY

- *If $\mathcal{C}$ has an initial object Y and $\overline{F} : \mathcal{C}^{\triangleleft} \to \mathcal{D}$ is a functor which carries $\{Y\}^{\triangleleft} \simeq \Delta^1$ to an isomorphism in $\mathcal{D}$, then $\overline{F}$ is a U-limit diagram.*
- *If $\mathcal{C}$ has a final object Y and $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ is a functor which carries $\{Y\}^{\triangleright} \simeq \Delta^1$ to an isomorphism in $\mathcal{D}$, then $\overline{F}$ is a U-colimit diagram.*

Proof. Combine Corollary 7.2.2.6 with Proposition 7.1.5.14. □

Corollary 7.2.3.7. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:* 02P3

(1) *If F is a left adjoint, then it is left cofinal.*

(2) *If F is a right adjoint, then it is right cofinal.*

Proof. We will prove (1); the proof of (2) is similar. Suppose that F admits a right $G : \mathcal{D} \to \mathcal{C}$. For every object $X \in \mathcal{D}$, Corollary 6.2.4.2 guarantees that the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/X}$ has a final object. In particular, the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/X}$ is weakly contractible (Corollary 4.6.7.25). Allowing X to vary and applying Theorem 7.2.3.1, we conclude that F is left cofinal. □

03J7 **Example 7.2.3.8.** Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}_0 \subseteq \mathcal{C}$ is a reflective subcategory (Definition 6.2.2.1), then the inclusion map $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ is right cofinal (this is a special case of Corollary 7.2.3.7, since Proposition 6.2.2.11 guarantees that ι has a left adjoint). Similarly, if $\mathcal{C}_0$ is a coreflective subcategory of $\mathcal{C}$, then the inclusion ι is left cofinal.

02P4 **Corollary 7.2.3.9.** *Let $\mathcal{C}$ be an ∞-category and let K be a simplicial sets. The following conditions are equivalent:*

(1) *The diagonal map $\delta : \mathcal{C} \to \operatorname{Fun}(K, \mathcal{C})$ is right cofinal.*

(2) *For every diagram $f : K \to \mathcal{C}$, the coslice ∞-category $\mathcal{C}_{f/}$ is weakly contractible.*

Proof. By virtue of Remark 7.2.3.2, condition (1) is equivalent to the requirement that for every diagram $f : K \to \mathcal{C}$, the oriented fiber product $\{f\} \tilde{\times}_{\operatorname{Fun}(K,\mathcal{C})} \mathcal{C}$ is weakly contractible. The equivalence of (1) and (2) now follows from Theorem 4.6.4.17. □

Our proof of Theorem 7.2.3.1 will require some preliminaries.

039D **Lemma 7.2.3.10.** *Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \operatorname{Set}_{\Delta}$ be a diagram of simplicial sets indexed by $\mathcal{C}$. Suppose we are given morphisms of simplicial sets $A \xrightarrow{f} B \xrightarrow{g} \mathrm{N}_{\bullet}(\mathcal{C})$, where f is right anodyne. Then the induced map $A \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\operatorname{holim}}(\mathscr{F}) \to B \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\operatorname{holim}}(\mathscr{F})$ is right anodyne.*

Proof. Without loss of generality, we may assume that f is the inclusion map $\Lambda^n_i \hookrightarrow \Delta^n$ for some $0 < i \leq n$. Using Remark 5.3.2.3, we can reduce to the case where $\mathcal{C}$ is the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$ and g is the identity map. In this case, Remark 5.3.2.12 supplies a pushout diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^n_i \times \mathscr{F}(0) \arrow[r] \arrow[d] & \Lambda^n_i \times_{\Delta^n} \underrightarrow{\operatorname{holim}}(\mathscr{F}) \arrow[d] \\
\Delta^n \times \mathscr{F}(0) \arrow[r] & \underrightarrow{\operatorname{holim}}(\mathscr{F}).
\end{tikzcd}$$

It will therefore suffice to show that the left vertical map is right anodyne, which follows from Proposition 4.2.5.3. □

039E **Example 7.2.3.11.** Let $\mathscr{F} : \mathcal{C} \to \operatorname{Set}_{\Delta}$ be a diagram of simplicial sets, and suppose that the category $\mathcal{C}$ contains a final object C. Combining Lemma 7.2.3.10 with Corollary 4.6.7.24, we deduce that the inclusion map

$$\mathscr{F}(C) \simeq \{C\} \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\operatorname{holim}}(\mathscr{F}) \hookrightarrow \underrightarrow{\operatorname{holim}}(\mathscr{F})$$

is right anodyne.

Proposition 7.2.3.12. *Suppose we are given a pullback diagram of simplicial sets* 02P6

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "F"] \arrow[d] & \mathcal{C} \arrow[d, "\pi"] \\
\mathcal{D}' \arrow[r, "\overline{F}"] & \mathcal{D}.
\end{tikzcd}$$

If π is a cocartesian fibration and $\overline{F}$ is right cofinal, then F is right cofinal.

Proof. By virtue of Corollary 7.2.1.15, it will suffice to prove Proposition 7.2.3.12 in the special case where $\overline{F}$ is right anodyne. Let S be the collection of all morphisms of simplicial sets $\overline{F} : \mathcal{D}' \to \mathcal{D}$ having the property that, for every cocartesian fibration $\pi : \mathcal{C} \to \mathcal{D}$, the induced map $F : \mathcal{D}' \times_{\mathcal{D}} \mathcal{C} \to \mathcal{C}$ is right anodyne. We wish to show show that every right anodyne morphism belongs to S. It follows immediately from the definitions that S is weakly saturated, in the sense of Definition 1.5.4.12. It will therefore suffice to show that S contains every horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ for $0 < i \leq n$. In other words, we are reduced to proving Proposition 7.2.3.12 in the special case where $\mathcal{D} = \Delta^n$ is a standard simplex and $\overline{F}$ is the inclusion of the horn $\Lambda^n_i \subseteq \Delta^n$.

Applying Corollary 5.3.4.9, we deduce that there exists a diagram of ∞-categories $\mathscr{G} : [n] \to \mathrm{QCat}$ and a scaffold $\lambda : \underrightarrow{\mathrm{holim}}(\mathscr{G}) \to \mathcal{C}$ for the cocartesian fibration π. We then have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\Lambda^n_i \times_{\Delta^n} \underrightarrow{\mathrm{holim}}(\mathscr{G}) \arrow[r] \arrow[d, "F'"] & \Lambda^n_i \times_{\Delta^n} \mathcal{C} \arrow[d, "F"] \\
\underrightarrow{\mathrm{holim}}(\mathscr{G}) \arrow[r, "\lambda"] & \mathcal{C},
\end{tikzcd}$$

where F' is right anodyne (Lemma 7.2.3.10) and therefore right cofinal (Proposition 7.2.1.3). Lemma 5.3.6.4 guarantees that horizontal maps are categorical equivalences, so that F is also right cofinal (Corollary 7.2.1.22). □

Corollary 7.2.3.13. *Suppose we are given a pullback diagram of simplicial sets* 02P7

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "F"] \arrow[d] & \mathcal{C} \arrow[d, "\pi"] \\
\mathcal{D}' \arrow[r, "\overline{F}"] & \mathcal{D}.
\end{tikzcd}$$

If π is a cocartesian fibration and $\overline{F}$ is right anodyne, then F is right anodyne.

Proof. Combine Propositions 7.2.3.12 and 7.2.1.3. □

03J8 **Example 7.2.3.14.** Let $\pi : \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration of ∞-categories, let X be an object of $\mathcal{D}$, and set $\mathcal{C}_X = \{X\} \times_{\mathcal{D}} \mathcal{C}$. If X is a final object of $\mathcal{D}$, then the inclusion map $\mathcal{C}_X \hookrightarrow \mathcal{C}$ is right anodyne, and therefore right cofinal. This follows by combining Corollaries 7.2.3.13 and 4.6.7.24.

Proof of Theorem 7.2.3.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, where $\mathcal{D}$ is an ∞-category. We will show that F is right cofinal if and only if, for every object $X \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/}$ is weakly contractible; the analogous characterization of left cofinal morphisms follows by a similar argument.

Suppose first that F is right cofinal. For every object $X \in \mathcal{D}$, the projection map $\mathcal{D}_{X/} \to \mathcal{D}$ is a left fibration (Proposition 4.3.6.1), and therefore a cocartesian fibration (Proposition 5.1.4.14). Applying Proposition 7.2.3.12, we conclude that the projection map $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/} \to \mathcal{D}_{X/}$ is also right cofinal. In particular, it is a weak homotopy equivalence (Proposition 7.2.1.5). Since the ∞-category $\mathcal{D}_{X/}$ has an initial object (Proposition 4.6.7.22), it is weakly contractible, so that the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/}$ is also weakly contractible.

We now prove the converse. Assume that, for every object $X \in \mathcal{D}$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/}$ is weakly contractible. We wish to show that F is right cofinal. Using Proposition 4.1.3.2, we can factor F as a composition

$$\mathcal{C} \xrightarrow{F'} \mathcal{C}' \xrightarrow{F''} \mathcal{D},$$

where F' is inner anodyne and F'' is an inner fibration. Since F' is right cofinal (Proposition 7.2.1.3), it will suffice to show that F'' is right cofinal (Proposition 7.2.1.6). For every object $X \in \mathcal{D}$, Proposition 5.3.6.1 guarantees that the induced map $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/} \hookrightarrow \mathcal{C}' \times_{\mathcal{D}} \mathcal{D}_{X/}$ is a categorical equivalence. In particular, it is a weak homotopy equivalence (Remark 4.5.3.4), so that $\mathcal{C}' \times_{\mathcal{D}} \mathcal{D}_{X/}$ is also weakly contractible. We may therefore replace $\mathcal{C}$ by $\mathcal{C}'$ and thereby reduce to the case where $F : \mathcal{C} \to \mathcal{D}$ is an inner fibration, so that $\mathcal{C}$ is also an ∞-category (Remark 4.1.1.9).

Let $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathcal{D}$ denote the functors given by evaluation at the vertices $0, 1 \in \Delta^1$, and let $\delta : \mathcal{D} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{D})$ be the diagonal map. Note that there is a unique natural transformation from id_{Δ^1} to the constant map $\Delta^1 \twoheadrightarrow \{1\} \hookrightarrow \Delta^1$, which induces a natural transformation $h : \mathrm{id}_{\mathrm{Fun}(\Delta^1, \mathcal{D})} \to \delta \circ \mathrm{ev}_1$. Let $\mathcal{M}$ denote the oriented fiber product $\mathcal{D} \tilde{\times}_{\mathcal{D}} \mathcal{C} = \mathrm{Fun}(\Delta^1, \mathcal{D}) \times_{\mathrm{Fun}(\{1\}, \mathcal{D})} \mathrm{Fun}(\{1\}, \mathcal{C})$ of Construction 4.6.4.1, so that ev_0 and ev_1 lift to functors

$$\mathcal{D} \xleftarrow{\widetilde{\mathrm{ev}}_0} \mathcal{M} \xrightarrow{\widetilde{\mathrm{ev}}_1} \mathcal{C},$$

the diagonal map δ lifts to a functor $\widetilde{\delta} : \mathcal{C} \hookrightarrow \mathcal{M}$, and h lifts to a natural transformation $\widetilde{h} : \mathrm{id}_{\mathcal{M}} \to \widetilde{\delta} \circ \widetilde{\mathrm{ev}}_1$. Note that $\widetilde{h}$ can be identified with a morphism of simplicial sets $\Delta^1 \times \mathcal{M} \to \mathcal{M}$ which fits into a commutative diagram

$$\begin{tikzcd}
\{0\} \times \mathcal{C} \arrow[r] \arrow[d, "\widetilde{\delta}"] & (\Delta^1 \times \mathcal{C}) \coprod_{(\{1\} \times \mathcal{C})} (\{1\} \times \mathcal{M}) \arrow[r] \arrow[d, "\iota"] & \mathcal{C} \arrow[d, "\widetilde{\delta}"] \\
\{0\} \times \mathcal{M} \arrow[r] & \Delta^1 \times \mathcal{M} \arrow[r, "\widetilde{h}"] & \mathcal{M},
\end{tikzcd}$$

where the horizontal compositions are the identity. It follows that $\widetilde{\delta}$ is a retract of ι. Since ι is right anodyne (Proposition 4.2.5.3), $\widetilde{\delta}$ is also right anodyne, and therefore right cofinal (Proposition 7.2.1.3).

The functor $\widetilde{\mathrm{ev}}_0 : \mathcal{M} \to \mathcal{D}$ is a cartesian fibration (Corollary 5.3.7.3). Moreover, for each object $X \in \mathcal{D}$, the fiber $\widetilde{\mathrm{ev}}_0^{-1}\{X\} \simeq \{X\} \tilde{\times}_{\mathcal{D}} \mathcal{C}$ is equivalent to the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{X/}$ (Example 5.1.7.7), and is therefore weakly contractible. Applying Corollary 6.3.5.3, we deduce that the functor $\widetilde{\mathrm{ev}}_1$ exhibits $\mathcal{D}$ as a localization of $\mathcal{D} \tilde{\times}_{\mathcal{D}} \mathcal{C}$, and is therefore right cofinal (Proposition 7.2.1.10). We now observe that the functor $F : \mathcal{C} \to \mathcal{D}$ factors as a composition

$$\mathcal{C} \xrightarrow{\widetilde{\delta}} \mathcal{M} \xrightarrow{\widetilde{\mathrm{ev}}_1} \mathcal{D},$$

and is therefore also right cofinal (Proposition 7.2.1.6). □

Combining Theorem 7.2.3.1 with Proposition 7.2.3.12, we obtain the following:

Corollary 7.2.3.15 (Fiberwise Cofinality Criterion)**.** *Suppose we are given a commutative diagram of simplicial sets* 03LR

$$\begin{tikzcd}
\mathcal{E}' \arrow[rr, "F"] \arrow[dr, "U'"'] & & \mathcal{E} \arrow[dl, "U"] \\
& \mathcal{C} &
\end{tikzcd}$$

where U and U' are cartesian fibrations, and the morphism F carries U'-cartesian edges of $\mathcal{E}'$ to U-cartesian edges of $\mathcal{E}$. The following conditions are equivalent:

(1) *The morphism F is right cofinal.*

(2) *For every vertex $C \in \mathcal{C}$, the induced map of fibers $F_C : \mathcal{E}'_C \to \mathcal{E}_C$ is right cofinal.*

Proof. We first reduce to the case where $\mathcal{C}$ is an ∞-category. Using Corollary 4.1.3.3, we can choose an inner anodyne morphism $\iota : \mathcal{C} \hookrightarrow \overline{\mathcal{C}}$, where $\overline{\mathcal{C}}$ is an ∞-category. Using Proposition 5.6.7.2, we can extend U and U' to cocartesian fibrations of ∞-categories $\overline{U} : \overline{\mathcal{E}} \to \overline{\mathcal{C}}$ and $\overline{U}' : \overline{\mathcal{E}}' \to \overline{\mathcal{C}}$. Then the inclusion maps $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ and $\mathcal{E}' \hookrightarrow \overline{\mathcal{E}}'$ are categorical equivalences (Lemma 5.3.6.5). Since $\overline{U}$ is an isofibration (Proposition 5.1.4.8), we can extend F to a functor $\overline{F} : \overline{\mathcal{E}}' \to \overline{\mathcal{E}}$ satisfying $\overline{U} \circ \overline{F} = \overline{U}'$ (Proposition 4.5.5.1). It follows from Remark 5.3.1.12 that the functor $\overline{F}$ carries $\overline{U}'$-cartesian morphisms of $\overline{\mathcal{E}}'$ to $\overline{U}$-cartesian morphisms of $\overline{\mathcal{E}}$. Moreover, the morphism F is right cofinal if and only if $\overline{F}$ is right cofinal (Corollary 7.2.1.22). Consequently, we can replace $\mathcal{C}$ by $\overline{\mathcal{C}}$ and thereby reduce to proving Corollary 7.2.3.15 in the case where $\mathcal{C}$ is an ∞-category.

Fix an object $X \in \mathcal{E}$, let $C = U(X)$ denote its image in $\mathcal{C}$, and wlet $\mathcal{E}_C$ and $\mathcal{E}'_C$ denote the fibers $\{C\} \times_{\mathcal{C}} \mathcal{E}$ and $\{C\} \times_{\mathcal{C}} \mathcal{E}'$, respectively. We will prove that the following conditions are equivalent:

(1_X) The ∞-category $\mathcal{E}' \times_{\mathcal{E}} \mathcal{E}_{X/}$ is weakly contractible.

(2_X) The ∞-category $\mathcal{E}'_C \times_{\mathcal{E}_C} (\mathcal{E}_C)_{X/}$ is weakly contractible.

Corollary 7.2.3.15 will then follow by allowing the object X to vary and applying the criterion of Theorem 7.2.3.1.

To complete the proof, it will suffice to show that the inclusion map

$$\mathcal{E}'_C \times_{\mathcal{E}_C} (\mathcal{E}_C)_{X/} \hookrightarrow \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}_{X/}$$

is a weak homotopy equivalence. In fact, we will show that it is left anodyne. Unwinding the definitions, we have a pullback diagram

$$\begin{tikzcd}
\mathcal{E}'_C \times_{\mathcal{E}_C} (\mathcal{E}_C)_{X/} \arrow[r] \arrow[d] & \mathcal{E}' \times_{\mathcal{E}} \mathcal{E}_{X/} \arrow[d] \\
\{\mathrm{id}_C\} \arrow[r] & \mathcal{C}_{C/},
\end{tikzcd}$$

where the right vertical map is a cartesian fibration (Corollary 5.1.4.21). By virtue of Proposition 7.2.3.12, we are reduced to showing that the inclusion map $\{\mathrm{id}_C\} \hookrightarrow \mathcal{C}_{C/}$ is left anodyne, or equivalently that $\{\mathrm{id}_C\}$ is an initial object of the ∞-category $\mathcal{C}_{C/}$ (Corollary 4.6.7.23). This is a special case of Proposition 4.6.7.22. □

7.2.4 Filtered ∞-Categories

02P8 We begin by recalling the classical notion of a filtered category.

Definition 7.2.4.1. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *filtered* if it satisfies the following conditions: 02P9

- The category $\mathcal{C}$ is nonempty.
- For every pair of objects $X, Y \in \mathcal{C}$, there exists an object $Z \in \mathcal{C}$ and a pair of morphisms $u : X \to Z$ and $v : Y \to Z$.
- For every pair of objects $X, Y \in \mathcal{C}$ and every pair of morphisms $f_0, f_1 : X \to Y$, there exists a morphism $v : Y \to Z$ in $\mathcal{C}$ satisfying $v \circ f_0 = v \circ f_1$.

Exercise 7.2.4.2. We say that a partially ordered set $(A, \leq)$ is *directed* if every finite subset $A_0 \subseteq A$ has an upper bound. Show that $(A, \leq)$ is directed if and only if it is filtered, when regarded as a category. 02PA

Our goal in this section is to introduce an ∞-categorical counterpart of Definition 7.2.4.1:

Definition 7.2.4.3. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *filtered* if, for every finite simplicial set K, every diagram $f : K \to \mathcal{C}$ admits an extension $\overline{f} : K^{\triangleright} \to \mathcal{C}$. 02PB

In §7.2.5, we will show that Definition 7.2.4.3 is a generalization of Definition 7.2.4.1: that is, a category $\mathcal{C}$ is filtered if and only if the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$ is filtered (Corollary 7.2.5.8).

Variant 7.2.4.4. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *cofiltered* if, for every finite simplicial set K, every diagram $f : K \to \mathcal{C}$ admits an extension $\overline{f} : K^{\triangleleft} \to \mathcal{C}$. Equivalently, $\mathcal{C}$ is cofiltered if the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ is filtered. 02PC

Example 7.2.4.5. Let $\mathcal{C}$ be an ∞-category which contains a final object X. Then every morphism of simplicial sets $f : K \to \mathcal{C}$ can be extended to a morphism $\overline{f} : K^{\triangleright} \to \mathcal{C}$ which carries the cone point of $K^{\triangleright}$ to the object X. In particular, the ∞-category $\mathcal{C}$ is filtered. For a more general statement, see Proposition 7.2.7.1. 02PD

Remark 7.2.4.6. Let $\{\mathcal{C}_{\alpha}\}$ be a filtered diagram of simplicial sets, where each $\mathcal{C}_{\alpha}$ is a filtered ∞-category. Then the colimit $\mathcal{C} = \varinjlim_{\alpha} \mathcal{C}_{\alpha}$ is also a filtered ∞-category. To prove this, we first observe that $\mathcal{C}$ is an ∞-category (Remark 1.4.0.9). If K is a finite simplicial set, then any morphism $f : K \to \mathcal{C}$ factors through $f_{\alpha} : K \to \mathcal{C}_{\alpha}$ for some index α (see Proposition 3.6.1.9). Our assumption that $\mathcal{C}_{\alpha}$ is filtered guarantees that f_{α} extends to a diagram $\overline{f}_{\alpha} : K^{\triangleright} \to \mathcal{C}_{\alpha}$, from which it follows that f extends to a diagram $\overline{f} : K^{\triangleright} \to \mathcal{C}$. 02PE

Remark 7.2.4.7. Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent: 02PF

(1) The ∞-category $\mathcal{C}$ is filtered.

(2) For every finite simplicial set K and every diagram $f : K \to \mathcal{C}$, the coslice ∞-category $\mathcal{C}_{f/}$ is nonempty.

(3) For every finite simplicial set K and every diagram $f : K \to \mathcal{C}$, the oriented fiber product $\{f\} \tilde{\times}_{\mathrm{Fun}(K,\mathcal{C})} \mathcal{C}$ is nonempty.

(4) For every finite simplicial set K and every diagram $f : K \to \mathcal{C}$, there exists a morphism $f \to f'$ in the ∞-category $\mathrm{Fun}(K, \mathcal{C})$, where $f' : K \to \mathcal{C}$ is a constant diagram.

The equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) follow immediately from the definitions, and the equivalence (2) $\Leftrightarrow$ (3) follows from Theorem 4.6.4.17.

02PG **Proposition 7.2.4.8.** *Let $\mathcal{C}$ be a filtered ∞-category and let $f : K \to \mathcal{C}$ be a diagram, where K is a finite simplicial set. Then the ∞-category $\mathcal{C}_{f/}$ is also filtered.*

Proof. By virtue of Remark 7.2.4.7, it will suffice to show that for every finite simplicial set L and every morphism $g : L \to \mathcal{C}_{f/}$, the ∞-category $(\mathcal{C}_{f/})_{g/}$ is nonempty. Unwinding the definitions, we can identify g with a morphism of simplicial sets $\overline{f} : K \star L \to \mathcal{C}$ satisfying $\overline{f}|_K = f$. This identification supplies an isomorphism $(\mathcal{C}_{f/})_{g/} \simeq \mathcal{C}_{\overline{f}/}$. We are therefore reduced to showing that the coslice ∞-category $\mathcal{C}_{\overline{f}/}$ is nonempty. This follows from Remark 7.2.4.7, since the simplicial set $K \star L$ is finite (Remark 4.3.3.21). $\square$

02PH **Proposition 7.2.4.9.** *Let $\mathcal{C}$ be a filtered ∞-category. Then $\mathcal{C}$ is weakly contractible.*

Proof. By virtue of Proposition 3.1.7.1, there exists a functor $Q : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta}$ and a natural transformation $u : \mathrm{id}_{\mathrm{Set}_{\Delta}} \to Q$ with the following properties:

- The functor Q commutes with filtered colimits.
- For every simplicial set X, the simplicial set $Q(X)$ is a Kan complex.
- For every simplicial set X, the morphism $u_X : X \to Q(X)$ is a weak homotopy equivalence.

To show that $\mathcal{C}$ is weakly contractible, it will suffice to show that the Kan complex $Q(\mathcal{C})$ is contractible. Note that $\mathcal{C}$ is nonempty, so that $Q(\mathcal{C})$ is also nonempty. It will therefore suffice to show that for every integer $n \geq 0$, every morphism of simplicial sets $\sigma : \partial \Delta^n \to Q(\mathcal{C})$ is nullhomotopic (see Variant 3.2.4.12). Since the simplicial set $\partial \Delta^n$ is finite and the functor Q commutes with filtered colimits, the morphism σ factors as a composition $\partial \Delta^n \to Q(K) \xrightarrow{Q(\iota)} Q(\mathcal{C})$, where K is a finite simplicial subset of $\mathcal{C}$ and $\iota : K \hookrightarrow \mathcal{C}$ denotes the inclusion map. We will complete the proof by showing that $Q(\iota)$ is nullhomotopic. Since $u_K : K \to Q(K)$ is a weak homotopy equivalence, this is equivalent to assertion that the composite morphism $Q(\iota) \circ u_K = u_{\mathcal{C}} \circ \iota$ is nullhomotopic. This is clear: our assumption that $\mathcal{C}$ is filtered guarantees that there exists a natural transformation from ι to a constant diagram $K \to \mathcal{C}$ (Remark 7.2.4.7). $\square$

Proposition 7.2.4.10. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:* 02PJ

(1) *The ∞-category $\mathcal{C}$ is filtered.*

(2) *For every finite simplicial set K and every morphism $f : K \to \mathcal{C}$, the ∞-category $\mathcal{C}_{f/}$ is filtered.*

(3) *For every finite simplicial set K and every morphism $f : K \to \mathcal{C}$, the ∞-category $\mathcal{C}_{f/}$ is weakly contractible.*

(4) *For every finite simplicial set K, the diagonal map $\delta : \mathcal{C} \to \operatorname{Fun}(K, \mathcal{C})$ is right cofinal.*

Proof. The implication (1) ⇒ (2) follows from Proposition 7.2.4.8, the implication (2) ⇒ (3) from Proposition 7.2.4.9, and the implication (3) ⇒ (1) is immediate from the definitions (Remark 7.2.4.7). The equivalence (3) ⇔ (4) is a special case of Corollary 7.2.3.9. □

Corollary 7.2.4.11. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories. Then $\mathcal{C}$ is filtered if and only if $\mathcal{D}$ is filtered.* 02PK

Proof. By virtue of Proposition 7.2.4.10, it will suffice to show that for every (finite) simplicial set K, the diagonal map $\delta_{\mathcal{C}} : \mathcal{C} \to \operatorname{Fun}(K, \mathcal{C})$ is right cofinal if and only if the diagonal map $\delta_{\mathcal{D}} : \mathcal{D} \to \operatorname{Fun}(K, \mathcal{D})$ is right cofinal. This follows by applying Corollary 7.2.1.22 to the commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\delta_{\mathcal{C}}"] \arrow[d, "F"] & \operatorname{Fun}(K, \mathcal{C}) \arrow[d, "F \circ"] \\
\mathcal{D} \arrow[r, "\delta_{\mathcal{D}}"] & \operatorname{Fun}(K, \mathcal{D}).
\end{tikzcd}$$

□

Corollary 7.2.4.12. *Let $\mathcal{C}$ be a Kan complex. Then $\mathcal{C}$ is filtered if and only if it is contractible.* 02PL

Proof. If $\mathcal{C}$ is a contractible Kan complex, then there exists a categorical equivalence $\mathcal{C} \to \Delta^0$, so that $\mathcal{C}$ is filtered by virtue of Corollary 7.2.4.11. The converse is a special case of Proposition 7.2.4.9. □

7.2.5 Local Characterization of Filtered ∞-Categories

02PM Let $\mathcal{C}$ be an ∞-category and let $\mathrm{h}\mathcal{C}$ denote the homotopy category of $\mathcal{C}$, which we view as enriched over the homotopy category hKan of Kan complexes (see Construction 4.6.9.13). In this section, we show that the condition that $\mathcal{C}$ is filtered can be formulated entirely in terms of $\mathrm{h}\mathcal{C}$, together with its hKan-enrichment (Theorem 7.2.5.5).

02PN **Definition 7.2.5.1.** Let hKan denote the homotopy category of Kan complexes (Construction 7.2.1.22), and let $\mathcal{C}$ be a category which is enriched over hKan. We will say that $\mathcal{C}$ is *homotopy filtered* if it is nonempty and satisfies the following condition for each $n \geq 1$:

$(*_n)$ For every pair of objects $X, Y \in \mathcal{C}$ and for every morphism of simplicial sets $\sigma : \partial\Delta^{n-1} \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$, there exists a morphism $v : Y \to Z$ for which the composite morphism

$$\partial\Delta^{n-1} \xrightarrow{\sigma} \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \xrightarrow{v\circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z)$$

is nullhomotopic.

02PP **Warning 7.2.5.2.** In the formulation of condition $(*_n)$ of Definition 7.2.5.1, postcomposition with v defines a map of Kan complexes $V : \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Z)$ which is only well-defined up to homotopy. However, the condition that $V \circ \sigma$ is nullhomotopic depends only on the homotopy class of V.

02PQ **Example 7.2.5.3.** Let $\mathcal{C}$ be an ordinary category, which we regard as an hKan-enriched category in which each of the Kan complexes $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$ is equal to $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ (regarded as a constant simplicial set). In this case, condition $(*_n)$ of Definition 7.2.5.1 is automatically satisfied for $n \geq 3$. Moreover, we can state conditions $(*_1)$ and $(*_2)$ more concretely as follows:

$(*_1)$ For every pair of objects $X, Y \in \mathcal{C}$, there exists an object $Z \in \mathcal{C}$ equipped with morphisms $u : X \to Z$ and $v : Y \to Z$.

$(*_2)$ For every pair of objects $X, Y \in \mathcal{C}$ and every pair of morphisms $f_0, f_1 : X \to Y$, there exists a morphism $v : Y \to Z$ satisfying $v \circ f_0 = v \circ f_1$.

It follows that $\mathcal{C}$ is homotopy filtered (in the sense of Definition 7.2.5.1) if and only if is filtered (in the sense of Definition 7.2.4.1).

02PR **Remark 7.2.5.4.** Let $\mathcal{C}$ be an hKan-enriched category. If $\mathcal{C}$ is homotopy filtered (in the sense of Definition 7.2.5.1), then it is filtered when regarded as an ordinary category (in the sense of Definition 7.2.4.1). Beware that the converse is false in general (see Warning 7.2.5.7).

We can now state the main result of this section:

Theorem 7.2.5.5. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is filtered (in the sense of Definition 7.2.4.3) if and only if the homotopy category $\mathrm{h}\mathcal{C}$ is homotopy filtered (in the sense of Definition 7.2.5.1), when regarded as an hKan-enriched category by means of Construction 4.6.9.13.* 02PS

Before giving the proof of Theorem 7.2.5.5, let us note some of its consequences.

Corollary 7.2.5.6. *Let $\mathcal{C}$ be a filtered ∞-category (in the sense of Definition 7.2.4.3). Then $\mathrm{h}\mathcal{C}$ is a filtered category (in the sense of Definition 7.2.4.1).* 02PT

Proof. Combine Theorem 7.2.5.5 with Remark 7.2.5.4. □

Warning 7.2.5.7. The converse of Corollary 7.2.5.6 is false. For example, if $\mathcal{C}$ is a simply connected Kan complex, then the homotopy category $\mathrm{h}\mathcal{C}$ is automatically filtered. However, $\mathcal{C}$ is filtered if and only if it is contractible (Corollary 7.2.4.12). 02PU

Corollary 7.2.5.8. *Let $\mathcal{C}$ be a category. Then the category $\mathcal{C}$ is filtered (in the sense of Definition 7.2.4.1) if and only if the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ is filtered (in the sense of Definition 7.2.4.3).* 02PV

Proof. Combine Theorem 7.2.5.5 with Example 7.2.5.3. □

Example 7.2.5.9. Let $(A, \leq)$ be a partially ordered set. Combining Exercise 7.2.4.2 with Corollary 7.2.5.8, we see that the ∞-category $\mathrm{N}_\bullet(A)$ is filtered if and only if the partially ordered set $(A, \leq)$ is directed. 02PW

Corollary 7.2.5.10. *Let $\mathcal{C}$ be a locally Kan simplicial category. Then the ∞-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ is filtered if and only if the homotopy category $\mathrm{h}\mathcal{C}$ is homotopy filtered, when regarded as an hKan-enriched category.* 02PX

Proof. Combine Theorem 7.2.5.5 with Corollary 4.6.9.20. □

Exercise 7.2.5.11. Let $\mathcal{C}$ be a $(2,1)$-category (Definition 2.2.8.5). Show that the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$ is a filtered ∞-category if and only if $\mathcal{C}$ satisfies the following conditions: 02PY

- The 2-category $\mathcal{C}$ is nonempty.
- For every pair of objects $X, Y \in \mathcal{C}$, there exists an object $Z \in \mathcal{C}$ and a pair of 1-morphisms $f : X \to Z$ and $g : Y \to Z$.
- For every pair of objects $X, Y \in \mathcal{C}$ and every pair of 1-morphisms $f, g : X \to Y$, there exists a 1-morphism $h : Y \to Z$ such that the 1-morphisms $h \circ f$ and $h \circ g$ are isomorphic (when viewed as objects of the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Z)$).
- For every 1-morphism $f : X \to Y$ in $\mathcal{C}$ and every 2-morphism $\gamma : f \Rightarrow f$, there exists a 1-morphism $g : Y \to Z$ for which the horizontal composition $\mathrm{id}_g \circ \gamma$ is equal to the identity 2-morphism $\mathrm{id}_{g \circ f}$.

We now turn to the proof of Theorem 7.2.5.5. The easy part is to show that if $\mathcal{C}$ is a filtered ∞-category, then the homotopy category $\mathrm{h}\mathcal{C}$ is homotopy filtered. Condition $(*_n)$ of Definition 7.2.5.1 is a special case of the following assertion:

02PZ **Lemma 7.2.5.12.** *Let $\mathcal{C}$ be a filtered ∞-category containing objects X and Y, and let K be a finite simplicial set equipped with a morphism $f : K \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Then there exists a morphism $v : Y \to Z$ of $\mathcal{C}$ for which the composition $K \xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{v\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is nullhomotopic.*

Proof. Let $\Sigma(K)$ denote the iterated coproduct

$$\{x\}\coprod_{(\{0\}\times K)}(\Delta^1 \times K)\coprod_{(\{1\}\times K)}\{y\},$$

so that we can identify f with a morphism of simplicial sets $F : \Sigma(K) \to \mathcal{C}$ satisfying $F(x) = X$ and $F(y) = Y$. Our assumption that $\mathcal{C}$ is filtered guarantees that we can extend F to a morphism of simplicial set $\overline{F} : \Sigma(K) \star \{z\} \to \mathcal{C}$. Set $Z = \overline{F}(z)$. Then $\overline{F}$ carries $\{x\} \star \{z\}$ and $\{y\} \star \{z\}$ to morphisms $u : X \to Z$ and $v : Y \to Z$ in $\mathcal{C}$. Moreover, the natural map $\Delta^1 \times K \to \Sigma(K)$ admits a unique extension $q : \Delta^2 \times K \to \Sigma(K) \star \{z\}$ carrying $\{2\} \times K$ to the vertex z, and the composition

$$\Delta^2 \times K \xrightarrow{q} \Sigma(K) \star \{z\} \xrightarrow{\overline{F}} \mathcal{C}$$

determines a morphism of simplicial sets $g : K \to \mathrm{Hom}_{\mathcal{C}}(X, Y, Z)$. Unwinding the definitions, we see that the diagram of simplicial sets

$$\begin{tikzcd}
K \arrow[r, "{(v,f)}"] \arrow[dr, "g"] \arrow[dd] & \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \\
 & \mathrm{Hom}_{\mathcal{C}}(X, Y, Z) \arrow[u] \arrow[d] \\
\Delta^0 \arrow[r, "u"] & \mathrm{Hom}_{\mathcal{C}}(X, Z)
\end{tikzcd}$$

is strictly commutative, from which we immediately deduce (from the definition of the composition law on $\mathcal{C}$) that the composition $K \xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{v\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ is homotopic to the constant map taking the value u. □

The difficult half of Theorem 7.2.5.5 will require some further preliminaries. We first note that, to verify that an ∞-category $\mathcal{C}$ is filtered, it suffices to verify the extension condition of Definition 7.2.4.3 in the special case where $K = \partial \Delta^n$ is the boundary of a simplex.

Lemma 7.2.5.13. *An ∞-category $\mathcal{C}$ is filtered if and only if it satisfies the following condition for every integer $n \geq 0$:* 02Q0

$(*'_n)$ *Every morphism of simplicial sets $\partial\Delta^n \to \mathcal{C}$ can be extended to a morphism $(\partial\Delta^n)^{\triangleright} \to \mathcal{C}$.*

Proof. The necessity of condition $(*'_n)$ is clear. For the converse, suppose that $\mathcal{C}$ satisfies $(*'_n)$ for each $n \geq 0$. We wish to prove that $\mathcal{C}$ is filtered. Let $f : K \to \mathcal{C}$ be a diagram where K is a finite simplicial set; we wish to show that the ∞-category $\mathcal{C}_{f/}$ is nonempty. If $K = \emptyset$, then this follows immediately from assumption $(*'_0)$. Otherwise, the simplicial set K has dimension m for some integer $m \geq 0$. We proceed by induction on m and on the number of nondegenerate m-simplices of K. Choose a nondegenerate m-simplex $\sigma : \Delta^m \to K$. Using Proposition 1.1.4.12, we can choose a pushout diagram

$$\begin{tikzcd}
\partial\Delta^m \arrow[r] \arrow[d] & \Delta^m \arrow[d, "\sigma"] \\
K' \arrow[r] & K
\end{tikzcd}$$

where $K' \subseteq K$ is a simplicial subset having a smaller number of nondegenerate m-simplices. Set $f' = f|_{K'}$, $f_0 = f \circ \sigma$, and $f'_0 = f \circ \sigma|_{\partial\Delta^m}$, so that we have a pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C}_{f/} \arrow[r] \arrow[d] & \mathcal{C}_{f'/} \arrow[d, "\Phi"] \\
\mathcal{C}_{f_0/} \arrow[r, "\Psi"] & \mathcal{C}_{f'_0/} .
\end{tikzcd}$$

Applying our inductive hypothesis, we deduce that the ∞-category $\mathcal{C}_{f'/}$ is nonempty. Choose an object X of $\mathcal{C}_{f'/}$, so that $\Phi(X) \in \mathcal{C}_{f_{\partial\Delta}/}$ can be identified with a morphism of simplicial sets $g : (\partial\Delta^m)^{\triangleright} \to \mathcal{C}$. Amalgamating $f \circ \sigma$ with g, we obtain a morphism of simplicial sets

$$\overline{g} : \partial\Delta^{m+1} \simeq (\partial\Delta^m)^{\triangleright} \coprod_{\partial\Delta^m} \Delta^m \to \mathcal{C} .$$

Invoking $(*_{m+1})$, we conclude that $\overline{g}$ can be extended to a morphism of simplicial sets $(\partial\Delta^{m+1})^{\triangleright} \to \mathcal{C}$. Unwinding the definitions, we see that this extension supplies an object $Y \in \mathcal{C}_{f_0/}$ together with a morphism $u : \Phi(X) \to \Psi(Y)$ in the ∞-category $\mathcal{C}_{f'_0/}$.

Note that the projection maps $\mathcal{C}_{f'/} \to \mathcal{C} \leftarrow \mathcal{C}_{f'_0/}$ are left fibrations (Proposition 4.3.6.1). Let $\overline{X}$ denote the image of X in the ∞-category $\mathcal{C}$, so that Corollary 4.3.7.13 guarantees that the vertical maps in the diagram

$$\begin{tikzcd}
(\mathcal{C}_{f'/})_{X/} \arrow[rr, "\Phi_{X/}"] \arrow[dr] & & (\mathcal{C}_{f'_0/})_{U(X)/} \arrow[dl] \\
& \mathcal{C}_{\overline{X}/} &
\end{tikzcd}$$

are trivial Kan fibrations. In particular, they are equivalences of ∞-categories, so that the functor $\Phi_{X/}$ is also an equivalence of ∞-categories. It follows that we can choose a morphism $w : X \to Z$ in the ∞-category $\mathcal{C}_{f'/}$ and a 2-simplex

$$\begin{tikzcd}
& \Psi(Y) \arrow[dr, "v"] & \\
\Phi(X) \arrow[ur, "u"] \arrow[rr, "\Phi(w)"] & & \Phi(Z)
\end{tikzcd}$$

in the ∞-category $\mathcal{C}_{f'_0/}$, where v is an isomorphism. Since Ψ is a left fibration (Corollary 4.3.6.11), we can lift v to a morphism $\widetilde{v} : Y \to \widetilde{Z}$ of the ∞-category $\mathcal{C}_{f_0/}$. The pair $(Z, \widetilde{Z})$ can then be regarded as an object of the ∞-category $\mathcal{C}_{f/} = \mathcal{C}_{f'/} \times_{\mathcal{C}_{f'_0/}} \mathcal{C}_{f_0/}$. □

02Q1 **Remark 7.2.5.14.** Let $\mathcal{C}$ be an ∞-category and let $n \geq 0$ be a nonnegative integer. Condition $(*'_n)$ of Lemma 7.2.5.13 is equivalent to the assertion that, for every morphism of simplicial sets $f : \partial\Delta^n \to \mathcal{C}$, the coslice ∞-category $\mathcal{C}_{f/}$ is nonempty. By virtue of Theorem 4.6.4.17, this is equivalent to the requirement that the oriented fiber product $\{f\} \tilde{\times}_{\mathrm{Fun}(\partial\Delta^n, \mathcal{C})} \mathcal{C}$ is nonempty. We can therefore reformulate $(*'_n)$ as follows:

$(*'_n)$ For every diagram $f : \partial\Delta^n \to \mathcal{C}$, there exists an object $C \in \mathcal{C}$ and a natural transformation $f \to \underline{C}$, where $\underline{C} : \partial\Delta^n \to \mathcal{C}$ is the constant morphism taking the value C.

For each integer $n \geq 1$, let us identify the standard simplex Δ^{n-1} with its image in $\partial\Delta^n \subset \Delta^n$ (given by the face opposite the nth vertex).

02Q2 **Lemma 7.2.5.15.** *Let $\mathcal{C}$ be an ∞-category and let $n \geq 1$ be an integer. Then condition $(*'_n)$ of Lemma 7.2.5.13 is equivalent to the following:*

$(*''_n)$ *Let $f : \partial\Delta^n \to \mathcal{C}$ be a morphism of simplicial sets for which the restriction $f|_{\Delta^{n-1}}$ is constant. Then f can be extended to a morphism $\overline{f} : (\partial\Delta^n)^{\triangleright} \to \mathcal{C}$.*

Proof. The implication $(*'_n) \Rightarrow (*''_n)$ is immediate. We will prove the converse. Assume that $(*''_n)$ is satisfied, and let $g : \partial \Delta^n \to \mathcal{C}$ be an arbitrary morphism of simplicial sets; we wish to show that g can be extended to a morphism $\overline{g} : (\partial \Delta^n) \to \mathcal{C}$. If $n = 1$, this follows immediately form $(*''_n)$; we will therefore assume that $n \geq 2$. Note that we can write $\partial \Delta^n$ as the union of Δ^{n-1} and the horn Λ^n_n, whose intersection is the simplicial subset $\partial \Delta^{n-1} \subset \Delta^{n-1}$. Set

$$g_- = g|_{\Delta^{n-1}} \qquad g_\pm = g|_{\partial \Delta^{n-1}} \qquad g_+ = g|_{\Lambda^n_n}.$$

Let $X = g(0)$ and $Y = g(n)$ and let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ denote the projection map, so that we can identify g_+ with a morphism $\widetilde{g}_\pm : \partial \Delta^{n-1} \to \mathcal{C}_{/Y}$ satisfying $\pi \circ \widetilde{g}_\pm = g_\pm$.

Let $f_- : \Delta^{n-1} \to \mathcal{C}$ be the constant morphism taking the value X, and let $h_- : f_- \to g_-$ be the natural transformation given by the composite map

$$\Delta^1 \times \Delta^{n-1} \xrightarrow{(i,j) \mapsto ij} \Delta^{n-1} \xrightarrow{g_-} \mathcal{C}\,.$$

Set $f_\pm = f_-|_{\partial \Delta^{n-1}}$ and $h_\pm = h_-|_{\Delta^1 \times \partial \Delta^{n-1}}$, so that $h_\pm$ can be regarded as a natural transformation from $f_\pm$ to $g_\pm$. Since π is a right fibration, we can lift $h_\pm$ to a natural transformation $\widetilde{h}_\pm : \widetilde{f}_\pm \to \widetilde{g}_\pm$ in the ∞-category $\operatorname{Fun}(\partial \Delta^{n-1}, \mathcal{C}_{/Y})$. Let us identify $\widetilde{f}_\pm$ with a morphism of simplicial sets $f_+ : \Lambda^n_n \to \mathcal{C}$ satisfying $f_+(n) = Y$. Then $\widetilde{h}_\pm$ determines a natural transformation $h_+ : f_+ \to g_+$, given by the composition

$$\Delta^1 \times \Lambda^n_n \simeq \Delta^1 \times (\partial \Delta^{n-1})^{\triangleright} \to (\Delta^1 \times \partial \Delta^{n-1})^{\triangleright} \xrightarrow{\widetilde{h}_\pm} (\mathcal{C}_{/Y})^{\triangleright} \to \mathcal{C}\,.$$

Note that f_- and f_+ can be amalgamated to a morphism $f : \partial \Delta^n \to \mathcal{C}$, and that h_- and h_+ can be amalgamated to a natural transformation $h : f \to g$ in $\operatorname{Fun}(\partial \Delta^n, \mathcal{C})$.

Invoking hypothesis $(*''_n)$, we see that f can be extended to a morphism $\overline{f} : (\partial \Delta^n)^{\triangleright} \to \mathcal{C}$. Let $Z \in \mathcal{C}$ denote the image under $\overline{f}$ of the cone point and let $\varphi : \mathcal{C}_{/Z} \to \mathcal{C}$ denote the projection map, so that $\overline{f}$ can be identified with a morphism of simplicial sets $f' : \partial \Delta^n \to \mathcal{C}_{/Z}$ satisfying $\varphi \circ f' = f$. Let us identify the vertex $f'(n) \in \mathcal{C}_{/Z}$ with a morphism $v : Y \to Z$ in the ∞-category $\mathcal{C}$, so that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{/v} \arrow[r, "\varphi'"] \arrow[d, "\pi'"] & \mathcal{C}_{/Y} \arrow[d, "\pi"] \\
\mathcal{C}_{/Z} \arrow[r, "\varphi"] & \mathcal{C}\,.
\end{tikzcd}$$

Set $f'_+ = f'|_{\Lambda^n_n}$ and $f'_\pm = f'|_{\partial \Delta^{n-1}}$, so that we can identify f'_+ with a morphism $\widetilde{f}'_\pm : \partial \Delta^{n-1} \to \mathcal{C}_{/v}$ satisfying $\pi' \circ \widetilde{f}'_\pm = f'_+$. Since the inclusion $\{0\} \hookrightarrow \Delta^1$ is left anodyne, the morphism $\varphi' : \mathcal{C}_{/v} \to \mathcal{C}_{/Y}$ is a trivial Kan fibration (Corollary 4.3.6.13). We can therefore lift $\widetilde{h}_{\pm}$ to a natural transformation $\widetilde{h}'_{\pm} : \widetilde{f}'_{\pm} \to \widetilde{g}'_{\pm}$ for some morphism $\widetilde{g}'_{\pm} : \partial\Delta^{n-1} \to \mathcal{C}_{/v}$. Let us identify $\widetilde{g}'_{\pm}$ with a morphism $g'_{+} : \Lambda^n_n \to \mathcal{C}_{/Z}$ satisfying $\varphi \circ g'_{+} = g_{+}$. Then $\widetilde{h}'_{\pm}$ determines a natural transformation $h'_{+} : f'_{+} \to g'_{+}$, given by the composition

$$\Delta^1 \times \Lambda^n_n \simeq \Delta^1 \times (\partial\Delta^{n-1})^{\triangleright} \to (\Delta^1 \times \partial\Delta^{n-1})^{\triangleright} \xrightarrow{\widetilde{h}'_{\pm}} (\mathcal{C}_{/v})^{\triangleright} \to \mathcal{C}_{/Z}\,.$$

Let e denote the restriction $h'_{+}|_{\Delta^1 \times \{0\}}$, which we regard as an edge of the simplicial set $\mathcal{C}_{/Z}$. By construction, $\varphi(e)$ is the degenerate edge id_X of $\mathcal{C}$. Since φ is a right fibration (Proposition 4.3.6.1), it follows that e is an isomorphism in $\mathcal{C}_{/Z}$ (Proposition 4.4.2.11). Applying Proposition 4.4.5.8, we deduce that the lifting problem

$$\begin{tikzcd}
(\Delta^1 \times \Lambda^n_n)\coprod_{(\{0\}\times\Lambda^n_n)}(\{0\} \times \partial\Delta^n) \arrow[r, "{(h'_{+}, f')}"] \arrow[d] & \mathcal{C}_{/Z} \arrow[d, "\varphi"] \\
\Delta^1 \times \partial\Delta^n \arrow[r, "h"] \arrow[ur, dashed, "h'"] & \mathcal{C}
\end{tikzcd}$$

admits a solution. The morphism h' is then a natural transformation from f' to a morphism $g' : \partial\Delta^n \to \mathcal{C}_{/Z}$, which we can identify with a map $\overline{g} : (\partial\Delta^n)^{\triangleright} \to \mathcal{C}$ satisfying $\overline{g}|_{\partial\Delta^n} = g$. □

Proof of Theorem 7.2.5.5. Let $\mathcal{C}$ be an ∞-category and suppose that the homotopy category $\mathrm{h}\mathcal{C}$ is homotopy filtered; we wish to show that $\mathcal{C}$ is filtered (the reverse implication follows from Lemma 7.2.5.12). By virtue of Lemma 7.2.5.13, it will suffice to show that for every integer $n \geq 0$, every morphism of simplicial sets $f : \partial\Delta^n \to \mathcal{C}$ can be extended to a morphism $\overline{f} : (\partial\Delta^n)^{\triangleright} \to \mathcal{C}$. For $n = 0$, this follows from our assumption that $\mathrm{h}\mathcal{C}$ is nonempty. We will therefore assume that $n > 0$. By virtue of Lemma 7.2.5.15, we may assume without loss of generality that the restriction $f_{-} = f|_{\Delta^{n-1}}$ is the constant map taking the value X for some object $X \in \mathcal{C}$. Set $Y = f(n)$ and let $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y) = \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Y}$ denote the right-pinched morphism space of Construction 4.6.5.1, so that we can identify $f|_{\Lambda^n_n}$ with a morphism of simplicial sets $g : \partial\Delta^{n-1} \to \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y)$. Invoking assumption $(*_n)$ of Definition 7.2.5.1, we deduce that there exists a morphism $v : Y \to Z$ in $\mathcal{C}$ for which the composite map

$$\partial\Delta^{n-1} \xrightarrow{g} \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{[v]\circ} \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

is nullhomotopic. Since the projection map $\mathcal{C}_{/f} \to \mathcal{C}_{/Y}$ is a trivial Kan fibration (Corollary 4.3.6.13), we can lift g to a morphism $\widetilde{g} : \partial\Delta^{n-1} \to \{X\} \times_{\mathcal{C}} \mathcal{C}_{/f}$. Combining Propositions

5.2.8.7 and 4.6.9.16, we deduce that the diagram of Kan complexes

$$\begin{tikzcd}
\{X\} \times_{\mathcal{C}} \mathcal{C}_{/Y} \arrow[d, "\iota^{\mathrm{R}}_{X,Y}"] & \{X\} \times_{\mathcal{C}} \mathcal{C}_{/f} \arrow[l] \arrow[r] & \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Z} \arrow[d, "\iota^{\mathrm{R}}_{X,Z}"] \\
\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[rr, "{[v]\circ}"] & & \mathrm{Hom}_{\mathcal{C}}(X,Z)
\end{tikzcd}$$

commutes up to homotopy, where $\iota^{\mathrm{R}}_{X,Y}$ and $\iota^{\mathrm{R}}_{X,Z}$ are the right-pinch inclusion morphisms of Construction 4.6.5.7. Since $\iota^{\mathrm{R}}_{X,Z}$ is a homotopy equivalence (Proposition 4.6.5.10), it follows that the composite map $\partial\Delta^{n-1} \xrightarrow{\widetilde{g}} \{X\} \times_{\mathcal{C}} \mathcal{C}_{/f} \to \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Z}$ is nullhomotopic, and can therefore be extended to an $(n-1)$-simplex $g' : \Delta^{n-1} \to \{X\} \times_{\mathcal{C}} \mathcal{C}_{/Z}$ (Variant 3.2.4.12). Unwinding the definitions, we can identify $\widetilde{g}$ and g' with morphisms $(\Lambda^n_n)^{\triangleright} \to \mathcal{C}$ and $(\Delta^{n-1})^{\triangleright} \to \mathcal{C}$, which can be amalgamated to a single morphism $\overline{f} : (\partial\Delta^n)^{\triangleright} \to \mathcal{C}$ extending f. □

Exercise 7.2.5.16. Let $\mathcal{C}$ be an ∞-category and let $n \geq 1$ be an integer. Show that the homotopy category $\mathrm{h}\mathcal{C}$ satisfies condition $(*_n)$ of Definition 7.2.5.1 if and only if $\mathcal{C}$ satisfies condition $(*'_n)$ of Lemma 7.2.5.13. 02Q3

7.2.6 Left Fibrations over Filtered ∞-Categories

Our goal in this section is to prove the following: 02Q4

Theorem 7.2.6.1. *Let $U : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a left fibration of ∞-categories, where the ∞-category $\mathcal{C}$ is filtered. For each object $X \in \mathcal{C}$, let $\widetilde{\mathcal{C}}_X$ denote the fiber $\{X\} \times_{\mathcal{C}} \widetilde{\mathcal{C}}$. The following conditions are equivalent:* 02Q5

(1) *The ∞-category $\widetilde{\mathcal{C}}$ is filtered.*

(2) *The ∞-category $\widetilde{\mathcal{C}}$ is weakly contractible.*

(3) *For every object $X \in \mathcal{C}$ and every diagram $e : K \to \widetilde{\mathcal{C}}_X$ where K is a finite simplicial set, there exists a morphism $f : X \to Y$ in $\mathcal{C}$ for which the composite map $K \xrightarrow{e} \widetilde{\mathcal{C}}_X \xrightarrow{f_!} \widetilde{\mathcal{C}}_Y$ is nullhomotopic; here $f_! : \widetilde{\mathcal{C}}_X \to \widetilde{\mathcal{C}}_Y$ is given by covariant transport along f (see Notation 5.2.2.9).*

(4) *For every object $X \in \mathcal{C}$, every integer $n \geq 0$, and every diagram $e : \partial\Delta^n \to \widetilde{\mathcal{C}}_X$, there exists a morphism $f : X \to Y$ in $\mathcal{C}$ for which the composite map $\partial\Delta^n \xrightarrow{e} \widetilde{\mathcal{C}}_X \xrightarrow{f_!} \widetilde{\mathcal{C}}_Y$ is nullhomotopic.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 7.2.4.9 and the implication (3) $\Rightarrow$ (4) is immediate. We next show that (4) implies (1). Assume that condition (4) is satisfied; we wish to prove that $\widetilde{\mathcal{C}}$ is filtered. By virtue of Lemma 7.2.5.13 (and Remark 7.2.5.14), it will suffice to show that for every integer $n \geq 0$ and every diagram $e : \partial \Delta^n \to \widetilde{\mathcal{C}}$, there exists a natural transformation from e to a constant diagram. Set $\overline{e} = U \circ e$, which we regard as an object of the ∞-category $\mathrm{Fun}(\partial \Delta^n, \mathcal{C})$. Since $\mathcal{C}$ is filtered, there exists an object $X \in \mathcal{C}$ and a morphism $\overline{\alpha} : \overline{e} \to \underline{X}$ in the ∞-category $\mathrm{Fun}(\partial \Delta^n, \mathcal{C})$, where $\underline{X} : \partial \Delta^n \to \mathcal{C}$ denotes the constant morphism taking the value X. Since U is a left fibration, we can lift $\overline{\alpha}$ to a morphism $\alpha : e \to e'$ in $\mathrm{Fun}(\partial \Delta^n, \widetilde{\mathcal{C}})$, where e' is a morphism from $\partial \Delta^n$ to the Kan complex $\widetilde{\mathcal{C}}_X$ (see Remark 4.2.6.3). Invoking assumption (4), we can choose a morphism $f : X \to Y$ in $\mathcal{C}$ and a covariant transport functor $f_! : \widetilde{\mathcal{C}}_X \to \widetilde{\mathcal{C}}_Y$ for which the composite map $f_! \circ u'$ is nullhomotopic. It follows that there exists a natural transformation $\beta : e' \to e''$ in $\mathrm{Fun}(\partial \Delta^n, \widetilde{\mathcal{C}})$, where $e'' : \partial \Delta^n \to \widetilde{\mathcal{C}}_Y$ is a constant map. Any choice of composition of α and β then determines a natural transformation from e to the constant diagram e''.

We now complete the proof by showing that (2) implies (3). Assume that the ∞-category $\widetilde{\mathcal{C}}$ is weakly contractible, and suppose that we are given an object $X \in \mathcal{C}$ and a diagram $e : K \to \widetilde{\mathcal{C}}_X$, where the simplicial set K is finite. We wish to show that there exists a morphism $f : X \to Y$ in $\mathcal{C}$ for which the composite map $K \xrightarrow{e} \widetilde{\mathcal{C}}_X \xrightarrow{f_!} \widetilde{\mathcal{C}}_Y$ is nullhomotopic. Choose an embedding $K \hookrightarrow L$, where L is another finite simplicial set which is weakly contractible (for example, we can take $L = K^{\triangleright}$). Let $\mathrm{Ex}^{\infty}(\widetilde{\mathcal{C}})$ be the simplicial set given by Construction 3.3.6.1, so that $\mathrm{Ex}^{\infty}(\widetilde{\mathcal{C}})$ is a Kan complex (Proposition 3.3.6.9). Let $\rho^{\infty} : \widetilde{\mathcal{C}} \to \mathrm{Ex}^{\infty}(\widetilde{\mathcal{C}})$ be the weak homotopy equivalence of Proposition 3.3.6.7. Since $\widetilde{\mathcal{C}}$ is weakly contractible, the Kan complex $\mathrm{Ex}^{\infty}(\widetilde{\mathcal{C}})$ is contractible. It follows that the composite map $K \xrightarrow{e} \widetilde{\mathcal{C}}_X \xrightarrow{\rho^{\infty}} \mathrm{Ex}^{\infty}(\widetilde{\mathcal{C}})$ can be extended to a map $e^{+} : L \to \mathrm{Ex}^{\infty}(\widetilde{\mathcal{C}})$. Since the simplicial set L is finite, the morphism $\overline{e}$ factors through $\mathrm{Ex}^{m}(\widetilde{\mathcal{C}})$ for some $m \gg 0$ (see Proposition 3.6.1.9). By virtue of Proposition 3.3.4.8, we can replace K and L by the iterated subdivisions subdivision $\mathrm{Sd}^m(K)$ and $\mathrm{Sd}^m(L)$ (and e by the composite map $\mathrm{Sd}^m(K) \twoheadrightarrow K \xrightarrow{e} \widetilde{\mathcal{C}}_X$) and thereby reduce to the case $m = 0$, so that e admits an extension $e^{+} : L \to \widetilde{\mathcal{C}}$.

Set $\overline{e}^{+} = U \circ e^{+}$, which we regard as an object of the ∞-category $\mathrm{Fun}(L, \mathcal{C})$. Since $\mathcal{C}$ is filtered, there exists an object $Y \in \mathcal{C}$ and a natural transformation $\overline{\alpha} : \overline{e}^{+} \to \underline{Y}$, where $\underline{Y} \in \mathrm{Fun}(L, \mathcal{C})$ denotes the constant diagram taking the value Y (Remark 7.2.4.7). Let $\overline{\alpha}_0$ denote the image of $\overline{\alpha}$ in $\mathrm{Fun}(K, \mathcal{C})$. Then $\overline{\alpha}_0$ can be identified with a morphism from K to the morphism space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Since $\mathcal{C}$ is filtered, Theorem 7.2.5.5 guarantees the existence of a morphism $g : Y \to Z$ of $\mathcal{C}$ for which the composite map

$$K \xrightarrow{\overline{\alpha}_0} \mathrm{Hom}_{\mathcal{C}}(X, U) \xrightarrow{g \circ} \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

is nullhomotopic. Let $\underline{Z} : L \to \mathcal{C}$ denote the constant diagram taking the value Z, so that g determines a morphism $\underline{g} : \underline{Y} \to \underline{Z}$ in the ∞-category $\mathrm{Fun}(L, \mathcal{C})$. Replacing Y by Z and $\overline{\alpha}$ by its composition with $\underline{g}$, we can reduce to the case where the morphism $\overline{\alpha}_0 : K \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is nullhomotopic. Note that the restriction map $\mathrm{Fun}(L, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$ is an isofibration of ∞-categories (Corollary 4.4.5.3), and therefore induces a Kan fibration of morphism spaces $\mathrm{Hom}_{\mathrm{Fun}(L,\mathcal{C})}(\overline{e}^{+}, \underline{Y}) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\overline{e}^{+}|_{K}, \underline{Y}_{K})$ (Exercise 4.6.1.24). We may therefore modify $\overline{\alpha}$ by a homotopy and thereby reduce to the case where $\overline{\alpha}_0 : K \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is the constant map taking some value $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Since U is a left fibration, we can lift $\overline{\alpha}$ to a natural transformation $\alpha : e^{+} \to e'^{+}$, for some diagram $e'^{+} : L \to \widetilde{\mathcal{C}}_Y \subseteq \widetilde{\mathcal{C}}$. Set $e' = e'^{+}|_{K}$, so that α restricts to a natural transformation $\alpha_0 : e \to e'$ which witnesses e' as given by covariant transport along f, in the sense of Definition 5.2.2.4. To complete the proof, it will suffice to show that the morphism $e' : K \to \widetilde{\mathcal{C}}_Y$ is nullhomotopic. This is clear: already the morphism $e'^{+} : L \to \widetilde{\mathcal{C}}_Y$ is nullhomotopic, since L is weakly contractible and $\widetilde{\mathcal{C}}_Y$ is a Kan complex (see Remark 3.2.4.18). □

Corollary 7.2.6.2. *Suppose we are given a pullback diagram of simplicial sets* 02Q6

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r, "V"] & \mathcal{C},
\end{tikzcd}$$

where U and V are left fibrations. If $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{E}$ are filtered ∞-categories, then $\mathcal{E}'$ is also a filtered ∞-category.

Proof. Since $U' : \mathcal{E}' \to \mathcal{C}'$ is a pullback of U, it is a left fibration. It will therefore suffice to show that U' satisfies condition (4) of Theorem 7.2.6.1. Suppose we are given an object $X' \in \mathcal{C}'$ and a morphism of simplicial sets $e : \partial \Delta^n \to \mathcal{E}'_{X'} = \{X'\} \times_{\mathcal{C}'} \mathcal{E}'$. Set $X = V(X')$, so that we can identify e with a morphism from $\partial \Delta^n$ to the fiber $\mathcal{E}_X = \{X\} \times_{\mathcal{C}} \mathcal{E}$. Since $\mathcal{E}$ and $\mathcal{C}$ are filtered, Theorem 7.2.6.1 guarantees that we can choose a morphism $f : X \to Y$ in $\mathcal{C}$ for which the composite map $\partial \Delta^n \xrightarrow{e} \mathcal{E}_X \xrightarrow{f_!} \mathcal{E}_Y$ is nullhomotopic, where $f_!$ is given by covariant transport along f. Since V is a left fibration, we can write $f = V(f')$ for some morphism $f' : X' \to Y'$ in the ∞-category $\mathcal{C}'$. Under the canonical isomorphisms $\mathcal{E}'_{X'} \simeq \mathcal{E}_X$ and $\mathcal{E}'_{Y'} \simeq \mathcal{E}_Y$, the morphism $f_! : \mathcal{E}_X \to \mathcal{E}_Y$ corresponds to a functor $f'_! : \mathcal{E}'_{X'} \to \mathcal{E}'_{Y'}$ given by covariant transport along f' (Remark 5.2.8.5), so that the composition $(f'_! \circ e) : \partial \Delta^n \to \mathcal{E}'_{Y'}$ is also nullhomotopic. □

Using Corollary 7.2.6.2, we obtain another characterization of the class of filtered ∞-categories:

Corollary 7.2.6.3. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is filtered if and only if it satisfies the following pair of conditions:* 02Q7

(a) *The* ∞*-category* $\mathcal{C}$ *is weakly contractible.*

(b) *Let* $U : \widetilde{\mathcal{C}} \to \mathcal{C}$, $V_0 : \widetilde{\mathcal{C}}_0 \to \widetilde{\mathcal{C}}$, *and* $V_1 : \widetilde{\mathcal{C}}_1 \to \widetilde{\mathcal{C}}$ *be left fibrations of* ∞*-categories. If* $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{C}}_0$, *and* $\widetilde{\mathcal{C}}_1$ *are weakly contractible, then the fiber product* $\widetilde{\mathcal{C}}_0 \times_{\widetilde{\mathcal{C}}} \widetilde{\mathcal{C}}_1$ *is also weakly contractible.*

Proof. Suppose first that $\mathcal{C}$ is filtered. Assertion (a) follows from Proposition 7.2.4.9. To prove (b), suppose we are given left fibrations $U : \widetilde{\mathcal{C}} \to \mathcal{C}$, $V_0 : \widetilde{\mathcal{C}}_0 \to \widetilde{\mathcal{C}}$, and $V_1 : \widetilde{\mathcal{C}}_1 \to \widetilde{\mathcal{C}}$, where $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{C}}_0$, and $\widetilde{\mathcal{C}}_1$ are weakly contractible. Applying Theorem 7.2.6.1, we deduce that the ∞-categories $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{C}}_0$, and $\widetilde{\mathcal{C}}_1$ are filtered. Applying Corollary 7.2.6.2 to the diagram of left fibrations

$$\begin{tikzcd}
\widetilde{\mathcal{C}}_0 \times_{\widetilde{\mathcal{C}}} \widetilde{\mathcal{C}}_1 \arrow[r] \arrow[d] & \widetilde{\mathcal{C}}_0 \arrow[d, "V_0"] \\
\widetilde{\mathcal{C}}_1 \arrow[r, "V_1"] & \widetilde{\mathcal{C}},
\end{tikzcd}$$

we conclude that the fiber product $\widetilde{\mathcal{C}}_0 \times_{\widetilde{\mathcal{C}}} \widetilde{\mathcal{C}}_1$ is also filtered; in particular, it is weakly contractible (Proposition 7.2.4.9).

We now prove the converse. Assume that $\mathcal{C}$ satisfies conditions (a) and (b); we wish to show that $\mathcal{C}$ is filtered. We will prove this using the criterion of Lemma 7.2.5.13. Fix an integer $n \geq 0$ and a diagram $e : \partial\Delta^n \to \mathcal{C}$; we wish to show that the coslice ∞-category $\mathcal{C}_{e/}$ is nonempty. In fact, we will prove the following stronger assertion: for every simplicial subset $K \subseteq \partial\Delta^n$, the coslice ∞-category $\mathcal{C}_{e_K/}$ is weakly contractible, where e_K denotes the restriction $e|_K$. Our proof proceeds by induction on the number of nondegenerate simplices of K. If $K = \emptyset$, then the desired result follows from assumption (a). If K is not isomorphic to a standard simplex, then we can use Proposition 1.1.4.12 to write K as a union $K(0) \cup K(1)$, where $K(0), K(1) \subsetneq K$ are proper simplicial subsets. Setting $K(01) = K(0) \cap K(1)$, we have a pullback diagram of left fibrations

$$\begin{tikzcd}
\mathcal{C}_{e_K/} \arrow[r] \arrow[d] & \mathcal{C}_{e_{K(0)}/} \arrow[d] \\
\mathcal{C}_{e_{K(1)}/} \arrow[r] & \mathcal{C}_{e_{K(01)}/},
\end{tikzcd}$$

where the ∞-categories $\mathcal{C}_{e_{K(0)}/}$, $\mathcal{C}_{e_{K(1)}/}$, and $\mathcal{C}_{e_{K(01)}/}$ are weakly contractible by virtue of our inductive hypothesis. Applying (b), we deduce that $\mathcal{C}_{e_K/}$ is weakly contractible. We may therefore assume without loss of generality that $K \simeq \Delta^m$ is a standard simplex. In particular, K contains a final vertex v for which the inclusion $\{v\} \hookrightarrow K$ is right anodyne (Example 4.3.7.11), so that the restriction map $\mathcal{C}_{e_K/} \to \mathcal{C}_{e(v)/}$ is a trivial Kan fibration (Corollary 4.3.6.13). It will therefore suffice to show that the ∞-category $\mathcal{C}_{e(v)/}$ is weakly contractible. This follows from Corollary 4.6.7.25, since the ∞-category $\mathcal{C}_{e(v)/}$ has an initial object (Proposition 4.6.7.22). □

7.2.7 Cofinal Approximation

Let $\mathcal{C}$ be an ∞-category. Recall that an object $X \in \mathcal{C}$ is final if and only if the inclusion map $\{X\} \hookrightarrow \mathcal{C}$ is right cofinal (Corollary 4.6.7.24). If this condition is satisfied, then the ∞-category $\mathcal{C}$ is filtered (Example 7.2.4.5). We now establish a generalization: 02Q8

Proposition 7.2.7.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. If $\mathcal{C}$ is filtered and F is right cofinal, then $\mathcal{D}$ is filtered.* 02Q9

Proof. We will show that the ∞-category $\mathcal{D}$ satisfies conditions (a) and (b) of Corollary 7.2.6.3. Since $\mathcal{C}$ is weakly contractible (Proposition 7.2.4.9) and F is a weak homotopy equivalence (Proposition 7.2.1.5), we deduce immediately that $\mathcal{D}$ is weakly contractible. Suppose we are given left fibrations $U : \widetilde{\mathcal{D}} \to \mathcal{D}$, $V_0 : \widetilde{\mathcal{D}}_0 \to \widetilde{\mathcal{D}}$, and $V_1 : \widetilde{\mathcal{D}}_1 \to \widetilde{\mathcal{D}}$, where the ∞-categories $\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}_0$, and $\widetilde{\mathcal{D}}_1$ are weakly contractible. We wish to show that the fiber product $\widetilde{\mathcal{D}}_0 \times_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{D}}_1$ is also weakly contractible. Set $\widetilde{\mathcal{C}} = \mathcal{C} \times_{\mathcal{D}} \widetilde{\mathcal{D}}$, and define $\widetilde{\mathcal{C}}_0$ and $\widetilde{\mathcal{C}}_1$ similarly. Applying Proposition 7.2.3.12, we deduce that the projection maps

$$\widetilde{\mathcal{C}}_0 \to \widetilde{\mathcal{D}}_0 \qquad \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}} \qquad \widetilde{\mathcal{C}}_1 \to \widetilde{\mathcal{D}}_1$$

are right cofinal; in particular, they are weak homotopy equivalences (Proposition 7.2.4.9). It follows that the ∞-categories $\widetilde{\mathcal{C}}$, $\widetilde{\mathcal{C}}_0$, and $\widetilde{\mathcal{C}}_1$ are weakly contractible. Since $\mathcal{C}$ is filtered, Corollary 7.2.6.3 guarantees that the fiber product $\widetilde{\mathcal{C}}_0 \times_{\widetilde{\mathcal{C}}} \widetilde{\mathcal{C}}_1$ is weakly contractible. The projection map

$$\widetilde{\mathcal{C}}_0 \times_{\widetilde{\mathcal{C}}} \widetilde{\mathcal{C}}_1 \to \widetilde{\mathcal{D}}_0 \times_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{D}}_1$$

is also right cofinal (Proposition 7.2.3.12) and therefore a weak homotopy equivalence (Proposition 7.2.4.9). It follows that $\widetilde{\mathcal{D}}_0 \times_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{D}}_1$ is also weakly contractible, as desired. □

We now establish a partial converse of Proposition 7.2.7.1.

Theorem 7.2.7.2. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:* 02QA

- *The ∞-category $\mathcal{C}$ is filtered.*
- *There exists a directed partially ordered set $(A, \leq)$ and a right cofinal functor $F : \mathrm{N}_{\bullet}(A) \to \mathcal{C}$.*

We first prove the following:

02QB **Lemma 7.2.7.3.** *Let $\mathcal{C}$ be a filtered ∞-category. Then there exists a trivial Kan fibration of simplicial sets $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$, where $\widetilde{\mathcal{C}}$ is an ∞-category having the following property:*

- $(*)$ *For every finite simplicial subset $K \subseteq \widetilde{\mathcal{C}}$, the inclusion map $K \hookrightarrow \widetilde{\mathcal{C}}$ extends to a monomorphism $K^{\triangleright} \hookrightarrow \mathcal{C}$.*

Proof. Let J be an infinite set, and let $\mathcal{J}$ be the corresponding indiscrete category (that is, the category having object set $\mathrm{Ob}(\mathcal{J}) = J$ and $\mathrm{Hom}_{\mathcal{J}}(j, j') = *$ for every pair of elements $j, j' \in J$). Then the nerve $\mathrm{N}_{\bullet}(\mathcal{J})$ is a contractible Kan complex. Setting $\widetilde{\mathcal{C}} = \mathrm{N}_{\bullet}(\mathcal{J}) \times \mathcal{C}$, it follows that the projection map $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ is a trivial Kan fibration. We will complete the proof by showing that $\widetilde{\mathcal{C}}$ satisfies condition $(*)$. Let K be a finite simplicial subset of $\widetilde{\mathcal{C}}$, so that the inclusion map $K \hookrightarrow \widetilde{\mathcal{C}}$ can be identified with a pair of diagrams

$$f : K \to \mathrm{N}_{\bullet}(\mathcal{J}) \qquad g : K \to \mathcal{C}\,.$$

Since J is infinite, we can choose an element $j \in J$ which is not of the form $f(x)$ for any vertex $x \in K$. It follows that f admits a unique extension $\overline{f} : K^{\triangleright} \to \mathrm{N}_{\bullet}(\mathcal{J})$ which carries the cone point of $K^{\triangleright}$ to the element $j \in J$. Our assumption that $\mathcal{C}$ is filtered guarantees that g admits an extension $\overline{g} : K^{\triangleright} \to \mathcal{C}$. We complete the proof by observing that the pair $(\overline{f}, \overline{g})$ determines a monomorphism of simplicial sets $K^{\triangleright} \to \widetilde{\mathcal{C}}$. □

Proof of Theorem 7.2.7.2. Let $\mathcal{C}$ be a filtered ∞-category; we wish to show that there exists a directed partially ordered set $(A, \leq)$ and a right cofinal functor $\mathrm{N}_{\bullet}(A) \to \mathcal{C}$ (the reverse implication follows from Proposition 7.2.7.1 and Example 7.2.5.9). Choose a trivial Kan fibration $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ which satisfies condition $(*)$ of Lemma 7.2.7.3. Then π is right cofinal (Corollary 7.2.1.13). Since the collection of right cofinal morphisms is closed under composition (Proposition 7.2.1.6), we can replace $\mathcal{C}$ by $\widetilde{\mathcal{C}}$ and thereby reduce to proving Theorem 7.2.1.6 in the special case where the ∞-category $\mathcal{C}$ satisfies condition $(*)$ of Lemma 7.2.7.3.

Let A be the collection of all simplicial subsets $L \subseteq \mathcal{C}$ which are isomorphic to $K^{\triangleright}$, for some finite simplicial set K. To avoid confusion, we use the symbol α to represent an element of A, and we will write L_{α} for the corresponding simplicial subset of $\mathcal{C}$. By assumption, we can write L_{α} as a join $K_{\alpha} \star \{C_{\alpha}\}$, where K_{α} is a finite simplicial subset of $\widetilde{\mathcal{C}}$ and C_{α} is an object of $\mathcal{C}$.

Note that condition $(*)$ of Lemma 7.2.7.3 can be restated as follows:

- $(*')$ Every finite simplicial subset $K \subseteq \mathcal{C}$ is equal to K_{α}, for some element $\alpha \in A$.

Let us regard A as a partially ordered set, where elements $\alpha, \beta \in A$ satisfy $\alpha \leq \beta$ if and only if L_{α} is contained in L_{β} (as simplicial subsets of $\mathcal{C}$). If A_0 is any finite subset of A, it follows from $(*')$ that we have $\bigcup_{\alpha \in A_0} L_{\alpha} = K_{\beta} \subset L_{\beta}$ for some element $\beta \in A$. In particular, we have $\alpha < \beta$ for each $\alpha \in A_0$. Allowing A_0 to vary, we conclude that the partially ordered set A is directed.

To every n-simplex $\sigma = (\alpha_0 \leq \cdots \leq \alpha_n)$ of $\mathrm{N}_\bullet(A)$, we associate an n-simplex $F(\sigma)$ of $L_{\alpha_n} \subseteq \mathcal{C}$ by the following recursive procedure:

- If $n = 0$, so that σ can be identified with an element $\alpha \in A$, then $F(\sigma)$ is the object $C_\alpha \in \mathcal{C}$.
- Suppose that $n > 0$, and let $\sigma' = d^n_n(\sigma)$ denote the $(n-1)$-simplex $(\alpha_0 \leq \cdots \leq \alpha_{n-1})$ of $\mathrm{N}_\bullet(A)$. Then $F(\sigma)$ is the *unique* n-simplex $\Delta^n \to L_{\alpha_n}$ whose restriction to Δ^{n-1} coincides with $F(\sigma')$ and which carries vertex $n \in \Delta^n$ to the cone point $C_{\alpha_n} \in L_{\alpha_n}$.

Regarding each $F(\sigma)$ as a simplex of the ∞-category $\mathcal{C}$, we observe that the construction $\sigma \mapsto F(\sigma)$ is compatible with face and degeneracy operators and therefore determines a functor of ∞-categories $F : \mathrm{N}_\bullet(A) \to \mathcal{C}$.

We will complete the proof by showing that the functor F is right cofinal. To verify this, we will use the criterion of Theorem 7.2.3.1. Let C be an object of $\mathcal{C}$; we wish to show that the ∞-category $\mathrm{N}_\bullet(A) \times_{\mathcal{C}} \mathcal{C}_{C/}$ is weakly contractible. We will prove something a bit stronger: the ∞-category $\mathrm{N}_\bullet(A) \times_{\mathcal{C}} \mathcal{C}_{C/}$ is filtered (this is sufficient, by virtue of Proposition 7.2.4.9). To prove this, let S be any finite simplicial set and suppose that we are given a diagram $g : S \to \mathrm{N}_\bullet(A) \times_{\mathcal{C}} \mathcal{C}_{C/}$; we wish to show that g can be extended to a morphism $\overline{g} : S^{\triangleright} \to \mathrm{N}_\bullet(A) \times_{\mathcal{C}} \mathcal{C}_{C/}$. Unwinding the definitions, we can identify g with a pair of diagrams

$$g_0 : S \to \mathrm{N}_\bullet(A) \qquad g_1 : S^{\triangleleft} \to \mathcal{C}$$

satisfying $g_1|_S = F \circ g_0$, where g_1 carries the cone point of $S^{\triangleleft}$ to the object $C \in \mathcal{C}$. Note that the union $K = \mathrm{im}(g_1) \cup \bigcup_{s \in S} L_{g_0(s)}$ is a finite simplicial subset of $\mathcal{C}$. Since $\mathcal{C}$ satisfies condition $(*)$ of Lemma 7.2.7.3, we can write $K = K_\alpha$ for some element $\alpha \in A$. Since the image of g_1 is contained in K_α, it admits a canonical extension

$$\overline{g}_1 : (S^{\triangleleft})^{\triangleright} \to K_\alpha^{\triangleright} = L_\alpha \subseteq \mathcal{C}\,.$$

Similarly, the inclusion $L_{g_0(s)} \subseteq K_\alpha \subset L_\alpha$ guarantees that g_0 can be extended uniquely to a morphism $\overline{g}_0 : S^{\triangleright} \to \mathrm{N}_\bullet(A)$ carrying the cone point of $S^{\triangleright}$ to the element $\alpha \in A$. We conclude by observing that the pair $(\overline{g}_0, \overline{g}_1)$ determines a diagram $\overline{g} : S^{\triangleright} \to \mathrm{N}_\bullet(A) \times_{\mathcal{C}} \mathcal{C}_{C/}$ satisfying $\overline{g}|_S = g$. □

Definition 7.2.7.4. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ *admits small filtered colimits* it admits $\mathcal{K}$-indexed colimits, for every small filtered ∞-category $\mathcal{K}$. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ *preserves small filtered colimits* if it preserves $\mathcal{K}$-indexed colimits, for every small filtered ∞-category $\mathcal{K}$. 039F

039G **Corollary 7.2.7.5.** *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ admits small filtered colimits.*

(2) *For every small filtered category $\mathcal{K}$, the ∞-category $\mathcal{C}$ admits $\mathrm{N}_\bullet(\mathcal{K})$-indexed colimits.*

(3) *For every directed partially ordered set $(A, \leq)$, the ∞-category $\mathcal{C}$ admits $\mathrm{N}_\bullet(A)$-indexed colimits.*

Proof. The implication (1) ⇒ (2) follows from Corollary 7.2.5.8 and the implication (2) ⇒ (3) follows from Exercise 7.2.4.2. The implication (3) ⇒ (1) follows from Theorem 7.2.7.2 and Corollary 7.2.2.3. □

039H **Variant 7.2.7.6.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:

(1) The functor F preserves small filtered colimits.

(2) For every small filtered category $\mathcal{K}$, the functor F preserves $\mathrm{N}_\bullet(\mathcal{K})$-indexed colimits.

(3) For every directed partially ordered set $(A, \leq)$, the functor F preserves $\mathrm{N}_\bullet(A)$-indexed colimits.

We close this section by recording another consequence of Lemma 7.2.7.3.

02QC **Proposition 7.2.7.7.** *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(1) *There exists a filtered diagram of simplicial sets $\{\mathcal{C}_\alpha\}$, where each $\mathcal{C}_\alpha$ is an ∞-category with a final object, and an equivalence of ∞-categories $F : \mathcal{C} \to \varinjlim_\alpha \mathcal{C}_\alpha$.*

(2) *There exists a filtered diagram of simplicial sets $\{\mathcal{C}_\alpha\}$, where each $\mathcal{C}_\alpha$ is a filtered ∞-category, and an equivalence of ∞-categories $F : \mathcal{C} \to \varinjlim_\alpha \mathcal{C}_\alpha$.*

(3) *There exists an equivalence of ∞-categories $F : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}'$ is filtered.*

(4) *The ∞-category $\mathcal{C}$ is filtered.*

Proof. The implication (1) ⇒ (2) follows from Example 7.2.4.5, the implication (2) ⇒ (3) from Remark 7.2.4.6, and the implication (3) ⇒ (4) from Corollary 7.2.4.11. We will complete the proof by showing that every filtered ∞-category $\mathcal{C}$ satisfies condition (1). Without loss of generality, we may assume that $\mathcal{C}$ satisfies condition $(*)$ of Lemma 7.2.7.3. Let A be the directed partially ordered set defined in the proof of Theorem 7.2.7.2. For each $\alpha \in A$, let $L_\alpha \subseteq \mathcal{C}$ denote the corresponding subset of $\mathcal{C}$. By virtue of Corollary 4.1.3.3, we can choose an ∞-category $\mathcal{C}_\alpha$ and an inner anodyne morphism $F_\alpha : L_\alpha \hookrightarrow \mathcal{C}_\alpha$, which depend functorially on α. Applying Corollary 4.5.7.2, we see that the morphisms F_α induce an equivalence of ∞-categories

$$\mathcal{C} \simeq \varinjlim_{\alpha \in A} L_\alpha \xrightarrow{\{F_\alpha\}_{\alpha \in A}} \varinjlim_{\alpha \in A} \mathcal{C}_\alpha .$$

To complete the proof, it will suffice to show that each of the ∞-categories $\mathcal{C}_\alpha$ contains a final object. By construction, there exists an isomorphism of simplicial sets $u : L_\alpha \simeq K^{\triangleright}$, for some finite simplicial set K. Using Corollary 4.1.3.3, we can choose a categorical equivalence $v : K \to \mathcal{D}$, where $\mathcal{D}$ is an ∞-category. Applying Corollary 4.5.8.9, we deduce that the map $v^{\triangleright} : K^{\triangleright} \to \mathcal{D}^{\triangleright}$ is also a categorical equivalence of simplicial sets. Since F_α is inner anodyne, there exists a functor $G : \mathcal{C}_\alpha \to \mathcal{D}^{\triangleright}$ satisfying $G \circ F_\alpha = v^{\triangleright} \circ u$. Applying the two-out-of-three property (Remark 4.5.3.5), we see that G is an equivalence of ∞-categories. Since the ∞-category $\mathcal{D}^{\triangleright}$ has a final object (given by the cone point; see Example 4.6.7.5), it follows that $\mathcal{C}_\alpha$ also has a final object (Corollary 4.6.7.21). □

7.2.8 Sifted Simplicial Sets

We now introduce a useful enlargement of the class of filtered ∞-categories. 02QD

Definition 7.2.8.1. Let K be a simplicial set. We say that K is *sifted* if, for every finite set I, the diagonal map $K \to K^I$ is right cofinal. If $\mathcal{C}$ is an ∞-category, we say that a diagram $K \to \mathcal{C}$ is *sifted* if the simplicial set K is sifted. 02QE

Warning 7.2.8.2. Definition 7.2.8.1 has a counterpart in classical category theory. In [1], Adámek and Rosický define a *sifted category* to be a nonempty category $\mathcal{C}$ which satisfies the following condition: 02QF

(∗) For every pair of objects $X, Y \in \mathcal{C}$, the nerve of the category $\mathcal{C}_{X/} \times_{\mathcal{C}} \mathcal{C}_{Y/}$ is connected.

It follows from Corollary 7.2.8.9 below that if the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C})$ is sifted (in the sense of Definition 7.2.8.1), then the category $\mathcal{C}$ satisfies condition (∗). Beware that the converse is false (see Exercise 7.2.8.11). In other words, Definition 7.2.8.1 is not a generalization of the classical notion of a sifted category (instead, it generalizes the notion of a *homotopy sifted* category, introduced by Rosický in [48]).

Variant 7.2.8.3. Let K be a simplicial set. We say that K is *cosifted* if, for every finite set I, the diagonal map $K \to K^I$ is left cofinal. Equivalently, K is cosifted if and only if the opposite simplicial set K^{op} is sifted. 02QG

Example 7.2.8.4. Every filtered ∞-category $\mathcal{C}$ is sifted (see Proposition 7.2.4.10). In particular, if $\mathcal{C}$ is an ∞-category which contains a final object, then $\mathcal{C}$ is sifted (see Example 7.2.4.5). 02QH

02QJ **Proposition 7.2.8.5.** *Let $f : K \to K'$ be a right cofinal morphism of simplicial sets. If K is sifted, then K' is also sifted.*

Proof. Fix a finite set I. We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
K \arrow[r, "\delta_K"] \arrow[d, "f"] & K^I \arrow[d, "f^I"] \\
K' \arrow[r, "\delta_{K'}"] & K'^I,
\end{tikzcd}$$

where the vertical maps are right cofinal (Corollary 7.2.1.20). Our assumption that K is sifted guarantees that δ_K is right cofinal, so that $\delta_{K'}$ is also right cofinal (Proposition 7.2.1.6). □

02QK **Proposition 7.2.8.6.** *Let $f : K \to K'$ be a categorical equivalence of simplicial sets. Then K is sifted if and only if K' is sifted.*

Proof. It will suffice to show that, for every finite set I, the diagonal map $\delta_K : K \to K^I$ is right cofinal if and only if the diagonal map $\delta_{K'} : K \to K'^I$ is right cofinal. This follows by applying Corollary 7.2.1.22 to the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\delta_K"] \arrow[d, "f"] & K^I \arrow[d, "f^I"] \\
K' \arrow[r, "\delta_{K'}"] & K'^I.
\end{tikzcd}$$

□

02QL **Proposition 7.2.8.7.** *Every sifted simplicial set is weakly contractible.*

Proof. Let K be a sifted simplicial set. Taking $I = \emptyset$ in Definition 7.2.8.1, we conclude that the projection map $K \to \Delta^0$ is right cofinal, so that K is weakly contractible by virtue of Proposition 7.2.1.5. □

02QM **Proposition 7.2.8.8.** *Let K be a simplicial set. Then K is sifted if and only if it is nonempty and the diagonal map $\delta : K \hookrightarrow K \times K$ is right cofinal.*

Proof. It follows immediately from the definition that if K is sifted, then the diagonal map $\delta : K \hookrightarrow K \times K$ is right cofinal. Moreover, Proposition 7.2.1.5 guarantees that K is weakly contractible, and therefore nonempty.

For the converse, assume that K is nonempty and that δ is right cofinal. We wish to prove that, for every finite set I, the map $\delta_I : K \to K^I$ is right cofinal. The proof proceeds by induction on the cardinality of I. We first treat the case where $I = \emptyset$. Note that our assumption that δ is right cofinal guarantees in particular that it is a weak homotopy equivalence (Proposition 7.2.1.5). Since K is nonempty, it follows that K is weakly contractible (Corollary 3.5.1.33). Applying Proposition 7.2.1.5 again, we deduce that the projection map $K \to \Delta^0$ is right cofinal, as desired.

We now carry out the inductive step. Assume that the set I is nonempty. Choose an element $i \in I$, and set $J = I \setminus \{i\}$. Unwinding the definitions, we see that δ_I can be identified with the composition

$$K \xrightarrow{\delta} K \times K \xrightarrow{\mathrm{id}_K \times \delta_J} K \times K^J.$$

Our inductive hypothesis guarantees that δ_J is right cofinal, so that the product map $\mathrm{id}_K \times \delta_J$ is also right cofinal (Corollary 7.2.1.19). Since the collection of right cofinal morphisms is closed under composition (Proposition 7.2.1.6), it follows that δ_I is also right cofinal. □

Corollary 7.2.8.9. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is sifted if and only if it is nonempty and, for every pair of objects $X, Y \in \mathcal{C}$, the ∞-category $\mathcal{C}_{X/} \times_{\mathcal{C}} \mathcal{C}_{Y/}$ is weakly contractible.* 02QN

Proof. Combine Proposition 7.2.8.8 with Theorem 7.2.3.1. □

We now consider an important example.

Proposition 7.2.8.10. *Let $\mathbf{\Delta}$ be the simplex category (Definition 1.1.0.2). Then the ∞-category $\mathrm{N}_\bullet(\mathbf{\Delta})$ is cosifted.* 02QP

Proof. We use the criterion of Corollary 7.2.8.9. Since the category $\mathbf{\Delta}$ is nonempty, it will suffice to show that for every pair of nonnegative integers $m, n \geq 0$, the simplicial set

$$\mathrm{N}(\mathbf{\Delta})_{/[m]} \times_{\mathrm{N}(\mathbf{\Delta})} \mathrm{N}(\mathbf{\Delta})_{/[n]} \simeq \mathrm{N}(\mathbf{\Delta}_{/[m]} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]})$$

is weakly contractible. Unwinding the definitions, we can identify $\mathbf{\Delta}_{/[m]} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]}$ with the category of simplices $\mathbf{\Delta}_S$ of Construction 1.1.3.9, where S is the product $\Delta^m \times \Delta^n$. Note that S can be identified with the nerve of a partially ordered set, and is therefore a braced simplicial set (Exercise 3.3.1.2). Let $\mathbf{\Delta}_S^{\mathrm{nd}}$ denote the full subcategory of $\mathbf{\Delta}_S$ spanned by the nondegenerate simplices of S (Notation 3.3.3.11), so that the inclusion $\mathbf{\Delta}_S^{\mathrm{nd}} \hookrightarrow \mathbf{\Delta}_S$ admits a left adjoint (Exercise 3.3.3.15). It follows that the inclusion map $\mathrm{N}_\bullet(\mathbf{\Delta}_S^{\mathrm{nd}}) \hookrightarrow \mathrm{N}_\bullet(\mathbf{\Delta}_S)$ is a homotopy equivalence of simplicial sets (Proposition 3.1.6.9). It will therefore suffice to show that the nerve $\mathrm{N}_\bullet(\mathbf{\Delta}_S^{\mathrm{nd}})$ is weakly contractible. Using Proposition 3.3.3.16, we can identify $\mathrm{N}_\bullet(\mathbf{\Delta}_S^{\mathrm{nd}})$ with the subdivision $\mathrm{Sd}(S)$, so that Construction 3.3.4.3 supplies a weak homotopy equivalence $\lambda_S : \mathrm{N}_\bullet(\mathbf{\Delta}_S^{\mathrm{nd}}) \to S$. We conclude by observing that the simplicial set $S = \Delta^m \times \Delta^n$ is weakly contractible (in fact, it is contractible, since it is the nerve of a partially ordered set having a smallest element). □

02QQ **Exercise 7.2.8.11.** Let $\mathbf{\Delta}_{\leq 1}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by the objects $[0]$ and $[1]$, which we depict informally as a diagram

$$\begin{tikzcd} {[0]} \arrow[r, shift left=2] \arrow[r, shift right=2] & {[1]}. \arrow[l] \end{tikzcd}$$

Show that:

- The opposite category $\mathbf{\Delta}^{\mathrm{op}}_{\leq 1}$ satisfies condition $(*)$ of Warning 7.2.8.2 (that is, it is a sifted category in the sense of [1]).
- The simplicial set $\mathrm{N}_{\bullet}(\mathbf{\Delta}^{\mathrm{op}}_{\leq 1})$ is not sifted.

7.3 Kan Extensions

02Y1 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. In practice, it is often possible to reconstruct the functor F (at least up to isomorphism) from its restriction to a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. To make this more precise, it will be convenient to introduce some terminology.

02Y2 **Definition 7.3.0.1.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. We say that F is *left Kan extended from* $\mathcal{C}^0$ if, for every object $C \in \mathcal{C}$, the collection of morphisms $\{F(u) : F(C_0) \to F(C)\}_{u: C_0 \to C}$ exhibits $F(C)$ as a colimit of the diagram

$$(\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C}) \to \mathcal{C}^0 \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}\,.$$

The central features of Definition 7.3.0.1 can be summarized as follows:

02Y3 **Exercise 7.3.0.2** (Uniqueness of Kan Extensions)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors between categories, and suppose that F is left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Show that the restriction map

$$\begin{tikzcd} \{\text{Natural transformations from } F \text{ to } G\} \arrow[d] \\ \{\text{Natural transformations from } F|_{\mathcal{C}^0} \text{ to } G|_{\mathcal{C}^0}\} \end{tikzcd}$$

is a bijection. In particular, the functor F can be recovered (up to canonical isomorphism) from the restriction $F|_{\mathcal{C}^0}$.

02Y4 **Exercise 7.3.0.3** (Existence of Kan Extensions)**.** Let $\mathcal{C}$ be a category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and let $F_0 : \mathcal{C}^0 \to \mathcal{D}$ be a functor between categories. Show that the following conditions are equivalent;

(1) There exists a functor $F : \mathcal{C} \to \mathcal{D}$ which is left Kan extended from $\mathcal{C}^0$ and satisfies $F|_{\mathcal{C}^0} = F_0$.

(2) For every object $C \in \mathcal{C}$, the diagram

$$(\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C}) \to \mathcal{C}^0 \xrightarrow{F_0} \mathcal{D} \tag{7.13}$$ 02Y5

has a colimit in $\mathcal{D}$.

Stated more informally, if the diagram (7.13) has a colimit in $\mathcal{D}$, then that colimit depends functorially on the object $C \in \mathcal{C}$.

In this section, we adapt the theory of Kan extensions to the ∞-categorical setting. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. We will say that F is *left Kan extended from* $\mathcal{C}^0$ if it satisfies an ∞-categorical analogue of the condition appearing in Definition 7.3.0.1, which we formulate in §7.3.2 (see Definition 7.3.2.1). Our main results are ∞-categorical counterparts of Exercises 7.3.0.2 and 7.3.0.3, which we prove in §7.3.6 and §7.3.5, respectively (see Corollary 7.3.6.9 and Corollary 7.3.5.8).

For many applications, it will be useful to consider a different generalization of Definition 7.3.0.1, where we replace the inclusion map $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ by an arbitrary functor $\delta : \mathcal{K} \to \mathcal{C}$. Suppose we are given functors $F : \mathcal{C} \to \mathcal{D}$, $\delta : \mathcal{K} \to \mathcal{C}$, and $F_0 : \mathcal{K} \to \mathcal{D}$, together with a natural transformation $\beta : F_0 \to F \circ \delta$, as indicated in the diagram

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "F"] & \\
\mathcal{K} \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & {} \arrow[u, Rightarrow, "\beta"'] & \mathcal{D}.
\end{tikzcd}$$

We will say that β *exhibits* F *as a left Kan extension of* F_0 *along* δ if, for every object $C \in \mathcal{C}$, the collection of morphisms $\{(F(u) \circ \beta_X) : F_0(X) \to F(C)\}_{u : \delta(X) \to C}$ exhibits $F(C)$ as a colimit of the diagram $\mathcal{K} \times_{\mathcal{C}} \mathcal{C}_{/C} \to \mathcal{K} \xrightarrow{F_0} \mathcal{D}$. This notion also has an ∞-categorical generalization which we introduce in §7.3.1 (Variant 7.3.1.5), for which we have counterparts of Exercises 7.3.0.2 and 7.3.0.3 (see Propositions 7.3.6.1 and 7.3.5.1). In the special case where $\mathcal{K} = \mathcal{C}^0$ is a full subcategory of $\mathcal{C}$ and δ is the inclusion map, the Kan extension condition guarantees that β is an isomorphism, and therefore essentially reduces to the notion of Kan extension introduced in Definition 7.3.0.1 (see Corollary 7.3.2.7 for a precise statement). In §7.3.4 we study a different extreme, where the functor δ is assumed to be a cocartesian fibration: in this case, the left Kan extension F of a functor $F_0 : \mathcal{K} \to \mathcal{D}$ along δ is given concretely by the formula

$$F(C) = \varinjlim_{\delta(X) = C} (F_0(X))$$

where the colimit is taken over the *fiber* $\mathcal{K}_C = \mathcal{K} \times_{\mathcal{C}} \{C\}$ (see Proposition 7.3.4.1 and Corollary 7.3.4.2).

In §7.3.3 we consider another variant of Definition 7.3.0.1, where we replace colimits in $\mathcal{D}$ by the more general notion of U-colimit for an auxiliary functor $U : \mathcal{D} \to \mathcal{E}$ (see §7.1.5). The extra generality afforded by the relative setting is quite convenient in practice: for example, in §7.3.6 we show that relative Kan extensions satisfy a universal property (Proposition 7.3.6.7, analogous to Exercise 7.3.0.2) which can be formally deduced from an existence criterion (Proposition 7.3.5.5, analogous to Exercise 7.3.0.3).

In §7.3.8, we study the transitivity properties of Kan extensions. Let $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ be a functor of ∞-categories, and suppose we are given full subcategories $\mathcal{C}^0 \subseteq \mathcal{C} \subseteq \overline{\mathcal{C}}$ such that $F = \overline{F}|_{\mathcal{C}}$ is left Kan extended from $\mathcal{C}_0$. We will show that $\overline{F}$ is left Kan extended from $\mathcal{C}$ if and only if it is left Kan extended from $\mathcal{C}^0$ (Corollary 7.3.8.8). Moreover, we prove analogous statements for relative left Kan extensions (Proposition 7.3.8.6) and for Kan extensions along more general functors (Proposition 7.3.8.18). In §7.3.9, we apply these ideas to give a characterization of U-colimit diagrams in the special case where $U : \mathcal{D} \to \mathcal{E}$ is a cocartesian fibration of ∞-categories.

02Y6 **Remark 7.3.0.4.** In the summary above, we considered only the notion of *left* Kan extensions. There is also a dual theory of *right* Kan extensions, which can be obtained from the theory of left Kan extensions by passing to opposite categories.

7.3.1 Kan Extensions along General Functors

02Y7 We begin by introducing some notation.

02Y8 **Notation 7.3.1.1.** Let $\mathcal{C}$ be an ∞-category and let $\delta : K \to \mathcal{C}$ be a diagram. For each object $C \in \mathcal{C}$, we let $K_{/C}$ denote the fiber product $K \times_{\mathcal{C}} \mathcal{C}_{/C}$. Note that the slice diagonal of Construction 4.6.4.13 determines a map $K_{/C} \to K \tilde{\times}_{\mathcal{C}} \{C\}$, which we can identify with a natural transformation of diagrams $\gamma : \delta|_{K_{/C}} \to \underline{C}$; here $\delta|_{K_{/C}}$ denotes the composition $K_{/C} \to K \xrightarrow{\delta} \mathcal{C}$, while $\underline{C}$ denotes the constant diagram $K_{/C} \to \mathcal{C}$ taking the value C. Similarly, we let $K_{C/}$ denote the fiber product $\mathcal{C}_{C/} \times_{\mathcal{C}} K$, so that the coslice diagonal of Construction 4.6.4.13 determines a natural transformation $\gamma' : \underline{C} \to \delta|_{K_{C/}}$.

02Y9 **Definition 7.3.1.2** (Right Kan Extensions)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Suppose we are given a simplicial set K together with diagrams $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$

and a natural transformation $\alpha : F \circ \delta \to F_0$, as indicated in the diagram

$$\begin{tikzcd}
& \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \alpha"] & \\
K \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & & \mathcal{D}.
\end{tikzcd}$$

We will say that α *exhibits* F *as a right Kan extension of* F_0 *along* δ if, for every object $C \in \mathcal{C}$, the following condition is satisfied:

$(*_C)$ Let α_C denote a composition of the natural transformations

$$\underline{F(C)} \xrightarrow{F(\gamma')} (F \circ \delta)|_{K_{C/}} \xrightarrow{\alpha} F_0|_{K_{C/}}$$

(formed in the ∞-category $\mathrm{Fun}(K_{C/}, \mathcal{D})$), where $\gamma' : \underline{C} \to \delta|_{K_{C/}}$ is defined in Notation 7.3.1.1. Then α_C exhibits $F(C)$ as a limit of the diagram

$$K_{C/} = \mathcal{C}_{C/} \times_{\mathcal{C}} K \to K \xrightarrow{F_0} \mathcal{D},$$

in the sense of Definition 7.1.1.1.

Remark 7.3.1.3. Stated more informally, a diagram 02YA

$$\begin{tikzcd}
& \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \alpha"] & \\
K \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & & \mathcal{D}.
\end{tikzcd}$$

exhibits F as a right Kan extension of F_0 along δ if, for every object $C \in \mathcal{C}$, we can calculate the value $F(C) \in \mathcal{D}$ as a limit of the diagram

$$K_{C/} = \mathcal{C}_{C/} \times_{\mathcal{C}} K \to K \xrightarrow{F_0} \mathcal{D}.$$

Note that this requirement characterizes the object $F(C) \in \mathcal{D}$ up to isomorphism (see Proposition 7.1.1.12). We will later prove a stronger assertion: if the diagrams $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$ are fixed, then a right Kan extension of F_0 along δ is uniquely determined (up to isomorphism) as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (Remark 7.3.6.6).

Warning 7.3.1.4. In the situation of Definition 7.3.1.2, the natural transformation α_C appearing in condition $(*_C)$ is defined as a composition of morphisms in the ∞-category $\mathrm{Fun}(K_{/C}, \mathcal{D})$, which is only well-defined up to homotopy. However, the condition that α_C exhibits $F(C)$ as a colimit of the diagram $F_0|_{K_{/C}}$ depends only on the homotopy class $[\beta_C]$ (Remark 7.1.1.7). 02YB

02YC **Variant 7.3.1.5** (Left Kan Extensions)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Suppose we are given a simplicial set K together with diagrams $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$ and a natural transformation $\beta : F_0 \to F \circ \delta$, as indicated in the diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "F"] & \\ K \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & \Uparrow \beta & \mathcal{D}. \end{tikzcd}$$

We will say that β *exhibits* F *as a left Kan extension of* F_0 *along* δ if, for every object $C \in \mathcal{C}$, the following condition is satisfied:

$(*_C)$ Let β_C denote a composition of the natural transformations

$$F_0|_{K_{/C}} \xrightarrow{\beta} (F \circ \delta)|_{K_{/C}} \xrightarrow{F(\gamma)} \underline{F(C)}$$

(formed in the ∞-category $\mathrm{Fun}(K_{/C}, \mathcal{D})$), where $\gamma : \delta|_{K_{/C}} \to \underline{C}$ is defined in Notation 7.3.1.1. Then β_C exhibits $F(C)$ as a colimit of the diagram

$$K_{/C} = K \times_{\mathcal{C}} \mathcal{C}_{/C} \to K \xrightarrow{F_0} \mathcal{D},$$

in the sense of Definition 7.1.1.1.

02YD **Remark 7.3.1.6.** In the situation of Variant 7.3.1.5, the natural transformation $\beta : F_0 \to F \circ \delta$ exhibits F as a left Kan extension of F_0 along δ if and only if it exhibits F^{op} as a right Kan extension of F_0^{op} along δ^{op}, when regarded as a morphism in the ∞-category $\mathrm{Fun}(K^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}) \simeq \mathrm{Fun}(K, \mathcal{D})^{\mathrm{op}}$.

02YE **Example 7.3.1.7.** Let $\mathcal{D}$ be an ∞-category, let $F_0 : K \to \mathcal{D}$ be a diagram. Let $\delta : K \to \Delta^0$ be the projection map and let $F : \Delta^0 \to \mathcal{D}$ be the functor corresponding to an object $Y \in \mathcal{D}$. Then:

- A natural transformation $\alpha : \underline{Y} = (F \circ \delta) \to F_0$ exhibits Y as a limit of F_0 (in the sense of Definition 7.1.1.1) if and only if it exhibits F as a right Kan extension of F_0 along δ (in the sense of Definition 7.3.1.2).
- A natural transformation $\beta : F_0 \to (F \circ \delta) = \underline{Y}$ exhibits Y as a colimit of F_0 (in the sense of Definition 7.1.1.1) if and only if it exhibits F as a left Kan extension of F_0 along δ (in the sense of Variant 7.3.1.5).

02YF **Example 7.3.1.8.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $\alpha : F \to G$ be a morphism in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The following conditions are equivalent:

(1) The natural transformation α is an isomorphism in the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$.

(2) The natural transformation α exhibits F as a right Kan extension of G along the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

(3) The natural transformation α exhibits G as a left Kan extension of F along the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$.

To prove the equivalence of (1) and (2), fix an object $C \in \mathcal{C}$. Since the identity morphism id_C is an initial object of the ∞-category $\mathcal{C}_{C/}$ (Proposition 4.6.7.22), the natural transformation α satisfies condition $(*_C)$ of Definition 7.3.1.2 if and only if the induced map $\alpha_C : F(C) \to G(C)$ is an isomorphism in $\mathcal{D}$ (Corollary 7.2.2.6). The equivalence (1) $\Leftrightarrow$ (2) now follows from the criterion of Theorem 4.4.4.4. The equivalence (1) $\Leftrightarrow$ (3) follows by a similar argument.

Remark 7.3.1.9. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$ be diagrams. Then: `02YG`

- The condition that a natural transformation $\alpha : F \circ \delta \to F_0$ exhibits F as a right Kan extension of F_0 along δ depends only on the homotopy class $[\alpha]$ (as a morphism in the ∞-category $\operatorname{Fun}(K, \mathcal{D})$).
- The condition that a natural transformation $\beta : F_0 \to F \circ \delta$ exhibits F as a left Kan extension of F_0 along δ depends only on the homotopy class $[\beta]$ (as a morphism in the ∞-category $\operatorname{Fun}(K, \mathcal{D})$).

See Remark 7.1.1.7.

Remark 7.3.1.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $\delta : K \to \mathcal{C}$ be a diagram, and let $\rho : F_0 \to F_0'$ be an isomorphism in the ∞-category $\operatorname{Fun}(K, \mathcal{D})$. Then: `02YH`

- A natural transformation $\alpha : F \circ \delta \to F_0$ exhibits F as a right Kan extension of F_0 along δ if and only if the composite natural transformation

  $$F \circ \delta \xrightarrow{\alpha} F_0 \xrightarrow{\rho} F_0'$$

  exhibits F as a right Kan extension of F_0' along δ (note that this condition is independent of the composition chosen, by virtue of Remark 7.3.1.9).

- A natural transformation $\beta : F_0' \to F \circ \delta$ exhibits F as a left Kan extension of F_0' along δ if and only if the composite natural transformation

  $$F_0 \xrightarrow{\rho} F_0' \xrightarrow{\beta} F \circ \delta$$

  exhibits F as a left Kan extension of F_0 along δ.

See Remark 7.1.1.8.

03J9 **Remark 7.3.1.11.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $F_0 : K \to \mathcal{D}$ be a diagram, and let $\rho : \delta' \to \delta$ be an isomorphism in the ∞-category $\mathrm{Fun}(K, \mathcal{C})$. Then:

- A natural transformation $\alpha : F \circ \delta \to F_0$ exhibits F as a right Kan extension of F_0 along δ if and only if the composite natural transformation

  $$F \circ \delta' \xrightarrow{\rho} F \circ \delta \xrightarrow{\alpha} F_0$$

  exhibits F as a right Kan extension of F_0 along δ' (note that this condition is independent of the composition chosen, by virtue of Remark 7.3.1.9).

- A natural transformation $\beta : F_0 \to F \circ \delta'$ exhibits F as a left Kan extension of F_0 along δ' if and only if the composite natural transformation

  $$F_0 \xrightarrow{\beta} F \circ \delta' \xrightarrow{\rho} F \circ \delta$$

  exhibits F as a left Kan extension of F_0 along δ.

See Remark 7.1.1.8.

02YJ **Remark 7.3.1.12.** Suppose we are given a diagram

$$\begin{tikzcd}
& \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \alpha"] & \\
K \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & {} & \mathcal{D}
\end{tikzcd}$$

as in Definition 7.3.1.2. Let $\rho : F' \to F$ be a morphism in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Then any two of the following conditions imply the third:

- The natural transformation α exhibits F as a right Kan extension of F_0 along δ.

- The composite natural transformation

  $$\delta \circ F' \xrightarrow{\rho} \delta \circ F \xrightarrow{\alpha} F_0$$

  exhibits F' as a right Kan extension of F_0 along δ (note that this condition does not depend on the composition chosen, by virtue of Remark 7.3.1.9).

- The morphism ρ is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

This follows by combining Remark 7.1.1.9 with Theorem 4.4.4.4.

Remark 7.3.1.13 (Change of Target)**.** Suppose we are given a diagram 02YK

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \alpha"] & \\ K \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & {} & \mathcal{D} \end{tikzcd}$$

as in Definition 7.3.1.2, and let $G : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories. Then:

- If G is fully faithful and $G(\alpha) : (G \circ F) \circ \delta \to G \circ F_0$ exhibits $G \circ F$ as a right Kan extension of $G \circ F_0$ along δ, then α exhibits F as a right Kan extension of F_0 along δ.

- If G is an equivalence of ∞-categories and α exhibits F as a right Kan extension of F_0 along δ, then $G(\alpha)$ exhibits $G \circ F$ as a right Kan extension of $G \circ F_0$ along δ.

See Remark 7.1.1.10.

Proposition 7.3.1.14 (Change of Diagram)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$ be diagrams, and let $\epsilon : K' \to K$ be a categorical equivalence of simplicial sets. Then:* 02YL

(1) *A natural transformation $\alpha : F \circ \delta \to F_0$ exhibits F as a right Kan extension of F_0 along δ if and only if the induced transformation $\alpha' : F \circ (\delta \circ \epsilon) \to F_0 \circ \epsilon$ exhibits F as a right Kan extension of $F_0 \circ \epsilon$ along $\delta \circ \epsilon$.*

(2) *A natural transformation $\beta : F_0 \to F \circ \delta$ exhibits F as a left Kan extension of F_0 along δ if and only if the induced transformation $\beta' : F_0 \circ \epsilon \to F \circ (\delta \circ \epsilon)$ exhibits F as a left Kan extension of $F_0 \circ \epsilon$ along $\delta \circ \epsilon$.*

Proof. We will prove (1); the proof of (2) is similar. Fix an object $C \in \mathcal{C}$. Since ϵ is a categorical equivalence and the projection map $\mathcal{C}_{C/} \to \mathcal{C}$ is a left fibration (Proposition 4.3.6.1), it follows that the induced map $\epsilon_{C/} : K' \times_{\mathcal{C}} \mathcal{C}_{C/} \to K \times_{\mathcal{C}} \mathcal{C}_{C/}$ is also a categorical equivalence of simplicial sets (Corollary 5.6.7.6). In particular, $\epsilon_{C/}$ is left cofinal (Corollary 7.2.1.13). Applying Corollary 7.2.2.3, we see that the natural transformation α satisfies condition $(*_C)$ of Definition 7.3.1.2 if and only if α' satisfies condition $(*_C)$. The desired result now follows by allowing the object $C \in \mathcal{C}$ to vary. □

02YM **Proposition 7.3.1.15.** *Suppose we are given a diagram*

$$\begin{tikzcd} & \mathcal{C} \arrow[dr, "F"] \arrow[d, phantom, "\Downarrow \alpha"] & \\ K \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & & \mathcal{D} \end{tikzcd}$$

as in Definition 7.3.1.2, where δ factors as a composition

$$K \xrightarrow{\delta^0} \mathcal{C}^0 \xrightarrow{G} \mathcal{C}$$

for some ∞-category $\mathcal{C}^0$. Then:

(1) *If G is fully faithful and α exhibits F as a right Kan extension of F_0 along δ, then it also exhibits $F \circ G$ as a right Kan extension of F_0 along δ^0.*

(2) *If G is an equivalence of ∞-categories and α exhibits $F \circ G$ as a right Kan extension of F_0 along δ^0, then it exhibits F as a right Kan extension of F_0 along δ.*

Proof. Assume that G is fully faithful. Then, for every pair of objects $X, Y \in \mathcal{C}^0$, the induced map of left-pinched morphism spaces

$$\mathcal{C}^0_{X/} \times_{\mathcal{C}} \{Y\} = \operatorname{Hom}^{\mathrm{L}}_{\mathcal{C}^0}(X,Y) \to \operatorname{Hom}^{\mathrm{L}}_{\mathcal{C}}(G(X), G(Y)) = \mathcal{C}_{G(X)/} \times_{\mathcal{C}} \{G(Y)\}$$

is a homotopy equivalence. Allowing Y to vary and applying Corollary 5.1.7.16, we see that the natural map $\mathcal{C}^0_{X/} \to \mathcal{C}_{G(X)/} \times_{\mathcal{C}} \mathcal{C}^0$ is an equivalence of left fibrations over $\mathcal{C}^0$. It follows that the induced map

$$\mathcal{C}^0_{X/} \times_{\mathcal{C}^0} K \to \mathcal{C}_{G(X)/} \times_{\mathcal{C}} K$$

is an equivalence of left fibrations over K. In particular it is a categorical equivalence of simplicial sets (Proposition 5.1.7.5) and therefore left cofinal (Corollary 7.2.1.13). Applying Corollary 7.2.2.3, we see that the natural transformation α satisfies condition $(*_X)$ of Definition 7.3.1.2 if and only if it satisfies condition $(*_{G(X)})$. Assertion (1) now follows by allowing the object $X \in \mathcal{C}^0$ to vary.

We now prove (2). Assume that G is an equivalence of ∞-categories and that α exhibits $F \circ G$ as a right Kan extension of F_0 along δ^0; we wish to show that α exhibits F as a right Kan extension of F_0 along δ. Let $H : \mathcal{C} \to \mathcal{C}^0$ be a homotopy inverse of G. Then H is left adjoint to G, so we can choose natural transformations

$$\eta : \mathrm{id}_{\mathcal{C}} \to G \circ H \qquad \epsilon : H \circ G \to \mathrm{id}_{\mathcal{C}^0}$$

which are compatible up to homotopy in the sense of Definition 6.2.1.1. Note that η and ϵ are isomorphisms (Proposition 6.1.4.1). Let α' denote a composition of the natural transformations

$$F \circ G \circ H \circ G \circ \delta^0 \xrightarrow{\epsilon} F \circ G \circ \delta^0 \xrightarrow{\alpha} F_0.$$

Using Remark 7.3.1.11, we see that α' exhibits $F \circ G$ as a right Kan extension of F_0 along $H \circ G \circ \delta^0 = H \circ \delta$. Applying assertion (1) to the fully faithful functor $H : \mathcal{C} \to \mathcal{C}^0$, we deduce that α' also exhibits δ as a right Kan extension of F_0 along $F \circ G \circ H$. The compatibility of η and ϵ guarantees that α is a composition of the natural transformations

$$F \circ \delta \xrightarrow{\eta} F \circ G \circ H \circ \delta \xrightarrow{\alpha'} F_0.$$

Applying Remark 7.3.1.12, we conclude that α exhibits F as a right Kan extension of F_0 along δ, as desired. □

Corollary 7.3.1.16. *Let $G : \mathcal{C}^0 \to \mathcal{C}$, $F_0 : \mathcal{C}^0 \to \mathcal{D}$, and $F : \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories, where G is fully faithful. Then:* 02YN

- *If $\alpha : F \circ G \to F_0$ is a natural transformation which exhibits F as a right Kan extension of F_0 along G, then α is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$.*
- *If $\beta : F_0 \to F \circ G$ is a natural transformation which exhibits F as a left Kan extension of F_0 along G, then β is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$.*

Proof. Let $\alpha : F \circ G \to F_0$ be a natural transformation which exhibits F as a right Kan extension of F_0 along G. Applying Proposition (in the special case where $K = \mathcal{C}^0$), we deduce that α also exhibits $F \circ G$ as a right Kan extension of F_0 along the identity functor $\mathrm{id}_{\mathcal{C}^0} : \mathcal{C}^0 \to \mathcal{C}^0$. Invoking Example 7.3.1.8, we see that α is an isomorphism. This proves the first assertion; the second follows by a similar argument. □

Proposition 7.3.1.17. *Suppose we are given a diagram* 039N

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "F"] \arrow[d, Rightarrow, "\alpha"] & \\
K \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & {} & \mathcal{D}
\end{tikzcd}$$

as in Definition 7.3.1.2. Assume that δ exhibits $\mathcal{C}$ as a localization of K (with respect to some collection of edges of K) and that α is an isomorphism in the ∞-category $\mathrm{Fun}(K, \mathcal{D})$. Then α exhibits F as a right Kan extension of F_0 along δ.

Proof. Fix an object $C \in \mathcal{C}$. Since α is an isomorphism, it will suffice to show that the tautological map $\underline{F(C)} \to (F \circ \delta)|_{K_{C/}}$ exhibits $F(C)$ as a limit of the diagram $(F \circ \delta)|_{K_{C/}}$. Since the projection map $\mathcal{C}_{C/} \to \mathcal{C}$ is a left fibration (Proposition 4.3.6.1), the map $\delta_{C/} : K_{C/} \to \mathcal{C}_{C/}$ exhibits the ∞-category $\mathcal{C}_{C/}$ as a localization of the simplicial set $K_{C/}$ (Corollary 6.3.5.5). In particular, $\delta_{C/}$ is left cofinal (Proposition 7.2.1.10). We can therefore replace K by $\mathcal{C}$ (Corollary 7.2.2.7), in which case the desired result follows from the criterion of Corollary 7.2.2.6. □

7.3.2 Kan Extensions along Inclusions

`02YP` Let $\mathcal{C}$ be an ∞-category and let $\delta : K \to \mathcal{C}$ be a diagram. In §7.3.1, we introduced the notion of a functor $F : \mathcal{C} \to \mathcal{D}$ being a left Kan extension of another diagram $F_0 : K \to \mathcal{D}$ along δ (Variant 7.3.1.5). Beware that this terminology is potentially misleading: if F is a left Kan extension of F_0 along δ, then the composition $F \circ \delta$ need not be equal to F_0. Instead, it is equipped with a natural transformation $\beta : F_0 \to F \circ \delta$ satisfying a certain universal property. In this section, we specialize to the case where $K = \mathcal{C}^0$ is a full subcategory of $\mathcal{C}$ and $\delta : \mathcal{C}^0 \hookrightarrow \mathcal{C}$ is the inclusion map. In this case, the natural transformation β is necessarily an isomorphism (Corollary 7.3.1.16). Consequently, the Kan extension condition can be substantially simplified: it can be regarded as a *property* of the functor F, which can be formulated without reference to the diagram F_0 or the natural transformation β.

`02YQ` **Definition 7.3.2.1.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Fix an object $C \in \mathcal{C}$. We will say that F *is left Kan extended from* $\mathcal{C}^0$ *at* C if the composite map

$$(\mathcal{C}^0_{/C})^{\triangleright} \hookrightarrow (\mathcal{C}_{/C})^{\triangleright} \xrightarrow{c} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a colimit diagram in the ∞-category $\mathcal{D}$. Here $\mathcal{C}^0_{/C}$ denotes the fiber product $\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C}$ (Notation 7.3.1.1), and c is the slice contraction morphism of Construction 4.3.5.12. Similarly, we say that F *is right Kan extended from* $\mathcal{C}^0$ *at* C if the composite map

$$(\mathcal{C}^0_{C/})^{\triangleleft} \hookrightarrow (\mathcal{C}_{C/})^{\triangleleft} \xrightarrow{c'} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a limit diagram in $\mathcal{D}$. We say that F *is left Kan extended from* $\mathcal{C}^0$ if it is left Kan extended from $\mathcal{C}^0$ at every object $C \in \mathcal{C}$. We say that F *is right Kan extended from* $\mathcal{C}^0$ if it is right Kan extended from $\mathcal{C}^0$ at every object $C \in \mathcal{C}$.

`02YR` **Remark 7.3.2.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Then F is right Kan extended from $\mathcal{C}^0$ if and only if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is left Kan extended from $\mathcal{C}^0$.

`02YS` **Exercise 7.3.2.3.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Show that, for every object $C \in \mathcal{C}^0$, the functor F is both left and right Kan extended from $\mathcal{C}^0$ at C. For a more general statement, see Proposition 7.3.3.7.

Remark 7.3.2.4. Let $F : \mathcal{B} \star \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and set $G = F|_{\mathcal{B}}$, so that F can be identified with a functor $f : \mathcal{C} \to \mathcal{D}_{G/}$. If $\mathcal{C}^0$ is a full subcategory of $\mathcal{C}$, then f is left Kan extended from $\mathcal{C}^0$ at an object $C \in \mathcal{C}$ if and only if F is left Kan extended from $\mathcal{B} \star \mathcal{C}^0$ at C (see Remark 7.1.2.11). Combining this observation with Exercise 7.3.2.3, we see that f is left Kan extended from $\mathcal{C}^0$ if and only if F is left Kan extended from $\mathcal{B} \star \mathcal{C}^0$. 04K2

Example 7.3.2.5. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ordinary categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Then F is left Kan extended from $\mathcal{C}^0$ (in the sense of Definition 7.3.0.1) if and only if the induced functor of ∞-categories $\mathrm{N}_\bullet(F) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{D})$ is left Kan extended from $\mathrm{N}_\bullet(\mathcal{C}^0)$ (in the sense of Definition 7.3.2.1). 02YT

We now show that Definition 7.3.2.1 can be regarded as a special case of the notions introduced in §7.3.1:

Proposition 7.3.2.6. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let F_0 denote the restriction of F to a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$, and let $\iota : \mathcal{C}^0 \hookrightarrow \mathcal{C}$ denote the inclusion functor. Then:* 02YU

- *The functor F is left Kan extended from $\mathcal{C}^0$ (in the sense of Definition 7.3.2.1) if and only if the identity transformation $\mathrm{id} : F_0 \to F \circ \iota$ exhibits F as a left Kan extension of F_0 along ι (in the sense of Variant 7.3.1.5).*
- *The functor F is right Kan extended from $\mathcal{C}^0$ (in the sense of Definition 7.3.2.1) if and only if the identity transformation $\mathrm{id}_{F_0} : F \circ \iota \to F_0$ exhibits F as a right Kan extension of F_0 along ι (in the sense of Definition 7.3.1.2).*

Proof. Fix an object $C \in \mathcal{C}$. It follows from Remark 7.1.2.6 that the composition

$$(\mathcal{C}^0_{/C})^{\triangleright} \hookrightarrow (\mathcal{C}_{/C})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a colimit diagram in $\mathcal{D}$ if and only if the natural transformation id_{F_0} satisfies condition $(*_C)$ of Variant 7.3.1.5. The first assertion follows by allowing the object C to vary, and the second follows by a similar argument. □

Corollary 7.3.2.7. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and let $\beta : F_0 \to F|_{\mathcal{C}^0}$ be a natural transformation of functors from $\mathcal{C}^0$ to $\mathcal{D}$. Then β exhibits F as a left Kan extension of F_0 along the inclusion map $\iota : \mathcal{C}^0 \hookrightarrow \mathcal{C}$ (in the sense of Variant 7.3.1.5) if and only if the following pair of conditions is satisfied:* 02YV

(1) *The functor F is left Kan extended from $\mathcal{C}^0$ (in the sense of Definition 7.3.2.1).*

(2) *The natural transformation β is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$.*

Proof. By virtue of Corollary 7.3.1.16, we may assume that condition (2) is satisfied. Using Remark 7.3.1.10, we can reduce further to the special case where $F_0 = F|_{\mathcal{C}^0}$ and β is the identity transformation, in which case the desired result is a restatement of Proposition 7.3.2.6. □

02YW **Corollary 7.3.2.8.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and let $F_0 : \mathcal{C}^0 \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:*

(1) *There exists a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\beta : F_0 \to F|_{\mathcal{C}^0}$ which exhibits F as a left Kan extension of F_0 along the inclusion functor $\mathcal{C}^0 \hookrightarrow \mathcal{C}$.*

(2) *There exists a functor $F : \mathcal{C} \to \mathcal{D}$ which is left Kan extended from $\mathcal{C}^0$ and satisfies $F_0 = F|_{\mathcal{C}^0}$.*

Proof. We will show that (1) implies (2); the converse is an immediate consequence of Proposition 7.3.2.6. Let $\beta : F_0 \to F'|_{\mathcal{C}^0}$ exhibit F' as a left Kan extension of F_0 along the inclusion functor $\mathcal{C}^0 \hookrightarrow \mathcal{C}$. Then β is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$ (Corollary 7.3.1.16). Using Corollary 4.4.5.9, we can lift β to an isomorphism $\widetilde{\beta} : F \to F'$ in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where F satisfies $F|_{\mathcal{C}^0} = F_0$. Applying Remark 7.3.1.12, we deduce that the identity transformation id_{F_0} exhibits F as a left Kan extension of F_0 along the inclusion map $\mathcal{C}^0 \hookrightarrow \mathcal{C}$. Invoking Proposition 7.3.2.6, we conclude that F is left Kan extended from $\mathcal{C}^0$. □

02YX **Definition 7.3.2.9.** Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and suppose we are given functors $F : \mathcal{C} \to \mathcal{D}$ and $F_0 : \mathcal{C}^0 \to \mathcal{D}$. We will say that F *is a left Kan extension of* F_0 if F is left Kan extended from $\mathcal{C}^0$ and satisfies $F|_{\mathcal{C}^0} = F_0$. We will say that F *is a right Kan extension of* F_0 if F is right Kan extended from $\mathcal{C}^0$ and satisfies $F|_{\mathcal{C}^0} = F_0$.

02YY **Warning 7.3.2.10.** Let $\mathcal{C}$ be an ∞-category, let $\iota : \mathcal{C}^0 \hookrightarrow \mathcal{C}$ be the inclusion of a full subcategory, and let $F_0 : \mathcal{C}^0 \to \mathcal{D}$. be a functor. We have given two definitions for the notion of Kan extension:

(a) A functor $F : \mathcal{C} \to \mathcal{D}$ is a left Kan extension of F_0 if it is left Kan extended from $\mathcal{C}^0$ and satisfies $F|_{\mathcal{C}^0} = F_0$ (Definition 7.3.2.9).

(b) A functor $F : \mathcal{C} \to \mathcal{D}$ is a left Kan extension of F_0 along ι if there exists a natural transformation $\beta : F_0 \to F|_{\mathcal{C}^0}$ which exhibits F as a left Kan extension of F_0 along ι, in the sense of Variant 7.3.1.5.

These definitions are not quite equivalent. By virtue of Proposition 7.3.2.6, a functor $F : \mathcal{C} \to \mathcal{D}$ satisfies condition (a) if and only if it satisfies a stronger version of condition (b), where β is required to be an identity natural transformation. In particular, condition (a)

implies condition (b). However, the converse is false: if F is a left Kan extension of F_0 along ι, then the restriction $F|_{\mathcal{C}^0}$ need not be *equal* to F_0. However, it is necessarily isomorphic to F_0, by virtue of Corollary 7.3.2.7.

Let $\delta : \mathcal{K} \to \mathcal{C}$ be a functor of ∞-categories. The preceding results show that, if δ is an isomorphism from $\mathcal{K}$ to a full subcategory of $\mathcal{C}$, then the theory of Kan extensions along δ (in the sense of §7.3.1) can be reformulated in terms of Definition 7.3.2.1. We now extend this observation to the case of a general functor, by identifying $\mathcal{K}$ with a full subcategory of the relative join $\mathcal{K} \star_{\mathcal{C}} \mathcal{C}$ of Construction 5.2.3.1.

Proposition 7.3.2.11. *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a functor of ∞-categories, let $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ be another functor having restrictions $F_0 = F|_{\mathcal{K}}$ and $F_1 = F|_{\mathcal{C}}$, so that the composition* 02YZ

$$\Delta^1 \times \mathcal{K} \simeq \mathcal{K} \star_{\mathcal{K}} \mathcal{K} \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

determines a natural transformation $\beta : F_0 \to F_1 \circ \delta$. The following conditions are equivalent:

(1) *The functor F is left Kan extended from the full subcategory $\mathcal{K} \subseteq \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$ (in the sense of Definition 7.3.2.1).*

(2) *The natural transformation β exhibits F_1 as a left Kan extension of F_0 along δ (in the sense of Variant 7.3.1.5).*

Proof. By virtue of Exercise 7.3.2.3, it will suffice to show that for every object $C \in \mathcal{C}$, the following conditions are equivalent:

(1_C) The functor F is left Kan extended from $\mathcal{K}$ at C (in the sense of Definition 7.3.2.1).

(2_C) The natural transformation β satisfies condition $(*_C)$ of Variant 7.3.1.5.

For the remainder of the proof, let us regard the object $C \in \mathcal{C}$ as fixed, and set $\mathcal{K}_{/C} = \mathcal{K} \times_{\mathcal{C}} \mathcal{C}_{/C}$. Let $\pi : \Delta^2 \times \mathcal{K}_{/C} \to (\Delta^1 \times \mathcal{K}_{/C})^{\triangleright}$ be the functor which is the identity on $\Delta^1 \times \mathcal{K}_{/C}$ and carries $\{2\} \times \mathcal{K}_{/C}$ to the cone point of $(\Delta^1 \times \mathcal{K}_{/C})^{\triangleright}$. Let σ denote the composite map

$$\begin{aligned} \Delta^2 \times \mathcal{K}_{/C} & \xrightarrow{\pi} (\Delta^1 \times \mathcal{K}_{/C})^{\triangleright} \\ & \simeq ((\mathcal{K} \times \Delta^1) \times_{\mathcal{K} \star_{\mathcal{C}} \mathcal{C}} (\mathcal{K} \star_{\mathcal{C}} \mathcal{C})_{/C})^{\triangleright} \\ & \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \\ & \xrightarrow{F} \mathcal{D}. \end{aligned}$$

We will regard σ as a 2-simplex in the ∞-category $\operatorname{Fun}(\mathcal{K}_{/C}, \mathcal{D})$, which we display as a

diagram

$$\begin{tikzcd}
& (F_1 \circ \delta)|_{\mathcal{K}_{/C}} \arrow[dr] & \\
F_0|_{\mathcal{K}_{/C}} \arrow[ur] \arrow[rr] & & \underline{F_1(C)}
\end{tikzcd}$$

which witnesses the bottom horizontal map as the natural transformation β_C appearing in condition $(*_C)$. By construction, this natural transformation β_C is given by the composite map

$$\mathrm{N}_{\bullet}(\{0 < 2\}) \times \mathcal{K}_{/C} \to (\mathcal{K}_{/C})^{\triangleright} \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \xrightarrow{F} \mathcal{D},$$

so the equivalence $(1_C) \Leftrightarrow (2_C)$ is a special case of Remark 7.1.2.6. □

02Z0 **Warning 7.3.2.12.** For a general diagram

$$\begin{tikzcd}
& \mathcal{C} \arrow[dr, "F_1"] \arrow[d, phantom, "\Uparrow \beta"] & \\
\mathcal{K} \arrow[ur, "\delta"] \arrow[rr, "F_0"'] & {} & \mathcal{D},
\end{tikzcd}$$

we cannot always arrange that there exists a functor $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ satisfying the requirements of Proposition 7.3.2.11. However, we can always find a functor $F' : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ which satisfies $F'|_{\mathcal{K}} = F_0$, $F'|_{\mathcal{C}} = F_1$, and the map

$$\Delta^1 \times \mathcal{K} \simeq \mathcal{K} \star_{\mathcal{K}} \mathcal{K} \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \xrightarrow{F'} \mathcal{D}$$

determines a natural transformation $\beta' : F_0 \to F_1 \circ \delta$ which is homotopic to β. To see this, set $M = (\Delta^1 \times \mathcal{K}) \coprod_{(\{1\} \times \mathcal{K})} \mathcal{C}$, so that the pair (β, F_1) determines a morphism of simplicial sets $f : M \to \mathcal{D}$. Proposition 5.2.4.5 supplies a categorical equivalence of simplicial sets $\theta : M \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$, so the induced map

$$\mathrm{Fun}_{\mathcal{K} \coprod \mathcal{C}/}(\mathcal{K} \star_{\mathcal{C}} \mathcal{C}, \mathcal{D}) \xrightarrow{\circ \theta} \mathrm{Fun}_{\mathcal{K} \coprod \mathcal{C}/}(M, \mathcal{D})$$

is an equivalence of ∞-categories (Corollary 4.5.4.5). It follows that there exists a functor $F' : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ such that $F'|_{\mathcal{K}} = F_0$, $F'|_{\mathcal{C}} = F_1$, and $F' \circ \theta$ is isomorphic to f as an object of the ∞-category $\mathrm{Fun}_{\mathcal{K} \coprod \mathcal{C}/}(M, \mathcal{D})$. The last requirement is a reformulation of the condition that $\beta' = F'|_{\Delta^1 \times \mathcal{K}}$ is homotopic to β.

02Z1 **Corollary 7.3.2.13.** *Let $\delta : \mathcal{K} \to \mathcal{C}$, $F_0 : \mathcal{K} \to \mathcal{D}$, and $F_1 : \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories. The following conditions are equivalent:*

(1) *There exists a functor* $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ *which is left Kan extended from* $\mathcal{K}$ *which satisfies* $F_0 = F|_{\mathcal{K}}$ *and* $F_1 = F|_{\mathcal{C}}$.

(2) *There exists a natural transformation* $\beta : F_0 \to F_1 \circ \delta$ *which exhibits* F_1 *as a left Kan extension of* F_0 *along* δ.

Proof. The implication (1) $\Leftrightarrow$ (2) follows immediately from Proposition 7.3.2.11. Conversely, suppose that there exists a natural transformation $\beta : F_0 \to F_1 \circ \delta$ which exhibits F_1 as a left Kan extension of F_0 along δ. By virtue of Remark 7.3.1.9, we can modify β by a homotopy and thereby arrange that there exists a functor $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ satisfying $F|_{\mathcal{K}} = F_0$, $F|_{\mathcal{K}} = F_1$ and for which the induced map

$$\Delta^1 \times \mathcal{K} \simeq \mathcal{K} \star_{\mathcal{K}} \mathcal{K} \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

coincides with β (Warning 7.3.2.12). Applying Proposition 7.3.2.11, we see that F is left Kan extended from $\mathcal{K}$. □

For later use, we record a slightly more general version of Proposition 7.3.2.11.

Corollary 7.3.2.14. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor of* ∞*-categories, and let* $U : \mathcal{C} \to \Delta^1$ *be a cocartesian fibration having fibers* $\mathcal{C}_0 = \{0\} \times_{\Delta^1} \mathcal{C}$ *and* $\mathcal{C}_1 = \{1\} \times_{\Delta^1} \mathcal{C}$. *Choose a functor* $G : \mathcal{C}_0 \to \mathcal{C}_1$ *and a natural transformation* $\beta : \mathrm{id}_{\mathcal{C}_0} \to G$ *which exhibits* G *as given by covariant transport along the nondegenerate edge of* Δ^1 *(see Definition 5.2.2.4). The following conditions are equivalent:* 03U3

(1) *The functor* F *is left Kan extended from* $\mathcal{C}_0$.

(2) *The natural transformation* $F(\beta) : F|_{\mathcal{C}_0} \to F|_{\mathcal{C}_1} \circ G$ *exhibits* $F|_{\mathcal{C}_1}$ *as a left Kan extension of* $F|_{\mathcal{C}_0}$ *along* G.

Proof. Let us regard the functor G as fixed. Let $M = (\Delta^1 \times \mathcal{C}_0) \coprod_{(\{1\} \times \mathcal{C}_0)} \mathcal{C}_1$ be the mapping cylinder of G, and let us abuse notation by identifying $\mathcal{C}_0 \simeq \{0\} \times \mathcal{C}_0$ and $\mathcal{C}_1$ with (disjoint) simplicial subsets of M. We can then identify α with a morphism of simplicial sets $\mu : M \to \mathcal{C}$ which is the identity when restricted to $\mathcal{C}_0$ and $\mathcal{C}_1$.

Note that the tautological map

$$\Delta^1 \times \mathcal{C}_0 \simeq \mathcal{C}_0 \star_{\mathcal{C}_0} \mathcal{C}_0 \to \mathcal{C}_0 \star_{\mathcal{C}_1} \mathcal{C}_1$$

extends to a morphism of simplicial sets $\lambda : M \to \mathcal{C}_0 \star_{\mathcal{C}_1} \mathcal{C}_1$ which is the identity on $\mathcal{C}_1$; moreover, λ is a categorical equivalence (Proposition 5.2.4.5). It follows that precomposition with λ induces an equivalence of ∞-categories

$$\mathrm{Fun}_{\mathcal{C}_0 \coprod \mathcal{C}_1 /}(\mathcal{C}_0 \star_{\mathcal{C}_1} \mathcal{C}_1, \mathcal{C}) \to \mathrm{Fun}_{\mathcal{C}_0 \coprod \mathcal{C}_1 /}(M, \mathcal{C}).$$

We can therefore choose a functor $G : \mathcal{C}_0 \star_{\mathcal{C}_1} \mathcal{C}_1 \to \mathcal{C}$ satisfying $G|_{\mathcal{C}_0} = \mathrm{id}_{\mathcal{C}_0}$ and $G|_{\mathcal{C}_1} = \mathrm{id}_{\mathcal{C}_1}$, where $G \circ \lambda$ is isomorphic to μ as an object of the ∞-category $\mathrm{Fun}_{\mathcal{C}_0 \coprod \mathcal{C}_1 /}(M, \mathcal{C})$ Since condition (2) depends only on the homotopy class of the natural transformation β (Remark 7.3.1.9), we are free to modify β and may therefore assume that $G \circ \lambda = \mu$. In this case, Proposition 7.3.2.11 allows us to reformulate condition (2) as follows:

($2'$) The functor $(F \circ G) : \mathcal{C}_0 \star_{\mathcal{C}_1} \mathcal{C}_1 \to \mathcal{D}$ is left Kan extended from $\mathcal{C}_0$.

Since λ and μ are categorical equivalences of simplicial sets (Proposition 5.2.4.5), the functor G is an equivalence of ∞-categories (Remark 4.5.3.5). The equivalence of (1) and ($2'$) is now a special case of Proposition 7.3.3.18. □

7.3.3 Relative Kan Extensions

02Z2 For many applications, it will be convenient to work with a generalization of Definition 7.3.2.1. In what follows, we assume that the reader is familiar with the theory of relative (co)limit diagrams introduced in §7.1.5.

02Z3 **Definition 7.3.3.1** (Relative Kan Extensions)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. For each object $C \in \mathcal{C}$, we will say that F *is U-left Kan extended from $\mathcal{C}^0$ at C* if the composite map

$$(\mathcal{C}^0_{/C})^{\triangleright} \hookrightarrow (\mathcal{C}_{/C})^{\triangleright} \xrightarrow{c} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a U-colimit diagram in the ∞-category $\mathcal{D}$. We say that F *is U-right Kan extended from $\mathcal{C}^0$ at C* if the composite map

$$(\mathcal{C}^0_{C/})^{\triangleleft} \hookrightarrow (\mathcal{C}_{C/})^{\triangleleft} \xrightarrow{c'} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a U-limit diagram in $\mathcal{D}$. Here c and c' denote the slice and coslice contraction morphisms of Construction 4.3.5.12. We say that F *is U-left Kan extended from $\mathcal{C}^0$* if it is U-left Kan extended from $\mathcal{C}^0$ at every object $C \in \mathcal{C}$. We say that F *is U-right Kan extended from $\mathcal{C}^0$* if it is U-right Kan extended from $\mathcal{C}^0$ at every object $C \in \mathcal{C}$.

02Z4 **Remark 7.3.3.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Then F is left Kan extended from $\mathcal{C}^0$ (in the sense of Definition 7.3.2.1) if and only if it is U-left Kan extended from $\mathcal{C}^0$ (in the sense of Definition 7.3.3.1), where $U : \mathcal{D} \to \Delta^0$ is the projection map. Similarly, F is right Kan extended from $\mathcal{C}^0$ if and only if it is U-right Kan extended from $\mathcal{C}^0$. See Example 7.1.5.3.

03Y2 **Remark 7.3.3.3.** Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and let $U : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories. Consider the evaluation functor

$$\mathrm{ev} : \mathcal{C} \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D} \qquad (C, F) \mapsto F(C).$$

For every object $C \in \mathcal{C}$ and every functor $F : \mathcal{C} \to \mathcal{D}$, the following conditions are equivalent:

(a) The functor F is U-left Kan extended from $\mathcal{C}^0$ at C.

(b) The evaluation functor ev is U-left Kan extended from $\mathcal{C}^0 \times \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ at (C, F).

To prove this, it will suffice to show that the inclusion map

$$\mathcal{C}^0_{/C} \times \{\mathrm{id}_F\} \hookrightarrow \mathcal{C}^0_{/C} \times \operatorname{Fun}(\mathcal{C}, \mathcal{D})_{/F}$$

is right cofinal (Corollary 7.2.2.2). This follows from Corollary 7.2.1.19, since the inclusion map $\{\mathrm{id}_F\} \hookrightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D})_{/F}$ is right cofinal (the identity morphism id_F is an isomorphism in $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$, and therefore final when regarded as an object of the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})_{/F}$ by virtue of Proposition 4.6.7.22).

Remark 7.3.3.4. In the situation of Definition 7.3.3.1, the morphism $F : \mathcal{C} \to \mathcal{D}$ is U-right Kan extended from $\mathcal{C}^0$ if and only if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is U^{op}-left Kan extended from $(\mathcal{C}^0)^{\mathrm{op}}$. `02Z5`

Example 7.3.3.5. Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. If U is fully faithful, then F is U-left Kan extended and U-right Kan extended from any full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ (see Example 7.1.5.4). `02Z7`

Example 7.3.3.6. Let $U : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories. Then a functor $F : \mathcal{C} \to \mathcal{D}$ is U-left Kan extended from the empty subcategory $\emptyset \subseteq \mathcal{C}$ if and only if it carries each object of $\mathcal{C}$ to a U-initial object of $\mathcal{D}$. Similarly, F is U-right Kan extended from the empty subcategory if and only if it carries each object of $\mathcal{C}$ to a U-final object of $\mathcal{D}$. `043F`

To verify the Kan extension conditions of Definition 7.3.3.1, it suffices to consider objects C which do *not* belong to the full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$.

Proposition 7.3.3.7. *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory and let $C \in \mathcal{C}$ be an object which is isomorphic to an object of $\mathcal{C}^0$. Then F is both U-left Kan extended from $\mathcal{C}^0$ and U-right Kan extended from $\mathcal{C}^0$ at C.* `02Z8`

Proof. We will show that F is U-left Kan extended from $\mathcal{C}^0$ at C; the analogous statement for the right Kan extension condition follows by a similar argument. Let $c : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{C}$ be the slice contraction morphism; we wish to show that the composition $(F \circ c) : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{D}$ is a U-colimit diagram. Choose an object $C' \in \mathcal{C}^0$ and an isomorphism $u : C' \to C$ in the ∞-category $\mathcal{C}$. Our assumption that u is an isomorphism guarantees that it is final when viewed as an object of the slice ∞-category $\mathcal{C}_{/C}$ (Proposition 4.6.7.22), and therefore also when viewed as an object of the ∞-category $\mathcal{C}^0_{/C}$. The desired result now follows from Corollary 7.2.3.6, since $F(u)$ is an isomorphism in the ∞-category $\mathcal{D}$. □

Example 7.3.3.8. Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Then F is U-left Kan extended and U-right Kan extended from the full subcategory $\mathcal{C} \subseteq \mathcal{C}$. `03U4`

02Z9 **Example 7.3.3.9.** Let $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, and set $F = \overline{F}|_{\mathcal{C}}$. Then F is U-left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ if and only if $\overline{F}$ is U-left Kan extended from the cone $(\mathcal{C}^0)^{\triangleright} \subseteq \mathcal{C}^{\triangleright}$. To prove this, it suffices (by virtue of Proposition 7.3.3.7) to show that F is U-left Kan extended from $\mathcal{C}^0$ at an object $C \in \mathcal{C}$ if and only if $\overline{F}$ is U-left Kan extended from $(\mathcal{C}^0)^{\triangleright}$ at C, which follows immediately from the definition.

02ZA **Example 7.3.3.10.** Let $\mathcal{C}$ be an ∞-category and let $U : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories. It follows from Proposition 7.3.3.7 that a functor $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ is a U-colimit diagram (in the sense of Definition 7.1.5.1) if and only if it is U-left Kan extended from $\mathcal{C}$.

039R **Proposition 7.3.3.11.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $U : \mathcal{D} \to \mathcal{E}$ be an inner fibration of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a coreflective subcategory of $\mathcal{C}$. The following conditions are equivalent:*

(1) *The functor F is U-left Kan extended from $\mathcal{C}^0$.*

(2) *Let $e : X \to Y$ be a morphism in $\mathcal{C}$ which exhibits X as a $\mathcal{C}^0$-coreflection of Y (Definition 6.2.2.1). Then $F(e)$ is a U-cocartesian morphism of $\mathcal{D}$.*

(3) *Let $T : \mathcal{C} \to \mathcal{C}_0$ be a right adjoint to the inclusion. If e is a morphism in $\mathcal{C}$ and $T(e)$ is an isomorphism in $\mathcal{C}_0$, then $F(e)$ is a U-cocartesian morphism of $\mathcal{D}$.*

Proof. Let Y be an object of $\mathcal{C}$. By assumption, there exists an object $X \in \mathcal{C}^0$ and a morphism $e : X \to Y$ which exhibits X as a $\mathcal{C}^0$-coreflection of Y. Then e is final when viewed as an object of the ∞-category $\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/Y}$. It follows that F is U-left Kan extended from $\mathcal{C}^0$ at Y if and only if $F(e)$ is U-cocartesian morphism of $\mathcal{D}$; in particular, this condition is independent of the choice of e. Allowing the object Y to vary, we deduce the equivalence (1) $\Leftrightarrow$ (2).

Using Lemma 6.2.2.14, we can choose a functor $T : \mathcal{C} \to \mathcal{C}^0$ and a natural transformation $\epsilon : T \to \mathrm{id}_{\mathcal{C}}$ which exhibits T as a $\mathcal{C}^0$-coreflection functor, so that T is right adjoint to the inclusion of $\mathcal{C}^0$ into $\mathcal{C}$ (Proposition 6.2.2.15). Let $e : X \to Y$ be a morphism in $\mathcal{C}$. If e exhibits X as a $\mathcal{C}^0$-coreflection of Y, then $T(e)$ is an isomorphism in $\mathcal{C}^0$, which shows immediately that (3) implies (2). Conversely, suppose that (2) is satisfied and that $T(e)$ is an isomorphism in $\mathcal{C}^0$. We then have a commutative diagram

$$\begin{tikzcd}
(F \circ T)(X) \arrow[r, "(F\circ T)(e)"] \arrow[d, "F(\epsilon_X)"] & (F \circ T)(Y) \arrow[d, "F(\epsilon_Y)"] \\
F(X) \arrow[r, "F(e)"] & F(Y)
\end{tikzcd}$$

in the ∞-category $\mathcal{D}$, where the upper horizontal map is an isomorphism and the vertical maps are U-cocartesian. Using Corollary 5.1.2.4, we see that $F(e)$ is also U-cocartesian. □

Corollary 7.3.3.12. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a coreflective subcategory. The following conditions are equivalent:* 039S

(1) *The functor F is left Kan extended from $\mathcal{C}^0$.*

(2) *Let $e : X \to Y$ be a morphism in $\mathcal{C}$ which exhibits X as a $\mathcal{C}^0$-coreflection of Y (Definition 6.2.2.1). Then $F(e)$ is an isomorphism in $\mathcal{D}$.*

(3) *Let $T : \mathcal{C} \to \mathcal{C}_0$ be a right adjoint to the inclusion. If e is a morphism in $\mathcal{C}$ and $T(e)$ is an isomorphism in $\mathcal{C}_0$, then $F(e)$ is an isomorphism in $\mathcal{D}$.*

Proof. Combine Proposition 7.3.3.11 with Example 5.1.1.4 (for a closely related statement, see Proposition 7.3.1.17). □

Corollary 7.3.3.13. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $U : \mathcal{D} \to \mathcal{E}$ be an inner fibration of ∞-categories, and suppose that $\mathcal{C}$ contains an initial object. The following conditions are equivalent:* 043G

(1) *The functor F is U-left Kan extended from the full subcategory $\mathcal{C}^{\mathrm{init}} \subseteq \mathcal{C}$ spanned by the initial objects.*

(2) *The functor F carries every morphism of $\mathcal{C}$ to a U-cocartesian morphism of $\mathcal{D}$.*

Proof. Combine Proposition 7.3.3.11 with Example 6.2.2.5. □

Corollary 7.3.3.14 (Constant Diagrams)**.** *Let $\mathcal{C}$ be an ∞-category which contains an initial object, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:* 05E0

(1) *The functor F is left Kan extended from the full subcategory $\mathcal{C}^{\mathrm{init}} \subseteq \mathcal{C}$ spanned by the initial objects.*

(2) *The functor F carries each morphism in $\mathcal{C}$ to an isomorphism in the ∞-category $\mathcal{D}$.*

(3) *The functor F is isomorphic to a constant functor.*

Proof. The equivalence (1) ⇔ (2) follows from Corollary 7.3.3.13 by taking $\mathcal{E} = \Delta^0$, and the implication (3) ⇒ (2) is immediate. To prove the converse, we observe that condition (2) guarantees that F can be regarded as a morphism from $\mathcal{C}$ to the Kan complex $\mathcal{D}^{\simeq}$. Since $\mathcal{C}$ has an initial object, it is weakly contractible (Corollary 4.6.7.25), so this morphism is automatically nullhomotopic (Remark 3.2.4.18). □

We now record some basic stability properties enjoyed by the class of relative Kan extensions, which follow easily from the analogous stability properties of relative (co)limit diagrams.

02ZB **Remark 7.3.3.15.** Suppose we are given a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "G"] \arrow[d, "U"] & \mathcal{D}' \arrow[d, "U'"] \\
\mathcal{E} \arrow[r] & \mathcal{E}',
\end{tikzcd}$$

where the horizontal functors are equivalence of ∞-categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Then F is U-left Kan extended from $\mathcal{C}^0$ if and only if $G \circ F$ is U'-left Kan extended from $\mathcal{C}^0$ (see Remark 7.1.5.6). Similarly, F is U-right Kan extended from $\mathcal{C}^0$ if and only if $G \circ F$ is U'-right Kan extended from $\mathcal{C}^0$.

02ZC **Remark 7.3.3.16.** Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Let $V : \mathcal{D} \to \mathcal{E}$ be a functor which is isomorphic to U (as an object of the ∞-category $\operatorname{Fun}(\mathcal{D}, \mathcal{E})$). Then F is U-left Kan extended from $\mathcal{C}^0$ if and only if it is V-left Kan extended from $\mathcal{C}^0$ (see Remark 7.1.5.7). Similarly, F is U-right Kan extended from $\mathcal{C}^0$ if and only if it is V-right Kan extended from $\mathcal{C}^0$.

02ZD **Remark 7.3.3.17.** Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, let $G : \mathcal{C} \to \mathcal{D}$ be a functor which is isomorphic to F (as an object of the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$), and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Then F is U-left Kan extended from $\mathcal{C}^0$ if and only if G is U-left Kan extended from $\mathcal{C}^0$ (see Proposition 7.1.5.13). Similarly, F is U-right Kan extended from $\mathcal{C}^0$ if and only if G is U-right Kan extended from $\mathcal{C}^0$.

03U5 **Proposition 7.3.3.18** (Change of Source). *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a replete full subcategory. Let $G : \mathcal{B} \to \mathcal{C}$ be an equivalence of ∞-categories, and set $\mathcal{B}^0 = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{B}$. Then F is U-left Kan extended from $\mathcal{C}^0$ if and only if $F \circ G$ is U-left Kan extended from $\mathcal{B}^0$.*

Proof. Assume first that F is U-left Kan extended from $\mathcal{C}^0$; we will show that $F \circ G$ is U-left Kan extended from $\mathcal{B}^0$. Fix an object $B \in \mathcal{B}$ and set $\mathcal{B}^0_{/B} = \mathcal{B}^0 \times_{\mathcal{B}} \mathcal{B}_{/B}$; we wish to show that the composite map

$$\theta : (\mathcal{B}^0_{/B})^{\triangleright} \hookrightarrow \mathcal{B}^{\triangleright}_{/B} \to \mathcal{B} \xrightarrow{F \circ G} \mathcal{D}$$

is a U-colimit diagram. Set $C = G(B)$ and $\mathcal{C}^0_{/C} = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}^0$. Since G is an equivalence of ∞-categories, the induced map $G_{/B} : \mathcal{B}_{/B} \to \mathcal{C}_{/C}$ is also an equivalence of ∞-categories

(Corollary 4.6.4.19). Our assumption that $\mathcal{C}^0$ is a replete subcategory of $\mathcal{C}$ guarantees that $\mathcal{C}^0_{/C}$ is a replete subcategory of $\mathcal{C}_{/C}$. In particular, the inclusion map $\mathcal{C}^0_{/C} \hookrightarrow \mathcal{C}_{/C}$ is an isofibration, so that $G_{/B}$ restricts to an equivalence of ∞-categories $G^0_{/B} : \mathcal{B}^0_{/B} \to \mathcal{C}^0_{/C}$. By construction, the morphism θ is the composition of $(G^0_{/B})^{\triangleright}$ with the map

$$\theta' : (\mathcal{C}^0_{/C})^{\triangleright} \hookrightarrow \mathcal{C}^{\triangleright}_{/C} \to \mathcal{C} \xrightarrow{F} \mathcal{D},$$

which is a U-colimit diagram by virtue of our assumption that F is U-left Kan extended from $\mathcal{C}^0$. Applying Corollary 7.2.2.2, we deduce that θ is also a U-colimit diagram.

We now prove the converse. Assume that $F \circ G$ is U-left Kan extended from $\mathcal{B}^0$; we wish to show that F is U-left Kan extended from $\mathcal{C}^0$. Let $H : \mathcal{C} \to \mathcal{B}$ be a homotopy inverse to G, so that $(G \circ H) : \mathcal{C} \to \mathcal{C}$ is isomorphic to the identity functor $\mathrm{id}_{\mathcal{C}}$. Since $\mathcal{C}^0 \subseteq \mathcal{C}$ is replete, it coincides with the inverse image $(G \circ H)^{-1}\,\mathcal{C}^0 = H^{-1}\,\mathcal{B}^0$. Applying the first part of the proof, we deduce that the functor $(F \circ G \circ H) : \mathcal{C} \to \mathcal{D}$ is U-left Kan extended from $\mathcal{C}^0$. The functor F is isomorphic to $F \circ G \circ H$, and is therefore also U-left Kan extended from $\mathcal{C}^0$ (Remark 7.3.3.17). □

Remark 7.3.3.19 (Transitivity)**.** Let $F : \mathcal{C} \to \mathcal{D}$, $U : \mathcal{D} \to \mathcal{E}$, and $V : \mathcal{E} \to \mathcal{E}'$ be functors of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that $U \circ F$ is V-left Kan extended from $\mathcal{C}^0$. Then F is U-left Kan extended from $\mathcal{C}^0$ if and only if it is $(V \circ U)$-left Kan extended from $\mathcal{C}^0$ (see Proposition 7.1.5.14). Similarly, if $U \circ F$ is V-right Kan extended from $\mathcal{C}^0$, then F is U-right Kan extended from $\mathcal{C}^0$ if and only if it is $(V \circ U)$-right Kan extended from $\mathcal{C}^0$. 02ZE

Remark 7.3.3.20. Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that $U \circ F$ is left Kan extended from $\mathcal{C}^0$. Then F is left Kan extended from $\mathcal{C}^0$ if and only if it is U-left Kan extended from $\mathcal{C}^0$; this follows by applying Remark 7.3.3.19 in the special case $\mathcal{E}' = \Delta^0$. Similarly, if $U \circ F$ is right Kan extended from $\mathcal{C}^0$, then F is right Kan extended from $\mathcal{C}^0$ if and only if it is U-right Kan extended from $\mathcal{C}^0$. 02ZF

Proposition 7.3.3.21 (Base Change)**.** *Suppose we are given a commutative diagram of* 02ZG

∞-categories

02ZH
$$\begin{tikzcd}
\mathcal{D}' \arrow[rr, "H'"] \arrow[dr] \arrow[dd, "G"] & & \mathcal{E}' \arrow[dl] \arrow[dd] \\
& \mathcal{B}' \arrow[dd] & \\
\mathcal{D} \arrow[rr, "H"] \arrow[dr, "U"'] & & \mathcal{E} \arrow[dl, "V"] \\
& \mathcal{B} &
\end{tikzcd} \tag{7.14}$$

where each square is a pullback and the diagonal maps are inner fibrations. Let $F : \mathcal{C} \to \mathcal{D}'$ *be a functor of ∞-categories and* $\mathcal{C}^0 \subseteq \mathcal{C}$ *be a full subcategory. Then:*

(1) *If* $G \circ F$ *is* H*-left Kan extended from* $\mathcal{C}^0$*, then* F *is* H'*-left Kan extended from* $\mathcal{C}^0$*.*

(2) *Assume that* U *and* V *are cartesian fibrations and that the functor* G *carries* U*-cartesian morphisms of* $\mathcal{D}$ *to* V*-cartesian morphisms of* $\mathcal{E}$*. If* F *is* H'*-left Kan extended from* $\mathcal{C}^0$*, then* $G \circ F$ *is an* H*-left Kan extended from* $\mathcal{C}^0$*.*

Proof. Use Proposition 7.1.5.19. ☐

02ZJ **Corollary 7.3.3.22.** *Suppose we are given a pullback diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{D}' \arrow[r, "G"] \arrow[d, "U'"] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{E}' \arrow[r] & \mathcal{E},
\end{tikzcd}$$

where the vertical maps are inner fibrations. Let $F : \mathcal{C} \to \mathcal{D}'$ *be a functor of ∞-categories and let* $\mathcal{C}^0 \subseteq \mathcal{C}$ *be a full subcategory. If* $G \circ F$ *is* U*-left Kan extended from* $\mathcal{C}^0$*, then* F *is* U'*-left Kan extended from* $\mathcal{C}^0$*. The converse holds if* U *is a cartesian fibration.*

Proof. Apply Proposition 7.3.3.21 in the special case $\mathcal{B} = \mathcal{E}$. ☐

Corollary 7.3.3.23. *Let $U : \mathcal{D} \to \mathcal{E}$ be an inner fibration of ∞-categories, let $\mathcal{D}_E = \{E\} \times_{\mathcal{E}} \mathcal{D}$ be the fiber of U over an object $E \in \mathcal{E}$, let $F : \mathcal{C} \to \mathcal{D}_E$ be a functor of ∞-categories, and $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. If F is U-left Kan extended from $\mathcal{C}^0$ (when regarded as a functor from $\mathcal{C}$ to $\mathcal{D}$), then it is left Kan extended from $\mathcal{C}^0$ (when regarded as a functor from $\mathcal{C}$ to $\mathcal{D}_E$). The converse holds if U is a cartesian fibration.* 02ZK

Proof. Apply Corollary 7.3.3.22 in the special case $\mathcal{E}' = \{E\}$. □

Remark 7.3.3.24. In the situation of Corollary 7.3.3.23, the functor F is U-left Kan extended from $\mathcal{C}^0$ if and only if, for every morphism $f : E \to E'$ in the ∞-category $\mathcal{E}$, the composite map 05J9

$$\mathcal{C} \xrightarrow{F} \{E\} \times_{\mathcal{E}} \mathcal{D} \hookrightarrow \Delta^1 \times_{\mathcal{E}} \mathcal{D}$$

is left Kan extended from the full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. See Remark 7.1.5.23.

7.3.4 Kan Extensions along Fibrations

In this section, we study the formation of left Kan extension along cocartesian fibrations. We can state a preliminary version of our main result as follows: 02ZL

Proposition 7.3.4.1. *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. Suppose we are given functors of ∞-categories $F_0 : \mathcal{K} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\beta : F_0 \to F \circ \delta$. The following conditions are equivalent:* 02ZM

(1) *The natural transformation β exhibits F as a left Kan extension of F_0 along δ.*

(2) *For each object $C \in \mathcal{C}$, the restriction of β to the fiber $\mathcal{K}_C = \{C\} \times_{\mathcal{C}} \mathcal{K}$ determines a natural transformation $F_0|_{\mathcal{K}_C} \to \underline{F(C)}$ which exhibits $F(C)$ as a colimit of the diagram $F_0|_{\mathcal{K}_C}$ in the ∞-category $\mathcal{D}$.*

Proof. By virtue of Corollary 7.2.2.7, it will suffice to show that for each object $C \in \mathcal{C}$, the tautological map

$$\mathcal{K}_C = \mathcal{K} \times_{\mathcal{C}} \{C\} \hookrightarrow \mathcal{K} \times_{\mathcal{C}} \mathcal{C}_{/C}$$

is right cofinal. Since δ is a cocartesian fibration, it will suffice to show that the inclusion map $\{\mathrm{id}_C\} \hookrightarrow \mathcal{C}_{/C}$ is right cofinal (Proposition 7.2.3.12). This follows from Corollary 4.6.7.24, since id_C is a final object of the ∞-category $\mathcal{C}_{/C}$ (Proposition 4.6.7.22). □

Corollary 7.3.4.2. *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:* 02ZN

(1) *The functor F is left Kan extended from $\mathcal{K}$.*

(2) *For every object $C \in \mathcal{C}$, the functor*

$$F_C : \mathcal{K}_C^{\triangleright} \simeq \mathcal{K}_C \star_{\{C\}} \{C\} \hookrightarrow \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a colimit diagram.

Proof. Combine Propositions 7.3.4.1 and 7.3.2.11. □

Corollary 7.3.4.2 generalizes to the setting of relative Kan extensions:

`02ZP` **Proposition 7.3.4.3.** *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors. The following conditions are equivalent:*

(1) *The functor F is U-left Kan extended from $\mathcal{K}$.*

(2) *For every object $C \in \mathcal{C}$, the functor*

$$F_C : \mathcal{K}_C^{\triangleright} \simeq \mathcal{K}_C \star_{\{C\}} \{C\} \hookrightarrow \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a U-colimit diagram.

Proof. By virtue of Proposition 7.3.3.7, it will suffice to show that for each object $C \in \mathcal{C}$, the following conditions are equivalent:

(1_C) The functor F is U-left Kan extended from $\mathcal{K}$ at C.

(2_C) The functor F_C is a U-colimit diagram.

This follows from Corollary 7.2.2.3, since the tautological map

$$\mathcal{K}_C \simeq \{\mathrm{id}_C\} \times_{\mathcal{C}_{/C}} \mathcal{K}_{/C} \hookrightarrow \mathcal{K}_{/C}$$

is right cofinal (as noted in the proof of Proposition 7.3.4.1). □

Our next goal is establish a companion to Proposition 7.3.4.1, which provides necessary and sufficient conditions for the *existence* of a left Kan extension.

`02ZQ` **Proposition 7.3.4.4.** *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let $F_0 : \mathcal{K} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:*

(1) *The functor F_0 admits a left Kan extension along δ.*

(2) *For every object $C \in \mathcal{C}$, the induced diagram*

$$\mathcal{K}_C = \{C\} \times_{\mathcal{C}} \mathcal{K} \hookrightarrow \mathcal{K} \xrightarrow{F_0} \mathcal{D}$$

has a colimit in the ∞-category $\mathcal{D}$.

Note that the implication (1) ⇒ (2) of Proposition 7.3.4.4 follows immediately from Proposition 7.3.4.1. To prove the converse, it will be convenient to again translate to a question about the inclusion map $\mathcal{K} \hookrightarrow \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$, which we will address in a more general form. First, we need a variant of Corollary 7.1.6.6.

Lemma 7.3.4.5. *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a simplicial subset which contains every vertex of $\mathcal{C}$, and set $\mathcal{K}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{K}$. Suppose we are given a lifting problem* `02ZR`

$$\begin{tikzcd}
\mathcal{K} \coprod_{\mathcal{K}_0} (\mathcal{K}_0 \star_{\mathcal{C}_0} \mathcal{C}_0) \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{K} \star_{\mathcal{C}} \mathcal{C} \arrow[r] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd} \tag{7.15}$$

`02ZS`

which satisfies the following condition:

- $(*)$ *Let $\sigma : \Delta^n \to \mathcal{C}$ be an n-simplex which is not contained in $\mathcal{C}_0$ and set $C = \sigma(0)$. Then the composite map*
$$\mathcal{K}_C^{\triangleright} \simeq \mathcal{K}_C \star_{\{C\}} \{C\} \hookrightarrow \mathcal{K}_0 \star_{\mathcal{C}_0} \mathcal{C}_0 \xrightarrow{F_0} \mathcal{D}$$
is a U-colimit diagram in the ∞-category $\mathcal{D}$.

Then the lifting problem (7.15) admits a solution.

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is an ∞-category (working one simplex at a time, we could even assume that $\mathcal{C} = \Delta^m$ is a standard simplex and that $\mathcal{C}_0 = \partial \Delta^m$ is its boundary). Set $\overline{\mathcal{K}} = \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$, so that δ extends to a map

$$\overline{\delta} : \overline{\mathcal{K}} = \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{C} \star_{\mathcal{C}} \mathcal{C} \simeq \Delta^1 \times \mathcal{C} \to \mathcal{C}.$$

Since δ is a cocartesian fibration, Lemma 5.2.3.17 guarantees that $\overline{\delta}$ is also a cocartesian fibration. Applying Corollary 5.3.6.8, we obtain a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}(\overline{\mathcal{K}} / \mathcal{C}, \mathcal{D}) \arrow[rr] \arrow[dd, "U\circ"] \arrow[dr, "T"] & & \mathrm{Fun}(\mathcal{K} / \mathcal{C}, \mathcal{D}) \arrow[dd, "U\circ"] \arrow[dl] \\
& \mathcal{C} & \\
\mathrm{Fun}(\overline{\mathcal{K}} / \mathcal{C}, \mathcal{E}) \arrow[rr] \arrow[ur] & & \mathrm{Fun}(\mathcal{K} / \mathcal{C}, \mathcal{E}), \arrow[ul]
\end{tikzcd} \tag{7.16}$$

`02ZT`

where the diagonal arrows are cartesian fibrations and the morphisms on the outside of the diagram preserve cartesian morphisms. Applying Proposition 5.1.4.20, we see that the induced map

$$T' : \mathrm{Fun}(\mathcal{K} \,/\, \mathcal{C}, \mathcal{D})) \times_{\mathrm{Fun}(\mathcal{K} \,/\, \mathcal{C}, \mathcal{E})} \mathrm{Fun}(\overline{\mathcal{K}}/\, \mathcal{C}, \mathcal{E}) \to \mathcal{C}$$

is also a cartesian fibration, and that the outer square of the diagram (7.16) determines a functor

$$V : \mathrm{Fun}(\overline{\mathcal{K}}/\, \mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{K} \,/\, \mathcal{C}, \mathcal{D})) \times_{\mathrm{Fun}(\mathcal{K} \,/\, \mathcal{C}, \mathcal{E})} \mathrm{Fun}(\overline{\mathcal{K}}/\, \mathcal{C}, \mathcal{E}) \to \mathcal{C}$$

which carries T-cartesian morphisms to T'-cartesian morphisms.

We next claim that V is an isofibration. Fix a monomorphism of simplicial sets $i : A \hookrightarrow B$ which is also a categorical equivalence; we wish to show that every diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathrm{Fun}(\overline{\mathcal{K}}/\, \mathcal{C}, \mathcal{D}) \arrow[d, "V"] \\
B \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\mathcal{K} \,/\, \mathcal{C}, \mathcal{D})) \times_{\mathrm{Fun}(\mathcal{K} \,/\, \mathcal{C}, \mathcal{E})} \mathrm{Fun}(\overline{\mathcal{K}}/\, \mathcal{C}, \mathcal{E})
\end{tikzcd}$$

admits a solution. Note that this lifting problem determines a morphism of simplicial sets $B \to \mathcal{C}$. Invoking the universal property of Proposition 4.5.9.5, we can rewrite this as a lifting problem

$$\begin{tikzcd}
(A \times_{\mathcal{C}} \overline{\mathcal{K}}) \coprod_{(A \times_{\mathcal{C}} \mathcal{K})} (B \times_{\mathcal{C}} \mathcal{K}) \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
B \times_{\mathcal{C}} \overline{\mathcal{K}} \arrow[r] \arrow[ur, dashed] & \mathcal{E} .
\end{tikzcd}$$

Since U is an isofibration, it will suffice to show that the left vertical map is a categorical equivalence of simplicial sets, or equivalently that the diagram

$$\begin{tikzcd}
A \times_{\mathcal{C}} \mathcal{K} \arrow[r] \arrow[d] & B \times_{\mathcal{C}} \mathcal{K} \arrow[d] \\
A \times_{\mathcal{C}} \overline{\mathcal{K}} \arrow[r] & B \times_{\mathcal{C}} \overline{\mathcal{K}}
\end{tikzcd}$$

is a categorical pushout square (Proposition 4.5.4.11). This follows from Proposition 4.5.4.10, since the horizontal maps are categorical equivalences (Corollary 5.6.7.6).

Unwinding the definitions, we can rewrite (7.15) as a lifting problem

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "G_0"] \arrow[d] & \mathrm{Fun}(\overline{\mathcal{K}}/\,\mathcal{C}, \mathcal{D}) \arrow[d, "V"] \\
\mathcal{C} \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\mathcal{K}\,/\,\mathcal{C}, \mathcal{D})) \times_{\mathrm{Fun}(\mathcal{K}\,/\,\mathcal{C}, \mathcal{E})} \mathrm{Fun}(\overline{\mathcal{K}}/\,\mathcal{C}, \mathcal{E}).
\end{tikzcd}$$

By virtue of Corollary 7.1.6.6, to show that this lifting problem admits a solution, it will suffice to verify the following:

$(*')$ Let $\sigma : \Delta^n \to \mathcal{C}$ be an n-simplex which is not contained in $\mathcal{C}_0$ and set $C = \sigma(0)$. Then $G_0(C)$ is a V-initial object of the ∞-category $\mathrm{Fun}(\overline{\mathcal{K}}/\,\mathcal{C}, \mathcal{D})$.

Unwinding the definitions, we see that the functor $T^{-1}\{C\} \to T'^{-1}\{C\}$ induced by V can be identified with the restriction map

$$V_C : \mathrm{Fun}(\mathcal{K}_C^{\triangleright}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{K}_C, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{K}_C, \mathcal{E})} \mathrm{Fun}(\mathcal{K}_C^{\triangleright}, \mathcal{E}).$$

Combining assumption $(*)$ with Proposition 7.1.6.3, we see that $G_0(C)$ is a V_C-initial object of the ∞-category $\mathrm{Fun}(K_C^{\triangleright}, \mathcal{D})$. Proposition 7.1.4.19 then guarantees that $G_0(C)$ is also V-initial when regarded as an object of the ∞-category $\mathrm{Fun}(\overline{\mathcal{K}}/\,\mathcal{C}, \mathcal{D})$. □

Lemma 7.3.4.6. *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories, and suppose we are given a lifting problem* `02ZU`

`02ZV`

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{K} \star_{\mathcal{C}} \mathcal{C} \arrow[r, "G"] \arrow[ur, dashed, "F"] & \mathcal{E}
\end{tikzcd} \qquad (7.17)$$

with the following property:

$(*)$ *For each vertex $C \in \mathcal{C}$, the induced lifting problem*

$$\begin{tikzcd}
\mathcal{K}_C \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{K}_C \star_{\{C\}} \{C\} \arrow[r] \arrow[ur, dashed, "F_C"] & \mathcal{E}
\end{tikzcd}$$

admits a solution $F_C : \mathcal{K}_C^{\triangleright} \to \mathcal{D}$ which is a U-colimit diagram.

Then (7.17) admits a solution $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ satisfying $F|_{\mathcal{K}_C^{\triangleright}} = F_C$ for each vertex $C \in \mathcal{C}$.

Proof. Let $\mathcal{C}_0 = \mathrm{sk}_0(\mathcal{C})$ be the 0-skeleton of $\mathcal{C}$ and set $\mathcal{K}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{K} = \coprod_{C \in \mathcal{C}} \mathcal{K}_C$, so that we can amalgamate F_0 with the morphisms $\{F_C\}_{C \in \mathcal{C}}$ to obtain a map $F_1 : \mathcal{K} \coprod_{\mathcal{K}_0} (\mathcal{K}_0 \star_{\mathcal{C}_0} \mathcal{C}_0) \to \mathcal{D}$. To prove Lemma 7.3.4.6, we must show that the lifting problem

$$\begin{tikzcd}
\mathcal{K} \coprod_{\mathcal{K}_0} (\mathcal{K}_0 \star_{\mathcal{C}_0} \mathcal{C}_0) \arrow[r, "F_1"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C} \star_{\mathcal{C}} \mathcal{C} \arrow[r, "G"] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd}$$

has a solution, which is a special case of Lemma 7.3.4.5. □

02ZW **Proposition 7.3.4.7.** *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories, and suppose we are given a lifting problem*

02ZX
$$\begin{tikzcd}
\mathcal{K} \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{K} \star_{\mathcal{C}} \mathcal{C} \arrow[r] \arrow[ur, dashed, "F"] & \mathcal{E} .
\end{tikzcd} \tag{7.18}$$

The following conditions are equivalent:

(1) *The lifting problem (7.18) has a solution $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{K}$.*

(2) *For every object $C \in \mathcal{C}$, the associated lifting problem*

$$\begin{tikzcd}
\mathcal{K}_C \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{K}_C^{\triangleright} \arrow[r] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd}$$

has a solution $\mathcal{K}_C^{\triangleright} \to \mathcal{D}$ which is a U-colimit diagram.

Proof. Combine Lemma 7.3.4.6 with Proposition 7.3.4.3. □

02ZY **Corollary 7.3.4.8.** *Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let $F_0 : \mathcal{K} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:*

(1) *There exists a functor* $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ *which is left Kan extended from* $\mathcal{K}$ *and satisfies* $F|_{\mathcal{K}} = F_0$.

(2) *For every object* $C \in \mathcal{C}$, *the diagram*

$$\mathcal{K}_C = \{C\} \times_{\mathcal{C}} \mathcal{K} \hookrightarrow \mathcal{K} \xrightarrow{F_0} \mathcal{D}$$

admits a colimit in the ∞*-category* $\mathcal{D}$.

Proof of Proposition 7.3.4.4. Let $\delta : \mathcal{K} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let $F_0 : \mathcal{K} \to \mathcal{D}$ be a functor of ∞-categories. Suppose that, for every object $C \in \mathcal{C}$, the diagram

$$\mathcal{K}_C = \{C\} \times_{\mathcal{C}} \mathcal{K} \hookrightarrow \mathcal{K} \xrightarrow{F_0} \mathcal{D}$$

has a colimit in the ∞-category $\mathcal{D}$. Applying Corollary 7.3.4.8, we deduce that there exists a functor $F : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ which is left Kan extended from $\mathcal{K}$ and satisfies $F|_{\mathcal{K}} = F_0$. Applying Proposition 7.3.1.15, we see that the restriction $F|_{\mathcal{C}}$ is a left Kan extension of F_0 along δ. □

7.3.5 Existence of Kan Extensions

Our goal in this section is to establish the following existence criterion for Kan extensions: 02ZZ

Proposition 7.3.5.1. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* ∞*-categories, and suppose we are given diagrams* $\delta : K \to \mathcal{C}$ *and* $F_0 : K \to \mathcal{D}$. *Then:* 0300

- *The diagram* F_0 *admits a left Kan extension along* δ *if and only if, for every object* $C \in \mathcal{C}$, *the diagram*
$$K_{/C} = K \times_{\mathcal{C}} \mathcal{C}_{/C} \to K \xrightarrow{F_0} \mathcal{D}$$
has a colimit in the ∞*-category* $\mathcal{D}$.

- *The diagram* F_0 *admits a right Kan extension along* δ *if and only if, for every object* $C \in \mathcal{C}$, *the diagram*
$$K_{C/} = K \times_{\mathcal{C}} \mathcal{C}_{C/} \to K \xrightarrow{F_0} \mathcal{D}$$
has a limit in the ∞*-category* $\mathcal{D}$.

Corollary 7.3.5.2. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* ∞*-categories and let* $\delta : K \to \mathcal{C}$ *be a diagram. Assume that, for every object* $C \in \mathcal{C}$, *the* ∞*-category* $\mathcal{D}$ *admits* $K_{/C}$*-indexed colimits. Then every diagram* $F_0 : K \to \mathcal{D}$ *admits a left Kan extension along* δ. 0301

Corollary 7.3.5.3. *Let* $\mathcal{C}$ *be a category, let* $\mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ *be a diagram of simplicial sets, let* $\mathcal{D}$ *be an* ∞*-category, and let* $F_0 : \underrightarrow{\mathrm{holim}}(\mathscr{G}) \to \mathcal{D}$ *be a diagram. The following conditions are equivalent:* 039T

(1) *The diagram F_0 admits a left Kan extension along the projection map $U : \underrightarrow{\mathrm{holim}}(\mathscr{G}) \to \mathrm{N}_\bullet(\mathcal{C})$.*

(2) *For every object $C \in \mathcal{C}$, the diagram*

$$\mathscr{G}(C) \simeq \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{G}) \hookrightarrow \underrightarrow{\mathrm{holim}}(\mathscr{G}) \xrightarrow{F_0} \mathcal{D}$$

admits a colimit in the ∞-category $\mathcal{D}$.

Proof. For each object $C \in \mathcal{C}$, the inclusion map

$$\mathscr{G}(C) \hookrightarrow \mathrm{N}_\bullet(\mathcal{C}_{/C}) \times_{\mathrm{N}_\bullet(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{G}) \simeq \underrightarrow{\mathrm{holim}}(\mathscr{G}|_{\mathcal{C}_{/C}})$$

is right anodyne (Example 7.2.3.11), and therefore right cofinal. The desired result now follows by combining Proposition 7.3.5.1 with Corollary 7.2.2.10. □

039U **Remark 7.3.5.4.** In the situation of Corollary 7.3.5.3, suppose we are given a functor $F : \mathrm{N}_\bullet(\mathcal{C}) \to \mathcal{D}$ and a natural trnasformation $\beta : F_0 \to F \circ U$. Then β exhibits F as a left Kan extension of F along U if and only if, for every object $C \in \mathcal{C}$, the induced natural transformation $\beta_C : F_0|_{\mathscr{G}(C)} \to \underline{F(C)}$ exhibits $F(C)$ as a colimit of the diagram $F_0|_{\mathscr{G}(C)}$

In the special case where δ is a cocartesian fibration, Proposition 7.3.5.1 is essentially a reformulation of Proposition 7.3.4.4. We will proceed in general by reducing to this special case (see [52] for a similar approach). With an eye toward future applications, we first consider a variant of Proposition 7.3.5.1 in the setting of *relative* Kan extensions.

0302 **Proposition 7.3.5.5.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories, and suppose we are given a lifting problem*

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$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\ \mathcal{C} \arrow[r, "G"] \arrow[ur, dashed, "F"] & \mathcal{E} \end{tikzcd} \tag{7.19}$$

Then (7.19) admits a solution $F : \mathcal{C} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{C}^0$ if and only if, for every object $C \in \mathcal{C}$, the following condition is satisfied:

($*_C$) *The induced lifting problem*

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$$\begin{tikzcd} \mathcal{C}^0_{/C} \arrow[rr] \arrow[d] & & \mathcal{D} \arrow[d, "U"] \\ (\mathcal{C}^0_{/C})^{\triangleright} \arrow[r] \arrow[urr, dashed, "F_C"] & \mathcal{C} \arrow[r, "G"] & \mathcal{E} \end{tikzcd} \tag{7.20}$$

admits a solution $F_C : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{D}$ *which is a* U*-colimit diagram.*

Proof. Assume that condition $(*_C)$ is satisfied for every object $C \in \mathcal{C}$; we will show that the lifting problem (7.19) admits a solution $F : \mathcal{C} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{C}^0$ (the converse follows immediately from the definitions). Let $\mathcal{K}$ denote the oriented fiber product $\mathcal{C}^0 \tilde{\times}_{\mathcal{C}} \mathcal{C}$: that is, the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms $e : X \to Y$ of $\mathcal{C}$ such that X belongs to the subcategory $\mathcal{C}^0$. Let $\pi : \mathcal{K} \to \mathcal{C}^0$ and $\pi' : \mathcal{K} \to \mathcal{C}$ be the evaluation maps, given on objects by $\pi(e) = X$ and $\pi'(e) = Y$, respectively. We then have a natural transformation $\alpha : \pi \to \pi'$ (which carries each morphism $e : X \to Y$ to itself). Regarding $\mathcal{K}$ as an object of $(\mathrm{Set}_{\Delta})_{/\mathcal{C}}$ via the functor π', let $\mathcal{K} \star_{\mathcal{C}} \mathcal{C}$ denote the relative join of Construction 5.2.3.1. We will write $\iota_{\mathcal{K}} : \mathcal{K} \hookrightarrow \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$ and $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$ for the inclusion maps, and $\iota_{\mathcal{C}^0}$ for the restriction of $\iota_{\mathcal{C}}$ to the full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. The natural transformation α then determines a functor $S : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{C}$ satisfying $S \circ \iota_{\mathcal{K}} = \pi$ and $S \circ \iota_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$. Consider the lifting problem

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$$\begin{tikzcd}
\mathcal{K} \arrow[rr, "F_0 \circ \pi"] \arrow[d] & & \mathcal{D} \arrow[d, "U"] \\
\mathcal{K} \star_{\mathcal{C}} \mathcal{C} \arrow[r, "S"] \arrow[urr, "\overline{F}"] & \mathcal{C} \arrow[r, "G"] & \mathcal{E} .
\end{tikzcd} \tag{7.21}$$

For each object $C \in \mathcal{C}$, write $\mathcal{K}_C$ for the fiber $\pi'^{-1}\{C\}$, so that (7.21) restricts to a lifting problem

0306

$$\begin{tikzcd}
\mathcal{K}_C \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{K}_C \star_{\{C\}} \{C\} \arrow[r] \arrow[ur, dashed, "\overline{F}_C"] & \mathcal{E} .
\end{tikzcd} \tag{7.22}$$

Note that $\mathcal{K}_C$ can be identified with the oriented fiber product $\mathcal{C}^0 \tilde{\times}_{\mathcal{C}} \{C\}$. Moreover, after precomposing with the slice diagonal equivalence $\mathcal{C}^0_{/C} \to \mathcal{C}^0 \tilde{\times}_{\mathcal{C}} \{C\}$ of Theorem 4.6.4.17, (7.22) recovers the lifting problem (7.20). Combining assumption $(*_C)$ with Proposition 7.2.2.9, we deduce that the lifting problem (7.22) admits a solution $\overline{F}_C : \mathcal{K}_C^{\triangleright} \to \mathcal{D}$ which is a U-colimit diagram. Since $\pi' : \mathcal{K} \to \mathcal{C}$ is a cocartesian fibration (Corollary 5.3.7.3), Proposition 7.3.4.7 guarantees that the lifting problem (7.21) admits a solution $\overline{F} : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{K}$.

Note that the diagonal inclusion $\mathcal{C} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$ restricts to a map $\delta : \mathcal{C}^0 \hookrightarrow \mathcal{K}$. Let β denote the composite map

$$\Delta^1 \times \mathcal{C}^0 \simeq \mathcal{C}^0 \star_{\mathcal{C}^0} \mathcal{C}^0 \xrightarrow{\delta \star \mathrm{id}} \mathcal{K} \star_{\mathcal{C}} \mathcal{C},$$

which we regard as a natural transformation from $\iota_{\mathcal{K}} \circ \delta$ to $\iota_{\mathcal{C}^0}$. This natural transformation carries each object $X \in \mathcal{C}^0$ to a morphism $\beta_X : \iota_{\mathcal{K}}(\mathrm{id}_X) \to \iota_{\mathcal{C}}(X)$ in the ∞-category $\mathcal{K} \star_{\mathcal{C}} \mathcal{C}$. Since id_X is a final object of the ∞-category $\mathcal{K}_X \simeq \mathcal{C}^0 \tilde{\times}_{\mathcal{C}^0} \{X\}$ (Proposition 4.6.7.22) and $\overline{F}|_{\mathcal{K}_X^{\triangleright}}$ is a U-colimit diagram (Proposition 7.3.4.3), the image $\overline{F}(\beta_X)$ is a U-cocartesian morphism of $\mathcal{D}$ (Corollary 7.2.2.5). Since $U(\overline{F}(\beta_X)) = \mathrm{id}_{G(X)}$ is an isomorphism in $\mathcal{E}$, we conclude that $\overline{F}(\beta_X)$ is an isomorphism in $\mathcal{D}$. Applying Corollary 4.4.5.9, we deduce that $\overline{F}(\beta)$ can be lifted to an isomorphism $F \to \overline{F} \circ \iota_{\mathcal{C}}$ in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, where $F : \mathcal{C} \to \mathcal{D}$ is a solution to the lifting problem (7.19). We will show $\overline{F} \circ \iota_{\mathcal{C}}$ is U-left Kan extended from $\mathcal{C}^0$, so that F is also U-left Kan extended from $\mathcal{C}^0$ (Remark 7.3.3.17).

Fix an object $C \in \mathcal{C}$, let $c : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{C}$ be the slice contraction map and set $T^+ = \iota_{\mathcal{C}} \circ c$; we wish to show that $T^+ : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{D}$ is a U-colimit diagram. Let $\psi : \mathcal{C}_{/C} \hookrightarrow \mathcal{C} \tilde{\times}_{\mathcal{C}} \{C\}$ be the slice diagonal of Construction 4.6.4.13. Note that ψ is an equivalence of right fibrations over $\mathcal{C}$ (Theorem 4.6.4.17 and Proposition 5.1.7.5), and therefore restricts to an equivalence of full subcategories $\psi_0 : \mathcal{C}^0_{/C} \to \mathcal{C}^0 \tilde{\times}_{\mathcal{C}} \{C\} = \mathcal{K}_C$. Let T^- denote the composite functor

$$(\mathcal{C}^0_{/C})^{\triangleright} \xrightarrow{\psi_0^{\triangleright}} \mathcal{K}_C^{\triangleright} = \mathcal{K}_C \star_{\{C\}} \{C\} \hookrightarrow \mathcal{K} \star_{\mathcal{C}} \mathcal{C}.$$

Because $\overline{F}$ is U-left Kan extended from $\mathcal{K}$, the $\overline{F}|_{\mathcal{K}_C^{\triangleright}}$ is a U-colimit diagram in $\mathcal{D}$ (Proposition 7.3.4.3). Since the functor ψ_0 is right cofinal (Corollary 7.2.1.13), the functor $\overline{F} \circ T^-$ is also a U-colimit diagram (Corollary 7.2.2.2). Beware that the functors $T^-, T^+ : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$ are not isomorphic: if $\widetilde{X}$ is an object of the ∞-category $\mathcal{C}_{/C}$ given by a morphism $e : X \to C$ in $\mathcal{C}$, then we have $T^+(\widetilde{X}) = \iota_{\mathcal{C}}(X)$ and $T^-(\widetilde{X}) = \iota_{\mathcal{K}}(e)$. However, we will show that the functors $\overline{F} \circ T^-$ and $\overline{F} \circ T^+$ are isomorphic when regarded as objects of the ∞-category $\mathrm{Fun}((\mathcal{C}^0_{/C})^{\triangleright}, \mathcal{D})$, so that $\overline{F} \circ G^+$ a U-colimit diagram by virtue of Proposition 7.1.5.13.

Let $b : (\mathcal{C}^0_{/C})^{\triangleright} \to \Delta^1$ be the map carrying $\mathcal{C}^0_{/C}$ to the vertex $0 \in \Delta^1$ and the cone point of $(\mathcal{C}^0_{/C})^{\triangleright}$ to the vertex $1 \in \Delta^1$. Note that the map $(b, c) : (\mathcal{C}^0_{/C})^{\triangleright} \to \Delta^1 \times \mathcal{C}$ factors through the full subcategory

$$\mathcal{C}^0 \star_{\mathcal{C}} \mathcal{C} \subseteq \mathcal{C} \star_{\mathcal{C}} \mathcal{C} \subseteq \Delta^1 \times \mathcal{C}.$$

We let $T : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$ denote the composite functor

$$(\mathcal{C}^0_{/C})^{\triangleright} \xrightarrow{(b,c)} \mathcal{C}^0 \star_{\mathcal{C}} \mathcal{C} \xrightarrow{\delta \star \mathrm{id}} \mathcal{K} \star_{\mathcal{C}} \mathcal{C}.$$

Concretely, the functor T carries the cone point of $(\mathcal{C}^0_{/C})^{\triangleright}$ to the object $\iota_{\mathcal{C}}(C) \in \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$, and carries an object $(e : X \to C)$ of $\mathcal{C}^0_{/C}$ to the object $\iota_{\mathcal{K}}(\mathrm{id}_X) \in \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$. We will complete the proof by verifying the following:

(a) There exists a natural transformation of functors $\gamma^+ : T \to T^+$, which carries the cone point of $(\mathcal{C}^0_{/C})^{\triangleright}$ to the identity morphism $\iota_{\mathcal{C}}(\mathrm{id}_C)$, and carries each object $(e : X \to C) \in \mathcal{C}^0_{/C}$ to the morphism β_X.

(b) There exists a natural transformation of functors $\gamma^{-} : T \to T^{-}$, which carries the cone point of $(\mathcal{C}^0_{/C})^{\triangleright}$ to the identity morphism $\iota_{\mathcal{C}}(\mathrm{id}_C)$ and carries each object $(e : X \to C)$ to the morphism of $\mathcal{K} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ given by a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"] & X \arrow[d, "e"] \\
X \arrow[r, "e"] & C
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$.

Assuming this has been done, we observe that the natural transformations $\overline{F}(\gamma^{-})$ and $\overline{F}(\gamma^{+})$ carry each object of $(\mathcal{C}^0_{/C})^{\triangleright}$ to an isomorphism in the ∞-category $\mathcal{D}$ and therefore supply isomorphisms $\overline{F} \circ T^{-} \xleftarrow{\sim} \overline{F} \circ T \xrightarrow{|sim} \overline{F} \circ T^{+}$ in the ∞-category $\mathrm{Fun}((\mathcal{C}^0_{/C})^{\triangleright}, \mathcal{D})$.

We begin by constructing the natural transformation γ^{+}. Let $b' : (\mathcal{C}^0_{/C})^{\triangleright} \to \Delta^1$ be the constant map taking the value 1, so that there is a unique natural transformation $\xi : b \to b'$. Note that ξ induces a natural transformation from (b, c) to (b', c) in the ∞-category $\mathrm{Fun}((\mathcal{C}^0_{/C})^{\triangleright}, \mathcal{C}^0 \star_{\mathcal{C}} \mathcal{C})$. Composing with the map $(\delta \star \mathrm{id}) : \mathcal{C}^0 \star_{\mathcal{C}} \mathcal{C} \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C}$, we obtain a natural transformation $\gamma^{+} : T \to T^{+}$ satisfying the requirements of (a).

We now construct the natural transformation γ^{-}. Note that T and T_{-} both carry $\mathcal{C}^0_{/C}$ into $\mathcal{K}$ and the cone point of $(\mathcal{C}^0_{/C})^{\triangleright}$ to the object $\iota_{\mathcal{C}}(C)$ and can therefore be identified with functors $T_0, T_0^{-} : \mathcal{C}^0_{/C} \to \mathcal{K} \times_{\mathcal{C}} \mathcal{C}_{/C}$. Let σ be an n-simplex of the product $\Delta^1 \times \mathcal{C}^0_{/C}$, which we identify with a pair (ϵ, τ) where $\epsilon : [n] \to [1]$ is a nondecreasing function and $\tau : \Delta^{n+1} \to \mathcal{C}$ has the property that $\tau|_{\Delta^n}$ factors through $\mathcal{C}^0$ and $\tau(n+1) = C$. Let $\rho : \Delta^1 \times \Delta^n \to \Delta^{n+1}$ and $\rho' : \Delta^{n+1} \to \Delta^{n+1}$ denote the maps given on vertices by the formulae

$$\rho(i,j) = \begin{cases} n+1 & \text{if } i = 1 = \epsilon(j) \\ j & \text{otherwise} \end{cases} \qquad \rho'(j) = \begin{cases} j & \text{if } j \leq n \text{ and } \epsilon(j) = 0 \\ n+1 & \text{otherwise.} \end{cases}$$

Then $(\rho \circ \tau) : \Delta^1 \times \Delta^n \to \mathcal{C}$ can be identified with an n-simplex of the simplicial set $\mathcal{K} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$, so that $(\rho \circ \tau, \rho' \circ \tau)$ is an n-simplex of $\mathcal{K} \times_{\mathcal{C}} \mathcal{C}_{/C}$. The construction $\sigma \mapsto (\rho \circ \tau, \rho' \circ \tau)$ depends functorially on $[n]$, and therefore determines a morphism of simplicial sets

$$\Delta^1 \times \mathcal{C}^0_{/C} \to \mathcal{K} \times_{\mathcal{C}} \mathcal{C}_{/C}$$

We can identify this map with a natural transformation $\gamma_0^{-} : T_0 \to T_0^{-}$, which then determines a natural transformation $\gamma^{-} : T \to T^{-}$ satisfying the requirements of (b). $\square$

Example 7.3.5.6. Let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories and let $G : \mathcal{C} \to \mathcal{E}$ be 043H
a functor. Suppose that, for every object $C \in \mathcal{C}$, the image $G(C) \in \mathcal{E}$ can be lifted to a

U-initial object of $\mathcal{D}$. Applying Proposition 7.3.5.5 (in the special case $\mathcal{C}^0 = \emptyset$), we deduce that G can be lifted to a functor $F : \mathcal{C} \to \mathcal{D}$ which carries each object of $\mathcal{C}$ to a U-initial object of $\mathcal{D}$ (see Example 7.3.3.6).

043J **Corollary 7.3.5.7.** *Let $U : \mathcal{D} \to \mathcal{C}$ be a cartesian fibration of ∞-categories. Suppose that, for every object $C \in \mathcal{C}$, the fiber $\mathcal{D}_C = \{C\} \times_{\mathcal{C}} \mathcal{D}$ has an initial object. Then the ∞-category* $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ *has an initial object. Moreover, an object $F \in \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{D})$ is initial if and only if it satisfies the following condition:*

$(*)$ For each object $C \in \mathcal{C}$, the image $F(C)$ is an initial object of $\mathcal{D}_C$.

Proof. Since U is a cartesian fibration, an object $D \in \mathcal{D}$ is U-initial if and only if it is initial when viewed as an object of the ∞-category $\mathcal{D}_C$ for $C = U(D)$ (Corollary 7.1.4.21). It follows from Example 7.3.5.6 that there exists a functor $F \in \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{D})$ which satisfies condition $(*)$. Proposition 7.1.6.9 then guarantees that F is an initial object of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{D})$. Any other initial object of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{D})$ is isomorphic to F, and therefore also satisfies condition $(*)$. □

0307 **Corollary 7.3.5.8.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, let $F_0 : \mathcal{C}^0 \to \mathcal{D}$ be a functor of ∞-categories. Then:*

- *The functor F_0 admits a left Kan extension $F : \mathcal{C} \to \mathcal{D}$ if and only if, for every object $C \in \mathcal{C}$, the diagram*
$$\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C} \to \mathcal{C}^0 \xrightarrow{F_0} \mathcal{D}$$
has a colimit in the ∞-category $\mathcal{D}$.

- *The functor F_0 admits a right Kan extension $F : \mathcal{C} \to \mathcal{D}$ if and only if, for every object $C \in \mathcal{C}$, the diagram*
$$\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{C/} \to \mathcal{C}^0 \xrightarrow{F_0} \mathcal{D}$$
has a limit in the ∞-category $\mathcal{D}$.

Proof. The first assertion follows by applying the criterion of Proposition 7.3.5.5 in the special case $\mathcal{E} = \Delta^0$, and the second assertion follows by a similar argument. □

047U **Corollary 7.3.5.9.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a coreflective full subcategory and let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories. Suppose we are given a lifting problem*

039W
$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "G"] \arrow[ur, dashed, "F"] & \mathcal{E} .
\end{tikzcd} \qquad (7.23)$$

The following conditions are equivalent:

(1) *The lifting problem (7.23) admits a solution F which is U-left Kan extended from $\mathcal{C}^0$.*

(2) *For every morphism $u : C' \to C$ of $\mathcal{C}$ which exhibits C' as a $\mathcal{C}^0$-coreflection of $\mathcal{C}$, the image $G(u)$ can be lifted to a U-cocartesian morphism $F_0(C') \to D$ of $\mathcal{D}$.*

Proof. Fix an object $C \in \mathcal{C}$, and choose a morphism $u : C' \to C$ which exhibits C' as a $\mathcal{C}^0$-coreflection of $\mathcal{C}$. By virtue of Proposition 7.3.5.5, it will suffice to show that the lifting problem

$$\begin{tikzcd}
\mathcal{C}^0_{/C} \arrow[rr] \arrow[d] & & \mathcal{D} \arrow[d, "U"] \\
(\mathcal{C}^0_{/C})^{\triangleright} \arrow[r] \arrow[urr, dashed, "F_C"] & \mathcal{C} \arrow[r, "G"] & \mathcal{E}
\end{tikzcd}$$

admits a solution which is a U-colimit diagram if and only if $G(u)$ can be lifted to a U-cocartesian morphism $F_0(C') \to D$. This follows from Corollary 7.2.2.14, since u is final when viewed as an object of the ∞-category $\mathcal{C}^0_{/C}$. □

Corollary 7.3.5.10. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a coreflective full subcategory, and let $U : \mathcal{D} \to \mathcal{E}$ be a cocartesian fibration of ∞-categories. Then every lifting problem* 039V

$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "G"] \arrow[ur, dashed, "F"] & \mathcal{E}
\end{tikzcd}$$

admits a solution $F : \mathcal{C} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{C}^0$.

Corollary 7.3.5.11. *Let $\mathcal{C}$ be an ∞-category which has an initial object and let $U : \mathcal{D} \to \mathcal{E}$ be a cocartesian fibration of ∞-categories. Then every lifting problem* 043K

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{init}} \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "G"] \arrow[ur, dashed, "F"] & \mathcal{E}
\end{tikzcd}$$

admits a solution $F : \mathcal{C} \to \mathcal{D}$ which is U-left Kan extended from the full subcategory $\mathcal{C}^{\mathrm{init}} \subseteq \mathcal{C}$ spanned by the initial objects.

Proof. Combine Corollary 7.3.5.10 with Example 6.2.2.5. □

Proof of Proposition 7.3.5.1. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and suppose we are given diagrams $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$ with the property that, for every object $C \in \mathcal{C}$, the composite map

$$K_{/C} = K \times_{\mathcal{C}} \mathcal{C}_{/C} \to K \xrightarrow{F_0} \mathcal{D}$$

has a colimit in the ∞-category $\mathcal{D}$. We wish to show that F_0 has a left Kan extension along δ (the converse assertion is immediate from the definitions, and the analogous assertion for right Kan extensions will follow by a similar argument). Using Corollary 4.1.3.3, we can choose an inner anodyne morphism $\iota : K \hookrightarrow \mathcal{K}$, where $\mathcal{K}$ is an ∞-category. Since $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, we can extend δ and F_0 to functors $\overline{\delta} : \mathcal{K} \to \mathcal{C}$ and $\overline{F}_0 : \mathcal{K} \to \mathcal{D}$, respectively (Proposition 4.1.3.1). For every object $C \in \mathcal{C}$, the induced map $K \times_{\mathcal{C}} \mathcal{C}_{/C} \hookrightarrow \mathcal{K} \times_{\mathcal{C}} \mathcal{C}_{/C}$ is a categorical equivalence (Corollary 5.6.7.6), and therefore right cofinal (Corollary 7.2.1.13). Applying Proposition 7.2.2.9, we deduce that the composite map

$$\mathcal{K} \times_{\mathcal{C}} \mathcal{C}_{/C} \to \mathcal{K} \xrightarrow{\overline{F}_0} \mathcal{D}$$

has a colimit in $\mathcal{D}$. Corollary 7.3.5.8 now guarantees that the functor $\overline{F}_0$ admits a left Kan extension $\overline{F} : \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{D}$. Set $F = \overline{F}|_{\mathcal{C}}$. Applying Proposition 7.3.2.11, we obtain a natural transformation $\overline{\beta} : \overline{F}_0 \to F \circ \overline{\delta}$ which exhibits F as a left Kan extension of $\overline{F}_0$ along $\overline{\delta}$. Since ι is a categorical equivalence, it follows that $\overline{\beta}$ restricts to a natural transformation $F_0 \to F \circ \delta$ which exhibits F as a left Kan extension of F_0 along δ (Proposition 7.3.1.14). □

7.3.6 The Universal Property of Kan Extensions

0308 The goal of this section is to show that Kan extensions (when they exist) can be characterized by a universal mapping property.

0309 **Proposition 7.3.6.1.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$ be diagrams, and let $\beta : F_0 \to F \circ \delta$ be a natural transformation which exhibits F as a left Kan extension of F_0 along δ. Then, for every functor $G : \mathcal{C} \to \mathcal{D}$, the composite map*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(F \circ \delta, G \circ \delta) \xrightarrow{\circ[\beta]} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(F_0, G \circ \delta)$$

is a homotopy equivalence of Kan complexes.

We will give the proof of Proposition 7.3.6.1 at the end of this section.

030A **Warning 7.3.6.2.** In classical category theory, some authors take the universal property of Proposition 7.3.6.1 as the definition of a Kan extension. Beware that this is a slightly different notion in general: it is possible for a natural transformation $\beta : F_0 \to F \circ \delta$ to satisfy the universal property of Proposition 7.3.6.1 without exhibiting F as a left Kan extension of F_0 along δ (in which case F_0 cannot admit any other left Kan extension along δ; see Corollary 7.3.6.5).

Corollary 7.3.6.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and let $\delta : K \to \mathcal{C}$ be a diagram. Suppose that every diagram $F_0 : K \to \mathcal{D}$ has a left Kan extension along δ. Then the restriction functor* 030B

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\circ\delta} \mathrm{Fun}(K, \mathcal{D})$$

has a left adjoint, which carries each diagram $F_0 : K \to \mathcal{D}$ to a left Kan extension of F_0 along δ.

Proof. Combine Propositions 7.3.6.1 and 6.2.4.1. □

Corollary 7.3.6.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and let $\delta : K \to \mathcal{C}$ be a diagram. Suppose that, for every object $C \in \mathcal{C}$, the ∞-category $\mathcal{D}$ admits colimits indexed by the simplicial set $K_{/C} = K \times_{\mathcal{C}} \mathcal{C}_{/C}$. Then the restriction functor* 030C

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\circ\delta} \mathrm{Fun}(K, \mathcal{D})$$

has a left adjoint, which carries each diagram $F_0 : K \to \mathcal{D}$ to a left Kan extension of F_0 along δ.

Proof. Combine Corollaries 7.3.6.3 and 7.3.5.2. □

Corollary 7.3.6.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories equipped with diagrams $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$, and suppose that F_0 admits a left Kan extension along δ. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let $\beta : F_0 \to F \circ \delta$ be a natural transformation. The following conditions are equivalent:* 030D

(1) *The natural transformation β exhibits F as a left Kan extension of F_0 along δ.*

(2) *For every functor $G : \mathcal{C} \to \mathcal{D}$, the composite map*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(F \circ \delta, G \circ \delta) \xrightarrow{\circ[\beta]} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(F_0, G \circ \delta)$$

is a homotopy equivalence of Kan complexes.

(3) *For every functor $G : \mathcal{C} \to \mathcal{D}$, the composite map*

$$\mathrm{Hom}_{\mathrm{hFun}(\mathcal{C},\mathcal{D})}(F, G) \to \mathrm{Hom}_{\mathrm{hFun}(K,\mathcal{D})}(F \circ \delta, G \circ \delta)) \xrightarrow{\circ[\beta]} \mathrm{Hom}_{\mathrm{hFun}(K,\mathcal{D})}(F_0, G \circ \delta)$$

is a bijection of sets.

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 7.3.6.1 and the implication (2) $\Rightarrow$ (3) is immediate. We will complete the proof by showing that (3) $\Rightarrow$ (1). By assumption, there exists a functor $F' : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\beta' : F_0 \to F' \circ \delta$ which exhibits F' as a left Kan extension of F along δ. Applying Proposition 7.3.6.1, we deduce that there exists a natural transformation $\gamma : F' \to F$ for which β is a composition of β' with the induced transformation $\gamma|_K : (F' \circ \delta) \to (F \circ \delta)$. For each object $G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{hFun}(\mathcal{C},\mathcal{D})}(F, G) \arrow[rr, "\circ[\gamma]"] \arrow[dr] & & \mathrm{Hom}_{\mathrm{hFun}(\mathcal{C},\mathcal{D})}(F', G) \arrow[dl] \\
& \mathrm{Hom}_{\mathrm{hFun}(K,\mathcal{D})}(F_0, G \circ \delta), &
\end{tikzcd}$$

where the right vertical map is bijective. If condition (3) is satisfied, then the left vertical map is also bijective. Allowing the functor G to vary, it follows that the homotopy class $[\gamma]$ is an isomorphism in the homotopy category $\mathrm{hFun}(\mathcal{C}, \mathcal{D})$, so that γ is an isomorphism in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Invoking Remark 7.3.1.12, we conclude that β exhibits F as a left Kan extension of F_0 along δ. □

030E **Remark 7.3.6.6.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories equipped with diagrams $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$. It follows from Corollary 7.3.6.5 that if F_0 admits a left Kan extension $F : \mathcal{C} \to \mathcal{D}$ along δ, then the isomorphism class of the functor F is uniquely determined: it is characterized by the requirement that it corepresents the functor

$$\mathrm{hFun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Set} \qquad G \mapsto \mathrm{Hom}_{\mathrm{hFun}(K,\mathcal{D})}(F_0, G \circ \delta).$$

We will deduce Proposition 7.3.6.1 from a more general statement about *relative* Kan extensions.

030F **Proposition 7.3.6.7.** *Let $\mathcal{C}$ be an ∞-category, let $U : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors having restrictions $F_0 = F|_{\mathcal{C}^0}$ and $G_0 = G|_{\mathcal{C}^0}$, so that we have a commutative diagram of Kan complexes*

030G
$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{D})}(F_0, G_0) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(U \circ F, U \circ G) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{E})}(U \circ F_0, U \circ G_0).
\end{tikzcd} \tag{7.24}$$

If F is U-left Kan extended from $\mathcal{C}^0$ or G is U-right Kan extended from $\mathcal{C}^0$, then (7.24) is a homotopy pullback square.

Remark 7.3.6.8. In the situation of Proposition 7.3.6.7, the horizontal maps in the diagram (7.24) are Kan fibrations (Corollary 4.1.4.2 and Proposition 4.6.1.21). Consequently, the diagram (7.24) is a homotopy pullback square if and only if the induced map 030H

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \arrow[d, "\theta"] \\
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{D})}(F_0, G_0) \times_{\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{E})}(UF_0, UG_0)} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(UF, UG)
\end{tikzcd}$$

is a homotopy equivalence (Example 3.4.1.3). Writing $\mathcal{M}$ for the fiber product

$$\mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{E})} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

and $V : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{M}$ for the functor given by $V(H) = (H|_{\mathcal{C}^0}, U \circ H)$, we can identify θ with the map $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \mathrm{Hom}_{\mathcal{M}}(V(F), V(G))$ determined by V. We can therefore restate Proposition 7.3.6.7 as follows:

- If the functor $F : \mathcal{C} \to \mathcal{D}$ is U-left Kan extended from $\mathcal{C}^0 \subseteq \mathcal{C}$, then it is V-initial when viewed as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.
- If the functor $G : \mathcal{C} \to \mathcal{D}$ is U-right Kan extended from $\mathcal{C}^0 \subseteq \mathcal{C}$, then it is V-final when viewed as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proof of Proposition 7.3.6.7. We will assume that the functor F is U-left Kan extended from $\mathcal{C}^0$ (the proof in the case where G is U-right Kan extended from $\mathcal{C}^0$ is similar). Using Corollary 4.5.2.23, we can factor the functor U as a composition $\mathcal{D} \xrightarrow{T} \mathcal{D}' \xrightarrow{U'} \mathcal{E}$, where U' is an isofibration and T is an equivalence of ∞-categories. Note that the functor $T \circ F$ is U'-left Kan extended from $\mathcal{C}^0$ (Remark 7.3.3.15), and that the natural maps

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D}')}(T \circ F, T \circ G)$$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{D})}(F_0, G_0) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{D}')}(T \circ F_0, T \circ G_0)$$

are homotopy equivalences. Consequently, we can replace $\mathcal{D}$ by $\mathcal{D}'$ and thereby reduce to proving Proposition 7.3.6.7 in the special case where the functor $U : \mathcal{D} \to \mathcal{E}$ is an isofibration of ∞-categories.

Let $V : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{E})} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ be as in Remark 7.3.6.8; we wish to show that F is a V-initial object of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Note that V is also an isofibration (Proposition 4.4.5.1). By virtue of Corollary 7.1.4.17, it will suffice to show that every lifting problem

030J
$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d, "V"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0, \mathcal{E})} \mathrm{Fun}(\mathcal{C}, \mathcal{E})
\end{tikzcd} \tag{7.25}$$

has a solution, provided that $n \geq 0$ and $\sigma_0(0) = F$. Unwinding the definitions, we can rewrite (7.25) as a lifting problem

030K
$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r, "G_0"] \arrow[d] & \mathrm{Fun}(\Delta^n, \mathcal{D}) \arrow[d, "V'"] \\
\mathcal{C} \arrow[r] \arrow[ur, dashed, "G"] & \mathrm{Fun}(\partial\Delta^n, \mathcal{D}) \times_{\mathrm{Fun}(\partial\Delta^n, \mathcal{E})} \mathrm{Fun}(\Delta^n, \mathcal{E})
\end{tikzcd} \tag{7.26}$$

Note that V' is also an isofibration of ∞-categories (Proposition 4.4.5.1).

We will complete the proof by showing that the lifting problem (7.26) admits a solution $G : \mathcal{C} \to \mathrm{Fun}(\Delta^n, \mathcal{D})$ which is V'-left Kan extended from $\mathcal{C}^0$. By virtue of Proposition 7.3.5.5, it will suffice to show that for each object $C \in \mathcal{C}$, the induced lifting problem

030L
$$\begin{tikzcd}
\mathcal{C}^0_{/C} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n, \mathcal{D}) \arrow[d, "V'"] \\
(\mathcal{C}^0_{/C})^{\triangleright} \arrow[r] \arrow[ur, dashed, "Q"] & \mathrm{Fun}(\partial\Delta^n, \mathcal{D}) \times_{\mathrm{Fun}(\partial\Delta^n, \mathcal{E})} \mathrm{Fun}(\Delta^n, \mathcal{E})
\end{tikzcd} \tag{7.27}$$

admits a solution $Q : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathrm{Fun}(\Delta^n, \mathcal{D})$ which is a V'-colimit diagram. Our assumption that $\sigma_0(0) = F$ is U-left Kan extended from $\mathcal{C}^0$ guarantees that the composite map

$$(\mathcal{C}^0_{/C})^{\triangleright} \to \mathrm{Fun}(\partial\Delta^n, \mathcal{D}) \to \mathrm{Fun}(\{0\}, \mathcal{D}) = \mathcal{D}$$

is a U-colimit diagram. Applying Corollary 7.1.6.6, we conclude that the lifting problem (7.27) admits a solution Q, and Proposition 7.1.6.9 guarantees that Q is automatically a V'-colimit diagram. □

Corollary 7.3.6.9. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. If F is left Kan extended from $\mathcal{C}^0$ or G is right Kan extended from $\mathcal{C}^0$, then the restriction map* 030M

$$\theta : \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^0,\mathcal{D})}(F|_{\mathcal{C}^0}, G|_{\mathcal{C}^0})$$

is a trivial Kan fibration.

Proof. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories and let $\mathcal{C}^0$ be a full subcategory of $\mathcal{C}$. Assume either that F is left Kan extended from $\mathcal{C}^0$ or that G is right Kan extended from $\mathcal{C}^0$. Applying Proposition 7.3.6.7 in the special case $\mathcal{E} = \Delta^0$, we deduce that the restriction map

$$\theta : \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{D})}(F, G) \to \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^0,\mathcal{D})}(F|_{\mathcal{C}^0}, G|_{\mathcal{C}^0})$$

is a homotopy equivalence of Kan complexes. Since the restriction map $\operatorname{Fun}(\mathcal{C}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}^0, \mathcal{D})$ is an inner fibration of ∞-categories (Corollary 4.1.4.2), the map θ is also a Kan fibration (Proposition 4.6.1.21), and therefore a trivial Kan fibration (Proposition 3.3.7.6). □

Corollary 7.3.6.10. *Let $\mathcal{C}$ be an ∞-category containing an initial object C, let $U : \mathcal{D} \to \mathcal{E}$ be a cocartesian fibration of ∞-categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be a pair of functors satisfying $U \circ F = U \circ G$. Suppose that F carries each morphism of $\mathcal{C}$ to a U-cocartesian morphism of $\mathcal{D}$. Then evaluation at C induces a homotopy equivalence* 043M

$$\theta : \operatorname{Hom}_{\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C},\mathcal{D})}(F, G) \to \operatorname{Hom}_{\operatorname{Fun}_{/\mathcal{E}}(\{C\},\mathcal{D})}(F(C), G(C)).$$

Proof. Let $\mathcal{C}^{\text{init}}$ denote the full subcategory of $\mathcal{C}$ spanned by its initial objects. The morphism θ then factors as a composition

$$\begin{aligned} \operatorname{Hom}_{\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C},\mathcal{D})}(F, G) &\xrightarrow{\theta'} \operatorname{Hom}_{\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}^{\text{init}},\mathcal{D})}(F|_{\mathcal{C}^{\text{init}}}, G|_{\mathcal{C}^{\text{init}}}) \\ &\xrightarrow{\theta''} \operatorname{Hom}_{\operatorname{Fun}_{/\mathcal{E}}(\{C\},\mathcal{D})}(F(C), G(C)). \end{aligned}$$

Our assumption guarantees that F is U-left Kan extended from $\mathcal{C}^{\text{init}}$ (Corollary 7.3.3.13), so that the morphism θ' is a homotopy equivalence (Proposition 7.3.6.7). Since $\mathcal{C}^{\text{init}}$ is a contractible Kan complex (Corollary 4.6.7.14), the inclusion map $\{C\} \hookrightarrow \mathcal{C}^{\text{init}}$ is a categorical equivalence of simplicial sets, which implies that θ'' is also a homotopy equivalence. □

Corollary 7.3.6.11. *Let $\overline{F} : \mathcal{C} \to \mathcal{E}$ be a functor of ∞-categories and let $U : \mathcal{D} \to \mathcal{E}$ be a cocartesian fibration. Suppose that $\mathcal{C}$ contains an initial object C having image $E = \overline{F}(C)$ and that the ∞-category $\mathcal{D}_E = \{E\} \times_{\mathcal{E}} \mathcal{D}$ has an initial object. Then the ∞-category $\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$ has an initial object. Moreover, an object $F \in \operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$ is initial if and only if it satisfies the following conditions:* 043N

(1) *The image $F(C)$ is an initial object of the ∞-category $\mathcal{D}_E$.*

(2) *The functor F carries each morphism of $\mathcal{C}$ to a U-cocartesian morphism of $\mathcal{D}$.*

Proof. It follows from Corollary 7.3.6.10 that any object of $\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$ which satisfies conditions (1) and (2) is initial. It will therefore suffice to show that there exists an object $F \in \operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$ satisfying (1) and (2) (any other initial object of $\operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$ will be isomorphic to F, and will therefore also satisfy (1) and (2)).

Let $\mathcal{C}^{\mathrm{init}}$ denote the full subcategory of $\mathcal{C}$ spanned by its initial objects, and let D be an initial object of the ∞-category $\mathcal{D}_E$. Since $\mathcal{C}^{\mathrm{init}}$ is a contractible Kan complex, we can lift $\overline{F}|_{\mathcal{C}^{\mathrm{init}}}$ to a functor $F_0 : \mathcal{C}^{\mathrm{init}} \to \mathcal{D}$ satisfying $F_0(C) = D$. Corollary 7.3.5.11 then guarantees that F_0 admits a U-left Kan extension $F \in \operatorname{Fun}_{/\mathcal{E}}(\mathcal{C}, \mathcal{D})$, which satisfies condition (2) by virtue of Corollary 7.3.3.13. ☐

Note that relative Kan extensions are characterized by the mapping property described in Proposition 7.3.6.7:

030N **Corollary 7.3.6.12.** *Suppose we are given a commutative diagram of ∞-categories*

030P
$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "\overline{F}"] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd} \tag{7.28}$$

where $\mathcal{C}^0$ is a full subcategory of $\mathcal{C}$. Assume that the lifting problem (7.28) admits a solution given by a functor $\mathcal{C} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{C}^0$. Let $F : \mathcal{C} \to \mathcal{D}$ be an arbitrary solution to the lifting problem (7.28). Then the following conditions are equivalent:

(1) *The functor F is U-left Kan extended from $\mathcal{C}^0$.*

(2) *For every functor $G : \mathcal{C} \to \mathcal{D}$, the diagram of Kan complexes*

$$\begin{tikzcd}
\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{D})}(F, G) \arrow[r] \arrow[d] & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^0,\mathcal{D})}(F|_{\mathcal{C}^0}, G|_{\mathcal{C}^0}) \arrow[d] \\
\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(U \circ F, U \circ G) \arrow[r] & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^0,\mathcal{E})}(U \circ F|_{\mathcal{C}^0}, U \circ G|_{\mathcal{C}^0}).
\end{tikzcd}$$

is a homotopy pullback square.

Proof. The implication (1) ⇒ (2) follows from Proposition 7.3.6.7. To prove the converse, let $F' : \mathcal{C} \to \mathcal{D}$ be a solution to the lifting problem (7.28) which is U-left Kan extended from $\mathcal{C}^0$, and let $V : \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\operatorname{Fun}(\mathcal{C}^0, \mathcal{E})} \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ be as in Remark 7.3.6.8. If condition (2) is satisfied, then F and F' are both V-initial objects of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ satisfying $V(F) = V(F')$. Applying Corollary 7.1.4.12, we see that F and F' are isomorphic as objects of the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$, so that F is also U-left Kan extended from $\mathcal{C}^0$ (Remark 7.3.3.17). □

Corollary 7.3.6.13. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $F_0 = F|_{\mathcal{C}^0}$ be the restriction of F to a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Suppose that the functor F_0 admits a left Kan extension to $\mathcal{C}$. The following conditions are equivalent:* 030Q

(1) *The functor F is left Kan extended from $\mathcal{C}^0$.*

(2) *For every functor $G : \mathcal{C} \to \mathcal{D}$, the restriction map*

$$\theta : \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathcal{D})}(F, G) \to \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^0, \mathcal{D})}(F|_{\mathcal{C}^0}, G|_{\mathcal{C}^0})$$

is a homotopy equivalence of Kan complexes.

(3) *For every functor $G : \mathcal{C} \to \mathcal{D}$, the restriction map*

$$\theta : \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathcal{D})}(F, G) \to \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^0, \mathcal{D})}(F|_{\mathcal{C}^0}, G|_{\mathcal{C}^0})$$

is a trivial Kan fibration of simplicial sets.

Proof. The equivalence (1) ⇔ (2) follows by applying Corollary 7.3.6.12 in the special case $\mathcal{E} = \Delta^0$. The equivalence (2) ⇔ (3) is a special case of Proposition 3.3.7.6, since the morphism θ is automatically a Kan fibration (see Corollary 4.1.4.2 and Proposition 4.6.1.21). □

Combining Proposition 7.3.6.7 with the existence criterion of Proposition 7.3.5.5, we obtain the following:

Theorem 7.3.6.14. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories. Let $\operatorname{Fun}'(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which are U-left Kan extended from $\mathcal{C}^0$, and let $\mathcal{B}$ denote the full subcategory of $\operatorname{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\operatorname{Fun}(\mathcal{C}^0, \mathcal{E})} \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ whose objects correspond to lifting problems* 030R

$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C} \arrow[r] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd}$$

with the following property:

$(*)$ *For every object $C \in \mathcal{C}$, the induced lifting problem*

$$\begin{tikzcd}
\mathcal{C}^0_{/C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
(\mathcal{C}^0_{/C})^{\triangleright} \arrow[r] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd}$$

admits a solution which is a U-colimit diagram $(\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{D}$.

Then the restriction map

$$V : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^0, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}^0, \mathcal{E})} \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

restricts to a trivial Kan fibration $\mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \to \mathcal{B}$.

Stated more informally, Theorem 7.3.6.14 asserts that if we are given a lifting problem

$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\mathcal{C} \arrow[r] \arrow[ur, dashed, "F"] & \mathcal{E}
\end{tikzcd}$$

which has a *possibility* to be solved by a functor $F : \mathcal{C} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{C}^0$, then the functor F exists and is unique up to a contractible space of choices.

Proof of Theorem 7.3.6.14. Note that the functor V is an isofibration of ∞-categories (Proposition 4.4.5.1). It follows from Proposition 7.3.5.5 that $\mathcal{B}$ is the essential image of the functor $V|_{\mathrm{Fun}'(\mathcal{C}, \mathcal{D})}$, and from Proposition 7.3.6.7 (together with Remark 7.3.6.8) that every object of $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ is V-initial when regarded as an object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Applying Corollary 7.1.4.18, we see that the functor $V|_{\mathrm{Fun}(\mathcal{C}, \mathcal{D})} : \mathrm{Fun}'(\mathcal{C}, \mathcal{D}) \to \mathcal{B}$ is a trivial Kan fibration. □

030S **Corollary 7.3.6.15.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Let* $\mathrm{Fun}'(\mathcal{C}, \mathcal{D})$ *denote the full subcategory of* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ *spanned by those functors which are left Kan extended from $\mathcal{C}^0$, and let* $\mathrm{Fun}'(\mathcal{C}^0, \mathcal{D})$ *denote the full subcategory of* $\mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$ *spanned by those functors F_0 which satisfy the following condition:*

$(*)$ *For every object $C \in \mathcal{C}$, the diagram*

$$\mathcal{C}^0_{/C} = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C} \to \mathcal{C}^0 \xrightarrow{F_0} \mathcal{D}$$

has a colimit in the ∞-category $\mathcal{D}$.

Then the restriction map $\mathrm{Fun}'(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}'(\mathcal{C}^0,\mathcal{D})$ *is a trivial Kan fibration of simplicial sets.*

Proof. Apply Theorem 7.3.6.14 in the special case $\mathcal{E} = \Delta^0$. □

We now return to the proof of Proposition 7.3.6.1.

Proof of Proposition 7.3.6.1. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors of ∞-categories. Suppose we are given a simplicial set K equipped with diagrams $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$, together with a natural transformation $\beta : F_0 \to F \circ \delta$ which exhibits F as a left Kan extension of F_0 along δ. Let θ denote the composite map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F,G) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(F\circ\delta, G\circ\delta) \xrightarrow{\circ[\beta]} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(F_0, G\circ\delta).$$

We wish to show that θ is a homotopy equivalence.

It follows from Corollary 4.1.3.3 that there exists an inner anodyne morphism $K \hookrightarrow \mathcal{K}$, where $\mathcal{K}$ is an ∞-category. Since $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, we can extend δ and F_0 to functors $\delta' : \mathcal{K} \to \mathcal{C}$ and $F_0' : \mathcal{K} \to \mathcal{D}$, respectively (Proposition 1.5.6.7). Moreover, the restriction functor $\mathrm{Fun}(\mathcal{K},\mathcal{D}) \to \mathrm{Fun}(K,\mathcal{D})$ is a trivial Kan fibration (Proposition 1.5.7.6). We can therefore extend β to a natural transformation $\beta' : F_0' \to F \circ \delta'$, which induces a map of Kan complexes $\theta' : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F,G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0', G\circ\delta')$. By construction, the map θ is obtained (up to homotopy) by composing θ' with the restriction map $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0', G\circ\delta') \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{D})}(F_0, G\circ\delta)$, which is a trivial Kan fibration. Consequently, to show that θ is a homotopy equivalence, it will suffice to show that θ' is a homotopy equivalence. We may therefore replace K by $\mathcal{K}$ and thereby reduce to proving Proposition 7.3.6.1 in the special case where $K = \mathcal{K}$ is an ∞-category.

Let $\overline{\mathcal{C}}$ denote the relative join $\mathcal{K} \star_{\mathcal{C}} \mathcal{C}$. Note that the definition of θ (as a morphism in the homotopy category hKan) depends only on the homotopy class of β. We may therefore assume without loss of generality that there exists a functor $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ for which $\overline{F}|_{\mathcal{K}} = F_0$, $\overline{F}|_{\mathcal{C}} = F$, and the natural transformation β is given by the composition

$$\Delta^1 \times \mathcal{K} \simeq \mathcal{K} \star_{\mathcal{K}} \mathcal{K} \to \mathcal{K} \star_{\mathcal{C}} \mathcal{C} \xrightarrow{\overline{F}} \mathcal{D}.$$

Let $\overline{G} : \overline{\mathcal{C}} \to \mathcal{D}$ denote the functor given by the composition

$$\mathcal{K} \star_{\mathcal{C}} \mathcal{C} \to \mathcal{C} \star_{\mathcal{C}} \mathcal{C} \simeq \Delta^1 \times \mathcal{C} \to \mathcal{C} \xrightarrow{G} \mathcal{D}.$$

Our assumption on β guarantees that $\overline{F}$ is left Kan extended from the full subcategory $\mathcal{K} \subseteq \overline{\mathcal{C}}$ (Proposition 7.3.2.11). Applying Corollary 7.3.6.9, we deduce that precomposition with the inclusion $\mathcal{K} \hookrightarrow \overline{\mathcal{C}}$ determines a trivial Kan fibration

$$\varphi_{-} : \mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F}, \overline{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, G\circ\delta).$$

We claim that $\overline{G}$ is right Kan extended from the full subcategory $\mathcal{C} \subseteq \overline{\mathcal{C}}$. To prove this, it will suffice to show that for every object $X \in \mathcal{K}$, the functor $\overline{G}$ is right Kan extended from $\mathcal{C}$ at X (see Proposition 7.3.3.7). Let $e_X : X \to \delta(X)$ denote the morphism in $\overline{\mathcal{C}}$ given by the edge

$$\Delta^1 \simeq \{X\} \star_{\{\delta(X)\}} \{\delta(X)\} \hookrightarrow \mathcal{K} \star_{\mathcal{C}} \mathcal{C} = \overline{\mathcal{C}}.$$

Note that e_X is cocartesian with respect to the projection map $\overline{\mathcal{C}} \to \Delta^1$ (Proposition 5.2.3.15), and therefore exhibits $\delta(X)$ as a $\mathcal{C}$-reflection of X in the ∞-category $\overline{\mathcal{C}}$ (Lemma 6.2.3.1). It will therefore suffice to show that $\overline{G}$ carries e_X to an isomorphism in the ∞-category $\mathcal{D}$, which is clear (by construction, $\overline{G}(e_X)$ is the identity morphism id_D for $D = G(\delta(X))$). Applying Corollary 7.3.6.9 again, we deduce that precomposition with the inclusion map $\mathcal{C} \hookrightarrow \overline{\mathcal{C}}$ determines a trivial Kan fibration

$$\varphi_+ : \mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F}, \overline{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G).$$

Let $\varphi_\pm : \mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F}, \overline{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F \circ \delta, G \circ \delta)$ be given by precomposition with the functor $\mathcal{K} \xrightarrow{\delta} \mathcal{C} \hookrightarrow \overline{\mathcal{C}}$. Consider the diagram of Kan complexes

030T
$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F}, \overline{G}) \arrow[dl, "\varphi_+"'] \arrow[d, "\varphi_\pm"] \arrow[dr, "\varphi_-"] & \\
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{C})}(F \circ \delta, G \circ \delta) \arrow[r, "\circ[\beta]"] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, G \circ \delta).
\end{tikzcd} \tag{7.29}$$

Note that the diagonal maps are homotopy equivalences, and the triangle on the left is commutative. Consequently, to show that θ is a homotopy equivalence, it will suffice to show that the triangle on the right commutes up to homotopy.

Let $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, F \circ \delta, G \circ \delta)$ be the Kan complex introduced in Notation 4.6.9.1. To verify the homotopy commutativity of the right triangle in the diagram (7.29), it will suffice to show that there is exists map of Kan complexes $\rho : \mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F}, \overline{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, F \circ \delta, G \circ \delta)$ satisfying the following conditions:

- The composition

  $$\mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F}, \overline{G}) \xrightarrow{\rho} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, F \circ \delta, G \circ \delta) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, F \circ \delta)$$

  is the constant map taking the value β.

- The composition

  $$\mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F}, \overline{G}) \xrightarrow{\rho} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, F \circ \delta, G \circ \delta) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, G \circ \delta)$$

  is equal to φ_-.

- The composition

  $$\mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F},\overline{G}) \xrightarrow{\rho} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, F\circ\delta, G\circ\delta) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F\circ\delta, G\circ\delta)$$

  is equal to $\varphi_{\pm}$.

Let σ denote the 2-simplex of $\Delta^1 \times \Delta^1$ given on vertices by the formulae

$$\sigma(0) = (0,0) \qquad \sigma(1) = (0,1) \qquad \sigma(2) = (1,1),$$

and let $T : \Delta^2 \times \mathcal{K} \to \Delta^1 \times \overline{\mathcal{C}}$ be the functor given by the composition

$$\begin{aligned} \Delta^2 \times \mathcal{K} &\xrightarrow{\sigma\times\mathrm{id}_{\mathcal{K}}} \Delta^1 \times \Delta^1 \times \mathcal{K} \\ &\simeq \Delta^1 \times (\mathcal{K} \star_{\mathcal{K}} \mathcal{K}) \\ &\to \Delta^1 \times (\mathcal{K} \star_{\mathcal{C}} \mathcal{C}) \\ &= \Delta^1 \times \overline{\mathcal{C}}. \end{aligned}$$

More concretely, the functor T is given on objects by the formulae

$$T(0,X) = (0,X) \qquad T(1,X) = (0,\delta(X)) \qquad T(2,X) = (1,\delta(X)).$$

We conclude by observing that precomposition with T induces a map of Kan complexes

$$\rho : \mathrm{Hom}_{\mathrm{Fun}(\overline{\mathcal{C}},\mathcal{D})}(\overline{F},\overline{G}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{K},\mathcal{D})}(F_0, F\circ\delta, G\circ\delta)$$

having the desired properties. □

7.3.7 Kan Extensions in Functor ∞-Categories

Let $U : \mathcal{C} \to \mathcal{B}$ be an exponentiable inner fibration of ∞-categories (Definition 4.5.9.10). For every ∞-category $\mathcal{D}$, Corollary 4.5.9.19 guarantees that the simplicial set $\mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B},\mathcal{D})$ is an ∞-category (see Construction 4.5.9.1). The goal of this section is to describe (relative) Kan extensions of functors which take values in the ∞-category $\mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B},\mathcal{D})$. We can state our main result as follows: 047V

Theorem 7.3.7.1. *Let $U : \mathcal{C} \to \mathcal{B}$ be an exponentiable inner fibration of ∞-categories, let $V : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories, and let $V' : \mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B},\mathcal{E})$ denote the functor given by postcomposition with V. Let $f : \mathcal{A} \to \mathrm{Fun}(\mathcal{C}\,/\,\mathcal{B},\mathcal{D})$ be a functor of ∞-categories, corresponding to a morphism $\mathcal{A} \to \mathcal{B}$ and a functor $F : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$. Let $\mathcal{A}^0 \subseteq \mathcal{A}$ be a full subcategory. If F is V-left Kan extended from $\mathcal{A}^0 \times_{\mathcal{B}} \mathcal{C}$, then f is V'-left Kan extended from $\mathcal{A}^0$.* 047W

We will give the proof of Theorem 7.3.7.1 at the end of this section.

047X **Remark 7.3.7.2.** In the situation of Theorem 7.3.7.1, suppose that the functor $F^0 = F|_{\mathcal{A}^0 \times_{\mathcal{B}} \mathcal{C}}$ admits a V-left Kan extension $F' : \mathcal{A} \times_{\mathcal{B}} \mathcal{C}$ satisfying $V \circ F' = V \circ F$. Then the converse of Theorem 7.3.7.1 is also true: if f is V'-left Kan extended from $\mathcal{A}^0$, then F is V-left Kan extended from $\mathcal{A}^0 \times_{\mathcal{B}} \mathcal{C}$. To prove this, it will suffice to show that the functors F and F' are isomorphic (Remark 7.3.3.17). This is clear: we can identify F' with a functor $f' : \mathcal{A} \to \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$ satisfying $V' \circ f' = V' \circ f$, and Theorem 7.3.7.1 guarantees that f' is V'-left Kan extended from $\mathcal{A}^0$. Since the functors f and f' coincide on $\mathcal{A}^0$, they are isomorphic by virtue of Theorem 7.3.6.14.

047Y **Corollary 7.3.7.3.** *Let $U : \mathcal{C} \to \mathcal{B}$ be an exponentiable inner fibration of ∞-categories, let $\mathcal{D}$ be an ∞-category, and let $\pi : \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathcal{B}$ be the projection map. Let $f : \mathcal{A} \to \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$ be a functor of ∞-categories and let $\mathcal{A}^0 \subseteq \mathcal{A}$ be a full subcategory. If the induced map $\mathcal{A} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ is left Kan extended from $\mathcal{A}^0 \times_{\mathcal{B}} \mathcal{C}$, then f is π-left Kan extended from $\mathcal{A}^0$.*

Proof. Apply Theorem 7.3.7.1 in the special case $\mathcal{E} = \Delta^0$. □

047Z **Corollary 7.3.7.4.** *Let $\mathcal{C}$ be an ∞-category, let $V : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories, and let $V' : \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ be the functor given by postcomposition with V'. Suppose we are given another functor $f : \mathcal{A} \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ and a full subcategory $\mathcal{A}^0 \subseteq \mathcal{A}$. If the induced map $\mathcal{A} \times \mathcal{C} \to \mathcal{D}$ is V-left Kan extended from $\mathcal{A}^0 \times \mathcal{C}$, then f is V'-left Kan extended from $\mathcal{A}^0$.*

Proof. Apply Theorem 7.3.7.1 in the special case $\mathcal{B} = \Delta^0$. □

0480 **Corollary 7.3.7.5.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and let $f : \mathcal{A} \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ be a functor of ∞-categories, corresponding to a functor $F : \mathcal{A} \times \mathcal{C} \to \mathcal{D}$. Let $\mathcal{A}^0 \subseteq \mathcal{A}$ be a full subcategory. If F is left Kan extended from $\mathcal{A}^0 \times \mathcal{C}$, then f is left Kan extended from $\mathcal{A}^0$.*

Proof. Apply Corollary 7.3.7.4 in the special case $\mathcal{E} = \Delta^0$. □

0481 **Corollary 7.3.7.6.** *Let $U : \mathcal{C} \to \mathcal{B}$ be an exponentiable inner fibration of ∞-categories, let $V : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories, and let $V' : \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})$ be the isofibration given by postcomposition with V (see Proposition 4.5.9.18). Let e be a morphism of the ∞-category $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$, corresponding to a pair $(\overline{e}, f)$ where $\overline{e}$ is a morphism of $\mathcal{B}$ and $f : \Delta^1 \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories. If f is V-left Kan extended from $\{0\} \times_{\mathcal{B}} \mathcal{C}$, then the morphism e is V'-cocartesian.*

Proof. Apply Theorem 7.3.7.1 in the special case $\mathcal{A} = \Delta^1$ and $\mathcal{A}^0 = \{0\}$ (see Example 7.1.5.9). □

Example 7.3.7.7. In the situation of Corollary 7.3.7.6, suppose that $\mathcal{B} = \Delta^0$. Corollary 7.3.7.6 then asserts that a morphism $e : X \to Y$ in the functor ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ is V'-cocartesian if, for every object $C \in \mathcal{C}$, the induced map $e_C : X(C) \to Y(C)$ is a V-cocartesian morphism of $\mathcal{D}$. This is a special case of Lemma 5.2.1.5. 0482

Example 7.3.7.8. In the situation of Corollary 7.3.7.6, suppose that U is a cartesian fibration. Let $U_{\overline{e}} : \Delta^1 \times_{\mathcal{B}} \mathcal{C} \to \Delta^1$ denote the cartesian fibration given by projection onto the first factor. By virtue of Proposition 7.3.3.11, the functor f is V-left Kan extended from $\{0\} \times_{\mathcal{B}} \mathcal{C}$ if and only if it carries $U_{\overline{e}}$-cartesian morphisms of $\Delta^1 \times_{\mathcal{B}} \mathcal{C}$ to V-cocartesian morphisms of $\mathcal{D}$. In this case, Corollary 7.3.7.6 is a special case of (the dual of) Lemma 5.3.6.11. 0483

Corollary 7.3.7.9. *Let $U : \mathcal{C} \to \mathcal{B}$ be an exponentiable inner fibration of ∞-categories, let $\mathcal{D}$ be an ∞-category, and let $\pi : \operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathcal{B}$ be the projection map. Let e be a morphism of the ∞-category $\operatorname{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$, corresponding to a pair $(\overline{e}, f)$ where $\overline{e}$ is a morphism of $\mathcal{B}$ and $f : \Delta^1 \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories. If f is left Kan extended from $\{0\} \times_{\mathcal{B}} \mathcal{C}$, then the morphism e is π-cocartesian.* 0484

Proof. Apply Corollary 7.3.7.6 in the special case $\mathcal{E} = \Delta^0$. □

The proof of Theorem 7.3.7.1 will require some preliminaries.

Lemma 7.3.7.10. *Let $V : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories, let K be a simplicial set equipped with a diagram $\overline{f} : K^{\triangleright} \to \mathcal{E}$, and let $\operatorname{Fun}'_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D}) \subseteq \operatorname{Fun}_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D})$ be a full subcategory. Let $\operatorname{Fun}'_{/\mathcal{E}}(K, \mathcal{D})$ denote the essential image of $\operatorname{Fun}'_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D})$ under the restriction map $\operatorname{Fun}_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D}) \to \operatorname{Fun}_{/\mathcal{E}}(K, \mathcal{D})$. Suppose that every simplicial set A satisfies the following condition:* 0485

($*_A$) *For every extension of $\overline{f}$ to a morphism $K^{\triangleright} \star A \to \mathcal{E}$, the restriction functor*

$$\begin{tikzcd}
\operatorname{Fun}'_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D}) \times_{\operatorname{Fun}_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D})} \operatorname{Fun}_{/\mathcal{E}}(K^{\triangleright} \star A, \mathcal{D}) \arrow[d, "\theta_A"] \\
\operatorname{Fun}'_{/\mathcal{E}}(K, \mathcal{D}) \times_{\operatorname{Fun}_{/\mathcal{E}}(K, \mathcal{D})} \operatorname{Fun}_{/\mathcal{E}}(K \star A, \mathcal{D})
\end{tikzcd}$$

is an equivalence of ∞-categories.

Then every object of $\operatorname{Fun}'_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D})$ is a V-colimit diagram in the ∞-category $\mathcal{D}$.

Proof. Without loss of generality, we may assume that the full subcategory $\operatorname{Fun}'_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D})$ is replete. For every extension of $\overline{f}$ to a morphism $K^{\triangleright} \star A \to \mathcal{E}$, let $\operatorname{Fun}'_{/\mathcal{E}}(K^{\triangleright} \star A, \mathcal{D}) \subseteq \operatorname{Fun}_{/\mathcal{E}}(K^{\triangleright} \star A, \mathcal{D})$ and $\operatorname{Fun}'_{/\mathcal{E}}(K \star A, \mathcal{D}) \subseteq \operatorname{Fun}_{/\mathcal{E}}(K \star A, \mathcal{D})$ denote the inverse images of $\operatorname{Fun}'_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D})$ and $\operatorname{Fun}'_{/\mathcal{E}}(K, \mathcal{D})$, respectively. For every monomorphism of simplicial sets $A \hookrightarrow B$, Proposition 4.4.5.1 guarantees that the restriction map $\mathrm{Fun}_{/\mathcal{E}}(K \star B, \mathcal{D}) \to \mathrm{Fun}_{/\mathcal{E}}(K \star A, \mathcal{D})$ is an isofibration, and therefore induces an isofibration $\mathrm{Fun}'_{/\mathcal{E}}(K \star B, \mathcal{D}) \to \mathrm{Fun}'_{/\mathcal{E}}(K \star A, \mathcal{D})$ Combining Corollary 4.5.2.30 with assumptions $(*_A)$ and $(*_B)$, we conclude that the restriction map

$$\theta_{A,B} : \mathrm{Fun}'_{/\mathcal{E}}(K^{\triangleright} \star B, \mathcal{D}) \to \mathrm{Fun}'_{/\mathcal{E}}(K^{\triangleright} \star A, \mathcal{D}) \times_{\mathrm{Fun}'_{/\mathcal{E}}(K \star A, \mathcal{D})} \mathrm{Fun}'_{/\mathcal{E}}(K \star B, \mathcal{D})$$

is an equivalence of ∞-categories. Proposition 4.4.5.1 implies that $\theta_{A,B}$ is also an isofibration, and is therefore a trivial Kan fibration (Proposition 4.5.5.20). In particular, $\theta_{A,B}$ is surjective on vertices. Unwinding the definitions, we conclude that every lifting problem

$$\begin{tikzcd}
(K^{\triangleright} \star A)\coprod_{(K \star A)}(K \star B) \arrow[r, "g"] \arrow[d] & \mathcal{D} \arrow[d, "V"] \\
K^{\triangleright} \star B \arrow[r] \arrow[ur, dashed] & \mathcal{E}
\end{tikzcd}$$

admits a solution, provided that the restriction $g|_{K^{\triangleright}}$ belongs to $\mathrm{Fun}'_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D})$. The desired result now follows by invoking the criterion of Remark 7.1.5.11. □

0486 **Exercise 7.3.7.11.** Prove the converse of Lemma 7.3.7.10.

0487 **Remark 7.3.7.12.** In the situation of Lemma 7.3.7.10, suppose we are given an inner anodyne morphism of simplicial sets $A \hookrightarrow B$. Then every diagram $K^{\triangleright} \star A \to \mathcal{E}$ can be extended to a morphism $K^{\triangleright} \star B \to \mathcal{E}$, so that condition $(*_A)$ is satisfied if and only if condition $(*_B)$ is satisfied. Consequently, to show that every object of $\mathrm{Fun}'_{/\mathcal{E}}(K^{\triangleright}, \mathcal{D})$ is a V-colimit diagram, it suffices to verify condition $(*_A)$ in the special case where A is an ∞-category (see Corollary 4.1.3.3).

We will deduce Theorem 7.3.7.1 from the following special case:

0488 **Proposition 7.3.7.13.** *Let $U : \mathcal{C} \to \mathcal{B}$ be an exponentiable inner fibration of ∞-categories, let $V : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories, and let $V' : \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})$ denote the functor given by postcomposition with V. Let $\mathcal{K}$ be an ∞-category and let $f : \mathcal{K}^{\triangleright} \to \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})$ be a functor, corresponding to a morphism $\mathcal{K}^{\triangleright} \to \mathcal{B}$ and a functor $F : \mathcal{K}^{\triangleleft} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$. If F is V-left Kan extended from $\mathcal{K} \times_{\mathcal{B}} \mathcal{C}$, then f is a V'-colimit diagram.*

Proof. Without loss of generality, we may assume that V is an isofibration, so that V' is also an isofibration (Proposition 4.5.9.18). Fix a morphism $\overline{f} : \mathcal{K}^{\triangleright} \to \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})$, and let

$$\mathrm{Fun}'_{/\mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})}(\mathcal{K}^{\triangleright}, \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D})) \subseteq \mathrm{Fun}_{/\mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{E})}(\mathcal{K}^{\triangleright}, \mathrm{Fun}(\mathcal{C} / \mathcal{B}, \mathcal{D}))$$

denote the full subcategory spanned by those morphisms $\mathcal{K}^{\triangleright} \to \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$ which correspond to functors $\mathcal{K}^{\triangleleft} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ which are V-left Kan extended from $\mathcal{K}$, and let $\operatorname{Fun}'_{/\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})}(\mathcal{K}, \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}))$ denote its essential image under the restriction map

$$\operatorname{Fun}_{/\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})}(\mathcal{K}^{\triangleright}, \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})) \to \operatorname{Fun}_{/\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})}(\mathcal{K}, \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})).$$

We will complete the proof by showing that every object of $\operatorname{Fun}'_{/\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})}(\mathcal{K}^{\triangleright}, \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}))$ is a V'-colimit diagram in the ∞-category $\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$. By virtue of Remark 7.3.7.12, it will suffice to verify condition $(*_{\mathcal{A}})$ of Lemma 7.3.7.10 for every ∞-category $\mathcal{A}$.

Fix a morphism $\mathcal{K}^{\triangleright} \star \mathcal{A} \to \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})$ extending $\overline{f}$, which we identify with a diagram $\mathcal{K}^{\triangleright} \star \mathcal{A} \to \mathcal{B}$ and a functor

$$\overline{G} : (\mathcal{K}^{\triangleright} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C} \to \mathcal{E}\,.$$

Let $\operatorname{Fun}'_{/\mathcal{E}}((\mathcal{K}^{\triangleright} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}, \mathcal{D})$ denote denote the full subcategory of $\operatorname{Fun}_{/\mathcal{E}}((\mathcal{K}^{\triangleright} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}, \mathcal{D})$ given by the inverse image of $\operatorname{Fun}'_{/\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})}(\mathcal{K}^{\triangleright}, \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}))$, and define $\operatorname{Fun}'_{/\mathcal{E}}((\mathcal{K} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}, \mathcal{D})$ similarly. We wish to show that the restriction map

$$\theta : \operatorname{Fun}'_{/\mathcal{E}}((\mathcal{K}^{\triangleright} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}, \mathcal{D}) \to \operatorname{Fun}'_{/\mathcal{E}}((\mathcal{K} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}, \mathcal{D})$$

is an equivalence of ∞-categories. Unwinding the definitions (and using the existence criterion of Proposition 7.3.5.5), we see that a functor $G \in \operatorname{Fun}_{/\mathcal{E}}((\mathcal{K}^{\triangleright} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}, \mathcal{D})$ belongs to the subcategory $\operatorname{Fun}'_{/\mathcal{E}}((\mathcal{K}^{\triangleright} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}, \mathcal{D})$ if and only if it is V-left Kan extended from $(\mathcal{K} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}$, and that a functor $G_0 \in \operatorname{Fun}_{/\mathcal{E}}((\mathcal{K}^{\triangleright} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C}, \mathcal{D})$ belongs to $\operatorname{Fun}'_{/\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})}(\mathcal{K}^{\triangleright}, \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}))$ if and only if admits a left Kan extension $G : (\mathcal{K}^{\triangleright} \star \mathcal{A}) \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ satisfying $V \circ G = \overline{G}$. Applying Theorem 7.3.6.14, we conclude that θ is a trivial Kan fibration. □

Example 7.3.7.14. In the situation of Proposition 7.3.7.13, suppose that $\mathcal{B} = \Delta^0$, so that we can identify F with a functor from $\mathcal{K}^{\triangleleft} \times \mathcal{C}$ to $\mathcal{D}$. Let X denote the cone point of $\mathcal{K}^{\triangleright}$. For each object $C \in \mathcal{C}$, the inclusion map $\mathcal{K} \times \{\mathrm{id}_C\} \hookrightarrow \mathcal{K} \times \mathcal{C}_{/C}$ is right cofinal (see Corollary 7.2.1.19). Applying Corollary 7.2.2.2, we deduce that F is V-left Kan extended from $\mathcal{K} \times \mathcal{C}$ if and only if the induced map 0489

$$\mathcal{K}^{\triangleleft} \simeq \mathcal{K}^{\triangleleft} \times \{C\} \hookrightarrow \mathcal{K}^{\triangleleft} \times \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a V-colimit diagram for each $C \in \mathcal{C}$. Proposition 7.3.7.13 asserts that, if this condition is satisfied, then F determines a V'-colimit diagram $\mathcal{K}^{\triangleleft} \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$; this is a special case of Corollary 7.1.6.11.

Proof of Theorem 7.3.7.1. Let $U : \mathcal{C} \to \mathcal{B}$ be an inner fibration of ∞-categories, let $V : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories, and let $V' : \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{E})$ be the functor given by postcomposition with V'. Suppose we are given a functor $f : \mathcal{A} \to \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$

and a full subcategory $\mathcal{A}^0 \subseteq \mathcal{A}$. Assume that the induced map $F : \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{D}$ is V-left Kan extended from $\mathcal{A}^0 \times_{\mathcal{B}} \mathcal{C}$; we ish show that f is V'-left Kan extended from $\mathcal{A}^0$. Fix an object $A \in \mathcal{A}$ and set $\mathcal{A}^0_{/A} = \mathcal{A}^0 \times_{\mathcal{A}} \mathcal{A}_{/A}$; we wish to show that the composite map

$$(\mathcal{A}^0_{/A})^{\triangleright} \hookrightarrow (\mathcal{A}_{/A})^{\triangleright} \to \mathcal{A} \xrightarrow{f} \operatorname{Fun}(\mathcal{C} \,/\, \mathcal{B}, \mathcal{D})$$

is a V-colimit diagram. Let F_A denote the composition

$$(\mathcal{A}^0_{/A})^{\triangleright} \times_{\mathcal{B}} \mathcal{C} \to \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \xrightarrow{F} \mathcal{D}.$$

By virtue of Proposition 7.3.7.13, it will suffice to show that F_A is V-left Kan extended from $\mathcal{A}^0_{/A} \times_{\mathcal{B}} \mathcal{C}$. Let X denote the cone point of $(\mathcal{A}^0_{/A})^{\triangleright}$, let B denote its image in $\mathcal{B}$, and let $C \in \mathcal{C}$ be an object satisfying $U(C) = B$. Unwinding the definitions, we see that F_A is V-left Kan extended from $\mathcal{A}^0_{/A} \times_{\mathcal{B}} \mathcal{C}$ at the object (X, C) if and only if the diagram

$$(\mathcal{A}^0_{/A} \times_{\mathcal{B}_{/B}} \mathcal{C}_{/C})^{\triangleright} \to (\mathcal{A}^0_{/A})^{\triangleright} \times_{(\mathcal{B}_{/B})^{\triangleright}} (\mathcal{C}_{/C})^{\triangleright} \to \mathcal{A} \times_{\mathcal{B}} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a V-colimit diagram. This follows from our assumption that F is V-left Kan extended from the full subcategory $\mathcal{A}^0 \times_{\mathcal{B}} \mathcal{C}$ at the object (A, C). □

7.3.8 Transitivity of Kan Extensions

030U Let $\overline{\mathcal{C}}$ be an ∞-category equipped with full subcategories $\mathcal{C}^0 \subseteq \mathcal{C} \subseteq \overline{\mathcal{C}}$. Our goal in this section is to show that a functor of ∞-categories $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ is left Kan extended from $\mathcal{C}^0$ if and only if it is left Kan extended from $\mathcal{C}$ and $\overline{F}_{\mathcal{C}}$ is left Kan extended from $\mathcal{C}^0$ (Corollary 7.3.8.8). We begin by analyzing the case special case where the ∞-category $\overline{\mathcal{C}}$ has the form $\mathcal{C}^{\triangleright}$.

030V **Proposition 7.3.8.1.** *Let $\mathcal{C}$ be an ∞-category, let $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ be a functor of ∞-categories, and let $U : \mathcal{D} \to \mathcal{E}$ be another functor of ∞-categories. Assume that $F = \overline{F}|_{\mathcal{C}}$ is U-left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Then $\overline{F}$ is a U-colimit diagram if and only if the composite map*

$$(\mathcal{C}^0)^{\triangleright} \hookrightarrow \mathcal{C}^{\triangleright} \xrightarrow{\overline{F}} \mathcal{D}$$

is a U-colimit diagram.

Proof. For each object $D \in \mathcal{D}$, let $\underline{D} \in \operatorname{Fun}(\mathcal{C}^{\triangleright}, \mathcal{D})$ denote the constant functor taking the value D. By virtue of Proposition 7.1.5.12, the functor $\overline{F}$ is a U-colimit diagram if and only if, for each $D \in \mathcal{D}$, the upper half of the diagram

`030W`

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\triangleright},\mathcal{D})}(\overline{F}, \underline{D}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\triangleright},\mathcal{E})}(U \circ \overline{F}, U \circ \underline{D}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, \underline{D}|_{\mathcal{C}}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(U \circ F, U \circ \underline{D}|_{\mathcal{C}}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{D})}(F|_{\mathcal{C}^0}, \underline{D}|_{\mathcal{C}^0}) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{E})}(U \circ F|_{\mathcal{C}^0}, U \circ \underline{D}|_{\mathcal{C}^0})
\end{tikzcd} \tag{7.30}$$

is a homotopy pullback square. Since F is U-left Kan extended from $\mathcal{C}^0$, Proposition 7.3.6.7 shows that the right half of the diagram is a homotopy pullback square. It follows that $\overline{F}$ is a U-colimit diagram if and only if the outer rectangle of (7.30) is a homotopy pullback square for each $D \in \mathcal{D}$ (Proposition 3.4.1.11).

Let v denote the cone point of $\mathcal{C}^{\triangleright}$. Let $\mathcal{C}^1$ denote the cone $(\mathcal{C}^0)^{\triangleright}$, which we regard as a full subcategory of $\mathcal{C}^{\triangleright}$. Note that the functors $\underline{D}$, $\underline{D}|_{\mathcal{C}^1}$, $U \circ \underline{D}$ and $U \circ \underline{D}|_{\mathcal{C}^1}$ are right Kan extended from the cone point, so Corollary 7.3.6.9 implies that the restriction maps

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\triangleright},\mathcal{D})}(\overline{F}, \underline{D}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^1,\mathcal{D})}(\overline{F}|_{\mathcal{C}^1}, \underline{D}|_{\mathcal{C}^1}) \to \mathrm{Hom}_{\mathcal{D}}(\overline{F}(v), D)$$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\triangleright},\mathcal{E})}(U \circ \overline{F}, U \circ \underline{D}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^1,\mathcal{E})}(U \circ \overline{F}|_{\mathcal{C}^1}, U \circ \underline{D}|_{\mathcal{C}^1}) \to \mathrm{Hom}_{\mathcal{E}}((U \circ \overline{F}(v))), U(D))$$

are homotopy equivalences. It follows that the restriction map from the outer rectangle of (7.30) to the diagram

`030X`

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^1,\mathcal{D})}(\overline{F}|_{\mathcal{C}^1}, \underline{D}|_{\mathcal{C}^1}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{D})}(F|_{\mathcal{C}^0}, \underline{D}|_{\mathcal{C}^0}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^1,\mathcal{E})}(U \circ \overline{F}|_{\mathcal{C}^1}, U \circ \underline{D}|_{\mathcal{C}^1}) \arrow[r] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^0,\mathcal{E})}(U \circ F, U \circ \underline{D}|_{\mathcal{C}^0})
\end{tikzcd} \tag{7.31}$$

is a levelwise homotopy equivalence. In particular, the outer rectangle of (7.30) is a homotopy pullback square if and only if (7.31) is a homotopy pullback square (Corollary 3.4.1.12). By virtue of Proposition 7.1.5.12, this is satisfied for every object $D \in \mathcal{D}$ if and only if F^1 is a U-colimit diagram. □

030Y **Corollary 7.3.8.2.** *Let $\mathcal{C}$ be an ∞-category and let $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ be a functor of ∞-categories. Suppose that $F = \overline{F}|_{\mathcal{C}}$ is left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Then $\overline{F}$ is a colimit diagram if and only if the composite map*

$$(\mathcal{C}^0)^{\triangleright} \hookrightarrow \mathcal{C}^{\triangleright} \xrightarrow{\overline{F}} \mathcal{D}$$

is a colimit diagram.

Proof. Apply Proposition 7.3.8.1 in the special case $\mathcal{E} = \Delta^0$. □

030Z **Proposition 7.3.8.3.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Suppose we are given a right fibration of ∞-categories $V : \mathcal{B} \to \mathcal{C}$ and set $\mathcal{B}^0 = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{B}$. Then, for every object $B \in \mathcal{B}$, the functor $F \circ V$ is U-left Kan extended from $\mathcal{B}^0$ at B if and only if F is U-left Kan extended from $\mathcal{C}^0$ at $V(B)$.*

Proof. Set $C = V(B)$, and let F_C denote the composite map

$$(\mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C})^{\triangleright} \to (\mathcal{C}_{/C})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}.$$

We wish to show that F_C is a U-colimit diagram if the composite map

$$(\mathcal{B}^0 \times_{\mathcal{B}} \mathcal{B}_{/B})^{\triangleright} \to (\mathcal{B}_{/B})^{\triangleright} \to \mathcal{B} \xrightarrow{V} \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a U-colimit diagram. By virtue of Corollary 7.2.2.2, it will suffice to show that the natural map

$$\theta : \mathcal{B}^0 \times_{\mathcal{B}} \mathcal{B}_{/B} \to \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C}$$

is right cofinal. By construction, θ is a pullback of the map $V_{/B} : \mathcal{B}_{/B} \to \mathcal{C}_{/V(B)}$. Our assumption that V is a right fibration guarantees that $V_{/B}$ is a trivial Kan fibration (Corollary 4.3.7.13). It follows that θ is also a trivial Kan fibration, and therefore right cofinal by virtue of Corollary 7.2.1.13. □

0310 **Corollary 7.3.8.4.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Suppose we are given a right fibration of ∞-categories $V : \mathcal{B} \to \mathcal{C}$ and set $\mathcal{B}^0 = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{B}$. If F is U-left Kan extended from $\mathcal{C}^0$, then $F \circ V$ is U-left Kan extended from $\mathcal{B}^0$. The converse holds if every fiber of V is nonempty.*

0311 **Corollary 7.3.8.5.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Suppose we are given a right fibration of ∞-categories $V : \mathcal{B} \to \mathcal{C}$ and set $\mathcal{B}^0 = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{B}$. If F is left Kan extended from $\mathcal{C}^0$, then $F \circ V$ is left Kan extended from $\mathcal{B}^0$. The converse holds if every fiber of V is nonempty.*

Proof. Apply Corollary 7.3.8.4 in the special case $\mathcal{E} = \Delta^0$. □

Proposition 7.3.8.6 (Transitivity for Kan Extensions)**.** *Let $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Let $\mathcal{C}^0 \subseteq \mathcal{C} \subseteq \overline{\mathcal{C}}$ be full subcategories. Then $\overline{F}$ is U-left Kan extended from $\mathcal{C}^0$ if and only if it satisfies the following pair of conditions:* 0312

(1) *The functor $\overline{F}$ is U-left Kan extended from $\mathcal{C}$.*

(2) *The restriction $\overline{F}|_{\mathcal{C}}$ is U-left Kan extended from $\mathcal{C}^0$.*

Remark 7.3.8.7. In the special case $\overline{\mathcal{C}} = \mathcal{C}^{\triangleright}$, Proposition 7.3.8.6 is essentially a restatement of Proposition 7.3.8.1 (see Example 7.3.3.10). 0313

Proof of Proposition 7.3.8.6. It follows immediately from the definitions that if $\overline{F}$ is U-left Kan extended from $\mathcal{C}^0$, then the functor $F = \overline{F}|_{\mathcal{C}}$ has the same property. We may therefore assume that condition (2) is satisfied. Fix an object $X \in \overline{\mathcal{C}}$. We will complete the proof by showing that $\overline{F}$ is U-left Kan extended from $\mathcal{C}^0$ at X if and only if it is U-left Kan extended from $\mathcal{C}$ at X. Let $\overline{F}_X$ denote the composite map

$$(\mathcal{C} \times_{\overline{\mathcal{C}}} \overline{\mathcal{C}}_{/X})^{\triangleright} (\overline{\mathcal{C}}_{/X})^{\triangleright} \to \overline{\mathcal{C}} \xrightarrow{\overline{F}} \mathcal{D}.$$

We wish to show that $\overline{F}_X$ is a U-colimit diagram if and only if its restriction to $(\mathcal{C}^0 \times_{\overline{\mathcal{C}}} \overline{\mathcal{C}}_{/X})^{\triangleright}$ is a U-colimit diagram. Let F_X denote the restriction of $\overline{F}_X$ to $\mathcal{C} \times_{\overline{\mathcal{C}}} \overline{\mathcal{C}}_{/X}$. By virtue of Proposition 7.3.8.1, it will suffice to show that F_X is U-left Kan extended from $\mathcal{C}^0 \times_{\overline{\mathcal{C}}} \overline{\mathcal{C}}_{/X}$. This follows by applying Corollary 7.3.8.4 to the right fibration $\mathcal{C} \times_{\overline{\mathcal{C}}} \overline{\mathcal{C}}_{/X} \to \mathcal{C}$. □

Corollary 7.3.8.8. *Let $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ be a functor of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C} \subseteq \overline{\mathcal{C}}$ be full subcategories. Then $\overline{F}$ is left Kan extended from $\mathcal{C}^0$ if and only if it satisfies the following pair of conditions:* 0314

(1) *The functor $\overline{F}$ is left Kan extended from $\mathcal{C}$.*

(2) *The restriction $\overline{F}|_{\mathcal{C}}$ is left Kan extended from $\mathcal{C}^0$.*

Proof. Apply Proposition 7.3.8.6 in the special case $\mathcal{E} = \Delta^0$. □

Corollary 7.3.8.9. *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and let $C, C' \in \mathcal{C}$ be objects which are isomorphic. If F is U-left Kan extended from $\mathcal{C}^0$ at C, then it is U-left Kan extended from $\mathcal{C}^0$ at C'.* 0315

Proof. Let $\mathcal{C}^1 \subseteq \mathcal{C}$ be the full subcategory spanned by the objects of $\mathcal{C}^0$ together with the object C, and let $\mathcal{C}^2 \subseteq \mathcal{C}$ be the full subcategory spanned by the objects of $\mathcal{C}$ together with the objects C and C'. If F is U-left Kan extended from $\mathcal{C}^0$ at C, then the functor $F|_{\mathcal{C}^1}$ is U-left Kan extended from $\mathcal{C}^0$. Since every object of $\mathcal{C}^2$ is isomorphic to an object of $\mathcal{C}^1$. the functor $F|_{\mathcal{C}^2}$ is automatically U-left Kan extended from $\mathcal{C}^1$ (Proposition 7.3.3.7). Applying Proposition 7.3.8.6, we see that $F|_{\mathcal{C}^2}$ is also U-left Kan extended from $\mathcal{C}^0$. In particular, F is U-left Kan extended from $\mathcal{C}^0$ at the object $C' \in \mathcal{C}^2$. □

We now prove a variant of Proposition 7.3.8.6, which gives a criterion for the *existence* of relative Kan extensions.

0316 **Proposition 7.3.8.10.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, and suppose that F is U-left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Set $F_0 = F|_{\mathcal{C}^0}$. Then the restriction map*

$$\theta : \mathcal{D}_{F/} \to \mathcal{D}_{F_0/} \times_{\mathcal{E}_{(U \circ F_0)/}} \mathcal{E}_{(U \circ F)/}$$

is an equivalence of ∞-categories.

Proof. Note that the restriction maps

$$\mathcal{D}_{F/} \to \mathcal{D}_{F_0/} \qquad \mathcal{D}_{F_0/} \to \mathcal{D} \qquad \mathcal{E}_{(U \circ F)/} \to \mathcal{E}_{(U \circ F_0)/}$$

are left fibrations of simplicial sets (Corollary 4.3.6.11). It follows that we can regard θ as a functor of ∞-categories which are left-fibered over $\mathcal{D}$. Consequently, to show that θ is an equivalence of ∞-categories, it will suffice to show that for every object $D \in \mathcal{D}$, the commutative diagram

0317
$$\begin{tikzcd}
\mathcal{D}_{F/} \times_{\mathcal{D}} \{D\} \arrow[r] \arrow[d] & \mathcal{D}_{F_0/} \times_{\mathcal{D}} \{D\} \arrow[d] \\
\mathcal{E}_{(U \circ F)/} \times_{\mathcal{E}} \{U(D)\} \arrow[r] & \{U(D)\} \times_{\mathcal{E}_{(U \circ F_0)/}} \times_{\mathcal{E}} \{U(D)\}
\end{tikzcd} \tag{7.32}$$

induces a homotopy equivalence of Kan complexes

$$\mathcal{D}_{F/} \times_{\mathcal{D}} \{D\} \to (\mathcal{D}_{F_0/} \times_{\mathcal{E}_{(U \circ F_0)/}} \mathcal{E}_{(U \circ F)/}) \times_{\mathcal{D}} \{D\}.$$

Note that the horizontal maps in the diagram (7.32) are left fibrations between Kan complexes (Corollary 4.3.6.11), and therefore Kan fibrations (Corollary 4.4.3.8). We are therefore reduced to showing that the diagram (7.32) is a homotopy pullback square (Example 3.4.1.3).

Let $\underline{D} \in \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ denote the constant functor taking the value D. Using Theorem 4.6.4.17, we obtain a (termwise) homotopy equivalence from (7.32) to the diagram of morphism spaces

0318
$$\begin{tikzcd}
\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{D})}(F, \underline{D}) \arrow[r] \arrow[d] & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^0,\mathcal{E})}(F_0, \underline{D}|_{\mathcal{C}^0}) \arrow[d] \\
\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(U \circ F, U \circ \underline{D}) \arrow[r] & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^0,\mathcal{E})}(U \circ F_0, U \circ \underline{D}|_{\mathcal{C}^0}).
\end{tikzcd} \tag{7.33}$$

Using Corollary 3.4.1.12, we are reduced to showing that the diagram (7.33) is a homotopy pullback square, which is a special case of Proposition 7.3.6.7. □

Corollary 7.3.8.11. *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, where U is an inner fibration and F is U-left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Set $F_0 = F|_{\mathcal{C}^0}$. Then the restriction map* 0319

$$\theta : \mathcal{D}_{F/} \to \mathcal{D}_{F_0/} \times_{\mathcal{E}_{(U \circ F_0)/}} \mathcal{E}_{(U \circ F)/}$$

is a trivial Kan fibration.

Proof. It follows from Proposition 4.3.6.8 that θ is a left fibration, and therefore an isofibration (Example 4.4.1.11). By virtue of Proposition 4.5.5.20, it will suffice to show that θ is an equivalence of ∞-categories, which follows from Proposition 7.3.8.10. □

Corollary 7.3.8.12. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which is Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$, and set $F_0 = F|_{\mathcal{C}^0}$. Then the restriction functor $\theta : \mathcal{C}_{F/} \to \mathcal{C}_{F_0/}$ is a trivial Kan fibration.* 031A

Proof. Apply Corollary 7.3.8.11 in the special case $\mathcal{E} = \Delta^0$. □

Corollary 7.3.8.13. *Let $\mathcal{C}$ be an ∞-category, let $U : \mathcal{D} \to \mathcal{E}$ be an inner fibration of ∞-categories, and suppose we are given a lifting problem* 031B

031C

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\ \mathcal{C}^{\triangleright} \arrow[r] \arrow[ur, dashed, "\overline{F}"] & \mathcal{E}. \end{tikzcd} \qquad (7.34)$$

Assume that F is U-left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. The following conditions are equivalent:

(1) *The lifting problem (7.34) admits a solution $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ which is a U-colimit diagram.*

(2) *The induced lifting problem*

031D

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F|_{\mathcal{C}^0}"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\ (\mathcal{C}^0)^{\triangleright} \arrow[r] \arrow[ur, dashed, "\overline{F}_0"] & \mathcal{E}. \end{tikzcd} \qquad (7.35)$$

admits a solution $\overline{F}_0 : (\mathcal{C}^0)^{\triangleright} \to \mathcal{D}$ which is a U-colimit diagram.

Proof. The implication (1) ⇒ (2) follows immediately from Proposition 7.3.8.1. For the converse, suppose that $\overline{F}_0 : (\mathcal{C}^0)^{\triangleright} \to \mathcal{D}$ is a U-colimit diagram which solves the lifting problem (7.35). Applying Corollary 7.3.8.11, we see that $\overline{F}_0$ can be extended to a functor $\overline{F} : \mathcal{C}^{\triangleright} \to \mathcal{D}$ which solves the lifting problem (7.34). It then follows from Proposition 7.3.8.1 that $\overline{F}$ is a U-colimit diagram. □

031E **Corollary 7.3.8.14.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which is left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Then F has a colimit in $\mathcal{D}$ if and only if the restriction $F|_{\mathcal{C}^0}$ has a colimit in $\mathcal{D}$.*

Proof. Apply Corollary 7.3.8.13 in the special case $\mathcal{E} = \Delta^0$. □

03E5 **Remark 7.3.8.15.** In the situation of Corollary 7.3.8.14, an object of $\mathcal{D}$ is a colimit of the diagram F if and only if it is a colimit of the diagram $F|_{\mathcal{C}^0}$. This follows by combining Corollaries 7.3.8.14 and 7.3.8.2.

031F **Proposition 7.3.8.16.** *Let $\overline{\mathcal{C}}$ be an ∞-category, let $\mathcal{C} \subseteq \overline{\mathcal{C}}$ be a full subcategory, and let $U : \mathcal{D} \to \mathcal{E}$ be an isofibration of ∞-categories. Suppose we are given a lifting problem*

031G
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\overline{\mathcal{C}} \arrow[r] \arrow[ur, dashed, "\overline{F}"] & \mathcal{E},
\end{tikzcd} \tag{7.36}$$

where F is U-left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. The following conditions are equivalent:

(1) *The lifting problem (7.36) admits a solution $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{C}$.*

(2) *The induced lifting problem*

031H
$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r, "F|_{\mathcal{C}^0}"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
\overline{\mathcal{C}} \arrow[r] \arrow[ur, dashed, "\overline{F}"] & \mathcal{E},
\end{tikzcd} \tag{7.37}$$

admits a solution $\overline{F} : \overline{\mathcal{C}} \to \mathcal{D}$ which is U-left Kan extended from $\mathcal{C}^0$.

Proof. The implication (1) ⇒ (2) follows immediately from Proposition 7.3.8.6. For the converse, assume that (2) is satisfied. To prove (1), it will suffice to show that for each object $C \in \overline{\mathcal{C}}$, the induced lifting problem

031J
$$\begin{tikzcd}
\mathcal{C}_{/C} \arrow[r, "F_C"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
(\mathcal{C}_{/C})^{\triangleright} \arrow[r] \arrow[ur, dashed, "\overline{F}_C"] & \mathcal{E}
\end{tikzcd} \tag{7.38}$$

admits a solution $\overline{F}_C : (\mathcal{C}_{/C})^{\triangleright} \to \mathcal{D}$ which is a U-colimit diagram (Proposition 7.3.5.5). Arguing as in the proof of Proposition 7.3.8.6, we see that F_C is U-left Kan extended from the full subcategory $\mathcal{C}^0_{/C} \subseteq \mathcal{C}_{/C}$. Let F^0_C denote the restriction of F_C to the subcategory $\mathcal{C}^0_{/C} \subseteq \mathcal{C}_{/C}$. By virtue of Corollary 7.3.8.13, it will suffice to show that the induced lifting problem

031K
$$\begin{tikzcd}
\mathcal{C}^0_{/C} \arrow[r, "F^0_C"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\
(\mathcal{C}^0_{/C})^{\triangleright} \arrow[r] \arrow[ur, dashed, "\overline{F}^0_C"] & \mathcal{E}
\end{tikzcd} \tag{7.39}$$

has a solution $\overline{F}^0_C : (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{D}$ which is a U-colimit diagram, which follows immediately from assumption (2). □

Corollary 7.3.8.17. *Let $\overline{\mathcal{C}}$ be an ∞-category, let $\mathcal{C} \subseteq \overline{\mathcal{C}}$ be a full subcategory, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which is left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Then F admits a left Kan extension $\overline{\mathcal{C}} \to \mathcal{D}$ if and only if the restriction $F|_{\mathcal{C}^0}$ admits a left Kan extension $\overline{\mathcal{C}} \to \mathcal{D}$.* 031L

Proof. Apply Proposition 7.3.8.16 in the special case $\mathcal{E} = \Delta^0$. □

We close this section by establishing counterparts of Corollaries 7.3.8.8 and 7.3.8.14 for Kan extensions along more general functors.

Proposition 7.3.8.18. *Let $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{D}$ be ∞-categories. Suppose we are given functors $F_i : \mathcal{C}_i \to \mathcal{D}$ for $0 \leq i \leq 2$, functors $G : \mathcal{C}_0 \to \mathcal{C}_1$ and $H : \mathcal{C}_1 \to \mathcal{C}_2$, and natural transformations* 031M

$$\alpha : F_0 \to F_1 \circ G \qquad \beta : F_1 \to F_2 \circ H,$$

where α exhibits F_1 as a left Kan extension of F_0 along G. The following conditions are equivalent:

(1) *The natural transformation β exhibits F_2 as a left Kan extension of F_1 along H.*

(2) *Let $\gamma : F_0 \to F_2 \circ H \circ G$ be a composition of α with $\beta|_{\mathcal{C}^0}$ (formed in the ∞-category $\mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$). Then γ exhibits F_2 as a left Kan extension of F_0 along $H \circ G$.*

Proof. Let $\mathcal{C}$ denote the iterated relative join $(\mathcal{C}_0 \star_{\mathcal{C}_1} \mathcal{C}_1) \star_{\mathcal{C}_2} \mathcal{C}_2$, so that we have a cocartesian fibration of ∞-categories $\pi : \mathcal{C} \to \Delta^2$ having fibers $\pi^{-1}\{i\} = \mathcal{C}_i$ for $0 \leq i \leq 2$ (see Lemma 5.2.3.17). For $0 \leq i < j \leq 2$, let $\mathcal{C}_{ij}$ denote the fiber product $\mathrm{N}_{\bullet}(\{i < j\}) \times_{\Delta^2} \mathcal{C}$, which we will identify with $\mathcal{C}_i \star_{\mathcal{C}_j} \mathcal{C}_j$. By virtue of Remark 7.3.1.9, we are free to replace α and β by homotopic natural transformations. We can therefore assume that there exist functors

$$F_{01} : \mathcal{C}_{01} \to \mathcal{D} \qquad F_{12} : \mathcal{C}_{12} \to \mathcal{D}$$

satisfying $F_{01}|_{\mathcal{C}_0} = F_0$, $F_{01}|_{\mathcal{C}_1} = F_1 = F_{12}|_{\mathcal{C}_1}$, and $F_{12}|_{\mathcal{C}_2} = F_2$, where α and β are given by the composite maps

$$\Delta^1 \times \mathcal{C}_0 \simeq \mathcal{C}_0 \star_{\mathcal{C}_0} \mathcal{C}_0 \to \mathcal{C}_0 \star_{\mathcal{C}_1} \mathcal{C}_1 \xrightarrow{F_{01}} \mathcal{D}$$

$$\Delta^1 \times \mathcal{C}_1 \simeq \mathcal{C}_1 \star_{\mathcal{C}_1} \mathcal{C}_1 \to \mathcal{C}_1 \star_{\mathcal{C}_2} \mathcal{C}_2 \xrightarrow{F_{12}} \mathcal{D}$$

(see Warning 7.3.2.12). Note that F_{01} and F_{12} can be amalgamated to a morphism of simplicial sets $F' : \Lambda^2_1 \times_{\Delta^1} \mathcal{C} \to \mathcal{D}$. Since π is a cocartesian fibration, the inclusion map $\Lambda^2_1 \times_{\Delta^1} \mathcal{C} \hookrightarrow \mathcal{C}$ is a categorical equivalence (Proposition 5.3.6.1). Applying Lemma 4.5.5.2, we can extend F' to a functor $F : \mathcal{C} \to \mathcal{D}$.

Let F_{02} denote the restriction of F to $\mathcal{C}_{02}$, and let $\gamma : F_0 \to F_2 \circ H \circ G$ denote the natural transformation given by the composite map

$$\Delta^1 \times \mathcal{C}_0 \simeq \mathcal{C}_0 \star_{\mathcal{C}_0} \mathcal{C}_0 \to \mathcal{C}_0 \star_{\mathcal{C}_2} \mathcal{C}_2 \xrightarrow{F_{02}} \mathcal{D}.$$

Note that the composite map

$$\Delta^2 \times \mathcal{C}_0 \simeq (\mathcal{C}_0 \star_{\mathcal{C}_0} \mathcal{C}_0) \star_{\mathcal{C}_0} \mathcal{C}_0 \to (\mathcal{C}_0 \star_{\mathcal{C}_1} \mathcal{C}_1) \star_{\mathcal{C}_2} \mathcal{C}_2 \xrightarrow{F} \mathcal{D}$$

can be regarded as a 2-simplex of the ∞-category $\mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$, which witnesses γ as a composition of α with $\beta|_{\mathcal{C}_0}$. Applying Proposition 7.3.2.11, we see that (1) and (2) can be reformulated as follows:

($1'$) The functor $F_{12} : \mathcal{C}_{12} \to \mathcal{D}$ is left Kan extended from $\mathcal{C}_1$.

($2'$) The functor $F_{02} : \mathcal{C}_{02} \to \mathcal{D}$ is left Kan extended from $\mathcal{C}_0$.

By assumption, the natural transformation α exhibits F_1 as a left Kan extension of F_0 along G. Applying Proposition 7.3.2.11, we see that the functor F_{01} is left Kan extended from $\mathcal{C}_0$. In particular, F is left Kan extended from $\mathcal{C}^0$ at every object of the full subcategory $\mathcal{C}_1 \subseteq \mathcal{C}$. It follows that ($2'$) is equivalent to the following:

$(2'')$ The functor $F : \mathcal{C} \to \mathcal{D}$ is left Kan extended from $\mathcal{C}_0$.

Using Corollary 7.3.8.8, we see that $(2'')$ is equivalent to the following:

$(1'')$ The functor $F : \mathcal{C} \to \mathcal{D}$ is left Kan extended from $\mathcal{C}_{01}$.

To complete the proof, it will suffice to show that conditions $(1')$ and $(1'')$ are equivalent. We will prove something slightly more precise: for every object $X \in \mathcal{C}_2$, the conditions are equivalent:

$(1'_X)$ The functor $F_{12} : \mathcal{C}_{12} \to \mathcal{D}$ is left Kan extended from $\mathcal{C}_1$ at X.

$(1''_X)$ The functor $F : \mathcal{C} \to \mathcal{D}$ is left Kan extended from $\mathcal{C}_{01}$ at X.

Let us regard the object X as fixed, and let F_X denote the composite map

$$(\mathcal{C}_{01} \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright} \hookrightarrow (\mathcal{C}_{/X})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}.$$

We wish to show that F_X is a colimit diagram in $\mathcal{D}$ if and only if its restriction to $(\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright}$ is a colimit diagram in $\mathcal{D}$. By virtue of Corollary 7.2.2.3, it will suffice to show that the inclusion map $\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_{/X} \hookrightarrow \mathcal{C}_{01} \times_{\mathcal{C}} \mathcal{C}_{/X}$ is right cofinal. This follows by applying Proposition 7.2.3.12 to the upper square of the pullback diagram

$$\begin{tikzcd}
\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_{/X} \arrow[r] \arrow[d] & \{1\} \arrow[d] \\
\mathcal{C}_{01} \times_{\mathcal{C}} \mathcal{C}_{/X} \arrow[r] \arrow[d] & \Delta^1 \arrow[d] \\
\mathcal{C}_{/X} \arrow[r, "\pi'"] & \Delta^2,
\end{tikzcd}$$

where π' denotes the composite map $\mathcal{C}_{/X} \to \mathcal{C} \to \Delta^2$ (which is a cocartesian fibration by virtue of Proposition 5.1.4.19). □

Proposition 7.3.8.19. *Let $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{D}$ be ∞-categories. Suppose we are given functors $F_0 : \mathcal{C}_0 \to \mathcal{D}$, $F_1 : \mathcal{C}_1 \to \mathcal{D}$, $G : \mathcal{C}_0 \to \mathcal{C}_1$, and $H : \mathcal{C}_1 \to \mathcal{C}_2$, where F_1 is a left Kan extension of F_0 along G. The following conditions are equivalent:* 031N

(1) *The functor F_1 admits a left Kan extension along H.*

(2) *The functor F_0 admits a left Kan extension along $H \circ G$.*

Proof. The implication (1) ⇒ (2) is immediate from Proposition 7.3.8.18. To prove the converse, assume that (2) is satisfied. Define $\mathcal{C}$ as in the proof of Proposition 7.3.8.18. Using the criterion of Corollary 7.3.5.8, we see that F_0 admits a left Kan extension $F : \mathcal{C} \to \mathcal{D}$. It follows from Proposition 7.3.2.11 that $F|_{\mathcal{C}_1}$ is a left Kan extension of F_0 along G, and is therefore isomorphic to F_1 (Remark 7.3.6.6). We may therefore assume without loss of generality that $F_1 = F|_{\mathcal{C}_1}$ (Remark 7.3.1.10). We will complete the proof by showing that $F_{12} = F|_{\mathcal{C}_{12}}$ is left Kan extended from $\mathcal{C}_1$, and therefore exhibits $F|_{\mathcal{C}_2}$ as a left Kan extension of F_1 along H (Proposition 7.3.2.11).

Fix an object $X \in \mathcal{C}_2$, and let F_X denote the composite map

$$(\mathcal{C}_{01} \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright} \hookrightarrow (\mathcal{C}_{/X})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}.$$

We wish to show that the composite map

$$(\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright} \hookrightarrow (\mathcal{C}_{01} \times_{\mathcal{C}} \mathcal{C}_{/X})^{\triangleright} \xrightarrow{F_X} \mathcal{D}$$

is a colimit diagram in $\mathcal{D}$. As in the proof of Proposition 7.3.8.18, the inclusion map $\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_{/X} \hookrightarrow \mathcal{C}_{01} \times_{\mathcal{C}} \mathcal{C}_{/X}$ is right cofinal. It will therefore suffice to show that F_X is a colimit diagram in $\mathcal{D}$ (Corollary 7.2.2.3). This is clear: by construction, the functor F is left Kan extended from the full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$, and is therefore also left Kan extended from the larger subcategory $\mathcal{C}_{01} \subseteq \mathcal{C}$ (Proposition 7.3.8.6). □

039X **Corollary 7.3.8.20.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$ be diagrams, and let $\alpha : F_0 \to F \circ \delta$ be a natural transformation which exhibits F as a left Kan extension of F_0 along δ (see Variant 7.3.1.5). Then:*

(1) *The diagram F admits a colimit in $\mathcal{D}$ if and only if F_0 admits a colimit in $\mathcal{D}$.*

(2) *Let X be an object of $\mathcal{D}$, let $\underline{X} : \mathcal{C} \to \mathcal{D}$ denote the constant functor taking the value X. Then a natural transformation $\beta : F \to \underline{X}$ exhibits X as a colimit of the diagram F if and only if the composite natural transformation*

$$F_0 \xrightarrow{\alpha} F \circ \delta \xrightarrow{\beta|_K} \underline{X}|_K$$

exhibits X as a colimit of the diagram F_0.

Proof. Using Corollary 4.1.3.3, we can choose an inner anodyne morphism $i : K \hookrightarrow \mathcal{K}$, where $\mathcal{K}$ is an ∞-category. Since $\mathcal{C}$ is an ∞-category, we extend δ and F_0 to functors $\overline{\delta} : \mathcal{K} \to \mathcal{C}$ and $\overline{F}_0 : \mathcal{K} \to \mathcal{D}$, respectively. Similarly, we can extend α to a natural transformation $\overline{\alpha} : \overline{F}_0 \to F \circ \overline{\delta}$. It follows from Proposition 7.3.1.14 that we $\overline{\alpha}$ exhibits F as a left Kan extension of $\overline{F}_0$ along $\overline{\delta}$. We may therefore replace K by $\mathcal{K}$ and thereby reduce to proving Corollary 7.3.8.20 in the special case where K is an ∞-category. In this case, assertion (1) is a special case of Proposition 7.3.8.19, and assertion (2) is a special case of Proposition 7.3.8.18 (see Example 7.3.1.7). □

Exercise 7.3.8.21. Show that the conclusions of Propositions 7.3.8.18 and 7.3.8.19 hold if we drop the assumption that the simplicial set $\mathcal{C}_0$ is an ∞-category. 031P

7.3.9 Relative Colimits for Cocartesian Fibrations

031Q Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let $D \in \mathcal{D}$ be an object, and suppose we are given a morphism

$$f : K^{\triangleright} \to \mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C} \subseteq \mathcal{C}.$$

If f is a U-colimit diagram in the ∞-category $\mathcal{C}$, then it is a colimit diagram in the ∞-category $\mathcal{C}_D$. The converse holds if U is a cartesian fibration (Corollary 7.1.5.20), but not in general. In this section, we study the dual situation where U is a cocartesian fibrations. Our main result asserts that f is a U-colimit diagram in $\mathcal{C}$ if and only if it is a *transport-stable* colimit diagram in the ∞-category $\mathcal{C}_D$: that is, for every morphism $e : D \to D'$ in $\mathcal{D}$, the covariant transport functor $e_! : \mathcal{C}_D \to \mathcal{C}_{D'}$ carries f to a colimit diagram in the ∞-category $\mathcal{C}_{D'}$ (Proposition 7.3.9.2). We begin by showing that the collection of U-colimit diagrams is stable under covariant transport.

Proposition 7.3.9.1. *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, let K be a simplicial set, and let $\alpha : F_0 \to F_1$ be a natural transformation between diagrams $F_0, F_1 : K^{\triangleright} \to \mathcal{C}$. Suppose that, for every vertex $x \in K^{\triangleright}$, the morphism $\alpha_x : F_0(x) \to F_1(x)$ is U-cocartesian. Then:* 031R

(1) *If F_0 is a U-colimit diagram, then F_1 is also a U-colimit diagram.*

(2) *If F_1 is a U-colimit diagram and the natural transformation α carries the cone point $v \in K^{\triangleright}$ to an isomorphism $\alpha_v : F_0(v) \to F_1(v)$, then F_0 is a U-colimit diagram.*

Proof. Using Corollary 4.1.3.3, we can choose an inner anodyne morphism $i : K \hookrightarrow \mathcal{K}$, where $\mathcal{K}$ is an ∞-category. It follows that the induced map $i^{\triangleright} : K^{\triangleright} \hookrightarrow \mathcal{K}^{\triangleright}$ is also inner anodyne (Example 4.3.6.7), so that the restriction map $\operatorname{Fun}(\mathcal{K}^{\triangleright}, \mathcal{C}) \to \operatorname{Fun}(K^{\triangleright}, \mathcal{D})$ is a trivial Kan fibration of simplicial sets (Proposition 1.5.7.6). We can therefore lift α to a natural transformation $\overline{\alpha} : \overline{F}_0 \to \overline{F}_1$ between natural transformations $\overline{F}_0, \overline{F}_1 : \mathcal{K}^{\triangleright} \to \mathcal{C}$. Since $i^{\triangleright}$ is bijective on vertices, the natural transformation $\overline{\alpha}$ carries each object of $\mathcal{K}^{\triangleright}$ to a U-cocartesian morphism of $\mathcal{C}$. The morphism $i^{\triangleright}$ is right cofinal (Corollary 7.2.1.13), so Corollary 7.2.2.2 guarantees that F_0 is a U-colimit diagram if and only if $\overline{F}_0$ is a U-colimit diagram. Similarly, F_1 is a U-colimit diagram if and only if $\overline{F}_1$ is a U-colimit diagram. We may therefore replace α by $\overline{\alpha}$ in the statement of Proposition 7.3.9.1, and thereby reduce to the case where $K = \mathcal{K}$ is an ∞-category.

Let us identify α with a functor of ∞-categories $F : \Delta^1 \times \mathcal{K}^{\triangleright} \to \mathcal{C}$. For each object $x \in \mathcal{K}^{\triangleright}$, we can regard $\Delta^1 \times \{x\}$ with a morphism $e_x : (0, x) \to (1, x)$ in the ∞-category

$\Delta^1 \times \mathcal{K}^{\triangleright}$. By construction, the functor F carries each e_x to the U-cocartesian morphism α_x of $\mathcal{C}$. By virtue of Proposition 4.6.7.22, e_x is final when viewed as an object of the ∞-category

$$(\{0\} \times \mathcal{K}^{\triangleright}) \times_{(\Delta^1 \times \mathcal{K}^{\triangleright})} (\Delta^1 \times \mathcal{K}^{\triangleright})_{/(1,x)} \simeq (\mathcal{K}^{\triangleright})_{/x},$$

so that F is U-left Kan extended from $\{0\} \times \mathcal{K}^{\triangleright}$ at $(1,x)$ (Corollary 7.2.2.5). Allowing the object x to vary, we see that the functor F is U-left Kan extended from $\{0\} \times \mathcal{K}^{\triangleright}$.

We now prove (1). Suppose that F_0 is a U-colimit diagram. Then F_0 is U-left Kan extended from $\mathcal{K}$ (Example 7.3.3.9). Applying Proposition 7.3.8.6, we see that the functor F is U-left Kan extended from $\{0\} \times \mathcal{K}$, and therefore from the larger subcategory $\Delta^1 \times \mathcal{K} \subseteq \Delta^1 \times \mathcal{K}^{\triangleright}$. It follows that the composite map

$$(\Delta^1 \times \mathcal{K}) \star \{(1,v)\} \hookrightarrow \Delta^1 \times (\mathcal{K} \star \{v\}) \xrightarrow{F} \mathcal{C}$$

is a U-colimit diagram. Since the inclusion map $\{1\} \times \mathcal{K} \hookrightarrow \Delta^1 \times \mathcal{K}$ is right cofinal (Proposition 7.2.1.3), Corollary 7.2.2.2 guarantees that $F_1 = F|_{\{1\} \times \mathcal{K}^{\triangleright}}$ is also a U-colimit diagram.

We now prove (2). Let $\pi : \mathcal{K}^{\triangleright} \to \Delta^1$ be the functor carrying $\mathcal{K}$ to the vertex $0 \in \Delta^1$ and the cone point $v \in \mathcal{K}^{\triangleright}$ to the vertex $1 \in \Delta^1$, and let $G : \mathcal{K}^{\triangleright} \to \mathcal{C}$ be the functor given by the composition

$$\mathcal{K}^{\triangleright} \xrightarrow{(\pi, \mathrm{id})} \Delta^1 \times \mathcal{K}^{\triangleright} \xrightarrow{F} \mathcal{C}.$$

Note that there is a natural transformation $\beta : F_0 \to G$ which is the identity when restricted to $\mathcal{K}$ and which carries the cone point v to the morphism $\alpha_v : F_0(v) \to F_1(v) = G(v)$. If α_v is an isomorphism, then the natural transformation β is also an isomorphism (Theorem 4.4.4.4). Consequently, to show that F_0 is a U-colimit diagram, it will suffice to show that G is a U-colimit diagram (Proposition 7.1.5.13). Arguing as above, we see that the functor $F|_{\Delta^1 \times \mathcal{K}}$ is U-left Kan extended from the full subcategory $\{0\} \times \mathcal{K} \subseteq \Delta^1 \times \mathcal{K}$. Applying Proposition 7.3.8.1, we see that G is a U-colimit diagram if and only if the composite map

$$(\Delta^1 \times \mathcal{K}) \star \{(1,v)\} \hookrightarrow \Delta^1 \times (\mathcal{K} \star \{v\}) \xrightarrow{F} \mathcal{C}$$

is a U-colimit diagram. By virtue of Corollary 7.2.2.2, this is equivalent to the requirement that F_1 is a U-colimit diagram. □

031S **Proposition 7.3.9.2.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a locally cocartesian fibration of ∞-categories, let $D \in \mathcal{D}$ be an object, and let $f : K^{\triangleright} \to \mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ be a diagram. Then f is a U-colimit diagram in the ∞-category $\mathcal{C}$ if and only if it satisfies the following condition:*

(∗) *Let $e : D \to D'$ be a morphism in the ∞-category $\mathcal{D}$ and let $e_! : \mathcal{C}_D \to \mathcal{C}_{D'}$ be the covariant transport functor of Notation 5.2.2.9. Then $(e_! \circ f) : K^{\triangleright} \to \mathcal{C}_{D'}$ is a colimit diagram in the ∞-category*

Example 7.3.9.3. In the situation of Proposition 7.3.9.2, suppose that that U is also a cartesian fibration. Then, for every morphism $e : D \to D'$ of $\mathcal{D}$, the covariant transport functor $e_!$ has a right adjoint e^*, given by contravariant transport along e (Proposition 6.2.3.4). In particular, the functor $e_!$ automatically preserves K-indexed colimits (Corollary 7.1.3.21). We therefore recover the criterion of Corollary 7.1.5.20: the morphism f is a U-colimit diagram in $\mathcal{C}$ if and only if it is a colimit diagram in the ∞-category $\mathcal{C}_D$. 031T

Proof of Proposition 7.3.9.2. For every morphism $e : D \to D'$ in $\mathcal{D}$, we can choose a natural transformation $\alpha : f \to e_! \circ f$ carrying each vertex of $K^{\triangleright}$ to a U-cocartesian morphism of $\mathcal{C}$. It follows from Proposition 7.3.9.1 that if f is a U-colimit diagram, then $e_! \circ f$ is also a U-colimit diagram, and therefore a colimit diagram in the ∞-category $\mathcal{C}_{D'}$ (Corollary 7.1.5.20). This proves the necessity of condition $(*)$. For the converse, suppose that f satisfies condition $(*)$; we wish to show that f is a U-colimit diagram. By virtue of Proposition 7.1.5.12, this is equivalent to the assertion that for every object $C \in \mathcal{C}$, the diagram of Kan complexes

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$$\begin{tikzcd}
\operatorname{Hom}_{\operatorname{Fun}(K^{\triangleright},\mathcal{C})}(f,\underline{C}) \arrow[r] \arrow[d] & \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(f|_K,\underline{C}|_K) \arrow[d] \\
\operatorname{Hom}_{\operatorname{Fun}(K^{\triangleright},\mathcal{D})}(U\circ f,U\circ\underline{C}) \arrow[r] & \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{D})}(U\circ f|_K,U\circ\underline{C}|_K)
\end{tikzcd} \tag{7.40}$$

is a homotopy pullback square, where $\underline{C} \in \operatorname{Fun}(K^{\triangleright},\mathcal{C})$ is the constant diagram taking the value C. Since U is an inner fibration, the vertical maps in this diagram are Kan fibrations (Proposition 4.6.1.21 and Corollary 4.1.4.3). Using the criterion of Example 3.4.1.4, it will suffice to show that for every vertex $u \in \operatorname{Hom}_{\operatorname{Fun}(K^{\triangleright},\mathcal{D})}(U\circ f, U\circ\underline{C})$, the induced map

$$\begin{tikzcd}
\{u\} \times_{\operatorname{Hom}_{\operatorname{Fun}(K^{\triangleright},\mathcal{D})}(U\circ f,U\circ\underline{C})} \operatorname{Hom}_{\operatorname{Fun}(K^{\triangleright},\mathcal{C})}(f,\underline{C}) \arrow[d, "\theta_u"] \\
\{u\} \times_{\operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{D})}(U\circ f|_K,U\circ\underline{C}|_K)} \operatorname{Hom}_{\operatorname{Fun}(K,\mathcal{C})}(f|_K,\underline{C}|_K)
\end{tikzcd}$$

is a homotopy equivalence of Kan complexes. Set $D' = U(C)$, so that u can be identified with a morphism of simplicial sets $K^{\triangleright} \to \operatorname{Hom}_{\mathcal{D}}(D,D')$, and the condition that θ_u is a homotopy equivalence depends only on the homotopy class of u. Since the simplicial set $K^{\triangleright}$ is weakly contractible (Example 4.3.7.11), we may assume without loss of generality that $u : K^{\triangleright} \to \operatorname{Hom}_{\mathcal{D}}(D,D')$ is the constant map taking the value e, for some morphism

$e : D \to D'$ in $\mathcal{D}$. In this case, we can use Proposition 5.1.3.11 to replace θ_u by the restriction map

$$\mathrm{Hom}_{\mathrm{Fun}(K^{\triangleright}, \mathcal{C}_{D'})}(e_! \circ f, \underline{D}) \to \mathrm{Hom}_{\mathrm{Fun}(K, \mathcal{C}_{D'})}(e_! \circ f|_K, \underline{D}|_K),$$

which is a homotopy equivalence by virtue of assumption $(*)$ (see Proposition 7.1.5.12). □

Using Proposition 7.3.9.2, we obtain a relative version of Corollary 7.2.3.5:

`031V` **Corollary 7.3.9.4.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration of ∞-categories, let K be a weakly contractible simplicial set, and let $\overline{f} : K^{\triangleright} \to \mathcal{C}$ be a diagram. The following conditions are equivalent:*

(1) *The diagram $\overline{f}$ carries each edge of $K^{\triangleright}$ to a U-cocartesian morphism of $\mathcal{C}$.*

(2) *The restriction $f = \overline{f}|_K$ carries each edge of K to a U-cocartesian morphism of $\mathcal{C}$, and $\overline{f}$ is a U-colimit diagram.*

Proof. Without loss of generality, we may assume that f carries each edge of K to a U-cocartesian morphism of $\mathcal{C}$. Let $\pi : \Delta^1 \times K^{\triangleright} \to K^{\triangleright}$ be the morphism which is the identity on $\{0\} \times K^{\triangleright}$ and which carries $\{1\} \times K^{\triangleright}$ to the cone point $v \in K^{\triangleright}$. Set $C = f(v) \in \mathcal{C}$ and $D = U(C) \in \mathcal{D}$. Proposition 5.2.1.3 guarantees that the lifting problem

$$\begin{tikzcd}
\{0\} \times K^{\triangleright} \arrow[r, "\overline{f}"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\Delta^1 \times K^{\triangleright} \arrow[r, "U \circ \overline{f} \circ \pi"] \arrow[ur, dashed, "\alpha"] & \mathcal{D}
\end{tikzcd}$$

admits a solution $\alpha : \Delta^1 \times K^{\triangleright} \to \mathcal{C}$ which carries $\Delta^1 \times \{x\}$ to a U-cocartesian morphism of $\mathcal{C}$, for each vertex $x \in K^{\triangleright}$. Set $\overline{g} = \alpha|_{\{1\} \times K^{\triangleright}}$, which we regard as a morphism from $K^{\triangleright}$ to the ∞-category $\mathcal{C}_D$, and let us identify α with a natural transformation from $\overline{f}$ to $\overline{g}$. Note that $\alpha_v : \overline{f}(v) \to \overline{g}(v)$ is a U-cocartesian morphism of $\mathcal{C}$ satisfying $U(\alpha_v) = \mathrm{id}_D$, and is therefore an isomorphism (Proposition 5.1.1.8). Applying Proposition 7.3.9.1, we can reformulate (2) as follows:

(2′) The morphism $\overline{g} : K^{\triangleright} \to \mathcal{C}_D$ is a U-colimit diagram in $\mathcal{C}$.

Set $g = \overline{g}|_K$. For every edge $u : x \to y$ of K, we have a commutative diagram

$$\begin{tikzcd}
f(x) \arrow[r, "f(u)"] \arrow[d, "\alpha_x"] & f(y) \arrow[d, "\alpha_y"] \\
g(x) \arrow[r, "g(u)"] & g(y)
\end{tikzcd}$$

where $f(u)$, α_x, and α_y are U-cocartesian. Applying Corollary 5.1.2.4, we deduce that $g(u)$ is U-cocartesian when viewed as a morphism of $\mathcal{C}$, and is therefore an isomorphism in the ∞-category $\mathcal{C}_D$ (Proposition 5.1.1.8). Similarly, for every vertex $x \in K$, the unique edge $c_x : x \to v$ of $K^{\triangleright}$ determines a commutative diagram

$$\begin{tikzcd}
f(x) \arrow[r, "\overline{f}(c_x)"] \arrow[d, "\alpha_x"] & \overline{f}(v) \arrow[d, "\alpha_v"] \\
g(x) \arrow[r, "\overline{g}(c_x)"] & \overline{g}(v),
\end{tikzcd}$$

where α_x is U-cocartesian and α_v is an isomorphism. Combining Corollary 5.1.2.4, Corollary 5.1.2.5, and Proposition 5.1.1.8, we see that $\overline{f}(c_x)$ is U-cocartesian if and only if $\overline{g}(c_x)$ is an isomorphism in the ∞-category $\mathcal{C}_D$. We can therefore reformulate condition (1) as follows:

($1'$) The diagram $\overline{g}$ carries each edge of $K^{\triangleright}$ to an isomorphism in the ∞-category $\mathcal{C}_D$.

By virtue of Corollary 7.2.3.5, ($1'$) is equivalent to the requirement that $\overline{g}$ is a colimit diagram in the ∞-category $\mathcal{C}_D$. In particular, the implication ($2'$) $\Rightarrow$ ($1'$) follows from Corollary 7.1.5.20. To prove the converse, it will suffice to show that condition ($1'$) is satisfied, then for every morphism $e : D \to D'$ in $\mathcal{D}$, the covariant transport functor $e_! : \mathcal{C}_D \to \mathcal{C}_{D'}$ carries $\overline{g}$ to a colimit diagram in the ∞-category $\mathcal{C}_{D'}$ (Proposition 7.3.9.2). This follows immediately from Corollary 7.2.3.5 (applied to the composite diagram $K^{\triangleleft} \xrightarrow{\overline{g}} \mathcal{C}_D \xrightarrow{e_!} \mathcal{C}_{D'}$). $\square$

The criterion of Proposition 7.3.9.2 has a counterpart for the *existence* of U-colimit diagrams.

Proposition 7.3.9.5. *Let $U : \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration of ∞-categories, and suppose we are given a lifting problem* 031W

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$$\begin{tikzcd}
K \arrow[r, "f_0"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
K^{\triangleright} \arrow[r, "g"] \arrow[ur, dashed, "\overline{f}_0"] & \mathcal{D}
\end{tikzcd} \tag{7.41}$$

Let $v \in K^{\triangleright}$ be the cone point and set $D = g(v)$. Then there exists a diagram $f_1 : K \to \mathcal{C}_D \subseteq \mathcal{C}$ and a natural transformation $\alpha : f_0 \to f_1$ which carries each vertex $x \in K$ to a U-cocartesian morphism $\alpha_x : f_0(x) \to f_1(x)$ of $\mathcal{C}$, where$U \circ \alpha$ is given by the composition $\Delta^1 \times K \xrightarrow{c} K^{\triangleright} \xrightarrow{g} \mathcal{D}$. Moreover, the lifting problem (7.41) admits a solution $\overline{f}_0 : K^{\triangleright} \to \mathcal{C}$ which is a U-colimit diagram if and only if the following pair of conditions is satisfied:

(1) *The diagram f_1 admits a colimit $\overline{f}_1 : K^{\triangleright} \to \mathcal{C}_D$ in the ∞-category $\mathcal{C}_D$.*

(2) *Let $e : D \to D'$ be a morphism in the ∞-category $\mathcal{D}$ and let $e_! : \mathcal{C}_D \to \mathcal{C}_{D'}$ be the covariant transport functor of Notation 5.2.2.9. Then $(e_! \circ \overline{f}_1) : K^{\triangleright} \to \mathcal{C}_{D'}$ is a colimit diagram in the ∞-category $\mathcal{C}_{D'}$.*

Proof. The existence (and essential uniqueness) of the diagram f_1 and the natural transformation $\alpha : f_0 \to f_1$ follow from Proposition 5.2.1.3. Let us first show that conditions (1) and (2) are necessary. Suppose that the lifting problem (7.41) admits a solution $\overline{f}_0 : K^{\triangleright} \to \mathcal{C}$ which is a U-colimit diagram. Using Proposition 5.2.1.3, we can extend f_1 to a diagram $\overline{f}_1 : K^{\triangleright} \to \mathcal{C}_D$ and α to a natural transformation $\overline{\alpha} : \overline{f}_0 \to \overline{f}_1$ which carries each vertex $x \in K^{\triangleright}$ to a U-cocartesian morphism $\overline{\alpha}_x : \overline{f}_0(x) \to \overline{f}_1(x)$. Proposition 7.3.9.1 guarantees that $\overline{f}_1$ is a U-colimit diagram in the ∞-category $\mathcal{C}$, and therefore satisfies conditions (1) and (2) by virtue of Proposition 7.3.9.2.

Now suppose that conditions (1) and (2) are satisfied. Let $\overline{f}_1 : K^{\triangleright} \to \mathcal{C}_D$ be a colimit diagram extending f_1. It follows from (2) that $\overline{f}_1$ is a U-colimit diagram in the ∞-category $\mathcal{C}$. Let $\pi : (\Delta^1 \times K)^{\triangleright} \to K^{\triangleright}$ denote the morphism which is the identity when restricted to $\{0\} \times K$, and which carries $(\{1\} \times K)^{\triangleright}$ to the cone point of $K^{\triangleright}$. Since the inclusion map $\{1\} \times K \hookrightarrow \Delta^1 \times K$ is right cofinal (Proposition 7.2.1.3), Proposition 7.2.2.9 guarantees that the lifting problem

$$\begin{tikzcd}
\Delta^1 \times K \arrow[r, "\alpha"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
(\Delta^1 \times K)^{\triangleright} \arrow[r, "g\circ\pi"] \arrow[ur, dashed, "\overline{\alpha}"] & \mathcal{D}
\end{tikzcd}$$

admits a solution $\overline{\alpha} : (\Delta^1 \times K)^{\triangleright} \to \mathcal{C}$ which is a U-colimit diagram. Note that in this case $\overline{f}'_1 = \overline{\alpha}|_{(\{1\}\times K)^{\triangleright}}$ is also a U-colimit diagram (Corollary 7.2.2.2). Setting $\overline{f}_0 = \overline{\alpha}|_{(\{0\}\times K)^{\triangleright}}$, we note that $\overline{\alpha}$ determines a natural transformation of functors $\overline{f}_0 \to \overline{f}'_1$ which carries each vertex of x to a U-cocartesian morphism of $\mathcal{C}$ and carries the cone point to an identity morphism of $\mathcal{C}$. Applying the criterion of Proposition 7.3.9.1, we conclude that $\overline{f}_0$ is a U-colimit diagram which solves the lifting problem (7.41). □

031Y **Corollary 7.3.9.6.** *Let $U : \mathcal{C} \to \mathcal{D}$ be a cocartesian fibration of ∞-categories and let K be a simplicial set. The following conditions are equivalent:*

(1) *For every object $D \in \mathcal{D}$, the ∞-category $\mathcal{C}_D = \{D\} \times_{\mathcal{D}} \mathcal{C}$ admits K-indexed colimits. Moreover, for every morphism $e : D \to D'$ in $\mathcal{D}$, the covariant transport functor $e_! : \mathcal{C}_D \to \mathcal{C}_{D'}$ preserves K-indexed colimits.*

(2) *Every lifting problem*

031Z
$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\ K^{\triangleright} \arrow[r] \arrow[ur, dashed, "\overline{f}"] & \mathcal{D} \end{tikzcd} \tag{7.42}$$

admits a solution $\overline{f} : K^{\triangleright} \to \mathcal{C}$ *which is a* U*-colimit diagram.*

Proof. The implication (1) ⇒ (2) follows immediately from Proposition 7.3.9.5. Conversely, suppose that (2) is satisfied. For each object $D \in \mathcal{D}$, condition (2) guarantees that every diagram $f : K \to \mathcal{C}_D$ admits an extension $\overline{f} : K^{\triangleright} \to \mathcal{C}_D$ which is a U-colimit diagram in $\mathcal{C}$. In particular, $\overline{f}$ is a colimit diagram in $\mathcal{C}_D$ (Corollary 7.1.5.20) having the property that for every morphism $e : D \to D'$ in $\mathcal{D}$, the composition $e_! \circ \overline{f}$ is a colimit diagram in $\mathcal{C}_{D'}$ (Proposition 7.3.9.2). To complete the proof, we observe that if $\overline{f}' : K^{\triangleright} \to \mathcal{C}_D$ is any other colimit diagram satisfying $\overline{f}'|_K = f$, then $\overline{f}'$ is isomorphic to $\overline{f}$ as an object of the ∞-category $\operatorname{Fun}(K^{\triangleright}, \mathcal{C}_D)$, so that $e_! \circ \overline{f}'$ is also a colimit diagram in $\mathcal{C}_{D'}$ (Corollary 7.1.2.14). □

Corollary 7.3.9.7. *Let* $U : \mathcal{D} \to \mathcal{E}$ *be a cocartesian fibration of* ∞*-categories, let* $\mathcal{C}$ *be an* ∞*-category, and let* $\mathcal{C}^0 \subseteq \mathcal{C}$ *be a full subcategory. Suppose that the following conditions are satisfied:* 0320

- *For every object* $C \in \mathcal{C}$ *and every object* $E \in \mathcal{E}$*, the* ∞*-category* $\mathcal{D}_E = \{E\} \times_{\mathcal{E}} \mathcal{D}$ *admits* $\mathcal{C}^0_{/C}$*-indexed colimits.*
- *For every object* $C \in \mathcal{C}$ *and every morphism* $e : E \to E'$ *in* $\mathcal{E}$*, the covariant transport functor* $e_! : \mathcal{D}_E \to \mathcal{D}_{E'}$ *preserves* $\mathcal{C}^0_{/C}$*-indexed colimits.*

Then every lifting problem

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d, "U"] \\ \mathcal{C} \arrow[r] \arrow[ur, dashed, "\overline{F}"] & \mathcal{E} \end{tikzcd}$$

admits a solution $\overline{F} : \mathcal{C} \to \mathcal{D}$ *which is* U*-left Kan extended from* $\mathcal{C}^0$*.*

Proof. Combine Proposition 7.3.5.5 with Corollary 7.3.9.6. □

7.4 Limits and Colimits of ∞-Categories

02SX Recall that the collection of (small) ∞-categories can be organized into a (large) ∞-category $\mathcal{QC}$ (see Construction 5.5.4.1). Our goal in this section is to study limits and colimits in the ∞-category $\mathcal{QC}$. Fix a small ∞-category $\mathcal{C}$, and suppose we are given a diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$. We will show that the diagram $\mathscr{F}$ admits both a limit $\varprojlim(\mathscr{F})$ and a colimit $\varinjlim(\mathscr{F})$, which can be described explicitly in terms of the ∞-category of elements $\int_{\mathcal{C}} \mathscr{F}$ introduced in Definition 5.6.2.1:

(1) Let $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ be the forgetful functor, and let $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \int_{\mathcal{C}} \mathscr{F})$ denote the full subcategory of $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}, \int_{\mathcal{C}} \mathscr{F})$ spanned by those functors $F : \mathcal{C} \to \int_{\mathcal{C}} \mathscr{F}$ which satisfy $U \circ F = \mathrm{id}_{\mathcal{C}}$ and which carry each morphism of $\mathcal{C}$ to a U-cocartesian morphism of $\int_{\mathcal{C}} \mathscr{F}$. In §7.4.1, we show that the ∞-category $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \int_{\mathcal{C}} \mathscr{F})$ is a limit of the diagram $\mathscr{F}$ (Corollary 7.4.1.10).

(2) Let W be the collection of all U-cocartesian morphisms of $\int_{\mathcal{C}} \mathscr{F}$, and let $(\int_{\mathcal{C}} \mathscr{F})[W^{-1}]$ denote a localization of $\int_{\mathcal{C}} \mathscr{F}$ with respect to W (Definition 6.3.1.9). In §7.4.3, we show that $(\int_{\mathcal{C}} \mathscr{F})[W^{-1}]$ is a colimit of the diagram $\mathscr{F}$ (Corollary 7.4.3.12).

For many applications, it is not enough to describe the limit $\varprojlim(\mathscr{F})$ and colimit $\varinjlim(\mathscr{F})$ as *abstract* ∞-categories: we also need to understand their relationship to the diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$. In other words, we would like to have criteria which can be used to detect when an extension $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathcal{QC}$ is a limit diagram, and when an extension $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathcal{QC}$ is a colimit diagram. To formulate these criteria, it will be convenient to slightly shift our perspective. Fix a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ having covariant transport representation $\mathscr{F}$ (that is, a cocartesian fibration which is equivalent to the forgetful functor $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$).

- Suppose U is obtained as the pullback of a cocartesian fibration $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleleft}$, and let $\overline{\mathcal{E}}_{\mathbf{0}}$ denote the fiber of $\overline{U}$ over the cone point $\mathbf{0} \in \mathcal{C}^{\triangleleft}$. In §7.4.1, we introduce a map

  $$\mathrm{Df} : \overline{\mathcal{E}}_{\mathbf{0}} \to \operatorname{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}),$$

  which we will refer to as the *covariant diffraction functor* (Construction 7.4.1.3). Roughly speaking, it is characterized by the requirement that for every object $X \in \overline{\mathcal{E}}_{\mathbf{0}}$ and every object $C \in \mathcal{C}$, there is a $\overline{U}$-cocartesian morphism $X \to \mathrm{Df}(X)(C)$ (depending functorially on X and C).

- Suppose U is obtained as the pullback of a cocartesian fibration $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$, and let $\overline{\mathcal{E}}_{\mathbf{1}}$ denote the fiber of $\overline{U}$ over the cone point $\mathbf{1} \in \mathcal{C}^{\triangleright}$. In §7.4.3, we introduce a map

  $$\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}},$$

which we will refer to as the *covariant refraction functor* (Definition 7.4.3.1). Roughly speaking, it is characterized by the requirement that for every object $X \in \mathcal{E}$, there is a $\overline{U}$-cocartesian morphism $X \to \mathrm{Rf}(X)$ (depending functorially on X).

We will deduce (1) and (2) from the following more precise assertions:

Diffraction Criterion: Suppose we are given a pullback diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. Then the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{E}^{\triangleleft}} : \mathcal{C}^{\triangleleft} \to \mathcal{QC}$ is a limit diagram (in the ∞-category $\mathcal{QC}$) if and only if the covariant diffraction functor $\mathrm{Df} : \overline{\mathcal{E}}_{\mathbf{0}} \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is a fully faithful embedding, whose essential image is the the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ of cocartesian sections of U (see Theorem 7.4.1.1 and Remark 7.4.1.5).

Refraction Criterion: Suppose we are given a pullback diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. Then the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{E}^{\triangleright}} : \mathcal{C}^{\triangleright} \to \mathcal{QC}$ is a colimit diagram (in the ∞-category $\mathcal{QC}$) if and only if the covariant refraction functor $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to the collection of U-cocartesian morphisms (Theorem 7.4.3.6).

We will establish the diffraction and refraction criteria in §7.4.2 and §7.4.3, respectively. In §7.4.5, we restrict our attention to the special case where $U : \mathcal{E} \to \mathcal{C}$ is a left fibration, and apply the results described above to describe limits and colimits in the ∞-category $\mathcal{S}$ of spaces.

Remark 7.4.0.1. In the outline above, we have implicitly suggested that $\mathcal{C}$ is an ∞-category. This is not important: all of the results of this section can be applied to diagrams $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ indexed by an arbitrary (small) simplicial set $\mathcal{C}$. 02SY

02SZ **Remark 7.4.0.2.** For any cocartesian fibration $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleleft}$, the associated covariant diffraction functor $\mathrm{Df} : \overline{\mathcal{E}}_{\mathbf{0}} \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ automatically factors through the full subcategory $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ (see Construction 7.4.1.3). Similarly, for any cocartesian fibration $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$, the covariant refraction functor $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ automatically carries U-cocartesian edges of $\mathcal{E}$ to isomorphisms in the ∞-category $\overline{\mathcal{E}}_{\mathbf{1}}$ (Remark 7.4.3.5).

7.4.1 Limits of ∞-Categories

02T0 Let $\mathcal{QC}$ denote the ∞-category of (small) ∞-categories (Construction 5.5.4.1). Our goal in this section (and §7.4.2) is to show that the ∞-category $\mathcal{QC}$ admits small limits (Corollary 7.4.1.11). In fact, we will prove something more precise: if $\mathcal{C}$ is a small ∞-category, then the limit of any diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ can be realized as explicitly as a full subcategory of the ∞-category of sections of the cocartesian fibration $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ of Proposition 5.6.2.2 (Corollary 7.4.1.10).

Recall that, if $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ are cocartesian fibrations of simplicial sets, then $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}')$ denotes the full subcategory of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}')$ spanned by those functors $F : \mathcal{E} \to \mathcal{E}'$ which carry U-cocartesian edges of $\mathcal{E}$ to U'-cocartesian edges of $\mathcal{E}'$ (Notation 5.3.1.10). Our main result can be stated as follows:

02T8 **Theorem 7.4.1.1** (Diffraction Criterion)**.** *Suppose we are given a pullback diagram of small simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. The following conditions are equivalent:

(1) *The restriction map*

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$$

is an equivalence of ∞-categories.

(2) *The covariant transport representation*

$$\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}} : \mathcal{C}^{\triangleleft} \to \mathcal{QC}$$

of Notation 5.6.5.14 is a limit diagram in the ∞-category $\mathcal{QC}$.

02T9 **Remark 7.4.1.2.** In the situation of Theorem 7.4.1.1, the restriction map $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is automatically an isofibration of ∞-categories (Remark 5.3.1.18). Using Proposition 4.5.5.20, we see that condition (1) of Theorem 7.4.1.1 is equivalent to the following *a priori* stronger condition:

$(1')$ The restriction map

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$$

is a trivial Kan fibration of simplicial sets.

Construction 7.4.1.3 (Covariant Diffraction)**.** Suppose we are given a pullback diagram of simplicial sets 02TD

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. Let $\overline{\mathcal{E}}_{\mathbf{0}}$ denote the fiber of $\overline{U}$ over the cone point $\mathbf{0} \in \mathcal{C}^{\triangleleft}$. We then have restriction maps

$$\overline{\mathcal{E}}_{\mathbf{0}} \xleftarrow{\mathrm{ev}} \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \xrightarrow{\theta} \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}),$$

where ev is a trivial Kan fibration (Corollary 5.3.1.23). Composing θ with a section of ev, we obtain a functor of ∞-categories $\mathrm{Df} : \overline{\mathcal{E}}_{\mathbf{0}} \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ which is well-defined up to isomorphism. We will refer to Df as the *covariant diffraction functor* associated to the cocartesian fibration $\overline{U}$.

Remark 7.4.1.4. In the situation of Construction 7.4.1.3, let $C \in \mathcal{C}$ be a vertex and let $\mathrm{ev}_C : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \to \mathcal{E}_C$ be the evaluation functor, given on objects by $\mathrm{ev}_C(F) = F(C)$. Then the composition 02TE

$$\overline{\mathcal{E}}_{\mathbf{0}} \xrightarrow{\mathrm{Df}} \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \xrightarrow{\mathrm{ev}_C} \mathcal{E}_C$$

is given by covariant transport along the unique edge $\mathbf{0} \to C$ of $\mathcal{C}^{\triangleleft}$.

Remark 7.4.1.5. Suppose we are given a pullback diagram of small simplicial sets 02TF

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft}.
\end{tikzcd}$$

Then the covariant diffraction functor $\mathrm{Df} : \overline{\mathcal{E}}_{\mathbf{0}} \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ of Construction 7.4.1.3 is an equivalence of ∞-categories if and only if the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}} : \mathcal{C}^{\triangleleft} \to \mathcal{QC}$ is a limit diagram in the ∞-category $\mathcal{QC}$ (this is a restatement of Theorem 7.4.1.1).

We now show that there exists a good supply of cocartesian fibrations which satisfy the hypotheses of Theorem 7.4.1.1.

02TG **Proposition 7.4.1.6.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then there exists a pullback diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where $\overline{U}$ is a cocartesian fibration and the restriction map

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$$

is an equivalence of ∞-categories.

Proof. Let $\mathrm{ev} : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C} \to \mathcal{E}$ denote the evaluation morphism (given on vertices by the formula $\mathrm{ev}(F, C) = F(C)$), and let

$$\mathcal{E}' = (\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C}) \star_{\mathcal{E}} \mathcal{E}$$

denote the relative join of Construction 5.2.3.1. Note that we have a canonical map

$$U' : \mathcal{E}' = (\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C}) \star_{\mathcal{E}} \mathcal{E} \to \mathcal{C} \star_{\mathcal{C}} \mathcal{C} \simeq \Delta^1 \times \mathcal{C}\,.$$

Let $\pi : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C} \to \mathcal{C}$ be given by projection onto the second factor. Note that π is a cocartesian fibration, and that an edge of the product $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C}$ is π-cocartesian if and only if its image in $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is an isomorphism. It follows that the ev carries π-cocartesian edges of $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C}$ to U-cocartesian edges of $\mathcal{E}$. Applying Lemma 5.2.3.17, we deduce that U' is a cocartesian fibration. By construction, we can identify $\mathcal{E}$ with the inverse image of $\{1\} \times \mathcal{C}$ under U'.

Let $\mathcal{E}''$ denote the pushout

$$(\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C}^{\triangleleft}) \coprod_{(\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C}^{\triangleleft})} \mathcal{E}'\,.$$

Amalgamating U' with the projection map $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \times \mathcal{C}^{\triangleleft} \to \mathcal{C}^{\triangleleft}$, we obtain a morphism of simplicial sets $U'' : \mathcal{E}'' \to K$, where K denotes the pushout $(\{0\} \times \mathcal{C})^{\triangleleft} \coprod_{(\{0\} \times \mathcal{C})} (\Delta^1 \times \mathcal{C})$. It follows from Proposition 5.1.4.7 that U'' is also a cocartesian fibration.

Let us abuse notation by identifying K with its image in the simplicial set $(\Delta^1 \times \mathcal{C})^{\triangleleft}$. Since the inclusion map $\{0\} \times \mathcal{C} \hookrightarrow \Delta^1 \times \mathcal{C}$ is left anodyne (Proposition 4.2.5.3), the inclusion

$K \hookrightarrow (\Delta^1 \times \mathcal{C})^{\triangleleft}$ is inner anodyne (Example 4.3.6.5). Applying Proposition 5.6.7.2, we can write U'' as the pullback of a cocartesian fibration $U''' : \mathcal{E}''' \to (\Delta^1 \times \mathcal{C})^{\triangleleft}$. We then have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E}'' \arrow[r] \arrow[d, "U''"] & \mathcal{E}''' \arrow[d, "U'''"] \\
\{1\} \times \mathcal{C} \arrow[r] & \Delta^1 \times \mathcal{C} \arrow[r] & K \arrow[r] & (\Delta^1 \times \mathcal{C})^{\triangleleft},
\end{tikzcd}$$

where each square is a pullback and each vertical map is a cocartesian fibration. Let $\overline{\mathcal{E}}$ denote the pullback $(\{1\} \times \mathcal{C})^{\triangleleft} \times_{(\Delta^1 \times \mathcal{C})^{\triangleleft}} \mathcal{E}'''$, so that U''' restricts to a cocartesian fibration $\overline{U} : \overline{\mathcal{E}} \to (\{1\} \times \mathcal{C})^{\triangleleft}$. We will complete the proof by showing that the commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\{1\} \times \mathcal{C} \arrow[r] & (\{1\} \times \mathcal{C})^{\triangleleft}
\end{tikzcd}$$

satisfies the requirements of Proposition 7.4.1.6.

For every simplicial subset $A \subseteq (\Delta^1 \times \mathcal{C})^{\triangleleft}$, let $\mathcal{D}(A)$ denote the ∞-category

$$\mathrm{Fun}^{\mathrm{CCart}}_{/A}(A, A \times_{(\Delta^1 \times \mathcal{C})^{\triangleleft}} \mathcal{E}''').$$

Let $\mathbf{0}$ denote the cone point of $(\Delta^1 \times \mathcal{C})^{\triangleleft}$. Note that we have a commutative diagram of restriction functors

$$\begin{tikzcd}
\mathcal{D}((\Delta^1 \times \mathcal{C})^{\triangleleft}) \arrow[r, "\alpha'"] \arrow[d, "\beta"] & \mathcal{D}((\{1\} \times \mathcal{C})^{\triangleleft}) \arrow[d, "\alpha"] \\
\mathcal{D}(K) \arrow[r, "\beta'"] \arrow[d, "\gamma"] & \mathcal{D}(\{1\} \times \mathcal{C}) \\
\mathcal{D}(\{\mathbf{0}\}). &
\end{tikzcd}$$

We wish to show that α is an equivalence of ∞-categories. Since the inclusion $K \hookrightarrow (\Delta^1 \times \mathcal{C})^{\triangleleft}$ is inner anodyne (as noted above) and the inclusion $(\{1\} \times \mathcal{C})^{\triangleleft} \hookrightarrow (\Delta^1 \times \mathcal{C})^{\triangleleft}$ is left anodyne

(Lemma 4.3.7.8), the morphisms α' and β are trivial Kan fibrations (Proposition 5.3.1.21). It will therefore suffice to show that β' is an equivalence of ∞-categories.

Amalgamating the map

$$\begin{aligned}
\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) \times \Delta^1 \times \mathcal{C} &\simeq (\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) \times \mathcal{C}) \star_{(\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E})\times\mathcal{C})} (\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) \times \mathcal{C}) \\
&\to (\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) \times \mathcal{C}) \star_{\mathcal{E}} \mathcal{E} \\
&= \mathcal{E}'
\end{aligned}$$

with the identity on $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) \times \mathcal{C}^{\triangleleft}$, we obtain a morphism of simplicial sets $F : \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) \times K \to \mathcal{E}''$. If e is an edge of the product $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) \times K$ whose image in $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E})$ is an isomorphism, then $F(e)$ is a U''-cocartesian edge of $\mathcal{E}''$. We can therefore identify F with a morphism of simplicial sets $f : \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) \to \mathcal{D}(K)$. Unwinding the definitions, we see that $\beta' \circ f$ is an isomorphism of simplicial sets. Consequently, to show that β' is an equivalence of ∞-categories, it will suffice to show that f is an equivalence of ∞-categories. Similarly, the composite map $\gamma \circ f$ is an isomorphism, so we are reduced to proving that γ is an equivalence of ∞-categories. Since β is a trivial Kan fibration, this is equivalent to the assertion that $\gamma \circ \beta$ is an equivalence of ∞-categories, which is a special case of Corollary 5.3.1.23. □

02TH **Remark 7.4.1.7.** If $U : \mathcal{E} \to \mathcal{C}$ is a cocartesian fibration of small simplicial sets, then the simplicial set $\overline{\mathcal{E}}$ constructed in the proof of Proposition 7.4.1.6 will also be small.

03E6 **Remark 7.4.1.8.** In the situation of Proposition 7.4.1.6, suppose that $U : \mathcal{E} \to \mathcal{C}$ is a left fibration. Then the extension $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleleft}$ is also a left fibration. To prove this, it will suffice to show that the fiber $\overline{\mathcal{E}}_{\mathbf{0}}$ is a Kan complex (Proposition 5.1.4.14). This follows from the fact that the covariant diffraction functor

$$\operatorname{Df} : \overline{\mathcal{E}}_{\mathbf{0}} \to \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E}) = \operatorname{Fun}_{/\mathcal{C}}(\mathcal{C},\mathcal{E})$$

is an equivalence of ∞-categories, since the simplicial set $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{C},\mathcal{E})$ is a Kan complex by (Corollary 4.4.2.5).

02TJ **Corollary 7.4.1.9.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of small simplicial sets and let $\operatorname{Tr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathcal{QC}$ be a covariant transport representation for U. Then the diagram $\operatorname{Tr}_{\mathcal{E}/\mathcal{C}}$ has a limit in the ∞-category $\mathcal{QC}$, given by the ∞-category $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C},\mathcal{E})$ of cocartesian sections of U.*

Proof. Using Proposition 7.4.1.6 (and Remark 7.4.1.7), we see that there exists a pullback diagram of small simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where $\overline{U}$ is a cocartesian fibration and the restriction map $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is a trivial Kan fibration. Using Corollary 5.6.5.11, we can extend $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ to a diagram $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}} : \mathcal{C}^{\triangleleft} \to \mathcal{QC}$ which is a covariant transport representation for $\overline{U}$. Let $\mathbf{0}$ denote the cone point of $\mathcal{C}^{\triangleleft}$. It follows from Theorem 7.4.1.1 that $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}}$ is a limit diagram in the ∞-category $\mathcal{QC}$, and therefore exhibits the ∞-category $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}}(\mathbf{0}) \simeq \overline{\mathcal{E}}_{\mathbf{0}}$ as a limit of the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$. Using Remark 7.4.1.5, we see that covariant diffraction supplies an equivalence of ∞-categories $\overline{\mathcal{E}}_{\mathbf{0}} \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$, so that $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is also a limit of the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ (Proposition 7.1.1.12). □

Corollary 7.4.1.10. *Let $\mathcal{C}$ be a small simplicial set and let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a diagram in the ∞-category $\mathcal{QC}$. Then the ∞-category of cocartesian sections $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \int_{\mathcal{C}} \mathscr{F})$ is a limit of the diagram $\mathscr{F}$.* 02TK

Proof. Apply Corollary 7.4.1.9 to the cocartesian fibration $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$. □

Corollary 7.4.1.11. *The ∞-category $\mathcal{QC}$ is complete: that is, it admits small limits.* 02TL

By inspecting the proof of Corollary 7.4.1.11, we can obtain more precise information.

Corollary 7.4.1.12. *Let n be an integer, let $\mathcal{C}$ be a simplicial set and let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a diagram. Suppose that, for every vertex $C \in \mathcal{C}$, the ∞-category $\mathscr{F}(C)$ is locally n-truncated. Then the limit $\varprojlim(\mathscr{F})$ is a locally n-truncated ∞-category.* 05E1

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is an ∞-category and that $n \geq -2$. Let $\mathcal{E} = \int_{\mathcal{C}} \mathscr{F}$ denote the ∞-category of elements of $\mathscr{F}$. It follows from Variant 5.1.5.17 that the projection map $U : \mathcal{E} \to \mathcal{C}$ is an essentially $(n+1)$-categorical cocartesian fibration. Applying Corollary 4.8.6.22, we see that the ∞-category of sections $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is locally n-truncated. Since $\varprojlim(\mathscr{F})$ can be identified with a full subcategory of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ (Corollary 7.4.1.10), it is also locally n-truncated (Remark 4.8.2.3). □

Corollary 7.4.1.13. *Let λ be an uncountable cardinal and let $\kappa = \mathrm{ecf}(\lambda)$ be the exponential cofinality of λ. Suppose we are given a diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$, where $\mathcal{C}$ is a κ-small simplicial set. If the ∞-category $\mathscr{F}(C)$ is essentially λ-small for each $C \in \mathcal{C}$, then the limit $\varprojlim(\mathscr{F})$ is also essentially λ-small.* 03U6

Proof. Using Proposition 4.7.5.5, we can choose a categorical equivalence $G : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a λ-small ∞-category (if κ is uncountable, we can even arrange that $\mathcal{D}$ is κ-small). Without loss of generality, we may assume that $\mathscr{F}$ is obtained as the restriction of the covariant transport representation of some cocartesian fibration $U : \mathcal{E} \to \mathcal{D}$. Using Corollary 7.4.1.9, we can identify $\varprojlim(\mathscr{F})$ with a full subcategory of the ∞-category $\mathrm{Fun}_{/\mathcal{D}}(\mathcal{C}, \mathcal{E})$. It will therefore suffice to show that the ∞-category $\mathrm{Fun}_{/\mathcal{D}}(\mathcal{C}, \mathcal{E})$ is essentially λ-small (Corollary 4.7.5.13). By construction, we have a pullback diagram of simplicial sets

03U7
$$\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{D}}(\mathcal{C}, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \arrow[d, "U\circ"] \\
\{G\} \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{D})
\end{tikzcd} \tag{7.43}$$

where the vertical maps are cocartesian fibrations (Theorem 5.2.1.1), and therefore isofibrations (Proposition 5.1.4.8). It follows that (7.43) is also a categorical pullback square (Corollary 4.5.2.27). Using Corollary 4.7.5.16, we are reduced to proving that the ∞-categories $\mathrm{Fun}(\mathcal{C}, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are essentially λ-small, which follows from Remark 4.7.5.10. □

7.4.2 Proof of the Diffraction Criterion

02UG The goal of this section is to prove Theorem 7.4.1.1. We begin by treating a special case (which is already sufficient for most applications).

02UM **Proposition 7.4.2.1.** *Suppose we are given a pullback diagram of small ∞-categories*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations and the restriction map $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is an equivalence of ∞-categories. Then the covariant transport representation

$$\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}} : \mathcal{C}^{\triangleleft} \to \mathcal{QC}$$

is a limit diagram in the ∞-category $\mathcal{QC}$.

Proof. Suppose we are given an integer $n \geq 1$ and a diagram $\mathscr{F}_0 : \partial\Delta^n \star \mathcal{C} \to \mathcal{QC}$ with the property that $\mathscr{F}_0|_{\{n\}\star\mathcal{C}} : \{n\} \star \mathcal{C} \to \mathcal{QC}$ is a covariant transport representation for the cocartesian fibration $\overline{U}$; here we abuse notation by identifying $\{n\} \star \mathcal{C}$ with the cone $\mathcal{C}^{\triangleleft}$. We wish to show that $\mathscr{F}_0$ can be extended to a diagram $\mathscr{F} : \Delta^n \star \mathcal{C} \to \mathcal{QC}$. Using Lemma 5.6.7.1, we can choose a pullback diagram

$$\begin{tikzcd}
\overline{\mathcal{E}} \arrow[r] \arrow[d, "\overline{U}"] & \overline{\mathcal{E}}^{+} \arrow[d, "\overline{U}^{+}"] \\
\{n\} \star \mathcal{C} \arrow[r] & \partial\Delta^n \star \mathcal{C},
\end{tikzcd}$$

where $\overline{U}^{+}$ is a cocartesian fibration having covariant transport representation $\mathscr{F}_0$. Fix an auxiliary symbol c, so that the projection map $\mathcal{C} \to \{c\}$ induces a cartesian fibration of ∞-categories $T : \Delta^n \star \mathcal{C} \to \Delta^n \star \{c\}$ (this follows by repeated application of Lemma 5.2.3.17). Note that T restricts to a to a morphism of simplicial sets $T_0 : \partial\Delta^n \star \mathcal{C} \to \partial\Delta^n \star \{c\}$ which is a pullback of T, and is therefore also a cartesian fibration. Let $\mathcal{D}$ be the cocartesian direct image $\operatorname{Res}^{\mathrm{CCart}}_{\partial\Delta^n \star \mathcal{C} \,/\, \partial\Delta^n \star \{c\}}(\overline{\mathcal{E}}^{+})$ introduced in Notation 5.3.7.7, so that the projection map $\pi : \mathcal{D} \to \partial\Delta^n \star \{c\}$ is a cocartesian fibration of simplicial sets (Proposition 5.3.7.9).

Applying Corollary 5.6.5.12, we can choose a covariant transport representation $\mathscr{G}_0 : \partial\Delta^n \star \{c\} \to \mathcal{QC}$ for the cocartesian fibration π. Note that the value of $\mathscr{G}_0$ on the edge $e = \{n\} \star \{c\} \subseteq \partial\Delta^n \star \{c\}$ can be identified with the composition

$$\begin{aligned}
\mathscr{G}_0(\{n\}) &\simeq \pi^{-1}\{n\} \\
&\xrightarrow{s} \operatorname{Fun}^{\mathrm{CCart}}_{/e}(e, e \times_{\partial\Delta^n \star \{c\}} \mathcal{D}) \\
&\xrightarrow{u} \pi^{-1}\{c\},
\end{aligned}$$

where u is given by evaluation on the final vertex $\{c\} \subseteq e$, and s is a section of the trivial Kan fibration $\operatorname{Fun}^{\mathrm{CCart}}_{/e}(e, e \times_{\partial\Delta^n \star \{c\}}, \mathcal{D}) \to \pi^{-1}\{n\}$ given by evaluation at the initial vertex $\{n\} \subseteq e$. Using Proposition 5.3.7.10, we can identify u with the restriction map $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \mathcal{E}) \to \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$, which is an equivalence of ∞-categories (by assumption). It follows that the diagram $\mathscr{G}_0$ carries the edge e to an isomorphism in the ∞-category $\mathcal{QC}$. Identifying $\partial\Delta^n \star \{c\}$ with the outer horn Λ^{n+1}_{n+1} and applying Theorem 4.4.2.6, we deduce that $\mathscr{G}_0$ can be extended to a diagram $\mathscr{G} : \Delta^n \star \{c\} \to \mathcal{QC}$.

Note that we have a commutative diagram of simplicial sets

$$\begin{tikzcd}
(\partial\Delta^n \star \mathcal{C}) \times_{(\partial\Delta^n \star \{c\})} \mathcal{D} \arrow[rr, "\mathrm{ev}"] \arrow[dr, "\pi'"'] & & \overline{\mathcal{E}}^{+} \arrow[dl, "\overline{U}^{+}"] \\
& \partial\Delta^n \star \mathcal{C}, &
\end{tikzcd}$$

where π' is given given by projection onto the first factor and ev is the restriction of the evaluation map described in Construction 4.5.9.1. Note that ev carries π'-cocartesian edges of $(\partial\Delta^n \star \mathcal{C}) \times_{(\partial\Delta^n \star \{c\})} \mathcal{D}$ to $\overline{U}^{+}$-cocartesian edges of $\overline{\mathcal{E}}^{+}$. Let $\overline{\mathcal{E}}^{++}$ denote the relative join

$$(\partial\Delta^n \star \mathcal{C}) \times_{(\partial\Delta^n \star \{c\})} \mathcal{D}) \star_{\overline{\mathcal{E}}^{+}} \overline{\mathcal{E}}^{+}$$

of Construction 5.2.3.1. Applying Lemma 5.2.3.17, we see that π' and $\overline{U}^{+}$ induce a cocartesian fibration

$$\overline{U}^{++} : \overline{\mathcal{E}}^{++} \to (\partial\Delta^n \star \mathcal{C}) \star_{(\partial\Delta^n \star \mathcal{C})} (\partial\Delta^n \star \mathcal{C}) \simeq \Delta^1 \times (\partial\Delta^n \star \mathcal{C}).$$

Applying Corollary 5.6.5.11, we deduce that $\overline{U}^{++}$ admits a covariant transport representation $\mathscr{H}_0 : \Delta^1 \times (\partial\Delta^n \star \mathcal{C}) \to \mathcal{QC}$ having the property that $\mathscr{H}_0|_{\{0\}\times(\partial\Delta^n \star \mathcal{C})} = \mathscr{G}_0 \circ T_0$ and $\mathscr{H}_0|_{\{1\}\times(\partial\Delta^n \star \mathcal{C})} = \mathscr{F}_0$. Note that, for $0 \leq i \leq n$, the evaluation map ev restricts to an isomorphism of ∞-categories $\{i\} \times_{(\partial\Delta^n \star \{c\})} \mathcal{D} \to \{i\} \times_{(\partial\Delta^n \star \mathcal{C})} \overline{\mathcal{E}}^{+}$, so that the diagram $\mathscr{H}_0$ carries the edge ge $\Delta^1 \times \{i\}$ to an isomorphism in the ∞-category $\mathcal{QC}$. Moreover, if $\sigma : \Delta^m \to \Delta^n \star \mathcal{C}$ is any simplex which does not factor through $\partial\Delta^n \star \mathcal{C}$, then the vertex $\sigma(0)$ must belong to $\partial\Delta^n$. Applying Proposition 4.4.5.8, we can extend $\mathscr{H}_0$ to a diagram $\mathscr{H} : \Delta^1 \times (\Delta^n \star \mathcal{C}) \to \mathcal{QC}$ satisfying $\mathscr{H}|_{\{0\}\times(\Delta^n \star \mathcal{C})} = \mathscr{G} \circ T$. We complete the proof by observing that the restriction $\mathscr{F} = \mathscr{H}|_{\{1\}\times(\Delta^n \star \mathcal{C})}$ provides the desired extension of the diagram $\mathscr{F}_0$. □

Proof of Theorem 7.4.1.1. Suppose we are given a pullback diagram of small simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. Assume first that the restriction map

$$\theta : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$$

is an equivalence of ∞-categories; we wish to show that the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}} : \mathcal{C}^{\triangleleft} \to \mathcal{QC}$ is a limit diagram in the ∞-category $\mathcal{QC}$.

Using Corollary 4.1.3.3, we can choose an inner anodyne morphism $\mathcal{C} \hookrightarrow \mathcal{C}'$, where $\mathcal{C}'$ is an ∞-category. Note that the induced map $\mathcal{C}^{\triangleleft} \hookrightarrow \mathcal{C}'^{\triangleleft}$ is also inner anodyne (Proposition 4.3.6.4). Applying Corollary 5.6.7.3, we can realize $\overline{U}$ as the pullback of a cocartesian fibration of ∞-categories $\overline{U}' : \overline{\mathcal{E}}' \to \mathcal{C}'^{\triangleleft}$. Set $\mathcal{E}' = \mathcal{C}' \times_{\mathcal{C}'^{\triangleleft}} \overline{\mathcal{E}}'$, so that we have a commutative diagram of restriction functors

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}'^{\triangleleft}}(\mathcal{C}'^{\triangleleft}, \overline{\mathcal{E}}') \arrow[r, "\theta'"] \arrow[d] & \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}'}(\mathcal{C}', \mathcal{E}') \arrow[d] \\
\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \arrow[r, "\theta"] & \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}),
\end{tikzcd}$$

where the vertical maps are trivial Kan fibrations (Proposition 5.3.1.21). It follows that θ' is also an equivalence of ∞-categories.

Using Corollary 5.6.5.11, we can extend $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}}$ to a functor

$$\mathrm{Tr}_{\overline{\mathcal{E}}'/\mathcal{C}'^{\triangleleft}} : \mathcal{C}'^{\triangleleft} \to \mathcal{QC}$$

which is a covariant transport representation for the cocartesian fibration $\overline{U}'$. Since $\mathcal{C}'$ is an ∞-category, Proposition 7.4.2.1 guarantees that $\mathrm{Tr}_{\overline{\mathcal{E}}'/\mathcal{C}'^{\triangleleft}}$ is a limit diagram in the ∞-category $\mathcal{QC}$. Since the inclusion map $\mathcal{C} \hookrightarrow \mathcal{C}'$ is left cofinal (Proposition 7.2.1.3), it follows that $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}}$ is also a limit diagram in $\mathcal{QC}$.

We now prove the converse. Assume that the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}}$ is a limit diagram in the ∞-category $\mathcal{QC}$; we wish to show that θ is an equivalence of ∞-categories. Using Proposition 7.4.1.6, we can choose another pullback diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \mathcal{E}^{+} \arrow[d, "U^{+}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where U^{+} is a cocartesian fibration for which the restriction map $\theta^{+} : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \mathcal{E}^{+}) \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}^{+})$ is an equivalence of ∞-categories. Applying Corollary 5.6.5.11, we see that U^{+} admits a covariant transport representation $\mathrm{Tr}_{\mathcal{E}^{+}/\mathcal{C}^{\triangleleft}} : \mathcal{C}^{\triangleleft} \to \mathcal{QC}$ satisfying $(\mathrm{Tr}_{\mathcal{E}^{+}/\mathcal{C}^{\triangleleft}})|_{\mathcal{C}} = (\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}})|_{\mathcal{C}}$. The first part of the proof shows that $\mathrm{Tr}_{\mathcal{E}^{+}/\mathcal{C}^{\triangleleft}}$ is also a limit diagram in the ∞-category $\mathcal{QC}$, and is therefore isomorphic to $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}}$ as an object of the ∞-category

$\operatorname{Fun}(\mathcal{C}^{\triangleleft}, \mathcal{QC})$. Applying Theorem 5.6.0.2, we deduce that there exists a morphism $F : \overline{\mathcal{E}} \to \overline{\mathcal{E}}^{+}$ which is an equivalence of cocartesian fibrations over $\mathcal{C}^{\triangleleft}$. We have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \mathcal{E}^{+}) \arrow[r, "\theta^{+}"] \arrow[d] & \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}') \arrow[d] \\
\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \arrow[r, "\theta"] & \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}),
\end{tikzcd}$$

where the vertical maps are given by precomposition with F and are therefore equivalences of ∞-categories. Since θ^{+} is an equivalence of ∞-categories, it follows that θ is also an equivalence of ∞-categories. □

7.4.3 Colimits of ∞-Categories

02UN Let $\mathcal{QC}$ denote the ∞-category of (small) ∞-categories (Construction 5.5.4.1). Our goal in this section is to show that the ∞-category $\mathcal{QC}$ admits small colimits (Corollary 7.4.3.13). In fact, we will prove something more precise: if $\mathcal{C}$ is a small ∞-category, then the colimit of any diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ can be described explicitly as the localization $(\int_{\mathcal{C}} \mathscr{F})[W^{-1}]$, where $\int_{\mathcal{C}} \mathscr{F}$ denotes the ∞-category of elements of $\mathscr{F}$ (Definition 5.6.2.4) and W is the collection of all morphisms of $\int_{\mathcal{C}} \mathscr{F}$ which are cocartesian with respect to the forgetful functor $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ (Corollary 7.4.3.12).

We begin with some general remarks. Let $\mathcal{C}^{\triangleright}$ denote the right cone on on a simplicial set $\mathcal{C}$ (Construction 4.3.3.26), and let $\mathbf{1} \in \mathcal{C}^{\triangleright}$ denote the cone point. For every vertex $C \in \mathcal{C}$, there is a unique edge $e_C : C \to \mathbf{1}$ in $\mathcal{C}^{\triangleright}$. If $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$ is a cocartesian fibration of simplicial sets, then covariant transport along e_C determines a functor

$$e_{C!} : \overline{\mathcal{E}}_C = \{C\} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}} \to \{\mathbf{1}\} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}} = \overline{\mathcal{E}}_{\mathbf{1}}.$$

In what follows, it will be convenient to amalgamate the functors $\{e_{C!}\}_{C \in \mathcal{C}}$ into a single morphism $\mathrm{Rf} : \mathcal{C} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}} \to \overline{\mathcal{E}}_{\mathbf{1}}$, which we will refer to as the *covariant refraction diagram.*

02UP **Definition 7.4.3.1.** Let $\mathcal{C}$ be a simplicial set, and let $\mathbf{1}$ denote the cone point of the simplicial set $\mathcal{C}^{\triangleright} \simeq \mathcal{C} \star \{\mathbf{1}\}$. Suppose that we are given a cocartesian fibration $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$, and set

$$\mathcal{E} = \mathcal{C} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}} \qquad \overline{\mathcal{E}}_{\mathbf{1}} = \{\mathbf{1}\} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}}.$$

We will say that a morphism $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ is a *covariant refraction diagram* if there exists a morphism of simplicial sets $H : \Delta^1 \times \mathcal{E} \to \overline{\mathcal{E}}$ satisfying the following conditions:

- The restriction $H|_{\{0\} \times \mathcal{E}}$ is the identity morphism from $\mathcal{E}$ to itself.

- The restriction $H|_{\{1\}\times\mathcal{E}}$ is equal to Rf.
- For every vertex $X \in \mathcal{E}$, the restriction $H|_{\Delta^1\times\{X\}}$ is a $\overline{U}$-cocartesian edge of $\overline{\mathcal{E}}$.

Remark 7.4.3.2. In the situation of Definition 7.4.3.1, suppose that $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ is a covariant refraction diagram. Then, for every vertex $C \in \mathcal{C}$, the restriction $\mathrm{Rf}|_{\mathcal{E}_C} : \mathcal{E}_C \to \overline{\mathcal{E}}_{\mathbf{1}}$ is given by covariant transport along the unique edge $e_C : C \to \mathbf{1}$ of $\mathcal{C}^{\triangleright}$, in the sense of Definition 5.2.2.4. 02UQ

Proposition 7.4.3.3. *Let $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$ be a cocartesian fibration of simplicial sets, set $\mathcal{E} = \mathcal{C} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}}$, and let $\mathbf{1}$ denote the cone point of $\mathcal{C}^{\triangleright}$. Then:* 02UR

(1) *There exists a covariant refraction diagram* $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ *(Definition 7.4.3.1).*

(2) *Let $F : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ be any morphism of simplicial sets. Then F is a covariant refraction diagram if and only if it is isomorphic to* Rf *as an object of the ∞-category* $\mathrm{Fun}(\mathcal{E}, \overline{\mathcal{E}}_{\mathbf{1}})$.

Proof. This is a special case of Lemma 5.2.2.13. □

Example 7.4.3.4. Let $\mathcal{C}$ be an ∞-category and let $\mathbf{1}$ denote the cone point of $\mathcal{C}^{\triangleright}$. Using Example 5.2.3.18, we see that the tautological map $V : \mathcal{C}^{\triangleright} \to (\Delta^0)^{\triangleright} \simeq \Delta^1$ is a cocartesian fibration. If $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$ is another cocartesian fibration, then the ∞-categories $\mathcal{E} = \mathcal{C} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}}$ and $\overline{\mathcal{E}}_{\mathbf{1}} = \{\mathbf{1}\} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}}$ can be identified with the fibers of the composite map 02US

$$(V \circ \overline{U}) : \overline{\mathcal{E}} \to \Delta^1,$$

which is also a cocartesian fibration (Proposition 5.1.4.13). In this case, the covariant refraction diagram $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ of Proposition 7.4.3.3 is given by covariant transport for the cocartesian fibration $V \circ \overline{U}$ (along the nondegenerate edge of Δ^1).

Remark 7.4.3.5. Suppose we are given a pullback diagram of simplicial sets 02UT

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. Let $\mathbf{1}$ denote the cone point of $\mathcal{C}^{\triangleright}$ and let $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ be a covariant refraction diagram. For every U-cocartesian edge $e : X \to Y$ of

$\mathcal{E}$, the image $\mathrm{Rf}(e)$ is an isomorphism in the ∞-category $\overline{\mathcal{E}}_{\mathbf{1}}$. To prove this, we observe that there is a morphism $\Delta^1 \times \Delta^1 \rightarrow \overline{\mathcal{E}}$ as indicated in the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "e"] & \mathrm{Rf}(X) \arrow[d, "\mathrm{Rf}(e)"] \\
Y \arrow[r] & \mathrm{Rf}(Y),
\end{tikzcd}$$

where the horizontal maps are $\overline{U}$-cocartesian. Applying Proposition 5.1.4.12, we deduce that $\mathrm{Rf}(e)$ is an $\overline{U}$-cocartesian edge of $\overline{\mathcal{E}}$, and therefore an isomorphism in the ∞-category $\overline{\mathcal{E}}_{\mathbf{1}}$ (Proposition 5.1.4.11).

Our study of colimits in the ∞-category $\mathcal{QC}$ will make use of the following recognition principle for colimits in the ∞-category $\mathcal{QC}$:

02UU **Theorem 7.4.3.6** (Refraction Criterion)**.** *Suppose we are given a pullback diagram of small simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. Let $\mathbf{1}$ denote the cone point of $\mathcal{C}^{\triangleright}$ and let W be the collection of all U-cocartesian edges of $\mathcal{E}$. The following conditions are equivalent:

(1) *The covariant refraction diagram* $\mathrm{Rf} : \mathcal{E} \rightarrow \overline{\mathcal{E}}_{\mathbf{1}}$ *of Proposition 7.4.3.3 exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect W.*

(2) *The covariant transport representation* $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \rightarrow \mathcal{QC}$ *of Notation 5.6.5.14 is a colimit diagram in the ∞-category $\mathcal{QC}$.*

02UV **Remark 7.4.3.7.** In the statement of Theorem 7.4.3.6, the covariant refraction diagram $F : \mathcal{E} \rightarrow \overline{\mathcal{E}}_{\mathbf{1}}$ and the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \rightarrow \mathcal{QC}$ are only well-defined up to isomorphism (as objects of the ∞-categories $\mathrm{Fun}(\mathcal{E}, \overline{\mathcal{E}}_{\mathbf{1}})$ and $\mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{QC})$, respectively). However, conditions (1) and (2) depend only on their isomorphism classes (see Exercise 6.3.1.11 and Corollary 7.1.2.14).

02UW **Exercise 7.4.3.8.** Let $\overline{U} : \overline{\mathcal{E}} \rightarrow \mathcal{C}^{\triangleright}$ and $\overline{U}' : \overline{\mathcal{E}}' \rightarrow \mathcal{C}^{\triangleright}$ be cocartesian fibrations of simplicial sets which are equivalent as inner fibrations over $\mathcal{C}^{\triangleright}$ (in the sense of Definition 5.1.7.1). Show that $\overline{U}$ satisfies condition (1) of Theorem 7.4.3.6 if and only if $\overline{U}'$ satisfies condition (1) of Theorem 7.4.3.6.

We will prove Theorem 7.4.3.6 in §7.4.4. The remainder of this section is devoted to explaining some of its consequences. We begin by showing that there is a good supply of cocartesian fibrations which satisfy the assumptions of Theorem 7.4.3.6.

Proposition 7.4.3.9. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let $\mathbf{1}$ denote the cone point of $\mathcal{C}^{\triangleright}$. Then there exists a pullback diagram* 02UX

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

where $\overline{U}$ is a cocartesian fibration and a covariant refraction diagram $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ which exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to the collection of all U-cocartesian edges of $\mathcal{E}$.

Proof. Let W be the collection of all U-cocartesian edges of $\mathcal{E}$. Applying Proposition 6.3.2.1, we deduce that there exists an ∞-category $\mathcal{E}[W^{-1}]$ and a diagram $\mathrm{Rf} : \mathcal{E} \to \mathcal{E}[W^{-1}]$ which exhibits $\mathcal{E}[W^{-1}]$ as a localization of $\mathcal{E}$ with respect to W. In particular, the diagram Rf carries each U-cocartesian edge of $\mathcal{E}$ to an isomorphism in $\mathcal{E}[W^{-1}]$. Let $\overline{\mathcal{E}}$ denote the relative join $\mathcal{E} \star_{\mathcal{E}[W^{-1}]} \mathcal{E}[W^{-1}]$ (Construction 5.2.3.1). Applying Lemma 5.2.3.17 to the commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\mathrm{Rf}"] \arrow[d, "U"] & \mathcal{E}[W^{-1}] \arrow[d] \\
\mathcal{C} \arrow[r] & \Delta^0,
\end{tikzcd}$$

we deduce that vertical maps induce a cocartesian fibration

$$\overline{U} : \overline{\mathcal{E}} = \mathcal{E} \star_{\mathcal{E}[W^{-1}]} \mathcal{E}[W^{-1}] \to \mathcal{C} \star_{\Delta^0} \Delta^0 \simeq \mathcal{C}^{\triangleright}.$$

By construction, we have a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

and the fiber of $\overline{U}$ over the cone point $\mathbf{1} \in \mathcal{C}^{\triangleright}$ can be identified with the ∞-category $\mathcal{E}[W^{-1}]$. Moreover, Rf induces a morphism of simplicial sets

$$H : \Delta^1 \times \mathcal{E} \simeq \mathcal{E} \star_{\mathcal{E}} \mathcal{E} \to \mathcal{E} \star_{\mathcal{E}[W^{-1}]} \mathcal{E}[W^{-1}] = \overline{\mathcal{E}}$$

for which $H|_{\{0\} \times \mathcal{E}}$ is the inclusion map $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$, and $H|_{\{1\} \times \mathcal{E}}$ is the diagram $\mathrm{Rf} : \mathcal{E} \to \mathcal{E}[W^{-1}]$. For every vertex $X \in \mathcal{E}$, the criterion of Lemma 5.2.3.17 guarantees that $H|_{\Delta^1 \times \{X\}}$ is a $\overline{U}$-cocartesian edge of $\overline{\mathcal{E}}$, so that H exhibits $\mathrm{Rf} : \mathcal{E} \to \mathcal{E}[W^{-1}]$ as a covariant refraction diagram. □

02UY **Remark 7.4.3.10.** In the situation of Proposition 7.4.3.9, suppose that the simplicial sets $\mathcal{E}$ and $\mathcal{C}$ are small. Then the localization $\mathcal{E}[W^{-1}]$ supplied by Proposition 6.3.2.1 can also be chosen to be small. It follows that the simplicial set $\overline{\mathcal{E}}$ constructed in the proof is also small.

02UZ **Corollary 7.4.3.11.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration between small simplicial sets, and let $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathcal{QC}$ be a covariant transport representation of U. Then the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ admits a colimit in $\mathcal{QC}$. Moreover, an object $\mathcal{D} \in \mathcal{QC}$ is a colimit of the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ if and only if it is equivalent to the localization $\mathcal{E}[W^{-1}]$, where W is the collection of all U-cocartesian morphisms of $\mathcal{E}$.*

Proof. Let $\mathbf{1}$ denote the cone point of $\mathcal{C}^{\triangleright}$. By virtue of Proposition 7.4.3.9 (and Remark 7.4.3.10), there exists a pullback diagram of small simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright}
\end{tikzcd}$$

where $\overline{U}$ is a cocartesian fibration, and a covariant refraction diagram $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ which exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to W. Applying Corollary 5.6.5.11, we see that $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ extends to a covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \to \mathcal{QC}$. By virtue of Theorem 7.4.3.6, this extension is a colimit diagram carrying $\mathbf{0}$ to the ∞-category $\overline{\mathcal{E}}_{\mathbf{1}} \simeq \mathcal{E}[W^{-1}]$. □

02V0 **Corollary 7.4.3.12.** *Let $\mathcal{C}$ be a small simplicial set, let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a diagram, let $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ denote the projection map, and let W be the collection of all U-cocartesian morphisms of $\int_{\mathcal{C}} \mathscr{F}$. Then the localization $(\int_{\mathcal{C}} \mathscr{F})[W^{-1}]$ is a colimit of the diagram $\mathscr{F}$ in the ∞-category $\mathcal{QC}$.*

Proof. Apply Corollary 7.4.3.11 to the cocartesian fibration $\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$. □

Corollary 7.4.3.13. *The ∞-category $\mathcal{QC}$ is cocomplete: that is, it admits small colimits.* `02V1`

By examining the proof of Corollary 7.4.3.13, we can obtain more precise information.

Corollary 7.4.3.14. *Let κ be an uncountable regular cardinal, let $\mathcal{C}$ be a simplicial set which is essentially κ-small, and suppose we are given a diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ with the property that, for each vertex $C \in \mathcal{C}$, the ∞-category $\mathscr{F}(C)$ is essentially κ-small. Then the colimit $\varinjlim(\mathscr{F})$ (formed in the ∞-category $\mathcal{QC}$) is essentially κ-small.* `03U8`

Proof. Without loss of generality, we may assume that $\mathscr{F}$ is a covariant transport representation for a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$, so that the colimit $\varinjlim(\mathscr{F})$ can be identified with the localization $\mathcal{E}[W^{-1}]$, where W is the collection of U-cocartesian morphisms of $\mathcal{E}$ (Corollary 7.4.3.11). By virtue of Variant 6.3.2.6, it will suffice to show that the simplicial set $\mathcal{E}$ is essentially κ-small, which follows from Corollary 5.6.7.7. □

Corollary 7.4.3.15. *Let λ be an uncountable cardinal and let $\kappa = \mathrm{cf}(\lambda)$ be the cofinality of λ. Let $\mathcal{C}$ be a κ-small simplicial set and let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a diagram. Suppose that, for each object $C \in \mathcal{C}$, the ∞-category $\mathscr{F}(C)$ is essentially λ-small. Then the colimit $\varinjlim(\mathscr{F})$ is essentially λ-small.* `03U9`

Proof. For each vertex $C \in \mathcal{C}$, the ∞-category $\mathscr{F}(C)$ is essentially λ-small, and is therefore essentially τ_C^+-small for some infinite cardinal $\tau_C < \lambda$ (Corollary 4.7.6.17). Since λ has cofinality κ, the supremum $\tau = \sup\{\tau_C\}_{C \in \mathcal{C}}$ satisfies $\tau < \lambda$. Replacing λ by the cardinal $\sup\{\tau^+, \kappa\}$, we are reduced to proving Corollary 7.4.3.15 in the special case where λ is regular. In this case, the desired result follows from Variant 7.4.3.14. □

For strictly commutative diagrams, we can use the results of §5.3 to give an alternative description of the colimit.

Corollary 7.4.3.16. *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a (strictly commutative) diagram of ∞-categories indexed by $\mathcal{C}$. Let $U : \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ be the cocartesian fibration of Definition 5.3.3.1, and let W be the collection of U-cocartesian morphisms of $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$. Then the localization $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})[W^{-1}]$ is a colimit of the diagram $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathscr{F}) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathcal{QC}$.* `039J`

Proof. Combine Corollary 7.4.3.16 with Example 5.6.5.6. □

Corollary 7.4.3.17. *Let $\mathcal{C}$ be an ∞-category, let $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$ be a cocartesian fibration, and set $\mathcal{E} = \mathcal{C} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}}$. Let $F : \overline{\mathcal{E}} \to \mathcal{D}$ be a functor of ∞-categories which carries $\overline{U}$-cocartesian morphisms of $\overline{\mathcal{E}}$ to isomorphisms in $\mathcal{D}$. If the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \to \mathcal{QC}$ is a colimit diagram, then F is left Kan extended from $\mathcal{E}$.* `039P`

Proof. Let $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ be a covariant refraction diagram, so that there exists a natural transformation $h : \mathrm{id}_{\mathcal{E}} \to \mathrm{Rf}$ (in the ∞-category $\mathrm{Fun}(\mathcal{E}, \overline{\mathcal{E}})$) which carries each object $X \in \mathcal{E}$ to an $\overline{U}$-cocartesian morphism $h_X : X \to \mathrm{Rf}(X)$. Our assumption that $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}}$ is a colimit diagram guarantees that the functor Rf exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ (Corollary 7.4.3.9). Moreover, for each object $X \in \mathcal{C}$, the functor F carries h_X to an isomorphism in the ∞-category $\mathcal{D}$. Set $F_0 = F|_{\mathcal{E}}$ and $F_1 = F|_{\overline{\mathcal{E}}_{\mathbf{1}}}$. Applying Proposition 7.3.1.17, we deduce that the natural transformation $F(h) : F_0 \to F_1 \circ \mathrm{Rf}$ exhibits the functor F_1 as a left Kan extension of F_0 along Rf. By virtue of Example 7.4.3.4, the natural transformation h exhibits Rf as a covariant transport functor for the cocartesian fibration

$$\overline{\mathcal{E}} \xrightarrow{\overline{U}} \mathcal{C}^{\triangleright} \to (\Delta^0)^{\triangleright} \simeq \Delta^1.$$

Applying Corollary 7.3.2.14, we conclude that the functor F is left Kan extended from $\mathcal{E}$. □

7.4.4 Proof of the Refraction Criterion

02V2 Our goal in this section is to prove Theorem 7.4.3.6. Our starting point is the following extension property for outer horns of the ∞-category $\mathcal{QC}$:

02V3 **Lemma 7.4.4.1.** *Let $n \geq 2$, let $X : \Lambda^n_0 \to \mathcal{QC}$ be a diagram, and let W be a collection of morphisms of the ∞-category $X(0)$ which satisfies the following pair of conditions:*

(1) *Let $1 \leq i \leq n$, and let $X(0 < i) : X(0) \to X(i)$ be the functor obtained by evaluating X on the edge $\mathrm{N}_{\bullet}(\{0 < i\}) \subseteq \Lambda^n_0$. Then $X(0 < i)$ carries each element of W to an isomorphism in the ∞-category $X(i)$.*

(2) *The functor $X(0 < 1) : X(0) \to X(1)$ exhibits $X(1)$ as a localization of $X(0)$ with respect to W.*

Then X can be extended to an n-simplex $\Delta^n \to \mathcal{QC}$.

Proof. Set $\mathcal{C} = X(0)$, $\mathcal{D} = X(1)$, and let $F : \mathcal{C} \to \mathcal{D}$ be the functor $X(0 < 1)$. Using the isomorphism $\Lambda^n_0 \simeq (\partial \Delta^{n-1})^{\triangleleft}$, we can identify X with a diagram $\sigma_0 : \partial \Delta^{n-1} \to \mathcal{QC}_{\mathcal{C}/}$. To complete the proof, it will suffice to show that σ_0 can be extended to an $(n-1)$-simplex of $\mathcal{QC}_{\mathcal{C}/}$. Let us identify the objects of the ∞-category $\mathcal{QC}_{\mathcal{C}/}$ with pairs $(\mathcal{E}, G)$, where $\mathcal{E}$ is a small ∞-category and $G : \mathcal{C} \to \mathcal{E}$ is a functor. Let $\mathcal{QC}^W_{\mathcal{C}/}$ denote the full subcategory of $\mathcal{QC}_{\mathcal{C}/}$ spanned by those pairs $(\mathcal{E}, G)$, where the functor G carries each element of W to an isomorphism in $\mathcal{E}$. It follows from assumption (1) that the diagram σ_0 factors through the subcategory $\mathcal{QC}^W_{\mathcal{C}/} \subseteq \mathcal{QC}_{\mathcal{C}/}$. To prove the existence of σ, it will suffice (by virtue of Corollary 4.6.7.13) to show that $\sigma_0(0) = (\mathcal{D}, F)$ is an initial object of the ∞-category $\mathcal{QC}^W_{\mathcal{C}/}$. Fix another object $(\mathcal{E}, G) \in \mathcal{QC}^W_{\mathcal{C}/}$; we wish to show that the morphism space $\mathrm{Hom}_{\mathcal{QC}^W_{\mathcal{C}/}}((\mathcal{D}, F), (\mathcal{E}, G)) = \mathrm{Hom}_{\mathcal{QC}_{\mathcal{C}/}}((\mathcal{D}, F), (\mathcal{E}, G))$ is a contractible Kan complex. Using

Corollary 4.6.9.18 and Remark 5.5.4.6, we can identify $\mathrm{Hom}_{\mathcal{QC}_{\mathcal{C}/}}((\mathcal{D}, F), (\mathcal{E}, G))$ with the homotopy fiber of the map of Kan complexes

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \xrightarrow{\circ F} \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$$

over the vertex $G \in \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$. Assumption (2) guarantees that this map is a homotopy equivalence onto the summand of $\mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$ spanned by those functors $\mathcal{C} \to \mathcal{E}$ which carry each element of W to an isomorphism in $\mathcal{E}$. It will therefore suffice to show that this summand contains the functor G, which follows from the definition of $\mathcal{QC}^{W}_{\mathcal{C}/}$. □

We now prove a weak form of Theorem 7.4.3.6 (which is already sufficient for most of our applications):

Proposition 7.4.4.2. *Suppose we are given a pullback diagram of small ∞-categories* 02V4

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. Let W be the collection of all U-cocartesian morphism of $\mathcal{E}$, let 0 denote the cone point of $\mathcal{C}^{\triangleright} \simeq \mathcal{C} \star \{0\}$, and assume that the covariant refraction diagram $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_0$ of Proposition 7.4.3.3 exhibits $\overline{\mathcal{E}}_0$ as a localization of $\mathcal{E}$ with respect to W. Then the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \to \mathcal{QC}$ is a colimit diagram in the ∞-category $\mathcal{QC}$.

Proof. Fix an integer $n > 0$, and suppose we are given a diagram $\mathscr{F}_0 : \mathcal{C} \star \partial \Delta^n \to \mathcal{QC}$ for which the restriction $\mathscr{F}_0|_{\mathcal{C} \star \{0\}}$ coincides with $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}}$. We wish to show that $\mathscr{F}_0$ can be extended to a functor $\mathscr{F} : \mathcal{C} \star \Delta^n \to \mathcal{QC}$. Applying Lemma 5.6.7.1, we can choose a pullback diagram

$$\begin{tikzcd}
\overline{\mathcal{E}} \arrow[r] \arrow[d, "\overline{U}"] & \overline{\mathcal{E}}^{-} \arrow[d, "\overline{U}^{-}"] \\
\mathcal{C} \star \{0\} \arrow[r] & \mathcal{C} \star \partial \Delta^n,
\end{tikzcd}$$

where $\overline{U}^{-}$ is a cocartesian fibration having covariant transport representation $\mathscr{F}_0$. For $0 \leq i \leq n$, let us write $\overline{\mathcal{E}}^{-}_{i}$ for the ∞-category given by the fiber of $\overline{U}^{-}$ on the vertex $i \in \partial \Delta^n$.

Fix an auxiliary symbol c, so that the projection map $\mathcal{C} \to \{c\}$ induces a cocartesian fibration of ∞-categories $V^{+} : \mathcal{C} \star \Delta^{n} \to \{c\} \star \Delta^{n}$ (this follows by repeated application of Lemma 5.2.3.17). Note that V^{+} restricts to a morphism of simplicial sets $V^{-} : \mathcal{C} \star \partial\Delta^{n} \to \{c\} \star \partial\Delta^{n}$ which is a pullback of V^{+}, and therefore also a cocartesian fibration (Remark 5.1.4.6). Applying Proposition 5.1.4.13, we deduce that the composite map $(V^{-} \circ \overline{U}^{-}) : \overline{\mathcal{E}}^{-} \to \{c\} \star \partial\Delta^{n} \simeq \Lambda^{n+1}_{0}$ is also a cocartesian fibration.

Let $\mathscr{G}_0 : \{c\} \star \partial\Delta^{n} \to \mathcal{QC}$ be a covariant transport representation for the cocartesian fibration $V^{-} \circ \overline{U}^{-}$. Let us identify $\mathscr{G}_0(c)$ with the ∞-category $\mathcal{E}$. For $0 \leq i \leq n$, we can identify $\mathscr{G}_0(i)$ with the ∞-category $\overline{\mathcal{E}}^{-}_{i}$ for $0 \leq i \leq n$, and the restriction of $\mathscr{G}_0$ to the edge $\{c\} \star \{i\}$ with a functor $G_i : \mathcal{E} \to \overline{\mathcal{E}}^{-}_{i}$. Applying Example 7.4.3.4 (and Remark 5.6.5.8), we see that G_i is a covariant refraction diagram for the cocartesian fibration

$$(\mathcal{C} \star \{i\}) \times_{\mathcal{C} \star \partial\Delta^{n}} \overline{\mathcal{E}}^{-} \to \mathcal{C} \star \{i\}.$$

In particular, each of the functors G_i carries elements of W to isomorphisms in the ∞-category $\overline{\mathcal{E}}^{-}_{i}$ (Remark 7.4.3.5). Moreover, the functor G_0 is isomorphic to Rf (Proposition 7.4.3.3), and therefore exhibits $\overline{\mathcal{E}}^{-}_{0} = \overline{\mathcal{E}}_{0}$ as a localization of $\mathcal{E}$ with respect to W (Exercise 6.3.1.11). Applying Lemma 7.4.4.1, we can extend $\mathscr{G}_0$ to a diagram $\mathscr{G} : \{c\} \star \Delta^{n} \to \mathcal{QC}$. Using Lemma 5.6.7.1, we can choose a pullback diagram

$$\begin{tikzcd}
\overline{\mathcal{E}}^{-} \arrow[r] \arrow[d, "V^{-} \circ \overline{U}^{-}"] & \overline{\mathcal{E}}^{+} \arrow[d, "T"] \\
\mathcal{C} \star \partial\Delta^{n} \arrow[r] & \mathcal{C} \star \Delta^{n},
\end{tikzcd}$$

where T is a cocartesian fibration having covariant transport representation $\mathscr{G}$. Note that we can write T uniquely as a composition

$$\overline{\mathcal{E}}^{+} \xrightarrow{\overline{U}^{+}} \mathcal{C} \star \Delta^{n} \xrightarrow{V^{+}} \{c\} \star \Delta^{n},$$

where $\overline{U}^{+}$ is a morphism of simplicial sets which fits into a pullback diagram

$$\begin{tikzcd}
\overline{\mathcal{E}}^{-} \arrow[r] \arrow[d, "\overline{U}^{-}"] & \overline{\mathcal{E}}^{+} \arrow[d, "\overline{U}^{+}"] \\
\mathcal{C} \star \partial\Delta^{n} \arrow[r] & \mathcal{C} \star \Delta^{n}.
\end{tikzcd}$$

We will show that the morphism $\overline{U}^{+}$ is a cocartesian fibration. Assuming this, we can complete the proof by applying Corollary 5.6.5.11 to extend $\mathscr{F}_0$ to a diagram $\mathscr{F} : \mathcal{C} \star \Delta^{n} \to \mathcal{QC}$ (which is a covariant transport representation for the cocartesian fibration $\overline{U}^{+}$).

We first prove that $\overline{U}^{+}$ is an inner fibration of simplicial sets. Suppose we are given integers $0 < i < m$; we wish to show that every lifting problem

02V5

$$\begin{tikzcd}
\Lambda^m_i \arrow[r, "\sigma_0"] \arrow[d] & \overline{\mathcal{E}}^{+} \arrow[d, "\overline{U}^{+}"] \\
\Delta^m \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed, "\sigma"] & \mathcal{C} \star \Delta^n
\end{tikzcd} \tag{7.44}$$

admits a solution. If $\overline{\sigma}$ factors through $\mathcal{C}$, then a solution exists by virtue of the fact that U is an inner fibration. Let us therefore assume that $\overline{\sigma}$ does not factor through $\mathcal{C}$. Since T is an inner fibration, we can extend σ_0 to an n-simplex σ of $\overline{\mathcal{E}}^{+}$ satisfying $T \circ \sigma = V^{+} \circ \overline{\sigma}$. We claim that the n-simplex σ solves the lifting problem (7.44). Set $\overline{\sigma}' = \overline{U}^{+} \circ \sigma$; we wish to show that $\overline{\sigma}'$ coincides with $\overline{\sigma}$ (as m-simplices of the simplicial set $\mathcal{C} \star \Delta^n$). Note that we have $V^{+} \circ \overline{\sigma} = V^{+} \circ \overline{\sigma}'$. It follows that $\overline{\sigma}$ and $\overline{\sigma}'$ both carry the final vertex $m \in \Delta^m$ to the same vertex of $\Delta^n \subseteq \mathcal{C} \star \Delta^n$. Consequently, it will suffice to show that $\overline{\sigma}$ and $\overline{\sigma}'$ agree when restricted to the face $\Delta^{m-1} \subseteq \Delta^m$. This follows from the commutativity of the diagram (7.44), since Δ^{m-1} is contained in the horn $\Lambda^m_i \subseteq \Delta^m$.

Fix an object X of the ∞-category $\overline{\mathcal{E}}^{+}$ having image $\overline{X} = \overline{U}^{+}(X)$ and a morphism $\overline{e} : \overline{X} \to \overline{Y}$ in the ∞-category $\mathcal{C} \star \Delta^m$. We will complete the proof by showing that $\overline{e}$ can be lifted to an $\overline{U}^{+}$-cocartesian morphism $e : X \to Y$ of $\mathcal{E}$. If $\overline{X}$ and $\overline{Y}$ belong to $\mathcal{C}$, then we take $e : X \to Y$ to be a U-cocartesian morphism of $\mathcal{E}$ satisfying $U(e) = \overline{e}$ (which exists by virtue of our assumption that U is a cocartesian fibration). Otherwise, we take $e : X \to Y$ to be a T-cocartesian morphism of $\overline{\mathcal{E}}^{+}$ satisfying $T(e) = V^{+}(\overline{e})$ (which exists by virtue of the fact that T is a cocartesian fibration). In either case, we will prove that the morphism e is $\overline{U}^{+}$-cocartesian by verifying the criterion of Proposition 5.1.2.1. Choose another object $Z \in \overline{\mathcal{E}}^{+}$ having image $\overline{Z} = \overline{U}^{+}(Z)$; we wish to show that the diagram of Kan complexes

02V6

$$\begin{tikzcd}
\{c\} \times_{\operatorname{Hom}_{\overline{\mathcal{E}}^{+}}(X,Y)} \operatorname{Hom}_{\overline{\mathcal{E}}^{+}}(X,Y,Z) \arrow[r] \arrow[d] & \operatorname{Hom}_{\overline{\mathcal{E}}^{+}}(X,Z) \arrow[d] \\
\{\overline{e}\} \times_{\operatorname{Hom}_{\mathcal{C} \star \Delta^n}(\overline{X},\overline{Y})} \operatorname{Hom}_{\mathcal{C} \star \Delta^n}(\overline{X},\overline{Y},\overline{Z}) \arrow[r] & \operatorname{Hom}_{\mathcal{C} \star \Delta^n}(\overline{X},\overline{Z})
\end{tikzcd} \tag{7.45}$$

is a homotopy pullback square. We consider several cases:

- Suppose first that the object $\overline{Z}$ belongs to $\mathcal{C}$. If $\overline{X}$ and $\overline{Y}$ belong to $\mathcal{C}$, then we deduce that (7.45) is a homotopy pullback square by applying Proposition 5.1.2.1 to the cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ (since, by construction, the morphism e is

U-cocartesian). Otherwise, each of the Kan complexes appearing in the diagram (7.45) is empty, so there is nothing to prove.

- Suppose that the objects $\overline{Y}$ and $\overline{Z}$ belong to Δ^n. In this case, we deduce that (7.45) is a homotopy pullback square by applying Proposition 5.1.2.1 to the cocartesian fibration $T : \overline{\mathcal{E}}^+ \to \{c\} \star \Delta^n$ (since, by construction, the morphism e is T-cocartesian).

- Suppose that the objects $\overline{X}$ and $\overline{Y}$ belong to $\mathcal{C}$, but the object $\overline{Z}$ belongs to Δ^n. In this case, the Kan complexes on the bottom row of (7.45) are contractible (see Example 4.6.1.6; in fact, they are both isomorphic to Δ^0). In particular, the bottom horizontal map is a homotopy equivalence. To show that (7.45) is a homotopy pullback square, we must show that the upper horizontal map is also a homotopy equivalence (Corollary 3.4.1.5). In other words, we must show that composition with the homotopy class $[e]$ induces an isomorphism $\theta : \mathrm{Hom}_{\overline{\mathcal{E}}^+}(Y, Z) \to \mathrm{Hom}_{\overline{\mathcal{E}}^+}(X, Z)$ in the homotopy category hKan (see Notation 4.6.9.15). Let

$$G : \mathcal{E} = \{c\} \times_{\{c\}\star\Delta^n} \overline{\mathcal{E}}^+ \to \{\overline{Z}\} \times_{\{c\}\star\Delta^n} \overline{\mathcal{E}}^+ = \overline{\mathcal{E}}^+_{\overline{Z}}$$

  be given by covariant transport for the cocartesian fibration T. Using Corollary 5.1.2.3, we can identify θ with the morphism $\mathrm{Hom}_{\overline{\mathcal{E}}^+_{\overline{Z}}}(G(Y), Z) \to \mathrm{Hom}_{\overline{\mathcal{E}}^+_{\overline{Z}}}(G(X), Z)$ given by precomposition with the morphism $G(e) : G(X) \to G(Y)$. Since the morphism e is U-cocartesian, its image $G(e)$ is an isomorphism in the ∞-category $\overline{\mathcal{E}}^+_{\overline{Z}}$, so that θ is a homotopy equivalence as desired.

□

To extend Proposition 7.4.4.2 to the case where $\mathcal{C}$ is not assumed to be an ∞-category, we will need the following variant of Corollary 5.6.7.6:

02V7 **Lemma 7.4.4.3.** *Suppose we are given a pullback diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "\widetilde{F}"] \arrow[d, "U_0"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}_0 \arrow[r, "F"] & \mathcal{C},
\end{tikzcd}$$

where U_0 and U are cocartesian fibrations. Let W_0 denote the collection of all U-cocartesian edges of $\mathcal{E}_0$, and let W denote the collection of all U-cocartesian morphisms of $\mathcal{E}$. If F is inner anodyne, then $\widetilde{F}$ induces an equivalence of ∞-categories $\mathcal{E}_0[W_0^{-1}] \to \mathcal{E}[W^{-1}]$.

Remark 7.4.4.4. Using Theorem 7.4.3.6, one can show that conclusion of Lemma 7.4.4.3 holds more generally under the assumption that $F : \mathcal{C}_0 \to \mathcal{C}$ is a left cofinal morphism of simplicial sets. For simplicity, let us assume that each of the simplicial sets appearing in the statement of Lemma 7.4.4.3 is small. Using Proposition 7.4.3.9, we can assume that U is the pullback of a cocartesian fibration $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$ for which the covariant refraction diagram $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ exhibits the ∞-category $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to W. Using Theorem 7.4.3.6, we deduce that the covariant transport representation $\mathrm{Tr} = \mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \to \mathcal{QC}$ is a colimit diagram. Since F is right cofinal, it follows that the restriction $\mathrm{Tr}|_{\mathcal{C}_0^{\triangleright}}$ is also a colimit diagram (Corollary 7.2.2.3). Applying Theorem 7.4.3.6 again, we conclude that $\mathrm{Rf}|_{\mathcal{E}_0}$ exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}_0$ with respect to W_0, so that $\widetilde{F}$ induces an equivalence $\mathcal{E}_0[W_0^{-1}] \xrightarrow{\sim} \mathcal{E}[W^{-1}]$. 02V8

Proof of Lemma 7.4.4.3. Fix an ∞-category $\mathcal{D}$; we wish to show that precomposition with $\widetilde{F}$ induces an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{E}[W^{-1}], \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}_0[W_0^{-1}], \mathcal{D})$ (see Notation 6.3.1.1). Corollary 5.6.7.6 guarantees that $\widetilde{F}$ is a categorical equivalence of simplicial sets, so that precomposition with $\widetilde{F}$ induces an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}_0, \mathcal{D})$. It will therefore suffice to prove the following:

$(*)$ Let $G : \mathcal{E} \to \mathcal{D}$ be a morphism of simplicial sets with the property that $G \circ \widetilde{F}$ carries every U_0-cocartesian edge of $\mathcal{E}_0$ to an isomorphism in $\mathcal{D}$. Then G carries each U-cocartesian edge of $\mathcal{E}$ to an isomorphism in $\mathcal{D}$.

Let us henceforth regard the ∞-category $\mathcal{D}$ and the functor $G : \mathcal{E} \to \mathcal{D}$ as fixed. For every morphism of simplicial sets $K \to \mathcal{C}$, let $\mathcal{E}_K$ denote the fiber product $K \times_{\mathcal{C}} \mathcal{E}$, let $U_K : \mathcal{E}_K \to K$ be the projection map, and let G_K denote the restriction of G to $\mathcal{E}_K$. Let us say that a monomorphism of simplicial sets $K' \hookrightarrow K$ is *good* if, for every morphism $K \to \mathcal{C}$ with the property that $G_{K'}$ carries $U_{K'}$-cocartesian morphisms of $\mathcal{E}_{K'}$ to isomorphisms in $\mathcal{D}$, the morphism G_K carries U_K-cocartesian morphisms of $\mathcal{E}_K$ to isomorphisms in $\mathcal{D}$. To prove $(*)$, it will suffice to show that $F : \mathcal{C}_0 \to \mathcal{C}$ is weakly saturated. It is not difficult to see that the collection of good morphisms is weakly saturated, in the sense of Definition 1.5.4.12. It will therefore suffice to show that the horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is good for $0 < i < n$. In other words, it will suffice to prove $(*)$ in the special case where $\mathcal{C} = \Delta^n$ is a standard simplex and $F : \Lambda^n_i \hookrightarrow \Delta^n$ is the inclusion of an inner horn.

If $n \geq 3$, then every edge of $\mathcal{C} = \Delta^n$ is contained in the horn Λ^n_i; it follows that the morphism $\widetilde{F} : \mathcal{E}_0 \to \mathcal{E}$ induces a bijection $W_0 \xrightarrow{\sim} W$, so there is nothing to prove. We may therefore assume without loss of generality that $n = 2$. Let $w : X \to Z$ be a U-cocartesian of $\mathcal{E}$ which does not belong to the simplicial subset $\mathcal{E}_0 = \Lambda^2_1 \times_{\Delta^2} \mathcal{E}$, so that $U(X) = 0$ and $U(Z) = 2$. Since U is a cocartesian fibration, we can choose a U-cocartesian morphism $u : X \to Y$ with $U(Y) = 1$. Our assumption that u is U-cocartesian guarantees that there exists a 2-simplex of $\mathcal{E}'$ whose boundary is indicated in the diagram

$$\begin{tikzcd} & Y \arrow[dr, "v"] & \\ X \arrow[ur, "u"] \arrow[rr, "w"] & & Z. \end{tikzcd}$$

Invoking Corollary 5.1.2.4, we see that v is also U-cocartesian, so that u and v can be regarded as elements of W_0. It now suffices to observe that if $G : \mathcal{E} \to \mathcal{D}$ is any functor which carries both u and v to isomorphisms in $\mathcal{D}$, then G also carries w to an isomorphism in $\mathcal{D}$.

□

Proof of Theorem 7.4.3.6. Suppose we are given a pullback diagram of small simplicial sets

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\ \mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright}, \end{tikzcd}$$

where U and $\overline{U}$ are cocartesian fibrations. Let W denote the collection of all U-cocartesian edges of $\mathcal{E}$, let $\mathbf{1}$ denote the cone point of $\mathcal{C}^{\triangleright}$, let $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ be a covariant refraction diagram (Definition 7.4.3.1). Assume first that Rf exhibits the ∞-category $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to W. We wish to show that the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \to \mathcal{QC}$ is a colimit diagram in the ∞-category $\mathcal{QC}$.

Using Corollary 4.1.3.3, we can choose an inner anodyne morphism $\mathcal{C} \hookrightarrow \mathcal{C}'$, where $\mathcal{C}'$ is an ∞-category. Note that the induced map $\mathcal{C}^{\triangleright} \hookrightarrow \mathcal{C}'^{\triangleright}$ is also inner anodyne (Proposition 4.3.6.4). Applying Corollary 5.6.7.3, we can realize $\overline{U}$ as the pullback of a cocartesian fibration of ∞-categories $\overline{U}' : \overline{\mathcal{E}}' \to \mathcal{C}'^{\triangleright}$. Form a pullback diagram

02V9
$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d, "U'"] & \overline{\mathcal{E}}' \arrow[d, "\overline{U}'"] \\ \mathcal{C}' \arrow[r] & \mathcal{C}'^{\triangleright}, \end{tikzcd} \tag{7.46}$$

and let W' denote the collection of all U'-cocartesian morphisms of $\mathcal{E}'$. Using Proposition 7.4.3.3, we can choose a covariant refraction diagram $\mathrm{Rf}' : \mathcal{E}' \to \overline{\mathcal{E}}'_{\mathbf{1}} = \overline{\mathcal{E}}_{\mathbf{1}}$ for the cocartesian fibration $\overline{U}'$. Note that the restriction $\mathrm{Rf}|_{\mathcal{E}}$ is a covariant refraction collapse diagram for the cocartesian fibration $\overline{U}$, and is therefore isomorphic to Rf as an object of the ∞-category $\mathrm{Fun}(\mathcal{E}, \overline{\mathcal{E}}_{\mathbf{1}}$. It follows that $\mathrm{Rf}'|_{\mathcal{E}}$ also exhibits the ∞-category $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to W (Exercise 6.3.1.11). Applying Lemma 7.4.4.3, we see that Rf exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}'$ with respect to W.

Using Corollary 5.6.5.11, we can extend $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}}$ to a functor

$$\mathrm{Tr}_{\overline{\mathcal{E}}'/\mathcal{C}'^{\triangleright}} : \mathcal{C}'^{\triangleright} \to \mathcal{QC}$$

which is a covariant transport representation for $\overline{U}'$. Applying Proposition 7.4.4.2 to the diagram of ∞-categories (7.46), we deduce that $\mathrm{Tr}_{\overline{\mathcal{E}}'/\mathcal{C}'^{\triangleright}}$ is a colimit diagram in the ∞-category $\mathcal{QC}$. Since the inclusion map $\mathcal{C} \hookrightarrow \mathcal{C}'$ is right cofinal (Proposition 7.2.1.3), it follows that $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}}$ is also a colimit diagram in $\mathcal{QC}$, as desired.

We now prove the converse. Assume that the covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}}$ is a colimit diagram in the ∞-category $\mathcal{QC}$; we wish to show that the covariant refraction diagram Rf exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to W. By virtue of Proposition 7.4.3.9 (and Remark 7.4.3.10), we can choose another pullback diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \mathcal{E}^{+} \arrow[d, "U^{+}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

where U^{+} is a cocartesian fibration for which the covariant refraction diagram $\mathrm{Rf}^{+} : \mathcal{E} \to \mathcal{E}^{+}_{\mathbf{1}}$ exhibits $\mathcal{E}^{+}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to W. Applying Corollary 5.6.5.11, we see that U^{+} admits a covariant transport representation $\mathrm{Tr}_{\mathcal{E}^{+}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \to \mathcal{QC}$ satisfying $(\mathrm{Tr}_{\mathcal{E}^{+}/\mathcal{C}^{\triangleright}})|_{\mathcal{C}} = (\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}})|_{\mathcal{C}}$. The first part of the proof shows that $\mathrm{Tr}_{\mathcal{E}^{+}/\mathcal{C}^{\triangleright}}$ is also a colimit diagram in the ∞-category $\mathcal{QC}$, and is therefore isomorphic to $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}}$ as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}^{\triangleright}, \mathcal{QC})$. Applying Theorem 5.6.0.2, we see $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleright}$ and $U^{+} : \mathcal{E}^{+} \to \mathcal{C}^{\triangleright}$ are equivalent as cocartesian fibrations over $\mathcal{C}^{\triangleright}$. Applying Exercise 7.4.3.8, we conclude that Rf also exhibits $\overline{\mathcal{E}}_{\mathbf{1}}$ as a localization of $\mathcal{E}$ with respect to W, as desired. □

7.4.5 Limits and Colimits of Spaces

Let $\mathcal{S}$ denote the ∞-category of spaces (Construction 5.5.1.1), which we regard as a full subcategory of the ∞-category $\mathcal{QC}$ (Remark 5.5.4.8). Our goal in this section is to describe limits and colimits in the ∞-category $\mathcal{S}$. Given the results of §7.4.1 and §7.4.3, this is a relatively formal exercise. We begin with an elementary observation: 02VA

`02VB` **Proposition 7.4.5.1.** *Let $f : K \to \mathcal{S}$ be a diagram. Then:*

- *An extension $\overline{f} : K^{\triangleleft} \to \mathcal{S}$ is a limit diagram if and only if it is a limit diagram in the ∞-category $\mathcal{QC}$.*
- *An extension $\overline{f} : K^{\triangleright} \to \mathcal{S}$ is a colimit diagram if and only if it is a colimit diagram in the ∞-category $\mathcal{QC}$.*

Proof. It follows immediately from the definitions that a diagram in $\mathcal{S}$ which is a limit (or colimit) diagram in the larger ∞-category $\mathcal{QC}$, then it is already a limit (or colimit) diagram in $\mathcal{S}$ (see Variant 7.1.3.10). To prove the converse implications, we must show that the inclusion functor $\iota : \mathcal{S} \to \mathcal{QC}$ preserves all limits and colimits. This follows from Corollary 7.1.3.21, since the functor ι admits both left and right adjoints (Example 6.2.2.16). □

`02VC` **Corollary 7.4.5.2.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a left fibration between small simplicial sets and let $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathcal{S}$ be a covariant transport representation for U. Then the simplicial set $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ of sections of U is a Kan complex, which is a limit of the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ in the ∞-category $\mathcal{S}$.*

Proof. Since U is a left fibration, Corollary 4.4.2.4 guarantees that the simplicial set $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is a Kan complex. Note that every edge of $\mathcal{E}$ is U-cocartesian (Example 5.1.1.3), so that $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ coincides with the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ of cocartesian sections of U. Applying Corollary 7.4.1.9, we see that the Kan complex $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ is a limit of the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ in the ∞-category $\mathcal{QC}$, and therefore also in the full subcategory $\mathcal{S} \subseteq \mathcal{QC}$ (Proposition 7.4.5.1). □

`02VD` **Corollary 7.4.5.3.** *Let $\mathcal{C}$ be a small simplicial set. Then any diagram $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ admits a limit in the ∞-category $\mathcal{S}$, given by the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \int_{\mathcal{C}} \mathscr{F})$.*

Proof. Apply Corollary 7.4.5.2 to the left fibration $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ of Example 5.6.2.9. □

`02VE` **Corollary 7.4.5.4.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a left fibration between small simplicial sets and let $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathcal{S}$ be a covariant transport representation for U. Then the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ admits a colimit in the ∞-category $\mathcal{S}$. Moreover, a Kan complex X is a colimit of $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ if and only if there exists a weak homotopy equivalence $\mathcal{E} \to X$.*

Proof. Since U is a left fibration, every edge of $\mathcal{E}$ is U-cocartesian (Example 5.1.1.3). Let W be the collection of all U-cocartesian edges of $\mathcal{E}$. By virtue of Corollary 7.4.3.11, an ∞-category X is a colimit of $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ in the ∞-category $\mathcal{QC}$ if and only if there exists a functor $f : \mathcal{E} \to X$ which exhibits X as a localization of $\mathcal{E}$ with respect to W. By virtue of Proposition 6.3.1.20, this is equivalent to the requirement that X is a Kan complex and that f is a weak homotopy equivalence. In this case, X is also a colimit of the diagram $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ in the full subcategory $\mathcal{S} \subseteq \mathcal{QC}$ (Proposition 7.4.5.1). □

Corollary 7.4.5.5. *Let $\mathcal{C}$ be a small simplicial set. Then any diagram $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ admits a colimit in the ∞-category $\mathcal{S}$. Moreover, a Kan complex X is a colimit of the diagram $\mathscr{F}$ if and only if there exists a weak homotopy equivalence $\int_{\mathcal{C}} \mathscr{F} \to X$.* 02VF

Proof. Apply Corollary 7.4.5.4 to the left fibration $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ of Example 5.6.2.9. □

Corollary 7.4.5.6. *The ∞-category $\mathcal{S}$ is complete and cocomplete.* 02VG

Proof. Combine Corollaries 7.4.5.5 and 7.4.5.3 □

Remark 7.4.5.7 (Size Estimates for Colimits)**.** Let λ be an uncountable cardinal and let $\kappa = \mathrm{cf}(\lambda)$ be its cofinality. Suppose we are given a diagram $\mathscr{F} : \mathcal{C} \to \mathcal{S}$, where $\mathcal{C}$ is a κ-small simplicial set, and that the Kan complex $\mathscr{F}(C)$ is essentially λ-small for each $C \in \mathcal{C}$. Then the colimit $\varinjlim(\mathscr{F})$ is also essentially λ-small. This follows from Corollary 7.4.3.15 and Proposition 7.4.5.1. 03UA

Variant 7.4.5.8 (Size Estimates for Limits)**.** Let λ be an uncountable cardinal and let $\kappa = \mathrm{ecf}(\lambda)$ be its exponential cofinality. Suppose we are given a diagram $\mathscr{F} : \mathcal{C} \to \mathcal{S}$, where $\mathcal{C}$ is a κ-small simplicial set, and that the Kan complex $\mathscr{F}(C)$ is essentially λ-small for each $C \in \mathcal{C}$. Then the limit $\varprojlim(\mathscr{F})$ is also essentially λ-small. This follows from Corollary 7.4.1.13 and Proposition 7.4.5.1. 03UB

Remark 7.4.5.9 (Limits of Truncated Spaces)**.** Let n be an integer. Suppose we are given a diagram $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ such that, for every vertex $C \in \mathcal{C}$, the Kan complex $\mathscr{F}(C)$ is n-truncated. Then the limit $\varprojlim(\mathscr{F})$ is n-truncated. For $n \geq -1$, this follows by combining Corollary 7.4.1.12 with Example 4.8.2.4. For $n \leq -2$, our assumption ensures that each of the Kan complexes $\mathscr{F}(C)$ is contractible, and we wish to show that the Kan complex $\varprojlim(\mathscr{F})$ is also contractible. This follows from the description given in Corollary 7.4.5.3, since the projection map $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ is a trivial Kan fibration (see Proposition 4.4.2.14). 05E2

Corollary 7.4.5.10. *Let n be an integer, let $\mathcal{C}$ be a simplicial set and let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a diagram. Suppose that, for every vertex $C \in \mathcal{C}$, the Kan complex $\mathscr{F}(C)$ is n-truncated. Then the limit $\varprojlim(\mathscr{F})$ is an n-truncated Kan complex.∞-category.* 05E3

Proposition 7.4.5.11. *Let $\mathcal{C}$ be an ∞-category and let $f : K \to \mathcal{C}$ be a morphism of simplicial sets. The following conditions are equivalent:* 04K3

(1) *The morphism f is right cofinal (Definition 7.2.1.1).*

(2) *For every corepresentable functor $h : \mathcal{C} \to \mathcal{S}$, the composite map $K \xrightarrow{f} \mathcal{C} \xrightarrow{h} \mathcal{S}$ has a contractible colimit.*

Proof. Fix an object $X \in \mathcal{C}$, and let $h^X : \mathcal{C} \to \mathcal{S}^{<\kappa}$ be a functor corepresented by X (Theorem 5.6.6.13). Using Proposition 5.6.6.21, we see that $f \circ h^X$ is a covariant transport representation for the left fibration $K \times_{\mathcal{C}} \mathcal{C}_{X/} \to K$. Using Corollary 7.4.5.4, we can reformulate condition (2) as follows:

$(2')$ For each object $X \in \mathcal{C}$, the simplicial set $K \times_{\mathcal{C}} \mathcal{C}_{X/}$ is weakly contractible.

The equivalence $(1) \Leftrightarrow (2')$ follows from Theorem 7.2.3.1. □

For strictly commutative diagrams, we can use the results of §5.3 to give an alternative construction.

039K **Corollary 7.4.5.12.** *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a (strictly commutative) diagram of Kan complexes indexed by $\mathcal{C}$. Then a Kan complex X is a colimit of the functor $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathscr{F}) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathcal{S}$ if and only if it is weakly homotopy equivalent to the weighted nerve $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ of Definition 5.3.3.1.*

Proof. Combine Corollary 7.4.5.4 with Example 5.6.5.6. □

For many applications, it will be useful to have more precise versions of the preceding results, which characterize limit and colimit *diagrams* in the ∞-category $\mathcal{S}$.

02VH **Corollary 7.4.5.13.** *Suppose we are given a pullback diagram of small simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleleft},
\end{tikzcd}$$

where U and $\overline{U}$ are left fibrations. The following conditions are equivalent:

(1) *The restriction map*

$$\mathrm{Fun}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$$

is a homotopy equivalence of Kan complexes.

(2) *The covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleleft}} : \mathcal{C}^{\triangleleft} \to \mathcal{S}$ is a limit diagram in the ∞-category $\mathcal{S}$.*

Proof. Since $\overline{U}$ is a left fibration, every edge of $\overline{\mathcal{E}}$ is $\overline{U}$-cocartesian (Example 5.1.1.3). We can therefore identify $\mathrm{Fun}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}})$ and $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$ with the ∞-categories $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}})$ and $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$, respectively. The desired result now follows by combining Theorem 7.4.1.1 with Proposition 7.4.5.1. □

As an application, we prove a converse of Corollary 7.2.2.3:

Corollary 7.4.5.14. *Let $e : \mathcal{C}' \to \mathcal{C}$ be a morphism of simplicial sets. Then e is left cofinal if and only if it satisfies the following condition:* 03E7

$(*)$ *For every limit diagram $\overline{F} : \mathcal{C}^{\triangleleft} \to \mathcal{S}$, the composition $(\overline{F} \circ e^{\triangleleft}) : \mathcal{C}'^{\triangleleft} \to \mathcal{S}$ is also a limit diagram.*

Proof. Assume that condition $(*)$ is satisfied; we will show that e is left cofinal (the reverse implication is a special case of Corollary 7.2.2.3). Fix a left fibration $U : \mathcal{E} \to \mathcal{C}$; we wish to show that the restriction map

$$e^* : \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}', \mathcal{E})$$

is a homotopy equivalence of Kan complexes. Using Proposition 7.4.1.6 (together with Remark 7.4.1.8), we can extend U to a left fibration $\overline{U} : \overline{\mathcal{E}} \to \mathcal{C}^{\triangleleft}$ for which the restriction map

$$T : \mathrm{Fun}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E})$$

is a homotopy equivalence.

Form a pullback diagram of simplicial sets

$$\begin{tikzcd}
\overline{\mathcal{E}}' \arrow[r] \arrow[d, "\overline{U}'"] & \overline{\mathcal{E}} \arrow[d, "U"] \\
\mathcal{C}'^{\triangleleft} \arrow[r, "e"] & \mathcal{C}^{\triangleleft}.
\end{tikzcd}$$

Let $\overline{F} : \mathcal{C}^{\triangleleft} \to \mathcal{S}$ be a covariant transport representation for the left fibration $\overline{U}$, so that $\overline{F} \circ e^{\triangleleft}$ is a covariant transport representation for the left fibration $\overline{U}'$. It follows from the criterion of Corollary 7.4.5.13 that $\overline{F}$ is a limit diagram in the ∞-category $\mathcal{S}$. Applying assumption $(*)$, we see that $\overline{F} \circ e^{\triangleleft}$ is also a limit diagram in the ∞-category $\mathcal{S}$. We therefore have a commutative diagram of restriction maps

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}'^{\triangleleft}, \overline{\mathcal{E}}) \arrow[r, "T'"] \arrow[d, "(e^{\triangleleft})^*"] & \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}', \mathcal{E}) \arrow[d, "e^*"] \\
\mathrm{Fun}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}) \arrow[r, "T"] & \mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}, \mathcal{E}),
\end{tikzcd}$$

where the horizontal maps are homotopy equivalences (Corollary 7.4.5.13). Consequently, to show that e^* is a homotopy equivalence, it will suffice to show that $(e^{\triangleleft})^*$ is a homotopy equivalence. We now observe that $(e^{\triangleleft})^*$ fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}'^{\triangleleft}, \overline{\mathcal{E}}) \arrow[dd, "(e^{\triangleleft})^*"] \arrow[dr] & \\
& \{\mathbf{0}\} \times_{\mathcal{C}^{\triangleleft}} \overline{\mathcal{E}} \\
\mathrm{Fun}_{/\mathcal{C}^{\triangleleft}}(\mathcal{C}^{\triangleleft}, \overline{\mathcal{E}}), \arrow[ur] &
\end{tikzcd}$$

where the horizontal maps are given by evaluation at the cone points of the simplicial sets $\mathcal{C}^{\triangleleft}$ and $\mathcal{C}'^{\triangleleft}$ are are therefore trivial Kan fibrations (Corollary 5.3.1.23). $\square$

`02VJ` **Corollary 7.4.5.15.** *Suppose we are given a pullback diagram of small simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \overline{\mathcal{E}} \arrow[d, "\overline{U}"] \\
\mathcal{C} \arrow[r] & \mathcal{C}^{\triangleright},
\end{tikzcd}$$

where U and $\overline{U}$ are left fibrations. The following conditions are equivalent:

(1) *The inclusion map $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ is a weak homotopy equivalence of simplicial sets.*

(2) *The inclusion map $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ is left cofinal.*

(3) *The covariant transport representation $\mathrm{Tr}_{\overline{\mathcal{E}}/\mathcal{C}^{\triangleright}} : \mathcal{C}^{\triangleright} \to \mathcal{S}$ is a colimit diagram in the ∞-category $\mathcal{S}$.*

Proof. Let $\mathbf{1}$ denote the cone point of $\mathcal{C}^{\triangleright}$, and let $\overline{\mathcal{E}}_{\mathbf{1}} = \{\mathbf{1}\} \times_{\mathcal{C}^{\triangleright}} \overline{\mathcal{E}}$ denote the corresponding fiber of $\overline{\mathcal{E}}$. Since the inclusion map $\{\mathbf{1}\} \hookrightarrow \mathcal{C}^{\triangleright}$ is right anodyne (Example 4.3.7.11), the inclusion $\iota : \overline{\mathcal{E}}_{\mathbf{1}} \hookrightarrow \overline{\mathcal{E}}$ is also right anodyne (Corollary 7.2.3.13). In particular, ι is a weak homotopy equivalence of simplicial sets. Let $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ be a covariant refraction diagram (Proposition 7.4.3.3), so that the inclusion map $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ is homotopic to the composition $\iota \circ \mathrm{Rf}$. It follows that condition (1) can be reformulated as follows:

$(1')$ The covariant refraction diagram $\mathrm{Rf} : \mathcal{E} \to \overline{\mathcal{E}}_{\mathbf{1}}$ is a weak homotopy equivalence.

The equivalence $(1') \Leftrightarrow (3)$ follows by combining Proposition 7.4.5.1, Theorem 7.4.3.6, and Proposition 6.3.1.20.

The implication $(2) \Rightarrow (1)$ follows from Proposition 7.2.1.5. We will complete the proof by showing that $(1')$ implies (2). Choose an inner anodyne monomorphism $\mathcal{C} \hookrightarrow \mathcal{C}'$, where K' is an ∞-category. Then the induced map $\mathcal{C}^{\triangleright} \to \mathcal{C}'^{\triangleright}$ is also inner anodyne (Corollary 4.3.6.6); in particular, it is a categorical equivalence. Using Proposition 5.6.7.2 (and Remark 5.6.7.4), we can assume that $\overline{U}$ is the pullback of a left fibration $\overline{U}' : \overline{\mathcal{E}}' \to \mathcal{C}'^{\triangleright}$. Setting $\mathcal{E}' = \mathcal{C}' \times_{\mathcal{C}'^{\triangleright}} \overline{\mathcal{E}}'$, we have a commutative diagram of inclusion maps

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d] & \overline{\mathcal{E}} \arrow[d] \\
\mathcal{E}' \arrow[r] & \overline{\mathcal{E}}',
\end{tikzcd}$$

where the vertical maps are categorical equivalences (Corollary 5.6.7.6). Consequently, to prove that the inclusion map $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ is left cofinal, it will suffice to show that the inclusion $\mathcal{E}' \hookrightarrow \overline{\mathcal{E}}'$ is left cofinal (Corollary 7.2.1.22). We may therefore replace $\overline{U}$ by the left fibration $\overline{U}' : \overline{\mathcal{E}}' \to \mathcal{C}'^{\triangleright}$, and thereby reduce to proving the implication $(1') \Rightarrow (2)$ under the assumption that $\mathcal{C}$ is an ∞-category.

Let $\mathcal{E} \tilde{\times}_{\overline{\mathcal{E}}} \overline{\mathcal{E}}_{\mathbf{1}}$ denote the oriented fiber product of Definition 4.6.4.1, and consider the projection maps

$$\mathcal{E} \xleftarrow{\pi} \mathcal{E} \tilde{\times}_{\overline{\mathcal{E}}} \overline{\mathcal{E}}_{\mathbf{1}} \xrightarrow{\pi'} \overline{\mathcal{E}}_{\mathbf{1}}.$$

The functor π is a trivial Kan fibration, and the refraction functor Rf is obtained by composing π' with a choice of section of π. Consequently, assumption $(1')$ guarantees that π' is a weak homotopy equivalence of simplicial sets. For each vertex $X \in \overline{\mathcal{E}}_{\mathbf{1}}$, we have a pullback diagram of simplicial sets

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$$\begin{tikzcd}
\mathcal{E} \tilde{\times}_{\overline{\mathcal{E}}} \{X\} \arrow[r] \arrow[d] & \{X\} \arrow[d] \\
\mathcal{E} \tilde{\times}_{\overline{\mathcal{E}}} \overline{\mathcal{E}}_{\mathbf{1}} \arrow[r, "\pi"] & \overline{\mathcal{E}}_{\mathbf{1}}.
\end{tikzcd} \tag{7.47}$$

Since π' is an isofibration of ∞-categories (Corollary 5.3.7.3), the diagram(7.47) is a categorical pullback square (Corollary 4.5.2.27). Because $\overline{\mathcal{E}}_{\mathbf{1}}$ is a Kan complex, the diagram (7.47) is also a homotopy pullback square (Variant 4.5.2.11). Our assumption that π' is a weak homotopy equivalence guarantees that the upper horizontal map is also a weak homotopy equivalence: that is, the simplicial set $\mathcal{E} \tilde{\times}_{\overline{\mathcal{E}}} \{X\}$ is weakly contractible (Corollary 3.4.1.5). Condition (2) now follows by allowing the object X to vary and applying the criterion of of Theorem 7.2.3.1 (together with Remark 7.2.3.2). □

05JA **Corollary 7.4.5.16.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a left fibration of ∞-categories, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, and set $\mathcal{E}^0 = \mathcal{E} \times_{\mathcal{C}} \mathcal{C}^0$. The following conditions are equivalent:*

(1) *The covariant transport representation $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathcal{S}$ is left Kan extended from $\mathcal{C}^0$.*

(2) *The inclusion functor $\mathcal{E}^0 \hookrightarrow \mathcal{E}$ is left cofinal.*

Proof. Fix an object $C \in \mathcal{C}$ and set $\mathcal{C}^0_{/C} = \mathcal{C}^0 \times_{\mathcal{C}} \mathcal{C}_{/C}$. Note that the inclusion map $\mathcal{C}^0_{/C} \to \mathcal{C}_{/C}$ factors as a composition

$$\mathcal{C}^0_{/C} \hookrightarrow (\mathcal{C}^0_{/C})^{\triangleright} \xrightarrow{\lambda} \mathcal{C}_{/C},$$

where λ carries the cone point $(\mathcal{C}^0_{/C})^{\triangleright}$ to the final object $\mathrm{id}_C \in \mathcal{C}_{/C}$. In particular, λ is right cofinal (Corollary 7.2.1.9). It follows that the induced map $\lambda_{\mathcal{E}} : \mathcal{E} \times_{\mathcal{C}} (\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{E} \times_{\mathcal{C}} \mathcal{C}_{/C}$ is also right cofinal (Proposition 7.2.3.12); in particular, $\lambda_{\mathcal{E}}$ is a weak homotopy equivalence (Proposition 7.2.1.5). Combining this observation with Corollary 7.4.5.15, we see that the following conditions are equivalent:

(1_C) The covariant transport representation $\mathrm{Tr}_{\mathcal{E}/\mathcal{C}}$ is left Kan extended from $\mathcal{C}^0$ at the object $C \in \mathcal{C}$.

($1'_C$) The inclusion map $\iota_C : \mathcal{E} \times_{\mathcal{C}} \mathcal{C}^0_{/C} \hookrightarrow \mathcal{E} \times_{\mathcal{C}} \mathcal{C}_{/C}$ is a weak homotopy equivalence.

Choose an object $X \in \mathcal{E}$ satisfying $U(X) = C$ and set $\mathcal{E}^0_{/X} = \mathcal{E}^0 \times_{\mathcal{E}} \mathcal{E}_{/X}$, so that we have a pullback diagram of simplicial sets

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$$\begin{tikzcd}
\mathcal{E}^0_{/X} \arrow[r] \arrow[d] & \mathcal{E}_{/X} \arrow[d] \\
\mathcal{E} \times_{\mathcal{C}} \mathcal{C}^0_{/C} \arrow[r, "\iota_C"] & \mathcal{E} \times_{\mathcal{C}} \mathcal{C}_{/C} .
\end{tikzcd} \tag{7.48}$$

Since U is a left fibration, the vertical maps in this diagram are Kan fibrations; it follows that (7.48) is also a homotopy pullback square (Example 3.4.1.3). In particular, if condition ($1'_C$) is satisfied, then the inclusion map $\mathcal{E}^0_{/X} \hookrightarrow \mathcal{E}_{/X}$ is a weak homotopy equivalence (Corollary 3.4.1.5), so that the ∞-category $\mathcal{E}^0_{/X}$ is weakly contractible. Conversely, if $\mathcal{E}^0_{/X}$ is weakly contractible for *every* object $X \in \mathcal{C}$ satisfying $U(X) = C$, then ι_C is a weak homotopy equivalence: this follows from the observation that every connected component of $\mathcal{E} \times_{\mathcal{C}} \mathcal{C}_{/C}$ has nonempty intersection with the fiber $\mathcal{E}_C = \mathcal{E} \times_{\mathcal{C}} \{C\}$. It follows that ($1'_C$) can be reformulated as follows:

(2_C) For every object $X \in \mathcal{E}$ satisfying $U(X) = C$, the ∞-category $\mathcal{E}^0_{/X}$ is weakly contractible.

The equivalence of (1) and (2) now follows from the equivalence $(1_C) \Leftrightarrow (2_C)$ by allowing the object $C \in \mathcal{C}$ to vary (and applying Theorem 7.2.3.1). □

We conclude this section with an application of Corollary 7.4.5.13.

Proposition 7.4.5.17. *Let $\mathcal{C}$ be a locally small ∞-category and let K be a small simplicial set. Then a morphism $F : K^{\triangleleft} \to \mathcal{C}$ is a limit diagram if and only if, for every object $X \in \mathcal{C}$, the composition* 039L

$$K^{\triangleleft} \xrightarrow{F} \mathcal{C} \xrightarrow{h^X} \mathcal{S}$$

is a limit diagram in the ∞-category of spaces; here h^X denotes the functor corepresented by X (Notation 5.6.6.14).

Proof. Applying Proposition 7.1.5.12, we see that F is a limit diagram if and only if, for every object $X \in \mathcal{C}$, the restriction map

$$\theta_X : \mathrm{Hom}_{\mathrm{Fun}(K^{\triangleleft},\mathcal{C})}(\underline{X}, F) \to \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{X}|_K, F|_K)$$

is a homotopy equivalence of Kan complexes. Let $\mathcal{E}$ denote the oriented fiber product $\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C}$ and let $U : \mathcal{E} \to \mathcal{C}$ be given by projection onto the second factor. Note that U is a left fibration (Proposition 4.6.4.11) and that θ_X can be identified with the restriction map

$$\mathrm{Fun}_{/\mathcal{C}}(K^{\triangleleft}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{C}}(K, \mathcal{E}).$$

The identity morphism id_X can be viewed as an initial object of $\mathcal{E}$ satisfying $U(\mathrm{id}_X) = X$ (Proposition 4.6.7.22), so the corepresentable functor $h^X : \mathcal{C} \to \mathcal{S}$ is a covariant transport representation for U (Proposition 5.6.6.21). Applying Corollary 7.4.5.13, we see that θ_X is a homotopy equivalence if and only if $h^X \circ F$ is a limit diagram in the ∞-category $\mathcal{S}$. □

Corollary 7.4.5.18. *Let $\mathcal{C}$ be a locally small ∞-category. For every object $X \in \mathcal{C}$, the functors* 039M

$$h^X : \mathcal{C} \to \mathcal{S} \qquad h_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$$

preserve K-indexed limits, for every small simplicial set K.

Remark 7.4.5.19. Let λ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is locally λ-small. Let $\kappa = \mathrm{ecf}(\lambda)$ be the exponential cofinality of λ and let K be a κ-small simplicial set. Then, in the statements of Proposition 7.4.5.17 and Corollary 7.4.5.18, we can replace $\mathcal{S}$ by the ∞-category $\mathcal{S}^{<\lambda}$ of λ-small spaces (see Variant 7.4.5.8). 03V0

7.5 Homotopy Limits and Colimits

039Y Let $\mathcal{C}$ be a small category, and let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes indexed by $\mathcal{C}$. Recall that the diagram $\mathscr{F}$ has a limit $\varprojlim(\mathscr{F})$ in the category of simplicial sets, given concretely by the formula

$$\varprojlim(\mathscr{F})(C)_n = \varprojlim_{C \in \mathcal{C}} \mathscr{F}(C)_n$$

(Remark 1.1.0.8). However, from the perspective of homotopy theory, the construction $\mathscr{F} \mapsto \varprojlim(\mathscr{F})$ is poorly behaved:

- Although each of the simplicial sets $\{\mathscr{F}(C)\}_{C \in \mathcal{C}}$ is assumed to be a Kan complex, the inverse limit $\varprojlim(\mathscr{F})$ need not be a Kan complex.

- If $\alpha : \mathscr{F} \to \mathscr{G}$ is a natural transformation between diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Kan}$ which is a levelwise homotopy equivalence (Remark 4.5.6.2), then the induced map $\varprojlim(\mathscr{F}) \to \varprojlim(\mathscr{G})$ need not be a (weak) homotopy equivalence (see Warning 3.4.0.1).

These deficiencies can be remedied by working in the framework of ∞-categories. By passing to the homotopy coherent nerve, every functor of ordinary categories $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ determines a functor of ∞-categories $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan}) = \mathcal{S}$. By virtue of Corollary 7.4.5.6, the ∞-category of spaces $\mathcal{S}$ admits all (small) limits and colimits. In particular, there exists a Kan complex X which is a limit of the diagram $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$. This construction has the advantage of being homotopy invariant: if $\alpha : \mathscr{F} \to \mathscr{G}$ is a levelwise homotopy equivalence, then X is also a limit of the diagram $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{G})$ (see Remark 7.1.1.8). However, it has the disadvantage of being somewhat inexplicit: the Kan complex X is *a priori* well-defined only up to homotopy equivalence, rather than up to isomorphism.

By combining the results of §7.4 and §5.3, we can obtain a more direct description of the Kan complex X. Let $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ denote the $\mathscr{F}$-weighted nerve of $\mathcal{C}$ (Definition 5.3.3.1). It follows from Example 5.6.5.6 that $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$ is a covariant transport representation for the left fibration $U : \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$. By virtue of Corollary 7.4.5.2, the Kan complex $\mathrm{Fun}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}), \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}))$ is a limit of the diagram $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$. We will denote this Kan complex by $\mathrm{ho}\varprojlim(\mathscr{F})$ and refer to it as the *homotopy limit* of the diagram $\mathscr{F}$ (Construction 7.5.1.1). In §7.5.1, we give review some elementary properties of this construction (which goes back to the work of Bousfield and Kan; see [6]).

In §7.5.2, we extend the definition of the homotopy limit $\mathrm{ho}\varprojlim(\mathscr{F})$ to the case where $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ is a diagram of ∞-categories (rather than a diagram of Kan complexes). In this case, the projection map $U : \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a cocartesian fibration (rather than a left fibration), and we define $\mathrm{ho}\varprojlim(\mathscr{F})$ to be the ∞-category of *cocartesian* sections of U (that is, sections which carry each morphism of $\mathcal{C}$ to a U-cocartesian morphism of $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$:

see Construction 7.5.2.1). It follows from the results of §7.4 that $\mathrm{ho}\varprojlim(\mathscr{F})$ is a limit of the diagram of ∞-categories $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{QC}$ (Proposition 7.5.2.6).

In §7.5.3, we consider another perspective on the homotopy limit construction $\mathscr{F} \mapsto \mathrm{ho}\varprojlim(\mathscr{F})$: it can be viewed as a *right derived functor* of the usual inverse limit $\mathscr{F} \mapsto \varprojlim(\mathscr{F})$. More precisely, for every diagram of ∞-categories $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$, there is a canonical isomorphism $\mathrm{ho}\varprojlim(\mathscr{F}) \simeq \varprojlim(\mathscr{F}^{+})$, where $\mathscr{F}^{+} : \mathcal{C} \to \mathrm{QCat}$ is an *isofibrant replacement* for the diagram $\mathscr{F}$ (see Construction 7.5.3.3 and Proposition 7.5.3.7). In particular, there is a tautological map $\varprojlim(\mathscr{F}) \hookrightarrow \mathrm{ho}\varprojlim(\mathscr{F})$ (see Remark 7.5.2.12), which is an equivalence of ∞-categories when the diagram $\mathscr{F}$ is already isofibrant (Proposition 7.5.3.12). This condition is satisfied, for example, when the diagram $\mathscr{F}$ corresponds to a tower of ∞-categories

$$\cdots \to \mathcal{E}(3) \to \mathcal{E}(2) \to \mathcal{E}(1) \to \mathcal{E}(0)$$

in which the transition functors are isofibrations (see Example 7.5.3.13).

Let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ be an extension of $\mathscr{F}$, carrying the initial object of $\mathcal{C}^{\triangleleft}$ to a Kan complex X. We say that $\overline{\mathscr{F}}$ is a *homotopy limit diagram* if the composite map

$$X \to \varprojlim(\mathscr{F}) \hookrightarrow \mathrm{ho}\varprojlim(\mathscr{F})$$

is a homotopy equivalence (Definition 7.5.4.1). In §7.5.4, we show that this condition is equivalent to the requirement that $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\overline{\mathscr{F}})$ is a limit diagram in the ∞-category $\mathcal{S}$ (Proposition 7.5.4.5). Moreover, we extend the definition of homotopy limit diagram to the case where $\overline{\mathscr{F}}$ is an arbitrary diagram of simplicial sets (Definition 7.5.4.8), and show that it generalizes the notion of homotopy pullback diagram introduced in §3.4.1 (Proposition 7.5.4.13). In §7.5.5, we introduce the parallel (and closely related) notion of *categorical limit diagram* (Definition 7.5.5.11), and show that it generalizes the notion of categorical pullback square introduced in §4.5.2 (Corollary 7.5.5.10).

There is a close relationship between the homotopy limit construction $\mathscr{F} \mapsto \mathrm{ho}\varprojlim(\mathscr{F})$ of this section and the homotopy colimit construction $\mathscr{F} \mapsto \mathrm{ho}\varinjlim(\mathscr{F})$ introduced in §5.3.2. If $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ is a diagram of simplicial sets and X is a Kan complex, then there is a canonical isomorphism of simplicial sets

$$\mathrm{ho}\varprojlim(X^{\mathscr{F}})^{\mathrm{op}} \simeq \mathrm{Fun}(\mathrm{ho}\varinjlim(\mathscr{F}^{\mathrm{op}}), X^{\mathrm{op}}),$$

where $X^{\mathscr{F}} : \mathcal{C} \to \mathrm{Kan}$ denotes the functor given by $C \mapsto \mathrm{Fun}(\mathscr{F}(C), X)$ (Example 7.5.1.7; see Example 7.5.2.11 for a generalization to the case where X is an ∞-category). Just as the homotopy limit construction can be viewed as a *right* derived functor of the limit functor $\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to \mathrm{Set}_{\Delta}$, the homotopy colimit construction can be viewed as a *left* derived functor of the colimit functor $\varinjlim : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to \mathrm{Set}_{\Delta}$. More precisely, we show in §7.5.6 that the homotopy colimit of a diagram $\mathscr{F}$ is isomorphic to the colimit $\varinjlim(\mathscr{G})$, where $\mathscr{G}$ is a projectively cofibrant diagram of simplicial sets equipped with a levelwise weak homotopy equivalence $\alpha : \mathscr{G} \to \mathscr{F}$ (Construction 7.5.6.8).

In §7.5.7, we show that the homotopy colimit construction has a close relationship with the formation of colimits in the ∞-category $\mathcal{S}$. If $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ is a diagram of Kan complexes, then a Kan complex is a colimit of the diagram $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$ if and only if it is weakly homotopy equivalent to $\mathrm{ho}\varinjlim(\mathscr{F})$ (Proposition 7.5.7.1). In fact, we can be more precise: if $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Kan}$ is a diagram extending $\mathscr{F}$ which carries the final object of $\mathcal{C}^{\triangleright}$ to a Kan complex X, then $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$ is a colimit diagram if and only if the composite map $\mathrm{ho}\varinjlim(\mathscr{F}) \twoheadrightarrow \varinjlim(\mathscr{F}) \to X$ is a weak homotopy equivalence (Corollary 7.5.7.7). If this condition is satisfied, we will say that $\overline{\mathscr{F}}$ is a *homotopy colimit diagram* (Definition 7.5.7.3). In §7.5.8, we introduce the parallel notion of *categorical colimit diagram* (Definition 7.5.8.2), which has a similar relationship with colimits in the ∞-category $\mathcal{QC}$ (Corollary 7.5.8.9).

7.5.1 Homotopy Limits of Kan Complexes

039Z In this section, we introduce the *homotopy limit* of a diagram of Kan complexes, following Bousfield and Kan (see [6]).

03A0 **Construction 7.5.1.1** (Homotopy Limits of Kan Complexes). Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes indexed by $\mathcal{C}$, and $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ denote the weighted nerve of $\mathscr{F}$ (Definition 5.3.3.1). We define

$$\mathrm{ho}\varprojlim(\mathscr{F}) = \mathrm{Fun}_{/\,\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}), \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}))$$

to be the simplicial set which parametrizes sections of the projection map $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$. We will refer to $\mathrm{ho}\varprojlim(\mathscr{F})$ as the *homotopy limit* of the diagram $\mathscr{F}$.

03A1 **Proposition 7.5.1.2.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes. Then the homotopy limit $\mathrm{ho}\varprojlim(\mathscr{F})$ is a Kan complex.*

Proof. This is a special case of Corollary 4.4.2.5, since the projection map $U : \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a left fibration (Corollary 5.3.3.19). □

03A2 **Remark 7.5.1.3** (Homotopy Invariance). Let $\mathcal{C}$ be a category and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a natural transformation between functors $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Kan}$. Then α induces a morphism of weighted nerves $T : \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathscr{G}}_{\bullet}(\mathcal{C})$, and therefore a morphism of Kan complexes $\mathrm{ho}\varprojlim(\alpha) : \mathrm{ho}\varprojlim(\mathscr{F}) \to \mathrm{ho}\varprojlim(\mathscr{G})$. If α is a levelwise homotopy equivalence, then T is an equivalence of left fibrations over $\mathrm{N}_{\bullet}(\mathcal{C})$ (Corollary 5.3.3.20), so $\mathrm{ho}\varprojlim(\alpha)$ is a homotopy equivalence.

Warning 7.5.1.4. In [6], Bousfield and Kan define the homotopy limit of an arbitrary diagram $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ to be the simplicial set $\mathrm{Fun}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}), \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}))$ appearing in Construction 7.5.1.1. We will avoid this convention for two reasons: 03A3

- Many important features of the Bousfield-Kan construction (such as the homotopy invariance property of Remark 7.5.1.3) are true for diagrams of Kan complexes, but not for general diagrams of simplicial sets.
- In the case where $\mathscr{F}$ is a diagram of ∞-categories, it will be convenient to adopt a slightly different definition of homotopy limit (Construction 7.5.2.1), which generally does not agree with the Bousfield-Kan construction.

Note that every (strictly commutative) diagram of Kan complexes $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ determines a diagram

$$\mathrm{N}_{\bullet}^{\mathrm{hc}} : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$$

in the ∞-category of spaces $\mathcal{S}$.

Proposition 7.5.1.5. *Let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes. Then the Kan complex $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ is a limit of the diagram* 03A4

$$\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$$

in the ∞-category $\mathcal{S}$.

Proof. This is a special case of Corollary 7.4.5.2, since the functor $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F})$ is a covariant transport representation for the projection map $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ (Example 5.6.5.6). □

We now give a more concrete description of the homotopy limit.

Remark 7.5.1.6. Let $\mathcal{C}$ be a category. For each object $C \in \mathcal{C}$, let $\mathscr{E}(C)$ denote the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C}_{/C})$. The construction $C \mapsto \mathscr{E}(C)$ determines a functor $\mathscr{E} : \mathcal{C} \to \mathrm{Set}_{\Delta}$, which we view as an object of the functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$. For every diagram of Kan complexes $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$, Proposition 5.3.3.24 supplies a canonical isomorphism 03A5

$$\underleftarrow{\mathrm{holim}}(\mathscr{F}) = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{E}, \mathscr{F})_{\bullet},$$

where the right hand side is defined using the simplicial enrichment of $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$ described in Example 2.4.2.2.

Stated more concretely, we can identify $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ with a simplicial subset of the product $\prod_{C \in \mathcal{C}} \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}_{/C}), \mathscr{F}(C))$, whose n-simplices are collections of maps $\{\sigma_C : \Delta^n \times \mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \to \mathscr{F}(C)\}$ which satisfy the following condition:

$(*)$ For every morphism $f : C \to D$ in the category $\mathcal{C}$, the diagram of simplicial sets

$$\begin{tikzcd}
\Delta^n \times \mathrm{N}_\bullet(\mathcal{C}_{/C}) \arrow[r, "\circ f"] \arrow[d, "\sigma_C"] & \Delta^n \times \mathrm{N}_\bullet(\mathcal{C}_{/D}) \arrow[d, "\sigma_D"] \\
\mathscr{F}(C) \arrow[r, "\mathscr{F}(f)"] & \mathscr{F}(D)
\end{tikzcd}$$

is commutative.

In particular, we have an equalizer diagram of simplicial sets

$$\underleftarrow{\mathrm{holim}}(\mathscr{F}) \to \prod_C \mathrm{Fun}(\mathrm{N}_\bullet(\mathcal{C}_{/C}), \mathscr{F}(C)) \rightrightarrows \prod_{f:C\to D} \mathrm{Fun}(\mathrm{N}_\bullet(\mathcal{C}_{/C}), \mathscr{F}(D)).$$

03A6 **Example 7.5.1.7** (Duality with Homotopy Colimits)**.** Let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets, let X be a Kan complex, and let $X^{\mathscr{F}} : \mathcal{C} \to \mathrm{Kan}$ be the diagram of Kan complexes given by the formula $X^{\mathscr{F}}(C) = \mathrm{Fun}(\mathscr{F}(C), X)$. Let us write $\mathscr{F}^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_\Delta$ for the functor given by the formula $\mathscr{F}^{\mathrm{op}}(C) = \mathscr{F}(C)^{\mathrm{op}}$, and let $\mathscr{E} : \mathcal{C} \to \mathrm{Set}_\Delta$ denote the functor given by $\mathscr{E}(C) = \mathrm{N}_\bullet(\mathcal{C}_{/C})$. Combining Remark 7.5.1.6 with Proposition 5.3.2.21, we obtain canonical isomorphisms

$$\begin{aligned}
\underleftarrow{\mathrm{holim}}(X^{\mathscr{F}})^{\mathrm{op}} &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_\Delta)}(\mathscr{E}, X^{\mathscr{F}})_\bullet^{\mathrm{op}} \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Set}_\Delta)}(\mathscr{F}, X^{\mathscr{E}})_\bullet^{\mathrm{op}} \\
&\simeq \mathrm{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}^{\mathrm{op}}), X^{\mathrm{op}}).
\end{aligned}$$

7.5.2 Homotopy Limits of ∞-Categories

03A7 We now extend the definition of homotopy limit to diagrams taking values in the category QCat.

03A8 **Construction 7.5.2.1** (Homotopy Limits of ∞-Categories)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a (strictly commutative) diagram of ∞-categories, let $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ denote the weighted nerve of $\mathscr{F}$ (Definition 5.3.3.1), and let $U : \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ the cocartesian fibration of Corollary 5.3.3.16. We let $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ denote the full subcategory

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})) \subseteq \mathrm{Fun}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}))$$

whose objects are functors $G : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ which satisfy $U \circ G = \mathrm{id}_{\mathrm{N}_\bullet(\mathcal{C})}$ and which carry each morphism of $\mathcal{C}$ to a U-cocartesian morphism of $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ (see Notation 5.3.1.10). We will refer to $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ as the *homotopy limit* of the diagram $\mathscr{F}$.

Example 7.5.2.2 (Homotopy Limits of Kan Complexes)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a (strictly commutative) diagram of Kan complexes. Then the projection map $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ is a left fibration of simplicial sets (Corollary 5.3.3.19). It follows that every morphism of the ∞-category $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ is U-cocartesian, so the homotopy limit $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ of Construction 7.5.2.1 coincides with the homotopy limit $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ of Construction 7.5.1.1. 03A9

Remark 7.5.2.3. Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a (strictly commutative) diagram of ∞-categories, let K be a simplicial set, and let $\mathscr{F}^K : \mathcal{C} \to \mathrm{QCat}$ denote the functor given on objects by the formula $\mathscr{F}^K(C) = \mathrm{Fun}(K, \mathscr{F}(C))$. Then there is a canonical isomorphism of simplicial sets $\underleftarrow{\mathrm{holim}}(\mathscr{F}^K) \simeq \mathrm{Fun}(K, \underleftarrow{\mathrm{holim}}(\mathscr{F}))$ (see Remarks 5.3.3.5 and 5.3.1.19). 03AA

Variant 7.5.2.4 (Homotopy Limits of Ordinary Categories)**.** Let Cat denote the (ordinary) category of categories, let $\mathscr{F} : \mathcal{C} \to \mathrm{Cat}$ be a functor, and let $\int_{\mathcal{C}} \mathscr{F}$ denote the category of elements of $\mathscr{F}$. We let $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ denote the category $\mathrm{Fun}_{/\mathcal{C}}^{\mathrm{CCart}}(\mathcal{C}, \int_{\mathcal{C}} \mathscr{F})$ whose objects are sections of the projection functor $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$ which carry each morphism of $\mathcal{C}$ to a U-cocartesian morphism of $\int_{\mathcal{C}} \mathscr{F}$. Let $\mathrm{N}_{\bullet}(\mathscr{F})$ denote the QCat-valued functor given by $C \mapsto \mathrm{N}_{\bullet}(\mathscr{F}(C))$. Combining Proposition 1.5.3.3 with Example 5.6.1.8, we obtain a canonical isomorphism of simplicial sets 03Y3

$$\underleftarrow{\mathrm{holim}}(\mathrm{N}_{\bullet}(\mathscr{F})) \simeq \mathrm{N}_{\bullet}(\underleftarrow{\mathrm{holim}}(\mathscr{F})).$$

In particular, the formation of homotopy limits preserves the full subcategory of QCat spanned by the (nerves of) ordinary categories.

Remark 7.5.2.5. Let $\mathscr{F} : \mathcal{C} \to \mathrm{Cat}$ be a functor of ordinary categories. Then the category $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ can be described more concretely as follows: 03Y4

(1) An $M \in \underleftarrow{\mathrm{holim}}(\mathscr{F})$ is a rule which assigns to each object $C \in \mathcal{C}$ an object $M(C) \in \mathscr{F})$, and to each morphism $u : C \to D$ in $\mathcal{C}$ an isomorphism $M(u) : \mathscr{F}(u)(M(C)) \xrightarrow{\sim} M(D)$, subject to the following constraints:

  - For each object $C \in \mathcal{C}$, $M(\mathrm{id}_C)$ is the identity morphism from $M(C)$ to itself.
  - For every pair of composable morphisms $u : C \to D$ and $v : D \to E$ of $\mathcal{C}$, we have $M(v \circ u) = M(v) \circ \mathscr{F}(v)(M(u))$.

(2) If M and N are objects of $\underleftarrow{\mathrm{holim}}(\mathscr{F})$, then a morphism $\alpha : M \to N$ in $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ is a rule which assigns to each object $C \in \mathcal{C}$ a morphism $\alpha_C : M(C) \to N(C)$ in the category $\mathscr{F}(C)$, subject to the following constraint:

- For every morphism $u : C \to D$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
\mathscr{F}(u)(M(C)) \arrow[r, "M(u)", "\sim"'] \arrow[d, "\mathscr{F}(u)(\alpha_C)"] & M(D) \arrow[d, "\alpha_D"] \\
\mathscr{F}(u)(N(C)) \arrow[r, "N(u)", "\sim"'] & N(D)
\end{tikzcd}$$

commutes (in the category $\mathscr{F}(D)$).

03AB **Proposition 7.5.2.6.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a (strictly commutative) diagram of ∞-categories. Then the homotopy limit $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ is an ∞-category. Moreover, $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ can be identified with a limit of the diagram*

$$\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{QCat}) = \mathcal{QC}$$

in the ∞-category $\mathcal{QC}$.

Proof. By virtue of Example 5.6.5.6, the functor $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F})$ is a covariant transport representation for the cocartesian fibration $U : \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$. Proposition 7.5.2.6 is therefore a special case of Corollary 7.4.1.9. □

03AC **Remark 7.5.2.7** (Homotopy Invariance)**.** Let $\mathcal{C}$ be a category and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a natural transformation between functors $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{QCat}$. Then α determines a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \arrow[rr, "T"] \arrow[dr, "U"'] & & \mathrm{N}_{\bullet}^{\mathscr{G}}(\mathcal{C}) \arrow[dl, "V"] \\
& \mathrm{N}_{\bullet}(\mathcal{C}). &
\end{tikzcd}$$

The functor T carries U-cocartesian morphisms of $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ to V-cocartesian morphisms of $\mathrm{N}_{\bullet}^{\mathscr{G}}(\mathcal{C})$ (see Corollary 5.3.3.16), and therefore induces a functor of ∞-categories $\underleftarrow{\mathrm{holim}}(\alpha) : \underleftarrow{\mathrm{holim}}(\mathscr{F}) \to \underleftarrow{\mathrm{holim}}(\mathscr{G})$. If α is a levelwise categorical equivalence, then T is an equivalence of cocartesian fibrations over $\mathrm{N}_{\bullet}(\mathcal{C})$ (Corollary 5.3.3.20), so $\underleftarrow{\mathrm{holim}}(\alpha)$ is an equivalence of ∞-categories.

03AD **Example 7.5.2.8** (Homotopy Limits of Cores)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories, and let $\mathscr{F}^{\simeq} : \mathcal{C} \to \mathrm{Kan}$ be the functor given on objects by the formula

$\mathscr{F}^{\simeq}(C) = \mathscr{F}(C)^{\simeq}$. Then the inclusion map $\mathscr{F}^{\simeq} \hookrightarrow \mathscr{F}$ induces a monomorphism of simplicial sets $\underleftarrow{\mathrm{holim}}(\mathscr{F}^{\simeq}) \to \underleftarrow{\mathrm{holim}}(\mathscr{F})$, whose image is the core of the ∞-category $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ (see Example 5.3.3.21 and Remark 5.3.1.20). In other words, there is a canonical isomorphism of Kan complexes $\underleftarrow{\mathrm{holim}}(\mathscr{F}^{\simeq}) \simeq \underleftarrow{\mathrm{holim}}(\mathscr{F})^{\simeq}$.

Remark 7.5.2.9. Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories and let $\mathscr{F}_0 = \mathscr{F}|_{\mathcal{C}_0}$ be the restriction of $\mathscr{F}$ to a subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$. Suppose that the inclusion $\mathrm{N}_{\bullet}(\mathcal{C}_0) \hookrightarrow \mathrm{N}_{\bullet}(\mathcal{C})$ is left anodyne (this condition is satisfied, for example, if the inclusion map $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ has a right adjoint: see Corollary 7.2.3.7). Then the restriction map $\underleftarrow{\mathrm{holim}}(\mathscr{F}) \to \underleftarrow{\mathrm{holim}}(\mathscr{F}_0)$ is a trivial Kan fibration of ∞-categories (see Proposition 5.3.1.21). `03AE`

Remark 7.5.2.10. Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories. Arguing as in Remark 7.5.1.6, we can identify the homotopy limit $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ with a simplicial subset of the product $\prod_{C \in \mathcal{C}} \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathcal{C}_{/C}), \mathscr{F}(C))$, whose n-simplices are collections of maps $\{\sigma_C : \Delta^n \times \mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \to \mathscr{F}(C)\}$ which satisfy the following pair of conditions: `03AF`

$(*)$ For every morphism $f : C \to D$ in the category $\mathcal{C}$, the diagram of simplicial sets

$$\begin{tikzcd}
\Delta^n \times \mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \arrow[r, "\circ f"] \arrow[d, "\sigma_C"] & \Delta^n \times \mathrm{N}_{\bullet}(\mathcal{C}_{/D}) \arrow[d, "\sigma_D"] \\
\mathscr{F}(C) \arrow[r, "\mathscr{F}(f)"] & \mathscr{F}(D)
\end{tikzcd}$$

is commutative.

$(*')$ For every object $C \in \mathcal{C}$ and every integer $0 \leq i \leq n$, the composite map

$$\{i\} \times \mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \hookrightarrow \Delta^n \times \mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \xrightarrow{\sigma_C} \mathscr{F}(C)$$

carries every morphism in the category $\mathcal{C}_{/C}$ to an isomorphism in the ∞-category $\mathscr{F}(C)$.

Example 7.5.2.11 (Duality with Homotopy Colimits)**.** Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, and let W denote the collection of horizontal edges of the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F}^{\mathrm{op}})$ (see Definition 5.3.4.1). Let $\mathcal{D}$ be an ∞-category and let $\mathcal{D}^{\mathscr{F}} : \mathcal{C} \to \mathrm{QCat}$ denote the functor given by the formula $\mathcal{D}^{\mathscr{F}}(C) = \mathrm{Fun}(\mathscr{F}(C), \mathcal{D})$. Arguing as in Example 7.5.1.7, we obtain a canonical isomorphism `03AG`

$$\theta : \mathrm{Fun}_{/\,\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}), \mathrm{N}_{\bullet}^{\mathcal{D}^{\mathscr{F}}}(\mathcal{C}))^{\mathrm{op}} \simeq \mathrm{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}^{\mathrm{op}}), \mathcal{D}^{\mathrm{op}}).$$

Unwinding the definitions, we see that θ restricts to an isomorphism of ∞-categories $\underleftarrow{\mathrm{holim}}(\mathcal{D}^{\mathscr{F}})^{\mathrm{op}} \simeq \mathrm{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}^{\mathrm{op}})[W^{-1}], \mathcal{D}^{\mathrm{op}})$.

03AH **Remark 7.5.2.12** (Comparison with the Limit)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories and let $X = \varprojlim(\mathscr{F})$ denote the limit of $\mathscr{F}$, formed in the category of simplicial sets. Let $\underline{X} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ denote the constant functor taking the value X. We then have a tautological map $\underline{X} \to \mathscr{F}$. The induced morphism of simplicial sets

$$X \times \mathrm{N}_{\bullet}(\mathcal{C}) \simeq \mathrm{N}_{\bullet}^{\underline{X}}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$$

determines a comparison map $\iota : X = \varprojlim(\mathscr{F}) \to \mathrm{ho}\varprojlim(\mathscr{F})$. Note that ι is a monomorphism of simplicial sets (since each of the projection maps $X = \varprojlim(\mathscr{F}) \to \mathscr{F}(C)$ factor through ι).

03AJ **Proposition 7.5.2.13.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories, and suppose that the category $\mathcal{C}$ has an initial object. Then the comparison map $\iota : \varprojlim(\mathscr{F}) \hookrightarrow \mathrm{ho}\varprojlim(\mathscr{F})$ of Remark 7.5.2.12 is an equivalence of ∞-categories.*

Proof. Let $C \in \mathcal{C}$ be an initial object, so that the inclusion map $\{C\} \to \mathrm{N}_{\bullet}(\mathcal{C})$ is left anodyne (Corollary 4.6.7.24). Applying Remark 7.5.2.9, we see that evaluation at C induces an equivalence of ∞-categories $\mathrm{ev}_C : \mathrm{ho}\varprojlim(\mathscr{F}) \to \mathscr{F}(C)$. Our assumption that C is initial also guarantees that the composition $(\mathrm{ev}_C \circ \iota) : \varprojlim(\mathscr{F}) \to \mathscr{F}(C)$ is an isomorphism of simplicial sets, so that $\varprojlim(\mathscr{F})$ is an ∞-category and ι is an equivalence of ∞-categories. $\square$

03AK **Example 7.5.2.14.** Let I be a set, which we regard as a category having only identity morphisms. Let $\mathscr{F} : I \to \mathrm{QCat}$ be a diagram, which we view as a collection of ∞-categories $\{\mathcal{C}_i\}_{i \in I}$ indexed by I. Then the comparison morphism

$$\textstyle\prod_{i \in I} \mathcal{C}_i = \varprojlim(\mathscr{F}) \to \mathrm{ho}\varprojlim(\mathscr{F})$$

of Remark 7.5.2.12 is an isomorphism.

03AL **Exercise 7.5.2.15** (Homotopy Limits of Sets)**.** Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$ be a diagram in the category of sets. Let us abuse notation by identifying Set with the full subcategory of Kan spanned by the constant simplicial sets. Show that the comparison map $\varprojlim(\mathscr{F}) \hookrightarrow \mathrm{ho}\varprojlim(\mathscr{F})$ of Remark 7.5.2.12 is an isomorphism.

Beware that the comparison morphism of Remark 7.5.2.12 is not an isomorphism in general.

03AM **Example 7.5.2.16.** Let $[1]$ denote the linearly ordered set $\{0 < 1\}$ and let $\mathscr{F} : [1] \to \mathrm{QCat}$ be a diagram, which we identify with a functor of ∞-categories $T : \mathcal{C} \to \mathcal{D}$. Then the homotopy limit $\mathrm{ho}\varprojlim(\mathscr{F})$ of Construction 7.5.1.1 can be identified with the homotopy fiber product

$$\mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D} = \mathcal{C} \times_{\mathrm{Fun}(\{0\}, \mathcal{D})} \mathrm{Isom}(\mathcal{D})$$

of Construction 4.5.2.1. Under this identification, the comparison morphism $\varprojlim(\mathscr{F}) \to \operatorname{ho}\varprojlim(\mathscr{F})$ of Remark 7.5.2.12 corresponds to the monomorphism

$$\mathcal{C} \simeq \mathcal{C} \times_{\mathcal{D}} \mathcal{D} \hookrightarrow \mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{D}$$

of Proposition 3.4.0.7. This morphism is usually not an isomorphism of simplicial sets, though it is always an equivalence of ∞-categories (Proposition 7.5.2.13).

Example 7.5.2.17. Let $\mathcal{K}$ be the partially ordered set depicted in the diagram 03AN

$$\bullet \to \bullet \leftarrow \bullet$$

and suppose we are given a functor $\mathscr{F} : \mathcal{K} \to \mathrm{QCat}$, which we depict as a diagram of ∞-categories

$$\mathcal{C}_0 \xrightarrow{T_0} \mathcal{C} \xleftarrow{T_1} \mathcal{C}_1 .$$

Then the homotopy limit $\operatorname{ho}\varprojlim(\mathscr{F})$ can be identified with the iterated homotopy pullback $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} (\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C})$. Applying Corollary 4.5.2.20, we see that the equivalence $\mathcal{C}_1 \hookrightarrow \mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}$ of Example 7.5.2.16 induces an equivalence of ∞-categories

$$\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \hookrightarrow \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} (\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}) \simeq \operatorname{ho}\varprojlim(\mathscr{F}).$$

In particular, the comparison map $\varprojlim(\mathscr{F}) \to \operatorname{ho}\varprojlim(\mathscr{F})$ is a categorical equivalence of simplicial sets if and only if the inclusion $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}_1 \hookrightarrow \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is a categorical equivalence of simplicial sets. This condition is satisfied if either T_0 or T_1 is a isofibration of ∞-categories (Corollary 4.5.2.28), but not in general.

7.5.3 The Homotopy Limit as a Derived Functor

Let $\mathcal{C}$ be a small category. In general, the inverse limit functor $\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{QCat}) \to \mathrm{Set}_{\Delta}$ 03AP does not respect categorical equivalence: that is, if $\alpha : \mathscr{F} \to \mathscr{G}$ is a levelwise categorical equivalence of diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{QCat}$, then the induced map $\varprojlim(\alpha) : \varprojlim(\mathscr{F}) \to \varprojlim(\mathscr{G})$ need not be a categorical equivalence of simplicial sets. In §7.5.2 and §4.5.6, we discussed two different ways of addressing this point:

- We can replace the limit $\varprojlim(\mathscr{F})$ by the homotopy limit $\operatorname{ho}\varprojlim(\mathscr{F})$ of Construction 7.5.2.1. If $\alpha : \mathscr{F} \to \mathscr{G}$ is a levelwise categorical equivalence of diagrams $\mathscr{F}, \mathscr{G}$, then Remark 7.5.2.7 guarantees that the induced map $\operatorname{ho}\varprojlim(\alpha) : \operatorname{ho}\varprojlim(\mathscr{F}) \to \operatorname{ho}\varprojlim(\mathscr{G})$ is an equivalence of ∞-categories.

- We can restrict our attention to *isofibrant* diagrams of ∞-categories (Definition 4.5.6.3). If $\alpha : \mathscr{F} \to \mathscr{G}$ is a levelwise categorical equivalence between isofibrant diagrams, then Corollary 4.5.6.16 guarantees that the induced map $\varprojlim(\alpha) : \varprojlim(\mathscr{F}) \to \varprojlim(\mathscr{G})$ is an equivalence of ∞-categories.

In this section, we will show that these perspectives are closely related: if $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ is a diagram of ∞-categories, then the homotopy limit $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ can be identified with the limit of an *isofibrant replacement* for $\mathscr{F}$. More precisely, we show that there exists a canonical isomorphism $\underleftarrow{\mathrm{holim}}(\mathscr{F}) \simeq \varprojlim(\mathscr{F}^+)$, where $\mathscr{F}^+ : \mathcal{C} \to \mathrm{QCat}$ is an isofibrant diagram of simplicial sets equipped with a levelwise categorical equivalence $\alpha : \mathscr{F} \hookrightarrow \mathscr{F}^+$ (Construction 7.5.3.3 and Proposition 7.5.3.7). Moreover, we show that for *any* isofibrant diagram $\mathscr{G} : \mathcal{C} \to \mathrm{QCat}$, the inclusion map $\varprojlim(\mathscr{G}) \hookrightarrow \underleftarrow{\mathrm{holim}}(\mathscr{G})$ is an equivalence of ∞-categories (Proposition 7.5.3.12). Consequently, if $\beta : \mathscr{F} \to \mathscr{G}$ is any levelwise categorical equivalence from $\mathscr{F}$ to an isofibrant diagram $\mathscr{G}$, then the maps

$$\underleftarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\underleftarrow{\mathrm{holim}}(\beta)} \underleftarrow{\mathrm{holim}}(\mathscr{G}) \leftarrow \varprojlim(\mathscr{G})$$

are equivalences of ∞-categories; in particular, the ∞-categories $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ and $\varprojlim(\mathscr{G})$ are equivalent (see Remark 7.5.3.15).

We begin with some elementary observations.

03AQ **Proposition 7.5.3.1.** *Let $\mathcal{C}$ be a small category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be an isofibration of ∞-categories. Then the weak transport representation*

$$\mathrm{wTr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathrm{Set}_\Delta \qquad C \mapsto \mathrm{Fun}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E})$$

of Construction 5.3.1.1 is an isofibrant diagram of simplicial sets.

Proof. Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a functor and let $\mathscr{F}_0 \subseteq \mathscr{F}$ be a subfunctor for which the inclusion $\mathscr{F}_0 \hookrightarrow \mathscr{F}$ is a levelwise categorical equivalence. We wish to show that every natural transformation $\alpha_0 : \mathscr{F}_0 \to \mathrm{wTr}_{\mathcal{E}/\mathcal{C}}$ admits an extension $\alpha : \mathscr{F} \to \mathrm{wTr}_{\mathcal{E}/\mathcal{C}}$. Using Corollary 5.3.2.23, we can reformulate this as a lifting problem

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}_0) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[r] \arrow[ur, dashed] & \mathrm{N}_\bullet(\mathcal{C})
\end{tikzcd}$$

in the category of simplicial sets. Since U is an isofibration, we are reduced to showing that the inclusion map $\underrightarrow{\mathrm{holim}}(\mathscr{F}_0) \hookrightarrow \underrightarrow{\mathrm{holim}}(\mathscr{F})$ is a categorical equivalence (Proposition 4.5.5.1), which is a special case of Variant 5.3.2.19. □

03AR **Corollary 7.5.3.2.** *Let $\mathcal{C}$ be a small category and let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories. Then the strict transport representation*

$$\mathrm{sTr}_{\mathcal{E}/\mathcal{C}} : \mathcal{C} \to \mathrm{Set}_\Delta \qquad C \mapsto \mathrm{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E})$$

of Construction 5.3.1.5 is an isofibrant diagram of simplicial sets.

Proof. Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a functor and let $\mathscr{F}_0 \subseteq \mathscr{F}$ be a subfunctor for which the inclusion $\mathscr{F}_0 \hookrightarrow \mathscr{F}$ is a levelwise categorical equivalence. Suppose we are given a natural transformation $\alpha_0 : \mathscr{F}_0 \to \mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$. It follows from Proposition 7.5.3.1 that α_0 can be extended to a natural transformation $\alpha : \mathscr{F} \to \mathrm{wTr}_{\mathcal{E}/\mathcal{C}}$. To complete the proof, it will suffice to show that α factors through the subfunctor $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}$. Equivalently, we must show that for each object $C \in \mathcal{C}$, the lifting problem

$$\begin{tikzcd}
\mathscr{F}_0(C) \arrow[r] \arrow[d] & \mathrm{sTr}_{\mathcal{E}/\mathcal{C}}(C) \arrow[d] \\
\mathscr{F}(C) \arrow[r] \arrow[ur, dashed] & \mathrm{wTr}_{\mathcal{E}/\mathcal{C}}(C)
\end{tikzcd}$$

admits a (unique) solution. This is clear, since the left vertical map is a categorical equivalence and $\mathrm{sTr}_{\mathcal{E}/\mathcal{C}}(C)$ is a replete subcategory of $\mathrm{wTr}_{\mathcal{E}/\mathcal{C}}(C)$ (see Remark 5.3.1.15). □

Construction 7.5.3.3 (Explicit Isofibrant Replacement)**.** Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a (strictly commutative) diagram of ∞-categories. Let $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ denote the $\mathscr{F}$-weighted nerve of $\mathcal{C}$ (Definition 5.3.3.1), and let $\mathscr{F}^{+} = \mathrm{sTr}_{\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})/\mathcal{C}}$ denote the strict transport representation of the projection map $\mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$. It follows from Remark 5.3.4.10 that there is a unique natural transformation $\alpha : \mathscr{F} \to \mathscr{F}^{+}$ for which the diagram of simplicial sets `03AS`

$$\begin{tikzcd}
 & \underrightarrow{\mathrm{holim}}(\mathscr{F}^{+}) \arrow[dr, "\lambda_u"] & \\
\underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[ur, "\underrightarrow{\mathrm{holim}}(\alpha)"] \arrow[rr, "\lambda_t"] & & \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}),
\end{tikzcd}$$

is commutative, where λ_u denotes the universal scaffold of Construction 5.3.4.7 and λ_t denotes the taut scaffold of Construction 5.3.4.11. We will refer to $\mathscr{F}^{+}$ as the *isofibrant replacement* of $\mathscr{F}$.

Proposition 7.5.3.4. *Let $\mathcal{C}$ be a small category, let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories, and let $\alpha : \mathscr{F} \to \mathscr{F}^{+}$ be the natural transformation of Construction 7.5.3.3. Then $\mathscr{F}^{+} : \mathcal{C} \to \mathrm{QCat}$ is an isofibrant diagram, and α is a levelwise categorical equivalence. Moreover, α is also a monomorphism.* `03AT`

Proof. It follows from Corollary 7.5.3.2 that the diagram $\mathscr{F}^+$ is isofibrant. To see that α is a monomorphism, we observe that for each object $C \in \mathcal{C}$, the functor

$$\alpha_C : \mathscr{F}(C) \to \mathscr{F}^+(C) = \operatorname{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}))$$

has a left inverse, given by the evaluation map

$$\mathrm{ev}_C : \operatorname{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})) \to \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \simeq \mathscr{F}(C).$$

Since ev_C is a trivial Kan fibration (Proposition 5.3.1.7), it follows that α_C is an equivalence of ∞-categories. ☐

03AU **Corollary 7.5.3.5** (Existence of Isofibrant Replacements)**.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets. Then there exists a monomorphism of diagrams $\alpha : \mathscr{F} \hookrightarrow \mathscr{G}$, where α is a levelwise categorical equivalence and $\mathscr{G} : \mathcal{C} \to \mathrm{QCat}$ is an isofibrant diagram of ∞-categories.*

Proof. Using Proposition 4.1.3.2, we can reduce to the case where $\mathscr{F}$ is a diagram of ∞-categories. In this case, we can take α to be the natural transformation $\mathscr{F} \hookrightarrow \mathscr{F}^+$ of Construction 7.5.3.3 (Proposition 7.5.3.4). ☐

03AV **Variant 7.5.3.6.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets. Then there exists a monomorphism of diagrams $\alpha : \mathscr{F} \hookrightarrow \mathscr{G}$, where α is a levelwise weak homotopy equivalence and $\mathscr{G} : \mathcal{C} \to \mathrm{Kan}$ is an isofibrant diagram of Kan complexes.

Proof. Using Proposition 3.1.7.1, we can reduce to the case where $\mathscr{F}$ is a diagram of Kan complexes. In this case, we can again take α to be the natural transformation $\mathscr{F} \hookrightarrow \mathscr{F}^+$ of Construction 7.5.3.3 (since α is a levelwise categorical equivalence, it follows that $\mathscr{F}^+$ is also a diagram of Kan complexes: see Remark 4.5.1.21). ☐

03AW **Proposition 7.5.3.7.** *Let $\mathcal{C}$ be a small category, let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram, and let $\mathscr{F}^+ : \mathcal{C} \to \mathrm{QCat}$ be the isofibrant replacement of Construction 7.5.3.3. Then there is a canonical isomorphism of simplicial sets $\theta : \underleftarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\sim} \underleftarrow{\lim}(\mathscr{F}^+)$, which is characterized by the following requirement: for each object $C \in \mathcal{C}$, the composition*

$$\begin{aligned} \operatorname{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}), \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})) &= \underleftarrow{\mathrm{holim}}(\mathscr{F}) \\ &\xrightarrow{\theta} \underleftarrow{\lim}(\mathscr{F}^+) \\ &\to \mathscr{F}^+(C) \\ &= \operatorname{Fun}^{\mathrm{CCart}}_{/\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})) \end{aligned}$$

is given by precomposition with the projection map $\mathcal{C}_{C/} \to \mathcal{C}$.

Proposition 7.5.3.7 is a consequence of the following concrete assertion:

Lemma 7.5.3.8. *Let $\mathcal{C}$ be a category. Then the collection of projection maps $\{\mathrm{N}_\bullet(\mathcal{C}_{C/}) \to \mathrm{N}_\bullet(\mathcal{C})\}_{C \in \mathcal{C}}$ exhibit $\mathrm{N}_\bullet(\mathcal{C})$ as the colimit of the diagram* `03AX`

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_\Delta \qquad C \mapsto \mathrm{N}_\bullet(\mathcal{C}_{C/}).$$

Proof. Fix an integer $n \geq 0$; we wish to show that the canonical map

$$\rho : \varinjlim_{C \in \mathcal{C}^{\mathrm{op}}} \mathrm{N}_n(\mathcal{C}_{C/}) \to \mathrm{N}_n(\mathcal{C})$$

is an isomorphism in the category of sets. Let σ be an n-simplex of $\mathrm{N}_n(\mathcal{C})$, given by a diagram

$$X_0 \to X_1 \to \cdots \to X_n$$

in the category $\mathcal{C}$. Then the fiber $\rho^{-1}\{\sigma\}$ can be identified with the colimit

$$\varinjlim_{C \in \mathcal{C}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(C, X_0),$$

formed in the category of sets. This colimit consists of a single element, represented by the identity morphism $\mathrm{id}_{X_0} \in \mathrm{Hom}_{\mathcal{C}}(X_0, X_0)$. □

Remark 7.5.3.9. Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ is a morphism of simplicial sets, and let $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathcal{C} \to \mathrm{Set}_\Delta$ denote the weak transport representation of Construction 5.3.1.1, given on objects by the formula $\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) = \mathrm{Fun}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E})$. Then Lemma 7.5.3.8 supplies a canonical isomorphism of simplicial sets `03AY`

$$\mathrm{Fun}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}), \mathcal{E}) \xrightarrow{\sim} \varprojlim(\mathrm{wTr}_{\mathcal{E}\,/\,\mathcal{C}}).$$

Variant 7.5.3.10. Let $\mathcal{C}$ be a category, let $U : \mathcal{E} \to \mathrm{N}_\bullet(\mathcal{C})$ be a cocartesian fibration of ∞-categories, and let $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathcal{C} \to \mathrm{Set}_\Delta$ denote the strict transport representation of Construction 5.3.1.5, given on objects by the formula $\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}(C) = \mathrm{Fun}^{\mathrm{CCart}}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}_{C/}), \mathcal{E})$. Then the isomorphism of Remark 7.5.3.9 restricts to an isomorphism of simplicial sets `03AZ`

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\,\mathrm{N}_\bullet(\mathcal{C})}(\mathrm{N}_\bullet(\mathcal{C}), \mathcal{E}) \xrightarrow{\sim} \varprojlim(\mathrm{sTr}_{\mathcal{E}\,/\,\mathcal{C}}).$$

Proof of Proposition 7.5.3.7. Apply Variant 7.5.3.10 in the special case where $\mathcal{E} = \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$ is the $\mathscr{F}$-weighted nerve of the category $\mathcal{C}$. □

Remark 7.5.3.11. In the situation of Proposition 7.5.3.7, the isomorphism $\theta : \operatorname{ho}\varprojlim(\mathscr{F}) \xrightarrow{\sim} \varprojlim(\mathscr{F}^+)$ fits into a commutative diagram `03B0`

$$\begin{tikzcd}
 & \operatorname{ho}\varprojlim(\mathscr{F}) \arrow[dr, "\theta", "\sim"'] & \\
\varprojlim(\mathscr{F}) \arrow[ur, "\iota"] \arrow[rr, "\varprojlim(\alpha)"] & & \varprojlim(\mathscr{F}^+),
\end{tikzcd}$$

where ι is the comparison map of Remark 7.5.2.12 and $\alpha : \mathscr{F} \hookrightarrow \mathscr{F}^{+}$ is the natural transformation appearing in Construction 7.5.3.3.

03B1 **Proposition 7.5.3.12.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be an isofibrant diagram of ∞-categories. Then the inclusion map $\iota : \varprojlim(\mathscr{F}) \hookrightarrow \operatorname{ho}\varprojlim(\mathscr{F})$ is an equivalence of ∞-categories.*

Proof. Let $\alpha : \mathscr{F} \hookrightarrow \mathscr{F}^{+}$ be the isofibrant replacement of Construction 7.5.3.3. By virtue of Proposition 7.5.3.7 (and Remark 7.5.3.11), it will suffice to show that the limit $\varprojlim(\alpha) : \varprojlim(\mathscr{F}) \hookrightarrow \varprojlim(\mathscr{F}^{+})$ is an equivalence of ∞-categories. This is a special case of Corollary 4.5.6.17, since α is a levelwise categorical equivalence between isofibrant diagrams (Proposition 7.5.3.4). □

03B2 **Example 7.5.3.13** (Towers of Isofibrations)**.** Suppose we are given a tower of ∞-categories

$$\cdots \to \mathcal{C}(3) \to \mathcal{C}(2) \to \mathcal{C}(1) \to \mathcal{C}(0),$$

which we identify with a functor $\mathscr{F} : \mathbf{Z}_{\geq 0}^{\mathrm{op}} \to \mathrm{QCat}$. If each of the transition functors $\mathcal{C}(n{+}1) \to \mathcal{C}(n)$ is an isofibration, then the comparison map $\varprojlim_{n} \mathcal{C}(n) = \varprojlim(\mathscr{F}) \hookrightarrow \operatorname{ho}\varprojlim(\mathscr{F})$ is an equivalence of ∞-categories. This follows by combining Example 4.5.6.8 with Proposition 7.5.3.12.

03B3 **Warning 7.5.3.14.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a strictly commutative diagram of ∞-categories and let $\alpha : \mathscr{F} \hookrightarrow \mathscr{F}^{+}$ denote the isofibrant replacement of Construction 7.5.3.3, and let $\theta : \operatorname{ho}\varprojlim(\mathscr{F}) \xrightarrow{\sim} \varprojlim(\mathscr{F}^{+})$ be the isomorphism of Proposition 7.5.3.7. We then have a diagram of simplicial sets

$$\begin{tikzcd}
\varprojlim(\mathscr{F}) \arrow[r, "\varprojlim(\alpha)"] \arrow[d] & \varprojlim(\mathscr{F}^{+}) \arrow[d] \\
\operatorname{ho}\varprojlim(\mathscr{F}) \arrow[r, "\operatorname{ho}\varprojlim(\alpha)"'] \arrow[ur, "\theta", "\sim"'] & \operatorname{ho}\varprojlim(\mathscr{F}^{+}),
\end{tikzcd}$$

where the outer square and the upper left triangle are commutative (Remark 7.5.3.11). Beware that the lower right triangle is usually *not* commutative. That is, $\operatorname{ho}\varprojlim(\mathscr{F})$ and $\varprojlim(\mathscr{F}^{+})$ are isomorphic when viewed as *abstract* simplicial sets, but do not coincide when identified with simplicial subsets of $\operatorname{ho}\varprojlim(\mathscr{F}^{+})$.

03B4 **Remark 7.5.3.15** (The Homotopy Limit as a Right Derived Functor)**.** The results of this section can be interpreted in the language of model categories. For every small category $\mathcal{C}$, the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$ can be equipped with a model structure in which the cofibrations are monomorphisms and the weak equivalences are levelwise categorical equivalences (see Example [?]). The inverse limit functor

$$\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to \mathrm{Set}_{\Delta}$$

then admit a *right derived functor* $\mathrm{R}\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta}) \to \mathrm{Set}_{\Delta}$, which carries a diagram $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ to the limit of a fibrant replacement of $\mathscr{F}$. It follows from Propositions 7.5.3.4 and 7.5.3.7 that, when restricted to the subcategory $\mathrm{Fun}(\mathcal{C}, \mathrm{QCat}) \subset \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$, the functor $\mathrm{R}\varprojlim$ is (categorically) equivalent to the homotopy limit functor $\mathrm{ho}\varprojlim : \mathrm{Fun}(\mathcal{C}, \mathrm{QCat}) \to \mathrm{QCat}$ of Construction 7.5.2.1. We will return to this point in §[?].

7.5.4 Homotopy Limit Diagrams

Let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram in the category of Kan complexes, and let 03B5

$$\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$$

be the induced functor of ∞-categories. Then the homotopy limit $\varprojlim(\mathscr{F})$ is a Kan complex, which can be regarded as a limit of the diagram $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F})$ in the ∞-category of spaces $\mathcal{S}$ (Proposition 7.5.1.5). For many applications, this assertion is insufficiently precise: we would like to have not only a Kan complex X which is known *abstractly* to be a limit of the diagram $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F})$, but also a diagram $\mathrm{N}_{\bullet}(\mathcal{C}^{\triangleleft}) \to \mathcal{S}$ which *exhibits* X as a limit of $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F})$.

Definition 7.5.4.1. Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ be a functor having restriction $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$. We will say that $\overline{\mathscr{F}}$ is a *homotopy limit diagram* if the composite map 03B6

$$\overline{\mathscr{F}}(\mathbf{0}) \to \varprojlim(\mathscr{F}) \hookrightarrow \mathrm{ho}\varprojlim(\mathscr{F})$$

is a homotopy equivalence of Kan complexes; here $\mathbf{0}$ denotes the initial object of the cone $\mathcal{C}^{\triangleleft} \simeq \{\mathbf{0}\} \star \mathcal{C}$, and the morphism on the right is the comparison map of Remark 7.5.2.12).

Example 7.5.4.2 (Limits of Isofibrant Diagrams)**.** Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ be a limit diagram in the category of simplicial sets. Suppose that the diagram $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$ is isofibrant (Definition 4.5.6.3) and that, for each object $C \in \mathcal{C}$, the simplicial set $\mathscr{F}(C)$ is a Kan complex. Then $\overline{\mathscr{F}}$ is a homotopy limit diagram of Kan complexes: this follows by combining Corollary 4.5.6.20 with Proposition 7.5.3.12. 03B7

Warning 7.5.4.3. For every diagram of Kan complexes $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$, the homotopy limit $\mathrm{ho}\varprojlim(\mathscr{F})$ of Construction 7.5.1.1 is well-defined. However, one cannot always extend $\mathscr{F}$ to a homotopy limit *diagram* $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ (see Warning 3.4.1.8). This is possible only if the tautological map $\varprojlim(\mathscr{F}) \hookrightarrow \mathrm{ho}\varprojlim(\mathscr{F})$ has a left homotopy inverse. However, we can always choose a levelwise homotopy equivalence $\alpha : \mathscr{F} \hookrightarrow \mathscr{G}$, where $\mathscr{G}$ is an isofibrant diagram of 03B8

Kan complexes (Variant 7.5.3.6). We can then extend $\mathscr{G}$ can be extended to a limit diagram $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$, which is also a homotopy limit diagram (Example 7.5.4.2). Moreover, if take $\mathscr{G} = \mathscr{F}^{+}$ to be the isofibrant replacement of Construction 7.5.3.3, then $\overline{\mathscr{G}}$ carries the initial object of $\mathcal{C}^{\triangleleft}$ to the homotopy limit $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ (Proposition 7.5.3.7).

03B9 **Proposition 7.5.4.4** (Homotopy Invariance)**.** *Let $\mathcal{C}$ be a category and let $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ be a natural transformation between diagrams $\overline{\mathscr{F}}, \overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$. Assume that, for every object $C \in \mathcal{C}$, the induced map $\alpha_C : \overline{\mathscr{F}}(C) \to \overline{\mathscr{G}}(C)$ is a homotopy equivalence of Kan complexes. Then any two of the following conditions imply the third:*

(1) *The functor $\overline{\mathscr{F}}$ is a homotopy limit diagram.*

(2) *The functor $\overline{\mathscr{G}}$ is a homotopy limit diagram.*

(3) *The natural transformation α induces a homotopy equivalence $\overline{\mathscr{F}}(\mathbf{0}) \to \overline{\mathscr{G}}(\mathbf{0})$, where $\mathbf{0}$ denotes the cone point of $\mathcal{C}^{\triangleleft}$.*

Proof. Setting $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$ and $\mathscr{G} = \overline{\mathscr{G}}|_{\mathcal{C}}$, we observe that α determines a commutative diagram of Kan complexes

$$\begin{tikzcd}
\overline{\mathscr{F}}(\mathbf{0}) \arrow[r] \arrow[d] & \overline{\mathscr{G}}(\mathbf{0}) \arrow[d] \\
\underleftarrow{\mathrm{holim}}(\mathscr{F}) \arrow[r] & \underleftarrow{\mathrm{holim}}(\mathscr{G}),
\end{tikzcd}$$

where the bottom horizontal map is a homotopy equivalence (Remark 7.5.1.3). The desired result now follows from the two-out-of-three property (Remark 3.1.6.7). □

03BA **Proposition 7.5.4.5.** *Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ be a diagram of Kan complexes. Then $\overline{\mathscr{F}}$ is a homotopy limit diagram (in the sense of Definition 7.5.4.1) if and only if the induced functor of ∞-categories*

$$\mathrm{N}_{\bullet}^{\mathrm{hc}}(\overline{\mathscr{F}}) : \mathrm{N}_{\bullet}(\mathcal{C}^{\triangleleft}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$$

is a limit diagram (in the sense of Definition 7.1.2.4).

Proof. Let $\mathrm{N}_{\bullet}^{\overline{\mathscr{F}}}(\mathcal{C}^{\triangleleft})$ be the weighted nerve of the functor $\overline{\mathscr{F}}$ (Definition 5.3.3.1) and let $U : \mathrm{N}_{\bullet}^{\overline{\mathscr{F}}}(\mathcal{C}^{\triangleleft}) \to \mathrm{N}_{\bullet}(\mathcal{C}^{\triangleleft})$ be the projection map. Then U is a left fibration, and $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\overline{\mathscr{F}})$ is a covariant transport representation for U (Example 5.6.5.6). Set $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$. Applying Corollary 7.4.5.13, we deduce that $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\overline{\mathscr{F}})$ is a limit diagram in the ∞-category $\mathcal{S}$ if and only if the restriction map

$$\rho : \mathrm{Fun}_{/\,\mathrm{N}_{\bullet}(\mathcal{C}^{\triangleleft})}(\mathrm{N}_{\bullet}(\mathcal{C}^{\triangleleft}), \mathrm{N}_{\bullet}^{\overline{\mathscr{F}}}(\mathcal{C}^{\triangleleft})) \xrightarrow{\simeq} \mathrm{Fun}_{/\,\mathrm{N}_{\bullet}(\mathcal{C})}(\mathrm{N}_{\bullet}(\mathcal{C}), \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C}))$$

is a homotopy equivalence of Kan complexes. We then have a commutative diagram ρ fits into a commutative diagram

$$\begin{tikzcd}
\varprojlim(\overline{\mathscr{F}}) \arrow[r, "\rho'"] \arrow[d, "\overline{\iota}"] & \varprojlim(\mathscr{F}) \arrow[d, "\iota"] \\
\operatorname{holim}_{\longleftarrow}(\overline{\mathscr{F}}) \arrow[r, "\rho"] & \operatorname{holim}_{\longleftarrow}(\mathscr{F}),
\end{tikzcd}$$

where ι and $\overline{\iota}$ are the comparison maps of Remark 7.5.2.12. Since the category $\mathcal{C}^{\triangleleft}$ has an initial object, the morphism $\overline{\iota}$ is a homotopy equivalence (Proposition 7.5.2.13). It follows that ρ is a homotopy equivalence if and only if the composition $\rho \circ \overline{\iota} = \iota \circ \rho'$ is a homotopy equivalence. We conclude by observing that the composition $\iota \circ \rho'$ can be identified with the map $\overline{\mathscr{F}}(\mathbf{0}) \to \operatorname{holim}_{\longleftarrow}(\mathscr{F})$ appearing in Definition 7.5.4.1. □

Corollary 7.5.4.6. *Let $\mathcal{C}$ be a small category, let $\mathcal{D}$ be a locally Kan simplicial category, and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathcal{D}$ be a functor. The following conditions are equivalent:* `03BB`

(1) *The functor*

$$\mathrm{N}_{\bullet}^{\mathrm{hc}}(\overline{\mathscr{F}}) : \mathrm{N}_{\bullet}(\mathcal{C})^{\triangleleft} \simeq \mathrm{N}_{\bullet}(\mathcal{C}^{\triangleleft}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{D})$$

is a limit diagram in the ∞-category $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{D})$.

(2) *For every object $D \in \mathcal{D}$, the functor*

$$\mathcal{C}^{\triangleleft} \to \mathrm{Kan} \qquad C \mapsto \mathrm{Hom}_{\mathcal{D}}(D, \mathscr{F}(C))_{\bullet}$$

is a homotopy limit diagram of Kan complexes.

Proof. By virtue of Proposition 7.4.5.17, condition (1) is satisfied if and only if, for every object $D \in \mathcal{D}$, the composition $(h^D \circ \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathscr{F})) : \mathrm{N}_{\bullet}(\mathcal{C})^{\triangleleft} \to \mathcal{S}$ is a limit diagram in the ∞-category $\mathcal{S}$, where $h^D : \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{D}) \to \mathcal{S}$ denotes a functor corepresented by D. Using Proposition 5.6.6.17, we can take h^D to be the homotopy coherent nerve of the simplicial functor $\mathrm{Hom}_{\mathcal{D}}(D, \bullet) : \mathcal{D} \to \mathrm{Kan}$. In this case, $h^D \circ \mathrm{N}_{\bullet}^{\mathrm{hc}}(X)$ is the homotopy coherent nerve of the functor $C \mapsto \mathrm{Hom}_{\mathcal{D}}(D, \mathscr{F}(C))_{\bullet}$. The equivalence (1) $\Leftrightarrow$ (2) now follows from the criterion of Proposition 7.5.4.5. □

Corollary 7.5.4.7. *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to$* Kan *be an isofibrant diagram of Kan complexes. Then $\mathscr{F}$ has a limit in the category* Kan*, which is preserved by the inclusion functor* $\mathrm{N}_{\bullet}(\mathrm{Kan}) \hookrightarrow \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$. `03BC`

Proof. Combine Example 7.5.4.2 with Proposition 7.5.4.5. □

For some applications, it is useful to extend Definition 7.5.4.1 to diagrams of simplicial sets which do not take values in the full subcategory $\mathrm{Kan} \subset \mathrm{Set}_{\Delta}$ of Kan complexes.

03BD **Definition 7.5.4.8** (Homotopy Limit Diagrams of Simplicial Sets)**.** Let $\mathcal{C}$ be a small category. We say that a functor $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ is a *homotopy limit diagram* if there exists a levelwise weak homotopy equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$, where $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ is a homotopy limit diagram of Kan complexes (in the sense of Definition 7.5.4.1).

03BE **Remark 7.5.4.9.** Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ be a functor. The following conditions are equivalent:

(1) The functor $\overline{\mathscr{F}}$ is a homotopy limit diagram in the sense of Definition 7.5.4.1 (that is, it induces a homotopy equivalence $\overline{\mathscr{F}}(\mathbf{0}) \to \underleftarrow{\mathrm{holim}}(\overline{\mathscr{F}}|_{\mathcal{C}})$, where $\mathbf{0}$ denotes the cone point of $\mathcal{C}^{\triangleleft}$).

(2) The functor $\overline{\mathscr{F}}$ is a homotopy limit diagram in the sense of Definition 7.5.4.8: that is, there exists a homotopy limit diagram of Kan complexes $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ and a levelwise weak homotopy equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$.

The implication (1) ⇒ (2) is immediate, and the reverse implication follows from Proposition 7.5.4.4.

03BF **Proposition 7.5.4.10.** *Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\overline{\mathscr{F}}$ is a homotopy limit diagram. That is, there exists a homotopy limit diagram $\overline{\mathscr{F}}' : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ and a levelwise weak homotopy equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{F}}'$.*

(2) *Let $\overline{\mathscr{F}}' : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ be any functor. If there exists a levelwise weak homotopy equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{F}}'$, then $\overline{\mathscr{F}}'$ is a homotopy limit diagram.*

Proof. Using Proposition 3.1.7.1, we can choose a functor $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ and a natural transformation $\beta : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ which carries each object $C \in \mathcal{C}^{\triangleleft}$ to an anodyne morphism of simplicial sets $\beta_C : \overline{\mathscr{F}}(C) \to \overline{\mathscr{G}}(C)$. We will show that (1) and (2) are equivalent to the following:

(3) The functor $\overline{\mathscr{G}}$ is a homotopy limit diagram.

The implications (2) ⇒ (3) ⇒ (1) are immediate. To prove the reverse implications, suppose we are given another functor $\overline{\mathscr{F}}' : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ and a levelwise weak homotopy equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{F}}'$. We will show that $\overline{\mathscr{F}}'$ is a homotopy limit diagram if and only if $\overline{\mathscr{G}}$ is a homotopy limit diagram.

Applying Proposition 3.1.7.1 again, we can choose a functor $\overline{\mathscr{G}}' : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ and a commutative diagram

$$\begin{tikzcd}
\overline{\mathscr{F}} \arrow[r, "\alpha"] \arrow[d, "\beta"] & \overline{\mathscr{F}}' \arrow[d, "\beta'"] \\
\overline{\mathscr{G}} \arrow[r, "\alpha'"] & \overline{\mathscr{G}}'
\end{tikzcd}$$

with the property that, for every object $C \in \mathcal{C}^{\triangleleft}$, the induced map

$$\overline{\mathscr{F}}'(C) \coprod_{\overline{\mathscr{F}}(C)} \overline{\mathscr{G}}(C) \to \overline{\mathscr{G}}'(C)$$

is anodyne (and, in particular, a weak homotopy equivalence). Applying Proposition 3.4.2.11, we see that the diagram of simplicial sets

$$\begin{tikzcd}
\overline{\mathscr{F}}(C) \arrow[r, "\alpha_C"] \arrow[d, "\beta_C"] & \overline{\mathscr{F}}'(C) \arrow[d, "\beta'_C"] \\
\overline{\mathscr{G}}(C) \arrow[r, "\alpha'_C"] & \overline{\mathscr{G}}'(C)
\end{tikzcd}$$

is a homotopy pushout square. Since α_C and β_C are weak homotopy equivalences, it follows that α'_C and β'_C are also weak homotopy equivalences (Proposition 3.4.2.10). Applying Proposition 7.5.4.4, we see that $\overline{\mathscr{F}}'$ and $\overline{\mathscr{G}}$ are homotopy limit diagrams if and only if $\overline{\mathscr{G}}'$ is a homotopy limit diagram. □

Corollary 7.5.4.11 (Homotopy Invariance)**.** *Let $\mathcal{C}$ be a category, let $\overline{\mathscr{F}}, \overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ be functors, and let $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ be a natural transformation. Suppose that, for every object $C \in \mathcal{C}$, the induced map $\alpha_C : \overline{\mathscr{F}}(C) \to \overline{\mathscr{G}}(C)$ is a weak homotopy equivalence. Then any two of the following conditions imply the third:* `03BG`

(1) *The functor $\overline{\mathscr{F}}$ is a homotopy limit diagram.*

(2) *The functor $\overline{\mathscr{G}}$ is a homotopy limit diagram.*

(3) *The natural transformation α induces a weak homotopy equivalence $\overline{\mathscr{F}}(\mathbf{0}) \to \overline{\mathscr{G}}(\mathbf{0})$, where $\mathbf{0}$ denotes the cone point of $\mathcal{C}^{\triangleleft}$.*

Proof. Using Proposition 3.1.7.1, we can choose functors $\overline{\mathscr{F}}', \overline{\mathscr{G}}' : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ and a commutative diagram

$$\begin{tikzcd}
\overline{\mathscr{F}} \arrow[r, "\alpha"] \arrow[d] & \overline{\mathscr{G}} \arrow[d] \\
\overline{\mathscr{F}}' \arrow[r, "\alpha'"] & \overline{\mathscr{G}}',
\end{tikzcd}$$

where the vertical maps are levelwise weak homotopy equivalences. Using Proposition 7.5.4.10, we can replace α by the natural transformation $\alpha' : \overline{\mathscr{F}}' \to \overline{\mathscr{G}}'$, in which case the desired result follows from Proposition 7.5.4.4. □

03BH **Corollary 7.5.4.12.** *Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ be a functor. Let $\overline{\mathscr{F}}^{\mathrm{op}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ be the functor given on objects by $\overline{\mathscr{F}}^{\mathrm{op}}(C) = \overline{\mathscr{F}}(C)^{\mathrm{op}}$. Then $\overline{\mathscr{F}}$ is a homotopy limit diagram if and only if $\overline{\mathscr{F}}^{\mathrm{op}}$ is a homotopy limit diagram.*

Proof. For each object $C \in \mathcal{C}^{\triangleleft}$, let $|\overline{\mathscr{F}}(C)|$ denote the geometric realization of the simplicial set $\overline{\mathscr{F}}(C)$ (Definition 1.2.3.1). Then the construction $C \mapsto \mathrm{Sing}_{\bullet}(|\overline{\mathscr{F}}(C)|)$ determines a functor $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$, and the unit maps $\overline{\mathscr{F}}(C) \to \mathrm{Sing}_{\bullet}(|\overline{\mathscr{F}}(C)|)$ determine a levelwise weak homotopy equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ (Theorem 3.6.4.1). By virtue of Corollary 7.5.4.11, it will suffice to show that the functor $\overline{\mathscr{G}}$ is a homotopy limit diagram if and only if $\overline{\mathscr{G}}^{\mathrm{op}}$ is a homotopy limit diagram. This is clear, since the functors $\overline{\mathscr{G}}$ and $\overline{\mathscr{G}}^{\mathrm{op}}$ are isomorphic (see Example 1.4.2.5). □

The notion of homotopy pullback square (see §3.4.1) can be regarded as a special case of the notion of homotopy limit diagram:

03BJ **Proposition 7.5.4.13.** *Suppose we are given a commutative diagram of simplicial sets*

03BK
$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X,
\end{tikzcd} \tag{7.49}$$

which we identify with a functor $\mathscr{F} : [1] \times [1] \to \mathrm{Set}_{\Delta}$. Then (7.49) is a homotopy pullback square (in the sense of Definition 3.4.1.1) if and only if $\mathscr{F}$ is a homotopy limit diagram (in the sense of Definition 7.5.4.8).

Proof. Using Proposition 3.1.7.1, we can choose a levelwise weak homotopy equivalence $\alpha : \mathscr{F} \to \mathscr{F}'$, where $\mathscr{F}'$ is a diagram of Kan complexes. Using Corollaries 3.4.1.12 and

7.5.4.11, we can replace $\mathscr{F}$ by $\mathscr{F}'$ and thereby reduce to the case where (7.49) is a diagram of Kan complexes. By virtue of Corollary 3.4.1.6, the diagram (7.49) is a homotopy pullback square if and only if it induces a homotopy equivalence $f : X_{01} \to X_0 \times^{\mathrm{h}}_X X_1$, where $X_0 \times^{\mathrm{h}}_X X_1$ is the homotopy fiber product of Construction 3.4.0.3. On the other hand, $\mathscr{F}$ is a homotopy limit diagram if and only if the composition $\iota \circ f$ is a homotopy equivalence, where

$$\iota : X_0 \times^{\mathrm{h}}_X X_1 \hookrightarrow X_0 \times^{\mathrm{h}}_X (X_1 \times^{\mathrm{h}}_X X) \simeq \underleftarrow{\mathrm{holim}}(F)$$

is the comparison map described in Example 7.5.2.17. The desired result now follows from the observation that ι is a homotopy equivalence (see Example 7.5.2.17). □

7.5.5 Categorical Limit Diagrams

The theory of homotopy limit diagrams introduced in §7.5.4 should be regarded as belonging to the "classical" homotopy theory of simplicial sets: for example, it is invariant under weak homotopy equivalence (Corollary 7.5.4.11). When using simplicial sets to model higher category theory (rather than homotopy theory), it is useful to work with slightly different class of diagrams. 03BL

Definition 7.5.5.1 (Categorical Limit Diagrams of ∞-Categories)**.** Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ be a functor having restriction $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$. We will say that $\overline{\mathscr{F}}$ is a *categorical limit diagram* if the composite map 03BM

$$\overline{\mathscr{F}}(\mathbf{0}) \to \underleftarrow{\lim}(\mathscr{F}) \hookrightarrow \underleftarrow{\mathrm{holim}}(\mathscr{F})$$

is an equivalence of ∞-categories; here $\mathbf{0}$ denotes the initial object of the cone $\mathcal{C}^{\triangleleft} \simeq \{\mathbf{0}\} \star \mathcal{C}$, and the morphism on the right is the comparison map of Remark 7.5.2.12.

Example 7.5.5.2. Let $\mathcal{C}$ be a category. A diagram of Kan complexes $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ is a categorical limit diagram (in the sense of Definition 7.5.5.1) if and only if it is a homotopy limit diagram (in the sense of Definition 7.5.4.1). 03BN

Example 7.5.5.3 (Limits of Isofibrant Diagrams)**.** Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ be a limit diagram in the category of simplicial sets. Suppose that the diagram $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$ is isofibrant (Definition 4.5.6.3). Then $\overline{\mathscr{F}}$ is a categorical limit diagram of ∞-categories: this follows by combining Corollary 4.5.6.13 with Proposition 7.5.3.12. 03BP

Warning 7.5.5.4. Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ be a diagram of ∞-categories. In general, the condition that $\overline{\mathscr{F}}$ is a categorical limit diagram (in the sense of Definition 7.5.5.1) is independent of the condition that it is a homotopy limit diagram (in the sense of Definition 7.5.4.8): see Exercises 4.5.2.12 and 4.5.2.13. 03BQ

03BR **Remark 7.5.5.5.** Let $\mathcal{C}$ be a category, let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ be a categorical limit diagram of ∞-categories, and define $\overline{\mathscr{F}}^{\simeq} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ by the formula $\overline{\mathscr{F}}^{\simeq}(C) = \overline{\mathscr{F}}(C)^{\simeq}$. Then $\overline{\mathscr{F}}^{\simeq}$ is a homotopy limit diagram. This follows by combining Example 7.5.2.8 with Remark 4.5.1.20.

03BS **Remark 7.5.5.6** (Homotopy Invariance)**.** Let $\mathcal{C}$ be a small category and let $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ be a natural transformation between diagrams $\overline{\mathscr{F}}, \overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$. Assume that, for every object $C \in \mathcal{C}$, the induced map $\alpha_C : \overline{\mathscr{F}}(C) \to \overline{\mathscr{G}}(C)$ is an equivalence of ∞-categories. Then α determines a commutative diagram of ∞-categories

$$\begin{tikzcd}
\overline{\mathscr{F}}(\mathbf{0}) \arrow[r] \arrow[d] & \underleftarrow{\mathrm{holim}}(\overline{\mathscr{F}}|_{\mathcal{C}}) \arrow[d] \\
\overline{\mathscr{G}}(\mathbf{0}) \arrow[r] & \underleftarrow{\mathrm{holim}}(\overline{\mathscr{G}}|_{\mathcal{C}}),
\end{tikzcd}$$

where the right vertical map is an equivalence (Remark 7.5.2.7). It follows that any two of the following conditions imply the third:

(1) The functor $\overline{\mathscr{F}}$ is a categorical limit diagram.

(2) The functor $\overline{\mathscr{G}}$ is a categorical limit diagram.

(3) The natural transformation α induces an equivalence of ∞-categories $\overline{\mathscr{F}}(\mathbf{0}) \to \overline{\mathscr{G}}(\mathbf{0})$, where $\mathbf{0}$ denotes the cone point of $\mathcal{C}^{\triangleleft}$.

03BT **Proposition 7.5.5.7.** *Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\overline{\mathscr{F}}$ is a categorical limit diagram, in the sense of Definition 7.5.5.1.*

(2) *For every simplicial set K, the functor*

$$\overline{\mathscr{F}}^{K} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat} \qquad C \mapsto \mathrm{Fun}(K, \overline{\mathscr{F}}(C))$$

is a categorical limit diagram.

(3) *For every simplicial set K, the functor*

$$(\overline{\mathscr{F}}^{K})^{\simeq} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan} \qquad C \mapsto \mathrm{Fun}(K, \overline{\mathscr{F}}(C))^{\simeq}$$

is a homotopy limit diagram.

(4) *The functor $(\overline{\mathscr{F}}^{\Delta^1})^{\simeq} : \mathcal{C}^{\triangleleft} \to \mathrm{Kan}$ is a homotopy limit diagram.*

Proof. The implication (1) ⇒ (2) follows from Remarks 7.5.2.3 and 4.5.1.16, the implication (2) ⇒ (3) from Remark 7.5.5.5, and the implication (3) ⇒ (4) is immediate. Set $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$, and let $\mathbf{0}$ denote the initial object of $\mathcal{C}^{\triangleleft}$. Using Remark 7.5.2.3 and Example 7.5.2.8, we see that condition (4) is equivalent to the requirement that the map $\overline{\mathscr{F}}(\mathbf{0}) \to \underset{\longleftarrow}{\mathrm{holim}}(\mathscr{F})$ induces a homotopy equivalence of Kan complexes

$$\mathrm{Fun}(\Delta^1, \overline{\mathscr{F}}(\mathbf{0}))^{\simeq} \to \mathrm{Fun}(\Delta^1, \underset{\longleftarrow}{\mathrm{holim}}(\mathscr{F}))^{\simeq} \simeq \underset{\longleftarrow}{\mathrm{holim}}((\overline{\mathscr{F}}^{\Delta^1})^{\simeq}).$$

The implication (4) ⇒ (1) now follows from Theorem 4.5.7.1. □

Corollary 7.5.5.8. *Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ be a functor. Then $\overline{\mathscr{F}}$ is a categorical limit diagram if and only if the induced functor of ∞-categories* 03BU

$$\mathrm{N}^{\mathrm{hc}}_{\bullet}(\overline{\mathscr{F}}) : \mathrm{N}_{\bullet}(\mathcal{C}^{\triangleleft}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}) = \mathcal{QC}$$

is a limit diagram in the ∞-category $\mathcal{QC}$ (in the sense of Definition 7.1.2.4).

Proof. By virtue of Corollary 7.5.4.6, the diagram $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\overline{\mathscr{F}})$ is a limit diagram in the ∞-category $\mathcal{QC}$ if and only if, for every ∞-category $\mathcal{E}$, the diagram of Kan complexes

$$\mathcal{C}^{\triangleleft} \to \mathrm{Kan} \qquad C \mapsto \mathrm{Hom}_{\mathrm{QCat}}(\mathcal{E}, \overline{\mathscr{F}}(C))_{\bullet} = \mathrm{Fun}(\mathcal{E}, \overline{\mathscr{F}}(C))^{\simeq}$$

is a homotopy limit diagram. Using Proposition 7.5.5.7, we see that this is equivalent to the requirement that $\overline{\mathscr{F}}$ is a categorical limit diagram. □

Corollary 7.5.5.9. *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be an isofibrant diagram of ∞-categories Then $\mathscr{F}$ has a limit in the category* QCat*, which is preserved by the inclusion functor* $\mathrm{N}_{\bullet}(\mathrm{QCat}) \hookrightarrow \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}) = \mathcal{QC}$. 03BV

Proof. Combine Example 7.5.5.3 with Corollary 7.5.5.8. □

Corollary 7.5.5.10. *Suppose we are given a commutative diagram of ∞-categories* 03BW

03BX

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C},
\end{tikzcd} \tag{7.50}$$

which we identify with a functor $\overline{\mathscr{F}} : [1] \times [1] \to \mathcal{QC}$. The following conditions are equivalent:

(1) *The diagram (7.50) is a categorical pullback square, in the sense of Definition 4.5.2.8.*

(2) *The functor $\overline{\mathscr{F}}$ is a categorical limit diagram, in the sense of Definition 7.5.5.1.*

Proof. Using Proposition 4.5.2.14, we can restate (2) as follows:

(2′) For every simplicial set K, the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Fun}(K, \mathcal{C}_{01})^{\simeq} \arrow[r] \arrow[d] & \mathrm{Fun}(K, \mathcal{C}_0)^{\simeq} \arrow[d] \\
\mathrm{Fun}(K, \mathcal{C}_1)^{\simeq} \arrow[r] & \mathrm{Fun}(K, \mathcal{C})^{\simeq}
\end{tikzcd}$$

is a homotopy pullback square.

The equivalence (1) ⇔ (2′) follows by combining Propositions 7.5.4.13, and 7.5.5.7. □

We now extend the scope of Definition 7.5.5.1 to arbitrary diagrams of simplicial sets.

03BY **Definition 7.5.5.11** (Categorical Limit Diagrams of Simplicial Sets)**.** Let $\mathcal{C}$ be a category. We say that a functor $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ is a *categorical limit diagram* if there exists a levelwise categorical equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$, where $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ is a categorical limit diagram (in the sense of Definition 7.5.5.1).

03BZ **Remark 7.5.5.12.** Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ be a functor. The following conditions are equivalent:

(1) The functor $\overline{\mathscr{F}}$ is a categorical limit diagram in the sense of Definition 7.5.5.1: that is, it induces an equivalence of ∞-categories $\overline{\mathscr{F}}(\mathbf{0}) \to \underleftarrow{\mathrm{holim}}(\overline{\mathscr{F}}|_{\mathcal{C}})$.

(2) The functor $\overline{\mathscr{F}}$ is a categorical limit diagram in the sense of Definition 7.5.5.11: that is, there exists a levelwise categorical equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$, where $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ induces an equivalence of ∞-categories $\overline{\mathscr{G}}(\mathbf{0}) \to \underleftarrow{\mathrm{holim}}(\overline{\mathscr{G}}|_{\mathcal{C}})$.

The implication (1) ⇒ (2) is immediate, and the reverse implication follows from Remark 7.5.5.6.

03C0 **Proposition 7.5.5.13.** *Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\overline{\mathscr{F}}$ is a categorical limit diagram. That is, there exists a categorical limit diagram $\overline{\mathscr{F}}' : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ and a levelwise categorical equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{F}}'$.*

(2) *Let $\overline{\mathscr{F}}' : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ be any functor. If there exists a levelwise categorical equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{F}}'$, then $\overline{\mathscr{F}}'$ is a categorical limit diagram.*

Proof. We proceed as in the proof of Proposition 7.5.4.10. Using Proposition 4.1.3.2, we can choose a functor $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ and a natural transformation $\beta : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ for which the morphism of simplicial sets $\beta_C : \overline{\mathscr{F}}(C) \to \overline{\mathscr{G}}(C)$ is inner anodyne for each object $C \in \mathcal{C}^{\triangleleft}$. We will show that (1) and (2) are equivalent to the following:

(3) The functor $\overline{\mathscr{G}}$ is a categorical limit diagram.

The implications (2) ⇒ (3) ⇒ (1) are immediate. To prove the reverse implications, suppose we are given another functor $\overline{\mathscr{F}}' : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ and a levelwise categorical equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{F}}'$. We will show that $\overline{\mathscr{F}}'$ is a categorical limit diagram if and only if $\overline{\mathscr{G}}$ is a categorical limit diagram.

Applying Proposition 4.1.3.2 again, we can choose a functor $\overline{\mathscr{G}}' : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ and a commutative diagram

$$\begin{tikzcd}
\overline{\mathscr{F}} \arrow[r, "\alpha"] \arrow[d, "\beta"] & \overline{\mathscr{F}}' \arrow[d, "\beta'"] \\
\overline{\mathscr{G}} \arrow[r, "\alpha'"] & \overline{\mathscr{G}}'
\end{tikzcd}$$

with the property that, for every object $C \in \mathcal{C}^{\triangleleft}$, the induced morphism of simplicial sets $\overline{\mathscr{F}}'(C) \coprod_{\overline{\mathscr{F}}(C)} \overline{\mathscr{G}}(C) \to \overline{\mathscr{G}}'(C)$ is inner anodyne (and, in particular, a categorical equivalence). Applying Proposition 4.5.4.11, we see that the diagram of simplicial sets

$$\begin{tikzcd}
\overline{\mathscr{F}}(C) \arrow[r, "\alpha_C"] \arrow[d, "\beta_C"] & \overline{\mathscr{F}}'(C) \arrow[d, "\beta'_C"] \\
\overline{\mathscr{G}}(C) \arrow[r, "\alpha'_C"] & \overline{\mathscr{G}}'(C)
\end{tikzcd}$$

is a categorical pushout square. Since α_C and β_C are categorical equivalences, it follows that α'_C and β'_C are also categorical equivalences (Proposition 4.5.4.10). Applying Remark 7.5.5.6, we see that $\overline{\mathscr{F}}'$ and $\overline{\mathscr{G}}$ are categorical limit diagrams if and only if $\overline{\mathscr{G}}'$ is a categorical limit diagram. □

Corollary 7.5.5.14. *Let $\mathcal{C}$ be a category, let $\overline{\mathscr{F}}, \overline{\mathscr{F}}' : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ be functors, and let $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{F}}'$ be a natural transformation. Suppose that, for every object $C \in \mathcal{C}$, the induced map $\alpha_C : \overline{\mathscr{F}}(C) \to \overline{\mathscr{F}}'(C)$ is a categorical equivalence of simplicial sets. Then any two of the following conditions imply the third:* 03C1

(1) *The functor $\overline{\mathscr{F}}$ is a categorical limit diagram.*

(2) *The functor* $\overline{\mathscr{F}}'$ *is a categorical limit diagram.*

(3) *The natural transformation* α *induces a categorical equivalence of simplicial sets* $\overline{\mathscr{F}}(\mathbf{0}) \to \overline{\mathscr{F}}'(\mathbf{0})$*, where* $\mathbf{0}$ *denotes the initial object of* $\mathcal{C}^{\triangleleft}$*.*

Proof. Using Proposition 4.1.3.2, we can choose functors $\overline{\mathscr{G}}, \overline{\mathscr{G}}' : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ and a commutative diagram

$$\begin{tikzcd}
\overline{\mathscr{F}} \arrow[r, "\alpha"] \arrow[d] & \overline{\mathscr{F}}' \arrow[d] \\
\overline{\mathscr{G}} \arrow[r, "\beta"] & \overline{\mathscr{G}}',
\end{tikzcd}$$

where the vertical maps are levelwise categorical equivalences. By virtue of Proposition 7.5.5.13, we can replace α by the natural transformation $\beta : \overline{\mathscr{G}} \to \overline{\mathscr{G}}'$. In this case, the desired result follows from Remark 7.5.5.6. □

03C2 **Corollary 7.5.5.15.** *Let* $\mathcal{C}$ *be a small category and let* $\overline{\mathscr{F}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ *be a functor. Let* $\overline{\mathscr{F}}^{\mathrm{op}} : \mathcal{C}^{\triangleleft} \to \mathrm{Set}_{\Delta}$ *be the functor given on objects by* $\overline{\mathscr{F}}^{\mathrm{op}}(C) = \overline{\mathscr{F}}(C)^{\mathrm{op}}$*. Then* $\overline{\mathscr{F}}$ *is a categorical limit diagram if and only if* $\overline{\mathscr{F}}^{\mathrm{op}}$ *is a categorical limit diagram.*

Proof. Using Proposition 4.1.3.2, we can choose a functor $\overline{\mathscr{G}} : \mathcal{C}^{\triangleleft} \to \mathrm{QCat}$ and a levelwise categorical equivalence $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$. By virtue of Corollary 7.5.5.14, it will suffice to show that $\overline{\mathscr{G}}$ is a categorical limit diagram if and only if $\overline{\mathscr{G}}^{\mathrm{op}}$ is a categorical limit diagram. This follows by combining Proposition 7.5.5.7 with Corollary 7.5.4.12. □

7.5.6 The Homotopy Colimit as a Derived Functor

03C3 Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a diagram of ∞-categories. In §7.5.3, we showed that the homotopy limit $\underleftarrow{\mathrm{holim}}(\mathscr{F})$ can be identified with the limit of an *isofibrant replacement* for $\mathscr{F}$: that is, there exists an isomorphism $\underleftarrow{\mathrm{holim}}(\mathscr{F}) \simeq \underleftarrow{\lim}(\mathscr{F}^{+})$, where $\mathscr{F}^{+} : \mathcal{C} \to \mathrm{QCat}$ is an isofibrant diagram equipped with a levelwise categorical equivalence $\mathscr{F} \hookrightarrow \mathscr{F}^{+}$ (Construction 7.5.3.3 and Proposition 7.5.3.7). Our goal in this section is to present a parallel treatment of the homotopy colimit functor of Construction 5.3.2.1. More precisely, we show that the homotopy colimit of a diagram $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ can be identified with the colimit of an auxiliary diagram $\mathscr{F}_{+} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ which is equipped with a levelwise weak homotopy equivalence $\mathscr{F}_{+} \twoheadrightarrow \mathscr{F}$ (Proposition 7.5.6.12).

We begin by introducing some terminology. Recall that a natural transformation $\beta : \widetilde{\mathscr{G}} \to \mathscr{G}$ is a *levelwise trivial Kan fibration* if, for each object $C \in \mathcal{C}$, the morphism $\beta_C : \widetilde{\mathscr{G}}(C) \to \mathscr{G}(C)$ is a trivial Kan fibration of simplicial sets.

Definition 7.5.6.1. Let $\mathcal{C}$ be a small category. We say that a diagram of simplicial sets $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ is *projectively cofibrant* if, for every levelwise trivial Kan fibration $\beta : \mathscr{G}' \to \mathscr{G}$, the induced map 03C4

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G}') \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G})$$

is surjective. That is, every natural transformation $\alpha : \mathscr{F} \to \mathscr{G}$ factors through β.

Example 7.5.6.2. Let $\mathcal{C}$ be a category and let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be a morphism of simplicial sets. Then the diagram 03C5

$$\mathscr{F}_{\mathcal{E}} : \mathcal{C} \to \mathrm{Set}_{\Delta} \qquad \mathscr{F}_{\mathcal{E}}(C) = \mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathcal{E}$$

is projectively cofibrant, in the sense of Definition 7.5.6.1. To prove this, we must show that for every levelwise trivial Kan fibration $\mathscr{G}' \to \mathscr{G}$ between functors $\mathscr{G}', \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$, the induced map

$$\theta : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{F}_{\mathcal{E}}, \mathscr{G}') \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{G}_{\mathcal{E}}, \mathscr{G})$$

is surjective. Using Proposition 5.3.3.24, we can identify θ with a pullback of the map $\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathcal{E}, \mathrm{N}^{\mathscr{G}'}_{\bullet}(\mathcal{C})) \to \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathcal{E}, \mathrm{N}^{\mathscr{G}}_{\bullet}(\mathcal{C}))$, which is surjective by virtue of Exercise 5.3.3.11.

Exercise 7.5.6.3 (Well-Founded Diagrams)**.** Let $(Q, \leq)$ be a well-founded partially ordered set. Show that a diagram of simplicial sets $\mathscr{F} : Q \to \mathrm{Set}_{\Delta}$ is projectively cofibrant if and only if, for each element $q \in Q$, the associated map $\varinjlim_{p<q} \mathscr{F}(p) \to \mathscr{F}(q)$ is a monomorphism of simplicial sets (compare with Proposition 4.5.6.6). 03C6

Example 7.5.6.4 (Projectively Cofibrant Sequences)**.** A sequential diagram of simplicial sets 03C7

$$X(0) \to X(1) \to X(2) \to X(3) \to \cdots$$

is projectively cofibrant (when regarded as a functor $\mathbf{Z}_{\geq 0} \to \mathrm{Set}_{\Delta}$) if and only if each of the transition maps $X(n) \to X(n+1)$ is a monomorphism.

Example 7.5.6.5 (Projectively Cofibrant Squares)**.** A commutative diagram of simplicial sets 03C8

03C9
$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_1"] \\ A_1 \arrow[r, "f_0"] & A_{01} \end{tikzcd} \qquad (7.51)$$

is projectively cofibrant (when regarded as a functor $[1] \times [1] \to \mathrm{Set}_{\Delta}$) if and only if the morphisms

$$f_0 : A \to A_0 \qquad f_1 : A \to A_1 \qquad (f_1', f_0') : A_0 \coprod_A A_1 \to A_{01}$$

are monomorphisms of simplicial sets. Equivalently, (7.51) is projectively cofibrant if it is a pullback square consisting of monomorphisms.

03CA **Remark 7.5.6.6** (Relationship to Isofibrant Diagrams)**.** Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, let $\mathcal{D}$ be an ∞-category, and let $\mathcal{D}^{\mathscr{F}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ denote the functor given by the construction $C \mapsto \mathrm{Fun}(\mathscr{F}(C), \mathcal{D})$. If $\mathscr{F}$ is projectively cofibrant (in the sense of Definition 7.5.6.1), then $\mathcal{D}^{\mathscr{F}}$ is isofibrant (in the sense of Definition 4.5.6.3). That is, if $\mathscr{E} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}_{\Delta}$ is a diagram of simplicial sets and $\mathscr{E}_0 \subseteq \mathscr{E}$ is a subfunctor for which the equivalence $\mathscr{E}_0 \hookrightarrow \mathscr{E}$ is a levelwise categorical equivalence, then the restriction map

$$\theta : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}_{\Delta})}(\mathscr{E}, \mathcal{D}^{\mathscr{F}}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}_{\Delta})}(\mathscr{E}_0, \mathcal{D}^{\mathscr{F}})$$

is surjective. This follows from the observation that θ can be identified with the map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{F}, \mathcal{D}^{\mathscr{E}}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})}(\mathscr{F}, \mathcal{D}^{\mathscr{E}_0})$$

given by composition with the restriction map $\mathcal{D}^{\mathscr{E}} \to \mathcal{D}^{\mathscr{E}_0}$, which is a levelwise trivial Kan fibration by virtue of Corollary 4.5.5.19.

03CB **Proposition 7.5.6.7.** *Let $\mathcal{C}$ be a small category and let $\alpha : \mathscr{F} \to \mathscr{G}$ be a natural transformation between projectively cofibrant diagrams $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. If α is a levelwise categorical equivalence, then the induced map $\varinjlim(\alpha) : \varinjlim(\mathscr{F}) \to \varinjlim(\mathscr{G})$ is a categorical equivalence of simplicial sets. If α is a levelwise weak homotopy equivalence, then $\varinjlim(\alpha)$ is a weak homotopy equivalence.*

Proof. We will prove the first assertion; the second follows by a similar argument. Assume that α is levelwise categorical equivalence and let $\mathcal{D}$ be an ∞-category; we wish to show that precomposition with $\varprojlim(\alpha)$ induces an equivalence of ∞-categories $\alpha^* : \mathrm{Fun}(\varinjlim(\mathscr{G}), \mathcal{D}) \to \mathrm{Fun}(\varinjlim(\mathscr{F}), \mathcal{D})$. α is a levelwise categorical equivalence, precomposition with α induces a levelwise categorical equivalence $\beta : \mathcal{D}^{\mathscr{G}} \to \mathcal{D}^{\mathscr{F}}$ in the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}_{\Delta})$. Unwinding the definitions, we see that α^* can be identified with the limit $\varprojlim(\beta)$. Since $\mathcal{D}^{\mathscr{F}}$ and $\mathcal{D}^{\mathscr{G}}$ are isofibrant diagrams (Remark 7.5.6.6), the functor $\varprojlim(\beta)$ is an equivalence of ∞-categories (Corollary 4.5.6.17). □

We now show that every diagram of simplicial sets $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ admits a weak homotopy equivalence from a projectively cofibrant diagram (for a stronger statement, see Proposition 7.5.9.7).

03CC **Construction 7.5.6.8** (Explicit Cofibrant Replacement)**.** Let $\mathcal{C}$ be a small category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, and let $\mathrm{ho}\varinjlim(\mathscr{F})$ denote the homotopy colimit of $\mathscr{F}$ (Construction 5.3.2.1). For each object $C \in \mathcal{C}$, we let $\mathscr{F}_+(C)$ denote the simplicial set given by the fiber product

$$\mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{ho}\varinjlim(\mathscr{F}) = \mathrm{ho}\varinjlim(\mathscr{F}|_{\mathcal{C}_{/C}}).$$

The construction $C \mapsto \mathscr{F}_{+}(C)$ determines a diagram of simplicial sets $\mathscr{F}_{+} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. This diagram is equipped with a natural transformation $\alpha : \mathscr{F}_{+} \twoheadrightarrow \mathscr{F}$, which carries each object $C \in \mathcal{C}$ to the comparison map

$$\mathscr{F}_{+}(C) = \underrightarrow{\mathrm{holim}}(\mathscr{F}|_{\mathcal{C}_{/C}}) \twoheadrightarrow \underrightarrow{\lim}(\mathscr{F}|_{\mathcal{C}_{/C}}) \simeq \mathscr{F}(C)$$

of Remark 5.3.2.9.

Proposition 7.5.6.9. *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets. Then the diagram $\mathscr{F}_{+} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ of Construction 7.5.6.8 is projectively cofibrant, and the natural transformation $\alpha : \mathscr{F}_{+} \to \mathscr{F}$ is a levelwise weak homotopy equivalence. Moreover, α is also an epimorphism.* 03CD

Proof. Example 7.5.6.2 shows that the diagram $\mathscr{F}_{+}$ is projectively cofibrant and Remark 5.3.2.9 shows that α is an epimorphism. To complete the proof, it will suffice to show that for each object $C \in \mathcal{C}$, the map $\alpha_C : \mathscr{F}_{+}(C) \to \mathscr{F}(C)$ is a weak homotopy equivalence of simplicial sets. Replacing $\mathcal{C}$ by the slice category $\mathcal{C}_{/C}$, we can reduce to the case where C is a final object of $\mathcal{C}$; in this case, we wish to prove that the comparison map

$$\underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \underrightarrow{\lim}(\mathscr{F}) \simeq \mathscr{F}(C)$$

is a weak homotopy equivalence. Note that this map admits a section, given by the inclusion map

$$\iota : \mathscr{F}(C) \simeq \{C\} \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \underrightarrow{\mathrm{holim}}(\mathscr{F}).$$

We complete the proof by that our assumption that $C \in \mathcal{C}$ is a final object guarantees that ι is right anodyne (Example 7.2.3.11). □

Warning 7.5.6.10. In the situation of Proposition 7.5.6.9, the natural transformation $\alpha : \mathscr{F}_{+} \twoheadrightarrow \mathscr{F}$ is usually not a levelwise categorical equivalence. For example, if $\mathscr{F}$ is the constant functor taking the value Δ^0, then $\mathscr{F}_{+}$ is given by the construction $C \mapsto \mathrm{N}_{\bullet}(\mathcal{C}_{/C})$. 03CE

Remark 7.5.6.11. Constructions 7.5.6.8 and 7.5.3.3 are closely related. Let $\mathcal{C}$ be a small category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, and let $\mathscr{G} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes. Combining Corollary 5.3.2.24 with Proposition 5.3.3.24, we obtain canonical isomorphisms of Kan complexes 03CF

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathscr{G}^{+})_{\bullet} &= \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}, \mathrm{sTr}_{\mathrm{N}^{\mathscr{G}}_{\bullet}(\mathcal{C})/\mathcal{C}}) \\ &\simeq \mathrm{Fun}_{/\mathrm{N}_{\bullet}(\mathcal{C})}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathrm{N}^{\mathscr{G}}_{\bullet}(\mathcal{C})) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}_{+}, \mathscr{G})_{\bullet}. \end{aligned}$$

More generally, if $\mathscr{G}$ is a diagram of ∞-categories, we can identify $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F},\mathscr{G}^{+})_{\bullet}$ with the full subcategory of $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set}_{\Delta})}(\mathscr{F}_{+},\mathscr{G})_{\bullet}$ spanned by those natural transformations $\alpha : \mathscr{F}_{+} \rightarrow \mathscr{G}$ having the property that, for each object $C \in \mathcal{C}$, the diagram

$$\alpha_C : \mathscr{F}_{+}(C) = \underrightarrow{\mathrm{holim}}(\mathscr{F}|_{\mathcal{C}_{/C}}) \rightarrow \mathscr{G}(C)$$

carries horizontal edges of $\underrightarrow{\mathrm{holim}}(\mathscr{F}|_{\mathcal{C}_{/C}})$ to isomorphisms in the ∞-category $\mathscr{G}(C)$.

`03CG` **Proposition 7.5.6.12.** *Let $\mathcal{C}$ be a small category, let $\mathscr{F} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, and let $\mathscr{F}_{+} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ be the diagram of Construction 7.5.6.8. Then there is a canonical isomorphism of simplicial sets $\lambda : \varinjlim(\mathscr{F}_{+}) \rightarrow \underrightarrow{\mathrm{holim}}(\mathscr{F})$ which is characterized by the following requirement: for each object $C \in \mathcal{C}$, the composition*

$$\begin{aligned} \mathrm{N}_{\bullet}(\mathcal{C}_{/C}) \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) &= \mathscr{F}_{+}(C) \\ &\rightarrow \varinjlim(\mathscr{F}_{+}) \\ &\xrightarrow{\lambda} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \end{aligned}$$

is given by projection onto the second factor.

Proof. It follows from the definition of the colimit that there is a unique morphism of simplicial sets $\lambda : \varinjlim(\mathscr{F}_{+}) \rightarrow \underrightarrow{\mathrm{holim}}(\mathscr{F})$ having the desired property. Using the dual of Lemma 7.5.3.8, we deduce that λ is an isomorphism. □

`03CH` **Remark 7.5.6.13.** Let $\mathscr{F} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, let $\theta : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \twoheadrightarrow \varinjlim(\mathscr{F})$ be the comparison map of Remark 5.3.2.9, and let $\lambda : \varinjlim(\mathscr{F}_{+}) \xrightarrow{\sim} \underrightarrow{\mathrm{holim}}(\mathscr{F})$ be the isomorphism of Proposition 7.5.6.12. Then the composition $(\theta \circ \lambda) : \varinjlim(\mathscr{F}_{+}) \rightarrow \varinjlim(\mathscr{F})$ is induced by the natural transformation $\alpha : \mathscr{F}_{+} \twoheadrightarrow \mathscr{F}$ appearing in Construction 7.5.6.8.

`03CJ` **Corollary 7.5.6.14.** *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \rightarrow \mathrm{Set}_{\Delta}$ be a projectively cofibrant diagram of simplicial sets. Then the comparison map $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \twoheadrightarrow \varinjlim(\mathscr{F})$ of Remark 5.3.2.9 is a weak homotopy equivalence.*

Proof. By virtue of Remark 7.5.6.13, it will suffice to show that the natural transformation $\alpha : \mathscr{F}_{+} \twoheadrightarrow \mathscr{F}$ of Construction 7.5.6.8 induces a weak homotopy equivalence $\varinjlim(\alpha) : \varinjlim(\mathscr{F}_{+}) \rightarrow \varinjlim(\mathscr{F})$. This is a special case of Proposition 7.5.6.7, since α is a levelwise weak homotopy equivalence between projectively cofibrant diagrams (Proposition 7.5.6.9). □

`03CK` **Warning 7.5.6.15.** Let $\mathscr{F} : \mathcal{C} \rightarrow \mathrm{QCat}$ be a diagram of simplicial sets, let $\alpha : \mathscr{F}_{+} \twoheadrightarrow \mathscr{F}$ be the natural transformation of Construction 7.5.6.8, and let $\lambda : \varinjlim(\mathscr{F}_{+}) \xrightarrow{\sim} \underrightarrow{\mathrm{holim}}(\mathscr{F})$ be the isomorphism of Proposition 7.5.6.12. Then we have a diagram of simplicial sets

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}_+) \arrow[r, "\underrightarrow{\mathrm{holim}}(\alpha)"] \arrow[d] & \underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[d] \\
\underrightarrow{\lim}(\mathscr{F}_+) \arrow[r, "\underrightarrow{\lim}(\alpha)"'] \arrow[ur, "\lambda", "\sim"'] & \underrightarrow{\lim}(\mathscr{F}),
\end{tikzcd}$$

where the outer square and the lower right triangle are commutative (Remark 7.5.6.13). Beware that the upper left triangle is usually *not* commutative. That is, $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ and $\underrightarrow{\lim}(\mathscr{F}_+)$ are isomorphic when viewed as abstract simplicial sets, but not when viewed as quotients of the simplicial set $\underrightarrow{\mathrm{holim}}(\mathscr{F}_+)$ (compare with Warning 7.5.3.14).

Remark 7.5.6.16 (The Homotopy Colimit as a Left Derived Functor)**.** The preceding results can be interpreted in the language of model categories. For every small category $\mathcal{C}$, the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)$ can be equipped with a model structure in which the fibrations are levelwise Kan fibrations and weak equivalences are levelwise weak homotopy equivalences (see Example [?]). Combining Propositions 7.5.6.9 and 7.5.6.12, we deduce that the homotopy colimit functor $\underrightarrow{\mathrm{holim}} : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta) \to \mathrm{Set}_\Delta$ can be viewed as a *left derived* functor of the usual colimit $\underrightarrow{\lim} : \mathrm{Fun}(\mathcal{C}, \mathrm{Set}_\Delta) \to \mathrm{Set}_\Delta$ (see Definition [?]). `03CL`

7.5.7 Homotopy Colimit Diagrams

Let $\mathcal{C}$ be a (small) category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a (strictly commutative) diagram of Kan complexes indexed by $\mathcal{C}$. Passing to the homotopy coherent nerve, we obtain a functor of ∞-categories `03CM`

$$\mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F}) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan}) = \mathcal{S}.$$

By virtue of Corollary 7.4.5.6, this functor admits a colimit in the ∞-category $\mathcal{S}$. This colimit admits a classical description, using the *homotopy colimit* of Construction 5.3.2.1.

Proposition 7.5.7.1. *Let $\mathcal{C}$ be a small category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a (strictly commutative) diagram of ∞-categories indexed by $\mathcal{C}$. Then a Kan complex X is a colimit of the functor $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathscr{F})$ if and only if it is weakly homotopy equivalent to the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$.* `03CN`

Proof. Let $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$ be the taut scaffold of Construction 5.3.4.11. Then λ_t is a categorical equivalence of simplicial sets (Corollary 5.3.5.9), and therefore a weak homotopy equivalence (Remark 4.5.3.4). The desired result now follows from Corollary 7.4.5.12. □

03CP **Example 7.5.7.2.** Let $\mathcal{C}$ be a groupoid and let $\mathscr{F} : \mathcal{C} \to \mathrm{Kan}$ be a diagram of Kan complexes indexed by $\mathcal{C}$. Then the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ is a Kan complex (Corollary 5.3.4.23). In this case, Proposition 7.5.7.1 guarantees that $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ is a colimit of the diagram $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$ in the ∞-category $\mathcal{S}$. For example, if X is a Kan complex equipped with an action of a group G, then the homotopy quotient $X_{\mathrm{h}G}$ is a colimit of the associated diagram $B_{\bullet}G \to \mathcal{S}$ (Example 5.3.4.24).

Our goal in this section is to formulate a companion to Proposition 7.5.7.1, which provides concrete models for colimit *diagrams* in the ∞-category $\mathcal{S}$ (rather than colimits in the abstract).

03CQ **Definition 7.5.7.3.** Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets restriction $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$. We will say that $\overline{\mathscr{F}}$ is a *homotopy colimit diagram* if the composite map

$$\underrightarrow{\mathrm{holim}}(\mathscr{F}) \twoheadrightarrow \underrightarrow{\lim}(\mathscr{F}) \to \overline{\mathscr{F}}(\mathbf{1})$$

is a weak homotopy equivalence of simplicial sets. Here $\mathbf{1}$ denotes the final object of the cone $\mathcal{C}^{\triangleright} \simeq \mathcal{C} \star \{\mathbf{1}\}$, and the morphism on the left is the comparison map of Remark 5.3.2.9.

03CR **Example 7.5.7.4.** Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a colimit diagram in the category of simplicial sets. If the diagram $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$ is projectively cofibrant, then $\overline{\mathscr{F}}$ is a homotopy colimit diagram: this is a reformulation of Corollary 7.5.6.14 (for a stronger statement, see Corollary 7.5.8.7).

03CS **Proposition 7.5.7.5** (Homotopy Invariance)**.** *Let $\mathcal{C}$ be a category and let $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ be a natural transformation between diagrams $\overline{\mathscr{F}}, \overline{\mathscr{G}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$. Assume that, for every object $C \in \mathcal{C}$, the induced map $\alpha_C : \overline{\mathscr{F}}(C) \to \overline{\mathscr{G}}(C)$ is a weak homotopy equivalence of simplicial sets. Then any two of the following conditions imply the third:*

(1) *The functor $\overline{\mathscr{F}}$ is a homotopy colimit diagram.*

(2) *The functor $\overline{\mathscr{G}}$ is a homotopy colimit diagram.*

(3) *The natural transformation α induces a weak homotopy equivalence $\overline{\mathscr{F}}(\mathbf{1}) \to \overline{\mathscr{G}}(\mathbf{1})$, where $\mathbf{1}$ denotes the cone point of $\mathcal{C}^{\triangleright}$.*

Proof. Setting $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$ and $\mathscr{G} = \overline{\mathscr{G}}|_{\mathcal{C}}$, we observe that α determines a commutative diagram of simplicial sets

$$\begin{tikzcd}
\underrightarrow{\mathrm{holim}}(\mathscr{F}) \arrow[r] \arrow[d] & \underrightarrow{\mathrm{holim}}(\mathscr{G}) \arrow[d] \\
\overline{\mathscr{F}}(\mathbf{1}) \arrow[r] & \overline{\mathscr{G}}(\mathbf{1})
\end{tikzcd}$$

where the upper horizontal map is a weak homotopy equivalence (Proposition 5.3.2.18). The desired result now follows from the two-out-of-three property (Remark 3.1.6.16). □

There is a close relationship between homotopy colimit diagrams (in the sense of Definition 7.5.7.3) and homotopy limit diagrams (in the sense of Definition 7.5.4.1).

Proposition 7.5.7.6. *Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets. Then $\overline{\mathscr{F}}$ is a homotopy colimit diagram if and only if, for every Kan complex X, the functor* 03CT

$$X^{\overline{\mathscr{F}}} : (\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \mathrm{Kan} \qquad C \mapsto \mathrm{Fun}(\overline{\mathscr{F}}(C), X)$$

is a homotopy limit diagram.

Proof. Set $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$, let $\mathbf{1}$ denote the final object of $\mathcal{C}^{\triangleright}$, and let $\theta : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \overline{\mathscr{F}}(\mathbf{1})$ be the map appearing in Definition 7.5.7.3. Then $\overline{\mathscr{F}}$ is a homotopy colimit diagram if and only if, for every Kan complex X, precomposition with θ induces a homotopy equivalence of Kan complexes

$$\theta^{*} : \mathrm{Fun}(\overline{\mathscr{F}}(\mathbf{1}), X) \to \mathrm{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), X).$$

Setting $\mathscr{G} = \mathscr{F}^{\mathrm{op}}$, $\overline{\mathscr{G}} = \overline{\mathscr{F}}^{\mathrm{op}}$, and $Y = X^{\mathrm{op}}$, Example 7.5.1.7 identifies θ^{*} with the opposite of the restriction map $Y^{\overline{\mathscr{G}}}(\mathbf{1}) \to \underleftarrow{\mathrm{holim}}(Y^{\mathscr{G}})$ appearing in Definition 7.5.4.1. In particular, θ^{*} is a homotopy equivalence if and only if $Y^{\overline{\mathscr{G}}}$ is a homotopy limit diagram of Kan complexes. By virtue of Corollary 7.5.4.12, this is equivalent to the requirement that $X^{\overline{\mathscr{F}}}$ is a homotopy limit diagram. □

Corollary 7.5.7.7. *Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Kan}$ be a diagram of Kan complexes. Then $\overline{\mathscr{F}}$ is a homotopy colimit diagram (in the sense of Definition 7.5.7.3) if and only if the induced functor of ∞-categories* 03CU

$$\mathrm{N}^{\mathrm{hc}}_{\bullet}(\overline{\mathscr{F}}) : \mathrm{N}_{\bullet}(\mathcal{C}^{\triangleright}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan}) = \mathcal{S}$$

is a colimit diagram (in the sense of Variant 7.1.2.5).

Proof. Combine Proposition 7.5.7.6 with Corollary 7.5.4.6 (applied to the simplicial category $\mathrm{Kan}^{\mathrm{op}}$). □

Corollary 7.5.7.8. *Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a functor. Let $\overline{\mathscr{F}}^{\mathrm{op}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be the functor given on objects by $\overline{\mathscr{F}}^{\mathrm{op}}(C) = \overline{\mathscr{F}}(C)^{\mathrm{op}}$. Then $\overline{\mathscr{F}}$ is a homotopy colimit diagram if and only if $\overline{\mathscr{F}}^{\mathrm{op}}$ is a homotopy colimit diagram.* 03CV

Proof. Combine Proposition 7.5.7.6 with Corollary 7.5.7.8. □

03CW **Corollary 7.5.7.9.** *Suppose we are given a commutative diagram of simplicial sets*

03CX
$$\begin{tikzcd} A \arrow[r] \arrow[d] & A_0 \arrow[d] \\ A_1 \arrow[r] & A_{01}, \end{tikzcd} \tag{7.52}$$

which we identify with a functor $\mathscr{F} : [1] \times [1] \to \mathrm{Set}_{\Delta}$. *Then (7.52) is a homotopy pushout square (in the sense of Definition 3.4.2.1) if and only if* $\mathscr{F}$ *is a homotopy colimit diagram (in the sense of Definition 7.5.7.3).*

Proof. Combine Propositions 7.5.7.6 and 7.5.4.13. □

7.5.8 Categorical Colimit Diagrams

03CY In §7.5.7, we introduced the notion of a *homotopy colimit diagram* (Definition 7.5.7.3), and showed that one can use homotopy colimit diagrams to compute colimits in the ∞-category $\mathcal{S}$ of spaces (Corollary 7.5.7.7). In this section, we introduce the closely related notion of *categorical colimit diagram*, which can be used to compute colimits in the larger ∞-category $\mathcal{QC} \supset \mathcal{S}$.

03CZ **Proposition 7.5.8.1.** *Let* $\mathcal{C}$ *be a small category and let* $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ *be a (strictly commutative) diagram of* ∞*-categories indexed by* $\mathcal{C}$*, and let* W *denote the collection of horizontal edges of the homotopy colimit* $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ *(Definition 5.3.4.1). Then an* ∞*-category* $\mathcal{D}$ *is a colimit of the diagram*

$$\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}) = \mathcal{QC}$$

if and only if it is a localization of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ *with respect to* W*, in the sense of Remark 6.3.2.2.*

Proof. Let $U : \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ be the projection map of Definition 5.3.3.1 and let W' denote the collection of all U-cocartesian morphisms of $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$. Choose a functor of ∞-categories $T : \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \to \mathcal{D}$ which exhibits $\mathcal{D}$ as a localization of $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ with respect to W'. Let $\lambda_t : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ denote the taut scaffold of Construction 5.3.4.11. Then λ_t is a categorical equivalence of simplicial sets (Corollary 5.3.5.9). Moreover, a morphism of $\mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C})$ belongs to W' if and only if it is isomorphic (as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}))$ to an element of $\lambda_t(W)$ (see Corollary 5.3.3.16). It follows that the composite map $\underrightarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{\lambda_t} \mathrm{N}^{\mathscr{F}}_{\bullet}(\mathcal{C}) \xrightarrow{T} \mathcal{D}$ exhibits $\mathcal{D}$ as a localization of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ with respect to W. We conclude by observing that $\mathcal{D}$ is a colimit of the diagram $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F})$ (Corollary 7.4.3.16). □

Motivated by Proposition 7.5.8.1, we introduce the following variant of Definition 7.5.7.3:

Definition 7.5.8.2. Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets. Set $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$, and let W denote the collection of horizontal edges of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ (Definition 5.3.4.1). We will say that $\overline{\mathscr{F}}$ is a *categorical colimit diagram* if the composite map 03D0

$$\underrightarrow{\mathrm{holim}}(\mathscr{F}) \twoheadrightarrow \underrightarrow{\lim}(\mathscr{F}) \to \overline{\mathscr{F}}(\mathbf{1})$$

exhibits $\overline{\mathscr{F}}(\mathbf{1})$ as a localization of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ with respect to W. (see Definition 6.3.1.9). Here $\mathbf{1}$ denotes the final object of the cone $\mathcal{C}^{\triangleright} \simeq \mathcal{C} \star \{\mathbf{1}\}$, and the morphism on the left is the comparison map of Remark 5.3.2.9.

Remark 7.5.8.3. Let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a categorical colimit diagram of simplicial sets. Then $\overline{\mathscr{F}}$ is also a homotopy colimit diagram of simplicial sets, in the sense of Definition 7.5.7.3. This follows from the observation that every localization of simplicial sets is a weak homotopy equivalence (Remark 6.3.1.16). 03D1

Proposition 7.5.8.4. *Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets. The following conditions are equivalent:* 03D2

(1) *The diagram $\overline{\mathscr{F}}$ is a categorical colimit diagram.*

(2) *For every ∞-category $\mathcal{D}$, the diagram of ∞-categories*

$$(\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \mathrm{QCat} \qquad C \mapsto \mathrm{Fun}(\overline{\mathscr{F}}(C), \mathcal{D})$$

is a categorical limit diagram (Definition 7.5.5.1).

(3) *For every ∞-category $\mathcal{D}$, the diagram of Kan complexes*

$$(\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \mathrm{Kan} \qquad C \mapsto \mathrm{Fun}(\overline{\mathscr{F}}(C), \mathcal{D})^{\simeq}$$

is a homotopy limit diagram (Definition 7.5.4.1).

Proof. The equivalence of (1) and (2) follows by combining Example 7.5.2.11 with Corollary 7.5.5.15. The equivalence with (3) follows by combining the same results with Proposition 6.3.1.13 and Example 7.5.2.8. □

Corollary 7.5.8.5. *Suppose we are given a commutative diagram of simplicial sets* 03D3

03D4

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01},
\end{tikzcd} \tag{7.53}$$

which we identify with a functor $\mathscr{F} : [1] \times [1] \to \mathrm{Set}_{\Delta}$*. Then (7.53) is a categorical pushout square (in the sense of Definition 4.5.4.1) if and only if* $\mathscr{F}$ *is a categorical colimit diagram (in the sense of Definition 7.5.8.2).*

03D5 **Corollary 7.5.8.6** (Homotopy Invariance)**.** *Let* $\mathcal{C}$ *be a category and let* $\alpha : \overline{\mathscr{F}} \to \overline{\mathscr{G}}$ *be a natural transformation between diagrams* $\overline{\mathscr{F}}, \overline{\mathscr{G}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$*. Assume that, for every object* $C \in \mathcal{C}$*, the induced map* $\alpha_C : \overline{\mathscr{F}}(C) \to \overline{\mathscr{G}}(C)$ *is a categorical equivalence of simplicial sets. Then any two of the following conditions imply the third:*

(1) *The functor* $\overline{\mathscr{F}}$ *is a categorical colimit diagram.*

(2) *The functor* $\overline{\mathscr{G}}$ *is a categorical colimit diagram.*

(3) *The natural transformation* α *induces a categorical equivalence* $\overline{\mathscr{F}}(\mathbf{1}) \to \overline{\mathscr{G}}(\mathbf{1})$*, where* $\mathbf{1}$ *denotes the cone point of* $\mathcal{C}^{\triangleright}$*.*

Proof. By virtue of Proposition 7.5.8.4 (and Proposition 4.5.3.8), it will suffice to show that for every ∞-category $\mathcal{D}$, any two of the following conditions imply the third:

($1_{\mathcal{D}}$) The functor

$$(\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \mathrm{QCat} \qquad C \mapsto \mathrm{Fun}(\overline{\mathscr{F}}(C), \mathcal{D})$$

is a categorical limit diagram.

($2_{\mathcal{D}}$) The functor

$$(\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \mathrm{QCat} \qquad C \mapsto \mathrm{Fun}(\overline{\mathscr{G}}(C), \mathcal{D})$$

is a categorical limit diagram.

($3_{\mathcal{D}}$) The natural transformation α induces an equivalence of ∞-categories $\mathrm{Fun}(\overline{\mathscr{G}}(\mathbf{1}), \mathcal{D}) \to \mathrm{Fun}(\overline{\mathscr{F}}(\mathbf{1}), \mathcal{D})$.

This follows from Remark 7.5.5.6. □

03D6 **Corollary 7.5.8.7.** *Let* $\mathcal{C}$ *be a small category and let* $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ *be a colimit diagram in the category of simplicial sets. If the diagram* $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$ *is projectively cofibrant, then* $\overline{\mathscr{F}}$ *is a categorical colimit diagram.*

Proof. Let $\mathcal{D}$ be an ∞-category and define $\overline{\mathscr{G}} : (\mathcal{C}^{\triangleright})^{\mathrm{op}} \to \mathrm{QCat}$ by the formula $\overline{\mathscr{G}}(C) = \mathrm{Fun}(\overline{\mathscr{F}}(C), \mathcal{D})$. By virtue of Proposition 7.5.8.4, it will suffice to show that the diagram of Kan complexes $\overline{\mathscr{G}}^{\simeq}$ is a homotopy limit diagram. Setting $\mathscr{G} = \overline{\mathscr{G}}|_{\mathcal{C}^{\mathrm{op}}}$, our assumption that $\mathscr{F}$ is projectively cofibrant guarantees that the diagram $\mathscr{G}$ is isofibrant (Remark 7.5.6.6). It follows that the diagram of Kan complexes $\mathscr{G}^{\simeq}$ is also isofibrant, and that $\overline{\mathscr{G}}^{\simeq}$ is a limit diagram (Corollary 4.5.6.21). The desired result now follows from Example 7.5.4.2. □

Corollary 7.5.8.8. *Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets, let $\theta : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \underrightarrow{\lim}(\mathscr{F})$ be the comparison map of Remark 5.3.2.9, and let W denote the collection of all horizontal edges of the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ (Definition 5.3.4.1). If $\mathscr{F}$ is projectively cofibrant (Definition 7.5.6.1), then θ exhibits $\underrightarrow{\lim}(\mathscr{F})$ as a localization of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ with respect to W.* 03D7

Proof. This is a restatement of Corollary 7.5.8.7. □

Corollary 7.5.8.9. *Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{QCat}$ be a diagram of ∞-categories. Then $\overline{\mathscr{F}}$ is a categorical colimit diagram (in the sense of Definition 7.5.7.3) if and only if the induced functor of ∞-categories* 03D8

$$\mathrm{N}_{\bullet}^{\mathrm{hc}}(\overline{\mathscr{F}}) : \mathrm{N}_{\bullet}(\mathcal{C}^{\triangleright}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{QCat}) = \mathcal{QC}$$

is a colimit diagram (in the sense of Variant 7.1.2.5).

Proof. Combine Proposition 7.5.8.4 with Corollary 7.5.4.6 (applied to the simplicial category $\mathrm{QCat}^{\mathrm{op}}$). □

Corollary 7.5.8.10. *Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Kan}$ be a diagram of Kan complexes. Then $\overline{\mathscr{F}}$ is a categorical colimit diagram if and only if it is a homotopy colimit diagram.* 03D9

Proof. Combine Corollary 7.5.8.9, Corollary 7.5.7.7, and Proposition 7.4.5.1. □

Corollary 7.5.8.11. *Let $\mathcal{C}$ be a category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a functor. Let $\overline{\mathscr{F}}^{\mathrm{op}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be the functor given on objects by $\overline{\mathscr{F}}^{\mathrm{op}}(C) = \overline{\mathscr{F}}(C)^{\mathrm{op}}$. Then $\overline{\mathscr{F}}$ is a categorical colimit diagram if and only if $\overline{\mathscr{F}}^{\mathrm{op}}$ is a categorical colimit diagram.* 03DA

Proof. Combine Proposition 7.5.8.4 with Corollary 7.5.5.15. □

We close this section with an application of the formalism of categorical colimit diagrams.

Proposition 7.5.8.12 (Rewriting Colimits)**.** *Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a categorical colimit diagram which carries the final object of $\mathcal{C}^{\triangleright}$ to a simplicial set K. Let $\mathcal{D}$ be an ∞-category equipped with a diagram $q : K \to \mathcal{D}$ satisfying the following condition:* 03DB

(∗) *For each object $C \in \mathcal{C}$, the composite map*

$$q_C : \overline{\mathscr{F}}(C) \to K \xrightarrow{q} \mathcal{D}$$

admits a colimit in the ∞-category $\mathcal{D}$.

Then there exists a functor $Q : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{D}$ *with the following properties:*

(1) *For each object* $C \in \mathcal{C}$*, the object* $Q(C) \in \mathcal{D}$ *is a colimit of the diagram* q_C*.*

(2) *An object* $X \in \mathcal{D}$ *is a colimit of the diagram* q *if and only if it is a colimit of* Q*. In particular, the diagram* q *has a colimit in* $\mathcal{D}$ *if and only if the diagram* Q *has a colimit in* $\mathcal{D}$*.*

(3) *Let* $G : \mathcal{D} \to \mathcal{E}$ *be a functor of* ∞*-categories which preserves the colimit of each of the diagrams* q_C*, and suppose that the diagrams* q *and* Q *admit colimits in* $\mathcal{D}$*. Then* G *preserves the colimit of* q *if and only if it preserves the colimit of* Q*.*

Proof. Set $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$, let $U : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to \mathrm{N}_{\bullet}(\mathcal{C})$ be the projection map, and let W be the collection of all horizontal edges of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$. The diagram $\overline{\mathscr{F}}$ then determines a morphism of simplicial sets $T : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \to K$ which exhibits K as a localization of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ with respect to W. It follows from assumption $(*)$ that for each object $C \in \mathcal{C}$, the composite map

$$\mathscr{F}(C) \simeq \{C\} \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \underrightarrow{\mathrm{holim}}(\mathscr{F}) \hookrightarrow \underrightarrow{\mathrm{holim}}(\mathscr{F}) \xrightarrow{T} K \xrightarrow{q} \mathcal{D}$$

admits a colimit in $\mathcal{D}$. Applying Corollary 7.3.5.3, we conclude that there is a functor $Q : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{D}$ and a natural transformation $\beta : T \circ q \to Q \circ U$ which exhibits Q as a left Kan extension of $T \circ q$ along U. We will complete the proof by showing that Q satisfies conditions (1), (2), and (3) of Proposition 7.5.8.12. Condition (1) follows immediately from Remark 7.3.5.4.

We now prove (2). Assume first that $X \in \mathcal{D}$ is a colimit of the diagram Q. For every simplicial set S, we let $\underline{X}_S$ denote the image of X in the ∞-category $\mathrm{Fun}(S, \mathcal{D})$. Choose a natural transformation $\alpha : Q \to \underline{X}_{\mathrm{N}_{\bullet}(\mathcal{C})}$ which exhibits $X \in \mathcal{D}$ as a colimit of the diagram Q, let $\widetilde{\alpha} : Q \circ U \to \underline{X}_{\underrightarrow{\mathrm{holim}}(\mathscr{F})}$ denote the image of α in $\mathrm{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{D})$, and let $\widetilde{\gamma} : q \circ T \to \underline{X}_{\underrightarrow{\mathrm{holim}}(\mathscr{F})} = \underline{X}_K \circ T$ be a composition of β with $\widetilde{\alpha}$ in $\mathrm{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{D})$. Since precomposition with T induces a fully faithful functor $\mathrm{Fun}(K, \mathcal{D}) \to \mathrm{Fun}(\underrightarrow{\mathrm{holim}}(\mathscr{F}), \mathcal{D})$, we may assume without loss of generality that $\widetilde{\gamma}$ is the image of a natural transformation $\gamma : q \to \underline{X}_K$. Note that $\widetilde{\gamma}$ exhibits X as a colimit of the diagram $q \circ T$ (Corollary 7.3.8.20). Since T is right cofinal (Proposition 7.2.1.10), it follows that γ exhibits X as a colimit of the diagram q (Corollary 7.2.2.7).

To prove the reverse implication, it will suffice to show that if the diagram $q : K \to \mathcal{D}$ admits a colimit, then Q also admits a colimit. Since T is right cofinal, the diagram $q \circ T$ also admits a colimit in $\mathcal{D}$ (Corollary 7.2.2.11), so the desired result is immediate from Corollary 7.3.8.20.

We now prove (3). Let $G : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories which preserves the colimit of the diagram q_C, for each object $C \in \mathcal{C}$. Let $\alpha : Q \to \underline{X}_{\mathrm{N}_{\bullet}(\mathcal{C})}$ and $\gamma : q \to \overline{X}_K$ be defined as above; we wish to show that $G(\alpha)$ exhibits $G(X)$ as a colimit of the diagram $G \circ Q$ if and only if $G(\gamma)$ exhibits $G(X)$ as a colimit of the diagram $G \circ q$. Using Corollary 7.2.2.7, we see that latter condition is equivalent to the requirement that $G(\widetilde{\gamma})$ exhibits $G(X)$ as a colimit of the diagram $G \circ q \circ T$. By virtue of Corollary 7.3.8.20, we are reduced to showing that the natural transformation $G(\beta)$ exhibits $G \circ Q$ as a left Kan extension of $G \circ q \circ T$ along U. This follows from the criterion of Remark 7.3.5.4. □

Corollary 7.5.8.13. *Let $\mathcal{C}$ be a small category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a categorical colimit diagram carrying the final object of $\mathcal{C}^{\triangleright}$ to a simplicial set K. Let $\mathcal{D}$ be an ∞-category which admits $\mathrm{N}_{\bullet}(\mathcal{C})$-indexed colimits and $\overline{\mathscr{F}}(C)$-indexed colimits, for each object $C \in \mathcal{C}$. Then $\mathcal{D}$ also admits K-indexed colimits. Moreover, if $G : \mathcal{D} \to \mathcal{E}$ is a functor of ∞-categories which preserves $\mathrm{N}_{\bullet}(\mathcal{C})$-indexed colimits and $\overline{\mathscr{F}}(C)$-indexed colimits for each $C \in \mathcal{C}$, then G also preserves K-indexed colimits.* 03DC

7.5.9 Application: Filtered Colimits of ∞-Categories

Let $\mathcal{C}$ be a small filtered category, let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram, and let $\mathcal{E} = \varinjlim(\mathscr{F})$ denote the colimit of $\mathscr{F}$ in the category of simplicial sets. If each of the simplicial sets $\mathscr{F}(C)$ is an ∞-category, then the simplicial set $\mathcal{E}$ is also an ∞-category (Remark 1.4.0.9). Our goal in this section is to show that, in this case, we can also regard $\mathcal{E}$ as a colimit of the diagram $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathscr{F}) : \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathcal{QC}$. This is a consequence of the following more general result: 03DD

Proposition 7.5.9.1. *Let $\mathcal{C}$ be a small filtered category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_{\Delta}$ be a colimit diagram in the category of simplicial sets. Then $\overline{\mathscr{F}}$ is a categorical colimit diagram.* 03DE

Remark 7.5.9.2. Let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ be a diagram of simplicial sets and let W denote the collection of horizontal edges of the homotopy colimit $\underrightarrow{\mathrm{holim}}(\mathscr{F})$. Proposition 7.5.9.1 asserts that, if the category $\mathcal{C}$ is filtered, then the comparison map $\theta : \underrightarrow{\mathrm{holim}}(\mathscr{F}) \twoheadrightarrow \varinjlim(\mathscr{F})$ exhibits $\varinjlim(\mathscr{F})$ as a localization of $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ with respect to W. In particular, θ is a weak homotopy equivalence. 03DF

Before giving the proof of Proposition 7.5.9.1, let us record some of its consequences.

Corollary 7.5.9.3. *Let $\mathcal{C}$ be a small filtered category. Then the inclusion map* 03DG

$$\mathrm{N}_{\bullet}(\mathrm{QCat}) \hookrightarrow \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}) = \mathcal{QC}$$

preserves $\mathrm{N}_{\bullet}(\mathcal{C})$-indexed colimits.

Proof. We first observe that the full subcategory $\mathrm{QCat} \subseteq \mathrm{Set}_{\Delta}$ is closed under filtered colimits (Remark 1.4.0.9), so the category QCat admits $\mathcal{C}$-indexed colimits. Fix a colimit diagram $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{QCat}$ in the ordinary category QCat. We wish to show that the induced map $\mathrm{N}_\bullet^{\mathrm{hc}}(\overline{\mathscr{F}}) : \mathrm{N}_\bullet(\mathcal{C}) \to \mathcal{QC}$ is a colimit diagram in the ∞-category $\mathcal{QC}$. By virtue of Corollary 7.5.8.9, this is equivalent to the requirement that $\overline{\mathscr{F}}$ is a categorical colimit diagram, which follows from Proposition 7.5.9.1. □

03DH **Variant 7.5.9.4.** Let $\mathcal{C}$ be a small filtered category. Then the inclusion map

$$\mathrm{N}_\bullet(\mathrm{Kan}) \hookrightarrow \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$$

preserves $\mathrm{N}_\bullet(\mathcal{C})$-indexed colimits.

03DJ **Corollary 7.5.9.5.** *Let $\mathcal{C}$ be a small filtered category and let $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets having colimit $K = \varinjlim(\mathscr{F})$. Let $\mathcal{D}$ be an ∞-category which admits $\mathrm{N}_\bullet(\mathcal{C})$-indexed colimits and which admits $\mathscr{F}(C)$-indexed colimits, for each $C \in \mathcal{C}$. Then $\mathcal{D}$ also admits K-indexed colimits. Moreover, if $G : \mathcal{D} \to \mathcal{E}$ is a functor which preserves both $\mathrm{N}_\bullet(\mathcal{C})$-indexed colimits and $\mathscr{F}(C)$-indexed colimits for each $C \in \mathcal{C}$, then G also preserves K-indexed colimits.*

Proof. Combine Proposition 7.5.9.1 with Corollary 7.5.8.13. □

03DK **Corollary 7.5.9.6.** *Let $\mathcal{D}$ be an ∞-category which admits finite colimits and small filtered colimits. Then $\mathcal{D}$ admits all small colimits. Moreover, if $G : \mathcal{D} \to \mathcal{E}$ is a functor of ∞-categories which preserves finite colimits and small filtered colimits, then G preserves all small colimits.*

Proof. This is a special case of Corollary 7.5.9.5, since every small simplicial set K can be realized as a (small) filtered colimit of finite simplicial sets. For example, we can write K as the union of all finite simplicial subsets of itself. □

Our proof of Proposition 7.5.9.1 will require a brief digression. Let $\mathcal{C}$ be a small category and let $\mathscr{G} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets. In §7.5.6, we showed that there exists a projectively cofibrant diagram $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ equipped with a levelwise weak homotopy equivalence $\alpha : \mathscr{F} \to \mathscr{G}$ (Proposition 7.5.6.9). Using a somewhat less explicit construction, we can obtain a better approximation to $\mathscr{G}$:

03DL **Proposition 7.5.9.7.** *Let $\mathcal{C}$ be a small category and let $\mathscr{G} : \mathcal{C} \to \mathrm{Set}_\Delta$ be a diagram of simplicial sets. Then there exists a projectively cofibrant diagram $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_\Delta$ and a levelwise trivial Kan fibration $\alpha : \mathscr{F} \to \mathscr{G}$.*

Proof of Proposition 7.5.9.1 from Proposition 7.5.9.7. Let $\mathcal{C}$ be a small filtered category and let $\overline{\mathscr{F}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_\Delta$ be a colimit diagram in the category of simplicial sets; we wish to show that $\overline{\mathscr{F}}$ is a categorical colimit diagram. Set $\mathscr{F} = \overline{\mathscr{F}}|_{\mathcal{C}}$. Using Proposition 7.5.9.7, we can choose a levelwise categorical equivalence $\alpha : \mathscr{E} \to \mathscr{F}$, where $\mathscr{E} : \mathcal{C} \to \mathrm{Set}_\Delta$ is projectively cofibrant. Let $\overline{\mathscr{E}} : \mathcal{C}^{\triangleright} \to \mathrm{Set}_\Delta$ be a colimit diagram extending $\mathscr{E}$, so that α extends uniquely to a natural transformation $\overline{\alpha} : \overline{\mathscr{E}} \to \overline{\mathscr{F}}$. Applying Corollary 4.5.7.2, we deduce that $\overline{\alpha}$ is also a levelwise categorical equivalence. Consequently, to show that $\overline{\mathscr{F}}$ is a categorical colimit diagram, it will suffice to show that $\overline{\mathscr{E}}$ is a categorical colimit diagram (Corollary 7.5.8.6). This follows from Corollary 7.5.8.7, since $\mathscr{E}$ is projectively cofibrant. □

It will be useful to formulate a slightly stronger version of Proposition 7.5.9.7. First, we need some terminology.

Definition 7.5.9.8. Let $\mathcal{C}$ be a small category and let $\alpha : \mathscr{F}' \to \mathscr{F}$ be a natural transformation between diagrams $\mathscr{F}, \mathscr{F}' : \mathcal{C} \to \mathrm{Set}_{\Delta}$. We say that α is a *projective cofibration* if it is left semiorthgonal to all levelwise trivial Kan fibrations (see Remark 4.5.6.2). That is, α is a projective cofibration if every lifting problem 03DM

$$\begin{tikzcd}
\mathscr{F}' \arrow[r] \arrow[d, "\alpha"] & \mathscr{G}' \arrow[d, "\beta"] \\
\mathscr{F} \arrow[r] \arrow[ur, dashed] & \mathscr{G}
\end{tikzcd}$$

admits a solution, under the assumption that β is a levelwise trivial Kan fibration between diagrams $\mathscr{G}, \mathscr{G}' : \mathcal{C} \to \mathrm{Set}_{\Delta}$.

Example 7.5.9.9. Let $\mathcal{C}$ be a small category. Then a diagram of simplicial sets $\mathscr{F} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ is projectively cofibrant (in the sense of Definition 7.5.6.1) if and only if the unique natural transformation $\underline{\emptyset} \to \mathscr{F}$ is a projective cofibration (in the sense of Definition 7.5.9.8). Here $\underline{\emptyset} : \mathcal{C} \to \mathrm{Set}_{\Delta}$ denotes the initial object of the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$, which carries every object of $\mathcal{C}$ to the empty simplicial set. 03DN

Example 7.5.9.10. Let $\mathcal{C}$ be a small category. For each object $C \in \mathcal{C}$, let $h^C : \mathcal{C} \to \mathrm{Set}$ denote the functor corepresented by C (given on objects by the formula $h^C(D) = \mathrm{Hom}_{\mathcal{C}}(C, D)$). If $A \hookrightarrow B$ is a monomorphism of simplicial sets, then the natural transformation $\underline{A} \times h^C \hookrightarrow \underline{B} \times h^C$ is a projective cofibration in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$; here $\underline{A}$ and $\underline{B}$ denote the constant simplicial sets taking the values A and B, respectively. 03DP

Remark 7.5.9.11. Let $\mathcal{C}$ be a small category. Then the collection of projective cofibrations in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set}_{\Delta})$ is weakly saturated, in the sense of Definition 1.5.4.12. That is, it is closed under retracts, pushouts, and transfinite composition. See Proposition 1.5.4.13. 03DQ

Proposition 7.5.9.12. *Let $\mathcal{C}$ be a small category and let $\alpha_0 : \mathscr{F}_0 \to \mathscr{G}$ be a natural transformation between diagrams $\mathscr{F}_0, \mathscr{G} : \mathcal{C} \to \mathrm{Set}_{\Delta}$. Then α_0 factors as a composition* 03DR

$$\mathscr{F}_0 \xrightarrow{\beta} \mathscr{F} \xrightarrow{\alpha} \mathscr{G},$$

where β is a projective cofibration and α is a levelwise trivial Kan fibration.

Proof. We will construct $\mathscr{F}$ as the colimit of a diagram of projective cofibrations

$$\mathscr{F}_0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to \cdots$$

in the category $\operatorname{Fun}(\mathcal{C}, \operatorname{Set}_{\Delta})_{/\mathscr{G}}$. Fix $n \geq 0$, and suppose that we have constructed an object $\mathscr{F}_n \in \operatorname{Fun}(\mathcal{C}, \operatorname{Set}_{\Delta})_{/\mathscr{G}}$, which we identify with a natural transformation $\alpha_n : \mathscr{F}_n \to \mathscr{G}$. For each object $C \in \mathcal{C}$, Exercise 3.1.7.11 guarantees that $\alpha_{n,C}$ factors as a composition

$$\mathscr{F}_n(C) \xrightarrow{\alpha'_{n,C}} \mathscr{F}'_n(C) \xrightarrow{\alpha''_{n,C}} \mathscr{G}(C),$$

where $\alpha'_{n,C}$ is a monomorphism and $\alpha''_{n,C}$ is a trivial Kan fibration (beware that $\mathscr{F}'_n(C)$ does not depend functorially on C). Form a pushout diagram

$$\begin{tikzcd}
\coprod_{C \in \mathcal{C}} \underline{\mathscr{F}_n(C)} \times h^C \arrow[r] \arrow[d] & \coprod_{C \in \mathcal{C}} \underline{\mathscr{F}'_n(C)} \times h^C \arrow[d] \\
\mathscr{F}_n \arrow[r] & \mathscr{F}_{n+1}
\end{tikzcd}$$

in the category $\operatorname{Fun}(\mathcal{C}, \operatorname{Set}_{\Delta})_{/\mathscr{G}}$, where the upper horizontal map is the coproduct of the projective cofibrations described in Example 7.5.9.10. Using Remark 7.5.9.11, we see that each of the maps

$$\mathscr{F}_0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to \cdots$$

is a projective cofibration. Setting $\mathscr{F} = \varinjlim_n \mathscr{F}_n$, we obtain a factorization of α_0 as a composition $\mathscr{F}_0 \xrightarrow{\beta} \mathscr{F} \xrightarrow{\alpha} \mathscr{G}$, where β is a projective cofibration. We complete the proof by observing that for each object $C \in \mathcal{C}$, the morphism $\alpha_C : \mathscr{F}(C) \to \mathscr{G}(C)$ is a trivial Kan fibration, since it can be written as a filtered colimit (in the arrow category $\operatorname{Fun}([1], \operatorname{Set}_{\Delta})$) of the trivial cofibrations $\alpha''_{n,C} : \mathscr{F}'_n(C) \to \mathscr{G}(C)$ (see Remark 1.5.5.3). □

Proof of Proposition 7.5.9.7. Apply Proposition 7.5.9.12 in the special case $\mathscr{F}_0 = \underline{\emptyset}$ (see Example 7.5.9.9). □

03DS **Corollary 7.5.9.13.** *Let $\mathcal{C}$ be a small category, and let S be the collection of all projective cofibrations in the category $\operatorname{Fun}(\mathcal{C}, \operatorname{Set}_{\Delta})$. Then S is the smallest weakly saturated collection of morphisms which contains each of the inclusion maps $\iota_{n,C} : \underline{\partial \Delta^n} \times h^C \hookrightarrow \underline{\Delta^n} \times h^C$, for each $n \geq 0$ and each object $C \in \mathcal{C}$.*

Proof. It follows from Remark 7.5.9.11 that S is weakly saturated. Let S' be the smallest weakly saturated collection of morphisms of $\operatorname{Fun}(\mathcal{C}, \operatorname{Set}_{\Delta})$ which contains each $\iota_{n,C}$. Using Example 7.5.9.10, we see that S' is contained in S. For every monomorphism of simplicial sets $A \hookrightarrow B$ and every object $C \in \mathcal{C}$, Proposition 1.5.5.14 guarantees that the projective cofibration $\underline{A} \times h^C \hookrightarrow \underline{B} \times h^C$ is contained in S'. It follows from the proof of Proposition 7.5.9.12 that every morphism $\alpha_0 : \mathscr{F}_0 \to \mathscr{G}$ in $\operatorname{Fun}(\mathcal{C}, \mathrm{Set}_\Delta)$ factors as a composition $\mathscr{F}_0 \xrightarrow{\beta} \mathscr{F} \xrightarrow{\alpha} \mathscr{G}$, where β belongs to S' and α is a trivial Kan fibration. If α_0 is projective cofibration, then the lifting problem

$$\begin{tikzcd}
\mathscr{F}_0 \arrow[r, "\beta"] \arrow[d, "\alpha_0"] & \mathscr{F} \arrow[d, "\alpha"] \\
\mathscr{G} \arrow[r, equal] \arrow[ur, dashed] & \mathscr{G}
\end{tikzcd}$$

admits a solution. It follows that α_0 is a retract of the morphism β, and therefore belongs to S'. □

7.6 Examples of Limits and Colimits

Let $\mathcal{C}$ be an ∞-category. In §7.1, we introduced the notion of limit and colimit for an arbitrary morphism of simplicial sets $\sigma : K \to \mathcal{C}$. Our goal in this section is to make the general theory more explicit for some special classes of diagrams which arise frequently in practice. 03E8

We begin in §7.6.1 by considering the case where K is a *discrete* simplicial set. In this case, specifying a functor $\sigma : K \to \mathcal{C}$ is equivalent to specifying a collection of objects $\{Y_k \in \mathcal{C}\}_{k \in K}$, indexed by the collection of vertices of K. We say that an object of $\mathcal{C}$ is a *product* of the collection $\{Y_k\}_{k \in K}$ if it is a limit of the diagram σ, and a *coproduct* of the collection $\{Y_k\}_{k \in K}$ if it is a colimit of the diagram σ. These conditions can be formulated purely in terms of the homotopy category $\mathrm{h}\mathcal{C}$, provided that we regard $\mathrm{h}\mathcal{C}$ as enriched over the homotopy category of Kan complexes hKan (see Remark 7.6.1.5). In particular, the forgetful functor from $\mathcal{C}$ to (the nerve of) its homotopy category $\mathrm{h}\mathcal{C}$ preserves products and coproducts (Warning 7.6.1.2).

In §7.6.2, we allow K to be an arbitrary simplicial set, but require $\sigma : K \to \mathcal{C}$ to be a constant diagram taking some value $Y \in \mathcal{C}$. In this case, we will denote a limit of σ (if it exists) by Y^K and a colimit of σ (if it exists) by $K \otimes Y$ (Notation 7.6.2.5). We refer to Y^K as a *power of Y by K*, and $K \otimes Y$ as a *tensor product of Y by K*. These notions can again be formulated purely at the level of the homotopy category $\mathrm{h}\mathcal{C}$, regarded as an hKan-enriched category (Definition 7.6.2.1 and Remark 7.6.2.6).

In §7.6.3, we study limit and colimit diagrams indexed by the simplicial set $K = \Delta^1 \times \Delta^1$.

Let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a functor of ∞-categories, which we depict as a diagram

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X.
\end{tikzcd}$$

We say that σ is a *pullback square* if it is a limit diagram, and a *pushout square* if it is a colimit diagram (Definition 7.6.3.1). Beware that these conditions cannot be formulated at the level of the homotopy category $\mathrm{h}\mathcal{C}$, even if its hKan-enrichment is accounted for: see Warning 7.6.3.3 Example 7.6.3.4.

It follows from Proposition 7.5.4.13 that a (strictly commutative) diagram of Kan complexes

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

determines a pullback square in the ∞-category $\mathcal{S}$ if and only if it is a homotopy pullback square. However, not every pullback square in the ∞-category $\mathcal{S}$ arises in this way. In §7.6.4, we give a detailed classification of *all* pullback squares in the ∞-category $\mathcal{S}$ (Corollary 7.6.4.10). In particular, for every pair of morphisms of Kan complexes $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$, we construct a pullback diagram

$$\begin{tikzcd}
X_0 \times^{\mathrm{h}}_{X} X_1 \arrow[r] \arrow[d] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X
\end{tikzcd}$$

in the ∞-category $\mathcal{S}$ (Example 7.6.4.12); beware that this diagram usually does not commute in the ordinary category of simplicial sets. Our analysis can be applied more generally to any ∞-category which arises as the homotopy coherent nerve of a locally Kan simplicial category (Corollary 7.6.4.14); in particular, it can be applied to the ∞-category $\mathcal{C} = \mathcal{QC}$ of small ∞-categories (see Proposition 7.6.4.8 and Corollary 7.6.4.9).

Let $(\bullet \rightrightarrows \bullet)$ denote the simplicial set given by the coproduct $\Delta^1 \coprod_{\partial \Delta^1} \Delta^1$ (Notation 7.6.5.1). In §7.6.5, we study limits and colimits of diagrams indexed by $(\bullet \rightrightarrows \bullet)$. For any ∞-category $\mathcal{C}$, functors $\sigma : (\bullet \rightrightarrows \bullet) \to \mathcal{C}$ can be identified with pairs $f_0, f_1 : Y \to X$ of morphisms in $\mathcal{C}$ having the same source and target. In this case, we denote a limit of σ (if it exists) by $\mathrm{Eq}(f_0, f_1)$, and a colimit of σ (if it exists) by $\mathrm{Coeq}(f_0, f_1)$ (Notation 7.6.5.5). We refer to $\mathrm{Eq}(f_0, f_1)$ as an *equalizer* of the pair (f_0, f_1), and to $\mathrm{Coeq}(f_0, f_1)$ as a *coequalizer* of (f_0, f_1) (Definition 7.6.5.4). Beware that, as with pullbacks and pushouts, the notions of equalizer and coequalizer cannot be formulated purely in terms of the homotopy category $\mathrm{h}\mathcal{C}$; in particular, the forgetful functor from $\mathcal{C}$ to (the nerve of) its homotopy category need not preserve equalizers and coequalizers.

Let $\mathbf{Z}_{\geq 0}$ denote the set of nonnegative integers, endowed with its usual linear ordering. In §7.6.6, we study colimits of diagram $X : \mathrm{N}_{\bullet}(\mathbf{Z}_{\geq 0}) \to \mathcal{C}$, which we represent informally as

$$X(0) \xrightarrow{f_0} X(1) \xrightarrow{f_1} X(2) \xrightarrow{f_2} X(3) \xrightarrow{f_3} X(4) \to \cdots$$

In the special case where $\mathcal{C} = \mathcal{S}$ is the ∞-category of spaces, we show that the colimit $\varinjlim X(n)$ (formed in the ordinary category of simplicial sets, using the transition morphisms f_i) is also a colimit in the ∞-category $\mathcal{S}$ (Variant 7.6.6.9). Similarly, if $Y : \mathrm{N}_{\bullet}(\mathbf{Z}_{\geq 0}^{\mathrm{op}}) \to \mathcal{S}$ is a diagram which we depict informally as

$$\cdots \to Y(4) \xrightarrow{g_3} Y(3) \xrightarrow{g_2} Y(2) \xrightarrow{g_1} Y(1) \xrightarrow{g_0} Y(0),$$

then the usual inverse limit $\varprojlim Y(n)$ (formed in the category of simplicial sets, using the transition morphisms g_n) is also a limit in the ∞-category $\mathcal{S}$, provided that each of the maps g_n is a Kan fibration (Variant 7.6.6.11). These assertions have counterparts for sequential limits and colimits in the ∞-category $\mathcal{QC}$: see Examples 7.6.6.8 and 7.6.6.10.

Though the classes of diagrams we study in this section are of a very restricted type, they are nonetheless useful for analyzing limits and colimits in general. If K is a complicated simplicial set which can be decomposed into simpler constituents, then we can often use Proposition 7.5.8.12 to reduce questions about K-indexed (co)limits to questions about (co)limits indexed by those constituents. We will consider several variants on this theme:

- If a simplicial set K decomposes as a disjoint union $\coprod_{j \in J} K_j$, then we can often rewrite K-indexed limits as products; see Proposition 7.6.1.18.
- If a simplicial set K fits into a categorical pushout diagram

$$\begin{tikzcd} K_{01} \arrow[r] \arrow[d] & K_0 \arrow[d] \\ K_1 \arrow[r] & K, \end{tikzcd}$$

  then we can often rewrite K-indexed limits as pullbacks; see Proposition 7.6.3.26.

- If $\mathcal{C}$ is an ∞-category which admits finite products, then an equalizer of a pair of morphisms $f_0, f_1 : Y \to X$ (if it exists) is characterized by the existence of a pullback diagram

  $$\begin{tikzcd}
  \mathrm{Eq}(f_0, f_1) \arrow[r] \arrow[d] & Y \arrow[d, "{(f_0,f_1)}"] \\
  X \arrow[r, "\delta_X"] & X \times X;
  \end{tikzcd}$$

  see Proposition 7.6.5.22.

- If $\mathcal{C}$ is an ∞-category which admits finite products, then a pullback of a diagram $X_0 \xrightarrow{f_0} X \xleftarrow{f_1} X_1$ can be rewritten as the equalizer of a diagram $X_0 \times X_1 \rightrightarrows X$ (Proposition 7.6.5.23).

- If $\mathcal{C}$ is an ∞-category which admits countable products, then the limit of a tower

  $$\cdots \to X(3) \xrightarrow{f_2} X(2) \xrightarrow{f_1} X(1) \xrightarrow{f_0} X(0)$$

  can be rewritten as an equalizer $\mathrm{Eq}(f, \mathrm{id}_X)$, where X is the product $\prod_{n \geq 0} X(n)$ and $f : X \to X$ is the endomorphism of X determined by the sequence $\{f_n\}_{n \geq 0}$; see Proposition 7.6.6.16.

- If K is a simplicial set which can be written as the colimit of a sequence

  $$K(0) \to K(1) \to K(2) \to K(3) \to \cdots,$$

  then we can often rewrite K-indexed limits as sequential limits (Corollary 7.6.6.14).

By applying these observations iteratively, one can build arbitrarily complicated limits (and colimits) out of the constructions studied in this section. For example, we show that an ∞-category $\mathcal{C}$ admits finite limits if and only if it admits pullbacks and has a final object (Corollary 7.6.3.27).

7.6.1 Products and Coproducts

03E9 We now study limits and colimits of diagrams which are indexed by *discrete* simplicial sets. In this case, the definitions of limit and colimit can be formulated entirely at the level of the (enriched) homotopy category.

03EA **Definition 7.6.1.1.** Let hKan denote the homotopy category of Kan complexes and let $\mathcal{C}$ be an hKan-enriched category. We say that a collection of morphisms $\{q_i : Y \to Y_i\}_{i \in I}$ of $\mathcal{C}$

exhibits Y as an hKan-*enriched product of the collection* $\{Y_i\}_{i \in I}$ if, for every object $X \in \mathcal{C}$, the collection of maps $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \xrightarrow{q_i \circ} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y_i)$ induces an isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \prod_{i \in I} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y_i)$$

in the homotopy category hKan.

We say that a collection of morphisms $\{e_i : Y_i \to Y\}_{i \in I}$ *exhibits Y as an* hKan-*enriched coproduct of the collection* $\{Y_i\}_{i \in I}$ if, for every object $Z \in \mathcal{C}$, the collection of maps $\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ e_i} \underline{\mathrm{Hom}}_{\mathcal{C}}(Y_i, Z)$ induces an isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \prod_{i \in I} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y_i)$$

Warning 7.6.1.2. Let $\mathcal{C}$ be an hKan-enriched category, and let $\{q_i : Y \to Y_i\}_{i \in I}$ be a collection of morphisms in $\mathcal{C}$. If $\{q_i\}_{i \in I}$ exhibits Y as an hKan-enriched product of $\{Y_i\}_{i \in I}$, then it also exhibits Y as a product of the collection $\{Y_i\}_{i \in I}$ in the underlying category $\mathcal{C}$ (where we neglect its hKan-enrichment). Beware that the converse is false in general (see Warning 7.6.1.11). 03EB

Definition 7.6.1.3. Let $\mathcal{C}$ be an ∞-category. We say that a collection of morphisms $\{q_i : Y \to Y_i\}_{i \in I}$ in $\mathcal{C}$ *exhibits Y as a product of the collection* $\{Y_i\}_{i \in I}$ if the collection of homotopy classes $\{[q_i] : Y \to Y_i\}_{i \in I}$ exhibits Y as an hKan-enriched product $\{Y_i\}_{i \in I}$ in the homotopy category $\mathrm{h}\mathcal{C}$ (equipped with the hKan-enrichment described in Construction 4.6.9.13). In other words, the collection of morphisms $\{q_i\}_{i \in I}$ exhibits Y as a product of the collection of objects $\{Y_i\}_{i \in I}$ if, for every object $X \in \mathcal{C}$, the induced map 03EC

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X, Y_i)$$

is a homotopy equivalence of Kan complexes. Similarly, we say that a collection of morphisms $\{e_i : Y_i \to Y\}_{i \in I}$ of $\mathcal{C}$ *exhibits Y as a coproduct of the collection* $\{Y_i\}_{i \in I}$ if, for every object $Z \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(Y_i, Z)$$

is a homotopy equivalence of Kan complexes.

Remark 7.6.1.4. Let $\{f_i : Y \to Y_i\}_{i \in I}$ be a collection of morphisms in an ∞-category $\mathcal{C}$. Then the collection $\{q_i\}_{i \in I}$ exhibits Y as a product of the collection $\{Y_i\}_{i \in I}$ in the category $\mathcal{C}$ if and only if it exhibits Y as a coproduct of the collection $\{Y_i\}_{i \in I}$ in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. 03ED

Remark 7.6.1.5. Let $\mathcal{C}$ be an ∞-category and let $\{Y_i\}_{i \in I}$ be a collection of objects of $\mathcal{C}$, which we will identify with a diagram 03EE

$$F : I \to \mathcal{C} \qquad F(i) = Y_i$$

indexed by the constant simplicial set associated to I (Remark 1.1.5.3). Suppose we are given another object $Y \in \mathcal{C}$ together with a collection of morphisms $\{q_i : Y \to Y_i\}_{i \in I}$. The following conditions are equivalent:

(1) The collection of morphisms $\{q_i\}_{i \in I}$ exhibits Y as a product of the collection $\{Y_i\}_{i \in I}$, in the sense of Definition 7.6.1.3.

(2) Let $\underline{Y} : I \to \mathcal{C}$ denote the constant diagram taking the value Y, so that the collection $\{q_i\}_{i \in I}$ can be identified with a natural transformation $q : \underline{Y} \to F$. Then q exhibits Y as a limit of the diagram F, in the sense of Definition 7.1.1.1.

(3) Let $\overline{F} : I^{\triangleleft} \to \mathcal{C}$ be the diagram carrying each edge $\{i\}^{\triangleleft} \subseteq I^{\triangleleft}$ to the morphism q_i. Then $\overline{F}$ is a limit diagram in $\mathcal{C}$, in the sense of Definition 7.1.2.4.

The equivalence (1) ⇔ (2) is immediate from the definitions (see Remark 4.6.1.9) and the equivalence (2) ⇔ (3) follows from Remark 7.1.2.6.

`03EF` **Remark 7.6.1.6.** Let $\mathcal{C}$ be an ordinary category, and let $\{q_i : Y \to Y_i\}_{i \in I}$ be a collection of morphisms in $\mathcal{C}$. Then $\{q_i\}_{i \in I}$ exhibits Y as a product of the collection $\{Y_i\}_{i \in I}$ in the category $\mathcal{C}$ (in the sense of classical category theory) if and only if it exhibits Y as a product of the collection $\{Y_i\}_{i \in I}$ in the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$ (in the sense of Definition 7.6.1.3).

`03EG` **Notation 7.6.1.7.** Let $\mathcal{C}$ be an ∞-category and let $\{Y_i\}_{i \in I}$ be a collection of objects of $\mathcal{C}$. We will say that an object $Y \in \mathcal{C}$ is a *product* of the collection $\{Y_i\}_{i \in I}$ if there exists a collection of morphisms $\{q_i : Y \to Y_i\}$ which exhibits Y as a product of $\{Y_i\}_{i \in I}$. If this condition is satisfied, then the object Y is uniquely determined up to isomorphism (see Proposition 7.1.1.12). To emphasize this uniqueness, we will sometimes denote the object Y by $\prod_{i \in I} Y_i$, and refer to it as *the* product of the collection $\{Y_i\}_{i \in I}$. Similarly, we say that Y is a *coproduct* of the collection $\{Y_i\}_{i \in I}$ if there exists a collection of morphisms $\{e_i : Y_i \to Y\}_{\in I}$ which exhibits Y as a coproduct of $\{Y_i\}_{i \in I}$. In this case, we sometimes denote the object Y by $\coprod_{i \in I} Y_i$ and refer to it as *the* coproduct of the collection $\{Y_i\}_{i \in I}$.

`03EH` **Example 7.6.1.8** (Initial and Final Objects)**.** Let $\mathcal{C}$ be an ∞-category. An object $Y \in \mathcal{C}$ is initial (in the sense of Definition 4.6.7.1) if and only if it is the coproduct of the empty collection of objects of $\mathcal{C}$ (see Example 7.1.1.6). Similarly, Y is final if and only if it is a product of the empty collection of objects.

`03EJ` **Example 7.6.1.9** (Isomorphisms)**.** Let $f : X \to Y$ be a morphism in an ∞-category $\mathcal{C}$. The following conditions are equivalent:

(1) The morphism f is an isomorphism.

(2) The morphism f exhibits X as a product of the one-element collection of objects $\{Y\}$.

(3) The morphism f exhibits Y as a coproduct of the one-element collection of objects $\{X\}$.

Notation 7.6.1.10. In practice, we will use Definition 7.6.1.3 most often in the case where the set I has exactly two elements, so that the collection $\{Y_i\}_{i \in I}$ can be identified with an ordered pair (Y_0, Y_1) of objects of $\mathcal{C}$. In this case, we say that morphisms $q_0 : Y \to Y_0$ and $q_1 : Y \to Y_1$ *exhibit* Y *as a product of* Y_0 *with* Y_1 if they satisfy the requirement of Definition 7.6.1.3: that is, for every object $X \in \mathcal{C}$, the induced map `03EK`

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y_0) \times \mathrm{Hom}_{\mathcal{C}}(X, Y_1)$$

is a homotopy equivalence. If this condition is satisfied, then we will often denote the object Y by $Y_0 \times Y_1$ and refer to it as *the* product of Y_0 with Y_1. Similarly, we say that a pair of morphisms $e_0 : Y_0 \to Y$ and $e_1 : Y_1 \to Y$ *exhibit* Y *as a coproduct of* Y_0 *with* Y_1 if, for every object $Z \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(Y_0, Z) \times \mathrm{Hom}_{\mathcal{C}}(Y_1, Z)$$

is a homotopy equivalence; in this case, we denote Y by $Y_0 \coprod Y_1$ and refer to it as *the* coproduct of Y_0 with Y_1.

Warning 7.6.1.11. Let $\mathcal{C}$ be an ∞-category. if $\{q_i : Y \to Y_i\}_{i \in I}$ is a collection of morphisms of $\mathcal{C}$ which exhibits Y as a product of the collection of objects $\{Y_i\}_{i \in I}$ in the ∞-category $\mathcal{C}$, then the collection of homotopy classes $\{[q_i] : Y \to Y_i\}_{i \in I}$ exhibits Y as a product of the collection $\{Y_i\}_{i \in I}$ in the ordinary category $\mathrm{h}\mathcal{C}$. The converse holds if the collection $\{Y_i\}_{i \in I}$ admits a products in the ∞-category $\mathcal{C}$. However, the converse need not hold in general, even in the special case where the set I is empty: see Warning 4.6.7.18. `03EL`

Proposition 7.6.1.12. *Let $\mathcal{C}$ be an ∞-category containing an object X. The following conditions are equivalent:* `04Q7`

(1) *For every object $Y \in \mathcal{C}$, there exists a product of $X \times Y$ in the ∞-category $\mathcal{C}$.*

(2) *The forgetful functor $U : \mathcal{C}_{/Y} \to \mathcal{C}$ admits a right adjoint.*

If these conditions are satisfied, then the right adjoint of U is given on objects by the construction $Y \mapsto X \times Y$.

Proof. If Y is an object of $\mathcal{C}$, then a product $X \times Y$ (if it exists) can be identified with a final object of the ∞-category $\mathcal{C}_{/X} \times_{\mathcal{C}} \mathcal{C}_{/Y}$. The equivalence of (1) and (2) is therefore a special case of the criterion of Corollary 6.2.4.2. □

Example 7.6.1.13 (Homotopy Products). Let $\mathcal{C}$ be a locally Kan simplicial category, and let $\{q_i : Y \to Y_i\}_{i \in I}$ be a collection of morphisms in $\mathcal{C}$. By virtue of Theorem 4.6.8.5 (and Proposition 4.6.9.19), the following conditions are equivalent: `03EM`

(1) The morphisms q_i exhibit Y as a product of the collection $\{Y_i\}_{i \in I}$ in the ∞-category $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$.

(2) For every object $X \in \mathcal{C}$, composition with the morphisms q_i determines a homotopy equivalence of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)_{\bullet} \to \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X, Y_i)_{\bullet}.$$

03EN **Example 7.6.1.14** (Products in $\mathcal{S}$)**.** Let $\{Y_i\}_{i \in I}$ be a collection of Kan complexes and let $Y = \prod_{i \in I} Y_i$ denote their product, formed in the ordinary category of simplicial sets. For each $i \in I$, let $q_i : Y \to Y_i$ denote the projection map. Applying Example 7.6.1.13 to the simplicial category category Kan, we deduce that the morphisms q_i also exhibit Y as a product of the collection $\{Y_i\}_{i \in I}$ in the ∞-category of spaces $\mathcal{S} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{Kan})$. Similarly, if $Y' = \coprod_{i \in I} Y_i$ is the coproduct of the collection $\{Y_i\}_{i \in I}$ in the ordinary category of simplicial sets, then the inclusion maps $Y_i \hookrightarrow Y'$ exhibit Y' as a coproduct of $\{Y_i\}_{i \in I}$ in the ∞-category $\mathcal{S}$.

03EP **Example 7.6.1.15** (Products in $\mathcal{QC}$)**.** Let $\{\mathcal{C}_i\}_{i \in I}$ be a collection of ∞-categories and let $\mathcal{C} = \prod_{i \in I} \mathcal{C}_i$ denote their product, formed in the ordinary category of simplicial sets. For each $i \in I$, let $q_i : \mathcal{C} \to \mathcal{C}_i$ denote the projection map. Applying Example 7.6.1.13 to the simplicial category QCat (see Construction 5.5.4.1), we deduce that the morphisms q_i also exhibit $\mathcal{C}$ as a product of the collection $\{\mathcal{C}_i\}_{i \in I}$ in the ∞-category $\mathcal{QC} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat})$ (this is a special case of the diffraction criterion of Theorem 7.4.1.1). Similarly, if $\mathcal{C}' = \coprod_{i \in I} \mathcal{C}_i$ is the coproduct of the collection $\{\mathcal{C}_i\}_{i \in I}$ in the ordinary category of simplicial sets, then the inclusion maps $\mathcal{C}_i \hookrightarrow \mathcal{C}'$ exhibit $\mathcal{C}'$ as a coproduct of $\{\mathcal{C}_i\}_{i \in I}$ in the ∞-category QCat (this is a special case of the refraction criterion of Theorem 7.4.3.6).

03EQ **Example 7.6.1.16** (Products in a Duskin Nerve)**.** Let $\mathcal{C}$ be a $(2,1)$-category and let $\{q_i : Y \to Y_i\}$ be a collection of 1-morphisms in $\mathcal{C}$. Then the following conditions are equivalent:

(1) The morphisms q_i exhibit Y as a product of the collection $\{Y_i\}_{i \in I}$ in the ∞-category $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$.

(2) For every object $X \in \mathcal{C}$, horizontal composition with the 1-morphisms q_i induces an equivalence of categories

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \to \prod_{i \in I} \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y_i).$$

This follows from the explicit description of pinched morphism spaces in $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ supplied by Example 4.6.5.13.

Example 7.6.1.17 (Products in a Differential Graded Nerve)**.** Let $\mathcal{C}$ be a differential graded category and let $\{q_i : Y \to Y_i\}$ be a collection of morphisms in the underlying category of $\mathcal{C}$ (that is, each q_i is a 0-cycle of the chain complex $\mathrm{Hom}_{\mathcal{C}}(Y, Y_i)_*$). Using Example 4.6.5.15 (together with Exercise 3.2.2.22), we see that the following conditions are equivalent: 03ER

(1) The morphisms q_i exhibit Y as a product of the collection $\{Y_i\}_{i \in I}$ in the ∞-category $\mathrm{N}^{\mathrm{dg}}_{\bullet}(\mathcal{C})$.

(2) For every object $X \in \mathcal{C}$, the map of chain complexes

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)_* \to \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X, Y_i)_*$$

induces an isomorphism on homology in degrees ≥ 0.

Proposition 7.6.1.18 (Rewriting Limits as Products)**.** *Let $\mathcal{C}$ be an ∞-category, and let $\{f_i : K_i \to \mathcal{C}\}_{i \in I}$ be a collection of diagrams, each of which admits a limit $X_i = \varprojlim(f_i)$. Set $K = \coprod_{i \in I} K_i$, so that the collection $\{f_i\}_{i \in I}$ determines a diagram $f : K \to \mathcal{C}$. Then an object of $\mathcal{C}$ is a limit of the diagram f if it is a product of the collection of objects $\{X_i\}_{i \in I}$.* 03ES

Proof. This is a special case of (the dual of) Proposition 7.5.8.12. □

Remark 7.6.1.19. In the situation of Proposition 7.6.1.18, let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves the limits of each of the diagrams f_i. Suppose that the collection $\{X_i\}_{i \in I}$ admits a product in $\mathcal{C}$. Then the product of $\{X_i\}_{i \in I}$ is preserved by the functor F if and only if the limit of f is preserved by the functor F. 03ET

Corollary 7.6.1.20. *Let $\{K_i\}_{i \in I}$ be a collection of simplicial sets having coproduct $K = \coprod_{i \in I} K_i$, and let $\mathcal{C}$ be an ∞-category. Suppose that $\mathcal{C}$ admits I-indexed products and K_i-indexed limits for each $i \in I$. Then $\mathcal{C}$ admits K-indexed limits. Moreover, if $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves I-indexed products and K_i-indexed colimits for each $i \in I$, then F also preserves K-indexed limits.* 03EU

Corollary 7.6.1.21. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ admits finite products if and only if it satisfies the following pair of conditions:* 03EV

(1) *The ∞-category $\mathcal{C}$ has a final object* $\mathbf{1}$.

(2) *The ∞-category $\mathcal{C}$ admits pairwise products. That is, every pair of objects $X, Y \in \mathcal{C}$ have a product $X \times Y$ in $\mathcal{C}$.*

Proof. The necessity of conditions (1) and (2) is clear (see Example 7.6.1.8). Conversely, suppose that (1) and (2) are satisfied, and let I be a finite set. We wish to show that $\mathcal{C}$ admits I-indexed limits. We proceed by induction on the cardinality of I. If I is empty, then the desired result follows from assumption (1). If I is a singleton, then the desired result is obvious (see Example 7.6.1.9). Otherwise, we can write I as a disjoint union of proper subsets $I_-, I_+ \subset I$. Our inductive hypothesis then guarantees that $\mathcal{C}$ admits I_--indexed limits and I_+-indexed limits. Combining assumption (2) with Corollary 7.6.1.20, we deduce that $\mathcal{C}$ admits limits indexed by $I = I_- \coprod I_+$. $\square$

03EW **Remark 7.6.1.22.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, where $\mathcal{C}$ admits finite products. Then F preserves finite products if and only if it preserves final objects and pairwise products.

7.6.2 Powers and Tensors

03EX We now study limits and colimits which are indexed by *constant* diagrams of simplicial sets. Like products and coproducts, these can be characterized by universal properties in the (enriched) homotopy category.

03EY **Definition 7.6.2.1.** Let $\mathcal{C}$ be an ∞-category containing a pair of objects X and Y, and let $e : K \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ be a morphism of simplicial sets. We will say that *e exhibits X as a power of Y by K* if, for every object $W \in \mathcal{C}$, the composition law $\circ : \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y)$ of Construction 4.6.9.9 induces a homotopy equivalence of Kan complexes $\mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(W, Y))$.

We will say that *e exhibits Y as a tensor product of X by K* if, for every object $Z \in \mathcal{C}$, the composition law $\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ induces a homotopy equivalence of Kan complexes $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(X, Z))$.

03EZ **Warning 7.6.2.2.** In the situation of Definition 7.6.2.1, the composition law

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

is only well-defined up to homotopy. However, the requirement that it induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(X, Z))$ depends only on its homotopy class.

03F0 **Remark 7.6.2.3.** In the situation of Definition 7.6.2.1, the condition that $e : K \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ exhibits X as a power of Y by K (or Y as a tensor product of X by K) depends only on the homotopy class $[e] \in \pi_0(\mathrm{Fun}(K, \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)))$.

03F1 **Remark 7.6.2.4** (Duality)**.** In the situation of Definition 7.6.2.1, the morphism $e : K \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ exhibits X as a power of Y by K in the ∞-category $\mathcal{C}$ if and only if the morphism

$$e^{\mathrm{op}} : K^{\mathrm{op}} \to \mathrm{Hom}_{\mathcal{C}}(X, Y)^{\mathrm{op}} \simeq \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, X)$$

exhibits X as a tensor product of Y by K^{op} in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$.

Notation 7.6.2.5. Let $\mathcal{C}$ be an ∞-category, let Y be an object of $\mathcal{C}$, and let K be a simplicial set. Suppose that there exists an object $X \in \mathcal{C}$ and a morphism $e : K \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$ which exhibits X as a power of Y by K. In this case, the object X is uniquely determined up to isomorphism. To emphasize this uniqueness, we will sometimes denote the object X by Y^K. 03F2

Similarly, if there exists an object $Z \in \mathcal{C}$ and a morphism $e : K \to \underline{\operatorname{Hom}}_{\mathcal{C}}(Y, Z)$ which exhibits Z as a tensor product of Y by K, then Z is uniquely determined up to isomorphism. We will sometimes emphasize this dependence by denoting the object Z by $K \otimes Y$.

Remark 7.6.2.6 (Powers as Limits)**.** Let $\mathcal{C}$ be an ∞-category containing objects X and Y. Then a morphism of simplicial sets $e : K \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$ can be identified with a natural transformation $\alpha : \underline{X} \to \underline{Y}$, where $\underline{X}, \underline{Y} : K \to \mathcal{C}$ denote the constant diagrams taking the values X and Y, respectively. In this case: 03F3

- The natural transformation α exhibits the object X as a limit of the diagram $\underline{Y}$ (in the sense of Definition 7.1.1.1) if and only if e exhibits X as a power of Y by K (in the sense of Definition 7.6.2.1).
- The natural transformation α exhibits the object Y as a colimit of the diagram $\underline{X}$ (in the sense of Definition 7.1.1.1) if and only if e exhibits Y as a tensor product of X by K (in the sense of Definition 7.6.2.1).

Example 7.6.2.7. Let $\mathcal{C}$ be an ∞-category containing objects X and Y, and suppose we are given a collection of morphisms $\{f_j : X \to Y\}_{j \in J}$ indexed by a set J. If we abuse notation by identifying J with the corresponding discrete simplicial set, then the collection $\{f_j\}_{j \in J}$ can be identified with a map $e : J \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$. In this case: 03F4

- The morphism e exhibits X as a power of Y by J (in the sense of Definition 7.6.2.1) if and only if the collection $\{f_j\}_{j \in J}$ exhibits X as a product of the collection $\{Y\}_{j \in J}$ (in the sense of Definition 7.6.1.3). Stated more informally, we have a canonical isomorphism $Y^J \simeq \prod_{j \in J} Y$ (provided that either side is defined).
- The morphism e exhibits Y as a tensor product of X by J (in the sense of Definition 7.6.2.1) if and only if the collection $\{f_j\}_{j \in J}$ exhibits Y as a coproduct of the collection $\{X\}_{j \in J}$ (in the sense of Definition 7.6.1.3). Stated more informally, we have a canonical isomorphism $J \otimes X \simeq \coprod_{j \in J} X$ (provided that either side is defined).

Example 7.6.2.8. Let $\mathcal{C}$ be an ∞-category containing objects X and Y. Then the unique morphism $e : \emptyset \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$ exhibits X as a power of Y by the empty simplicial set if and only if X is a final object of $\mathcal{C}$. Similarly, e exhibits Y as a tensor product of X by the empty simplicial set if and only if Y is an initial object of $\mathcal{C}$. 03F5

Notation 7.6.2.9 (Diagonal Morphisms)**.** Let $\mathcal{C}$ be an ∞-category containing a pair of objects X and Y, and let $e : K \to \operatorname{Hom}_{\mathcal{C}}(X, Y)$ be a morphism of simplicial sets which 04K4 exhibits X as a power of Y by K. Then there exists a morphism $\delta : Y \to X$ which is characterized (up to homotopy) by the requirement that the diagram of simplicial sets

$$\begin{tikzcd}
K \arrow[r, "e"] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[d, "\circ[\delta]"] \\
\{\mathrm{id}_Y\} \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Y,Y)
\end{tikzcd}$$

commutes up to homotopy. We will refer to δ as the *diagonal morphism.*

We will be particularly interested in the special case where $K = \partial\Delta^1$, so that X can be identified with the product $Y \times Y$ (Example 7.6.2.7). In this case, we will often denote δ by $\delta_Y : Y \to Y \times Y$ and refer to it as the *diagonal of* Y.

03F6 **Proposition 7.6.2.10.** *Let $\mathcal{C}$ be a locally Kan simplicial category, let X and Y be objects of $\mathcal{C}$, and let $e : K \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ be a morphism of simplicial sets. Let $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ denote the homotopy coherent nerve of $\mathcal{C}$, and let $\theta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(X,Y)$ denote the comparison map of Remark 4.6.8.6. Then:*

(1) *The morphism $\theta_{X,Y} \circ e$ exhibits X as a power of Y by K in the ∞-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ if and only if, for every object $W \in \mathcal{C}$, composition with e induces a homotopy equivalence of Kan complexes*

$$c_W : \mathrm{Hom}_{\mathcal{C}}(W,X)_\bullet \to \mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(W,Y)_\bullet).$$

(2) *The morphism $\theta_{X,Y} \circ e$ exhibits Y as a tensor product of X by K in the ∞-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ if and only if, for every object $Z \in \mathcal{C}$, precomposition with e induces a homotopy equivalence of Kan complexes*

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z)_\bullet \to \mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(X,Z)_\bullet).$$

Proof. We will prove (1); the proof of (2) is similar. Fix an object $W \in \mathcal{C}$, so that the composition law

$$\circ : \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(X,Y) \times \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(W,X) \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(W,Y)$$

of Construction 4.6.9.9 determines a morphism of Kan complexes $c'_W : \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(W,X) \to \mathrm{Fun}(K, \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(W,Y)_\bullet)$ (which is well-defined up to homotopy). To prove Proposition 7.6.2.10, it will suffice to show that c'_W is a homotopy equivalence if and only if c_W is a homotopy equivalence. Proposition 4.6.9.19 guarantees that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(W,X)_\bullet \arrow[r, "c_W"] \arrow[d, "\theta_{W,X}"] & \mathrm{Fun}(K, \mathrm{Hom}_{\mathcal{C}}(W,Y)_\bullet) \arrow[d, "\theta_{W,Y}\circ"] \\
\mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(W,X) \arrow[r, "c'_W"] & \mathrm{Fun}(K, \mathrm{Hom}_{\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})}(W,Y)_\bullet
\end{tikzcd}$$

commutes up to homotopy. We conclude by observing that the horizontal maps are homotopy equivalences, by virtue of Theorem 4.6.8.5 (and Remark 4.6.8.6). □

Example 7.6.2.11. Let X and Y be essentially small Kan complexes, let $e_0 : K \to \operatorname{Fun}(X, Y)$ be a morphism of simplicial sets, and let $e : K \to \operatorname{Hom}_{\mathcal{S}}(X, Y)$ denote the composition of e_0 with the homotopy equivalence $\operatorname{Fun}(X, Y) \to \operatorname{Hom}_{\mathcal{S}}(X, Y)$ of Remark 5.5.1.5. Then: 03F7

- The morphism e exhibits X as a power of Y by K in the ∞-category $\mathcal{S}$ and only the induced map $X \to \operatorname{Fun}(K, Y)$ is a homotopy equivalence of Kan complexes.
- The morphism e exhibits Y as a tensor product of X by K in the ∞-category $\mathcal{S}$ if and only if the induced map $K \times X \to Y$ is a weak homotopy equivalence of simplicial sets.

Example 7.6.2.12. Let Y be an essentially small Kan complex. Suppose we are given a morphism of simplicial sets $f : K \to \operatorname{Hom}_{\mathcal{S}}(\Delta^0, Y)$, which we identify with a morphism $\widetilde{f} : \underline{\Delta^0}_K \to \underline{Y}_K$ in the ∞-category $\operatorname{Fun}(K, \mathcal{S})$. Then f is a weak homotopy equivalence if and only if $\widetilde{f}$ exhibits Y as a tensor product of Δ^0 by K (in the ∞-category $\mathcal{S}$). To prove this, we are free to modify the morphism f by a homotopy (see Remark 7.6.2.3). We may therefore assume without loss of generality that f factors through the homotopy equivalence $e : \operatorname{Fun}(\Delta^0, Y) \to \operatorname{Hom}_{\mathcal{S}}(\Delta^0, Y)$ of Remark 5.5.1.5, in which case the desired result follows from the criterion of Example 7.6.2.11 (applied in the case $X = \Delta^0$). Taking $K = Y$ and $f = e$, we see that every Kan complex Y can be viewed as a colimit of the constant diagram $Y \to \{\Delta^0\} \hookrightarrow \mathcal{S}$ (see Remark 7.6.2.6). 03F8

Remark 7.6.2.13 (Cofinality and Kan Extensions)**.** Let $\mathcal{C}$ be an ∞-category and let $\delta : K \to \mathcal{C}$ be a morphism of simplicial sets. The following conditions are equivalent: 03LS

(1) The morphism δ is left cofinal.

(2) The identity transformation $\mathrm{id} : \underline{\Delta^0}_K \to \underline{\Delta^0}_{\mathcal{C}} \circ \delta$ exhibits the constant functor $\underline{\Delta^0}_{\mathcal{C}} : \mathcal{C} \to \mathcal{S}$ as a left Kan extension of the constant diagram $\underline{\Delta^0}_K : K \to \mathcal{S}$ along δ.

By virtue of Theorem 7.2.3.1 and Example 7.6.2.12, both conditions are equivalent to the requirement that, for every object $C \in \mathcal{C}$, the simplicial set $K_{/C} = K \times_{\mathcal{C}} \mathcal{C}_{/C}$ is weakly contractible.

Example 7.6.2.14. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, let $e_0 : K \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$ be a morphism of Kan complexes, and let e denote the composition of e_0 with the homotopy equivalence $\operatorname{Fun}(\mathcal{C}, \mathcal{D})^{\simeq} \to \operatorname{Hom}_{\mathcal{QC}}(\mathcal{C}, \mathcal{D})$ of Remark 5.5.4.5. Combining Propositions 7.6.2.10 and 4.4.3.22, we obtain the following: 03F9

- The morphism e exhibits $\mathcal{C}$ as a power of $\mathcal{D}$ by K in the ∞-category $\mathcal{QC}$ if and only if the induced map $\mathcal{C} \to \operatorname{Fun}(K, \mathcal{D})$ is an equivalence of ∞-categories.

- The morphism e exhibits $\mathcal{C}$ as a tensor product of $\mathcal{D}$ by K in the ∞-category $\mathcal{QC}$ if and only if the induced map $K \times \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories.

03FA **Warning 7.6.2.15.** In the statement of Example 7.6.2.14, the assumption that K is a Kan complex cannot be omitted.

Examples 7.6.2.11 and 7.6.2.14 show that the ∞-categories $\mathcal{S}$ and $\mathcal{QC}$ admit powers and tensor products by any small simplicial set K. Beware that it is very rare for a *small* ∞-category to have the same property:

048B **Proposition 7.6.2.16.** *Let S be an infinite set of cardinality κ and let $\mathcal{C}$ be an ∞-category which is locally κ^+-small. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is equivalent to the nerve of a partially ordered set.*

(2) *For every nonempty simplicial set K and every object $X \in \mathcal{C}$, the constant map*

$$K \to \{\mathrm{id}_X\} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, X)$$

exhibits X as a power of itself by K.

(3) *Every object $X \in \mathcal{C}$ admits a power by S.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are immediate from the definitions. We will show that (3) implies (1). Assume that condition (3) is satisfied and fix a pair of objects $X, Y \in \mathcal{C}$; we wish to show that the morphism space $M = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ is either empty or contractible. Assume otherwise: then there exists a morphism $f : Y \to X$ in $\mathcal{C}$ and an integer $n \geq 0$ such that the homotopy set $\pi_n(M, f)$ has at least two elements. Using assumption (3), we can choose an object $X' \in \mathcal{C}$ and a collection of morphisms $\{g_s : X' \to X\}$ which exhibit X' as a power of X by S. Choose a morphism $f' : Y \to X$ such that $g_s \circ f'$ is homotopic to f for each $s \in S$. Then $\pi_n(M', f')$ can be identified with the product $\prod_{s \in S} \pi_n(M, f)$. This set has cardinality larger than κ (Proposition 4.7.2.8), contradicting our assumption that $\mathcal{C}$ is locally κ^+-small. □

We can use Example 7.6.2.11 to give an alternative proof of the univerality of the left fibration $\mathcal{S}_* \to \mathcal{S}$ (see Corollary 5.6.0.6).

03LT **Proposition 7.6.2.17** (Covariant Transport as a Kan Extension)**.** *Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small left fibration of ∞-categories, let $\underline{\Delta^0}_{\mathcal{E}}$ denote the constant functor $\mathcal{E} \to \mathcal{S}$ taking the value Δ^0, and let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be any functor. Suppose we are given a natural transformation $\beta : \underline{\Delta^0}_{\mathcal{E}} \to \mathscr{F} \circ U$. The following conditions are equivalent:*

(1) *The natural transformation β exhibits $\mathscr{F}$ as a left Kan extension of $\underline{\Delta^0}_{\mathcal{E}}$ along U (in the sense of Variant 7.3.1.5).*

(2) *The commutative diagram*

03LU

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\beta"] \arrow[d, "U"] & \{\Delta^0\} \tilde{\times}_{\mathcal{S}} \mathcal{S} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{F}"] & \mathcal{S}
\end{tikzcd} \tag{7.54}$$

is a categorical pullback square.

Proof. Fix an object $C \in \mathcal{C}$ and let $\mathcal{E}_C$ denote the fiber $\{C\} \times_{\mathcal{C}} \mathcal{E}$, so that the restriction of β to $\mathcal{E}_C$ can be identified with a morphism of Kan complexes $e_C : \mathcal{E}_C \to \mathrm{Hom}_{\mathcal{S}}(\Delta^0, \mathscr{F}(C))$. By virtue of Proposition 7.3.4.1 and Corollary 5.1.7.16, it will suffice to show that the following conditions are equivalent:

(1_C) The morphism e_C exhibits $\mathscr{F}(C)$ as a tensor product of Δ^0 by $\mathscr{E}_C$ (as an object of the ∞-category $\mathcal{S}$).

(2_C) The morphism e_C is a homotopy equivalence.

This is a special case of Example 7.6.2.12. □

Corollary 7.6.2.18. *Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small left fibration of ∞-categories. Then a functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ is a covariant transport representation for U (in the sense of Definition 5.6.5.1) if and only if it is a left Kan extension of the constant functor $\underline{\Delta^0}_{\mathcal{E}}$ along U.* 03LV

Proof. Combine Proposition 7.6.2.17 with the equivalence $\mathcal{S}_* \hookrightarrow \{\Delta^0\} \tilde{\times}_{\mathcal{S}} \mathcal{S}$ of Theorem 4.6.4.17. □

Variant 7.6.2.19. Let $U : \mathcal{E} \to \mathcal{C}$ be a left fibration of ∞-categories, and suppose that the fibers of U are essentially κ-small for some uncountable cardinal κ. Then, in the statements of Proposition 7.6.2.17 and Corollary 7.6.2.18, we can replace $\mathcal{S}$ by the ∞-category $\mathcal{S}^{<\kappa}$ of κ-small spaces (see Variant 5.5.4.12). 03V1

We now consider a variant of Proposition 7.6.2.10. Suppose we are given a differential graded category $\mathcal{C}$ containing objects X and Y. Let

$$\rho_{X,Y} : \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*) \hookrightarrow \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y)$$

denote the composition of the isomorphism $\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*) \simeq \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y)$ of Example 4.6.5.15 with the pinch inclusion morphism $\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y) \hookrightarrow \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y)$ of Construction 4.6.5.7.

03FB **Proposition 7.6.2.20.** *Let $\mathcal{C}$ be a differential graded category, let X and Y be objects of $\mathcal{C}$, and suppose we are given a morphism of simplicial sets $e_0 : S \to \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Y)_*)$, which we identify with a morphism of chain complexes $f : \mathrm{N}_*(S;\mathbf{Z}) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)_*$. Let $e : S \to \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Y)$ denote the composition of e_0 with the morphism $\rho_{X,Y}$. The following conditions are equivalent:*

(1) *The morphism e exhibits Y as a tensor product of X by S in the ∞-category $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$.*

(2) *Let Z be an object of $\mathcal{C}$, so that f induces a morphism of chain complexes*

$$\theta : \mathrm{Hom}_{\mathcal{C}}(Y,Z)_* \to \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(S;\mathbf{Z}), \mathrm{Hom}_{\mathcal{C}}(X,Z)_*)_*.$$

Then θ is an isomorphism on homology in degrees ≥ 0.

Proof of Proposition 7.6.2.20. Fix an object $Z \in \mathcal{C}$. Using Proposition 4.6.9.21, we see that the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(Y,Z)_*) \arrow[r, "\mathrm{K}(\theta)"] \arrow[dd, "\rho_{Y,Z}"] & \mathrm{K}(\mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(S;\mathbf{Z}), \mathrm{Hom}_{\mathcal{C}}(X,Z)_*)_*) \arrow[d, "\psi"] \\
 & \mathrm{Fun}(S, \mathrm{K}(\mathrm{Hom}_{\mathcal{C}}(X,Z)_*)) \arrow[d, "\rho_{X,Z}\circ"] \\
\mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(Y,Z) \arrow[r] & \mathrm{Fun}(S, \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(X,Z))
\end{tikzcd}$$

commutes up to homotopy, where ψ is the homotopy equivalence of Example 3.1.6.11 and the bottom horizontal map is given by combining e with the composition law on the ∞-category $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$. Note that condition (1) is equivalent to the requirement that the bottom horizontal map is a homotopy equivalence (for each object $Z \in \mathcal{C}$). Since the map $\rho_{Y,Z}$ and $\rho_{X,Z}$ are also homotopy equivalences (Proposition 4.6.5.10), this is equivalent to the requirement that $\mathrm{K}(\theta)$ is a homotopy equivalence (for each object $Z \in \mathcal{C}$). The equivalence of (1) and (2) now follows from the criterion of Corollary 3.2.7.4. □

03FC **Example 7.6.2.21** (Homology as a Colimit)**.** Let $\mathcal{C} = \mathrm{Ch}(\mathbf{Z})$ denote the category of chain complexes of abelian groups, which we regard as a differential graded category (see Example 2.5.2.5). Let A be an abelian group, and let us abuse notation by identifying A with its image in $\mathcal{C}$ (by regarding it as a chain complex concentrated in degree zero). For every simplicial set S, let $\mathrm{N}_*(S;A)$ denote the normalized chain complex of S with coefficients in A, given by the tensor product $\mathrm{N}_*(S;\mathbf{Z}) \boxtimes A$. Then the tautological map

$$f : \mathrm{N}_*(S;\mathbf{Z}) \to \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(A, \mathrm{N}_*(S;A))_*$$

satisfies condition (2) of Proposition 7.6.2.20: in fact, for every object $M_* \in \mathcal{C}$, precomposition with f induces an *isomorphism* of chain complexes

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{N}_*(S;A), M_*)_* \to \mathrm{Hom}_{\mathcal{C}}(\mathrm{N}_*(S;\mathbf{Z}), \mathrm{Hom}_{\mathcal{C}}(A, M_*)_*)_*.$$

It follows that the induced map $S \to \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})}(A, \mathrm{N}_*(S;A))$ exhibits $\mathrm{N}_*(S;A)$ as a tensor product of A by S in the ∞-category $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathcal{C})$. In particular, the chain complex $\mathrm{N}_*(S;A)$ can be viewed as a colimit of the constant diagram $S \to \{A\} \hookrightarrow \mathrm{N}_\bullet^{\mathrm{dg}}(\mathrm{Ch}(\mathbf{Z}))$.

Variant 7.6.2.22 (Cohomology as a Limit)**.** Let A be an abelian group, let S be a simplicial set, and let 03FD

$$\mathrm{N}^*(S;A) = \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}_*(S;\mathbf{Z}), A)$$

denote the normalized *cochain* complex of S with coefficients in A. Applying Proposition 7.6.2.20 to the differential graded category $\mathrm{Ch}(\mathbf{Z})^{\mathrm{op}}$ (and using Remark 7.6.2.4), we see that the tautological chain map $\mathrm{N}_*(S;\mathbf{Z}) \to \mathrm{Hom}_{\mathrm{Ch}(\mathbf{Z})}(\mathrm{N}^*(S;A), A)_*$ induces a morphism of simplicial sets

$$e : S \to \mathrm{Hom}_{\mathrm{N}_\bullet^{\mathrm{dg}}(\mathrm{Ch}(\mathbf{Z}))}(\mathrm{N}^*(S;A), A)$$

which exhibits $\mathrm{N}^*(S;A)$ as a power of A by S in the ∞-category $\mathrm{N}_\bullet^{\mathrm{dg}}(\mathrm{Ch}(\mathbf{Z}))$. In particular, $\mathrm{N}^*(S;A)$ can be viewed as a limit of the constant diagram $S \to \{A\} \hookrightarrow \mathrm{N}_\bullet^{\mathrm{dg}}(\mathrm{Ch}(\mathbf{Z}))$.

7.6.3 Pullbacks and Pushouts

Let $\mathcal{C}$ be an ∞-category. Recall that a *commutative square* in $\mathcal{C}$ is a morphism of simplicial sets $\Delta^1 \times \Delta^1 \to \mathcal{C}$ which we represent informally by a diagram 03FE

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

(see Example 1.5.2.15). Note that the simplicial set $\Delta^1 \times \Delta^1 \simeq \mathrm{N}_\bullet([1] \times [1])$ can be regarded both as a left cone (on the nerve of the partially ordered set $[1] \times [1] \setminus \{(0,0)\}$) and as a right cone (on the nerve of the partially ordered set $[1] \times [1] \setminus \{(1,1)\}$).

03FF **Definition 7.6.3.1.** Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative square. We say that σ is a *pullback square* if it is a limit diagram in $\mathcal{C}$ (see Definition 7.1.2.4), and that σ is a *pushout square* if it is a colimit diagram in $\mathcal{C}$.

03FG **Example 7.6.3.2.** Let $\mathcal{C}$ be an ordinary category. Then diagram $\sigma : [1] \times [1] \to \mathcal{C}$ is a pullback square in $\mathcal{C}$ (in the sense of classical category theory) if and only if the induced map

$$\mathrm{N}_\bullet(\sigma) : \Delta^1 \times \Delta^1 \to \mathrm{N}_\bullet(\mathcal{C})$$

is a pullback square in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ (in the sense of Definition 7.6.3.1); this follows from Example 7.1.1.4 and Remark 7.1.2.6. Similarly, σ is a pushout square in $\mathcal{C}$ if and only if $\mathrm{N}_\bullet(\sigma)$ is a pushout square in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$.

03FH **Warning 7.6.3.3.** Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a morphism, which we depict as a diagram

$$\begin{tikzcd}
X_{01} \arrow[r, "g_0"] \arrow[d, "g_1"] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X.
\end{tikzcd}$$

Beware that, if σ is a pullback square in the ∞-category $\mathcal{C}$, then the associated diagram

$$\begin{tikzcd}
X_{01} \arrow[r, "{[g_0]}"] \arrow[d, "{[g_1]}"] & X_0 \arrow[d, "{[f_0]}"] \\
X_1 \arrow[r, "{[f_1]}"] & X
\end{tikzcd}$$

need not be a pullback square in the homotopy category $\mathrm{h}\mathcal{C}$ (see Example 7.6.3.4 and Exercise 7.6.3.5). If Y is an object of $\mathcal{C}$, then the map of sets

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X_{01}) \xrightarrow{([g_0],[g_1])\circ} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X_0) \times_{\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y,X)} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X_1)$$

is surjective, but need not be injective. Given a commutative diagram

03FJ
$$\begin{tikzcd}
Y \arrow[r, "{[g_0]}"] \arrow[d, "{[g_1]}"] & X_0 \arrow[d, "{[f_0]}"] \\
X_1 \arrow[r, "{[f_1]}"] & X
\end{tikzcd} \qquad (7.55)$$

in the homotopy category $\mathrm{h}\mathcal{C}$, we can always find a morphism $g_{01} : Y \to X_{01}$ satisfying $g_{01} : Y \to X_{01}$ satisfying $[g_0] = [f_0'] \circ [g_{01}]$ and $[g_1] = [f_1'] \circ [g_{01}]$. However, the homotopy class $[g_{01}]$ is not uniquely determined: roughly speaking, to construct g_{01}, we need to lift (7.55) to a commutative diagram in the ∞-category $\mathcal{C}$. Such a lift always exists (Exercise 1.5.2.10), but is not unique (even up to homotopy).

Example 7.6.3.4. Let $q : X \to S$ be a Kan fibration between Kan complexes, let $s \in S$ be a vertex, and let X_s denote the fiber $\{s\} \times_S X$. Then the commutative diagram of simplicial sets 03FK

03FL
$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ \{s\} \arrow[r] & S \end{tikzcd} \qquad (7.56)$$

is a homotopy pullback square (Example 3.4.1.3), and therefore induces a pullback square in the ∞-category $\mathcal{S} = \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan})$ (see Example 7.6.4.2). However, if X is contractible and X_s is not, then (7.56) is not a pullback square in the homotopy category hKan.

Exercise 7.6.3.5. Let G be a group and let $H \subseteq G$ be a commutative normal subgroup, so that we have a commutative diagram of Kan complexes 03FM

03FN
$$\begin{tikzcd} B_\bullet H \arrow[r] \arrow[d] & B_\bullet G \arrow[d] \\ \Delta^0 \arrow[r] & B_\bullet(G/H) \end{tikzcd} \qquad (7.57)$$

- Show that (7.57) is a pullback diagram in the ordinary category of Kan complexes, and that it determines a pullback diagram in the ∞-category $\mathcal{S} = \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan})$ (see Example 7.6.4.2).
- Show that, if H is contained in the center of G, then the diagram (7.57) is also pullback square in the homotopy category hKan.
- Show that, if H is not contained in the center of G, then the diagram $B_\bullet G \to B_\bullet(G/H) \leftarrow \Delta^0$ does not have a limit in the homotopy category hKan. In particular, the diagram (7.57) is not a pullback square in hKan.

Variant 7.6.3.6. Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. We say that a diagram $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ is a *U-pullback square* if it is a U-limit diagram in $\mathcal{C}$ (Definition 7.1.5.1). We say that σ is a *U-pushout square* if it is a U-colimit diagram in the ∞-category $\mathcal{C}$. 048C

048D **Remark 7.6.3.7.** Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative square in $\mathcal{C}$. Then σ is a pullback square if and only if it is U-pullback square, where $U : \mathcal{C} \to \Delta^0$ is the projection map (see Example 7.1.5.3).

03FP **Remark 7.6.3.8** (Symmetry)**.** Let $\mathcal{C}$ be an ∞-category, let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative square in $\mathcal{C}$, and let $\sigma' : \Delta^1 \times \Delta^1 \to \mathcal{C}$ denote the commutative square which is obtained from σ by precomposing with the automorphism of $\Delta^1 \times \Delta^1$ given by permuting the factors. Then σ is a pullback square if and only if σ' is a pullback square, and σ is a pushout square if and only if σ' is a pushout square.

More generally, if $U : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories, then σ is a U-pullback square if and only if σ' is a U-pullback square, and σ is a U-pushout square if and only if σ' is a U-pushout square.

03FQ **Remark 7.6.3.9.** Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative diagram in $\mathcal{C}$. Then σ is a pushout diagram in $\mathcal{C}$ if and only if the opposite diagram $\sigma^{\mathrm{op}} : \Delta^1 \times \Delta^1 \to \mathcal{C}^{\mathrm{op}}$ is a pullback diagram in the ∞-category $\mathcal{C}^{\mathrm{op}}$; here we implicitly identify the simplicial set $\Delta^1 \times \Delta^1$ with its opposite (beware that there are two possible identifications we could choose, but the choice does not matter by virtue of Remark 7.6.3.8).

More generally, if $U : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories, then σ is a U-pushout diagram if and only if σ^{op} is a U^{op}-pullback diagram.

03FR **Remark 7.6.3.10.** Let $\mathcal{C}$ be an ∞-category and let $\sigma, \sigma' : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be square diagrams which are isomorphic (when viewed as objects of the ∞-category $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$). Then σ is a pullback square if and only if σ' is a pullback square, and a pushout square if and only if σ' is a pushout square.

More generally, if $U : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories, then then σ is a U-pullback square if and only if σ' is a U-pullback square, and σ is a U-pushout square if and only if σ' is a U-pushout square (see Proposition 7.1.5.13).

03FS **Notation 7.6.3.11** (Fiber Products)**.** Let $\mathcal{C}$ be an ∞-category, and suppose we are given a pair of morphisms $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ of $\mathcal{C}$ having the same target. It follows from Proposition 7.1.1.12 that if there exists a pullback diagram

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X
\end{tikzcd}$$

in $\mathcal{C}$, then the object X_{01} is determined up to isomorphism by f_0 and f_1. To emphasize this, we will often denote the object X_{01} by $X_0 \times_X X_1$ and refer to it as the *fiber product of* X_0

with X_1 *over* X. Similarly, if there exists a pushout diagram

$$\begin{tikzcd}
Y \arrow[r, "g_0"] \arrow[d, "g_1"] & Y_0 \arrow[d] \\
Y_1 \arrow[r] & Y_{01}
\end{tikzcd}$$

in $\mathcal{C}$, then the object Y_{01} is determined up to isomorphism by g_0 and g_1. To emphasize this, we often denote the object Y_{01} by $Y_0 \coprod_Y Y_1$ and refer to it as the *pushout of* Y_0 *with* Y_1 *along* Y.

Definition 7.6.3.12. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ *admits pullbacks* if, for every pair of morphisms $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ having the same target, there exists a pullback diagram 03FT

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X.
\end{tikzcd}$$

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ *preserves pullbacks* if, for every pullback square $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ in the ∞-category $\mathcal{C}$, the composition $(F \circ \sigma) : \Delta^1 \times \Delta^1 \to \mathcal{D}$ is a pullback square in the ∞-category $\mathcal{D}$.

We say that $\mathcal{C}$ *admits pushouts* if, for every pair of morphisms $g_0 : Y \to Y_0$ and $g_1 : Y \to Y_1$ having the same source, there exists a pushout diagram

$$\begin{tikzcd}
Y \arrow[r, "g_0"] \arrow[d, "g_1"] & Y_0 \arrow[d] \\
Y_1 \arrow[r] & Y_{01}.
\end{tikzcd}$$

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ *preserves pushouts* if, for every pushout square $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ in the ∞-category $\mathcal{C}$, the composition $(F \circ \sigma) : \Delta^1 \times \Delta^1 \to \mathcal{D}$ is a pushout square in the ∞-category $\mathcal{D}$.

Remark 7.6.3.13. Let $U : \mathcal{C} \to \mathcal{D}$ be a right fibration of ∞-categories, and suppose that $\mathcal{D}$ admits pullbacks. Then $\mathcal{C}$ also admits pullbacks, and the functor U preserves pullbacks. See Corollary 7.1.5.18. 04Q8

03FU **Proposition 7.6.3.14.** *Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative square, which we represent by a diagram*

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X.
\end{tikzcd}$$

Then σ is a pullback diagram in $\mathcal{C}$ if and only if it exhibits X_{01} as a product of X_0 with X_1 in the slice ∞-category $\mathcal{C}_{/X}$.

Proof. This is a special case of Remark 7.1.2.11. □

We now give an alternative characterization of the fiber product construction.

04Q9 **Definition 7.6.3.15.** Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. We will say that a functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ is *given by pullback along f* if it is a right adjoint to the functor $\mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ given by postcomposition with f (see Example 4.3.6.14). Note that this condition characterizes the functor f^* up to isomorphism (see Remark 6.2.1.19).

04QA **Proposition 7.6.3.16.** *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:*

(1) *There exists a functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ given by pullback along f (in the sense of Definition 7.6.3.15).*

(2) *For every morphism $u : Y' \to Y$, there exists a pullback diagram*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[d, "u"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$.

Moreover, if these conditions are satisfied, then the pullback functor f^* carries each object $Y' \in \mathcal{C}_{/Y}$ to the fiber product $X \times_Y Y'$.

Proof. Let $e_0 : \mathcal{C}_{/f} \to \mathcal{C}_{/X}$ and $e_1 : \mathcal{C}_{/f} \to \mathcal{C}_{/Y}$ denote the restriction map. Then e_0 is a trivial Kan fibration (Corollary 4.3.6.13), and postcomposition with f is defined as the composition of e_1 with a section of e_0 (Example 4.3.6.14). We can therefore reformulate (1) as follows:

(1′) The restriction functor $e_1 : \mathcal{C}_{/f} \to \mathcal{C}_{/Y}$ admits a right adjoint.

Let us identify the morphism f with an object $\widetilde{X} \in \mathcal{C}_{/Y}$. Using Proposition 7.6.3.14, we can reformulate condition (2) as follows:

$(2')$ For every object $\widetilde{Y}' \in \mathcal{C}_{/Y}$, there exists a product of $\widetilde{X}$ with $\widetilde{Y}'$ in $\mathcal{C}_{/Y}$.

The equivalence of $(1')$ and $(2')$ now follows from Proposition 7.6.1.12, applied to the slice ∞-category $\mathcal{C}_{/Y}$. □

Corollary 7.6.3.17. *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ admits fiber products if and only if, for every morphism $f : X \to Y$ in $\mathcal{C}$, the postcomposition functor* 04QB

$$\mathcal{C}_{/X} \to \mathcal{C}_{/Y} \qquad e \mapsto (f \circ e)$$

of Example 4.3.6.14 admits a right adjoint.

Notation 7.6.3.18 (Relative Diagonals)**.** Let $\mathcal{C}$ be an ∞-category and let $f : Y \to X$ be a morphism in $\mathcal{C}$. Suppose that there exists a pullback square 04K5

04K6
$$\begin{tikzcd} Y \times_X Y \arrow[r, "\pi"] \arrow[d, "\pi'"] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \tag{7.58}$$

in the ∞-category $\mathcal{C}$. Let us abuse notation by identifying Y with an object of the slice ∞-category $\mathcal{C}_{/X}$, so that $Y \times_X Y$ can be viewed as a product of Y with itself in $\mathcal{C}_{/X}$ (Proposition 7.6.3.14). Applying the construction of Notation 7.6.2.9, we obtain a morphism $\delta_{Y/X} : Y \to Y \times_X Y$, which we will refer to as the *relative diagonal of f*. It is characterized (up to homotopy) by the requirement that (7.58) can be extended to a commutative diagram

$$\begin{tikzcd} Y \arrow[dr, "\delta_{Y/X}"] \arrow[drr, "\mathrm{id}_Y"] \arrow[ddr, "\mathrm{id}_Y"'] & & \\ & Y \times_X Y \arrow[r, "\pi"] \arrow[d, "\pi'"] & Y \arrow[d, "f"] \\ & Y \arrow[r, "f"] & X, \end{tikzcd}$$

where the outer square is the commutative diagram given by the composition

$$\Delta^1 \times \Delta^1 \xrightarrow{(i,j) \mapsto ij} \Delta^1 \xrightarrow{f} \mathcal{C}.$$

Variant 7.6.3.19 (Relative Codiagonals)**.** Let $\mathcal{C}$ be an ∞-category, let $f : Y \to X$ be a morphism of $\mathcal{C}$, and suppose that there exists a pushout square 04K7

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "f"] & X \arrow[d] \\ X \arrow[r] & X \coprod_Y X. \end{tikzcd}$$

Applying the construction of Notation 7.6.3.18 in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$, we obtain a morphism $\gamma_{Y/X} : X \coprod_Y X \to X$ which we will refer to as the *relative codiagonal* of the morphism f.

Stated more informally, a fiber product $X_0 \times_X X_1$ (formed in an ∞-category $\mathcal{C}$) is a product of X_0 with X_1 in the ∞-category $\mathcal{C}_{/X}$.

03FV **Corollary 7.6.3.20.** *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ admits pullbacks if and only if, for each object $X \in \mathcal{C}$, the slice ∞-category $\mathcal{C}_{/X}$ admits finite products.*

Proof. By virtue of Proposition 7.6.3.14, the ∞-category $\mathcal{C}$ admits pullbacks if and only if, for every object $X \in \mathcal{C}$, the ∞-category $\mathcal{C}_{/X}$ admits pairwise products. Since $\mathcal{C}_{/X}$ has an initial object (given by the identity morphism $\mathrm{id}_X : X \to X$; see Proposition 4.6.7.22), this is equivalent to the requirement that $\mathcal{C}_{/X}$ admits finite products (Corollary 7.6.1.21). □

03FW **Remark 7.6.3.21.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, where $\mathcal{C}$ admits pullbacks. Then F preserves pullbacks if and only if, for each object $X \in \mathcal{C}$, the induced functor $\mathcal{C}_{/X} \to \mathcal{D}_{/F(X)}$ preserves finite products.

03FX **Corollary 7.6.3.22.** *Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative square, which we represent by a diagram*

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, "f_1"] & X_0 \arrow[d] \\
X_1 \arrow[r] & \mathbf{1}.
\end{tikzcd}$$

Suppose that $\mathbf{1}$ is a final object of $\mathcal{C}$. Then σ is a pullback square if and only if the morphisms f_0 and f_1 exhibit X as a coproduct of X_0 with X_1 in the ∞-category $\mathcal{C}$.

Proof. The assumption that $\mathbf{1}$ is final guarantees that the projection map $\mathcal{C}_{/\mathbf{1}} \to \mathcal{C}$ is a trivial Kan fibration (Proposition 4.6.7.10), so that the desired result follows from the criterion of Proposition 7.6.3.14. □

048E **Proposition 7.6.3.23.** *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative square, represented informally by the diagram*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Then:

(1) *If f is U-cartesian, then σ is a U-pullback square if and only if f' is also U-cartesian.*

(2) *If f' is U-cocartesian, then σ is a U-pushout square if and only if f is also U-cocartesian.*

Proof. We will prove (1); the proof of (2) is similar. Note that σ restricts to a diagram

$$\sigma_0 : \mathrm{N}_\bullet(\{(0,1) < (1,1) > (1,0)\}) \to \mathcal{C}$$

satisfying $\sigma_0(0,1) = X$, $\sigma_0(1,1) = Y$, and $\sigma_0(1,0) = Y'$. The assumption that f is U-cartesian guarantees that σ_0 is U-right Kan extended from the full subcategory

$$\{1\} \times \Delta^1 \subseteq \mathrm{N}_\bullet(\{(0,1) < (1,1) > (1,0)\}).$$

It follows that σ is a U-pullback diagram if and only if the restriction $\sigma|_{\mathrm{N}_\bullet(\{(0,0)<(1,0)<(1,1)\})}$ is a U-limit diagram (Proposition 7.3.8.1) By virtue of Corollary 7.2.2.5, this is equivalent to the requirement that f' is U-cartesian. □

Corollary 7.6.3.24. *Let $\mathcal{C}$ be an ∞-category and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative square, represented informally by the diagram* 048F

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Then:

(1) *If f is an isomorphism, then σ is a pullback square if and only if f' is also an isomorphism.*

(2) *If f' is an isomorphism, then σ is a pushout square if and only if f is also an isomorphism.*

Proof. Combine Proposition 7.6.3.23 with Remark 7.6.3.7 (and Example 5.1.1.4). □

Proposition 7.6.3.25 (Transitivity). *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\sigma : \Delta^2 \times \Delta^1 \to \mathcal{C}$ be diagram, which we depict informally as* 03FZ

03G0

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\
X \arrow[r] & Y \arrow[r] & Z.
\end{tikzcd} \tag{7.59}$$

Then:

(1) *Assume that the right square of (7.59) is a U-pullback. Then the left square is a U-pullback if and only if the outer rectangle is a U-pullback.*

(2) *Assume that the left square of (7.59) is a U-pushout. Then the left square is a U-pushout if and only if the outer rectangle is a U-pushout.*

Proof. We will prove (1); the proof of (2) is similar. Let A denote the partially ordered set $([2] \times [1]) \setminus \{(0,0)\}$. Note that the inclusion maps

$$A \setminus \{(2,0),(2,1)\} \hookrightarrow A \qquad A \setminus \{(1,0),(1,1)\} \hookrightarrow A \setminus \{(1,0)\}$$

admit right adjoints, and therefore induce left cofinal morphisms

$$\mathrm{N}_\bullet(A \setminus \{(2,0),(2,1)\}) \hookrightarrow \mathrm{N}_\bullet(A) \qquad \mathrm{N}_\bullet(A \setminus \{(1,0),(1,1)\}) \hookrightarrow \mathrm{N}_\bullet(A \setminus \{(1,0)\})$$

(Corollary 7.2.3.7). Applying Corollary 7.2.2.2, we obtain the following:

- The left square of (7.59) is a U-pullback diagram if and only if σ is a U-limit diagram.
- The outer rectangle of (7.59) is a U-pullback diagram if and only if the restriction $\sigma|_{\mathrm{N}_\bullet(([2]\times[1])\setminus\{(1,0)\})}$ is a U-limit diagram.

If the right square of (7.59) is a U-pullback diagram, then $\sigma|_{\mathrm{N}_\bullet(A)}$ is U-right Kan extended from $\sigma|_{\mathrm{N}_\bullet(A\setminus\{(1,0)\})}$, so the desired equivalence follows from Proposition 7.3.8.1. □

03G1 **Proposition 7.6.3.26** (Rewriting Limits as Pullbacks). *Suppose we are given a categorical pushout square of simplicial sets*

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & K_0 \arrow[d] \\
K_1 \arrow[r] & K_{01}.
\end{tikzcd}$$

Let $\mathcal{C}$ be an ∞-category which admits pullbacks. If $\mathcal{C}$ admits K-indexed limits, K_0-indexed limits, and K_1-indexed limits, then it also admits K_{01}-indexed limits. Moreover, if $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories which preserves pullback squares, K-indexed limits, K_0-indexed limits, and K_1-indexed limits, then F also preserves K-indexed limits.

Proof. Combine Corollary 7.5.8.5 with (the dual of) Corollary 7.5.8.13. □

03G2 **Corollary 7.6.3.27.** *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ admits finite limits if and only if it admits pullbacks and has a final object. If these conditions are satisfied, then a functor $F : \mathcal{C} \to \mathcal{D}$ preserves finite limits if and only if it preserves pullbacks and final objects.*

Proof. We will prove the first assertion; the second follows by a similar argument. Assume that the ∞-category $\mathcal{C}$ admits pullbacks and has a final object; we wish to show that $\mathcal{C}$ admits K-indexed limits for every finite simplicial set K (the converse is immediate from the definitions). We proceed by induction on the dimension of K. If K is empty, then the desired result follows from our assumption that $\mathcal{C}$ has a final object. Let us therefore assume that K has dimension $n \geq 0$, and proceed also by induction on the number of nondegenerate n-simplices of K. It follows from Proposition 1.1.4.12 that there exists a pushout square of simplicial sets

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\
K' \arrow[r] & K,
\end{tikzcd}$$

where K' is a simplicial subset of K. Since the horizontal maps are monomorphisms, this pushout square is also a categorical pushout square (Example 4.5.4.12). By virtue of Proposition 7.6.3.26, it will suffice to show that the ∞-category $\mathcal{C}$ admits K'-indexed limits, $\partial\Delta^n$-indexed limits, and Δ^n-indexed limits. In the first two cases, this follows from our inductive hypothesis. To handle the third case, we observe that the inclusion $\{0\} \hookrightarrow \Delta^n$ is left cofinal (Example 4.3.7.11). Using Corollary 7.2.2.12, we are reduced to proving that $\mathcal{C}$ admits Δ^0-indexed limits, which is immediate (see Example 7.1.1.5). □

Example 7.6.3.28. Let $\mathcal{C}$ be an ∞-category which admits pullbacks. Then, for every object $X \in \mathcal{C}$, the slice ∞-category $\mathcal{C}_{/X}$ admits finite limits. This follows from Corollary 7.6.3.27, since $\mathcal{C}_{/X}$ also admits finite pullbacks (Remark 7.6.3.13), and has a final object given by the identity morphism $\mathrm{id}_X : X \to X$ (Proposition 4.6.7.22). Similarly, if $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves pullbacks, then the induced functor $F_{/X} : \mathcal{C}_{/X} \to \mathcal{D}_{/F(X)}$ preserves finite limits. 05E4

7.6.4 Examples of Pullback and Pushout Squares

We now give some examples of ∞-categorical pullback diagrams. 03G3

Proposition 7.6.4.1. *Let $\mathcal{C}$ be a locally Kan simplicial category and let σ :* 03G4

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$. The following conditions are equivalent:

(1) *The composite map*

$$\Delta^1 \times \Delta^1 \xrightarrow{\mathrm{N}_\bullet(\sigma)} \mathrm{N}_\bullet(\mathcal{C}) \hookrightarrow \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$$

is a pullback square in the ∞-category $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ (in the sense of Definition 7.6.3.1).

(2) *For every object $Y \in \mathcal{C}$, the diagram of Kan complexes*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(Y, X_{01})_\bullet \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(Y, X_0)_\bullet \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}}(Y, X_1)_\bullet \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(Y, X)_\bullet
\end{tikzcd}$$

is a homotopy pullback square (in the sense of Definition 3.4.1.1).

Proof. Combine Corollary 7.5.4.6 with Proposition 7.5.4.13. □

03G5 **Example 7.6.4.2.** A (strictly) commutative diagram of Kan complexes

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback square (in the sense of Definition 3.4.1.1) if and only if the induced diagram $\Delta^1 \times \Delta^1 \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$ is a pullback square in the ∞-category of spaces $\mathcal{S}$. This follows by combining Propositions 7.5.4.13 and 7.5.4.5.

03G6 **Example 7.6.4.3.** A (strictly) commutative diagram of Kan complexes

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_0 \arrow[d] \\
A_1 \arrow[r] & A_{01}
\end{tikzcd}$$

is a homotopy pushout square (in the sense of Definition 3.4.2.1) if and only if the induced diagram $\Delta^1 \times \Delta^1 \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}$ is a pushout square in the ∞-category of spaces $\mathcal{S}$. This follows by combining Corollaries 7.5.7.7 and 7.5.7.9.

Example 7.6.4.4. A (strictly) commutative diagram of ∞-categories 03G7

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}
\end{tikzcd}$$

is a categorical pullback square (in the sense of Definition 4.5.2.8) if and only if the induced diagram $\Delta^1 \times \Delta^1 \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{QCat}) = \mathcal{QC}$ is a pullback square in the ∞-category $\mathcal{QC}$. This follows by combining Corollaries 7.5.5.8 and 7.5.5.10.

Example 7.6.4.5. A (strictly) commutative diagram of ∞-categories 03G8

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d] \\
\mathcal{C}_1 \arrow[r] & \mathcal{C}_{01}
\end{tikzcd}$$

is a categorical pushout square (in the sense of Definition 4.5.4) if and only if the induced diagram $\Delta^1 \times \Delta^1 \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{QCat}) = \mathcal{QC}$ is a pushout square in the ∞-category $\mathcal{QC}$. This follows by combining Corollaries 7.5.8.5 and 7.5.8.9.

Recall that the ∞-category of spaces $\mathcal{S}$ admits small limits and colimits (Corollary 7.4.5.6). In particular, if $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ are morphisms of Kan complexes, then there exists a pullback diagram σ :

$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X
\end{tikzcd}$$

in the ∞-category $\mathcal{S}$. However, it is not always possible to obtain σ from a commutative diagram in the ordinary category Kan. It will therefore be useful to have a generalization of Proposition 7.6.4.1, which applies to *homotopy coherent* squares.

Remark 7.6.4.6 (Homotopy Coherent Squares)**.** Let $\mathcal{C}$ be a simplicial category and let $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ denote the homotopy coherent nerve of $\mathcal{C}$. Combining Examples 1.5.2.9, 2.4.3.9, and 2.4.3.10, we see that morphisms from $\Delta^1 \times \Delta^1$ to $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ can be identified with the following data: 03G9

(a) A collection of objects X_{01}, X_0, X_1, and X of the category $\mathcal{C}$.

(b) A collection of morphisms $f_0 : X_0 \to X$, $f_1 : X_1 \to X$, $g_0 : X_{01} \to X_0$, $g_1 : X_{01} \to X_1$.

(c) A morphism $h : X_{01} \to X$ in $\mathcal{C}$ together with a pair of edges $\alpha_0 : f_0 \circ g_0 \to h$ and $\alpha_1 : f_1 \circ g_1 \to h$ in the simplicial set $\operatorname{Hom}_{\mathcal{C}}(X_{01}, X)_\bullet$.

We can summarize this data in a diagram

$$\begin{tikzcd}
X_{01} \arrow[r, "g_0"] \arrow[d, "g_1"'] \arrow[dr, "h" description] & X_0 \arrow[d, "f_0"] \arrow[dl, phantom, "\overset{\alpha_0}{\Swarrow}"] \\
X_1 \arrow[r, "f_1"'] \arrow[ur, phantom, "\overset{\alpha_1}{\Nearrow}"] & X.
\end{tikzcd}$$

Here we can regard (a) and (b) as supplying a (potentially) non-commutative square diagram in the category $\mathcal{C}$, and (c) as supplying a witness to the fact that it commutes up to homotopy.

03GA **Example 7.6.4.7** (Square Diagrams in $\mathcal{QC}$)**.** Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories. Using Remark 7.6.4.6, we see that the data of a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d, "F_0"] \\
\mathcal{C}_1 \arrow[r, "F_1"] & \mathcal{C}
\end{tikzcd}$$

in the ∞-category $\mathcal{QC}$ is equivalent to the data of an ∞-category $\mathcal{C}_{01}$ equipped with functors

$$G_0 : \mathcal{C}_{01} \to \mathcal{C}_0 \qquad G_1 : \mathcal{C}_{01} \to \mathcal{C}_1 \qquad H : \mathcal{C}_{01} \to \mathcal{C}$$

together with natural isomorphisms $\alpha_0 : (F_0 \circ G_0) \xrightarrow{\sim} H$ and $\alpha_1 : (F_1 \circ G_1) \xrightarrow{\sim} H$. In this case, we can identify the data of the tuple $(G_0, \alpha_0, G_1, \alpha_1, H)$ with a single functor of ∞-categories

$$G : \mathcal{C}_{01} \to \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} (\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}).$$

Proposition 7.6.4.8. *Suppose we are given a commutative diagram* 03GB

03GC
$$\begin{tikzcd}
\mathcal{C}_{01} \arrow[r] \arrow[d] & \mathcal{C}_0 \arrow[d, "F_0"] \\
\mathcal{C}_1 \arrow[r, "F_1"] & \mathcal{C}
\end{tikzcd} \tag{7.60}$$

in the ∞-category $\mathcal{QC}$, corresponding to a functor

$$G : \mathcal{C}_{01} \to \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} (\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}).$$

Then (7.60) is a pullback square in $\mathcal{QC}$ if and only if the functor G is an equivalence of ∞-categories.

Proof. Let us identify the diagram (7.60) with a functor of simplicial categories $\mathscr{F} : \mathrm{Path}[[1] \times [1]]_{\bullet} \to \mathrm{QCat}$. Using Corollary 4.5.2.23, we can factor the functor F_0 as a composition $\mathcal{C}_0 \xrightarrow{T} \mathcal{C}'_0 \xrightarrow{F'_0} \mathcal{C}$, where T is an equivalence of ∞-categories and F'_0 is an isofibration. Let $\mathcal{C}'_{01}$ denote the iterated homotopy fiber product $\mathcal{C}'_0 \times^{\mathrm{h}}_{\mathcal{C}} (\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C})$. Then Example 7.6.4.7 supplies a commutative diagram

03GD
$$\begin{tikzcd}
\mathcal{C}'_{01} \arrow[r] \arrow[d] & \mathcal{C}'_0 \arrow[d, "F'_0"] \\
\mathcal{C}_1 \arrow[r, "F_1"] & \mathcal{C}
\end{tikzcd} \tag{7.61}$$

in the ∞-category $\mathcal{QC}$, which we view as a functor of simplicial categories $\mathscr{F}' : \mathrm{Path}[[1] \times [1]]_{\bullet} \to \mathrm{QCat}$. The morphisms G and T determine a natural transformation of simplicial functors $\mathscr{F} \to \mathscr{F}'$, which induces a natural transformation from the diagram (7.60) to the diagram (7.61) in the ∞-category $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{QC})$. By virtue of Corollary 4.5.2.20, this natural transformation is an isomorphism of diagrams if and only if the functor G is an equivalence of ∞-categories. Consequently, Proposition 7.6.4.8 is equivalent to the assertion that (7.61) is a pullback square in the ∞-category $\mathcal{QC}$ (see Proposition 7.1.2.13).

Note that we have a (strictly) commutative diagram of simplicial sets

03GE
$$\begin{tikzcd}
\mathcal{C}'_0 \times_{\mathcal{C}} \mathcal{C}_1 \arrow[r] \arrow[d] & \mathcal{C}'_0 \arrow[d, "F'_0"] \\
\mathcal{C}_1 \arrow[r, "F'_1"] & \mathcal{C},
\end{tikzcd} \tag{7.62}$$

which determines a subfunctor $\mathscr{F}'' \subseteq \mathscr{F}'$. Since F_0' is an isofibration, it follows from Corollaries 4.5.2.28, 4.5.2.22, and 4.5.2.29 that the inclusion maps

$$\begin{aligned} \mathcal{C}_0' \times_{\mathcal{C}} \mathcal{C}_1 &\hookrightarrow \mathcal{C}_0' \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \\ &\hookrightarrow \mathcal{C}_0' \times^{\mathrm{h}}_{\mathcal{C}} (\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}) \end{aligned}$$

are equivalences of ∞-categories. Consequently, the inclusion $\mathscr{F}'' \hookrightarrow \mathscr{F}'$ is a levelwise categorical equivalence of simplicial functors and therefore induces an isomorphism from the diagram (7.62) to the diagram (7.61) in the ∞-category $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$. By virtue of Proposition 7.1.2.13, it will suffice to show that the diagram (7.62) is a pullback square in the ∞-category $\mathcal{QC}$. This is a special case of Example 7.6.4.4, since (7.62) is a categorical pullback square (see Corollary 4.5.2.27). □

`03GF` **Corollary 7.6.4.9.** *Let $F_0 : \mathcal{C}_0 \to \mathcal{C}$ and $F_1 : \mathcal{C}_1 \to \mathcal{C}$ be functors of ∞-categories, let let $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ denote the homotopy fiber product of Construction 4.5.2.1, and let*

$$G_0 : \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \to \mathcal{C}_0 \qquad G_1 : \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \to \mathcal{C}_1$$

denote the projection maps, so that we have a canonical isomorphism $\alpha : F_0 \circ G_0 \to F_1 \circ G_1$ in the ∞-category $\mathrm{Fun}(\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1, \mathcal{C})$. Then the diagram

$$\begin{tikzcd}
\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \arrow[r, "G_0"] \arrow[d, "G_1"'] \arrow[dr, "F_1 \circ G_1", ""{name=D}] & \mathcal{C}_0 \arrow[d, "F_0"] \\
\mathcal{C}_1 \arrow[r, "F_1"'] & \mathcal{C}
\arrow[from=1-2, to=D, Rightarrow, "\alpha"', shorten=8mm]
\arrow[from=2-1, to=D, Rightarrow, "\mathrm{id}", shorten=8mm]
\end{tikzcd}$$

corresponds to a pullback square in the ∞-category $\mathcal{QC}$. In particular, $\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1$ is a fiber product of $\mathcal{C}_0$ with $\mathcal{C}_1$ over $\mathcal{C}$ in the ∞-category $\mathcal{QC}$.

Proof. By virtue of Proposition 7.6.4.8, it will suffice to show that the inclusion

$$\delta : \mathcal{C}_1 \simeq \mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C} \hookrightarrow \mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}$$

induces an equivalence of homotopy fiber products

$$\mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}_1 \hookrightarrow \mathcal{C}_0 \times^{\mathrm{h}}_{\mathcal{C}} (\mathcal{C}_1 \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C}).$$

This is a special case of Corollary 4.5.2.20, since δ is an equivalence of ∞-categories (Corollary 4.5.2.22). □

Corollary 7.6.4.10. *Suppose we are given a commutative diagram* 03GG

03GH
$$\begin{tikzcd}
X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X
\end{tikzcd} \tag{7.63}$$

in the ∞-category $\mathcal{S}$, classified by a map of Kan complexes

$$g : X_{01} \to X_0 \times^{\mathrm{h}}_{X} (X_1 \times^{\mathrm{h}}_{X} X).$$

Then (7.63) is a pullback square in $\mathcal{S}$ if and only if g is a homotopy equivalence.

Proof. Combine Propositions 7.6.4.8 and 7.4.5.1. □

Corollary 7.6.4.11. *Let n be an integer and suppose we are given a pullback diagram* 05E5

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

in the ∞-category $\mathcal{S}$. If f is n-truncated, then f' is n-truncated. If f is n-connective, then f' is n-connective.

Proof. Combine Corollaries 7.6.4.10, 3.5.9.12, and 3.5.1.25. □

Example 7.6.4.12. Let $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ be morphisms of Kan complexes. 03GJ Applying the construction of Corollary 7.6.4.9, we obtain a pullback square

$$\begin{tikzcd}
X_0 \times^{\mathrm{h}}_{X} X_1 \arrow[r] \arrow[d] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X
\end{tikzcd}$$

in the ∞-category $\mathcal{S}$.

Exercise 7.6.4.13. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, so that Corollary 7.6.4.9 04K8 supplied an identification of $\mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}$ with the fiber product of $\mathcal{C}$ with itself over $\mathcal{D}$ in the

∞-category $\mathcal{QC}$. Show that, under this identification, the relative diagonal of F (in the sense of Notation 7.6.3.18) is represented by the inclusion map $\mathcal{C} \hookrightarrow \mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}$. Moreover, if F is an isofibration, then we can replace the homotopy fiber product $\mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}$ with the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ (formed in the ordinary category of simplicial sets); see Corollary 4.5.2.27.

03GK **Corollary 7.6.4.14.** *Let $\mathcal{C}$ be a locally Kan simplicial category, and suppose we are given a commutative diagram $\sigma : \Delta^1 \times \Delta^1 \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$, corresponding to a diagram*

$$\begin{tikzcd}
X_{01} \arrow[r, "g_0"] \arrow[d, "g_1"'] \arrow[dr, "h" description] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"'] & X
\arrow[from=1-2, to=2-1, Rightarrow, shorten <=12mm, shorten >=28mm, "\alpha_0"']
\arrow[from=2-1, to=1-2, Rightarrow, shorten <=12mm, shorten >=28mm, "\alpha_1"]
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$ (see Remark 7.6.4.6). Then σ is a pullback square in the ∞-category $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ if and only if, for every object $Y \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(Y, X_{01})_{\bullet} \to \mathrm{Hom}_{\mathcal{C}}(Y, X_0)_{\bullet} \times^{\mathrm{h}}_{\mathrm{Hom}_{\mathcal{C}}(Y,X)_{\bullet}} (\mathrm{Hom}_{\mathcal{C}}(Y, X_1)_{\bullet} \times^{\mathrm{h}}_{\mathrm{Hom}_{\mathcal{C}}(Y,X)_{\bullet}} \mathrm{Hom}_{\mathcal{C}}(Y, X)_{\bullet})$$

is a homotopy equivalence of Kan complexes.

Proof. Combine Corollary 7.6.4.10 with Proposition 7.4.5.17. □

7.6.5 Equalizers and Coequalizers

03GL We now study (co)limits of a particularly simple shape.

03GM **Notation 7.6.5.1.** Let $(\bullet \rightrightarrows \bullet)$ denote the simplicial set given by the pushout $\Delta^1 \coprod_{\partial \Delta^1} \Delta^1$. For any ∞-category $\mathcal{C}$, we will identify morphisms from $(\bullet \rightrightarrows \bullet)$ to $\mathcal{C}$ with pairs (f_0, f_1), where $f_0 : Y \to X$ and $f_1 : Y \to X$ are morphisms of $\mathcal{C}$ having the same source and target.

03GN **Remark 7.6.5.2.** The simplicial set $(\bullet \rightrightarrows \bullet)$ of Notation 7.6.5.1 is isomorphic to the nerve of its homotopy category $\mathcal{J}$, which can be described concretely as follows:

- The category $\mathcal{J}$ has exactly two objects Y and X.
- There are exactly two non-identity morphisms in $\mathcal{J}$, both of which have source Y and target X.

Remark 7.6.5.3. There is a tautological epimorphism of simplicial sets 03GP

$$\begin{aligned} (\bullet \rightrightarrows \bullet) &= \Delta^1 \coprod_{\partial \Delta^1} \Delta^1 \\ &\twoheadrightarrow \Delta^1 \coprod_{\partial \Delta^1} \Delta^0 \\ &= \Delta^1 / \partial \Delta^1 . \end{aligned}$$

It follows from Example 6.3.4.4 that this epimorphism exhibits $\Delta^1 / \partial \Delta^1$ as a localization of $(\bullet \rightrightarrows \bullet)$. In particular, it is both left and right cofinal (Proposition 7.2.1.10).

Definition 7.6.5.4 (Equalizers and Coequalizers)**.** Let $\mathcal{C}$ be an ∞-category and let $f_0, f_1 : Y \to X$ be morphisms of $\mathcal{C}$ having the same source and target, which we identify with functor $\sigma : (\bullet \rightrightarrows \bullet) \to \mathcal{C}$. An *equalizer of* f_0 *and* f_1 is a limit of the diagram σ. A *coequalizer of* f_0 *and* f_1 is a colimit of the diagram σ. We say that the ∞-category $\mathcal{C}$ *admits equalizers* if every pair of morphisms $f_0, f_1 : Y \to X$ have an equalizer in $\mathcal{C}$, and that $\mathcal{C}$ *admits coequalizers* if every pair of morphisms $f_0, f_1 : Y \to X$ have a coequalizer in $\mathcal{C}$. 03GQ

Notation 7.6.5.5. Let $\mathcal{C}$ be an ∞-category and let $f_0, f_1 : Y \to X$ be morphisms of $\mathcal{C}$ having the same source and target. If there exists an object $Z \in \mathcal{C}$ which is an equalizer of f_0 and f_1, then Z is uniquely determined up to isomorphism (Proposition 7.1.1.12). To emphasize this uniqueness, we denote the object Z (if it exists) by $\mathrm{Eq}(f_0, f_1)$. Similarly, if there exists an object $W \in \mathcal{C}$ which is a coequalizer of f_0 and f_1, then W is uniquely determined up to isomorphism; to emphasize this, we denote W by $\mathrm{Coeq}(f_0, f_1)$. 03GR

Remark 7.6.5.6 (Duality)**.** The simplicial set $(\bullet \rightrightarrows \bullet)$ is canonically isomorphic to its opposite $(\bullet \rightrightarrows \bullet)^{\mathrm{op}}$. Consequently, if $f_0, f_1 : Y \to X$ are morphisms in an ∞-category $\mathcal{C}$ which admit an equalizer $Z = \mathrm{Eq}(f_0, f_1)$, then Z can be regarded as a coequalizer of f_0 and f_1 in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. 03GS

Remark 7.6.5.7 (Symmetry)**.** The simplicial set $(\bullet \rightrightarrows \bullet)$ has a unique nontrivial automorphism, which exchanges its nondegenerate edges. It follows that, if $f_0, f_1 : Y \to X$ are a pair of morphisms in an ∞-category $\mathcal{C}$, then we can identify (co)equalizers of the pair (f_0, f_1) with (co)equalizers of the pair (f_1, f_0). 03GT

Example 7.6.5.8 (Fixed Points of Endomorphisms)**.** Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. An *endomorphism of* X is a morphism $f : X \to X$ from the object X to itself. Note that the pair (X, f) can be identified with a morphism of simplicial sets $\sigma : (\Delta^1 / \partial \Delta^1) \to \mathcal{C}$. It follows from Remark 7.6.5.3 (together with Corollary 7.2.2.11) that an object of $\mathcal{C}$ is a limit of the diagram σ if and only if it is an equalizer of the pair of morphisms $f, \mathrm{id}_X : X \to X$. Similarly, an object of $\mathcal{C}$ is a colimit of σ if and only if it is a coequalizer of the pair (f, id_X). 03GU

03GV **Variant 7.6.5.9.** Let $\mathbf{Z}_{\geq 0}$ denote the collection of nonnegative integers, which we regard as a commutative monoid under addition, and let $B_\bullet \mathbf{Z}_{\geq 0}$ denote the classifying simplicial set of Construction 1.3.2.5. The simplicial set $B_\bullet \mathbf{Z}_{\geq 0}$ is an ∞-category which contains a (unique) object X, and the generator $1 \in \mathbf{Z}_{\geq 0}$ determines an endomorphism $e : X \to X$. We can regard $B_\bullet \mathbf{Z}_{\geq 0}$ as *freely generated* by the endomorphism e: more precisely, the pair (X, e) determines a morphism of simplicial sets $\sigma : \Delta^1 / \partial \Delta^1 \hookrightarrow B_\bullet \mathbf{Z}_{\geq 0}$ which is inner anodyne (see Example 1.5.7.11), and therefore induces a trivial Kan fibration $\mathrm{Fun}(B_\bullet \mathbf{Z}_{\geq 0}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C})$ for every ∞-category $\mathcal{C}$. In particular, the morphism σ is both left and right cofinal (Proposition 7.2.1.3).

If $F : B_\bullet \mathbf{Z}_{\geq 0} \to \mathcal{C}$ is a functor of ∞-categories, then Corollary 7.2.2.11 guarantees that an object of $\mathcal{C}$ is a limit of the functor F if and only if it is a limit of the diagram $F \circ \sigma$: that is, if and only if it is an equalizer of the pair of morphisms $F(e), \mathrm{id}_{F(X)} : F(X) \to F(X)$ (see Example 7.6.5.8). Similarly, an object of $\mathcal{C}$ is a colimit of the functor F if and only if it is a coequalizer of the pair $(F(e), \mathrm{id}_{F(X)})$.

03GW **Definition 7.6.5.10** (Equalizer and Coequalizer Diagrams)**.** Let $\mathcal{C}$ be an ∞-category. An *equalizer diagram in* $\mathcal{C}$ is a limit diagram $(\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathcal{C}$. A *coequalizer diagram* is a colimit diagram $(\bullet \rightrightarrows \bullet)^{\triangleright} \to \mathcal{C}$.

03GX **Warning 7.6.5.11.** Let $\mathcal{C}$ be an ∞-category and suppose we are given an equalizer diagram

03GY
$$\begin{tikzcd} Z \arrow[r, "g"] & Y \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & X \end{tikzcd} \tag{7.64}$$

in $\mathcal{C}$. Then the image of (7.64) in the homotopy category $\mathrm{h}\mathcal{C}$ need not be an equalizer diagram. In other words, the forgetful functor $\mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ does not preserve equalizer diagrams in general.

03GZ **Example 7.6.5.12.** Let X be a Kan complex containing vertices x and y. Then there exists an equalizer diagram

03H0
$$\begin{tikzcd} \{x\} \times^{\mathrm{h}}_X \{y\} \arrow[r, "f"] & \Delta^0 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & X \end{tikzcd} \tag{7.65}$$

in the ∞-category $\mathcal{S}$ (for a more general statement, see Corollary 7.6.5.21). However, unless the homotopy fiber product $\{x\} \times^{\mathrm{h}}_X \{y\}$ is either empty or contractible, the image of (7.65) in the homotopy category hKan is not an equalizer diagram (since the homotopy class $[f]$ is a monomorphism in hKan).

03H1 **Exercise 7.6.5.13.** Let $\mathcal{C}$ be an ∞-category and suppose we are given an equalizer diagram

03H2
$$\begin{tikzcd} Z \arrow[r, "g"] & Y \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & X \end{tikzcd} \tag{7.66}$$

in $\mathcal{C}$. Show that, for every object $C \in \mathcal{C}$, the map of sets

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, Z) \xrightarrow{[g]\circ} \mathrm{Eq}(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, Y) \rightrightarrows \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, X))$$

is surjective (though it is generally not injective).

We now give some examples of (co)equalizer diagrams.

Proposition 7.6.5.14. *Let $F_0, F_1 : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories, and let $G : \mathcal{E} \to \mathcal{D}$ be a functor of ∞-categories satisfying $F_0 \circ G = F_1 \circ G$. The following conditions are equivalent:* 03H3

(1) *The resulting diagram of ∞-categories $(\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathcal{QC}$ is an equalizer diagram.*

(2) *The commutative diagram*

03H4

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "G"] \arrow[d] & \mathcal{D} \arrow[d, "(F_0, F_1)"] \\
\mathcal{C} \arrow[r, "(\mathrm{id}, \mathrm{id})"] & \mathcal{C} \times \mathcal{C}
\end{tikzcd} \qquad (7.67)$$

is a categorical pullback square (Definition 4.5.2.8).

Proof. Let us identify the pair (F_0, F_1) with a functor of ordinary categories $\mathscr{F} : \mathcal{J} \to \mathrm{QCat}$, where $\mathcal{J}$ is the category described in Remark 7.6.5.2. The functor G then induces a map $\mathcal{E} \to \underleftarrow{\mathrm{holim}}(\mathscr{F})$, which can be identified with the map

$$\mathcal{E} \to \mathcal{C} \times^{\mathrm{h}}_{(\mathcal{C} \times \mathcal{C})} \mathcal{D}$$

determined by the diagram (7.67). Proposition 7.6.5.14 now follows from the criterion of Corollary 7.5.5.8. □

Corollary 7.6.5.15. *Let $f_0, f_1 : Y \to X$ be morphisms of Kan complexes and let $g : Z \to Y$ be a morphism of Kan complexes satisfying $f_0 \circ g = f_1 \circ g$. The following conditions are equivalent:* 03H5

(1) *The resulting diagram of ∞-categories $(\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathcal{S}$ is an equalizer diagram.*

(2) *The commutative diagram of Kan complexes*

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d] & Y \arrow[d, "(f_0, f_1)"] \\
X \arrow[r, "(\mathrm{id}, \mathrm{id})"] & X \times X
\end{tikzcd}$$

is a homotopy pullback square.

Proof. Combine Propositions 7.6.5.14 and 7.4.5.1. □

03H6 **Corollary 7.6.5.16.** *Let $\mathcal{C}$ be a locally Kan simplicial category and suppose we are given morphisms*

$$\begin{tikzcd} Z \arrow[r, "g"] & Y \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & X \end{tikzcd}$$

in $\mathcal{C}$ satisfying $f_0 \circ g = f_1 \circ g$. The following conditions are equivalent:

(1) *The induced diagram $(\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$ is an equalizer diagram in the ∞-category $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathcal{C})$, in the sense of Definition 7.6.5.10.*

(2) *For every object $C \in \mathcal{C}$, the diagram of Kan complexes*

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, Z)_{\bullet} \arrow[r] \arrow[d, "g"] & \mathrm{Hom}_{\mathcal{C}}(C, Y)_{\bullet} \arrow[d] \\ \mathrm{Hom}_{\mathcal{C}}(C, X)_{\bullet} \arrow[r, "(f_0, f_1)"] & \mathrm{Hom}_{\mathcal{C}}(C, X)_{\bullet} \times \mathrm{Hom}_{\mathcal{C}}(C, X)_{\bullet} \end{tikzcd}$$

is a homotopy pullback square.

Proof. Combine Corollary 7.6.5.15 with Proposition 7.4.5.17. □

Let $f_0, f_1 : Y \to X$ be morphisms of Kan complexes. By virtue of Corollary 7.4.5.6, the morphisms f_0 and f_1 have an equalizer in the ∞-category $\mathcal{S}$. Beware that this equalizer generally cannot be obtained from Corollary 7.6.5.15. For example, if f_0 and f_1 have disjoint images, then the existence of a morphism $g : Z \to Y$ satisfying $f_0 \circ g = f_1 \circ g$ guarantees that the simplicial set Z is empty. In such cases, to extend the pair (f_0, f_1) to an equalizer diagram in $\mathcal{S}$, we are forced to consider homotopy coherent diagrams which do not strictly commute.

03H7 **Remark 7.6.5.17.** Let $F_0, F_1 : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories, which we identify with a diagram $\sigma : (\bullet \rightrightarrows \bullet) \to \mathcal{QC}$. Unwinding the definitions, we see that extensions of σ to a diagram $\overline{\sigma} : (\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathcal{QC}$ can be identified with the following data:

- An ∞-category $\mathcal{E}$ equipped with functors $G : \mathcal{E} \to \mathcal{D}$ and $H : \mathcal{E} \to \mathcal{C}$.
- Isomorphisms $\alpha_0 : F_0 \circ G \xrightarrow{\sim} H$ and $\alpha_1 : F_1 \circ G \xrightarrow{\sim} H$ in the ∞-category $\mathrm{Fun}(\mathcal{E}, \mathcal{C})$.

In this case, we can identify the quadruple $(G, H, \alpha_0, \alpha_1)$ with a single functor of ∞-categories

$$U : \mathcal{E} \to \mathcal{D} \times^{\mathrm{h}}_{(\mathcal{C} \times \mathcal{C})} \mathcal{C}.$$

Proposition 7.6.5.18. *Let $F_0, F_1 : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories, which we identify with a diagram $\sigma : (\bullet \rightrightarrows \bullet) \to \mathcal{QC}$. Suppose we are given an extension $\overline{\sigma} : (\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathcal{QC}$ of σ, corresponding to a functor of ∞-categories* 03H8

$$T : \mathcal{E} \to \mathcal{D} \times^{\mathrm{h}}_{(\mathcal{C} \times \mathcal{C})} \mathcal{C}.$$

Then $\overline{\sigma}$ is an equalizer diagram in the ∞-category $\mathcal{QC}$ if and only if T is an equivalence of ∞-categories.

Proof. We proceed as in the proof of Proposition 7.6.4.8, with minor modifications. Let $\mathcal{A}$ denote the simplicial path category of $(\bullet \rightrightarrows \bullet)^{\triangleleft}$, so that we can identify $\overline{\sigma}$ with a simplicial functor $\mathscr{F} : \mathcal{A} \to \mathrm{QCat}$. Using Corollary 4.5.2.23, we can factor the functor $(F_0, F_1) : \mathcal{D} \to \mathcal{C} \times \mathcal{C}$ as a composition

$$\mathcal{D} \xrightarrow{U} \mathcal{D}' \xrightarrow{(F_0', F_1')} \mathcal{C} \times \mathcal{C},$$

where U is an equivalence of ∞-categories and (F_0', F_1') is an isofibration. The pair (F_0', F_1') can be identified with a morphism of simplicial sets $\sigma' : (\bullet \rightrightarrows \bullet) \to \mathcal{QC}$. Applying Remark 7.6.5.17, we can extend σ' to a diagram $\overline{\sigma}' : (\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathcal{QC}$, carrying the cone point to the ∞-category $\mathcal{E}' = \mathcal{D}' \times^{\mathrm{h}}_{(\mathcal{C} \times \mathcal{C})} \mathcal{C}$. The diagram $\overline{\sigma}'$ corresponds to a simplicial functor $\mathscr{F}' : \mathcal{A} \to \mathrm{QCat}$. The morphisms T and U determine a natural transformation of simplicial functors $\mathscr{F} \to \mathscr{F}'$, hence also a morphism $\overline{\sigma} \to \overline{\sigma}'$ in the ∞-category $\mathrm{Fun}((\bullet \rightrightarrows \bullet)^{\triangleleft}, \mathcal{QC})$. By virtue of Corollary 4.5.2.20, this natural transformation is an isomorphism of diagrams if and only if the functor T is an equivalence of ∞-categories. Consequently, Proposition 7.6.5.18 is equivalent to the assertion that $\overline{\sigma}'$ is an equalizer diagram in $\mathcal{QC}$ (Proposition 7.1.2.13).

Invoking Remark 7.6.5.17 again, we obtain another diagram $\overline{\sigma}''$ extending σ', which carries the cone point of $(\bullet \rightrightarrows \bullet)^{\triangleleft}$ to the equalizer $\mathrm{Eq}(F_0', F_1') = \mathcal{D}' \times_{(\mathcal{C} \times \mathcal{C})} \mathcal{C}$ (formed in the category of simplicial sets). The diagram $\overline{\sigma}''$ corresponds to another simplicial functor $\mathscr{F}'' : \mathcal{A} \to \mathrm{QCat}$. Note that there is a natural inclusion map $\mathscr{F}'' \hookrightarrow \mathscr{F}'$, which carries the cone point to the inclusion

$$\iota : \mathcal{D}' \times_{(\mathcal{C} \times \mathcal{C})} \mathcal{C} \subseteq \mathcal{D}' \times^{\mathrm{h}}_{(\mathcal{C} \times \mathcal{C})} \mathcal{C}.$$

Since (F_0', F_1') is an isofibration, the functor ι is an equivalence of ∞-categories (Corollary 4.5.2.28). It follows that the inclusion $\mathscr{F}'' \hookrightarrow \mathscr{F}'$ induces an isomorphism $\overline{\sigma}'' \to \overline{\sigma}'$ in the ∞-category $\mathrm{Fun}((\bullet \rightrightarrows \bullet)^{\triangleleft}, \mathcal{QC})$. By virtue of Proposition 7.1.2.13, we are reduced to showing that $\overline{\sigma}''$ is an equalizer diagram in $\mathcal{QC}$. This follows from the criterion of Proposition 7.6.5.14 (since ι is an equivalence of ∞-categories). □

It follows from Proposition 7.6.5.18 that, if $F_0, F_1 : \mathcal{D} \to \mathcal{C}$ are functors of ∞-categories, then the homotopy fiber product $\mathcal{D} \times^{\mathrm{h}}_{(\mathcal{C} \times \mathcal{C})} \mathcal{C}$ is an equalizer of F_0 and F_1 in the ∞-category

$\mathcal{QC}$ (this can also be viewed as a special case of Proposition 7.5.2.6). However, it is possible to be more efficient.

03H9 **Construction 7.6.5.19** (The Homotopy Equalizer)**.** Let $F_0, F_1 : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories, and form a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{hEq}(F_0, F_1) \arrow[r] \arrow[d, "G"] & \mathrm{Isom}(\mathcal{C}) \arrow[d] \\
\mathcal{D} \arrow[r, "(F_0, F_1)"] & \mathcal{C} \times \mathcal{C} .
\end{tikzcd}$$

Note that the right vertical map is an isofibration (Corollary 4.4.5.5), so the left vertical map is also an isofibration; in particular, $\mathrm{hEq}(F_0, F_1)$ is an ∞-category. We will refer to $\mathrm{hEq}(F_0, F_1)$ as the *homotopy equalizer* of the functors F_0 and F_1. By construction, objects of $\mathrm{hEq}(F_0, F_1)$ can be identified with pairs (X, u), where X is an object of $\mathcal{D}$ and $u : F_0(X) \to F_1(X)$ is an isomorphism in the ∞-category $\mathcal{C}$.

Set $H = G \circ F_1$, so that the construction $(X, u) \mapsto u$ determines an isomorphism $\alpha_0 : G \circ F_0 \xrightarrow{\sim} H$ in the ∞-category $\mathrm{Fun}(\mathrm{Eq}(F_0, F_1), \mathcal{C})$. Taking α_1 to be the identity morphism $\mathrm{id} : G \circ F_1 \xrightarrow{\sim} H$, we see that the quadruple $(G, H, \alpha_0, \alpha_1)$ determines a diagram $\overline{\sigma} : (\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathcal{QC}$, carrying the cone point to the homotopy equalizer $\mathrm{hEq}(F_0, F_1)$ (see Remark 7.6.5.17).

03HA **Corollary 7.6.5.20.** *Let $F_0, F_1 : \mathcal{D} \to \mathcal{C}$ and $F_1 : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. Then the morphism $\overline{\sigma} : (\bullet \rightrightarrows \bullet)^{\triangleleft} \to \mathcal{QC}$ of Construction 7.6.5.19 is an equalizer diagram. In particular, the homotopy equalizer* $\mathrm{hEq}(F_0, F_1)$ *is an equalizer of F_0 and F_1 in the ∞-category $\mathcal{QC}$.*

Proof. The diagram $\overline{\sigma}$ can be identified with a functor $U : \mathrm{hEq}(F_0, F_1) \to \mathcal{D} \times^{\mathrm{h}}_{(\mathcal{C} \times \mathcal{C})} \mathcal{C}$. By virtue of Proposition 7.6.5.18, it will suffice to show that U is an equivalence of ∞-categories. Unwinding the definitions, we see that U fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{hEq}(F_0, F_1) \arrow[r, "U"] \arrow[d] & \mathcal{D} \times^{\mathrm{h}}_{(\mathcal{C} \times \mathcal{C})} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C} \arrow[d] \\
\mathcal{D} \arrow[r] & \mathcal{D} \times^{\mathrm{h}}_{\mathcal{C}} \mathcal{C} \arrow[r] & \mathcal{D} \times \mathcal{C},
\end{tikzcd}$$

where the homotopy fiber product on the upper right is formed using the functor F_0, and the homotopy fiber product on the lower middle is formed using the functor F_1. Each of the squares in this diagram is a pullback, and the right vertical map is an isofibration (Remark 4.5.2.2). It follows that the left side of the diagram is a categorical pullback square (Corollary 4.5.2.27). Since the functor on the lower left is an equivalence of ∞-categories (Corollary 4.5.2.22), it follows that U is an equivalence of ∞-categories. □

Corollary 7.6.5.21. *Let $f_0, f_1 : Y \to X$ be morphisms of Kan complexes. Then the homotopy equalizer* $\mathrm{hEq}(f_0, f_1)$ *is a Kan complex, which is an equalizer of f_0 and f_1 in the ∞-category $\mathcal{S}$.* 03HB

Proof. Combine Corollary 7.6.5.20 with Proposition 7.4.5.1. □

Corollaries 7.6.5.20 and 7.6.5.21 illustrate a general phenomenon: if $\mathcal{C}$ is an ∞-category which admits pairwise products, then equalizers in $\mathcal{C}$ can be viewed as a special kind of fiber product.

Proposition 7.6.5.22 (Rewriting Equalizers as Pullbacks)**.** *Let $\mathcal{C}$ be an ∞-category, let $f_0, f_1 : Y \to X$ be morphisms of $\mathcal{C}$. Let $X \times X$ be a product of X with itself in the ∞-category $\mathcal{C}$, so that f_0 and f_1 determine a morphism $(f_0, f_1) : Y \to X \times X$, and let $\delta_X : X \to X \times X$ be the diagonal map (Notation 7.6.2.9). Then an object of $\mathcal{C}$ is an equalizer of f_0 and f_1 if and only if it is a fiber product of Y with X over $X \times X$.* 03HC

Proof. Let $\mathcal{K}$ denote the simplicial set given by the product $(\bullet \rightrightarrows \bullet)^{\triangleleft} \times \Delta^1$. Then $\mathcal{K}$ is an ∞-category, which we depict informally by the diagram

$$\begin{tikzcd}
z \arrow[r] \arrow[d] & y \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & x \arrow[d] \\
z' \arrow[r] & y' \arrow[r, shift left] \arrow[r, shift right] & x'.
\end{tikzcd}$$

We now proceed in several steps.

- Let $\mathcal{K}_0$ denote the full subcategory of $\mathcal{K}$ spanned by the objects x and y. Then $\mathcal{K}_0$ is isomorphic to the simplicial set $(\bullet \rightrightarrows \bullet)$. In particular, the pair of morphisms $f_0, f_1 : Y \to X$ can be identified with a functor $\sigma_0 : \mathcal{K}_0 \to \mathcal{C}$, satisfying $\sigma_0(x) = X$ and $\sigma_0(y) = Y$. By definition, an object of $\mathcal{C}$ is an equalizer of the pair (f_0, f_1) if and only if it is a limit of the diagram σ_0.

- Let $\mathcal{K}_1$ denote the full subcategory of $\mathcal{K}$ spanned by the objects x, x', and y. Note that the identity map $\mathrm{id}_{\mathcal{K}_0}$ extends uniquely to a retraction $r : \mathcal{K}_1 \to \mathcal{K}_0$, carrying the object $x' \in \mathcal{K}_1$ to $x \in \mathcal{K}_0$. Let $\sigma_1 : \mathcal{K}_1 \to \mathcal{C}$ be the composition $\sigma_0 \circ r$. Note that that the inclusion map $\mathcal{K}_0 \hookrightarrow \mathcal{K}_1$ admits a right adjoint (given by the retraction r), and is therefore left cofinal (Corollary 7.2.3.7). It follows that an object of $\mathcal{C}$ is a limit of the diagram σ_0 if and only if it is a limit of the diagram σ_1 (Corollary 7.2.2.11).

- Choose a pair of morphisms $\pi_0, \pi_1 : X \times X \to X$ in the ∞-category $\mathcal{C}$ which exhibit $X \times X$ as a product of X with itself. The morphism $(f_0, f_1) : Y \to X$ is characterized (up to homotopy) by the requirement that there exist 2-simplices σ_0 and σ_1 of $\mathcal{C}$, where σ_i exhibits f_i as a composition of π_i with (f_0, f_1). Let $\mathcal{K}_2$ denote the full subcategory of $\mathcal{K}$ spanned by the objects x, x', y, and y'. Then the pair (σ_0, σ_1) determines an extension of σ_1 to a functor $\sigma_2 : \mathcal{K}_2 \to \mathcal{C}$ satisfying $\sigma_2(y') = X \times X$.

- The diagonal morphism $\delta_X : X \to X \times X$ is characterized (up to homotopy) by the requirement that there exist 2-simplices τ_0 and τ_i of $\mathcal{C}$, where τ_i exhibits id_X as a composition of π_i with δ_X. Let $\mathcal{K}_3$ denote the full subcategory of $\mathcal{K}$ spanned by the objects x, x', y, and y', and z'. Then the pair (τ_0, τ_1) determines an extension of σ_2 to a functor $\sigma_3 : \mathcal{K}_3 \to \mathcal{C}$ satisfying $\sigma_3(z') = X$. The diagram σ_3 can be represented informally by the diagram

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[d, "{(f_0,f_1)}"] & X \arrow[d, "\mathrm{id}_X"] \\
X \arrow[r, "\delta_X"] & X \times X \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & X.
\end{tikzcd}$$

Note that σ_3 is right Kan extended from the full subcategory $\mathcal{K}_1 \subseteq \mathcal{K}_3$. Consequently, an object of $\mathcal{C}$ is a limit of the diagram σ_1 if and only if it is a limit of the diagram σ_3 (Remark 7.3.8.15).

- Let $\mathcal{K}_4$ denote the full subcategory of $\mathcal{K}$ spanned by the objects x', y, y', and z'. Note that the functor σ_3 is right Kan extended from $\mathcal{K}_4$. It follows that an object of $\mathcal{C}$ is a limit of the functor σ_3 if and only if it is a limit of the functor $\sigma_4 = \sigma_3|_{\mathcal{K}_4}$.

- Let $\mathcal{K}_5$ denote the full subcategory of $\mathcal{K}$ spanned by the objects y, y', and z'. Using the criterion of Theorem 7.2.3.1, we see that the inclusion $\mathcal{K}_5 \hookrightarrow \mathcal{K}_4$ is left cofinal. It follows that an object of $\mathcal{C}$ is a limit of the diagram σ_4 if and only if it is a limit of the diagram $\sigma_5 = \sigma_4|_{\mathcal{K}_5}$ (Corollary 7.2.2.11).

Combining these steps, we deduce that an object of $\mathcal{C}$ is an equalizer of f_0 and f_1 if and only if it is a limit of the diagram σ_5: that is, if and only if it is a fiber product of Y with X over $X \times X$ (along the morphisms (f_0, f_1) and δ_X). $\square$

In an ∞-category which admits finite products, we can use a similar argument to describe pullbacks in terms of equalizers.

Proposition 7.6.5.23 (Rewriting Pullbacks as Equalizers)**.** *Let $\mathcal{C}$ be an ∞-category and let $f_0 : X_0 \to X$ and $f_1 : X_1 \to X$ be morphisms of $\mathcal{C}$. Suppose that X_0 and X_1 admit a product $X_0 \times X_1$, and let $\pi_0 : X_0 \times X_1 \to X_0$ and $\pi_1 : X_0 \times X_1 \to X_1$ denote the projection maps. For $i \in \{0, 1\}$, let $g_i : X_0 \times X_1 \to X$ denote a composition of π_i with f_i in the ∞-category $\mathcal{C}$. Then an object of $\mathcal{C}$ is a pullback of X_0 with X_1 over X if and only if it is an equalizer of the pair of morphisms (g_0, g_1).* `03HD`

Proof. Let $\mathcal{K}$ denote the category which is freely generated by a non-commutative square, as indicated in the diagram

$$\begin{tikzcd}
Y_{01} \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & Y_0 \arrow[d] \\
Y_1 \arrow[r] & Y.
\end{tikzcd}$$

Note that the upper right and lower left regions of this diagram determine monomorphisms $\tau_0, \tau_1 : \Delta^2 \hookrightarrow \mathrm{N}_\bullet(\mathcal{K})$. The images of τ_0 and τ_1 are simplicial subsets of $\mathrm{N}_\bullet(\mathcal{K})$, whose union is $\mathrm{N}_\bullet(\mathcal{K})$ and whose intersection is the discrete simplicial set $\{Y_{01}, Y\}$. It follows that τ_0 and τ_1 induce an isomorphism of simplicial sets $(\tau_0, \tau_1) : \Delta^2 \coprod_{\{0,2\}} \Delta^2 \simeq \mathrm{N}_\bullet(\mathcal{K})$.

For $i \in \{0, 1\}$, let σ_i be a 2-simplex of $\mathcal{C}$ which witnesses g_i as a composition of π_i with f_i (in the sense of Definition 1.4.4.1). Then there is a unique morphism of simplicial sets $q : \mathrm{N}_\bullet(\mathcal{K}) \to \mathcal{C}$ satisfying $q \circ \tau_i = \sigma_i$, which we indicate as a diagram

$$\begin{tikzcd}
X_0 \times X_1 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] \arrow[dr, shift left, "g_0"] \arrow[dr, shift right, "g_1"'] & X_0 \arrow[d, "f_0"] \\
X_1 \arrow[r, "f_1"] & X.
\end{tikzcd}$$

Let $\mathcal{K}_+ \subseteq \mathcal{K}$ denote the full subcategory spanned by the objects Y_{01} and Y. Then the nerve $\mathrm{N}_\bullet(\mathcal{K}_+)$ can be identified with the simplicial set $(\bullet \rightrightarrows \bullet)$ of Notation 7.6.5.1, and the restriction $q_+ = q|_{\mathrm{N}_\bullet(\mathcal{K}_+)}$ corresponds to the pair of morphisms $g_0, g_1 : X_0 \times X_1 \to X$. Note that the full subcategory $\mathrm{N}_\bullet(\mathcal{K}_+) \subset \mathrm{N}_\bullet(\mathcal{K})$ is coreflective, so the inclusion map $\mathrm{N}_\bullet(\mathcal{K}_+) \hookrightarrow \mathrm{N}_\bullet(\mathcal{K})$ is left cofinal (Corollary 7.2.3.7). It follows that an object of $\mathcal{C}$ is an equalizer of g_0 and g_1 if and only if it is a limit of the diagram q (Corollary 7.2.2.11).

To complete the proof, it will suffice to show that an object of $\mathcal{C}$ is a limit of q if and only if it is a fiber product of X_0 with X_1 over X. Let $\mathcal{K}_- \subseteq \mathcal{K}$ denote the full subcategory spanned by the objects Y_0, Y_1, and Y. By virtue of Corollaries 7.3.8.2 and 7.3.8.14, it will suffice to show that the functor q is right Kan extended from $\mathrm{N}_\bullet(\mathcal{K}_-)$. E quivalently, we wish to show that the natural map

$$\mathrm{N}_\bullet(\mathcal{K}_- \times_{\mathcal{K}} \mathcal{K}_{Y_{01}/})^{\triangleleft} \to \mathrm{N}_\bullet(\mathcal{K}) \xrightarrow{q} \mathcal{C}$$

is a limit diagram in $\mathcal{C}$. Unwinding the definitions, we see that $\mathcal{K}_- \times_{\mathcal{K}} \mathcal{K}_{Y_{01}/}$ can be written as a disjoint union of subcategories having initial objects Y_0 and Y_1, respectively. In particular, the inclusion map

$$\{Y_0, Y_1\} \hookrightarrow \mathrm{N}_\bullet(\mathcal{K}_- \times_{\mathcal{K}} \mathcal{K}_{Y_{01}/})$$

is left cofinal. The desired result now follows from Corollary 7.2.2.3, together with our assumption that the maps π_0 and π_1 exhibit $X_0 \times X_1$ as a product of X_0 with X_1. □

03HE **Exercise 7.6.5.24.** In the situation of Proposition 7.6.5.23, suppose that X_0 and X_1 admit a fiber product over X. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves the product of X_0 and X_1 (that is, $F(\pi_0)$ and $F(\pi_1)$ exhibit $F(X_0 \times X_1)$ as a product of $F(X_0)$ with $F(X_1)$ in the ∞-category $\mathcal{D}$). Show that F preserves the fiber product of X_0 with X_1 over X if and only if it preserves the equalizer of the morphisms g_0 and g_1.

03HF **Corollary 7.6.5.25.** *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ admits finite limits if and only if it admits finite products and equalizers. If these conditions are satisfied, then a functor $F : \mathcal{C} \to \mathcal{D}$ preserves finite limits if and only if it preseves finite products and equalizers.*

Proof. Combine Corollary 7.6.3.27 with Proposition 7.6.5.23 (and Exercise 7.6.5.24). □

7.6.6 Sequential Limits and Colimits

03HG Throughout this section, we let $\mathbf{Z}_{\geq 0}$ denote the set of nonnegative integers, endowed with its usual ordering.

03HH **Definition 7.6.6.1** (Towers)**.** Let $\mathcal{C}$ be an ∞-category. A *tower* in $\mathcal{C}$ is a functor $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}^{\mathrm{op}}) \to \mathcal{C}$. We say that $\mathcal{C}$ *admits sequential limits* if every tower in $\mathcal{C}$ has a limit, and that $\mathcal{C}$ *admits sequential colimits* if every diagram $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}) \to \mathcal{C}$ has a colimit. We say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *preserves sequential limits* if it preserves limits indexed by the simplicial set $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}^{\mathrm{op}})$, and that it *preserves sequential colimits* if it preserves colimits indexed by the simplicial set $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0})$.

03HJ **Notation 7.6.6.2.** Let $\mathcal{C}$ be an ∞-category. We will generally abuse notation by identifying a functor $X : \mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}) \to \mathcal{C}$ with the collection of objects $\{X(n)\}_{n \geq 0}$ and morphisms $f_n : X(n+1) \to X(n)$ obtained by the evaluating X on the edges of $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0})$ corresponding to ordered pairs of the form $(n, n+1)$; we will depict the pair $(\{X(n)\}_{n \geq 0}, \{f_n\}_{n \geq 0})$ as a diagram

$$X(0) \xrightarrow{f_0} X(1) \xrightarrow{f_1} X(2) \xrightarrow{f_2} X(3) \xrightarrow{f_3} X(4) \rightarrow \cdots$$

Similarly, we abuse notation by identifying towers $X : \mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}^{\mathrm{op}}) \rightarrow \mathcal{C}$ with diagrams

$$\cdots \rightarrow X(4) \xrightarrow{f_3} X(3) \xrightarrow{f_2} X(2) \xrightarrow{f_1} X(1) \xrightarrow{f_0} X(0).$$

Beware that the convention of Notation 7.6.6.2 is slightly abusive: the simplicial set $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0})$ has nondegenerate simplices in every dimension, so a functor $X : \mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}) \rightarrow \mathcal{C}$ is not literally determined by its underlying diagram

$$X(0) \xrightarrow{f_0} X(1) \xrightarrow{f_1} X(2) \xrightarrow{f_2} X(3) \xrightarrow{f_3} X(4) \rightarrow \cdots$$

However, the abuse is essentially harmless:

Remark 7.6.6.3. Let $\mathrm{Spine}[\mathbf{Z}_{\geq 0}]$ denote the simplicial subset of $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0})$ whose k-simplices are sequences of nonnegative integers $(n_0, n_1, \cdots, n_k)$ satisfying $n_0 \leq n_1 \leq \cdots \leq n_k \leq n_0 + 1$. Then $\mathrm{Spine}[\mathbf{Z}_{\geq 0}]$ is a 1-dimensional simplicial set, which corresponds (under the equivalence of Proposition 1.1.6.9) to the directed graph G indicated in the diagram 03HK

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots .$$

Moreover, the linearly ordered set $(\mathbf{Z}_{\geq 0}, \leq)$ can be identified with the path category $\mathrm{Path}[G]$ of Construction 1.3.7.1. It follows that the inclusion map $\mathrm{Spine}[\mathbf{Z}_{\geq 0}] \hookrightarrow \mathrm{N}_\bullet(\mathbf{Z}_{\geq 0})$ is inner anodyne (Proposition 1.5.7.3).

In particular, for any ∞-category $\mathcal{C}$, the restriction map

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}), \mathcal{C}) \rightarrow \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}_{\geq 0}], \mathcal{C})$$

is a trivial Kan fibration (Theorem 1.5.7.1). Stated more informally, every sequence of composable morphisms

$$X(0) \xrightarrow{f_0} X(1) \xrightarrow{f_1} X(2) \xrightarrow{f_2} X(3) \xrightarrow{f_3} X(4) \rightarrow \cdots$$

admits an *essentially unique* extension to a functor $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}) \rightarrow \mathcal{C}$.

Example 7.6.6.4. Let $\mathcal{C}$ be a locally Kan simplicial category, and suppose we are given a collection of objects $\{X(n)\}_{n \geq 0}$ and morphisms $f_n : X(n) \rightarrow X(n+1)$ in $\mathcal{C}$. It follows from Remark 7.6.6.3 that the diagram 03HL

$$X(0) \xrightarrow{f_0} X(1) \xrightarrow{f_1} X(2) \xrightarrow{f_2} X(3) \xrightarrow{f_3} X(4) \rightarrow \cdots$$

can be extended to a functor $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}) \rightarrow \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$. In fact, there is a preferred choice of such an extension, which is uniquely determined by the requirement that it factors through the inclusion map $\mathrm{N}_\bullet(\mathcal{C}) \hookrightarrow \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$.

03HM **Remark 7.6.6.5.** Let $\mathcal{C}$ be an ∞-category, and suppose we are given a tower $X : \mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}^{\mathrm{op}}) \to \mathcal{C}$, which we depict as a diagram

$$\cdots \to X(4) \xrightarrow{f_3} X(3) \xrightarrow{f_2} X(2) \xrightarrow{f_1} X(1) \xrightarrow{f_0} X(0),$$

having a limit $\varprojlim(X)$. Then, for every object $Y \in \mathcal{C}$, the map of sets

$$\theta : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, \varprojlim(X)) \to \varprojlim(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X(n)))$$

is surjective. To prove this, suppose we are given a collection of morphisms $g_n : Y \to X(n)$ satisfying $[f_n] \circ [g_{n+1}] = [g_n]$ in the homotopy category $\mathrm{h}\mathcal{C}$. Then, for each $n \geq 0$, we can choose a 2-simplex σ_n in $\mathcal{C}$ as indicated in the diagram

$$\begin{tikzcd} & X(n+1) \arrow[dr, "f_n"] & \\ Y \arrow[ur, "g_{n+1}"] \arrow[rr, "g_n"] & & X(n). \end{tikzcd}$$

Let X_0 denote the restriction of X to the spine $\mathrm{Spine}[\mathbf{Z}_{\geq 0}^{\mathrm{op}}] \subset \mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}^{\mathrm{op}})$. Then the collection of 2-simplices $\{\sigma_n\}_{n \geq 0}$ determines an extension of X_0 to a diagram $\overline{X}_0 : \mathrm{Spine}[\mathbf{Z}_{\geq 0}^{\mathrm{op}}]^{\triangleleft} \to \mathcal{C}$ carrying the cone point to the object Y. The isomorphism class of this extension can be identified with a morphism $[g] : Y \to \varprojlim(X_0) \simeq \varprojlim(X)$ in the homotopy category $\mathrm{h}\mathcal{C}$, which is a preimage of the sequence $\{[g_n]\}_{n \geq 0}$ under the function θ.

03HN **Warning 7.6.6.6.** In the situation of Remark 7.6.6.5, the map

$$\theta : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, \varprojlim(X)) \to \varprojlim(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, X(n)))$$

need not be injective. That is, the forgetful functor $\mathcal{C} \to \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})$ generally does not preserve sequential limits (or colimits).

03HP **Example 7.6.6.7.** Fix a prime number p. For every integer $n \geq 0$, let $p^n\,\mathbf{Z}$ denote the cyclic subgroup of $\mathbf{Z}$ generated by p^n, so that we have a tower of classifying simplicial sets

03HQ
$$\begin{tikzcd} \cdots \arrow[r] & B_\bullet(p^3\,\mathbf{Z}) \arrow[r] & B_\bullet(p^2\,\mathbf{Z}) \arrow[r] & B_\bullet(p\,\mathbf{Z}) \arrow[r] & B_\bullet(\mathbf{Z}). \end{tikzcd} \quad (7.68)$$

Then:

- The tower (7.68) has a limit in the ordinary category of simplicial sets, given by the simplicial set Δ^0 (which we can identify with the classifying simplicial set for the trivial group $(0) = \bigcap_{n \geq 0} p^n\,\mathbf{Z}$).

- The simplicial set Δ^0 is also a limit of the tower (7.68) in the homotopy category hKan.
- In the ∞-category $\mathcal{S}$, the tower (7.68) has a different limit, which has uncountably many connected components (see Remark [?]).

We now give some easy examples of sequential limits and colimits.

Example 7.6.6.8 (Sequential Colimits in $\mathcal{QC}$)**.** Suppose we are given a collection of ∞-categories $\{\mathcal{C}(n)\}_{n \geq 0}$ and functors $F_n : \mathcal{C}(n) \to \mathcal{C}(n+1)$, which we view as a diagram 03HR

$$\mathcal{C}(0) \xrightarrow{F_0} \mathcal{C}(1) \xrightarrow{F_1} \mathcal{C}(2) \xrightarrow{F_2} \mathcal{C}(3) \to \cdots$$

Let $\varinjlim_n \mathcal{C}(n)$ denote the colimit of this diagram (formed in the ordinary category of simplicial sets). Then $\varinjlim_n \mathcal{C}(n)$ is also an ∞-category, which is also a colimit of the associated diagram $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}) \to \mathcal{QC}$. This is a special case of Corollary 7.5.9.3.

Variant 7.6.6.9 (Sequential Colimits in $\mathcal{S}$)**.** Suppose we are given a collection of Kan complexes $\{X(n)\}_{n \geq 0}$ and morphisms $f_n : X(n) \to X(n+1)$, which we view as a diagram 03HS

$$X(0) \xrightarrow{f_0} X(1) \xrightarrow{f_1} X(2) \xrightarrow{f_2} X(3) \to \cdots$$

Let $\varinjlim_n X(n)$ denote the colimit of this diagram (formed in the ordinary category of simplicial sets). Then $\varinjlim_n X(n)$ is also a Kan complex, which is also a colimit of the associated diagram $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}) \to \mathcal{S}$. See Variant 7.5.9.4.

Example 7.6.6.10 (Towers of Isofibrations)**.** Suppose we are given a collection of ∞-categories $\{\mathcal{C}(n)\}_{n \geq 0}$ and functors $F_n : \mathcal{C}(n+1) \to \mathcal{C}(n)$, which we view as a tower 03HT

$$\cdots \to \mathcal{C}(4) \xrightarrow{F_3} \mathcal{C}(3) \xrightarrow{F_2} \mathcal{C}(2) \xrightarrow{F_1} \mathcal{C}(1) \xrightarrow{F_0} \mathcal{C}(0)$$

If each of the functors F_n is an isofibration, then the limit $\varprojlim_n \mathcal{C}(n)$ (formed in the ordinary category of simplicial sets) is also an ∞-category, which can be also be viewed as a limit of the associated tower $\mathrm{N}_\bullet(\mathbf{Z}^{\mathrm{op}}_{\geq 0}) \to \mathcal{QC}$. This follows by combining Example 4.5.6.8, Example 7.5.5.3, and Proposition 7.5.5.7.

Variant 7.6.6.11 (Towers of Kan Fibrations)**.** Suppose we are given a collection of Kan complexes $\{X(n)\}_{n \geq 0}$ and morphisms $f_n : X(n+1) \to X(n)$, which we view as a tower 03HU

$$\cdots \to X(4) \xrightarrow{f_3} X(3) \xrightarrow{f_2} X(2) \xrightarrow{f_1} X(1) \xrightarrow{f_0} X(0).$$

If each of the morphisms f_n is a Kan fibration, then the limit $\varprojlim_n X(n)$ (formed in the ordinary category of simplicial sets) is also a Kan complex, which can be also be viewed as a limit of the associated tower $\mathrm{N}_\bullet(\mathbf{Z}^{\mathrm{op}}_{\geq 0}) \to \mathcal{S}$ (combine Example 7.6.6.10 with Proposition 7.4.5.1).

03HV **Variant 7.6.6.12** (Limits of General Towers)**.** Suppose we are given a sequence of ∞-categories $\{\mathcal{C}(n)\}_{n \geq 0}$ and functors $F_n : \mathcal{C}(n+1) \to \mathcal{C}(n)$, which we view as a tower

03HW
$$\cdots \to \mathcal{C}(4) \xrightarrow{F_3} \mathcal{C}(3) \xrightarrow{F_2} \mathcal{C}(2) \xrightarrow{F_1} \mathcal{C}(1) \xrightarrow{F_0} \mathcal{C}(0) \tag{7.69}$$

If the functors F_n are not assumed to be isofibrations, then the limit $\varprojlim_n \mathcal{C}(n)$ (formed in the ordinary category of simplicial sets) might not be a limit of the associated tower in $\mathcal{QC}$ (for example, $\varprojlim_n \mathcal{C}(n)$ might fail to be an ∞-category). Nevertheless, we can always compute the relevant limit in $\mathcal{QC}$ by replacing (7.69) by a levelwise equivalent diagram of ∞-categories in which the transition functors are isofibrations. For example, we can replace (7.69) by the isofibrant tower of iterated homotopy fiber products

$$\cdots \to \mathcal{C}(2) \times^{\mathrm{h}}_{\mathcal{C}(1)} (\mathcal{C}(1) \times^{\mathrm{h}}_{\mathcal{C}(0)} \mathcal{C}(0)) \to \mathcal{C}(1) \times^{\mathrm{h}}_{\mathcal{C}(0)} \mathcal{C}(0) \to \mathcal{C}(0).$$

Let us denote the limit of this tower (in the category of simplicial sets) by

$$\cdots \times^{\mathrm{h}}_{\mathcal{C}(3)} \mathcal{C}(3) \times^{\mathrm{h}}_{\mathcal{C}(2)} \mathcal{C}(2) \times^{\mathrm{h}}_{\mathcal{C}(1)} \mathcal{C}(1) \times^{\mathrm{h}}_{\mathcal{C}(0)} \mathcal{C}(0).$$

It is an ∞-category whose objects can be identified with sequences of pairs $\{(C_n, \alpha_n)\}_{n \geq 0}$, where each C_n is an object of the ∞-category $\mathcal{C}(n)$ and each $\alpha_n : F_n(C_{n+1}) \xrightarrow{\sim} C_n$ is an isomorphism in the ∞-category $\mathcal{C}(n)$. Combining Example 7.6.6.10 with Remark 7.1.1.8, we see that it can be identified with a limit of the diagram (7.69) in the ∞-category $\mathcal{QC}$.

Sequential limits are useful for building more complicated types of limits.

03HX **Proposition 7.6.6.13.** *Suppose we are given a diagram of simplicial sets*

$$K(0) \to K(1) \to K(2) \to K(3) \to \cdots$$

having colimit K. Let $\mathcal{C}$ be an ∞-category and let $f : K \to \mathcal{C}$ be a diagram, corresponding to a compatible sequence of diagrams $f_n : K(n) \to \mathcal{C}$. Suppose that each of the diagrams f_n admits a limit in $\mathcal{C}$. Then there exists a tower $X : \mathrm{N}_\bullet(\mathbf{Z}^{\mathrm{op}}_{\geq 0}) \to \mathcal{C}$ with the following properties:

(1) *For each $n \geq 0$, the object $X(n) \in \mathcal{C}$ is a limit of the diagram f_n.*

(2) *An object of $\mathcal{C}$ is a limit of the diagram f if and only if it is a limit of the tower X. In particular, the diagram f has a limit if and only if the tower X has a limit.*

(3) *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves the limits of each of the diagrams f_n. Then F preserves limits of the diagram f if and only if it preserves limits of the tower X.*

Proof. Combine Propositions 7.5.8.12 and 7.5.9.1. □

Corollary 7.6.6.14. *Suppose we are given a diagram of simplicial sets* 03HY

$$K(0) \to K(1) \to K(2) \to K(3) \to \cdots$$

having colimit K, and let $\mathcal{C}$ be an ∞-category which admits sequential limits and $K(n)$-indexed limits, for each $n \geq 0$. Then $\mathcal{C}$ admits K-indexed colimits. If $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories which preserves sequential limits and $K(n)$-indexed limits for each $n \geq 0$, then it also preserves K-indexed limits.

Example 7.6.6.15. Let $\mathcal{C}$ be an ∞-category which admits finite products. If $\mathcal{C}$ admits sequential limits, then it also admits countable products. More precisely, for any countable collection of objects $\{X_n\}_{n \geq 0}$ of $\mathcal{C}$, the product $\prod_{n \geq 0} X_n$ can be computed as the limit of a tower 03HZ

$$\cdots \to X_2 \times X_1 \times X_0 \to X_1 \times X_0 \to X_0.$$

We now establish a partial converse to Example 7.6.6.15. Let $\mathcal{C}$ be an ∞-category which admits countable products, and suppose that we are given a tower

$$\cdots \to X(3) \xrightarrow{f_2} X(2) \xrightarrow{f_1} X(1) \xrightarrow{f_0} X(0)$$

in $\mathcal{C}$. Then the collection of morphisms $\{f_n\}_{n \geq 0}$ determine an endomorphism f of the product $P = \prod_{n \geq 0} X(n)$, given informally by the composition

$$\begin{array}{ccc} P & = & \prod_{n \geq 0} X(n) \\ & \to & \prod_{n > 0} X(n) \\ & = & \prod_{m \geq 0} X(m+1) \\ & \xrightarrow{\prod_{m \geq 0} f_m} & \prod_{m \geq 0} X(m) \\ & = & P. \end{array}$$

In this case, we can identify limits of the tower X with equalizers of the pair of morphisms $f, \mathrm{id}_P : P \to P$. We can formulate this assertion more precisely as follows:

Proposition 7.6.6.16 (Sequential Limits as Equalizers)**.** *Let $\mathcal{C}$ be an ∞-category and let $X : \mathrm{N}_{\bullet}(\mathbf{Z}^{\mathrm{op}}_{\geq 0}) \to \mathcal{C}$ be a tower, which we identify with the diagram* 03J0

$$\cdots \to X(3) \xrightarrow{f_2} X(2) \xrightarrow{f_1} X(1) \xrightarrow{f_0} X(0).$$

Suppose that there exists an object $P \in \mathcal{C}$ equipped with morphisms $\{q_n : P \to X(n)\}_{n \geq 0}$ which exhibits P as a product of the collection $\{X(n)\}_{n \geq 0}$. Then:

(1) *There exists a morphism $f : P \to P$ with the property that, for each $n \geq 0$, the diagram*

03J1
$$\begin{tikzcd}
P \arrow[r, "{[f]}"] \arrow[d, "{[q_{n+1}]}"] & P \arrow[d, "{[q_n]}"] \\
X(n+1) \arrow[r, "{[f_n]}"] & X(n)
\end{tikzcd} \tag{7.70}$$

commutes in the homotopy category $\mathrm{h}\mathcal{C}$. *Moreover, the morphism f is uniquely determined up to homotopy.*

(2) *An object of $\mathcal{C}$ is a limit of the tower X if and only if it is an equalizer of the pair of morphisms $f, \mathrm{id}_P : P \to P$.*

Proof. Assertion (1) follows immediately from the definitions (see Warning 7.6.1.11). To prove (2), let $M = \mathbf{Z}_{\geq 0}$ denote the set of nonnegative integers, which we regard as a commutative monoid with respect to addition. Let BM denote the associated category (consisting of a single object E having endomorphism monoid $\mathrm{Hom}_{BM}(E, E) = M$) and let $B_\bullet M$ denote the nerve of BM (Construction 1.3.2.5). There is a functor of ordinary categories $(\mathbf{Z}_{\geq 0}, \leq)^{\mathrm{op}} \to BM$ which is characterized by the requirement that, for every pair of nonnegative integers $m \leq n$, the induced map

$$\mathrm{Hom}_{\mathbf{Z}_{\geq 0}}(m, n) \to \mathrm{Hom}_{BM}(E, E) = M$$

carries the unique element of $\mathrm{Hom}_{\mathbf{Z}_{\geq 0}}(m, n)$ to the difference $n - m \in M$. Passing to nerves, we obtain a functor of ∞-categories $U : \mathrm{N}_\bullet(\mathbf{Z}_{\geq 0}^{\mathrm{op}}) \to B_\bullet M$. The functor U is a cartesian fibration, whose fiber over the vertex $E \in B_\bullet M$ can be identified with the discrete simplicial set $\{0, 1, 2, \cdots\}$. Applying Corollary 7.3.4.8, we deduce that there exists a functor $Y : B_\bullet M \to \mathcal{C}$ and a natural transformation $\alpha : Y \circ U \to X$ which exhibits Y as a right Kan extension of X along U.

For every nonnegative integer n, α induces a morphism $\alpha_n : Y(E) \to X(n)$ in the ∞-category $\mathcal{C}$. Using the criterion of Proposition 7.3.4.1, we see that the collection of morphisms $\{\alpha_n\}_{n \geq 0}$ exhibit $Y(E)$ as a product of the collection of objects $\{X(n)\}_{n \geq 0}$. We may therefore assume without loss of generality that $P = Y(E)$ and $q_n = \alpha_n$, for each $n \geq 0$. Let $f : P \to P$ be the morphism obtained by evaluating the functor Y on the generator $1 \in M$. For each $n \geq 0$, the natural transformation α carries the edge $n + 1 \to n$ of $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0})$

to a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "q_{n+1}"] & P \arrow[d, "q_n"] \\
X(n+1) \arrow[r, "f_n"] & X(n)
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, which witnesses the commutativity of the diagram (7.70) in the homotopy category $\mathrm{h}\mathcal{C}$. Moreover, an object $C \in \mathcal{C}$ is an equalizer of the pair of morphisms $f, \mathrm{id}_P : P \to P$ if and only if it is a limit of the diagram Y (Variant 7.6.5.9). To prove (2), it suffices to observe that this is equivalent to the requirement that C is a limit of the tower X, which follows from Corollary 7.3.8.20. □

Remark 7.6.6.17. In the situation of Proposition 7.6.6.16, suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories which preserves the product of the collection $\{X(n)\}_{n \geq 0}$. Then F preserves limits of the tower X if and only if it preserves equalizers of the pair of morphisms $f, \mathrm{id}_P : P \to P$. 03J2

7.6.7 Small Limits

We now study limits and colimits indexed by diagrams of bounded size. 03UC

Definition 7.6.7.1. Let κ be an infinite cardinal and let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *κ-complete* if admits K-indexed limits, for every κ-small simplicial set K. 03UD

We say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *preserves κ-small limits* if it preserves K-indexed limits, for every κ-small simplicial set K (Definition 7.1.3.4).

Remark 7.6.7.2. Let $\mathcal{C}$ be an ∞-category and let λ be an infinite cardinal. If $\mathcal{C}$ is λ-complete, then it is also κ-complete for each infinite cardinal $\kappa < \lambda$. Similarly, if a functor $F : \mathcal{C} \to \mathcal{D}$ preserves λ-small limits, then it also preserves κ-small limits for each $\kappa < \lambda$. In both cases, the converse holds if λ is an uncountable limit cardinal (since, in that case, every λ-small simplicial set K is κ-small for some $\kappa < \lambda$). 03UE

Example 7.6.7.3. An ∞-category $\mathcal{C}$ is $\aleph_0$-complete if and only if it admits finite limits. Similarly, a functor $F : \mathcal{C} \to \mathcal{D}$ preserves $\aleph_0$-small limits if and only if it preserves finite limits. 03UF

Example 7.6.7.4. Let λ be an uncountable regular cardinal and let $\kappa = \mathrm{ecf}(\lambda)$ be its exponential cofinality (Definition 4.7.3.16). Let $\mathcal{S}^{<\lambda}$ denote the ∞-category of λ-small spaces (Variant 5.5.4.12) and let $\mathcal{QC}^{<\lambda}$ denote the ∞-category of λ-small ∞-categories (Variant 03UG

5.5.4.10). Then the ∞-categories $\mathcal{S}^{<\lambda}$ and $\mathcal{QC}^{<\lambda}$ are κ-complete. Moreover, the inclusion maps

$$\mathcal{S}^{<\lambda} \hookrightarrow \mathcal{S} \qquad \mathcal{QC}^{<\lambda} \hookrightarrow \mathcal{QC}$$

preserve κ-small limits. See Corollary 7.4.1.13 and Variant 7.4.5.8. In particular, if $\kappa = \lambda$ is a strongly inaccessible cardinal, then the ∞-categories $\mathcal{S}^{<\kappa}$ and $\mathcal{QC}^{<\kappa}$ are κ-complete.

048G **Remark 7.6.7.5.** Let $\mathcal{C}$ be an ∞-category which is κ-complete for some infinite cardinal κ. Then, for every simplicial set K, the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ is also κ-complete. See Corollary 7.1.6.2.

03UH **Remark 7.6.7.6.** Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:

- The ∞-category $\mathcal{C}$ is κ-complete: that is, it admits K-indexed limits for every κ-small simplicial set K.
- The ∞-category $\mathcal{C}$ admits K-indexed limits, for every simplicial set K which is essentially κ-small.

Moreover, in either case, it suffices to consider the case where K is an ∞-category. See Remark 7.1.1.15 and Proposition 4.7.5.5. Similarly, a functor $F : \mathcal{C} \to \mathcal{D}$ preserves κ-small limits if and only if it preserves K-indexed limits, for every simplicial set K which is essentially κ-small (and it again suffices to consider the case where K is an ∞-category).

03UJ **Variant 7.6.7.7.** Let κ be an infinite cardinal. We say that an ∞-category $\mathcal{C}$ is *κ-cocomplete* if it admits K-indexed colimits, for every κ-small simplicial set K. Equivalently, the ∞-category $\mathcal{C}$ is κ-cocomplete if the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ is κ-complete.

We say that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ *preserves κ-small colimits* if it preserves K-indexed colimits, for every κ-small simplicial set K. Equivalently, F preserves κ-small colimits if the opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ preserves κ-small limits.

03UK **Example 7.6.7.8.** Let λ be an uncountable regular cardinal and let $\kappa = \mathrm{cf}(\lambda)$ denote its cofinality. Then the ∞-categories $\mathcal{S}^{<\lambda}$ and $\mathcal{QC}^{<\lambda}$ are κ-cocomplete. Moreover, the inclusion maps

$$\mathcal{S}^{<\lambda} \hookrightarrow \mathcal{S} \qquad \mathcal{QC}^{<\lambda} \hookrightarrow \mathcal{QC}$$

preserve κ-small colimits. See Corollary 7.4.3.15 and Remark 7.4.5.7. In particular, if $\kappa = \lambda$ is an uncountable regular cardinal, then the ∞-categories $\mathcal{S}^{<\kappa}$ and $\mathcal{QC}^{<\kappa}$ are κ-cocomplete.

03UL **Proposition 7.6.7.9.** *Let $\mathcal{C}$ be an ∞-category and let κ be an infinite cardinal. Then $\mathcal{C}$ is κ-complete if and only if it satisfies the following conditions:*

(1) *The ∞-category $\mathcal{C}$ admits κ-small products. That is, every collection of objects $\{X_j\}_{j \in J}$ indexed by a κ-small set J admits a product in $\mathcal{C}$.*

(2) *The ∞-category $\mathcal{C}$ admits finite limits.*

Proof. Assume that $\mathcal{C}$ satisfies conditions (1) and (2); we wish to show that $\mathcal{C}$ is κ-complete (the converse is immediate from the definitions). Let S be a κ-small simplicial set; we wish to show that $\mathcal{C}$ admits S-indexed limits. If $\kappa = \aleph_0$, this follows immediately from assumption (2) (Example 7.6.7.3). We may therefore assume that κ is uncountable, so that $\mathcal{C}$ admits countable products.

For each $n \geq 0$, let $\mathrm{sk}_n(S)$ denote the n-skeleton of S (Construction 1.1.4.1), so that $S = \bigcup_n \mathrm{sk}_n(S)$. It follows from Proposition 7.6.6.16 that $\mathcal{C}$ admits sequential limits. Consequently, to show that $\mathcal{C}$ admits S-indexed limits, it will suffice to show that it admits $\mathrm{sk}_n(S)$-indexed limits, for each $n \geq 0$ (Corollary 7.6.6.14). We may therefore assume without loss of generality that the simplicial set S has finite dimension. We proceed by induction on the dimension n of S. If $n = -1$, then S is empty and the desired result is immediate. Assume that $n \geq 0$ and let $\{\sigma_j\}_{j \in J}$ denote the collection of nondegenerate n-simplices of S, so that Proposition 1.1.4.12 supplies a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{j \in J} \partial \Delta^n \arrow[r] \arrow[d] & \coprod_{j \in J} \Delta^n \arrow[d] \\
\mathrm{sk}_{n-1}(S) \arrow[r] & \mathrm{sk}_n(S).
\end{tikzcd}$$

Since the horizontal maps in this diagram are monomorphisms, it is also a categorical pushout square (Example 4.5.4.12). By virtue of Proposition 7.6.3.26, it will suffice to show that $\mathcal{C}$ admits limits indexed by the simplicial sets $\mathrm{sk}_{n-1}(S)$, $J \times \partial \Delta^n$, and $J \times \Delta^n$. In the first two cases, this follows from our inductive hypothesis. To handle the third case, we can use assumption (1) and Corollary 7.6.1.20 to reduce to showing that the ∞-category $\mathcal{C}$ admits Δ^n-indexed limits. This is clear, since the simplicial set Δ^n is an ∞-category containing an initial object (see Corollary 7.2.2.12). □

Remark 7.6.7.10. In the situation of Proposition 7.6.7.9, we can replace (2) by either of the following *a priori* weaker conditions: 03UM

(2′) The ∞-category $\mathcal{C}$ admits pullbacks.

(2″) The ∞-category $\mathcal{C}$ admits equalizers.

See Corollary 7.6.3.27 and 7.6.5.25.

03J4 **Exercise 7.6.7.11.** Let κ be an infinite cardinal, let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and suppose that $\mathcal{C}$ is κ-complete. Show that F preserves κ-small limits if and only if it preserves finite limits and κ-small products.

03UN **Corollary 7.6.7.12.** *Let $\mathcal{C}$ be an ∞-category and let λ be an infinite cardinal which is not regular. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is λ-complete.*

(2) *For every infinite cardinal $\kappa < \lambda$, the ∞-category $\mathcal{C}$ is κ-complete.*

(3) *The ∞-category $\mathcal{C}$ is λ^+-complete, where λ^+ denotes the successor cardinal of λ.*

By virtue of Corollary 7.6.7.12, little information is lost by restricting the use of Definition 7.6.7.1 to the case where κ is a regular cardinal.

Proof of Corollary 7.6.7.12. The equivalence (1) $\Leftrightarrow$ (2) and the implication (3) $\Rightarrow$ (1) follow from Remark 7.6.7.2. We will complete the proof by showing that (1) implies (3). Assume that $\mathcal{C}$ is λ-complete; we wish to show that it is λ^+-complete. By virtue of Proposition 7.6.7.9, it will suffice to show that every collection of objects $\{X_i\}_{i \in I}$ admits a product in $\mathcal{C}$, provided that the index set I has cardinality $\leq \lambda$. Our assumption that λ is not regular guarantees that we can decompose I as a disjoint union of λ-small subsets $\{I_j \subseteq I\}_{j \in J}$, where the index set J is λ-small. It follows from (1) that $\mathcal{C}$ admits J-indexed products and also that it admits I_j-indexed products for each $j \in J$, and therefore admits I-indexed products by virtue of Corollary 7.6.1.20. $\square$

The existence of κ-small limits can be used to prove the existence of a large class of Kan extensions.

03UP **Proposition 7.6.7.13.** *Let κ be an uncountable regular cardinal and let K be a simplicial set which is essentially κ-small. Suppose we are given a pair of ∞-categories $\mathcal{C}$ and $\mathcal{D}$, together with diagrams $\delta : K \to \mathcal{C}$ and $F_0 : K \to \mathcal{D}$. Suppose that $\mathcal{C}$ is locally κ-small and that $\mathcal{D}$ is κ-complete. Then F_0 admits a right Kan extension along δ.*

Proof. By virtue of Proposition 7.3.5.1, it will suffice to show that for every object $C \in \mathcal{C}$, the composite map

$$K \times_{\mathcal{C}} \mathcal{C}_{C/} \to K \xrightarrow{F_0} \mathcal{D}$$

admits a limit in the ∞-category $\mathcal{D}$. Note that the projection map $K \times_{\mathcal{C}} \mathcal{C}_{C/}$ is a left fibration of simplicial sets (Proposition 4.3.6.1), whose fiber over each vertex $x \in K$ can be identified with the Kan complex $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(C, \delta(x))$. Invoking Proposition 4.6.5.10, we see that $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(C, \delta(x)$ is homotopy equivalent to the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, \delta(x))$, and is therefore essentially κ-small (by virtue of our assumption that $\mathcal{C}$ is locally κ-small). Since K

is essentially κ-small, Corollary 5.6.7.7 implies that the simplicial set $K \times_{\mathcal{C}} \mathcal{C}_{C/}$ is essentially κ-small. The desired result now follows from our assumption that $\mathcal{D}$ is κ-complete (Remark 7.6.7.6). □

Chapter 8

The Yoneda Embedding

03JA 8.1 Twisted Arrows and Cospans

03JB Let $\mathcal{C}$ be an ∞-category. In §4.6.1, we associated to every pair of objects $X, Y \in \mathcal{C}$ a Kan complex $\operatorname{Hom}_{\mathcal{C}}(X, Y)$, whose vertices are morphisms from X to Y. In §8.3.3, we will see that the construction $(X, Y) \mapsto \operatorname{Hom}_{\mathcal{C}}(X, Y)$ can be refined to a functor of ∞-categories

$$\operatorname{Hom}_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}\mathrm{et} = \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}).$$

It is somewhat cumbersome to give an explicit description of this functor. It will therefore be more convenient to specify it *implicitly* by realizing it as the covariant transport representation of a left fibration over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. We begin by discussing the counterpart of this fibration in the setting of classical category theory.

00AZ **Construction 8.1.0.1** (The Twisted Arrow Category)**.** Let $\mathcal{C}$ be a category. We define a new category $\operatorname{Tw}(\mathcal{C})$ as follows:

- An object of $\operatorname{Tw}(\mathcal{C})$ is a morphism $f : X \to Y$ in $\mathcal{C}$.
- Let $f : X \to Y$ and $f' : X' \to Y'$ be objects of $\operatorname{Tw}(\mathcal{C})$. A morphism from f to f' in $\operatorname{Tw}(\mathcal{C})$ is a pair of morphisms $u : X' \to X$, $v : Y \to Y'$ in $\mathcal{C}$ satisfying $f' = v \circ f \circ u$, so that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & X' \arrow[l, "u"'] \arrow[d, "f'"] \\
Y \arrow[r, "v"] & Y'.
\end{tikzcd}$$

- Let $f : X \to Y$, $f' : X' \to Y'$, and $f'' : X'' \to Y''$ be objects of $\mathrm{Tw}(\mathcal{C})$. If (u, v) is a morphism from f to f' in $\mathrm{Tw}(\mathcal{C})$ and (u', v') is a morphism from f' to f'' in $\mathcal{C}$, then the composition $(u', v') \circ (u, v)$ in $\mathrm{Tw}(\mathcal{C})$ is the pair $(u \circ u', v' \circ v)$.

We will refer to $\mathrm{Tw}(\mathcal{C})$ as the *twisted arrow category* of $\mathcal{C}$.

Remark 8.1.0.2. Let $\mathcal{C}$ be a category. Then the construction $(f : X \to Y) \mapsto (X, Y)$ determines a forgetful functor $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Moreover, λ is a left covering functor, in the sense of Definition 4.2.3.1. `00B1`

Remark 8.1.0.3 ($\mathrm{Tw}(\mathcal{C})$ as a Category of Elements)**.** Let $\mathcal{C}$ be a category. Then the construction $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$ determines a functor $\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$. The twisted arrow category $\mathrm{Tw}(\mathcal{C})$ of Construction 8.1.0.1 can be identified with the category of elements $\int_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$ (see Construction 5.2.6.1). `03JC`

It follows that the functor $\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet)$ is determined (up to canonical isomorphism) by the datum of the twisted arrow category $\mathrm{Tw}(\mathcal{C})$ together with the forgetful functor $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ of Remark 8.1.0.2 (see Corollary 5.2.7.5).

Warning 8.1.0.4 (Untwisted Arrow Categories)**.** Let $[1] = \{0 < 1\}$ denote a linearly ordered set with two elements. For any category $\mathcal{C}$, we can identify morphisms of $\mathcal{C}$ with functors $F : [1] \to \mathcal{C}$. The collection of such functors can be organized into a category $\mathrm{Fun}([1], \mathcal{C})$, which we refer to as the *arrow category* of $\mathcal{C}$. The arrow category $\mathrm{Fun}([1], \mathcal{C})$ has the same objects as the twisted arrow category $\mathrm{Tw}(\mathcal{C})$. However, the morphisms are different: if $f : X \to Y$ and $f' : X' \to Y'$ are morphisms of $\mathcal{C}$, then morphisms from f to f' in $\mathrm{Fun}([1], \mathcal{C})$ can be identified with commutative diagrams `03JD`

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y \arrow[r] & Y',
\end{tikzcd}$$

where the horizontal maps are oriented in the same direction.

Example 8.1.0.5. Let Q be a partially ordered set, which we regard as a category. Then the twisted arrow category $\mathrm{Tw}(Q)$ can be identified (via the forgetful functor of Remark 8.1.0.2) with the partially ordered set `00B2`

$$\{(p, q) \in Q^{\mathrm{op}} \times Q : p \leq q\} \subseteq Q^{\mathrm{op}} \times Q.$$

Remark 8.1.0.6. Let $\mathcal{C}$ be a category. For every object $X \in \mathcal{C}$, the fiber $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ can be identified with the coslice category $\mathcal{C}_{X/}$ of Variant 4.3.1.4. Similarly, the fiber $\mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{X\}$ can be identified with the *opposite* of the slice category $\mathcal{C}_{/X}$ of Construction 4.3.1.1. `03JE`

In §8.1.1, we generalize Construction 8.1.0.1 to the setting of ∞-categories. To every simplicial set $\mathcal{C}$, we associate another simplicial set $\mathrm{Tw}(\mathcal{C})$, whose n-simplices can be identified with $(2n+1)$-simplices of $\mathcal{C}$ (Construction 8.1.1.1). This construction has the following features:

- If $\mathcal{C} = \mathrm{N}_{\bullet}(\mathcal{C}_0)$ is (the nerve of) an ordinary category $\mathcal{C}_0$, then $\mathrm{Tw}(\mathcal{C})$ can be identified with the nerve of the twisted arrow category $\mathrm{Tw}(\mathcal{C}_0)$ (Proposition 8.1.1.10). Consequently, the twisted arrow construction of §8.1.1 can be regarded as a generalization of Construction 8.1.0.1.

- The simplicial set $\mathrm{Tw}(\mathcal{C})$ is equipped with a projection map $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. If $\mathcal{C}$ is an ∞-category, then λ is a left fibration (Proposition 8.1.1.11); in particular, $\mathrm{Tw}(\mathcal{C})$ is also an ∞-category (Corollary 8.1.1.12).

Let $\mathcal{C}$ be an ∞-category. In §8.1.2, we study the fibers of the left fibration $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Our main result asserts that if $f : X \to Y$ is an isomorphism in the ∞-category $\mathcal{C}$, then f is initial when viewed as an object of the ∞-category $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ (see Proposition 8.1.2.1, and Corollary 8.1.2.21 for the converse). From this, we deduce an analogue of Remark 8.1.0.6: there is a canonical equivalence of ∞-categories $\mathcal{C}_{X/} \hookrightarrow \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ (Proposition 8.1.2.9), which induces a homotopy equivalence of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}$$

for each object $Y \in \mathcal{C}$ (Notation 8.1.2.14). Moreover, we show that these homotopy equivalences are compatible with covariant transport for the left fibration λ (Corollary 8.1.2.18).

The twisted arrow construction $S \mapsto \mathrm{Tw}(S)$ determines a functor from the category of simplicial sets to itself. In particular, to every simplicial set T we can associate a new simplicial set $\mathrm{Cospan}(T)$, whose n-simplices are given by maps $\mathrm{Tw}(\Delta^n) \to T$. We will refer to $\mathrm{Cospan}(T)$ as the *simplicial set of cospans in* T (Construction 8.1.3.1). This construction has the following features:

- The construction $T \mapsto \mathrm{Cospan}(T)$ determines a functor from the category of simplicial sets to itself, which is right adjoint to the twisted arrow functor $S \mapsto \mathrm{Tw}(S)$ (Corollary 8.1.3.8).

- Let $\mathcal{C}$ be an ordinary category which admits pushouts, and let $\mathrm{Cospan}(\mathcal{C})$ denote the 2-category of cospans in $\mathcal{C}$ (Example 2.2.2.1). Then there is a canonical isomorphism of simplicial sets
$$\mathrm{Cospan}(\mathrm{N}_{\bullet}(\mathcal{C})) \xrightarrow{\sim} \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathrm{Cospan}(\mathcal{C})),$$
which we construct in §8.1.3 (see Corollary 8.1.3.15).

- Let $\mathcal{C}$ be a 2-category containing a pair of objects X and Y, and let $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ denote the category of 1-morphisms from X to Y. Then there is a canonical isomorphism of simplicial sets
$$\mathrm{Cospan}(\mathrm{N}_{\bullet}\, \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X, Y),$$
which we construct in §8.1.8 (see Corollary 8.1.8.6).

- If $\mathcal{C}$ is an ∞-category which admits pushouts, then the simplicial set $\mathrm{Cospan}(\mathcal{C})$ is an $(\infty, 2)$-category (Proposition 8.1.4.1). We prove this in §8.1.4 using an explicit characterization of the collection of thin 2-simplices of $\mathrm{Cospan}(\mathcal{C})$ (Proposition 8.1.4.2), which we prove in §8.1.5.

8.1.1 The Twisted Arrow Construction

We now describe an ∞-categorical generalization of Construction 8.1.0.1. 03JF

Construction 8.1.1.1 (Twisted Arrows in Simplicial Sets)**.** Let $\boldsymbol{\Delta}$ denote the simplex category (Definition 1.1.0.2) and let $\mathcal{C}$ be a simplicial set. We let $\mathrm{Tw}(\mathcal{C}) : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathrm{Set}$ denote the functor given by the construction 03JG

$$(J \in \boldsymbol{\Delta}^{\mathrm{op}}) \mapsto \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{N}_{\bullet}(J^{\mathrm{op}} \star J), \mathcal{C}).$$

We will refer to $\mathrm{Tw}(\mathcal{C})$ as the *simplicial set of twisted arrows of* $\mathcal{C}$.

Remark 8.1.1.2. For every integer $n \geq 0$, there is a unique isomorphism of simplicial sets $\mathrm{N}_{\bullet}([n]^{\mathrm{op}} \star [n]) \simeq \Delta^{2n+1}$. It follows that, for every simplicial set $\mathcal{C}$, we can identify n-simplices σ of $\mathrm{Tw}(\mathcal{C})$ with $(2n+1)$-simplices $\overline{\sigma}$ of $\mathcal{C}$. In terms of these identifications, the face and degeneracy operators of $\mathrm{Tw}(\mathcal{C})$ are given explicitly by the formulae 03JH

$$\overline{d^n_i \sigma} = d^{2n}_{n-i} d^{2n+1}_{n+1+i} \overline{\sigma} \qquad \overline{s^n_i \sigma} = s^{2n+2}_{n-i} s^{2n+1}_{n+1+i} \overline{\sigma}.$$

Remark 8.1.1.3. Let $\mathcal{C}$ be a simplicial set. We will generally use Remark 8.1.1.2 to identify vertices of the simplicial set $\mathrm{Tw}(\mathcal{C})$ with edges $f : X \to Y$ of $\mathcal{C}$. More generally, it will be useful to think of n-simplices of $\mathrm{Tw}(\mathcal{C})$ as encoding diagrams 03JJ

$$\begin{tikzcd}
X_0 \arrow[d, "f_0"] & X_1 \arrow[l] \arrow[d, "f_1"] & X_2 \arrow[l] \arrow[d, "f_2"] & \cdots \arrow[l] \arrow[d] & X_n \arrow[l] \arrow[d, "f_n"] \\
Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_n.
\end{tikzcd}$$

Remark 8.1.1.4. The construction $\mathcal{C} \mapsto \mathrm{Tw}(\mathcal{C})$ determines a functor from the category of simplicial sets to itself, which preserves all limits and colimits (this follows from Remark 8.1.1.2, since limits and colimits in the category Set_{Δ} are computed levelwise). 00B3

03Y5 **Remark 8.1.1.5.** Let κ be an uncountable cardinal. If $\mathcal{C}$ is a κ-small simplicial set, then $\mathrm{Tw}(\mathcal{C})$ is also κ-small. To prove this, we may assume without loss of generality that κ is the smallest uncountable cardinal for which $\mathcal{C}$ is κ-small. In particular, κ is regular. It will therefore suffice to show that, for every integer n, the set of n-simplices of $\mathrm{Tw}(\mathcal{C})$ is κ-small (Proposition 4.7.4.9). This follows from the κ-smallness of the set of $(2n+1)$-simplices of $\mathcal{C}$ (Remark 8.1.1.2).

03JK **Notation 8.1.1.6** (Projection Maps)**.** Let $\mathcal{C}$ be a simplicial set. Then the simplicial set $\mathrm{Tw}(\mathcal{C})$ is equipped with projection maps

$$\lambda_- : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \qquad \lambda_+ : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}.$$

Here λ_+ carries an n-simplex σ of $\mathrm{Tw}(\mathcal{C})$ to the n-simplex of $\mathcal{C}$ given by the composition

$$\Delta^n = \mathrm{N}_\bullet([n]) \hookrightarrow \mathrm{N}_\bullet([n]^{\mathrm{op}} \star [n]) \xrightarrow{\sigma} \mathcal{C},$$

while λ_- carries σ to the n-simplex of $\mathcal{C}^{\mathrm{op}}$ given by the composite map

$$(\Delta^n)^{\mathrm{op}} = \mathrm{N}_\bullet([n]^{\mathrm{op}}) \hookrightarrow \mathrm{N}_\bullet([n]^{\mathrm{op}} \star [n]) \xrightarrow{\sigma} \mathcal{C}.$$

Concretely, λ_- and λ_+ are given on vertices by the formulae $\lambda_-(f : X \to Y) = X$ and $\lambda_+(f : X \to Y) = Y$.

03JL **Remark 8.1.1.7** (Duality)**.** Let $\mathcal{C}$ be a simplicial set. Then there is a canonical isomorphism of simplicial sets $\iota : \mathrm{Tw}(\mathcal{C}) \simeq \mathrm{Tw}(\mathcal{C}^{\mathrm{op}})$, given on n-simplices by precomposition with the unique isomorphism

$$\mathrm{N}_\bullet([n]^{\mathrm{op}} \star [n])^{\mathrm{op}} \simeq \mathrm{N}_\bullet([n]^{\mathrm{op}} \star [n]).$$

The isomorphism ι interchanges the projection maps λ_- and λ_+ of Notation 8.1.1.6.

03JM **Exercise 8.1.1.8** (Slices of Twisted Arrows)**.** Let $\mathcal{C}$ be a simplicial set and let $f : X \to Y$ be an edge of $\mathcal{C}$, which we regard as a vertex of the simplicial set $\mathrm{Tw}(\mathcal{C})$. Show that there is a canonical isomorphism of simplicial sets

$$\mathrm{Tw}(\mathcal{C})_{/f} \simeq \mathrm{Tw}(\mathcal{C}_{X/\,/Y}).$$

Here $\mathcal{C}_{X/\,/Y}$ denotes the simplicial set $(\mathcal{C}_{X/})_{/Y} \simeq (\mathcal{C}_{/Y})_{X/}$, obtained either by promoting Y to a vertex of $\mathcal{C}_{X/}$ or X to a vertex of $\mathcal{C}_{/Y}$ by means of the edge f (see Remark 4.6.6.2).

03LW **Warning 8.1.1.9.** Let $\mathcal{C}$ be a simplicial set. Then there is a tautological map $T : \mathcal{C} \to \mathrm{Tw}(\mathcal{C}^{\mathrm{op}} \star \mathcal{C})$, which carries an n-simplex $\sigma : \Delta^n \to \mathcal{C}$ to the n-simplex $T(\sigma)$ of $\mathrm{Tw}(\mathcal{C}^{\mathrm{op}} \star \mathcal{C})$ given by the composition

$$\mathrm{N}_\bullet([n]^{\mathrm{op}} \star [n]) \xrightarrow{\sigma^{\mathrm{op}} \star \sigma} \mathcal{C}^{\mathrm{op}} \star \mathcal{C}.$$

If $\mathcal{D}$ is another simplicial set, then precomposition with T induces a comparison map

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}^{\mathrm{op}} \star \mathcal{C}, \mathcal{D}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Tw}(\mathcal{C}^{\mathrm{op}} \star \mathcal{C}), \mathrm{Tw}(\mathcal{D})) \xrightarrow{\circ T} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{C}, \mathrm{Tw}(\mathcal{D})).$$

Beware that, in general, this map is not a bijection. However, it is a bijection whenever $\mathcal{C}$ is isomorphic to the nerve of a linearly ordered set Q. To prove this, we can write Q as a filtered colimit of its finite subsets and thereby reduce to the case where Q is finite. In this case, the linearly ordered set Q is either empty (in which case the desired result is obvious) or isomorphic to $[n]$ for some integer $n \geq 0$ (in which case the desired result follows from the definition of the $\mathrm{Tw}(\mathcal{D})$).

We now show that Construction 8.1.1.1 can be regarded as a generalization of Construction 8.1.0.1.

Proposition 8.1.1.10. *Let $\mathcal{C}$ be a category. Then there is a canonical isomorphism of simplicial sets $T : \mathrm{N}_\bullet(\mathrm{Tw}(\mathcal{C})) \xrightarrow{\sim} \mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C}))$, which is uniquely determined by the following requirements:* 03JN

(1) *For every morphism $f : C \to D$ in the category $\mathcal{C}$, the map T carries f (regarded as an object of $\mathrm{Tw}(\mathcal{C})$) to itself (regarded as a vertex of $\mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C}))$).*

(2) *The diagram*

$$\begin{tikzcd}
\mathrm{N}_\bullet(\mathrm{Tw}(\mathcal{C})) \arrow[r, "T"] \arrow[d] & \mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C})) \arrow[d, "(\lambda_-, \lambda_+)"] \\
\mathrm{N}_\bullet(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \arrow[r, "\sim"] & \mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}} \times \mathrm{N}_\bullet(\mathcal{C})
\end{tikzcd}$$

commutes, where the right vertical map is given by Notation 8.1.1.6 and the left vertical map is the nerve of the forgetful functor

$$\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \qquad (f : X \to Y) \mapsto (X, Y).$$

Proof. Let σ be an n-simplex of the simplicial set $\mathrm{N}_\bullet(\mathrm{Tw}(\mathcal{C}))$, which we identify with a diagram

$$(f_0 : X_0 \to Y_0) \xrightarrow{(u_1, v_1)} (f_1 : X_1 \to Y_1) \xrightarrow{(u_2, v_2)} \cdots \xrightarrow{(u_n, v_n)} (f_n : X_n \to Y_n)$$

in the category $\mathrm{Tw}(\mathcal{C})$. Here each $f_i : X_i \to Y_i$ denotes a morphism in $\mathcal{C}$, and each (u_i, v_i) is a pair of morphisms in $\mathcal{C}$ which determine a commutative diagram

$$\begin{tikzcd}
X_{i-1} \arrow[d, "f_{i-1}"] & X_i \arrow[l, "u_i"'] \arrow[d, "f_i"] \\
Y_{i-1} \arrow[r, "v_i"] & Y_i.
\end{tikzcd}$$

In this case, we can regard the chain of morphisms

03JP
$$\begin{tikzcd}
X_0 \arrow[d, "f_0"] & X_1 \arrow[l, "u_1"'] & X_2 \arrow[l, "u_2"'] & \cdots \arrow[l] & X_n \arrow[l, "u_n"'] \\
Y_0 \arrow[r, "v_1"] & Y_1 \arrow[r, "v_2"] & Y_2 \arrow[r] & \cdots \arrow[r, "v_n"] & Y_n
\end{tikzcd} \tag{8.1}$$

as a $(2n+1)$-simplex of $\mathrm{N}_\bullet(\mathcal{C})$, which we identify with an n-simplex $T(\sigma)$ of $\mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C}))$. The construction $\sigma \mapsto T(\sigma)$ then determines a morphism of simplicial sets $T : \mathrm{N}_\bullet(\mathrm{Tw}(\mathcal{C})) \to \mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C}))$, which satisfies conditions (1) and (2) by construction.

We now claim that T is an isomorphism of simplicial sets. Let τ be an n-simplex of $\mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C}))$; we wish to show that there is a unique n-simplex σ of $\mathrm{N}_\bullet(\mathrm{Tw}(\mathcal{C}))$ satisfying $T(\sigma) = \tau$. Let us identify τ with a diagram of the form (8.1) in the category $\mathcal{C}$. We wish to show that there is a unique collection of morphisms $\{f_i : X_i \to Y_i\}_{1 \leq i \leq n}$ satisfying the identities $f_i = v_i \circ f_{i-1} \circ u_i$, which follows immediately by induction on i.

We now complete the proof by establishing the uniqueness of T. Suppose that $T' : \mathrm{N}_\bullet(\mathrm{Tw}(\mathcal{C})) \xrightarrow{\sim} \mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C}))$ is another morphism of simplicial sets satisfying conditions (1) and (2). Then $T^{-1} \circ T'$ determines a functor F from the twisted arrow category $\mathrm{Tw}(\mathcal{C})$ to itself. Because T and T' both satisfy condition (1), the functor F carries each object of $\mathrm{Tw}(\mathcal{C})$ to itself. Since the forgetful functor $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is faithful, condition (2) guarantees that F also carries each morphism of $\mathrm{Tw}(\mathcal{C})$ to itself. It follows that F is the identity functor, so that $T' = T$. □

Let $\mathcal{C}$ be a simplicial set. It follows from Proposition 8.1.1.10 that if $\mathcal{C}$ is isomorphic to the nerve of a category, then the simplicial set $\mathrm{Tw}(\mathcal{C})$ is also isomorphic to the nerve of a category. Moreover, the projection maps of Notation 8.1.1.6 determine a left covering map $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ (see Remark 8.1.0.2). This observation has an ∞-categorical counterpart:

03JQ **Proposition 8.1.1.11.** *Let $\mathcal{C}$ be an ∞-category. Then the projection maps of Notation 8.1.1.6 determine a left fibration of simplicial sets*

$$(\lambda_-, \lambda_+) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}\,.$$

Corollary 8.1.1.12. *Let $\mathcal{C}$ be an ∞-category. Then the simplicial set* $\mathrm{Tw}(\mathcal{C})$ *is also an ∞-category.* 03JR

Proof. Combine Proposition 8.1.1.11 with Remark 4.1.1.9. □

Corollary 8.1.1.13. *Let $\mathcal{C}$ be a Kan complex. Then the projection map* $(\lambda_-, \lambda_+) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ *is a Kan fibration. In particular,* $\mathrm{Tw}(\mathcal{C})$ *is a Kan complex.* 048H

Proof. Combine Proposition 8.1.1.11 with Corollary 4.4.3.8. □

In the situation of Corollary 8.1.1.12, we will refer to $\mathrm{Tw}(\mathcal{C})$ as the *twisted arrow ∞-category of $\mathcal{C}$.*

Corollary 8.1.1.14. *Let $\mathcal{C}$ be an ∞-category. Then the projection maps* $\lambda_- : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ *and* $\lambda_+ : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}$ *are cocartesian fibrations of ∞-categories. Moreover, a morphism f of* $\mathrm{Tw}(\mathcal{C})$ *is λ_--cocartesian if and only if $\lambda_+(f)$ is an isomorphism, and λ_+-cocartesian if and only if $\lambda_-(f)$ is an isomorphism.* 03JS

Proof. Let $\pi_- : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and $\pi_+ : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ denote the projection maps. Then π_- and π_+ are cocartesian fibrations of simplicial sets. Moreover, a morphism (e_-, e_+) of $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is π_--cocartesian if and only if e_+ is an isomorphism in $\mathcal{C}$, and π_+-cocartesian if and only if e_- is an isomorphism in $\mathcal{C}^{\mathrm{op}}$ (this follows immediately from Remark 5.1.4.6 and Example 5.1.1.4). Corollary 8.1.1.14 now follows by applying Proposition 8.1.1.11 to left and right sides of the diagram

$$\begin{tikzcd}
 & \mathrm{Tw}(\mathcal{C}) \arrow[dl, "\lambda_-"'] \arrow[d, "(\lambda_-, \lambda_+)"] \arrow[dr, "\lambda_+"] & \\
\mathcal{C}^{\mathrm{op}} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[l, "\pi_-"'] \arrow[r, "\pi_+"] & \mathcal{C},
\end{tikzcd}$$

since the vertical map in the center is a left fibration (Proposition 8.1.1.11). □

Proposition 8.1.1.11 is a special case of the following more general assertion:

Proposition 8.1.1.15. *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets. Then the projection maps of Notation 8.1.1.6 determine a left fibration of simplicial sets* 03JT

$$\mathrm{Tw}(\mathcal{C}) \to (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \times_{(\mathcal{D}^{\mathrm{op}} \times \mathcal{D})} \mathrm{Tw}(\mathcal{D}).$$

Proof. Fix a pair of integers $0 < i \leq n$; we wish to show that every lifting problem

03JU
$$\begin{tikzcd}
\Lambda^n_{n-i} \arrow[r] \arrow[d] & \mathrm{Tw}(\mathcal{C}) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \times_{(\mathcal{D}^{\mathrm{op}} \times \mathcal{D})} \mathrm{Tw}(\mathcal{D})
\end{tikzcd} \tag{8.2}$$

admits a solution.

For each nonempty subset $S \subseteq [2n+1] = \{0 < 1 < \cdots < 2n+1\}$, let σ_S denote the corresponding nondegenerate simplex of Δ^{2n+1}. Let us say that S is *basic* if it satisfies one of the following conditions:

(a) The set S is contained in $\{0 < 1 < \cdots < n\}$.

(b) The set S is contained in $\{n+1 < n+2 < \cdots < 2n+1\}$.

(c) There exists an integer $j \neq i$ such that $0 \leq j \leq n$ and $S \cap \{j, 2n+1-j\} = \emptyset$.

Let $K_0 \subseteq \Delta^{2n+1}$ be the simplicial subset whose nondegenerate simplices have the form σ_S, where S is basic. Unwinding the definitions, we can rewrite (8.2) as a lifting problem

$$\begin{tikzcd}
K_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\Delta^{2n+1} \arrow[r] \arrow[ur, dashed] & \mathcal{D}.
\end{tikzcd}$$

Since U is an inner fibration, it will suffice to show that the inclusion $K_0 \hookrightarrow \Delta^{2n+1}$ is an inner anodyne map of simplicial sets.

We now introduce two more collections of subsets of $[2n+1]$.

- We say that a subset $S \subseteq [2n+1]$ is *primary* if it is not basic, the intersection $S \cap \{0, 1, \cdots, i-1\}$ is empty, and $2n+1-i \in S$.
- We say that a subset $S \subseteq [2n+1]$ is *secondary* if it is not basic, the intersection $S \cap \{0, 1, \cdots, i-1\}$ is nonempty, and $i \in S$.

Let $\{S_1, S_2, \cdots, S_m\}$ be an ordering of the collection of all subsets of $[2n+1]$ which are either primary or secondary, satisfying the following conditions:

- The sequence of cardinalities $|S_1|, |S_2|, \cdots, |S_m|$ is nondecreasing. That is, for $1 \leq p \leq q \leq m$, we have $|S_p| \leq |S_q|$.

- If $|S_p| = |S_q|$ for $p \leq q$ and S_q is primary, then S_p is also primary.

For $1 \leq q \leq m$, let $\sigma_q \subseteq \Delta^{2n+1}$ denote the simplex spanned by the vertices of S_q, and let K_q denote the union of K_0 with the simplices $\{\sigma_1, \sigma_2, \cdots, \sigma_q\}$. We have inclusion maps

$$K_0 \hookrightarrow K_1 \hookrightarrow K_2 \hookrightarrow \cdots \hookrightarrow K_m.$$

Note that we have $\sigma_m = K_m = \Delta^{2n+1}$ (since the set $[2n+1]$ is secondary). It will therefore suffice to show that for $1 \leq q \leq m$, the inclusion map $K_{q-1} \hookrightarrow K_q$ is inner anodyne.

In what follows, we regard q as fixed. Let d be the dimension of the simplex σ_q. Let us abuse notation by identifying σ_q with a morphism of simplicial sets $\Delta^d \to K_q \subseteq \Delta^{2n+1}$, and set $L = \sigma_q^{-1} K_{q-1} \subseteq \Delta^d$. To complete the proof, it will suffice to show that L is an inner horn of Δ^d, so that the diagram of simplicial sets

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & K_{q-1} \arrow[d] \\
\Delta^d \arrow[r, "\sigma_q"] & K_q.
\end{tikzcd}$$

is a pushout square by virtue of Lemma 3.1.2.11.

We first consider the case where the set $S_q = \{j_0 < j_1 < \cdots < j_d\}$ is primary, so that we have $j_0 \geq i$ and $j_k = 2n+1-i$ for some $0 \leq k \leq d$. Note that we must have $k > 0$ (otherwise S_q satisfies condition (b)) and $k < d$ (otherwise, S_q satisfies condition (c), since it is disjoint from $\{0, 2n+1\}$). In this case, we will show that L coincides with the inner horn $\Lambda^d_k \subset \Delta^d$. This can be restated as follows:

$(*)$ Let j be an element of S_q, and set $S' = S_q \setminus \{j\}$. Then $\sigma_{S'}$ is contained in K_{q-1} if and only if $j \neq 2n+1-i$.

Assume first that $j \neq 2n+1-i$. Then the set S' contains $2n+1-i$ and satisfies $S' \cap \{0, 1, \cdots, i-1\} = \emptyset$. Consequently, the set S' is either primary (and therefore coincides with $S_{q'}$ for some $q' < q$) or basic. In either case, the simplex $\sigma_{S'}$ belongs to the simplicial subset $K_{q-1} \subseteq \Delta^{2n+1}$.

We now prove $(*)$ in the case $j = 2n+1-i$. Since S_q does not satisfy conditions (b) or (c), the set S' also does not satisfy conditions (b) or (c). It also cannot satisfy condition (a): if S' were contained in the set $\{0, 1, \cdots, n\}$, then S_q would be contained in the set $\{i, i+1, \cdots, n, 2n+1-i\}$, and would therefore satisfy condition (c). It follows that S' is not basic. Assume, for a contradiction, that $\sigma_{S'}$ is contained in K_{q-1}. We then have $\sigma_{S'} \subseteq \sigma_{q'}$ for some $q' < q$. Since S' is neither primary nor secondary, this must be a proper inclusion: that is, we must have

$$\dim(\sigma_q) - 1 = \dim(\sigma_{S'}) < \dim(\sigma_{q'}) \leq \dim(\sigma_q).$$

It follows that the second inequality must be an equality: that is, we have $|S_{q'}| = |S_q|$ and therefore $S_{q'}$ is also primary. In particular, the set $S_{q'}$ contains $2n+1-i$, and therefore contains the union $S' \cup \{2n+1-i\} = S_q$. Since S_q and $S_{q'}$ have the same cardinality, it follows that $S_q = S_{q'}$ and therefore $q = q'$, contradicting our assumption that $q' < q$.

We now consider the case where $S_q = \{j_0 < j_1 < \cdots < j_d\}$ is secondary, so that we have $j_0 < i$ and $j_k = i$ for some $0 < k \leq d$. Note that we must have $k < d$ (otherwise, S_q satisfies condition (a)). In this case, we will show that L coincides with the inner horn $\Lambda^d_k \subset \Delta^d$. This can be restated as follows:

$(*')$ Let $j \in S_q$ and set $S' = S_q \setminus \{j\}$. Then the simplex $\sigma_{S'}$ is contained in K_{q-1} if and only if $j = i$.

We first treat the case where $j \neq i$, so that $i \in S'$. If S' is basic, then $\sigma_{S'} \subseteq K \subseteq K_{q-1}$. We may therefore assume that S' is not basic. If the intersection $S' \cap \{0, 1, \cdots, i-1\}$ is nonempty, then S' is secondary and has smaller cardinality than S_q. It follows that $S' = S_{q'}$ for some $q' < q$, so that $\sigma_{S'} \subseteq K_{q'} \subseteq K_{q-1}$. We may therefore assume that the intersection $S' \cap \{0, 1, \cdots, i-1\}$ is empty. In this case, the union $S' \cup \{2n+1-i\}$ is a primary set of cardinality $\leq |S_q|$, and therefore has the form $S_{q'}$ for some $q' < q$. From this, we again conclude that $\sigma_{S'} \subseteq K_{q'} \subseteq K_{q-1}$.

We now prove $(*')$ in the case $j = i$. Since S_q does not satisfy conditions (a) or (c), it follows that S' also does not satisfy conditions (a) or (c). The set S' also does not satisfy condition (b), since the intersection $S' \cap \{0, \cdots, i-1\}$ is nonempty. It follows that S' is not basic. Assume, for a contradiction, that $\sigma_{S'}$ is contained in K_{q-1}. We then have $\sigma_{S'} \subseteq \sigma_{q'}$ for some $q' < q$. Since the intersection $S_{q'} \cap \{1, \cdots, i-1\}$ is nonempty, the set $S_{q'}$ cannot be primary and is therefore secondary. In particular, the set $S_{q'}$ contains the element i and therefore contains the union $S' \cup \{i\} = S_q$. Combining this observation with the inequality $|S_{q'}| \leq |S_q|$, we deduce that $S_{q'} = S_q$ and therefore $q' = q$, contradicting our assumption that $q' < q$. □

8.1.2 Homotopy Transport for Twisted Arrows

03JV Let $\mathcal{C}$ be an ∞-category and let $\mathrm{Tw}(\mathcal{C})$ denote its twisted arrow ∞-ategory. For every pair of objects $X, Y \in \mathcal{C}$, Proposition 8.1.1.11 guarantees that the fiber product

$$\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}$$

is a Kan complex, whose vertices can be identified with morphisms $f : X \to Y$. Our goal in this section is to show that this identification can be promoted to a homotopy equivalence of Kan complexes

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y) \hookrightarrow \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\},$$

where $\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) = \mathcal{C}_{X/} \times_{\mathcal{C}} \{Y\}$ denotes the left-pinched morphism space of Construction 4.6.5.1 (see Corollary 8.1.2.10). Our starting point is the following result:

Proposition 8.1.2.1. *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be an isomorphism in $\mathcal{C}$. Then f is initial when viewed as an object of the ∞-category $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$.* 03JW

Remark 8.1.2.2. The converse of Proposition 8.1.2.1 is also true: see Corollary 8.1.2.21. 03JX

We will prove Proposition 8.1.2.1 at the end of this section. First, let us record some consequences.

Corollary 8.1.2.3. *Let $\mathcal{C}$ be a Kan complex. Then the projection maps* 048J

$$\mathcal{C}^{\mathrm{op}} \xleftarrow{\lambda_-} \mathrm{Tw}(\mathcal{C}) \xrightarrow{\lambda_+} \mathcal{C}$$

are trivial Kan fibrations of simplicial sets.

Proof. It follows from Corollary 8.1.1.13 that λ_- and λ_+ are Kan fibrations. By virtue of Proposition 3.3.7.6, it will suffice to show that the fibers of λ_- and λ_+ are contractible Kan complexes, which is an immediate consequence of Proposition 8.1.2.1 (see Corollary 4.6.7.11). □

Corollary 8.1.2.4. *Let $\mathcal{C}$ be a simplicial set. Then the projection map $\lambda_+ : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}$ is universally localizing (see Definition 6.3.6.1).* 048K

Proof. Writing $\mathcal{C}$ as the filtered colimit of its skeleta $\mathrm{sk}_n(\mathcal{C})$ and using Proposition 6.3.6.12, we can reduce to the case where $\mathcal{C}$ has dimension $\leq n$ for some integer $n \geq 0$. We proceed by induction on n. If $n = 0$, the morphism λ_+ is an isomorphism. Let us therefore assume that n is positive. Let S denote the collection of nondegenerate n-simplices of $\mathcal{C}$, so that Proposition 1.1.4.12 supplies a pushout square

$$\begin{tikzcd}
S \times \partial \Delta^n \arrow[r] \arrow[d] & S \times \Delta^n \arrow[d] \\
\mathrm{sk}_{n-1}(\mathcal{C}) \arrow[r] & \mathcal{C},
\end{tikzcd}$$

where the horizontal maps are monomorphisms. Combining our inductive hypothesis with Proposition 6.3.6.13, we can replace $\mathcal{C}$ by $S \times \Delta^n$ and thereby reduce to the case where $\mathcal{C}$ is an ∞-category. In this case, λ_+ is a cocartesian fibration (Corollary 8.1.1.14) having weakly contractible fibers (Proposition 8.1.2.1 and Corollary 4.6.7.25), and is therefore universally localizing by virtue of Example 6.3.6.2. □

048L **Corollary 8.1.2.5.** *Let $\mathcal{C}$ be a simplicial set. Then the projection map $\lambda_+ : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}$ is a weak homotopy equivalence.*

Proof. Combine Corollary 8.1.2.4 with Remark 6.3.6.5. □

048M **Corollary 8.1.2.6.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then F is a weak homotopy equivalence if and only if the induced map $\mathrm{Tw}(F) : \mathrm{Tw}(\mathcal{C}) \to \mathrm{Tw}(\mathcal{D})$ is a weak homotopy equivalence.*

Proof. We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}) \arrow[r, "\mathrm{Tw}(F)"] \arrow[d] & \mathrm{Tw}(\mathcal{D}) \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{D},
\end{tikzcd}$$

where the vertical maps are weak homotopy equivalences by virtue of Corollary 8.1.2.4. □

03JY **Construction 8.1.2.7.** Let $\mathcal{C}$ be a simplicial set and let X be a vertex of $\mathcal{C}$. Let σ be an n-simplex of the coslice simplicial set $\mathcal{C}_{X/}$, which we identify with a morphism of simplicial sets $\{x\} \star \Delta^n \to \mathcal{C}$ satisfying $\sigma(x) = X$. Then the composite map

$$(\Delta^n)^{\mathrm{op}} \star \Delta^n \twoheadrightarrow \{x\} \star \Delta^n \xrightarrow{\sigma} \mathcal{C}$$

can be identified with an n-simplex of the twisted arrow simplicial set $\mathrm{Tw}(\mathcal{C})$, which we will denote by $\iota_X(\sigma)$. The construction $\sigma \mapsto \iota_X(\sigma)$ is compatible with the formation of face and degeneracy operators, and therefore determines a morphism of simplicial sets $\iota_X : \mathcal{C}_{X/} \to \mathrm{Tw}(\mathcal{C})$. Moreover, the diagram

$$\begin{tikzcd}
\mathcal{C}_{X/} \arrow[r, "\iota_X"] \arrow[d] & \mathrm{Tw}(\mathcal{C}) \arrow[d, "\lambda_-"] \\
\{X\} \arrow[r] & \mathcal{C}^{\mathrm{op}}
\end{tikzcd}$$

commutes, where λ_- is the projection map of Notation 8.1.1.6. It follows that ι_X can be regarded as a morphism of simplicial sets from $\mathcal{C}_{X/}$ to the fiber $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. We will refer to this morphism as the *coslice inclusion*.

Remark 8.1.2.8. Let $\mathcal{C}$ be a simplicial set and let $X \in \mathcal{C}$ be a vertex. Then an n-simplex σ of $\mathcal{C}_{X/}$ can be identified with an $(n+1)$-simplex of $\mathcal{C}$, which we represent informally as a diagram 03JZ

$$X \xrightarrow{f} Y_0 \xrightarrow{v_1} Y_1 \xrightarrow{v_2} Y_2 \rightarrow \cdots \xrightarrow{v_n} Y_n.$$

The morphism ι_X of Construction 8.1.2.7 carries σ to a $(2n+1)$-simplex τ of $\mathcal{C}$, which can be represented informally by the diagram

$$\begin{tikzcd}
X \arrow[d, "f"] & X \arrow[l, "\mathrm{id}"'] & X \arrow[l, "\mathrm{id}"'] & \cdots \arrow[l, "\mathrm{id}"'] & X \arrow[l, "\mathrm{id}"'] \\
Y_0 \arrow[r, "v_1"] & Y_1 \arrow[r, "v_2"] & Y_2 \arrow[r] & \cdots \arrow[r, "v_n"] & Y_n.
\end{tikzcd}$$

Note that σ can be recover from τ (by composing with the inclusion map $\Delta^{n+1} \hookrightarrow \Delta^{2n+1}$, given on vertices by $i \mapsto i+n$). It follows that ι_X is a monomorphism of simplicial sets $\mathcal{C}_{X/} \hookrightarrow \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ (as suggested by our terminology).

Proposition 8.1.2.9. *Let $\mathcal{C}$ be an ∞-category. For every object $X \in \mathcal{C}$, the coslice inclusion* 03K0

$$\iota_X : \mathcal{C}_{X/} \hookrightarrow \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$$

is an equivalence of ∞-categories.

Proof. By construction, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{X/} \arrow[rr, "\iota_X"] \arrow[dr] & & \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[dl, "\lambda_+"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where the vertical maps are left fibrations of ∞-categories (Propositions 4.3.6.1 and 8.1.1.11). Moreover, the ∞-category $\mathcal{C}_{X/}$ has an initial object $\widetilde{X}$, given by the identity morphism $\mathrm{id}_X : X \to X$ (Proposition 4.6.7.22). Proposition 8.1.2.1 guarantees that $\iota_X(\widetilde{X})$ is an initial object of the ∞-category $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$, so that ι_X is an equivalence of ∞-categories by virtue of Corollary 5.6.6.20. □

Corollary 8.1.2.10. *Let $\mathcal{C}$ be an ∞-category. For every pair of objects $X, Y \in \mathcal{C}$, the coslice inclusion ι_X restricts to a homotopy equivalence of Kan complexes* 03K1

$$\mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X,Y) \hookrightarrow \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}.$$

Proof. Combine Proposition 8.1.2.9 with Corollary 5.1.6.4. □

03K2 **Corollary 8.1.2.11.** *Let $\mathcal{C}$ be an ∞-category and let $f, f' : X \to Y$ be morphisms of $\mathcal{C}$. Then f and f' are homotopic (in the sense of Definition 1.4.3.1) if and only they belong to the same connected component of the Kan complex $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}$. Consequently, we have a canonical isomorphism of sets*

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \simeq \pi_0(\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}).$$

03K3 **Exercise 8.1.2.12.** Prove Corollary 8.1.2.11 directly from the definitions.

03K4 **Exercise 8.1.2.13.** Let $\mathcal{C}$ be an ∞-category containing morphisms $u : X' \to X$ and $v : Y \to Y'$, so that covariant transport for the left fibration $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ determines a morphism of Kan complexes

$$T : \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\} \to \{X'\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y'\}.$$

Show that, under the identifications supplied by Corollary 8.1.2.11, the induced map of connected components $\pi_0(T) : \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X', Y')$ is given by the construction $[f] \mapsto [v] \circ [f] \circ [u]$.

We now apply Proposition 8.1.2.9 to describe the left fibration $(\lambda_-, \lambda_+) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ of Proposition 8.1.1.15.

03K5 **Notation 8.1.2.14.** Let $\mathcal{C}$ be an ∞-category. For every pair of objects $X, Y \in \mathcal{C}$, Proposition 4.6.5.10 and Corollary 8.1.2.10 supply homotopy equivalences of Kan complexes

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \hookleftarrow \mathrm{Hom}^{\mathrm{L}}_{\mathcal{C}}(X, Y) \hookrightarrow \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}.$$

Passing to homotopy, we obtain an isomorphism

$$\alpha_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}$$

in the homotopy category hKan.

03LX **Corollary 8.1.2.15.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is fully faithful if and only if the diagram*

03LY
$$\begin{tikzcd} \mathrm{Tw}(\mathcal{C}) \arrow[r, "\mathrm{Tw}(F)"] \arrow[d] & \mathrm{Tw}(\mathcal{D}) \arrow[d] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "F^{\mathrm{op}} \times F"] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \end{tikzcd} \qquad (8.3)$$

is a categorical pullback square.

Proof. Since the vertical maps in the diagram (8.3) are left fibrations (Proposition 8.1.1.11), it is a categorical pullback square if and only if, for every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\} \to \{F(X)\} \times_{\mathcal{D}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{D}) \times_{\mathcal{D}} \{F(Y)\}$$

is a homotopy equivalence of Kan complexes (Corollary 5.1.7.16). Using Notation 8.1.2.14, we see that this is equivalent to the requirement that F induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$. □

Corollary 8.1.2.16. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories. Then the induced map $\mathrm{Tw}(F) : \mathrm{Tw}(\mathcal{C}) \to \mathrm{Tw}(\mathcal{D})$ is also an equivalence of ∞-categories.* 03K9

Proof. Combine Corollary 8.1.2.15 with Proposition 4.5.2.21. □

Corollary 8.1.2.17. *Let $\mathcal{C}$ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ is essentially κ-small, then $\mathrm{Tw}(\mathcal{C})$ is essentially κ-small.* 03Y6

Proof. Choose an equivalence of ∞-categories $\mathcal{C}' \to \mathcal{C}$, where $\mathcal{C}'$ is a κ-small simplicial set. It follows from Corollary 8.1.2.16 that the induced map $\mathrm{Tw}(\mathcal{C}') \to \mathrm{Tw}(\mathcal{C})$ is also an equivalence of ∞-categories. We conclude by observing that $\mathrm{Tw}(\mathcal{C}')$ is also a κ-small simplicial set (Remark 8.1.1.5). □

Corollary 8.1.2.18. *Let $\mathcal{C}$ be an ∞-category, let $\mathrm{h}\mathcal{C}$ be its homotopy category, and let* 03K6

$$\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}} : \mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{h}\mathcal{C} \to \mathrm{hKan} \qquad (X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

denote the functor determined by the hKan*-enrichment of Construction 4.6.9.13. Then the assignment $(X, Y) \mapsto \alpha_{X,Y}$ of Notation 8.1.2.14 determines an isomorphism from $\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}$ to the homotopy transport representation of the left fibration $(\lambda_{-}, \lambda_{+}) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.*

Proof. Let $H : \mathrm{h}(\mathcal{C}^{\mathrm{op}}) \times \mathrm{h}\mathcal{C} \to \mathrm{hKan}$ denote the homotopy transport representation for the left fibration $(\lambda_{-}, \lambda_{+})$, given on objects by the formula $H(X, Y) = \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}$. For every pair of objects $X, Y \in \mathcal{C}$, Notation 8.1.2.14 determines an isomorphism

$$\alpha_{X,Y} : \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Y) \xrightarrow{\sim} H(X, Y)$$

in the homotopy category hKan. We wish to show that $\alpha_{X,Y}$ depends functorially on X and Y.

We first establish a strong form of functoriality in Y. Fix an object $X \in \mathcal{C}$, and let $h^X : \mathrm{h}\mathcal{C} \to \mathrm{hKan}$ denote the hKan-enriched functor corepresented by X, given concretely by the formula $h^X(Y) = \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Let $H^X : \mathrm{h}\mathcal{C} \to \mathrm{hKan}$ denote the restriction $H|_{\{X\} \times \mathrm{h}\mathcal{C}}$, which we also regard as an hKan-enriched functor (using Variant 5.2.8.11). Note that h^X can be identified with the (enriched) homotopy transport representation of the left fibration $\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \to \mathcal{C}$ (see Example 5.2.8.13). Corollary 4.6.4.18 and Proposition 8.1.2.9 supply equivalences

$$\{X\} \tilde{\times}_{\mathcal{C}} \mathcal{C} \hookleftarrow \mathcal{C}_{X/} \hookrightarrow \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$$

of left fibrations over $\mathcal{C}$, which induce an isomorphism of hKan-enriched functors $\alpha_{X,-} : h^X \xrightarrow{\sim} H^X$. By construction, this isomorphism carries each object $Y \in \mathrm{h}\mathcal{C}$ to the isomorphism $\alpha_{X,Y} : \underline{\mathrm{Hom}}(X,Y) \xrightarrow{\sim} H(X,Y)$ of Notation 8.1.2.14, which proves that $\alpha_{X,Y}$ depends functorially on Y.

We now show that $\alpha_{X,Y}$ depends functorially on X. Fix a morphism $f : W \to X$ in the ∞-category $\mathcal{C}$. We then have a diagram of hKan-enriched functors

03K7
$$\begin{tikzcd}
h^X \arrow[r, "\alpha_{X,-}"] \arrow[d] & H^X \arrow[d] \\
h^W \arrow[r, "\alpha_{X,-}"] & H^W,
\end{tikzcd} \tag{8.4}$$

where the vertical maps are induced by the homotopy class $[f] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}^{\mathrm{op}}}(X,W)$. To complete the proof, it will suffice to show that this diagram commutes. Using the corepresentability of the hKan-enriched functor h^X, we are reduced to showing that clockwise and counterclockwise composition around the diagram (8.4) carry $[\mathrm{id}_X] \in \pi_0(h^X(X)) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X,X)$ to the same element of $\pi_0(H^W(X))$. We conclude by observing that under the identification $\pi_0(H^W(X)) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(W,X)$ supplied by Corollary 8.1.2.11, both constructions carry $[\mathrm{id}_X]$ to $[f]$ (Exercise 8.1.2.13). □

03K8 **Warning 8.1.2.19.** Let $\mathcal{C}$ be an ∞-category. Our proof of Corollary 8.1.2.18 shows that the isomorphism $\alpha_{X,Y} : \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X,Y) \to H(X,Y)$ is compatible with the hKan-enrichment in the second variable. Beware that things are a bit more subtle if we wish to view $\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}(X,Y)$ and $H(X,Y)$ as hKan-enriched functors of the *first* variable. The functor $\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{C}}$ is defined using the enrichment of the category $\mathrm{h}\mathcal{C}$, and can therefore be viewed an hKan-enriched functor

$$(\mathrm{h}\mathcal{C})^{\mathrm{op}} \times \mathrm{h}\mathcal{C} \to \mathrm{hKan}.$$

On the other hand, the functor H is defined as the enriched homotopy transport representation of the left fibration $(\lambda_-, \lambda_+) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, which is an hKan-enriched functor

$$\mathrm{h}(\mathcal{C}^{\mathrm{op}}) \times \mathrm{h}\mathcal{C} \to \mathrm{hKan}.$$

The hKan-enriched categories $(\mathrm{h}\mathcal{C})^{\mathrm{op}}$ and $\mathrm{h}(\mathcal{C}^{\mathrm{op}})$ are *a priori* different objects: to a pair of objects $X, Y \in \mathcal{C}$, they assign morphism spaces $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y)^{\mathrm{op}}$, respectively.

It is possible to address this point (since $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\mathrm{Hom}_{\mathcal{C}}(X,Y)^{\mathrm{op}}$ are canonically isomorphic as objects of the homotopy category hKan), but we will not pursue the matter here.

We can use Proposition 8.1.2.9 to deduce a stronger form of Proposition 8.1.2.1.

Corollary 8.1.2.20. *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories and let $f : X \to Y$ be a morphism of $\mathcal{C}$, which we regard as an object of the twisted arrow ∞-category* $\mathrm{Tw}(\mathcal{C})$. *Then:* 03KA

- *The morphism f is U-cocartesian if and only if it is V-initial, where V denotes the induced map*
$$\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \{U(X)\} \times_{\mathcal{D}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{D}).$$
- *The morphism f is U-cartesian if and only if it is V'-initial, where V' denotes the induced map*
$$\mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\} \to \mathrm{Tw}(\mathcal{D}) \times_{\mathcal{D}} \{U(Y)\}.$$

Proof. We will prove the first assertion; the proof of the second is similar. Construction 8.1.2.7 supplies a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C}_{X/} \arrow[r, "\iota_X"] \arrow[d, "U_{X/}"] & \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[d, "V"] \\
\mathcal{D}_{U(X)/} \arrow[r, "\iota_{U(X)}"] & \{U(X)\} \times_{\mathcal{D}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{D}),
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories (Proposition 8.1.2.9). By virtue of Remark 7.1.4.9, it will suffice to show that f is U-cocartesian if and only if it is a $U_{X/}$-initial object of the ∞-category $\mathcal{C}_{X/}$, which is a special case of Example 7.1.5.9. □

Corollary 8.1.2.21. *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:* 03KB

(1) *The morphism f is an isomorphism in the ∞-category $\mathcal{C}$.*

(2) *The morphism f is initial when regarded as an object of the ∞-category* $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$.

(3) *The morphism f is initial when regarded as an object of the ∞-category* $\mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}$.

Proof. Apply Corollary 8.1.2.20 in the special case $\mathcal{D} = \Delta^0$ (together with Examples 7.1.4.2 and 5.1.1.4). □

Proof of Proposition 8.1.2.1. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be an isomorphism in $\mathcal{C}$; we wish to show that f is initial when viewed as an object of the ∞-category $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. Fix an integer $n > 0$ and a morphism $\rho_0 : \partial \Delta^n \to \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ satisfying $\rho_0(0) = f$; we wish to show that ρ_0 can be extended to an n-simplex of $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$.

We now use a variation on the proof of Proposition 8.1.1.15. For every nonempty subset $S \subseteq [2n+1]$, let σ_S denote the corresponding nondegenerate simplex of Δ^{2n+1}. Let us say that S is *basic* if it satisfies one of the following conditions:

(a) The set S is contained in $\{0 < 1 < \cdots < n\}$.

(b) There exists an integer $0 \leq i \leq n$ such that $S \cap \{i, 2n+1-i\} = \emptyset$.

Let $K_0 \subseteq \Delta^{2n+1}$ be the simplicial subset whose nondegenerate simplices have the form σ_S, where S is basic. Unwinding the definitions, we can identify ρ_0 with a morphism of simplicial sets $\theta_0 : K \to \mathcal{C}$, where the composition $\Delta^n \hookrightarrow K \xrightarrow{\theta_0} \mathcal{C}$ is the constant map taking the value X and the composition

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{n < n+1\}) \hookrightarrow K \xrightarrow{\theta_0} \mathcal{C}$$

is the morphism f. To complete the proof, we must show that θ_0 admits an extension $\theta : \Delta^{2n+1} \to \mathcal{C}$.

Let S be a nonempty subset of $[2n+1]$ which is not basic. Then there exists an integer $0 \leq i \leq n$ such that $2n+1-i$ belongs to S. We denote the largest such integer by $\mathrm{pr}(S)$ and refer to it as the *priority* of S. We say that S is *prioritized* if it also contained the integer $\mathrm{pr}(S)$. Let $\{S_1, S_2, \cdots, S_m\}$ be an ordering of the collection of all prioritized (non-basic) subsets of $[2n+1]$ which satisfies the following conditions:

- The sequence of priorities $\mathrm{pr}(S_1), \mathrm{pr}(S_2), \cdots, \mathrm{pr}(S_m)$ is nondecreasing. That is, if $1 \leq p \leq q \leq m$, then we have $\mathrm{pr}(S_p) \leq \mathrm{pr}(S_q)$.
- If $\mathrm{pr}(S_p) = \mathrm{pr}(S_q)$ for $p \leq q$, then $|S_p| \leq |S_q|$.

For $1 \leq q \leq m$, let $\sigma_q \subseteq \Delta^{2n+1}$ denote the simplex spanned by the vertices of S_q, and let $K_q \subseteq \Delta^{2n+1}$ denote the union of K_0 with the simplices $\{\sigma_1, \sigma_2, \cdots, \sigma_q\}$, so that we have inclusion maps

$$K_0 \hookrightarrow K_1 \hookrightarrow K_2 \hookrightarrow \cdots \hookrightarrow K_m.$$

Note that the set $S = [2n+1]$ is prioritized (with priority n), and is therefore equal to S_m. It follows that $K_m = \Delta^{2n+1}$. We will complete the proof by showing that θ_0 admits a compatible sequence of extensions $\{\theta_q : K_q \to \mathcal{C}\}_{0 \leq q \leq m}$, so that $\theta = \theta_m$ is an extension of θ_0 to Δ^{2n+1}.

For the remainder of the proof, we fix an integer $1 \leq q \leq m$, and suppose that the morphism $\theta_{q-1} : K_{q-1} \to \mathcal{C}$ has already been constructed. Let d denote the dimension of the simplex σ_q, let us abuse notation by identifying σ_q with a morphism of simplicial sets $\Delta^d \to K_q \subseteq \Delta^{2n+1}$, and set $L = \sigma_q^{-1} K_{q-1} \subseteq \Delta^d$. Let $i = \mathrm{pr}(S_q)$ denote the priority of S_q, so that S_q contains both i and $2n+1-i$. Write $S_q = \{j_0 < j_1 < \cdots < j_d\}$, so that $i = j_k$ for some integer $0 \leq k \leq d$. We will prove below that L is equal to the horn $\Lambda^d_k \subseteq \Delta^d$, so that the diagram of simplicial sets

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & K_{q-1} \arrow[d] \\
\Delta^d \arrow[r, "\sigma_q"] & K_q
\end{tikzcd}$$

is a pushout square (Lemma 3.1.2.11). Let τ_0 denote the composite map $L \xrightarrow{\sigma_q} K_{q-1} \xrightarrow{\theta_{q-1}} \mathcal{C}$. We will complete the proof by showing that τ_0 admits an extension $\tau : \Delta^d \to \mathcal{C}$ (which then determines a morphism $\theta_q : K_q \to \mathcal{C}$ extending θ_{q-1}). The proof splits into four cases:

- Suppose that $0 < k < d$. Then $\Lambda^d_k \subseteq \Delta^d$ is an inner horn, so that τ_0 admits an extension $\tau : \Delta^d \to \mathcal{C}$ by virtue of our assumption that $\mathcal{C}$ is an ∞-category.

- Suppose that $k = d$. Then S_q is contained in $\{0, 1, \cdots, n\}$, contradicting our assumption that S_q is not basic.

- Suppose $k = 0$ and $i < n$, so that i is the least element of S_q. Our assumption $\mathrm{pr}(S_q) = i$ guarantees that $2n - i \notin S$. Since S_q does not satisfy (b), we must also have $i + 1 \in S_q$. It follows that $d \geq 2$ (otherwise, S_q would satisfy (a)), and that $\tau_0 : \Lambda^d_0 \to \mathcal{C}$ carries the initial edge $\mathrm{N}_\bullet(\{0 < 1\})$ to the identity morphism id_X. In this case, the existence of the extension τ follows from Theorem 4.4.2.6.

- Suppose $k = 0$ and $i = n$, so that $i = n$ is the least element of S_q. Since S_q has priority n, the element $n + 1$ also belongs to S_q. We must then have $d \geq 2$ (otherwise, S_q would satisfy condition (b)). It follows that $\tau_0 : \Lambda^d_0 \to \mathcal{C}$ carries the initial edge $\mathrm{N}_\bullet(\{0 < 1\})$ to the morphism f, which is an isomorphism in $\mathcal{C}$. In this case, the existence of the extension τ again follows from Theorem 4.4.2.6.

It remains to prove that $L = \Lambda^d_k$, which we can formulate more concretely as follows:

$(*)$ Let j be an element of S_q, and set $S' = S_q \setminus \{j\}$. Then $\sigma_{S'}$ is contained in K_{q-1} if and only if $j \neq i$.

We first treat the case $j = i$; in this case, we wish to show that $\sigma_{S'}$ is not contained in K_{q-1}. Note that S' cannot be basic: it cannot be contained in $\{0, 1, \cdots, n\}$ (otherwise

$S_q = S' \cup \{i\}$ would have the same property) and cannot have empty intersection with a set of the form $\{i', 2n+1-i'\}$ (otherwise S_q would have the same property; here we use the fact that $2n+1-i$ is contained in S_q). Moreover, we have $\mathrm{pr}(S') = i \notin S'$, so that S' is not prioritized. Assume, for a contradiction, that $\sigma_{S'}$ is contained in K_{q-1}. Then we must have $S' \subseteq S_{q'}$, for some $1 \leq q' < q$. Note that $2n+1-i \in S' \subseteq S_{q'}$, so that $S_{q'}$ has priority $\geq i$. Since $q' < q$, it follows that $S_{q'}$ has priority i and that $|S_{q'}| \leq |S_q|$. Since $S_{q'}$ is prioritized, it contains the element i, and therefore contains $S_q = S' \cup \{i\}$. It follows that $S_{q'} = S_q$, contradicting our assumption that $q' < q$.

We now treat the case $j \neq i$; in this case, we wish to show that $\sigma_{S'}$ is contained in K_{q-1}. We may assume without loss of generality that S' is not basic (otherwise, the simplex $\sigma_{S'}$ is already contained in K_0). Let $i' = \mathrm{pr}(S')$ denote the priority of S'; note that the inclusion $S' \subseteq S_q$ guarantees that $i' \leq i$. If $i' < i$, then $S' \cup \{i'\}$ is a prioritized set of priority $< i$, and therefore of the form $S_{q'}$ for some $q' < q$. If $i' = i$, then S' is a prioritized set of priority i and cardinality $|S_q| - 1$, and therefore of the form $S_{q'}$ for some $q' < q$. In either case, we obtain $\sigma_{S'} \subseteq \sigma_{q'} \subseteq K_{q-1}$. □

8.1.3 The Cospan Construction

00AY Let $\mathcal{C}_0$ be a category which admits pushouts. In §2.2.1, we introduced a 2-category $\mathrm{Cospan}(\mathcal{C}_0)$ having the same objects, where 1-morphisms from X to Y in $\mathrm{Cospan}(\mathcal{C}_0)$ are *cospans* from X to Y: that is, diagrams $X \xrightarrow{f} B \xleftarrow{g} Y$ in the category $\mathcal{C}_0$ (see Example 2.2.2.1). In this section, we introduce a generalization of this construction, which will allow us to replace the ordinary category $\mathcal{C}_0$ by an ∞-category. More precisely, we will associate to every simplicial set $\mathcal{C}$ a simplicial set $\mathrm{Cospan}(\mathcal{C})$ of *cospans in* $\mathcal{C}$ (Construction 8.1.3.1). In the special case where $\mathcal{C} = \mathrm{N}_{\bullet}(\mathcal{C}_0)$ is the nerve of a category $\mathcal{C}_0$ which admits pushouts, we show that $\mathrm{Cospan}(\mathcal{C})$ can be identified with the Duskin nerve of the 2-category $\mathrm{Cospan}(\mathcal{C}_0)$ (Corollary 8.1.3.15).

03KN **Construction 8.1.3.1.** Let $\mathcal{C}$ be a simplicial set. For every integer $n \geq 0$, we let $\mathrm{Cospan}_n(\mathcal{C})$ denote the collection of morphisms $\mathrm{Tw}(\Delta^n) \to \mathcal{C}$ in the category of simplicial sets. The construction $[n] \mapsto \mathrm{Cospan}_n(\mathcal{C})$ depends functorially on the set $[n] = \{0 < 1 < \cdots < n\}$ as an object of the category $\mathbf{\Delta}^{\mathrm{op}}$, and can therefore be viewed as a simplicial set. We will denote this simplicial set by $\mathrm{Cospan}(\mathcal{C})$ and refer to it as the *simplicial set of cospans in* $\mathcal{C}$.

03KP **Remark 8.1.3.2.** Let $n \geq 0$ be an integer. Then the simplicial set $\mathrm{Tw}(\Delta^n)$ can be identified with the nerve of the partially ordered set $Q = \{(i,j) \in [n]^{\mathrm{op}} \times [n] : i \leq j\}$ (see Example 8.1.0.5). Consequently, if $\mathcal{C}$ is an arbitrary simplicial set, then n-simplices of $\mathrm{Cospan}(\mathcal{C})$ can be identified with morphisms $\mathrm{N}_\bullet(Q) \to \mathcal{C}$, which we depict informally as diagrams

$$\begin{tikzcd}
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl] \arrow[dr] & \cdots & X_{n-1,n-1} \arrow[dl] \arrow[dr] & & X_{n,n} \arrow[dl] \\
& \cdots \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & \\
& & X_{0,n-1} \arrow[dr] & & X_{1,n} \arrow[dl] & & \\
& & & X_{0,n}. & & &
\end{tikzcd}$$

Example 8.1.3.3. Let $\mathcal{C}$ be a simplicial set. Then: 03KQ

- Vertices of the simplicial set $\mathrm{Cospan}(\mathcal{C})$ can be identified with vertices of $\mathcal{C}$.
- Let X and Y be vertices of $\mathcal{C}$. Then edges of $\mathrm{Cospan}(\mathcal{C})$ joining X to Y can be identified with pairs (f, g), where $f : X \to B$ and $g : Y \to B$ are edges of $\mathcal{C}$ having the same target.

Remark 8.1.3.4 (Symmetry)**.** Let $\mathcal{C}$ be a simplicial set and let σ be an n-simplex of $\mathrm{Cospan}(\mathcal{C})$, which we identify with a morphism of simplicial sets $\mathrm{Tw}(\Delta^n) \to \mathcal{C}$. Composing with the automorphism 03KR

$$\mathrm{Tw}(\Delta^n) \xrightarrow{\sim} \mathrm{Tw}(\Delta^n) \qquad (i, j) \mapsto (n - j, n - i),$$

we obtain a new n-simplex $\overline{\sigma}$ of $\mathrm{Cospan}(\mathcal{C})$. The construction $\sigma \mapsto \overline{\sigma}$ determines an isomorphism of simplicial sets $\tau : \mathrm{Cospan}(\mathcal{C}) \simeq \mathrm{Cospan}(\mathcal{C})^{\mathrm{op}}$, which can be described concretely as follows:

- For every vertex $X \in \mathcal{C}$, the morphism τ carries X (regarded as a vertex of $\mathrm{Cospan}(\mathcal{C})$) to itself.
- Let X and Y be vertices of $\mathcal{C}$, and let $e : X \to Y$ be an edge of $\mathrm{Cospan}(\mathcal{C})$, given by a pair of edges $(f : X \to B, g : Y \to B)$ of $\mathcal{C}$. Then $\tau(e) : Y \to X$ is the edge of $\mathrm{Cospan}(\mathcal{C})$ given by the pair (g, f).

Note that τ is involutive: that is, the composition

$$\mathrm{Cospan}(\mathcal{C}) \xrightarrow{\tau} \mathrm{Cospan}(\mathcal{C})^{\mathrm{op}} \xrightarrow{\tau^{\mathrm{op}}} \mathrm{Cospan}(\mathcal{C})$$

is the identity automorphism of $\mathrm{Cospan}(\mathcal{C})$.

03KT **Construction 8.1.3.5.** Let $\mathcal{D}$ be a simplicial set and let $\sigma : \Delta^n \to \mathcal{D}$ be an n-simplex of $\mathcal{D}$. Invoking the functoriality of the twisted arrow construction, we obtain a map $\mathrm{Tw}(\Delta^n) \xrightarrow{\mathrm{Tw}(\sigma)} \mathrm{Tw}(\mathcal{D})$, which we can identify with an n-simplex $u(\sigma)$ of the simplicial set $\mathrm{Cospan}(\mathrm{Tw}(\mathcal{D}))$. The construction $\sigma \mapsto u(\sigma)$ is compatible with face and degeneracy operators, and therefore determines a morphism of simplicial sets $u : \mathcal{D} \to \mathrm{Cospan}(\mathrm{Tw}(\mathcal{D}))$ which we will refer to as the *unit map.*

048N **Example 8.1.3.6** (Tautological Cospans)**.** Let $\mathcal{D}$ be a simplicial set and let $e : X \to Y$ be an edge of $\mathcal{D}$, which we also view as a vertex of the simplicial set $\mathrm{Tw}(\mathcal{D})$. Then the morphism $\mathcal{D} \to \mathrm{Cospan}(\mathrm{Tw}(\mathcal{D}))$ carries e to an edge of the simplicial set $\mathrm{Cospan}(\mathrm{Tw}(\mathcal{D}))$, which we can identify with a pair of edges

$$\mathrm{id}_X \xrightarrow{e_L} e \xleftarrow{e_R} \mathrm{id}_Y$$

in the simplicial set $\mathrm{Tw}(\mathcal{D})$. Here e_L and e_R can be identified with degenerate 3-simplices of $\mathcal{D}$, which we depict informally in the diagrams

$$\begin{tikzcd} X \arrow[d, "\mathrm{id}_X"] & X \arrow[l, "\mathrm{id}_X"] & & Y \arrow[d, "\mathrm{id}_Y"] & X \arrow[l, "f"] \\ X \arrow[r, "f"] & Y & & Y \arrow[r, "\mathrm{id}_Y"] & Y. \end{tikzcd}$$

03KU **Proposition 8.1.3.7.** *Let $\mathcal{D}$ be a simplicial set and let $u : \mathcal{D} \to \mathrm{Cospan}(\mathrm{Tw}(\mathcal{D}))$ be the unit map of Construction 8.1.3.5. For every simplicial set $\mathcal{C}$, the composite map*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Tw}(\mathcal{D}), \mathcal{C}) &\to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Cospan}(\mathrm{Tw}(\mathcal{D})), \mathrm{Cospan}(\mathcal{C})) \\ &\xrightarrow{\circ u} \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{D}, \mathrm{Cospan}(\mathcal{C})) \end{aligned}$$

is a bijection.

Proof. Let us regard the simplicial set $\mathcal{C}$ as fixed. For every simplicial set $\mathcal{D}$, the unit map u of Construction 8.1.3.5 determines a function

$$\theta_{\mathcal{D}} : \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathrm{Tw}(\mathcal{D}), \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Set}_\Delta}(\mathcal{D}, \mathrm{Cospan}(\mathcal{C})).$$

Using Remark 8.1.1.4, we see that the construction $\mathcal{D} \mapsto \theta_{\mathcal{D}}$ carries colimits (in the category of simplicial sets) to limits (in the arrow category $\mathrm{Fun}([1], \mathrm{Set})$). Consequently, to show that $\theta_{\mathcal{D}}$ is a bijection, we may assume without loss of generality that $\mathcal{D} = \Delta^n$ is a standard simplex (see Remark 1.1.3.13). In this case, the desired result follows immediately from the definition of the simplicial set $\mathrm{Cospan}(\mathcal{C})$. □

Corollary 8.1.3.8. *The twisted arrow functor* 03KV

$$\mathrm{Tw} : \mathrm{Set}_{\Delta} \to \mathrm{Set}_{\Delta} \qquad \mathcal{D} \mapsto \mathrm{Tw}(\mathcal{D})$$

has a right adjoint, given on objects by the construction $\mathcal{C} \mapsto \mathrm{Cospan}(\mathcal{C})$.

Remark 8.1.3.9. Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial sets. Using Proposition 8.1.3.7, we obtain bijections 048P

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathrm{Fun}(\mathcal{C}, \mathrm{Cospan}(\mathcal{D}))) &\simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n \times \mathcal{C}, \mathrm{Cospan}(\mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Tw}(\Delta^n \times \mathcal{C}), \mathcal{D}) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Tw}(\Delta^n) \times \mathrm{Tw}(\mathcal{C}), \mathcal{D}) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\mathrm{Tw}(\Delta^n), \mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathcal{D})) \\
&\simeq \mathrm{Hom}_{\mathrm{Set}_{\Delta}}(\Delta^n, \mathrm{Cospan}(\mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathcal{D}))).
\end{aligned}$$

These bijections depend functorially on $[n] \in \mathbf{\Delta}$, and therefore determine an isomorphism of simplicial sets $\mathrm{Fun}(\mathcal{C}, \mathrm{Cospan}(\mathcal{D})) \simeq \mathrm{Cospan}(\mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathcal{D}))$.

Corollary 8.1.3.10. *Let* $U : \mathcal{C} \to \mathcal{D}$ *be a Kan fibration of simplicial sets. Then the induced map* $\mathrm{Cospan}(U) : \mathrm{Cospan}(\mathcal{C}) \to \mathrm{Cospan}(\mathcal{D})$ *is also a Kan fibration.* 048Q

Proof. Let $i : A \hookrightarrow B$ be a monomorphism of simplicial sets which is a weak homotopy equivalence. We wish to show that every lifting problem

048R

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & \mathrm{Cospan}(\mathcal{C}) \arrow[d, "\mathrm{Cospan}(U)"] \\
B \arrow[r] \arrow[ur, dashed] & \mathrm{Cospan}(\mathcal{D})
\end{tikzcd} \tag{8.5}$$

admits a solution. Using Proposition 8.1.3.7, we can rewrite (8.5) as a lifting problem of the form

$$\begin{tikzcd}
\mathrm{Tw}(A) \arrow[r] \arrow[d, "\mathrm{Tw}(i)"] & \mathcal{C} \arrow[d, "U"] \\
\mathrm{Tw}(B) \arrow[r] \arrow[ur, dashed] & \mathcal{D}.
\end{tikzcd}$$

Our assumption that U is a Kan fibration guarantees that this lifting problem has a solution, since the monomorphism $\mathrm{Tw}(i) : \mathrm{Tw}(A) \hookrightarrow \mathrm{Tw}(B)$ is also a weak homotopy equivalence (Corollary 8.1.2.6). □

048S **Corollary 8.1.3.11.** *Let $\mathcal{C}$ be a Kan complex. Then the simplicial set* $\mathrm{Cospan}(\mathcal{C})$ *is also a Kan complex.*

Proof. Apply Corollary 8.1.3.10 in the special case $\mathcal{D} = \Delta^0$. □

We now study the relationship between Construction 8.1.3.1 with the classical cospan construction (Example 2.2.2.1).

00B4 **Construction 8.1.3.12.** Let $\mathcal{C}$ be a category which admits pushouts, and let $\mathrm{Cospan}(\mathcal{C})$ denote the 2-category of Example 2.2.2.1. Suppose we are given another category $\mathcal{D}$ and a functor $F : \mathrm{Tw}(\mathcal{D}) \to \mathcal{C}$. We define a strictly unitary lax functor $F^+ : \mathcal{D} \to \mathrm{Cospan}(\mathcal{C})$ as follows:

- For each $X \in \mathcal{D}$, we define $F^+(X) = F(\mathrm{id}_X)$; here we regard the identity morphism $\mathrm{id}_X : X \to X$ as an object of the twisted arrow category $\mathrm{Tw}(\mathcal{C})$.
- For each morphism $f : X \to Y$ in $\mathcal{D}$, we define $F^+(f)$ to be the 1-morphism of $\mathrm{Cospan}(\mathcal{C})$ given by the cospan

$$F(\mathrm{id}_X) \xrightarrow{F(\mathrm{id}_X, f)} F(f) \xleftarrow{F(f, \mathrm{id}_Y)} F(\mathrm{id}_Y).$$

  Note that this determines the values of F^+ on 2-morphisms, since every 2-morphism in $\mathcal{D}$ is an identity 2-morphism.

- For every pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{D}$, the composition constraint $\mu_{g,f} : F^+(g) \circ F^+(f) \Rightarrow F^+(g \circ f)$ is the 2-morphism of $\mathrm{Cospan}(\mathcal{C})$ corresponding to the map $F(f) \amalg_{F(\mathrm{id}_Y)} F(g) \to F(g \circ f)$ classifying the commutative diagram

$$\begin{tikzcd}
F(\mathrm{id}_Y) \arrow[r, "F(f, \mathrm{id}_Y)"] \arrow[d, "F(\mathrm{id}_Y, g)"] & F(f) \arrow[d, "F(\mathrm{id}_X, g)"] \\
F(g) \arrow[r, "F(f, \mathrm{id}_Z)"] & F(g \circ f)
\end{tikzcd}$$

  in the category $\mathcal{C}$.

03KW **Example 8.1.3.13.** Let $\mathcal{C}$ be a category which admits pushouts and let n be a nonnegative integer. Applying Construction 8.1.3.12 in the special case where $\mathcal{D} = [n]$, we obtain a function

$$\{\text{Functors } \mathrm{Tw}([n]) \to \mathcal{C}\} \xrightarrow{\sim} \{\text{Strictly unitary lax functors } [n] \to \mathrm{Cospan}(\mathcal{C})\}.$$

Using Propositions 1.3.3.1 and 8.1.1.10, we can identify the left hand side with the collection of n-simplices of the simplicial set $\mathrm{Cospan}(\mathrm{N}_\bullet(\mathcal{C}))$. This construction depends functorially on n, and therefore determines a morphism of simplicial sets from $\mathrm{Cospan}(\mathrm{N}_\bullet(\mathcal{C}))$ to the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{Cospan}(\mathcal{C}))$.

We can now formulate our main result.

Theorem 8.1.3.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, where $\mathcal{C}$ admits pushouts. Then Construction 8.1.3.12 induces a bijection of sets* 00B5

$$\{\text{Functors } F : \mathrm{Tw}(\mathcal{D}) \to \mathcal{C}\} \xrightarrow{\sim} \{\text{Strictly unitary lax functors } F^{+} : \mathcal{D} \to \mathrm{Cospan}(\mathcal{C})\}.$$

Corollary 8.1.3.15. *Let $\mathcal{C}$ be a category which admits pushouts. Then the comparison map of Example 8.1.3.13 determines an isomorphism of simplicial sets* 00B6

$$\mathrm{Cospan}(\mathrm{N}_\bullet(\mathcal{C})) \to \mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{Cospan}(\mathcal{C})).$$

Exercise 8.1.3.16. Show that Corollary 8.1.3.15 implies Theorem 8.1.3.14. That is, to prove Theorem 8.1.3.14 in general, it suffices to treat the special case where $\mathcal{D}$ is a category of the form $[n] = \{0 < 1 < \cdots < n\}$ for $n \geq 0$. 03KX

Remark 8.1.3.17. Let $\mathcal{C}$ be a category which admits pushouts. The construction of the 2-category $\mathrm{Cospan}(\mathcal{C})$ of Example 2.2.2.1 involves some auxiliary choices: if $X \to B \leftarrow Y$ and $Y \to C \leftarrow Z$ are cospans in $\mathcal{C}$, then their composition (as 1-morphisms of $\mathrm{Cospan}(\mathcal{C})$) is given by $X \to (B \amalg_Y C) \leftarrow Z$, where the pushout $B \amalg_Y C$ is only well-defined up to (canonical) isomorphism. Corollary 8.1.3.15 supplies a description of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{Cospan}(\mathcal{C}))$ which does not depend on these choices. This shows, in particular, that the 2-category $\mathrm{Cospan}(\mathcal{C})$ is well-defined up to (non-strict) isomorphism; see Example 2.2.6.13. 03KY

Example 8.1.3.18. Let $\mathcal{C}$ be a category which admits pushouts. Then 2-simplices σ of the Duskin nerve $\mathrm{N}^{\mathrm{D}}_\bullet(\mathrm{Cospan}(\mathcal{C}))$ can be identified with commutative diagrams 00B8

$$\begin{tikzcd}
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl] \arrow[dr] & & X_{2,2} \arrow[dl] \\
& X_{0,1} \arrow[dr] & & X_{1,2} \arrow[dl] & \\
& & X_{0,2} & &
\end{tikzcd}$$

in the category $\mathcal{C}$. It follows from Theorem 2.3.2.5 that the 2-simplex σ is thin (in the sense of Definition 2.3.2.3) if and only if the square appearing in the diagram is a pushout: that is, it induces an isomorphism $X_{0,1} \amalg_{X_{1,1}} X_{1,2} \to X_{0,2}$ in the category $\mathcal{C}$.

Proof of Theorem 8.1.3.14. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, where $\mathcal{C}$ admits pushouts, and let $G : \mathcal{D} \to \operatorname{Cospan}(\mathcal{C})$ be a strictly unitary lax functor of 2-categories. For every morphism $f : X \to Y$ in the category $\mathcal{D}$, we can identify $G(f)$ with a cospan from $G(X)$ to $G(Y)$ in the category $\mathcal{C}$, given by a diagram we will denote by $G(X) \xrightarrow{b_-(f)} B(f) \leftarrow b_+(f) G(Y)$. Our assumption that G is strictly unitary guarantees the following:

($*$) For each object $X \in \mathcal{D}$, the object $B(\mathrm{id}_X)$ is equal to $G(X)$, and the maps $b_-(\mathrm{id}_X) : G(X) \to B(\mathrm{id}_X)$ and $b_+(\mathrm{id}_X) : G(X) \to B(\mathrm{id}_X)$ are the identity morphisms from $G(X)$ to itself in the category $\mathcal{C}$.

For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the composition constraint $\mu_{g,f}$ for the lax functor G can be identified with a morphism from the pushout $B(f) \amalg_{G(Y)} B(g)$ to $B(g \circ f)$, or equivalently with a pair of morphisms

$$p(g,f) : B(f) \to B(g \circ f) \qquad q(g,f) : B(g) \to B(g \circ f)$$

satisfying $p(g,f) \circ b_+(f) = q(g,f) \circ b_-(g)$. The axioms for a lax functor (Definition 2.2.4.5) then translate to the following additional conditions:

(a) For every morphism $f : X \to Y$ in the category $\mathcal{D}$, $p(\mathrm{id}_Y, f)$ is the identity morphism from $B(f)$ to itself.

(b) For every morphism $f : X \to Y$ in the category $\mathcal{D}$, $q(f, \mathrm{id}_X)$ is the identity morphism from $B(f)$ to itself.

(c) For every composable triple of 1-morphisms $W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ in the category $\mathcal{D}$, we have

$$p(h \circ g, f) = p(h, g \circ f) \circ p(g,f) \qquad q(h, g \circ f) = q(h \circ g, f) \circ q(h,g)$$

$$p(h, g \circ f) \circ q(g,f) = q(h \circ g, f) \circ p(h,g).$$

We wish to show that there exists a unique functor of ordinary categories $F : \operatorname{Tw}(\mathcal{D}) \to \mathcal{C}$ such that $G = F^+$, where F^+ is the lax functor associated to F by Construction 8.1.3.12. For this condition to be satisfied, the functor F must satisfy the following conditions:

(0) For each object $X \in \mathcal{D}$, we have $F(\mathrm{id}_X) = G(X)$ (this guarantees that G and F^+ coincide on objects).

(1) For each morphism $f : X \to Y$ in $\mathcal{D}$ (regarded as an object of $\mathrm{Tw}(\mathcal{D})$), we have $F(f) = B(f)$, and the morphisms $b_-(f)$ and $b_+(f)$ are given by $F(\mathrm{id}_X, f)$ and $F(f, \mathrm{id}_Y)$, respectively (this guarantees that G and F^+ coincide on 1-morphisms, and therefore also on 2-morphisms).

(2) For every pair of composable 1-morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, the morphisms $p(g, f) : B(f) \to B(g \circ f)$ and $q(g, f) : B(g) \to B(g \circ f)$ are given by $F(\mathrm{id}_X, g) : F(f) \to F(g \circ f)$ and $F(f, \mathrm{id}_Z) : F(g) \to F(g \circ f)$, respectively (this guarantees that the composition constraints on G and F^+ coincide).

Note that the value of F on each object of $\mathrm{Tw}(\mathcal{D})$ is determined by condition (1). Moreover, if (u, v) is a morphism from $f : X \to Y$ to $f' : X' \to Y'$ in the category $\mathrm{Tw}(\mathcal{D})$, then condition (2) guarantees that $F(u, v)$ must be equal to the composition

$$F(f) = B(f) \xrightarrow{q(f,u)} B(f \circ u) \xrightarrow{p(v, f \circ u)} B(v \circ f \circ u) = B(f') = F(f').$$

This proves the uniqueness of the functor F.

To prove existence, we define F on objects f of $\mathrm{Tw}(\mathcal{D})^{\mathrm{op}}$ by the formula $F(f) = B(f)$, and on morphisms $(u, v) : f \to f'$ by the formula $F(u, v) = p(v, f \circ u) \circ q(f, u)$. For any morphism $f : X \to Y$ in $\mathcal{C}$, we can use (a) and (b) to compute

$$F(\mathrm{id}_X, \mathrm{id}_Y) = p(\mathrm{id}_X, f) \circ q(f, \mathrm{id}_Y) = \mathrm{id}_{B(f)} \circ \mathrm{id}_{B(f)} = \mathrm{id}_{B(f)},$$

so that F carries identity morphisms in $\mathrm{Tw}(\mathcal{D})$ to identity morphisms in $\mathcal{C}$. To complete the proof that F is a functor, we note that for every pair of composable morphisms

$$(f : X \to Y) \xrightarrow{(u,v)} (f' : X' \to Y') \xrightarrow{(u',v')} (f'' : X'' \to Y'')$$

in the twisted arrow category $\mathrm{Tw}(\mathcal{D})$, the identities given in (c) allow us to compute

$$\begin{aligned}
F(u', v') \circ F(u, v) &= p(v', f' \circ u') \circ q(f', u') \circ p(v, f \circ u) \circ q(f, u) \\
&= p(v', v \circ f \circ u \circ u') \circ q(v \circ f \circ u, u') \circ p(v, f \circ u) \circ q(f, u) \\
&= p(v', v \circ f \circ u \circ u') \circ p(v, f \circ u \circ u') \circ q(f \circ u, u') \circ q(f, u) \\
&= p(v' \circ v, f \circ u \circ u') \circ q(f, u \circ u') \\
&= F(u \circ u', v' \circ v).
\end{aligned}$$

We now complete the proof by showing that the functor F satisfies conditions (0), (1), and (2). Condition (0) is an immediate consequence of $(*)$. To prove (2), we note that for any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{D}$, identities (a) and (b) yield equalities

$$F(\mathrm{id}_X, g) = p(g, f) \circ q(f, \mathrm{id}_X) = p(g, f) \qquad F(f, \mathrm{id}_Z) = p(\mathrm{id}_Z, g \circ f) \circ q(g, f) = q(g, f).$$

To prove (1), we note that if $f : X \to Y$ is a morphism in $\mathcal{D}$, then we have

$$\begin{aligned} F(\mathrm{id}_X, f) &= p(f, \mathrm{id}_X \circ \mathrm{id}_X) \circ q(\mathrm{id}_X, \mathrm{id}_X) \\ &= p(f, \mathrm{id}_X) \circ \mathrm{id}_{G(X)} \\ &= p(f, \mathrm{id}_X) \circ b_+(\mathrm{id}_X) \\ &= q(f, \mathrm{id}_X) \circ b_-(f) \\ &= \mathrm{id}_{B(f)} \circ b_-(f) \\ &= b_-(f), \end{aligned}$$

and a similar calculation yields $F(f, \mathrm{id}_Y) = b_+(f)$. □

8.1.4 Cospans in ∞-Categories

03L4 Let $\mathcal{C}$ be an ∞-category, and let $\mathrm{Cospan}(\mathcal{C})$ denote the simplicial set of cospans in $\mathcal{C}$ (Construction 8.1.3.1). In the special case where $\mathcal{C} = \mathrm{N}_\bullet(\mathcal{C}_0)$ is the nerve of an ordinary category $\mathcal{C}_0$ which admits pushouts, Corollary 8.1.3.15 supplies an isomorphism of $\mathrm{Cospan}(\mathcal{C})$ with the Duskin nerve of the 2-category $\mathrm{Cospan}(\mathcal{C}_0)$ of Example 2.2.2.1. In particular, $\mathrm{Cospan}(\mathcal{C})$ is an $(\infty, 2)$-category (see Proposition 5.4.1.5). Our goal in this section is to prove an ∞-categorical generalization of this result.

03L5 **Proposition 8.1.4.1.** *Let $\mathcal{C}$ be an ∞-category. Then the simplicial set* $\mathrm{Cospan}(\mathcal{C})$ *is an $(\infty, 2)$-category if and only if $\mathcal{C}$ admits pushouts.*

Our proof of Proposition 8.1.4.1 will require several steps. The main ingredient is the following characterization of thin 2-simplices of $\mathrm{Cospan}(\mathcal{C})$, which we will establish in §8.1.5:

03L6 **Proposition 8.1.4.2.** *Let $\mathcal{C}$ be an ∞-category and let σ be a 2-simplex of* $\mathrm{Cospan}(\mathcal{C})$*, which we identify with a diagram*

03L7

$$\begin{tikzcd} X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl] \arrow[dr] & & X_{2,2} \arrow[dl] \\ & X_{0,1} \arrow[dr] & & X_{1,2} \arrow[dl] & \\ & & X_{0,2} & & \end{tikzcd} \tag{8.6}$$

in the ∞-category $\mathcal{C}$. Then σ is thin (in the sense of Definition 2.3.2.3) if and only if the inner region is a pushout square in the ∞-category $\mathcal{C}$.

Corollary 8.1.4.3. *Let $\mathcal{C}$ be an ∞-category. Then every degenerate 2-simplex of* $\operatorname{Cospan}(\mathcal{C})$ *is thin.* 03L8

Proof. Let σ be a 1-simplex of $\operatorname{Cospan}^{L,R}(\mathcal{C})$, corresponding to a diagram $X \xrightarrow{f} B \xleftarrow{g} Y$ in the ∞-category $\mathcal{C}$ where f belongs to L and g belongs to R. We will show that the left-degenerate 2-simplex $s^1_0(\sigma)$ is thin; a similar argument will show that the right-degenerate 2-simplex $s^1_1(\sigma)$ is thin (see Remark 8.1.3.4). Unwinding the definitions, we see that $s^1_0(\sigma)$ corresponds to a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{id}_X"'] & & X \arrow[dl, "\mathrm{id}_X"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& X \arrow[dr, "f"'] & & B \arrow[dl, "\mathrm{id}_B"] & \\
& & B. & &
\end{tikzcd}$$

By virtue of Proposition 8.1.4.2, it will suffice to show that the inner region of the diagram is a pushout square in $\mathcal{C}$. This follows from Corollary 7.6.3.24, since id_B and id_X are isomorphisms in $\mathcal{C}$. □

Lemma 8.1.4.4. *Let $\mathcal{C}$ be an ∞-category and let $\sigma_0 : \Lambda^2_1 \to \operatorname{Cospan}(\mathcal{C})$ be a morphism of simplicial sets, corresponding to a diagram* 048T

03LA

$$\begin{tikzcd}
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl, "u"] \arrow[dr, "v"'] & & X_{2,2} \arrow[dl] \\
& X_{0,1} & & X_{1,2} &
\end{tikzcd} \tag{8.7}$$

in the ∞-category $\mathcal{C}$. Then any commutative diagram

048U

$$\begin{tikzcd}
X_{1,1} \arrow[r, "u"] \arrow[d, "v"] & X_{1,2} \arrow[d] \\
X_{0,1} \arrow[r] & X_{0,2}
\end{tikzcd} \tag{8.8}$$

can be obtained from an extension of σ_0 to a 2-simplex of $\operatorname{Cospan}(\mathcal{C})$*. In particular, if $X_{1,2}$ and $X_{0,1}$ admit a pushout along $X_{1,1}$, then σ_0 can be extended to a thin 2-simplex of $\mathcal{C}$.*

Proof. Let us identify $\mathrm{Tw}(\Delta^2)$ with the simplicial set $\mathrm{N}_\bullet(\overline{Q})$, where $\overline{Q}$ denotes the partially ordered set $\{(i,j) \in [2]^{\mathrm{op}} \times [2] : i \leq j\}$. Under this identification, $\mathrm{Tw}(\Lambda^2_1)$ corresponds to the simplicial subset $\mathrm{N}_\bullet(Q) \subseteq \mathrm{N}_\bullet(\overline{Q})$, where $Q = \overline{Q} \setminus \{(0,2)\}$, so σ_0 determines a diagram $\tau : \mathrm{N}_\bullet(Q) \to \mathcal{C}$.

Set $Q_0 = Q \setminus \{(0,0),(2,2)\}$ and $\tau_0 = \tau|_{\mathrm{N}_\bullet(Q_0)}$. Lemma 8.1.4.4 is equivalent to the assertion that the restriction map $\mathcal{C}_{\tau/} \to \mathcal{C}_{\tau_0/}$ is surjective on vertices. To prove this, it will suffice to show that the inclusion map $\mathrm{N}_\bullet(Q_0) \hookrightarrow \mathrm{N}_\bullet(Q)$ is right anodyne (Corollary 4.3.6.13), or equivalently that it is right cofinal (Proposition 7.2.1.3). This is a special case of Corollary 7.2.3.7, since the inclusion map $Q_0 \hookrightarrow Q$ has a left adjoint (given by $(0,0) \mapsto (0,1)$ and $(2,2) \mapsto (1,2)$). □

03LB **Remark 8.1.4.5.** Let $\mathcal{C}$ be an ∞-category and suppose we are given a morphism of simplicial sets $\varphi : \mathrm{Tw}(\Delta^2) \to \mathcal{C}$, which we display as a diagram

$$\begin{tikzcd}
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl] \arrow[dr] & & X_{2,2} \arrow[dl] \\
& X_{0,1} \arrow[dr] & & X_{1,2} \arrow[dl] & \\
& & X_{0,2}. & &
\end{tikzcd}$$

The proof of Lemma 8.1.4.4 shows that φ is a colimit diagram if and only if the inner region is a pushout square.

We now study the problem of filling outer horns in simplicial sets of the form $\mathrm{Cospan}(\mathcal{C})$.

048V **Lemma 8.1.4.6.** *Let $U : \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets. Suppose we are given an integer $n \geq 3$ and a lifting problem*

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$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{Cospan}(\mathcal{C}) \arrow[d, "\mathrm{Cospan}(U)"] \\
\Delta^n \arrow[r, "\overline{\sigma}"] \arrow[ur, dashed] & \mathrm{Cospan}(\mathcal{D}),
\end{tikzcd} \tag{8.9}$$

where σ_0 corresponds to a morphism of simplicial sets $F_0 : \mathrm{Tw}(\Lambda^n_0) \to \mathcal{C}$ with the following properties:

(a) The morphism F_0 carries the edge $(0,0) \to (0,1)$ of $\mathrm{Tw}(\Lambda^n_0)$ to a U-cocartesian edge of $\mathcal{C}$.

(b) The morphism F_0 carries the edge $(1,n) \to (0,n)$ to a U-cartesian edge of $\mathcal{C}$.

Then the lifting problem (8.9) admits a solution.

Proof. Using Proposition 8.1.3.7, we can rewrite (8.9) as a lifting problem

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$$\begin{tikzcd}
\mathrm{Tw}(\Lambda^n_0) \arrow[r, "F_0"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\mathrm{Tw}(\Delta^n) \arrow[r, "\overline{F}"] \arrow[ur, dashed] & \mathcal{D}.
\end{tikzcd} \tag{8.10}$$

Let P denote the set of all ordered pairs (i,j), where i and j are integers satisfying $0 \leq i \leq j \leq n$. We regard P as a partially ordered set by identifying it with its image in the product $[n]^{\mathrm{op}} \times [n]$ (so that $(i,j) \leq (i',j')$ if and only if $i' \leq i$ and $j \leq j'$). In what follows, we will identify $\mathrm{Tw}(\Delta^n)$ with the nerve $\mathrm{N}_\bullet(P)$; under this identification, $\mathrm{Tw}(\Lambda^n_0)$ corresponds to a simplicial subset $K_0 \subseteq \mathrm{N}_\bullet(P)$.

Let $S = \{(i_0,j_0) < (i_1,j_1) < \cdots < (i_d,j_d)\}$ be a nonempty linearly ordered subset of P, so that we have inequalities $0 \leq i_d \leq i_{d-1} \leq \cdots \leq i_0 \leq j_0 \leq j_1 \leq \cdots \leq j_d \leq n$. In this case, we write τ_S for the corresponding nondegenerate d-simplex of $\mathrm{N}_\bullet(P)$. We will say that S is *basic* if τ_S is contained in K_0. Equivalently, S is basic if the set $\{i_0, i_1, \cdots, i_d, j_0, j_1, \cdots, j_d\}$ does not contain $\{1 < 2 < \cdots < n\}$. If S is not basic, we let $\mathrm{pr}(S)$ denote the largest integer j such that S contains the pair (i,j) for some $i \neq 0$. If no such integer exists, we define $\mathrm{pr}(S) = 0$. We will refer to $\mathrm{pr}(S)$ as the *priority* of S. We say that S is *prioritized* if it is not basic and contains the pair $(0, \mathrm{pr}(S))$.

Let $\{S_1, S_2, \cdots, S_m\}$ be an enumeration of the collection of all prioritized linearly ordered subsets of P which satisfies the following conditions:

- The sequence of priorities $\mathrm{pr}(S_1), \mathrm{pr}(S_2), \cdots, \mathrm{pr}(S_m)$ is nondecreasing. That is, if $1 \leq k \leq \ell \leq m$, then we have $\mathrm{pr}(S_k) \leq \mathrm{pr}(S_\ell)$.
- If $\mathrm{pr}(S_k) = \mathrm{pr}(S_\ell)$ for $k \leq \ell$, then $|S_k| \leq |S_\ell|$.

For $1 \leq \ell \leq m$, let $\tau_\ell \subseteq \mathrm{N}_\bullet(P)$ denote the simplex τ_{S_ℓ} and let $K_\ell \subseteq \mathrm{N}_\bullet(P)$ denote the union of K_0 with the simplices $\{\tau_1, \tau_2, \cdots, \tau_\ell\}$, so that we have inclusion maps

$$K_0 \hookrightarrow K_1 \hookrightarrow K_2 \hookrightarrow \cdots \hookrightarrow K_m.$$

We claim that $K_m = \mathrm{N}_\bullet(P)$: that is, K_m contains τ_S for every nonempty linearly ordered subset $S \subseteq P$. If S is basic, there is nothing to prove. We may therefore assume that S

has priority p for some integer $p \geq 0$. The union $S \cup \{(0,p)\}$ is then a prioritized linearly ordered subset of P, and therefore coincides with S_ℓ for some $1 \leq \ell \leq m$. In this case, we have $\tau_S \subseteq \tau_\ell \subseteq K_\ell \subseteq K_m$.

We will complete the proof by constructing a compatible sequence of maps $F_\ell : K_\ell \to \mathcal{C}$ extending F_0 and satisfying $U \circ F_\ell = \overline{F}|_{K_\ell}$. Fix an integer $1 \leq \ell \leq m$, and suppose that $F_{\ell-1}$ has already been constructed. Write $S_\ell = \{(i_0, j_0) < (i_1, j_1) < \cdots < (i_d, j_d)\}$, so that the simplex τ_ℓ has dimension d. Let p be the priority of S_ℓ. Since S_ℓ is prioritized, it contains $(0,p)$; we can therefore write $(0,p) = (i_{d'}, j_{d'})$ for some integer $0 \leq d' \leq d$. Let $L \subseteq \Delta^d$ denote the inverse image of $K_{\ell-1}$ under the map $\tau_\ell : \Delta^d \to \mathrm{N}_\bullet(P)$. We will show that L coincides with the horn $\Lambda^d_{d'}$, so that the pullback diagram of simplicial sets

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & K_{\ell-1} \arrow[d] \\
\Delta^d \arrow[r, "\tau_\ell"] & K_\ell.
\end{tikzcd}$$

is also a pushout square (Lemma 3.1.2.11). This can be stated more concretely as follows:

$(*)$ Let (i,j) be an element of S_ℓ, and set $S' = S_\ell \setminus \{(i,j)\}$. Then the simplex $\tau_{S'}$ is contained in $K_{\ell-1}$ if and only if $(i,j) \neq (0,p)$.

We first prove $(*)$ in the case where $(i,j) \neq (0,p)$; in this case, we wish to show that $\tau_{S'}$ is contained in $K_{\ell-1}$. If S' is basic, then $\tau_{S'}$ is contained in K_0 and there is nothing to prove. Let us therefore assume that S' is not basic. Let $p' = \mathrm{pr}(S')$ denote the priority of S'. Then the union $S' \cup \{(0,p')\}$ is a prioritized subset of P, and therefore has the form S_k for some $1 \leq k \leq m$. By construction, we have $\mathrm{pr}(S_k) = p' \leq p = \mathrm{pr}(S_\ell)$. Moreover, if $p' = p$, then our assumption $(i,j) \neq (0,p)$ guarantees that $S_k = S'$, so that $|S_k| < |S_\ell|$. It follows that $k < \ell$, so that we have $\tau_{S'} \subseteq \tau_k \subseteq K_k \subseteq K_{\ell-1}$.

We now prove $(*)$ in the case $(i,j) = (0,p)$; in this case, we wish to show that $\tau_{S'}$ is not contained in $K_{\ell-1}$. Assume otherwise. Then, since S' is not basic, it is contained in S_k for some $k < \ell$. The inequalities

$$p = \mathrm{pr}(S') \leq \mathrm{pr}(S_k) \leq \mathrm{pr}(S_\ell) = p.$$

ensure that S_k has priority p. Since S_k is prioritized, it contains $(0,p)$, and therefore contains the union $S_\ell = S' \cup \{(0,p)\}$. The inequality $|S_k| \leq |S_\ell|$ then forces $k = \ell$, contradicting our assumption that $k < \ell$. This completes the proof of $(*)$.

Let ρ_0 denote the composite map $\Lambda^d_{d'} = L \xrightarrow{\tau_\ell} K_{\ell-1} \xrightarrow{F_{\ell-1}} \mathcal{C}$. To complete the proof, it will suffice to show that the lifting problem

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$$\begin{tikzcd}
\Lambda^d_{d'} \arrow[r, "\rho_0"] \arrow[d] & \mathcal{C} \arrow[d, "U"] \\
\Delta^d \arrow[r, "\overline{F} \circ \tau_\ell"] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd} \tag{8.11}$$

admits a solution. We consider three cases:

- If $0 < d' < d$, then the lifting problem (8.11) admits a solution by virtue of our assumption that U is an inner fibration of simplicial sets.
- Suppose that $d' = 0$: that is, the pair $(0, p)$ is the smallest element of S_ℓ. Then S_ℓ does not contain any pairs (i, j) with $i \neq 0$, so we have $p = 0$. Since the set S_ℓ is not basic, we must have $S_\ell = \{(0,0) < (0,1) < \cdots < (0, n-1) < (0, n)\}$. In this case, the lifting problem (8.11) admits a solution by virtue of assumption (a).
- Suppose that $d' = d$: that is, the pair $(0, p)$ is the largest element of S_ℓ. Our assumption that S_ℓ is not basic then guarantees that $p = n$ and $(1, n) \in S_\ell$: that is, we have $S_\ell = \{(i_0, j_0) < (i_1, j_1) < \cdots < (1, n) < (0, n)\}$. In this case, the lifting problem (8.11) admits a solution by virtue of assumption (b).

□

Specializing Lemma 8.1.4.6 to the case $\mathcal{D} = \Delta^0$, we obtain the following:

Lemma 8.1.4.7. *Let $\mathcal{C}$ be an ∞-category and let $\sigma_0 : \Lambda^n_0 \to \mathrm{Cospan}(\mathcal{C})$ be a morphism of simplicial sets, which we identify with a diagram $X : \mathrm{Tw}(\Lambda^n_0) \to \mathrm{Cospan}(\mathcal{C})$. Assume that $n \geq 3$ and that the morphisms $X(0,0) \to X(0,1)$ and $X(1,n) \to X(0,n)$ are isomorphisms in $\mathcal{C}$. Then σ_0 can be extended to an n-simplex of* Cospan($\mathcal{C}$). 03LC

Proof of Proposition 8.1.4.1. Let $\mathcal{C}$ be an ∞-category. By virtue of Lemma 8.1.4.7 and Corollary 8.1.4.3, the simplicial set Cospan($\mathcal{C}$) satisfies conditions (2) and (3) of Definition 5.4.1.1. Since Cospan($\mathcal{C}$) is isomorphic to $\mathrm{Cospan}(\mathcal{C})^{\mathrm{op}}$ (Remark 8.1.3.4), it also satisfies condition (4) of Definition 5.4.1.1. It follows that Cospan($\mathcal{C}$) is an $(\infty, 2)$-category if and only if it satisfies the following condition:

$(*)$ Every morphism of simplicial sets $\Lambda^2_1 \to \mathrm{Cospan}(\mathcal{C})$ can be extended to a thin 2-simplex of Cospan($\mathcal{C}$).

Using Lemma 8.1.4.4, we can rewrite condition $(*)$ as follows:

$(*')$ For every diagram

$$\begin{tikzcd}
X \arrow[dr] & & Y \arrow[dl] \arrow[dr] & & Z \arrow[dl] \\
& B & & C &
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, there exists a pushout of B and C along Y.

It is clear that if the ∞-category $\mathcal{C}$ admits pushouts, then it satisfies condition $(*')$. The converse follows by applying condition $(*')$ to diagrams of the form

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{id}_X"] & & Y \arrow[dl] \arrow[dr] & & Z \arrow[dl, "\mathrm{id}_Z"'] \\
& X & & Z. &
\end{tikzcd}$$

□

03LD **Corollary 8.1.4.8.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, where $\mathcal{C}$ and $\mathcal{D}$ admit pushouts. The following conditions are equivalent:*

(1) *The functor F carries pushout squares in $\mathcal{C}$ to pushout squares in $\mathcal{D}$.*

(2) *The induced map $\mathrm{Cospan}(F) : \mathrm{Cospan}(\mathcal{C}) \to \mathrm{Cospan}(\mathcal{D})$ is a functor of $(\infty, 2)$-categories, in the sense of Definition 5.4.7.1.*

Proof. The implication (1) ⇒ (2) follows immediately from the criterion of Proposition 8.1.4.2. For the converse implication, suppose that $\mathrm{Cospan}(F) : \mathrm{Cospan}(\mathcal{C}) \to \mathrm{Cospan}(\mathcal{D})$ is a functor of 2-categories, and let $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a pushout square, which we display as a diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X_0 \arrow[d] \\
X_1 \arrow[r] & X_{01}.
\end{tikzcd}$$

Let $\rho : \mathrm{Tw}(\Delta^2) \to \Delta^1 \times \Delta^1$ denote the morphism of simplicial sets given on vertices by the formula $\rho(i,j) = (\max(0, 1-i), \max(0, j-1))$. Then $\sigma \circ \rho$ can be identified with a 2-simplex

τ of $\operatorname{Cospan}(\mathcal{C})$, corresponding to a diagram

$$\begin{tikzcd}
X_0 \arrow[dr, "\mathrm{id}"'] & & X \arrow[dl] \arrow[dr] & & X_1 \arrow[dl, "\mathrm{id}"] \\
& X_0 \arrow[dr] & & X_1 \arrow[dl] & \\
& & X_{01} & &
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. It follows from the criterion of Proposition 8.1.4.2 that τ is a thin 2-simplex of $\operatorname{Cospan}(\mathcal{C})$. If $\operatorname{Cospan}(F)$ is a functor of $(\infty, 2)$-categories, then it carries τ to a thin 2-simplex of $\operatorname{Cospan}(\mathcal{D})$. Applying the criterion of Proposition 8.1.4.2 again, we conclude that $F(\sigma)$ is a pushout square in $\mathcal{D}$. □

8.1.5 Thin 2-Simplices of $\operatorname{Cospan}(\mathcal{C})$

Let $\mathcal{C}$ be an ∞-category. Our goal in this section is to prove Proposition 8.1.4.2, which provides necessary and sufficient conditions for a 2-simplex of the simplicial set $\operatorname{Cospan}(\mathcal{C})$ to be thin. By virtue of Remark 8.1.4.5, it will suffice to prove the following pair of assertions: `03LE`

Lemma 8.1.5.1. *Let $\mathcal{C}$ be an ∞-category and let σ be a 2-simplex of $\operatorname{Cospan}(\mathcal{C})$, which we identify with a diagram $\varphi : \operatorname{Tw}(\Delta^2) \to \mathcal{C}$. If σ is thin, then φ is a colimit diagram.* `03LF`

Lemma 8.1.5.2. *Let $\mathcal{C}$ be an ∞-category and let σ be a 2-simplex of $\operatorname{Cospan}(\mathcal{C})$, which we identify with a diagram $\varphi : \operatorname{Tw}(\Delta^2) \to \mathcal{C}$. If φ is a colimit diagram, then σ is thin.* `03LG`

Proof of Lemma 8.1.5.1. Let Q denote the partially ordered set appearing in the proof of Lemma 8.1.4.4, so that we can identify $\operatorname{Tw}(\Lambda^2_1)$ with the nerve of Q. Set $\varphi_0 = \varphi|_{\operatorname{N}_\bullet(Q)}$. Assume that σ is thin. We wish to show that the restriction map $\mathcal{C}_{\varphi/} \to \mathcal{C}_{\varphi_0/}$ is a trivial Kan fibration: that is, every lifting problem

`03LH`

$$\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{\varphi/} \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{\varphi_0/}
\end{tikzcd} \tag{8.12}$$

admits a solution.

Let K denote the coproduct $(\mathrm{Tw}(\Delta^2) \star \partial \Delta^n) \coprod_{(\mathrm{N}_\bullet(Q) \star \partial \Delta^n)} (\mathrm{N}_\bullet(Q) \star \Delta^n)$, which we regard as a simplicial subset of $\mathrm{Tw}(\Delta^2) \star \Delta^n$. Unwinding the definitions, we can identify the lifting problem (8.12) with a morphism of simplicial sets $\tau_0 : K \to \mathcal{C}$ satisfying $\tau_0|_{\mathrm{Tw}(\Delta^2)} = \varphi$. We wish to show that τ_0 can be extended to a morphism $\tau : \mathrm{Tw}(\Delta^2) \star \Delta^n \to \mathcal{C}$.

Let $\iota : \mathrm{Tw}(\Delta^2) \star \Delta^n \to \mathrm{Tw}(\Delta^{n+3})$ be the morphism of simplicial sets given on vertices by the formula

$$\iota(x) = \begin{cases} (i,j) & \text{if } x = (i,j) \in \mathrm{Tw}(\Delta^2) \\ (0, x+3) & \text{if } x \in \Delta^n. \end{cases}$$

The morphism ι has a left inverse $\rho : \mathrm{Tw}(\Delta^{n+3}) \to \mathrm{Tw}(\Delta^2) \star \Delta^n$, given on vertices by the formula

$$\rho(i,j) = \begin{cases} (i,j) \in \mathrm{Tw}(\Delta^2) & \text{if } j \leq 2 \\ j - 3 \in \Delta^n & \text{otherwise.} \end{cases}$$

We observe that ι and ρ restrict to morphisms of simplicial subsets

$$\iota_0 : K \to \mathrm{Tw}(\Lambda^{n+3}_1) \qquad \rho_0 : \mathrm{Tw}(\Lambda^{n+3}_1) \to K,$$

so that we have a commutative diagram of simplicial sets

$$\begin{tikzcd}
K \arrow[r, "\iota_0"] \arrow[d] & \mathrm{Tw}(\Lambda^{n+3}_1) \arrow[r, "\rho_0"] \arrow[d] & K \arrow[d] \\
\mathrm{Tw}(\Delta^2) \star \Delta^n \arrow[r, "\iota"] & \mathrm{Tw}(\Delta^{n+3}) \arrow[r, "\rho"] & \mathrm{Tw}(\Delta^2) \star \Delta^n
\end{tikzcd}$$

where the horizontal compositions are equal to the identity.

The composition $\tau_0 \circ \rho_0$ can be identified with a morphism of simplicial sets $\psi_0 : \Lambda^{m+3}_1 \to \mathrm{Cospan}(\mathcal{C})$ having the property that the composition

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{0 < 1 < 2\}) \xrightarrow{\psi_0} \mathrm{Cospan}(\mathcal{C})$$

coincides with σ. Since σ is thin, we can extend ψ_0 to an $(n+3)$-simplex ψ of $\mathrm{Cospan}(\mathcal{C})$, which we identify with a map $\tau' : \mathrm{Tw}(\Delta^{n+3}) \to \mathcal{C}$. It follows that the composition $\tau = \tau' \circ \iota$ is a morphism $\mathrm{Tw}(\Delta^2) \star \Delta^n \to \mathcal{C}$ satisfying $\tau|_K = \tau_0$. □

Proof of Lemma 8.1.5.2. Let $\mathcal{C}$ be an ∞-category and let σ be a 2-simplex of $\mathrm{Cospan}(\mathcal{C})$, which we identify with a diagram $\varphi : \mathrm{Tw}(\Delta^2) \to \mathcal{C}$. Assume that φ is a colimit diagram. We wish to show that σ is thin. We proceed by a (somewhat more complicated) variation on the proof of Lemma 8.1.4.7.

Fix integers $0 < q < n$ with $n \geq 3$, and suppose that we are given a morphism $f_0 : \Lambda^n_q \to \mathrm{Cospan}(\mathcal{C})$ for which the composition

$$\Delta^2 \simeq \mathrm{N}_\bullet(\{q-1 < q < q+1\}) \hookrightarrow \Lambda^n_q \xrightarrow{f_0} \mathrm{Cospan}(\mathcal{C})$$

is equal to σ; we wish to show that f_0 can be extended to an n-simplex of $\mathrm{Cospan}(\mathcal{C})$. Using Proposition 8.1.3.7, we can identify f_0 with a morphism of simplicial sets $F_0 : \mathrm{Tw}(\Lambda^n_q) \to \mathcal{C}$; we wish to show that F_0 admits an extension $\mathrm{Tw}(\Delta^n) \to \mathcal{C}$.

Let P denote the set of all ordered pairs (i, j), where i and j are integers satisfying $0 \leq i \leq j \leq n$. We regard P as a partially ordered set by identifying it with its image in the product $[n]^{\mathrm{op}} \times [n]$ (so that $(i, j) \leq (i', j')$ if and only if $i' \leq i$ and $j \leq j'$). In what follows, we will identify $\mathrm{Tw}(\Delta^n)$ with the nerve $\mathrm{N}_\bullet(P)$; under this identification, $\mathrm{Tw}(\Lambda^n_q)$ corresponds to a simplicial subset $K_0 \subseteq \mathrm{N}_\bullet(P)$.

Let $S = \{(i_0, j_0) < (i_1, j_1) < \cdots < (i_d, j_d)\}$ be a nonempty linearly ordered subset of P, so that we have inequalities $0 \leq i_d \leq i_{d-1} \leq \cdots \leq i_0 \leq j_0 \leq j_1 \leq \cdots \leq j_d \leq n$. In this case, we write τ_S for the corresponding nondegenerate d-simplex of $\mathrm{N}_\bullet(P)$. Let $C(S)$ denote the set of integers $\{i_0, i_1, \cdots, i_d, j_0, j_1, \cdots, j_d\}$, which we regard as a subset of $[n] = \{0 < 1 < \cdots < n\}$; we will refer to $C(S)$ as the *content* of S.

- We will say that S is *basic* if $C(S) \cup \{q\} \neq [n]$. Equivalently, S is basic if the simplex τ_S is contained in K_0.

- Suppose that S is not basic and that it contains an ordered pair of the form $(p, q+1)$. We will say that S is *low* if the largest such integer p satisfies $p \leq q-2$. In this case, we will denote p by $\mathrm{pr}(S)$ and refer to it as the *priority* of S.

- Suppose that S is not basic and that it contains an ordered pair of the form $(q-1, r)$. We will say that S is *high* if the smallest such integer r satisfies $r \geq q+2$.

Note that the set S cannot be both low and high, since the elements $(p, q+1)$ and $(q-1, r)$ are incomparable in P when $p < q < r$. Moreover, if S is low, then any nonempty subset $S' \subseteq S$ is either low (and satisfies $\mathrm{pr}(S') \leq \mathrm{pr}(S)$) or satisfies $q+1 \notin C(S')$, so that S' is basic (the set S' cannot contain any elements of the form $(q+1, r)$, since these are incomparable with $(p, q+1)$). Consequently, the collection of simplices of the form τ_S where S is either basic or low determine a simplicial subset $K_{\mathrm{low}} \subseteq \mathrm{N}_\bullet(P)$. Similarly, the collection of simplices of the form τ_S where S is either basic or high determine a simplicial subset $K_{\mathrm{high}} \subseteq \mathrm{N}_\bullet(P)$, and the intersection $K_{\mathrm{low}} \cap K_{\mathrm{high}}$ is equal to K_0.

We will prove the following:

$(*)$ The inclusion maps $K_0 \hookrightarrow K_{\mathrm{low}}$ and $K_0 \hookrightarrow K_{\mathrm{high}}$ are inner anodyne.

We will prove that the inclusion map $K_0 \hookrightarrow K_{\text{low}}$ is inner anodyne; the analogous assertion for the inclusion $K_0 \hookrightarrow K_{\text{high}}$ follows by a similar argument. Let us say that a linearly ordered subset $S \subseteq P$ is *prioritized* if it low and contains the ordered pair (p, q), where $p = \mathrm{pr}(S)$ denotes the priority of S.

Let $\{S_1, S_2, \cdots, S_m\}$ be an enumeration of the collection of all prioritized low subsets of P which satisfies the following conditions:

- The sequence of priorities $\mathrm{pr}(S_1), \mathrm{pr}(S_2), \cdots, \mathrm{pr}(S_m)$ is nondecreasing. That is, if $1 \leq k \leq \ell \leq m$, then we have $\mathrm{pr}(S_k) \leq \mathrm{pr}(S_\ell)$.
- If $\mathrm{pr}(S_k) = \mathrm{pr}(S_\ell)$ for $k \leq \ell$, then $|S_k| \leq |S_\ell|$.

For $1 \leq \ell \leq m$, let $\tau_\ell \subseteq \mathrm{N}_\bullet(P)$ denote the simplex τ_{S_ℓ} and let K_ℓ denote the union of K_0 with the simplices $\{\tau_1, \tau_2, \cdots, \tau_\ell\}$, so that we have inclusion maps

$$K_0 \hookrightarrow K_1 \hookrightarrow K_2 \hookrightarrow \cdots \hookrightarrow K_m.$$

We claim that $K_m = K_{\text{low}}$: that is, every low subset $S \subseteq P$ is contained in S_ℓ for some $1 \leq \ell \leq m$. This is clear: if $p = \mathrm{pr}(S)$ is the priority of S, then the union $S \cup \{(p, q)\}$ is a prioritized low subset of P (having the same priority p).

We will prove $(*)$ by showing that, for $1 \leq \ell \leq m$, the inclusion map $K_{\ell-1} \hookrightarrow K_\ell$ is inner anodyne. Set $S_q = \{(i_0, j_0) < (i_1, j_1) < \cdots < (i_d, j_d)\}$. Let p denote the priority of S_ℓ. Since S_ℓ is prioritized, it contains the ordered pair (p, q). We therefore have $(p, q) = (i_c, j_c)$ for some $0 \leq c \leq d$. Note that since $p \leq q - 2$, we must have $c > 0$: otherwise, we have $q - 1 \notin C(S_\ell)$, contradicting our assumption that S_ℓ is not basic. Since S_ℓ also contains $(p, q+1)$, we must also have $c < d$. Let $L \subseteq \Delta^d$ denote the inverse image of $K_{\ell-1}$ under the map $\tau_\ell : \Delta^d \to \mathrm{N}_\bullet(P)$. We will complete the proof of $(*)$ by showing that L coincides with the inner horn Λ^d_c, so that the pullback diagram of simplicial sets

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & K_{\ell-1} \arrow[d] \\
\Delta^d \arrow[r, "\tau_\ell"] & K_\ell
\end{tikzcd}$$

is also a pushout square (Lemma 3.1.2.11). This can be stated more concretely as follows:

$(*')$ Let (i, j) be an element of S_ℓ, and set $S' = S_\ell \setminus \{(i, j)\}$. Then the simplex $\tau_{S'}$ is contained in $K_{\ell-1}$ if and only if $(i, j) \neq (p, q)$.

We first prove $(*')$ in the case where $(i, j) \neq (p, q)$; in this case, we wish to show that $\tau_{S'}$ is contained in $K_{\ell-1}$. If S' is basic, then $\tau_{S'}$ is contained in K_0 and there is nothing to prove.

We may therefore assume that S' is not basic, and is therefore low. If $(i,j) \neq (p,q+1)$, then S' is a prioritized low subset of P satisfying $\mathrm{pr}(S') = p = \mathrm{pr}(S_\ell)$ and $|S'| < |S_\ell|$. It follows that $S' = S_k$ for some $k < \ell$, so that $\tau_{S'}$ is contained in $K_k \subseteq K_{\ell-1}$. In the case $(i,j) = (p,q+1)$, the set S' has priority $p' = \mathrm{pr}(S') < p$. It follows that $S' \cup \{(p',q)\}$ is a prioritized low subset of P having priority $p' < \mathrm{pr}(S_q)$, and is therefore of the form S_k for some $k < \ell$. In this case, we again conclude that $\tau_{S'}$ is contained in $K_k \subseteq K_{\ell-1}$.

We now prove $(*')$ in the case where $(i,j) = (p,q)$; in this case, we wish to show that $\tau_{S'}$ is not contained in $K_{\ell-1}$. Note that, since S' contains $(p,q+1)$, we have $C(S') \cup \{q\} = C(S_\ell) \cup \{q\}$. Since S_ℓ is not basic, it follows that S' is not basic. Assume, for a contradiction, that $\tau_{S'}$ is contained in $K_{\ell-1}$; it follows that we have $S' \subseteq S_k$ for some $1 \leq k \leq< \ell$. We then have $\mathrm{pr}(S_k) \leq \mathrm{pr}(S_\ell) = p$. Since S_k contains $(p,q+1)$, we must have $\mathrm{pr}(S_k) = p$. Since S_k is prioritized, it contains (p,q), and therefore contains $S' \cup \{(p,q)\} = S_\ell$. The inequality $k < \ell$ guarantees that $|S_k| \leq |S_\ell|$. It follows that $S_k = S_\ell$, contradicting our assumption that $k < \ell$. This completes the proof of $(*)$.

Since $\mathcal{C}$ is an ∞-category, assertion $(*)$ guarantees that the morphism $F_0 : K_0 \to \mathcal{C}$ admits an extension $F_{\mathrm{low}} : K_{\mathrm{low}} \to \mathcal{C}$ (Proposition 1.5.6.7). Similarly, the morphism F_0 admits an extension $F_{\mathrm{high}} : K_{\mathrm{high}} \to \mathcal{C}$. Let K denote the union of K_{low} with K_{high} (as simplicial subsets of $\mathrm{N}_\bullet(P)$). Since the intersection $K_{\mathrm{low}} \cap K_{\mathrm{high}}$ coincides with K_0, we can amalgamate F_{low} with F_{high} to obtain a morphism of simplicial sets $F : K \to \mathcal{C}$. We will complete the proof of Proposition 8.1.4.2 by showing that F can be extended to a morphism $\mathrm{N}_\bullet(P) \to \mathcal{C}$.

Set $P_- = \{(i,j) \in P : (i,j) < (q-1,q+1)\}$ and $P_+ = \{(i,j) \in P : (i,j) > (q-1,q+1)\}$. Let us say that a nonempty linearly ordered subset $S \subseteq P$ is *decomposable* if the union $S \cup \{(q-1,q+1)\}$ is also linearly ordered. In this case, we can write S (uniquely) as a union $S_- \cup S_0 \cup S_+$, where $S_- \subseteq P_-$, $S_0 \subseteq \{(q-1,q+1)\}$, and $S_+ \subseteq P_+$. The collection of simplices τ_S, where S is decomposable, span a simplicial subset of $\mathrm{N}_\bullet(P)$ which will identify with the join $\mathrm{N}_\bullet(P_-) \star \{(q-1,q+1)\} \star \mathrm{N}_\bullet(P_+)$.

We next claim that $\mathrm{N}_\bullet(P)$ is the union of K with the join $\mathrm{N}_\bullet(P_-) \star \{(q-1,q+1)\} \star \mathrm{N}_\bullet(P_+)$. In other words, if a nonempty linearly ordered subset $S \subseteq P$ is not decomposable, then τ_S is contained in K. Choose an element $(i,j) \in S$ which is incomparable with $(q-1,q+1)$ in the partially ordered set P. Without loss of generality, we may assume that $i < q-1$ and $j < q+1$. If S is basic, there is nothing to prove. We may therefore assume that $q+1$ belongs to the content $C(S)$. Note that ordered pairs of the form $(q+1,r)$ are incomparable with (i,j), and therefore cannot be contained in S. It follows that S contains an element of the form $(p,q+1)$. Since $(p,q+1)$ is comparable with (i,j) in P, we must have $p \leq i \leq q-2$. It follows that S is low, so that τ_S is contained in $K_{\mathrm{low}} \subseteq K$.

Let $K' \subseteq K$ denote the intersection of K with the join $\mathrm{N}_\bullet(P_-) \star \{(q-1,q+1)\} \star \mathrm{N}_\bullet(P_+)$, so that we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
K' \arrow[r] \arrow[d] & K \arrow[d] \\
\mathrm{N}_\bullet(P_-) \star \{(q-1,q+1)\} \star \mathrm{N}_\bullet(P_+) \arrow[r] & \mathrm{N}_\bullet(P).
\end{tikzcd}$$

Set $F' = F|_{K'}$. To complete the proof, it will suffice to show that F' can be extended to a morphism of simplicial sets $\mathrm{N}_\bullet(P_-) \star \{(q-1,q+1)\} \star \mathrm{N}_\bullet(P_+) \to \mathcal{C}$.

We now give a more explicit description of K'. Let us say that a linearly ordered subset $S_+ \subseteq P_+$ is *old* if the simplex $\tau_{S_+ \cup \{(q-1,q+1)\}}$ is contained in K. Let $S \subseteq P$ be an arbitrary decomposable subset, and write $S = S_- \cup S_0 \cup S_+$ as above. We then make the following observations:

(*a*) Suppose that $S_0 = \{(q-1,q+1)\}$. Then replacing S by the subset $S_0 \cup S_+$ does not change the set $C(S) \cup \{q\}$. In particular, S is basic if and only if $S_0 \cup S_+$ is basic. Moreover, since S_- does not contain any pairs of the form $(p, q+1)$ for $p \leq q-1$, it follows that S is low if and only if $S_0 \cup S_+$ is low. Similarly, S is high if and only if $S_0 \cup S_+$ is high. It follows that τ_S is contained in K if and only if $\tau_{S_0 \cup S_+}$ is contained in K: that is, if and only if S_+ is old.

(*b*) Suppose that $S_0 = \emptyset$. In this case, we claim that τ_S is automatically contained in K. Assume otherwise. Then S is not basic, so the set $C(S)$ contains the element $q+1$. Since ordered pairs of the form $(q+1, q')$ are incomparable with $(q-1, q+1)$ for $q' > q+1$, it follows that S contains an ordered pair of the form $(p, q+1)$ for some $p \leq q+1$. Since S is not low, we must have $p \geq q-1$. By assumption, S does not contain $(q-1, q+1)$, so we must have $p \in \{q, q+1\}$. By the same reasoning (using the fact that $C(S)$ contains $q-1$ and S is not high), we conclude that S contains an element of the form $(q-1, r)$ for $r \in \{q-1, q\}$. This is a contradiction, since S is linearly ordered and the ordered pairs $(p, q+1)$ and $(q-1, r)$ are incomparable in P.

Let $A \subseteq \mathrm{N}_\bullet(P_+)$ denote the simplicial subset spanned by the simplices τ_{S_+}, where S_+ is old. Combining (*a*) and (*b*), we deduce that K' can be identified with the pushout

$$(\mathrm{N}_\bullet(P_-) \star \{(q-1,q+1)\} \star A) \coprod_{\mathrm{N}_\bullet(P_-) \star A} (\mathrm{N}_\bullet(P_-) \star \mathrm{N}_\bullet(P_+)).$$

Note that, since the restriction of f_0 to the simplex $\mathrm{N}_\bullet(\{q-1 < q < q+1\})$ coincides with σ, the restriction of F' to the simplicial subset $\mathrm{N}_\bullet(P_-) \star \{(q-1,q+1)\} \subseteq K$ can be identified with the diagram $\varphi : \mathrm{Tw}(\Delta^2) \to \mathcal{C}$. Let φ_0 denote the restriction of φ to

$\mathrm{N}_\bullet(P_-)$. Unwinding the definitions, we see that the problem of extension of F' to a morphism $\mathrm{N}_\bullet(P_-) \star \{(q-1, q+1)\} \star \mathrm{N}_\bullet(P_+) \rightarrow \mathcal{C}$ can be rewritten as a lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \mathcal{C}_{\varphi/} \arrow[d] \\
\mathrm{N}_\bullet(P_+) \arrow[r] \arrow[ur, dashed] & \mathcal{C}_{\varphi_0/} .
\end{tikzcd}$$

This lifting problem admits a solution by virtue of our assumption that φ is a colimit diagram in $\mathcal{C}$. □

8.1.6 Restricted Cospans

Let $\mathcal{C}$ be an ∞-category, and let $\mathrm{Cospan}(\mathcal{C})$ be the simplicial set introduced in Construction 8.1.3.1. By definition, the edges of $\mathrm{Cospan}(\mathcal{C})$ correspond to cospans in the ∞-category $\mathcal{C}$: that is, pairs of morphisms $X \xrightarrow{f} B \xleftarrow{g} Y$ having a common target. In practice, it will sometimes be useful to consider a variant of this construction, where we place additional restrictions on the morphisms f and g. 048Z

Definition 8.1.6.1 (Restricted Cospans)**.** Let $\mathcal{C}$ be a simplicial set and let L and R be collections of edges of $\mathcal{C}$. We let $\mathrm{Cospan}^{L,R}(\mathcal{C})$ denote the simplicial subset of $\mathrm{Cospan}(\mathcal{C})$ whose n-simplices are given by diagrams $X : \mathrm{Tw}(\Delta^n) \rightarrow \mathcal{C}$ which satisfy the following condition: 0490

- For every pair of integers $0 \leq i \leq j \leq n$, the edge $X_{i,i} \rightarrow X_{i,j}$ belongs to L and the edge $X_{j,j} \rightarrow X_{i,j}$ belongs to R.

Remark 8.1.6.2 (Symmetry)**.** Let $\mathcal{C}$ be a simplicial set and let L and R be collections of edges of $\mathcal{C}$. Then the isomorphism $\mathrm{Cospan}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Cospan}(\mathcal{C})^{\mathrm{op}}$ of Remark 8.1.3.4 restricts to an isomorphism of simplicial subsets $\mathrm{Cospan}^{L,R}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Cospan}^{R,L}(\mathcal{C})^{\mathrm{op}}$. 0491

Remark 8.1.6.3. Let $\mathcal{C}$ be a simplicial set and let L and R be collections of edges of $\mathcal{C}$. Suppose we are given a morphism of simplicial sets $f : \mathcal{D} \rightarrow \mathrm{Cospan}(\mathcal{C})$, corresponding to a morphism $F : \mathrm{Tw}(\mathcal{D}) \rightarrow \mathcal{C}$ (see Proposition 8.1.3.7). For every edge $e : X \rightarrow Y$ of $\mathcal{D}$, let $e_L : \mathrm{id}_X \rightarrow e$ and $e_R : \mathrm{id}_Y \rightarrow e$ be the edges of $\mathrm{Tw}(\mathcal{D})$ described in Example 8.1.3.6. Then f factors through the simplicial subset $\mathrm{Cospan}^{L,R}(\mathcal{C})$ if and only if the edge $F(e_L)$ belongs to L and the edge $F(e_R)$ belongs to R, for every edge e of $\mathcal{D}$. 0492

Remark 8.1.6.4. Let $\mathcal{C}$ be a simplicial set, let L and R be collections of edges of $\mathcal{C}$, and let $\mathrm{Cospan}^{L,R}(\mathcal{C})$ denote the restricted cospan construction of Definition 8.1.6.1. Note that a 0493 morphism of simplicial sets $K \to \mathrm{Cospan}(\mathcal{C})$ factors through $\mathrm{Cospan}^{L,R}(\mathcal{C})$ if and only if its restriction to the 1-skeleton $\mathrm{sk}_1(K)$ factors through $\mathrm{Cospan}^{L,R}(\mathcal{C})$. In particular, if σ is a 2-simplex of $\mathrm{Cospan}^{L,R}(\mathcal{C})$ which is thin when viewed as a 2-simplex of $\mathrm{Cospan}(\mathcal{C})$, then it is also thin when viewed as a 2-simplex of $\mathrm{Cospan}^{L,R}(\mathcal{C})$. For a partial converse, see Corollary 8.1.6.8.

We now formulate a criterion which guarantees that the simplicial set $\mathrm{Cospan}^{L,R}(\mathcal{C})$ is an $(\infty, 2)$-category.

0494 **Definition 8.1.6.5.** Let $\mathcal{C}$ be an ∞-category and let L and R be collections of morphisms of $\mathcal{C}$. We will say that L and R are *pushout-compatible* if, for every morphism $f_0 : X \to X_0$ of $\mathcal{C}$ which belongs to L and every morphism $f_1 : X \to X_1$ of $\mathcal{C}$ which belongs to R, there exists a pushout diagram

$$\begin{tikzcd}
X \arrow[r, "f_0"] \arrow[d, "f_1"] & X_0 \arrow[d, "f_1'"] \\
X_1 \arrow[r, "f_0'"] & X_{01}
\end{tikzcd}$$

where f_0' belongs to L and f_1' belongs to R.

0495 **Example 8.1.6.6.** Let $\mathcal{C}$ be an ∞-category, let L be a collection of morphisms of $\mathcal{C}$, and let R be the collection of all isomorphisms in $\mathcal{C}$. Assume that L is stable under isomorphism (that is, if f if a morphism of $\mathcal{C}$ which is isomorphic to an element of L in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$, then f also belongs to L). Then L and R are pushout-compatible.

0496 **Proposition 8.1.6.7.** *Let $\mathcal{C}$ be an ∞-category and let L and R denote collections of morphisms of $\mathcal{C}$ which are closed under composition. If L and R are pushout-compatible (in the sense of Definition 8.1.6.5), then the simplicial set* $\mathrm{Cospan}^{L,R}(\mathcal{C})$ *is an* $(\infty, 2)$*-category.*

Proof. We will show that $\mathrm{Cospan}^{L,R}(\mathcal{C})$ satisfies each condition of Definition 5.4.1.1:

(1) Let $\sigma_0 : \Lambda^2_1 \to \mathrm{Cospan}^{L,R}(\mathcal{C})$ be a morphism of simplicial sets; we wish to show that σ_0 can be extended to a thin 2-simplex of $\mathrm{Cospan}^{L,R}(\mathcal{C})$. Let us identify σ_0 with a diagram

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$$\begin{tikzcd}
X_{0,0} \arrow[dr, "u"] & & X_{1,1} \arrow[dl, "u'"'] \arrow[dr, "v"] & & X_{2,2} \arrow[dl, "v'"'] \\
& X_{0,1} & & X_{1,2} &
\end{tikzcd} \tag{8.13}$$

in the ∞-category $\mathcal{C}$, where the morphisms u and v belong to L and the morphisms u' and v' belong to R. Combining our assumption that L and R are pushout-compatible with Lemma 8.1.4.4, we can enlarge (8.13) to a commutative diagram

$$\begin{tikzcd}
X_{0,0} \arrow[dr, "u"] & & X_{1,1} \arrow[dl, "u'"'] \arrow[dr, "v"] & & X_{2,2} \arrow[dl, "v'"'] \\
& X_{0,1} \arrow[dr, "w"] & & X_{1,2} \arrow[dl, "w'"'] & \\
& & X_{0,2}, & &
\end{tikzcd}$$

where w belongs to L, w' belongs to R, and the lower square is a pushout. By virtue of Proposition 8.1.4.2, this extension can be viewed as a thin 2-simplex of the simplicial set $\operatorname{Cospan}(\mathcal{C})$. Using our assumption that L and R are closed under composition, we see that σ belongs to the simplicial subset $\operatorname{Cospan}^{L,R}(\mathcal{C})$, and is therefore also thin when regarded as a 2-simplex of $\operatorname{Cospan}^{L,R}(\mathcal{C})$ (Remark 8.1.6.4).

(2) Every degenerate 2-simplex of $\operatorname{Cospan}^{L,R}(\mathcal{C})$ is thin; this follows from Corollary 8.1.4.3 and Remark 8.1.6.4.

(3) Let $n \geq 3$ and let $\sigma_0 : \Lambda^n_0 \to \operatorname{Cospan}^{L,R}(\mathcal{C})$ be a morphism of simplicial sets with the property that the 2-simplex $\sigma_0|_{\mathrm{N}_\bullet(\{0<1<n\})}$ is left-degenerate. Using Lemma 8.1.4.7, we can extend σ_0 to an n-simplex σ of $\operatorname{Cospan}(\mathcal{C})$. Since every edge of Δ^n is contained in Λ^n_0, the extension σ is automatically contained in $\operatorname{Cospan}^{L,R}(\mathcal{C})$ (Remark 8.1.6.4).

(4) Let $n \geq 3$ and let $\sigma_0 : \Lambda^n_n \to \operatorname{Cospan}^{L,R}(\mathcal{C})$ be a morphism of simplicial sets with the property that the 2-simplex $\sigma_0|_{\mathrm{N}_\bullet(\{0<n-1<n\})}$ is right-degenerate. Then σ_0 can be extended to an n-simplex of $\operatorname{Cospan}^{L,R}(\mathcal{C})$; this follows by applying (3) to the opposite simplicial set $\operatorname{Cospan}^{L,R}(\mathcal{C})^{\mathrm{op}} \simeq \operatorname{Cospan}^{R,L}(\mathcal{C})$ (see Remark 8.1.6.2).

□

Corollary 8.1.6.8. *Let $\mathcal{C}$ be an ∞-category and let L and R denote collections of morphisms of $\mathcal{C}$ which are closed under composition. If L and R are pushout-compatible, then a 2-simplex of $\operatorname{Cospan}^{L,R}(\mathcal{C})$ is thin if and only if it is thin when viewed as a 2-simplex of the simplicial set $\operatorname{Cospan}(\mathcal{C})$.* 0498

Proof. Let σ be a thin 2-simplex of $\mathrm{Cospan}^{L,R}(\mathcal{C})$; we will show that σ is also thin when viewed as a 2-simplex of $\mathrm{Cospan}(\mathcal{C})$ (the reverse implication follows from Remark 8.1.6.4). Choose a fully faithful functor $f : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is an ∞-category which admits pushouts and the functor f preserves all pushout squares which exist in $\mathcal{C}$ (see Corollary 8.3.3.17). Then f induces a morphism of simplicial sets $F : \mathrm{Cospan}^{L,R}(\mathcal{C}) \to \mathrm{Cospan}(\mathcal{D})$. The proof of Proposition 8.1.6.7 shows that every morphism $\tau_0 : \Lambda^2_1 \to \mathrm{Cospan}^{L,R}(\mathcal{C})$ can be extended to a 2-simplex τ of $\mathrm{Cospan}^{L,R}(\mathcal{C})$ which is thin when viewed as a 2-simplex of $\mathrm{Cospan}(\mathcal{C})$. Since f preserves pushout squares, the criterion of Proposition 8.1.4.2 guarantees that $F(\tau)$ is a thin 2-simplex of $\mathrm{Cospan}(\mathcal{D})$. Allowing τ_0 to vary and invoking Proposition 5.4.7.9, we deduce that F is a functor of $(\infty, 2)$-categories. In particular, $F(\sigma)$ is a thin 2-simplex of $\mathrm{Cospan}(\mathcal{D})$. Using the criterion of Proposition 8.1.4.2 and the assumption that f is fully faithful, we deduce that σ is also thin when viewed as a 2-simplex of $\mathrm{Cospan}(\mathcal{C})$.

□

0499 **Remark 8.1.6.9.** In the situation of Proposition 8.1.6.7, let σ be an n-simplex of the $(\infty, 2)$-category $\mathrm{Cospan}^{L,R}(\mathcal{C})$, corresponding to a diagram

049A

$$\begin{tikzcd}
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl] \arrow[dr] & \cdots & X_{n-1,n-1} \arrow[dl] \arrow[dr] & & X_{n,n} \arrow[dl] \\
& \cdots \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & \\
& & X_{0,n-1} \arrow[dr] & & X_{1,n} \arrow[dl] & & \\
& & & X_{0,n} & & &
\end{tikzcd} \tag{8.14}$$

in the ∞-category $\mathcal{C}$. Then σ is contained in the pith $\mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C}))$ if and only if each of the rectangular regions in the diagram (8.14) is a pushout square in the ∞-category $\mathcal{C}$. This follows from the thinness criterion of Corollary 8.1.6.8 (together with Proposition 7.6.3.25).

049B **Corollary 8.1.6.10** (Invertible Cospans)**.** *Let $\mathcal{C}$ be an ∞-category. Let L and R be collections of morphisms of $\mathcal{C}$ which contain all isomorphisms and are closed under composition. Assume that L and R are pushout-compatible and let $e : X \to Y$ be a morphism in the $(\infty, 2)$-category $\mathrm{Cospan}^{L,R}(\mathcal{C})$, which we identify with a diagram $X \xrightarrow{f} B \xleftarrow{g} Y$ in the ∞-category $\mathcal{C}$. Then e is an isomorphism in $\mathrm{Cospan}^{L,R}(\mathcal{C})$ if and only if f and g are isomorphisms in $\mathcal{C}$.*

Proof. Let $\mathcal{C}^{\simeq}$ denote the core of $\mathcal{C}$ (Construction 1.3.5.4). If f and g are isomorphisms, then e can be regarded as an edge of the simplicial set $\mathrm{Cospan}(\mathcal{C}^{\simeq})$. Since $\mathcal{C}^{\simeq}$ is a Kan complex (Corollary 4.4.3.11), the simplicial set $\mathrm{Cospan}(\mathcal{C}^{\simeq})$ is also a Kan complex (Corollary 8.1.3.11), so e is automatically an isomorphism when viewed as a morphism of $\mathrm{Cospan}(\mathcal{C}^{\simeq})$ (Proposition 1.4.6.10). Since the inclusion map $\mathrm{Cospan}(\mathcal{C}^{\simeq}) \hookrightarrow \mathrm{Cospan}^{L,R}(\mathcal{C})$ is a functor of $(\infty, 2)$-categories (Corollary 8.1.6.8), it follows that e is also an isomorphism when regarded as a morphism of $\mathrm{Cospan}^{L,R}(\mathcal{C})$ (Remark 5.4.7.5).

Now suppose that e is an isomorphism in the $(\infty, 2)$-category $\mathrm{Cospan}^{L,R}(\mathcal{C})$. Arguing as in the proof of Proposition 5.4.6.6, we can produce a 3-simplex $\sigma : \Delta^3 \to \mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C}))$, where $\sigma|_{\mathrm{N}_{\bullet}(\{0<1\})}$ is the morphism e, $\sigma|_{\mathrm{N}_{\bullet}(\{0<2\})}$ is the identity morphism id_X, and $\sigma|_{\mathrm{N}_{\bullet}(\{1<3\})}$ is the identity morphism id_Y. Let us identify σ with a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"] & & Y \arrow[dl, "g"', "\sim"] \arrow[dr] & & X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \\
& B \arrow[dr, "u"] & & C \arrow[dl] \arrow[dr] & & D \arrow[dl] & \\
& & X \arrow[dr, "v"] & & Y \arrow[dl] & & \\
& & & Z & & &
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. This diagram exhibits the identity morphism id_X as a composition of u with f. Since $\sigma|_{\mathrm{N}_{\bullet}(\{0<1<3\})}$ is a thin 2-simplex of $\mathrm{Cospan}^{L,R}(\mathcal{C})$, the outer rectangular region on the left is a pushout square in $\mathcal{C}$ (Corollary 8.1.6.8). It follows that the composition of v with u is an isomorphism in $\mathcal{C}$ (since it is a pushout of the identity morphism id_Y; see Corollary 7.6.3.24). Applying the two-out-of-six property to the 3-simplex of $\mathcal{C}$ given by the left edge of the diagram, we conclude that f is an isomorphism in $\mathcal{C}$ (see Proposition 5.4.6.5). A similar argument shows that g is an isomorphism in $\mathcal{C}$. □

Exercise 8.1.6.11. Show that, if the conditions of Corollary 8.1.6.10 are satisfied, then the diagram $Y \xrightarrow{g} B \xleftarrow{f} X$ is a homotopy inverse of e, when regarded as a morphism from Y to X in the $(\infty, 2)$-category $\mathrm{Cospan}^{L,R}(\mathcal{C})$. 049C

Corollary 8.1.6.12. *Let $\mathcal{C}$ be an ∞-category containing objects X and Y. Let L and R be collections of morphisms of $\mathcal{C}$ which are pushout-compatible, contain all isomorphisms of $\mathcal{C}$,* 049D *and are closed under composition. Then X and Y are isomorphic as objects of the ∞-category $\mathcal{C}$ if and only if they are isomorphic as objects of the $(\infty, 2)$-category* $\mathrm{Cospan}^{L,R}(\mathcal{C})$.

8.1.7 Comparing $\mathcal{C}$ with $\mathrm{Cospan}(\mathcal{C})$

`049E` Let $\mathcal{C}$ be a category which admits pushouts. Then there is a functor from $\mathcal{C}$ to the 2-category $\mathrm{Cospan}(\mathcal{C})$ of Example 2.2.2.1, which carries each object of $\mathcal{C}$ to itself and each morphism $f : X \to Y$ to the cospan $X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y} Y$. This observation has an ∞-categorical counterpart:

`049F` **Construction 8.1.7.1.** Let $\mathcal{C}$ be a simplicial set and let $\lambda_+ : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}$ be the projection map of Notation 8.1.1.6, carrying each vertex $(f : X \to Y)$ of $\mathrm{Tw}(\mathcal{C})$ to the vertex $Y \in \mathcal{C}$. Under the bijection supplied by Proposition 8.1.3.7, we can identify λ_+ with a morphism of simplicial sets $\rho_+ : \mathcal{C} \to \mathrm{Cospan}(\mathcal{C})$. If σ is an n-simplex of $\mathcal{C}$, which we display informally as a diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \xrightarrow{f_n} X_n,$$

then $\rho_+(\sigma)$ is an n-simplex of $\mathrm{Cospan}(\mathcal{C})$ which can be depicted informally as a diagram

$$\begin{tikzcd}
X_0 \arrow[dr, "f_1"] & & X_1 \arrow[dl, "\mathrm{id}"] \arrow[dr, "f_2"] & & \cdots \arrow[dl] \arrow[dr] & & X_{n-1} \arrow[dl, "\mathrm{id}"] \arrow[dr, "f_n"] & & X_n \arrow[dl, "\mathrm{id}"] \\
& X_1 \arrow[dr, "f_2"] & & X_2 \arrow[dl, "\mathrm{id}"] \arrow[dr, "f_3"] & \cdots & X_{n-1} \arrow[dl, "\mathrm{id}"] \arrow[dr, "f_n"] & & X_n \arrow[dl, "\mathrm{id}"] & \\
& & \cdots \arrow[dr, "f_{n-1}"] & & \cdots \arrow[dl, "\mathrm{id}"] \arrow[dr, "f_n"] & & \cdots \arrow[dl, "\mathrm{id}"] & & \\
& & & X_{n-1} \arrow[dr, "f_n"] & & X_n \arrow[dl, "\mathrm{id}"] & & & \\
& & & & X_n. & & & &
\end{tikzcd}$$

Note that ρ_+ is a monomorphism of simplicial sets.

Our goal in this section is to study the behavior of Construction 8.1.7.1 in the case where the simplicial set $\mathcal{C}$ is an ∞-category. In this case, we will show that ρ_+ induces an equivalence of $\mathcal{C}$ with a certain restricted cospan construction (Proposition 8.1.7.6).

`049G` **Construction 8.1.7.2.** Let $\mathcal{C}$ be an ∞-category. We let $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ denote the simplicial subset of $\operatorname{Cospan}(\mathcal{C})$ whose n-simplices are diagrams

$$\begin{tikzcd}
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl, "\sim"'] \arrow[dr] & \cdots & X_{n-1,n-1} \arrow[dl, "\sim"'] \arrow[dr] & & X_{n,n} \arrow[dl, "\sim"'] \\
& \cdots \arrow[dr] & & \cdots \arrow[dl, "\sim"'] \arrow[dr] & & \cdots \arrow[dl, "\sim"'] & \\
& & X_{0,n-1} \arrow[dr] & & X_{1,n} \arrow[dl, "\sim"'] & & \\
& & & X_{0,n}, & & &
\end{tikzcd}$$

where each of the leftward-directed arrows is an isomorphism in $\mathcal{C}$.

Remark 8.1.7.3. Let $\mathcal{C}$ be an ∞-category. Then the simplicial set $\operatorname{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ of Construction 8.1.7.2 coincides with the restricted cospan construction $\operatorname{Cospan}^{L,R}(\mathcal{C})$ of Definition 8.1.6.1, where we take L to be the collection of all morphisms of $\mathcal{C}$ and R to be the collection of all isomorphisms in $\mathcal{C}$ (see Example 8.1.7.10 for a more general statement). `049H`

Variant 8.1.7.4. Let $\mathcal{C}$ be a simplicial set and let R be a collection of edges of $\mathcal{C}$. We let $\operatorname{Cospan}^{\mathrm{all},R}(\mathcal{C})$ denote the simplicial subset $\operatorname{Cospan}^{A,R}(\mathcal{C}) \subseteq \operatorname{Cospan}(\mathcal{C})$, where A is the collection of all edges of $\mathcal{C}$. Similarly, if L is a collection of edges of $\mathcal{C}$, we let $\operatorname{Cospan}^{L,\mathrm{all}}(\mathcal{C})$ denote the simplicial subset $\operatorname{Cospan}^{L,A}(\mathcal{C}) \subseteq \operatorname{Cospan}(\mathcal{C})$. Note that the simplicial set $\operatorname{Cospan}^{L,R}(\mathcal{C})$ of Definition 8.1.6.1 can be recovered as the intersection $\operatorname{Cospan}^{L,\mathrm{all}}(\mathcal{C}) \cap \operatorname{Cospan}^{\mathrm{all},R}(\mathcal{C})$. `049J`

Proposition 8.1.7.5. *Let $\mathcal{C}$ be an ∞-category. Then the simplicial set $\operatorname{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ is an ∞-category.* `049K`

Proof. Let σ be a 2-simplex of $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$, which we identify with a diagram

$$\begin{tikzcd}
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl, "\sim"'] \arrow[dr] & & X_{2,2} \arrow[dl, "\sim"'] \\
& X_{0,1} \arrow[dr] & & X_{1,2} \arrow[dl, "\sim"'] & \\
& & X_{0,2} & &
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$ where the leftward-directed morphisms are isomorphisms. Using Corollary 7.6.3.24, we deduce that the inner region is a pushout square in $\mathcal{C}$. It follows that σ is automatically thin when regarded as a 2-simplex of $\mathrm{Cospan}(\mathcal{C})$ (Proposition 8.1.4.2), and therefore also when regarded as a 2-simplex of $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ (Remark 8.1.6.4). To complete the proof, it will suffice to show that every diagram $\Lambda^2_1 \to \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ can be extended to a 2-simplex of $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ (see Example 2.3.2.4). Using Lemma 8.1.4.4, we can restate this as follows: for every pair of morphisms $f : X_{1,1} \to X_{1,2}$ and $u : X_{1,1} \to X_{0,1}$ of $\mathcal{C}$ where u is an isomorphism, there exists a commutative diagram

$$\begin{tikzcd}
& X_{1,1} \arrow[dl, "u"'] \arrow[dr, "f"] & \\
X_{0,1} \arrow[dr] & & X_{1,2} \arrow[dl, "v"'] \\
& X_{0,2} &
\end{tikzcd}$$

where v is also an isomorphism. This follows immediately from the definitions (or from Corollary 4.4.5.9). □

Let $\mathcal{C}$ be an ∞-category and let $\rho_+ : \mathcal{C} \hookrightarrow \mathrm{Cospan}(\mathcal{C})$ the morphism described in Construction 8.1.7.1. Note that ρ_+ carries each object of $\mathcal{C}$ to itself, and each morphism $f : X \to Y$ of $\mathcal{C}$ to the edge of $\mathrm{Cospan}(\mathcal{C})$ given by the diagram $X \xrightarrow{f} Y \xleftarrow{\mathrm{id}_Y}$. Since every identity morphism in $\mathcal{C}$ is an isomorphism, ρ_+ factors through the ∞-category $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$. The remainder of this section is devoted to the proof of the following:

Proposition 8.1.7.6. *Let $\mathcal{C}$ be an ∞-category. Then the functor $\rho_+ : \mathcal{C} \hookrightarrow \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ is an equivalence of ∞-categories.* 049L

Example 8.1.7.7. Let X be a Kan complex. Applying Proposition 8.1.7.6 (and noting that every edge of X is an isomorphism), we see that $\rho_+ : X \to \mathrm{Cospan}(X)$ is a homotopy equivalence of Kan complexes (see Corollary 8.1.3.11). 049M

Our proof of Proposition 8.1.7.6 will require some preliminaries.

Definition 8.1.7.8. Let $\mathcal{C}$ be a simplicial set and let W be a collection of morphisms of $\mathcal{C}$. We say that W *has the left two-out-of-three property* if, for every 2-simplex of $\mathcal{C}$ with boundary indicated in by the diagram 049N

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\
& Y, \arrow[ur, "g"'] &
\end{tikzcd}$$

where f belongs to W, g belongs to W if and only if h belongs to W.

Lemma 8.1.7.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories and let $F : \mathrm{Tw}(\mathcal{D}) \to \mathcal{C}$ be a functor, corresponding to a morphism of simplicial sets $f : \mathcal{D} \to \mathrm{Cospan}(\mathcal{C})$. Let L and R be collections of morphisms of $\mathcal{C}$ which satisfy the left two-out-of-three property. Then f factors through $\mathrm{Cospan}^{L,R}(\mathcal{C})$ if and only if F satisfies the following pair of conditions:* 049P

(1) *For each object X in $\mathcal{D}$, the morphism F carries every morphism of $\{X\} \times_{\mathcal{D}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{D})$ to a morphism of $\mathcal{C}$ which belongs to L.*

(2) *For each vertex Y in $\mathcal{D}$, the morphism F carries every morphism of $\mathrm{Tw}(\mathcal{D}) \times_{\mathcal{D}} \{Y\}$ to a morphism of $\mathcal{C}$ which belongs to R.*

Proof. For every morphism $e : X \to Y$ of $\mathcal{D}$, let $\mathrm{id}_X \xrightarrow{e_L} e \xleftarrow{e_R} \mathrm{id}_Y$ be the tautological cospan in $\mathrm{Tw}(\mathcal{D})$ described in Example 8.1.3.6. By virtue of Remark 8.1.6.3, the morphism f factors through $\mathrm{Cospan}^{L,R}(\mathcal{C})$ if and only if it satisfies the following pair of conditions:

(1′) For every morphism $e : X \to Y$ of $\mathcal{D}$, the functor F carries e_L to a morphism of $\mathcal{C}$ which belongs to L.

(2′) For every morphism $e : X \to Y$ of $\mathcal{D}$, the functor F carries e_R to a morphism of $\mathcal{C}$ which belongs to R.

The implications (1) ⇒ (1′) and (2) ⇒ (2′) are immediate (if $e : X \to Y$ is any morphism of $\mathcal{D}$, then e_L is contained to the fiber $\{X\} \times_{\mathcal{D}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{D})$, and e_R is contained in $\mathrm{Tw}(\mathcal{D}) \times_{\mathcal{D}} \{Y\}$). We will complete the proof by showing that (1′) implies (1); a similar argument shows that (2′) implies (2).

Assume that condition (1′) is satisfied, let X be an object of $\mathcal{D}$, and let $u : X \to Y$ and $v : X \to Z$ be morphisms of $\mathcal{D}$. Suppose we are given a morphism $g : u \to v$ in the ∞-category $\mathcal{E} = \{X\} \times_{\mathcal{D}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{D})$; we wish to show that $F(g)$ belongs to L. Since id_X is initial when viewed as an object of $\mathcal{E}$ (Proposition 8.1.2.1), there is a 2-simplex of $\mathcal{E}$ whose boundary is indicated in the diagram

$$\begin{tikzcd}
& \mathrm{id}_X \arrow[dl, "u_L"'] \arrow[dr, "v_L"] & \\
u \arrow[rr, "g"] & & v.
\end{tikzcd}$$

Assumption (1′) guarantees that $F(u_L)$ and $F(v_L)$ belong to L. Since L satisfies the left two-out-of-three property, it follows that $F(g)$ also belongs to L. □

049Q **Example 8.1.7.10.** Let $\mathcal{C}$ be an ∞-category and let σ be an n-simplex of the simplicial set $\mathrm{Cospan}(\mathcal{C})$, corresponding to a diagram

049R

$$\begin{tikzcd}
X_{0,0} \arrow[dr] & & X_{1,1} \arrow[dl] \arrow[dr] & \cdots & X_{n-1,n-1} \arrow[dl] \arrow[dr] & & X_{n,n} \arrow[dl] \\
& \cdots \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] & \\
& & X_{0,n-1} \arrow[dr] & & X_{1,n} \arrow[dl] & & \\
& & & X_{0,n}. & & &
\end{tikzcd} \qquad (8.15)$$

Let L and R be collections of morphisms of $\mathcal{C}$ which contain all identity morphisms and have the left two-out-of-three property. Then σ belongs to $\mathrm{Cospan}^{L,R}(\mathcal{C})$ if and only if each of the rightward-pointing morphisms displayed in (8.15) belong to L, and each of the leftward-pointing morphisms displayed in (8.15) belong to R.

049S **Remark 8.1.7.11.** Let $\mathcal{C}$ be an ∞-category, let R be a collection of morphisms of $\mathcal{C}$ which has the left two-out-of-three property, and let $\mathcal{D}$ be a simplicial set. Suppose we are given a pair of morphisms $f, g : \mathcal{D} \to \mathrm{Fun}^{\mathrm{all},R}(\mathcal{C})$, corresponding to diagrams $F, G : \mathrm{Tw}(\mathcal{D}) \to \mathcal{C}$. If $\alpha : F \to G$ is a natural transformation of diagrams, then the following conditions are equivalent:

(a) For every edge $u : D' \to D$ of $\mathcal{D}$, the morphism $\alpha_u : F(u) \to G(u)$ belongs to R.

(b) For every degenerate edge $u : D \to D$ of $\mathcal{D}$, the morphism $\alpha_u : F(u) \to G(u)$ belongs to R.

The implication (a) ⇒ (b) is immediate. To prove the reverse implication, let $u : D' \to D$ be an edge of $\mathcal{D}$ and let $u_R : \mathrm{id}_D \to u$ be the edge of $\mathrm{Tw}(\mathcal{D})$ described in Example 8.1.3.6. Evaluating α on the morphism u_R, we obtain a commutative diagram

$$\begin{tikzcd}
F(\mathrm{id}_D) \arrow[r, "\alpha_{\mathrm{id}_D}"] \arrow[d] & G(\mathrm{id}_D) \arrow[d] \\
F(u) \arrow[r, "\alpha_u"] & G(u)
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, where the vertical maps belong to R by virtue of our assumption that f and g factor through $\mathrm{Cospan}^{\mathrm{all},R}(\mathcal{C})$. Applying the left two-out-of-three property, we conclude that if the upper horizontal map belongs to R, then the lower horizontal map also belongs to R.

Lemma 8.1.7.12. *Let $\mathcal{C}$ be an ∞-category, let L and R be collections of morphisms of $\mathcal{C}$ which have the left two-out-of-three property, and let $\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$ spanned by those objects which correspond to diagrams $\mathcal{D} \to \mathrm{Cospan}^{L,R}(\mathcal{C}) \subseteq \mathrm{Cospan}(\mathcal{C})$. Let R' be the collection of morphisms in $\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$ which satisfy the equivalent conditions of Remark 8.1.7.11, and define L' similarly. Then the isomorphism $\mathrm{Fun}(\mathcal{D}, \mathrm{Cospan}(\mathcal{C})) \simeq \mathrm{Cospan}(\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{C}))$ of Remark 8.1.3.9 restricts to an isomorphism of simplicial subsets $\mathrm{Fun}(\mathcal{D}, \mathrm{Cospan}^{L,R}(\mathcal{C})) \simeq \mathrm{Cospan}^{L',R'}(\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C}))$.* 049T

Proof. Writing $\mathrm{Cospan}^{L,R}(\mathcal{C})$ as the intersection $\mathrm{Cospan}^{L,\mathrm{all}}(\mathcal{C}) \cap \mathrm{Cospan}^{\mathrm{all},R}(\mathcal{C})$ (see Variant 8.1.7.4), we can reduce to the case where either L or R is the collection of all morphisms of $\mathcal{C}$. Let us assume that L is the collection of all morphisms of $\mathcal{C}$, so that L' is the collection of all morphisms of $\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$. Suppose we are given another simplicial set $\mathcal{E}$ and a diagram $F : \mathrm{Tw}(\mathcal{D}) \times \mathrm{Tw}(\mathcal{E}) \to \mathcal{C}$. We can identify F with a morphism of simplicial sets $\mathcal{E} \to \mathrm{Fun}(\mathcal{D}, \mathrm{Cospan}(\mathcal{C}))$. By virtue of Remark 8.1.6.3, this morphism factors through the simplicial subset $\mathrm{Fun}(\mathcal{D}, \mathrm{Cospan}^{L,R}(\mathcal{C}))$ if and only if, for every edge $u : D' \to D$ of $\mathcal{D}$ and every edge $v : E' \to E$ of $\mathcal{E}$, the morphism F satisfies the following condition:

($1_{u,v}$) Let $u_R : \mathrm{id}_D \to u$ and $v_R : \mathrm{id}_E \to v$ be the edges of $\mathrm{Tw}(\mathcal{D})$ and $\mathrm{Tw}(\mathcal{E})$ described in Example 8.1.3.6. Then the morphism $F(u_R, v_R)$ belongs to R.

Identifying F with a morphism $f : \mathcal{E} \to \mathrm{Cospan}(\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{C}))$, we see that f factors through $\mathrm{Cospan}(\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C}))$ if and only if it satisfies condition ($1_{u,v}$) whenever v is a degenerate edge of $\mathcal{E}$. Under this assumption, f factors through $\mathrm{Cospan}^{L',R'}(\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C}))$ if and only if, for every edge $u : D' \to D$ of $\mathcal{D}$ and every edge $v : E' \to E$ of $\mathcal{E}$, the diagram F satisfies the following condition:

($2_{u,v}$) The morphism $F(\mathrm{id}_u, v)$ belongs to R.

To complete the proof, it suffices to observe that if condition ($1_{u,\mathrm{id}_E}$) is satisfied, then condition ($1_{u,v}$) is equivalent to condition ($2_{u,v}$). This follows by applying the left two-out-of-three property to the upper triangle appearing in the diagram

$$\begin{tikzcd}
F(\mathrm{id}_D, \mathrm{id}_E) \arrow[r, "F(u_R,\mathrm{id})"] \arrow[d, "F(\mathrm{id},v_R)"] \arrow[dr, "F(u_R,v_R)"] & F(u, \mathrm{id}_E) \arrow[d, "F(\mathrm{id}_u,v_R)"] \\
F(\mathrm{id}_D, v) \arrow[r, "F(u_R,\mathrm{id}_v)"] & F(u, v).
\end{tikzcd}$$

□

049U **Lemma 8.1.7.13.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{D}$ be a simplicial set, and suppose we are given a pair of diagrams $f, g : \mathcal{D} \to \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{C})$, corresponding to diagrams $F, G : \mathrm{Tw}(\mathcal{D}) \to \mathcal{C}$. The following conditions are equivalent:*

(1) *The diagrams f and g are isomorphic when viewed as objects of the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{C}))$.*

(2) *The diagrams F and G are isomorphic when viewed as objects of the ∞-category $\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$.*

Proof. Let $\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$ spanned by those functors $\mathrm{Tw}(\mathcal{D}) \to \mathcal{C}$ which correspond to diagrams $\mathcal{D} \to \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{C})$. Lemma 8.1.7.12, identifies $\mathrm{Fun}(\mathcal{D}, \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{C}))$ with the ∞-category $\mathrm{Cospan}^{\mathrm{all,iso}}(\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C}))$. We are therefore reduced to proving that F and G are isomorphic when viewed as objects of the ∞-category $\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$ if and only if they are isomorphic when viewed as objects of the ∞-category $\mathrm{Cospan}^{\mathrm{all,iso}}(\mathrm{Fun}'(\mathrm{Tw}(\mathcal{D}), \mathcal{C}))$. This is a special case of Corollary 8.1.6.12. □

Proof of Proposition 8.1.7.6. Let $\mathcal{C}$ be an ∞-category. We wish to show that the comparison map $\rho_+ : \mathcal{C} \hookrightarrow \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{C})$ of Construction 8.1.7.1. Let $\mathcal{D}$ be a simplicial set; we will show that composition with ρ_+ induces a bijection $\theta : \pi_0(\mathrm{Fun}(\mathcal{D},\mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathcal{D}, \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C}))^{\simeq})$.

Let $\lambda_+ : \mathrm{Tw}(\mathcal{D}) \to \mathcal{D}$ denote the projection map, and let W be the collection of all edges e of $\mathrm{Tw}(\mathcal{D})$ such that $\lambda_+(e)$ is a degenerate edge of $\mathcal{D}$. Let $\mathrm{Fun}(\mathrm{Tw}(\mathcal{D})[W^{-1}], \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{C})$ spanned by those diagrams $F : \mathrm{Tw}(\mathcal{D}) \to \mathcal{C}$ which carry each edge of W to an isomorphism in $\mathcal{C}$ (Notation 6.3.1.1). Using Lemmas 8.1.7.9 and 8.1.7.13, we can identify θ with the map $\pi_0(\mathrm{Fun}(\mathcal{D},\mathcal{C})^{\simeq}) \to \pi_0(\mathrm{Fun}(\mathrm{Tw}(\mathcal{D})[W^{-1}],\mathcal{C})^{\simeq})$ given by composition λ_+. To complete the proof, it will suffice to show that λ_+ exhibits $\mathcal{D}$ as a localization of $\mathrm{Tw}(\mathcal{D})$ with respect to W, in the sense of Definition 6.3.1.9. This follows from Corollary 6.3.6.4, since the morphism λ_+ is universally localizing (Corollary 8.1.2.4). □

Variant 8.1.7.14. Let $\mathcal{C}$ be a simplicial set. Then the projection map $\lambda_- : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ determines a morphism of simplicial sets $\rho_- : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cospan}(\mathcal{C})$. If σ is an n-simplex of $\mathcal{C}^{\mathrm{op}}$, which we display informally as a diagram 049V

$$X_0 \xleftarrow{f_1} X_1 \xleftarrow{f_2} X_2 \leftarrow \cdots \xleftarrow{f_n} X_n,$$

then $\rho_-(\sigma)$ is an n-simplex of $\mathrm{Cospan}(\mathcal{C})$ which is depicted informally by the diagram

$$\begin{tikzcd}
X_0 \arrow[dr, "\mathrm{id}"] & & X_1 \arrow[dl, "f_1"] \arrow[dr, "\mathrm{id}"] & & \cdots \arrow[dl] \arrow[dr] & & X_{n-1} \arrow[dl, "f_{n-1}"] \arrow[dr, "\mathrm{id}"] & & X_n \arrow[dl, "f_n"] \\
& X_0 \arrow[dr, "\mathrm{id}"] & & X_1 \arrow[dl, "f_1"] \arrow[dr, "\mathrm{id}"] & \cdots & X_{n-2} \arrow[dl, "f_{n-2}"] \arrow[dr, "\mathrm{id}"] & & X_{n-1} \arrow[dl, "f_{n-1}"] & \\
& & \cdots \arrow[dr, "\mathrm{id}"] & & \cdots \arrow[dl, "f_1"] \arrow[dr, "\mathrm{id}"] & & \cdots \arrow[dl, "f_2"] & & \\
& & & X_0 \arrow[dr, "\mathrm{id}"] & & X_1 \arrow[dl, "f_1"] & & & \\
& & & & X_0. & & & &
\end{tikzcd}$$

If $\mathcal{C}$ is an ∞-category, then ρ_- restricts an equivalence $\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{C})$, where $\mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{C})$ is the simplicial subset $\mathrm{Cospan}^{L,R}(\mathcal{C}) \subseteq \mathrm{Cospan}(\mathcal{C})$ where L is the collection of isomorphisms in $\mathcal{C}$, and R is the collection of all morphisms of $\mathcal{C}$.

8.1.8 Morphisms in the Duskin Nerve

Let S be a simplicial set. Recall that, for every pair of vertices $X, Y \in S$, the morphism space $\mathrm{Hom}_S(X,Y)$ is defined by the formula 03KZ

$$\mathrm{Hom}_S(X,Y) = \{X\} \tilde{\times}_S \{Y\} = \{X\} \times_{\mathrm{Fun}(\{0\},S)} \mathrm{Fun}(\Delta^1, S) \times_{\mathrm{Fun}(\{1\},S)} \{Y\}.$$

In this section, we specialize to the case where $S = \mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is the Duskin nerve of a 2-category $\mathcal{C}$. In this case, we will see that there is close relationship between the simplicial set $\mathrm{Hom}_S(X, Y)$ and the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ of 1-morphisms from X to Y. More precisely, we will construct a comparison map

$$\mathrm{Cospan}(\mathrm{N}_{\bullet}\, \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)) \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X, Y),$$

and show that it is an isomorphism of simplicial sets (Corollary 8.1.8.6).

03L0 **Warning 8.1.8.1.** Let $\mathcal{C}$ be a 2-category. If every 2-morphism in $\mathcal{C}$ is invertible, then the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$ is an ∞-category (Theorem 2.3.2.1). It follows that, for every pair of objects $X, Y \in \mathcal{C}$, the simplicial set $\mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X, Y)$ is a Kan complex. Beware that, in the case where $\mathcal{C}$ contains non-invertible 2-morphisms, the simplicial set $\mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X, Y)$ is generally not an ∞-category (in fact, it is not even an $(\infty, 2)$-category unless the category $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ admits pushouts: see Proposition 8.1.4.1). In such cases, it may be more useful to consider the *pinched* morphism spaces of $\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})$: see Example 4.6.5.13 and Remark 8.1.8.8.

01G9 **Construction 8.1.8.2.** Let $\mathcal{A}$ be a category, let $\mathcal{C}$ be a 2-category containing objects X and Y, and let $F : \mathrm{Tw}(\mathcal{A}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ be a functor. We define a strictly unitary lax functor $U_F : [1] \times \mathcal{A} \to \mathcal{C}$ as follows:

(1) The lax functor U_F is given on objects by $U_F(0, A) = X$ and $U_F(1, A) = Y$ for each object $A \in \mathcal{A}$.

(2) Let $f : A \to B$ be a morphism in the category $\mathcal{A}$, which we also regard as an object of the twisted arrow category $\mathrm{Tw}(\mathcal{A})$. For $0 \leq i \leq j \leq 1$, we let f_{ji} denote the corresponding morphism from (i, A) to (j, B) in the product category $[1] \times \mathcal{A}$. Then the lax functor U_F is given on 1-morphisms by the formula

$$U_F(f_{ji}) = \begin{cases} \mathrm{id}_X & \text{if } i = j = 0 \\ \mathrm{id}_Y & \text{if } i = j = 1 \\ F(f) & \text{if } 0 = i < j = 1. \end{cases}$$

(3) Let $f : A \to B$ and $v : B \to C$ be composable morphisms in the category $\mathcal{A}$, and let $0 \leq i \leq j \leq k \leq 1$. Then the composition constraint $\mu_{g_{kj}, f_{ji}}$ for the lax functor U_F is given as follows:

- If $i = j = k = 0$, then $\mu_{g_{kj}, f_{ji}}$ is the unit constraint $\upsilon_X : \mathrm{id}_X \circ \mathrm{id}_X \xRightarrow{\sim} \mathrm{id}_X$ of the 2-category $\mathcal{C}$.
- If $i = 0$ and $j = k = 1$, then $\mu_{g_{kj}, f_{ji}}$ is given by the composition

$$\mathrm{id}_Y \circ F(f) \xRightarrow{\lambda_{F(f)}} F(f) \xRightarrow{F(\mathrm{id}_A, g)} F(g \circ f),$$

where $\lambda_{F(f)}$ is the left unit constraint of Construction 2.2.1.11 and we regard the pair (id_A, g) as an element of $\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}(f, g \circ f)$.

- If $i = j = 0$ and $k = 1$, then $\mu_{g_{kj}, f_{ji}}$ is given by the composition

$$F(g) \circ \mathrm{id}_X \xRightarrow{\rho_{F(g)}} F(g) \xRightarrow{F(f, \mathrm{id}_C)} F(g \circ f),$$

where $\rho_{F(g)}$ is the right unit constraint of Construction 2.2.1.11 and we regard the pair (f, id_C) as an element of $\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}(g, g \circ f)$.

- If $i = j = k = 1$, then $\mu_{g_{kj}, f_{ji}}$ is equal to the unit constraint $\upsilon_Y : \mathrm{id}_Y \circ \mathrm{id}_Y \overset{\sim}{\Longrightarrow} \mathrm{id}_Y$ of the 2-category $\mathcal{C}$.

Exercise 8.1.8.3. Show that Construction 8.1.8.2 is well-defined. That is, given a functor $F : \mathrm{Tw}(\mathcal{A}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$ as in Construction 8.1.8.2, show that there is a unique strictly unitary lax functor U_F satisfying properties (1), (2), and (3) of Construction 8.1.8.2. `01GA`

We can now formulate the main result of this section.

Theorem 8.1.8.4. *Let $\mathcal{A}$ be a category and let $\mathcal{C}$ be a 2-category containing objects X and Y. Then the assignment $F \mapsto U_F$ of Construction 8.1.8.2 induces a monomorphism of sets* `01GB`

$$\begin{tikzcd}
\{\text{Functors } F : \mathrm{Tw}(\mathcal{A}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)\} \arrow[d] \\
\{\text{Strictly unitary lax functors } U : [1] \times \mathcal{A} \to \mathcal{C}\}.
\end{tikzcd}$$

The image of this monomorphism consists of those strictly unitary lax functors $U : [1] \times \mathcal{C} \to \mathcal{D}$ having the property that $U|_{\{0\} \times \mathcal{A}}$ and $U|_{\{1\} \times \mathcal{A}}$ are the constant functors taking the values X and Y, respectively.

Remark 8.1.8.5. Let $\mathcal{C}$ be a 2-category containing objects X and Y. For every category $\mathcal{A}$, we can use Theorem 2.3.4.1 to identify strictly unitary lax functors $U : [1] \times \mathcal{A} \to \mathcal{C}$ with morphisms of simplicial sets $G : \Delta^1 \times \mathrm{N}_\bullet(\mathcal{A}) \to \mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})$. Consequently, Theorem 8.1.8.4 supplies a bijection `03L1`

$$\begin{tikzcd}
\{\text{Functors } F : \mathrm{Tw}(\mathcal{A}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)\} \arrow[d, "\sim"] \\
\{\text{Morphisms of simplicial sets } \mathrm{N}_\bullet(\mathcal{A}) \to \mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})}(X, Y)\}.
\end{tikzcd}$$

Note that the bijection of Remark 8.1.8.5 depends functorially on the simplicial set $\mathcal{A}$. Specializing to categories of the form $\mathcal{A} = [n]$, we obtain the following:

01GC **Corollary 8.1.8.6.** *Let $\mathcal{C}$ be a 2-category containing objects X and Y. Then Construction 8.1.8.2 induces an isomorphism of simplicial sets*

$$\mathrm{Cospan}(\mathrm{N}_\bullet \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})}(X,Y).$$

03L2 **Exercise 8.1.8.7.** Show that Theorem 8.1.8.4 follows from Corollary 8.1.8.6. In other words, to prove Theorem 8.1.8.4, there is no loss of generality in assuming that $\mathcal{A}$ has the form $\{0 < 1 < \cdots < n\}$ for some integer $n \geq 0$.

03L3 **Remark 8.1.8.8.** Let $\mathcal{C}$ be a 2-category containing a pair of objects X and Y. Then we have a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{N}_\bullet \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) \arrow[r, "\rho_+"] \arrow[d, "\sim"] & \mathrm{Cospan}(\mathrm{N}_\bullet \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)) \arrow[d, "\sim"] & \mathrm{N}_\bullet \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)^{\mathrm{op}} \arrow[l, "\rho_-"'] \arrow[d, "\sim"] \\
\mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})}(X,Y) \arrow[r, "\iota^{\mathrm{L}}"] & \mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})}(X,Y) & \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})}(X,Y) \arrow[l, "\iota^{\mathrm{R}}"']
\end{tikzcd}$$

where the upper horizontal maps are the inclusions of Construction 8.1.7.1 and Variant 8.1.7.14, the lower horizontal maps are the pinch inclusion maps of Construction 4.6.5.7, the outer vertical maps are the isomorphisms of Example 4.6.5.13, and the inner vertical map is the isomorphism of Corollary 8.1.8.6.

Stated more concretely, Corollary 8.1.8.6 asserts that we can identify n-simplices of the simplicial set $\mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})}(X,Y)$ with commutative diagrams

$$\begin{tikzcd}[column sep=small]
f_{0,0} \arrow[dr, Rightarrow] & & f_{1,1} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \cdots \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & f_{n-1,n-1} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & f_{n,n} \arrow[dl, Rightarrow] \\
& \cdots \arrow[dr, Rightarrow] & & \cdots \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \cdots \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & \cdots \arrow[dl, Rightarrow] & \\
& & f_{0,n-2} \arrow[dr, Rightarrow] & & f_{1,n-1} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & & f_{2,n} \arrow[dl, Rightarrow] & & \\
& & & f_{0,n-1} \arrow[dr, Rightarrow] & & f_{1,n} \arrow[dl, Rightarrow] & & & \\
& & & & f_{0,n} & & & &
\end{tikzcd}$$

in the category of 1-morphisms $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$. The image of the left-pinch inclusion morphism

$$\iota^{\mathrm{L}} : \mathrm{Hom}^{\mathrm{L}}_{\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_\bullet(\mathcal{C})}(X,Y)$$

consists of those simplices which correspond (under this identification) to commutative diagrams in which each of the leftward pointing 2-morphisms $f_{i,j} \Rightarrow f_{i-1,j}$ is an identity map. In this case, the entire diagram is determined by the sequence of composable morphisms $f_{0,0} \Rightarrow f_{0,1} \Rightarrow f_{0,2} \Rightarrow \cdots \Rightarrow f_{0,n}$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$. Similarly, the image of the right-pinch inclusion morphism

$$\iota^{\mathrm{R}} : \mathrm{Hom}^{\mathrm{R}}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X,Y) \hookrightarrow \mathrm{Hom}_{\mathrm{N}^{\mathrm{D}}_{\bullet}(\mathcal{C})}(X,Y)$$

consists of those simplices which correspond to commutative diagrams in which the rightward pointing 2-morphisms $f_{i,j} \Rightarrow f_{i,j+1}$ are identity maps, which ensures that the entire diagram is determined by the sequence of composable morphisms $f_{n,n} \Rightarrow f_{n-1,n} \Rightarrow f_{n-2,n} \Rightarrow \cdots \Rightarrow f_{0,n}$ in $\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$.

Proof of Theorem 8.1.8.4. Let $\mathcal{A}$ be an ordinary category, let $\mathcal{C}$ be a 2-category containing objects X and Y, and let $U : [1] \times \mathcal{A} \to \mathcal{C}$ be a strictly unitary lax functor having the property that $U|_{\{0\}\times\mathcal{A}}$ and $U|_{\{1\}\times\mathcal{A}}$ are the constant functors taking the values X and Y, respectively. We wish to show that there exists a unique functor of ordinary categories $F : \mathrm{Tw}(\mathcal{A}) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$ such that U is equal to the strictly unitary lax functor U_F given by Construction 8.1.8.2. To prove this, we may assume without loss of generality that the 2-category $\mathcal{C}$ is strictly unitary (Proposition 2.2.7.7). Given a morphism $f : A \to B$ in the category $\mathcal{A}$ and a pair of integers $0 \leq i \leq j \leq 1$, we write $f_{ji} : (i,A) \to (j,B)$ for the corresponding morphism in the product category $[1] \times \mathcal{A}$. Unwinding the definitions, we see that the identity $U = U_F$ imposes the following requirements on the functor F:

(1) Let $f : A \to B$ be a morphism in the category $\mathcal{C}$, which we identify with an object of the twisted arrow category $\mathrm{Tw}(\mathcal{A})$. Then $F(f)$ is equal to $U(f_{10}) \in \underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y)$.

(2) Let $f : A \to B$ and $g : B \to C$ be composable morphisms in the category $\mathcal{A}$, and regard the pairs (id_A, g) and (f, id_C) as elements of $\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}(f, g\circ f)$ and $\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}(g, g\circ f)$, respectively. Then $F(\mathrm{id}_A, g)$ and $F(f, \mathrm{id}_C)$ are equal to the composition constraints $\mu_{g_{11}, f_{10}}$ and $\mu_{g_{10}, f_{00}}$ for the lax functor U, respectively.

We now establish the uniqueness of the functor F. The value of F on objects is determined by condition (1). If $f : A \to B$ and $f' : A' \to B'$ are objects of the twisted arrow category $\mathrm{Tw}(\mathcal{A})$, then an element of $\mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}(f, f')$ can be identified with a pair (u, v) where $u \in \mathrm{Hom}_{\mathcal{A}}(A', A)$ and $v \in \mathrm{Hom}_{\mathcal{A}}(B, B')$ satisfy $f' = v \circ f \circ u$. In this case, the morphism (u, v) factors as a composition $(u, \mathrm{id}_{B'}) \circ (\mathrm{id}_A, v)$, so condition (2) guarantees the identity

$$F(u,v) = F(u, \mathrm{id}_{B'}) \circ F(\mathrm{id}_A, v) = \mu_{(vf)_{10}, u_{00}} \circ \mu_{v_{11}, f_{10}}.$$

This proves the uniqueness of F on morphisms.

To prove existence, we define the functor F on objects $f \in \mathrm{Tw}(\mathcal{A})$ by setting $F(f) = U(f_{10})$, and on morphisms $(u, v) \in \mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}(f, f')$ by the formula

$$F(u, v) = \mu_{(vf)_{10}, u_{00}} \circ \mu_{v_{11}, f_{10}}.$$

Note that this prescription automatically satisfies condition (1). Since U is a strictly unitary functor between strictly unitary 2-categories, its composition constraints $\mu_{g,h}$ are the identity whenever either g or h is an identity morphism (Remark 2.2.7.5), which shows that F satisfies condition (2) and that it carries identity morphisms to identity morphisms. We will complete the proof by showing that F is compatible with composition. Let $f : A \to B$, $f' : A' \to B'$, and $f'' : A'' \to B''$ be objects of the twisted arrow category $\mathrm{Tw}(\mathcal{A})$, and suppose we are given morphisms $(u, v) \in \mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}(f, f')$ and $(u', v') \in \mathrm{Hom}_{\mathrm{Tw}(\mathcal{A})}(f', f'')$. We wish to prove an equality $F(u \circ u', v' \circ v) = F(u', v') \circ F(u, v)$ of morphisms from $F(f)$ to $F(f'')$ in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$. Unwinding the definitions, this is equivalent to the commutativity of the outer cycle of the diagram

$$\begin{tikzcd}
& & F(vfu) \arrow[dr, Rightarrow, "\mu_{v'_{11},(vfu)_{10}}"] & & \\
& F(vf) \arrow[ur, Rightarrow, "\mu_{(vf)_{10},u_{00}}"] \arrow[dr, Rightarrow, "\mu_{v'_{11},(vf)_{10}}"] & & F(v'vfu) \arrow[dr, Rightarrow, "\mu_{(v'vfu)_{10},u'_{00}}"] & \\
F(f) \arrow[ur, Rightarrow, "\mu_{v_{11},f_{10}}"] \arrow[rr, Rightarrow, "\mu_{(v'v)_{11},f_{10}}"] & & F(v'vf) \arrow[ur, Rightarrow, "\mu_{(v'vf)_{10},u_{00}}"] \arrow[rr, Rightarrow, "\mu_{(v'vf)_{10},(uu')_{00}}"] & & F(v'vfuu')
\end{tikzcd}$$

in the category $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y)$. In fact, the entire diagram commutes. The commutativity of the upper square follows by applying property (c) of Definition 2.2.4.5 to the composable triple of morphisms

$$(0, A') \xrightarrow{u_{00}} (0, A) \xrightarrow{(vf)_{10}} (1, B') \xrightarrow{v'_{11}} (1, B'')$$

in the product category $[1] \times \mathcal{A}$. The commutativity of the lower left triangle follows by applying property (c) to the composable triple of morphisms

$$(0, A) \xrightarrow{f_{10}} (1, B) \xrightarrow{v_{11}} (1, B') \xrightarrow{v'_{11}} (1, B'')$$

and noting that the composition constraint $\mu_{v'_{11}, v_{11}}$ is equal to the identity (by virtue of our assumption that the lax functor $U|_{\{1\} \times \mathcal{A}}$ is constant). Similarly, the commutativity of the lower right triangle follows by applying (c) to the compostable triple of morphisms

$$(0, A'') \xrightarrow{u'_{00}} (0, A') \xrightarrow{u_{00}} (0, A) \xrightarrow{(v'vf)_{10}} (1, B'')$$

and noting that the composition constraint $\mu_{u_{00},u'_{00}}$ is equal to the identity (by virtue of our assumption that the lax functor $U|_{\{0\}\times\mathcal{A}}$ is constant). □

8.1.9 Cospan Fibrations

Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. Beware that the induced map $\mathrm{Cospan}(U) : \mathrm{Cospan}(\mathcal{E}) \to \mathrm{Cospan}(\mathcal{C})$ is usually not an inner fibration. For example, in the special case $\mathcal{C} = \Delta^0$, the morphism U is an inner fibration if and only if $\mathcal{E}$ is an ∞-category. In this case, the simplicial set $\mathrm{Cospan}(\mathcal{E})$ is usually not an ∞-category (unless $\mathcal{E}$ is a Kan complex). However, it contains an ∞-category $\mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{E}) \subseteq \mathrm{Cospan}(\mathcal{E})$ (Construction 8.1.7.2), which is canonically equivalent to the ∞-category $\mathcal{E}$ (Proposition 8.1.7.6). In this section, we describe a generalization which applies to any simplicial set $\mathcal{C}$. Our main result can be stated as follows: `049W`

Proposition 8.1.9.1. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets and let W denote the collection of all U-cocartesian edges of $\mathcal{E}$. Then the induced map* $\mathrm{Cospan}^{\mathrm{all},W}(\mathcal{E}) \to \mathrm{Cospan}(\mathcal{C})$ *is an inner fibration of simplicial sets.* `049X`

We will give the proof of Proposition 8.1.9.1 at the end of this section.

Remark 8.1.9.2. In the situation of Proposition 8.1.9.1, let C be a vertex of $\mathcal{C}$ and let $\mathcal{E}_C$ denote the fiber $\{C\} \times_{\mathcal{C}} \mathcal{E}$. Note that a morphism u in the ∞-category $\mathcal{E}_C$ is an isomorphism if and only if it is U-cocartesian when viewed as a morphism in $\mathcal{E}$ (Proposition 5.1.4.11). It follows that the fiber $\{C\} \times_{\mathrm{Cospan}(\mathcal{C})} \mathrm{Cospan}^{\mathrm{all},W}(\mathcal{E})$ can be identified with the ∞-category $\mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{E}_C)$. In particular, Proposition 8.1.7.6 supplies an equivalence of ∞-categories `049Y`

$$\rho_+ : \{C\} \times_{\mathcal{C}} \mathcal{E} \hookrightarrow \{C\} \times_{\mathrm{Cospan}(\mathcal{C})} \mathrm{Cospan}^{\mathrm{all},W}(\mathcal{E}).$$

Remark 8.1.9.3. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let W be the collection of all U-cocartesian morphisms of $\mathcal{E}$. Then we have a commutative diagram `049Z`

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Cospan}^{\mathrm{all},W}(\mathcal{E}) \arrow[d] \\
\mathcal{C} \arrow[r] & \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{C}) \arrow[r] & \mathrm{Cospan}(\mathcal{C}),
\end{tikzcd}$$

where the horizontal maps on the left are equivalences of ∞-categories (Proposition 8.1.7.6), and the right half of the diagram is a pullback square (Proposition 5.1.1.8). It follows that from Proposition 8.1.9.1 that map $\mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{C})$ is an inner fibration of ∞-categories. In fact, it is even an isofibration: this follows easily from the description of isomorphisms in the ∞-categories $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ and $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{E})$ supplied by Corollary 8.1.6.10 (together with the fact that U is an isofibration; see Proposition 5.1.4.8). Applying Theorem 5.1.6.1 to the right side of the diagram, we conclude that the map $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$ is also a cocartesian fibration of simplicial sets. Moreover, Corollary 4.5.2.29 guarantees that the induced map

$$\mathcal{E} \hookrightarrow \mathcal{C} \times_{\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})} \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{E}) \simeq \mathcal{C} \times_{\mathrm{Cospan}(\mathcal{C})} \mathrm{Cospan}^{\mathrm{all},W}(\mathcal{E})$$

is an equivalence of ∞-categories.

04FN **Remark 8.1.9.4.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let W be the collection of all U-cocartesian morphisms of $\mathcal{E}$. It follows from Proposition 8.1.9.1 that U also induces an inner fibration $\mathrm{Cospan}^{W,\mathrm{all}}(\mathcal{E}) \to \mathrm{Cospan}(\mathcal{C})$, whose fiber over an object $C \in \mathcal{C}$ is equivalent to the *opposite* of the ∞-category $\mathcal{E}_C$ (see Variant 8.1.7.14). This construction will play an important role in §8.6.

For later use, it will be convenient to have a generalization of Proposition 8.1.9.1, where we impose some additional constraints on the cospans that we consider.

04A0 **Definition 8.1.9.5.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets and let $\overline{f} : \overline{X} \to \overline{Y}$ be an edge of $\mathcal{C}$. We say that $\overline{f}$ *admits U-cartesian lifts* if, for every vertex $Y \in \mathcal{E}$ satisfying $U(Y) = \overline{Y}$, there is a U-cartesian edge $f : X \to Y$ of $\mathcal{E}$ satisfying $U(f) = \overline{f}$. We say that $\overline{f}$ *admits U-cocartesian lifts* if, for every vertex $X \in \mathcal{E}$ satisfying $U(X) = \overline{X}$, there is a U-cocartesian edge $f : X \to Y$ of $\mathcal{E}$ satisfying $U(f) = \overline{f}$.

04A1 **Remark 8.1.9.6.** In the situation of Definition 8.1.9.5, the edge $\overline{f}$ admits U-cocartesian lifts if and only if it it admits U^{op}-cartesian lifts, when regarded as an edge of the opposite simplicial set $\mathcal{E}^{\mathrm{op}}$.

04A2 **Example 8.1.9.7.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. Then U is a cocartesian fibration if and only if every edge of $\mathcal{C}$ admits U-cocartesian lifts. Similarly, U is a cartesian fibration if and only if every edge of $\mathcal{C}$ admits U-cartesian lifts.

04A3 **Example 8.1.9.8.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories. The following conditions are equivalent:

- The morphism U is an isofibration of ∞-categories.
- Every isomorphism of $\mathcal{C}$ admits U-cocartesian lifts.
- Every isomorphism of $\mathcal{C}$ admits U-cartesian lifts.

04A4 **Proposition 8.1.9.9.** *Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets, let R be a collection of edges of $\mathcal{C}$ which admits U-cocartesian lifts, and let $\widetilde{R}$ denote the collection of all U-cocartesian edges f of $\mathcal{E}$ such that $U(f)$ belongs to $\overline{R}$. Then the induced map $\mathrm{Cospan}^{\mathrm{all},\widetilde{R}}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{all},R}(\mathcal{C})$ is an inner fibration of simplicial sets.*

Proof. Replacing $\mathcal{C}$ by a full simplicial subset if necessary, we may assume that R contains every degenerate edge of $\mathcal{C}$. Choose integers $0 < i < n$; we wish to show that every lifting problem

04A5

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f_0"] \arrow[d] & \mathrm{Cospan}^{\mathrm{all},\widetilde{R}}(\mathcal{E}) \arrow[d] \\
\Delta^n \arrow[r, "\overline{f}"] \arrow[ur, dashed] & \mathrm{Cospan}^{\mathrm{all},R}(\mathcal{C})
\end{tikzcd} \tag{8.16}$$

admits a solution. Using Proposition 8.1.3.7, we can rewrite (8.16) as a lifting problem

$$\begin{tikzcd}
\mathrm{Tw}(\Lambda^n_i) \arrow[r, "F_0"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathrm{Tw}(\Delta^n) \arrow[r, "\overline{F}"] \arrow[ur, dashed, "F"] & \mathcal{C},
\end{tikzcd}$$

where the morphism F is required to satisfy the following additional condition:

$(*)$ For every pair of integers $0 \leq i \leq j \leq n$, the morphism F carries $(j, j) \to (i, j)$ to a U-cocartesian edge of $\mathcal{E}$.

Replacing $\mathcal{E}$ by the fiber product $\mathrm{Tw}(\Delta^n) \times_{\mathcal{C}} \mathcal{E}$, we can reduce to the case where $\mathcal{C} = \mathrm{Tw}(\Delta^n)$, and $\overline{F}$ is the identity morphism. Since U is an inner fibration, it follows that $\mathcal{E}$ is an ∞-category.

Suppose first that $n \geq 3$. In this case, F_0 determines a commutative diagram

04A6

$$\begin{tikzcd}
F_0(i,i) \arrow[r] \arrow[d] & F_0(i-1,i) \arrow[d] \\
F_0(i,i+1) \arrow[r] & F_0(i-1,i+1)
\end{tikzcd} \tag{8.17}$$

in the ∞-category $\mathcal{E}$. Using our assumption that f_0 factors through $\mathrm{Cospan}^{\mathrm{all},\widetilde{R}}(\mathcal{E}) \subseteq \mathrm{Cospan}(\mathcal{E})$ (together with Corollary 5.1.2.4), we deduce that the horizontal maps the diagram (8.17) are U-cocartesian. In particular, (8.17) is a U-colimit diagram (Proposition 7.6.3.23). Since the image of (8.17) in $\mathrm{Tw}(\Delta^n)$ is a pushout square, it is a pushout diagram in $\mathcal{E}$ (Corollary 7.1.5.15). Applying Proposition 8.1.4.2, we deduce that $f_0|_{\mathrm{N}_{\bullet}(\{i-1<i<i+1\})}$ is a thin 2-simplex of $\mathrm{Cospan}(\mathcal{E})$, and therefore also of $\mathrm{Cospan}^{\mathrm{all},\widetilde{R}}(\mathcal{E})$ (Remark 8.1.6.4). It

follows that f_0 can be extended to an n-simplex σ of $\mathrm{Cospan}^{\mathrm{all},\widetilde{R}}(\mathcal{E})$, which we can identify with a functor $F : \mathrm{Tw}(\Delta^n) \to \mathcal{E}$ satisfying condition $(*)$ and the identity $F|_{\mathrm{Tw}(\Lambda^n_i)} = F_0$. The equality $U \circ F = \overline{F}$ is automatic, since $\mathrm{Tw}(\Delta^n)$ is the nerve of a partially ordered set and $\mathrm{Tw}(\Lambda^n_i)$ contains every vertex of $\mathrm{Tw}(\Delta^n)$.

We now treat the case $n = 2$ (so that $i = 1$). In this case, we can identify F_0 with a diagram

$$X_{0,0} \xrightarrow{r} X_{0,1} \xleftarrow{u} X_{1,1} \xrightarrow{s} X_{1,2} \leftarrow v X_{2,2}$$

in the ∞-category $\mathcal{E}$, where the morphisms u and v are U-cocartesian. Our assumption that $\overline{f}$ factors through $\mathrm{Cospan}^{\mathrm{all},R}(\mathcal{C})$ guarantees that the morphism $(2,2) \to (0,2)$ belongs to R. Since morphisms of R admit U-cocartesian lifts, we can choose a U-cocartesian morphism $w' : X_{2,2} \to X_{0,2}$ in $\mathcal{E}$, where $X_{0,2}$ belongs to the fiber over the object $(0,2) \in \mathcal{C}$. Since v is also U-cocartesian, we can choose a 2-simplex σ_0 of $\mathcal{E}$ with boundary indicated in the diagram

$$\begin{tikzcd}
X_{2,2} \arrow[rr, "w'"] \arrow[dr, "v"'] & & X_{0,2} \\
& X_{1,2}. \arrow[ur, "w"'] &
\end{tikzcd}$$

Since $\mathcal{E}$ is an ∞-category, we can choose another 2-simplex σ_1 of $\mathcal{E}$ with boundary indicated in the diagram

$$\begin{tikzcd}
X_{1,1} \arrow[rr, "q"] \arrow[dr, "s"'] & & X_{0,2} \\
& X_{1,2}. \arrow[ur, "w'"'] &
\end{tikzcd}$$

Invoking our assumption that u is U-cocartesian, we can choose another 2-simplex σ_2 of $\mathcal{E}$ with boundary indicated in the diagram

$$\begin{tikzcd}
X_{1,1} \arrow[rr, "q"] \arrow[dr, "u"'] & & X_{0,2} \\
& X_{0,1}. \arrow[ur, "t"'] &
\end{tikzcd}$$

Using the fact that $\mathcal{E}$ is an ∞-category, we obtain another 2-simplex σ_3 of $\mathcal{E}$ with boundary indicated in the diagram

$$\begin{tikzcd}
X_{0,0} \arrow[rr] \arrow[dr, "r"] & & X_{0,2} \\
& X_{0,1}. \arrow[ur, "t"] &
\end{tikzcd}$$

The 2-simplices σ_0, σ_1, σ_2, and σ_3 determine a functor $F : \mathrm{Tw}(\Delta^2) \to \mathcal{C}$ extending F_0, which we display informally as a diagram

$$\begin{tikzcd}
X_{0,0} \arrow[dr, "r"] & & X_{1,1} \arrow[dl, "u"'] \arrow[dr, "s"] & & X_{2,2} \arrow[dl, "v"'] \\
& X_{0,1} \arrow[dr, "t"] & & X_{1,2} \arrow[dl, "w"'] & \\
& & X_{0,2}. & &
\end{tikzcd}$$

Since the morphism w' is U-cocartesian, the functor F satisfies condition $(*)$ and can therefore be viewed as a solution to the lifting problem (8.16). □

Proof of Proposition 8.1.9.1. Combine Proposition 8.1.9.9 with Example 8.1.9.7. □

Proposition 8.1.9.10. *Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories, let L and R be collections of morphisms of $\mathcal{C}$ which are pushout-compatible, and assume that morphisms of R admit U-cocartesian lifts. Let $\widetilde{L}$ denote the collection of all morphisms f of $\mathcal{E}$ such that $U(f) \in L$, and let $\widetilde{R}$ denote the collection of all U-cocartesian morphisms f of $\mathcal{E}$ such that $U(f) \in R$. Then the collections $\widetilde{L}$ and $\widetilde{R}$ are also pushout-compatible.* 04A7

Proof. Let $f : X \to X_1$ be a morphism of $\mathcal{E}$ which belongs to $\widetilde{L}$, and let $g' : X \to X_0$ be a morphism of $\mathcal{E}$ which belongs to $\widetilde{R}$. We wish to show that there exists a pushout diagram

04A8

$$\begin{tikzcd}
X \arrow[r, "g'"] \arrow[d, "f"] & X_0 \arrow[d, "f'"] \\
X_1 \arrow[r, "g"] & X_{01}
\end{tikzcd} \qquad (8.18)$$

in the ∞-category $\mathcal{E}$, where f' belongs to $\widetilde{L}$ and g belongs to $\widetilde{R}$. Since L and R are pushout compatible, there exists a pushout diagram

04A9
$$\begin{tikzcd}
U(X) \arrow[r, "U(g')"] \arrow[d, "U(f)"] & U(X_0) \arrow[d, "\overline{f}'"] \\
U(X_1) \arrow[r, "\overline{g}"] & \overline{X}_{01}
\end{tikzcd} \tag{8.19}$$

in the ∞-category $\mathcal{C}$, where $\overline{f}'$ belongs to L and $\overline{g}$ belongs to R. Our assumption on U guarantees that $\overline{g}$ can be lifted to a U-cocartesian morphism $g : X_0 \to X_{01}$ of $\mathcal{E}$. Since U is an inner fibration, the lower left half of (8.19) can be lifted to a 2-simplex σ of $\mathcal{E}$ which we display as a diagram

$$\begin{tikzcd}
X \arrow[d, "f"] \arrow[dr, "h"] & \\
X_1 \arrow[r, "g"] & X_{01}.
\end{tikzcd}$$

Since g' is U-cocartesian, we can then lift the upper right half of (8.19) to a 2-simplex τ of $\mathcal{E}$ which we display as a diagram

$$\begin{tikzcd}
X \arrow[r, "g'"] \arrow[dr, "h"] & X_0 \arrow[d, "f'"] \\
& X_{01}.
\end{tikzcd}$$

Amalgamating σ and τ, we obtain a diagram of the form (8.18), where $f' \in \widetilde{L}$ and $g \in \widetilde{R}$. We will complete the proof by showing that this diagram is a pushout square in the ∞-category $\mathcal{E}$. Since (8.19) is a pushout square in $\mathcal{C}$, it will suffice to show that 8.18 is a U-pushout square (Corollary 7.1.5.16). This is a special case of Proposition 7.6.3.23, since the horizontal morphisms appearing in the diagram are U-cocartesian. □

04AA **Remark 8.1.9.11.** In the situation of Proposition 8.1.9.10, suppose that the collections L and R are closed under composition. Then $\widetilde{L}$ and $\widetilde{R}$ are also closed under composition (see Corollary 5.1.2.4). Applying Proposition 8.1.6.7, we deduce that the simplicial sets $\mathrm{Cospan}^{L,R}(\mathcal{C})$ and $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ are $(\infty,2)$-categories. Moreover, it follows from the proof of Proposition 8.1.9.10 that for every pushout diagram σ :

$$\begin{tikzcd}
X \arrow[r, "g'"] \arrow[d, "f"] & X_0 \arrow[d, "f'"] \\
X_1 \arrow[r, "g"] & X_{01}
\end{tikzcd}$$

in $\mathcal{E}$ where f belongs to $\widetilde{L}$ and g belongs to $\widetilde{R}$, the image $U(\sigma)$ is a pushout diagram in $\mathcal{C}$. Combining this observation with Corollary 8.1.6.8 and Proposition 8.1.4.2, we see that a 2-simplex of $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ is thin if and only if its image in $\mathrm{Cospan}^{L,R}(\mathcal{C})$ is thin. In particular:

- The induced map $\overline{V} : \mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}) \to \mathrm{Cospan}^{L,R}(\mathcal{C})$ is a functor of $(\infty, 2)$-categories.
- The functor $\overline{V}$ is an inner fibration (since it is a pullback of the inner fibration $\mathrm{Cospan}^{\mathrm{all},\widetilde{R}}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{all},R}(\mathcal{C})$ of Proposition 8.1.9.9).
- The underlying functor $V : \mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})) \to \mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C}))$ is also an inner fibration (since it is a pullback of $\overline{V}$).

8.1.10 Beck-Chevalley Fibrations

Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories. For each object $C \in \mathcal{C}$, we let $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ denote the corresponding fiber of U. If U is a cartesian fibration, then every morphism $f : C \to B$ in $\mathcal{C}$ determines a functor $f^* : \mathcal{E}_B \to \mathcal{E}_C$, given by contravariant transport along f (Definition 5.2.2.15). If U is a cocartesian fibration, then every morphism morphism $f' : C' \to B$ determines a functor $f'_! : \mathcal{E}_{C'} \to \mathcal{E}_B$, given by covariant transport along f' (Definition 5.2.2.4). If both of these conditions are satisfied, then every cospan $C \xrightarrow{f} B \xleftarrow{f'} C'$ in $\mathcal{C}$ determines a functor from $\mathcal{E}_{C'}$ to $\mathcal{E}_C$, given by the composition $f^* \circ f'_!$. Our goal in this section is to show that, under some mild assumptions, the construction $(f, f') \mapsto f^* \circ f'_!$ is compatible with the composition law on cospans, up to coherent homotopy. More precisely, we will show that this construction is given by contravariant transport for a certain cartesian fibration between cospan constructions (Theorem 8.1.10.3 and Remark 8.1.10.4). First, we need some terminology. 04AB

Definition 8.1.10.1. Let $\mathcal{C}$ be an ∞-category which admits pushouts. We will say that an inner fibration $U : \mathcal{E} \to \mathcal{C}$ is a *dual Beck-Chevalley fibration* if the following conditions are satisfied: 04AC

(1) The morphism U is a cartesian fibration.

(2) The morphism U is a cocartesian fibration.

(3) Suppose we are given a morphism $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{E}$, which we display informally as a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & X_0 \arrow[d, "g'"] \\
X_1 \arrow[r, "f'"] & X_{01}.
\end{tikzcd}$$

Assume that f is U-cartesian, that g' is U-cocartesian, and that $U(\sigma)$ is a pushout square in $\mathcal{C}$. Then f' is U-cartesian if and only if g is U-cocartesian.

04AD **Example 8.1.10.2.** Let $\mathcal{E} = \mathrm{Mod(Ab)}$ denote the category of pairs (A, M), where A is a commutative ring and M is an A-module (see Example 5.0.0.2). Let $\mathcal{C}$ denote the category of commutative rings and let $U : \mathcal{E} \to \mathcal{C}$ be the forgetful functor $(A, M) \mapsto A$. Then (the nerve of) U is a dual Beck-Chevalley fibration, in the sense of Definition 8.1.10.1. To see this, suppose we are given a commutative diagram σ :

$$\begin{tikzcd}
(A, M) \arrow[r, "f"] \arrow[d, "g"] & (A_0, M_0) \arrow[d, "g'"] \\
(A_1, M_1) \arrow[r, "f'"] & (A_{01}, M_{01})
\end{tikzcd}$$

in the category $\mathcal{E}$. Then:

- The morphism f is U-cartesian if and only if the underlying map $M \to M_1$ is an isomorphism of A-modules.
- The morphism g' is U-cocartesian if and only if it exhibits M_{01} as obtained from M_1 by extending scalars along the ring homomorphism $A_1 \to A_{01}$: that is, if and only if it induces an isomorphism $A_{01} \otimes_{A_1} M_1 \to M_{01}$.
- The image of σ is a pushout diagram in $\mathcal{C}$ if and only if the induced map $A_0 \otimes_A A_1 \to A_{01}$ is an isomorphism.

If all three of these conditions are satisfied, then the composite map $A_0 \otimes_A M \to M_0 \to M_{01}$ is an isomorphism. Using the two-out-of-three property, we conclude that f' is U-cartesian if and only if g is U-cocartesian.

We can now formulate our main result, which we prove at the end of this section.

Theorem 8.1.10.3. *Let $\mathcal{C}$ be an ∞-category which admits pushouts, let $U : \mathcal{E} \to \mathcal{C}$ be a dual Beck-Chevalley fibration and let R denote the collection of all U-cocartesian morphisms of $\mathcal{E}$. Then the map* $\mathrm{Cospan}(U) : \mathrm{Cospan}(\mathcal{E}) \to \mathrm{Cospan}(\mathcal{C})$ *restricts to a cartesian fibration of ∞-categories* 04AE

$$V : \mathrm{Pith}(\mathrm{Cospan}^{\mathrm{all},R}(\mathcal{E})) \to \mathrm{Pith}(\mathrm{Cospan}(\mathcal{C})).$$

Moreover, a morphism of $\mathrm{Pith}(\mathrm{Cospan}^{\mathrm{all},R}(\mathcal{E}))$ *is V-cartesian if and only if it corresponds to a cospan $X \xrightarrow{f} B \xleftarrow{g} Y$ in the ∞-category $\mathcal{E}$, where f is U-cartesian and g is U-cocartesian.*

Remark 8.1.10.4 (Contravariant Transport for Cospan Fibrations)**.** In the situation of Theorem 8.1.10.3, let C and C' be objects of $\mathcal{C}$, so that Proposition 8.1.7.6 supplies equivalences of ∞-categories 04AF

$$\rho_{+}^{C} : \mathcal{E}_{C} \hookrightarrow \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{E}_{C}) = V^{-1}\{C\} \qquad \rho_{+}^{C'} : \mathcal{E}_{C'} \hookrightarrow \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{E}_{C'}) = V^{-1}\{C'\}.$$

Let e be a morphism from C to C' in the $(\infty,2)$-category $\mathrm{Cospan}(\mathcal{C})$, which we identify with a pair of morphisms $C \xrightarrow{f} B \xleftarrow{f'} C'$ in the ∞-category $\mathcal{C}$. Choose functors $f^{*} : \mathcal{E}_{B} \to \mathcal{E}_{C}$, $f'_{!} : \mathcal{E}_{C'} \to \mathcal{E}_{B}$ and diagrams

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E}_B \arrow[r, "H"] \arrow[d] & \mathcal{E} \arrow[d, "U"] & \Delta^1 \times \mathcal{E}_{C'} \arrow[l, "H'"'] \arrow[d] \\
\Delta^1 \arrow[r, "f"] & \mathcal{C} & \Delta^1 \arrow[l, "f'"']
\end{tikzcd}$$

which exhibit f^{*} and $f'_{!}$ as given by contravariant and covariant transport along f and f', respectively (see Definitions 5.2.2.4 and 5.2.2.15). Then the composition

$$\begin{array}{lll}
\mathrm{Tw}(\Delta^1 \times \mathcal{E}_{C'}) & \simeq & \mathrm{Tw}(\Delta^1) \times \mathrm{Tw}(\mathcal{E}_{C'}) \\
 & \to & \mathrm{Tw}(\Delta^1) \times \mathcal{E}_{C'} \\
 & \simeq & (\mathrm{N}_{\bullet}(\{(0,0) < (0,1)\}) \amalg_{\mathrm{N}_{\bullet}(\{(0,1)\})} \mathrm{N}_{\bullet}(\{(1,1) < (0,1)\})) \times \mathcal{E}_{C'} \\
 & \xrightarrow{(H \circ (\mathrm{id} \times f'_{!}), H')} & \mathcal{E}
\end{array}$$

can be identified with a functor $T : \Delta^1 \times \mathcal{E}_{C'} \to \mathrm{Cospan}(\mathcal{E})$ which fits into a commutative diagram

$$\begin{tikzcd}
\Delta^1 \times \mathcal{E}_{C'} \arrow[r, "T"] \arrow[d] & \mathrm{Pith}(\mathrm{Cospan}^{\mathrm{all},W}(\mathcal{E})) \arrow[d, "V"] \\
\Delta^1 \arrow[r, "e"] & \mathrm{Pith}(\mathrm{Cospan}(\mathcal{C})).
\end{tikzcd}$$

For each object $X \in \mathcal{E}_{C'}$, the characterization of V-cartesian morphisms given in Theorem 8.1.10.3 shows that $T|_{\Delta^1 \times \{X\}}$ is a V-cartesian morphism of $\mathrm{Pith}(\mathrm{Cospan}^{\mathrm{all},R}(\mathcal{E}))$. It follows that the diagram

$$\begin{tikzcd}
\mathcal{E}_{C'} \arrow[r, "f'_!"] \arrow[d, "\rho_+^{C'}"] & \mathcal{E}_B \arrow[r, "f^*"] & \mathcal{E}_C \arrow[d, "\rho_+^{C}"] \\
V^{-1}\{C'\} \arrow[rr, "e^*"] & & V^{-1}\{C\}
\end{tikzcd}$$

commutes up to homotopy, where the functor e^* is given by contravariant transport along e (for the cartesian fibration V).

04AG **Corollary 8.1.10.5.** *Let $\mathcal{C}$ be an ∞-category which admits pushouts, let $U : \mathcal{E} \to \mathcal{C}$ be a dual Beck-Chevalley fibration, let L be the collection of all U-cartesian morphisms of $\mathcal{C}$, and let R be the collection of all U-cocartesian morphisms of $\mathcal{C}$. Then* $\mathrm{Cospan}^{L,R}(\mathcal{E})$ *is an* $(\infty, 2)$*-category, and U induces a right fibration of ∞-categories* $\mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{E})) \to \mathrm{Pith}(\mathrm{Cospan}(\mathcal{C}))$.

Proof. The collections L and R are closed under composition (Corollary 5.1.2.4) and pushout-compatible by virtue of our assumption that U is a Beck-Chevalley fibration. Using Proposition 8.1.6.7, we see that $\mathrm{Cospan}^{L,R}(\mathcal{E})$ is an $(\infty, 2)$-category. Moreover, the pith of $\mathrm{Cospan}^{L,R}(\mathcal{E})$ can be identified with the subcategory of $\mathrm{Pith}(\mathrm{Cospan}^{\mathrm{all},R}(\mathcal{E}))$ spanned by those morphisms which are cartesian with respect to the fibration $V : \mathrm{Pith}(\mathrm{Cospan}^{\mathrm{all},R}(\mathcal{E})) \to \mathrm{Pith}(\mathrm{Cospan}(\mathcal{C}))$ of Theorem 8.1.10.3. The desired result now follows from Corollary 5.1.4.15. □

For later use, we will prove a more general form of Theorem 8.1.10.3, where we place some restrictions on the cospans under consideration. This will allow us to loosen the requirements of Definition 8.1.10.1.

04AH **Definition 8.1.10.6.** Let $\mathcal{C}$ be an ∞-category and let L and R be collections of morphisms of $\mathcal{C}$ which are pushout-compatible (Definition 8.1.6.5). We will say that an inner fibration $U : \mathcal{E} \to \mathcal{C}$ is a *dual Beck-Chevalley fibration relative to* (L, R) if the following conditions are satisfied:

(1) Every morphism of $\mathcal{C}$ which belongs to L admits U-cartesian lifts (Definition 8.1.9.5).

(2) Every morphism of $\mathcal{C}$ which belongs to R admits U-cocartesian lifts.

(3) Suppose we are given a morphism $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{E}$, which we display informally as a

diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"] & X_0 \arrow[d, "g'"] \\
X_1 \arrow[r, "f'"] & X_{01}.
\end{tikzcd}$$

Assume that f is U-cartesian, that g is U-cocartesian, that $U(f)$ belongs to L, that $U(g)$ belongs to R, and that $U(\sigma)$ is a pushout square in $\mathcal{C}$. Then f' is U-cartesian if and only if g is U-cocartesian.

Example 8.1.10.7. Let $\mathcal{C}$ be an ∞-category which admits pushouts and let A denote the collection of all morphisms of $\mathcal{C}$. Then an inner fibration $U : \mathcal{E} \to \mathcal{C}$ is a dual Beck-Chevalley fibration (in the sense of Definition 8.1.10.6) if and only if if it is a dual Beck-Chevalley fibration relative to (A, A) (in the sense of Definition 8.1.10.6). 04AJ

Example 8.1.10.8. In the situation of Definition 8.1.10.6, suppose that L is the collection of all isomorphisms in $\mathcal{C}$. Then condition (1) is equivalent to the requirement that U is an isofibration (Example 8.1.9.8), and condition (3) is automatic. Similarly, if R is the collection of all isomorphisms in $\mathcal{C}$, then condition (2) is the requirement that U is an isofibration, and condition (3) is automatic. 04AK

Theorem 8.1.10.9. *Let $\mathcal{C}$ be an ∞-category, let L and R be collections of morphisms of $\mathcal{C}$ which are closed under composition and pushout-compatible. Let $U : \mathcal{E} \to \mathcal{C}$ be a dual Beck-Chevalley fibration with respect to (L, R) and define $\widetilde{L}$ and $\widetilde{R}$ as in Proposition 8.1.9.10. Then the functor $V : \mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})) \to \mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{E}))$ of Remark 8.1.9.11 is a cartesian fibration of ∞-categories. Moreover, a morphism e of $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ is V-cartesian if and only if it satisfies the following condition:* 04AL

$(*)$ *The morphism e corresponds to a cospan $X \xrightarrow{f} B \xleftarrow{g} Y$ in the ∞-category $\mathcal{E}$, where f is U-cartesian and g is U-cocartesian.*

Proof. Let us say that a morphism e of $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ is *special* if it satisfies condition $(*)$. We first show that every special morphism e of $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ is V-cartesian. Suppose we are given an integer $n \geq 2$ and a lifting problem

04AM

$$\begin{tikzcd}
\Lambda^n_n \arrow[r, "h_0"] \arrow[d] & \mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})) \arrow[d, "U'"] \\
\Delta^n \arrow[r, "\overline{h}"] \arrow[ur, dashed, "h"] & \mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C}),
\end{tikzcd} \qquad (8.20)$$

where the composition

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{n-1 < n\}) \subseteq \Lambda^n_n \xrightarrow{h_0} \mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$$

coincides with the edge e; we wish to show that (8.20) admits a solution. Let us identify $\overline{h}$ with a diagram $\overline{F} : \mathrm{Tw}(\Delta^n) \to \mathcal{E}$ and h_0 with a diagram $H_0 : \mathrm{Tw}(\Lambda^n_n) \to \mathcal{C}$ satisfying $U \circ H_0 = \overline{H}|_{\mathrm{Tw}(\Lambda^n_n)}$. We first treat the case $n = 2$. In this case, we can identify F_0 with a pair of cospans

$$X_{0,0} \xrightarrow{f'} X_{0,2} \xleftarrow{g'} X_{2,2} \qquad X_{1,1} \xrightarrow{f} X_{1,2} \xleftarrow{g} X_{2,2}$$

in the ∞-category $\mathcal{E}$, where $f, f' \in \widetilde{L}$ and $g, g' \in \widetilde{R}$. Since g is U-cocartesian, we can lift $\overline{H}((2,2) \to (1,2) \to (0,2))$ to a 2-simplex σ_0 of $\mathcal{E}$ whose boundary we display in the diagram

$$\begin{tikzcd}
X_{2,2} \arrow[rr, "g'"] \arrow[dr, "g"'] & & X_{0,2} \\
& X_{1,2}. \arrow[ur, "g''"'] &
\end{tikzcd}$$

Since g and g' are U-cocartesian by assumption, Corollary 5.1.2.4 guarantees that g'' is also U-cocartesian. Since U is an inner fibration, we can lift $\overline{H}((1,1) \to (1,2) \to (0,2))$ to a 2-simplex σ_1 of $\mathcal{E}$, whose boundary we display in the diagram

$$\begin{tikzcd}
X_{1,1} \arrow[rr, "s"] \arrow[dr, "f"'] & & X_{0,2} \\
& X_{1,2}. \arrow[ur, "g''"'] &
\end{tikzcd}$$

Since morphisms of R admit U-cocartesian lifts, we can lift $\overline{H}((1,1) \to (0,1) \to (0,2))$ to a 2-simplex σ_2 of $\mathcal{E}$ displayed in the diagram

$$\begin{tikzcd}
X_{1,1} \arrow[rr, "s"] \arrow[dr, "g'''"'] & & X_{0,2} \\
& X_{0,1}, \arrow[ur, "f''"'] &
\end{tikzcd}$$

where g''' is U-cocartesian. Applying condition (3) of Definition 8.1.10.6 to the diagram

$$\begin{tikzcd}
X_{1,1} \arrow[r, "f"] \arrow[d, "g'''"] & X_{1,2} \arrow[d, "g''"] \\
X_{0,1} \arrow[r, "f''"] & X_{0,2},
\end{tikzcd}$$

we deduce that the morphism f'' is U-cartesian. We can therefore lift $\overline{H}((0,0) \rightarrow (0,1) \rightarrow (0,2))$ to a 2-simplex σ_3 of $\mathcal{E}$ which we display as a diagram

$$\begin{tikzcd}
X_{0,0} \arrow[rr, "f'"] \arrow[dr, "f'''"'] & & X_{0,2} \\
& X_{0,1}, \arrow[ur, "f''"'] &
\end{tikzcd}$$

The 2-simplices σ_0, σ_1, σ_2, and σ_3 can then be amalgamed into a functor $H : \mathrm{Tw}(\Delta^2) \rightarrow \mathcal{E}$ which we display informally as a diagram

$$\begin{tikzcd}
X_{0,0} \arrow[dr, "f'''"] & & X_{1,1} \arrow[dl, "g'''"'] \arrow[dr, "f"] & & X_{2,2} \arrow[dl, "g"'] \\
& X_{0,1} \arrow[dr, "f''"] & & X_{1,2} \arrow[dl, "g''"] & \\
& & X_{0,2}, & &
\end{tikzcd}$$

which is a solution to the lifting problem (8.20).

We now treat the case $n \geq 3$. By virtue of Lemma 8.1.4.6, it will suffice to show that the following conditions are satisfied:

(a) The functor F_0 carries the edge $(n,n) \rightarrow (n-1,n)$ of $\mathrm{Tw}(\Lambda^n_n)$ to a U-cocartesian edge of $\mathcal{E}$.

(b) The functor F_0 carries the edge $(0,n-1) \rightarrow (0,n)$ of $\mathrm{Tw}(\Lambda^n_n)$ to a U-cartesian edge of $\mathcal{E}$.

Assertion (a) follows immediately from our requirement that f_0 factors through $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$. To prove (b), we observe that F_0 determines a commutative diagram τ :

$$\begin{tikzcd}
F_0(n-1,n-1) \arrow[r, "f"] \arrow[d, "g"] & F_0(n-1,n) \arrow[d, "g'"] \\
F_0(0,n-1) \arrow[r, "f'"] & F_0(0,n)
\end{tikzcd}$$

in the ∞-category $\mathcal{E}$, where $f \in \widetilde{L}$ and $g \in \widetilde{R}$. Our assumption that e is special guarantees that f is U-cartesian, and our assumption that $\overline{h}$ factors through the pith of $\mathrm{Cospan}^{L,R}(\mathcal{C})$ guarantees that $U(\tau)$ is a pushout diagram in $\mathcal{C}$. Applying Corollary 5.1.2.4 to the diagram

$$\begin{tikzcd}
F_0(n,n) \arrow[rr] \arrow[dr] & & F_0(0,n) \\
& F_0(n-1,n), \arrow[ur, "g'"] &
\end{tikzcd}$$

we see that g' is U-cocartesian. Condition (3) of Definition 8.1.10.6 then guarantees that f' is U-cartesian, as desired. This completes the proof that every special morphism of $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ is V-cartesian.

It follows from Remark 8.1.9.11 that V is an inner fibration of ∞-categories. To show that V is a cartesian fibration, it will suffice to show that for every object $Y \in \mathcal{E}$ and every morphism $\overline{e} : \overline{X} \to U(Y)$ in the ∞-category $\mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C}))$, there exists a special morphism $e : X \to Y$ in $\mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}))$ satisfying $V(e) = \overline{e}$. Let us identify $\overline{e}$ with a cospan $\overline{X} \xrightarrow{\overline{f}} \overline{B} \xleftarrow{\overline{g}} U(Y)$ in the ∞-category $\mathcal{C}$. Then $\overline{g}$ belongs to R, and can therefore be lifted to a U-cocartesian morphism $g : Y \to B$ in the ∞-category $\mathcal{E}$. Since $\overline{f}$ belongs to L, it can be lifted to a U-cartesian morphism $f : X \to B$ in the ∞-category $\mathcal{E}$. The cospan $X \xrightarrow{f} B \xleftarrow{g} Y$ then determines a special morphism $e : X \to Y$ of $\mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}))$ satisfying $V(e) = \overline{e}$.

We now complete the proof of Theorem 8.1.10.3 by showing that every V-cartesian morphism $e : X \to Y$ of $\mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}))$ is special. Let $\overline{e}$ denote the image of e in $\mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C}))$. Arguing as above, we can lift $\overline{e}$ to a special morphism $e' : X' \to Y$ of $\mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}))$. Then e' is also V-cartesian. Applying Remark 5.1.3.8, we can choose a 2-simplex σ of $\mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C})$ which exhibits e as the composition of e' with an isomorphism in the ∞-category $\mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}))$. Let us identify σ with a diagram

$$\begin{tikzcd}
X \arrow[dr, "u"] & & X' \arrow[dl, "v"'] \arrow[dr, "f"] & & Y \arrow[dl] \\
& X'' \arrow[dr, "f'"] & & B' \arrow[dl, "w"] & \\
& & B & &
\end{tikzcd}$$

in the ∞-category $\mathcal{E}$. Corollary 8.1.6.10 implies that u and v are isomorphisms in $\mathcal{E}$. Since the inner region is a pushout diagram in $\mathcal{E}$, it follows that w is also an isomorphism (Corollary 7.6.3.24). Our assumption that e' is special guarantees that f is U-cartesian. Applying Corollary 5.1.2.5, we deduce that f' is U-cartesian. It follows that any composition of f' with u is U-cartesian (Corollary 5.1.2.4), so that the morphism e is also special. □

Proof of Theorem 8.1.10.3. Apply Theorem 8.1.10.9 in the special case $L = A = R$, where A is the collection of all morphisms in the ∞-category $\mathcal{C}$ (Example 8.1.10.7). □

Remark 8.1.10.10. In the situation of Theorem 8.1.10.9, the morphism 04AN

$$\overline{V} : \mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}) \to \mathrm{Cospan}^{L,R}(\mathcal{C})$$

is a locally cartesian fibration. To prove this, we observe that Remark 8.1.9.11 guarantees that $\overline{V}$ is an inner fibration of $(\infty, 2)$-categories and that the diagram

$$\begin{tikzcd}
\mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})) \arrow[r] \arrow[d, "V"] & \mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}) \arrow[d, "\overline{V}"] \\
\mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C})) \arrow[r] & \mathrm{Cospan}^{L,R}(\mathcal{C})
\end{tikzcd}$$

is a pullback square. Since every morphism of $\mathrm{Cospan}^{L,R}(\mathcal{C})$ is contained in the pith $\mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C}))$, the desired result follows from Theorem 8.1.10.9 (see Remark 5.1.5.6).

Remark 8.1.10.11. In the situation of Theorem 8.1.10.9, suppose that $\mathrm{Cospan}^{L,R}(\mathcal{C})$ is an 04AP ∞-category (this condition is satisfied, for example, if either L or R consists of isomorphisms; see Proposition 8.1.7.5). Then every 2-simplex of $\mathrm{Cospan}^{L,R}(\mathcal{C})$ is thin. Applying Remark

8.1.9.11, we deduce that every 2-simplex of $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ is thin, so that $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ is also an ∞-category (Example 2.3.2.4). In this case, Theorem 8.1.10.9 asserts that the projection map $V : \mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}) \to \mathrm{Cospan}^{L,R}(\mathcal{E})$ is a cartesian fibration.

We also have the following variant of Corollary 8.1.10.5:

`04AQ` **Corollary 8.1.10.12.** *Let $\mathcal{C}$ be an ∞-category, let L and R be collections of morphisms of $\mathcal{C}$ which are closed under composition and pushout-compatible. Let $U : \mathcal{E} \to \mathcal{C}$ be a dual Beck-Chevalley fibration with respect to (L, R), let $\widetilde{L}$ denote the collection of all U-cartesian morphisms f of $\mathcal{E}$ such that $U(f) \in L$, and let $\widetilde{R}$ denote the collection of all U-cocartesian morphisms f of $\mathcal{E}$ such that $U(f) \in R$. Then:*

(1) *The simplicial set* $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ *is an* $(\infty, 2)$*-category.*

(2) *The morphism U induces a right fibration* $\mathrm{Pith}(\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})) \to \mathrm{Pith}(\mathrm{Cospan}^{L,R}(\mathcal{C}))$.

(3) *If* $\mathrm{Cospan}^{L,R}(\mathcal{C})$ *is an ∞-category, then* $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E})$ *is an ∞-category, and U induces a right fibration* $\mathrm{Cospan}^{\widetilde{L},\widetilde{R}}(\mathcal{E}) \to \mathrm{Cospan}^{L,R}(\mathcal{C})$.

`04AR` **Example 8.1.10.13.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories and let L and R be collections of morphisms of $\mathcal{C}$ which are closed under composition and pushout-compatible. Assume that the following conditions are satisfied:

(0) Let $f : X \to Y$ be a morphism of $\mathcal{E}$. If $U(f)$ belongs to L, then f is U-cartesian. If $U(f)$ belongs to R, then f is U-cocartesian.

(1) For every object $Y \in \mathcal{E}$ and every morphism $\overline{f} : \overline{X} \to U(Y)$ of $\mathcal{C}$ which belongs to L, there exists a morphism $f : X \to Y$ of $\mathcal{E}$ satisfying $U(f) = \overline{f}$.

(2) For every object $X \in \mathcal{E}$ and every morphism $\overline{f} : U(X) \to \overline{Y}$ of $\mathcal{C}$ which belongs to R, there exists a morphism $f : X \to Y$ of $\mathcal{E}$ satisfying $U(f) = \overline{f}$.

Then U is a dual Beck-Chevalley fibration relative to (L, R). Applying Remark 8.1.10.10, we deduce that the projection map

$$V : \mathrm{Cospan}(\mathcal{E}) \times_{\mathrm{Cospan}(\mathcal{C})} \mathrm{Cospan}^{L,R}(\mathcal{C}) \to \mathrm{Cospan}^{L,R}(\mathcal{C})$$

is a locally cartesian fibration. Corollary 8.1.10.12 guarantees that each fiber of V is a Kan complex, so that V is a right fibration (Corollary 5.1.5.12).

8.2 Couplings of ∞-Categories

We now axiomatize an essential feature of the twisted arrow construction introduced in §8.1. 043P

Definition 8.2.0.1. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories. A *coupling of* $\mathcal{C}_+$ *with* $\mathcal{C}_-$ is an ∞-category $\mathcal{C}$ equipped with a left fibration $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$. 043Q

In the situation of Definition 8.2.0.1, we will often refer to the functor $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ as a *coupling of* ∞*-categories*. This terminology signifies both that λ is a left fibration and that its target is equipped with a *specified* factorization as a product of ∞-categories $\mathcal{C}_-^{\mathrm{op}}$ and $\mathcal{C}_+$.

Example 8.2.0.2 (The Twisted Arrow Coupling)**.** Let $\mathcal{C}$ be an ∞-category. Then the map $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ of Notation 8.1.1.6 is a coupling of $\mathcal{C}$ with itself (Proposition 8.1.1.11). We will refer to λ as the *twisted arrow coupling* of the ∞-category $\mathcal{C}$. 043R

Construction 8.2.0.3. Let $G : \mathcal{C}_+ \to \mathcal{C}_-$ be a functor of ∞-categories. Pulling back the left fibration $\mathrm{Tw}(\mathcal{C}_-) \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_-$ of Proposition 8.1.1.11, we obtain a left fibration of ∞-categories 043S

$$\lambda_G : \mathrm{Tw}(\mathcal{C}_-) \times_{\mathcal{C}_-} \mathcal{C}_+ \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+,$$

which we regard as a coupling of $\mathcal{C}_+$ with $\mathcal{C}_-$. We will refer to λ_G as the *coupling associated to the functor* G.

We say that a coupling $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ is *representable* if, for every object $C_+ \in \mathcal{C}_+$, the ∞-category $\mathcal{C} \times_{\mathcal{C}_+} \{C_+\}$ has an initial object (Definition 8.2.1.3). It is not difficult to show that, for every functor $G : \mathcal{C}_+ \to \mathcal{C}_-$, the coupling λ_G of Construction 8.2.0.3 is representable (Variant 8.2.1.6). Our primary goal in this section is to prove the converse:

Theorem 8.2.0.4. *Let* $\mathcal{C}_-$ *and* $\mathcal{C}_+$ *be* ∞*-categories. Then the assignment* $G \mapsto \lambda_G$ *of Construction 8.2.0.3 induces a bijection* 043T

$$\begin{tikzcd}
\{\text{Functors } G : \mathcal{C}_+ \to \mathcal{C}_-\}/\text{Isomorphism} \arrow[d] \\
\{\text{Representable couplings } \lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+\}/\text{Equivalence.}
\end{tikzcd}$$

Remark 8.2.0.5. Let $\mathcal{C}_-$ be a (locally small) ∞-category. For every ∞-category $\mathcal{C}_+$, Corollary 5.6.0.6 supplies an identification of equivalence classes of couplings $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ (having essentially small fibers) with isomorphism classes of functors $T : \mathcal{C}_+ \to$ 043U

$\mathrm{Fun}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{S})$. Moreover, λ is representable if and only if T factors through the full subcategory $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{S})$ spanned by the representable functors (see Proposition 5.6.6.21). Consequently, Theorem 8.2.0.4 supplies a bijection

$$\mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C}_+, \mathcal{C}_-) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C}_+, \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{S})).$$

It is not hard to see that this bijection depends functorially on $\mathcal{C}_+$, and is therefore induced by an isomorphism $\mathcal{C}_- \simeq \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{S})$ in the homotopy category hQCat. We can therefore regard Theorem 8.2.0.4 as an “implicit” version of Yoneda’s lemma. We will give a more precise formulation in §8.3 (see Theorem 8.3.3.13).

Let us outline our approach to Theorem 8.2.0.4. Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories. For a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$, we say that λ is *representable by G* if it is equivalent to the coupling λ_G of Construction 8.2.0.3 (Definition 8.2.3.1). Theorem 8.2.0.4 asserts that every representable coupling is representable by some functor $G : \mathcal{C}_+ \to \mathcal{C}_-$, which is uniquely determined up to isomorphism. To prove this, we need to construct a commutative diagram

043V

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{G}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_-) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "\mathrm{id} \times G"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_-
\end{tikzcd} \tag{8.21}$$

which is a categorical pullback square; in this case, we say that (8.21) *exhibits λ as represented by G* (Definition 8.2.3.5).

It will be useful to place this problem in a somewhat larger context. Suppose that $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ is another coupling of ∞-categories. We will refer to a commutative diagram

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$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[d, "\mu"] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+
\end{tikzcd} \tag{8.22}$$

as a *morphism of couplings* from λ to μ (Definition 8.2.2.1). The collection of such diagrams can be organized into an ∞-category $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$, which is equipped with a forgetful functor

$$\Phi : \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+).$$

It is not difficult to see that Φ is also a left fibration: that is, it can be regarded as a coupling of the ∞-category $\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)$ with the ∞-category $\mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)$ (Proposition

8.2.2.2). Suppose now that the coupling λ is representable, and that the coupling μ is corepresentable (that is, for every object $D_- \in \mathcal{D}_-$, the ∞-category $\{D_-\} \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$ has an initial object). In §8.2.2, we show these these assumptions imply that the coupling Φ is also corepresentable (Theorem 8.2.2.11). In particular, every functor $F_- : \mathcal{C}_- \to \mathcal{D}_-$ has a canonical promotion to a commutative diagram of the form (8.22), which is characterized (up to isomorphism) by the requirement that it represents an initial object of the ∞-category $\{F_-\} \times_{\mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$. In §8.2.3, we specialize this assertion to the situation where μ is the twisted arrow coupling $\mathrm{Tw}(\mathcal{C}_-) \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and F_- is the identity functor from $\mathcal{C}_-$ to itself. In this case, we obtain a diagram of the form (8.21) and use it to deduce Theorem 8.2.0.4.

Every assertion in the preceding discussion has a dual counterpart, where the assumption of representability is replaced by corepresentability (and vice versa). If $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ is a corepresentable coupling of ∞-categories, then Theorem 8.2.0.4 guarantees the existence of a categorical pullback square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_+) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ .
\end{tikzcd}$$

for some functor $F : \mathcal{C}_- \to \mathcal{C}_+$, which is uniquely determined up to isomorphism; in this case, we say that the coupling λ is *corepresentable by* F (Variant 8.2.3.8). In §8.2.5, we study couplings which are simultaneously representable and corepresentable. Our main result asserts that if a coupling $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ is representable by a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$, then it is corepresentable by a functor $F : \mathcal{C}_- \to \mathcal{C}_+$ if and only if F is left adjoint to G (Theorem 8.2.5.1). Our proof is based on an alternative characterization of the corepresenting functor F, which we explain in §8.2.4.

Let $\mathcal{C}$ be an ∞-category. Then the twisted arrow coupling $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ has the following features:

(*a*) The coupling λ is corepresentable. That is, for every object $X \in \mathcal{C}$, the ∞-category $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ has an initial object.

(*b*) The coupling λ is representable. That is, for every object $Y \in \mathcal{C}$, the ∞-category $\mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}$ has an initial object.

(*c*) Let f be an object of the ∞-category $\mathrm{Tw}(\mathcal{C})$, which we regard as a morphism $X \to Y$ in the ∞-category $\mathcal{C}$. Then f is initial when viewed as an object of the ∞-category $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ if and only if it is initial when viewed as an object of the ∞-category

$\mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\}$ (by virtue of Corollary 8.1.2.21, both conditions are equivalent to the requirement that f corresponds to an isomorphism in the ∞-category $\mathcal{C}$).

In §8.2.6, we show that twisted arrow couplings are characterized (up to equivalence) by these properties. More precisely, we show that a coupling $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ satisfies conditions (a), (b) and (c) if and only if it is representable (or corepresentable) by an equivalence of ∞-categories (Theorem 8.2.6.5), and therefore equivalent to the twisted arrow ∞-category associated to the ∞-category $\mathcal{D}_-$ (or $\mathcal{D}_+$). In this case, we say that the coupling λ is *balanced* (Definition 8.2.6.1).

8.2.1 Representable Couplings

043X We now axiomatize an essential feature of the couplings that can be obtained from Construction 8.2.0.3.

043Y **Definition 8.2.1.1.** Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let C be an object of $\mathcal{C}$, having image $\lambda(C) = (C_-, C_+) \in \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$. We say that X is *universal* if it is an initial object of the ∞-category $\mathcal{C} \times_{\mathcal{C}_+} \{C_+\}$, and *couniversal* if it is an initial object of the ∞-category $\{C_-\} \times_{\mathcal{C}_-^{\mathrm{op}}} \mathcal{C}$.

043Z **Remark 8.2.1.2** (Uniqueness)**.** Let $\lambda = (\lambda_+, \lambda_-) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories, let C be a universal object of $\mathcal{C}$, and let D be another object of $\mathcal{C}$. The following conditions are equivalent:

- The object C is isomorphic to D (as an object of the ∞-category $\mathcal{C}$).
- The object D is universal, and $\lambda_+(D)$ is isomorphic to $\lambda_+(C)$ (as an object of the ∞-category $\mathcal{C}_+$).

0440 **Definition 8.2.1.3.** Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories.

- We say that λ is *representable* if, for every object $C_+ \in \mathcal{C}_+$, there exists a universal object $C \in \mathcal{C}$ satisfying $\lambda_+(C) = C_+$.
- We say that λ is *corepresentable* if, for every object $C_- \in \mathcal{C}_-$, there exists a couniversal object $C \in \mathcal{C}$ satisfying $\lambda_-(C) = C_-$.

0441 **Remark 8.2.1.4** (Symmetry)**.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories, and let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of $\mathcal{C}_+$ with $\mathcal{C}_-$. Then the transposition $\lambda' = (\lambda_+, \lambda_-)$ can be regarded as a coupling of $\mathcal{C}_-^{\mathrm{op}}$ with $\mathcal{C}_+^{\mathrm{op}}$. In this situation:

- An object $C \in \mathcal{C}$ is universal for the coupling λ if and only if it is couniversal for the coupling λ'.

- The coupling λ is representable if and only if the coupling λ' is corepresentable.

Example 8.2.1.5. Let $\mathcal{C}$ be an ∞-category and let $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ be the twisted arrow coupling of Example 8.2.0.2. For every morphism $f : X \to Y$ in the ∞-category $\mathcal{C}$, Corollary 8.1.2.21 asserts that the following conditions are equivalent: `0442`

(1) The morphism f is an isomorphism in $\mathcal{C}$.

(2) As an object of $\mathrm{Tw}(\mathcal{C})$, f is couniversal with respect to the coupling λ.

(3) As an object of $\mathrm{Tw}(\mathcal{C})$, f is universal with respect to the coupling λ.

In particular, the coupling λ is both representable and corepresentable.

Variant 8.2.1.6. Let $G : \mathcal{C}_+ \to \mathcal{C}_-$ be a functor of ∞-categories, set $\mathcal{C} = \mathrm{Tw}(\mathcal{C}_-) \times_{\mathcal{C}_-} \mathcal{C}_+$, and let $\lambda_G : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ denote the coupling of Construction 8.2.0.3. Unwinding the definitions, we see that objects of $\mathcal{C}$ can be identified with pairs (e, C_+), where C_+ is an object of the ∞-category $\mathcal{C}_+$ and $e : C_- \to G(C_+)$ is a morphism in the ∞-category $\mathcal{C}_-$. It follows from Example 8.2.1.5 that an object $(e, C_+) \in \mathcal{C}$ is universal if and only if e is an isomorphism in the ∞-category $\mathcal{C}_-$. Note that every object $C_+ \in \mathcal{C}_+$ can be lifted to a universal object of $\mathcal{C}$ (for example, we can choose e to be the identity morphism from $G(C_+)$ to itself), so that the coupling λ_G is representable. In §8.2.3, we will prove the converse: every representable coupling of ∞-categories can be obtained (up to equivalence) from Construction 8.2.0.3(Theorem 8.2.0.4). `0443`

Our goal in this section is to establish a universal mapping property of (co)representable couplings (Proposition 8.2.1.8). First, we give a reformulation of Definition 8.2.0.1 (compare with Corollary 8.1.1.14).

Proposition 8.2.1.7. *Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories. Then a morphism of simplicial sets $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ is a left fibration if and only if it satisfies the following conditions:* `0444`

(1) *The morphism λ is an isofibration; in particular, $\mathcal{C}$ is an ∞-category.*

(2) *The functor $\lambda_- : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}}$ is a cocartesian fibration. Moreover, a morphism u of $\mathcal{C}$ is λ_--cocartesian if and only if $\lambda_+(u)$ is an isomorphism in $\mathcal{C}_+$.*

(3) *The functor $\lambda_+ : \mathcal{C} \to \mathcal{C}_+$ is a cocartesian fibration. Moreover, a morphism u of $\mathcal{C}$ is λ_+-cocartesian if and only if $\lambda_-(u)$ is an isomorphism in $\mathcal{C}_-^{\mathrm{op}}$.*

Proof. Suppose first that λ is a left fibration. Then λ is a cocartesian fibration, and every morphism of $\mathcal{C}$ is λ-cocartesian (Proposition 5.1.4.14). In particular, λ is an isofibration. Let $\pi_- : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{C}_-^{\mathrm{op}}$ and $\pi_+ : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{C}_+$ denote the projection maps. Note that

π_- is a cocartesian fibration, and that a morphism (u_-, u_+) of $\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ is π_--cocartesian if and only if $u_+ = \pi_+(u_-, u_+)$ is an isomorphism in the ∞-category $\mathcal{C}_+$ (see Remark 5.1.4.6). Applying Proposition 5.1.4.13, we see that $\lambda_- = \pi_- \circ \lambda$ is also a cocartesian fibration, and that a morphism u of $\mathcal{C}$ is λ_--cocartesian if and only if $\pi_+(\lambda(u)) = \lambda_+(u)$ is an isomorphism in $\mathcal{C}_+$. This proves assertion (2), and assertion (3) follows by a similar argument.

We now prove the converse. Suppose that λ satisfies conditions (1), (2), and (3); we wish to show that λ is a left fibration. We first show that λ is a cocartesian fibration. Fix an object $X \in \mathcal{C}$ having image $\overline{X} = \lambda(X)$, together with a morphism $\overline{w} : \overline{X} \to \overline{Z}$ in the product $\mathcal{C}_- \times \mathcal{C}_+$. We wish to show that we can write $\overline{u} = \lambda(w)$ for some λ-cocartesian morphism $w : X \to Z$ in $\mathcal{C}$. Invoking assumption (2), we can choose a λ_--cocartesian morphism $u : X \to Y$ of $\mathcal{C}$ satisfying $\lambda_-(u) = \pi_-(\overline{w})$. Set $\overline{Y} = \lambda(Y)$ and $\overline{u} = \lambda(u)$. Note that the morphism $\lambda_+(u) = \pi_+(\overline{u})$ is an isomorphism in the ∞-category $\mathcal{C}_+$. We can therefore choose a 2-simplex $\overline{\sigma}$ of $\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ as indicated in the diagram

$$\begin{tikzcd}
& \overline{Y} \arrow[dr, "\overline{v}"] & \\
\overline{X} \arrow[ur, "\overline{u}"] \arrow[rr, "\overline{w}"] & & \overline{Z},
\end{tikzcd}$$

for which the image $\pi_-(\overline{\sigma})$ is a right-degenerate 2-simplex of $\mathcal{C}_-^{\mathrm{op}}$. Note that, if v can be lifted to a λ-cocartesian morphism $v : Y \to Z$ of $\mathcal{C}$, then assumption (1) guarantees that we can lift $\overline{\sigma}$ to a diagram

$$\begin{tikzcd}
& Y \arrow[dr, "v"] & \\
X \arrow[ur, "u"] \arrow[rr, "w"] & & Z
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, where w is λ-cocartesian by virtue of Proposition 5.1.4.12. Consequently, to prove the existence of w, we can replace $\overline{w}$ by $\overline{v}$ and thereby reduce to the case where $\lambda_-(\overline{w})$ is an isomorphism in the ∞-category $\mathcal{C}_-^{\mathrm{op}}$. Repeating this argument with the roles of $\mathcal{C}_-^{\mathrm{op}}$ and $\mathcal{C}_+$ interchanged, we may also assume that $\lambda_+(\overline{w})$ is an isomorphism in the ∞-category $\mathcal{C}_+$. In this case, assumption (1) guarantees that we can lift $\overline{w}$ to an isomorphism $w : X \to Z$ in the ∞-category $\mathcal{C}$, which is λ-cocartesian by virtue of Proposition 5.1.1.8. This completes the proof that λ is a cocartesian fibration.

To complete the proof that λ is a left fibration, it will suffice to show that every morphism $w : X \to Z$ in $\mathcal{C}$ is λ-cocartesian (see Proposition 5.1.4.14). Arguing as in Remark 5.1.3.8, we can choose a diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "v"] & \\
X \arrow[ur, "u"] \arrow[rr, "w"] & & Z
\end{tikzcd}$$

in $\mathcal{C}$, where u is λ-cocartesian and $\lambda(v)$ is an isomorphism in $\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$. It follows from (2) that v is λ_--cocartesian. Since $\lambda_-(v)$ is an isomorphism in $\mathcal{C}_-^{\mathrm{op}}$, it follows that v is an isomorphism in $\mathcal{C}$ (Proposition 5.1.1.8). In particular, v is λ-cocartesian (Proposition 5.1.1.8), so that $w = v \circ u$ is also λ-cocartesian (Proposition 5.1.4.12). □

Proposition 8.2.1.8. *Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a corepresentable coupling and let $U : \mathcal{E} \to \mathcal{C}_+$ be a cocartesian fibration of ∞-categories. Suppose that, for every object $Y \in \mathcal{C}_+$, the fiber $\{Y\} \times_{\mathcal{C}_+} \mathcal{E}$ has an initial object. Then the ∞-category $\mathrm{Fun}_{/\mathcal{C}_+}(\mathcal{C}, \mathcal{E})$ has an initial object. Moreover, an object $F \in \mathrm{Fun}_{/\mathcal{C}_+}(\mathcal{C}, \mathcal{E})$ is initial if and only if it satisfies the following pair of conditions:* 0445

(1) *For every couniversal object $C \in \mathcal{C}$, the image $F(C)$ is initial when viewed as an object of the ∞-category $\{\lambda_+(C)\} \times_{\mathcal{C}_+} \mathcal{E}$.*

(2) *The functor F carries λ_+-cocartesian morphisms of $\mathcal{C}$ to U-cocartesian morphisms of $\mathcal{E}$.*

Remark 8.2.1.9. By virtue of Proposition 8.2.1.7, we can restate condition (2) of Proposition 8.2.1.8 as follows: 0446

(2′) Let e be a morphism of $\mathcal{C}$ having the property that $\lambda_-(e)$ is an isomorphism in the ∞-category $\mathcal{C}_-^{\mathrm{op}}$. Then $F(e)$ is a U-cocartesian morphism of $\mathcal{E}$.

Proof of Proposition 8.2.1.8. The functor $\lambda_- : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}}$ is a cocartesian fibration (Proposition 8.2.1.7) and is therefore exponentiable (Proposition 5.3.6.1). Let $\mathrm{Fun}(\mathcal{C} / \mathcal{C}_-^{\mathrm{op}}, \mathcal{E})$ and $\mathrm{Fun}(\mathcal{C} / \mathcal{C}_-^{\mathrm{op}}, \mathcal{C}_+)$ be the relative exponentials introduced in Construction 4.5.9.1. Composition with U induces a functor $V : \mathrm{Fun}(\mathcal{C} / \mathcal{C}_-^{\mathrm{op}}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C} / \mathcal{C}_-^{\mathrm{op}}, \mathcal{C}_+)$, which is an isofibration by virtue of Proposition 4.5.9.18. Let us identify the functor $\lambda_+ : \mathcal{C} \to \mathcal{C}_+$ with a section s of the projection map $\mathrm{Fun}(\mathcal{C} / \mathcal{C}_-^{\mathrm{op}}, \mathcal{C}_+) \to \mathcal{C}_-^{\mathrm{op}}$, and form a pullback diagram

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d, "V'"] & \mathrm{Fun}(\mathcal{C} / \mathcal{C}_-^{\mathrm{op}}, \mathcal{E}) \arrow[d, "V"] \\
\mathcal{C}_-^{\mathrm{op}} \arrow[r, "s"] & \mathrm{Fun}(\mathcal{C} / \mathcal{C}_-^{\mathrm{op}}, \mathcal{C}_+).
\end{tikzcd}$$

Since V' is a pullback of V, it is also an isofibration (Remark 4.5.5.11). Moreover, the ∞-category $\operatorname{Fun}_{/\mathcal{C}_+}(\mathcal{C}, \mathcal{E})$ can be identified with the ∞-category $\operatorname{Fun}_{/\mathcal{C}_-^{\mathrm{op}}}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{D})$ of sections of V'.

For each object $X \in \mathcal{C}_-$, let $\mathcal{C}_X$ denote the fiber $\{X\} \times_{\mathcal{C}_-^{\mathrm{op}}} \mathcal{C}$. Unwinding the definitions, we can identify objects of $\mathcal{D}$ with pairs (X, F_X), where X is an object of $\mathcal{C}_-$ and $F_X : \mathcal{C}_X \to \mathcal{E}$ is a functor satisfying $U \circ f = \lambda_+|_{\mathcal{C}_X}$. For fixed $X \in \mathcal{C}_-$, our assumption that λ is corepresentable guarantees that the ∞-category $\mathcal{C}_X$ has an initial object C. Set $Y = \lambda_+(C)$. By assumption, the ∞-category $\{Y\} \times_{\mathcal{C}_+} \mathcal{E}$ also has an initial object. Invoking the criterion of Corollary 7.3.6.11, we see that the ∞-category $\{X\} \times_{\mathcal{C}_-^{\mathrm{op}}} \mathcal{D} \simeq \operatorname{Fun}_{/\mathcal{C}_+}(\mathcal{C}_X, \mathcal{E})$ also has an initial object. Moreover, an object $F_X \in \operatorname{Fun}_{/\mathcal{C}_+}(\mathcal{C}_X, \mathcal{E})$ is initial if and only if it satisfies the following pair of conditions:

(1_X) For every initial object $C \in \mathcal{C}_X$, the image $F_X(C)$ is an initial object of the ∞-category $\{Y\} \times_{\mathcal{C}_+} \mathcal{E}$.

(2_X) The functor F_X carries each morphism in $\mathcal{C}_X$ to a U-cocartesian morphism of $\mathcal{E}$.

We will prove below that the functor V' is a cartesian fibration. Assuming this Corollary 7.3.5.7, we guarantees that the ∞-category

$$\operatorname{Fun}_{/\mathcal{C}_+}(\mathcal{C}, \mathcal{E}) \simeq \operatorname{Fun}_{/\mathcal{C}_-^{\mathrm{op}}}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{D})$$

has an initial object. Moreover, an object $F \in \operatorname{Fun}_{/\mathcal{C}_+}(\mathcal{C}, \mathcal{E})$ is initial if and only if, for every object $X \in \mathcal{C}_-$, the restriction $F_X = F|_{\mathcal{C}_X}$ satisfies conditions (1_X) and (2_X) above. Unwinding the definitions, this is equivalent to the requirement that F satisfies condition (1) and the following variant of condition $(2')$ of Remark 8.2.1.9:

$(2'')$ If e is a morphism of $\mathcal{C}$ such that $\lambda_-(e)$ is an identity morphism of $\mathcal{C}_-^{\mathrm{op}}$, then $F(e)$ is a U-cocartesian morphism of $\mathcal{E}$.

The implication $(2') \Rightarrow (2'')$ is immediate. The reverse implication follows from the observation that if $\lambda_-(e)$ is an isomorphism in $\mathcal{C}_-^{\mathrm{op}}$, then e is isomorphic (as an object of $\operatorname{Fun}(\Delta^1, \mathcal{C})$) to a morphism e' such that $\lambda_-(e')$ is an identity morphism of $\mathcal{C}_-^{\mathrm{op}}$.

We now complete the proof by showing that V' is a cartesian fibration. Fix an object $(X, F_X) \in \mathcal{D}$, and a morphism $u : X' \to X$ in the ∞-category $\mathcal{C}_-^{\mathrm{op}}$. We wish to show that u can be lifted to a V'-cartesian morphism $\widetilde{u} : (X', F_{X'}) \to (X, F_X)$ in the ∞-category $\mathcal{D}$. We will prove a slightly stronger assertion: we can arrange that the image of $\widetilde{u}$ in the ∞-category $\operatorname{Fun}(\mathcal{C} \,/\, \mathcal{C}_-^{\mathrm{op}}, \mathcal{E})$ is V-cartesian. Let us identify u with a morphism $\Delta^1 \to \mathcal{C}_-^{\mathrm{op}}$ and set $\mathcal{C}_u = \Delta^1 \times_{\mathcal{C}_-^{\mathrm{op}}} \mathcal{C}$, so that $\mathcal{C}_X$ can be identified with the fiber $\{1\} \times_{\Delta^1} \mathcal{C}_u$. By virtue of Corollary 7.3.7.6, it will suffice to show that the lifting problem

$$\begin{tikzcd}
\mathcal{C}_X \arrow[r, "F_X"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}_u \arrow[r] \arrow[ur, dashed, "F_u"] & \mathcal{C}_+
\end{tikzcd}$$

admits a solution having the property that F_u is U-right Kan extended from $\mathcal{C}_X$.

Let $\pi : \mathcal{C}_u \to \Delta^1$ denote the projection map. Since π is a pullback of λ_-, it is a cocartesian fibration of ∞-categories (Proposition 8.2.1.7). In particular, $\mathcal{C}_X$ is a reflective subcategory of $\mathcal{C}_u$. Moreover, if C is an object of $\mathcal{C}_X$, then a morphism $v : C' \to C$ in $\mathcal{C}_u$ is π-cocartesian if and only if it exhibits C as a $\mathcal{C}_X$-reflection of C' (see Proposition 6.2.2.22). By virtue of Corollary 7.3.5.9, it will suffice to show that if this condition is satisfied, then $\lambda_+(v)$ can be lifted to a U-cartesian morphism $E \to F_X(C)$ in $\mathcal{E}$. This is clear: our assumption that v is π-cocartesian guarantees that $\lambda_+(v)$ is an isomorphism in the ∞-category $\mathcal{C}_+$ (Proposition 8.2.1.7), and can therefore be lifted to an isomorphism in $\mathcal{E}$ by virtue of the fact that U is an isofibration (Proposition 5.1.4.8). □

Corollary 8.2.1.10. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. Suppose that, for every object $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \mathcal{E} \times_{\mathcal{C}} \{C\}$ has an initial object. Then the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}), \mathcal{E})$ has an initial object. Moreover, an object $F \in \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}), \mathcal{E})$ is initial if and only if it satisfies the following pair of conditions:* 0447

(1) *For every object $C \in \mathcal{C}$, the image $F(\mathrm{id}_C)$ is an initial object of the ∞-category $\mathcal{E}_C$.*

(2) *Let e be a morphism of $\mathrm{Tw}(\mathcal{C})$ whose image in $\mathcal{C}^{\mathrm{op}}$ is an isomorphism. Then $F(e)$ is a U-cocartesian morphism of $\mathcal{E}$.*

Stated more informally, Corollary 8.2.1.10 asserts that the twisted arrow ∞-category $\mathrm{Tw}(\mathcal{C})$ is *universal* among ∞-categories $\mathcal{E}$ equipped with a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ having the property that each fiber of U has an initial object.

8.2.2 Morphisms of Couplings

We begin by introducing a companion of Definition 8.2.0.1. 0448

Definition 8.2.2.1. Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be couplings of ∞-categories. A *morphism of couplings* from λ to μ is a triple of functors 0449

$$F_- : \mathcal{C}_- \to \mathcal{D}_- \qquad F : \mathcal{C} \to \mathcal{D} \qquad F_+ : \mathcal{C}_+ \to \mathcal{D}_+$$

for which the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[d, "\mu"] \\
\mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+} \arrow[r, "F_{-}^{\mathrm{op}} \times F_{+}"] & \mathcal{D}_{-}^{\mathrm{op}} \times \mathcal{D}_{+}
\end{tikzcd}$$

is commutative. Note that such diagrams can be identified with the vertices of a simplicial set $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$, defined by the formula

$$\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}_{-}, \mathcal{D}_{-})^{\mathrm{op}} \times_{\mathrm{Fun}(\mathcal{C}, \mathcal{D}_{-}^{\mathrm{op}})} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}, \mathcal{D}_{+})} \mathrm{Fun}(\mathcal{C}_{+}, \mathcal{D}_{+}).$$

044A **Proposition 8.2.2.2.** *Let $\lambda : \mathcal{C} \to \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}$ and $\mu : \mathcal{D} \to \mathcal{D}_{-}^{\mathrm{op}} \times \mathcal{D}_{+}$ be couplings of ∞-categories. Then the projection maps*

$$\Phi_{-} : \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_{-}, \mathcal{D}_{-})^{\mathrm{op}} \qquad \Phi_{+} : \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_{+}, \mathcal{D}_{+})$$

induce a left fibration

$$(\Phi_{-}, \Phi_{+}) : \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_{-}, \mathcal{D}_{-})^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_{+}, \mathcal{D}_{+}).$$

Proof. By construction, there is a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d, "(\Phi_{-}, \Phi_{+})"] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d, "\mu\circ"] \\
\mathrm{Fun}(\mathcal{C}_{-}, \mathcal{D}_{-})^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_{+}, \mathcal{D}_{+}) \arrow[r, "\circ\lambda"] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}_{-}^{\mathrm{op}} \times \mathcal{D}_{+}).
\end{tikzcd}$$

It will therefore suffice to show that the right vertical map is a left fibration (Remark 4.2.1.8), which follows from our assumption that μ is a left fibration (Corollary 4.2.5.2). ☐

044B **Remark 8.2.2.3** (Functor Couplings)**.** Let $\lambda : \mathcal{C} \to \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}$ and $\mu : \mathcal{D} \to \mathcal{D}_{-}^{\mathrm{op}} \times \mathcal{D}_{+}$ be couplings of ∞-categories. Proposition 8.2.2.2 asserts that the induced map

$$\Phi = (\Phi_{-}, \Phi_{+}) : \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_{-}, \mathcal{D}_{-})^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_{+}, \mathcal{D}_{+})$$

is also a coupling of ∞-categories. Moreover, it is characterized by a universal property: for every coupling of ∞-categories $\kappa : \mathcal{B} \to \mathcal{B}_{-}^{\mathrm{op}} \times \mathcal{B}_{+}$, there is a canonical isomorphism of simplicial sets

$$\mathrm{Fun}_{\pm}(\mathcal{B}, \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}_{\pm}(\mathcal{B} \times \mathcal{C}, \mathcal{D}),$$

where the right hand side is defined using the product coupling

$$\mathcal{B} \times \mathcal{C} \xrightarrow{\kappa \times \lambda} (\mathcal{B}_{-} \times \mathcal{C}_{-})^{\mathrm{op}} \times (\mathcal{B}_{+} \times \mathcal{C}_{+}).$$

Remark 8.2.2.4. Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and $\mu = (\mu_-, \mu_+) : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be couplings of ∞-categories, and suppose we are given a morphism of couplings 044C

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[d, "\mu"] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+ .
\end{tikzcd}$$

It follows from Proposition 8.2.1.7 that the functor F carries λ_--cocartesian morphisms of $\mathcal{C}$ to μ_--cocartesian morphisms of $\mathcal{D}$, and λ_+-cocartesian morphisms of $\mathcal{C}$ to μ_+-cocartesian morphisms of $\mathcal{D}$.

Corollary 8.2.2.5. *Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be couplings of ∞-categories. Then the simplicial set $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$ is an ∞-category.* 044D

Proof. Combine Proposition 8.2.2.2 with Remark 4.2.1.4. □

Example 8.2.2.6. The canonical isomorphism $\lambda : \Delta^0 \xrightarrow{\sim} (\Delta^0)^{\mathrm{op}} \times \Delta^0$ can be regarded as a coupling of the 0-simplex Δ^0 with itself. For every coupling of ∞-categories $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$, the ∞-category $\mathrm{Fun}_{\pm}(\Delta^0, \mathcal{D})$ can be identified with the ∞-category $\mathcal{D}$. 044E

Exercise 8.2.2.7. Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be couplings of ∞-categories, and suppose we are given a morphism of couplings 04AS

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$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[d, "\mu"] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+
\end{tikzcd} \qquad (8.23)$$

Show that the following conditions are equivalent:

- The functors F_-, F, and F_+ are equivalences of ∞-categories.
- There exists a morphism of couplings $(G_-, G, G_+) \in \mathrm{Fun}_{\pm}(\mathcal{D}, \mathcal{C})$ which is a homotopy inverse to (F_-, F, F_+), in the sense that the compositions $(G_- \circ F_-, G \circ F, G_+ \circ F_+)$ and $(F_- \circ G_-, F \circ G, F_+ \circ G_+)$ are isomorphic to $(\mathrm{id}_{\mathcal{C}_-}, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}_+})$ and $(\mathrm{id}_{\mathcal{D}_-}, \mathrm{id}_{\mathcal{D}}, \mathrm{id}_{\mathcal{D}_+})$ as objects of the ∞-categories $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{C})$ and $\mathrm{Fun}_{\pm}(\mathcal{D}, \mathcal{D})$, respectively.

If these conditions are satisfied, we will say that the diagram (8.23) is an *equivalence of couplings.*

04AU **Remark 8.2.2.8.** Suppose we are given a morphism of couplings

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[d, "\mu"] \\
\mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+} \arrow[r, "F_{-}^{\mathrm{op}} \times F_{+}"] & \mathcal{D}_{-}^{\mathrm{op}} \times \mathcal{D}_{+}
\end{tikzcd}$$

which is an equivalence (in the sense of Exercise 8.2.2.7). Then:

- The coupling λ is representable if and only if the coupling μ is representable.
- The coupling λ is corepresentable if and only if the coupling μ is corepresentable.
- An object $C \in \mathcal{C}$ is universal (with respect to the coupling λ) if and only if $F(C)$ is universal $\mathcal{D}$ (with respect to the coupling μ).
- An object $C \in \mathcal{C}$ is universal (with respect to the coupling λ) if and only if $F(C)$ is universal $\mathcal{D}$ (with respect to the coupling μ).

See Corollaries 4.6.7.21 and 4.6.7.20.

Beware that, in the situation of Definition 8.2.2.1, the ∞-category $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$ depends not only on $\mathcal{C}$ and $\mathcal{D}$, but also on the left fibrations $\lambda : \mathcal{C} \to \mathcal{C}_{-} \times \mathcal{C}_{+}$ and $\mu : \mathcal{D} \to \mathcal{D}_{-} \times \mathcal{D}_{+}$. Our goal in this section is to show that, nevertheless, it can often be identified with a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ (Proposition 8.2.2.9).

044F **Proposition 8.2.2.9.** *Let $\lambda = (\lambda_{-}, \lambda_{+}) : \mathcal{C} \to \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}$ and $\mu = (\mu_{-}, \mu_{+}) : \mathcal{D} \to \mathcal{D}_{-}^{\mathrm{op}} \times \mathcal{D}_{+}$ be couplings of ∞-categories. Let $\mathcal{E}_{-} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the full subcategory spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ which carry λ_{+}-cocartesian morphisms of $\mathcal{C}$ to μ_{+}-cocartesian morphisms of $\mathcal{D}$, define $\mathcal{E}_{+} \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ similarly, and set $\mathcal{E}_{\pm} = \mathcal{E}_{-} \cap \mathcal{E}_{+}$. Then:*

(1) *Suppose that, for every object $C_{-} \in \mathcal{C}_{-}$, the ∞-category $\{C_{-}\} \times_{\mathcal{C}_{-}^{\mathrm{op}}} \mathcal{C}$ is weakly contractible. Then the forgetful functor*

$$\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathcal{E}_{-} \times_{\mathrm{Fun}(\mathcal{C}, \mathcal{D}_{+})} \mathrm{Fun}(\mathcal{C}_{+}, \mathcal{D}_{+})$$

is an equivalence of ∞-categories.

(2) *Suppose that, for every object $C_{+} \in \mathcal{C}_{+}$, the ∞-category $\mathcal{C} \times_{\mathcal{C}_{+}} \{C_{+}\}$ is weakly contractible. Then the forgetful functor*

$$\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_{-}, \mathcal{D}_{-})^{\mathrm{op}} \times_{\mathrm{Fun}(\mathcal{C}, \mathcal{D}_{-}^{\mathrm{op}})} \mathcal{E}_{+}$$

is an equivalence of ∞-categories.

(3) *If the hypotheses of both* (1) *and* (2) *are satisfied, then the forgetful functor* $\mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \to \mathcal{E}_{\pm}$ *is an equivalence of* ∞*-categories.*

Proof. We first prove (1). Let W be the collection of all λ_+-cocartesian morphisms of $\mathcal{C}$. Note that a morphism u of $\mathcal{C}$ belongs to W if and only if $\lambda_-(u)$ is an isomorphism in the ∞-category $\mathcal{C}_-^{\mathrm{op}}$ (Proposition 8.2.1.7). Suppose that, for every object $C_- \in \mathcal{C}_-$, the ∞-category $\{C_-\} \times_{\mathcal{C}_-} \mathcal{C}$ is weakly contractible. Applying Corollary 6.3.5.3, we deduce that the functor λ_- exhibits $\mathcal{C}_-^{\mathrm{op}}$ as a localization of $\mathcal{C}$ with respect to W. It follows that precomposition with λ_- induces an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{C}_-,\mathcal{D}_-)^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}[W^{-1}],\mathcal{D}_-^{\mathrm{op}})$, where $\mathrm{Fun}(\mathcal{C}[W^{-1}],\mathcal{D}_-^{\mathrm{op}})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D}_-^{\mathrm{op}})$ spanned by those functors which carry each element of W to an isomorphism in $\mathcal{D}_-^{\mathrm{op}}$ (Notation 6.3.1.1). We have a commutative diagram of ∞-categories

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$$\begin{tikzcd}
\mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}_-,\mathcal{D}_-)^{\mathrm{op}} \arrow[d, "\circ\lambda_-"] \\
\mathcal{E}_- \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D}_+)} \mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}[W^{-1}],\mathcal{D}_-^{\mathrm{op}}) \arrow[d] \\
\mathrm{Fun}(\mathcal{C},\mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D}_+)} \mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+) \arrow[r, "\theta"] & \mathrm{Fun}(\mathcal{C},\mathcal{D}_-^{\mathrm{op}})
\end{tikzcd} \quad (8.24)$$

in which both squares are pullbacks. To prove (1), it will suffice to show that the upper square is a categorical pullback diagram (Proposition 4.5.2.21). In fact, we will show that θ is an isofibration, so that both squares are categorical pullback diagrams (Corollary 4.5.2.27). This follows by observing that θ factors as a composition

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D}_+)} \mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+) \xrightarrow{\theta'} \mathrm{Fun}(\mathcal{C},\mathcal{D}_-^{\mathrm{op}}) \times \mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+) \xrightarrow{\theta''} \mathrm{Fun}(\mathcal{C},\mathcal{D}_-^{\mathrm{op}}),$$

where θ' is a pullback of the composition map $\mathrm{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{\mu\circ} \mathrm{Fun}(\mathcal{C},\mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+)$ (hence a left fibration by virtue of Corollary 4.2.5.2) and θ'' is a pullback of the projection map $\mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+) \to \Delta^0$. This completes the proof of assertion (1).

Assertion (2) follows by a similar argument. We now prove (3). Suppose that λ satisfies the hypotheses of both (1) and (2); we wish to prove that the forgetful functor $T : \mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \to \mathcal{E}_{\pm}$ is an equivalence of ∞-categories. Note that T factors as a composition

$$\mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \xrightarrow{T'} \mathrm{Fun}(\mathcal{C}_-,\mathcal{D}_-)^{\mathrm{op}} \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D}_-^{\mathrm{op}})} \mathcal{E}_+ \xrightarrow{T''} \mathcal{E}_{\pm},$$

where T' is an equivalence of ∞-categories by virtue of (2). It will therefore suffice to show that T'' is an equivalence of ∞-categories. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}} \times_{\mathrm{Fun}(\mathcal{C}, \mathcal{D}_-^{\mathrm{op}})} \mathcal{E}_+ \arrow[r] \arrow[d, "T''"] & \mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}} \arrow[d, "\circ \lambda_-"] \\
\mathcal{E}_\pm \arrow[r, "\mu_- \circ"] \arrow[d] & \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}_-^{\mathrm{op}}) \arrow[d] \\
\mathcal{E}_+ \arrow[r, "\rho"] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}_-^{\mathrm{op}}),
\end{tikzcd}$$

where both squares are pullbacks and the upper right vertical map is an equivalence of ∞-categories. It will therefore suffice to show that the upper square is a categorical pullback diagram (Proposition 4.5.2.21). In fact, we claim that ρ is an isofibration, so that both squares are categorical pullback diagrams (Corollary 4.5.2.27). This follows by observing that ρ is the restriction of the map $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \xrightarrow{\mu_- \circ} \mathrm{Fun}(\mathcal{C}, \mathcal{D}_-^{\mathrm{op}})$ (which is a cocartesian fibration by virtue of Proposition 8.2.1.7 and Theorem 5.2.1.1) to a replete subcategory $\mathcal{E}_+ \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. □

044H **Corollary 8.2.2.10.** *Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and $\mu = (\mu_-, \mu_+) : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be couplings of ∞-categories. Fix a functor $F_+ : \mathcal{C}_+ \to \mathcal{D}_+$. If λ is corepresentable, then the forgetful functor*

$$\mathrm{Fun}_\pm(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)} \{F_+\} \to \mathrm{Fun}_{/\mathcal{D}_+}(\mathcal{C}, \mathcal{D})$$

is fully faithful, and its essential image is the full subcategory $\mathrm{Fun}^0_{/\mathcal{D}_+}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Fun}_{/\mathcal{D}_+}(\mathcal{C}, \mathcal{D})$ spanned by those functors which carry λ_+-cocartesian morphisms of $\mathcal{C}$ to μ_+-cocartesian morphisms of $\mathcal{D}$.

Proof. Let $\mathcal{E}_- \subseteq \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the full subcategory defined in Proposition 8.2.2.9. We then have a commutative diagram of ∞-categories

044J

$$\begin{tikzcd}
\mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+)} \{F_+\} \arrow[r] \arrow[d] & \mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \arrow[d] \\
\mathrm{Fun}^0_{/\mathcal{D}_+}(\mathcal{C},\mathcal{D}) \arrow[r] \arrow[d] & \mathcal{E}_- \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D}_+)} \mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+) \arrow[d] \\
\{F_+\} \arrow[r] & \mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+),
\end{tikzcd} \tag{8.25}$$

where both squares are pullback diagrams. Note that the vertical map on the lower right is a pullback of the functor $\mathcal{E}_- \to \mathrm{Fun}(\mathcal{C},\mathcal{D}_+)$ obtained by restricting the cocartesian fibration $\mathrm{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{\mu_+\circ} \mathrm{Fun}(\mathcal{C},\mathcal{D}_+)$ (see Proposition 8.2.1.7 and Theorem 5.2.1.1) to the replete subcategory $\mathcal{E}_- \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{D})$, and is therefore an isofibration. Moreover, the right vertical composition $\mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+)$ is a cocartesian fibration (see Proposition 8.2.1.7 and Remark 8.2.2.3), and therefore an isofibration. It follows that the bottom square and outer rectangle of (8.25) are categorical pullback diagrams (Corollary 4.5.2.27), so that the upper square is also a categorical pullback diagram (Proposition 4.5.2.18). Our assumption that λ is corepresentable guarantees that for each object $X_- \in \mathcal{C}_-$, the fiber $\{X_-\} \times_{\mathcal{C}_-^{\mathrm{op}}} \mathcal{C}$ has an initial object, and is therefore weakly contractible. Applying Proposition 8.2.2.9, we deduce that the vertical map on the upper right is an equivalence of ∞-categories. Invoking Proposition 4.5.2.21, we conclude that the forgetful functor $\mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+)} \{F_+\} \to \mathrm{Fun}^0_{/\mathcal{D}_+}(\mathcal{C},\mathcal{D})$ is also an equivalence of ∞-categories. □

We can now formulate the main result of this section.

Theorem 8.2.2.11. *Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a representable coupling of ∞-categories and let $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be a corepresentable coupling of ∞-categories. Then the functor coupling* 044K

$$\Phi = (\Phi_-, \Phi_+) : \mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_-,\mathcal{D}_-)^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_+,\mathcal{D}_+)$$

is corepresentable. Moreover, an object $(F_-, F, F_+) \in \mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{D})$ is couniversal if and only if the functor $F : \mathcal{C} \to \mathcal{D}$ carries universal objects of $\mathcal{C}$ to couniversal objects of $\mathcal{D}$.

Proof. Fix a functor $F_- : \mathcal{C}_- \to \mathcal{D}_-$; we wish to show that it can be extended to a couniversal object $(F_-, F, F_+) \in \mathrm{Fun})_{\pm}(\mathcal{C},\mathcal{D})$. Set $\mathcal{E} = \mathcal{C}_-^{\mathrm{op}} \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$. Then projection onto the first factor determines a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}_-^{\mathrm{op}}$ (Proposition 8.2.1.7). Let $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_-^{\mathrm{op}}}(\mathcal{C},\mathcal{E})$

denote the full subcategory of $\operatorname{Fun}_{/\mathcal{C}_-^{\mathrm{op}}}(\mathcal{C},\mathcal{E})$ spanned by those functors which carry λ_--cocartesian morphisms of $\mathcal{C}$ to U-cocartesian of $\mathcal{E}$ (Notation 5.3.1.10). Corollary 8.2.2.10 guarantees that the forgetful functor

$$\{F_-\} \times_{\operatorname{Fun}(\mathcal{C}_-,\mathcal{D}_-)^{\mathrm{op}}} \operatorname{Fun}_{\pm}(\mathcal{C},\mathcal{D}) \to \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_-^{\mathrm{op}}}(\mathcal{C},\mathcal{E})$$

is an equivalence of ∞-categories. Theorem 8.2.2.11 can therefore be restated as follows:

(1) The ∞-category $\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_-^{\mathrm{op}}}(\mathcal{C},\mathcal{E})$ has an initial object.

(2) An object $F \in \operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}_-^{\mathrm{op}}}(\mathcal{C},\mathcal{E})$ is initial if and only if, for universal object $C \in \mathcal{C}$, the image $F(C)$ is an initial object of the ∞-category

$$\mathcal{E}_{\lambda_-(C)} \simeq \{F_-(\lambda_-(C))\} \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}.$$

Our assumption that μ is corepresentable guarantees that, for each object $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_{\lambda_-(C)}$ has an initial object. Consequently, assertions (1) and (2) follow from (the dual of) Proposition 8.2.1.8. □

044L **Example 8.2.2.12.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then the commutative diagram

044M
$$\begin{tikzcd}
\operatorname{Tw}(\mathcal{C}) \arrow[r, "\operatorname{Tw}(F)"] \arrow[d] & \operatorname{Tw}(\mathcal{D}) \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "F^{\mathrm{op}} \times F"] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D}
\end{tikzcd} \tag{8.26}$$

is a morphism of couplings. It follows from Theorem 8.2.2.11 that this morphism is initial when viewed as an object of the ∞-category $\{F\} \times_{\operatorname{Fun}(\mathcal{C},\mathcal{D})^{\mathrm{op}}} \operatorname{Fun}_{\pm}(\operatorname{Tw}(\mathcal{C}), \operatorname{Tw}(\mathcal{D}))$.

044N **Corollary 8.2.2.13.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, and suppose we are given a pair of functors $F_-, F_+ : \mathcal{C} \to \mathcal{D}$. Then F_- and F_+ are isomorphic (as objects of the ∞-category* $\operatorname{Fun}(\mathcal{C},\mathcal{D})$*) if and only if there exists a morphism of couplings*

044P
$$\begin{tikzcd}
\operatorname{Tw}(\mathcal{C}) \arrow[r, "\widetilde{F}"] \arrow[d] & \operatorname{Tw}(\mathcal{D}) \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D}
\end{tikzcd} \tag{8.27}$$

having the property that the functor $\widetilde{F}$ carries isomorphisms of $\mathcal{C}$ (regarded as objects of $\operatorname{Tw}(\mathcal{C})$*) to isomorphisms of $\mathcal{D}$ (regarded as objects of* $\operatorname{Tw}(\mathcal{D})$*).*

Proof. Suppose first that there exists an isomorphism of functors $\alpha : F_- \to F_+$. Since the projection map $\mathrm{Tw}(\mathcal{D}) \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ is an isofibration, we can use Corollary 4.4.5.6 to lift the natural transformation

$$(\mathrm{id} \times \alpha) : F_-^{\mathrm{op}} \times F_+ \to F_-^{\mathrm{op}} \times F_-$$

to an isomorphism $\widetilde{F} \to \mathrm{Tw}(F_-)$ in the ∞-category $\mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathrm{Tw}(\mathcal{D}))$, so that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}) \arrow[r, "\widetilde{F}"] \arrow[d] & \mathrm{Tw}(\mathcal{D}) \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D}
\end{tikzcd}$$

where $\widetilde{F}$ carries isomorphisms of $\mathcal{C}$ to isomorphisms of $\mathcal{D}$.

We now prove the converse. Suppose we are given a commutative diagram (8.27), where $\widetilde{F}$ carries isomorphisms of $\mathcal{C}$ to isomorphisms of $\mathcal{D}$. Applying Theorem 8.2.2.11, we deduce that the triple $(F_-, \widetilde{F}, F_+)$ is initial when viewed as an object of the ∞-category $\{F_-\} \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}), \mathrm{Tw}(\mathcal{D}))$. Applying Example 8.2.2.12 (and Corollary 4.6.7.15), we deduce that $(F_-, \widetilde{F}, F_+)$ is isomorphic to $(F_-, \mathrm{Tw}(F_-), F_-)$ as an object of the ∞-category $\{F_-\} \times_{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}), \mathrm{Tw}(\mathcal{D}))$. In particular, F_+ is isomorphic to F_- as an object of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. □

8.2.3 Representations of Couplings

We now apply Theorem 8.2.2.11 to give a classification of representable couplings. 044Q

Definition 8.2.3.1. Let $G : \mathcal{C}_+ \to \mathcal{C}_-$ be a functor of ∞-categories. We will say that a coupling $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ is *representable by G* if it is equivalent (as a left fibration over $\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$) to the coupling λ_G of Construction 8.2.0.3. 044R

Remark 8.2.3.2. Let $G, G' : \mathcal{C}_+ \to \mathcal{C}_-$ be functors which are isomorphic (as objects of the ∞-category $\mathrm{Fun}(\mathcal{C}_+, \mathcal{C}_-)$). Then a coupling $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ is representable by G if and only if it is representable by G'. See Proposition 5.1.7.5. 044S

Example 8.2.3.3. Let $\mathcal{C}$ be an ∞-category. Then the twisted arrow coupling $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ of of Example 8.2.0.2 is representable by the identity functor $\mathrm{id} : \mathcal{C} \to \mathcal{C}$. 044T

Our goal is to prove the following restatement of Theorem 8.2.0.4:

Theorem 8.2.3.4. *Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories. Then λ is representable (in the sense of Definition 8.2.1.3) if and only there exists a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$ such that λ is representable by G (in the sense of Definition 8.2.3.1). If this condition is satisfied, then the functor G is uniquely determined up to isomorphism.* 044U

Before giving the proof of Theorem 8.2.3.4, it will be useful to formulate a more precise version of Definition 8.2.3.1.

044V **Definition 8.2.3.5.** Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let $G : \mathcal{C}_+ \to \mathcal{C}_-$ be a functor. We say that a morphism of couplings

044W
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{G}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_-) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "\mathrm{id} \times G"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_-,
\end{tikzcd} \tag{8.28}$$

exhibits the coupling λ as represented by G if it is a categorical pullback square. Note that λ is representable by G if and only if there exists a morphism of couplings which exhibits λ as represented by G.

We now describe an alternative formulation of Definition 8.2.3.5.

044X **Lemma 8.2.3.6.** *Suppose we are given a morphism of couplings*

044Y
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[d, "\mu"] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+,
\end{tikzcd} \tag{8.29}$$

where λ is representable and F_- is an equivalence of ∞-categories. The following conditions are equivalent:

(1) *The diagram (8.29) is a categorical pullback square.*

(2) *For every object $Y \in \mathcal{C}_+$, the functor F induces an equivalence of ∞-categories*

$$F_Y : \mathcal{C} \times_{\mathcal{C}_+} \{Y\} \to \mathcal{D} \times_{\mathcal{D}_+} \{F_+(Y)\}.$$

(3) *For every universal object $C \in \mathcal{C}$, the image $F(C) \in \mathcal{D}$ is universal.*

Proof. The equivalence (1) ⇔ (2) follows from Theorem 5.1.6.1 and Remark 8.2.2.4. To complete the proof, it will suffice to show that for each object $Y \in \mathcal{C}_+$, the following conditions are equivalent:

(2_Y) The functor F_Y is an equivalence of ∞-categories.

(3_Y) The functor F_Y carries initial objects of $\mathcal{C} \times_{\mathcal{C}_+} \{Y\}$ to initial objects of $\mathcal{D} \times_{\mathcal{D}_+} \{F_+(Y)\}$.

Replacing $\mathcal{D}$ by the ∞-category $\mathcal{C}_-^{\mathrm{op}} \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$, we can assume that F_- is an isomorphism. In this case, the equivalence of (2_Y) and (3_Y) is a special case of Corollary 4.6.7.21. □

Proposition 8.2.3.7. *Suppose we are given a morphism of couplings* 044Z

0450
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{G}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_-) \arrow[d, "\mu"] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "\mathrm{id} \times G"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_-,
\end{tikzcd} \tag{8.30}$$

The following conditions are equivalent:

(1) *The diagram (8.30) exhibits the coupling λ as represented by the functor G (in the sense of Definition 8.2.3.1).*

(2) *For every object $C \in \mathcal{C}_+$, the functor $\widetilde{G}$ induces an equivalence of ∞-categories*

$$\widetilde{G}_C : \mathcal{C} \times_{\mathcal{C}_+} \{C\} \to \mathrm{Tw}(\mathcal{C}_-) \times_{\mathcal{C}_-} \{G(C)\}.$$

(3) *The coupling λ is representable and, for every universal object $C \in \mathcal{C}$, the image $\widetilde{G}(C) \in \mathrm{Tw}(\mathcal{C}_-)$ is an isomorphism (when viewed as a morphism of the ∞-category $\mathcal{C}_-$).*

(4) *The coupling λ is representable and the triple $(\mathrm{id}, \widetilde{G}, G)$ is initial when viewed as an object of the ∞-category $\{\mathrm{id}\} \times_{\mathrm{Fun}(\mathcal{C}_-,\mathcal{C}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathcal{C}, \mathrm{Tw}(\mathcal{C}_-))$.*

Proof. The implication (1) ⇒ (2) is immediate. Note that, if condition (2) is satisfied, then the coupling λ is representable; the implications (2) ⇒ (3) ⇒ (1) then follow from Lemma 8.2.3.6 (using the characterization of universal objects of $\mathrm{Tw}(\mathcal{C}_-)$ given by Example 8.2.1.5). The equivalence (3) ⇔ (4) follows from Theorem 8.2.2.11. □

Proof of Theorem 8.2.3.4. Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories. It follows from Proposition 8.2.3.7 that λ is representable by a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$ if and only if it is representable and G can be lifted to an initial object of the ∞-category $\mathcal{E} = \{\mathrm{id}\} \times_{\mathrm{Fun}(\mathcal{C}_-,\mathcal{C}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathcal{C}, \mathrm{Tw}(\mathcal{C}_-))$. This immediately shows that G is uniquely determined up to isomorphism. To prove existence, it suffices to show that if λ is representable then $\mathcal{E}$ has an initial object. This follows from Theorem 8.2.2.11. □

0451 **Variant 8.2.3.8.** Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let $F : \mathcal{C}_- \to \mathcal{C}_+$ be a functor. We say that λ is *corepresentable by F* if there exists a categorical pullback square

0452
$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_+) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ .
\end{tikzcd} \tag{8.31}$$

In this case, we will say that the diagram (8.31) *exhibits the coupling λ as corepresented by F*. It follows from Theorem 8.2.3.4 that λ is corepresentable (in the sense of Definition 8.2.1.3) if and only if it is corepresentable by F, for some functor $F : \mathcal{C}_- \to \mathcal{C}_+$. Moreover, if this condition is satisfied, then the functor F is uniquely determined up to isomorphism.

Couplings representable by a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$ can be characterized by a universal mapping property.

0453 **Proposition 8.2.3.9.** *Let $\mu = (\mu_-, \mu_+) : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be a coupling of ∞-categories, let $G : \mathcal{D}_+ \to \mathcal{D}_-$ be a functor, and suppose we are given a morphism of couplings*

0454
$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\widetilde{G}"] \arrow[d] & \mathrm{Tw}(\mathcal{D}_-) \arrow[d] \\
\mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+ \arrow[r, "\mathrm{id} \times G"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_- .
\end{tikzcd} \tag{8.32}$$

The following conditions are equivalent:

(1) *The diagram (8.32) exhibits μ as represented by G (in the sense of Definition 8.2.3.5).*

(2) *For every coupling of ∞-categories $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and every pair of functors $F_- : \mathcal{C}_- \to \mathcal{D}_-$ and $F_+ : \mathcal{C}_+ \to \mathcal{D}_+$, composition with $\widetilde{G}$ induces a homotopy equivalence of Kan complexes*

$$\begin{tikzcd}
\{F_-\} \times_{\mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)} \{F_+\} \arrow[d] \\
\{F_-\} \times_{\mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathcal{C}, \mathrm{Tw}(\mathcal{D}_-)) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_-)} \{G \circ F_+\}
\end{tikzcd}$$

(3) *For every coupling of ∞-categories $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and every functor $F_+ : \mathcal{C}_+ \to \mathcal{D}_+$, composition $\widetilde{G}$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)} \{F_+\} \to \mathrm{Fun}_{\pm}(\mathcal{C}, \mathrm{Tw}(\mathcal{D}_-)) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_-)} \{G \circ F_+\}.$$

Proof. We first show that (1) ⇒ (2). Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let $F_- : \mathcal{C}_- \to \mathcal{D}_-$ and $F_+ : \mathcal{C}_+ \to \mathcal{D}_+$ be functors. If condition (1) is satisfied, then Remark 4.5.2.9 guarantees that the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[r, "\widetilde{G}\circ"] \arrow[d, "\mu\circ"] & \mathrm{Fun}(\mathcal{C}, \mathrm{Tw}(\mathcal{D}_-)) \arrow[d] \\
\mathrm{Fun}(\mathcal{C}, \mathcal{D}_-^{\mathrm{op}}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}_+) \arrow[r] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}_-^{\mathrm{op}}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}_-)
\end{tikzcd}$$

is a categorical pullback square, where the vertical maps are left fibrations (Corollary 4.2.5.2). Assertion (2) now follows by applying Corollary 4.5.2.31 to the object $(F_-^{\mathrm{op}} \circ \lambda_-, F_+ \circ \lambda_+) \in \mathrm{Fun}(\mathcal{C}, \mathcal{D}_-^{\mathrm{op}}) \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}_+)$. The implication (2) ⇒ (3) follows by applying Corollary 5.1.6.4 to the commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)} \{F_+\} \arrow[rr] \arrow[dr] & & \mathrm{Fun}_{\pm}(\mathcal{C}, \mathrm{Tw}(\mathcal{D}_-)) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_-)} \{G \circ F_+\} \arrow[dl] \\
& \mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}}, &
\end{tikzcd}$$

since the vertical maps are left fibrations (Proposition 8.2.2.2). We complete the proof by showing that (3) implies (1). Specializing assertion (3) to the coupling $\lambda : \Delta^0 \xrightarrow{\sim} (\Delta^0)^{\mathrm{op}} \times \Delta^0$, we deduce that $\widetilde{G}$ induces an equivalence of ∞-categories $\mathcal{D} \times_{\mathcal{D}_+} \{D\} \to \mathrm{Tw}(\mathcal{D}_-) \times_{\mathcal{D}_-} \{G(D)\}$ for each object $D \in \mathcal{D}_+$, so that (8.32) is a categorical pullback square by virtue of Proposition 8.2.3.7. □

8.2.4 Presentations of Representable Couplings

For some applications, it is convenient to work with a variant of Definition 8.2.3.5. 0455

0456 **Definition 8.2.4.1.** Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let $G : \mathcal{C}_+ \to \mathcal{C}_-$ be a functor. We will say that a morphism of couplings

0457
$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{G}"] \arrow[d] & \mathcal{C} \arrow[d, "\lambda"] \\
\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "G^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+
\end{tikzcd} \tag{8.33}$$

exhibits λ *as represented by* G if, for every object $C \in \mathcal{C}_+$, the image $\widetilde{G}(\mathrm{id}_C)$ is a universal object of $\mathcal{C}$.

0458 **Remark 8.2.4.2.** In the situation of Definition 8.2.4.1, if the diagram (8.33) exhibits λ as represented by G if and only if the functor $\widetilde{G}$ carries each isomorphism in $\mathcal{C}_+$ (regarded as an object of the ∞-category $\mathrm{Tw}(\mathcal{C}_+)$) to a universal object of $\mathcal{C}$. This follows from Remark 8.2.1.2, since every isomorphism in $\mathcal{C}_+$ is isomorphic to an identity morphism (when viewed as an object of $\mathrm{Tw}(\mathcal{C}_+)$).

0459 **Proposition 8.2.4.3.** *Let* $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ *be a coupling of* ∞*-categories and let* $G : \mathcal{C}_+ \to \mathcal{C}_-$ *be a functor. The following conditions are equivalent:*

(1) *The coupling* λ *is representable by* G *(in the sense of Definition 8.2.3.1).*

(2) *There exists a morphism of couplings*

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{G}"] \arrow[d] & \mathcal{C} \arrow[d, "\lambda"] \\
\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "G^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+
\end{tikzcd}$$

which exhibits λ *as represented by* G *(in the sense of Definition 8.2.4.1).*

Proof. We first show that (2) implies (1). Suppose that there exists a morphism of couplings

045A
$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{G}"] \arrow[d] & \mathcal{C} \arrow[d, "\lambda"] \\
\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "G^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+
\end{tikzcd} \tag{8.34}$$

which exhibits λ as represented by G (in the sense of Definition 8.2.4.1). For each object $C_+ \in \mathcal{C}_+$, the functor $\widetilde{G}$ carries id_C to a universal object of $C \in \mathcal{C}$ satisfying $\lambda_+(C) = C_+$. It follows that the coupling λ is representable. Theorem 8.2.3.4, guarantees that there exists a functor $G' : \mathcal{C}_+ \to \mathcal{C}_-$ such that λ is representable by G'. Choose morphism of couplings

045B

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{G'}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_-) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "\mathrm{id} \times G'"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_-,
\end{tikzcd} \tag{8.35}$$

which exhibits λ as represented by G' (in the sense of Definition 8.2.3.5). Composing (8.35) with (8.34), we obtain a morphism of twisted arrow couplings

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{G'} \circ \widetilde{G}"] \arrow[d] & \mathrm{Tw}(\mathcal{C}_-) \arrow[d] \\
\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "G^{\mathrm{op}} \times G'"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+
\end{tikzcd}$$

where the functor $\widetilde{G'} \circ \widetilde{G}$ carries isomorphisms of $\mathcal{C}_+$ to isomorphisms of $\mathcal{C}_-$. Invoking Corollary 8.2.2.13, we deduce that the functors G and G' are isomorphic, so that λ is also representable by G (Remark 8.2.3.2).

We now show that (1) implies (2). Assume that λ is representable by G. Setting $G' = G$, we can choose a diagram (8.35) which exhibits λ as represented by G. Applying Proposition 8.2.3.9, we deduce that composition with $\widetilde{G'}$ induces a homotopy equivalence of Kan complexes

$$\begin{tikzcd}
\{G\} \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{C}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathcal{C}) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{C}_+)} \{\mathrm{id}\} \arrow[d, "\theta"] \\
\{G\} \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{C}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathrm{Tw}(\mathcal{C}_-)) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{C}_-)} \{G\}.
\end{tikzcd}$$

In particular, there exists an object $(G, \widetilde{G}, \mathrm{id}) \in \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathcal{C})$ such that $(\mathrm{id}, \widetilde{G'}, G) \circ (G, \widetilde{G}, \mathrm{id})$ is isomorphic to $(G, \mathrm{Tw}(G), G)$ in the ∞-category $\mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathrm{Tw}(\mathcal{C}_-))$. In particular, the functor $\widetilde{G'} \circ \widetilde{G} : \mathrm{Tw}(\mathcal{C}_+) \to \mathrm{Tw}(\mathcal{C}_-)$ is isomorphic to the functor $\mathrm{Tw}(G)$, and therefore carries isomorphisms of $\mathcal{C}_+$ to isomorphisms of $\mathcal{C}_-$. It follows that the functor

$\widetilde{G} : \mathrm{Tw}(\mathcal{C}_+) \to \mathcal{C}$ carries isomorphisms in $\mathcal{C}_+$ to universal objects of $\mathcal{C}$, so that the diagram (8.34) exhibits λ as represented by G. □

045C **Corollary 8.2.4.4.** *Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be couplings of ∞-categories which are representable by functors $G : \mathcal{C}_+ \to \mathcal{C}_-$ and $H : \mathcal{D}_+ \to \mathcal{D}_-$, respectively. Let $F_- : \mathcal{C}_- \to \mathcal{D}_-$ and $F_+ : \mathcal{C}_+ \to \mathcal{D}_+$ be functors. The following conditions are equivalent:*

(1) *The functors $H \circ F_+$ and $F_- \circ G$ are isomorphic (as objects of the ∞-category* $\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_-)$*). That is, the diagram of ∞-categories*

$$\begin{tikzcd} \mathcal{C}_+ \arrow[r, "F_+"] \arrow[d, "G"] & \mathcal{D}_+ \arrow[d, "H"] \\ \mathcal{C}_- \arrow[r, "F_-"] & \mathcal{D}_- \end{tikzcd}$$

commutes up to isomorphism.

(2) *There is a morphism of couplings*

045D
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[d, "\mu"] \\ \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}^+ \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+, \end{tikzcd} \tag{8.36}$$

where the functor $\widetilde{F}$ carries universal objects of $\mathcal{C}$ to universal objects of $\mathcal{D}$.

Proof. Choose morphisms of couplings

045E
$$\begin{tikzcd} \mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{G}"] \arrow[d] & \mathcal{C} \arrow[d, "\lambda"] & \mathcal{D} \arrow[r, "\widetilde{H}"] \arrow[d, "\mu"] & \mathrm{Tw}(\mathcal{D}_-) \arrow[d] \\ \mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "G^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+ \arrow[r, "\mathrm{id} \times H"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_-, \end{tikzcd} \tag{8.37}$$

which exhibit λ and μ as represented by G and H, respectively. We first prove that (2) implies (1). Suppose there exists a diagram (8.36), where $\widetilde{F}$ carries universal objects of $\mathcal{C}$ to universal objects of $\mathcal{D}$. Composing with the morphisms (8.37), we obtain a morphism of twisted arrow couplings

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{H}\circ\widetilde{F}\circ\widetilde{G}"] \arrow[d] & \mathrm{Tw}(\mathcal{D}_-) \arrow[d] \\
\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "(F_-\circ G)^{\mathrm{op}}\times(H\circ F_+)"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_-,
\end{tikzcd}$$

where the functor $\widetilde{H} \circ \widetilde{F} \circ \widetilde{G}$ carries isomorphisms in $\mathcal{C}_+$ to isomorphisms in $\mathcal{D}_-$. Applying Corollary 8.2.2.13, we deduce that the functors $F_- \circ G$ and $H \circ F_+$ are isomorphic.

We now show that (1) implies (2). Since λ is representable and the twisted arrow pairing $\mathrm{Tw}(\mathcal{D}_-) \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_-$ is corepresentable, Theorem 8.2.2.11 guarantees that there exists a morphism of pairings

045F

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{T}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{D}_-) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}}\times T_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_-,
\end{tikzcd} \tag{8.38}$$

where $\widetilde{T}$ carries universal objects of $\mathcal{C}$ to isomorphisms in the ∞-category $\mathcal{D}_-$. Composing with the pairing on the left half of (8.37), we obtain a morphism of twisted arrow pairings

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{T}\circ\widetilde{G}"] \arrow[d] & \mathrm{Tw}(\mathcal{D}_-) \arrow[d] \\
\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "(F_-\circ G)^{\mathrm{op}}\times T_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_- .
\end{tikzcd}$$

Applying Corollary 8.2.2.13, we conclude that T_+ is isomorphic to the functor $F_- \circ G$. If condition (1) is satisfied, then T_+ is also isomorphic to the functor $H \circ F_+$. Replacing (8.38) by an isomorphic objects of ∞-category $\{F_-\} \times_{\mathrm{Fun}(\mathcal{C}_-,\mathcal{D}_-)^{\mathrm{op}}} \times \mathrm{Fun}_{\pm}(\mathcal{C}, \mathrm{Tw}(\mathcal{D}_-))$, we may assume without loss of generality that T_+ is equal to $H \circ F_+$. Invoking the universal property of Proposition 8.2.3.9, we can further assume that (8.38) factors as a composition

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[r, "\widetilde{H}"] \arrow[d, "\mu"] & \mathrm{Tw}(\mathcal{D}_-) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+ \arrow[r, "\mathrm{id} \times H"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_- .
\end{tikzcd}$$

Since $\widetilde{T} = \widetilde{H} \circ \widetilde{F}$ carries universal objects of $\mathcal{D}$ to isomorphisms in $\mathcal{D}_-$, the functor $\widetilde{F}$ carries universal objects of $\mathcal{C}$ to universal objects of $\mathcal{D}$. □

045G **Variant 8.2.4.5.** Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let $F : \mathcal{C}_- \to \mathcal{C}_+$ be a functor. We will say that a morphism of couplings

045H
$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_-) \arrow[r, "\widetilde{F}"] \arrow[d] & \mathcal{C} \arrow[d, "\lambda"] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_- \arrow[r, "\mathrm{id} \times F"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+
\end{tikzcd} \qquad (8.39)$$

exhibits λ as corepresented by F if, for every object $X_- \in \mathcal{C}_-$, the image $\widetilde{F}(\mathrm{id}_{X_-})$ is a couniversal object of $\mathcal{C}$. Equivalently, the diagram (8.39) exhibits λ as corepresented by F if exhibits the coupling $\lambda' : \mathcal{C} \to \mathcal{C}_+ \times \mathcal{C}_-^{\mathrm{op}}$ of Remark 8.2.1.4 as represented by the functor $F^{\mathrm{op}} : \mathcal{C}_-^{\mathrm{op}} \to \mathcal{C}_+^{\mathrm{op}}$.

We now apply these ideas to prove a more precise version of Theorem 8.2.2.11. To (slightly) simplify the notation, we state the result in a dual form.

045J **Theorem 8.2.4.6.** *Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ and $\mu : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be couplings of ∞-categories. Assume that λ is corepresentable by a functor $F : \mathcal{C}_- \to \mathcal{C}_+$ and that μ is representable by a functor $G : \mathcal{D}_+ \to \mathcal{D}_-$. Then:*

(1) *The coupling*

$$\Phi : \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)$$

of Remark 8.2.2.3 is representable by the functor

$$\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+) \to \mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-) \qquad T_+ \mapsto G \circ T_+ \circ F.$$

(2) *An object $(T_-, T, T_+) \in \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$ is universal if and only if the functor T carries couniversal objects of $\mathcal{C}$ to universal objects of $\mathcal{D}$.*

Remark 8.2.4.7. Stated more informally, Theorem 8.2.4.6 states that if a coupling $\lambda : \mathcal{C} \to \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}$ is corepresented by a functor $F : \mathcal{C}_{-} \to \mathcal{C}_{+}$ and a coupling $\mu : \mathcal{D} \to \mathcal{D}_{-}^{\mathrm{op}} \times \mathcal{D}_{+}$ is represented by a functor $G : \mathcal{D}_{+} \to \mathcal{D}_{-}$, then the objects of $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$ can be identified with triples (T_{-}, T_{+}, α) where T_{+} is a functor from $\mathcal{C}_{+}$ to $\mathcal{D}_{+}$, and $\alpha : T_{-} \to G \circ T_{+} \circ F$ is a natural transformation of functors from $\mathcal{C}_{-}$ to $\mathcal{D}_{-}$. Moreover, the natural transformation α is an isomorphism if and only if the corresponding functor $\mathcal{C} \to \mathcal{D}$ carries couniversal objects to universal objects. 045K

Example 8.2.4.8. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories, let $(\lambda_{-}, \lambda_{+}) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ and $(\mu_{-}, \mu_{+}) : \mathrm{Tw}(\mathcal{D}) \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ be the twisted arrow couplings of Example 8.2.0.2, and let 045L

$$\mathrm{ev} : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \to \mathcal{D} \qquad (F, C) \mapsto F(C)$$

be the evaluation functor. Passing to twisted arrow ∞-categories, we obtain a map

$$\mathrm{Tw}(\mathrm{ev}) : \mathrm{Tw}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \times \mathrm{Tw}(\mathcal{C}) \to \mathrm{Tw}(\mathcal{D}),$$

which we can identify with a functor $\widetilde{E} : \mathrm{Tw}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathrm{Tw}(\mathcal{D}))$. By construction, the functor $\widetilde{E}$ fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \arrow[d, "\circ\lambda_{-}"] & \mathrm{Tw}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \arrow[l] \arrow[r] \arrow[d, "\widetilde{T}"] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d, "\circ\lambda_{+}"] \\
\mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathcal{D}^{\mathrm{op}}) & \mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathrm{Tw}(\mathcal{D})) \arrow[l, "\mu_{-}\circ"'] \arrow[r, "\mu_{+}\circ"] & \mathrm{Fun}(\mathrm{Tw}(\mathcal{C}), \mathcal{D}),
\end{tikzcd}$$

and therefore determines a functor $E : \mathrm{Tw}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}), \mathrm{Tw}(\mathcal{D}))$. The commutative diagram

$$\begin{tikzcd}
\mathrm{Tw}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \arrow[r, "E"] \arrow[d] & \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}), \mathrm{Tw}(\mathcal{D})) \arrow[d, "\Phi"] \\
\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[r, "\mathrm{id} \times \mathrm{id}"] & \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}, \mathcal{D})
\end{tikzcd}$$

exhibits the coupling Φ as represented by the identity functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proof of Theorem 8.2.4.6. Let $\lambda : \mathcal{C} \to \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}$ be a coupling which is corepresented by a functor $F : \mathcal{C}_{-} \to \mathcal{C}_{+}$, and let $\mu : \mathcal{D} \to \mathcal{D}_{-}^{\mathrm{op}} \times \mathcal{D}_{+}$ be a coupling which is represented by a functor $G : \mathcal{D}_{+} \to \mathcal{D}_{-}$. It follows from Theorem 8.2.2.11 that the coupling

$$\Phi : \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_{-}, \mathcal{D}_{-})^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_{+}, \mathcal{D}_{+})$$

of Remark 8.2.2.3 is representable, and that an object $(T_-, T, T_+) \in \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$ is universal if and only if the functor T carries couniversal objects of $\mathcal{C}$ to universal objects of $\mathcal{D}$. We will complete the proof by showing that the coupling Φ is representable by the functor

$$H : \mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+) \to \mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-) \qquad T_+ \mapsto G \circ T_+ \circ F.$$

Choose a morphism of couplings

045M

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{F}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_+) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+
\end{tikzcd} \tag{8.40}$$

which exhibits λ as corepresented by F, and a morphism of couplings

045N

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{D}_+) \arrow[r, "\widetilde{G}"] \arrow[d] & \mathcal{D} \arrow[d, "\mu"] \\
\mathcal{D}_+^{\mathrm{op}} \times \mathcal{D}_+ \arrow[r, "G^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+
\end{tikzcd} \tag{8.41}$$

which exhibits μ as represented by G.

Let $E : \mathrm{Tw}(\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)) \to \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathrm{Tw}(\mathcal{D}_+))$ be the comparison map of Example 8.2.4.8. Precomposition with (8.40) and postcomposition with (8.41) determines a functor $E' : \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathrm{Tw}(\mathcal{D}_+)) \to \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$ for which the diagram

$$\begin{tikzcd}
\mathrm{Tw}(\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)) \arrow[r, "E' \circ E"] \arrow[d] & \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D}) \arrow[d] \\
\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+) \arrow[r, "H^{\mathrm{op}} \times \mathrm{id}"] & \mathrm{Fun}(\mathcal{C}_-, \mathcal{D}_-)^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)
\end{tikzcd}$$

is commutative. We will complete the proof by showing that this diagram exhibits the coupling Φ as represented by H.

Fix a functor $T_+ : \mathcal{C}_+ \to \mathcal{D}_+$; we wish to show that the composite functor

$$\mathrm{Tw}(\mathrm{Fun}(\mathcal{C}_+, \mathcal{D}_+)) \xrightarrow{E} \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathrm{Tw}(\mathcal{D}_+)) \xrightarrow{E'} \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$$

carries id_{T_+} to a universal object of $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$. Unwinding the definitions, we see that the image of id_{T_+} is given by the triple $(G \circ T_+ \circ F, \widetilde{G} \circ \mathrm{Tw}(T_+) \circ \widetilde{F}, T_+) \in \mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{D})$. Using the criterion of Theorem 8.2.2.11, we are reduced to showing that the composite functor

$$\mathcal{C} \xrightarrow{\widetilde{F}} \mathrm{Tw}(\mathcal{C}_+) \xrightarrow{\mathrm{Tw}(T_+)} \mathrm{Tw}(\mathcal{D}_+) \xrightarrow{\widetilde{G}} \mathcal{D}$$

carries every couniversal object $X \in \mathcal{C}$ to a universal object of $\mathcal{D}$. Proposition 8.2.3.7 guarantees that $\widetilde{F}(X) \in \mathrm{Tw}(\mathcal{C}_+)$ corresponds to an isomorphism in $\mathcal{C}_+$, so its image under $\mathrm{Tw}(T_+)$ corresponds to an isomorphism in $\mathcal{D}_+$; the desired result now follows from our hypothesis that the functor $\widetilde{G}$ carries isomorphisms in $\mathcal{D}_+$ to universal objects of $\mathcal{D}$. □

We close this section by recording an alternative formulation of Definition 8.2.4.1:

Proposition 8.2.4.9. *Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories, let $G : \mathcal{C}_+ \to \mathcal{C}_-$ be a functor, and suppose we are given a morphism of couplings* 045P

045Q

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{G}"] \arrow[d, "\mu"] & \mathcal{C} \arrow[d, "\lambda"] \\
\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "G^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+
\end{tikzcd} \qquad (8.42)$$

where $\mu = (\mu_-, \mu_+)$ is the twisted arrow coupling of Example 8.2.0.2. The following conditions are equivalent:

(1) *The diagram (8.42) exhibits λ as represented by G, in the sense of Definition 8.2.4.1. That is, the functor $\widetilde{G}$ carries isomorphisms of $\mathcal{C}_+$ to universal objects of $\mathcal{C}$.*

(2) *The functor $\widetilde{G}$ is left cofinal.*

Proof. By virtue of Proposition 8.2.1.7, the functors λ_+ and μ_+ are cocartesian fibrations, and the functor $\widetilde{G}$ carries μ_+-cocartesian morphisms of $\mathrm{Tw}(\mathcal{C}_+)$ to λ_+-cocartesian morphisms of $\mathcal{C}$. By virtue of Corollary 7.2.3.15, the functor $\widetilde{G}$ is left cofinal if and only if, for every object $C \in \mathcal{C}_+$, the induced map $\widetilde{G}_C : \mathrm{Tw}(\mathcal{C}_+) \times_{\mathcal{C}_+} \{C\} \to \mathcal{C} \times_{\mathcal{C}_+} \{C\}$ is left cofinal. It will therefore suffice to show that the following conditions are equivalent, for each object $C \in \mathcal{C}_+$.

(1_C) The image $\widetilde{G}(\mathrm{id}_C)$ is a universal object of $\mathcal{C}$: that is, it is initial when viewed as an object of the ∞-category $\mathcal{C} \times_{\mathcal{C}_+} \{C\}$.

(2_C) The functor $\widetilde{G}_C$ is left cofinal.

The equivalence $(1_C) \Leftrightarrow (2_C)$ is a special case of Corollary 7.2.1.9, since id_C is initial when viewed as an object of the ∞-category $\mathrm{Tw}(\mathcal{C}_+) \times_{\mathcal{C}_+} \{C\}$ (see Proposition 8.1.2.1). □

8.2.5 Adjunctions as Couplings

045R Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories which is both representable and corepresentable. Using Theorem 8.2.3.4 (and its dual), we can choose a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$ which represents λ and a functor $F : \mathcal{C}_- \to \mathcal{C}_+$ which corepresents λ. Either of these functors determines the pairing λ up to equivalence, and therefore determines the other up to isomorphism. Our goal in this section is to establish the following more precise result:

045S **Theorem 8.2.5.1.** *Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories which is representable by a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$. Then a functor $F : \mathcal{C}_- \to \mathcal{C}_+$ corepresents the coupling λ if and only if it is left adjoint to G.*

We first establish a weaker version of Theorem 8.2.5.1.

045T **Proposition 8.2.5.2.** *Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be coupling of ∞-categories which is representable by a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$. The following conditions are equivalent:*

(1) *The coupling λ is corepresentable.*

(2) *The functor G admits a left adjoint.*

Proof. By virtue of the criterion of Corollary 6.2.4.2, it will suffice to show that for each object $X \in \mathcal{C}_-$, the following conditions are equivalent:

(1_X) There exists a couniversal object $\widetilde{X} \in \mathcal{C}$ satisfying $\lambda_-(\widetilde{X}) = X$.

(2_X) The ∞-category $(\mathcal{C}_-)_{X/} \times_{\mathcal{C}_-} \mathcal{C}_+$ has an initial object.

Note that Proposition 8.1.2.9 supplies an equivalence $(\mathcal{C}_-)_{X/} \hookrightarrow \{X\} \times_{\mathcal{C}_-^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}_-)$ of ∞-categories which are left-fibered over $\mathcal{C}_-$. Restricting along the functor G, we obtain an equivalence of ∞-categories

$$(\mathcal{C}_-)_{X/} \times_{\mathcal{C}_-} \mathcal{C}_+ \hookrightarrow \{X\} \times_{\mathcal{C}_-}^{\mathrm{op}} \mathrm{Tw}(\mathcal{C}_-) \times_{\mathcal{C}_-} \mathcal{C}_+ .$$

Since λ is representable by G, there exists a categorical pullback square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{G}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_-) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "\mathrm{id} \times G"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_-
\end{tikzcd}$$

which induces an equivalence of ∞-categories

$$\{X\} \times_{\mathcal{C}_-^{\mathrm{op}}} \mathcal{C} \to \{X\} \times_{\mathcal{C}_-}^{\mathrm{op}} \mathrm{Tw}(\mathcal{C}_-) \times_{\mathcal{C}_-} \mathcal{C}_+ .$$

The equivalence of (1_X) and (2_X) now follows from Corollary 4.6.7.21. $\square$

Proof of Theorem 8.2.5.1. Let $\lambda : \mathcal{C} \to \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}$ be a coupling of ∞-categories which is representable by a functor $G : \mathcal{C}_{-}^{\mathrm{op}} \to \mathcal{C}_{+}$. By virtue of Proposition 8.2.5.2, the functor G admits a left adjoint if and only if the coupling λ is corepresentable. If this condition is satisfied, then there exists a functor $F : \mathcal{C}_{-} \to \mathcal{C}_{+}$ which corepresents the coupling λ; moreover, F is uniquely determined up to isomorphism (Theorem 8.2.0.4). We will complete the proof by showing that F is a left adjoint of G.

Choose a diagram

045U

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{G}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_{-}) \arrow[d] \\
\mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+} \arrow[r, "\mathrm{id} \times G"] & \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{-},
\end{tikzcd} \tag{8.43}$$

which exhibits λ as represented by G (see Definition 8.2.3.5). Then, for each universal object $C \in \mathcal{C}$, the image $\widetilde{G}(C) \in \mathrm{Tw}(\mathcal{C}_{-})$ corresponds to an isomorphism in the ∞-category $\mathcal{C}_{-}$ (Proposition 8.2.3.7). Using Proposition 8.2.4.3, we can choose a commutative diagram

045V

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_{-}) \arrow[r, "\widetilde{F}"] \arrow[d] & \mathcal{C} \arrow[d, "\lambda"] \\
\mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{-} \arrow[r, "\mathrm{id} \times F"] & \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}
\end{tikzcd} \tag{8.44}$$

which exhibits λ as corepresented by F (see Variant 8.2.4.5). It follows that the composite functor $\widetilde{F} \circ \widetilde{G} : \mathcal{C} \to \mathcal{C}$ carries universal objects of $\mathcal{C}$ to couniversal objects of $\mathcal{C}$. Applying Theorem 8.2.4.6, we deduce that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\widetilde{F} \circ \widetilde{G}"] \arrow[d, "\lambda"] & \mathcal{C} \arrow[d, "\lambda"] \\
\mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+} \arrow[r, "\mathrm{id} \times (F \circ G)"] & \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}
\end{tikzcd}$$

is couniversal when viewed as an object of the ∞-category $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathcal{C})$.

In particular, there exists an (essentially unique) morphism

$$\widetilde{\epsilon} : (\mathrm{id}_{\mathcal{C}_{-}}, \widetilde{F} \circ \widetilde{G}, F \circ G) \to (\mathrm{id}_{\mathcal{C}_{-}}, \mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}_{+}})$$

in the ∞-category $\{\mathrm{id}_{\mathcal{C}_-}\} \times_{\mathrm{Fun}(\mathcal{C}_-,\mathcal{C}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{C})$. Let $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{C}_+}$ denote the image of $\widetilde{\epsilon}$ under the forgetful functor $\mathrm{Fun}_{\pm}(\mathcal{C},\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}_+,\mathcal{C}_+)$. We will show that ϵ is the counit of an adjunction between F and G.

Using Example 8.2.2.12, we see that the diagram

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_-) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{Tw}(\mathcal{C}_-) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_- \arrow[r, "\mathrm{id} \times \mathrm{id}"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_-
\end{tikzcd}$$

is couniversal when viewed as an object of the ∞-category $\mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_-),\mathrm{Tw}(\mathcal{C}_-))$. In particular, there exists an (essentially unique) morphism

$$\widetilde{\eta} : (\mathrm{id}_{\mathcal{C}_-^{\mathrm{op}}}, \mathrm{id}_{\mathrm{Tw}(\mathcal{C}_-)}, \mathrm{id}_{\mathcal{C}_-}) \to (\mathrm{id}_{\mathcal{C}_-^{\mathrm{op}}}, \widetilde{G} \circ \widetilde{F}, G \circ F)$$

in the ∞-category $\{\mathrm{id}_{\mathcal{C}_-}\} \times_{\mathrm{Fun}(\mathcal{C}_-,\mathcal{C}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_-),\mathrm{Tw}(\mathcal{C}_-))$. Let $\eta : \mathrm{id}_{\mathcal{C}_+} \to G \circ F$ denote the image of $\widetilde{\eta}$ under the forgetful functor $\mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_-),\mathrm{Tw}(\mathcal{C}_-)) \to \mathrm{Fun}(\mathcal{C}_-,\mathcal{C}_-)$. We will complete the proof by showing that η is compatible with ϵ up to homotopy, in the sense of Definition 6.2.1.1. For this, we must verify the following:

($Z1$) The identity isomorphism id_F is a composition of the natural transformations

$$F = F \circ \mathrm{id}_{\mathcal{C}_-} \xrightarrow{\mathrm{id} \circ \eta} F \circ G \circ F \xrightarrow{\epsilon \circ \mathrm{id}} \mathrm{id}_{\mathcal{C}_+} \times F = F$$

in the ∞-category $\mathrm{Fun}(\mathcal{C}_-,\mathcal{C}_+)$.

($Z2$) The identity isomorphism id_G is a composition of the natural transformations

$$G = \mathrm{id}_{\mathcal{C}_-} \circ G \xrightarrow{\eta \circ \mathrm{id}} G \circ F \circ G \xrightarrow{\mathrm{id} \circ \epsilon} G \times \mathrm{id}_{\mathcal{C}_+} = G$$

in the ∞-category $\mathrm{Fun}(\mathcal{C}_+,\mathcal{C}_-)$.

Using Theorem 8.2.4.6, we deduce that the diagram (8.43) is a couniversal object of $\mathrm{Fun}_{\pm}(\mathcal{C},\mathrm{Tw}(\mathcal{C}_-)$: that is, it is initial when viewed as an object of the ∞-category $\mathcal{E} = \{\mathrm{id}_{\mathcal{C}_-}\} \times_{\mathrm{Fun}(\mathcal{C}_-,\mathcal{C}_-)^{\mathrm{op}}} \mathrm{Fun}_{\pm}(\mathcal{C},\mathrm{Tw}(\mathcal{C}_-))$. It follows that the diagram

$$\begin{tikzcd}
 & (\mathrm{id}_{\mathcal{C}_-}, \widetilde{G} \circ \widetilde{F} \circ \widetilde{G}, G \circ F \circ G) \arrow[dr, "\mathrm{id} \circ \widetilde{\epsilon}"] & \\
(\mathrm{id}_{\mathcal{C}_-}, \widetilde{G}, G) \arrow[ur, "\widetilde{\eta} \circ \mathrm{id}"] \arrow[rr, "\mathrm{id}"] & & (\mathrm{id}_{\mathcal{C}_-}, \widetilde{G}, G)
\end{tikzcd}$$

commutes up to homotopy in $\mathcal{E}$. Assertion $(Z2)$ follows by applying the forgetful functor $\mathrm{Fun}_{\pm}(\mathcal{C}, \mathrm{Tw}(\mathcal{C}_-)) \to \mathrm{Fun}(\mathcal{C}_+, \mathcal{C}_-)$. Assertion $(Z1)$ follows by a similar argument, using the observation that the diagram (8.44) is a couniversal object of $\mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_-), \mathcal{C})$ (Theorem 8.2.2.11). □

8.2.6 Balanced Couplings

Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories. In this section, we formulate a concrete criterion to determine if λ is representable (or corepresentable) by an equivalence of ∞-categories. 045W

Definition 8.2.6.1. Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \to \mathcal{C}_+$ be a coupling of ∞-categories. We say that λ is *balanced* if it satisfies the following conditions: 045X

(1) The coupling λ is representable. That is, for each object $C_+ \in \mathcal{C}_+$, there exists a universal object $C \in \mathcal{C}$ satisfying $\lambda_+(C) = C_+$.

(2) The coupling λ is corepresentable. That is, for each object $C_- \in \mathcal{C}_-$, there exists a couniversal object $C \in \mathcal{C}$ satisfying $\lambda_-(C) = C_-$.

(3) An object $C \in \mathcal{C}$ is universal if and only if it is couniversal.

Example 8.2.6.2. For every ∞-category $\mathcal{C}$, the twisted arrow coupling $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ of Example 8.2.0.2 is balanced. See Example 8.2.1.5. 045Y

Example 8.2.6.3. Let X be a Kan complex, let $\lambda_- : \mathrm{Fun}(\Delta^1, X) \to X$ be the morphism given by evaluation at the vertex $0 \in \Delta^1$, and let $\lambda_+ : \mathrm{Fun}(\Delta^1, X) \to X$ be the morphism given by evaluation at the vertex $1 \in \Delta^1$. It follows from Corollary 3.1.3.3 that the map 04AV

$$\lambda = (\lambda_-, \lambda_+) : \mathrm{Fun}(\Delta^1, X) \to X \times X$$

is a Kan fibration; in particular, we can view it as a coupling of X with itself. For each vertex $x \in X$, the path spaces $\lambda_-^{-1}\{x\} = \{x\} \tilde{\times}_X X$ and $\lambda_+^{-1}\{x\} = X \tilde{\times}_X \{x\}$ are contractible Kan complexes (Example 3.4.1.13), so that every object of $\mathrm{Fun}(\Delta^1, X)$ is both universal and couniversal for the coupling λ. In particular, λ is a balanced coupling.

Remark 8.2.6.4. Suppose we are given a morphism of couplings 04AW

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathcal{D} \arrow[d, "\mu"] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+
\end{tikzcd}$$

which is an equivalence (in the sense of Exercise 8.2.2.7). Then λ is balanced if and only if μ is balanced. See Remark 8.2.2.8.

We can now formulate the main result of this section.

045Z **Theorem 8.2.6.5.** *Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories. The following conditions are equivalent:*

(1) *The coupling λ is balanced.*

(2) *The coupling λ is representable by an equivalence of ∞-categories $G : \mathcal{C}_+ \to \mathcal{C}_-$.*

(3) *The coupling λ is corepresentable by an equivalence of ∞-categories $F : \mathcal{C}_- \to \mathcal{C}_+$.*

04AX **Corollary 8.2.6.6.** *Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories. Then $\mathcal{C}_-$ and $\mathcal{C}_+$ are equivalent if and only if there exists a balanced coupling $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$.*

0460 **Corollary 8.2.6.7.** *Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories. Then λ is balanced if and only if there exists an equivalence of couplings*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{D}) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}} \times F_+"] & \mathcal{D}^{\mathrm{op}} \times \mathcal{D}.
\end{tikzcd}$$

Proof. Suppose that λ is balanced. By virtue of Theorem 8.2.6.5, the coupling λ is corepresentable by a functor $F_- : \mathcal{C}_- \to \mathcal{C}_+$ which is an equivalence of ∞-categories. We can therefore choose a categorical pullback square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_+) \arrow[d] \\
\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "F_-^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+
\end{tikzcd}$$

which exhibits λ as corepresented by F_-. Since F_- is an equivalence of ∞-categories, it follows that F is also an equivalence of ∞-categories (Proposition 4.5.2.21). The reverse implication is an immediate consequence of Example 8.2.6.2 (and Remark 8.2.6.4). $\square$

We will deduce Theorem 8.2.6.5 from the following more general result.

Proposition 8.2.6.8. *Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories which is representable by a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$. The following conditions are equivalent:* 0461

(1) *Every universal object of $\mathcal{C}$ is couniversal.*

(2) *The functor G is fully faithful.*

Proof. Using Proposition 8.2.4.3, we can choose a commutative diagram

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$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}_+) \arrow[r, "\widetilde{G}'"] \arrow[d] & \mathcal{C} \arrow[r, "\widetilde{G}"] \arrow[d, "\lambda"] & \mathrm{Tw}(\mathcal{C}_-) \arrow[d] \\
\mathcal{C}_+^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "G^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \arrow[r, "\mathrm{id} \times G"] & \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_-,
\end{tikzcd} \tag{8.45}$$

where the left square exhibits λ as represented by G in the sense of Definition 8.2.4.1, and the right square exhibits λ as represented by G in the sense of Definition 8.2.3.5. Invoking (the dual of) Lemma 8.2.3.6, we see that (1) is equivalent to the following:

($1'$) The left square of (8.45) is a categorical pullback diagram.

For each object $C \in \mathcal{C}_+$, Proposition 8.2.3.7 guarantees that the composite functor $\widetilde{G} \circ \widetilde{G}'$ carries id_C to an isomorphism of $\mathcal{C}_-$ (regarded as an object of $\mathrm{Tw}(\mathcal{C}_-)$). It follows from Theorem 8.2.2.11 that $(G, \widetilde{G}' \circ \widetilde{G}, G)$ and $(G, \mathrm{Tw}(G), G)$ are isomorphic when viewed as objects of $\mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathrm{Tw}(\mathcal{C}_-))$ (since both are initial objects of the ∞-category $\mathrm{Fun}_{\pm}(\mathrm{Tw}(\mathcal{C}_+), \mathrm{Tw}(\mathcal{C}_-)) \times_{\mathrm{Fun}(\mathcal{C}_+, \mathcal{C}_-)} \{G\}$). In particular, for every pair of objects $X, Y \in \mathcal{C}_+$, the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}_+}(X, Y) \arrow[r] \arrow[d, "G"] & \{X\} \times_{\mathcal{C}_+^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}_+) \times_{\mathcal{C}_+} \{Y\} \arrow[d, "\widetilde{G}' \circ \widetilde{G}"] \\
\mathrm{Hom}_{\mathcal{C}_-}(G(X), G(Y)) \arrow[r] & \{G(X)\} \times_{\mathcal{C}_-^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}_-) \times_{\mathcal{C}_-} \{G(Y)\}
\end{tikzcd}$$

commutes up to homotopy, where the horizontal maps are the homotopy equivalences of Notation 8.1.2.14. Using Corollary 5.1.7.16, we see that (2) is equivalent to the following:

($2'$) The outer rectangle of (8.45) is a categorical pullback diagram.

The equivalence of ($1'$) and ($2'$) is a special case of Proposition 4.5.2.18, since the right square of (8.45) is a categorical pullback square by assumption. □

Proof of Theorem 8.2.6.5. We will prove the equivalence (1) ⇔ (2); the equivalence (1) ⇔ (3) follows by a similar argument. Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ is a coupling of ∞-categories which is representable by a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$. Combining Theorem 8.2.5.1 with Proposition 8.2.6.8, we see that λ is balanced if and only if the following conditions are satisfied:

- The functor G is fully faithful.
- The functor G admits a left adjoint $F : \mathcal{C}_- \to \mathcal{C}_+$.
- The functor F is fully faithful.

It follows from Corollary 6.2.2.19 that these conditions are satisfied if and only if G is an equivalence of ∞-categories. □

8.3 The Yoneda Embedding

03LZ Let $\mathcal{C}$ be a category. For every object $X \in \mathcal{C}$, we let $h^X : \mathcal{C} \to \mathrm{Set}$ denote the functor corepresented by X, given on objects by the formula $h^X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The construction $X \mapsto h^X$ determines a functor from $\mathcal{C}^{\mathrm{op}}$ to the functor category $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$, which we refer to as the *(contravariant) Yoneda embedding.* This terminology is justified by the following:

03M0 **Proposition 8.3.0.1** (Yoneda's Lemma, Weak Form)**.** *For any (locally small) category $\mathcal{C}$, the Yoneda embedding*

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \qquad X \mapsto h^X$$

is fully faithful.

The goal of this section is to extend Proposition 8.3.0.1 to the setting of ∞-categories. Our first step is to construct an analogue of the functor $X \mapsto h^X$. For every pair of objects $X, Y \in \mathcal{C}$, the morphism space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a Kan complex (Proposition 4.6.1.10), which we can regard as an object of the ∞-category $\mathcal{S}$. In §8.3.3, we show that the construction $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$ can be upgraded to a functor from the product $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to the ∞-category $\mathcal{S}$. More precisely, every locally small ∞-category $\mathcal{C}$ admits a Hom-*functor* $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$, which is characterized (up to isomorphism) by the requirement that it is a covariant transport representation for the twisted arrow fibration $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ of Proposition 8.3.3.2. This condition guarantees that for every object $X \in \mathcal{C}$, the functor $\mathscr{H}(X, -) : \mathcal{C} \to \mathcal{S}$ is corepresentable by X. We can therefore identify $\mathscr{H}$ with a functor

$$h^{\bullet} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \qquad X \mapsto \mathscr{H}(X, -)$$

carrying each object of $\mathcal{C}$ to a functor that it corepresents; we will refer to $h^{\bullet}$ as a *contravariant Yoneda embedding* for $\mathcal{C}$ (Definition 8.3.3.9).

To show that the Yoneda embedding is fully faithful, we will need an additional ingredient. Let us return to the situation where $\mathcal{C}$ is an ordinary category. Proposition 8.3.0.1 asserts that for every pair of objects $X, Y \in \mathcal{C}$, the natural map $\mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set})}(h^X, h^Y)$ is a bijection. It is easy to see that this map is injective: in fact, it has a left inverse $T : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set})}(h^X, h^Y) \to \mathrm{Hom}_{\mathcal{C}}(Y, X)$, which carries a natural transformation $\alpha : h^X \to h^Y$ to the element $\alpha_X(\mathrm{id}_X) \in h^Y(X) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. It will therefore suffice to show that T is bijective. This is a consequence of the following *strong* version of Yoneda's lemma: for every functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$, the evaluation map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}, \mathrm{Set})}(h^X, \mathscr{F}) \to \mathscr{F}(X) \qquad \alpha \mapsto \alpha_X(\mathrm{id}_X)$$

is a bijection (Proposition 8.3.1.1). This assertion also has a counterpart in the setting of ∞-categories (Proposition 8.3.1.3), which we formulate and prove in §8.3.1.

To exploit the universal mapping property of (co)representable functors, it will be convenient to introduce some terminology. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. We define a *profunctor from $\mathcal{D}$ to $\mathcal{C}$* to be a functor $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ (Definition 8.3.2.1). We say that a profunctor $\mathscr{K}$ is *corepresentable* if, for every object $X \in \mathcal{C}$, the functor $\mathscr{K}(X, -) : \mathcal{D} \to \mathcal{S}$ is corepresentable. In this case, the construction $X \mapsto \mathscr{K}(X, -)$ determines a functor from $\mathcal{C}^{\mathrm{op}}$ to the full subcategory $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{D}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{D}, \mathcal{S})$ spanned by the corepresentable functors. In §8.3.2, we establish a criterion for this functor to be an equivalence of ∞-categories (Corollary 8.3.2.20). Our ∞-categorical version of Yoneda's lemma then follows by specializing this criterion to the situation where $\mathcal{C} = \mathcal{D}$ and $\mathscr{K}$ is a Hom-functor (Theorem 8.3.3.13).

Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Assume that $\mathcal{C}$ is locally small, so that it admits a Hom-functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. For every functor $G : \mathcal{D} \to \mathcal{C}$, the composition

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{id} \times G} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathscr{H}} \mathcal{S},$$

can be regarded as a profunctor $\mathscr{K}_G$ from $\mathcal{D}$ to $\mathcal{C}$, given informally by the construction $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{D}}(X, G(Y))$. In §8.3.4, we show that this construction determines a fully faithful functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S})$, whose essential image is spanned by the *representable* profunctors (Proposition 8.3.4.1). Using the results of §8.2, we give an alternative characterization of representable profunctors (Proposition 8.3.4.15) and show that they satisfy a universal mapping property (Corollary 8.3.4.21). As an application, we show that morphism spaces in the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ can be computed as limits indexed by the twisted arrow ∞-category $\mathrm{Tw}(\mathcal{D})$ (Example 8.3.4.22).

Warning 8.3.0.2. If $\mathcal{C}$ is an ordinary category, the Yoneda embedding 03M1

$$h^{\bullet} : \mathcal{C}^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \qquad X \mapsto h^X$$

is given by a completely explicit construction. Beware that in the ∞-categorical setting, the Yoneda embedding depends on a choice of covariant transport representation for the twisted arrow fibration $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, which is well-defined only up to isomorphism. However, it is sometimes possible to eliminate this ambiguity. Suppose that $\mathcal{C} = \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}_0)$ is the homotopy coherent nerve of a locally Kan simplicial category $\mathcal{C}_0$. In this case, the simplicial enrichment of $\mathcal{C}_0$ determines a functor of simplicial categories

$$\mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_0 \to \mathrm{Kan} \qquad (X,Y) \mapsto \mathrm{Hom}_{\mathcal{C}_0}(X,Y)_{\bullet}.$$

Passing to the homotopy coherent nerve, we obtain a functor of ∞-categories $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ (Construction 8.3.6.1). In §8.3.6, we show that $\mathscr{H}$ is a Hom-functor for the ∞-category $\mathcal{C}$ (Proposition 8.3.6.2). Our proof uses a recognition principle for Hom-functors, which we formulate and prove in §8.3.5.

8.3.1 Yoneda's Lemma

03M2 Let $\mathcal{C}$ be a category. Every object $X \in \mathcal{C}$ determines a corepresentable functor $h^X : \mathcal{C} \to \mathrm{Set}$, given on objects by the formula $h^X(Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$. This functor can be characterized by a universal mapping property:

03M3 **Proposition 8.3.1.1** (Yoneda's Lemma, Strong Form)**.** *Let $\mathcal{C}$ be a category containing an object X. For every functor $\mathscr{F} : \mathcal{C} \to \mathrm{Set}$, evaluation on the identity morphism $\mathrm{id}_X \in h^X(X)$ induces a bijection*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathrm{Set})}(h^X, \mathscr{F}) \to \mathscr{F}(X).$$

Proof. Fix an element $x \in \mathscr{F}(X)$. We wish to show that there is a unique natural transformation $\alpha : h^X \to \mathscr{F}$ which carries $\mathrm{id}_X \in h^X(X)$ to the element $x \in \mathscr{F}(X)$.

For any object $Y \in \mathcal{C}$, every element $f \in h^X(Y) \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ can be obtained by evaluating the function $h^X(f) : h^X(X) \to h^X(Y)$ on the object id_X. It follows that, if $\alpha : h^X \to \mathscr{F}$ is a natural transformation satisfying $\alpha_X(\mathrm{id}_X) = x$, then it must satisfy the identity

$$\alpha_Y(f) = \alpha_Y(h^X(f)(\mathrm{id}_X)) = \mathscr{F}(f)(h_X(\mathrm{id}_X)) = \mathscr{F}(f)(x).$$

This proves uniqueness. To establish existence, it will suffice to show that the collection of functions

$$\alpha_Y : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathscr{F}(Y) \qquad f \mapsto \mathscr{F}(f)(x)$$

determine a natural transformation from h^X to $\mathscr{F}$. In other words, we must show that for each morphism $g : Y \to Z$ in $\mathcal{C}$, the diagram of sets

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "g\circ"] \arrow[d, "\alpha_Y"] & \mathrm{Hom}_{\mathcal{C}}(X,Z) \arrow[d, "\alpha_Z"] \\
\mathscr{F}(Y) \arrow[r, "\mathscr{F}(g)"] & \mathscr{F}(Z)
\end{tikzcd}$$

is commutative. This follows from the observation that, for every morphism $f : X \to Y$ of $\mathcal{C}$, we have an equality $\mathscr{F}(g \circ f)(x) = (\mathscr{F}(g) \circ \mathscr{F}(f))(x)$ in the set $\mathscr{F}(Z)$. □

Our goal in this section is prove a generalization of Yoneda's lemma, where we replace $\mathcal{C}$ by an ∞-category and Set by the ∞-category $\mathcal{S}$ of spaces (Proposition 8.3.1.3). In the ∞-categorical setting, the proof is more subtle: to construct a natural transformation α between functors $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathcal{S}$, it is not enough to specify a collection of morphisms $\{\alpha_Y : \mathscr{G}(Y) \to \mathscr{F}(Y)\}_{Y \in \mathcal{C}}$ and to verify a compatibility condition. To address this difficulty, we will use the formalism of Kan extensions developed in §7.3 (see Lemma 8.3.1.7).

Notation 8.3.1.2. Let $\mathcal{S}$ denote the ∞-category of spaces (Construction 5.5.1.1). Let $\mathcal{C}$ be an ∞-category and suppose we are given a pair of functors $\mathscr{F}, \mathscr{G} : \mathcal{C} \to \mathcal{S}$. Fix an object $X \in \mathcal{C}$ and a vertex $\eta \in \mathscr{F}(X)$. We then obtain a comparison morphism 03M4

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{F},\mathscr{G}) &\xrightarrow{\mathrm{ev}_X} \mathrm{Hom}_{\mathcal{S}}(\mathscr{F}(X),\mathscr{G}(X)) \\
&\xrightarrow{\circ[\eta]} \mathrm{Hom}_{\mathcal{S}}(\Delta^0,\mathscr{G}(X)) \\
&\simeq \mathscr{G}(X)
\end{aligned}$$

in the homotopy category hKan, where the first map is given by evaluation on the object X, the second by the composition law of Notation 4.6.9.15, and the third is (the inverse of) the homotopy equivalence of Remark 5.5.1.5.

Proposition 8.3.1.3 (∞-Categorical Yoneda Lemma)**.** *Let $\mathcal{C}$ be an ∞-category containing an object X, let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor, and let $\eta \in \mathscr{F}(X)$ be a vertex which exhibits the functor $\mathscr{F}$ as corepresented by X (see Definition 5.6.6.1). Then, for every functor $\mathscr{G} : \mathcal{C} \to \mathcal{S}$, the comparison map* 03M5

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{F},\mathscr{G}) \to \mathscr{G}(X)$$

of Notation 8.3.1.2 is an isomorphism in the homotopy category hKan*.*

Remark 8.3.1.4. In the special case where $\mathcal{C}$ is (the nerve of) an ordinary category and $\mathscr{G}$ is a set-valued functor, Proposition 8.3.1.3 reduces to Proposition 8.3.1.1. 03M6

03M7 **Remark 8.3.1.5.** Let $\mathcal{C}$ be a locally small ∞-category and let X be an object of $\mathcal{C}$. In §5.6.6, we proved that there exists a functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ which is corepresented by X, and that $\mathscr{F}$ is uniquely determined up to isomorphism (Theorem 5.6.6.13). Proposition 8.3.1.3 can be regarded as a more refined version of this uniqueness assertion: the functor $\mathscr{F}$ is characterized, up to isomorphism, by the requirement that it corepresents the evaluation functor

$$\mathrm{ev}_X : \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \to \mathcal{S} \qquad \mathscr{G} \mapsto \mathscr{G}(X).$$

03M8 **Corollary 8.3.1.6.** *Let $\mathcal{C}$ be an ∞-category containing an object X. Suppose that, for every object $Y \in \mathcal{C}$, the Kan complex $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is essentially small (this condition is satisfied, for example, if $\mathcal{C}$ is small). Let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor, and let η be a vertex of the Kan complex $\mathscr{F}(X)$. The following conditions are equivalent:*

(1) *The vertex η exhibits the functor $\mathscr{F}$ as corepresented by X, in the sense of Definition 5.6.6.1.*

(2) *For every functor $\mathscr{G} : \mathcal{C} \to \mathcal{S}$, the comparison map of Notation 8.3.1.2 is a homotopy equivalence $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{F}, \mathscr{G}) \to \mathscr{G}(X)$.*

(3) *For every functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$, the comparison map of Notation 8.3.1.2 induces a bijection $\pi_0(\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{F}, \mathscr{G})) \to \pi_0(\mathscr{G}(X))$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 8.3.1.3, and the implication (2) $\Rightarrow$ (3) is immediate. We will complete the proof by showing that (3) implies (1). Our assumption that the morphism space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is essentially small for each $Y \in \mathcal{C}$ guarantees that there exists a functor $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ and a vertex $\eta' \in \mathscr{F}'(X)$ which exhibits $\mathscr{F}'$ as corepresented by X (see Theorem 5.6.6.13). Applying assumption (3), we deduce that there exists a natural transformation $\alpha : \mathscr{F}' \to \mathscr{F}$ such that $\alpha_X(\eta')$ and η lie in the same connected component of $\mathscr{F}'$. Since the pair $(\mathscr{F}', \eta')$ also satisfies condition (3), composition with α induces a bijection $\pi_0(\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{F}, \mathscr{G})) \to \pi_0(\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{F}', \mathscr{G}))$ for each object $\mathscr{G} \in \mathrm{Fun}(\mathcal{C}, \mathcal{S})$. It follows that α is an isomorphism. Applying Remark 5.6.6.4, we deduce that $\alpha_X(\eta') \in \mathscr{F}(X)$ exhibits the functor $\mathscr{F}$ as corepresented by X. Since η and $\alpha_X(\eta)$ belong to the same connected component of $\mathscr{F}(X)$, it follows that η has the same property (Remark 5.6.6.3). $\square$

Proposition 8.3.1.3 is an easy consequence of the following:

03M9 **Lemma 8.3.1.7.** *Let $\mathcal{C}$ be an ∞-category containing an object X, let κ be an uncountable cardinal, let $\mathscr{F} : \mathcal{C} \to \mathcal{S}^{<\kappa}$ be a functor, and let $\eta \in \mathscr{F}(X)$ be a vertex. The following conditions are equivalent:*

(1) *The vertex η exhibits $\mathscr{F}$ as corepresented by the object X, in the sense of Definition 5.6.6.1.*

(2) *Let $\iota : \{X\} \hookrightarrow \mathcal{C}$ denote the inclusion map and let $\mathscr{F}_0 : \{X\} \to \mathcal{S}$ denote the constant functor taking the value Δ^0, so that η can be regarded as a natural transformation from $\mathscr{F}_0$ to the composite functor $\mathscr{F} \circ \iota$. Then η exhibits $\mathscr{F}$ as a left Kan extension of $\mathscr{F}_0$ along ι, in the sense of Variant 7.3.1.5. Moreover, for each object $Y \in \mathcal{C}$, the mapping space* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ *is essentially κ-small.*

Proof. Fix an object $Y \in \mathcal{C}$ and set $M = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. We may assume without loss of generality that M is essentially κ-small (this follows immediately from condition (2), and also follows from (1) since the Kan complex $\mathscr{F}(Y)$ is essentially small). For every Kan complex K, let $\underline{K}_M$ denote the constant functor $M \to \mathcal{S}$ taking the value K, so that the functor $\mathscr{F}$ determines a natural transformation $\gamma : \underline{\mathscr{F}(X)}_M \to \underline{\mathscr{F}(Y)}_M$. We will show that the following pair of conditions is equivalent:

(1_Y) The composite map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{S}}(\mathscr{F}(X), \mathscr{F}(Y)) \xrightarrow{\circ[\eta]} \mathrm{Hom}_{\mathcal{S}}(\Delta^0, \mathscr{F}(Y))$$

is a homotopy equivalence of Kan complexes.

(2_Y) The composite natural transformation

$$\underline{\Delta^0}_M \xrightarrow{\eta} \underline{\mathscr{F}(X)}_M \xrightarrow{\gamma} \underline{\mathscr{F}(Y)}_M$$

exhibits $\mathscr{F}(Y)$ as a colimit of the constant diagram $\underline{\Delta^0}|_M$ in the ∞-category $\mathcal{S}$.

The equivalence of (1_Y) and (2_Y) is a special case of Proposition 7.6.2.10 (see Example 7.6.2.12). Lemma 8.3.1.7 follows by allowing the object $Y \in \mathcal{C}$ to vary. □

Proof of Proposition 8.3.1.3. Combine Lemma 8.3.1.7 and 7.3.6.1. □

For later use, let us record another consequence of Lemma 8.3.1.7.

Corollary 8.3.1.8. *Let κ be an uncountable cardinal, let $T : \mathcal{C} \to \mathcal{D}$ be a functor between locally κ-small ∞-categories, let $\mathscr{F} : \mathcal{C} \to \mathcal{S}^{<\kappa}$ and $\mathscr{G} : \mathcal{D} \to \mathcal{S}^{<\kappa}$ be functors, and let $\beta : \mathscr{F} \to \mathscr{G} \circ T$ be a natural transformation of functors from $\mathcal{C}$ to $\mathcal{S}$. Fix an object $C \in \mathcal{C}$ and a vertex $\eta \in \mathscr{F}(C)$ which exhibits the functor $\mathscr{F}$ as corepresented by $\mathcal{C}$. The following conditions are equivalent:* 04B5

(1) *The natural transformation β carries η to a vertex of $\mathscr{G}(T(C))$ which exhibits the functor $\mathscr{G}$ as corepresented by the object $T(C) \in \mathcal{D}$.*

(2) *The natural transformation β exhibits $\mathscr{G}$ as a left Kan extension of $\mathscr{F}$ along the functor T (see Variant 7.3.1.5).*

Proof. Combine Lemma 8.3.1.7 with Proposition 7.3.8.18. □

04B6 **Corollary 8.3.1.9.** *Let κ be an uncountable cardinal, let $T : \mathcal{C} \to \mathcal{D}$ be a functor between locally κ-small ∞-categories and let $\mathscr{F} : \mathcal{C} \to \mathcal{S}$ be a functor which is corepresented by an object $C \in \mathcal{C}$. Then a functor $\mathscr{G} : \mathcal{D} \to \mathcal{S}$ is a left Kan extension of $\mathscr{F}$ along T if and only if it is corepresentable by the object $T(C) \in \mathcal{D}$.*

Proof. Assume that $\mathscr{G}$ is corepresentable by $T(C)$; we will show that it is a left Kan extension of $\mathscr{F}$ along T (the reverse implication follows immediately from Corollary 8.3.1.8). Fix a vertex $\eta \in \mathscr{F}(C)$ which exhibits $\mathscr{F}$ as corepresented by C. It follows from Proposition 8.3.1.3 that evaluation at η induces a homotopy equivalence of Kan complexes

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{S})}(\mathscr{F}, \mathscr{G} \circ T) \to \mathscr{G}(T(C)).$$

We can therefore choose a natural transformation $\beta : \mathscr{F} \to \mathscr{G} \circ T$ which carries η to a vertex which exhibits $\mathscr{G}$ as corepresented by $T(C)$. Applying Corollary 8.3.1.8, we see that β exhibits $\mathscr{G}$ as a left Kan extension of $\mathscr{F}$ along T. □

8.3.2 Profunctors of ∞-Categories

03MA Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be categories. A *profunctor* from $\mathcal{C}_+$ to $\mathcal{C}_-$ is a Set-valued functor on the product category $\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$. This notion has an evident ∞-categorical analogue, where we replace the ordinary category of sets by the ∞-category $\mathcal{S}$ of spaces (see Construction 5.5.1.1).

03MB **Definition 8.3.2.1.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories. A *profunctor from $\mathcal{C}_+$ to $\mathcal{C}_-$* is a functor $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$.

03MC **Example 8.3.2.2.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ordinary categories. Then every functor $K : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathrm{Set}$ determines a morphism of simplicial sets

$$\mathrm{N}_\bullet(K) : \mathrm{N}_\bullet(\mathcal{C}_-)^{\mathrm{op}} \times \mathrm{N}_\bullet(\mathcal{C}_+) \to \mathrm{N}_\bullet(\mathrm{Set}) \subset \mathcal{S}\,.$$

This construction determines a monomorphism from the collection of profunctors from $\mathcal{C}_+$ to $\mathcal{C}_-$ (in the sense of classical category theory) to the collection of profunctors from $\mathrm{N}_\bullet(\mathcal{C}_+)$ to $\mathrm{N}_\bullet(\mathcal{C}_-)$ (in the sense of Definition 8.3.2.1). Beware that this map is (usually) not bijective: its image consists of those profunctors

$$\mathscr{K} : \mathrm{N}_\bullet(\mathcal{C}_-)^{\mathrm{op}} \times \mathrm{N}_\bullet(\mathcal{C}_+) \to \mathcal{S}$$

having the property that for every pair of objects $X \in \mathcal{C}_-$ and $Y \in \mathcal{C}_+$, the Kan complex $\mathscr{K}(X, Y)$ is a constant simplicial set (see Proposition 1.3.3.1 and Remark 5.5.1.7).

Remark 8.3.2.3 (Symmetry)**.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a profunctor from $\mathcal{C}_+$ to $\mathcal{C}_-$. Then, by transposing its arguments, we can also regard $\mathscr{K}$ as a profunctor from $\mathcal{C}_-^{\mathrm{op}}$ to $\mathcal{C}_+^{\mathrm{op}}$. 03MD

Example 8.3.2.4 (From Profunctors to Couplings)**.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories, let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a profunctor from $\mathcal{C}_+$ to $\mathcal{C}_-$ (Definition 8.3.2.1). Applying the construction of Definition 5.6.2.1, we obtain an ∞-category $\int_{\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+} \mathscr{K}$ whose objects are triples (X, Y, η) where X is an object of $\mathcal{C}_-$, Y is an object of $\mathcal{C}_+$, and η is a vertex of the Kan complex $\mathscr{K}(X, Y)$. This ∞-category is equipped with a left fibration 0463

$$\lambda : \int_{\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+} \mathscr{K} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+,$$

given on objects by the construction $\lambda(X, Y, \eta) = (X, Y)$ (see Example 5.6.2.9). The left fibration λ is a coupling of $\mathcal{C}_+$ with $\mathcal{C}_-$, in the sense of Definition 8.2.0.1; we will refer to it as the *coupling associated to the profunctor* $\mathscr{K}$.

Modulo set-theoretic issues, every coupling can be obtained from the construction of Example 8.3.2.4:

Remark 8.3.2.5 (From Couplings to Profunctors)**.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories. By virtue of Corollary 5.6.0.6, the construction of Example 8.3.2.4 induces a monomorphism 0464

$$\begin{tikzcd}
\{\text{Profunctors } \mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}\}/\text{Isomorphism} \arrow[d] \\
\{\text{Couplings } \lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+\}/\text{Equivalence,}
\end{tikzcd}$$

whose image consists of equivalence classes of couplings $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ having essentially small fibers.

In particular, to every coupling $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ with essentially small fibers, we can associate a profunctor $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$, which is characterized (up to isomorphism) by the requirement that it is a covariant transport representation for λ (see Definition 5.6.5.1).

Variant 8.3.2.6. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories, and let κ be an uncountable cardinal. Then the construction of Example 8.3.2.4 induces a monomorphism 0465

$$\begin{tikzcd}
\{\text{Profunctors } \mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}^{<\kappa}\}/\text{Isomorphism} \arrow[d] \\
\{\text{Couplings } \lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+\}/\text{Equivalence,}
\end{tikzcd}$$

whose image consists of equivalence classes of couplings $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ whose fibers are essentially κ-small.

Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories. A profunctor $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ can be identified with a functor from $\mathcal{C}_-^{\mathrm{op}}$ to the ∞-category $\mathrm{Fun}(\mathcal{C}_+, \mathcal{S})$. Our primary goal in this section is to formulate a condition which guarantees that this functor is fully faithful. First, it will be convenient to introduce some terminology.

0466 **Definition 8.3.2.7.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a profunctor. Let X be an object of $\mathcal{C}_-$, let Y be an object of $\mathcal{C}_+$, and let η be a vertex of the Kan complex $\mathscr{K}(X, Y)$. We will say that η is *universal* if it exhibits the functor $\mathscr{K}(-, Y) : \mathcal{C}_-^{\mathrm{op}} \to \mathcal{S}$ as represented by the object $X \in \mathcal{C}_-$. We say that η is *couniversal* if it exhibits the functor $\mathscr{K}(X, -) : \mathcal{C}_+ \to \mathcal{S}$ as corepresented by the object $Y \in \mathcal{C}_+$.

0467 **Remark 8.3.2.8.** Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a covariant transport representation for λ. Let C be an object of $\mathcal{C}$ and set $\lambda(C) = (X, Y)$. Then the isomorphism class of C (as an object of the fiber $\lambda^{-1}\{(X, Y)\}$) can be identified with a connected component $[\eta]$ of the Kan complex $\mathscr{K}(X, Y)$. Invoking Proposition 5.6.6.21, we deduce the following:

- The object $C \in \mathcal{C}$ is universal (in the sense of Definition 8.2.1.1) if and only if the vertex $\eta \in \mathscr{K}(X, Y)$ is universal (in the sense of Definition 8.3.2.7).
- The object $C \in \mathcal{C}$ is couniversal (in the sense of Definition 8.2.1.1) if and only if the vertex $\eta \in \mathscr{K}(X, Y)$ is couniversal (in the sense of Definition 8.3.2.7).

03ME **Definition 8.3.2.9.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories, and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a profunctor from $\mathcal{C}_+$ to $\mathcal{C}_-$. We say that $\mathscr{K}$ is *representable* if, for each object $Y \in \mathcal{C}_+$, the functor $\mathscr{K}(-, Y) : \mathcal{C}_-^{\mathrm{op}} \to \mathcal{S}$ is representable (in the sense of Variant 5.6.6.2). We will say that $\mathscr{K}$ is *corepresentable* if, for each object $X \in \mathcal{C}_-$, the functor $\mathscr{K}(X, -) : \mathcal{C}_+ \to \mathcal{S}$ is corepresentable (in the sense of Definition 5.6.6.1).

03MF **Warning 8.3.2.10.** The terminology of Definition 8.3.2.9 is potentially confusing. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories, let $\mathcal{C}$ denote the product $\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$, and let $\mathscr{K} : \mathcal{C} \to \mathcal{S}$ be a morphism of simplicial sets. In general, there is no relationship between the corepresentability of $\mathscr{K}$ as a $\mathcal{S}$-valued functor on $\mathcal{C}$ (in the sense of Definition 5.6.6.1) and the corepresentability of $\mathscr{K}$ as a *profunctor* from $\mathcal{C}_+$ to $\mathcal{C}_-$ (in the sense of Definition 8.3.2.9). However, these notions of corepresentability coincide when $\mathcal{C}_-$ is a contractible Kan complex (see Example 8.3.2.13).

03MG **Remark 8.3.2.11** (Symmetry)**.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a profunctor from $\mathcal{C}_+$ to $\mathcal{C}_-$. Then $\mathscr{K}$ is representable if and only if it is corepresentable when regarded as a profunctor from $\mathcal{C}_-^{\mathrm{op}}$ to $\mathcal{C}_+^{\mathrm{op}}$ (see Remark 8.3.2.3).

Remark 8.3.2.12. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories and let $\mathscr{K}$ and $\mathscr{K}'$ be profunctors from $\mathcal{C}_+$ to $\mathcal{C}_-$ which are isomorphic (as objects of the ∞-category $\mathrm{Fun}(\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+, \mathcal{S})$). Then $\mathscr{K}$ is representable if and only if $\mathscr{K}'$ is representable. Similarly, $\mathscr{K}$ is corepresentable if and only if $\mathscr{K}'$ is corepresentable. See Remark 5.6.6.4. 03MH

Example 8.3.2.13. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a profunctor. If $\mathcal{C}_- = \Delta^0$, then the profunctor $\mathscr{K}$ is corepresentable (in the sense of Definition 8.3.2.9) if and only if it is corepresentable when regarded as a functor $\mathcal{C}_+ \to \mathcal{S}$ (in the sense of Definition 5.6.6.1). Similarly, if $\mathcal{C}_+ = \Delta^0$, then the profunctor $\mathscr{K}$ is representable (in the sense of Definition 8.3.2.9) if and only if it is representable when viewed as a functor $\mathcal{C}_-^{\mathrm{op}} \to \mathcal{S}$ (in the sense of Variant 5.6.6.2). 03MJ

Exercise 8.3.2.14. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ordinary categories. Show that a profunctor 03MK

$$\mathscr{K} : \mathrm{N}_\bullet(\mathcal{C}_-)^{\mathrm{op}} \times \mathrm{N}_\bullet(\mathcal{C}_+) \to \mathcal{S}$$

is representable (in the sense of Definition 8.3.2.9) if and only if it it is isomorphic to the profunctor $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}_-}(X, G(Y))$, for some functor $G : \mathcal{C}_+ \to \mathcal{C}_-$. See Proposition 8.3.4.1 for a more general result.

Remark 8.3.2.15. Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a profunctor. Then $\mathscr{K}$ is representable if and only if, for every object $Y \in \mathcal{C}_+$, there exists an object $X \in \mathcal{C}_-$ and a universal vertex $\eta \in \mathscr{K}(X, Y)$. Similarly, $\mathscr{K}$ is corepresentable if and only if, for every object $X \in \mathcal{C}_-$, there exists an object $Y \in \mathcal{C}_+$ and a couniversal vertex $\eta \in \mathscr{K}(X, Y)$. 0468

Remark 8.3.2.16. Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a covariant transport representation for λ. Using Remark 8.3.2.8, we deduce the following: 0469

- The coupling λ is representable (in the sense of Definition 8.2.1.3) if and only if the profunctor $\mathscr{K}$ is representable (in the sense of Definition 8.3.2.9).
- The coupling λ is corepresentable (in the sense of Definition 8.2.1.3) if and only if the profunctor $\mathscr{K}$ is corepresentable (in the sense of Definition 8.3.2.9).

The main result of this section is the following variant of Proposition 8.2.6.8:

Proposition 8.3.2.17. *Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories, and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a corepresentable profunctor from $\mathcal{C}_+$ to $\mathcal{C}_-$. The following conditions are equivalent:* 03ML

(1) *The profunctor $\mathscr{K}$ determines a fully faithful functor*

$$\mathcal{C}_-^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}_+, \mathcal{S}) \qquad X \mapsto \mathscr{K}(X, -).$$

(2) *Let X be an object of $\mathcal{C}_-$ and let Y be an object of $\mathcal{C}_+$. Then every couniversal vertex $\eta \in \mathscr{K}(X,Y)$ is also universal.*

Proof. Choose an object $X \in \mathcal{C}_-$. Since the functor $\mathscr{K}(X,-) : \mathcal{C}_+ \to \mathcal{S}$ is corepresentable, we can choose an object $Y \in \mathcal{C}_+$ and a couniversal vertex $\eta \in \mathscr{K}(X,Y)$. We will show that the following conditions are equivalent:

(1_X) For every object $X' \in \mathcal{C}_-$, the profunctor $\mathscr{K}$ induces a homotopy equivalence

$$\mathrm{Hom}_{\mathcal{C}_-^{\mathrm{op}}}(X,X') \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_+,\mathcal{S})}(\mathscr{K}(X,-),\mathscr{K}(X',-)).$$

(2_X) The vertex η is universal.

Proposition 8.3.2.17 will then follow by allowing the triple (X,Y,η) to vary.

Condition (2_X) is the assertion that, for each object $X' \in \mathcal{C}_-$, the composite map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_-^{\mathrm{op}}}(X,X') &\to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_+,\mathcal{S})}(\mathscr{K}(X,-),\mathscr{K}(X',-))\\ &\to \mathrm{Hom}_{\mathcal{S}}(\mathscr{K}(X,Y),\mathscr{K}(X',Y))\\ &\xrightarrow{\circ[\eta]} \mathrm{Hom}_{\mathcal{S}}(\Delta^0,\mathscr{K}(X',Y))\\ &\simeq \mathscr{K}(X',Y)\end{aligned}$$

is an isomorphism in the homotopy category hKan. The equivalence of this assertion with (1_X) follows immediately from Proposition 8.3.1.3. □

03MM **Definition 8.3.2.18** (Balanced Profunctors)**.** Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories. We say that a profunctor $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ is *balanced* if it satisfies the following conditions:

- The profunctor $\mathscr{K}$ is representable and corepresentable (Definition 8.3.2.9).
- Let X be an object of $\mathcal{C}_-$, let Y be an object of $\mathcal{C}_+$, and let η be a vertex of the Kan complex $\mathscr{K}(X,Y)$. Then η is universal if and only if it is couniversal.

In other words, $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ is balanced if it satisfies the hypotheses of Proposition 8.3.2.17 both when regarded as a profunctor from $\mathcal{C}_+$ to $\mathcal{C}_-$ and when regarded as a profunctor from $\mathcal{C}_-^{\mathrm{op}}$ to $\mathcal{C}_+^{\mathrm{op}}$.

046A **Remark 8.3.2.19.** Let $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ be a coupling of ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a covariant transport representation for λ. Then the coupling λ is balanced (in the sense of Definition 8.2.6.1) if and only if the profunctor $\mathscr{K}$ is balanced (in the sense of Definition 8.3.2.18). See Remark 8.3.2.8.

03MN **Corollary 8.3.2.20.** *Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ be a profunctor from $\mathcal{C}_-$ to $\mathcal{C}_+$. The following conditions are equivalent:*

(1) *The profunctor $\mathscr{K}$ is balanced (in the sense of Definition 8.3.2.18).*

(2) *The ∞-category $\mathcal{C}_+$ is locally small and $\mathscr{K}$ induces a fully faithful functor*

$$\mathcal{C}_-^{\mathrm{op}} \to \operatorname{Fun}(\mathcal{C}_+, \mathcal{S}) \qquad X \mapsto \mathscr{K}(X, -),$$

whose essential image is spanned by the corepresentable functors $\mathcal{C}_+ \to \mathcal{S}$.

(3) *The ∞-category $\mathcal{C}_-$ is locally small and $\mathscr{K}$ induces a fully faithful functor*

$$\mathcal{C}_+ \to \operatorname{Fun}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{S}) \qquad Y \mapsto \mathscr{K}(-, Y),$$

whose essential image is spanned by the representable functors $\mathcal{C}_-^{\mathrm{op}} \to \mathcal{S}$.

Proof. We will prove the equivalence of (1) and (2); the equivalence of (1) and (3) follows by a similar argument. Assume first that $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ is a balanced profunctor. Invoking Proposition 8.3.2.17, we see that the functor

$$\Phi : \mathcal{C}_-^{\mathrm{op}} \to \operatorname{Fun}(\mathcal{C}_+, \mathcal{S}) \qquad X \mapsto \mathscr{K}(X, -)$$

is fully faithful, and that the essential image of Φ consists of corepresentable functors from $\mathcal{C}_+$ to $\mathcal{S}$. Fix an object $Y \in \mathcal{C}_+$. Since $\mathscr{K}$ is representable, there exists an object $X \in \mathcal{C}_-$ and a universal vertex $\eta \in \mathscr{K}(X, Y)$. Our assumption that $\mathscr{K}$ is balanced guarantees that η is also couniversal. In particular, for every object $Y' \in \mathcal{C}_+$, η induces a homotopy equivalence $\operatorname{Hom}_{\mathcal{C}_+}(Y, Y') \xrightarrow{\sim} \mathscr{K}(X, Y')$, so that the Kan complex $\operatorname{Hom}_{\mathcal{C}_+}(X, Y')$ is essentially small. If $F : \mathcal{C}_+ \to \mathcal{S}$ is any functor corepresented by Y, then Theorem 5.6.6.13 guarantees that F is isomorphic to $\mathscr{K}(X, -)$ (as an object of the ∞-category $\operatorname{Fun}(\mathcal{C}_+, \mathcal{S})$), and therefore belongs to the essential image of Φ. Allowing the object Y to vary, we deduce that the profunctor $\mathscr{K}$ satisfies condition (2).

We now prove the converse. Assume that the functor Φ is fully faithful and that the essential image of Φ is spanned by the corepresentable functors $\mathcal{C}_+ \to \mathcal{S}$. We wish to show that the profunctor $\mathscr{K}$ is balanced. Since Φ takes values in the full subcategory of $\operatorname{Fun}(\mathcal{C}_+, \mathcal{S})$ spanned by the corepresentable functors, the profunctor $\mathscr{K}$ is corepresentable. We next show that $\mathscr{K}$ is representable. Fix an object $Y \in \mathcal{C}_+$; we wish to show that the functor $\mathscr{K}(-, Y) : \mathcal{C}_-^{\mathrm{op}} \to \mathcal{S}$ is representable. Since $\mathcal{C}_-$ is locally small, there exists a functor $F : \mathcal{C}_- \to \mathcal{S}$ which is corepresentable by Y (Theorem 5.6.6.13). Then F belongs to the essential image of Φ. We may therefore assume without loss of generality that $F = \mathscr{K}(X_0, -)$ for some object $X_0 \in \mathcal{C}_-$. Choose a couniversal vertex $\eta_0 \in \mathscr{K}(X_0, Y) = F(Y)$. Since Φ is fully faithful, Proposition 8.3.2.17 implies that η_0 is also universal, so that $\mathscr{K}(-, Y)$ is representable by X_0.

To complete the proof, we must show that the pairing $\mathscr{K}$ satisfies the second condition of Definition 8.3.2.18. Let $Y \in \mathcal{C}_+$ be as above, let X be any object of $\mathcal{C}_-$, and let η be a vertex of the Kan complex $\mathscr{K}(X,Y)$. Assume that η is universal; we wish to show that it is also couniversal (the reverse implication follows from Proposition 8.3.2.17). Choose $\eta_0 \in \mathscr{K}(X_0,Y)$ as above. Since η_0 is universal, there exists an isomorphism $u : X \to X_0$ in the ∞-category $\mathcal{C}_-$ such that $\mathscr{K}(u, \mathrm{id}_Y)(\eta_0)$ and η belong to the same connected component of the Kan complex $\mathscr{K}(X,Y)$ (Remark 5.6.6.6). We may therefore assume without loss of generality that $\eta = \mathscr{K}(u, \mathrm{id}_Y)(\eta_0)$ (Remark 5.6.6.3). The desired result now follows by applying Remark 5.6.6.4 to the isomorphism of functors $\mathscr{K}(u,-) : \mathscr{K}(X_0,-) \to \mathscr{K}(X,-)$. □

03MP **Corollary 8.3.2.21.** *Let $\mathcal{C}$ be a locally small ∞-category, and let $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{C},\mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{S})$ spanned by the corepresentable functors. Then the evaluation map*

$$\mathrm{ev} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{C},\mathcal{S}) \times \mathcal{C} \to \mathcal{S} \qquad (F,C) \mapsto F(C)$$

is a balanced profunctor.

03MQ **Remark 8.3.2.22.** Up to equivalence, every balanced profunctor $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}$ can be obtained from the construction of Corollary 8.3.2.21. More precisely, let $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{C}_+,\mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}_+,\mathcal{S})$ spanned by the corepresentable functors. If $\mathscr{K}$ is balanced, then it factors as a composition

$$\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \xrightarrow{\Phi\times\mathrm{id}} \mathrm{Fun}^{\mathrm{corep}}(\mathcal{C}_+,\mathcal{S}) \times \mathcal{C}_+ \xrightarrow{\mathrm{ev}} \mathcal{S},$$

where Φ is an equivalence of ∞-categories by virtue of Corollary 8.3.2.20.

8.3.3 Hom-Functors for ∞-Categories

03MR Let $\mathcal{C}$ be an ∞-category. In §4.6.1, we associated to every pair of objects $X, Y \in \mathcal{C}$ a Kan complex $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ parametrizing morphisms from X to Y. In this section, we will promote the construction $(X,Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)$ to a functor of ∞-categories.

046B **Definition 8.3.3.1.** Let $\mathcal{C}$ be an ∞-category. We say that a profunctor

$$\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

is a Hom-*functor for* $\mathcal{C}$ if it is a covariant transport representation for the twisted arrow coupling $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.

03N2 **Proposition 8.3.3.2** (Existence and Uniqueness)**.** *Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ admits a* Hom*-functor $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ if and only if it is locally small. If this condition is satisfied, then $\mathscr{H}$ is uniquely determined up to isomorphism.*

Proof. Combine Remark 8.3.5.3 with Corollary 5.6.0.6 (applied to the left fibration $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$). □

Remark 8.3.3.3. Let $\mathcal{C}$ be an ∞-category and let let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be a Hom-functor for $\mathcal{C}$. Passing to homotopy categories, we obtain a functor $H : \mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{h}\mathcal{C} \to \mathrm{hKan}$. It follows from Corollary 8.1.2.18 (and Remark 5.6.5.8) that H is isomorphic to the functor $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$ determined by the hKan enrichment of the homotopy category $\mathrm{h}\mathcal{C}$ (see Construction 4.6.9.13). See Remark 8.3.5.4 for a more precise statement. 03MX

Example 8.3.3.4. Let $\mathcal{C}$ be a category. The construction $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$ determines a functor 03MW

$$\mathscr{H} : \mathrm{N}_{\bullet}(\mathcal{C})^{\mathrm{op}} \times \mathrm{N}_{\bullet}(\mathcal{C}) \to \mathrm{N}_{\bullet}(\mathrm{Set}) \subset \mathcal{S} \qquad (X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y),$$

which is a Hom-functor for the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$. For a more general statement, see Proposition 8.3.6.2.

Variant 8.3.3.5. Let κ be an uncountable cardinal and let $\mathcal{S}^{<\kappa}$ denote the ∞-category of κ-small spaces (Variant 5.5.4.12). Then an ∞-category $\mathcal{C}$ admits a Hom-functor 03V2

$$\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}^{<\kappa}$$

if and only if it is locally κ-small. If this condition is satisfied, then $\mathscr{H}$ is uniquely determined up to isomorphism.

Remark 8.3.3.6 (Duality)**.** Let $\mathcal{C}$ be an ∞-category, let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be a functor, and let $\mathscr{H}' : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be the functor obtained from $\mathscr{H}$ by transposing its arguments. If $\mathscr{H}$ is a Hom-functor for $\mathcal{C}$, then $\mathscr{H}'$ is a Hom-functor for the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. 03MY

Notation 8.3.3.7. Let $\mathcal{C}$ be a locally small ∞-category. We will often use the notation $\mathrm{Hom}_{\mathcal{C}}(-,-)$ to denote a Hom-functor $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$. Beware that this convention introduces a slight potential for confusion. Given a pair of objects $X, Y \in \mathcal{C}$, we have two potentially different definitions of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$: 03N4

(a) The Kan complex $\{X\} \tilde{\times}_{\mathcal{C}} \{Y\}$ of Construction 4.6.1.1, which is well-defined up to canonical isomorphism.

(b) The Kan complex $\mathscr{H}(X, Y)$, which is only well-defined up to homotopy equivalence (since it depends on a choice of Hom-functor $\mathscr{H}$).

However, the danger is slight: Remark 8.3.3.3 guarantees the existence of homotopy equivalences $\{X\} \tilde{\times}_{\mathcal{C}} \{Y\} \simeq \mathscr{H}(X, Y)$, which can be chosen to depend functorially on X and Y (as morphisms in the homotopy category hKan). Consequently, we can always modify the choice of Hom-functor $\mathscr{H}$ to arrange that definitions (a) and (b) coincide (see Corollary 4.4.5.3).

046C **Proposition 8.3.3.8.** *Let $\mathcal{C}$ be an ∞-category and let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be a* Hom-*functor for $\mathcal{C}$. Then $\mathscr{H}$ is a balanced profunctor (see Definition 8.3.2.18).*

Proof. By virtue of Remark 8.3.2.19, it suffices to observe that the twisted arrow coupling $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ is balanced; see Example 8.2.6.2. □

03NF **Definition 8.3.3.9.** Let $\mathcal{C}$ be an ∞-category and let

$$h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \qquad Y \mapsto h_Y$$

be a functor. We say that $h_{\bullet}$ is a *covariant Yoneda embedding for $\mathcal{C}$* if the construction $(X, Y) \mapsto h_Y(X)$ is a Hom-functor for $\mathcal{C}$, in the sense of Definition 8.3.3.1. Similarly, we say that a functor

$$h^{\bullet} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \qquad X \mapsto h^X$$

is a *contravariant Yoneda embedding for $\mathcal{C}$* if the construction $(X, Y) \mapsto h^X(Y)$ is a Hom-functor for $\mathcal{C}$.

03NG **Remark 8.3.3.10** (Duality)**.** A functor $h^{\bullet} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ is a contravariant Yoneda embedding for $\mathcal{C}$ if and only if it is a covariant Yoneda embedding for the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$; see Remark 8.3.3.6.

03NH **Remark 8.3.3.11.** Let $\mathcal{C}$ be an ∞-category. By virtue of Proposition 8.3.3.2, the following conditions are equivalent:

- The ∞-category $\mathcal{C}$ is locally small.
- The ∞-category $\mathcal{C}$ admits a covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.
- The ∞-category $\mathcal{C}$ admits a contravariant Yoneda embedding $h^{\bullet} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S})$.

If these conditions are satisfied, then the functors $h_{\bullet}$ and $h^{\bullet}$ are uniquely determined up to isomorphism. Moreover, for every object $X \in \mathcal{C}$, the functor $h_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is representable by X, and the functor $h^X : \mathcal{C} \to \mathcal{S}$ is corepresentable by X (Proposition 8.3.5.5).

03V4 **Variant 8.3.3.12.** Let κ be an uncountable cardinal and let $\mathcal{S}^{<\kappa}$ denote the ∞-category of κ-small spaces (see Variant 5.5.4.12). For every ∞-category $\mathcal{C}$, the following conditions are equivalent:

- The ∞-category $\mathcal{C}$ is locally κ-small.
- The ∞-category $\mathcal{C}$ admits a covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.
- The ∞-category $\mathcal{C}$ admits a contravariant Yoneda embedding $h^{\bullet} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}^{<\kappa})$.

See Variant 8.3.5.7.

Theorem 8.3.3.13 (Yoneda's Lemma for ∞-Categories)**.** *Let $\mathcal{C}$ be a locally small ∞-category. Then the covariant and contravariant Yoneda embeddings* 03NJ

$$h_{\bullet} : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \qquad h^{\bullet} : \mathcal{C}^{\mathrm{op}} \to \operatorname{Fun}(\mathcal{C}, \mathcal{S})$$

are fully faithful functors, whose essential images are the full subcategories

$$\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subseteq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \qquad \operatorname{Fun}^{\mathrm{corep}}(\mathcal{C}, \mathcal{S}) \subseteq \operatorname{Fun}(\mathcal{C}, \mathcal{S})$$

spanned by the representable and corepresentable functors, respectively.

Proof. By virtue of Corollary 8.3.2.20, this is a reformulation of Proposition 8.3.3.8. □

We close this section by recording a simple observation about the Yoneda embedding.

Proposition 8.3.3.14. *Let $\mathcal{C}$ be a locally small ∞-category and let $h_{\bullet} : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ be a covariant Yoneda embedding for $\mathcal{C}$. Suppose we are given a diagram $\overline{f} : K^{\triangleleft} \to \mathcal{C}$, where K is a small simplicial set. The following conditions are equivalent:* 03NP

(1) *The morphism $\overline{f}$ is a limit diagram in $\mathcal{C}$.*

(2) *The composition $h_{\bullet} \circ \overline{f}$ is a limit diagram in the ∞-category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.*

Following the convention of Remark 4.7.0.5, we can regard Proposition 8.3.3.14 as a special case of the following more precise assertion (applied in the special case where $\kappa = \lambda$ is a strongly inaccessible cardinal):

Variant 8.3.3.15. Let λ be an uncountable cardinal, let $\mathcal{C}$ be a locally λ-small ∞-category, and let $h_{\bullet} : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$ be a covariant Yoneda embedding for $\mathcal{C}$. Let $\kappa = \operatorname{ecf}(\lambda)$ be the exponential cofinality of λ, let K be a κ-small simplicial set, and let $\overline{f} : K^{\triangleleft} \to \mathcal{C}$ be a diagram. Then the following conditions are equivalent: 03V5

(1) The morphism $\overline{f}$ is a limit diagram in $\mathcal{C}$.

(2) The composition $h_{\bullet} \circ \overline{f}$ is a limit diagram in the ∞-category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$.

Proof. Since K is κ-small, the ∞-category $\mathcal{S}^{<\lambda}$ admits K-indexed limits (Example 7.6.7.4). For each object $X \in \mathcal{C}$, let $\operatorname{ev}_X : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda}) \to \mathcal{S}^{<\lambda}$ denote the functor given by evaluation at X. By virtue of Proposition 7.1.6.1, condition (2) is equivalent to the requirement that for each object $X \in \mathcal{C}$, the composition

$$K^{\triangleleft} \xrightarrow{\overline{f}} \mathcal{C} \xrightarrow{h_{\bullet}} \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda}) \xrightarrow{\operatorname{ev}_X} \mathcal{S}^{<\lambda}$$

is a limit diagram in the ∞-category $\mathcal{S}^{<\lambda}$. Since the composite functor $(\operatorname{ev}_X \circ h_{\bullet}) : \mathcal{C} \to \mathcal{S}^{<\lambda}$ is corepresentable by X, the equivalence (1) $\Leftrightarrow$ (2) follows from Proposition 7.4.5.17 (and Remark 7.4.5.19). □

03V6 **Remark 8.3.3.16.** In the situation of Variant 8.3.3.15, suppose that the ∞-category $\mathcal{C}$ admits K-indexed limits. Then the ∞-category of representable functors $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$ also admits K-indexed limits, which are preserved by the inclusion functor $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$.

03Y7 **Corollary 8.3.3.17.** *Let $\mathcal{C}$ be an ∞-category and let κ be an infinite cardinal. Then there exists a fully faithful functor $F : \mathcal{C} \to \widehat{\mathcal{C}}$, where $\widehat{\mathcal{C}}$ is κ-complete and κ-cocomplete. Moreover, we can arrange that F preserves the limits of all κ-small diagrams which exist in $\mathcal{C}$.*

Proof. Using Remark 4.7.3.19, we can choose an uncountable cardinal λ of exponential cofinality $\geq \kappa$. Enlarging λ if necessary, we may assume that $\mathcal{C}$ is locally λ-small. Let $\widehat{\mathcal{C}}$ denote the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$ and let $F = h_{\bullet}$ be a covariant Yoneda embedding for $\mathcal{C}$. Since $\mathcal{S}^{<\lambda}$ is κ-complete and κ-cocomplete (Remark 7.4.5.7 and Variant 7.4.5.8), the ∞-category $\widehat{\mathcal{C}}$ has the same property (Remark 7.6.7.5). Moreover, the functor F is fully faithful (Theorem 8.3.3.13) and preserves limits of κ-small diagrams (Remark 8.3.3.16). $\square$

8.3.4 Representable Profunctors

03NQ Let $\mathcal{C}$ and $\mathcal{D}$ be categories. There is a fully faithful embedding from the category of functors $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ to the category of profunctors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathrm{Set})$, which assigns to each functor $G : \mathcal{D} \to \mathcal{C}$ the representable profunctor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set} \qquad (X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

This construction has an ∞-categorical counterpart:

03NR **Proposition 8.3.4.1** (Classification of Representable Profunctors)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Let κ be an uncountable cardinal for which $\mathcal{C}$ is locally κ-small, and let*

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}^{<\kappa}$$

be a Hom*-functor for $\mathcal{C}$ (see Notation 8.3.3.7). Then the construction $G \mapsto \mathrm{Hom}_{\mathcal{C}}(-, G(-))$ determines a fully faithful functor*

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S}^{<\kappa}),$$

whose essential image is spanned by the representable profunctors from $\mathcal{D}$ to $\mathcal{C}$.

Proof. Let $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by the representable functors. By virtue of Theorem 8.3.3.13, the construction $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(-, Y)$ determines an equivalence of ∞-categories $h^{\bullet} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. It follows that postcomposition with $h^{\bullet}$ induces an equivalence of ∞-categories

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})),$$

which is a restatement of Proposition 8.3.4.1. $\square$

Definition 8.3.4.2. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. Assume that $\mathcal{C}$ is locally κ-small and let $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}^{<\kappa}$ be a Hom-functor for $\mathcal{C}$. We say that a profunctor $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ is *representable by* G if it isomorphic to the composition 03NS

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{id} \times G} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,-)} \mathcal{S} \qquad (X,Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S}^{<\kappa})$. By virtue of Proposition 8.3.3.2, this condition does not depend on the choice of Hom-functor $\mathrm{Hom}_{\mathcal{C}}(-,-)$.

Example 8.3.4.3. Let $\mathcal{C}$ be a locally κ-small ∞-category, and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\kappa}$ be a functor. Then $\mathscr{F}$ is representable by an object $X \in \mathcal{C}$ (in the sense of Variant 5.6.6.2) if and only if, when regarded as a profunctor from Δ^0 to $\mathcal{C}$, it is representable by the functor $\Delta^0 \to \{X\} \hookrightarrow \mathcal{C}$ (in the sense of Definition 8.3.4.2). 03NT

Interchanging the roles of $\mathcal{C}$ and $\mathcal{D}$, we obtain the following dual notion:

Variant 8.3.4.4. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Assume that $\mathcal{D}$ is locally κ-small and let $\mathrm{Hom}_{\mathcal{D}}(-,-) : \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}^{<\kappa}$ be a Hom-functor for $\mathcal{D}$. We say that a profunctor $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ is *corepresentable by* F if it isomorphic to the composition 03NU

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{F^{\mathrm{op}} \times \mathrm{id}} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(-,-)} \mathcal{S} \qquad (X,Y) \mapsto \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$$

as an object of the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S}^{<\kappa})$. By virtue of Proposition 8.3.3.2, this condition does not depend on the choice of Hom-functor $\mathrm{Hom}_{\mathcal{C}}(-,-)$.

Example 8.3.4.5. Let $\mathcal{C}$ be a locally κ-small ∞-category and let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}^{<\kappa}$ be a profunctor from $\mathcal{C}$ to itself. The following conditions are equivalent: 03NV

- The profunctor $\mathscr{H}$ is a Hom-functor for $\mathcal{C}$.
- The profunctor $\mathscr{H}$ is representable by the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (Definition 8.3.4.2).
- The profunctor $\mathscr{H}$ is corepresentable by the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ (Variant 8.3.4.4).

Remark 8.3.4.6. Let $\lambda : \mathcal{C} \to \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+}$ be a coupling of ∞-categories which is essentially κ-small for some uncountable cardinal κ and let $\mathscr{K} : \mathcal{C}_{-}^{\mathrm{op}} \times \mathcal{C}_{+} \to \mathcal{S}^{<\kappa}$ be a covariant transport representation for λ. Then the profunctor $\mathscr{K}$ is representable by a functor $G : \mathcal{C}_{+} \to \mathcal{C}_{-}$ (in the sense of Definition 8.3.4.2) if and only if the coupling λ is representable by G (in the sense of Definition 8.2.3.1). Similarly, $\mathscr{K}$ is corepresentable by a functor $F : \mathcal{C}_{-} \to \mathcal{C}_{+}$ if and only if λ is corepresentable by F. 046D

03NW **Remark 8.3.4.7** (Uniqueness)**.** Let κ be an uncountable cardinal and let $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}^{<\kappa}$ be a profunctor of ∞-categories. If $\mathcal{C}$ is locally κ-small small, then Proposition 8.3.4.1 guarantees that $\mathscr{K}$ is representable (in the sense of Definition 8.3.2.9 if and only if it is representable by G, for some functor $G : \mathcal{D} \to \mathcal{C}$. Moreover, if this condition is satisfied, then the functor G is determined uniquely to up isomorphism. Similarly, if $\mathcal{D}$ is locally κ-small small, then $\mathscr{K}$ is corepresentable if and only if it is corepresentable by some functor $F : \mathcal{C} \to \mathcal{D}$. In this case, the functor F is also uniquely determined up to isomorphism.

046E **Example 8.3.4.8.** Let κ be an uncountable cardinal, let $\mathcal{C}_-$ and $\mathcal{C}_+$ be locally κ-small ∞-categories, and let $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}^{<\kappa}$ be a profunctor from $\mathcal{C}_+$ to $\mathcal{C}_-$. The following conditions are equivalent:

- The profunctor $\mathscr{K}$ is balanced (Definition 8.3.2.18).
- The profunctor $\mathscr{K}$ is representable by a functor $G : \mathcal{C}_+ \to \mathcal{C}_-$ which is an equivalence of ∞-categories.
- The profunctor $\mathscr{K}$ is corepresentable by a functor $F : \mathcal{C}_- \to \mathcal{C}_+$ which is an equivalence of ∞-categories.

By virtue of Theorem 8.3.3.13, this is a reformulation of Corollary 8.3.2.20.

03PA **Proposition 8.3.4.9** (Adjunctions as Profunctors)**.** *Let $G : \mathcal{C}_+ \to \mathcal{C}_-$ be a functor of ∞-categories which represents a profunctor $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}^{<\kappa}$. Then a functor $F : \mathcal{C}_- \to \mathcal{C}_+$ is left adjoint to G if and only if it corepresents the profunctor $\mathscr{K}$. In particular, $\mathscr{K}$ is corepresentable if and only if the functor G admits a left adjoint.*

Proof. Choose a realization of $\mathscr{K}$ as the covariant transport representation of a coupling of ∞-categories $\lambda : \mathcal{C} \to \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+$ (see Remark 8.3.2.5). By virtue of Remark 8.3.4.6, the coupling λ is representable by the functor G. By virtue of Theorem 8.2.5.1, a functor $F : \mathcal{C}_- \to \mathcal{C}_+$ is left adjoint to G if and only if it corepresents the coupling λ. Invoking Remark 8.3.4.6 again, we see that this is equivalent to the requirement that F corepresents the profunctor $\mathscr{K}$. □

Recall that, if $\mathcal{C}_-$ and $\mathcal{C}_+$ are ∞-categories, then we write $\mathrm{LFun}(\mathcal{C}_-, \mathcal{C}_+)$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}_-, \mathcal{C}_+)$ spanned by those functors $F : \mathcal{C}_- \to \mathcal{C}_+$ which are left adjoints (Notation 6.2.1.3). Similarly, we write $\mathrm{RFun}(\mathcal{C}_+, \mathcal{C}_-)$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}_+, \mathcal{C}_-)$ spanned by those functors $G : \mathcal{C}_+ \to \mathcal{C}_-$ which are right adjoints.

04B7 **Corollary 8.3.4.10.** *Let $\mathcal{C}_-$ and $\mathcal{C}_+$ be ∞-categories. Let κ be an uncountable cardinal for which $\mathcal{C}_-$ and $\mathcal{C}_+$ are locally κ-small, and let $\mathcal{E} \subseteq \mathrm{Fun}(\mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+, \mathcal{S}^{<\kappa})$ denote the full subcategory spanned by those profunctors $\mathscr{K} : \mathcal{C}_-^{\mathrm{op}} \times \mathcal{C}_+ \to \mathcal{S}^{<\kappa}$ which are both representable and corepresentable. Then:*

(1) *Composition with the covariant Yoneda embedding* $\mathcal{C}_- \to \mathrm{Fun}(\mathcal{C}_-^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ *induces an equivalence of* ∞*-categories* $\rho : \mathrm{RFun}(\mathcal{C}_+, \mathcal{C}_-) \to \mathcal{E}$.

(2) *Composition with the contravariant Yoneda embedding* $\mathcal{C}_+^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}_+, \mathcal{S}^{<\kappa})$ *induces an equivalence of* ∞*-categories* $\lambda : \mathrm{LFun}(\mathcal{C}_-, \mathcal{C}_+) \to \mathcal{E}^{\mathrm{op}}$.

(3) *The composition* $[\rho]^{-1} \circ [\lambda^{\mathrm{op}}]$ *determines a canonical isomorphism* $\mathrm{LFun}(\mathcal{C}_-, \mathcal{C}_+)^{\mathrm{op}} \simeq \mathrm{RFun}(\mathcal{C}_+, \mathcal{C}_-)$ *in the homotopy category* hQCat*, which carries each functor* $F \in \mathrm{LFun}(\mathcal{C}_-, \mathcal{C}_+)$ *to a functor* $G \in \mathrm{RFun}(\mathcal{C}_+, \mathcal{C}_-)$ *which is right adjoint to* F.

Proof. Assertion (1) follows by combining Propositions 8.3.4.1 and 8.3.4.9, and assertion (2) follows by a similar argument. Assertion (3) follows by combining (1) and (2) with Proposition 8.3.4.9. □

For many applications, Definition 8.3.4.2 is insufficiently precise. Given a functor of ∞-categories $G : \mathcal{D} \to \mathcal{C}$, we would like to be able to consider not only profunctors which are *representable* by G (meaning that they are *abstractly* isomorphic to the profunctor $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$) but profunctors which are *represented* by G (meaning that we have *chosen* an isomorphism with the profunctor $(X, Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$, or some essentially equivalent datum). Here it is inconvenient that the functor $\mathrm{Hom}_{\mathcal{C}}(-, -)$ is well-defined only up to isomorphism. To address this point, it is convenient to encode representability in a different way.

Notation 8.3.4.11. Let $\mathcal{S}$ denote the ∞-category of spaces (Construction 5.5.1.1). We will regard the contractible Kan complex Δ^0 as an object of $\mathcal{S}$. For every ∞-category $\mathcal{E}$, we let $\underline{\Delta^0}_{\mathcal{E}}$ denote the constant functor $\mathcal{E} \to \mathcal{S}$ taking the value Δ^0. 03MS

Definition 8.3.4.12. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories, let $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ be a profunctor from $\mathcal{D}$ to $\mathcal{C}$, and let $\mathscr{K}|_{\mathrm{Tw}(\mathcal{D})}$ denote the composite functor 03NX

$$\mathrm{Tw}(\mathcal{D}) \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{G^{\mathrm{op}} \times \mathrm{id}} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{\mathscr{K}} \mathcal{S}.$$

Suppose we are given a natural transformation $\beta : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})} \to \mathscr{K}|_{\mathrm{Tw}(\mathcal{D})}$, where $\underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})}$ denotes the constant functor $\mathrm{Tw}(\mathcal{D}) \to \mathcal{S}$ taking the value Δ^0. We say that β *exhibits the profunctor* $\mathscr{K}$ *as represented by* G if, for every object $D \in \mathcal{D}$, the evaluation of β at the object $\mathrm{id}_D \in \mathrm{Tw}(\mathcal{D})$ determines a vertex $\beta(\mathrm{id}_D) \in \mathscr{K}(G(D), D)$ which exhibits the functor $\mathscr{K}(-, D)$ as represented by the object $G(D) \in \mathcal{C}$ (see Variant 5.6.6.2).

Remark 8.3.4.13. In the situation of Definition 8.3.4.12, the natural transformation β can 046F be identified with a functor $\widetilde{G}$ which fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{D}) \arrow[r, "\widetilde{G}"] \arrow[d] & \{\Delta^0\} \tilde{\times}_{\mathcal{S}} (\mathcal{C}^{\mathrm{op}} \times \mathcal{D}) \arrow[d, "\lambda"] \\
\mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "G^{\mathrm{op}} \times \mathrm{id}"] & \mathcal{C}^{\mathrm{op}} \times \mathcal{D}.
\end{tikzcd}$$

Moreover, the natural transformation β exhibits $\mathscr{K}$ as represented by G (in the sense of Definition 8.3.4.12) if and only if $\widetilde{G}$ exhibits the coupling λ as represented by G (in the sense of Definition 8.2.4.1).

03NY **Example 8.3.4.14.** In the situation of Definition 8.3.4.12, suppose that $\mathcal{D} = \Delta^0$. In this case, we can identify the profunctor $\mathscr{K}$ with a functor $K : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, we can identify the functor G with an object $X \in \mathcal{C}$, and we can identify β with a vertex of the Kan complex $K(X)$. Then β exhibits the profunctor $\mathscr{K}$ as represented by the functor G (in the sense of Definition 8.3.4.12) if and only if it exhibits the functor K as represented by the object X (in the sense of Variant 5.6.6.2.

03P5 **Proposition 8.3.4.15.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories, where $\mathcal{C}$ is locally small, and let $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ be a profunctor. The following conditions are equivalent:*

(1) *The profunctor $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ is representable by G, in the sense of Definition 8.3.4.2.*

(2) *There exists a natural transformation $\beta : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})} \to \mathscr{K}|_{\mathrm{Tw}(\mathcal{D})}$ which exhibits $\mathscr{K}$ as represented by G, in the sense of Definition 8.3.4.12.*

Proof. By virtue of Remarks 8.3.4.6 and 8.3.4.13, this follows by applying Proposition 8.2.4.3 to the coupling

$$\{\Delta^0\} \tilde{\times}_{\mathcal{S}} (\mathcal{C}^{\mathrm{op}} \times \mathcal{D}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}.$$

□

03P8 **Variant 8.3.4.16** (Corepresentable Profunctors)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ be a profunctor. We say that a natural transformation $\beta : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})} \to \mathscr{K}|_{\mathrm{Tw}(\mathcal{C})}$ *exhibits $\mathscr{K}$ as corepresented by F* if, for every object $X \in \mathcal{C}$, the image $\beta(\mathrm{id}_X) \in \mathscr{K}(X, F(X))$ exhibits the functor $\mathscr{K}(X, -) : \mathcal{D} \to \mathcal{S}$ as corepresented by the object $F(X) \in \mathcal{D}$, in the sense of Definition 5.6.6.1. Equivalently, β exhibits $\mathscr{K}$ as corepresented by F if it exhibits $\mathscr{K}$ as represented by the opposite functor F^{op}, when regarded as a profunctor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}^{\mathrm{op}}$ (see Remark 8.3.2.3).

Remark 8.3.4.17 (Homotopy Invariance)**.** In the situation of Definition 8.3.4.12, the condition that β exhibits $\mathscr{K}$ as corepresented by G depends only on the homotopy class $[\beta]$ (as a morphism in the homotopy category $\mathrm{hFun}(\mathrm{Tw}(\mathcal{D}), \mathcal{S})$) (see Remark 5.6.6.3). `03P0`

Remark 8.3.4.18 (Change of $\mathscr{K}$)**.** Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. Suppose we are given a pair of profunctors $\mathscr{K}, \mathscr{K}' : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$, a natural transformation $\alpha : \mathscr{K} \to \mathscr{K}'$, and a commutative diagram `03P1`

$$\begin{tikzcd}
 & \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})} \arrow[dl, "\beta"'] \arrow[dr, "\beta'"] & \\
\mathscr{K}|_{\mathrm{Tw}(\mathcal{D})} \arrow[rr, "\alpha|_{\mathrm{Tw}(\mathcal{D})}"] & & \mathscr{K}'|_{\mathrm{Tw}(\mathcal{D})}
\end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{S})$. Then any two of the following conditions imply the third:

- The natural transformation β exhibits the profunctor $\mathscr{K}$ as represented by G.
- The natural transformation β' exhibits the profunctor $\mathscr{K}'$ as represented by G.
- The natural transformation α is an isomorphism.

See Remark 5.6.6.4.

Proposition 8.3.4.19. *Suppose we are given a functor of ∞-categories $G : \mathcal{D} \to \mathcal{C}$, a profunctor $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$, and a natural transformation $\beta : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})} \to \mathscr{K}|_{\mathrm{Tw}(\mathcal{D})}$. Then β exhibits $\mathscr{K}$ as represented by G (in the sense of Definition 8.3.4.12) if and only if the induced map $\mathrm{Tw}(\mathcal{D}) \to \{\Delta^0\} \tilde{\times}_{\mathcal{S}} (\mathcal{C}^{\mathrm{op}} \times \mathcal{D})$ is left cofinal.* `03P2`

Proof. By virtue of Remark 8.3.4.13, this is a special case of Proposition 8.2.4.9. □

Proposition 8.3.4.20 (Representable Profunctors as Kan Extensions)**.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories, let $\mathscr{K} : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ be a profunctor, and let $\beta : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})} \to \mathscr{K}|_{\mathrm{Tw}(\mathcal{D})}$ be a natural transformation which exhibits $\mathscr{K}$ as represented by G. Then β exhibits $\mathscr{K}$ as a left Kan extension of the constant diagram $\underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})}$ along the composite map* `03P3`

$$\mathrm{Tw}(\mathcal{D}) \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \xrightarrow{G^{\mathrm{op}} \times \mathrm{id}} \mathcal{C}^{\mathrm{op}} \times \mathcal{D}.$$

Proof. Let $\mathcal{E}$ denote the oriented fiber product $\{\Delta^0\} \tilde{\times}_{\mathcal{S}} (\mathcal{C}^{\mathrm{op}} \times \mathcal{D})$ and let $\mu : \mathcal{E} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ be the projection onto the second factor, so that we have a tautological natural transformation $\widetilde{\beta} : \underline{\Delta^0}_{\mathcal{E}} \to \mathscr{K} \circ \mu$. It follows from Proposition 7.6.2.17 that $\widetilde{\beta}$ exhibits $\mathscr{K}$ as a left Kan extension of $\underline{\Delta^0}_{\mathcal{E}}$ along μ. The natural transformation β then determines a functor

$T : \mathrm{Tw}(\mathcal{D}) \to \mathcal{E}$ such that precomposition with T carries $\widetilde{\beta}$ to β. By the transitivity of the formation of of Kan extensions (Proposition 7.3.8.18), we are reduced to showing that the identity transformation $\mathrm{id} : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})} \to \underline{\Delta^0}_{\mathcal{E}} \circ T$ exhibits $\underline{\Delta^0}_{\mathcal{E}}$ as a left Kan extension of $\underline{\mathrm{Tw}(\mathcal{D})}$ along T. This is a special case of Remark 7.6.2.13, since the functor T is left cofinal (Proposition 8.3.4.19). □

046K **Corollary 8.3.4.21** (The Universal Mapping Property of Representable Profunctors)**.** *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor of ∞-categories. Suppose we are given a pair of profunctors $\mathscr{K}, \mathscr{K}' : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$, and let $\beta : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})} \to \mathscr{K}|_{\mathrm{Tw}(\mathcal{D})}$ be a natural transformation which exhibits $\mathscr{K}$ as represented by G. Then precomposition with β induces a homotopy equivalence of Kan complexes*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S})}(\mathscr{K}, \mathscr{K}') \to \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{S})}(\underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})}, \mathscr{K}'|_{\mathrm{Tw}(\mathcal{D})}).$$

Proof. Combine Propositions 8.3.4.20 and 7.3.6.1. □

03P7 **Example 8.3.4.22** (Spaces of Natural Transformation)**.** Let $G, G' : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories and let $\mathscr{H}$ be a Hom-functor for $\mathcal{C}$. Combining Corollary 8.3.4.21 with Proposition 8.3.4.1, we obtain homotopy equivalences of Kan complexes

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D}, \mathcal{C})}(G, G') &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S})}(\mathscr{H} \circ (\mathrm{id} \times G), \mathscr{H} \circ (\mathrm{id} \times G')) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathcal{S})}(\underline{\Delta^0}_{\mathrm{Tw}(\mathcal{D})}, \mathscr{H}|_{\mathrm{Tw}(\mathcal{D})}) \\ &\simeq \varprojlim(\mathscr{H}|_{\mathrm{Tw}(\mathcal{D})}). \end{aligned}$$

Stated more informally, the space of natural transformations from G to G' can be viewed as a limit of the diagram

$$\mathrm{Tw}(\mathcal{D}) \to \mathcal{S} \qquad (f : X \to Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(G(X), G'(Y)).$$

8.3.5 Recognition of Hom-Functors

046H Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ is locally small, then Proposition 8.3.3.2 guarantees that it admits a Hom-functor $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$, which is uniquely determined up to isomorphism. Our goal in this section is to formulate a more precise statement, which characterizes the functor $\mathscr{H}$ up to *canonical* isomorphism (see Proposition 8.3.5.6).

03MT **Definition 8.3.5.1.** Let $\mathcal{C}$ be an ∞-category, let $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ denote the left fibration of Proposition 8.1.1.11, and let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be a profunctor. We say that a natural transformation $\alpha : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})} \to \mathscr{H}|_{\mathrm{Tw}(\mathcal{C})}$ *exhibits $\mathscr{H}$ as a* Hom-*functor for $\mathcal{C}$* if it satisfies the following condition:

$(*)$ For every pair of objects $X, Y \in \mathcal{C}$, the natural transformation α induces a homotopy equivalence of Kan complexes

$$\alpha_{X,Y} : \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\} \to \mathrm{Hom}_{\mathcal{S}}(\Delta^0, \mathscr{H}(X,Y)).$$

Remark 8.3.5.2. Let $\mathcal{C}$ be an ∞-category and let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be a profunctor. 03MZ The datum of a natural transformation $\alpha : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})} \to \mathscr{H}|_{\mathrm{Tw}(\mathcal{C})}$ can be identified with a commutative diagram of ∞-categories

03N0

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{C}) \arrow[r] \arrow[d] & \{\Delta^0\} \tilde{\times}_{\mathcal{S}} \mathcal{S} \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\mathscr{H}"] & \mathcal{S}.
\end{tikzcd} \tag{8.46}$$

In this case, the natural transformation α exhibits $\mathscr{H}$ as a Hom-functor for $\mathcal{C}$ if and only if the diagram (8.46) is a categorical pullback square (see Corollary 5.1.7.16).

Remark 8.3.5.3. Let $\mathcal{C}$ be an ∞-category. A profunctor $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ is a Hom-functor 03N1 for $\mathcal{C}$ (in the sense of Definition 8.3.3.1) if and only if there exists a natural transformation $\alpha : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})} \to \mathscr{H}|_{\mathrm{Tw}(\mathcal{C})}$ which exhibits $\mathscr{H}$ as a Hom-functor for $\mathcal{C}$ (in the sense of Definition 8.3.5.1).

Remark 8.3.5.4. Let $\mathcal{C}$ be an ∞-category, let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be a profunctor, and let 03MV $\alpha : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})} \to \mathscr{H}|_{\mathrm{Tw}(\mathcal{C})}$ be a natural transformation. For every pair of objects $X, Y \in \mathcal{C}$, Notation 8.1.2.14 and Remark 5.5.1.5 supply canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \simeq \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{Y\} \qquad \mathscr{H}(X,Y) \simeq \mathrm{Hom}_{\mathcal{S}}(\Delta^0, \mathscr{H}(X,Y))$$

in the homotopy category hKan. Consequently, the homotopy class of the morphism $\alpha_{X,Y}$ appearing in Definition 8.3.5.1 can be identified with a map $[\alpha_{X,Y}] : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathscr{H}(X,Y)$ in hKan, which depends functorially on X and Y (see Corollary 8.1.2.18). The natural transformation α exhibits $\mathscr{H}$ as a Hom-functor for $\mathcal{C}$ (in the sense of Definition 8.3.5.1) if and only if each $[\alpha_{X,Y}]$ is an isomorphism in the category hKan.

Proposition 8.3.5.5. *Let $\mathcal{C}$ be an ∞-category, let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be a profunctor, and let* 03NK *$\alpha : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})} \to \mathscr{H}|_{\mathrm{Tw}(\mathcal{C})}$ be a natural transformation. The following conditions are equivalent:*

(1) *The natural transformation α exhibits $\mathscr{H}$ as a* Hom*-functor for $\mathcal{C}$ (in the sense of Definition 8.3.5.1).*

(2) *The natural transformation α exhibits the profunctor $\mathscr{H}$ as represented by the identity functor* $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ *(in the sense of Definition 8.3.4.12). That is, for every object $X \in \mathcal{C}$, the vertex $\alpha(\mathrm{id}_X) \in \mathscr{H}(X, X)$ exhibits the functor $\mathscr{H}(-, X) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ as represented by the object X.*

(3) *The natural transformation α exhibits the profunctor $\mathscr{H}$ as corepresented by the identity functor* $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ *(in the sense of Variant 8.3.4.16). That is, for every object $X \in \mathcal{C}$, the vertex $\alpha(\mathrm{id}_X) \in \mathscr{H}(X, X)$ exhibits the functor $\mathscr{H}(X, -) : \mathcal{C} \to \mathcal{S}$ as corepresented by the object X.*

Proof. We will show that (1) $\Leftrightarrow$ (3); the proof of the equivalence (1) $\Leftrightarrow$ (2) is similar. The natural transformation α can be identified with a functor $T : \mathrm{Tw}(\mathcal{C}) \to \{\Delta^0\} \tilde{\times}_{\mathcal{S}} \mathcal{S}$. For each object $X \in \mathcal{C}$, let T_X denote the restriction of B to the simplicial subset $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \subseteq \mathrm{Tw}(\mathcal{C})$, and consider the following condition:

(1_X) The diagram of ∞-categories

03NL
$$\begin{tikzcd}
\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T_X"] \arrow[d] & \{\Delta^0\} \tilde{\times}_{\mathcal{S}} \mathcal{S} \arrow[d] \\
\mathcal{C} \arrow[r, "\mathscr{H}(X,-)"] & \mathcal{S}
\end{tikzcd} \tag{8.47}$$

is a categorical pullback square.

By virtue of Corollary 5.1.7.16, the natural transformation α exhibits $\mathscr{H}$ as a Hom-functor for $\mathcal{C}$ if and only if it satisfies condition (1_X) for every object $X \in \mathcal{C}$. To complete the proof, it will suffice to show that (1_X) is satisfied if and only if $\alpha(\mathrm{id}_X) \in \mathscr{H}(X, X)$ exhibits the functor $\mathscr{H}(X, -)$ as corepresented by X. This is a special case of Proposition 5.6.6.21, since the id_X is an initial object of the ∞-category $\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ (Proposition 8.1.2.1). □

03N3 **Proposition 8.3.5.6.** *Let $\mathcal{C}$ be a locally small ∞-category, let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ be a functor, and let $\alpha : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})} \to \mathscr{H}|_{\mathrm{Tw}(\mathcal{C})}$ be a natural transformation. The following conditions are equivalent:*

(1) *The natural transformation α exhibits $\mathscr{H}$ as a* Hom-*functor for $\mathcal{C}$: that is, it satisfies condition $(*)$ of Definition 8.3.5.1.*

(2) *The diagram*

$$\begin{tikzcd}
 & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[dr, "\mathscr{H}"] & \\
 \operatorname{Tw}(\mathcal{C}) \arrow[ur, "\lambda"] \arrow[rr, "\underline{\Delta^0}_{\operatorname{Tw}(\mathcal{C})}"'] & {} \arrow[u, Rightarrow, "\alpha"'] & \mathcal{S}.
\end{tikzcd}$$

exhibits $\mathscr{H}$ *as a left Kan extension of the constant functor* $\underline{\Delta^0}_{\operatorname{Tw}(\mathcal{C})}$ *along the left fibration* $\operatorname{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.

(3) *The pair* $(\mathscr{H}, \alpha)$ *is initial when viewed as an object of the oriented fiber product* $\{\underline{\Delta^0}_{\operatorname{Tw}(\mathcal{C})}\} \tilde{\times}_{\operatorname{Fun}(\operatorname{Tw}(\mathcal{C}),\mathcal{S})} \operatorname{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})$

Proof. The equivalence (1) ⇔ (2) follows from Proposition 7.6.2.17 and Remark 8.3.5.2. Since $\mathcal{C}$ is locally small, Proposition 8.3.3.2 guarantees that the functor $\underline{\Delta^0}_{\operatorname{Tw}(\mathcal{C})}$ admits a left Kan extension along λ, so the equivalence (2) ⇔ (3) follows from Corollary 7.3.6.5. □

Variant 8.3.5.7. Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is locally κ-small. Then, in the statement of Proposition 8.3.5.6, we can replace $\mathcal{S}$ with the ∞-category $\mathcal{S}^{<\kappa}$ of κ-small spaces (Variant 5.5.4.12). 03V3

Corollary 8.3.5.8 (Functoriality of Hom-Functors)**.** *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between ∞-categories. Choose natural transformations* 046J

$$\alpha : \underline{\Delta^0}_{\operatorname{Tw}(\mathcal{C})} \to \mathscr{H}_{\mathcal{C}}|_{\operatorname{Tw}(\mathcal{C})} \qquad \beta : \underline{\Delta^0}_{\operatorname{Tw}(\mathcal{D})} \to \mathscr{H}_{\mathcal{D}}|_{\operatorname{Tw}(\mathcal{D})}$$

which exhibit $\mathscr{H}_{\mathcal{C}}$ *and* $\mathscr{H}_{\mathcal{D}}$ *as* Hom-*functors for* $\mathcal{C}$ *and* $\mathcal{D}$, *respectively. Then there exists a natural transformation* $\gamma : \mathscr{H}_{\mathcal{C}}(-,-) \to \mathscr{H}_{\mathcal{D}}(F(-), F(-))$ *for which the diagram*

03PC

$$\begin{tikzcd}
 & \underline{\Delta^0}_{\operatorname{Tw}(\mathcal{C})} \arrow[dl, "{[\alpha]}"'] \arrow[dr, "{[\beta]}"] & \\
 \mathscr{H}_{\mathcal{C}}|_{\operatorname{Tw}(\mathcal{C})} \arrow[rr, "{[\gamma]}"] & & \mathscr{H}_{\mathcal{D}}|_{\operatorname{Tw}(\mathcal{C})}
\end{tikzcd} \tag{8.48}$$

commutes (in the homotopy category $\operatorname{hFun}(\operatorname{Tw}(\mathcal{C}), \mathcal{S})$*). Moreover, the natural transformation* γ *is uniquely determined up to homotopy.*

Proof. This is a special case of Proposition 7.3.6.1, since α exhibits $\mathscr{H}_{\mathcal{C}}$ as a left Kan extension of $\underline{\Delta^0}_{\operatorname{Tw}(\mathcal{C})}$ along the left fibration $\operatorname{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ (Proposition 8.3.5.6). □

03PD **Remark 8.3.5.9.** In the situation of Corollary 8.3.5.8, suppose that we are given a pair of objects $X, Y \in \mathcal{C}$. The commutativity of (8.48) guarantees that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[r, "F"] \arrow[d, "\sim"] & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \arrow[d, "\sim"] \\
\mathscr{H}_{\mathcal{C}}(X,Y) \arrow[r, "\gamma"] & \mathscr{H}_{\mathcal{D}}(F(X),F(Y))
\end{tikzcd}$$

commutes in the homotopy category hKan, where the vertical maps are the isomorphisms of Remark 8.3.5.4. We can summarize the situation more informally as follows: if $F : \mathcal{C} \to \mathcal{D}$ is a functor between (locally small) ∞-categories, then the induced map of Kan complexes $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ depends functorially on the pair (X,Y) (as an object of the ∞-category $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$).

8.3.6 Strict Models for Hom-Functors

03N5 Let $\mathcal{E}$ be a (locally small) ∞-category. Proposition 8.3.3.2 guarantees the existence of a Hom-functor $\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{S}$, which is well-defined up to isomorphism. Our goal in this section is to give an explicit construction of a Hom-functor in the special case where $\mathcal{E} = \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})$ arises as the homotopy coherent nerve of a (locally Kan) simplicial category $\mathcal{C}$.

03N6 **Construction 8.3.6.1.** Let $\mathcal{C}$ be a locally Kan simplicial category. Then the construction $(X,Y) \mapsto \mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet}$ determines a simplicial functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Kan}$. Passing to homotopy coherent nerves, we obtain a functor of ∞-categories

$$\mathscr{H}_{\mathcal{C}} : \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})^{\mathrm{op}} \times \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}) \to \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathrm{Kan}) = \mathcal{S}.$$

03N7 **Proposition 8.3.6.2.** *Let $\mathcal{C}$ be a locally Kan simplicial category. Then the functor $\mathscr{H}_{\mathcal{C}}$ of Construction 8.3.6.1 is a* Hom*-functor for the ∞-category $\mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C})$.*

03N8 **Remark 8.3.6.3.** Let $\mathcal{C}$ be an ordinary category, which we identify with the corresponding constant simplicial category (see Example 2.4.2.4). In this case, Proposition 8.3.6.2 reduces to Example 8.3.3.4.

03N9 **Remark 8.3.6.4.** By combining Proposition 8.3.6.2 with the rectification results of §[?], we can give an explicit construction of a Hom-functor for an arbitrary (small) ∞-category $\mathcal{E}$. Let $\mathrm{Path}[\mathcal{E}]_{\bullet}$ denote the simplicial path category of $\mathcal{E}$ (Definition 2.4.4.1) and let $\mathcal{C}$ be the locally Kan simplicial having the same objects, with morphism spaces given by $\mathrm{Hom}_{\mathcal{C}}(X,Y)_{\bullet} = \mathrm{Ex}^{\infty}(\mathrm{Hom}_{\mathrm{Path}[\mathcal{E}]}(X,Y)_{\bullet})$ (see Example [?]). It follows from Proposition 3.3.6.7 that the tautological map $\mathrm{Path}[\mathcal{E}]_{\bullet} \to \mathcal{C}$ is a weak equivalence of simplicial categories (in the sense of Definition 4.6.8.7), and therefore corresponds to an equivalence of ∞-categories $F : \mathcal{E} \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ (Theorem [?]). Using Proposition 8.3.6.2, we deduce that the composition

$$\mathcal{E}^{\mathrm{op}} \times \mathcal{E} \xrightarrow{F^{\mathrm{op}} \times F} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})^{\mathrm{op}} \times \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \xrightarrow{\mathscr{H}_{\mathcal{C}}} \mathcal{S}$$

is a Hom-functor for $\mathcal{E}$, given on objects by $(X, Y) \mapsto \mathrm{Ex}^\infty(\mathrm{Hom}_{\mathrm{Path}[\mathcal{E}]}(X, Y))$.

Beware that, although this construction is completely explicit in principle, it is hard to use in practice (since the operations $\mathcal{E} \mapsto \mathrm{Path}[\mathcal{E}]_\bullet$ and $S \mapsto \mathrm{Ex}^\infty(S)$ are both difficult to control).

Proposition 8.3.6.2 asserts that the functor $\mathscr{H}_{\mathcal{C}}$ is a covariant transport representation for the left fibration $\mathrm{Tw}(\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})^{\mathrm{op}} \times \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ (Remark 8.3.5.3). We will prove this by constructing a categorical pullback square of ∞-categories

$$\begin{tikzcd}
\mathrm{Tw}(\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})) \arrow[r, "\widetilde{\mathscr{H}_{\mathcal{C}}}"] \arrow[d] & \mathcal{S}_* \arrow[d, "U"] \\
\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})^{\mathrm{op}} \times \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \arrow[r, "\mathscr{H}_{\mathcal{C}}"] & \mathcal{S}.
\end{tikzcd}$$

To define the upper horizontal map, we will use a variant of Construction 8.3.6.1.

Construction 8.3.6.5. Let $\mathcal{C}$ be a locally Kan simplicial category, let $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})$ denote its homotopy coherent nerve. Let J be a linearly ordered set and let $\overline{J}$ denote its opposite; for each element $j \in J$, we write $\overline{j}$ for the corresponding element of $\overline{J}$. Suppose we are given a morphism of simplicial sets $\sigma : \mathrm{N}_\bullet(J) \to \mathrm{Tw}(\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}))$, which we identify with a simplicial functor $f : \mathrm{Path}[\overline{J} \star J]_\bullet \to \mathcal{C}$ (see Warning 8.1.1.9 and Proposition 2.4.4.15). Note that the composition `03NA`

$$\mathrm{N}_\bullet(J) \xrightarrow{\sigma} \mathrm{Tw}(\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})^{\mathrm{op}} \times \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \xrightarrow{\mathscr{H}_{\mathcal{C}}} \mathcal{S}$$

can be identified with a simplicial functor $F_\sigma : \mathrm{Path}[J]_\bullet \to \mathrm{Kan}$, given on objects by the formula $F_\sigma(j) = \mathrm{Hom}_{\mathcal{C}}(f(\overline{j}), f(j))_\bullet$ (see Proposition 2.4.4.15). Let $J^{\triangleleft} = \{x\} \star J$ denote the linearly ordered set obtained from J by adding a new smallest element x. We extend F_σ to a simplicial functor $\widetilde{F}_\sigma : \mathrm{Path}[J^{\triangleleft}]_\bullet \to \mathrm{Kan}$ as follows:

(a) The functor $\widetilde{F}_\sigma$ carries the element $x \in J^{\triangleleft}$ to the Kan complex Δ^0.

(b) Let j be an element of J. Let us identify $\mathrm{Hom}_{\mathrm{Path}[J^{\triangleleft}]}(x, j)_\bullet$ with the nerve $\mathrm{N}_\bullet(Q)$, where Q is the collection of finite subsets $I \subseteq J$ satisfying $\max(I) = j$ (partially ordered by reverse inclusion). Similarly, we identify $\mathrm{Hom}_{\mathrm{Path}[\overline{J} \star J]}(\overline{j}, j)_\bullet$ with the nerve $\mathrm{N}_\bullet(Q')$, where Q' is the collection of finite subsets $I' \subseteq \overline{J} \star J$ satisfying $\max(I') = j$

and $\min(I') = \overline{j}$ (partially ordered by reverse inclusion). Then $\widetilde{F}_\sigma$ is defined on the morphism space $\mathrm{Hom}_{\mathrm{Path}[J^{\triangleleft}]}(x, j)_\bullet$ by the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Path}[J^{\triangleleft}]}(x, j)_\bullet &\simeq \mathrm{N}_\bullet(Q) \\
&\xrightarrow{I \mapsto \overline{I} \cup I} \mathrm{N}_\bullet(Q') \\
&\simeq \mathrm{Hom}_{\mathrm{Path}[\overline{J} \star J]}(\overline{j}, j)_\bullet \\
&\xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(f(\overline{j}), f(j))_\bullet \\
&\simeq \mathrm{Fun}(\widetilde{F}_\sigma(x), \widetilde{F}_\sigma(j)).
\end{aligned}$$

In the special case where J is the linearly ordered set $[n] = \{0 < 1 < \cdots < n\}$, we can identify $\widetilde{F}_\sigma$ with an n-simplex of the ∞-category of pointed spaces $\mathcal{S}_* = \mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Kan})_{\Delta^0/}$. The assignment $\sigma \mapsto \widetilde{F}_\sigma$ depends functorially on $[n]$, and therefore determines a functor $\widetilde{\mathscr{H}}_{\mathcal{C}} : \mathrm{Tw}(\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})) \to \mathcal{S}_*$. By construction, this functor fits into a commutative diagram

03NB
$$\begin{tikzcd}
\mathrm{Tw}(\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})) \arrow[r, "\widetilde{\mathscr{H}}_{\mathcal{C}}"] \arrow[d] & \mathcal{S}_* \arrow[d, "U"] \\
\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C})^{\mathrm{op}} \times \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}) \arrow[r, "\mathscr{H}_{\mathcal{C}}"] & \mathcal{S},
\end{tikzcd} \tag{8.49}$$

where the left vertical map is the twisted arrow fibration of Proposition 8.1.1.11 and the right vertical map is the forgetful functor.

03NC **Exercise 8.3.6.6.** Verify that Construction 8.3.6.5 is well-defined. That is, for every linearly ordered set J and every morphism $\sigma : \mathrm{N}_\bullet(J) \to \mathrm{Tw}(\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{C}))$, show that the simplicial functor F_σ admits a unique extension $\widetilde{F}_\sigma : \mathrm{Path}[J^{\triangleleft}]_\bullet \to \mathrm{Kan}$ which satisfies conditions (a) and (b).

Proposition 8.3.6.2 is an immediate consequence of the following more precise result:

03ND **Proposition 8.3.6.7.** *Let $\mathcal{C}$ be a locally Kan simplicial category. Then the diagram (8.49) is a categorical pullback square.*

Proof. Note that the vertical maps in the diagram (8.49) are left fibrations (Propositions 8.1.1.11 and 5.5.3.2). It will therefore suffice to show that, for every pair of objects $X, Y \in \mathcal{C}$, the induced map of fibers

$$\begin{aligned}
\widetilde{\mathscr{H}}_{X,Y} : \{X\} \times_{\mathcal{E}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{E}) \times_{\mathcal{E}} \{Y\} &\to \{\mathscr{H}_{\mathcal{C}}(X, Y)\} \times_{\mathcal{S}} \mathcal{S}_* \\
&= \mathrm{Hom}^{\mathrm{L}}_{\mathcal{S}}(\Delta^0, \mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet)
\end{aligned}$$

is a homotopy equivalence of Kan complexes (see Corollary 5.1.7.16). Note that the coslice inclusion of Construction 8.1.2.7 induces a monomorphism of simplicial sets $\iota : \operatorname{Hom}^{\mathrm{L}}_{\mathcal{E}}(X,Y) \hookrightarrow \{X\} \times_{\mathcal{E}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{E}) \times_{\mathcal{E}} \{Y\}$. Unwinding the definitions, we see that the composite map

$$(\widetilde{\mathscr{H}}_{X,Y} \circ \iota) : \operatorname{Hom}^{\mathrm{L}}_{\mathcal{E}}(X,Y) \to \operatorname{Hom}^{\mathrm{L}}_{\mathcal{S}}(\Delta^0, \operatorname{Hom}_{\mathcal{C}}(X,Y)_{\bullet})$$

coincides with isomorphism described in Remark 4.6.8.18. It will therefore suffice to show that ι is a homotopy equivalence, which is a special case of Corollary 8.1.2.10. □

8.4 Cocompletion

Let $\mathcal{C}$ be a small ∞-category. It is very rare for $\mathcal{C}$ to admit small colimits: this is possible only if $\mathcal{C}$ is (equivalent to the nerve of) a partially ordered set (Proposition 7.6.2.16). However, it is always possible to *embed* $\mathcal{C}$ into a larger ∞-category which admits small colimits. Our goal in this section is to study the universal example of such an enlargement. 03V7

Definition 8.4.0.1. Let $h : \mathcal{C} \to \widehat{\mathcal{C}}$ be a functor of ∞-categories. We will say that h *exhibits $\widehat{\mathcal{C}}$ as a cocompletion of $\mathcal{C}$* if the following conditions are satisfied: 04B8

(1) The ∞-category $\widehat{\mathcal{C}}$ admits small colimits.

(2) Let $\mathcal{D}$ be an ∞-category which admits small colimits and let $\operatorname{Fun}'(\widehat{\mathcal{C}}, \mathcal{D})$ denote the full subcategory of $\operatorname{Fun}(\widehat{\mathcal{C}}, \mathcal{D})$ spanned by those functors which preserve small colimits. Then precomposition with h induces an equivalence of ∞-categories $\operatorname{Fun}'(\widehat{\mathcal{C}}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$.

Remark 8.4.0.2. Stated more informally, condition (2) of Definition 8.4.0.1 asserts that if $f : \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories where $\mathcal{D}$ admits small colimits, then f factors (up to isomorphism) as a composition $\mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}} \xrightarrow{F} \mathcal{D}$, where the functor F preserves small colimits; moreover, this factorization is required to be essentially unique. In other words, the ∞-category $\widehat{\mathcal{C}}$ should be "freely generated" by $\mathcal{C}$ under small colimits. 04B9

It follows immediately from the definition that if an ∞-category $\mathcal{C}$ admits a cocompletion $\widehat{\mathcal{C}}$, then $\widehat{\mathcal{C}}$ is determined uniquely up to equivalence. Our primary goal in this section is to prove the following existence result:

Theorem 8.4.0.3. *Let $\mathcal{C}$ be an essentially small ∞-category and let $h_{\bullet} : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ be a covariant Yoneda embedding for $\mathcal{C}$ (Definition 8.3.3.9). Then $h_{\bullet}$ exhibit $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ as a cocompletion of $\mathcal{C}$ (in the sense of Definition 8.4.0.1).* 03W9

Example 8.4.0.4. Let X be a contractible Kan complex, which we identify with a vertex x of the simplicial set $\mathcal{S}$. Applying Theorem 8.4.0.3 in the special case where $\mathcal{C} = \Delta^0$, we 03WC deduce that the map $x : \Delta^0 \to \mathcal{S}$ exhibits $\mathcal{S}$ as a cocompletion of the 0-simplex Δ^0. That is, for every ∞-category $\mathcal{D}$ which admits small colimits, the evaluation map

$$\operatorname{Fun}'(\mathcal{S}, \mathcal{D}) \to \mathcal{D} \qquad F \mapsto F(X)$$

is an equivalence of ∞-categories, where $\operatorname{Fun}'(\mathcal{S}, \mathcal{D})$ denotes the full subcategory of $\operatorname{Fun}(\mathcal{S}, \mathcal{D})$ spanned by the colimit-preserving functors. Note that this property characterizes the ∞-category $\mathcal{S}$ up to equivalence: it is "freely generated" under small colimits by the object Δ^0.

04BA **Warning 8.4.0.5.** In §8.4.5, we will show that *every* ∞-category $\mathcal{C}$ admits a cocompletion $\widehat{\mathcal{C}}$ (Proposition 8.4.5.3). Beware that, if $\mathcal{C}$ is not essentially small, then $\widehat{\mathcal{C}}$ cannot necessarily be identified with the ∞-category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ (Warning 8.4.3.4). However, if $\mathcal{C}$ is *locally* small, then it can be identified with a full subcategory of $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ (see Construction 8.4.5.5).

Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, where $\mathcal{C}$ is essentially small and $\mathcal{D}$ admits small colimits. Using the ∞-categorical version of Yoneda's lemma (Theorem 8.3.3.13), we see that f factors (up to isomorphism) as a composition

$$\mathcal{C} \xrightarrow{h_\bullet} \operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{F_0} \mathcal{D},$$

where $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denotes the full subcategory of $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by the representable functors. Theorem 8.4.0.3 asserts that F_0 admits an essentially unique extension to a functor $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$ which preserves small colimits. To prove this, it will be useful to characterize this extension in a different way. For any functor $F : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$, we will show that the following conditions are equivalent:

(a) The functor F preserves small colimits.

(b) The functor F is left Kan extended from the subcategory $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subseteq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.

Granting this equivalence, the proof of Theorem 8.4.0.3 is reduced to showing that every functor $F_0 : \operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$ admits an essentially unique left Kan extension $F : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$, which follows from the general results of §7.3.

To establish the equivalence of (a) and (b), we will proceed by reduction to an important special case. Suppose that $\mathcal{D} = \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and that $F : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$ is the identity functor. In this case, condition (a) is automatically satisfied. Condition (b) then asserts that the full subcategory $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subseteq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is *dense*: that is, every object $\mathscr{G} \in \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ can be recovered as the colimit of the diagram

$$\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}\,\mathrm{op}, \mathcal{S}) \times_{\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})} \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

of representable functors over $\mathscr{G}$ (see Definition 8.4.1.5). In §8.4.1, we discuss dense subcategories in general and provide a concrete criterion can be used to show that a subcategory is dense (Proposition 8.4.1.8). In §8.4.2, we apply this criterion to establish the density of the full subcategory $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ (Corollary 8.4.2.2). In §8.4.3, we use this result to establish the equivalence of (a) and (b) in general (Theorem 8.4.3.6), and deduce Theorem 8.4.0.3 as an easy consequence.

Let us now specialize the preceding discussion to the situation where the ∞-category $\mathcal{D}$ is locally small. In this case, we will show that conditions (a) and (b) above are equivalent to the following:

(c) The functor $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$ admits a right adjoint $G : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.

The implication $(c) \Rightarrow (a)$ is formal (by virtue of Corollary 7.1.3.21, every left adjoint preserves colimits). In §8.4.4, we prove the reverse implication by giving an explicit construction of the right adjoint G: it carries each object $D \in \mathcal{D}$ to the functor

$$\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S} \qquad C \mapsto \mathrm{Hom}_{\mathcal{D}}(f(C), D)$$

where $\mathrm{Hom}_{\mathcal{D}}(\bullet, \bullet)$ is a Hom-functor for the ∞-category $\mathcal{D}$ (see Proposition 8.4.4.1). The equivalence of (a) and (c) is a special case of the ∞-categorical adjoint functor theorem (Theorem [?]), which we discuss in §[?] .

If $\mathcal{C}$ is an essentially small ∞-category, then its cocompletion $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is a drastic enlargement of $\mathcal{C}$, obtained by (freely) adjoining a colimit for *every* small diagram. In practice, it will be useful to consider a variant of Definition 8.4.0.1, where we restrict our attention to diagrams indexed by some collection of simplicial sets $\mathbb{K}$. We say that a functor of ∞-categories $h : \mathcal{C} \to \widehat{\mathcal{C}}$ *exhibits* $\widehat{\mathcal{C}}$ *as a* $\mathbb{K}$*-cocompletion of* $\mathcal{C}$ if $\widehat{\mathcal{C}}$ admits K-indexed colimits for each $K \in \mathbb{K}$, and is universal with respect to this property (see Definition 8.4.5.1). In §8.4.5, we show that every ∞-category $\mathcal{C}$ admits a $\mathbb{K}$-cocompletion $\widehat{\mathcal{C}}$ (Proposition 8.4.5.3). Our proof proceeds by explicit construction. Assume for simplicity that the ∞-category $\mathcal{C}$ and each of the simplicial sets $K \in \mathbb{K}$ is essentially small; in this case, we show that we can take $\widehat{\mathcal{C}}$ to be the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ which contains all representable functors and is closed under K-indexed colimits, for each $K \in \mathbb{K}$ (Construction 8.4.5.5 and Proposition 8.4.5.7).

Let us isolate another important feature of the covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ associated to an essentially small ∞-category $\mathcal{C}$. For every pair of objects $X \in \mathcal{C}$ and $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, the ∞-categorical analogue of Yoneda's lemma supplies a homotopy equivalence

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(h_X, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(X)$$

(Proposition 8.3.1.3). It follows that h_X is an *atomic* object of the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$: that is, it corepresents a functor which preserves small colimits (Definition 8.4.6.1). The Yoneda embedding is essentially characterized by this property, together with the fact that it is dense and fully faithful. More precisely, suppose that we are given a functor of

∞-categories $h : \mathcal{C} \to \widehat{\mathcal{C}}$, where $\mathcal{C}$ is small and $\widehat{\mathcal{C}}$ admits small colimits. In §8.4.6, we show that h exhibits $\widehat{\mathcal{C}}$ as a cocompletion of $\mathcal{C}$ if and only if it is dense, fully faithful, and carries each object of $\mathcal{C}$ to an atomic object of $\widehat{\mathcal{C}}$ (Proposition 8.4.6.6). In §8.4.7, we apply this characterization to show that the formation of cocompletions is compatible with the formation of slice ∞-categories (Proposition 8.4.7.1). In particular, if $U : \widetilde{\mathcal{C}} \to \mathcal{C}$ is a right fibration between essentially small ∞-categories, we show that there is an equivalence of ∞-categories $\mathrm{Fun}(\widetilde{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}}$, where $\mathscr{F}$ denotes a covariant transport representation for the left fibration U^{op} (Corollary 8.4.7.2).

8.4.1 Dense Functors

03V8 To study the behavior of a (large) category $\mathcal{D}$, it is often useful to approximate $\mathcal{D}$ by well-chosen (small) subcategory $\mathcal{C} \subseteq \mathcal{D}$. The following condition guarantees that, for some purposes, passage from $\mathcal{D}$ to $\mathcal{C}$ does not lose too much information:

03V9 **Definition 8.4.1.1.** Let $\mathcal{D}$ be a (locally small) category. We say that a full subcategory $\mathcal{C} \subseteq \mathcal{D}$ is *dense* if the functor

$$\mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \qquad Y \mapsto \mathrm{Hom}_{\mathcal{D}}(\bullet, Y)$$

is fully faithful.

03VA **Remark 8.4.1.2.** Definition 8.4.1.1 was introduced by Isbell in [29]. Beware that Isbell uses the term *left adequate* subcategory for what we refer to as a *dense* subcategory.

03VB **Example 8.4.1.3.** Let Cat denote the ordinary category whose objects are small categories and whose morphisms are functors, and let $\mathbf{\Delta} \subset \mathrm{Cat}$ be the simplex category. Proposition 1.3.3.1 asserts that the restricted Yoneda embedding

$$\mathrm{Cat} \to \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set}) = \mathrm{Set}_{\Delta} \qquad \mathcal{C} \mapsto \mathrm{N}_{\bullet}(\mathcal{C})$$

is fully faithful, so that $\mathbf{\Delta}$ is a dense subcategory of Cat.

03VC **Exercise 8.4.1.4.** Let $\mathcal{C}$ denote the category of partially ordered sets, and let $\mathbf{\Delta}_{\leq 1}$ denote the full subcategory of $\mathcal{C}$ spanned by the objects $[0]$ and $[1]$. Show that $\mathbf{\Delta}_{\leq 1}$ is a dense subcategory of $\mathcal{C}$.

We now introduce an ∞-categorical counterpart of Definition 8.4.1.1.

03VD **Definition 8.4.1.5.** Let $\mathcal{D}$ be an ∞-category. We will say that a full subcategory $\mathcal{C} \subseteq \mathcal{D}$ is *dense* if, for every object $X \in \mathcal{D}$, the composition

$$(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/X})^{\triangleright} \hookrightarrow \mathcal{D}_{/X}^{\triangleright} \to \mathcal{D}$$

is a colimit diagram.

Remark 8.4.1.6. Let $\mathcal{D}$ be an ∞-category. Then a full subcategory $\mathcal{C} \subseteq \mathcal{D}$ is dense if and only if the identity functor $\mathrm{id}_{\mathcal{D}}$ is left Kan extended from $\mathcal{C}$. 03VE

Example 8.4.1.7. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is a dense full subcategory of itself (see Example 7.3.3.8). 03VF

In the situation of Definition 8.4.1.5, suppose that the ∞-category $\mathcal{D}$ is locally small, and let $h_{\bullet} : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})$ be a covariant Yoneda embedding for $\mathcal{D}$ (see Definition 8.3.3.9). Composing with the restriction functor $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, we obtain a functor

$$\mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \qquad Y \mapsto h_Y^{\circ}$$

which we will refer to as the *restricted Yoneda embedding.*

Proposition 8.4.1.8. *Let $\mathcal{D}$ be a locally small ∞-category. A full subcategory $\mathcal{C} \subseteq \mathcal{D}$ is dense if and only if the restricted Yoneda embedding $\mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful.* 03VG

We will deduce Proposition 8.4.1.8 from a more general result (Proposition 8.4.1.22), which we prove at the end of this section.

Corollary 8.4.1.9. *Let $\mathcal{D}$ be a locally small category. Then a full subcategory $\mathcal{C} \subseteq \mathcal{D}$ is dense (in the sense of Definition 8.4.1.1) if and only if $\mathrm{N}_{\bullet}(\mathcal{C})$ is a dense subcategory of the ∞-category $\mathrm{N}_{\bullet}(\mathcal{D})$ (in the sense of Definition 8.4.1.5).* 03VH

Warning 8.4.1.10. Let $\mathcal{D}$ be an ∞-category. Consider the following conditions on a full subcategory $\mathcal{C} \subseteq \mathcal{D}$: 03VJ

(1) The ∞-category $\mathcal{C} \subseteq \mathcal{D}$ is dense, in the sense of Definition 8.4.1.5.

(2) Every object $X \in \mathcal{D}$ can be realized as the colimit of a diagram taking values in the full subcategory $\mathcal{C} \subseteq \mathcal{D}$.

(3) The ∞-category $\mathcal{D}$ is generated by $\mathcal{C}$ under colimits. That is, if $\mathcal{D}_0 \subseteq \mathcal{D}$ is a full subcategory which contains $\mathcal{C}$ and is closed under the formation of colimits in $\mathcal{D}$, then $\mathcal{D}_0 = \mathcal{D}$.

It follows immediately from the definitions that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Beware that neither of this implications is reversible. See Exercises 8.4.1.11 and 8.4.1.12.

Exercise 8.4.1.11. Let $\mathcal{D}$ denote the category of free abelian groups, and let $\mathcal{C} \subseteq \mathcal{D}$ denote the full subcategory spanned by object $\mathbf{Z}$. Show that $\mathcal{C}$ is not a dense subcategory of $\mathcal{D}$. Consequently, the inclusion map $\mathrm{N}_{\bullet}(\mathcal{C}) \subset \mathrm{N}_{\bullet}(\mathcal{D})$ satisfies condition (2) of Warning 8.4.1.10, but does not satisfy condition (1). 03VK

03VL **Exercise 8.4.1.12.** Let Cat denote the (ordinary) category of small categories, and let $\mathbf{\Delta}_{\leq 1} \subset \mathrm{Cat}$ denote the full subcategory spanned by the objects $[0]$ and $[1]$. Show that:

- The full subcategory $\mathbf{\Delta}_{\leq 1}$ generates Cat under colimits.
- A small category $\mathcal{C}$ can be realized as the colimit (in Cat) of a diagram $\mathcal{K} \to \mathbf{\Delta}_{\leq 1}$ if and only if the category $\mathcal{C}$ is free, in the sense of Definition 1.3.7.7.

In particular, the inclusion $\mathrm{N}_{\bullet}(\mathbf{\Delta}_{\leq 1}) \subset \mathrm{N}_{\bullet}(\mathrm{Cat})$ satisfies condition (3) of Warning 8.4.1.10, but does not satisfy condition (2).

03VM **Warning 8.4.1.13** (Failure of Transitivity)**.** Let $\mathcal{E}$ be an ∞-category and let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{E}$ be full subcategories. Suppose that $\mathcal{C}$ is a dense subcategory of $\mathcal{E}$. Then $\mathcal{C}$ is also a dense subcategory of $\mathcal{D}$, and $\mathcal{D}$ is a dense subcategory of $\mathcal{E}$ (see Corollary 7.3.8.8). Beware that the converse is false (Example 8.4.1.14).

03VN **Example 8.4.1.14.** Let Cat denote the (ordinary) category of small categories. Then the simplex category $\mathbf{\Delta}$ is a dense full subcategory of Cat (Example 8.4.1.3), and $\mathbf{\Delta}_{\leq 1}$ is a dense full subcategory of $\mathbf{\Delta}$ (Exercise 8.4.1.4). However, $\mathbf{\Delta}_{\leq 1}$ is not a dense full subcategory of Cat (Exercise 8.4.1.12).

For some applications, it will be useful to consider the following generalization of Definition 8.4.1.5.

03VP **Definition 8.4.1.15.** Let $\mathcal{D}$ be an ∞-category and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. We say that F is *dense* if the identity transformation $\mathrm{id}_F : F \to \mathrm{id}_{\mathcal{D}} \circ F$ exhibits the identity functor $\mathrm{id}_{\mathcal{D}}$ as a left Kan extension of F along F (see Variant 7.3.1.5).

03VQ **Example 8.4.1.16.** Let $\mathcal{D}$ be an ∞-category. Then a full subcategory $\mathcal{C} \subseteq \mathcal{D}$ is dense (in the sense of Definition 8.4.1.5) if and only if the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{D}$ is dense (in the sense of Definition 8.4.1.15). See Proposition 7.3.2.6.

03VS **Remark 8.4.1.17** (Homotopy Invariance)**.** Let $\mathcal{D}$ be an ∞-category, let C be a simplicial set, and let $F, F' : \mathcal{C} \to \mathcal{D}$ be diagrams which are isomorphic (when viewed as objects of the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$). Then F is dense if and only if F' is dense. This follows by combining Remarks 7.3.1.10 and 7.3.1.11.

03VT **Remark 8.4.1.18** (Change of Source)**.** Let $\mathcal{D}$ be an ∞-category, let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, and let $G : \mathcal{B} \to \mathcal{C}$ be a categorical equivalence of simplicial sets. Then F is dense if and only if $F \circ G$ is dense. See Proposition 7.3.1.14.

03VU **Remark 8.4.1.19** (Change of Target)**.** Let $G : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then:

- If G is fully faithful and $G \circ F$ is dense, then F is dense.
- If G is an equivalence of ∞-categories and F is dense, then $G \circ F$ is dense.

See Remark 7.3.1.13.

Remark 8.4.1.20. Let $\mathcal{D}$ be an ∞-category and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then f is dense if and only if, for every object $Y \in \mathcal{D}$, the composite map 03VV

$$(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/Y})^{\triangleright} \to \mathcal{D}_{/Y}^{\triangleright} \to \mathcal{D}$$

is a colimit diagram in $\mathcal{D}$.

Remark 8.4.1.21. Let κ be an uncountable regular cardinal and let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Assume that $\mathcal{C}$ is essentially κ-small and that $\mathcal{D}$ is a locally κ-small ∞-category. Then, for each object $X \in \mathcal{D}$, the fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/X}$ is also essentially κ-small (Corollary 5.6.7.7). Let λ be an infinite cardinal satisfying $\mathrm{ecf}(\lambda) \geq \kappa$ (see Definition 4.7.3.16). Then F is dense if and only if, for every representable functor $h_Y : \mathcal{D}^{\mathrm{op}} \to \mathcal{S}^{<\lambda}$, the identity transformation $\mathrm{id} : h_Y \circ F^{\mathrm{op}} \to h_Y \circ F^{\mathrm{op}}$ exhibits the functor h_Y as a right Kan extension of $h_Y \circ F^{\mathrm{op}}$ along F^{op}. This follows by combining Remark 8.4.1.20 with Proposition 7.4.5.17 (together with Remark 7.4.5.19). 03VW

Proposition 8.4.1.22. *Let λ be an uncountable cardinal, let $\mathcal{D}$ be an ∞-category which is locally λ-small, and let* 03VX

$$h_{\bullet} : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda}) \qquad Y \mapsto h_Y$$

be a covariant Yoneda embedding for $\mathcal{D}$ (Definition 8.3.3.9). If $\mathcal{C}$ is a simplicial set, then a diagram $F : \mathcal{C} \to \mathcal{D}$ is dense if and only if the composite functor

$$\mathcal{D} \xrightarrow{h_{\bullet}} \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda}) \xrightarrow{\circ F^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$$

is fully faithful.

Proof. Choose an uncountable cardinal κ such that $\mathcal{C}$ is essentially κ-small and $\mathcal{D}$ is locally κ-small. Enlarging λ if necessary, we may assume that the exponential cofinality of λ is $\geq \kappa$ (see Remark 4.7.3.19). For each object $Y \in \mathcal{D}$, let $h_Y^{\circ} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\lambda}$ denote the composite functor $h_Y \circ F^{\mathrm{op}}$, given on objects by the construction $C \mapsto \mathrm{Hom}_{\mathcal{D}}(F(C), Y)$. By virtue of Remark 8.4.1.21, it will suffice to show that the following conditions are equivalent:

(1_Y) The identity transformation $\mathrm{id} : h_Y \circ F^{\mathrm{op}} \to h_Y$ exhibits h_Y as a right Kan extension of h_Y° along the functor F^{op}.

(2_Y) For each object $X \in \mathcal{C}$, the composite map

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})}(h_X, h_Y) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(h^{\circ}_X, h^{\circ}_Y)$$

is a homotopy equivalence of Kan complexes.

Since the covariant Yoneda embedding $X \mapsto h_X$ is fully faithful (Theorem 8.3.3.13), we can reformulate (2_Y) as follows:

$(2'_Y)$ The restriction map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda})}(h_X, h_Y) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})}(h^{\circ}_X, h^{\circ}_Y)$$

is a homotopy equivalence of Kan complexes.

The inequality $\kappa \leq \mathrm{ecf}(\lambda)$ guarantees that the ∞-category $\mathcal{S}^{<\lambda}$ admits κ-small limits (Corollary 7.4.1.13). Using Proposition 7.6.7.13, we can choose a functor $\mathscr{G} : \mathcal{D}^{\mathrm{op}} \to \mathcal{S}^{<\lambda}$ and a natural transformation $\alpha : \mathscr{G} \circ F^{\mathrm{op}} \to h^{\circ}_Y$ which exhibits $\mathscr{G}$ as a right Kan extension of h°_Y along the functor F^{op}. Invoking the universal mapping property of $\mathscr{G}$ (Proposition 7.3.6.1), we see that there exists a natural transformation $\beta : h_Y \to \mathscr{G}$ and a commutative diagram

03VY

$$\begin{tikzcd} & \mathscr{G} \circ F^{\mathrm{op}} \arrow[dr, "\alpha"] & \\ h_Y \circ F^{\mathrm{op}} \arrow[ur, "\beta"] \arrow[rr, "\mathrm{id}"] & & h^{\circ}_Y \end{tikzcd} \tag{8.50}$$

in the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$. Using Remark 7.3.1.12, we see that condition (1_Y) is satisfied if and only if the natural transformation β is an isomorphism: that is, it induces a homotopy equivalence of Kan complexes $\beta_X : h_Y(X) \to \mathscr{G}(X)$ for each object $X \in \mathcal{D}$ (Theorem 4.4.4.4). Combining this observation with Proposition 8.3.1.3, we can reformulate (1_Y) as follows:

$(1'_Y)$ For each object $X \in \mathcal{C}$, precomposition with β induces a homotopy equivalence

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda})}(h_X, h_Y) \xrightarrow{\circ[\beta]} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda})}(h_X, \mathscr{G}).$$

Using the commutativity of (8.50), we see that the diagram of Kan complexes

$$\begin{tikzcd} & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda})}(h_X, \mathscr{G}) \arrow[dr] & \\ \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda})}(h_X, h_Y) \arrow[ur, "\circ[\beta]"] \arrow[rr] & & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})}(h^{\circ}_X, h^{\circ}_Y) \end{tikzcd}$$

commutes up to homotopy, where the diagonal map on the right is the homotopy equivalence of Proposition 7.3.6.1. It follows that conditions $(1'_Y)$ and $(2'_Y)$ are equivalent. □

Proof of Proposition 8.4.1.8. Let $\mathcal{D}$ be a locally small ∞-category and let $\mathcal{C} \subseteq \mathcal{D}$ be a full subcategory. By virtue of Example 8.4.1.16, it will suffice to show that the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{D}$ is dense if and only if the restricted Yoneda embedding $\mathcal{D} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful. Following the convention of Remark 4.7.0.5, this is a special case of Proposition 8.4.1.22. □

Proposition 8.4.1.23. *Let $\mathcal{D}$ be an ∞-category, let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, and let $\mathcal{D}_0 \subseteq \mathcal{D}$ be a full subcategory which contains the image of F. If F is dense, then the subcategory $\mathcal{D}_0$ is dense.* `03VZ`

Proof. Using Proposition 4.1.3.2, we can factor F as a composition $\mathcal{C} \xrightarrow{F'} \mathcal{C}' \xrightarrow{F''} \mathcal{D}$, where F' is inner anodyne and F'' is an inner fibration. Using Remark 8.4.1.18, we see that the functor F' is dense. Replacing F by F'', we can reduce to proving Proposition 8.4.1.23 in the special case where $\mathcal{C}$ is an ∞-category.

Let γ denote the identity map id_F, which we regard as a natural transformation from F to $\mathrm{id}_{\mathcal{D}} \circ F$. Our assumption that F is dense guarantees that γ exhibits $\mathrm{id}_{\mathcal{D}}$ as a left Kan extension of F along F. To avoid confusion, let us write F_0 to denote the functor F, regarded as a functor from $\mathcal{C}$ to $\mathcal{D}_0$. Let $\iota : \mathcal{D}_0 \hookrightarrow \mathcal{D}$ denote the inclusion map, so that $F = \iota \circ F_0$. We can therefore also regard the identity map id_F as a natural transformation $\alpha : F \to \iota \circ F_0$. Our assumption that F is dense also guarantees that α exhibits ι as a left Kan extension of F along F_0. Note that $\gamma = \mathrm{id}_F$ is a composition of $\alpha = \mathrm{id}_F$ with $\beta|_{\mathcal{C}}$, where $\beta = \mathrm{id}_{\iota}$ is the identity transformation from ι to itself. Invoking the transitivity of Kan extensions (Proposition 7.3.8.18), we deduce that β exhibits the identity functor $\mathrm{id}_{\mathcal{D}}$ as a left Kan extension of ι along itself: that is, the functor ι is dense. Applying Example 8.4.1.16, we conclude that $\mathcal{D}_0$ is a dense subcategory of $\mathcal{D}$. □

Remark 8.4.1.24. Let $\mathcal{D}$ be an ∞-category and let $F : \mathcal{C} \to \mathcal{D}$ be a dense diagram. Choose another diagram $q : K \to \mathcal{D}$ and set $\widetilde{\mathcal{D}} = \mathcal{D}_{/q}$, so that the projection map $\pi : \widetilde{\mathcal{D}} \to \mathcal{D}$ is a right fibration. Then the projection map $\mathcal{C} \times_{\mathcal{D}} \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ is dense. To prove this, it will suffice to show that for every object $Y \in \widetilde{\mathcal{D}}$, the induced map `03W0`

$$\theta : (\mathcal{C} \times_{\mathcal{D}} \widetilde{\mathcal{D}}_{/Y})^{\triangleright} \to \widetilde{\mathcal{D}}^{\triangleright}_{/Y} \to \widetilde{\mathcal{D}}$$

is a colimit diagram in $\widetilde{\mathcal{D}}$ (Remark 8.4.1.20). By virtue of Proposition 7.1.3.19, this is equivalent to the requirement that $\pi \circ \theta$ is a colimit diagram in $\mathcal{D}$. Unwinding the definitions, we see that $\pi \circ \theta$ is given by the composition

$$(\mathcal{C} \times_{\mathcal{D}} \widetilde{\mathcal{D}}_{/Y})^{\triangleright} \to (\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/\pi(Y)})^{\triangleright} \to \mathcal{D}^{\triangleright}_{/\pi(Y)} \to \mathcal{D}.$$

Since π is a right fibration, the map $\widetilde{\mathcal{D}}_{/Y} \to \mathcal{D}_{/\pi(Y)}$ is a trivial Kan fibration (Proposition 4.3.7.12). Using Corollary 7.2.2.2, we are reduced to showing that the map $(\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/\pi(Y)})^{\triangleright} \to \mathcal{D}^{\triangleright}_{/\pi(Y)} \to \mathcal{D}$ is a colimit diagram, which follows from our assumption that F is dense (Remark 8.4.1.20).

8.4.2 Density of Yoneda Embeddings

03W1 Our goal in this section is to prove the following result, which supplies an important source of examples of dense functors:

03W2 **Theorem 8.4.2.1.** *Let $\mathcal{C}$ be a locally small ∞-category, and let $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ be a covariant Yoneda embedding (Definition 8.3.3.9). Then $h_{\bullet}$ is a dense functor.*

Since the covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful, Theorem 8.4.2.1 can be reformulated as follows:

03W3 **Corollary 8.4.2.2.** *Let $\mathcal{C}$ be a locally small ∞-category and let $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denote the full subcategory spanned by the representable functors. Then $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is a dense subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.*

Proof. By virtue of Example 8.4.1.16, it will suffice to show that the inclusion map $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is a dense functor. Since the covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is an equivalence of ∞-categories (Theorem 8.3.3.13), this is equivalent to the assertion that $h_{\bullet}$ is a dense functor from $\mathcal{C}$ to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ (Remark 8.4.1.18), which follows from Theorem 8.4.2.1. □

03W4 **Example 8.4.2.3.** Let $\mathcal{S}_{\mathrm{cont}}$ denote the full subcategory of $\mathcal{S}$ spanned by the contractible Kan complexes. Then $\mathcal{S}_{\mathrm{cont}}$ is a dense subcategory of $\mathcal{S}$. This follows by applying Corollary 8.4.2.2 in the special case $\mathcal{C} = \Delta^0$. Moreover, the same assertion holds if we replace $\mathcal{S}_{\mathrm{cont}}$ by any nonempty subcategory of itself; for example, the full subcategory of $\mathcal{S}$ spanned by the standard 0-simplex Δ^0.

By virtue of the convention of Remark 4.7.0.5, Theorem 8.4.2.1 can be regarded as a special case of the following:

03W5 **Variant 8.4.2.4.** Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is locally κ-small. Then the covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is a dense functor.

We will deduce Variant 8.4.2.4 from a more precise result. Recall that, if X is an object of a (locally small) ∞-category $\mathcal{C}$, then the representable functor $h_X \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ corepresents the evaluation functor $\mathrm{ev}_X : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$ (Remark 8.3.1.5). That is, for every functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, there is a canonical homotopy equivalence

$$\mathscr{F}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(h_X, \mathscr{F}),$$

which depends functorially on $\mathscr{F}$. The following result guarantees that this homotopy equivalence can also be chosen to depend functorially on X:

Proposition 8.4.2.5. *Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is locally κ-small. Then the profunctor* 03W6

$$\mathrm{ev} : \mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathcal{S}^{<\kappa} \qquad (X, \mathscr{F}) \mapsto \mathscr{F}(X)$$

is corepresentable by the covariant Yoneda embedding $h_\bullet : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.

Proof. Let $\mathrm{Tw}(\mathcal{C})$ denote the twisted arrow ∞-category of $\mathcal{C}$, let $\lambda : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ be the left fibration of Proposition 8.1.1.11, and let $\underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})}$ denote the constant functor $\mathrm{Tw}(\mathcal{C}) \to \mathcal{S}^{<\kappa}$. Let $\mathscr{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}^{<\kappa}$ denote the composition $\mathrm{ev} \circ (\mathrm{id} \times h_\bullet)$, so that $\mathscr{H}$ is a Hom-functor for $\mathcal{C}$. We can therefore choose a natural transformation

$$\alpha : \underline{\Delta^0}_{\mathrm{Tw}(\mathcal{C})} \to \mathscr{H} \circ \lambda = \mathrm{ev} \circ (\mathrm{id} \times h_\bullet) \circ \lambda$$

which exhibits $\mathscr{H}$ as a Hom-functor for $\mathcal{C}$, in the sense of Definition 8.3.5.1. By virtue of Proposition 8.3.4.15, it will suffice to show that the natural transformation α also exhibits the profunctor ev as corepresented by the functor $h_\bullet$, in the sense of Variant 8.3.4.16. Fix an object $X \in \mathcal{C}$, so that α carries the object $\mathrm{id}_X \in \mathrm{Tw}(\mathcal{C})$ to a vertex $\eta \in \mathscr{H}(X, X) = \mathrm{ev}(X, h_X)$. We wish to show that η exhibits evaluation functor $\mathrm{ev}_X : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathcal{S}^{<\kappa}$ as corepresented by h_X. This follows from Proposition 8.3.1.3, since η exhibits the functor h_X as represented by X. □

Example 8.4.2.6. Let $\mathcal{C}$ be a locally small category. Then the evaluation profunctor 03W7

$$\mathrm{ev} : \mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S} \qquad (X, \mathscr{F}) \mapsto \mathscr{F}(X)$$

is corepresentable by the covariant Yoneda embedding $h_\bullet : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.

Proof of Variant 8.4.2.4. Let κ be an uncountable cardinal and let $\mathcal{C}$ be an ∞-category which is locally κ-small. We wish to show that the covariant Yoneda embedding $h^{\mathcal{C}}_\bullet : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is dense. Choose a cardinal $\lambda \geq \kappa$ for which the ∞-category $\mathcal{D} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is locally λ-small, and let $h^{\mathcal{D}}_\bullet : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$ be a covariant Yoneda embedding for $\mathcal{D}$. By virtue of Proposition 8.4.1.22, it will suffice to show that the composite functor

$$\mathcal{D} \xrightarrow{h^{\mathcal{D}}_\bullet} \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\lambda}) \xrightarrow{\circ h^{\mathcal{C}}_\bullet} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$$

is fully faithful. Applying Proposition 8.4.2.5, we see that this functor is isomorphic to the inclusion of $\mathcal{D} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ as a full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$. □

Using Proposition 8.4.2.5, we can also give an alternative characterization of the covariant transport representation associated to a left fibration of ∞-categories.

04BB **Corollary 8.4.2.7.** *Let $\mathcal{C}$ be a locally κ-small ∞-category, let $U : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a right fibration, and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\kappa}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\mathscr{F}$ is a covariant transport representation for the left fibration $U^{\mathrm{op}} : \widetilde{\mathcal{C}}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.*

(2) *There exists a categorical pullback square*

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[r] \arrow[d, "U"] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}} \arrow[d] \\
\mathcal{C} \arrow[r, "h_\bullet"] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}),
\end{tikzcd}$$

where $h_\bullet$ is a covariant Yoneda embedding for $\mathcal{C}$.

Proof. Choose a cardinal $\lambda \geq \kappa$ such that $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is locally λ-small, and let $H : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})^{\mathrm{op}} \to \mathcal{S}^{<\lambda}$ be a functor represented by $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. It follows from Proposition 5.6.6.21 that H is a covariant transport representation for the left fibration $(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})^{\mathrm{op}}$. Consequently, if condition (2) is satisfied, then $H \circ h_\bullet^{\mathrm{op}}$ is a covariant transport representation for the left fibration U^{op}. Proposition 8.4.2.5 implies that $H \circ h_\bullet^{\mathrm{op}}$ is isomorphic to $\mathscr{F}$. This proves the implication (2) $\Rightarrow$ (1), and the reverse implication follows from the fact that the equivalence class of a left fibration is determined by its covariant transport representation (Corollary 5.6.0.6). $\square$

8.4.3 Cocompletion via the Yoneda Embedding

03W8 Let $\mathcal{C}$ be an essentially small ∞-category. Our goal in this section is to prove Theorem 8.4.0.3, which asserts that the covariant Yoneda embedding $h_\bullet : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ exhibits $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ as a cocompletion of $\mathcal{C}$. We begin by formulating a slightly more general assertion.

04BC **Notation 8.4.3.1.** Let κ be an uncountable regular cardinal. If $\mathcal{C}$ and $\mathcal{D}$ are κ-cocomplete ∞-categories, we let $\mathrm{Fun}^{\kappa}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors $F : \mathcal{C} \to \mathcal{D}$ which preserve κ-small colimits.

04BD **Definition 8.4.3.2.** Let κ be an uncountable regular cardinal. We say that a functor of ∞-categories $h : \mathcal{C} \to \widehat{\mathcal{C}}$ *exhibits $\widehat{\mathcal{C}}$ as a κ-cocompletion of* $\mathcal{C}$ if the following conditions are satisfied:

(1) The ∞-category $\widehat{\mathcal{C}}$ is κ-cocomplete.

(2) For every κ-cocomplete ∞-category $\mathcal{D}$, precomposition with h induces an equivalence of ∞-categories $\mathrm{Fun}^{\kappa}(\widehat{\mathcal{C}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Following the convention of Remark 4.7.0.5, we can regard Theorem 8.4.0.3 as a special case of the following more general assertion:

Theorem 8.4.3.3. *Let κ be an uncountable regular cardinal, let $\mathcal{C}$ be an ∞-category which is essentially κ-small, and let $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ be a covariant Yoneda embedding for $\mathcal{C}$. Then $h_{\bullet}$ exhibits* $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ *as a κ-cocompletion of $\mathcal{C}$.* `04BE`

Warning 8.4.3.4. The conclusion of Theorem 8.4.3.3 is not necessarily satisfied if we assume only that $\mathcal{C}$ is locally κ-small. For example, suppose that $\mathcal{C} = S$ is a set of cardinality κ (regarded as a discrete simplicial set), and let $\mathcal{D}$ be (the nerve of) the partially ordered set $\{0 < 1\}$. Then we can identify objects of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ with collections of κ-small Kan complexes $\{X_s\}_{s \in S}$. Define a functor $\lambda : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathcal{D}$ by the formula `03WF`

$$\lambda(\{X_s\}_{s \in S}) = \begin{cases} 0 & \text{if } |\{s \in S : X_s \neq \emptyset\}| < \kappa \\ 1 & \text{otherwise,} \end{cases}$$

and let $\lambda_0 : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathcal{D}$ be the constant functor taking the value 0. The functors λ and λ_0 both preserve κ-small colimits and coincide on the image of the Yoneda embedding $h_{\bullet}$, but do not coincide in general.

The proof of Theorem 8.4.3.3 will require some preliminaries. Let κ be an uncountable cardinal, and let $\mathcal{C}$ be an ∞-category which is locally κ-small. In what follows, we let $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ spanned by the representable functors $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\kappa}$. We will need the following elementary observation:

Lemma 8.4.3.5. *Let κ be an uncountable regular cardinal, let $\mathcal{C}$ be an ∞-category which is essentially κ-small, and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\kappa}$ be a functor. Then the ∞-category* `03WG`

$$\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}} = \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \times_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}}$$

is essentially κ-small.

Proof. The ∞-category $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is equivalent to $\mathcal{C}$ (Theorem 8.3.3.13), and is therefore essentially κ-small. Since κ is regular, it will suffice to show that each fiber of the right fibration $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}} \to \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is an essentially κ-small Kan complex (Corollary 5.6.7.7). Equivalently, we must show that for each object $\mathscr{G} \in \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$, the mapping space $X = \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})}(\mathscr{G}, \mathscr{F})$ is essentially κ-small. This follows from Proposition 8.3.1.3: if $\mathscr{G}$ is representable by the object $C \in \mathcal{C}$, then X is homotopy equivalent to the κ-small Kan complex $\mathscr{F}(C)$. □

We will deduce Theorem 8.4.3.3 from the following more precise assertion:

03WH **Theorem 8.4.3.6.** *Let κ be an uncountable regular cardinal, let $\mathcal{C}$ be an ∞-category which is essentially κ-small, and let $T : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:*

(1) *The functor T preserves κ-small colimits.*

(2) *The functor T is left Kan extended from the full subcategory $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \subseteq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.*

Proof. We first show that (1) implies (2). Assume that the functor T preserves κ-small colimits and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\kappa}$ be a functor; we wish to show that the composite functor

$$\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})^{\triangleright}_{/\mathscr{F}} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \xrightarrow{T} \mathcal{D}$$

is a colimit diagram in the ∞-category $\mathcal{D}$. Lemma 8.4.3.5 guarantees that the ∞-category $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}}$ is essentially κ-small. Since T preserves κ-small colimits, it will suffice to show that the map $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})^{\triangleright}_{/\mathscr{F}} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is a colimit diagram (Remark 7.6.7.6), which follows from Corollary 8.4.2.2.

We now show that (2) implies (1). Assume that T is left Kan extended from the ∞-category $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$; we wish to show that it preserves κ-small colimits. Choose a cardinal λ such that $\mathcal{D}$ is locally λ-small. Enlarging λ if necessary, we may assume that it has exponential cofinality $\geq \kappa$ (Remark 4.7.3.19). By virtue of Proposition 7.4.5.17 (and Remark 7.4.5.19), it will suffice to show that for every representable functor $H : \mathcal{D}^{\mathrm{op}} \to \mathcal{S}^{<\lambda}$, the composition $H^{\mathrm{op}} \circ T$ preserves κ-small colimits. Since H^{op} preserves κ-small colimits (Proposition 7.4.5.17 and Remark 7.4.5.19), the functor $H^{\mathrm{op}} \circ T$ is left Kan extended from $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. Consequently, to show that (2) implies (1), we may replace T by $H^{\mathrm{op}} \circ T$ and thereby reduce to the case where $\mathcal{D} = (\mathcal{S}^{<\lambda})^{\mathrm{op}}$, for some cardinal λ of exponential cofinality $\geq \kappa$.

Let $h_{\bullet} : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ be a covariant Yoneda embedding for $\mathcal{C}$, and let $\mathscr{F}$ denote the composite functor

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{h_{\bullet}^{\mathrm{op}}} \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})^{\mathrm{op}} \xrightarrow{T^{\mathrm{op}}} \mathcal{S}^{<\lambda}.$$

Using Remark 4.7.3.19 again, we can choose a cardinal $\lambda' \geq \lambda$ of exponential cofinality $\geq \kappa$ such that $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$ is locally λ'-small. In what follows, we abuse notation by identifying T with the composite functor $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \xrightarrow{T} (\mathcal{S}^{<\lambda})^{\mathrm{op}} \hookrightarrow (\mathcal{S}^{<\lambda'})^{\mathrm{op}}$. Note that, since the inclusion $\mathcal{S}^{<\lambda} \hookrightarrow \mathcal{S}^{<\lambda'}$ preserves κ-small limits (see Variant 7.4.5.8), this composite functor is also left Kan extended from $\operatorname{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.

Let $H' : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})^{\mathrm{op}} \to \mathcal{S}^{<\lambda'}$ be a functor represented by $\mathscr{F} \in \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda})$, and let U denote the composite functor

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \subseteq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\lambda}) \xrightarrow{H'^{\mathrm{op}}} (\mathcal{S}^{<\lambda'})^{\mathrm{op}}.$$

Applying Proposition 8.4.2.5, we see that the composition $U \circ h_\bullet$ is isomorphic to the functor $\mathscr{F} = T \circ h_\bullet$. Since the covariant Yoneda embedding $h_\bullet : \mathcal{C} \to \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is an equivalence of ∞-categories (Theorem 8.3.3.13), it follows that the functors U and T are isomorphic when restricted to $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. Proposition 7.4.5.17 and Remark 7.4.5.19 guarantee that the functor U preserves κ-small colimits. Invoking the implication (1) ⇒ (2), we see that U is left Kan extended from $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. Applying the universal property of Kan extensions (Corollary 7.3.6.13), we deduce that the functor T is isomorphic to U, and therefore also preserves κ-small colimits. □

Remark 8.4.3.7. In the statement of Theorem 8.4.3.6, it is not necessary to assume that the ∞-category $\mathcal{D}$ admits κ-small colimits (though we will primarily be interested in cases where this condition is satisfied). 03WJ

Example 8.4.3.8. Let $\mathcal{C}$ be a small ∞-category and let $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by the representable functors. Then a functor of ∞-categories $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$ preserves small colimits if and only if it is left Kan extended from $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. 03WK

Proof of Theorem 8.4.3.3. Let κ be an uncountable regular cardinal and let $\mathcal{C}$ be an ∞-category which is essentially κ-small. It follows from Example 7.6.7.8 that the ∞-category $\mathcal{S}^{<\kappa}$ is κ-cocomplete, so that the functor ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is also κ-cocomplete (Remark 7.6.7.5). Let $\mathcal{D}$ be an ∞-category which admits κ-small colimits. We wish to show that that the composite functor

$$\mathrm{Fun}^{\kappa}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D}) \to \mathrm{Fun}(\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D}) \xrightarrow{\circ h_\bullet} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence of ∞-categories. Theorem 8.3.3.13 guarantees that the covariant Yoneda embedding $\mathcal{C} \to \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is an equivalence of ∞-categories. We are therefore reduced to showing that the restriction functor

$$U : \mathrm{Fun}^{\kappa}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D}) \to \mathrm{Fun}(\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D}) \qquad F \mapsto F|_{\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})}$$

is an equivalence of ∞-categories. By virtue of Theorem 8.4.3.6, $\mathrm{Fun}^{\kappa}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D})$ is the full subcategory of $\mathrm{Fun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D})$ spanned by those functors which are left Kan extended from $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. Applying Corollary 7.3.6.15, we see that U restricts to a trivial Kan fibration

$$\mathrm{Fun}^{\kappa}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D}) \to \mathrm{Fun}'(\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D}),$$

where $\mathrm{Fun}'(\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}), \mathcal{D})$ spanned by those functors which admit a left Kan extension to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.

We will complete the proof by showing that every functor $f : \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathcal{D}$ admits a left Kan extension to $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. Fix an object $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ and let $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}}$ be as in the statement of Lemma 8.4.3.5. By virtue of Corollary 7.3.5.8, it will suffice to show the diagram

$$\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}} \to \mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \xrightarrow{f} \mathcal{D}$$

admits a colimit in the ∞-category $\mathcal{D}$. Since $\mathcal{D}$ admits κ-small colimits, we are reduced to showing that the ∞-category $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})_{/\mathscr{F}}$ is essentially κ-small (see Remark 7.6.7.6), which follows from Lemma 8.4.3.5. □

03WL **Corollary 8.4.3.9.** *Let κ be an uncountable regular cardinal, let $\mathcal{C}$ be an ∞-category which is essentially κ-small, and let $h_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ be a covariant Yoneda embedding for $\mathcal{C}$. Then every object $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ can be realized as the colimit of a diagram*

$$\mathcal{K} \to \mathcal{C} \xrightarrow{h_{\bullet}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}),$$

where $\mathcal{K}$ is a κ-small ∞-category.

Proof. Let $\mathcal{K}'$ denote the fiber product $\mathcal{C} \times_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}}$. Combining Lemma 8.4.3.5 with Theorem 8.3.3.13, we deduce that $\mathcal{K}'$ is essentially κ-small. We can therefore choose an equivalence of ∞-categories $e : \mathcal{K} \to \mathcal{K}'$, where $\mathcal{K}$ is κ-small. Applying Theorem 8.4.3.3, we deduce that $\mathscr{F}$ is a colimit of the composite functor

$$\mathcal{K} \xrightarrow{e} \mathcal{K}' = \mathcal{C} \times_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}} \to \mathcal{C} \xrightarrow{h_{\bullet}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}).$$

□

05JC **Corollary 8.4.3.10.** *Let κ be an uncountable regular cardinal and let $\mathcal{C}$ be an ∞-category which is essentially κ-small, and let $h^{\mathcal{C}}_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ be a covariant Yoneda embedding. Suppose we are given a functor*

$$T : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})^{\mathrm{op}} \to \mathcal{S}^{<\lambda},$$

where λ is a cardinal of exponential cofinality $< \kappa$. The following conditions are equivalent:

(1) *For each object $C \in \mathcal{C}$, the Kan complex $T(h^{\mathcal{C}}_{C})$ is essentially κ-small.*

(2) *The functor T preserves κ-small limits.*

Moreover, if these conditions are satisfied, then the functor T is representable by the object $\mathscr{F} = T \circ (h^{\mathcal{C}}_{\bullet})^{\mathrm{op}}$.

Proof. Assume first that T is representable by an object $\mathscr{G} \in \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. In this case, Corollary 7.4.5.18 guarantees that T satisfies condition (2). To verify (1), we note that for each object $C \in \mathcal{C}$, Proposition 8.3.1.3 supplies a homotopy equivalence

$$T(h^{\mathcal{C}}_{C}) \simeq \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})}(h^{\mathcal{C}}_{C}, \mathscr{G}) \xrightarrow{\sim} \mathscr{G}(C);$$

the desired result now follows from the observation that $\mathscr{G}(C)$ is essentially κ-small.

We now prove the converse. Assume that T satisfies conditions (1) and (2) and set $\mathscr{F} = T \circ (h^{\mathcal{C}}_{\bullet})^{\mathrm{op}}$. Condition (1) guarantees that we can view $\mathscr{F}$ as an object of the ∞-category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. Since λ has exponential cofinality $\geq \kappa$, the ∞-category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is locally λ-small (Corollary 4.7.8.8). We can therefore choose a functor $T' : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})^{\mathrm{op}} \to \mathcal{S}^{<\lambda}$ which is representable by $\mathscr{F}$ (Theorem 5.6.6.13). It follows from Proposition 8.4.2.5 that the composition $T' \circ (h^{\mathcal{C}}_{\bullet})^{\mathrm{op}}$ is isomorphic to the functor $\mathscr{F} = T \circ (h^{\mathcal{C}}_{\bullet})^{\mathrm{op}}$. Using condition (2) (and Corollary 7.4.5.18), we see that the functors T and T' both preserve κ-small limits. Applying Theorem 8.4.3.3, we conclude that T' is isomorphic to T, so that the functor T is also representable by the object $\mathscr{F}$. □

8.4.4 Example: Extensions as Adjoints

Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, where $\mathcal{C}$ is small and $\mathcal{D}$ admits small colimits. It follows from Theorem 8.4.0.3 that, up to isomorphism, the functor f factors as a composition 03WM

$$\mathcal{C} \xrightarrow{h_{\bullet}} \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{F} \mathcal{D},$$

where $h_{\bullet}$ denotes a covariant Yoneda embedding for $\mathcal{C}$ and F is a functor which preserves small colimits. The functor F is uniquely determined up to isomorphism: by virtue of Theorem 8.4.3.6, it can be characterized as a left Kan extension of f along $h_{\bullet}$. Our goal in this section is to show that, if the ∞-category $\mathcal{D}$ is locally small, then we can give another characterization of the functor F: it is left adjoint to the functor

$$\mathcal{D} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \qquad D \mapsto \operatorname{Hom}_{\mathcal{D}}(f(\bullet), D).$$

Proposition 8.4.4.1. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Assume that $\mathcal{C}$ is essentially small, that $\mathcal{D}$ is cocomplete and locally small, and let* 03WN

$$h^{\mathcal{C}}_{\bullet} : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \qquad h^{\mathcal{D}}_{\bullet} : \mathcal{D} \to \operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S})$$

be covariant Yoneda embeddings for $\mathcal{C}$ and $\mathcal{D}$, respectively. Let G denote the composite functor

$$\mathcal{D} \xrightarrow{h^{\mathcal{D}}_{\bullet}} \operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\circ f^{\mathrm{op}}} \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}).$$

Then the functor G admits a left adjoint $F : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$. Moreover, the composition $F \circ h^{\mathcal{C}}_{\bullet}$ is isomorphic to f.

04BF **Corollary 8.4.4.2.** *Let κ be a regular cardinal, let $\mathcal{C}$ be an essentially κ-small ∞-category, let $\mathcal{D}$ be an ∞-category which is cocomplete and locally small, and let $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$ be a functor. The following conditions are equivalent:*

(1) *The functor F preserves small colimits.*

(2) *The functor F admits a right adjoint $G : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.*

Proof. Assume that F preserves small colimits; we will show that it admits a right adjoint (the reverse implication follows from Corollary 7.1.3.21). Choose covariant Yoneda embeddings

$$h^{\mathcal{C}}_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \qquad h^{\mathcal{D}}_{\bullet} : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}),$$

set $f = F \circ h^{\mathcal{C}}_{\bullet}$, and let G denote the composite functor

$$\mathcal{D} \xrightarrow{h^{\mathcal{D}}_{\bullet}} \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\circ f^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}).$$

It follows from Proposition 8.4.4.1 that G admits a left adjoint $F' : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{D}$ such that $F' \circ h^{\mathcal{C}}_{\bullet}$ is isomorphic to $f = F \circ h^{\mathcal{C}}_{\bullet}$. Since the functor F' also preserves small colimits (Corollary 7.1.3.21), Theorem 8.4.0.3 implies that it is isomorphic to F. It follows that G is also a right adjoint of F. □

04BG **Corollary 8.4.4.3.** *Let $\mathcal{C}$ be an essentially small ∞-category and let $\mathscr{F} : \mathrm{Fun}(\mathcal{C}, \mathcal{S})^{\mathrm{op}} \to \mathcal{S}$ be a functor. The following conditions are equivalent:*

(1) *The functor $\mathscr{F}$ admits a left adjoint.*

(2) *The functor $\mathscr{F}$ is representable by an object of* $\mathrm{Fun}(\mathcal{C}, \mathcal{S})$.

(3) *The functor $\mathscr{F}$ preserves small limits.*

Proof. Since the identity functor $\mathrm{id} : \mathcal{S} \to \mathcal{S}$ is corepresentable (by the object $\Delta^0 \in \mathcal{S}$), the implication (1) ⇒ (2) follows from Corollary 6.2.4.2. The equivalence (2) ⇔ (3) is a special case of Corollary 8.4.3.10. The implication (3) ⇒ (1) follows by applying Corollary 8.4.4.2 to to the opposite functor $\mathscr{F}^{\mathrm{op}} : \mathrm{Fun}(\mathcal{C}, \mathcal{S}) \to \mathcal{S}^{\mathrm{op}}$. □

Following the convention of Remark 4.7.0.5, we will deduce Proposition 8.4.4.1 from the following more general assertion:

03WP **Variant 8.4.4.4.** Let κ be an uncountable regular cardinal and let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Assume that $\mathcal{C}$ is essentially κ-small, that $\mathcal{D}$ admits κ-small colimits, and that the morphism space $\mathrm{Hom}_{\mathcal{D}}(f(C), D)$ is essentially κ-small for every pair of objects $C \in \mathcal{C}$, $D \in \mathcal{D}$. Then the functor

$$G : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \qquad D \mapsto \mathrm{Hom}_{\mathcal{D}}(f(\bullet), D)$$

admits a left adjoint $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathcal{D}$ which preserves κ-small colimits. Moreover, the composite functor $\mathcal{C} \xrightarrow{h^{\mathcal{C}}_{\bullet}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \xrightarrow{F} \mathcal{D}$ is isomorphic to f.

Proof. We first prove the existence of the functor F. Fix a cardinal λ of exponential cofinality $\geq \kappa$, so that the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ is locally λ-small (see Corollary 4.7.8.8). By virtue of Proposition 6.2.4.1, it will suffice to show that for every functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\kappa}$, the composite functor

$$\mathcal{D} \xrightarrow{G} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \xrightarrow{\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})}(\mathscr{F}, \bullet)} \mathcal{S}^{<\lambda}$$

is corepresentable by an object of $\mathcal{D}$. Since $\mathcal{D}$ admits κ-small colimits, the collection of functors $\mathscr{F}$ which satisfy this condition is closed under κ-small colimits (Remark 8.3.3.16). Using Corollary 8.4.3.9, we can reduce to the case where the functor $\mathscr{F}$ is representable by an object $C \in \mathcal{C}$. In this case, the object $\mathscr{F} \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ corepresents the evaluation functor $\mathrm{ev}_C : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ (Remark 8.3.1.5). It now follows from the definition of the functor G that the composition $\mathrm{ev}_C \circ G$ is corepresentable by the object $f(C) \in \mathcal{D}$.

Choose functor $F : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathcal{D}$ and a natural transformation $\epsilon : (F \circ G) \to \mathrm{id}_{\mathcal{D}}$ which exhibits F as a left adjoint to the functor G. It follows from Corollary 7.1.3.21 that the functor F preserves all colimits which exist in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$; in particular, it preserves κ-small colimits. We will complete the proof by showing that $F \circ h^{\mathcal{C}}_{\bullet}$ is isomorphic to f.

For every pair of objects $X, Y \in \mathcal{C}$, let $\alpha_{X,Y}$ denote the morphism of Kan complexes

$$h^{\mathcal{C}}_{Y}(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(f(X), f(Y)) = G(f(Y))(X).$$

By virtue of Corollary 8.3.5.8, we can promote the construction $(X, Y) \mapsto \alpha_{X,Y}$ to a natural transformation of functors $\alpha : h^{\mathcal{C}}_{\bullet} \to G \circ f$. Let β denote a composition of the natural transformations

$$F \circ h^{\mathcal{C}}_{\bullet} \xrightarrow{F(\alpha)} F \circ G \circ f \xrightarrow{\epsilon} \mathrm{id}_{\mathcal{D}} \circ f = f.$$

We claim that β is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. By virtue of Theorem 4.4.4.4, it will suffice to show that β induces an isomorphism $\beta_X : F(h^{\mathcal{C}}_X) \to f(X)$ for each object $X \in \mathcal{C}$. Fix an object $D \in \mathcal{D}$; we wish to show that precomposition with β_X induces a homotopy equivalence of Kan complexes

$$\beta_{X,D} : \mathrm{Hom}_{\mathcal{D}}(f(X), D) \to \mathrm{Hom}_{\mathcal{D}}(F(h^{\mathcal{C}}_X), D) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})}(h^{\mathcal{C}}_X, G(D))$$

We conclude by observing that $\beta_{X,D}$ is left homotopy inverse to the morphism

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})}(h^{\mathcal{C}}_X, G(D)) \to G(D)(X) \simeq \mathrm{Hom}_{\mathcal{D}}(f(X), D)$$

given by evaluation at $\mathrm{id}_X \in h^{\mathcal{C}}_X(X)$, which is a homotopy equivalence by virtue of Proposition 8.3.1.3 □

03WQ **Example 8.4.4.5** (Functoriality of the Presheaf Construction)**.** Let κ be an uncountable regular cardinal, let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Assume that $\mathcal{C}$ is essentially κ-small and that $\mathcal{D}$ is locally κ-small, and fix covariant Yoneda embeddings

$$h_{\bullet}^{\mathcal{C}} : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \qquad h_{\bullet}^{\mathcal{D}} : \mathcal{D} \to \operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa}).$$

Let $G : \operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ be given by precomposition with f. Then the functor G admits a left adjoint $F : \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. Moreover, the diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "h_{\bullet}^{\mathcal{C}}"] \arrow[d, "f"] & \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \arrow[d, "F"] \\
\mathcal{D} \arrow[r, "h_{\bullet}^{\mathcal{D}}"] & \operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa}).
\end{tikzcd}$$

commutes up to isomorphism. This follows by applying Variant 8.4.4.4 to the composite functor $(h_{\bullet}^{\mathcal{D}} \circ f) : \mathcal{C} \to \operatorname{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.

8.4.5 Adjoining Colimits to ∞-Categories

04BH Let $\mathcal{C}$ be an essentially small ∞-category and set $\widehat{\mathcal{C}} = \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. Theorem 8.4.0.3 asserts that the covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a cocompletion of $\mathcal{C}$: that is, it is freely generated from $\mathcal{C}$ by adjoining colimits of small diagrams. In this section, we consider a variant of this construction, where we adjoint colimits of an arbitrary collection of diagrams (and we drop the assumption that $\mathcal{C}$ is essentially small).

04BJ **Definition 8.4.5.1.** Let $\mathbb{K}$ be a collection of simplicial sets. We say that an ∞-category $\mathcal{C}$ is $\mathbb{K}$-*cocomplete* if it admits K-indexed colimits, for each $K \in \mathbb{K}$. If $\mathcal{C}$ and $\mathcal{D}$ are $\mathbb{K}$-cocomplete ∞-categories, we let $\operatorname{Fun}^{\mathbb{K}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve K-indexed colimits, for each $K \in \mathbb{K}$. We say that a functor of ∞-categories $h : \mathcal{C} \to \widehat{\mathcal{C}}$ *exhibits* $\widehat{\mathcal{C}}$ *as a* $\mathbb{K}$-*cocompletion of* $\mathcal{C}$ if the following conditions are satisfied:

- The ∞-category $\widehat{\mathcal{C}}$ is $\mathbb{K}$-cocomplete.
- For every $\mathbb{K}$-cocomplete ∞-category $\mathcal{D}$, precomposition with h induces an equivalence of ∞-categories $\operatorname{Fun}^{\mathbb{K}}(\widehat{\mathcal{C}}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$.

04BK **Example 8.4.5.2.** Let κ be an uncountable regular cardinal, and let $\mathbb{K}$ denote the collection of all κ-small simplicial sets. Then an ∞-category $\mathcal{C}$ is $\mathbb{K}$-cocomplete if and only if it is κ-cocomplete, in the sense of Variant 7.6.7.7. A functor of ∞-categories $h : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits

$\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$ if and only if it exhibits $\widehat{\mathcal{C}}$ as a κ-cocompletion of $\mathcal{C}$, in the sense of Definition 8.4.3.2.

In particular, if $\mathbb{K}$ is the collection of all small simplicial sets, then an ∞-category $\mathcal{C}$ is a $\mathbb{K}$-cocomplete if and only if it is cocomplete, and a functor $h : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$ if and only if it exhibits $\widehat{\mathcal{C}}$ as a cocompletion of $\mathcal{C}$.

Our goal in this section is to prove the following existence result:

Proposition 8.4.5.3. *Let $\mathbb{K}$ be a collection of simplicial sets and let $\mathcal{C}$ be an ∞-category. Then there exists an ∞-category $\widehat{\mathcal{C}}$ and a functor $h : \mathcal{C} \to \widehat{\mathcal{C}}$ which exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$. Moreover, the functor h is dense and fully faithful.* 04BL

Warning 8.4.5.4. Let $\mathbb{K}$ be a collection of simplicial sets and let $h : \mathcal{C} \to \widehat{\mathcal{C}}$ be a functor of ∞-categories which exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$. In general, it is not true that every object of $\widehat{\mathcal{C}}$ can be recovered as the colimit of a diagram 04BM

$$K \to \mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}}$$

for some $K \in \mathbb{K}$.

Let κ be an uncountable regular cardinal. If $\mathbb{K}$ is the collection of all κ-small simplicial sets and the ∞-category $\mathcal{C}$ is essentially κ-small, then Proposition 8.4.5.3 follows from Theorem 8.4.3.3; in this case, we can take $\widehat{\mathcal{C}}$ to be the ∞-category of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. To prove Proposition 8.4.5.3 in general, we will build on this special case.

Construction 8.4.5.5. Let $\mathbb{K}$ be a collection of simplicial sets and let $\mathcal{C}$ be an ∞-category. Choose an uncountable regular cardinal κ such that $\mathcal{C}$ is locally κ-small and every simplicial set $K \in \mathbb{K}$ is essentially κ-small. We let $\widehat{\mathcal{C}}$ denote the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ which contains all representable functors and is closed under the formation of K-indexed colimits, for each $K \in \mathbb{K}$. Note that covariant Yoneda embedding for $\mathcal{C}$ determines a functor $h_{\bullet} : \mathcal{C} \to \widehat{\mathcal{C}}$, which is dense (by virtue of Variant 8.4.2.4 and Remark 8.4.1.19) and fully faithful (by virtue of Theorem 8.3.3.13). 04BN

Remark 8.4.5.6. In the situation of Construction 8.4.5.5, the ∞-category $\widehat{\mathcal{C}}$ is independent of the choice of κ (provided that κ is chosen large enough that $\mathcal{C}$ is locally κ-small and each $K \in \mathbb{K}$ is essentially κ-small). 04BP

Proposition 8.4.5.3 is an immediate consequence of the following more precise result:

Proposition 8.4.5.7. *Let $\mathbb{K}$ be a collection of simplicial sets, let $\mathcal{C}$ be an ∞-category, and define $\widehat{\mathcal{C}}$ as in Construction 8.4.5.5. Then the covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$.* 04BQ

The proof of Proposition 8.4.5.7 will require some preliminaries.

04BR **Lemma 8.4.5.8.** *Let $\mathbb{K}$ be a collection of simplicial sets, let $\mathcal{C}$ be an ∞-category, and define $\widehat{\mathcal{C}}$ as in Construction 8.4.5.5. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, where $\mathcal{D}$ is $\mathbb{K}$-cocomplete. Then there exists a functor $F : \widehat{\mathcal{C}} \to \mathcal{D}$ and an isomorphism $f \to F|_{\mathcal{C}}$, where the functor F is left Kan extended from the essential image of the Yoneda embedding $h_{\bullet} : \mathcal{C} \to \widehat{\mathcal{C}}$. Moreover, the functor F preserves K-indexed colimits, for each $K \in \mathbb{K}$.*

Proof. Fix an uncountable regular cardinal κ such that $\mathcal{C}$ is essentially κ-small and each $K \in \mathbb{K}$ is essentially κ-small. By virtue of Corollary 8.3.3.17, we may assume without loss of generality that $\mathcal{D}$ is a replete full subcategory of a κ-cocomplete ∞-category $\mathcal{D}'$, and that the inclusion map $\mathcal{D} \hookrightarrow \mathcal{D}'$ preserves all κ-small colimits which exist in $\mathcal{D}$. By virtue of Theorem 8.4.3.3, we can also assume that f factors as a composition

$$\mathcal{C} \xrightarrow{h_{\bullet}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \xrightarrow{F'} \mathcal{D}',$$

where F' preserves κ-small colimits. The full subcategory $F'^{-1}(\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ contains all representable functors and is closed under K-indexed colimits for each $K \in \mathbb{K}$, and therefore contains $\widehat{\mathcal{C}}$. It follows that F' restricts to a functor $F : \widehat{\mathcal{C}} \to \mathcal{D}$ which preserves K-indexed colimits for each $K \in \mathbb{K}$. Theorem 8.4.3.6 implies that F' is left Kan extended from the full subcategory $\mathrm{Fun}^{\mathrm{rep}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$, so the functor $F = F'|_{\widehat{\mathcal{C}}}$ has the same property. □

04BS **Lemma 8.4.5.9.** *Let $\mathbb{K}$ be a collection of simplicial sets, let $\mathcal{C}$ be an ∞-category, and define $\widehat{\mathcal{C}}$ as in Construction 8.4.5.5. Let $\mathcal{D}$ be a $\mathbb{K}$-cocomplete ∞-category and let $F : \widehat{\mathcal{C}} \to \mathcal{D}$ be a functor. The following conditions are equivalent:*

(1) *The functor F is left Kan extended from the essential image of the Yoneda embedding $h_{\bullet} : \mathcal{C} \to \widehat{\mathcal{C}}$.*

(2) *The functor F preserves K-indexed colimits, for each $K \in \mathbb{K}$.*

Proof. Let F_0 denote the restriction of F to the essential image of $h_{\bullet}$. Applying Lemma 8.4.5.8, we deduce that F_0 admits a left Kan extension $F' : \widehat{\mathcal{C}} \to \mathcal{D}$ which preserves K-indexed colimits for each $K \in \mathbb{K}$. Invoking the universal property of Kan extensions (Corollary 7.3.6.9), we see that there is an essentially unique natural transformation $\alpha : F' \to F$ which restricts to the identity transformation from F_0 to itself. We can then reformulate condition (1) as follows:

(1′) The natural transformation α is an isomorphism. That is, for each object $X \in \widehat{\mathcal{C}}$, the induced map $\alpha_X : F'(X) \to F(X)$ is an isomorphism in the ∞-category $\mathcal{D}$.

The implication (1′) ⇒ (2) follows from the fact that F' preserves K-indexed colimits for each $K \in \mathbb{K}$. To prove the converse, let $\widehat{\mathcal{C}}' \subseteq \widehat{\mathcal{C}}$ denote the full subcategory spanned by those objects X for which α_X is an isomorphism in the ∞-category $\mathcal{D}$. By construction, $\widehat{\mathcal{C}}'$ contains all representable functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\kappa}$. If condition (2) is satisfied, then $\widehat{\mathcal{C}}'$ is closed under the formation of K-indexed colimits for each $K \in \mathbb{K}$, and therefore coincides with $\widehat{\mathcal{C}}$. □

Proof of Proposition 8.4.5.7. Let $\mathbb{K}$ be a collection of simplicial sets, let $\mathcal{C}$ be an ∞-category, and let $\widehat{\mathcal{C}}$ be as in Construction 8.4.5.5. By construction, the ∞-category $\widehat{\mathcal{C}}$ is $\mathbb{K}$-cocomplete. To complete the proof, we must show that if $\mathcal{D}$ is any $\mathbb{K}$-cocomplete ∞-category, then composition with the covariant Yoneda embedding $h_\bullet : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ induces an equivalence of ∞-categories $\theta : \mathrm{Fun}^{\mathbb{K}}(\widehat{\mathcal{C}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Let $\mathcal{C}' \subseteq \widehat{\mathcal{C}}$ be the essential image of $h_\bullet$, so that θ factors as a composition

$$\mathrm{Fun}^{\mathbb{K}}(\widehat{\mathcal{C}}, \mathcal{D}) \xrightarrow{\theta'} \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \xrightarrow{\theta''} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

where θ'' is an equivalence of ∞-categories (Theorem 8.3.3.13). Using Lemma 8.4.5.9, we see that $\mathrm{Fun}^{\mathbb{K}}(\widehat{\mathcal{C}}, \mathcal{D})$ is the full subcategory of $\mathrm{Fun}(\widehat{\mathcal{C}}, \mathcal{D})$ spanned by those functors which are left Kan extended from $\mathcal{C}'$. It follows from Corollary 7.3.6.15 that θ' is a trivial Kan fibration onto a full subcategory of $\mathrm{Fun}(\mathcal{C}', \mathcal{D})$; in particular, it is fully faithful, so that θ is fully faithful. Lemma 8.4.5.8 implies that θ is essentially surjective, and therefore an equivalence of ∞-categories (Theorem 4.6.2.20). □

Corollary 8.4.5.10. *Let $\mathbb{K}$ be a collection of simplicial sets and let $h : \mathcal{C} \to \widehat{\mathcal{C}}$ be a functor of ∞-categories which exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$. Let $U : \mathcal{E} \to \widehat{\mathcal{C}}$ be a left fibration of ∞-categories. The following conditions are equivalent:* `05JD`

(1) *The covariant transport representation $\mathrm{Tr}_{\mathcal{E}/\widehat{\mathcal{C}}} : \widehat{\mathcal{C}} \to \mathcal{S}$ preserves K-indexed colimits, for each $K \in \mathbb{K}$.*

(2) *The projection map $\mathcal{E} \times_{\widehat{\mathcal{C}}} \mathcal{C} \to \mathcal{C}$ is left cofinal.*

Proof. It follows from Proposition 8.4.5.3 that the functor h is fully faithful. We can therefore replace $\mathcal{C}$ by its essential image and thereby reduce to the case where $\mathcal{C}$ is a replete full subcategory of $\widehat{\mathcal{C}}$ (and h is the inclusion functor). In this case, condition (1) is equivalent to the requirement that the functor $\mathrm{Tr}_{\mathcal{E}/\widehat{\mathcal{C}}}$ is left Kan extended from $\mathcal{C}$ (Lemma 8.4.5.9). The equivalence (1) ⇔ (2) is now a special case of Corollary 7.4.5.16. □

Remark 8.4.5.11. The $\mathcal{K}$-cocompletion construction of this section has been studied in more detail by Rezk; we refer the reader to [47] for more details. `04BT`

8.4.6 Recognition of Cocompletions

03WR Let $\mathcal{C}$ be an essentially small ∞-category, let $\widehat{\mathcal{C}}$ denote the ∞-category $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, and let $h : \mathcal{C} \to \widehat{\mathcal{C}}$ be a covariant Yoneda embedding for $\mathcal{C}$. Then:

(1) The functor h is fully faithful (Theorem 8.3.3.13).

(2) For each object $X \in \mathcal{C}$, the functor

$$\operatorname{Hom}_{\widehat{\mathcal{C}}}(h(X), \bullet) : \widehat{\mathcal{C}} \to \mathcal{S}$$

preserves small colimits (see Example 8.4.6.2 below).

(3) The ∞-category $\widehat{\mathcal{C}}$ is generated (under small colimits) by the essential image of h (in fact, the functor h is dense: see Theorem 8.4.2.1).

Our goal in this section is to prove the converse: if $h : \mathcal{C} \to \widehat{\mathcal{C}}$ is *any* functor which satisfies conditions (1) through (3), then h exhibits $\widehat{\mathcal{C}}$ as a cocompletion of $\mathcal{C}$ (Proposition 8.4.6.6) and is therefore equivalent to the covariant Yoneda embedding of $\mathcal{C}$. First, let us introduce a bit of terminology.

03WS **Definition 8.4.6.1.** Let $\mathcal{D}$ be a locally small ∞-category which admits small colimits. We say that an object $X \in \mathcal{D}$ is *atomic* if the corepresentable functor

$$h^X : \mathcal{D} \to \mathcal{S} \qquad Y \mapsto \operatorname{Hom}_{\mathcal{D}}(X, Y)$$

preserves small colimits.

03WT **Example 8.4.6.2** (Representable Functors are Atomic)**.** Let $\mathcal{C}$ be an essentially small ∞-category. Then every representable functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is atomic when regarded as an object of the ∞-category $\widehat{\mathcal{C}} = \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. To see this, suppose that $\mathscr{F}$ is representable by an object $C \in \mathcal{C}$. Using Remark 8.3.1.5, we see that $\mathscr{F}$ corepresents the evaluation functor

$$\operatorname{ev}_C : \widehat{\mathcal{C}} = \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S} \qquad \mathscr{G} \mapsto \mathscr{G}(C),$$

and therefore preserves small colimits by virtue of Proposition 7.1.6.1.

03WU **Definition 8.4.6.3.** Let $\widehat{\mathcal{C}}$ be an ∞-category. We say that a full subcategory $\mathcal{C} \subseteq \widehat{\mathcal{C}}$ is *weakly dense* if the following condition is satisfied:

- Let $f : Y \to Z$ be a morphism of $\widehat{\mathcal{C}}$ such that, for every object $X \in \mathcal{C}$, the induced map

  $$\operatorname{Hom}_{\widehat{\mathcal{C}}}(X, Y) \xrightarrow{[f]\circ} \operatorname{Hom}_{\widehat{\mathcal{C}}}(X, Z)$$

  is a homotopy equivalence of Kan complexes. Then f is an isomorphism.

We say that a collection of objects $\{X_i\}_{i \in I}$ of $\widehat{\mathcal{C}}$ is *weakly dense* if it spans a weakly dense full subcategory of $\widehat{\mathcal{C}}$.

Remark 8.4.6.4. Let $\widehat{\mathcal{C}}$ be a locally small ∞-category. A full subcategory $\mathcal{C} \subseteq \widehat{\mathcal{C}}$ is weakly dense if and only if the restricted Yoneda embedding $\widehat{\mathcal{C}} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is conservative. 03WV

Example 8.4.6.5. Let $\widehat{\mathcal{C}}$ be an ∞-category and let $\mathcal{C} \subseteq \widehat{\mathcal{C}}$ be a full subcategory which generates $\widehat{\mathcal{C}}$ under colimits (see Warning 8.4.1.10). Then the full subcategory $\mathcal{C}$ is weakly dense. In particular, every dense subcategory of $\widehat{\mathcal{C}}$ is weakly dense. 03WW

Proposition 8.4.6.6. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, where $\mathcal{C}$ is essentially small. Then f exhibits $\mathcal{D}$ as a cocompletion of $\mathcal{C}$ (in the sense of Definition 8.4.0.1) if and only if the following conditions are satisfied:* 03WX

(0) *The ∞-category $\mathcal{D}$ is locally small and cocomplete.*

(1) *The functor f is fully faithful.*

(2) *For each object $C \in \mathcal{C}$, the image $f(C) \in \mathcal{D}$ is atomic.*

(3) *The collection of objects $\{f(C)\}_{C \in \mathcal{C}}$ is weakly dense in $\mathcal{D}$.*

Corollary 8.4.6.7. *Let $\mathcal{D}$ be an ∞-category. The following conditions are equivalent:* 03WY

(a) *There exists an essentially small ∞-category $\mathcal{C}$ and a functor $h : \mathcal{C} \to \mathcal{D}$ which exhibits $\mathcal{D}$ as a cocompletion of $\mathcal{C}$.*

(b) *The ∞-category $\mathcal{D}$ is locally small and cocomplete. Moreover, it contains a small collection of atomic objects $\{X_i\}_{i \in I}$ which is weakly dense.*

Proof. We first show that (a) implies (b). Without loss of generality, we may assume that $\mathcal{D} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, where $\mathcal{C}$ is a small ∞-category. Since the ∞-category $\mathcal{S}$ is cocomplete (Corollary 7.4.5.6), the ∞-category $\mathcal{D}$ is also cocomplete (Remark 7.6.7.5). Corollary 4.7.8.9 guarantees that $\mathcal{D}$ is locally small. For each object $X \in \mathcal{C}$, let $h_X \in \mathcal{D}$ be a functor represented by X. The collection of objects $\{h_X\}_{X \in \mathcal{C}}$ span a full subcategory of $\mathcal{D}$ which is dense (Corollary 8.4.2.2), and therefore weakly dense (Example 8.4.6.5). We conclude by observing that each of the representable functors h_X is a atomic object of $\mathcal{D}$ (Example 8.4.6.2).

We now show that (b) implies (a). Assume that $\mathcal{D}$ is locally small and cocomplete. Let $\{X_i\}_{i \in I}$ be a small collection of atomic objects of $\mathcal{D}$, and let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the full subcategory that they span. It follows from Proposition 4.7.8.7 that the ∞-category $\mathcal{D}_0$ is essentially small. If $\mathcal{D}_0$ is weakly dense in $\mathcal{D}$, then the inclusion map $\mathcal{D}_0 \hookrightarrow \mathcal{D}$ satisfies the hypotheses of Proposition 8.4.6.6, and therefore exhibits $\mathcal{D}$ as a cocompletion of $\mathcal{D}_0$. □

Let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, where $\mathcal{C}$ is essentially small and $\mathcal{D}$ is cocomplete. Using Theorem 8.4.0.3, we see that f admits an essentially unique factorization as a composition

$$\mathcal{C} \xrightarrow{h_{\bullet}} \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{F} \mathcal{D},$$

where the functor F preserves small colimits. As a first step towards the proof of Proposition 8.4.6.6, we study conditions which guarantee that F is fully faithful. With an eye towards future applications, we consider a slightly more general situation.

03X1 **Lemma 8.4.6.8.** *Let $\mathbb{K}$ be a collection of simplicial sets, let $h : \mathcal{C} \to \widehat{\mathcal{C}}$ be a functor of ∞-categories which exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$, and let $F : \widehat{\mathcal{C}} \to \mathcal{D}$ a functor which satisfies the following conditions:*

(0) *The ∞-category $\mathcal{D}$ is $\mathbb{K}$-cocomplete and the functor F preserves K-indexed colimits, for each $K \in \mathbb{K}$.*

(1) *The functor $f = F \circ h$ is fully faithful.*

(2) *Let κ be an uncountable regular cardinal such that $\mathcal{D}$ is locally κ-small and each $K \in \mathbf{K}$ is essentially κ-small. Then, for each $C \in \mathcal{C}$, the corepresentable functor*

$$\mathcal{D} \to \mathcal{S}^{<\kappa} \qquad D \mapsto \operatorname{Hom}_{\mathcal{D}}(f(C), D)$$

preserves K-indexed colimits, for each $K \in \mathbb{K}$.

Then F is fully faithful.

Proof. By virtue of Proposition 8.4.5.7, we may assume without loss of generality that $\widehat{\mathcal{C}}$ is the smallest replete full subcategory of $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ which contains all representable functors and is closed under the formation of K-indexed colimits for $K \in \mathbb{K}$ (see Construction 8.4.5.5). For every pair of objects $\mathscr{G}, \mathscr{G}' \in \widehat{\mathcal{C}}$, the functor F induces a morphism of Kan complexes

$$\theta_{\mathscr{G},\mathscr{G}'} : \operatorname{Hom}_{\widehat{\mathcal{C}}}(\mathscr{G}, \mathscr{G}') \to \operatorname{Hom}_{\mathcal{D}}(F(\mathscr{G}), F(\mathscr{G}')).$$

By virtue of Corollary 8.3.5.8 (and Remark 8.3.5.9), we can promote the construction $(\mathscr{G}, \mathscr{G}') \mapsto \theta_{\mathscr{G},\mathscr{G}'}$ to a functor of ∞-categories

$$\theta : \widehat{\mathcal{C}}^{\mathrm{op}} \times \widehat{\mathcal{C}} \to \operatorname{Fun}(\Delta^1, \mathcal{S}^{<\kappa}).$$

We wish to show that, for every pair of objects $\mathscr{G}, \mathscr{G}' \in \widehat{\mathcal{C}}$, the morphism $\theta_{\mathscr{G},\mathscr{G}'}$ is a homotopy equivalence. Let us first regard the functor $\mathscr{G}'$ as fixed. Let $\widehat{\mathcal{C}}'$ denote the full subcategory of $\widehat{\mathcal{C}}$ spanned by those objects $\mathscr{G}$ for which $\theta_{\mathscr{G},\mathscr{G}'}$ is a homotopy equivalence. For each $K \in \mathbb{K}$, our assumption that F preserves K-indexed colimits guarantees that the functor $\mathscr{G} \mapsto \theta_{\mathscr{G},\mathscr{G}'}$ preserves K^{op}-indexed limits (Proposition 7.4.5.17). Consequently, the full subcategory $\widehat{\mathcal{C}}' \subseteq \widehat{\mathcal{C}}$ is closed under the formation of K-indexed colimits. It will therefore suffice to show that $\theta_{\mathscr{G},\mathscr{G}'}$ is a homotopy equivalence in the special case where $\mathscr{G} = h_C$ is the functor represented by some object $C \in \mathcal{C}$.

Let us now regard $\mathscr{G} = h_C$ as fixed. Combining Example 8.4.6.2 with assumption (2), we deduce that the functor

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathrm{Fun}(\Delta^1, \mathcal{S}^{<\lambda}) \qquad \mathscr{G}' \mapsto \theta_{\mathscr{G},\mathscr{G}'}$$

preserves K-indexed colimits, for each $K \in \mathbb{K}$. Invoking Corollary 8.4.3.9 again, we are reduced to proving that $\theta_{\mathscr{G},\mathscr{G}'}$ is a homotopy equivalence in the special case where $\mathscr{G}' = h_{C'}$ for some object $C' \in \mathcal{C}$. In this case, we have a commutative diagram of Kan complexes

$$\begin{tikzcd}
 & \mathrm{Hom}_{\mathcal{C}}(C, C') \arrow[dl] \arrow[dr] & \\
\mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathscr{G}, \mathscr{G}') \arrow[rr, "\theta_{\mathscr{G},\mathscr{G}'}"] & & \mathrm{Hom}_{\mathcal{D}}(F(\mathscr{G}), F(\mathscr{G}')),
\end{tikzcd}$$

where the left vertical map is a homotopy equivalence by virtue of Yoneda's lemma (Theorem 8.3.3.13) and the right vertical map is a homotopy equivalence by virtue of assumption (1). It follows that lower horizontal map is also a homotopy equivalence. □

Proof of Proposition 8.4.6.6. Let $\mathcal{C}$ be an essentially small ∞-category, let $\mathcal{D}$ be a cocomplete ∞-category, and let $f : \mathcal{C} \to \mathcal{D}$ be a functor. By virtue of Theorem 8.4.0.3, the functor f admits an essentially unique factorization as a composition

$$\mathcal{C} \xrightarrow{h_\bullet} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{F} \mathcal{D},$$

where $h_\bullet$ is the covariant Yoneda embedding for $\mathcal{C}$ and the functor F preserves small colimits. Moreover, f exhibits $\mathcal{D}$ as a cocompletion of $\mathcal{C}$ if and only if the functor F is an equivalence of ∞-categories. If this condition is satisfied, then $\mathcal{D}$ is locally small (Corollary 4.7.8.9), the functor f is fully faithful (Theorem 8.3.3.13), the essential image of f consists of atomic objects of $\mathcal{D}$ (Example 8.4.6.2). We may therefore assume without loss of generality that f satisfies conditions (0), (1), and (2) of Proposition 8.4.6.6, so that F is fully faithful (Lemma 8.4.6.8). Using Proposition 8.4.4.1, we see that the functor F admits a right adjoint $G : \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, given on objects by the formula $G(D)(C) = \mathrm{Hom}_{\mathcal{D}}(f(C), D)$. By virtue of Corollary 6.2.2.19, the functor F is an equivalence if and only if the functor G is conservative: that is, if and only if the collection of objects $\{f(C)\}_{C \in \mathcal{C}}$ is weakly dense. □

Proposition 8.4.6.6 has a counterpart for more general cocompletions:

03WZ **Variant 8.4.6.9.** Let $\mathbb{K}$ be a collection of simplicial sets and let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then f exhibits $\mathcal{D}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$ (in the sense of Definition 8.4.5.1) if and only if the following conditions are satisfied:

(0) The ∞-category $\mathcal{D}$ is $\mathbb{K}$-cocomplete.

(1) The functor f is fully faithful.

(2) Let κ be an uncountable regular cardinal such that $\mathcal{D}$ is locally κ-small and each $K \in \mathbf{K}$ is essentially κ-small. Then, for each $C \in \mathcal{C}$, the corepresentable functor

$$\mathcal{D} \to \mathcal{S}^{<\kappa} \qquad D \mapsto \mathrm{Hom}_{\mathcal{D}}(f(C), D)$$

preserves K-indexed colimits, for each $K \in \mathbb{K}$.

(3) The ∞-category $\mathcal{D}$ is generated by the objects $\{f(C)\}_{C \in \mathcal{C}}$ under the formation of K-indexed colimits for $K \in \mathbb{K}$.

Proof. Let κ be as in (2), and let $\widehat{\mathcal{C}} \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ denote the smallest replete full subcategory which contains all representable functors and is closed under the formation of K-indexed colimits, for each $K \in \mathbb{K}$ (see Construction 8.4.5.5). By virtue of Proposition 8.4.5.7, the covariant Yoneda embedding $h_{\bullet} : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$. Assume that $\mathcal{D}$ is $\mathbb{K}$-complete, so that f factors (up to isomorphism) as a composition $\mathcal{C} \xrightarrow{h_{\bullet}} \widehat{\mathcal{C}} \xrightarrow{F} \mathcal{D}$, where the functor F preserves K-indexed colimits for each $K \in \mathbb{K}$. To complete the proof, it will suffice to show that if f satisfies conditions (1), (2), and (3), then the functor F is an equivalence of ∞-categories (the reverse implication follows from Theorem 8.3.3.13 and Example 8.4.6.2). Applying Lemma 8.4.6.8, we see that the functor F is fully faithful and therefore restricts to an equivalence of $\widehat{\mathcal{C}}$ with a replete full subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$. For each $K \in \mathbb{K}$, our assumption that F preserves K-indexed colimits guarantees that the subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$ is closed under the formation of K-indexed colimits. Since $\mathcal{D}_0$ contains the essential image of the functor f, the equality $\mathcal{D}_0 = \mathcal{D}$ follows from assumption (3). □

05JE **Corollary 8.4.6.10.** *Let $\mathbb{K}$ be a collection of simplicial sets and let $f : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which exhibits $\mathcal{D}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory, and let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the smallest full subcategory which contains the essential image of $f_0 : f|_{\mathcal{C}_0}$ and is closed under the formation of K-indexed colimits, for each $K \in \mathbb{K}$. Then the functor $f_0 : \mathcal{C}_0 \to \mathcal{D}_0$ exhibits $\mathcal{D}_0$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}_0$.*

8.4.7 Slices of Cocompletions

04BU

Let $U : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a functor between essentially small ∞-categories. Using Example 8.4.4.5, we see that U admits an essentially unique extension $\operatorname{Fun}(\widetilde{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}) \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ which preserves small colimits. Our goal in this section is to show that, up to equivalence, this construction carries right fibrations to right fibrations. More precisely, if U is a right fibration, we show that $\operatorname{Fun}(\widetilde{\mathcal{C}}^{\mathrm{op}}, \mathcal{S})$ is equivalent to the slice ∞-category $\operatorname{Fun}(\widetilde{\mathcal{C}}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}}$, where $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is a covariant transport representation for the left fibration U^{op} (Corollary 8.4.7.2). This is a consequence of the following:

Proposition 8.4.7.1. *Let $h : \mathcal{C} \to \widehat{\mathcal{C}}$ be a functor of ∞-categories which exhibits $\widehat{\mathcal{C}}$ as a cocompletion of $\mathcal{C}$, let $\mathscr{F} \in \widehat{\mathcal{C}}$ be an object, and let* 04BV

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[r, "\widetilde{h}"] \arrow[d] & \widehat{\mathcal{C}}_{/\mathscr{F}} \arrow[d] \\
\mathcal{C} \arrow[r, "h"] & \widehat{\mathcal{C}}
\end{tikzcd}$$

be a categorical pullback square of ∞-categories. Then $\widetilde{h}$ exhibits $\widehat{\mathcal{C}}_{/\mathscr{F}}$ as a cocompletion of $\widetilde{\mathcal{C}}$.

Corollary 8.4.7.2. *Let $U : \widetilde{\mathcal{C}} \to \mathcal{C}$ be a right fibration between essentially small ∞-categories and let $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be a covariant transport representation for the left fibration U^{op}. Then there exists an equivalence of ∞-categories $T : \operatorname{Fun}(\widetilde{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}) \to \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}}$ for which the diagram of ∞-categories* 04BW

$$\begin{tikzcd}
\widetilde{\mathcal{C}} \arrow[r, "h^{\widetilde{\mathcal{C}}}_{\bullet}"] \arrow[dd, "U"] & \operatorname{Fun}(\widetilde{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}) \arrow[d, "T"] \\
 & \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}} \arrow[d] \\
\mathcal{C} \arrow[r, "h^{\mathcal{C}}_{\bullet}"] & \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})
\end{tikzcd}$$

commutes up to isomorphism. Here $h^{\mathcal{C}}_{\bullet}$ and $h^{\widetilde{\mathcal{C}}}_{\bullet}$ denote covariant Yoneda embeddings for $\mathcal{C}$ and $\widetilde{\mathcal{C}}$, respectively.

Proof. Using Corollary 8.4.2.7, we can choose categorical pullback square

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "\widetilde{h}"] \arrow[d, "U"] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}} \arrow[d] \\ \mathcal{C} \arrow[r, "h^{\mathcal{C}}_{\bullet}"] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}). \end{tikzcd}$$

It follows from Theorem 8.4.0.3 that the functor $\widetilde{h}$ factors (up to isomorphism) as a composition

$$\widetilde{\mathcal{C}} \xrightarrow{h^{\widetilde{\mathcal{C}}}_{\bullet}} \mathrm{Fun}(\widetilde{\mathcal{C}}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{T} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}},$$

where the functor T preserves small colimits. To complete the proof, it will suffice to show that T is an equivalence of ∞-categories. This is equivalent to the assertion that $\widetilde{h}$ exhibits $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})_{/\mathscr{F}}$ as a cocompletion of $\widetilde{\mathcal{C}}$, which is a special case of Proposition 8.4.7.1. □

Proposition 8.4.7.1 is a special case of the following more general assertion:

04BX **Proposition 8.4.7.3.** *Let $\mathbb{K}$ be a collection of simplicial sets, let $h : \mathcal{C} \to \widehat{\mathcal{C}}$ be a functor which exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$, let $f : L \to \widehat{\mathcal{C}}$ be any morphism of simplicial sets, and let*

04BY
$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r, "\widetilde{h}"] \arrow[d] & \widehat{\mathcal{C}}_{/f} \arrow[d] \\ \mathcal{C} \arrow[r, "h"] & \widehat{\mathcal{C}} \end{tikzcd} \tag{8.51}$$

be a categorical pullback square of ∞-categories. Then $\widetilde{h}$ exhibits $\widehat{\mathcal{C}}_{/f}$ as a $\mathbb{K}$-cocompletion of $\widetilde{\mathcal{C}}$.

Proof. We will show that $\widetilde{h}$ satisfies the hypotheses of Variant 8.4.6.9:

(0) The ∞-category $\widehat{\mathcal{C}}_{/f}$ is $\mathbb{K}$-cocomplete: that is, it admits K-indexed colimits, for each $K \in \mathbb{K}$. This follows from Corollary 7.1.3.20, since the ∞-category $\widehat{\mathcal{C}}$ is $\mathbb{K}$-cocomplete.

(1) The functor $\widetilde{h}$ is fully faithful. This is a special case of Remark 4.6.2.7, since the diagram (8.51) is a categorical pullback square and the functor h is fully faithful.

(2) Choose an uncountable regular cardinal κ such that $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{C}}_{/f}$ are locally κ-small, and each simplicial set $K \in \mathbb{K}$ is essentially κ-small. Choose an object $\widetilde{C} \in \widetilde{\mathcal{C}}$ having image $C \in \mathcal{C}$, and let

$$\mathscr{F} : \widehat{\mathcal{C}} \to \mathcal{S}^{<\kappa} \qquad \widetilde{\mathscr{F}} : \widehat{\mathcal{C}}_{/f} \to \mathcal{S}^{<\kappa}$$

be functors corepresented by the objects $X = h(C)$ and $\widetilde{X} = \widetilde{h}(\widetilde{C})$, respectively. For every simplicial set $K \in \mathbb{K}$, the functor $\mathscr{F}$ preserves K-indexed colimits, and we must show$\widetilde{\mathscr{F}}$ has the same property. Choose a colimit diagram $\widetilde{g} : K^{\triangleright} \to \widehat{\mathcal{C}}_{/f}$; we wish to show that $\widetilde{\mathscr{F}} \circ \widetilde{g}$ is a colimit diagram in the ∞-category $\mathcal{S}^{<\kappa}$. Let $g : K^{\triangleright} \to \widehat{\mathcal{C}}$ denote the composition of $\widetilde{g}$ with the projection map. Then g is a colimit diagram in $\widehat{\mathcal{C}}$ (Corollary 7.1.3.20), so $\mathscr{F} \circ g$ is a colimit diagram in the ∞-category $\mathcal{S}^{<\kappa}$. Define

$$\mathcal{E} = K^{\triangleright} \times_{\widehat{\mathcal{C}}} \widehat{\mathcal{C}}_{X/} \qquad \widetilde{\mathcal{E}} = K^{\triangleright} \times_{\widehat{\mathcal{C}}} \widehat{\mathcal{C}}_{X/}.$$

Using Proposition 5.6.6.21, we see that $\mathscr{F} \circ g$ and $\widetilde{\mathscr{F}} \circ \widetilde{g}$ are covariant transport representations for the left fibrations $\mathcal{E} \to K^{\triangleleft}$ and $\overline{\mathcal{E}} \to K^{\triangleleft}$, respectively. Our assumption that $\mathscr{F} \circ g$ is a colimit diagram guarantees that the inclusion map $K \times_{K^{\triangleleft}} \mathcal{E} \hookrightarrow \mathcal{E}$ is left cofinal, and we wish to show that the inclusion map $K \times_{K^{\triangleleft}} \widetilde{\mathcal{E}} \hookrightarrow \widetilde{\mathcal{E}}$ is also left cofinal (Corollary 7.4.5.15). This follows from Proposition 7.2.3.12, since the tautological map $\widetilde{\mathcal{E}} \to \mathcal{E}$ is a pullback of the projection map $(\widehat{\mathcal{C}}_{X/})_{/f} \to \widehat{\mathcal{C}}_{X/}$, and therefore a right fibration (Proposition 4.3.6.1).

(3) Let $\widehat{\mathcal{C}}'_{/f}$ denote the smallest replete full subcategory of $\widehat{\mathcal{C}}_{/f}$ which contains the essential image of $\widetilde{h}$ and is closed under the formation of K-indexed colimits for $K \in \mathbb{K}$. We wish to show that $\widehat{\mathcal{C}}'_{/f} = \widehat{\mathcal{C}}_{/f}$. Let $\widehat{\mathcal{C}}' \subseteq \widehat{\mathcal{C}}$ denote the full subcategory spanned by those objects $X \in \widehat{\mathcal{C}}$ having the property that every object $\widetilde{X} \in \widehat{\mathcal{C}}_{/f}$ lying over X belongs to $\widehat{\mathcal{C}}^{0}_{/f}$. We will complete the proof by showing that $\widehat{\mathcal{C}}' = \widehat{\mathcal{C}}$. Since the diagram (8.51) is a categorical pullback square, $\widehat{\mathcal{C}}'$ contains the essential image of the functor h. It will therefore suffice to show that $\widehat{\mathcal{C}}'$ is closed under the formation of K-indexed colimits, for each $K \in \mathbb{K}$. Fix a colimit diagram $g : K^{\triangleright} \to \widehat{\mathcal{C}}$ carrying the cone point of $K^{\triangleright}$ to an object $X \in \widehat{\mathcal{C}}$. Assume that $g|_K$ factors through $\widehat{\mathcal{C}}'$; we wish to show that X also belongs to $\widehat{\mathcal{C}}'$. Let $\widetilde{X}$ be an object of $\widehat{\mathcal{C}}_{/f}$ lying over X. Since the inclusion of the cone point into $K^{\triangleright}$ is right anodyne (Example 4.3.7.11), we can lift g to a diagram $\widetilde{g} : K^{\triangleright} \to \widehat{\mathcal{C}}_{/f}$ carrying the cone point to $\widetilde{X}$ (Proposition 4.2.4.5). The assumption that $g|_K$ factors through $\widehat{\mathcal{C}}'$ guarantees that $\widetilde{g}|_K$ factors through $\widehat{\mathcal{C}}^{0}_{/f}$. Since g is a colimit diagram, $\widetilde{g}$ is also a colimit diagram (Corollary 7.1.3.20). It follows that $\widetilde{X}$ belongs to $\widehat{\mathcal{C}}'_{/f}$. Allowing the object $\widetilde{X}$ to vary, we conclude that X belongs to $\widehat{\mathcal{C}}'$, as desired.

□

8.5 Retracts and Idempotents

Let $\mathcal{C}$ be a category containing an object X. Recall that an object $Y \in \mathcal{C}$ is a *retract of* 03Y9
X if there exist morphisms $i : Y \to X$ and $r : X \to Y$ satisfying $\mathrm{id}_Y = r \circ i$, so that we have

a commutative diagram

03YA
$$\begin{tikzcd}
& X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y.
\end{tikzcd} \tag{8.52}$$

In this case, we will refer to (8.52) as a *retraction diagram in $\mathcal{C}$*.

04BZ **Remark 8.5.0.1.** Let X and Y be objects of a category $\mathcal{C}$. Then a retraction diagram (8.52) can be viewed as a morphism from id_X to id_Y in the twisted arrow category $\mathrm{Tw}(\mathcal{C})$ of Construction 8.1.0.1. In particular, Y is a retract of X if and only if there exists a morphism $\mathrm{id}_X \to \mathrm{id}_Y$ in $\mathrm{Tw}(\mathcal{C})$.

There is a universal example of a retraction diagram:

03YB **Construction 8.5.0.2.** We define a category Ret as follows:

- The category Ret has exactly two objects, which we denote by $\widetilde{X}$ and $\widetilde{Y}$.
- The morphisms sets in Ret are given by

$$\mathrm{Hom}_{\mathrm{Ret}}(\widetilde{X}, \widetilde{X}) = \{\mathrm{id}_{\widetilde{X}}, \widetilde{e}\} \qquad \mathrm{Hom}_{\mathrm{Ret}}(\widetilde{X}, \widetilde{Y}) = \{\widetilde{r}\}$$
$$\mathrm{Hom}_{\mathrm{Ret}}(\widetilde{Y}, \widetilde{X}) = \{\widetilde{i}\} \qquad \mathrm{Hom}_{\mathrm{Ret}}(\widetilde{Y}, \widetilde{Y}) = \{\mathrm{id}_{\widetilde{Y}}\}.$$

- The composition law in Ret is given (on non-identity morphisms) by the formulae

$$\widetilde{r} \circ \widetilde{i} = \mathrm{id}_{\widetilde{Y}} \qquad \widetilde{i} \circ \widetilde{r} = \widetilde{e}$$
$$\widetilde{e} \circ \widetilde{i} = \widetilde{i} \qquad \widetilde{e} \circ \widetilde{e} = \widetilde{e} \qquad \widetilde{r} \circ \widetilde{e} = \widetilde{r}.$$

03YC **Exercise 8.5.0.3.** Let $\mathcal{C}$ be a category containing a retraction diagram

$$\begin{tikzcd}
& X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y.
\end{tikzcd}$$

Show that there is a unique functor $F : \mathrm{Ret} \to \mathcal{C}$ which is given on objects by $F(\widetilde{X}) = X$ and $F(\widetilde{Y}) = Y$, and also satisfies $F(\widetilde{i}) = i$ and $F(\widetilde{r}) = r$. We therefore obtain a bijection

$$\{\text{Functors } \mathrm{Ret} \to \mathcal{C}\} \xrightarrow{\sim} \{\text{Retraction diagrams in } \mathcal{C}\}.$$

In particular, an object $Y \in \mathcal{C}$ is a retract of another object $X \in \mathcal{C}$ if and only if there exists a functor $F : \mathrm{Ret} \to \mathcal{C}$ satisfying $F(\widetilde{X}) = X$ and $F(\widetilde{Y}) = Y$.

Our first goal in this section is to extend the theory of retracts to the setting of higher category theory. Here there are (at least) two ways that we might choose to proceed:

(a) Let $\mathcal{C}$ be an ∞-category containing an object X. We could define an object $Y \in \mathcal{C}$ to be a retract of X if there exist morphisms $i : Y \to X$ and $r : X \to Y$ such that the identity morphism id_Y is a composition of r with i, in the sense of Definition 1.4.4.1.

(b) Let $\mathcal{C}$ be an ∞-category containing an object X. We could define an object $Y \in \mathcal{C}$ to be a retract of X if there exists a functor of ∞-categories $F : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$ satisfying $F(\widetilde{X}) = X$ and $F(\widetilde{Y}) = Y$.

In §8.5.1, we show that these definitions are equivalent. Note that objects $X, Y \in \mathcal{C}$ satisfy condition (a) if and only if there exists a 2-simplex σ of $\mathcal{C}$ whose boundary is indicated in the diagram

$$\begin{tikzcd}
& X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y;
\end{tikzcd}$$

in this case, we will say that σ is a *retraction diagram in* $\mathcal{C}$ (Definition 8.5.1.19). We will establish the equivalence of (a) and (b) by showing that every retraction diagram in $\mathcal{C}$ can be extended to a functor $F : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$ (Corollary 8.5.1.28). In contrast with Exercise 8.5.0.3, the functor F is not necessarily unique; however, it is uniquely determined up to isomorphism (in fact, up to a contractible space of choices).

Our second goal in this section is to address the following:

Question 8.5.0.4. Given an ∞-category $\mathcal{C}$ and an object $X \in \mathcal{C}$, how can one classify the retracts of X? 03YD

In §8.5.2, we recall the answer to Question 8.5.0.4 in the situation where $\mathcal{C}$ is an ordinary category. We say that an endomorphism $e : X \to X$ is *idempotent* if it satisfies the identity $e \circ e = e$ (Definition 8.5.2.1). Every retraction diagram

03YE

$$\begin{tikzcd}
& X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y.
\end{tikzcd} \tag{8.53}$$

determines an idempotent endomorphism of X, given by the composition $e = i \circ r$ (Example 8.5.2.3). We say that an idempotent endomorphism is *split* if it can be obtained in this way

(Example 8.5.2.3). In this case, we can recover the retraction diagram (8.53) up to (unique) isomorphism from e; for example the object Y can be recovered as the equalizer of the pair of morphisms $(e, \mathrm{id}_X) : X \rightrightarrows X$ (see Corollary 8.5.2.5 and its proof). We therefore obtain a bijection

$$\{\text{Split idempotent endomorphisms of } X\} \simeq \{\text{Retraction Diagrams}\}/\text{Isomorphism}.$$

We can therefore reformulate Question 8.5.0.4 as follows:

03YF **Question 8.5.0.5.** What is the correct ∞-categorical counterpart of the notion of an idempotent endomorphism?

In §8.5.3, we propose an answer to Question 8.5.0.5. Let Idem denote the full subcategory of Ret spanned by the object $\widetilde{X}$ (Construction 8.5.2.7). We define an *idempotent in* $\mathcal{C}$ to be a functor of ∞-categories $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ (Definition 8.5.3.1). Every retraction diagram in $\mathcal{C}$ can be extended to a functor $\overline{F} : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$, which we can restrict to obtain an idempotent $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$. We will say that an idempotent in $\mathcal{C}$ is *split* if it can be obtained in this way. In this case, we will show that $\overline{F}$ can be recovered from the idempotent F up to isomorphism (Corollary 8.5.3.10). Consequently, we obtain a bijection

$$\{\text{Split idempotents in } \mathcal{C}\}/\text{Isomorphism} \simeq \{\text{Retraction diagrams in } \mathcal{C}\}/\text{Isomorphism}.$$

To fully address Question 8.5.0.4, we also need to provide a criterion to determine when an idempotent splits. Here we have several closely related results:

- If $\mathcal{C}$ is (the nerve of) an ordinary category, then an idempotent endomorphism $e : X \to X$ is split if and only if the pair of morphisms $(e, \mathrm{id}_X) : X \rightrightarrows X$ admits an equalizer (or a coequalizer); see Corollary 8.5.2.5. If this condition is satisfied, then the (co)equalizer is the associated retract of X.
- If $\mathcal{C}$ is an ∞-category, then an idempotent $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ is split if and only if it admits a limit (or a colimit); see Corollary 8.5.3.11. If this condition is satisfied, then the (co)limit is the associated retract of $X = F(\widetilde{X})$.
- Let $\mathcal{C}$ be an ∞-category and let $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ be an idempotent, carrying the morphism $\widetilde{e} : \widetilde{X} \to \widetilde{X}$ of Idem to a morphism $e : X \to X$ of $\mathcal{C}$. Then F is split if and only if the sequential diagram

  $$\cdots \to X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \to \cdots$$

  admits a limit (or a colimit); see Proposition 8.5.4.16. If this condition is satisfied, then the (co)limit is the associated retract of X.

In §8.5.4, we study ∞-categories $\mathcal{C}$ in which every idempotent is split; if this condition is satisfied, we say that $\mathcal{C}$ is *idempotent complete* (Definition 8.5.4.1). Many ∞-categories which arise in practice are idempotent complete. For example, an ∞-category which admits sequential limits or colimits is automatically idempotent complete (Corollary 8.5.4.17). In §8.5.5, we show that every ∞-category $\mathcal{C}$ admits an *idempotent completion* $\widehat{\mathcal{C}}$ which is characterized (up to equivalence) by the existence of a functor $H : \mathcal{C} \to \widehat{\mathcal{C}}$ having the following properties:

- The ∞-category $\widehat{\mathcal{C}}$ is idempotent complete.
- The functor H is fully faithful.
- Every object of $\widehat{\mathcal{C}}$ is a retract of $H(X)$, for some object $X \in \mathcal{C}$.

Moreover, the idempotent completion $\widehat{\mathcal{C}}$ can be characterized by a universal mapping property: for every idempotent complete ∞-category $\mathcal{D}$, composition with H induces an equivalence of ∞-categories $\operatorname{Fun}(\widehat{\mathcal{C}}, \mathcal{D}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ (see Proposition 8.5.5.2).

Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and let $e : X \to X$ be an endomorphism of X. We will say that e is *idempotent* if there exists a functor $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$ satisfying $F(\widetilde{X}) = X$ and $F(\widetilde{e}) = e$. If $\mathcal{C}$ is (the nerve of) an ordinary category, then the functor F is completely determined by the pair (X, e). In general, this need not be true: the simplicial set $\mathrm{N}_{\bullet}(\mathrm{Idem})$ contains a nondegenerate simplex of every dimension (Remark 8.5.3.3), so the specification of F requires an infinite quantity of data. Nevertheless, we prove in §8.5.6 that F is determined by the pair (X, e) up to (canonical) isomorphism (Corollary 8.5.6.5). This motivates the following:

Question 8.5.0.6. Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. How can one determine if e is idempotent? 03YG

Let us first record a necessary condition. We say that an endomorphism $e : X \to X$ is *homotopy idempotent* if the homotopy class $[e]$ is an idempotent endomorphism in the homotopy category $\mathrm{h}\mathcal{C}$ (Definition 8.5.7.1). It follows immediately from the definitions that every idempotent endomorphism in $\mathcal{C}$ is homotopy idempotent. In §8.5.7, we show that the converse is false in general (Proposition 8.5.7.15), though it is true in some important special cases (see Corollary 8.5.7.6 and Exercise 8.5.7.8).

For every integer $n \geq 0$, let $\mathrm{N}_{\leq n}(\mathrm{Idem})$ denote the n-skeleton of the simplicial set $\mathrm{N}_{\bullet}(\mathrm{Idem})$ (Variant 1.3.1.6). Note that an endomorphism $e : X \to X$ in $\mathcal{C}$ can be identified with a diagram $F : \mathrm{N}_{\leq 1}(\mathrm{Idem}) \to \mathcal{C}$ (Example 8.5.8.2), and that e is homotopy idempotent if and only if F admits an extension to $\mathrm{N}_{\leq 2}(\mathrm{Idem})$ (Example 8.5.8.3). In §8.5.8, we address Question 8.5.0.6 by showing that e is idempotent if and only if F admits an extension to $\mathrm{N}_{\leq 3}(\mathrm{Idem})$ (Corollary 8.5.8.8). As an application, we show that the construction $\mathcal{C} \mapsto \operatorname{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C})$ commutes with filtered colimits, up to equivalence (Corollary 8.5.8.9).

8.5.1 Retracts in ∞-Categories

03YH The notion of retract has an obvious counterpart in the setting of ∞-categories.

03YJ **Definition 8.5.1.1.** Let $\mathcal{C}$ be an ∞-category containing an object X. We say that an object $Y \in \mathcal{C}$ is a *retract of* X if there exist morphisms $i : Y \to X$ and $r : X \to Y$ for which the identity morphism id_Y is a composition of i and r, in the sense of Definition 1.4.4.1.

03YK **Remark 8.5.1.2.** Let $\mathcal{C}$ be an ∞-category containing an object X. Then an object $Y \in \mathcal{C}$ is a retract of X (in the sense of Definition 8.5.1.1) if and only if it is a retract of X when viewed as an object of the homotopy category $\mathrm{h}\mathcal{C}$.

04K9 **Variant 8.5.1.3.** Let $\mathcal{C}$ be an ∞-category containing morphisms $f : X \to X'$ and $g : Y \to Y'$. The following conditions are equivalent:

(1) The morphism g is a retract of f in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$.

(2) The homotopy class $[g]$ is a retract of $[f]$ in the ordinary category $\mathrm{Fun}([1], \mathrm{h}\mathcal{C})$.

The implication (1) $\Rightarrow$ (2) is immediate. Conversely, suppose that (2) is satisfied. Then we can choose a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "{[i]}"] \arrow[d, "{[g]}"] & X \arrow[r, "{[r]}"] \arrow[d, "{[f]}"] & Y \arrow[d, "{[g]}"] \\
Y' \arrow[r, "{[i']}"] & X' \arrow[r, "{[r']}"] & Y'
\end{tikzcd}$$

in the homotopy category $\mathrm{h}\mathcal{C}$, where the horizontal compositions are the identity morphisms $[\mathrm{id}_Y]$ and $[\mathrm{id}_{Y'}]$, respectively. By virtue of Exercise 1.5.2.10, the squares on the left and right of this diagram can be lifted to commutative diagrams in the ∞-category $\mathcal{C}$, which we can identify with morphisms $\alpha : g \to f$ and $\beta : f \to g$ in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. Beware that the composition $(\beta \circ \alpha) : g \to g$ need not be homotopic to the identity morphism id_g. However, the criterion of Theorem 4.4.4.4 guarantees that $\beta \circ \alpha$ is an isomorphism in $\mathrm{Fun}(\Delta^1, \mathcal{C})$. In particular, α admits a left homotopy inverse, and therefore exhibits g as a retract of f.

03YL **Remark 8.5.1.4.** Let $\mathcal{C}$ be a category containing an object X. Then an object $Y \in \mathcal{C}$ is retract of X if and only if it is a retract of X when viewed as an object of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ (in the sense of Definition 8.5.1.1). Consequently, Definition 8.5.1.1 can be viewed as a generalization of the classical notion of retract.

03YM **Example 8.5.1.5.** Let $\mathcal{C}$ be an ∞-category containing an object X. If an object $Y \in \mathcal{C}$ is isomorphic to X, then Y is a retract of X. In particular, the object X is a retract of itself.

Remark 8.5.1.6 (Transitivity)**.** Let $\mathcal{C}$ be an ∞-category containing objects X, Y, and Z. If Y is a retract of X and Z is a retract of Y, then Z is a retract of X. To prove this, it suffices to establish the analogous result for the homotopy category $\mathrm{h}\mathcal{C}$ (Remark 8.5.1.2), which follows immediately from Remark 8.5.0.1. `03YN`

In practice, many important properties of an object X of an ∞-category $\mathcal{C}$ are inherited by any retract of X. We record a few examples of this phenomenon which will be useful later.

Proposition 8.5.1.7 (Retracts of Isomorphisms)**.** *Let $\mathcal{C}$ be an ∞-category containing morphisms $f : X \to X'$ and $g : Y \to Y'$. Suppose that g is a retract of f (when regarded as objects of the arrow ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$). If f is an isomorphism, then g is also an isomorphism.* `03YP`

Proof. By virtue of Variant 8.5.1.3, we may assume that $\mathcal{C}$ is (the nerve of) an ordinary category. Choose a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, "g"] & X \arrow[r, "r"] \arrow[d, "f"] & Y \arrow[d, "g"] \\
Y' \arrow[r, "i'"] & X' \arrow[r, "r'"] & Y',
\end{tikzcd}$$

where the horizontal compositions are the identity morphisms id_Y and $\mathrm{id}_{Y'}$, respectively. If f is an isomorphism, then g is also an isomorphism, with inverse given by the composition $Y' \xrightarrow{i'} X' \xrightarrow{f^{-1}} X \xrightarrow{r} Y$. This follows from the calculations

$$g \circ r \circ f^{-1} \circ i' = r' \circ f \circ f^{-1} \circ i' = r' \circ \mathrm{id}_{X'} \circ i' = r' \circ i' = \mathrm{id}_{Y'}$$

$$r \circ f^{-1} \circ i' \circ g = r \circ f^{-1} \circ f \circ i = r \circ \mathrm{id}_X \circ i = r \circ i = \mathrm{id}_Y .$$

□

Proposition 8.5.1.8. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that every object $Y \in \mathcal{C}$ is a retract of some object $X \in \mathcal{C}^0$. Then F is left and right Kan extended from $\mathcal{C}^0$.* `03YQ`

Proof. We will show that F is left Kan extended from $\mathcal{C}^0$; the assertion that F is right Kan extended from $\mathcal{C}^0$ follows by a similar argument. Choose a regular cardinal κ for which $\mathcal{C}$ is essentially κ-small. Using Corollary 8.3.3.17, we can choose a fully faithful functor $\mathcal{D} \to \widehat{\mathcal{D}}$, where $\widehat{\mathcal{D}}$ admits κ-small colimits. By virtue of Remark 7.3.1.13, we can replace $\mathcal{D}$ by $\widehat{\mathcal{D}}$ and thereby reduce to proving Proposition 8.5.1.8 in the special case where $\mathcal{D}$ admits κ-small colimits.

Set $F^0 = F|_{\mathcal{C}^0}$. Using Proposition 7.6.7.13, we can extend F^0 to a functor $F' : \mathcal{C} \to \mathcal{D}$ which is left Kan extended from $\mathcal{C}^0$. Invoking the universal mapping property of Corollary 7.3.6.9, we deduce that there is a natural transformation $\alpha : F' \to F$ which restricts to the identity transformation from F^0 to itself. The natural transformation α carries each object $Y \in \mathcal{C}$ to a morphism $\alpha_Y : F'(Y) \to F(Y)$ in the ∞-category $\mathcal{D}$. By assumption, the object Y is a retract of some object $X \in \mathcal{C}^0$. It follows that α_Y is a retract of the morphism $\alpha_X = \mathrm{id}_{F(X)}$, and is therefore an isomorphism (Proposition 8.5.1.7). Invoking Theorem 4.4.4.4, we deduce that the natural transformation α is an isomorphism, so that F is left Kan extended from $\mathcal{C}^0$ by virtue of Remark 7.3.3.17. □

Proposition 8.5.1.8 immediately implies the following stronger version of Proposition 7.3.3.7:

03YR **Corollary 8.5.1.9.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that F is U-left Kan extended from $\mathcal{C}^0$ at an object $X \in \mathcal{C}$. If $Y \in \mathcal{C}$ is a retract of X, then F is also U-left Kan extended from $\mathcal{C}^0$ at Y.*

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is spanned by $\mathcal{C}^0$ together with the objects X and Y. By virtue of Proposition 7.3.8.6, we can further assume that $\mathcal{C}^0$ contains the object X. In this case, Proposition 8.5.1.8 implies that the functors F and $U \circ F$ are left Kan extended from $\mathcal{C}^0$, so that F is U-left Kan extended from $\mathcal{C}^0$ by virtue of Remark 7.3.3.20. □

03YS **Corollary 8.5.1.10.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{E}$ be functors of ∞-categories. Suppose that F is U-left Kan extended from a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$. Then any functor $G : \mathcal{C} \to \mathcal{D}$ which is a retract of F (in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$) is also U-left Kan extended from $\mathcal{C}^0$.*

Proof. Let $\mathrm{ev} : \mathcal{C} \times \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$ denote the evaluation functor. By virtue of Remark 7.3.3.3, the functor F is U-left Kan extended from $\mathcal{C}^0$ at an object $C \in \mathcal{C}$ if and only if the functor ev is U-left Kan extended from $\mathcal{C}^0 \times \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ at the object (C, F). If this condition is satisfied, then Corollary 8.5.1.9 guarantees that ev is also U-left Kan extended from $\mathcal{C}^0 \times \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ at the object (C, G), so that G is U-left Kan extended from $\mathcal{C}^0$ at C. The desired result now follows by allowing the object $C \in \mathcal{C}$ to vary. □

03YT **Corollary 8.5.1.11.** *Let $U : \mathcal{D} \to \mathcal{E}$ be a functor of ∞-categories, let K be a simplicial set, and suppose we are given a pair of diagrams $f, g : K^{\triangleright} \to \mathcal{D}$. If f is a U-colimit diagram and g is a retract of f (in the ∞-category $\mathrm{Fun}(K^{\triangleright}, \mathcal{D})$), then g is also a U-colimit diagram.*

Proof. Using Corollary 4.1.3.3, we can choose an inner anodyne morphism $K \hookrightarrow \mathcal{K}$, where $\mathcal{K}$ is an ∞-category. Using Remark 4.3.6.7, we see that the induced map $K^{\triangleright} \hookrightarrow \mathcal{K}^{\triangleright}$ is also inner anodyne. We may therefore extend f and g to functors $F, G : \mathcal{K}^{\triangleright} \to \mathcal{D}$. Since the restriction functor $\mathrm{Fun}(\mathcal{K}^{\triangleright}, \mathcal{D}) \to \mathrm{Fun}(K^{\triangleright}, \mathcal{D})$ is a trivial Kan fibration (Proposition 1.5.7.6), it follows that G is a retract of F. By virtue of Corollary 7.2.2.2, we can replace K by $\mathcal{K}$ and thereby reduce to proving Corollary 8.5.1.11 in the special case where K is an ∞-category. In this case, the desired result is a special case of Corollary 8.5.1.10 (see Example 7.3.3.9). □

Corollary 8.5.1.12. *Let $\mathcal{D}$ be an ∞-category and let $f, g : K^{\triangleright} \to \mathcal{D}$ be diagrams. If f is a colimit diagram and g is a retract of f, then g is also a colimit diagram.* 05E6

Proof. Apply Corollary 8.5.1.11 in the special case $\mathcal{E} = \Delta^0$. □

Corollary 8.5.1.13. *Let $U : \mathcal{D} \to \mathcal{E}$ be an inner fibration of ∞-categories and let f be a U-cocartesian morphism in $\mathcal{D}$. Then any retract of f (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$) is also U-cocartesian.* 04KA

Proof. Apply Corollary 8.5.1.11 in the special case $K = \Delta^0$ (see Example 7.1.5.9). □

Corollary 8.5.1.14. *Let K be a simplicial set and let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves K-indexed colimits. If $G : \mathcal{C} \to \mathcal{D}$ is a retract of F (in the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$), then G also preserves K-indexed colimits.* 03YU

Proposition 8.5.1.15. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-categories and let κ be an uncountable cardinal. Suppose that $\mathcal{Y}$ is a retract of $\mathcal{X}$ in the ∞-category $\mathcal{QC}$. If $\mathcal{X}$ is essentially κ-small, then $\mathcal{Y}$ is also essentially κ-small.* 03YV

Proof. By virtue of Proposition 4.7.6.15, we may assume that the ∞-categories $\mathcal{X}$ and $\mathcal{Y}$ are minimal, so that $\mathcal{X}$ is a κ-small simplicial set (Corollary 4.7.6.12). Choose functors $i : \mathcal{Y} \to \mathcal{X}$ and $r : \mathcal{X} \to \mathcal{Y}$ such that the composition $(r \circ i) : \mathcal{Y} \to \mathcal{Y}$ is isomorphic to the identity functor. Then $r \circ i$ is an equivalence of ∞-categories. Since $\mathcal{Y}$ is minimal, it follows that $r \circ i$ is an isomorphism of simplicial sets (Proposition 4.7.6.13). In particular, the functor $i : \mathcal{Y} \to \mathcal{X}$ is a monomorphism of simplicial sets. It follows that $\mathcal{Y}$ is κ-small (Remark 4.7.4.8), and therefore essentially κ-small. □

Remark 8.5.1.16. In the situation of Proposition 8.5.1.15, suppose that the ∞-category $\mathcal{X}$ is a Kan complex. Then $\mathcal{Y}$ is also a Kan complex, and is therefore a retract of X in the homotopy category hKan. To prove this, it will suffice to show that every morphism $f : Y \to Y'$ in the ∞-category $\mathcal{Y}$ is an isomorphism (Proposition 4.4.2.1). Since $\mathcal{X}$ is a Kan complex, the morphism $i(f) : i(Y) \to i(Y')$ is an isomorphism in $\mathcal{X}$ (Proposition 1.4.6.10). It follows that $(r \circ i)(f)$ is an isomorphism in $\mathcal{Y}$ (Remark 1.5.1.6). Since f is isomorphic to $(r \circ i)(f)$ (as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{Y})$), it is also an isomorphism (Example 4.4.1.14). 03YW

Corollary 8.5.1.17. *Let X and Y be Kan complexes and let κ be an uncountable cardinal. Suppose that Y is a retract of X in the ∞-category $\mathcal{S}$. If X is essentially κ-small, then Y is essentially κ-small.* 03YX

03YY **Warning 8.5.1.18.** In the statement of Corollary 8.5.1.17, the uncountability assumption on κ cannot be omitted. That is, if X is a Kan complex for which there exists a weak homotopy equivalence $K \to X$ for a *finite* simplicial set K, then a retract of X need not inherit the same property. See §[?].

We now make Definition 8.5.1.1 slightly more explicit.

03YZ **Definition 8.5.1.19.** Let $\mathcal{C}$ be an ∞-category. A *retraction diagram in* $\mathcal{C}$ is a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ for which the "long" face $d_1^2(\sigma)$ is an identity morphism of $\mathcal{C}$. In this case, we indicate σ by a diagram

$$\begin{tikzcd}
 & X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y,
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, and we say that σ *exhibits* Y *as a retract of* X.

03Z0 **Remark 8.5.1.20.** Let $\mathcal{C}$ be an ∞-category containing an object X. Then an object $Y \in \mathcal{C}$ is a retract of X (in the sense of Definition 8.5.1.1) if and only if there exists a retraction diagram which exhibits Y as a retract of X (in the sense of Definition 8.5.1.19).

03Z1 **Warning 8.5.1.21.** If $\mathcal{C}$ is (the nerve of) an ordinary category, then a retraction diagram in $\mathcal{C}$ can be identified with a pair of morphisms $i : Y \to X$ and $r : X \to Y$ satisfying the condition $r \circ i = \mathrm{id}_Y$. Beware that, if $\mathcal{C}$ is a general ∞-category, then a retraction diagram

$$\begin{tikzcd}
 & X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y
\end{tikzcd}$$

generally cannot be recovered (even up to isomorphism) from the morphisms i and r alone: one also needs a homotopy which witnesses the identity $[r] \circ [i] = [\mathrm{id}_Y]$ in the homotopy category $\mathrm{h}\mathcal{C}$.

03Z2 **Remark 8.5.1.22.** Let $\mathcal{C}$ be an ∞-category. A 2-simplex σ of $\mathcal{C}$ is a retraction diagram if and only if it is a retraction diagram when viewed as an object of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. Consequently, if X and Y are objects of $\mathcal{C}$, then Y is a retract of X in $\mathcal{C}$ if and only if it is a retract of X in the ∞-category $\mathcal{C}^{\mathrm{op}}$.

Remark 8.5.1.23 (Lifting Retraction Diagrams)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of ∞-categories. Suppose we are given a retraction diagram 04KB

04KC

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y \end{tikzcd} \tag{8.54}$$

in $\mathcal{C}$ and an object $\widetilde{Y} \in \mathcal{E}$ satisfying $U(\widetilde{Y}) = Y$. Our assumption that U is a cartesian fibration guarantees the existence of a U-cartesian morphism $\widetilde{r} : \widetilde{X} \to \widetilde{Y}$ in $\mathcal{E}$ satisfying $U(\widetilde{r}) = r$. Since $\widetilde{r}$ is U-cartesian, we can lift (8.54) to a retraction diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dr, "\widetilde{r}"] & \\ \widetilde{Y} \arrow[ur, "\widetilde{i}"] \arrow[rr, "\mathrm{id}_{\widetilde{Y}}"] & & \widetilde{Y} \end{tikzcd}$$

In particular, the object $\widetilde{Y}$ can be realized as a retract of an object $\widetilde{X}$ satisfying $U(\widetilde{X}) = X$.

Our goal in this section is carry out an ∞-categorical analogue of Exercise 8.5.0.3.

Notation 8.5.1.24. Let Ret denote the category introduced in Construction 8.5.0.2. By construction, the ∞-category $\mathrm{N}_\bullet(\mathrm{Ret})$ contains a retraction diagram $\sigma : \Delta^2 \to \mathrm{N}_\bullet(\mathrm{Ret})$, which we depict as 03Z3

$$\begin{tikzcd} & \widetilde{X} \arrow[dr, "\widetilde{r}"] & \\ \widetilde{Y} \arrow[ur, "\widetilde{i}"] \arrow[rr, "\mathrm{id}"] & & \widetilde{Y}. \end{tikzcd}$$

We let $\mathcal{R}$ denote the image of σ, which we regard as a simplicial subset of $\mathrm{N}_\bullet(\mathrm{Ret})$.

Remark 8.5.1.25. In the situation of Notation 8.5.1.24, the map $\sigma : \Delta^2 \twoheadrightarrow \mathcal{R}$ is an epimorphism of simplicial sets, which fits into a pushout square 03Z4

$$\begin{tikzcd} \mathrm{N}_\bullet(\{0 < 2\}) \arrow[r] \arrow[d] & \Delta^0 \arrow[d] \\ \Delta^2 \arrow[r, "\sigma"] & \mathcal{R}. \end{tikzcd}$$

It follows that, for every ∞-category $\mathcal{C}$, composition with σ induces a bijection from $\operatorname{Hom}_{\mathrm{Set}_\Delta}(\mathcal{R}, \mathcal{C})$ to the set of retraction diagrams in $\mathcal{C}$ (in the sense of Definition 8.5.1.19).

03Z5 **Remark 8.5.1.26.** Let $\sigma : \Delta^2 \twoheadrightarrow \mathcal{R}$ be the epimorphism of Notation 8.5.1.24. For every ∞-category $\mathcal{C}$, precomposition with σ induces a fully faithful functor $\operatorname{Fun}(\mathcal{R}, \mathcal{C}) \hookrightarrow \operatorname{Fun}(\Delta^2, \mathcal{C})$, whose essential image is the full subcategory $\operatorname{Fun}'(\Delta^2, \mathcal{C}) \subseteq \operatorname{Fun}(\Delta^2, \mathcal{C})$ spanned by those diagrams

$$\begin{tikzcd}
 & X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "u"] & & Y'
\end{tikzcd}$$

where u is an isomorphism. This follows by applying Corollary 4.5.2.29 to the pullback square

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{R}, \mathcal{C}) \arrow[r] \arrow[d] & \operatorname{Fun}'(\Delta^2, \mathcal{C}) \arrow[d] \\
\mathcal{C} \arrow[r] & \operatorname{Isom}(\mathcal{C}),
\end{tikzcd}$$

since the vertical maps are isofibrations (Corollary 4.4.5.3) and the lower horizontal map is an equivalence of ∞-categories by virtue of Corollary 4.5.3.13.

Our main result can now be stated as follows:

03Z6 **Proposition 8.5.1.27.** *The inclusion map* $\mathcal{R} \hookrightarrow \mathrm{N}_\bullet(\mathrm{Ret})$ *is an inner anodyne morphism of simplicial sets.*

03Z7 **Corollary 8.5.1.28.** *Let* $\mathcal{C}$ *be an* ∞*-category. Then composition with the inclusion map* $\mathcal{R} \hookrightarrow \mathrm{N}_\bullet(\mathrm{Ret})$ *induces a trivial Kan fibration*

$$\operatorname{Fun}(\mathrm{N}_\bullet(\mathrm{Ret}), \mathcal{C}) \to \operatorname{Fun}(\mathcal{R}, \mathcal{C}) \simeq \operatorname{Fun}(\Delta^2, \mathcal{C}) \times_{\operatorname{Fun}(\mathrm{N}_\bullet(\{0<2\}), \mathcal{C})} \mathcal{C}.$$

In particular, every retraction diagram in $\mathcal{C}$ *can be extended to a functor* $\mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$, *which is uniquely determined up to isomorphism.*

Proof. Combine Propositions 8.5.1.27 and 1.5.7.6. □

03Z8 **Remark 8.5.1.29.** Exercise 8.5.0.3 is an immediate consequence of Proposition 8.5.1.27 (see Variant 1.5.6.8).

Corollary 8.5.1.30. *Let $\mathcal{C}$ be an ∞-category. Then composition with the retraction diagram of Notation 8.5.1.24 induces a fully faithful functor* $\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}) \to \mathrm{Fun}(\Delta^2, \mathcal{C})$*, whose essential image is spanned by those diagrams* 03Z9

$$\begin{tikzcd}
 & X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "u"] & & Y'
\end{tikzcd}$$

where u is an isomorphism.

Proof. Combine Corollary 8.5.1.28 with Remark 8.5.1.26. □

Remark 8.5.1.31. Corollary 8.5.1.30 asserts that the map $\sigma : \Delta^2 \to \mathrm{N}_{\bullet}(\mathrm{Ret})$ exhibits $\mathrm{N}_{\bullet}(\mathrm{Ret})$ as a localization of the standard 2-simplex Δ^2 with respect to the "long edge" $0 \to 2$ (see Definition 6.3.1.9). 04KD

Corollary 8.5.1.32. *Let $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ be a diagram of simplicial sets indexed by a filtered category $\mathcal{I}$. Suppose that each $\mathcal{C}_i$ is an ∞-category. Then the tautological map* 03ZA

$$\theta : \varinjlim_{i \in \mathcal{I}} \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}_i) \to \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \varinjlim_{i \in \mathcal{I}} \mathcal{C}_i)$$

is an equivalence of ∞-categories.

Proof. The morphism θ fits into a commutative diagram

$$\begin{tikzcd}
\varinjlim_{i \in \mathcal{I}} \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}_i) \arrow[r, "\theta"] \arrow[d] & \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \varinjlim_{i \in \mathcal{I}} \mathcal{C}_i) \arrow[d] \\
\varinjlim_{i \in \mathcal{I}} \mathrm{Fun}(\mathcal{R}, \mathcal{C}_i) \arrow[r, "\theta'"] & \mathrm{Fun}(\mathcal{R}, \varinjlim_{i \in \mathcal{I}} \mathcal{C}_i),
\end{tikzcd}$$

where the vertical maps are trivial Kan fibrations (Corollary 8.5.1.30). It will therefore suffice to show that θ' is an equivalence of ∞-categories. In fact, θ' is an isomorphism of simplicial sets, since the simplicial set $\mathcal{R}$ is finite (Corollary 3.6.1.10). □

The proof of Proposition 8.5.1.27 will require the following:

Lemma 8.5.1.33 (Sparse Horns). *Let $n \geq 0$ be an integer and let S be a subset of $[n] = \{0 < 1 < \cdots < n\}$. Let $K \subseteq \Delta^n$ be the simplicial subset spanned by those nondegenerate simplices which do not contain every element of S. Suppose that there exist $0 \leq i < j < k \leq n$ such that $i, k \in S$, $j \notin S$. Then the inclusion $K \hookrightarrow \Delta^n$ is inner anodyne.* 03ZB

03ZC **Example 8.5.1.34.** In the situation of Lemma 8.5.1.33, suppose that $S = [n] \setminus \{j\}$ for some $0 \leq j \leq n$. Then K is the horn $\Lambda^n_j \subseteq \Delta^n$. The hypothesis of Lemma 8.5.1.33 guarantees that K is an inner horn, so that the inclusion map $K \hookrightarrow \Delta^n$ is inner anodyne by definition.

Proof of Lemma 8.5.1.33. Let P denote the collection of all subsets $S' \subseteq [n]$ which contain $S \cup \{j\}$. Choose a linear ordering

$$\{S(1) \leq \cdots \leq S(c)\}$$

of P with the property that if $S(a) \subseteq S(b)$, then $a \leq b$. Let For $0 \leq b \leq c$, let $K(b) \subseteq \Delta^n$ denote the union of K with the faces $\{\mathrm{N}_\bullet(S(a)) \subseteq \Delta^n\}_{1 \leq a \leq b}$. We then have inclusion maps

$$K = K(0) \subseteq K(1) \subseteq K(2) \subseteq \cdots \subseteq K(c-1) \subseteq K(c) = \Delta^n.$$

It will therefore suffice to show that, for every positive integer $b \leq c$, the inclusion map $K(b-1) \hookrightarrow K(b)$ is inner anodyne.

Let us identify $\mathrm{N}_\bullet(S(b))$ with the image of a nondegenerate simplex $\sigma : \Delta^m \hookrightarrow \Delta^n$. Let $L \subseteq \Delta^m$ be the inverse image $\sigma^{-1}(S(b-1))$, so that we have a pushout diagram of simplicial sets

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & \Delta^m \arrow[d, "\sigma"] \\
S(b-1) \arrow[r] & S(b).
\end{tikzcd}$$

It will therefore suffice to show that the inclusion map $L \subseteq \Delta^m$ is inner anodyne. Because $S(b)$ contains the integers $i < j < k$, we can write $j = \sigma(\overline{j})$ for some $0 < \overline{j} < n$. We conclude by observing that L can be identified with the inner horn $\Lambda^m_{\overline{j}} \subseteq \Delta^m$. □

Proof of Proposition 8.5.1.27. Let τ be a nondegenerate m-simplex of the simplicial set $\mathrm{N}_\bullet(\mathrm{Ret})$. We define the *weight* $w(\tau)$. to be the cardinality of the set $\{i \in [m] : \tau(i) = \widetilde{X}\}$. Note that, if τ' is any nondegenerate facet of τ, then $w(\tau') \leq w(\tau)$. For $n \geq 1$, the collection of nondegenerate simplices of weight $\leq n$ span a simplicial subset $\mathcal{R}(n) \subseteq \mathrm{N}_\bullet(\mathrm{Ret})$. It follows that we can write $\mathrm{N}_\bullet(\mathrm{Ret})$ as the union of an increasing sequence

$$\mathcal{R}(1) \hookrightarrow \mathcal{R}(2) \hookrightarrow \mathcal{R}(3) \hookrightarrow \cdots,$$

where $\mathcal{R}(1)$ coincides with the simplicial set $\mathcal{R}$ introduced in Notation 8.5.1.24. We will complete the proof by showing that, for each $n \geq 2$, the inclusion map $\mathcal{R}(n-1) \hookrightarrow \mathcal{R}(n)$ is inner anodyne.

Let $\sigma_n : \Delta^{2n} \to \mathrm{N}_\bullet(\mathrm{Ret})$ denote the simplex corresponding to the diagram

$$\widetilde{Y} \xrightarrow{\widetilde{i}} \widetilde{X} \xrightarrow{\widetilde{r}} \widetilde{Y} \xrightarrow{\widetilde{i}} \widetilde{X} \to \cdots \to \widetilde{X} \xrightarrow{\widetilde{r}} \widetilde{Y} \xrightarrow{\widetilde{i}} \widetilde{X} \xrightarrow{\widetilde{r}} \widetilde{Y}.$$

Note that σ_n is a nondegenerate simplex of weight n, and therefore factors through $\mathcal{R}(n)$. Let $K \subseteq \Delta^{2n}$ denote the inverse image $\sigma_n^{-1}\mathcal{R}(n-1)$, so that we have a commutative diagram of simplicial sets

03ZD

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & \Delta^{2n} \arrow[d, "\sigma_n"] \\
\mathcal{R}(n-1) \arrow[r] & \mathcal{R}(n).
\end{tikzcd} \tag{8.55}$$

Note that a nondegenerate simplex of Δ^{2n} belongs to K if and only if it does not contain $\mathrm{N}_\bullet(\{1 < 3 < \cdots < 2n-1\}) \subseteq \Delta^{2n}$. Applying Lemma 8.5.1.33, we deduce that the inclusion map $K \hookrightarrow \Delta^{2n}$ is inner anodyne. It will therefore suffice to show that the diagram (8.55) is a pushout square.

Let τ be an m-simplex of $\mathrm{N}_\bullet(\mathrm{Ret})$ which belongs to $\mathcal{R}(n)$, but does not belong to $\mathcal{R}(n-1)$. We wish to show that τ factors uniquely through σ_n. We first prove the existence of the desired factorization. For this, we may assume without loss of generality that τ is nondegenerate. Then τ has weight n, so we can write

$$\{i \in [m] : \tau(i) = X\} = \{d_1 < d_2 < \cdots < d_n\}.$$

Let $\alpha : \Delta^m \to \Delta^{2n}$ be the unique morphism of simplices which is given on vertices by the formula $\alpha(d_i) = 2i-1$ for $1 \leq i \leq n$. We claim that $\tau = \sigma_n \circ \alpha$. Note that τ and $\sigma_n \circ \alpha$ can both be regarded as functors from the linearly ordered set $[m]$ to the category Ret. By construction, these functors coincide on objects. It will therefore suffice to show that, for $0 \leq j < j' < m$, the functors τ and $\sigma_n \circ \alpha$ determine the same element of $\mathrm{Hom}_{\mathrm{Ret}}(\tau(j), \tau(j'))$. If $\tau(j) = \widetilde{Y}$ or $\tau(j') = \widetilde{Y}$, this condition is automatic (since the set $\mathrm{Hom}_{\mathrm{Ret}}(\tau(j), \tau(j'))$ has only one element). We may therefore assume without loss of generality that $\tau(j) = \widetilde{X} = \tau(j')$: that is, we have $j = d_i$ and $j' = d_{i'}$ for some $i < i'$. In this case, the functors τ and $\sigma_n \circ \alpha$ both carry the pair $(j < j')$ to the element $e \in \mathrm{Hom}_{\mathrm{Ret}}(X, X)$.

We now prove uniqueness. Suppose we are given a pair of maps $\alpha, \beta : \Delta^m \to \Delta^{2n}$ satisfying $\sigma_n \circ \alpha = \tau = \sigma_n \circ \beta$; we wish to show that $\alpha = \beta$. Suppose otherwise. Then there is some smallest integer $j \in [m]$ such that $\alpha(j) \neq \beta(j)$. Without loss of generality, we may assume that $\alpha(j) < \beta(j)$. Assume first that $\alpha(j)$ is odd. Since τ does not belong to K, $\alpha(j)$ is contained in the image of β; that is, we can write $\alpha(j) = \beta(i)$ for some $i < j$. Then minimality of j then guarantees that $\alpha(i) = \alpha(j)$, so that $\sigma_n \circ \alpha$ carries the pair $(i < j)$ to the identity morphism $\mathrm{id}_{\widetilde{X}}$ in the category Ret. Since $\sigma_n \circ \beta = \tau = \sigma_n \circ \alpha$, the morphism $\sigma_n \circ \beta$ also carries $(i < j)$ to the identity morphism $\mathrm{id}_{\widetilde{X}}$. It follows that $\beta(i) = \beta(j)$, contradicting our assumption that $\beta(i) = \alpha(j) < \beta(j)$.

We now treat the case where $\alpha(j)$ is even, so that $\tau(j) = (\sigma_n \circ \alpha)(j) = Y$. Using the equality $\sigma_n \circ \beta = \tau$, we deduce that $\beta(j)$ is also even. Since τ does not belong to K, the odd number $\beta(j) - 1$ belongs to the image of β. We therefore have $\beta(j) - 1 = \beta(i)$ for some integer $i < j$. We then have

$$\alpha(i) \leq \alpha(j) < \alpha(j) + 1 \leq \beta(j) - 1 = \beta(i),$$

contradicting the minimality of j. □

8.5.2 Idempotents in Ordinary Categories

03ZE Let M be a monoid. Recall that an element $e \in M$ is *idempotent* if it satisfies the equation $e^2 = e$. We now consider the special case where $M = \mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ is the set of endomorphisms of an object X of some category $\mathcal{C}$.

03ZF **Definition 8.5.2.1.** Let $\mathcal{C}$ be a category. An *idempotent endomorphism in* $\mathcal{C}$ is a pair (X, e), where X is an object of $\mathcal{C}$ and $e : X \to X$ is an endomorphism of X which satisfies the identity $e = e \circ e$, so that we have a commutative diagram

$$\begin{tikzcd} & X \arrow[dr, "e"] & \\ X \arrow[ur, "e"] \arrow[rr, "e"] & & X. \end{tikzcd}$$

In this situation, we will also say that e is an *idempotent endomorphism of* X.

03ZG **Example 8.5.2.2** (Identity Morphisms)**.** Let $\mathcal{C}$ be a category. For every object $X \in \mathcal{C}$, the identity morphism $\mathrm{id}_X : X \to X$ is an idempotent endomorphism in $\mathcal{C}$. Conversely, if $e : X \to X$ is an idempotent endomorphism in $\mathcal{C}$ which is also an isomorphism, then $e = \mathrm{id}_X$.

03ZH **Example 8.5.2.3** (Split Idempotents)**.** Let $\mathcal{C}$ be a category containing a retraction diagram

03ZJ

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y \end{tikzcd} \tag{8.56}$$

(see Definition 8.5.1.19). Then $e = i \circ r$ is an idempotent endomorphism of X. This follows from the calculation

$$e \circ e = (i \circ r) \circ (i \circ r) = i \circ \mathrm{id}_Y \circ r = i \circ r = e.$$

We will say that an idempotent endomorphism $e : X \to X$ is *split* if it can be obtained in this way (that is, if $e = i \circ r$, for some pair of morphisms $i : Y \to X$ an $r : X \to Y$ satisfying $r \circ i = \mathrm{id}_Y$.

In the situation of Example 8.5.2.3, the diagram (8.56) can be recovered (up to isomorphism) from the idempotent endomorphism $e : X \to X$, by virtue of the following:

Proposition 8.5.2.4. *Let $\mathcal{C}$ be a category containing a retraction diagram* 03ZK

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y, \end{tikzcd}$$

and let $e = i \circ r$ be the idempotent endomorphism Example 8.5.2.3. Then:

(1) *The morphism i exhibits Y as an equalizer of the pair of morphisms $(e, \mathrm{id}_X) : X \rightrightarrows X$.*

(2) *The morphism r exhibits Y as a coequalizer of the pair of morphisms $(e, \mathrm{id}_X) : X \rightrightarrows X$.*

Proof. We will prove (1); the proof of (2) is similar. Fix an object $Z \in \mathcal{C}$ and a morphism $f : Z \to X$ satisfying $e \circ f = \mathrm{id}_X \circ f$; we wish to show that there is a unique morphism $g : Z \to Y$ satisfying $i \circ g = f$. To prove uniqueness, we note that g is determined by the identity

$$g = \mathrm{id}_Y \circ g = (r \circ i) \circ g = r \circ (i \circ g) = r \circ f.$$

To establish existence, we observe that the composition $g = r \circ f$ satisfies the identity

$$i \circ g = i \circ (r \circ f) = (i \circ r) \circ f = e \circ f = \mathrm{id}_X \circ f = f.$$

□

Corollary 8.5.2.5. *Let $\mathcal{C}$ be a category and let $e : X \to X$ be an idempotent endomorphism in $\mathcal{C}$. The following conditions are equivalent:* 03ZL

(1) *The idempotent endomorphism e splits. That is, e admits a factorization $X \xrightarrow{r} Y \xrightarrow{i} X$, where $r \circ i = \mathrm{id}_Y$.*

(2) *The pair of morphisms $(e, \mathrm{id}_X) : X \rightrightarrows X$ admits an equalizer in $\mathcal{C}$.*

(3) *The pair of morphisms $(e, \mathrm{id}_X) : X \rightrightarrows X$ admits a coequalizer in $\mathcal{C}$.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) follow from Proposition 8.5.2.4. We will show that (2) implies (1); the proof of the implication (3) ⇒ (1) is similar. Suppose that there exists a morphism $i : Y \to X$ which exhibits Y as an equalizer of the pair of morphisms $(e, \mathrm{id}_X) : X \rightrightarrows X$. Since e is idempotent, we have $e \circ e = e = \mathrm{id}_X \circ e$. Invoking the universal property of Y, we deduce that there is a unique morphism $r : X \to Y$ satisfying $e = i \circ r$.

To complete the proof, it will suffice to show that $r \circ i$ is the identity morphism from Y to itself. Since i is a monomorphism, this follows from the calculation

$$i \circ (r \circ i) = (i \circ r) \circ i = e \circ i = \mathrm{id}_X \circ i = i = i \circ \mathrm{id}_Y .$$

□

03ZM **Corollary 8.5.2.6.** *Let $\mathcal{C}$ be a category which admits equalizers (or coequalizers). Then every idempotent endomorphism in $\mathcal{C}$ is split.*

03ZN **Construction 8.5.2.7** (The Universal Idempotent)**.** We define a category Idem as follows:

- The category Idem has single object $\widetilde{X}$.
- Morphisms in Idem are given by $\mathrm{Hom}_{\mathrm{Idem}}(\widetilde{X}, \widetilde{X}) = \{\mathrm{id}_{\widetilde{X}}, \widetilde{e}\}$.
- The composition law on Idem is given (on non-identity morphisms) by $\widetilde{e} \circ \widetilde{e} = \widetilde{e}$.

03ZP **Remark 8.5.2.8.** Let $\mathcal{C}$ be a category and let $e : X \to X$. be an idempotent endomorphism in $\mathcal{C}$. Then there is a unique functor $F : \mathrm{Idem} \to \mathcal{C}$ satisfying $F(\widetilde{X}) = X$ and $F(\widetilde{e}) = e$.

03ZQ **Exercise 8.5.2.9.** Show that the category Idem is filtered (see Definition 7.2.4.1).

03ZR **Remark 8.5.2.10.** Let Ret denote the category introduced in Construction 8.5.0.2. Then Idem can be identified with the full subcategory of Ret spanned by the object $\widetilde{X}$. Let $\mathcal{C}$ be a category and let $\overline{F} : \mathrm{Ret} \to \mathcal{C}$ be the functor determined by a retraction diagram in $\mathcal{C}$ (see Exercise 8.5.0.3). Then the restriction $F = \overline{F}|_{\mathrm{Idem}}$ corresponds (under the identification of Remark 8.5.2.8) to the idempotent endomorphism of Example 8.5.2.3.

8.5.3 Idempotents in ∞-Categories

03ZS We now consider an ∞-categorical counterpart of Definition 8.5.2.1.

03ZT **Definition 8.5.3.1.** Let $\mathcal{C}$ be an ∞-category. An *idempotent in* $\mathcal{C}$ is a functor of ∞-categories $\mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$. Here Idem denotes the category introduced in Construction 8.5.2.7.

03ZU **Remark 8.5.3.2.** Let $\mathcal{C}$ be a category. It follows from Remark 8.5.2.8 (and Proposition 1.3.3.1) that evaluation on the morphism $\widetilde{e} \in \mathrm{Hom}_{\mathrm{Idem}}(\widetilde{X}, \widetilde{X})$ supplies a bijection from the set of idempotents in the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$ (in the sense of Definition 8.5.3.1) to the set of idempotent endomorphisms (X, e) in the category $\mathcal{C}$ (in the sense of Definition 8.5.2.1). We can therefore view Definition 8.5.3.1 as a generalization of Definition 8.5.2.1.

Remark 8.5.3.3 (The Structure of $\mathrm{N}_\bullet(\mathrm{Idem})$)**.** For every integer $n \geq 0$, the simplicial set $\mathrm{N}_\bullet(\mathrm{Idem})$ contains a unique nondegenerate n-simplex σ_n, given by the diagram 03ZV

$$\widetilde{X} \xrightarrow{\widetilde{e}} \widetilde{X} \xrightarrow{\widetilde{e}} \widetilde{X} \xrightarrow{\widetilde{e}} \cdots \rightarrow \widetilde{X} \xrightarrow{\widetilde{e}} \widetilde{X}.$$

Moreover, the face morphisms of $\mathrm{N}_\bullet(\mathrm{Idem})$ satisfy $d_i^n(\sigma_n) = \sigma_{n-1}$ for $0 \leq i \leq n$. Applying Corollary 3.3.1.8, we obtain an isomorphism of $\mathrm{N}_\bullet(\mathrm{Idem})$ with the simplicial set $(\Delta^0)^+$ introduced in Construction 3.3.1.6. Here we abuse notation by identifying Δ^0 with its underlying semisimplicial set.

Remark 8.5.3.4. The simplicial set $\mathrm{N}_\bullet(\mathrm{Idem})$ is weakly contractible. This is a special case of Lemma 3.4.5.9, applied to the (discrete) category $[0]$. 03ZW

Definition 8.5.3.5 (Split Idempotents)**.** Let $\mathcal{C}$ be an ∞-category and let $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ be an idempotent in the ∞-category $\mathcal{C}$. A *splitting of* F is a functor $\overline{F} : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$ satisfying $\overline{F}|_{\mathrm{N}_\bullet(\mathrm{Idem})} = F$. We say that F is *split* if there exists a splitting of F. 03ZX

Example 8.5.3.6. Let $\mathcal{C}$ be a category and let $e : X \to X$ be an idempotent endomorphism in $\mathcal{C}$. Then e is split (in the sense of Example 8.5.2.3) if and only if the induced map $\mathrm{N}_\bullet(\mathrm{Idem}) \to \mathrm{N}_\bullet(\mathcal{C})$ is a split idempotent in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ (in the sense of Definition 8.5.3.5). 03ZY

Remark 8.5.3.7. Let $\mathcal{C}$ be an ∞-category and let $F, F' : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ be idempotents which are isomorphic (when regarded as objects of the ∞-category $\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C})$). Then F is split if and only if F' is split. See Corollary 4.4.5.3. 03ZZ

Let $\mathcal{C}$ be an ∞-category and let $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ be an idempotent in $\mathcal{C}$. If F is split, then the splitting is essentially unique.

Proposition 8.5.3.8. *Let $\mathcal{C}$ be an ∞-category and let $\overline{F} : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$ be a functor. Then $\overline{F}$ is both left and right Kan extended from the full subcategory* $\mathrm{N}_\bullet(\mathrm{Idem}) \subset \mathrm{N}_\bullet(\mathrm{Ret})$. 0400

Proof. This is a special case of Proposition 8.5.1.8. □

Remark 8.5.3.9. The category Ret of Construction 8.5.0.2 contains an initial object $\widetilde{Y}$. It follows that the inclusion map $\mathrm{Idem} \hookrightarrow \mathrm{Ret}$ has a unique extension $T : \mathrm{Idem}^{\triangleleft} \to \mathrm{Ret}$ carrying the cone point of $\mathrm{Idem}^{\triangleleft}$ to the object $\widetilde{Y}$. Unwinding the definitions, we see that a functor of ∞-categories $\overline{F} : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$ is right Kan extended from $\mathrm{N}_\bullet(\mathrm{Idem})$ if and only if the composition 0401

$$\mathrm{N}_\bullet(\mathrm{Idem})^{\triangleleft} \xrightarrow{\mathrm{N}_\bullet(T)} \mathrm{N}_\bullet(\mathrm{Ret}) \xrightarrow{\overline{F}} \mathcal{C}$$

is a limit diagram. Proposition 8.5.3.8 asserts that this condition is automatically satisfied. In particular, the object $\overline{F}(\widetilde{Y})$ is a limit of the underlying diagram $F = \overline{F}|_{\mathrm{N}_\bullet(\mathrm{Idem})}$. Similarly, $\overline{F}(\widetilde{Y})$ is a colimit of the diagram F.

0402 **Corollary 8.5.3.10** (Uniqueness of Splittings)**.** *Let $\mathcal{C}$ be an ∞-category. Then the restriction functor*

$$\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C})$$

is fully faithful, and its essential image is the full subcategory consists of the split idempotents in $\mathcal{C}$.

Proof. Combine Proposition 8.5.3.8 with Corollary 7.3.6.15. □

0403 **Corollary 8.5.3.11.** *Let $\mathcal{C}$ be an ∞-category and let $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$ be an idempotent in $\mathcal{C}$. The following conditions are equivalent:*

(1) *The idempotent F is split: that is, it can be extended to a functor $\mathrm{N}_{\bullet}(\mathrm{Ret}) \to \mathcal{C}$.*

(2) *The diagram F admits a limit in $\mathcal{C}$.*

(3) *The diagram F admits a colimit in $\mathcal{C}$.*

Proof. The implications (1) ⇒ (2) and (1) ⇒ (3) follow from Remark 8.5.3.9, and the reverse implications follow from Corollary 7.3.5.8. □

0404 **Corollary 8.5.3.12.** *Let $\mathcal{C}$ be an ∞-category and let $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$ be an idempotent in $\mathcal{C}$. If F admits a limit (or colimit) in $\mathcal{C}$, then it is preserved by any functor of ∞-categories $G : \mathcal{C} \to \mathcal{D}$.*

Proof. Suppose that F admits a limit in $\mathcal{C}$. Then F splits (Corollary 8.5.3.11): that is, it extends to a diagram $\overline{F} : \mathrm{N}_{\bullet}(\mathrm{Ret}) \to \mathcal{C}$. Let $T : \mathrm{Idem}^{\triangleleft} \to \mathrm{Ret}$ be as in Remark 8.5.3.9, so that $(\overline{F} \circ \mathrm{N}_{\bullet}(T)) : \mathrm{N}_{\bullet}(\mathrm{Idem})^{\triangleleft} \to \mathcal{C}$ is a limit diagram in the ∞-category $\mathcal{C}$. We wish to show that the functor $(G \circ \overline{F} \circ \mathrm{N}_{\bullet}(T)) : \mathrm{N}_{\bullet}(\mathrm{Idem})^{\triangleleft} \to \mathcal{D}$ is a limit diagram in the ∞-category $\mathcal{D}$. By virtue of Proposition 8.5.3.8, this is automatic (Remark 8.5.3.9). □

8.5.4 Idempotent Completeness

0405 We now study ∞-categories in which every idempotent splits.

0406 **Definition 8.5.4.1.** Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *idempotent complete* if every idempotent $\mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$ splits (see Definition 8.5.3.5).

0407 **Example 8.5.4.2.** Let $\mathcal{C}$ be a category. If $\mathcal{C}$ admits equalizers (or coequalizers), then the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$ is idempotent complete (this is a restatement of Corollary 8.5.2.6). In particular, if $\mathcal{C}$ admits finite limits or finite colimits, then $\mathrm{N}_{\bullet}(\mathcal{C})$ is idempotent complete.

0408 **Warning 8.5.4.3.** An ∞-category which admits finite limits (or colimits) need not be idempotent complete. See Example [?].

Example 8.5.4.4. Let X be a Kan complex. Since the simplicial set $\mathrm{N}_{\bullet}(\mathrm{Idem})$ is weakly contractible (Remark 8.5.3.4), every morphism of simplicial sets $\mathrm{N}_{\bullet}(\mathrm{Idem}) \to X$ is homotopic to a constant map. It follows that X is idempotent complete when viewed as an ∞-category. `0409`

Remark 8.5.4.5. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is idempotent complete if and only if the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ is idempotent complete. `040A`

Proposition 8.5.4.6. *Let $\mathcal{C}$ be an idempotent complete ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory. Suppose that, for every object $X \in \mathcal{C}_0$ and every object $Y \in \mathcal{C}$ which is a retract of X, there exists an object $Y' \in \mathcal{C}_0$ which is isomorphic to Y. Then $\mathcal{C}_0$ is idempotent complete.* `040B`

Proof. Let Ret denote the category of Construction 8.5.0.2. Suppose we are given an idempotent $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}_0$, carrying the object $\widetilde{X} \in \mathrm{Idem}$ to an object $X = F(\widetilde{X}) \in \mathcal{C}_0$. We wish to show that F is a split idempotent in $\mathcal{C}_0$. Since $\mathcal{C}$ is idempotent complete, we can extend F to a functor $\overline{F} : \mathrm{N}_{\bullet}(\mathrm{Ret}) \to \mathcal{C}$, carrying the object $\widetilde{Y} \in \mathrm{Ret}$ to an object $Y = \overline{F}(\widetilde{Y})$ which is a retract of X. By assumption, we can choose an isomorphism $\alpha_0 : Y \to Y'$, where Y' belongs to $\mathcal{C}_0$. Using Corollary 4.4.5.3, we can lift α_0 to an isomorphism of functors $\alpha : \overline{F} \to \overline{F}'$ in $\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C})$, whose image in $\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C})$ is the identity transformation from F to itself. Then $\overline{F}' : \mathrm{N}_{\bullet}(\mathrm{Ret}) \to \mathcal{C}_0$ is a splitting of the idempotent F. □

Proposition 8.5.4.7. *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:* `040C`

(1) *The ∞-category $\mathcal{C}$ is idempotent complete.*

(2) *The ∞-category $\mathcal{C}$ admits limits indexed by* $\mathrm{N}_{\bullet}(\mathrm{Idem})$.

(3) *The ∞-category $\mathcal{C}$ admits colimits indexed by* $\mathrm{N}_{\bullet}(\mathrm{Idem})$.

Proof. This is an immediate consequence of Corollary 8.5.3.11. □

Remark 8.5.4.8. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is idempotent complete if and only if the restriction functor `040D`

$$\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C})$$

is an equivalence of ∞-categories. This is an immediate consequence of Corollary 8.5.3.10.

Corollary 8.5.4.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which are equivalent. Then $\mathcal{C}$ is idempotent complete if and only if $\mathcal{D}$ is idempotent complete.* `040E`

Corollary 8.5.4.10. *Let $\mathcal{C}$ be an ∞-category and let K be a simplicial set. If $\mathcal{C}$ is idempotent complete, then* $\mathrm{Fun}(K, \mathcal{C})$ *is idempotent complete.* `040F`

Proof. Combine Propositions 8.5.4.7 and 7.1.6.1. □

040G **Corollary 8.5.4.11.** *Let $\mathcal{C}$ be an ∞-category and let $f : K \to \mathcal{C}$ be a morphism of simplicial sets. If $\mathcal{C}$ is idempotent complete, then the slice and coslice ∞-categories $\mathcal{C}_{/f}$ and $\mathcal{C}_{f/}$ are idempotent complete.*

Proof. Combine Proposition 8.5.4.7 with Corollary 7.1.3.20. □

To apply the criterion of Proposition 8.5.4.7, it is often useful to replace $\mathrm{N}_\bullet(\mathrm{Idem})$ by a simpler simplicial set.

040H **Notation 8.5.4.12.** Let $\mathrm{Spine}[\mathbf{Z}]$ denote the 1-dimensional simplicial set associated to the directed graph

$$\cdots \to -2 \to -1 \to 0 \to 1 \to 2 \to \cdots$$

We let $Q : \mathrm{Spine}[\mathbf{Z}] \to \mathrm{N}_\bullet(\mathrm{Idem})$ be the morphism of simplicial sets corresponding to the diagram

$$\cdots \to X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \to \cdots$$

in the category Idem.

040J **Remark 8.5.4.13.** Since the simplicial set $\mathrm{Spine}[\mathbf{Z}]$ is 1-dimensional, the morphism Q of Notation 8.5.4.12 factors (uniquely) through the 1-skeleton of $\mathrm{N}_\bullet(\mathrm{Idem})$, which we can identify with the simplicial circle $\Delta^1 / \partial\Delta^1$. Under this identification, Q corresponds to a morphism of simplicial sets $q : \mathrm{Spine}[\mathbf{Z}] \to \Delta^1 / \partial\Delta^1$. This is a covering map (see Definition 3.1.4.1), which exhibits the simplicial circle $\Delta^1 / \partial\Delta^1$ as the quotient of $\mathrm{Spine}[\mathbf{Z}]$ by a free action of the group $(\mathbf{Z}, +)$ by translations. The induced map of geometric realizations $|\mathrm{Spine}[\mathbf{Z}]| \to |\Delta^1 / \partial\Delta^1|$ can be identified with the standard covering map $\mathbf{R} \to S^1$ in the category of topological spaces.

040K **Remark 8.5.4.14.** In the situation of Notation 8.5.4.12, we can regard $\mathrm{Spine}[\mathbf{Z}]$ as a simplicial subset of the nerve $\mathrm{N}_\bullet(\mathbf{Z})$, where we regard the set of integers $\mathbf{Z} = \{\cdots < -2 < -1 < 0 < 1 < 2 < \cdots\}$ as equipped with its usual linear ordering. Moreover, the inclusion $\mathrm{Spine}[\mathbf{Z}] \hookrightarrow \mathrm{N}_\bullet(\mathbf{Z})$ is inner anodyne (this is a special case of Proposition 1.5.7.3).

040L **Remark 8.5.4.15.** The simplicial set $\mathrm{Spine}[\mathbf{Z}]$ is weakly contractible. This follows from Remark 8.5.4.14, since the ∞-category $\mathrm{N}_\bullet(\mathbf{Z})$ is filtered and therefore weakly contractible (Proposition 7.2.4.9). Alternatively, it can be deduced from Example 3.6.4.4, since the geometric realization $|\mathrm{Spine}[\mathbf{Z}]|$ is homeomorphic to the set of real numbers $\mathbf{R}$ (endowed with its usual topology).

040M **Proposition 8.5.4.16.** *The morphism $Q : \mathrm{Spine}[\mathbf{Z}] \to \mathrm{N}_\bullet(\mathrm{Idem})$ of Notation 8.5.4.12 is both left and right cofinal.*

Proof. We will show that Q is left cofinal; a similar argument will show that it is right cofinal. By virtue of Theorem 7.2.3.1, it will suffice to show that the simplicial set $K = \mathrm{Spine}[\mathbf{Z}] \times_{\mathrm{N}_\bullet(\mathrm{Idem})} \mathrm{N}_\bullet(\mathrm{Idem})_{/\widetilde{X}}$ is weakly contractible. Let us identify the vertices of K with pairs (n, f), where n is an integer and $f : \widetilde{X} \to \widetilde{X}$ is a morphism in the category Idem. Unwinding the definitions, we see that K is the 1-dimensional simplicial set associated to the direct graph given in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & (-1, \widetilde{e}) \arrow[r] \arrow[dr] & (0, \widetilde{e}) \arrow[r] \arrow[dr] & (1, \widetilde{e}) \arrow[r] \arrow[dr] & \cdots \\
\cdots & (-1, \mathrm{id}_{\widetilde{X}}) & (0, \mathrm{id}_{\widetilde{X}}) & (1, \mathrm{id}_{\widetilde{X}}) & \cdots .
\end{tikzcd}$$

The inclusion of the upper part of the diagram determines a monomorphism of simplicial sets $\mathrm{Spine}[\mathbf{Z}] \hookrightarrow K$ which is left anodyne (since it is a pushout of a coproduct of countably many copies of the inclusion map $\{0\} \hookrightarrow \Delta^1$), and therefore a weak homotopy equivalence (Proposition 3.1.6.14). The desired result now follows from the weak contractibility of the simplicial set $\mathrm{Spine}[\mathbf{Z}]$ (Remark 8.5.4.15). □

Corollary 8.5.4.17. *Let $\mathcal{C}$ be an ∞-category which admits sequential limits (or colimits). Then $\mathcal{C}$ is idempotent complete.* 040N

Proof. It follows from Proposition 8.5.4.16 (and Corollary 7.2.2.12) that the ∞-category $\mathcal{C}$ admits limits (or colimits) indexed by the ∞-category $\mathrm{N}_\bullet(\mathrm{Idem})$, and is therefore idempotent complete by virtue of Proposition 8.5.4.7. □

Remark 8.5.4.18. Broadly speaking, Proposition 8.5.4.16 will be useful to us because it shows that the ∞-category $\mathrm{N}_\bullet(\mathrm{Idem})$ admits a (left and right) cofinal diagram $Q : K \to \mathrm{N}_\bullet(\mathrm{Idem})$, where the simplicial set K is finite-dimensional. Beware that it is not possible to arrange that the simplicial set K is *finite*, since an ∞-category which admits finite colimits need not be idempotent complete (Warning 8.5.4.3). In particular, there does not exist a categorical equivalence $K \to \mathrm{N}_\bullet(\mathrm{Idem})$, where K is a finite simplicial set. 040P

Example 8.5.4.19. Let $\mathcal{QC}$ denote the ∞-category of (small) ∞-categories. Then $\mathcal{QC}$ is idempotent complete. More generally, for every uncountable cardinal κ, the ∞-category $\mathcal{QC}^{<\kappa}$ of κ-small ∞-categories is idempotent complete. To prove this, we can use Propositions 8.5.4.6 and 8.5.1.15 to reduce to the case where κ has uncountable cofinality. In this case, the ∞-category $\mathcal{QC}^{<\kappa}$ admits sequential colimits (Example 7.6.7.8), so the desired result follows from Corollary 8.5.4.17. 040Q

040R **Example 8.5.4.20.** Let $\mathcal{S}$ denote the ∞-category of spaces. Then $\mathcal{S}$ is idempotent complete. More generally, for every uncountable cardinal κ, the ∞-category $\mathcal{S}^{<\kappa}$ of κ-small spaces is idempotent complete. This follows from Example 8.5.4.19 and Proposition 8.5.4.6, since the full subcategory $\mathcal{S}^{<\kappa} \subseteq \mathcal{QC}^{<\kappa}$ is closed under the formation of retracts (Remark 8.5.1.16).

040S **Warning 8.5.4.21.** Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ is idempotent complete, then its homotopy category $\mathrm{h}\mathcal{C}$ need not be idempotent complete. For example, the ∞-category of spaces $\mathcal{S}$ is idempotent complete (Example 8.5.4.20), but its homotopy category $\mathrm{h}\mathcal{S} = \mathrm{hKan}$ is not (see Proposition 8.5.7.15).

8.5.5 Idempotent Completion

040T Let $\mathcal{C}$ be an ∞-category. It follows from Corollary 8.3.3.17 (together with the criterion of Proposition 8.5.4.7) that we can choose a fully faithful functor $H : \mathcal{C} \to \widehat{\mathcal{C}}$, where $\widehat{\mathcal{C}}$ is idempotent complete. Our goal in this section is to show that there is a canonical choice for the ∞-category $\widehat{\mathcal{C}}$, which is characterized (up to equivalence) by the requirement that it is as small as possible.

040U **Definition 8.5.5.1.** Let $\mathcal{C}$ be an ∞-category. We say that a functor of ∞-categories $H : \mathcal{C} \to \widehat{\mathcal{C}}$ *exhibits $\widehat{\mathcal{C}}$ as an idempotent completion of $\mathcal{C}$* if it satisfies the following conditions:

(1) The functor H is fully faithful.

(2) The ∞-category $\widehat{\mathcal{C}}$ is idempotent complete.

(3) For every object $Y \in \widehat{\mathcal{C}}$, there exists an object $X \in \mathcal{C}$ such that Y is a retract of $H(X)$.

We will say that an ∞-category $\widehat{\mathcal{C}}$ *is an idempotent completion of $\mathcal{C}$* if there exists a functor $H : \mathcal{C} \to \widehat{\mathcal{C}}$ which exhibits $\widehat{\mathcal{C}}$ as an idempotent completion of $\mathcal{C}$.

Our first goal is to show that the idempotent completion of an ∞-category $\mathcal{C}$ is uniquely determined up to equivalence. To prove this, we reformulate Definition 8.5.5.1 as a universal mapping property:

0411 **Proposition 8.5.5.2.** *Let $H : \mathcal{C} \to \widehat{\mathcal{C}}$ be a functor of ∞-categories, where $\widehat{\mathcal{C}}$ is idempotent complete. The following conditions are equivalent:*

(a) The functor H exhibits $\widehat{\mathcal{C}}$ as an idempotent completion of $\mathcal{C}$, in the sense of Definition 8.5.5.1.

(b) For every idempotent complete ∞-category $\mathcal{D}$, precomposition with H induces an equivalence of ∞-categories $\mathrm{Fun}(\widehat{\mathcal{C}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proof. By virtue of Proposition 8.5.4.7, an ∞-category $\mathcal{D}$ is idempotent complete if and only if it admits $\mathrm{N}_\bullet(\mathrm{Idem})$-indexed colimits: that is, if and only if it is $\mathbb{K}$-cocomplete, where $\mathbb{K} = \{\mathrm{N}_\bullet(\mathrm{Idem})\}$ (see Definition 8.4.5.1). Moreover, every functor of ∞-categories $F : \widehat{\mathcal{C}} \to \mathcal{D}$ automatically preserves $\mathrm{N}_\bullet(\mathrm{Idem})$-indexed colimits (Corollary 8.5.3.12). We can therefore restate (b) as follows:

(b') The functor H exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$, in the sense of Definition 8.4.5.1.

Using Variant 8.4.6.9, we see that this condition is satisfied if and only if H satisfies conditions (1) and (2) of Definition 8.5.5.1, together with the following variant of (3):

$(3')$ The ∞-category $\widehat{\mathcal{C}}$ is generated by the essential image of H under the formation of $\mathrm{N}_\bullet(\mathrm{Idem})$-indexed colimits. That is, if $\widehat{\mathcal{C}}' \subseteq \widehat{\mathcal{C}}$ is a replete full subcategory which contains the essential image of H and is closed under retracts, then $\widehat{\mathcal{C}}' = \widehat{\mathcal{C}}$.

The implication $(3) \Rightarrow (3')$ is immediate. To prove the converse, let $\widehat{\mathcal{C}}' \subseteq \widehat{\mathcal{C}}$ be the full subcategory spanned by those objects Y which are retracts of $H(X)$, for some $X \in \mathcal{C}$. Condition (3) of Definition 8.5.5.1 asserts that $\widehat{\mathcal{C}}' = \widehat{\mathcal{C}}$. This is a special case of $(3')$, since $\widehat{\mathcal{C}}'$ is closed under the formation of retracts (Remark 8.5.1.6). □

Corollary 8.5.5.3 (Existence)**.** *Let $\mathcal{C}$ be an ∞-category. Then there exists a functor $H : \mathcal{C} \to \widehat{\mathcal{C}}$ which exhibits $\widehat{\mathcal{C}}$ as an idempotent completion of $\mathcal{C}$.* 040Z

Proof. By virtue of Proposition 8.5.5.2 (and its proof), this is a special case of Proposition 8.4.5.3 □

Using the Yoneda embedding of §8.3, we can give an explicit construction of idempotent completions. For simplicity, let us assume first that $\mathcal{C}$ is an essentially small ∞-category. We let $\mathrm{Fun}^{\mathrm{atm}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by those functors $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ which are *atomic*, in the sense of Definition 8.4.6.1.

Proposition 8.5.5.4. *Let $\mathcal{C}$ be an essentially small ∞-category and let $h_\bullet : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ be a covariant Yoneda embedding for $\mathcal{C}$ (Definition 8.3.3.9). Then the functor $h_\bullet$ exhibits $\mathrm{Fun}^{\mathrm{atm}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ as an idempotent completion of $\mathcal{C}$.* 040X

Following the convention of Remark 4.7.0.5, we can regard Proposition 8.3.3.14 as a special case of the following more general assertion (which is essentially a special case of Proposition 8.4.5.7):

Proposition 8.5.5.5. *Let κ be an uncountable regular cardinal, let $\mathcal{C}$ be an ∞-category which is essentially κ-small, and let $\widehat{\mathcal{C}} \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ be the full subcategory spanned by those functors $\mathscr{F}$ for which the corepresentable functor $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})}(\mathscr{F}, \bullet)$ commutes with κ-small colimits. Then the covariant Yoneda embedding $h_\bullet : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ exhibits $\widehat{\mathcal{C}}$ as an idempotent completion of $\mathcal{C}$.* 040Y

Proof. To simplify the notation, set $\mathcal{D} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. For each object $C \in \mathcal{C}$, the representable functor $h_C \in \mathcal{D}$ corepresents the functor

$$\mathcal{D} \to \mathcal{S}^{<\kappa} \qquad \mathscr{F} \mapsto \mathscr{F}(C)$$

given by evaluation at C, which preserves κ-small colimits by virtue of Proposition 7.1.6.1. It follows that the covariant Yoneda embedding $h_{\bullet}$ factors through the subcategory $\widehat{\mathcal{C}} \subseteq \mathcal{D}$. Moreover, the functor $h_{\bullet}$ is fully faithful (Theorem 8.3.3.13).

The ∞-category $\mathcal{D}$ admits κ-small colimits, and is therefore idempotent complete by virtue of Proposition 8.5.4.7. It follows from Corollary 8.5.1.14 that the full category $\widehat{\mathcal{C}} \subseteq \mathcal{D}$ is closed under the formation of retracts, and is therefore also idempotent complete (Proposition 8.5.4.6).

To complete the proof, it will suffice to show that every object $\mathscr{F} \in \widehat{\mathcal{C}}$ is a retract of h_C for some object $C \in \mathcal{C}$. Applying Corollary 8.4.3.9, we deduce that $\mathscr{F}$ can be realized as the colimit of a diagram

$$\mathcal{K} \xrightarrow{T} \mathcal{C} \xrightarrow{h_{\bullet}} \mathcal{D},$$

where $\mathcal{K}$ is an essentially κ-small ∞-category. Since the functor $\mathrm{Hom}_{\mathcal{D}}(\mathscr{F}, \bullet)$ preserves κ-small colimits, it follows that the identity map $\mathrm{id}_{\mathscr{F}} \in \mathrm{Hom}_{\mathcal{D}}(\mathscr{F}, \mathscr{F})$ factors (up to homotopy) through $h_{T(B)}$ for some B. In particular, $\mathscr{F}$ is a retract of the representable functor $h_{T(B)}$. □

Let $\mathcal{C}$ be an ∞-category. Proposition 8.5.5.5 supplies an explicit description of its idempotent completion $\widehat{\mathcal{C}}$ which is somewhat transcendental in nature: it locates $\widehat{\mathcal{C}}$ as a full subcategory of an ∞-category which is much larger than $\mathcal{C}$. Let us remark that this is not necessary: the ∞-category $\widehat{\mathcal{C}}$ is essentially of the same size as $\mathcal{C}$ itself.

0410 **Proposition 8.5.5.6.** *Let $\mathcal{C}$ be an ∞-category, let $\widehat{\mathcal{C}}$ be an idempotent completion of $\mathcal{C}$, and let κ be an uncountable cardinal. Then:*

(1) *The ∞-category $\mathcal{C}$ is locally κ-small if and only if $\widehat{\mathcal{C}}$ is locally κ-small.*

(2) *The ∞-category $\mathcal{C}$ is essentially κ-small if and only if $\widehat{\mathcal{C}}$ is essentially κ-small.*

Proof. Choose a functor $H : \mathcal{C} \to \widehat{\mathcal{C}}$ which exhibits $\widehat{\mathcal{C}}$ as an idempotent completion of $\mathcal{C}$. We first prove (1). Assume that $\mathcal{C}$ is locally κ-small; we wish to show that $\widehat{\mathcal{C}}$ is locally κ-small (the reverse implication follows immediately from the definition). Fix a pair of objects $Y, Y' \in \widehat{\mathcal{C}}$; we wish to show that the morphism space $\mathrm{Hom}_{\widehat{\mathcal{C}}}(Y, Y')$ is essentially κ-small. By assumption, the object $Y \in \widehat{\mathcal{C}}$ is a retract of $H(X)$ for some object $X \in \mathcal{C}$. It follows that $\mathrm{Hom}_{\widehat{\mathcal{C}}}(Y, Y')$ is a retract of $\mathrm{Hom}_{\widehat{\mathcal{C}}}(H(X), Y')$ in the homotopy category hKan. By virtue of Corollary 8.5.1.17, it will suffice to show that the Kan complex $\mathrm{Hom}_{\widehat{\mathcal{C}}}(H(X), Y')$ is essentially κ-small. Applying the same argument to Y', we are reduced to showing that the mapping space $\mathrm{Hom}_{\widehat{\mathcal{C}}}(H(X), H(X'))$ is essentially κ-small for every pair of objects $X, X' \in \mathcal{C}$. Since the functor F is fully faithful, the canonical map $\mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}_{\widehat{\mathcal{C}}}(H(X), H(X'))$ is a homotopy equivalence. The desired result now follows from our assumption that the ∞-category $\mathcal{C}$ is essentially κ-small.

We now prove (2). Assume that $\mathcal{C}$ is essentially κ-small; we wish to show that $\widehat{\mathcal{C}}$ is also essentially κ-small (again, the reverse implication follows immediately from the definitions). Without loss of generality, we may assume that κ is the smallest cardinal for which $\mathcal{C}$ is essentially κ-small, and is therefore regular (Corollary 4.7.6.17). By virtue of the criterion of Proposition 4.7.8.7, it will suffice to show that the set of isomorphism classes $S = \pi_0(\widehat{\mathcal{C}}^{\simeq})$ is κ-small. For each object $X \in \mathcal{C}$, let $S_X \subseteq S$ be the collection of isomorphism classes of objects $Y \in \widehat{\mathcal{C}}$ which can be realized as a retract of $H(X)$. Note that we can write S as a union of the subsets S_X, where the X ranges over a set of representatives for the isomorphism classes in $\mathcal{C}$. Since κ is regular, and the set $\pi_0(\mathcal{C}^{\simeq})$ is κ-small, it will suffice to show that each of the sets S_X is κ-small. Let us henceforth regard the object $X \in \mathcal{C}$ as fixed, and let Y be any retract of $H(X)$ in the ∞-category $\widehat{\mathcal{C}}$. It follows from Proposition 8.5.2.4 that, as an object of the homotopy category $\mathrm{h}\mathcal{C}$, Y can be identified with the equalizer of a pair of morphisms $(\mathrm{id}, e) : H(X) \to H(X)$. It follows that the cardinality of the set of isomorphism classes S_X is bounded above by the cardinality of the set $\mathrm{Hom}_{\mathrm{h}\widehat{\mathcal{C}}}(H(X), H(X))$ of morphisms $e : H(X) \to H(X)$ in $\mathrm{h}\mathcal{C}$, which we can identify with the κ-small set $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, X)$. □

Let $\mathcal{QC}$ denote the ∞-category of (small) ∞-categories (Construction 5.5.4.1), and let $\mathcal{QC}^{\mathrm{ic}}$ denote the full subcategory of $\mathcal{QC}$ spanned by the idempotent complete ∞-categories. Proposition 8.5.5.2 asserts that a functor $F : \mathcal{C} \to \widehat{\mathcal{C}}$ exhibits $\widehat{\mathcal{C}}$ as an idempotent completion of $\mathcal{C}$ if and only if exhibits $\widehat{\mathcal{C}}$ as a $\mathcal{QC}^{\mathrm{ic}}$-reflection of $\mathcal{C}$, in the sense of Definition 6.2.2.1. Consequently, Proposition 8.5.5.6 is equivalent to the assertion that $\mathcal{QC}^{\mathrm{ic}} \subseteq \mathcal{QC}$ is reflective. Combining this observation with Proposition 6.2.2.11, we obtain the following:

Corollary 8.5.5.7. *Then the inclusion functor $\mathcal{QC}^{\mathrm{ic}} \hookrightarrow \mathcal{QC}$ admits a left adjoint, which carries each ∞-category $\mathcal{C}$ to an idempotent completion $\widehat{\mathcal{C}}$.* 0414

Corollary 8.5.5.8. *Let $\mathcal{C}$ be an ∞-category which can be realized as the limit of a small diagram $\mathscr{F} : \mathcal{D} \to \mathcal{QC}$. Suppose that, for each vertex $D \in \mathcal{D}$, the ∞-category $\mathscr{F}(D)$ is idempotent complete. Then $\mathcal{C}$ is idempotent complete.* 0415

Proof. Combine Corollary 8.5.5.7 with Variant 7.1.3.24. □

Corollary 8.5.5.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Suppose that $\mathcal{D}$ is a retract of $\mathcal{C}$ in the homotopy category* hQCat. *If $\mathcal{C}$ is idempotent complete, then $\mathcal{D}$ is also idempotent complete.* 0416

Proof. By virtue of Remark 8.5.3.9, we can identify $\mathcal{D}$ with the limit of a diagram $\mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{QC}$ carrying the unique object of Idem to the idempotent complete ∞-category $\mathcal{C}$. The desired result is now a special case of Corollary 8.5.5.8. □

0417 **Exercise 8.5.5.10.** Give a direct proof of Corollary 8.5.5.9.

8.5.6 Idempotent Endomorphisms

0418 Let $\mathcal{C}$ be an ∞-category and let $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ be an idempotent in $\mathcal{C}$. Then F carries the unique object of Idem to an object $X \in \mathcal{C}$, and the unique non-identity morphism of Idem to an endomorphism $e : X \to X$ in $\mathcal{C}$. If $\mathcal{C}$ is (the nerve of) an ordinary category, then the functor F is uniquely determined by the pair (X, e) (Remark 8.5.2.8). In more general situations, this is false: the simplicial set $\mathrm{N}_\bullet(\mathrm{Idem})$ contains a nondegenerate simplex of each dimension (Remark 8.5.3.3), so the specification of the functor F requires an infinite amount of data. Our goal in this section is to show that, nevertheless, the idempotent $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ can be recovered *up to isomorphism* from the underlying endomorphism (X, e). We begin by introducing some terminology.

0419 **Notation 8.5.6.1.** Let $\Delta^1 / \partial \Delta^1$ denote the simplicial circle (Example 1.5.7.11). For every ∞-category $\mathcal{C}$, we let $\mathrm{End}_{\mathcal{C}}$ denote the ∞-category of diagrams $\mathrm{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C})$. Note that objects of $\mathrm{End}_{\mathcal{C}}$ can be identified with pairs (X, e), where X is an object of $\mathcal{C}$ and $e : X \to X$ is an endomorphism of X. We will refer to $\mathrm{End}_{\mathcal{C}}$ as the *∞-category of endomorphisms in $\mathcal{C}$*.

041A **Remark 8.5.6.2.** Let $\mathcal{C}$ be an ∞-category. Evaluation on the unique vertex of $\Delta^1 / \partial \Delta^1$ induces an isofibration of ∞-categories $\mathrm{End}_{\mathcal{C}} \to \mathcal{C}$. Moreover, for each object $X \in \mathcal{C}$, the fiber $\{X\} \times_{\mathcal{C}} \mathrm{End}_{\mathcal{C}}$ can be identified with the endomorphism space $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ of Variant 4.6.1.3.

041B **Definition 8.5.6.3** (Idempotent Endomorphisms)**.** Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. We will say that e is *idempotent* if there exists a functor $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ satisfying $F(\widetilde{e}) = e$; here $\widetilde{e}$ denotes the (unique) non-identity morphism in the category Idem. We let $\mathrm{End}^{\mathrm{idm}}_{\mathcal{C}}$ denote the full subcategory of $\mathrm{End}_{\mathcal{C}}$ spanned by the idempotent endomorphisms.

We can now formulate our main result.

041C **Proposition 8.5.6.4.** *Let $\mathcal{C}$ be an ∞-category. Then the restriction functor*

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C}) \to \mathrm{End}^{\mathrm{idm}}_{\mathcal{C}}$$

has a left homotopy inverse.

Stated more informally, Proposition 8.5.6.4 asserts that if $e : X \to X$ is an endomorphism in the ∞-category $\mathcal{C}$ which can be extended to an idempotent $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$, then F is uniquely determined up to isomorphism and can be chosen to depend functorially on the pair (X, e).

Corollary 8.5.6.5. *For every ∞-category $\mathcal{C}$, evaluation on the non-identity morphism of* Idem *induces a bijection* 041D

$$\theta : \pi_0(\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C})^{\simeq}) \to \pi_0((\mathrm{End}_{\mathcal{C}}^{\mathrm{idm}})^{\simeq}).$$

Proof. The surjectivity of θ follows from the definition of an idempotent endomorphism, and the injectivity from Proposition 8.5.6.4. □

Warning 8.5.6.6. Let $\mathcal{C}$ be an ∞-category and let $R : \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}) \to \mathrm{End}_{\mathcal{C}}^{\mathrm{idm}}$ be the restriction functor. Proposition 8.5.6.4 asserts that there exists a functor $S : \mathrm{End}_{\mathcal{C}}^{\mathrm{idm}} \to \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C})$ for which the composition 041E

$$\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}) \xrightarrow{R} \mathrm{End}_{\mathcal{C}}^{\mathrm{idm}} \xrightarrow{S} \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}).$$

is isomorphic to the identity functor. Let $e : X \to X$ be an idempotent endomorphism in $\mathcal{C}$, so that e can be extended to a morphism $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$. Then $S(e) = (S \circ R)(F)$ is isomorphic to F, so there is an isomorphism of $(R \circ S)(e)$ with e in the category $\mathrm{End}_{\mathcal{C}}^{\mathrm{idm}}$. Beware that this isomorphism usually *cannot* be chosen to depend functorially on e. In general, the functor R is not an equivalence of ∞-categories, so the composition

$$\mathrm{End}_{\mathcal{C}}^{\mathrm{idm}} \xrightarrow{S} \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}) \xrightarrow{R} \mathrm{End}_{\mathcal{C}}^{\mathrm{idm}}$$

is not isomorphic to the identity functor on $\mathrm{End}_{\mathcal{C}}^{\mathrm{idm}}$.

Example 8.5.6.7. For any ∞-category $\mathcal{C}$, we have a commutative diagram 041F

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl] \arrow[dr, "\delta"] & \\
\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}) \arrow[rr] & & \mathrm{End}_{\mathcal{C}}^{\mathrm{idm}},
\end{tikzcd}$$

where the vertical maps are the diagonal embeddings. If $\mathcal{C}$ is a Kan complex, then the left vertical map is a homotopy equivalence of Kan complexes (since the simplicial set $\mathrm{N}_{\bullet}(\mathrm{Idem})$ is weakly contractible; see Remark 8.5.3.4). In this case, Proposition 8.5.6.4 reduces to the assertion that the diagonal map

$$\delta : \mathcal{C} \to \mathrm{End}_{\mathcal{C}}^{\mathrm{idm}} \subseteq \mathrm{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C}) \qquad X \mapsto (X, \mathrm{id}_X)$$

has a left homotopy inverse. This is clear: the map δ has a left inverse in the category of simplicial sets, given by evaluation at the vertex of $\Delta^1 / \partial \Delta^1$. Beware that δ is usually not a homotopy equivalence, since the simplicial set $\Delta^1 / \partial \Delta^1$ is not contractible.

We will give the proof of Proposition 8.5.6.4 at the end of this section. First, let us introduce an important class of idempotent endomorphisms.

`041G` **Definition 8.5.6.8.** Let $\mathcal{C}$ be an ∞-category. We say that an endomorphism $e : X \to X$ in $\mathcal{C}$ is *split idempotent* if the homotopy class $[e]$ is a split idempotent in the homotopy category $\mathrm{h}\mathcal{C}$ (see Example 8.5.2.3).

`041H` **Remark 8.5.6.9.** Let $\mathcal{C}$ be an ∞-category. Then an endomorphism $e : X \to X$ is split idempotent if and only there exists a retraction diagram

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y. \end{tikzcd}$$

in the ∞-category $\mathcal{C}$, where e factors as a composition $X \xrightarrow{r} Y \xrightarrow{i} X$.

`041J` **Proposition 8.5.6.10** (Lifting Split Idempotents)**.** *Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. Then e is split idempotent endomorphism if and only if it extends to a split idempotent $\mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$, in the sense of Definition 8.5.3.5. In particular, every split idempotent endomorphism is an idempotent endomorphism.*

Proof. Assume that the endomorphism e is split idempotent; we will show that e can be extended to a split idempotent $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$ (the reverse implication follows immediately from the definitions). Choose a retraction diagram

`041K`

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y \end{tikzcd} \tag{8.57}$$

in the ∞-category $\mathcal{C}$, where $[e] = [i] \circ [r]$ in the homotopy category $\mathrm{h}\mathcal{C}$. Using Corollary 8.5.1.28, we can extend the diagram (8.57) to a functor $\overline{F} : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$. By construction, $\overline{F}$ carries the unique non-identity morphism of Idem to a morphism $e' : X \to X$ of $\mathcal{C}$ which is homotopic to e. Replacing $\overline{F}$ by an isomorphic functor if necessary, we may assume that $e' = e$ (see Corollary 4.4.5.3). Then $F = \overline{F}|_{\mathrm{N}_\bullet(\mathrm{Idem})}$ is a split idempotent in $\mathcal{C}$ extending e. □

Let $\mathcal{C}$ be an ∞-category. When restricted to *split* idempotents, Proposition 8.5.6.4 asserts every retraction diagram

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y \end{tikzcd}$$

can be recovered (up to canonical isomorphism) from a choice of composition $e = (i \circ r)$ in the ∞-category $\mathcal{C}$. To prove this, we will exploit the observation that Y can be realized as the limit (and colimit) of the diagram

$$\cdots \rightarrow X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \rightarrow \cdots ,$$

indexed by the 1-dimensional simplicial set $\mathrm{Spine}[\mathbf{Z}]$ of Notation 8.5.4.12.

Notation 8.5.6.11. Let $q : \mathrm{Spine}[\mathbf{Z}] \rightarrow \Delta^1 / \partial \Delta^1$ be the covering map of Remark 8.5.4.13. For every ∞-category $\mathcal{C}$, precomposition with q induces a functor `041L`

$$T : \mathrm{End}_{\mathcal{C}} = \mathrm{Fun}(\Delta^1 / \partial \Delta^1, \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{C}) \qquad (X, e) \mapsto T_e.$$

More informally, the functor T carries each endomorphism $e : X \rightarrow X$ in the ∞-category $\mathcal{C}$ to the associated sequential diagram

$$\cdots \rightarrow X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \rightarrow \cdots$$

Proposition 8.5.6.12. *Let $\mathcal{C}$ be an ∞-category and let $e : X \rightarrow X$ be an idempotent endomorphism in $\mathcal{C}$. Then e splits if and only if the diagram $T_e : \mathrm{Spine}[\mathbf{Z}] \rightarrow \mathcal{C}$ admits a limit.* `041M`

Proof. Since e is idempotent, it can be extended to a functor $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \rightarrow \mathcal{C}$. Then $T_e = F \circ Q$, where $Q : \mathrm{Spine}[\mathbf{Z}] \rightarrow \mathrm{N}_{\bullet}(\mathrm{Idem})$ is the left cofinal morphism of Proposition 8.5.4.16. Using Corollary 7.2.2.10, we see that T_e has a limit in $\mathcal{C}$ if and only if F has a limit in $\mathcal{C}$. The desired result now follows from the criterion of Corollary 8.5.3.11 □

Remark 8.5.6.13. Let $\mathcal{C}$ be an ∞-category and let $e : X \rightarrow X$ be an split idempotent endomorphism in $\mathcal{C}$, so that the diagram `041N`

$$\cdots \rightarrow X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \xrightarrow{e} X \rightarrow \cdots$$

admits both a limit and colimit in $\mathcal{C}$. The limit and colimit of this diagram are automatically preserved by any functor of ∞-categories $\mathcal{C} \rightarrow \mathcal{D}$. This follows by combining Corollary 8.5.3.12 with Proposition 8.5.4.16.

Motivated by Proposition 8.5.6.12, we introduce a variant of Definition 8.5.6.8.

041P **Definition 8.5.6.14.** Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. We will say that e is *weakly split* if it satisfies the following conditions:

(1) The diagram T_e of Notation 8.5.6.11 can be extended to a limit diagram in $\mathcal{C}$, which we depict as

$$\begin{tikzcd}
 & & & Y \arrow[dll] \arrow[dl] \arrow[d, "i"] \arrow[dr] \arrow[drr] & & & \\
\cdots \arrow[r] & X \arrow[r, "e"] & X \arrow[r, "e"] & X \arrow[r, "e"] & X \arrow[r, "e"] & X \arrow[r, "e"] & \cdots
\end{tikzcd}$$

(2) The diagram T_e of Notation 8.5.6.11 can be extended to a colimit diagram in $\mathcal{C}$, which we depict as

$$\begin{tikzcd}
\cdots \arrow[r] & X \arrow[r, "e"] \arrow[drr] & X \arrow[r, "e"] \arrow[dr] & X \arrow[r, "e"] \arrow[d, "r"] & X \arrow[r, "e"] \arrow[dl] & X \arrow[r, "e"] \arrow[dll] & \cdots \\
 & & & Z. & & &
\end{tikzcd}$$

(3) The composition $Y \xrightarrow{i} X \xrightarrow{r} Z$ is an isomorphism in $\mathcal{C}$.

Our next goal is to show that every split idempotent endomorphism is weakly split.

041Q **Notation 8.5.6.15.** Let Ret denote the category of Construction 8.5.0.2. Then the object $\widetilde{Y} \in \mathrm{Ret}$ is both initial and final. It follows that the diagram $Q : \mathrm{Spine}[\mathbf{Z}] \to \mathrm{N}_\bullet(\mathrm{Idem})$ of Proposition 8.5.4.16 admits unique extensions

$$Q^- : \mathrm{Spine}[\mathbf{Z}]^{\triangleleft} \to \mathrm{N}_\bullet(\mathrm{Ret}) \qquad Q^+ : \mathrm{Spine}[\mathbf{Z}]^{\triangleright} \to \mathrm{N}_\bullet(\mathrm{Ret})$$

which carry the cone points to the object $\widetilde{Y}$.

041R **Lemma 8.5.6.16.** *Let $\mathcal{C}$ be an ∞-category and let $F : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$ be a functor. Then the composition $\mathrm{Spine}[\mathbf{Z}]^{\triangleleft} \xrightarrow{Q^-} \mathrm{N}_\bullet(\mathrm{Ret}) \xrightarrow{F} \mathcal{C}$ is a limit diagram in $\mathcal{C}$, and the composition $\mathrm{Spine}[\mathbf{Z}]^{\triangleright} \xrightarrow{Q^+} \mathrm{N}_\bullet(\mathrm{Ret}) \xrightarrow{F} \mathcal{C}$ is a colimit diagram in $\mathcal{C}$.*

Proof. Combine Remark 8.5.3.9, Corollary 7.2.2.3, and Proposition 8.5.4.16. □

041S **Proposition 8.5.6.17.** *Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be a split idempotent endomorphism in $\mathcal{C}$. Then e is weakly split.*

Proof. Let Ret denote the category introduced in Construction 8.5.0.2. Using Proposition 8.5.6.10, we can choose a functor $F : \mathrm{N}_\bullet(\mathrm{Ret}) \to \mathcal{C}$ satisfying $F(\widetilde{X}) = X$ and $F(\widetilde{e}) = e$.

Let $Q : \mathrm{Spine}[\mathbf{Z}] \to \mathrm{N}_\bullet(\mathrm{Idem})$ denote the (left and right) cofinal morphism of Proposition 8.5.4.16, and let $Q^- : \mathrm{Spine}[\mathbf{Z}]^{\triangleleft} \to \mathrm{N}_\bullet(\mathrm{Ret})$ and $Q^+ : \mathrm{Spine}[\mathbf{Z}]^{\triangleright} \to \mathrm{N}_\bullet(\mathrm{Ret})$ be the extensions of Notation 8.5.6.15. Lemma 8.5.6.16 guarantees that $F \circ Q^-$ is a limit diagram in $\mathcal{C}$ extending $F \circ Q = T_e$, so that e satisfies condition (1) of Definition 8.5.6.14. Similarly, $F \circ Q^+$ is a colimit diagram extending T_e, so that e satisfies condition (2) of Definition 8.5.6.14. Condition (3) follows from the observation that any composition of $F(\widetilde{i})$ with $F(\widetilde{r})$ is homotopic to the morphism $F(\widetilde{r} \circ \widetilde{i}) = F(\mathrm{id}_{\widetilde{Y}}) = \mathrm{id}_{F(\widetilde{Y})}$, and is therefore an isomorphism. □

Warning 8.5.6.18. The converse of Proposition 8.5.6.17 is false. For example, every isomorphism $e : X \to X$ is weakly split, but is split idempotent only if e is homotopic to the identity morphism id_X (see Example 8.5.2.2). 041T

Let $\mathcal{C}$ be an ∞-category, and let $\mathrm{End}^w_{\mathcal{C}}$ denote the full subcategory of $\mathrm{End}_{\mathcal{C}}$ spanned by the weakly split endomorphisms in $\mathcal{C}$. It follows from Proposition 8.5.6.17 that the restriction functor

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Ret}), \mathcal{C}) \to \mathrm{End}_{\mathcal{C}} \qquad F \mapsto F(e)$$

factors through $\mathrm{End}^w_{\mathcal{C}}$. We will deduce Proposition 8.5.6.4 from the following:

Proposition 8.5.6.19. *Let $\mathcal{C}$ be an ∞-category. Then the restriction functor* 041U

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Ret}), \mathcal{C}) \to \mathrm{End}^w_{\mathcal{C}}$$

admits a left homotopy inverse.

Proof. Let $\mathcal{D} \subseteq \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{C})$ denote the full subcategory spanned by those diagrams $S : \mathrm{Spine}[\mathbf{Z}] \to \mathcal{C}$ which admit both a limit and a colimit. Let u and v be auxiliary symbols, and let $\widetilde{\mathcal{D}}$ denote the full subcategory of $\mathrm{Fun}(\{u\} \star \mathrm{Spine}[\mathbf{Z}] \star \{v\}, \mathcal{C})$ spanned by those diagrams $S^{\pm} : \{u\} \star \mathrm{Spine}[\mathbf{Z}] \star \{v\} \to \mathcal{C}$ which satisfy the following conditions:

(1) The restriction $S^- = S^{\pm}|_{\{u\} \star \mathrm{Spine}[\mathbf{Z}]}$ is a limit diagram in $\mathcal{C}$.

(2) The restriction $S^+ = S^{\pm}|_{\mathrm{Spine}[\mathbf{Z}] \star \{v\}}$ is a colimit diagram in $\mathcal{C}$.

Note that the simplicial set $\mathrm{Spine}[\mathbf{Z}]$ is weakly contractible (Remark 8.5.4.15), so that the inclusion map $\mathrm{Spine}[\mathbf{Z}] \hookrightarrow \{u\} \star \mathrm{Spine}[\mathbf{Z}]$ is right anodyne (Proposition 4.3.7.9). Applying Corollary 7.2.2.3, we can replace (2) by the condition that $S^{\pm}$ is a colimit diagram in $\mathcal{C}$. Moreover, the functor S^- admits a colimit if and only if $S = S^{\pm}|_{\mathrm{Spine}[\mathbf{Z}]}$ admits a colimit (Corollary 7.2.2.10). Invoking Corollary 7.3.6.15 twice, we deduce that the restriction functor

$$R : \widetilde{\mathcal{D}} \to \mathcal{D} \qquad S^{\pm} \mapsto S^{\pm}|_{\mathrm{Spine}[\mathbf{Z}]}$$

is a trivial Kan fibration of ∞-categories.

Let $\widetilde{\mathcal{D}}^{w}$ denote the replete full subcategory of $\widetilde{\mathcal{D}}$ spanned by those functors $S^{\pm}$ for which the composition

$$\Delta^1 \simeq \{u\} \star \{v\} \hookrightarrow \{u\} \star \operatorname{Spine}[\mathbf{Z}] \star \{v\} \xrightarrow{S^{\pm}} \mathcal{C}$$

is an isomorphism in $\mathcal{C}$. Let $\mathcal{D}^w \subseteq \mathcal{D}$ be the essential image of $\widetilde{\mathcal{D}}^{w}$ under R, so that R restricts to a trivial Kan fibration $R^w : \widetilde{\mathcal{D}}^{w} \to \mathcal{D}^w$.

Let $T : \operatorname{End}_{\mathcal{C}} \to \operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}], \mathcal{C})$ be the functor given by precomposition with the covering map $\operatorname{Spine}[\mathbf{Z}] \to \Delta^1 / \partial \Delta^1$ (see Notation 8.5.4.12). By definition, and endomorphism e of $\mathcal{C}$ is weakly split if and only if the associated diagram $T_e : \operatorname{Spine}[\mathbf{Z}] \to \mathcal{C}$ is an object of $\mathcal{D}^w$. Consequently, the functor T restricts to a functor $T^w : \operatorname{End}_{\mathcal{C}} \to \mathcal{D}_w$.

Since the object $\widetilde{Y} \in \operatorname{Ret}$ is both initial and final, the diagram $Q : \operatorname{Spine}[\mathbf{Z}] \to \operatorname{N}_{\bullet}(\operatorname{Idem})$ admits an unique extension $Q^{\pm} : \{u\} \star \operatorname{Spine}[\mathbf{Z}] \star \{v\} \to \operatorname{N}_{\bullet}(\operatorname{Ret})$ carrying both u and v to the object $\widetilde{Y}$. It follows from Lemma 8.5.6.16 that precomposition with $Q^{\pm}$ induces a functor

$$\widetilde{T} : \operatorname{Fun}(\operatorname{N}_{\bullet}(\operatorname{Ret}), \mathcal{C}) \to \widetilde{\mathcal{D}}^{w} \subseteq \operatorname{Fun}(\{u\} \star \operatorname{Spine}[\mathbf{Z}] \star \{v\}, \mathcal{C}).$$

By construction, we have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\operatorname{Fun}(\operatorname{N}_{\bullet}(\operatorname{Ret}), \mathcal{C}) \arrow[r, "\widetilde{T}"] \arrow[d] & \widetilde{\mathcal{D}}^{w} \arrow[d, "\sim"', "R^w"] \\
\operatorname{End}_{\mathcal{C}}^{w} \arrow[r, "T^w"] & \mathcal{D}^w,
\end{tikzcd}$$

where the right vertical map is a trivial Kan fibration. Consequently, to show that the left vertical map has a left homotopy inverse, it will suffice to show that the functor $\widetilde{T}$ has a left homotopy inverse.

Note that precomposition with the map

$$\Delta^2 \simeq \{u\} \star \{0\} \star \{v\} \hookrightarrow \{u\} \star \operatorname{Spine}[\mathbf{Z}] \star \{v\}$$

determines an evaluation functor $\operatorname{ev} : \widetilde{\mathcal{D}} \to \operatorname{Fun}(\Delta^2, \mathcal{C})$. Let $\operatorname{Fun}^w(\Delta^2, \mathcal{C})$ denote the full subcategory of $\operatorname{Fun}(\Delta^2, \mathcal{C})$ spanned by those diagrams

$$\begin{tikzcd}
& X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "u"] & & Y'
\end{tikzcd}$$

where u is an isomorphism, so that ev restricts to a functor $\mathrm{ev}^w : \widetilde{\mathcal{D}}^w \to \mathrm{Fun}^w(\Delta^2, \mathcal{C})$. It will therefore suffice to show that the composite functor

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Ret}), \mathcal{C}) \xrightarrow{\widetilde{T}} \widetilde{\mathcal{D}}^w \xrightarrow{\mathrm{ev}^w} \mathrm{Fun}^w(\Delta^2, \mathcal{C})$$

has a left homotopy inverse. We conclude by observing that this composite functor is an equivalence of ∞-categories, by virtue of Corollary 8.5.1.30. □

Proof of Proposition 8.5.6.4. Let $\mathcal{C}$ be an ∞-category. We wish to show that the restriction functor

$$R : \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C}) \to \mathrm{End}^{\mathrm{idm}}_{\mathcal{C}}$$

has a left homotopy inverse. Using Corollary 8.5.5.3, we can choose a fully faithful functor $H : \mathcal{C} \to \mathcal{C}'$, where $\mathcal{C}'$ is idempotent complete. Replacing $\mathcal{C}$ by the essential image of H, we may assume without loss of generality that $\mathcal{C}$ is a full subcategory of $\mathcal{C}'$ (and H is the inclusion functor). Then R is the restriction of a functor $R' : \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C}') \to \mathrm{End}^{\mathrm{idm}}_{\mathcal{C}'}$. Since $\mathcal{C}'$ is idempotent complete, every idempotent endomorphism in $\mathcal{C}'$ is split, and therefore weakly split. Applying Proposition 8.5.6.19, we deduce that the composition

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Ret}), \mathcal{C}') \to \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C}') \xrightarrow{R'} \mathrm{End}^{\mathrm{idm}}_{\mathcal{C}'} \subseteq \mathrm{End}^{w}_{\mathcal{C}}$$

admits a left homotopy inverse. The restriction map $\mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Ret}), \mathcal{C}') \to \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C}')$ is an equivalence of ∞-categories (Remark 8.5.4.8), so R' admits a left homotopy inverse $S' : \mathrm{End}^{\mathrm{idm}}_{\mathcal{C}'} \to \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C}')$. Restricting to the full subcategory $\mathrm{End}^{\mathrm{idm}}_{\mathcal{C}} \subseteq \mathrm{End}^{\mathrm{idm}}_{\mathcal{C}'}$, we obtain a functor $S : \mathrm{End}^{\mathrm{idm}}_{\mathcal{C}} \to \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C})$ which is left homotopy inverse to R. □

8.5.7 Homotopy Idempotent Endomorphisms

Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. If e is idempotent (in the sense of Definition 8.5.6.3), then the homotopy class $[e]$ is an idempotent endomorphism in the homotopy category $\mathrm{h}\mathcal{C}$. One can ask if the converse is true: if the homotopy class $[e]$ is an idempotent endomorphism in $\mathrm{h}\mathcal{C}$, does it follow that e is an idempotent endomorphism of $\mathcal{C}$? In this section, we will show that this question has a negative answer in general (Proposition 8.5.7.15), but a positive answer under some additional assumptions (Corollary 8.5.7.5). Let us begin by introducing some terminology. 041V

Definition 8.5.7.1. Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. We say that e is *homotopy idempotent* if the homotopy class $[e]$ is an idempotent in the homotopy category $\mathrm{h}\mathcal{C}$, in the sense of Definition 8.5.2.1. 041W

041X **Remark 8.5.7.2.** Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. Then e is homotopy idempotent if and only if there exists a 2-simplex σ of $\mathcal{C}$ whose boundary is indicated in the diagram

$$\begin{tikzcd}
& X \arrow[dr, "e"] & \\
X \arrow[ur, "e"] \arrow[rr, "e"] & & X.
\end{tikzcd}$$

041Y **Example 8.5.7.3.** Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. If e is idempotent (that is, if it extends to a functor $\mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$), then it is homotopy idempotent.

We now provide a partial converse to Example 8.5.7.3.

041Z **Proposition 8.5.7.4.** *Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. The following conditions are equivalent:*

(1) *The homotopy class $[e]$ is a split idempotent in the homotopy category $\mathrm{h}\mathcal{C}$.*

(2) *The endomorphism e is a split idempotent in $\mathcal{C}$.*

(3) *The endomorphism e is homotopy idempotent and weakly split (Definition 8.5.6.14).*

(4) *The endomorphism e is homotopy idempotent and there exists a limit diagram*

$$\begin{tikzcd}
& & & Y \arrow[dlll] \arrow[dl, "i(-1)"'] \arrow[d, "i(0)"] \arrow[dr, "i(1)"] \arrow[drrr] & & & \\
\cdots \arrow[rr] & & X \arrow[r, "e"] & X \arrow[r, "e"] & X \arrow[rr, "e"] & & \cdots
\end{tikzcd}$$

in $\mathcal{C}$, where the morphism $i(0)$ has a left homotopy inverse.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is tautology, the implication (2) $\Rightarrow$ (3) follows from Proposition 8.5.6.17 (and Example 8.5.7.3), and the implication (3) $\Rightarrow$ (4) is immediate. We will complete the proof by showing that (4) implies (1). Assume that e is homotopy idempotent, so that there exists a 2-simplex σ of $\mathcal{C}$ whose boundary is indicated in the diagram

$$\begin{tikzcd}
 & X \arrow[dr, "e"] & \\
X \arrow[ur, "e"] \arrow[rr, "e"] & & X.
\end{tikzcd}$$

Let $\mathrm{Spine}[\mathbf{Z}]$ denote the 1-dimensional simplicial set of Notation 8.5.4.12, and let $T_{\sigma} : \mathrm{Spine}[\mathbf{Z}] \to \mathcal{C}_{X/}$ be the morphism of simplicial sets which carries each vertex of $\mathrm{Spine}[\mathbf{Z}]$ to e (regarded as an object of the coslice ∞-category $\mathcal{C}_{X/}$) and each nondegenerate edge of $\mathrm{Spine}[\mathbf{Z}]$ to σ (regarded as a morphism of the coslice ∞-category $\mathcal{C}_{X/}$). We will identify T_{σ} with a diagram $\overline{T}_e : \mathrm{Spine}[\mathbf{Z}]^{\triangleleft} \to \mathcal{C}$, where the restriction $\overline{T}_e|_{\mathrm{Spine}[\mathbf{Z}]}$ is the diagram $T_e : \mathrm{Spine}[\mathbf{Z}] \to \mathcal{C}$ of Notation 8.5.6.11. Let us further identify $\overline{T}_e$ with an object $\overline{X}$ of the slice ∞-category $\mathcal{C}_{/T_e}$, lying over the object $X \in \mathcal{C}$. By virtue of assumption (4), the ∞-category $\mathcal{C}_{/T_e}$ has a final object $\overline{Y}$, lying over a some object $Y \in \mathcal{C}$. We can therefore choose a morphism $\overline{r} : \overline{X} \to \overline{Y}$ in the ∞-category $\mathcal{C}_{/T_e}$, having some image $r : X \to Y$ in $\mathcal{C}$. The morphism $\overline{r}$ can be identified with a diagram $\Delta^1 \star \mathrm{Spine}[\mathbf{Z}] \to \mathcal{C}$, which we display informally as

$$\begin{tikzcd}
 & & X \arrow[d, "r"] & & \\
 & & Y \arrow[dl, "i(-1)"'] \arrow[d, "i(0)"] \arrow[dr, "i(1)"] & & \\
\cdots \arrow[r] & X \arrow[r, "e"] & X \arrow[r, "e"] & X \arrow[r, "e"] & \cdots
\end{tikzcd}$$

By construction, the restriction of this diagram to the middle column witnesses the equality $[i(0)] \circ [r] = [e]$ in the homotopy category $\mathrm{h}\mathcal{C}$. To show that the homotopy class $[e]$ is a split idempotent, it will suffice to show that $[r] \circ [i(0)]$ is the identity morphism $[\mathrm{id}_Y]$ in the homotopy category $\mathrm{h}\mathcal{C}$. Since the morphism $[i(0)]$ admits a left inverse, it will suffice to show that equality holds after postcomposition with $[i(0)]$. This follows from the calculation

$$\begin{aligned}
[i(0)] \circ ([r] \circ [i(0)]) &= ([i(0)] \circ [r]) \circ [i(0)] \\
&= [e] \circ [i(0)] \\
&= [e] \circ ([e] \circ [i(-1)]) \\
&= ([e] \circ [e]) \circ [i(-1)] \\
&= [e] \circ [i(-1)] \\
&= [i(0)].
\end{aligned}$$

□

0420 **Corollary 8.5.7.5.** *Let $\mathcal{C}$ be an ∞-category which admits sequential limits and colimits, and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. Then e is idempotent if and only if it is homotopy idempotent and the composite map*

$$\varprojlim(\cdots \to X \xrightarrow{e} X \to \cdots) \to X \to \varinjlim(\cdots \to X \xrightarrow{e} X \to \cdots)$$

is an isomorphism (see Definition 8.5.6.14).

Proof. This is a special case of Proposition 8.5.7.4, together with the observation that every idempotent in $\mathcal{C}$ is split (Proposition 8.5.6.12). □

Let us now present a sample application of Proposition 8.5.7.4.

0421 **Corollary 8.5.7.6.** *Let X be a connected Kan complex and let $x \in X$ be a vertex. Then every homotopy idempotent endomorphism $e : (X, x) \to (X, x)$ in the ∞-category $\mathcal{S}_*$ is (split) idempotent.*

Proof. Without loss of generality, we may assume that the morphism e is obtained from a morphism $(X, x) \to (X, x)$ in the ordinary category of pointed Kan complexes (see Proposition 5.5.3.8). Then the diagram $T_e : \mathrm{Spine}[\mathbf{Z}] \to \mathcal{S}_*$ of Notation 8.5.6.11 lifts to a functor of ordinary categories $\mathscr{F} : (\mathbf{Z}, \leq) \to \mathrm{Kan}_*$, which we display as

$$\cdots \to (X, x) \xrightarrow{e} (X, x) \xrightarrow{e} (X, x) \xrightarrow{e} (X, x) \xrightarrow{e} (X, x) \to \cdots$$

Let us abuse notation by identifying $\mathscr{F}$ with its image in the category of Kan complexes Kan. Applying Variant 7.5.3.6, we can choose a levelwise homotopy equivalence $\alpha : \mathscr{F} \to \mathscr{G}$, where $\mathscr{G} : (\mathbf{Z}, \leq) \to \mathrm{Kan}$ is an isofibrant diagram of Kan complexes. Note that we can also regard $\mathscr{G}$ as a diagram of pointed Kan complexes, by equipping each $\mathscr{G}(n)$ with the base point $y_n = \alpha(n)(x)$. Let us extend $\mathscr{G}$ to a functor $\mathscr{G}^{\pm} : (\mathbf{Z} \cup \{-\infty, \infty\}, \leq) \to \mathrm{Kan}_*$ by setting $\mathscr{G}^{\pm}(-\infty) = \varprojlim(\mathscr{G})$ and $\mathscr{G}^{\pm}(\infty) = \varinjlim(\mathscr{G})$, where the limit and colimit are formed in the category of (pointed) simplicial sets; we denote the base points of $\mathscr{G}^{\pm}(-\infty)$ and $\mathscr{G}^{\pm}(\infty)$

by $y_{-\infty}$ and y_{∞}, respectively. Passing to nerves, the functor $\mathscr{G}^{\pm}$ determines a diagram $S : \{-\infty\} \star \operatorname{Spine}[\mathbf{Z}] \star \{\infty\} \to \mathcal{S}_{*}$.

Let $U : \mathcal{S}_{*} \to \mathcal{S}$ be the forgetful functor (given on objects by $U(X, x) = X$). Since the diagram $\mathscr{G}$ is isofibrant and the inclusion $\operatorname{Spine}[\mathbf{Z}] \hookrightarrow \operatorname{N}_{\bullet}(\mathbf{Z})$ is left cofinal (Remark 8.5.4.14), the restriction $(U \circ S)|_{\{-\infty\} \star \operatorname{Spine}[\mathbf{Z}]}$ is a limit diagram in the ∞-category $\mathcal{S}$ (Corollary 7.5.4.7). Applying Corollary 7.1.3.20, we see that $S|_{\{-\infty\} \star \operatorname{Spine}[\mathbf{Z}]}$ is a limit diagram in the ∞-category $\mathcal{S}_{*}$. Since the ∞-category $\operatorname{N}_{\bullet}(\mathbf{Z})$ is filtered and the inclusion $\operatorname{Spine}[\mathbf{Z}] \hookrightarrow \operatorname{N}_{\bullet}(\mathbf{Z})$ is right cofinal, the restriction $(U \circ S)|_{\operatorname{Spine}[\mathbf{Z}] \star \{\infty\}}$ is a colimit diagram in the ∞-category $\mathcal{S}$ (Corollary 7.5.9.3). Since the spine $\operatorname{Spine}[\mathbf{Z}]$ is weakly contractible (Remark 8.5.4.15), it follows that $S|_{\operatorname{Spine}[\mathbf{Z}] \star \{\infty\}}$ is a colimit diagram in the ∞-category $\mathcal{S}_{*}$. Moreover, the natural transformation α induces an isomorphism $T_e \to S|_{\operatorname{Spine}[\mathbf{Z}]}$ in the ∞-category $\operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}], \mathcal{S}_{*})$. It follows that the morphism e is idempotent if and only if the composition

$$\Delta^1 \simeq \{-\infty\} \star \{\infty\} \hookrightarrow \{-\infty\} \star \operatorname{Spine}[\mathbf{Z}] \star \{\infty\} \xrightarrow{S} \mathcal{S}_{*}$$

is an isomorphism in $\mathcal{S}_{*}$: that is, if and only if the map of Kan complexes

$$\theta : \mathscr{G}^{\pm}(-\infty) = \varprojlim(\mathscr{G}) \to \varinjlim(\mathscr{G}) = \mathscr{G}^{\pm}(\infty)$$

is a homotopy equivalence of (pointed) Kan complexes (Corollary 8.5.7.6).

Since each $\mathscr{G}(n)$ is a connected Kan complex, it follows that the colimit $\varinjlim(\mathscr{G})$ is also connected. By virtue of Theorem 3.2.7.1, it will suffice to show that, for every integer $d \geq 0$, θ induces a bijection $\pi_d(\varprojlim(\mathscr{G}), y_{-\infty}) \to \pi_d(\varinjlim(\mathscr{G}), y_{\infty})$ (note that, in the case $d = 0$, this guarantees that the Kan complex $\varprojlim(\mathscr{G})$ is also connected, so that a similar conclusion holds for any choice of base point). Let $\overleftarrow{G}$ denote the diagram of sets

$$\cdots \to \pi_d(\mathscr{G}(-1), y_{-1}) \to \pi_d(\mathscr{G}(0), y_0) \to \pi_d(\mathscr{G}(1), y_1) \to \cdots$$

Note that α determines an isomorphism of $\overleftarrow{G}$ with the diagram

$$\cdots \to \pi_d(X, x) \xrightarrow{f_d} \pi_d(X, x) \xrightarrow{f_d} \pi_d(X, x) \to \cdots$$

where each of the transition maps is induced by e. Since e is homotopic to $e \circ e$ (in the homotopy category of pointed Kan complexes), it follows that $f_d = f_d \circ f_d$, so that the tautological map $v : \varprojlim(\overleftarrow{G}) \to \varinjlim(\overleftarrow{G})$ is a bijection. Unwinding the definition, we see that $\pi_d(\theta)$ factors as a composition

$$\pi_d(\varprojlim_n \mathscr{G}(n), y_{\infty}) \xrightarrow{u} \varprojlim_n \pi_d(\mathscr{G}(n), y_n)) \xrightarrow{v} \varinjlim_n \pi_d(\mathscr{G}(n), y_n) \xrightarrow{w} \pi_d(\mathscr{G}(\infty), y_{\infty}),$$

where the map w is also bijective (Remark 3.2.2.16). It will therefore suffice to show that the map u is bijective. By virtue of the Milnor exact sequence (Proposition [?]), this is equivalent to the assertion that the set

$$\varprojlim^1(\cdots \to \pi_{d+1}(\mathscr{G}(-2), y_{-2}) \to \pi_{d+1}(\mathscr{G}(-1), y_{-1}) \to \pi_{d+1}(\mathscr{G}(0), y_0))$$

has a single element. This is a special case of Proposition [?], since the inverse system of groups

$$\cdots \to \pi_{d+1}(X, x) \xrightarrow{f_{d+1}} \pi_{d+1}(X, x) \xrightarrow{f_{d+1}} \pi_{d+1}(X, x)$$

is Mittag-Leffler (since f_{d+1} is idempotent, its image coincides with the image of f_{d+1}^n for every integer $n > 0$). □

0422 **Corollary 8.5.7.7.** *Let X be a connected Kan complex and let $e : X \to X$ be a homotopy idempotent endomorphism in the ∞-category $\mathcal{S}$. Then e is idempotent if and only if it can be lifted to a homotopy endomorphism endomorphism $\widetilde{e} : (X, x) \to (X, x)$ in the ∞-category $\mathcal{S}_*$.*

Proof. Let $\widetilde{e} : (X, x) \to (X, x)$ be a lift of e to a morphism in the ∞-category $\mathcal{S}_*$. If $\widetilde{e}$ is homotopy idempotent, then it is idempotent (Corollary 8.5.7.6), so that e is also idempotent. For the converse, suppose that e is idempotent: that is, it can be extended to a functor $F : \mathrm{N}_\bullet(\mathrm{Idem}) \to \mathcal{C}$. Since the ∞-category $\mathcal{S}$ admits small colimits (Corollary 7.4.5.6), the idempotent F splits. Consequently, there is a retraction diagram

0423
$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"] & & Y \end{tikzcd} \tag{8.58}$$

in the ∞-category of spaces $\mathcal{S}$, where e is homotopic to the composition $(i \circ r) : X \to X$. Fix vertices $x \in X$ and $y \in Y$. Since X is connected, we can lift i to a morphism $\widetilde{i} : (Y, y) \to (X, x)$ in the ∞-category $\mathcal{S}_*$ (see Example 5.5.3.4). Since the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ is a left fibration, we can lift (8.58) to a retraction diagram

$$\begin{tikzcd} & (X, x) \arrow[dr, "\widetilde{r}"] & \\ (Y, y) \arrow[ur, "\widetilde{i}"] \arrow[rr, "\mathrm{id}"] & & (Y, y) \end{tikzcd}$$

in the ∞-category $\mathcal{S}_*$. It follows that e can be lifted to a (split) homotopy idempotent in $\mathcal{S}_*$, given by any composition of $\widetilde{r}$ with $\widetilde{i}$. □

Exercise 8.5.7.8. Let (X, x) be a pointed Kan complex and let $e : X \to X$ be a morphism from X to itself. Show that: 0424

- If X is connected, then e can be lifted to a morphism $\widetilde{e} : (X, x) \to (X, x)$ in the ∞-category $\mathcal{S}_*$.
- If X is simply connected, then e is homotopy idempotent (in the ∞-category $\mathcal{S}$) if and only if $\widetilde{e}$ is homotopy idempotent (in the ∞-category $\mathcal{S}_*$).

In particular, if X is simply connected, then every homotopy idempotent $e : X \to X$ is (split) idempotent (Corollary 8.5.7.7).

In the situation of Exercise 8.5.7.8, the simple connectivity assumption on X cannot be omitted. That is, not every homotopy idempotent in the ∞-category $\mathcal{S}$ is split. We present a counterexample, due originally to Freyd and Heller (see [21]).

Definition 8.5.7.9 (Dyadic Homeomorphisms)**.** Recall that a *dyadic rational number* is a real number of the form $\frac{a}{2^n}$, where a and n are integers. Let $s, t \geq 0$ be dyadic rational numbers. We say that a homeomorphism $f : [0, s] \xrightarrow{\sim} [0, t]$ is *dyadic* if it satisfies the following conditions: 0425

- The function f is piecewise linear; in particular, it is differentiable away from finitely many points of the closed interval $[0, s]$.
- If $x \in [0, s]$ is a point where f is not differentiable, then x is a dyadic rational number.
- For every point $x \in [0, s]$ where f is differentiable, the derivative $f'(x)$ is equal to 2^n for some integer n.

Note that the third condition implies that the homeomorphism f is strictly increasing, so that $f(0) = 0$ and $f(s) = t$.

Exercise 8.5.7.10 (Inverses of Dyadic Homeomorphisms)**.** Let $s, t \geq 0$ be dyadic rational numbers and let $f : [0, s] \xrightarrow{\sim} [0, t]$ be a dyadic homeomorphism. Show that the inverse homeomorphism $f^{-1} : [0, t] \xrightarrow{\sim} [0, s]$ is also dyadic. 0426

Exercise 8.5.7.11 (Composition of Dyadic Homeomorphisms)**.** Let $s, t, u \geq 0$ be dyadic rational numbers and let $f : [0, s] \xrightarrow{\sim} [0, t]$ and $g : [0, t] \xrightarrow{\sim} [0, u]$ be dyadic homeomorphisms. Show that the composition $(g \circ f) : [0, s] \xrightarrow{\sim} [0, u]$ is also a dyadic homeomorphism. 0427

Definition 8.5.7.12 (The Thompson Group)**.** Let $\mathrm{Aut}_{\mathrm{Dy}}([0, 1])$ denote the collection of all dyadic homeomorphisms from the unit interval $[0, 1]$ to itself. It follows from Exercises 8.5.7.10 and 8.5.7.11 that $\mathrm{Aut}_{\mathrm{Dy}}([0, 1])$ has the structure of a group (where the group law is given by composition of homeomorphisms). We will refer to $\mathrm{Aut}_{\mathrm{Dy}}([0, 1])$ as the *Thompson group*. 0428

0429 **Construction 8.5.7.13** (Speeding Up)**.** Let $f : [0,1] \to [0,1]$ be an orientation-preserving homeomorphism. We define $\alpha(f) : [0,1] \to [0,1]$ by the formula

$$\alpha(f)(x) = \begin{cases} f(2x)/2 & \text{if } 0 \leq x \leq 1/2 \\ x & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Then $\alpha(f)$ is also an orientation-preserving homeomorphism of $[0,1]$ with itself. Moreover, if f is dyadic, then $\alpha(f)$ is also dyadic. It follows that the construction $f \mapsto \alpha(f)$ determines a group homomorphism α from the Thompson group $\mathrm{Aut}_{\mathrm{Dy}}([0,1])$ to itself.

042A **Proposition 8.5.7.14.** *Let* $\mathrm{Aut}_{\mathrm{Dy}}([0,1])$ *be the Thompson group of Definition 8.5.7.12 and let* $X = B_{\bullet} \mathrm{Aut}_{\mathrm{Dy}}([0,1])$ *denote its classifying simplicial set (Construction 1.3.2.5). Then the homomorphism* α *of Construction 8.5.7.13 induces a homotopy idempotent endomorphism* $e : X \to X$ *in the* ∞*-category* $\mathcal{S}$*.*

Proof. We wish to show that the diagram of Kan complexes

$$\begin{tikzcd}
& X \arrow[dr, "e"] & \\
X \arrow[ur, "e"] \arrow[rr, "e"] & & X
\end{tikzcd}$$

commutes up to homotopy. By virtue of Proposition 1.5.3.3, this is equivalent to the assertion that the homomorphisms $\alpha, \alpha^2 : \mathrm{Aut}_{\mathrm{Dy}}([0,1]) \to \mathrm{Aut}_{\mathrm{Dy}}([0,1])$ are conjugate: that is, there exists an element $g \in \mathrm{Aut}_{\mathrm{Dy}}([0,1])$ satisfying the identity $\alpha(f) \circ g = g \circ \alpha^2(f)$ for every element $f \in \mathrm{Aut}_{\mathrm{Dy}}([0,1])$. Concretely, we can take g to be any dyadic homeomorphism satisfying the identity $g(x) = 2x$ for $0 \leq x \leq 1/4$. □

We now show that the homotopy idempotent of Proposition 8.5.7.14 cannot be refined to an idempotent in the ∞-category $\mathcal{S}$.

042B **Proposition 8.5.7.15.** *Let* $X = B_{\bullet} \mathrm{Aut}_{\mathrm{Dy}}([0,1])$*. Then homotopy idempotent endomorphism* $e : X \to X$ *of Proposition 8.5.7.14 is not idempotent.*

Proof. Let x denote the unique vertex of X. Suppose, for a contradiction, that e is idempotent. Then we can lift e to a homotopy idempotent morphism $\widetilde{e} : (X,x) \to (X,x)$ in the ∞-category $\mathcal{S}_*$ (Corollary 8.5.7.7). Passing to fundamental groups, we obtain an idempotent homomorphism β from the Thompson group $\mathrm{Aut}(_{\mathrm{Dy}}([0,1]) = \pi_1(X,x)$ to itself. Since the forgetful functor $\mathcal{S}_* \to \mathcal{S}$ carries $\widetilde{e}$ to e, β is conjugate to the homomorphism α of Construction 8.5.7.13. Since α is a monomorphism, it follows that β is also a monomorphism. The equation $\beta^2 = \beta$ then implies that β is the identity map. This is a contradiction, since β is conjugate to the homomorphism α (which is not the identity morphism). □

8.5.8 Partial Idempotents

Let Idem denote the category introduced in Construction 8.5.2.7. For each integer $n \geq 0$, we let $\mathrm{N}_{\leq n}(\mathrm{Idem})$ denote the n-skeleton of the simplicial set $\mathrm{N}_{\bullet}(\mathrm{Idem})$ (see Variant 1.3.1.6). If $\mathcal{C}$ is an ∞-category, we will refer to a morphism $\mathrm{N}_{\leq n}(\mathrm{Idem}) \to \mathcal{C}$ as a *partial idempotent* in $\mathcal{C}$. 042C

Example 8.5.8.1. The simplicial set $\mathrm{N}_{\leq 0}(\mathrm{Idem})$ is isomorphic to the standard simplex Δ^0. Consequently, if $\mathcal{C}$ is an ∞-category, then a morphism $\mathrm{N}_{\leq 0}(\mathrm{Idem}) \to \mathcal{C}$ can be identified with an object $X \in \mathcal{C}$. 042D

Example 8.5.8.2. The simplicial set $\mathrm{N}_{\leq 1}(\mathrm{Idem})$ can be identified with the simplicial circle $\Delta^1 / \partial \Delta^1$, obtained from the standard simplex Δ^1 by identifying its endpoints. Consequently, if $\mathcal{C}$ is an ∞-category, then a morphism $\mathrm{N}_{\leq 1}(\mathrm{Idem}) \to \mathcal{C}$ can be identified with a pair (X, e), where X is an object of $\mathcal{C}$ and e is an endomorphism of X. 042E

Example 8.5.8.3. Let $\mathcal{C}$ be an ∞-category. Then a morphism $\mathrm{N}_{\leq 2}(\mathrm{Idem}) \to \mathcal{C}$ can be identified with a triple (X, e, σ), where X is an object of $\mathcal{C}$, $e : X \to X$ is an endomorphism of X, and σ is a 2-simplex of $\mathcal{C}$ with boundary indicated in the diagram 042F

$$\begin{tikzcd}
 & X \arrow[dr, "e"] & \\
X \arrow[ur, "e"] \arrow[rr, "e"] & & X,
\end{tikzcd}$$

so that σ witnesses the identity $[e] = [e] \circ [e]$ in the homotopy category $\mathrm{h}\mathcal{C}$.

Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$, which we identify with a morphism $F_{\leq 1} : \mathrm{N}_{\leq 1}(\mathrm{Idem}) \to \mathcal{C}$. The endomorphism e is homotopy idempotent (in the sense of Definition 8.5.7.1) if and only if $F_{\leq 1}$ admits an extension $F_{\leq 2} : \mathrm{N}_{\leq 2}(\mathrm{Idem}) \to \mathcal{C}$. Proposition 8.5.7.15 shows that this condition does not guarantee the existence of an idempotent $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$ extending $F_{\leq 1}$. Our goal in this section is to show that a slightly stronger condition does suffice: namely, it is enough to assume that $F_{\leq 1}$ can be extended to a diagram $F_{\leq 3} : \mathrm{N}_{\leq 3}(\mathrm{Idem}) \to \mathcal{C}$. This is a consequence of the following:

Theorem 8.5.8.4. *Let $n \geq 3$ be an integer. The inclusion map $\mathrm{N}_{\leq n}(\mathrm{Idem}) \hookrightarrow \mathrm{N}_{\bullet}(\mathrm{Idem})$ admits a factorization* 042G

$$\mathrm{N}_{\leq n}(\mathrm{Idem}) \xrightarrow{\iota} \mathcal{E} \xrightarrow{U} \mathrm{N}_{\bullet}(\mathrm{Idem}),$$

where $\mathcal{E}$ is an ∞-category, ι is an inner anodyne morphism which is bijective on simplices of dimension $< n$, and the functor U admits a right inverse $V : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{E}$.

042H **Remark 8.5.8.5.** In the situation of Theorem 8.5.8.5, we can regard ι and $V|_{\mathrm{N}_{\leq n}(\mathrm{Idem})}$ as morphisms from $\mathrm{N}_{\leq n}(\mathrm{Idem})$ to $\mathcal{E}$. By construction, these morphisms coincide after composing with the functor $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathrm{Idem})$. Since U is bijective on simplices of dimensions $< n$, it follows that ι and $V|_{\mathrm{N}_{\leq n}(\mathrm{Idem})}$ coincide when on the $(n-1)$-skeleton of $\mathrm{N}_{\bullet}(\mathrm{Idem})$. Beware that ι and V do *not* coincide on the nondegenerate n-simplex of $\mathrm{N}_{\bullet}(\mathrm{Idem})$. In fact, we claim that ι and $V|_{\mathrm{N}_{\leq n}(\mathrm{Idem})}$ are not even isomorphic when viewed as an object of the ∞-category $\mathrm{Fun}(\mathrm{N}_{\leq n}(\mathrm{Idem}), \mathcal{E})$. Assume, for a contradiction, that there exists an isomorphism α of $\iota = \mathrm{id}_{\mathcal{E}} \circ \iota$ with $V|_{\mathrm{N}_{\leq n}(\mathrm{Idem})} = (V \circ U) \circ \iota$. Since ι is inner anodyne, we could then lift α to an isomorphism $\widetilde{\alpha} : \mathrm{id}_{\mathcal{E}} \to V \circ U$ in the ∞-category $\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C})$. It would follow that U is an equivalence of ∞-categories (with homotopy inverse given by V). Then $(U \circ \iota) : \mathrm{N}_{\leq n}(\mathrm{Idem}) \to \mathrm{N}_{\bullet}(\mathrm{Idem})$ would be a categorical equivalence of simplicial sets, which contradicts Remark 8.5.4.18.

We will give the proof of Theorem 8.5.8.4 at the end of this section. First, let us record some consequences.

042J **Corollary 8.5.8.6.** *Let $n \geq 3$ be an integer, and let $\mathcal{C}$ be an ∞-category equipped with a partial idempotent $F_{<n} : \mathrm{N}_{\leq n-1}(\mathrm{Idem}) \to \mathcal{C}$. The following conditions are equivalent:*

(1) *The morphism $F_{<n}$ extends to an idempotent $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$.*

(2) *The morphism $F_{<n}$ extends to a partial idempotent $F_{\leq n} : \mathrm{N}_{\leq n}(\mathrm{Idem}) \to \mathcal{C}$.*

Proof. The implication (1) ⇒ (2) is immediate. To prove the converse, suppose that $F_{\leq n} : \mathrm{N}_{\leq n}(\mathrm{Idem}) \to \mathcal{C}$ is an extension of $F_{<n}$. Let $\iota : \mathrm{N}_{\leq n}(\mathrm{Idem}) \hookrightarrow \mathcal{E}$, $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathrm{Idem})$, and $V : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{E}$ be as in Theorem 8.5.8.4. Since ι is inner anodyne, we can choose a functor $\overline{F} : \mathcal{E} \to \mathcal{C}$ satisfying $\overline{F} \circ \iota = F_{\leq n}$. Then $F = \overline{F} \circ V$ is a functor from $\mathrm{N}_{\bullet}(\mathrm{Idem})$ to $\mathcal{C}$, and Remark 8.5.8.5 shows that F coincides with $F_{<n}$ on the $(n-1)$-skeleton of $\mathrm{N}_{\bullet}(\mathrm{Idem})$. □

042K **Warning 8.5.8.7.** Let $\mathcal{C}$ be an ∞-category and let $F_{\leq n} : \mathrm{N}_{\leq n}(\mathrm{Idem}) \to \mathcal{C}$ be a partial idempotent in $\mathcal{C}$. Corollary 8.5.8.6 asserts that, if $n \geq 3$, then we can choose an idempotent $F : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{C}$ such that F and $F_{\leq n}$ coincide on the $(n-1)$-skeleton $\mathrm{N}_{\bullet}(\mathrm{Idem})$. Beware that we generally cannot arrange that $F|_{\mathrm{N}_{\leq n}(\mathrm{Idem})}$ coincides with $F_{\leq n}$. For example, this always fails in the (universal) case $F_{\leq n}$ is the inner anodyne morphism $\iota : \mathrm{N}_{\leq n}(\mathrm{Idem}) \hookrightarrow \mathcal{E}$ of Theorem 8.5.8.4 (see Remark 8.5.8.5).

042L **Corollary 8.5.8.8.** *Let $\mathcal{C}$ be an ∞-category and let $e : X \to X$ be an endomorphism in $\mathcal{C}$. Then e is idempotent if and only if it can be extended to a diagram $\mathrm{N}_{\leq 3}(\mathrm{Idem}) \to \mathcal{C}$.*

042M **Corollary 8.5.8.9.** *Let $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ be a diagram of simplicial sets indexed by a filtered category $\mathcal{I}$. Suppose that each $\mathcal{C}_i$ is an ∞-category. Then the tautological map*

$$\theta : \varinjlim_{i \in \mathcal{I}} \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}_i) \to \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \varinjlim_{i \in \mathcal{I}} \mathcal{C}_i)$$

is an equivalence of ∞-categories.

Proof. Choose any integer $n \geq 3$, and let $\iota : \mathrm{N}_{\leq n}(\mathrm{Idem}) \hookrightarrow \mathcal{E}$, $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathrm{Idem})$, and $V : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{E}$ be as in Theorem 8.5.8.4. We then have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\varinjlim \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}_i) \arrow[r, "\circ V"] \arrow[d, "\theta"] & \varinjlim \mathrm{Fun}(\mathcal{E}, \mathcal{C}_i) \arrow[r, "\circ U"] \arrow[d, "\theta'"] & \varinjlim \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}_i) \arrow[d, "\theta"] \\
\mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \varinjlim \mathcal{C}_i) \arrow[r, "\circ V"] & \mathrm{Fun}(\mathcal{E}, \varinjlim \mathcal{C}_i) \arrow[r, "\circ U"] & \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \varinjlim \mathcal{C}_i),
\end{tikzcd}$$

where the horizontal compositions are identity morphisms. Consequently, to show that θ is an equivalence of ∞-categories, it will suffice to show that θ' is an equivalence of ∞-categories (Proposition 8.5.1.7). The functor θ' fits into a commutative diagram

$$\begin{tikzcd}
\varinjlim \mathrm{Fun}(\mathcal{E}, \mathcal{C}_i) \arrow[r, "\circ \iota"] \arrow[d, "\theta'"] & \varinjlim \mathrm{Fun}(\mathrm{N}_{\leq n}(\mathrm{Idem}), \mathcal{C}_i) \arrow[d, "\theta''"] \\
\mathrm{Fun}(\mathcal{E}, \varinjlim \mathcal{C}_i) \arrow[r, "\circ \iota"] & \mathrm{Fun}(\mathrm{N}_{\leq n}(\mathrm{Idem}), \varinjlim(\mathcal{C}_i)).
\end{tikzcd}$$

Since ι is inner anodyne, the horizontal maps are trivial Kan fibrations (Proposition 1.5.7.6). We conclude by observing that θ'' is an isomorphism of simplicial sets, since the simplicial set $\mathrm{N}_{\leq n}(\mathrm{Idem})$ is finite (Corollary 3.6.1.10). □

Corollary 8.5.8.10. *Let $\{\mathcal{C}_i\}_{i \in \mathcal{I}}$ be a diagram of simplicial sets indexed by a filtered category $\mathcal{I}$. Suppose that each $\mathcal{C}_i$ is an idempotent complete ∞-category. Then the colimit $\mathcal{C} = \varinjlim_i \mathcal{C}_i$ is idempotent complete.* 042N

Proof. For each object $i \in \mathcal{I}$, our assumption that $\mathcal{C}_i$ is idempotent complete guarantees that the restriction functor $R_i : \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}_i) \to \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}_i)$ is an equivalence of ∞-categories (Remark 8.5.4.8). Passing to filtered colimits, we deduce that the induced map $\varinjlim \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}_i) \to \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}_i)$ is also an equivalence of ∞-categories (Corollary 4.5.7.2). This map fits into a commutative diagram

$$\begin{tikzcd}
\varinjlim \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}_i) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Ret}), \mathcal{C}) \arrow[d, "R"] \\
\varinjlim \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}_i) \arrow[r] & \mathrm{Fun}(\mathrm{N}_{\bullet}(\mathrm{Idem}), \mathcal{C}),
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories (Corollaries 8.5.8.9 and 8.5.1.32). It follows that the restriction functor $R : \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Ret}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Idem}), \mathcal{C})$ is also an equivalence of ∞-categories, so that $\mathcal{C}$ is idempotent complete (Remark 8.5.4.8). □

042P **Corollary 8.5.8.11.** *Let $\mathcal{C}$ be a small filtered ∞-category and let $\mathscr{F} : \mathcal{C} \to \mathcal{QC}$ be a functor. Suppose that, for every object $C \in \mathcal{C}$, the ∞-category $\mathscr{F}(C)$ is idempotent complete. Then the colimit $\varinjlim(\mathscr{F})$ (formed in the ∞-category $\mathcal{QC}$) is also idempotent complete.*

Proof. Using Theorem 7.2.7.2, we can choose a directed partially ordered set $(A, \leq)$ and a right cofinal functor $\mathrm{N}_\bullet(A) \to \mathcal{C}$. Using Corollary 7.2.2.3 we can replace $\mathcal{C}$ by $\mathrm{N}_\bullet(A)$ and thereby reduce to the case where $\mathcal{C}$ is (the nerve of) a directed partially ordered set. Replacing $\mathscr{F}$ by an isomorphic functor if necessary, we can assume that it obtained from an A-indexed diagram in the ordinary category QCat (Corollary 5.6.5.16). In this case, the colimit $\varinjlim(\mathscr{F})$ in the ∞-category $\mathcal{QC}$ can be identified with its colimit in the ordinary category $\mathrm{QCat} \subset \mathrm{Set}_\Delta$ (Corollary 7.5.9.3), so the desired result follows from Corollary 8.5.8.10. □

We now turn to the proof of Theorem 8.5.8.4. The existence of an inner anodyne morphism $\iota : \mathrm{N}_{\leq n}(\mathrm{Idem}) \to \mathcal{E}$ which is bijective on simplices of dimension $< n$ is essentially formal, by virtue of the following variant of Corollary 4.1.3.3:

042Q **Lemma 8.5.8.12.** *Let $n \geq 0$ be an integer and let K be a simplicial set which satisfies the following condition for $0 \leq m \leq n$:*

$(*_m)$ *For every integer $0 < i < m$, every morphism of simplicial sets $\sigma : \Lambda^m_i \to K$ can be extended to an m-simplex of K.*

Then there exists an inner anodyne morphism $\iota : K \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is an ∞-category and ι is bijective on simplices of dimension $< n$.

Proof. We construct $\mathcal{E}$ as the colimit of a sequence of inner anodyne maps

$$K = K(0) \hookrightarrow K(1) \hookrightarrow K(2) \hookrightarrow K(3) \hookrightarrow \cdots$$

Assume that $K(t)$ has been constructed for some $t \geq 0$, and let S be the collection of all maps $\sigma : \Lambda^m_i \to K(t)$ where $0 < i < m$ and $m > n$. For every $\sigma \in S$, let us write C_σ for the simplicial set Λ^m_i which is the source of σ, and D_σ for the simplex Δ^m. We then construct a pushout diagram of simplicial sets

$$\begin{tikzcd}
\coprod_{\sigma \in S} C_\sigma \arrow[r] \arrow[d] & K(t) \arrow[d] \\
\coprod_{\sigma \in S} D_\sigma \arrow[r] & K(t+1).
\end{tikzcd}$$

By construction, the morphism $K(t) \to K(t+1)$ is inner anodyne and bijective on simplices of dimension $< n$. To complete the proof, it will suffice to show that the colimit $\mathcal{E} = \varinjlim_t K(t)$ is an ∞-category. Fix a pair of integers $0 < i < m$ and a morphism $\sigma : \Lambda^m_i \to \mathcal{E}$; we wish to show that σ can be extended to an m-simplex of $\mathcal{E}$. Since Λ^m_i is a finite simplicial set, σ factors (uniquely) through $K(t)$ for some integer $t \gg 0$. By construction, if $m > n$ then σ can be extended to an m-simplex of $K(t+1)$. We may therefore assume that $m \leq n$. In this case, we can take $k = 0$, in which case the existence of the desired extension follows from assumption $(*_m)$. □

Example 8.5.8.13. Let $\mathcal{C}$ be an ∞-category. For every integer $n \geq 0$, the skeleton $\mathrm{sk}_n(\mathcal{C})$ satisfies condition $(*_m)$ of Lemma 8.5.8.12 for $0 \leq m \leq n$. We can therefore choose an inner anodyne morphism $\iota : \mathrm{sk}_n(\mathcal{C}) \hookrightarrow \mathcal{E}$, where $\mathcal{E}$ is an ∞-category and f is bijective on simplices of dimension $< n$. 042R

In what follows, we will write X for the unique object of the category Idem, and $e : X \to X$ for the unique non-identity morphism. The main content of Theorem 8.5.8.4 is contained in the following result:

Proposition 8.5.8.14. *Let $\mathcal{E}$ be an ∞-category and let $\iota : \mathrm{N}_{\leq 3}(\mathrm{Idem}) \to \mathcal{E}$ be a morphism of simplicial sets which is bijective on simplices of dimension ≤ 2. Then $\iota(\widetilde{e})$ is an idempotent morphism of $\mathcal{E}$.* 042S

Proof. Let us regard the linearly ordered set $\mathbf{Z} = \{\cdots < -2 < -1 < 0 < 1 < 2 < \cdots\}$ as a category. Let $X \in \mathrm{Fun}(\mathbf{Z}, \mathrm{Idem})$ denote the constant functor taking the value X, and let $Y \in \mathrm{Fun}(\mathbf{Z}, \mathrm{Idem})$ denote the functor which carries each non-identity morphism of $\mathbf{Z}$ to the morphism $\widetilde{e}$ of Idem. We then have natural transformations

$$e_X : X \to X \qquad i : Y \to X \qquad r : X \to Y \qquad e_Y : Y \to Y$$

which carry each element of $\mathbf{Z}$ to the morphism $\widetilde{e}$. Note that the linearly ordered set $(\mathbf{Z}, \leq)$ can be identified with the homotopy category of the simplicial set $\mathrm{Spine}[\mathbf{Z}]$ of Notation 8.5.4.12. Let $G : \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathrm{N}_{\leq 3}(\mathrm{Idem})) \to \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{E})$ be the morphism of simplicial sets given by composition with ι. Since the simplicial set $\mathrm{Spine}[\mathbf{Z}]$ is 1-dimensional, the inclusion map

$$\mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathrm{N}_{\leq 3}(\mathrm{Idem})) \hookrightarrow \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathrm{N}_{\bullet}(\mathrm{Idem})) \simeq \mathrm{N}_{\bullet}(\mathrm{Fun}(\mathbf{Z}, \mathrm{Idem}))$$

is bijective on simplices of dimension ≤ 2, and therefore induces an equivalence of homotopy categories. It follows that G induces a functor of homotopy categories $\mathrm{h}G : \mathrm{Fun}(\mathbf{Z}, \mathrm{Idem}) \to \mathrm{hFun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{E})$.

In what follows, we will identify the morphisms X and Y with vertices of the simplicial set $\mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathrm{N}_{\leq 3}(\mathrm{Idem}))$, and the morphisms i, r, e_X, and e_Y with edges of the simplicial set $\operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}], \mathrm{N}_{\leq 3}(\mathrm{Idem}))$. Let $\delta_{\mathcal{E}} : \mathcal{E} \to \operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}], \mathcal{E})$ be the diagonal map, so that we have $G(X) = \delta_{\mathcal{C}}(\iota(\widetilde{X}))$ and $G(e_X) = \delta_{\mathcal{C}}(\iota(\widetilde{e}))$. We will deduce Proposition 8.5.8.14 from the following:

$(*)$ There exists an ∞-category $\mathcal{C}$ and a functor $T : \operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}], \mathcal{E}) \to \mathcal{C}$ such that $(T \circ \delta_{\mathcal{E}}) : \mathcal{E} \to \mathcal{C}$ is fully faithful and $(T \circ G)(e_Y)$ is an isomorphism in $\mathcal{C}$.

Let us first assume $(*)$, and show that it implies Proposition 8.5.8.14. We wish to show that $\iota(\widetilde{e})$ is an idempotent endomorphism in the ∞-category $\mathcal{E}$. Since $T \circ \delta_{\mathcal{E}}$ is fully faithful, this is equivalent to the statement that $(T \circ \delta_C)(\iota(\widetilde{e})) = (T \circ G)(e_X))$ is an idempotent endomorphism in the ∞-category $\mathcal{C}$. By virtue of Proposition 8.5.6.10, it will suffice to show that the homotopy class $[T(G(e_X))] = (\mathrm{h}T \circ \mathrm{h}G)(e_X)$ is a split idempotent in the homotopy category $\mathrm{h}\mathcal{C}$. By construction, the morphism e_X factors as a composition $i \circ r$ in the category $\operatorname{Fun}(\mathbf{Z}, \mathrm{Idem})$. It will therefore suffice to show that the functor $\mathrm{h}T \circ \mathrm{h}G$ carries the commutative diagram

$$\begin{tikzcd}
 & X \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "e_Y"] & & Y
\end{tikzcd}$$

to a retraction diagram in $\mathrm{h}\mathcal{C}$: that is, that $(\mathrm{h}T \circ \mathrm{h}G)(e_Y)$ is an identity morphism. This follows from Remark 8.5.2.2, since the morphism $(\mathrm{h}T \circ \mathrm{h}G)(e_Y)$ is both idempotent (since e_Y is an idempotent in the category $\operatorname{Fun}(\mathbf{Z}, \mathrm{Idem})$) and an isomorphism (by virtue of assumption $(*)$).

We now prove $(*)$. Using Corollary 8.3.3.17, we can choose a fully faithful functor of ∞-categories $H : \mathcal{E} \to \mathcal{C}$, where the ∞-category $\mathcal{C}$ admits sequential colimits. Let $\mathcal{D}$ denote the full subcategory of $\operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}]^{\triangleright}, \mathcal{C})$ spanned by the colimit diagrams. Our assumption that $\mathcal{C}$ admits sequential colimits guarantees that the restriction functor

$$\mathcal{D} \to \operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}], \mathcal{C}) \qquad U \mapsto U|_{\operatorname{Spine}[\mathbf{Z}]}$$

is a trivial Kan fibration of ∞-categories (Corollary 7.3.6.15). Let $\delta_{\mathcal{C}} : \mathcal{C} \hookrightarrow \operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}], \mathcal{C})$ and $\widetilde{\delta}_{\mathcal{C}} : \mathcal{C} \hookrightarrow \operatorname{Fun}(\operatorname{Spine}[\mathbf{Z}]^{\triangleright}, \mathcal{C})$ be the diagonal embeddings. Since the simplicial set $\operatorname{Spine}[\mathbf{Z}]$ is weakly contractible, the morphism $\widetilde{\delta}_{\mathcal{C}}$ factors through $\mathcal{D}$ (Corollary 7.2.3.5). Let

$s : \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{C}) \to \mathcal{D}$ be a solution to the lifting problem

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\widetilde{\delta}_{\mathcal{C}}"] \arrow[d, "\delta_{\mathcal{C}}"] & & \mathcal{D} \arrow[d, "U \mapsto U|_{\mathrm{Spine}[\mathbf{Z}]}"] \\
\mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{C}) \arrow[rr, "\mathrm{id}"] \arrow[urr, dashed, "s"] & & \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{C}).
\end{tikzcd}$$

Let $\mathrm{ev} : \mathcal{D} \to \mathcal{C}$ be the functor given by evaluation at the cone point of $\mathrm{Spine}[\mathbf{Z}]^{\triangleright}$. We let $T : \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{E}) \to \mathcal{C}$ denote the functor given by the composition

$$\mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{E}) \xrightarrow{H\circ} \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{C}) \xrightarrow{s} \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}]^{\triangleright}, \mathcal{C}) \xrightarrow{\mathrm{ev}} \mathcal{C}.$$

Stated more informally, the functor T carries a diagram

$$\cdots \to C_{-2} \to C_{-1} \to C_0 \to C_1 \to C_2 \to \cdots$$

in the ∞-category $\mathcal{E}$ to a colimit of the diagram

$$\cdots \to H(C_{-2}) \to H(C_{-1}) \to H(C_0) \to H(C_1) \to H(C_2) \to \cdots$$

in the ∞-category $\mathcal{C}$. By construction, $T \circ \delta_{\mathcal{E}}$ coincides with the fully faithful functor H.

We now complete the proof by showing that the functor T carries $G(e_Y)$ to an isomorphism in $\mathcal{C}$. Let us regard $G(\widetilde{Y})$ as a diagram $\mathrm{Spine}[\mathbf{Z}] \to \mathcal{E}$. Since the inclusion $\mathrm{Spine}[\mathbf{Z}] \hookrightarrow \mathrm{N}_{\bullet}(\mathbf{Z})$ is inner anodyne (Remark 8.5.4.14), we can extend $G(Y)$ to a functor $C : \mathrm{N}_{\bullet}(\mathbf{Z}) \to \mathcal{E}$. Let $a : \{0 < 1\} \times \mathbf{Z} \to \mathbf{Z}$ be the morphism of partially ordered sets given by $a(i, n) = i + n$. Passing to nerves, we obtain a morphism of simplicial sets $A : \Delta^1 \times \mathrm{N}_{\bullet}(\mathbf{Z}) \to \mathrm{N}_{\bullet}(\mathbf{Z})$. The composition $M = C \circ A$ then corresponds to a diagram in $\mathcal{E}$ which we display informally as

$$\begin{tikzcd}
\cdots \arrow[r] & \iota(\widetilde{X}) \arrow[r, "\iota(\widetilde{e})"] \arrow[d, "\iota(\widetilde{e})"] & \iota(\widetilde{X}) \arrow[r, "\iota(\widetilde{e})"] \arrow[d, "\iota(\widetilde{e})"] & \iota(\widetilde{X}) \arrow[r, "\iota(\widetilde{e})"] \arrow[d, "\iota(\widetilde{e})"] & \cdots \\
\cdots \arrow[r] & \iota(\widetilde{X}) \arrow[r, "\iota(\widetilde{e})"] & \iota(\widetilde{X}) \arrow[r, "\iota(\widetilde{e})"] & \iota(\widetilde{X}) \arrow[r, "\iota(\widetilde{e})"] & \cdots
\end{tikzcd}$$

Let f denote the restriction $M|_{\Delta^1 \times \mathrm{Spine}[\mathbf{Z}]}$, which we regard as a morphism in the ∞-category $\mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{E})$. Since ι is bijective on simplices of dimension ≤ 2 and the simplicial set $\mathrm{Spine}[\mathbf{Z}]$ has dimension ≤ 1, the morphism G is bijective on simplices of dimension ≤ 1. We can therefore write $f = G(f_0)$ for a unique edge f_0 of $\mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathrm{N}_{\leq 3}(\mathrm{Idem}))$. It follows by inspection that f_0 must coincide with $e_{\widetilde{Y}}$. We are therefore reduced to showing that $T(f)$ is an isomorphism in the ∞-category $\mathcal{C}$.

Since the ∞-category $\mathcal{C}$ admits sequential colimits, we can extend $(H \circ C) : \mathrm{N}_{\bullet}(\mathbf{Z}) \to \mathcal{C}$ to a colimit diagram $\overline{C} : \mathrm{N}_{\bullet}(\mathbf{Z})^{\triangleright} \to \mathcal{C}$. Note that A extends uniquely to a morphism of simplicial sets $\overline{A} : \Delta^1 \times \mathrm{N}_{\bullet}(\mathbf{Z})^{\triangleright} \to \mathrm{N}_{\bullet}(\mathbf{Z})^{\triangleright}$ (given on vertices by $\overline{A}(i,v) = v$, where v is the cone point of $\mathrm{N}_{\bullet}(\mathbf{Z})^{\triangleright}$). Let us identify the restriction $(\overline{C} \circ \overline{A})|_{\Delta^1 \times \mathrm{Spine}[\mathbf{Z}]^{\triangleright}}$ with a morphism $\overline{f} : D \to D'$ in the ∞-category $\mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}]^{\triangleright}, \mathcal{C})$. Since the inclusion $\mathrm{Spine}[\mathbf{Z}] \hookrightarrow \mathrm{N}_{\bullet}(\mathbf{Z})$ is right cofinal (Proposition 7.2.1.3), both D and D' are colimit diagrams in $\mathcal{D}$ (Corollary 7.2.2.3). Consequently, we can view $\overline{f}$ as a morphism in the ∞-category $\mathcal{D}$. By construction, the restriction functor $\mathcal{D} \to \mathrm{Fun}(\mathrm{Spine}[\mathbf{Z}], \mathcal{C})$ carries $\overline{f}$ to $H(f)$. It follows that $T(f) = (\mathrm{ev} \circ s \circ H)(f)$ is isomorphic to $\mathrm{ev}(\overline{f})$ as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. We are therefore reduced to showing that $\mathrm{ev}(\overline{f})$ is an isomorphism in $\mathcal{C}$. This is clear: the morphism $\mathrm{ev}(\overline{f})$ is an identity morphism in $\mathcal{C}$, since the functor $\overline{A}$ carries $\Delta^1 \times \{v\}$ to a degenerate edge of $\mathrm{N}_{\bullet}(\mathbf{Z})^{\triangleright}$. □

Proof of Theorem 8.5.8.4. Fix an integer $n \geq 0$. Using Example 8.5.8.13, we can choose an ∞-category $\mathcal{E}$ and an inner anodyne morphism $\iota : \mathrm{N}_{\leq n}(\mathrm{Idem}) \hookrightarrow \mathcal{E}$ which is bijective on simplices of dimension $< n$. Since ι is inner anodyne, there is a unique functor $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathrm{Idem})$ for which $U \circ \iota$ is coincides with the inclusion map $\mathrm{N}_{\leq n}(\mathrm{Idem}) \hookrightarrow \mathrm{N}_{\bullet}(\mathrm{Idem})$. If $n \geq 3$, then Proposition 8.5.8.14 guarantees that $\iota(\widetilde{e})$ is an idempotent endomorphism in $\mathcal{E}$: that is, there exists a functor $V : \mathrm{N}_{\bullet}(\mathrm{Idem}) \to \mathcal{E}$ satisfying $V(\widetilde{e}) = \iota(\widetilde{e})$. To complete the proof, it will suffice to show that the composition $\mathrm{N}_{\bullet}(\mathrm{Idem}) \xrightarrow{V} \mathcal{E} \xrightarrow{U} \mathrm{N}_{\bullet}(\mathrm{Idem})$ is the identity functor. This follows from the universal property of Remark 8.5.2.8 (together with Proposition 1.3.3.1), it since the functor $U \circ V$ carries the morphism e to itself. □

8.5.9 The Thompson Groupoid

042T In §8.5.7, we constructed an example of a homotopy idempotent endomorphism $e : X \to X$ which is not idempotent. Our construction (following Heller and Freyd) involved the Thompson group $\mathrm{Aut}_{\mathrm{Dy}}([0,1])$. Our goal in this section is to show that this is no coincidence: there is a universal example of an ∞-category $\mathcal{C}$ containing a homotopy idempotent endomorphism, whose structure can be described explicitly in terms of $\mathrm{Aut}_{\mathrm{Dy}}([0,1])$. We begin with a variant of Definition 8.5.7.12.

042U **Definition 8.5.9.1** (The Thompson Groupoid)**.** We define a category Dy as follows:

- The objects of Dy are closed intervals of the form $[0,s]$, where $s \geq 0$ is a dyadic rational number.
- If $s,t \geq 0$ are dyadic rational numbers, then a morphism from $[0,s]$ to $[0,t]$ in the category Dy is a dyadic homeomorphism $[0,s] \xrightarrow{\sim} [0,t]$ (see Definition 8.5.7.9).

- The composition law on Dy is given by composition of dyadic homeomorphisms (which is well-defined by virtue of Exercise 8.5.7.11).

It follows from Exercise 8.5.7.10 that the category Dy is a groupoid. We will refer to Dy as the *Thompson groupoid*.

Remark 8.5.9.2. The Thompson groupoid Dy contains exactly two isomorphism classes: `042V`

- The isomorphism class of the degenerate interval $[0,0] = \{0\}$, whose automorphism group is the trivial group.
- The isomorphism class of the unit interval $[0,1]$, whose automorphism group is the Thompson group $\mathrm{Aut}_{\mathrm{Dy}}([0,1])$ of Definition 8.5.7.12.

By virtue of Remark 8.5.9.2, the Thompson groupoid Dy is equivalent to the full subcategory spanned by the objects $\{0\}$ and $[0,1]$, which can be described explicitly in terms of the Thompson group $\mathrm{Aut}_{\mathrm{Dy}}([0,1])$. However, allowing a larger class of intervals in the definition reveals some additional structure.

Construction 8.5.9.3 (Concatenation)**.** Let Dy denote the Thompson groupoid. We define a functor $\circledast : \mathrm{Dy} \times \mathrm{Dy} \to \mathrm{Dy}$ as follows: `042W`

- On objects, the functor $\circledast$ is given by the formula

$$[0,s] \circledast [0,t] = [0,s+t].$$

- On morphisms, the functor $\circledast$ is given by the formula

$$(f \circledast g)(x) = \begin{cases} g(x) & \text{if } 0 \leq x \leq t \\ f(x-t)+s & \text{if } t \leq x \leq s+t. \end{cases}$$

We will refer to $\circledast : \mathrm{Dy} \times \mathrm{Dy} \to \mathrm{Dy}$ as the *concatenation functor* on the Thompson groupoid Dy. Note that the operation $\circledast$ is strictly associative, and admits a (strict) unit given by the degenerate interval $\{0\} = [0,0]$. Consequently, $\circledast$ determines a strict monoidal structure on the category Dy (in the sense of Definition 2.1.2.1).

Notation 8.5.9.4. Let $B\mathrm{Dy}$ denote the (strict) 2-category obtained by delooping Dy (see Example 2.2.0.8). Since Dy is a groupoid, $B\mathrm{Dy}$ is a $(2,1)$-category. It follows that the Duskin nerve $\mathrm{N}^{\mathrm{D}}_{\bullet}(B\mathrm{Dy})$ is an ∞-category (Theorem 2.3.2.1). We can describe the low-dimensional simplices of $\mathrm{N}^{\mathrm{D}}_{\bullet}(B\mathrm{Dy})$ explicitly as follows: `042X`

- The ∞-category $\mathrm{N}^{\mathrm{D}}_{\bullet}(B\mathrm{Dy})$ has a unique object, which we will denote by $\overline{X}$.

- Morphisms from $\overline{X}$ to itself in the ∞-category $\mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy})$ can be identified with nonnegative dyadic rational numbers s (corresponding to the closed interval $[0, s]$, regarded as an object of the Thompson groupoid Dy).
- Suppose we are given dyadic rational numbers $s, t, u \geq 0$. Then 2-simplices of $\mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy})$ with boundary indicated in the diagram

$$\begin{tikzcd} & \overline{X} \arrow[dr, "s"] & \\ \overline{X} \arrow[ur, "t"] \arrow[rr, "u"] & & \overline{X} \end{tikzcd}$$

  can be identified with dyadic homeomorphisms $[0, s+t] \xrightarrow{\sim} [0, u]$.

Let $\overline{e} : \overline{X} \to \overline{X}$ denote the morphism in $\mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy})$ corresponding to the object $[0,1] \in \mathrm{Dy}$, and let $\overline{\sigma}$ be the 2-simplex of $\mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy})$ corresponding to the dyadic homeomorphism

$$[0,1] \circledast [0,1] = [0,2] \xrightarrow{\sim} [0,1] \qquad x \mapsto x/2.$$

Then the triple $(\overline{X}, \overline{e}, \overline{\sigma})$ can then be viewed as a partial idempotent $\iota : \mathrm{N}_{\leq 2}(\mathrm{Idem}) \to \mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy})$ (see Example 8.5.8.3).

We can now formulate our main result:

042Y **Theorem 8.5.9.5.** *The partial idempotent $\iota : \mathrm{N}_{\leq 2}(\mathrm{Idem}) \to \mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy})$ of Notation 8.5.9.4 is a categorical equivalence of simplicial sets.*

042Z **Corollary 8.5.9.6.** *Let $\mathcal{C}$ be an ∞-category. Then composition with the partial idempotent ι of Notation 8.5.9.4 induces a trivial Kan fibration of ∞-categories*

$$\mathrm{Fun}(\mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}_{\leq 2}(\mathrm{Idem}), \mathcal{C}).$$

Proof. Combine Theorem 8.5.9.5 with Corollary 4.5.5.19 (noting that ι is a monomorphism of simplicial sets). □

0430 **Corollary 8.5.9.7.** *Let $\mathcal{C}$ be an ∞-category containing an endomorphism $e : X \to X$. Then e is homotopy idempotent if and only if there is a functor of ∞-categories $F : \mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy}) \to \mathcal{C}$ satisfying $F(\overline{e}) = e$.*

0431 **Example 8.5.9.8.** Let $\mathrm{Dy}_{>0}$ denote the full subcategory of the Thompson groupoid Dy spanned by the intervals $[0, s]$ where $s > 0$. Note that the action of Dy on itself (via concatenation) restricts to an action of Dy on the groupoid $\mathrm{Dy}_{>0}$, and therefore determines a (strict) functor of 2-categories

$$B\mathrm{Dy} \to \{\text{Groupoids}\} \qquad X \mapsto \mathrm{Dy}_{>0}.$$

Passing to nerves, we obtain a functor of ∞-categories

$$\mathrm{N}_\bullet^{\mathrm{D}}(B\mathrm{Dy}) \to \mathcal{S} \qquad X \mapsto \mathrm{N}_\bullet(\mathrm{Dy}_{>0}),$$

which carries the 1-morphism $\overline{e}$ of $B\mathrm{Dy}$ to the homotopy idempotent endomorphism

$$\mathrm{N}_\bullet(\mathrm{Dy}_{>0}) \to \mathrm{N}_\bullet(\mathrm{Dy}_{>0}) \qquad [0,s] \mapsto [0,1] \circledast [0,s] = [0,1+s].$$

Note that, up to isomorphism, this coincides with the homotopy idempotent endomorphism constructed in Proposition 8.5.7.14; this follows from the observation that the diagram of categories

$$\begin{tikzcd}
B\,\mathrm{Aut}_{\mathrm{Dy}}([0,1]) \arrow[r, "\alpha"] \arrow[d, "\sim"] & B\,\mathrm{Aut}_{\mathrm{Dy}}([0,1]) \arrow[d, "\sim"] \\
\mathrm{Dy}_{>0} \arrow[r, "{[0,1]\circledast}"] & \mathrm{Dy}_{>0}
\end{tikzcd}$$

commutes up to isomorphism (where the vertical maps are the inclusion functors and α is the homomorphism of Construction 8.5.7.13).

Our first goal is to reduce Theorem 8.5.9.5 to a more concrete statement about simplicial monoids.

Notation 8.5.9.9. Let $\mathrm{N}_{\leq 2}(\mathrm{Idem})$ denote the simplicial set described in Example 8.5.8.3. 0432 By virtue of Proposition 2.4.4.3, we can choose a simplicial category $\mathcal{C}$ and a morphism of simplicial sets $u : \mathrm{N}_{\leq 2}(\mathrm{Idem}) \to \mathrm{N}_\bullet^{\mathrm{hc}}(\mathcal{C})$ which exhibits $\mathcal{C}$ as a simplicial path category for $\mathrm{N}_{\leq 2}(\mathrm{Idem})$. The morphism u carries the unique vertex $\widetilde{X}$ of $\mathrm{N}_{\leq 2}(\mathrm{Idem})$ to an object $X \in \mathcal{C}$. Set $E_\bullet = \mathrm{Hom}_{\mathcal{C}}(X,X)_\bullet$, which we regard as a simplicial monoid. Evaluating the morphism u on the nondegenerate edge $\widetilde{e}$ of $\mathrm{N}_{\leq 2}(\mathrm{Idem})$, we obtain a morphism $e : X \to X$ in the category $\mathcal{C}$, which we can view as a vertex of the simplicial set $E_\bullet$. Evaluating the morphism u on the nondegenerate 2-simplex of $\mathrm{N}_{\leq 2}(\mathrm{Idem})$, we obtain an edge $h : e^2 \to e$ in the simplicial monoid $E_\bullet$.

Let $\mathrm{N}_\bullet(\mathrm{Dy})$ denote the nerve of the Thompson groupoid. The concatenation functor of Notation 8.5.9.4 endows $\mathrm{N}_\bullet(\mathrm{Dy})$ with the structure of a simplicial monoid. We let $B\,\mathrm{N}_\bullet(\mathrm{Dy})$ denote the simplicial category given by delooping $\mathrm{N}_\bullet(\mathrm{Dy})$ (Example 2.4.2.3), so that the homotopy coherent nerve of $B\,\mathrm{N}_\bullet(\mathrm{Dy})$ can be identified with the Duskin nerve of the strict

2-category $B\mathrm{Dy}$ (see Example 2.4.3.11). It follows that the partial idempotent ι of Notation 8.5.9.4 can be identified with a morphism from $\mathrm{N}_{\leq 2}(\mathrm{Idem})$ to $\mathrm{N}_{\bullet}^{\mathrm{hc}}(B\,\mathrm{N}_{\bullet}(\mathrm{Dy}))$, which factors unique as a composition

$$\mathrm{N}_{\leq 2}(\mathrm{Idem}) \xrightarrow{u} \mathrm{N}_{\bullet}^{\mathrm{hc}}(\mathcal{C}) \xrightarrow{\mathrm{N}_{\bullet}^{\mathrm{hc}}(F)} \mathrm{N}_{\bullet}^{\mathrm{hc}}(B\,\mathrm{N}_{\bullet}(\mathrm{Dy}))$$

for some simplicial functor $F : \mathcal{C} \to B\,\mathrm{N}_{\bullet}(\mathrm{Dy})$, which we can identify with a homomorphism of simplicial monoids

$$\varphi : E_{\bullet} = \mathrm{Hom}_{\mathcal{C}}(X, X)_{\bullet} \xrightarrow{F} \mathrm{Hom}_{B\,\mathrm{N}_{\bullet}(\mathrm{Dy})}(F(X), F(X))_{\bullet} = \mathrm{N}_{\bullet}(\mathrm{Dy}).$$

By construction, the homomorphism φ carries the vertex e to the object $[0,1] \in \mathrm{Dy}$, and the edge $h : e^2 \to e$ to the dyadic homeomorphism

$$[0,1] \circledast [0,1] = [0,2] \xrightarrow{\sim} [0,1] \qquad x \mapsto x/2.$$

0433 **Remark 8.5.9.10.** Using the universal property of the path category $\mathcal{C} = \mathrm{Path}[\mathrm{N}_{\leq 2}(\mathrm{Idem})]_{\bullet}$, it is not difficult to see that $E_{\bullet}$ is freely generated (as a simplicial monoid) by the vertex e and the edge $h : e^2 \to e$. In particular, the homomorphism of simplicial monoids $\varphi : E_{\bullet} \to \mathrm{N}_{\bullet}(\mathrm{Dy})$ is uniquely determined by the requirement that it carries e to the unit interval $[0,1] \in \mathrm{Dy}$ and h to the dyadic homeomorphism $x \mapsto x/2$.

We will deduce Theorem 8.5.9.5 from the following:

0434 **Proposition 8.5.9.11.** *The homomorphism $\varphi : E_{\bullet} \to \mathrm{N}_{\bullet}(\mathrm{Dy})$ is a weak homotopy equivalence of simplicial sets.*

Proof of Theorem 8.5.9.5 from Proposition 8.5.9.11. We wish to show that the partial idempotent

$$\iota : \mathrm{N}_{\leq 2}(\mathrm{Idem}) \to \mathrm{N}_{\bullet}^{\mathrm{D}}(B\mathrm{Dy}) \simeq \mathrm{N}_{\bullet}^{\mathrm{hc}}(B\,\mathrm{N}_{\bullet}(\mathrm{Dy}))$$

is a categorical equivalence of simplicial sets. Since the category Dy is a groupoid, the simplicial monoid $\mathrm{N}_{\bullet}(\mathrm{Dy})$ is a Kan complex (Proposition 1.3.5.2). It follows that the simplicial category $B\,\mathrm{N}_{\bullet}(\mathrm{Dy})$ is locally Kan. Invoking Theorem [?], we are reduced to showing that ι induces a weak equivalence of simplicial categories $F : \mathrm{Path}[\mathrm{N}_{\leq 2}(\mathrm{Idem})]_{\bullet} \to B\,\mathrm{N}_{\bullet}(\mathrm{Dy})$, in the sense of Definition 4.6.8.7. Write X for the unique object of the path category $\mathrm{Path}[\mathrm{N}_{\leq 2}(\mathrm{Idem})]_{\bullet}$, so that $F(X)$ is the unique object of $B\,\mathrm{N}_{\bullet}(\mathrm{Dy})$. We are then reduced to showing that F induces a weak homotopy equivalence of simplicial monoids

$$\varphi : E_{\bullet} = \mathrm{Hom}_{\mathrm{Path}[\mathrm{N}_{\leq 2}(\mathrm{Idem})]}(X, X)_{\bullet} \to \mathrm{Hom}_{B\,\mathrm{N}_{\bullet}(\mathrm{Dy})}(F(X), F(X)) = \mathrm{N}_{\bullet}(\mathrm{Dy}),$$

which follows from Proposition 8.5.9.11. □

We will deduce Proposition 8.5.9.11 from a more refined result, which characterizes the simplicial monoid $E_\bullet$ up to categorical equivalence (rather than merely up to weak homotopy equivalence).

Notation 8.5.9.12. Let m and n be nonnegative integers. We will say that a homeomorphism $f : [0, m] \xrightarrow{\sim} [0, n]$ is a *dyadic contraction* if, for every integer $0 \leq k < m$, the restriction of f to the closed interval $[k, k+1]$ is given by the formula $f(x) = (x + a)/2^b$ for some integers a and b with $b \geq 0$. 0435

We let Dy_+ denote the subcategory of the Thompson groupoid Dy whose objects are intervals of the form $[0, m]$, where m is a nonnegative integer, and whose morphisms are dyadic contractions.

Exercise 8.5.9.13. Show that the subcategory $\mathrm{Dy}_+ \subset \mathrm{Dy}$ is well-defined: that is, the collection of dyadic contractions is closed under composition. 0436

Warning 8.5.9.14. The category Dy_+ of Notation 8.5.9.12 is not a groupoid. In fact, every isomorphism in the category Dy_+ is an identity morphism. 0437

Proposition 8.5.9.15. *The homomorphism $\varphi : E_\bullet \to \mathrm{N}_\bullet(\mathrm{Dy})$ factors as a composition* 0438

$$E_\bullet \xrightarrow{\varphi_+} \mathrm{N}_\bullet(\mathrm{Dy}_+) \subset \mathrm{N}_\bullet(\mathrm{Dy}),$$

where φ_+ is inner anodyne.

Proof of Proposition 8.5.9.11 from Proposition 8.5.9.15. By virtue of Proposition 8.5.9.15, it will suffice to show that the inclusion of categories $\mathrm{Dy}_+ \hookrightarrow \mathrm{Dy}$ induces a weak homotopy equivalence of simplicial sets $U : \mathrm{N}_\bullet(\mathrm{Dy}_+) \hookrightarrow \mathrm{N}_\bullet(\mathrm{Dy})$. Using Quillen's Theorem A (Example 7.2.3.3), we are reduced to proving the following: for every object $[0, s] \in \mathrm{Dy}$, the category $\mathcal{A} = \mathrm{Dy}_+ \times_{\mathrm{Dy}} \mathrm{Dy}_{[0,s]/}$ has weakly contractible nerve. We can describe the category $\mathcal{A}$ more concretely as follows:

- The objects of $\mathcal{A}$ are dyadic homeomorphisms $f : [0, s] \xrightarrow{\sim} [0, m]$, where m is an integer.
- Let $f : [0, s] \xrightarrow{\sim} [0, m]$ and $g : [0, s] \xrightarrow{\sim} [0, n]$ be dyadic homeomorphisms. Then there is a morphism from f to g (in the category $\mathcal{A}$) if and only if the homeomorphism $(g \circ f^{-1}) : [0, m] \to [0, n]$ is a dyadic contraction. If this condition is satisfied, then the morphism is unique.

It follows that the category $\mathcal{A}$ can be viewed as a partially ordered set. Moreover, every finite subset of $\mathcal{A}$ has a lower bound, given by the dyadic homeomorphism

$$[0, s] \xrightarrow{\sim} [0, 2^k s] \qquad x \mapsto 2^k x$$

for some integer $k \gg 0$. It follows that the category $\mathcal{A}^{\mathrm{op}}$ is filtered (Exercise 7.2.4.2), so that $\mathrm{N}_\bullet(\mathcal{A})$ is weakly contractible by virtue of Proposition 7.2.4.9. □

The proof of Proposition 8.5.9.15 will require some preliminaries.

0439 **Notation 8.5.9.16.** Let $n \geq 0$ be an integer and let J be a subset of $\{1, 2, \cdots, n\}$. We let $b_J : [0, n] \xrightarrow{\sim} [0, n + |J|]$ be the dyadic homeomorphism which is characterized by the following requirement: for every integer $1 \leq j \leq n$, the function b_I is differentiable at every point $x \in (j - 1, j)$, with derivative given by the formula

$$b'_J(x) = \begin{cases} 1 & \text{if } j \notin J \\ 2 & \text{if } j \in J. \end{cases}$$

Note that, if this condition is satisfied, then the inverse homeomorphism $b_J^{-1} : [0, n + |J|] \xrightarrow{\sim} [0, n]$ is a dyadic contraction. We say that a dyadic contraction is *elementary* if it has the form b_J^{-1}, for some integer $n \geq 0$ and some subset $J \subseteq \{1, 2, \cdots, n\}$.

043A **Remark 8.5.9.17.** Let $f : [0, m] \xrightarrow{\sim} [0, n]$ be a dyadic contraction. The following conditions are equivalent:

- The dyadic contraction f is elementary, in the sense of Notation 8.5.9.16.
- For every point $x \in [0, m]$ where f is differentiable, the derivative $f'(x)$ is either 1 or 1/2.
- The dyadic contraction f can be written as a concatenation $f_1 \circledast f_2 \circledast \cdots \circledast f_n$, where each f_i is either the identity function $\mathrm{id} : [0, 1] \xrightarrow{\sim} [0, 1]$ or the homeomorphism

$$H : [0, 2] \xrightarrow{\sim} [0, 1] \qquad x \mapsto x/2.$$

043B **Lemma 8.5.9.18.** *The collection of elementary dyadic contractions is a class of short morphisms for the ∞-category $\mathrm{N}_\bullet(\mathrm{Dy}_+)$, in the sense of Definition 6.2.5.4.*

Proof. Let S denote the collection of all elementary dyadic contractions. We verify that S satisfies conditions (1) through (4) of Definition 6.2.5.4:

(1) For every integer $n \geq 0$, the identity morphism $\mathrm{id} : [0, n] \xrightarrow{\sim} [0, n]$ is an elementary dyadic contraction. This is immediately from the definitions.

(2) Suppose we are given a commutative diagram of dyadic contractions

$$\begin{tikzcd} & {[0, m]} \arrow[dr, "g"] & \\ {[0, k]} \arrow[ur, "f"] \arrow[rr, "h"] & & {[0, n].} \end{tikzcd}$$

Assume that g and h are elementary; we wish to show that f is also elementary (in fact, the assumption that g is elementary will not be needed). Choose a point $x \in [0, k]$ at which f is differentiable; we wish to show that $f'(x) \geq 1/2$ (see Remark 8.5.9.17). Replacing x by a nearby point if necessary, we may assume that g is differentiable at the point $y = f(x)$. Since g is a dyadic contraction and h is an elementary dyadic contraction, we have $g'(y) \leq 1$ and $h'(x) \geq 1/2$. Applying the chain rule, we obtain inequalities $f'(x) \geq f'(x) \cdot g'(y) = h'(x) \geq 1/2$.

(3) Let $f : [0, m] \xrightarrow{\sim} [0, n]$ be a dyadic contraction. We wish to show that f admits an S-optimal factorization (in the sense of Definition 6.2.5.1). Let P denote the collection of all subsets $\{1, 2, \cdots, n\}$ having the property that the composition $(b_J \circ f) : [0, m] \to [0, n+|J|]$ is a dyadic contraction; here b_J denotes the dyadic homeomorphism introduced in Notation 8.5.9.16. Unwinding the definitions, we can identify P with the set of factorizations $f = s \circ g$, where g is a dyadic contraction and s is an elementary dyadic contraction (the identification carries a set $J \in P$ to the pair $(s, g) = (b_J^{-1}, b_J \circ f)$). Under this identification, a factorization $f = s \circ g$ is S-optimal if and only if J is a largest element of P. We conclude by observing that P has a largest element $J_{\max}$, given by the collection of those integers $j \in \{1, 2, \cdots, n\}$ having the property that the inverse homeomorphism f^{-1} has derivative ≥ 2 at every point $x \in [j-1, j]$ where f^{-1} is differentiable (alternatively, $J_{\max}$ can be described as the set of integers $1 \leq j \leq n$ which satisfy $f^{-1}(j) > f^{-1}(j-1) + 1$).

(4) Let $f : [0, m] \to \sim [0, n]$ be a dyadic contraction. Let us define the *length of* f to be the smallest nonnegative integer k such that $f'(x) \geq 1/2^k$ for every point $x \in [0, m]$ where f is differentiable. We claim that, if this condition is satisfied, then f can be written as a composition $s_1 \circ s_2 \circ \cdots \circ s_k$, where each s_i is an elementary dyadic contraction. Our proof proceeds by induction on k. If $k = 0$, then f is an identity morphism and there is nothing to prove. Let us therefore assume that $k > 0$, and let $f = s_1 \circ g$ be an S-optimal factorization of f. We claim that g can be written as a composition of elementary contractions $s_2 \circ \cdots \circ s_k$. By virtue of our inductive hypothesis, it will suffice to show that g has length $k - 1$, which follows from the proof of (3).

□

Proof of Proposition 8.5.9.15. Let $\mathrm{N}_\bullet(\mathrm{Dy}_+)^{\mathrm{short}}$ denote the simplicial subset of $\mathrm{N}_\bullet(\mathrm{Dy}_+)$ whose m-simplices are diagrams of dyadic contractions σ :

$$[0, n_0] \xrightarrow{\sim} [0, n_1] \xrightarrow{\sim} \cdots \xrightarrow{\sim} [0, n_m]$$

for which the composite map $[0, n_0] \xrightarrow{\sim} [0, n_m]$ is an elementary dyadic contraction (note that this guarantees that each intermediate composition $[0, n_i] \xrightarrow{\sim} [0, n_j]$ is also elementary). It follows from Lemma 8.5.9.18 and Theorem 6.2.5.10 that the inclusion map $\mathrm{N}_\bullet(\mathrm{Dy}_+)^{\mathrm{short}} \hookrightarrow \mathrm{N}_\bullet(\mathrm{Dy}_+)$ is inner anodyne. We will complete the proof by showing that the morphism $\varphi : E_\bullet \to \mathrm{N}_\bullet(\mathrm{Dy})$ induces of Notation 8.5.9.9 induces an isomorphism of $E_\bullet$ with the simplicial subset $\mathrm{N}_\bullet(\mathrm{Dy}_+)^{\mathrm{short}} \subseteq \mathrm{N}_\bullet(\mathrm{Dy})$.

Fix an integer $m \geq 0$, so that φ induces a monoid homomorphism $\varphi_m : E_m \to \mathrm{N}_m(\mathrm{Dy})$. We wish to show that φ_m is a monomorphism, whose image is the subset $\mathrm{N}_m(\mathrm{Dy}_+)^{\mathrm{short}} \subseteq \mathrm{N}_m(\mathrm{Dy})$. Note that $\mathrm{N}_m(\mathrm{Dy}_+)^{\mathrm{short}}$ is closed under concatenation, and therefore inherits the structure of a monoid. Let us say that an m-simplex σ of $\mathrm{N}_\bullet(\mathrm{Dy}_+)^{\mathrm{short}}$ is *indecomposable* if it corresponds to a diagram of dyadic contractions

$$[0, n_0] \xrightarrow{\sim} [0, n_1] \xrightarrow{\sim} \cdots \xrightarrow{\sim} [0, n_m]$$

with $n_m = 1$. In this case, we define the *index of* σ to be the smallest integer k such that $n_k = 1$. For every integer $0 \leq k \leq m$, the simplicial set $\mathrm{N}_\bullet(\mathrm{Dy}_+)^{\mathrm{short}}$ has a unique indecomposable m-simplex σ_k of index k, which can be described explicitly as follows:

- If $k = 0$, then σ_k is the diagram of identity morphisms
$$[0,1] \xrightarrow{\mathrm{id}} [0,1] \xrightarrow{\mathrm{id}} [0,1] \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} [0,1].$$
- If $k > 0$, then the diagram σ_k has the form
$$[0,2] \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} [0,2] \xrightarrow{x \mapsto x/2} [0,1] \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} [0,1].$$

Moreover, every m-simplex of $\mathrm{N}_\bullet(\mathrm{Dy}_+)^{\mathrm{short}}$ can be written uniquely as a concatenation of indecomposable m-simplices of $\mathrm{N}_\bullet(\mathrm{Dy}_+)^{\mathrm{short}}$: that is, $\mathrm{N}_m(\mathrm{Dy}_+)^{\mathrm{short}}$ can be identified with the free monoid generated by the set $\{\sigma_0, \sigma_1, \cdots, \sigma_m\}$.

Let $\mathcal{C}_\bullet = \mathrm{Path}[\mathrm{N}_{\leq 2}(\mathrm{Idem})]_\bullet$ denote the simplicial path category of $\mathrm{N}_{\leq 2}(\mathrm{Idem})$. Theorem 2.4.4.10 supplies an identification of $\mathcal{C}_m$ with the path category $\mathrm{Path}[G]$, where G is a directed graph having a single vertex X (corresponding to the unique vertex of the simplicial set $\mathrm{N}_{\leq 2}(\mathrm{Idem})$. It follows that $E_m = \mathrm{Hom}_{\mathcal{C}_m}(X, X)$ can be identified with the free monoid generated by the set of edges $\mathrm{Edge}(G)$ (Example 1.3.7.3). It will therefore suffice to prove the following:

$(*_m)$ The monoid homomorphism φ_m induces a bijection from the set $\mathrm{Edge}(G)$ to the collection $\{\sigma_0, \sigma_1, \cdots, \sigma_m\}$ of indecomposable m-simplices of $\mathrm{N}_\bullet(\mathrm{Dy}_+)^{\mathrm{short}}$.

To prove $(*_m)$, we recall that $\mathrm{Edge}(G)$ can be identified with the set of pairs $(\tau, \overrightarrow{I})$, where τ is a nondegenerate simplex of $\mathrm{N}_{\leq 2}(\mathrm{Idem})$ of dimension $n > 0$ and $\overrightarrow{I} = (I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{m-1} \supseteq I_m)$ is a chain of subsets of $[n] = \{0 < 1 < \cdots < n\}$ satisfying $I_0 = [n]$ and $I_m = \{0, n\}$. We consider two possibilities:

- The simplex τ has dimension $n = 1$. In this case, both τ and $\overrightarrow{I}$ are uniquely determined. We claim that the homomorphism φ_m carries $(\tau, \overrightarrow{I})$ to the indecomposable m-simplex σ_0. To prove this, we can invoke our assumption that φ is a morphism of simplicial monoids to reduce to the case $m = 0$, in which case it reduces to the identity $\varphi(e) = [0,1]$ of Remark 8.5.9.10.

- The simplex τ has dimension $n = 2$. In this case, τ is again uniquely determined, and the chain $\overrightarrow{I}$ is determined by a single integer $1 \leq k \leq m$, given by the formula $k = \min\{j : I_j = \{0,2\}\}$. We claim that the homomorphism φ_m carries $(\tau, \overrightarrow{I})$ to the indecomposable m-simplex σ_k. To prove this, we can again invoke our assumption that φ is a morphism of simplicial monoids to reduce to the case $m = 1$, in which case it reduces to the assertion that φ carries the edge $h : e^2 \to e$ of $E_\bullet$ to the dyadic contraction
$$[0,2] \xrightarrow{\sim} [0,1] \qquad x \mapsto x/2;$$
see Remark 8.5.9.10.

□

8.6 Conjugate and Dual Fibrations

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. For each object $C \in \mathcal{C}$, we let $\mathcal{E}_C$ denote the the fiber $U^{-1}\{C\} = \{C\} \times_{\mathcal{C}} \mathcal{E}$. In §5.2.5, we showed that this construction determines a functor 04C0

$$\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat} \qquad C \mapsto \mathcal{E}_C,$$

which we refer to as the *(covariant) homotopy transport representation* of the cocartesian fibration U (Construction 5.2.5.2). Similarly, if $U' : \mathcal{E}' \to \mathcal{C}'$ is a cartesian fibration of ∞-categories, then the assignment $C \mapsto \mathcal{E}'_C$ determines a functor

$$\mathrm{hTr}_{\mathcal{E}'/\mathcal{C}'} : \mathrm{h}\mathcal{C}'^{\mathrm{op}} \to \mathrm{hQCat} \qquad C \mapsto \mathcal{E}'_C$$

which we refer to as the *(contravariant) homotopy transport representaiton* of the cartesian fibration U (Construction 5.2.5.7). There is an obvious relationship between these constructions. If $U : \mathcal{E} \to \mathcal{C}$ is a cocartesian fibration, then the opposite map $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a cartesian fibration, and the homotopy transport representations $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$ and $\mathrm{hTr}_{\mathcal{E}^{\mathrm{op}}/\mathcal{C}^{\mathrm{op}}}$ are interchanged by composing with the automorphism

$$\sigma : \mathrm{hQCat} \to \mathrm{hQCat} \qquad \mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}.$$

In this section, we show that the passage from a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ to the opposite cartesian fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ can be broken into two steps:

- To every cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$, we will associate another cocartesian fibration $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ which we refer to as the *cocartesian dual of* U, whose (covariant) homotopy transport representation is given (up to isomorphism) by the construction
$$\mathrm{hTr}_{\mathcal{E}^{\vee}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat} \qquad C \mapsto \mathcal{E}_C^{\mathrm{op}} .$$
In particular, each fiber of $U^{\vee}$ is equivalent to the *opposite* of the corresponding fiber of U.

- To every cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$, we will associate a cartesian fibration $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ which we refer to as the *cartesian conjugate* of U, whose (contravariant) homotopy transport representation is given (up to isomorphism) by the construction
$$\mathrm{hTr}_{\mathcal{E}^{\dagger}/\mathcal{C}^{\mathrm{op}}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat} \qquad C \mapsto \mathcal{E}_C .$$
In particular, each fiber of $U^{\dagger}$ is equivalent to the corresponding fiber of U.

For a fixed ∞-category (or simplicial set) $\mathcal{C}$, the relationships between these constructions is summarized by the following diagram:

04C1

$$\begin{tikzcd}
\{\text{Cocartesian fibrations } U : \mathcal{E} \to \mathcal{C}\} \arrow[r, Leftrightarrow, "U \longleftrightarrow U^{\mathrm{op}}"] \arrow[d, Leftrightarrow, "U \longleftrightarrow U^{\dagger}"] \arrow[dr, Leftrightarrow, "U \longleftrightarrow U^{\vee}"] & \{\text{Cartesian fibrations } U' : \mathcal{E} \to \mathcal{C}^{\mathrm{op}}\} \arrow[d, Leftrightarrow, "V^{\dagger} \longleftrightarrow V"] \\
\{\text{Cartesian fibrations } V' : \mathcal{E} \to \mathcal{C}^{\mathrm{op}}\} \arrow[r, Leftrightarrow, "V^{\mathrm{op}} \longleftrightarrow V"] & \{\text{Cocartesian fibrations } V : \mathcal{E} \to \mathcal{C}\}
\end{tikzcd}$$

(8.59)

If $U : \mathcal{E} \to \mathcal{C}$ is a cocartesian fibration of ∞-categories, then the opposite fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is easy to describe at the level of simplicial sets (see §1.4.2). For the dual and conjugate fibrations $U^{\vee}$ and $U^{\dagger}$, this is somewhat more subtle. For example, the passage from a simplicial set $\mathcal{E}$ to its opposite $\mathcal{E}^{\mathrm{op}}$ is involutive, in the sense that there is a canonical isomorphism $\mathcal{E} \simeq (\mathcal{E}^{\mathrm{op}})^{\mathrm{op}}$ (in fact, if we adhere strictly to the convention of Construction 1.4.2.2, then the simplicial sets $\mathcal{E}$ and $(\mathcal{E}^{\mathrm{op}})^{\mathrm{op}}$ are *identical*). It is therefore natural to hope for the passage from a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ to its dual $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ to have a similar property: heuristically, $\mathcal{E}^{\vee}$ is obtained from $\mathcal{E}$ by applying the preceding construction to each fiber of U. Unfortunately, it does not seem possible to give a construction where this property is visible at the level of simplicial sets: the best we can expect is that cocartesian duality is involutive *up to equivalence*, in the sense that the double dual $(U^{\vee})^{\vee} : (\mathcal{E}^{\vee})^{\vee} \to \mathcal{C}$ is equivalent to the original cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ in some natural way. To address this point, it will be convenient to view duality as a *relationship* which can exist between cocartesian fibrations $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ over the same base, rather than as an *operation* which takes U as input and produces $U^{\vee}$ as an output. Similarly, we will view conjugacy as a relationship which can exist between a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ and a cartesian fibration $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ over the opposite base. Our first goal will be to describe these relationships more precisely:

- Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. We will say that a cartesian fibration $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ is a *cartesian conjugate* of U if there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

  satisfying two axioms (Definition 8.6.1.1), one of which requires that T restricts to an equivalence of ∞-categories $T_C : \mathcal{E}^{\dagger}_C \to \mathcal{E}_C$ for each vertex $C \in \mathcal{C}$. In §8.6.1, we develop the properties of this definition and give some examples.

- Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. We say that a cocartesian fibration $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ is a *cocartesian dual* of U if there exists a left fibration of simplicial sets $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ satisfying two axioms (Definition 8.6.4.1), one of which requires that for each vertex $C \in \mathcal{C}$, the induced map $\lambda_C : \widetilde{\mathcal{E}}_C \to \mathcal{E}^{\vee}_C \times \mathcal{E}_C$ is a balanced coupling of ∞-categories (Definition 8.2.6.1). This guarantees in particular that $\mathcal{E}^{\vee}_C$ is equivalent to the opposite ∞-category $\mathcal{E}^{\mathrm{op}}_C$ (Corollary 8.2.6.6). In §8.6.4, we develop the properties of this definition and give some examples.

Our next goal is to show that, if $U : \mathcal{E} \to \mathcal{C}$ is a cocartesian fibration of simplicial sets, then it admits a cartesian conjugate $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ and a cocartesian dual $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$, which are uniquely determined up to equivalence. In each case, we will prove existence (and ultimately uniqueness) using explicit constructions at the level of simplicial sets. To fix ideas, let us first assume that $\mathcal{C} = \Delta^0$. In this case, constructing the dual of cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ is tantamount to constructing an ∞-category $\mathcal{E}^{\vee}$ which is equivalent to the opposite of $\mathcal{E}$. We consider three different solutions to this problem:

(*a*) We can take $\mathcal{E}^{\vee}$ to be the opposite ∞-category $\mathcal{E}^{\mathrm{op}}$ itself, given concretely by Construction 1.4.2.2.

(b) We can take $\mathcal{E}^{\vee}$ to be the ∞-category $\mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{E})$ of Variant 8.1.7.14, whose morphisms are given by cospans $X \xrightarrow{f} B \xleftarrow{g} Y$ in the ∞-category $\mathcal{E}$ where f is an isomorphism. By virtue of Proposition 8.1.7.6, this is equivalent to the ∞-category $\mathcal{E}^{\mathrm{op}}$.

(c) We can take $\mathcal{E}^{\vee}$ to be the ∞-category of corepresentable functors $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E}, \mathcal{S})$. This is equivalent to $\mathcal{E}^{\mathrm{op}}$ by virtue of the ∞-categorical version of Yoneda's lemma (Theorem 8.3.3.13), at least if $\mathcal{E}$ is locally small.

Each of these approaches can be adapted to more general situations. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets.

- Assume that $\mathcal{C}$ is an ∞-category. In §8.6.2, we define a fibration

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{C}^{\mathrm{op}}$$

  and show that it is a cartesian conjugate of U (Proposition 8.6.2.3). In the special case $\mathcal{C} = \Delta^0$, this definition reproduces the original ∞-category $\mathcal{E}$; consequently, after passing to opposite fibrations, it can be viewed as a relative version of construction (a).

- Let $\mathrm{Cospan}^{\dagger}(\mathcal{E} / \mathcal{C})$ denote the fiber product $\mathrm{Cospan}^{\mathrm{all},R}(\mathcal{E}) \times_{\mathrm{Cospan}(\mathcal{C})} \mathcal{C}^{\mathrm{op}}$, where R denotes the collection of all U-cocartesian edges of $\mathcal{E}$. In §8.6.3, we show that projection onto the second factor determines a cartesian fibration $\mathrm{Cospan}^{\dagger}(\mathcal{E} / \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ which is conjugate to U (Proposition 8.6.3.5). In the special case $\mathcal{C} = \Delta^0$, this definition reproduces the ∞-category $\mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$. Consequently, after passing to opposite fibrations, it can be viewed as a relative version of construction (b).

- Assume that, for each vertex $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C$ is locally small. In §8.6.5, we define a fibration $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}) \to \mathcal{C}$ and show that it is a cocartesian dual of U (Proposition 8.6.5.8). In the special case $\mathcal{C} = \Delta^0$, this definition reproduces the ∞-category of corepresentable functors $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E}, \mathcal{S})$; consequently, it can be viewed as a relative version of construction (c).

04C2 **Remark 8.6.0.1.** Ultimately, each of the constructions described above gives rise to essentially the same object. However, it will be useful to consider all three, since each reveals different facets of the overall picture. Fix a cocartesian fibration of simplicial sets $U : \mathcal{E} \to \mathcal{C}$.

- The construction of §8.6.2 can be used to show that U admits a cartesian conjugate $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ (Corollary 8.6.2.4). However, from this point of view, it is not obvious that the cartesian fibration $U^{\dagger}$ is unique (unless $\mathcal{C}$ is assumed to be an ∞-category), or that every cartesian fibration can be obtained in this way.

- The construction of §8.6.3 produces a cartesian fibration $V : \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ having the property that *every* cartesian conjugate of U is equivalent to V (Proposition 8.6.3.11). This guarantees that the cartesian conjugate of U is unique up to equivalence. However, it does not by itself prove existence (see Warning 8.6.3.13).
- The construction of §8.6.5 can be used to show that U admits a cocartesian dual $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$, which is uniquely determined up to equivalence (Theorem 8.6.5.1). In §8.6.6, we show that the opposite fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ can be realized as a cartesian conjugate of $U^{\vee}$ (Proposition 8.6.6.1). It follows that every cartesian fibration can be realized as the conjugate of a cocartesian fibration. However, from this point of view, it is not obvious that this realization is unique.

Remark 8.6.0.2 (Duality via Transport Representations)**.** Let $\mathcal{QC}$ denote the ∞-category of small ∞-categories (Construction 5.5.4.1). Then $\mathcal{QC}$ admits an autoequivalence $\sigma : \mathcal{QC} \to \mathcal{QC}$, given on objects by the formula $\sigma(\mathcal{A}) = \mathcal{A}^{\mathrm{op}}$ (see Construction 8.6.7.6). Recall that an (essentially small) cocartesian fibrations $U : \mathcal{E} \to \mathcal{C}$ is determined, up to equivalence, by a functor $\mathrm{Tr}_{\mathcal{E}\,/\,\mathcal{C}} : \mathcal{C} \to \mathcal{QC}$, which we refer to as the *covariant transport representation* of U (Definition 5.6.5.1). In §8.6.7, we show that a cocartesian fibration $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ is a cocartesian dual of U if and only if its covariant transport representation is isomorphic to the composition $\mathcal{C} \xrightarrow{\mathrm{Tr}_{\mathcal{E}\,/\,\mathcal{C}}} \mathcal{QC} \xrightarrow{\sigma} \mathcal{QC}$ (Proposition 8.6.7.12). This gives another construction of the cocartesian dual of U (albeit one which is cumbersome to work with). 04C3

Remark 8.6.0.3. The commutativity of the diagram (8.59) is not immediately obvious: the notions of cartesian conjugacy and cocartesian duality have separate definitions that are *a priori* unrelated to one another. We will maintain this separation in our exposition: the portions of this section which discuss conjugate fibrations (§8.6.1, §8.6.2, and §8.6.3) can be read independently of those which discuss dual fibrations (§8.6.4, §8.6.5, and §8.6.7). Only in §8.6.6 will we consider both notions simultaneously. 04C4

Remark 8.6.0.4. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. The construction of the conjugate fibration $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ studied in §8.6.3 appears in work of Barwick-Glasman-Nardin; see [3]. 04C5

8.6.1 Conjugate Fibrations

Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. Our goal in this section is to formalize the requirement that U and $U^{\dagger}$ have "the same fibers": that is, that there exists a family of equivalences $\{T_C : \mathcal{E}^{\dagger}_C \to \mathcal{E}_C\}_{C \in \mathcal{C}}$ which in some sense depend functorially on the vertex $C \in \mathcal{C}$. 04C6

Definition 8.6.1.1 (Conjugate Fibrations)**.** Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. Let $\lambda_{-} : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ 04C7 and $\lambda_+ : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}$ be the projection maps of Notation 8.1.1.6. We say that a morphism of simplicial sets $T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ *exhibits* $U^{\dagger}$ *as a cartesian conjugate of* U if the following conditions are satisfied:

(0) The diagram

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r, "\lambda_+"] & \mathcal{C}
\end{tikzcd}$$

is commutative.

(1) For every vertex $C \in \mathcal{C}$, restricting T to the inverse image of the vertex $\mathrm{id}_C \in \mathrm{Tw}(\mathcal{C})$ determines an equivalence of ∞-categories $T_C : \mathcal{E}^{\dagger}_C \to \mathcal{E}_C$.

(2) Let e be an edge of the simplicial set $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. If the image of e in $\mathcal{E}^{\dagger}$ is $U^{\dagger}$-cartesian, then $T(e)$ is a U-cocartesian edge of $\mathcal{E}$.

We say that $U^{\dagger}$ *is a cartesian conjugate of* U if there exists a morphism $T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ which exhibits $U^{\dagger}$ as a cartesian conjugate of U.

04C8 **Warning 8.6.1.2** (Symmetry). Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. In §8.6.6, we will show that $U^{\dagger}$ is a cartesian conjugate of U if and only if U^{op} is a cocartesian conjugate of $U^{\dagger,\mathrm{op}}$ (Corollary 8.6.6.2). Beware that this is not obvious from Definition 8.6.1.1.

04C9 **Example 8.6.1.3.** Let $\mathcal{E}$ and $\mathcal{E}^{\dagger}$ be ∞-categories. Set $\mathcal{C} = \Delta^0$ and let $U : \mathcal{E} \to \mathcal{C}^{\mathrm{op}}$ and $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}$ denote the projection maps. Then a functor

$$T : \mathcal{E}^{\dagger} \simeq \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$$

exhibits $U^{\dagger}$ are a cartesian conjugate of U if and only if it an equivalence of ∞-categories. In particular, $U^{\dagger}$ is a cartesian conjugate of U if and only if the ∞-category $\mathcal{E}^{\dagger}$ is equivalent to $\mathcal{E}$.

04CA **Remark 8.6.1.4** (Base Change). Let $F : \mathcal{C}' \to \mathcal{C}$ be a morphism of simplicial sets. Suppose we are given pullback squares

$$\begin{tikzcd}
\mathcal{E}'^{\dagger} \arrow[r, "U'^{\dagger}"] \arrow[d] & \mathcal{C}'^{\mathrm{op}} \arrow[d, "F^{\mathrm{op}}"] & & \mathcal{E}' \arrow[r, "U'"] \arrow[d] & \mathcal{C}' \arrow[d, "F"] \\
\mathcal{E}^{\dagger} \arrow[r, "U^{\dagger}"] & \mathcal{C}^{\mathrm{op}} & & \mathcal{E} \arrow[r, "U"] & \mathcal{C},
\end{tikzcd}$$

where $U^{\dagger}$ is a cartesian fibration and U is a cocartesian fibration. If $T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ is a morphism which exhibits $U^{\dagger}$ as a cartesian conjugate of U, then the induced map $T' : \mathcal{E}'^{\dagger} \times_{\mathcal{C}'^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}') \to \mathcal{E}'$ exhibits $U'^{\dagger}$ as a cartesian conjugate of U'.

In the situation of Definition 8.6.1.1, we can regard condition (2) as a formulation of the requirement that the functor $T_C : \mathcal{E}^{\dagger}_C \to \mathcal{E}_C$ depends functorially on the vertex $C \in \mathcal{C}$. This heuristic can be articulated more precisely as follows:

Proposition 8.6.1.5. *Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. Let $e : C \to C'$ be an edge of $\mathcal{C}$, and let* 04CB

$$e^* : \mathcal{E}^{\dagger}_C \to \mathcal{E}^{\dagger}_{C'} \qquad e_! : \mathcal{E}_C \to \mathcal{E}_{C'}$$

be functors given by contravariant and covariant transport along e for the fibrations $U^{\dagger}$ and U, respectively. If $T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ is a morphism which satisfies conditions (0) and (2) of Definition 8.6.1.1, then the diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E}^{\dagger}_C \arrow[r, "e^*"] \arrow[d, "T_C"] & \mathcal{E}^{\dagger}_{C'} \arrow[d, "T_{C'}"] \\
\mathcal{E}_C \arrow[r, "e_!"] & \mathcal{E}_{C'}
\end{tikzcd}$$

commutes up to isomorphism.

Proof. The restriction of T to the inverse image of the vertex $\{e\} \subseteq \mathrm{Tw}(\mathcal{C})$ determines a functor of ∞-categories $T_e : \mathcal{E}^{\dagger}_C \to \mathcal{E}_{C'}$. To complete the proof, it will suffice to verify the following pair of assertions:

(a) The functor T_e is isomorphic to the composition $T_{C'} \circ e^*$.

(b) The functor T_e is isomorphic to the composition $e_! \circ T_C$.

We begin by proving (a). Choose a diagram

$$\begin{tikzcd}
\mathcal{E}^{\dagger}_C \times \Delta^1 \arrow[r, "H"] \arrow[d] & \mathcal{E}^{\dagger} \arrow[d, "U^{\dagger}"] \\
\Delta^1 \arrow[r, "e"] & \mathcal{C}^{\mathrm{op}}
\end{tikzcd}$$

which witnesses $e^* = H_{\mathcal{E}^{\dagger}_{C} \times \{0\}}$ as given by contravariant transport along e (see Definition 5.2.2.15). Let $e_L : \mathrm{id}_C \to e$ and $e_R : \mathrm{id}_{C'} \to e$ denote the edges of $\mathrm{Tw}(\mathcal{C})$ described in Example 8.1.3.6, and let $\widetilde{H}$ denote the product morphism

$$\mathcal{E}^{\dagger}_{C} \times \Delta^1 \xrightarrow{H \times e_R} \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}).$$

Then the composition $T \circ \widetilde{H}$ can be regarded as a natural transformation from the functor $T_{C'} \circ e^*$ to T_e. For each object X of the ∞-category $\mathcal{E}^{\dagger}_{C}$, the restriction $H|_{\{X\} \times \Delta^1}$ is a $U^{\dagger}$-cartesian edge of $\mathcal{E}^{\dagger}$. Using condition (2) of Definition 8.6.1.1, we conclude that $(T \circ \widetilde{H})|_{\{X\} \times \Delta^1}$ is a U-cocartesian edge of $\mathcal{E}$ lying over the degenerate edge $\mathrm{id}_{C'}$ of $\mathcal{C}$, and is therefore an isomorphism in the ∞-category $\mathcal{E}_{C'}$ (Proposition 5.1.4.11). Applying Theorem 4.4.4.4, we conclude that $T \circ \widetilde{H}$ is an isomorphism of functors from $T_{C'} \circ e^*$ to T_e.

We now prove (b). Let H' denote the composite map

$$\mathcal{E}^{\dagger}_{C} \times \Delta^1 \xrightarrow{\mathrm{id} \times e_L} \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \xrightarrow{T} \mathcal{E},$$

We then have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{\dagger}_{C} \times \Delta^1 \arrow[r, "H'"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^1 \arrow[r, "e"] & \mathcal{C}.
\end{tikzcd}$$

Condition (2) of Definition 8.6.1.1 guarantees that, for each object $X \in \mathcal{E}^{\dagger}_{C}$, the restriction $H'|_{\{X\} \times \Delta^1}$ is a U-cocartesian edge of $\mathcal{E}$. It follows that H' determines an isomorphism of $T_e = H'|_{\mathcal{E}^{\dagger}_{C} \times \{1\}}$ with the composition $e_! \circ (H'|_{\mathcal{E}^{\dagger}_{C} \times \{0\}}) = e_! \circ T_C$. □

04CC **Corollary 8.6.1.6.** *Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration having homotopy transport representation $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ (Construction 5.2.5.2), and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration having homotopy transport representation $\mathrm{hTr}_{\mathcal{E}^{\dagger}/\mathcal{C}^{\mathrm{op}}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$ (Construction 5.2.5.7). If $T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ exhibits $U^{\dagger}$ as a cartesian conjugate of U, then T induces an isomorphism of functors $\mathrm{hTr}_{\mathcal{E}^{\dagger}/\mathcal{C}^{\mathrm{op}}} \xrightarrow{\sim} \mathrm{hTr}_{\mathcal{E}/\mathcal{C}}$, carrying each vertex $C \in \mathcal{C}$ to (the isomorphism class of) the equivalence $T_C : \mathcal{E}^{\dagger}_{C} \to \mathcal{E}_C$.*

04CD **Corollary 8.6.1.7.** *Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. If $U^{\dagger}$ is a cartesian conjugate of U, then the homotopy transport representations*

$$\mathrm{hTr}_{\mathcal{E}^{\dagger}/\mathcal{C}^{\mathrm{op}}}, \mathrm{hTr}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}$$

are isomorphic.

We now give some concrete examples of conjugate fibrations.

Proposition 8.6.1.8 (Conjugates of Left Fibrations)**.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a left fibration of simplicial sets. Then:* 04CE

- (a) *The map $H : \mathrm{Tw}(\mathcal{E}) \to \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ is a trivial Kan fibration of simplicial sets.*
- (b) *Let T_0 be a section of H, and let $T : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ be the composition of T_0 with the projection map $\mathrm{Tw}(\mathcal{E}) \to \mathcal{E}$. Then T exhibits the opposite fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ as a cartesian conjugate of U.*

Proof. Note that the morphism H factors as a composition

$$\mathrm{Tw}(\mathcal{E}) \to \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}),$$

where the map on the left is a left fibration by virtue of Proposition 8.1.1.15, and the map on the right is pullback of U (and is therefore also a left fibration). It follows that H is a left fibration (Remark 4.2.1.11). To prove (a), it will suffice to show that every fiber of H is a contractible Kan complex (Proposition 4.4.2.14). For this, we may assume without loss of generality that $\mathcal{C} = \Delta^0$. In this case, $\mathcal{E}$ is a Kan complex (Proposition 4.4.2.1) and we can identify H with the projection map $\mathrm{Tw}(\mathcal{E}) \to \mathcal{E}^{\mathrm{op}}$, which is a trivial Kan fibration by virtue of Corollary 8.1.2.3.

Let $T : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ be as in (b); we wish to show that T satisfies conditions (0), (1), and (2) of Definition 8.6.1.1. Condition (0) follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Tw}(\mathcal{E}) \arrow[rr] \arrow[d, "H"] & & \mathcal{E} \arrow[d, "U"] \\
\mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C},
\end{tikzcd}$$

and condition (2) is vacuous (our assumption that U is a left fibration guarantees that every edge of $\mathcal{E}$ is U-cocartesian; see Example 5.1.1.3). To verify condition (1), we may again assume that $\mathcal{C} = \Delta^0$, in which case the desired result follows from the observation that the projection maps $\mathcal{E}^{\mathrm{op}} \leftarrow \mathrm{Tw}(\mathcal{E}) \to \mathcal{E}$ are homotopy equivalences of Kan complexes (see Corollary 8.1.2.3). □

Construction 8.6.1.9 (Conjugacy for Categories of Elements)**.** Let $\mathcal{C}$ be a category, let **Cat** denote the 2-category of (small) categories, and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories. Let 04CF

$$\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F} \qquad \int_{\mathcal{C}} \mathscr{F}$$

denote the contravariant and covariant categories of elements of $\mathscr{F}$, respectively (see Definitions 5.6.1.4 and 5.6.1.1). Recall that objects of either category can be identified with pairs (C, X), where C is an object of $\mathcal{C}$ and X is an object of the category $\mathscr{F}(C)$. However, morphisms are defined differently:

- A morphism from (C, X) to (D, Y) in the category $\int_{\mathcal{C}} \mathscr{F}$ is a pair (f, u) where $f : C \to D$ is a morphism in the category $\mathcal{C}$ and $u : \mathscr{F}(f)(X) \to Y$ is a morphism in the category $\mathscr{F}(D)$.
- A morphism from (C, X) to (D, Y) in the category $\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}$ is a pair (g, v), where $g : D \to C$ is a morphism in the category $\mathcal{C}$, and $v : X \to \mathscr{F}(g)(Y)$ is a morphism in the category $\mathscr{F}(C)$.

Let us identify the objects of the fiber product $(\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ with pairs $(s : C' \to C, X)$, where $s : C' \to C$ is a morphism in $\mathcal{C}$ and X is an object of the category $\mathscr{F}(C')$. We define a functor We define a functor

$$T : (\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \int_{\mathcal{C}} \mathscr{F}$$

as follows:

- On objects, T is given by the formula $T(s : C' \to C, X) = (C, \mathscr{F}(s)(X))$.
- Let $(s : C' \to C, X)$ and $(t : D' \to D, Y)$ be objects of the category $(\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. Unwinding the definitions, we see that a morphism from $(s : C' \to C, X)$ to $(t : D' \to D, Y)$ can be identified with triples (f, f', u), where $f : C \to D$ and $f' : D' \to C'$ are morphisms in $\mathcal{C}$ satisfying $t = f \circ s \circ f'$, and $u : X \to \mathscr{F}(f')(Y)$ is a morphism in the category $\mathscr{F}(C')$. In this case, we define $T(f, f', u)$ to be the morphism $(f, v) : (C, \mathscr{F}(s)(X)) \to (D, \mathscr{F}(t)(Y))$, where v is the morphism in $\mathscr{F}(D)$ given by the composition

$$\begin{array}{ccc} (\mathscr{F}(f) \circ \mathscr{F}(s))(X) & \xrightarrow{(\mathscr{F}(f) \circ \mathscr{F}(s))(u)} & (\mathscr{F}(f) \circ \mathscr{F}(s) \circ \mathscr{F}(f'))(Y) \\ & \simeq & \mathscr{F}(f \circ s \circ f')(Y) \\ & = & \mathscr{F}(t)(Y), \end{array}$$

  where the unlabeled isomorphism is supplied by the composition constraints for the functor $\mathscr{F}$.

04CG **Proposition 8.6.1.10.** *Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to$ **Cat** be a functor. Then, after passing to nerves, the functor*

$$T : (\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \int_{\mathcal{C}} \mathscr{F}$$

of Construction 8.6.1.9 exhibits the forgetful functor $U^{\dagger} : \int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F} \to \mathcal{C}^{\mathrm{op}}$ as a cartesian conjugate of the forgetful functor $U : \int_{\mathcal{C}} \mathscr{F} \to \mathcal{C}$.

Proof. Condition (0) of Definition 8.6.1.1 follows immediately from the construction. To verify condition (1), we observe that for each object $C \in \mathcal{C}$, the functor

$$T_C : (\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\} \to \{C\} \times_{\mathcal{C}} (\int_{\mathcal{C}} \mathscr{F})$$

can be identified with the functor $\mathscr{F}(\mathrm{id}_C) : \mathscr{F}(C) \to \mathscr{F}(C)$. The identity constraint of $\mathscr{F}$ supplies an isomorphism of functors $\mathrm{id}_{\mathscr{F}(C)} \xrightarrow{\sim} T_C$, so that T_C is an equivalence of categories. To verify condition (2), suppose we are given a morphism e in the category $(\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. We wish to show that, if the image of e in the category $\int^{\mathcal{C}^{\mathrm{op}}} \mathscr{F}$ is $U^{\dagger}$-cartesian, then $T(e)$ is a U-cocartesian morphism in the category $\int_{\mathcal{C}} \mathscr{F}$. Writing $e = (f, f', u)$ and $T(e) = (f, v)$ as in Construction 8.6.1.9, we are reduced to showing that if u is an isomorphism, then v is also an isomorphism (Proposition 5.6.1.15), which is immediate from the construction. □

Construction 8.6.1.11 (Conjugacy for Weighted Nerves)**.** Let QCat denote the category of ∞-categories (which we regard as a full subcategory of the category of simplicial sets). Let $\mathcal{C}$ be a category equipped with a functor $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$, and let $\mathcal{E} = \mathrm{N}_{\bullet}^{\mathscr{F}}(\mathcal{C})$ denote the weighted nerve of Definition 5.3.3.1. We will identify n-simplices of $\mathcal{E}$ with pairs (σ_+, τ_+), where $\sigma_+ = (C_0 \to C_1 \to \cdots \to C_n)$ is an n-simplex of the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C})$ and $\tau_+ = (\tau_0, \tau_1, \cdots, \tau_n)$ is the datum of a commutative diagram 04CH

$$\begin{tikzcd}
\Delta^0 \arrow[r, hook] \arrow[d, "\tau_0"'] & \Delta^1 \arrow[r, hook] \arrow[d, "\tau_1"] & \Delta^2 \arrow[r, hook] \arrow[d, "\tau_2"'] & \cdots \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, "\tau_n"'] \\
\mathscr{F}(C_0) \arrow[r] & \mathscr{F}(C_1) \arrow[r] & \mathscr{F}(C_2) \arrow[r] & \cdots \arrow[r] & \mathscr{F}(C_n).
\end{tikzcd}$$

Let $U : \mathcal{E} \to \mathrm{N}_{\bullet}(\mathcal{C})$ be the cocartesian fibration of Corollary 5.3.3.16, given on n-simplices by the formula $U(\sigma_+, \tau_+) = \sigma_+$.

Let $\mathscr{F}^{\mathrm{op}} : \mathcal{C} \to \mathrm{QCat}$ denote the functor given by the formula $\mathscr{F}^{\mathrm{op}}(C) = \mathscr{F}(C)^{\mathrm{op}}$, and let $\mathcal{E}^{\dagger}$ denote the ∞-category $\mathrm{N}_{\bullet}^{\mathscr{F}^{\mathrm{op}}}(\mathcal{C})^{\mathrm{op}}$. Unwinding the definitions, we see that n-simplices of $\mathcal{E}^{\dagger}$ can be identified with pairs (σ_-, τ_-), where $\sigma_- = (C'_n \to C'_{n-1} \to \cdots \to C'_0)$ is an n-simplex of the simplicial set $\mathrm{N}_{\bullet}(\mathcal{C})^{\mathrm{op}}$ and $\tau_- = (\tau'_n, \tau'_{n-1}, \cdots, \tau'_0)$ is the datum of a

commutative diagram

$$\begin{tikzcd}
\{n\} \arrow[r, hook] \arrow[d, "\tau'_n"'] & \mathrm{N}_\bullet(\{n-1<n\}) \arrow[r, hook] \arrow[d, "\tau'_{n-1}"] & \cdots \arrow[r, hook] & \Delta^n \arrow[d, "\tau'_0"] \\
\mathscr{F}(C'_n) \arrow[r] & \mathscr{F}(C'_{n-1}) \arrow[r] & \cdots \arrow[r] & \mathscr{F}(C'_0).
\end{tikzcd}$$

Corollary 5.3.3.16 supplies a cartesian fibration $U^\dagger : \mathcal{E}^\dagger \to \mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}}$, given on n-simplices by the formula $U^\dagger(\sigma_-, \tau_-) = \sigma_-$.

Let us identify n-simplices of the fiber product $\mathcal{E}^\dagger \times_{\mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}}} \mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C}))$ with quadruples $(\sigma_-, \tau_-, \sigma_+, e)$, where $\sigma_- = (C'_n \to \cdots \to C'_0)$ is an n-simplex of $\mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}}$, $\tau_- = (\tau'_n, \cdots, \tau'_0)$ is as above, $\sigma_+ = (C_0 \to \cdots \to C_n)$ is an n-simplex of $\mathrm{N}_\bullet(\mathcal{C})$, and $e : C'_0 \to C_0$ is a morphism in the category $\mathcal{C}$. For $0 \leq i \leq n$, we let τ_i denote the i-simplex of $\mathscr{F}(C_i)$ given by the composition

$$\Delta^i \hookrightarrow \Delta^n \xrightarrow{\tau'_0} \mathscr{F}(C'_0) \xrightarrow{\mathscr{F}(e)} \mathscr{F}(C_0) \to \mathscr{F}(C_i).$$

Set $\tau_+ = (\tau_0, \cdots, \tau_n)$, so that the pair (σ_+, τ_+) determines an n-simplex of the simplicial set $\mathcal{E}$. The construction $(\sigma_-, \tau_-, \sigma_+, e) \mapsto (\sigma_+, \tau_+)$ is compatible with face and degeneracy operators, and therefore determines a functor of ∞-categories

$$T : \mathcal{E}^\dagger \times_{\mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}}} \mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C})) \to \mathcal{E}\,.$$

04CJ **Proposition 8.6.1.12.** *Let $\mathcal{C}$ be a category equipped with a functor $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$, let $U : \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ denote the cocartesian fibration of Corollary 5.3.3.16, and define $U^\dagger : \mathrm{N}^{\mathscr{F}^{\mathrm{op}}}_\bullet(\mathcal{C})^{\mathrm{op}} \to \mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}}$ similarly. Then the functor*

$$T : \mathrm{N}^{\mathscr{F}^{\mathrm{op}}}_\bullet(\mathcal{C})^{\mathrm{op}} \times_{\mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}}} \mathrm{Tw}(\mathrm{N}_\bullet(\mathcal{C})) \to \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$$

of Construction 8.6.1.11 exhibits $U^\dagger$ as a cartesian conjugate of U.

Proof. Condition (0) of Definition 8.6.1.1 follows immediately from the construction. Condition (1) follows from the observation that, for each object $C \in \mathcal{C}$, the induced map

$$T_C : \mathrm{N}^{\mathscr{F}^{\mathrm{op}}}_\bullet(\mathcal{C})^{\mathrm{op}} \times_{\mathrm{N}_\bullet(\mathcal{C})^{\mathrm{op}}} \{C\} \to \{C\} \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}^{\mathscr{F}}_\bullet(\mathcal{C})$$

is an isomorphism of simplicial sets (under the identifications supplied by Example 5.3.3.8, it corresponds to the identity functor from the ∞-category $\mathscr{F}(C)$ to itself). Condition (2) follows from the characterization of U-cocartesian and $U^{\dagger,\mathrm{op}}$-cocartesian morphisms given in Corollary 5.3.3.16. □

We close this section with a technical result, which will be convenient for verifying hypothesis (2) of Definition 8.6.1.1. If $\mathcal{C}$ is a simplicial set and $e : C \to D$ is an edge of $\mathcal{C}$, we write $e_L : \mathrm{id}_C \to e$ and $e_R : \mathrm{id}_D \to e$ for the edges of $\mathrm{Tw}(\mathcal{C})$ described in Example 8.1.3.6.

Proposition 8.6.1.13. *Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration, and suppose we are given a commutative diagram* 04CK

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r, "\lambda_+"] & \mathcal{C} .
\end{tikzcd}$$

Then T satisfies condition (2) of Definition 8.6.1.1 if and only if it satisfies both of the following conditions:

(2′) *For every object $Y \in \mathcal{E}^{\dagger}$ having image $\overline{Y} = U^{\dagger}(Y)$ and every edge $e : \overline{Y} \to \overline{X}$ of $\mathcal{C}$, the morphism T carries $(\mathrm{id}_Y, e_L) : (Y, \mathrm{id}_{\overline{Y}}) \to (Y, e)$ to a U-cocartesian edge of $\mathcal{E}$.*

(2″) *Let $f : X \to Y$ be a $U^{\dagger}$-cartesian edge of $\mathcal{E}^{\dagger}$. Set $\overline{X} = U^{\dagger}(X)$ and $\overline{Y} = U^{\dagger}(Y)$, so that $U^{\dagger}(f)$ can be identified with an edge $e : \overline{Y} \to \overline{X}$ of the simplicial set $\mathcal{C}$. Then T carries $(f, e_R) : (X, \mathrm{id}_{\overline{X}}) \to (Y, e)$ to an isomorphism in the ∞-category $\mathcal{E}_{\overline{Y}}$.*

Proof. The implication (2) ⇒ (2′) is immediate from the definitions, and the implication (2) ⇒ (2″) follows from Proposition 5.1.4.11. For the converse, suppose that conditions (2′) and (2″) are satisfied. Let $f : X \to Y$ be a $U^{\dagger}$-cartesian edge of $\mathcal{E}^{\dagger}$, and let us identify $U^{\dagger}(f)$ with an edge $e : \overline{Y} \to \overline{X}$ of the simplicial set $\mathcal{C}$. Suppose we are given a lift of e to an edge $\widetilde{e} : u \to v$ of $\mathrm{Tw}(\mathcal{C})$, which we identify with a 3-simplex $\sigma : \Delta^3 \to \mathcal{C}$ depicted in the diagram

$$\begin{tikzcd}
\overline{X} \arrow[d, "u"] & \overline{Y} \arrow[l, "e"'] \arrow[d, "v"] \\
\overline{X}' \arrow[r] & \overline{Y}'.
\end{tikzcd}$$

We wish to show that $T(f, \widetilde{e})$ is a U-cocartesian edge of $\mathcal{E}$.

Let σ' denote the degenerate 5-simplex of $\mathcal{C}$ given by $\gamma^*(\sigma)$, where $\gamma : [5] \to [3]$ is given by $\gamma(0) = 0$, $\gamma(1) = \gamma(2) = \gamma(3) = 1$, $\gamma(4) = 2$, and $\gamma(5) = 3$. Let us abuse notation by identifying σ' with the 2-simplex of $\operatorname{Tw}(\mathcal{C})$ depicted in the diagram

04CL
$$\begin{tikzcd}
\overline{X} & \overline{X} \arrow[l, "\mathrm{id}"'] \arrow[d, "u"] & \overline{Y} \arrow[l, "e"'] \arrow[d, "v"] \\
\overline{X} \arrow[u, phantom] \arrow[r, "u"] & \overline{X}' \arrow[r] & \overline{Y}'.
\arrow[from=1-1, to=2-1, "\mathrm{id}"]
\end{tikzcd} \tag{8.60}$$

Evaluating T on the pair $(s_0^1(f), \sigma')$, we obtain a 2-simplex of $\mathcal{E}$ depicted in the diagram

$$\begin{tikzcd}
& T(X,u) \arrow[dr, "T(f{,}\widetilde{e})"] & \\
T(X, \mathrm{id}_X) \arrow[ur, "T(\mathrm{id}_X{,}u_L)"] \arrow[rr] & & T(Y,v),
\end{tikzcd}$$

where the left diagonal map is U-cocartesian by virtue of assumption $(2')$. Consequently, to show that the $T(f, \widetilde{e})$ is U-cocartesian, it will suffice to show that the horizontal edge is U-cocartesian (Proposition 5.1.4.12). Note that, in the special case where $\sigma = s_0^2(\sigma_0)$ for some 2-simplex σ_0 of $\mathcal{C}$, the horizontal edge coincides with $T(\mathrm{id}_X, v_L)$, which is also U-cocartesian by virtue of $(2')$.

To handle the general case, we can replace σ by the 3-simplex of $\mathcal{C}$ given by the outer rectangle of the diagram (8.60) (that is, by the 3-simplex $\sigma'|_{\mathrm{N}_\bullet(\{0<2<3<5\})}$, and thereby reduce to the special case where $\sigma = s_1^2(\sigma_1)$, for some 2-simplex σ_1 of $\mathcal{C}$ (so that $\overline{X} = \overline{X}'$ and u is a degenerate edge of $\mathcal{C}$). In this case, we let σ'' denote the 5-simplex of $\mathcal{C}$ given by $\beta^*(\sigma_1)$, where $\beta : [5] \to [2]$ is given by $\beta(0) = \beta(1) = 0$, $\beta(2) = \beta(3) = \beta(4) = 1$, and $\beta(5) = 2$. Let us view σ'' as a 2-simplex of $\operatorname{Tw}(\mathcal{C})$ depicted in the diagram

$$\begin{tikzcd}
\overline{X} \arrow[d, "u"] & \overline{Y} \arrow[l, "e"'] & \overline{Y} \arrow[l, "\mathrm{id}"'] \\
\overline{X}' \arrow[r, "\mathrm{id}"] & \overline{X}' \arrow[r] & \overline{Y}'.
\end{tikzcd}$$

Evaluating T on the pair $(s_1^1(f), \sigma'')$, we obtain a 2-simplex of $\mathcal{E}$ depicted in the diagram

$$\begin{tikzcd}
& T(Y,e) \arrow[dr] & \\
T(X,u) \arrow[ur, "T(f{,}e_R)"] \arrow[rr, "T(f{,}\widetilde{e})"] & & T(Y,v).
\end{tikzcd}$$

Here the left diagonal edge is U-cocartesian by virtue of assumption $(2'')$ and the right diagonal edge is U-cocartesian by virtue of the special case treated above. Applying Proposition 5.1.4.12, we conclude that $T(f, \widetilde{e})$ is also U-cocartesian. $\square$

Remark 8.6.1.14. In the situation of Proposition 8.6.1.13, suppose that T exhibits $U^{\dagger}$ as a cartesian conjugate of U. Then we have the following stronger version condition of $(2'')$: 05JF

$(*)$ Let $f : X \to Y$ be an edge of $\mathcal{E}^{\dagger}$. Set $\overline{X} = U^{\dagger}(X)$ and $\overline{Y} = U^{\dagger}(Y)$, so that $U^{\dagger}(f)$ can be identified with an edge $e : \overline{Y} \to \overline{X}$ of the simplicial set $\mathcal{C}$. Then f is $U^{\dagger}$-cartesian if and only if $T(f, e_R)$ is an isomorphism in the ∞-category $\mathcal{E}_{\overline{Y}}$.

The "only if" direction follows from Proposition 8.6.1.13. To prove the converse, choose a 2-simplex σ of $\mathcal{E}^{\dagger}$ corresponding to a diagram

$$\begin{tikzcd}
 & X' \arrow[dr, "f'"] & \\
X \arrow[ur, "u"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

of the simplicial set $\mathcal{E}^{\dagger}$, where $U^{\dagger}(\sigma)$ is a left-degenerate 2-simplex of $\mathcal{C}^{\mathrm{op}}$ and f' is $U^{\dagger}$-cartesian. We then obtain a 2-simplex

$$\begin{tikzcd}
 & T(X', f) \arrow[dr, "{T(f', e_R)}"] & \\
T(X, f) \arrow[ur, "T_{\overline{Y}}(u)"] \arrow[rr, "{T(f, e_R)}"] & & T(Y, \mathrm{id}_Y)
\end{tikzcd}$$

in the ∞-category $\mathcal{E}_{\overline{Y}}$. If both $T(f, e_R)$ and $T(f', e_R)$ are isomorphisms, then $T_{\overline{Y}}(u)$ is an isomorphism as well. Since the functor $T_{\overline{Y}} : \mathcal{E}^{\dagger}_{\overline{Y}} \to \mathcal{E}_{\overline{Y}}$ is an equivalence of ∞-categories, we conclude that u is an isomorphism in the ∞-category$\mathcal{E}^{\dagger}_{\overline{Y}}$, so that f is also $U^{\dagger}$-cartesian (see Remark 5.1.3.8).

8.6.2 Existence of Conjugate Fibrations

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. Our goal in this section is to show that U admits a cartesian conjugate $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ (Corollary 8.6.2.4). For this 04CM purpose, we will need to construct a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C}.
\end{tikzcd}$$

We begin by considering the universal example of such a diagram.

04CN **Notation 8.6.2.1.** Let $\mathcal{D}$ be a simplicial set equipped with a morphism $\lambda = (\lambda_-, \lambda_+) : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$. For every simplicial set $\mathcal{E}$, we let $\mathrm{Fun}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$ denote the relative exponential of Construction 4.5.9.1. For $n \geq 0$, we will identify n-simplices of $\mathrm{Fun}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$ with pairs (σ, f), where σ is an n-simplex of $\mathcal{D}_-^{\mathrm{op}}$ and $f : \Delta^n \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D} \to \mathcal{E}$ is a morphism of simplicial sets. Suppose that we are also given a morphism of simplicial sets $U : \mathcal{E} \to \mathcal{D}_+$. In this case, we let $\mathrm{Fun}_{/\mathcal{D}_+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$ denote the simplicial subset of $\mathrm{Fun}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$ whose n-simplices are pairs (σ, f) which satisfy the additional condition that the diagram

$$\begin{tikzcd}
\Delta^n \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D} \arrow[r, "f"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{D} \arrow[r, "\lambda_+"] & \mathcal{D}_+
\end{tikzcd}$$

is commutative.

04CP **Construction 8.6.2.2.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, and let $\mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ be the simplicial set given in Notation 8.6.2.1. Unwinding the definitions, we see that vertices of $\mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ can be identified with pairs (C, f_C), where C is an object of $\mathcal{C}$ and f_C is a morphism which fits into a commutative diagram

$$\begin{tikzcd}
\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[rr, "f_C"] \arrow[dr] & & \mathcal{E} \arrow[dl, "U"] \\
& \mathcal{C}. &
\end{tikzcd}$$

We let $\mathrm{Fun}_{/\mathcal{C}}^{\mathrm{CCart}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ denote the full simplicial subset of $\mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$, spanned by those pairs (C, f_C) where the morphism f_C carries each edge of $\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ to a U-cocartesian edge of $\mathcal{E}$. By construction, the simplicial set $\mathrm{Fun}_{/\mathcal{C}}^{\mathrm{CCart}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$

is equipped with a projection map $V : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{C}^{\mathrm{op}}$ and an evaluation map $\mathrm{ev} : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$, given on vertices by the construction $(C, f_C, u : C \to C') \mapsto f_C(u)$.

We can now formulate the main result of this section:

Proposition 8.6.2.3. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. Then the projection map $V : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{C}^{\mathrm{op}}$ of Construction 8.6.2.2 is a cartesian fibration of ∞-categories, and the evaluation functor* 04CQ

$$\mathrm{ev} : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E} \qquad (C, f_C, u : C \to C') \mapsto f_C(u)$$

exhibits V as a cartesian conjugate of U.

Corollary 8.6.2.4. *Every cocartesian fibration of simplicial sets $U : \mathcal{E} \to \mathcal{C}$ admits a cartesian conjugate.* 04CR

Proof. Using Corollary 5.6.7.3, we can choose a pullback diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r] & \mathcal{C}',
\end{tikzcd}$$

where U' is a cocartesian fibration of ∞-categories. By virtue of Remark 8.6.1.4, it will suffice to show that U' admits a cartesian conjugate, which follows from Proposition 8.6.2.3. □

Warning 8.6.2.5. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. If $\mathcal{C}$ is not an ∞-category, then the morphism $V : \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{C}^{\mathrm{op}}$ given by Construction 8.6.2.2 need not be a cartesian conjugate of U. In §8.6.3, we will give an alternative construction of a cartesian conjugate which works in complete generality (Proposition 8.6.3.5). 04CS

The proof of Proposition 8.6.2.3 will require some preliminaries.

Lemma 8.6.2.6. *Let $\lambda = (\lambda_-, \lambda_+) : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}} \times \mathcal{D}_+$ be a coupling of ∞-categories and let $U : \mathcal{E} \to \mathcal{D}_+$ be an isofibration. Then:* 04CT

(1) *The projection map $\overline{V} : \mathrm{Fun}_{/\mathcal{D}^+}(\mathcal{D} / \mathcal{D}_-^{\mathrm{op}}, \mathcal{E}) \to \mathcal{D}_-^{\mathrm{op}}$ is a cartesian fibration of ∞-categories.*

(2) *Let $\widetilde{e}$ be a morphism in the ∞-category $\mathrm{Fun}_{/\mathcal{D}^+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$, corresponding to a morphism e of $\mathcal{D}_-^{\mathrm{op}}$ and a functor $f_e : \Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D} \to \mathcal{E}$. Then $\widetilde{e}$ is $\overline{V}$-cartesian if and only if, for every morphism u of $\Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$ whose image in $\mathcal{D}_+$ is an isomorphism, the image $f_e(u)$ is an isomorphism in $\mathcal{E}$.*

Proof. Unwinding the definitions, we have a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{Fun}_{/\mathcal{D}_+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E}) \arrow[r] \arrow[d, "\overline{V}"] & \mathrm{Fun}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E}) \arrow[d, "\overline{V}'"] \\
\mathcal{D}_-^{\mathrm{op}} \arrow[r] & \mathrm{Fun}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{D}_+),
\end{tikzcd}$$

where the lower horizontal map classifies the morphism $\lambda_+ : \mathcal{D} \to \mathcal{D}_+$ and $\overline{V}'$ is given by composition with U. The functor $\lambda_- : \mathcal{D} \to \mathcal{D}_-^{\mathrm{op}}$ is a cocartesian fibration (Proposition 8.2.1.7) and therefore exponentiable (Proposition 5.3.6.1). It follows from Proposition 4.5.9.18 guarantees that $\overline{V}'$ is an isofibration, so that $\overline{V}$ is also an isofibration.

Let us say that a morphism $\widetilde{e}$ in the ∞-category $\mathrm{Fun}_{/\mathcal{D}^+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$ is *special* if it satisfies the condition described in (2). Let us identify $\widetilde{e}$ with a pair (e, f_e), where $e : D' \to D$ is a morphism in the ∞-category $\mathcal{D}_-^{\mathrm{op}}$ and $f_e : \Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D} \to \mathcal{E}$ is a functor of ∞-categories. Let $\pi : \Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D} \to \Delta^1$ be given by the projection map onto the first factor. Then π is a cocartesian fibration, and a morphism u of $\Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$ is π-cocartesian if and only if its image in $\mathcal{D}_+$ is an isomorphism (see Proposition 8.2.1.7). If this condition is satisfied, the assumption that $\widetilde{e}$ is special guarantees that $f_e(u)$ is an isomorphism in the ∞-category $\mathcal{E}$, and is therefore U-cartesian (Proposition 5.1.1.8). Applying Lemma 5.3.6.11, we deduce that $\widetilde{e}$ is $\overline{V}'$-cartesian when regarded as a morphism of $\mathrm{Fun}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$, and therefore also $\overline{V}$-cartesian when regarded as a morphism of $\mathrm{Fun}_{/\mathcal{D}_+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$.

To show that $\overline{V}$ is a cartesian fibration, it will suffice to show that if (D, f_D) is an object of $\mathrm{Fun}_{/\mathcal{D}^+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$, then every morphism $e : C \to D$ in the ∞-category $\mathcal{D}_-^{\mathrm{op}}$ can be lifted to a special morphism $\widetilde{e} : (C, f_C) \to (D, f_D)$ in $\mathrm{Fun}_{/\mathcal{D}^+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$. We first claim that the lifting problem

$$\begin{tikzcd}
\{D\} \times_{\mathcal{C}_-^{\mathrm{op}}} \mathcal{C} \arrow[r, "f_D"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D} \arrow[r] \arrow[ur, dashed, "f_e"] & \mathcal{D}_+
\end{tikzcd}$$

admits a solution f_e which is U-right Kan extended from $\{D\} \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$. Using the criterion of Corollary 7.3.5.9, we are reduced to showing that if $u : X' \to X$ is a π-cocartesian morphism of $\Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$ lying over the nondegenerate edge of Δ^1, then its image in $\mathcal{D}_+$ can be lifted to a U-cartesian morphism $E \to f_D(X)$ in $\mathcal{E}$. Since the image of u in $\mathcal{D}_+$ is an isomorphism (Proposition 8.2.1.7), this follows from our assumption that U is an isofibration. Note that the functor f_e carries *every* π-cocartesian morphism u of $\Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$ to an isomorphism in $\mathcal{D}$ (if the image of u in Δ^1 is degenerate, then u is an isomorphism and this condition is automatically satisfied), so that $\widetilde{e} = (e, f_e)$ is a special morphism of $\mathrm{Fun}_{/\mathcal{D}^+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$ having target (D, f_D). This completes the proof of (1).

To complete the proof of (2), it will suffice to show that every $\overline{V}$-cartesian morphism $\widetilde{e} = (C, f_C) \to (D, f_D)$ of $\mathrm{Fun}_{/\mathcal{D}^+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$ is special. Let $e : C \to D$ denote the image of $\widetilde{e}$ in the ∞-category $\mathcal{D}_-^{\mathrm{op}}$. Using the preceding argument, we can lift e to a special morphism $\widetilde{e}' : (C, f'_C) \to (D, f_D)$ of $\mathrm{Fun}_{/\mathcal{D}^+}(\mathcal{D} \,/\, \mathcal{D}_-^{\mathrm{op}}, \mathcal{E})$ Write $\widetilde{e} = (e, f_e)$ and $\widetilde{e}' = (e, f'_e)$. Since $\widetilde{e}'$ is also $\overline{V}$-cartesian, Remark 5.1.3.8 guarantees that the functors f_e and f'_e are isomorphic. In particular, if u is a morphism of $\Delta^1 \times_{\mathcal{D}_-^{\mathrm{op}}} \mathcal{D}$ such that $f'_e(u)$ is an isomorphism in $\mathcal{E}$, then $f_e(u)$ is also an isomorphism in $\mathcal{E}$. It follows that the morphism $\widetilde{e}$ is also special, as desired. □

Lemma 8.6.2.7. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let $\overline{V} : \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{C}^{\mathrm{op}}$ be the cartesian fibration of Lemma 8.6.2.6. Then:* 04CU

(1) *Let $\widetilde{e} : (C, f_C) \to (D, f_D)$ be a $\overline{V}$-cartesian morphism of $\mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E})$. If (D, f_D) belongs to the simplicial subset $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E})$ of Construction 8.6.2.2, then (C, f_C) also belongs to $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E})$.*

(2) *The morphism $\overline{V}$ restricts to a cartesian fibration of ∞-categories*

$$V : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{C}^{\mathrm{op}}.$$

(3) *A morphism in the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E})$ is V-cartesian if and only if it is $\overline{V}$-cartesian when regarded as a morphism of $\mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E})$.*

Proof. We will prove assertion (1); assertions (2) and (3) then follow as formal consequences (see Proposition 5.1.4.16). Let us identify $\widetilde{e}$ with a pair (e, f_e), where $e : C \to D$ is a morphism in the ∞-category $\mathcal{C}^{\mathrm{op}}$ and $f_e : \Delta^1 \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ is a functor. Let $u : \widetilde{C} \to \widetilde{C}'$ be a morphism in the fiber $\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$; we wish to show that $f_C(u)$ is a U-cocartesian morphism of $\mathcal{E}$. Since the projection map $\Delta^1 \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \Delta^1$ is a cocartesian fibration, we can choose a diagram

$$\begin{tikzcd}
\widetilde{C} \arrow[r] \arrow[d, "u"] & \widetilde{D} \arrow[d, "v"] \\
\widetilde{C}' \arrow[r] & \widetilde{D}'
\end{tikzcd}$$

in the ∞-category $\Delta^1 \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$, where v is a morphism of $\{D\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ and the horizontal maps are π-cocartesian. Applying the functor f_e, we obtain a diagram

$$\begin{tikzcd}
f_C(\widetilde{C}) \arrow[r] \arrow[d, "f_C(u)"] & f_D(\widetilde{D}) \arrow[d, "f_D(v)"] \\
f_C(\widetilde{C}') \arrow[r] & f_D(\widetilde{D}')
\end{tikzcd}$$

in the ∞-category $\mathcal{E}$, where the horizontal maps are isomorphisms (by virtue of our assumption that $\widetilde{e}$ is $\overline{V}$-cartesian; see Lemma 8.6.2.6). It will therefore suffice to show that $f_D(v)$ is U-cocartesian (Corollary 5.1.2.5), which follows from our assumption that (D, f_D) is an object of $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$. □

We will deduce Proposition 8.6.2.3 from the following more precise result:

04CV **Proposition 8.6.2.8.** *Let $\mathcal{C}$ be an ∞-category, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^\dagger : \mathcal{E}^\dagger \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. Suppose we are given a commutative diagram*

$$\begin{tikzcd}
\mathcal{E}^\dagger \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[rr, "T"] \arrow[dr] & & \mathcal{E} \arrow[dl, "U"] \\
& \mathcal{C}, &
\end{tikzcd}$$

which we identify with a functor $F : \mathcal{E}^\dagger \to \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$. The following conditions are equivalent:

(a) The functor T exhibits $U^\dagger$ as a cartesian conjugate of U (in the sense of Definition 8.6.1.1).

(b) The functor F restricts to an equivalence of $\mathcal{E}^\dagger$ with the full subcategory

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \subseteq \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$$

introduced in Construction 8.6.2.2.

Proof. Let $\lambda = (\lambda_-, \lambda_+) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ denote the twisted arrow fibration of Example 8.2.0.2. Recall that (a) is equivalent to the following pair of conditions:

(a_1) For every object $C \in \mathcal{C}$, the restriction of T to the fiber over the vertex $\{\mathrm{id}_C\} \subseteq \mathrm{Tw}(\mathcal{C})$ determines an equivalence of ∞-categories $T_C : \mathcal{E}^\dagger \times_{\mathcal{C}^{\mathrm{op}}} \{C\} \to \{C\} \times_{\mathcal{C}} \mathcal{E}$.

(a_2) Let (e', e) be an edge of the fiber product $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. If e' is a $U^{\dagger}$-cartesian morphism of $\mathcal{E}^{\dagger}$, then $T(e', e)$ is a U-cocartesian morphism of $\mathcal{E}$.

Unwinding the definitions, we see that F factors through $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ if and only if T satisfies the following weaker version of (a_2):

(b_0) Let (e', e) be an edge of the fiber product $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. If e' is a degenerate edge of $\mathcal{E}^{\dagger}$, then $T(e', e)$ is a U-cocartesian morphism of $\mathcal{E}$.

If this condition is satisfied, then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \arrow[rr, "F"] \arrow[dr, "U^{\dagger}"'] & & \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \arrow[dl, "V"] \\
& \mathcal{C}^{\mathrm{op}}, &
\end{tikzcd}$$

where the vertical maps are cartesian fibrations (Lemma 8.6.2.7). Using Theorem 5.1.6.1, we see that F is an equivalence if and only if it satisfies the following further conditions:

(b_1) For each object $C \in \mathcal{C}$, the functor F restricts to an equivalence of ∞-categories

$$F_C : \mathcal{E}^{\dagger}_C = \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \{C\} \to \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\}.$$

(b_2) The functor F carries $U^{\dagger}$-cartesian morphisms of $\mathcal{E}^{\dagger}$ to V-cartesian morphisms of $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$.

Let C be an object of $\mathcal{C}$. Unwinding the definitions, we can identify the fiber

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\}$$

with the ∞-category $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}), \mathcal{E})$. Since id_C is initial when viewed as an object of the ∞-category $\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ (Proposition 8.1.2.1), Proposition 5.3.1.21 guarantees that the evaluation map

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}), \mathcal{E}) \to \{C\} \times_{\mathcal{C}} \mathcal{E}$$

is a trivial Kan fibration. Moreover, the composition of this evaluation map with the functor F_C coincides with the functor T_C appearing in condition (a_1). It follows that conditions (a_1) and (b_1) are equivalent.

Using the characterization of V-cartesian morphisms supplied by Lemmas 8.6.2.6 and 8.6.2.7, we can reformulate (b_2) more concretely as follows:

(b_2') Let (e', e) be an edge of the fiber product $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. If e' is a $U^{\dagger}$-cartesian morphism of $\mathcal{E}^{\dagger}$ and $\lambda_{+}(e)$ is an isomorphism in $\mathcal{C}$, then $T(e', e)$ is an isomorphism in the ∞-category $\mathcal{E}$.

To complete the proof, it will suffice to show that the functor T satisfies (a_2) if and only if it satisfies both (b_0) and (b_2'). The implication $(a_2) \Rightarrow (b_0)$ is immediate, and the implication $(a_2) \Rightarrow (b_2')$ follows from Corollary 5.1.1.8. The reverse implication follows from Proposition 8.6.1.13. □

Proof of Proposition 8.6.2.3. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. It follows from Lemma 8.6.2.7 that the projection map $V : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{C}^{\mathrm{op}}$ is a cartesian fibration. We wish to show that the evaluation map

$$\mathrm{ev} : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$$

exhibits V as a cartesian conjugate of U. This follows from Proposition 8.6.2.8, since the identity automorphism of $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E})$ is an equivalence of ∞-categories. □

It follows from Proposition 8.6.2.8 that conjugate fibrations can be characterized by a universal mapping property:

05JG **Corollary 8.6.2.9.** *Let $\mathcal{C}$ be an ∞-category, let $V : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $V^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. Suppose we are given a commutative diagram*

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[rr, "T"] \arrow[dr] & & \mathcal{E} \arrow[dl, "V"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $V^{\dagger}$ as a cartesian conjugate of V. Then, for every morphism of simplicial sets $U^{\dagger} : \mathcal{D}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$, composition with T induces a fully faithful functor

$$\mathrm{Fun}_{/\mathcal{C}^{\mathrm{op}}}(\mathcal{D}^{\dagger}, \mathcal{E}^{\dagger}) \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}), \mathcal{E}).$$

The essential image is spanned by those diagrams $Q : \mathcal{D}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ having the property that, for every vertex $D \in \mathcal{D}^{\dagger}$, Q carries every edge of $\{D\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ to a V-cocartesian edge of $\mathcal{E}$.

Example 8.6.2.10 (Functoriality)**.** Suppose we are given a commutative diagram of ∞-categories 05JH

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E} \arrow[dl, "V"] \\
& \mathcal{C} &
\end{tikzcd}$$

where U and V are cocartesian fibrations, and the functor F carries U-cocartesian morphisms of $\mathcal{D}$ to V-cocartesian morphisms of $\mathcal{E}$. Let $U^{\dagger} : \mathcal{D}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ and $V^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be cartesian fibrations, and suppose we are given functors

$$T_0 : \mathcal{D}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{D} \qquad T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$$

which exhibit $U^{\dagger}$ and $V^{\dagger}$ as cartesian conjugates of U and V, respectively. Then there is a functor $F^{\dagger} : \mathcal{D}^{\dagger} \to \mathcal{E}^{\dagger}$ satisfying $V^{\dagger} \circ F^{\dagger} = U^{\dagger}$ for which the diagram

$$\begin{tikzcd}
\mathcal{D}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T_0"] \arrow[d, "F^{\dagger} \times \mathrm{id}"] & \mathcal{D} \arrow[d, "F"] \\
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] & \mathcal{E}
\end{tikzcd}$$

commutes up to isomorphism (in the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}), \mathcal{E})$). Moreover, the functor $F^{\dagger}$ is uniquely determined up to isomorphism (as an object of the ∞-category $\mathrm{Fun}_{/\mathcal{C}^{\mathrm{op}}}(\mathcal{D}^{\dagger}, \mathcal{E}^{\dagger})$).

Remark 8.6.2.11. In the situation of Example 8.6.2.10, the functor $F^{\dagger}$ automatically carries $U^{\dagger}$-cartesian morphisms of $\mathcal{D}^{\dagger}$ to $V^{\dagger}$-cartesian morphisms of $\mathcal{E}^{\dagger}$. See Remark 8.6.1.14. 05JJ

Let $\mathcal{C}$ be an ∞-category, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. It follows from Proposition 8.6.2.8 that if there exists a functor

$$T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$$

which exhibits $U^{\dagger}$ as a cartesian conjugate of U, then $U^{\dagger}$ can be recovered from U up to equivalence. We close this section by showing that, in the same situation, we can also recover U from the cartesian fibration $U^{\dagger}$.

Proposition 8.6.2.12. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration of ∞-categories, and suppose we are given a commutative* 04CY *diagram*

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

which exhibits $U^{\dagger}$ *as a cartesian conjugate of* U*. Then* T *also exhibits* $\mathcal{E}$ *as a localization of* $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ *with respect to* W*, where* W *is the collection of all morphisms* $w = (w', w'')$ *where* w' *is a* U'*-cartesian morphism of* $\mathcal{E}^{\dagger}$ *and* w'' *is a morphism of* $\mathrm{Tw}(\mathcal{C})$ *whose image in* $\mathcal{C}$ *is degenerate.*

04CZ **Remark 8.6.2.13.** The converse of Proposition 8.6.2.12 is also true; see Corollary 8.6.6.7.

Proof of Proposition 8.6.2.12. Let $\lambda = (\lambda_{-}, \lambda_{+}) : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ denote the twisted arrow coupling of Example 8.2.0.2, and let $V : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}$ denote the composition of λ_{+} with projection onto the second factor. Note that V factors as a composition

$$\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \xrightarrow{\mathrm{id} \times \lambda} \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \simeq \mathcal{E}^{\dagger} \times \mathcal{C} \to \mathcal{C},$$

where the first map is a left fibration (since it is a pullback of λ, which is a left fibration by virtue of Proposition 8.1.1.15), and the last map is a cocartesian fibration (since it is a pullback of the projection map $\mathcal{E}^{\dagger} \to \Delta^{0}$). It follows that V is a cocartesian fibration, and that a morphism of $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ is V-cocartesian if and only if its image in $\mathcal{E}^{\dagger}$ is an isomorphism. In particular, our hypotheses on T guarantees that it carries V-cocartesian morphisms of $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ to U-cocartesian morphisms of $\mathcal{E}$.

Fix an object $C \in \mathcal{C}$. Let $\mathcal{E}^{\dagger}(C)$ denote the fiber $V^{-1}\{C\} = \mathcal{E}' \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{C\}$, so that projection onto the middle factor gives a map $U^{\dagger}_{C} : \mathcal{E}^{\dagger}(C) \to \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \{C\}$. Note that $U^{\dagger}_{C}$ is a pullback of $U^{\dagger}$. It follows that $U^{\dagger}_{C}$ is a cartesian fibration, and that a morphism of $\mathcal{E}^{\dagger}(C)$ is $U^{\dagger}_{C}$-cartesian if and only if its image in $\mathcal{E}^{\dagger}$ is $U^{\dagger}$-cartesian (Remark 5.1.4.6). Let W_C denote the collection of morphisms of $\mathcal{E}^{\dagger}(C)$ which satisfy this condition, so that $W = \bigcup_{C \in \mathcal{C}} W_C$. Note that T restricts to a functor $T^{C} : \mathcal{E}^{\dagger}(C) \to \mathcal{E}_{C}$. By virtue of Proposition 6.3.5.2, it will suffice to verify the following (for each object $C \in \mathcal{C}$):

$(*_C)$ The functor T^{C} exhibits the ∞-category $\mathcal{E}_{C}$ as a localization of $\mathcal{E}^{\dagger}(C)$ with respect to W_C.

Let $\mathcal{K}$ denote the full subcategory of $\mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\}$ whose objects are isomorphisms $D \to C$. By virtue of Proposition 8.1.2.1, $\mathcal{K}$ can also be described as the full subcategory of $\mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\}$ spanned by its initial objects. It follows that $\mathcal{K}$ is a coreflective subcategory of $\mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\}$ (Example 6.2.2.5). Let $\mathcal{E}^{\dagger}_{0}(C) \subseteq \mathcal{E}^{\dagger}(C)$ denote the inverse image of $\mathcal{K}$

under $U^{\dagger}_{C}$, so that $\mathcal{E}^{\dagger}_{0}(C)$ is a coreflective subcategory of $\mathcal{E}^{\dagger}(C)$ (Proposition 6.2.2.22). Using Lemma 6.2.2.14, we can choose a functor $L : \mathcal{E}^{\dagger}(C) \to \mathcal{E}^{\dagger}_{0}(C)$ and a natural transformation $\epsilon : L \to \mathrm{id}_{\mathcal{E}^{\dagger}(C)}$ which exhibits L as a $\mathcal{E}^{\dagger}_{0}(C)$-coreflection functor. Our assumption on T guarantees that the functor T^{C} carries each element of W_{C} to an isomorphism in $\mathcal{E}^{\dagger}_{0}(C)$, so that ϵ induces an isomorphism of functors $(T^{C}|_{\mathcal{E}^{\dagger}_{0}(C)} \circ L) \to T^{C}$. Since the Kan complex $\mathcal{K}$ is contractible (Corollary 4.6.7.14), the inclusion map $\{\mathrm{id}_{C}\} \hookrightarrow \mathcal{K}$ is a homotopy equivalence of Kan complexes, and therefore induces an equivalence of ∞-categories $\mathcal{E}^{\dagger}(C) \times_{\mathcal{C}^{\mathrm{op}}} \{C\} \simeq \mathcal{E}^{\dagger}_{0}(C)$ (Corollary 4.5.2.29). Since the composition

$$\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \{C\} \hookrightarrow \mathcal{E}^{\dagger}_{0}(C) \xrightarrow{T^{C}} \mathcal{E}_{C}$$

is an equivalence of ∞-categories, we conclude that the functor $T^{C}|_{\mathcal{E}^{\dagger}_{0}(C)}$ is also an equivalence of ∞-categories. To complete the proof of (2_{C}), it will suffice to show that the functor L exhibits $\mathcal{E}^{\dagger}_{0}(C)$ as a localization of $\mathcal{E}^{\dagger}(C)$ with respect to W_{C} (see Remark 6.3.1.19). Let W^{+}_{C} denote the collection of morphisms v of $\mathcal{E}^{\dagger}(C)$ such that $L(v)$ is an isomorphism in $\mathcal{E}^{\dagger}_{0}(C)$. By virtue of the preceding arguments, this is equivalent to the requirement that $T(v)$ is an isomorphism in the ∞-category $\mathcal{E}_{C}$; in particular, assumption (1) guarantees that W_{C} is contained in W^{+}_{C}. Conversely, if $u : Y \to Z$ is a morphism of $\mathcal{E}^{\dagger}(C)$ which belongs to W^{+}_{C}, then we can choose a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "v"] & \\
X \arrow[ur, "u"] \arrow[rr, "w"] & & Z
\end{tikzcd}$$

where u and w exhibit X as $\mathcal{E}^{\dagger}_{0}(C)$-coreflections of the objects Y and Z, respectively, and therefore belong to W_{C}. We are therefore reduced to showing that the functor L exhibits $\mathcal{E}^{\dagger}_{0}(C)$ as a localization of $\mathcal{E}^{\dagger}(C)$ with respect to W^{+}_{C}, which is a special case of Example 6.3.3.7. □

8.6.3 Uniqueness of Conjugate Fibrations

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. It follows from Corollary 8.6.2.4 that U admits a cartesian conjugate $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$. When $\mathcal{C}$ is an ∞-category, Proposition 8.6.2.8 guarantees that $U^{\dagger}$ is unique up to equivalence: any cartesian conjugate of U is equivalent to the cartesian fibration 04EC

$$V : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathcal{C}^{\mathrm{op}}$$

of Construction 8.6.2.2. In this section, we will establish a more general uniqueness result, which applies even when $\mathcal{C}$ is not an ∞-category. Our proof will use an alternative construction of the conjugate fibration due to Barwick-Glasman-Nardin ([3]), which involves the restricted cospan construction of §8.1.6.

`04ED` **Notation 8.6.3.1.** Let $\mathcal{C}$ be a simplicial set and let $\rho_{-} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cospan}(\mathcal{C})$ be the comparison map of Variant 8.1.7.14. For every cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$, we let $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C})$ denote the fiber product $\mathrm{Cospan}(\mathcal{E})^{\mathrm{all},R}(\mathcal{E}) \times_{\mathrm{Cospan}(\mathcal{C})} \mathcal{C}^{\mathrm{op}}$, where R is the collection of all U-cocartesian edges of $\mathcal{E}$ (see Definition 8.1.6.1).

`04EE` **Remark 8.6.3.2.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Low-dimensional simplices of the simplicial set $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C})$ can be described as follows:

- Vertices of the simplicial set $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C})$ can be identified with vertices of the simplicial set $\mathcal{E}$.
- Let X and Y be vertices of $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C})$. Then edges $e : X \to Y$ of $\mathrm{Cospan}(\mathcal{E} \,/\, \mathcal{C})$ can be identified with pairs of edges $X \xrightarrow{f} B \xleftarrow{g} Y$ in the simplicial set $\mathcal{E}$ having the property that g is U-cocartesian and $U(f)$ is a degenerate edge of $\mathcal{C}$.

`04EF` **Example 8.6.3.3.** Let $\mathcal{E}$ be an ∞-category, so the projection map $\mathcal{E} \to \Delta^0$ is a cocartesian fibration of simplicial sets. Then the simplicial set $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/ \Delta^0)$ of Notation 8.6.3.1 can be identified with the simplicial set $\mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{E})$ of Construction 8.1.7.2. In particular, $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/ \Delta^0)$ is an ∞-category (Proposition 8.1.7.5), and Proposition 8.1.7.6 supplies an equivalence of ∞-categories $\rho_{+} : \mathcal{E} \to \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/ \Delta^0)$.

`04EG` **Remark 8.6.3.4** (Base Change)**.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r, "F"] & \mathcal{C},
\end{tikzcd}$$

where U and U' are cocartesian fibrations. Then we have a canonical isomorphism of simplicial sets

$$\mathrm{Cospan}^{\dagger}(\mathcal{E}' \,/\, \mathcal{C}') \simeq \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}}} \mathcal{C}'^{\,\mathrm{op}}.$$

In particular, for each vertex $C \in \mathcal{C}$, the fiber $\mathrm{Cospan}_{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\}$ is isomorphic to the ∞-category $\mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{E}_C)$, which is equivalent to the ∞-category $\mathcal{E}_C$.

We can now state a preliminary version of our main result:

Proposition 8.6.3.5. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then the projection map $V : \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ is a cartesian fibration, which is a cartesian conjugate of U.* 05JK

The proof of Proposition 8.6.3.5 will require some preliminaries. Our first step is to show that the projection map V is a cartesian fibration.

Lemma 8.6.3.6. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let R denote the collection of all U-cocartesian morphisms of $\mathcal{E}$, and let L denote the collection of all morphisms $e : X \to Y$ of $\mathcal{E}$ such that $U(e)$ is an isomorphism in $\mathcal{C}$. Then $\mathrm{Cospan}(U) : \mathrm{Cospan}(\mathcal{E}) \to \mathrm{Cospan}(\mathcal{C})$ restricts to a cartesian fibration of ∞-categories $V : \mathrm{Cospan}^{L,R}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{C})$. Moreover, an edge $e : X \to Y$ of $\mathrm{Cospan}^{L,R}(\mathcal{E})$ is V-ccartesian if and only if it satisfies the following condition:* 04EL

$(*)$ The edge e corresponds to a cospan $X \xrightarrow{\ell} B \xleftarrow{r} Y$ in $\mathcal{E}$, where ℓ is an isomorphism and r is U-cocartesian.

Proof. Let L_0 be the collection of all isomorphisms in $\mathcal{C}$, and let R_0 be the collection of all morphisms of $\mathcal{C}$. Then L_0 and R_0 are pushout-compatible, in the sense of Definition 8.1.6.5 (Example 8.1.6.6). Moreover, U is a Beck-Chevalley fibration relative to (R_0, L_0) (Example 8.1.10.8). Since $\mathrm{Cospan}^{L_0,R_0}(\mathcal{C}) = \mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{C})$ is an ∞-category, the desired result follows from Theorem 8.1.10.9 and Remark 8.1.10.11. □

Remark 8.6.3.7. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let R denote the collection of all U-cocartesian morphisms of $\mathcal{E}$, and let L denote the collection of all morphisms g of $\mathcal{E}$ such that $U(g)$ is an isomorphism in $\mathcal{C}$. We then have a commutative diagram of pullback squares 04EM

$$\begin{tikzcd}
\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Cospan}^{L,R}(\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Cospan}^{\mathrm{all},R}(\mathcal{E}) \arrow[d] \\
\mathcal{C}^{\mathrm{op}} \arrow[r, "\rho_{-}"] & \mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{C}) \arrow[r] & \mathrm{Cospan}(\mathcal{C}),
\end{tikzcd}$$

where the vertical map in the middle is a cartesian fibration (Lemma 8.6.3.6), and the horizontal map on the lower left is an equivalence of ∞-categories (Proposition 8.1.7.6). It follows that the projection map $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ is a cartesian fibration of ∞-categories. Moreover, Corollary 4.5.2.29 implies that the inclusion $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \hookrightarrow \mathrm{Cospan}^{L,R}(\mathcal{E})$ is an equivalence of ∞-categories.

Lemma 8.6.3.8. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then the projection map $V : \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ is also a cartesian fibration. Moreover, an* 04EN *edge* $X \to Y$ *of* $\operatorname{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C})$ *is* V*-cartesian if and only if it corresponds to a cospan* $X \xrightarrow{\ell} B \xleftarrow{r} Y$ *in* $\mathcal{E}$*, where* r *is* U*-cocartesian and* ℓ *is an isomorphism in the* ∞*-category* $\{U(X)\} \times_{\mathcal{C}} \mathcal{E}$*.*

Proof. Using Proposition 5.1.4.7 and Remark 8.6.3.4, we can reduce to the case where $\mathcal{C} = \Delta^n$ is a standard simplex. In particular, $\mathcal{C}$ is an ∞-category. In this case, the desired result follows by combining Remark 8.6.3.7 with Lemma 8.6.3.6. □

04EX **Construction 8.6.3.9.** Let $\mathcal{C}$ be a simplicial set, and suppose we are given a pair of morphisms $U : \mathcal{E} \to \mathcal{C}$ and $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$. Let $T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \to \mathcal{E}$ be a morphism of simplicial sets for which the diagram

04EY
$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\operatorname{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd} \tag{8.61}$$

is commutative. Let $\lambda_{+} : \operatorname{Tw}(\mathcal{E}^{\dagger}) \to \mathcal{E}^{\dagger}$ be the projection map of Notation 8.1.1.6 and let $\iota : \operatorname{Tw}(\mathcal{C}^{\mathrm{op}}) \xrightarrow{\sim} \operatorname{Tw}(\mathcal{C})$ be the isomorphism described in Remark 8.1.1.7. Then we can extend (8.61) to a commutative diagram

04EZ
$$\begin{tikzcd}
\operatorname{Tw}(\mathcal{E}^{\dagger}) \arrow[r, "(\lambda_{+}, \iota \circ \operatorname{Tw}(U^{\dagger}))"] \arrow[d, "\operatorname{Tw}(U^{\dagger})"] & \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\operatorname{Tw}(\mathcal{C}^{\mathrm{op}}) \arrow[r, "\iota"] & \operatorname{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C} .
\end{tikzcd} \tag{8.62}$$

Using Proposition 8.1.3.7, we can identify the outer rectangle with a diagram

04F0
$$\begin{tikzcd}
\mathcal{E}^{\dagger} \arrow[r] \arrow[d, "U^{\dagger}"] & \operatorname{Cospan}(\mathcal{E}) \arrow[d, "\operatorname{Cospan}(U)"] \\
\mathcal{C}^{\mathrm{op}} \arrow[r, "\rho_{-}"] & \operatorname{Cospan}(\mathcal{C}),
\end{tikzcd} \tag{8.63}$$

which we can identify with a comparison map $\Psi : \mathcal{E}^{\dagger} \to \operatorname{Cospan}(\mathcal{E}) \times_{\operatorname{Cospan}(\mathcal{C})} \mathcal{C}^{\mathrm{op}}$.

04F1 **Remark 8.6.3.10.** In the situation of Construction 8.6.3.9, the morphism Ψ can be described explicitly on low-dimensional simplices as follows:

- If X is a vertex of $\mathcal{E}^{\dagger}$ having image $C = U^{\dagger}(X)$, then $\Psi(X)$ is the vertex of $\mathrm{Cospan}(\mathcal{E})$ corresponding to the vertex $T(X, \mathrm{id}_C) \in \mathcal{E}$.

- Let X and Y be vertices of $\mathcal{E}^{\dagger}$, having images $C = U^{\dagger}(X)$ and $D = U^{\dagger}(Y)$. Let $f : X \to Y$ be an edge of $\mathcal{E}^{\dagger}$, and let us identify $U^{\dagger}(f)$ with an edge $e : D \to C$ in the simplicial set $\mathcal{C}$. Then $\Psi(f) : \Psi(X) \to \Psi(Y)$ is the edge of $\mathrm{Cospan}(\mathcal{E})$ corresponding to the pair of edges $T(X, \mathrm{id}_C) \xrightarrow{T(f, e_R)} T(Y, e) \xleftarrow{T(\mathrm{id}_Y, e_L)} T(Y, \mathrm{id}_D)$ in $\mathcal{E}$; here $e_L : \mathrm{id}_D \to e$ and $e_R : \mathrm{id}_C \to e$ denote the edges of $\mathrm{Tw}(\mathcal{C})$ described in Example 8.1.3.6.

We will deduce Proposition 8.6.3.5 from the following more precise result:

Proposition 8.6.3.11. *Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. Suppose we are given a morphism $T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ for which the diagram* 04F2

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

is commutative. The following conditions are equivalent:

(a) The morphism T exhibits $U^{\dagger}$ as a cartesian conjugate of U, in the sense of Definition 8.6.1.1.

(b) The comparison map Ψ of Construction 8.6.3.9 factors through the simplicial set $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C})$ of Notation 8.6.3.1, and the map $\mathcal{E}^{\dagger} \to \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C})$ is an equivalence of cartesian fibrations over $\mathcal{C}$.

Proof. Using Proposition 5.1.7.15, we see that (b) is equivalent to the following three conditions:

(b_0) The map Ψ factors through the simplicial subset $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \subseteq \mathrm{Cospan}(\mathcal{E}) \times_{\mathrm{Cospan}(\mathcal{C})} \mathcal{C}^{\mathrm{op}}$.

(b_1) Let $V : \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ be the cartesian fibration of Lemma 8.6.3.8. Then Ψ carries $U^{\dagger}$-cartesian edges of $\mathcal{E}^{\dagger}$ to V-cartesian edges of $\mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C})$.

(b_2) For each vertex $C \in \mathcal{C}$, the morphism Ψ restricts to an equivalence of ∞-categories

$$\Psi_C : \mathcal{E}^{\dagger}_C \to \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\} = \mathrm{Cospan}^{\mathrm{all,iso}}(\mathcal{E}_C).$$

For every edge $e : D \to C$ of $\mathcal{C}$, let $e_L : \mathrm{id}_D \to e$ and $e_R : \mathrm{id}_C \to e$ denote the edges of $\mathrm{Tw}(\mathcal{C})$ described in Example 8.1.3.6. Using Remark 8.6.3.10, we can rewrite condition (b_0) as follows:

(b_0') Let Y be a vertex of $\mathcal{E}^{\dagger}$ having image $D = U^{\dagger}(Y)$ in $\mathcal{C}$, and let $e : C \to D$ be an edge of $\mathcal{C}$. Then $T(\mathrm{id}_Y, e_L) : T(Y, \mathrm{id}_D) \to T(Y, e)$ is a U-cocartesian edge of $\mathcal{E}$.

Similarly, by combining Remark 8.6.3.10 with the characterization of V-cartesian edges supplied by Lemma 8.6.3.8, we can rewrite condition (b_1) as follows:

(b_1') Let $f : X \to Y$ be a $U^{\dagger}$-cartesian edge of $\mathcal{E}^{\dagger}$, and let us identify $U^{\dagger}(f)$ with an edge $e : D \to C$ of $\mathcal{C}$. Then $T(f, e_R) : T(X, \mathrm{id}_C) \to T(Y, e)$ is an isomorphism in the ∞-category $\mathcal{E}_C$.

Unwinding the definitions, we observe that for each vertex $C \in \mathcal{C}$, the functor Ψ_C factors as a composition

$$\mathcal{E}^{\dagger}_C \xrightarrow{T_C} \mathcal{E}_C \hookrightarrow \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{E}_C) \simeq \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}}} \{C\},$$

where the second map is the equivalence of Proposition 8.1.7.6. We can therefore rewrite (b_2) as follows:

(b_2') For each vertex $C \in \mathcal{C}$, the morphism T restricts to an equivalence of ∞-categories $T_C : \mathcal{E}^{\dagger}_C \to \mathcal{E}_C$.

The equivalence of (a) and (b) now follows from Proposition 8.6.1.13. □

04F5 **Example 8.6.3.12.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories. Applying Construction 8.6.3.9 to the evaluation functor

$$\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E} \qquad (C, f_C, u : C \to C') \mapsto f_C(u),$$

we we obtain a comparison functor

$$\Psi : \mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathrm{Cospan}^{\dagger}(\mathcal{E} \,/\, \mathcal{C}).$$

It follows from Propositions 8.6.3.11 and 8.6.2.3 that this functor is an equivalence of ∞-categories.

Proof of Proposition 8.6.3.5. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Using Corollary 8.6.2.4, we can choose a cartesian fibration $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ and a morphism

$T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ which exhibits $U^{\dagger}$ as a cartesian conjugate of U. Applying Proposition 8.6.3.11, we see that Construction 8.6.3.9 supplies a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \arrow[rr, "\Psi"] \arrow[dr, "U^{\dagger}"'] & & \mathrm{Cospan}^{\dagger}(\mathcal{E} / \mathcal{C}) \arrow[dl, "V"] \\
& \mathcal{C}^{\mathrm{op}}, &
\end{tikzcd}$$

where the horizontal map is an equivalence of cartesian fibrations over $\mathcal{C}^{\mathrm{op}}$. It follows that V is also a cartesian conjugate of U. □

Warning 8.6.3.13. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Proposition 8.6.3.5 guarantees the existence of a morphism 04F6

$$T : \mathrm{Cospan}^{\dagger}(\mathcal{E} / \mathcal{C}) \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$$

which exhibits the projection map $\mathrm{Cospan}^{\dagger}(\mathcal{E} / \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ as a cartesian conjugate of U. Beware that the construction of T requires making some auxiliary choices. For example, if $\mathcal{C}$ is an ∞-category, then we can construct the datum T by choosing a homotopy inverse to the equivalence $\mathrm{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \mathcal{E}) \to \mathrm{Cospan}^{\dagger}(\mathcal{E} / \mathcal{C})$ of Example 8.6.3.12.

Corollary 8.6.3.14 (Uniqueness)**.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then U admits a cartesian conjugate, which is uniquely determined up equivalence.* 04F4

Proof. Combining Propositions 8.6.3.11 and 8.6.3.5, we see that a cartesian fibration $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ is conjugate to U if and only if it is equivalent to the projection map $\mathrm{Cospan}^{\dagger}(\mathcal{E} / \mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$. □

We close this section with a simple application of Proposition 8.6.3.5.

Construction 8.6.3.15. Let e denote the nondegenerate edge of Δ^1, viewed as an object of the ∞-category $\mathrm{Tw}(\Delta^1)$. For every simplicial set $\mathcal{C}$, we let 05JL

$$\Xi : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Cospan}(\mathrm{Tw}(\mathcal{C}))$$

denote the morphism of simplicial sets which corresponds, under the bijection of Proposition 8.1.3.7, to the composite map

$$\begin{aligned}
\mathrm{Tw}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \quad & \simeq \quad \{e\} \times \mathrm{Tw}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \\
& \hookrightarrow \quad \mathrm{Tw}(\Delta^1) \times \mathrm{Tw}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \\
& \simeq \quad \mathrm{Tw}(\Delta^1 \times \mathrm{Fun}(\Delta^1, \mathcal{C})) \\
& \xrightarrow{\mathrm{Tw}(\mathrm{ev})} \quad \mathrm{Tw}(\mathcal{C}).
\end{aligned}$$

05JM **Remark 8.6.3.16.** Let $\mathcal{C}$ be a simplicial set and let

$$\lambda_- : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \qquad \lambda_+ : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}$$

be the projection maps of Notation 8.1.1.6. Then the morphism Ξ of Construction 8.6.3.15 fits into a commutative diagram

05JN
$$\begin{tikzcd}
\mathcal{C} \arrow[d, "\rho_-"] & \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[l, "\mathrm{ev}_0"'] \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\Xi"] & \mathcal{C} \arrow[d, "\rho_+"] \\
\mathrm{Cospan}(\mathcal{C}^{\mathrm{op}}) & \mathrm{Cospan}(\mathrm{Tw}(\mathcal{C})) \arrow[l, "\mathrm{Cospan}(\lambda_-)"'] \arrow[r, "\mathrm{Cospan}(\lambda_+)"] & \mathrm{Cospan}(\mathcal{C}),
\end{tikzcd} \tag{8.64}$$

where ρ_+ and ρ_- are the embeddings of Construction 8.1.7.1 and Variant 8.1.7.14. Moreover, the composition of Ξ with the diagonal map $\mathcal{C} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$ is the unit map $\mathcal{C} \to \mathrm{Cospan}(\mathrm{Tw}(\mathcal{C}))$ of Construction 8.1.3.5.

In the situation of Remark 8.6.3.16, suppose that the simplicial set $\mathcal{C}$ is an ∞-category. Then the projection map $\lambda_- : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$ is a cocartesian fibration of ∞-categories, and a morphism u of $\mathrm{Tw}(\mathcal{C})$ is λ_--cocartesian if and only if $\lambda_+(u)$ is an isomorphism in $\mathcal{C}$ (Corollary 8.1.1.14). Using the commutativity of the diagram (8.64), we see that the morphism Ξ factors through the simplicial subset $\mathrm{Cospan}^{\dagger}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}) \subseteq \mathrm{Cospan}(\mathrm{Tw}(\mathcal{C}))$ of Notation 8.6.3.1. In this case, we have the following:

05JP **Proposition 8.6.3.17.** *Let $\mathcal{C}$ be an ∞-category. Then the morphism $\Xi : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Cospan}^{\dagger}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}})$ is an equivalence of ∞-categories.*

Proof. Let $V : \mathrm{Cospan}^{\dagger}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}) \to \mathcal{C}$ be the projection map. It follows from Lemma 8.6.3.8 (and Corollary 8.1.1.14) that V is a cartesian fibration, and that a morphism in $\mathrm{Cospan}^{\dagger}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}})$ is V-cartesian if and only if its image under the composite map

$$\mathrm{Cospan}^{\dagger}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}) \hookrightarrow \mathrm{Cospan}(\mathrm{Tw}(\mathcal{C})) \xrightarrow{\mathrm{Cospan}(\lambda_+)} \mathrm{Cospan}(\mathcal{C})$$

is contained in $\mathrm{Cospan}^{\mathrm{iso,all}}(\mathcal{C})$. Recall that the evaluation functor $\mathrm{ev}_0 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is also a cartesian fibration, and that a morphism f of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ is ev_0-cartesian if and only if $\mathrm{ev}_1(f)$ is an isomorphism in $\mathcal{C}$ (Corollary 5.3.7.3). Invoking Remark 8.6.3.16, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[rr, "\Xi"] \arrow[dr, "\mathrm{ev}_0"'] & & \mathrm{Cospan}^{\dagger}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}) \arrow[dl, "V"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where Ξ carries ev_0-cartesian morphisms of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$ to V-cartesian morphisms of the ∞-category $\mathrm{Cospan}^{\dagger}(\mathrm{Tw}(\mathcal{C}), \mathcal{C}^{\mathrm{op}})$. By virtue of Proposition 5.1.7.15, it will suffice to show that for each object $X \in \mathcal{C}$, the functor Ξ restricts to an equivalence of fibers

$$\Xi_X : \{X\} \tilde{\times}_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \{X\} \times_{\mathcal{C}} \mathrm{Cospan}^{\dagger}(\mathrm{Tw}(\mathcal{C})/\,\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Cospan}^{\mathrm{all,iso}}(\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})).$$

We conclude by observing that Ξ_X fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{X/} \arrow[rr] \arrow[d] & & \{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[d, "\rho_+"] \\
\{X\} \otimes_{\mathcal{C}} \mathrm{Fun}(\Delta^1, \mathcal{C}) \arrow[r, "\Xi_X"] & \mathrm{Cospan}^{\mathrm{all,iso}}(\{X\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})) &
\end{tikzcd}$$

where the left vertical map is the equivalence of Corollary 4.6.4.18, the right vertical map is the equivalence of Proposition 8.1.7.6, and the upper horizontal map is the equivalence of Proposition 8.1.2.9. □

Corollary 8.6.3.18. *Let $\mathcal{C}$ be an ∞-category. Then the evaluation map* $\mathrm{ev}_0 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ *is a cartesian conjugate of the cocartesian fibration $\lambda_- : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$.* 05JQ

Proof. Combine Propositions 8.6.3.17 and 8.6.3.5. □

8.6.4 Dual Fibrations

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Corollary 8.6.3.14 asserts that U admits a cartesian conjugate $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$, which is uniquely determined up to equivalence. Setting $\mathcal{E}^{\vee} = \mathcal{E}^{\dagger,\mathrm{op}}$ and $U^{\vee} = U^{\dagger,\mathrm{op}}$, we obtain another cocartesian fibration $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$. Our goal in this section is to give a direct characterization of the relationship between U and $U^{\vee}$, which does not rely on the theory of conjugate fibrations developed in §8.6.1 and §8.6.2. To fix ideas, let us begin by considering the case $\mathcal{C} = \Delta^0$. In this case, the conjugate fibration $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ is characterized by the requirement that the ∞-category $\mathcal{E}^{\dagger}$ is equivalent to $\mathcal{E}$. Consequently, the cocartesian fibration $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ is characterized by the requirement that $\mathcal{E}^{\vee}$ is equivalent to the opposite ∞-category $\mathcal{E}^{\mathrm{op}}$. By virtue of Corollary 8.2.6.6, this is equivalent to the existence of a *balanced coupling* $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times \mathcal{E}$: that is, a left fibration which satisfies the following addition conditions: 04D0

- For every object $X \in \mathcal{E}$, there exists an object $\widetilde{X} \in \widetilde{\mathcal{E}}$ satisfying $\lambda_+(\widetilde{X}) = X$ which is *universal*: that is, it is an initial object of the ∞-category $\widetilde{\mathcal{E}} \times_{\mathcal{E}} \{X\}$.
- For every object $X^{\vee} \in \mathcal{E}^{\vee}$, there exists an object $\widetilde{X} \in \widetilde{\mathcal{E}}$ satisfying $\lambda_-(\widetilde{X}) = X^{\vee}$ which is *couniversal*: that is, it is an initial object of the ∞-category $\{X^{\vee}\} \times_{\mathcal{E}^{\vee}} \widetilde{\mathcal{E}}$.

- An object of $\widetilde{\mathcal{E}}$ is universal if and only if it is couniversal.

We now extend the notion of balanced coupling to the relative setting.

04D1 **Definition 8.6.4.1.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets, and let $\lambda = (\lambda_{-}, \lambda_{+}) : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ be a left fibration of simplicial sets. We will say that λ *exhibits* $U^{\vee}$ *as a cocartesian dual of* U if the following conditions are satisfied:

(a) For every vertex $C \in \mathcal{C}$, the left fibration

$$\lambda_C : \widetilde{\mathcal{E}}_C \to \mathcal{E}^{\vee}_C \times \mathcal{E}_C$$

is a balanced coupling of ∞-categories.

(b) Let $\widetilde{U} : \widetilde{\mathcal{E}} \to \mathcal{C}$ denote the projection map $U^{\vee} \circ \lambda_{-} = U \circ \lambda_{+}$, $f : \widetilde{X} \to \widetilde{X}'$ be a $\widetilde{U}$-cocartesian edge of $\widetilde{\mathcal{E}}$, and let $e : C \to C'$ be its image $\widetilde{U}(f)$ in the simplicial set $\mathcal{C}$. If the object $\widetilde{X} \in \widetilde{\mathcal{E}}_C$ is universal for the coupling λ_C, then the object $\widetilde{X}' \in \widetilde{\mathcal{E}}_{C'}$ is universal for the coupling $\lambda_{C'}$.

We say that $U^{\vee}$ *is a cocartesian dual of* U if there exists a left fibration $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ which exhibits $U^{\vee}$ as a cocartesian dual of U.

04D2 **Example 8.6.4.2.** Let $\mathcal{E}$ and $\mathcal{E}^{\vee}$ be ∞-categories, and let $U : \mathcal{E} \to \Delta^0$ and $U^{\vee} : \mathcal{E}^{\vee} \to \Delta^0$ denote the projection maps. Then a left fibration $\widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\Delta^0} \mathcal{E} = \mathcal{E}^{\vee} \times \mathcal{E}$ exhibits $U^{\vee}$ as a cocartesian dual of U if and only if it is a balanced coupling. In particular, $U^{\vee}$ is a cocartesian dual of U if and only if $\mathcal{E}^{\vee}$ is equivalent to the opposite ∞-category $\mathcal{E}^{\mathrm{op}}$ (Corollary 8.2.6.6).

04D3 **Remark 8.6.4.3** (Symmetry)**.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. Then a left fibration $\lambda = \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ exhibits $U^{\vee}$ as a cocartesian dual of U if and only if it exhibits U as a cocartesian dual of $U^{\vee}$, after identifying $\mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ with $\mathcal{E} \times_{\mathcal{C}} \mathcal{E}^{\vee}$. In particular, $U^{\vee}$ is a cocartesian dual of U if and only if U is a cocartesian dual of $U^{\vee}$.

04D4 **Remark 8.6.4.4** (Base Change)**.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets and $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ be a left fibration. The following conditions are equivalent:

(a) The left fibration λ exhibits $U^{\vee}$ as a cocartesian dual of U (in the sense of Definition 8.6.4.1).

(*b*) For every morphism of simplicial sets $\mathcal{C}_0 \to \mathcal{C}$, form a diagram of pullback squares

$$\begin{tikzcd}
\mathcal{E}_0^{\vee} \arrow[r, "U_0^{\vee}"] \arrow[d] & \mathcal{C}' \arrow[d] & \mathcal{E}_0 \arrow[l, "U_0"'] \arrow[d] \\
\mathcal{E} \arrow[r, "U^{\vee}"] & \mathcal{C} & \mathcal{E} . \arrow[l, "U"']
\end{tikzcd}$$

Then the induced map

$$\lambda_0 : (\mathcal{C}_0 \times_{\mathcal{C}} \widetilde{\mathcal{E}}) \to \mathcal{E}_0^{\vee} \times_{\mathcal{C}_0} \mathcal{E}_0$$

exhibits $U_0^{\vee}$ as a cocartesian dual of U_0.

Moreover, it suffices to verify condition (*b*) in the special case where $\mathcal{C}_0 = \Delta^1$ is the standard 1-simplex.

Remark 8.6.4.5. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, and let $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be a cocartesian dual of U. Then, for every morphism of simplicial sets $\mathcal{C}_0 \to \mathcal{C}$, the projection map $U_0^{\vee} : \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}^{\vee} \to \mathcal{C}_0$ is a cocartesian dual of the projection map $U_0 : \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E} \to \mathcal{C}_0$. In particular, for every object $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C^{\vee} = \{C\} \times_{\mathcal{C}} \mathcal{E}^{\vee}$ is equivalent to the opposite of the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ (Example 8.6.4.2). 04D5

Remark 8.6.4.6. Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$. In §8.6.6, we will show that $U^{\vee}$ is a cocartesian dual of U (in the sense of Definition 8.6.4.1) if and only if the opposite fibration $U^{\vee,\mathrm{op}} : \mathcal{E}^{\vee,\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a cartesian conjugate of U (in the sense of Definition 8.6.1.1). See Proposition 8.6.6.1. 04D6

Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets, and let $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ be a left fibration. Condition (*a*) of Definition 8.6.4.1 guarantees that, for each vertex $C \in \mathcal{C}$, the coupling $\lambda_C : \widetilde{\mathcal{E}}_C \to \mathcal{E}_C^{\vee} \times_{\mathcal{C}} \mathcal{E}_C$ is representable by an equivalence of ∞-categories $G_C : \mathcal{E}_C \to (\mathcal{E}_C^{\vee})^{\mathrm{op}}$ (Theorem 8.2.6.5). Heuristically, one can think of condition (*b*) as requiring that the equivalence G_C depends functorially on C. We can articulate this heuristic more precisely as follows:

Proposition 8.6.4.7. *Let $U : \mathcal{E} \to \Delta^1$ and $U^{\vee} : \mathcal{E}^{\vee} \to \Delta^1$ be cocartesian fibrations of ∞-categories, and let $F : \mathcal{E}_0 \to \mathcal{E}_1$ and $F^{\vee} : \mathcal{E}_0^{\vee} \to \mathcal{E}_1^{\vee}$ be functors given by covariant transport along the nondegenerate edge of Δ^1. Let $\lambda = (\lambda_-, \lambda_+) : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\Delta^1} \mathcal{E}$ be a left fibration, and suppose that the associated couplings* 04D7

$$\lambda_0 : \widetilde{\mathcal{E}}_0 \to \mathcal{E}_0^{\vee} \times \mathcal{E}_0 \qquad \lambda_1 : \widetilde{\mathcal{E}}_1 \to \mathcal{E}_1^{\vee} \times \mathcal{E}_1$$

are representable by functors $G_0 : \mathcal{E}_0 \to (\mathcal{E}_0^\vee)^{\mathrm{op}}$ *and* $G_1 : \mathcal{E}_1 \to (\mathcal{E}_1^\vee)^{\mathrm{op}}$*, respectively. If* λ *satisfies condition* (b) *of Definition 8.6.4.1, then the diagram of* ∞*-categories*

04D8
$$\begin{tikzcd} \mathcal{E}_0 \arrow[r, "F"] \arrow[d, "G_0"] & \mathcal{E}_1 \arrow[d, "G_1"] \\ (\mathcal{E}_0^\vee)^{\mathrm{op}} \arrow[r, "(F^\vee)^{\mathrm{op}}"] & (\mathcal{E}_1^\vee)^{\mathrm{op}} \end{tikzcd} \tag{8.65}$$

commutes up to isomorphism.

Proof. Let $\widetilde{U}$ denote the composite map

$$\widetilde{\mathcal{E}} \xrightarrow{\lambda} \mathcal{E}^\vee \times_{\Delta^1} \mathcal{E} \to \Delta^1.$$

Using Proposition 5.1.4.13, we see that λ is a cocartesian fibration, and that an edge e of $\widetilde{\mathcal{E}}$ is $\widetilde{U}$-cocartesian if and only if $\lambda_+(e)$ is a U-cocartesian edge of $\mathcal{E}$ and $\lambda_-(e)$ is a $U^\vee$-cocartesian edge of $\mathcal{E}^\vee$. Let $\widetilde{F} : \widetilde{\mathcal{E}}_0 \to \widetilde{\mathcal{E}}_1$ be given by covariant transport along the nondegenerate edge of Δ^1. Using Remark 5.2.8.5, we see that the diagram of ∞-categories

04D9
$$\begin{tikzcd} \widetilde{\mathcal{E}}_0 \arrow[r, "\widetilde{F}"] \arrow[d, "\lambda_0"] & \widetilde{\mathcal{E}}_1 \arrow[d, "\lambda_1"] \\ \mathcal{E}_0^\vee \times \mathcal{E}_0 \arrow[r, "F^\vee \times F"] & \mathcal{E}_1^\vee \times \mathcal{E}_1 \end{tikzcd} \tag{8.66}$$

commutes up to isomorphism. Since λ_1 is an isofibration, we can replace $\widetilde{F}$ by an isomorphic functor to arrange that the diagram (8.66) is strictly commutative (see Corollary 4.4.5.6). Condition (b) of Definition 8.6.4.1 guarantees that the functor $\widetilde{F}$ carries universal objects of $\widetilde{\mathcal{E}}_0$ (for the coupling λ_0) to universal objects of $\widetilde{\mathcal{E}}_1$ (for the coupling λ_1). The commutativity of the diagram (8.65) now follows from Corollary 8.2.4.4. □

04DA **Corollary 8.6.4.8.** *Let* $U : \mathcal{E} \to \mathcal{C}$ *and* $U^\vee : \mathcal{E}^\vee \to \mathcal{C}$ *be cocartesian fibrations of simplicial sets having homotopy transport representations*

$$\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}, \mathrm{hTr}_{\mathcal{E}^\vee/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat},$$

and let $\mathrm{hTr}^{\mathrm{op}}_{\mathcal{E}^\vee/\mathcal{C}}$ *denote the functor* $C \mapsto \mathrm{hTr}_{\mathcal{E}^\vee/\mathcal{C}}(C)^{\mathrm{op}} = (\mathcal{E}^\vee_C)^{\mathrm{op}}$*. Let* $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E}^\vee \times_{\mathcal{C}} \mathcal{E}$ *be a left fibration such that, for each vertex* $C \in \mathcal{C}$*, the coupling* $\lambda_C : \widetilde{\mathcal{E}}_C \to \mathcal{E}^\vee_C \times \mathcal{E}_C$ *is representable by a functor* $G_C : \mathcal{E}_C \to (\mathcal{E}^\vee_C)^{\mathrm{op}}$*. if* λ *satisfies condition* (b) *of Definition 8.6.4.1, then the construction* $C \mapsto [G_C]$ *determines a natural transformation of functors* $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} \to \mathrm{hTr}^{\mathrm{op}}_{\mathcal{E}^\vee/\mathcal{C}}$.

Corollary 8.6.4.9. *Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets having homotopy transport representations* 04DB

$$\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}, \mathrm{hTr}_{\mathcal{E}^{\vee}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}.$$

Let $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ be a left fibration which exhibits $U^{\vee}$ as a cocartesian dual of U. Then λ induces an isomorphism of functors $\mathrm{hTr}_{\mathcal{E}/\mathcal{C}} \xrightarrow{\sim} \mathrm{hTr}^{\mathrm{op}}_{\mathcal{E}^{\vee}/\mathcal{C}}$, which carries each vertex $C \in \mathcal{C}$ to (the isomorphism class of) a functor which represents the balanced coupling $\lambda_C : \widetilde{\mathcal{E}}_C \to \mathcal{E}^{\vee}_C \times \mathcal{E}_C$.

Proof. Combine Corollary 8.6.4.8 with Theorem 8.2.6.5. □

Corollary 8.6.4.10. *Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets having homotopy transport representations* 04DC

$$\mathrm{hTr}_{\mathcal{E}/\mathcal{C}}, \mathrm{hTr}_{\mathcal{E}^{\vee}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat}.$$

If $U^{\vee}$ is a cocartesian dual of U, then $\mathrm{hTr}_{\mathcal{E}^{\vee}/\mathcal{C}}$ is isomorphic to the functor

$$\mathrm{hTr}^{\mathrm{op}}_{\mathcal{E}/\mathcal{C}} : \mathrm{h}\mathcal{C} \to \mathrm{hQCat} \qquad C \mapsto \mathcal{E}^{\mathrm{op}}_C .$$

Remark 8.6.4.11. In §8.6.7, we will prove a stronger version of Corollary 8.6.4.10, which gives a reformulation of cocartesian duality in the language of transport representations (see Proposition 8.6.7.12). 04DD

For some applications, it will be convenient to work with a reformulation of Definition 8.6.4.1.

Definition 8.6.4.12. Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. We say that a morphism of simplicial sets $\mathscr{K} : \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}$ *exhibits $U^{\vee}$ as a cocartesian dual of U* if the following conditions are satisfied: 04DE

(a) For each vertex $C \in \mathcal{C}$, the induced map $\mathscr{K}_C : \mathcal{E}^{\vee}_C \times \mathcal{E}_C \to \mathcal{S}$ is a balanced profunctor (see Definition 8.3.2.18).

(b) Let $f : X \to Y$ be a U-cocartesian edge of $\mathcal{E}$ and let $f^{\vee} : X^{\vee} \to Y^{\vee}$ be a $U^{\vee}$-cocartesian edge of $\mathcal{E}^{\vee}$ having the same image $e : C \to D$ in $\mathcal{C}$. Then the map of Kan complexes

$$\mathscr{K}(f^{\vee}, f) : \mathscr{K}_C(X^{\vee}, X) \to \mathscr{K}_D(Y^{\vee}, Y)$$

carries universal vertices of $\mathscr{K}_C(X^{\vee}, X))$ to universal vertices of $\mathscr{K}_D(Y^{\vee}, Y)$ (see Definition 8.3.2.7).

04DF **Variant 8.6.4.13.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. In the formulation of Definition 8.6.4.12, we have implicitly assumed that for each vertex $C \in \mathcal{C}$, the ∞-categories $\mathcal{E}_C$ and $\mathcal{C}^{\vee}_C$ are locally small (if this condition is not satisfied, then a balanced profunctor $\mathscr{K}_C : \mathcal{E}^{\vee}_C \times \mathcal{E}_C \to \mathcal{S}$ cannot exist). However, we will sometimes apply the theory of cocartesian duality in situations where this condition is not satisfied. If κ is an uncountable cardinal (not necessarily small), we will say that a morphism $\mathscr{K} : \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ *exhibits* $U^{\vee}$ *as a cocartesian dual of* U if it satisfies conditions (a) and (b) of Definition 8.6.4.12. In this case, we can take κ to be any uncountable cardinal having the property that, for each vertex $C \in \mathcal{C}$, the ∞-categories $\mathcal{E}_C$ and $\mathcal{E}^{\vee}_C$ are locally κ-small.

04DG **Remark 8.6.4.14.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets, let $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}$ be a left fibration, and let $\mathscr{K} : \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ be a covariant transport representation for λ. Then λ exhibits $U^{\vee}$ as a cocartesian dual of U (in the sense of Definition 8.6.4.1) if and only if $\mathscr{K}$ exhibits $U^{\vee}$ as a cocartesian dual of U (in the sense of Variant 8.6.4.13). See Remarks 8.3.2.19 and 8.3.2.8.

Combining Remark 8.6.4.14 with the classification of left fibrations (Corollary 5.6.0.6), we obtain the following:

04DH **Proposition 8.6.4.15.** *Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of ∞-categories. Let κ be an uncountable cardinal with the property that, for each vertex $C \in \mathcal{C}$, the ∞-categories $\mathcal{E}_C$ and $\mathcal{E}^{\vee}_C$ are locally κ-small. Then $U^{\vee}$ is a cocartesian dual of U if and only if there exists a morphism $\mathscr{K} : \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ which exhibits $U^{\vee}$ as a cocartesian dual of U, in the sense of Definition 8.6.4.12.*

We now give some examples of cocartesian duality.

04DJ **Proposition 8.6.4.16.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a left fibration of simplicial sets, and set $\widetilde{\mathcal{E}} = \mathcal{C} \times_{\operatorname{Fun}(\Delta^1, \mathcal{E})} \operatorname{Fun}(\Delta^1, \mathcal{E})$. Then the evaluation maps $\mathrm{ev}_0, \mathrm{ev}_1 : \widetilde{\mathcal{E}} \to \mathcal{E}$ determine a left fibration $\lambda : \widetilde{\mathcal{E}} \to \mathcal{E} \times_{\mathcal{C}} \mathcal{E}$ which exhibits U as a cocartesian dual of itself.*

Proof. The morphism λ is a pullback of the restriction map

$$\operatorname{Fun}(\Delta^1, \mathcal{E}) \to \operatorname{Fun}(\partial \Delta^1, \mathcal{E}) \times_{\operatorname{Fun}(\partial \Delta^1, \mathcal{C})} \operatorname{Fun}(\Delta^1, \mathcal{C}),$$

and is therefore a left fibration by virtue of Proposition 4.2.5.1. For every vertex $C \in \mathcal{C}$, we can identify λ_C with the coupling

$$\operatorname{Fun}(\Delta^1, \mathcal{E}_C) \to \operatorname{Fun}(\{0\}, \mathcal{E}_C) \times \operatorname{Fun}(\{1\}, \mathcal{E}_C).$$

It follows from Example 8.2.6.3 that each λ_C is a balanced coupling, so that λ satisfies condition (a) of Definition 8.6.4.1. Moreover, every object of $\operatorname{Fun}(\Delta^1, \mathcal{E}_C)$ is universal for the coupling λ_C, so that condition (b) of Definition 8.6.4.1 is vacuous. □

Corollary 8.6.4.17. *Let $U : \mathcal{E} \to \mathcal{C}$ be a left fibration of simplicial sets. Then U is a cocartesian dual of itself.* 04DK

Example 8.6.4.18. In the special case $\mathcal{C} = \Delta^0$, Corollary 8.6.4.17 asserts that every Kan complex $X = \mathcal{E}$ is homotopy equivalent to the opposite Kan complex X^{op}. This can also be deduced from Theorem 3.6.0.1, since the geometric realizations $|X|$ and $|X^{\mathrm{op}}|$ are homeomorphic. 04DL

Proposition 8.6.4.19. *Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories, and let $\mathscr{F}' : \mathcal{C} \to \mathbf{Cat}$ be the functor given on objects by $C \mapsto \mathscr{F}(C)^{\mathrm{op}}$. Then (the nerves of) the fibrations* 04DM

$$\int_{\mathcal{C}} \mathscr{F} \to \mathcal{C} \qquad \int_{\mathcal{C}} \mathscr{F}' \to \mathcal{C}$$

are cocartesian dual to one another.

We will deduce Proposition 8.6.4.19 from a more precise result. To formulate it, we need to introduce a bit of notation.

Construction 8.6.4.20. Let $\mathcal{C}$ be a category, let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories, and let $\int_{\mathcal{C}} \mathscr{F}$ denote the category of elements of $\mathscr{F}$ (Definition 5.6.1.1); we identify objects of $\int_{\mathcal{C}} \mathscr{F}$ with pairs (C, X), where C is an object of $\mathcal{B}$ and X is an object of the category $\mathscr{F}(C)$. Let $\mathscr{F}' : \mathcal{C} \to \mathbf{Cat}$ denote the functor given on objects by $\mathscr{F}'(C) = \mathscr{F}(C)^{\mathrm{op}}$. We define a functor 04DN

$$\mathscr{K} : \int_{\mathcal{C}} \mathscr{F}' \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F} \to \mathrm{Set}$$

as follows:

- On objects, $\mathscr{K}$ is given by the formula $\mathscr{K}((C, X'), (C, X)) = \mathrm{Hom}_{\mathscr{F}(C)}(X', X)$.
- Let $f : (C, X) \to (D, Y)$ be a morphism in the category $\int_{\mathcal{C}} \mathscr{F}$ and let $f' : (C, X') \to (D, Y')$ be a morphism in the category $\int_{\mathcal{C}} \mathscr{F}'$ having the same image $u : C \to D$ in $\mathcal{B}$. Let us identify f and f' with morphisms $g : \mathscr{F}(u)(X) \to Y$ and $g' : Y' \to \mathscr{F}(u)(Y)$ in the category $\mathscr{F}(D)$. Then the function $\mathscr{K}(f', f) : \mathscr{K}((C, X'), (C, X)) \to \mathscr{K}((D, Y'), (D, Y))$ is given by the composition

$$\mathrm{Hom}_{\mathscr{F}(C)}(X', X) \xrightarrow{\mathscr{F}(u)} \mathrm{Hom}_{\mathscr{F}(D)}(\mathscr{F}(u)(X'), \mathscr{F}(u)(X)) \xrightarrow{g \circ \bullet \circ g'} \mathrm{Hom}_{\mathscr{F}(D)}(Y', Y).$$

Proposition 8.6.4.21. *Let $\mathcal{C}$ be a category and let $\mathscr{F} : \mathcal{C} \to \mathbf{Cat}$ be a functor of 2-categories. Then the functor* 04DP

$$\mathrm{N}_{\bullet}(\mathscr{K}) : \mathrm{N}_{\bullet}(\int_{\mathcal{C}} \mathscr{F}') \times_{\mathrm{N}_{\bullet}(\mathcal{C})} \mathrm{N}_{\bullet}(\int_{\mathcal{C}} \mathscr{F}) \to \mathrm{N}_{\bullet}(\mathrm{Set}) \subset \mathcal{S}$$

of Construction 8.6.4.20 exhibits the projection map $U' : \mathrm{N}_{\bullet}(\int_{\mathcal{C}} \mathscr{F}') \to \mathrm{N}_{\bullet}(\mathcal{C})$ as a cocartesian dual of the projection map $U : \mathrm{N}_{\bullet}(\int_{\mathcal{C}} \mathscr{F}) \to \mathrm{N}_{\bullet}(\mathcal{C})$.

Proof. For each object $C \in \mathcal{C}$, the restriction of $\mathscr{K}$ to the fiber over C is given concretely by the functor

$$\mathscr{K}_C : \mathscr{F}(C)^{\mathrm{op}} \times \mathscr{F}(C) \to \mathrm{Set} \qquad (X', X) \mapsto \mathrm{Hom}_{\mathscr{F}(C)}(X', X).$$

Example 8.3.3.4 implies that $\mathrm{N}_\bullet(\mathscr{K}_C)$ is a Hom-functor for the ∞-category $\mathrm{N}_\bullet(\mathscr{F}(C))$ and is therefore a balanced profunctor (Proposition 8.3.3.8). Let $u : C \to D$ be a morphism in the category $\mathcal{C}$, and let $f : (C, X) \to (D, Y)$ and $f' : (C, X') \to (D, Y')$ be lifts of u to the categories $\int_{\mathcal{C}} \mathscr{F}$ and $\int_{\mathcal{C}} \mathscr{F}'$, respectively. We wish to show that, if f is U-cocartesian and f' is U'-cocartesian, then the induced map

$$\mathscr{K}(f, f') : \mathscr{K}_C(X', X) \to \mathscr{K}_D(Y', Y)$$

carries universal elements of $\mathscr{K}_C(X', X)$ to universal elements of $\mathscr{K}_D(Y', Y)$. Let us identify f and f' with morphisms $g : \mathscr{F}(u)(X) \to Y$ and $g' : Y' \to \mathscr{F}(u)(Y)$ in the category $\mathscr{F}(D)$, so that $\mathscr{K}(f', f)$ is given by the composition

$$\mathrm{Hom}_{\mathscr{F}(C)}(X', X) \xrightarrow{\mathscr{F}(u)} \mathrm{Hom}_{\mathscr{F}(D)}(\mathscr{F}(u)(X'), \mathscr{F}(u)(X)) \xrightarrow{g \circ - \circ g'} \mathrm{Hom}_{\mathscr{F}(D)}(Y', Y).$$

Our assumption that f is U-cocartesian guarantees that g is an isomorphism in the category $\mathscr{F}(D)$, and our assumption that f' is a U'-cocartesian guarantees that g' is an isomorphism in the category $\mathscr{F}(D)$. The desired result now follows from the observation that if $e : X' \to X$ is an isomorphism in the category $\mathscr{F}(C)$, then the composition $g \circ \mathscr{F}(u)(e) \circ g'$ is an isomorphism in the category $\mathscr{F}(D)$. □

Proof of Proposition 8.6.4.19. Combine Propositions 8.6.4.15 and 8.6.4.21. □

Let QCat be the (ordinary) category of ∞-categories, which we regard as a full subcategory of Set_Δ. If $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ is a functor of ordinary categories, we let $\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$ denote the weighted nerve of Definition 5.3.3.1. According to Corollary 5.3.3.16, the projection map $U : \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ is a cocartesian fibration, whose fiber over an object $C \in \mathcal{C}$ can be identified with the ∞-category $\mathscr{F}(C)$. In this situation, it is easy to construct a cocartesian dual of U:

04DQ **Proposition 8.6.4.22.** *Let $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ be a functor of ordinary categories, and let $\mathscr{F}' : \mathcal{C} \to \mathrm{QCat}$ denote the functor given on objects by $C \mapsto \mathscr{F}(C)^{\mathrm{op}}$. Then the fibrations*

$$\mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C}) \leftarrow \mathrm{N}_\bullet^{\mathscr{F}'}(\mathcal{C})$$

are cocartesian dual to one another.

Proposition 8.6.4.22 is an immediate consequence of the following more precise result:

Proposition 8.6.4.23. *Let* $\mathscr{F} : \mathcal{C} \to \mathrm{QCat}$ *be a functor of ordinary categories and let* $\mathscr{F}', \mathrm{Tw}(\mathscr{F}) : \mathcal{C} \to \mathrm{QCat}$ *be the functors given on objects by the formulae* $\mathscr{F}'(C) = \mathscr{F}(C)^{\mathrm{op}}$ *and* $\mathrm{Tw}(\mathscr{F})(C) = \mathrm{Tw}(\mathscr{F}(C))$*. Then the tautological map* 04DR

$$\lambda = (\lambda_-, \lambda_+) : \mathrm{N}_\bullet^{\mathrm{Tw}(\mathscr{F})}(\mathcal{C}) \to \mathrm{N}_\bullet^{\mathscr{F}^{\mathrm{op}}}(\mathcal{C}) \times_{\mathrm{N}_\bullet(\mathcal{C})} \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C})$$

exhibits the fibration $U^\vee : \mathrm{N}_\bullet^{\mathscr{F}'}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ *as a cocartesian dual of the fibration* $U : \mathrm{N}_\bullet^{\mathscr{F}}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$*.*

Proof. For each object $C \in \mathcal{C}$, Proposition 8.1.1.11 guarantees that the morphism

$$\lambda_C : \mathrm{Tw}(\mathscr{F}(C)) \to \mathscr{F}(C)^{\mathrm{op}} \times \mathscr{F}(C)$$

is a left fibration of ∞-categories, which is a balanced coupling by virtue of Example 8.2.6.2. Applying Corollary 5.3.3.18, we deduce that λ is a left fibration of ∞-categories. Let $U : \mathrm{N}_\bullet^{\mathrm{Tw}(\mathscr{F})}(\mathcal{C}) \to \mathrm{N}_\bullet(\mathcal{C})$ denote the projection map, and let $f : X \to Y$ be a morphism in the ∞-category $\mathrm{N}_\bullet^{\mathrm{Tw}(\mathscr{F})}(\mathcal{C})$ having image $u : C \to D$ in $\mathcal{C}$. To complete the proof, it will suffice to show that if X is universal for the coupling λ_C and f is U-cocartesian, then Y is universal for the coupling λ_D. Our assumption that f is U-cocartesian guarantees that Y is isomorphic to the image of X under the functor $\mathrm{Tw}(\mathscr{F}(u)) : \mathrm{Tw}(\mathscr{F}(C)) \to \mathrm{Tw}(\mathscr{F}(D))$. The desired result now follows from Example 8.2.1.5, since the functor $\mathscr{F}(u)$ carries isomorphisms in the ∞-category $\mathscr{F}(C)$ to isomorphisms in the ∞-category $\mathscr{F}(D)$. □

8.6.5 Existence of Dual Fibrations

The goal of this section is to prove the following: 04DS

Theorem 8.6.5.1. *Let* $U : \mathcal{E} \to \mathcal{C}$ *be a cocartesian fibration of simplicial sets. Then* U *admits a cocartesian dual* $U^\vee : \mathcal{E}^\vee \to \mathcal{C}$*, which is uniquely determined up to equivalence.* 04DT

We will give the proof of Theorem 8.6.5.1 at the end of this section.

Corollary 8.6.5.2. *For every simplicial set* $\mathcal{C}$*, the formation of cocartesian duals induces a bijection* 04DU

$$\begin{tikzcd}
\{\text{Cocartesian fibrations } U : \mathcal{E} \to \mathcal{C}\}/\text{Equivalence} \arrow[d, "\theta"] \\
\{\text{Cocartesian fibrations } U^\vee : \mathcal{E}^\vee \to \mathcal{C}\}/\text{Equivalence.}
\end{tikzcd}$$

Proof. Theorem 8.6.5.1 implies that θ is well-defined, and Remark 8.6.4.3 implies that $\theta \circ \theta$ is the identity; in particular, θ is a bijection. □

04DV **Variant 8.6.5.3** (Cartesian Duality)**.** Let $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ be cartesian fibrations of simplicial sets. We say that U' *is a cartesian dual of* U if the cocartesian fibration $U'^{\mathrm{op}} : \mathcal{E}'^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a cocartesian dual of $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. It follows from Theorem 8.6.5.1 that every cartesian fibration $U : \mathcal{E} \to \mathcal{C}$ admits a cartesian dual $U' : \mathcal{E} \to \mathcal{C}$ which is uniquely determined up to equivalence. Moreover, Corollary 8.6.4.10 implies that the (contravariant) homotopy transport representation of U' is given by the composition

$$\mathrm{h}\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{hTr}_{\mathcal{E}\,/\,\mathcal{C}}} \mathrm{hQCat} \xrightarrow{\mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}} \mathrm{hQCat}.$$

In particular, for every vertex $C \in \mathcal{C}$, the fiber $\mathcal{E}'_C\{C\} \times_{\mathcal{C}} \mathcal{E}'$ is equivalent to the opposite of the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$.

04DW **Warning 8.6.5.4.** Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets which is both a cartesian fibration and a cocartesian fibration. Then U admits a both a cocartesian dual $U' : \mathcal{E}' \to \mathcal{C}$ and a cartesian dual $U'' : \mathcal{E}'' \to \mathcal{C}$. For every vertex $C \in \mathcal{C}$, there are equivalences of ∞-categories $\mathcal{E}'_C \simeq \mathcal{E}_C^{\mathrm{op}} \simeq \mathcal{E}''_C$. Beware that the fibrations U' and U'' are generally not equivalent to one another (see Example 8.6.5.5).

04DX **Example 8.6.5.5.** Let $U : \mathcal{E} \to \Delta^1$ be a cocartesian fibration of ∞-categories. By virtue of Remark 5.2.4.3, the cocartesian fibration U can be recovered (up to equivalence) from its homotopy transport representation, which we can identify with the functor $F : \mathcal{E}_0 \to \mathcal{E}_1$ given by covariant transport along the nondegenerate edge of Δ^1. The fibration U then a cocartesian dual $U' : \mathcal{E}' \to \Delta^1$, whose covariant transport functor can be identified with the composition

$$\mathcal{E}'_0 \simeq \mathcal{E}_0^{\mathrm{op}} \xrightarrow{F^{\mathrm{op}}} \mathcal{E}_1^{\mathrm{op}} \simeq \mathcal{E}'_1$$

(Corollary 8.6.4.10). Applying Proposition 6.2.3.5, we deduce the following:

(a) The cocartesian fibration U is a cartesian fibration if and only if the functor $F : \mathcal{E}_0 \to \mathcal{E}_1$ admits a right adjoint.

(b) The cocartesian fibration U' is a cartesian fibration if and only if the functor $F^{\mathrm{op}} : \mathcal{E}_0^{\mathrm{op}} \to \mathcal{E}_1^{\mathrm{op}}$ admits a right adjoint: that is, if and only if the functor F admits a left adjoint.

Note that conditions (a) and (b) are not equivalent. If (a) is satisfied and (b) is not, then U admits a cartesian dual $U'' : \mathcal{E}'' \to \Delta^1$ which cannot be equivalent to U' (since U'' is a cartesian fibration and U' is not).

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Theorem 8.6.5.1 implies that U admits a cocartesian dual $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$. To prove this, it will be convenient to use the formulation of cocartesian duality supplied by Definition 8.6.4.12. In the special case where $\mathcal{C} = \Delta^0$ and $\mathcal{E}$ is locally small, we wish to show that there exists an ∞-category $\mathcal{E}^{\vee}$ and a balanced profunctor $\mathscr{K} : \mathcal{E}^{\vee} \times \mathcal{E} \to \mathcal{S}$. This is a special case of Corollary 8.3.2.21: in fact, we can take $\mathcal{E}^{\vee}$ to be the ∞-category $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E}, \mathcal{S})$ of corepresentable functors from $\mathcal{E}$ to $\mathcal{S}$, and $\mathscr{K}$ to be the evaluation functor

$$\mathrm{ev} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E}, \mathcal{S}) \times \mathcal{E} \to \mathcal{S} \qquad (\mathscr{F}, X) \mapsto \mathscr{F}(X).$$

To handle the general case we will use a variant of this construction, defined using the relative exponential introduced in §4.5.9.

Construction 8.6.5.6. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let κ be an uncountable cardinal, and let let $\mathcal{S}^{<\kappa}$ denote the ∞-category of essentially κ-small spaces. We let $\mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ denote the relative exponential of Construction 4.5.9.1. By construction, we can identify vertices of $\mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ with pairs $(C, \mathscr{F}_C)$, where C is a vertex of $\mathcal{C}$ and $\mathscr{F}_C : \mathcal{E}_C \to \mathcal{S}^{<\kappa}$ is a functor of ∞-categories. We let $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ denote the full simplicial subset of $\mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ spanned by those vertices $(C, \mathscr{F}_C)$ where the functor $\mathscr{F}_C$ is corepresentable by an object of the ∞-category $\mathcal{E}_C$. In what follows, we will generally write $\pi : \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ for the projection map, and $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ for the restriction of π to the simplicial subset $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \subseteq \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$. 04DY

Remark 8.6.5.7. Construction 8.6.5.6 is independent of the choice of the cardinal κ, provided that each of the ∞-categories $\mathcal{E}_C$ is locally κ-small. If this condition is satisfied and $\lambda \geq \kappa$, then every corepresentable functor $\mathscr{F} : \mathcal{E}_C \to \mathcal{S}^{<\lambda}$ factors through $\mathcal{S}^{<\kappa}$. It follows that $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) = \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\lambda})$. 04DZ

The existence assertion of Theorem 8.6.5.1 is a consequence of the following more precise result:

Proposition 8.6.5.8. *Let κ be an uncountable cardinal and let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets which is locally κ-small (Variant 4.7.9.2). Then the evaluation map* 04E0

$$\mathrm{ev} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa} \qquad ((C, \mathscr{F}_C), X) \mapsto \mathscr{F}_C(X)$$

exhibits the projection map $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ as a cocartesian dual of U (in the sense of Variant 8.6.4.13).

Our first goal is to show that, in the situation of Proposition 8.6.5.8, the projection map π^{corep} is a cocartesian fibration of simplicial sets. We begin with some more general remarks.

Proposition 8.6.5.9. *Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets which is essentially κ-small, and let $\mathcal{D}$ be an ∞-category which is κ-cocomplete. Then the projection map $\pi : \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{D}) \to \mathcal{C}$ is a cocartesian fibration of simplicial sets.* 04E1

Proof. It follows from Corollary 5.3.6.8 that π is a cartesian fibration of simplicial sets. Let $e : C \to C'$ be an edge of the simplicial set $\mathcal{C}$, and let $e_! : \mathcal{E}_C \to \mathcal{E}_{C'}$ be the functor given by covariant transport along e (for the cocartesian fibration U). Then precomposition with $e_!$ determines a functor

$$e^* : \{C'\} \times_{\mathcal{C}} \operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D}) = \operatorname{Fun}(\mathcal{E}_{C'}, \mathcal{D}) \xrightarrow{\circ e_!} \operatorname{Fun}(\mathcal{E}_C, \mathcal{D}) = \{C\} \times_{\mathcal{C}} \operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D}).$$

Proposition 5.3.6.9 guarantees that the functor e^* is given by contravariant transport along e (for the cartesian fibration π). Using Proposition 6.2.3.5, we see that π is a cocartesian fibration if and only if the functor e^* has a left adjoint (for every edge e of $\mathcal{C}$). By virtue of Corollary 7.3.6.3, it will suffice to show that every functor $F : \mathcal{E}_C \to \mathcal{D}$ admits a left Kan extension along the functor $e_! : \mathcal{E}_C \to \mathcal{E}_{C'}$. This is a special case of Proposition 7.6.7.13, by virtue of our assumptions on the cardinal κ. □

04E2 **Remark 8.6.5.10.** In the situation of Proposition 8.6.5.9, let $e : C \to C'$ be an edge of $\mathcal{C}$ and let

$$e_! : \mathcal{E}_C \to \mathcal{E}_{C'} \qquad e'_! : \operatorname{Fun}(\mathcal{E}_C, \mathcal{D}) \to \operatorname{Fun}(\mathcal{E}_{C'}, \mathcal{D})$$

be functors given by covariant transport along e (for the cocartesian fibrations U and π, respectively). Then the functor $e'_!$ is given by left Kan extension along $e_!$.

04E3 **Variant 8.6.5.11.** Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \to \mathcal{C}$ be an exponentiable inner fibration which is essentially κ-small, and let $\mathcal{D}$ be an ∞-category which is κ-cocomplete. Then the projection map $\pi : \operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D}) \to \mathcal{C}$ is a cocartesian fibration. Moreover, an edge $\widetilde{e}$ of $\operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D})$ is π-cocartesian if and only if satisfies the following condition:

$(*)$ Write $\widetilde{e} = (e, F_e)$, where e is an edge of $\mathcal{C}$ and $F_e : \Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \mathcal{D}$ is a functor of ∞-categories. Then F_e is left Kan extended from the full subcategory $\{0\} \times_{\mathcal{C}} \mathcal{E}$.

Proof. Corollary 4.5.9.19 guarantees that π is an isofibration, and Corollary 7.3.7.9 guarantees that every edge of $\operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D})$ which satisfies condition $(*)$ is π-cocartesian. Suppose we are given a vertex $\widetilde{C} = (C, F_C)$ of $\operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D})$, where C is a vertex of $\mathcal{C}$ and $F_C : \mathcal{E}_C \to \mathcal{D}$ is a functor of ∞-categories. If $e : C \to C'$ is an edge of $\mathcal{C}$, then Proposition 7.6.7.13 guarantees that F_C admits a left Kan extension $F_e : \Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \mathcal{D}$, which we can identify with an edge $\widetilde{e}$ of $\operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D})$ satisfying $\pi(\widetilde{e}) = e$. By construction, the morphism $\widetilde{e}$ satisfies condition $(*)$, and is therefore π-cocartesian by virtue of Corollary 7.3.7.9. Allowing $\widetilde{C}$ and e to vary, we conclude that π is a cocartesian fibration. To complete the proof, it will suffice to show that every π-cocartesian edge $\widetilde{e}'$ of $\operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D})$ satisfies condition $(*)$. Let us identify $\widetilde{e}'$ with a pair (e, F'_e), where e is an edge of $\mathcal{B}$ and $F'_e : \Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \mathcal{D}$ is a functor. Using the preceding argument, we see that the restriction $F'_e|_{\{0\} \times_{\mathcal{C}} \mathcal{E}}$ admits a left Kan extension $F_e : \Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \mathcal{D}$, corresponding to another edge $\widetilde{e}$ of $\operatorname{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{D})$. By construction, $\widetilde{e}$

satisfies condition $(*)$ and is therefore π-cocartesian. Invoking the uniqueness of cocartesian lifts (Remark 5.1.3.8), we deduce that the functors F_e and F'_e are isomorphic. It follows that F'_e is also left Kan extended from $\{0\} \times_{\mathcal{C}} \mathcal{E}$ (Remark 7.3.3.17), so that $\widetilde{e}'$ satisfies condition $(*)$ as desired. □

Proposition 8.6.5.12. *Let κ be an uncountable regular cardinal and let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets which is essentially κ-small. Then:* 04E4

(1) *The projection map $\pi : \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ is both a cartesian fibration and a cocartesian fibration.*

(2) *Let $\widetilde{e}$ be a π-cocartesian edge of the simplicial set $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$. If the source of $\widetilde{e}$ belongs to the simplicial subset $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$, then the target of $\widetilde{e}$ also belongs to the simplicial subset $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$.*

(3) *The morphism π restricts to a cocartesian fibration $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$. Moreover, an edge of $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ is π^{corep}-cocartesian if and only if it is π-cocartesian.*

Proof. Assertion (1) follows from Corollary 5.3.6.8 and Proposition 8.6.5.9 (since the ∞-category $\mathcal{S}^{<\kappa}$ admits κ-small colimits; see Remark 7.4.5.7). We will prove (2). Let $\widetilde{e} : (C, \mathscr{F}_C) \to (C', \mathscr{F}_{C'})$ be an edge of the simplicial set $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ having image $e : C \to C'$ in $\mathcal{C}$. Let $e_! : \mathcal{E}_C \to \mathcal{E}_{C'}$ be given by covariant transport along e for the cocartesian fibration U. If $\widetilde{e}$ is π-cocartesian, then we can identify $\mathscr{F}_{C'}$ with a left Kan extension of $\mathscr{F}_C$ along the functor $e_!$ (Remark 8.6.5.10). In particular, if the functor $\mathscr{F}_C : \mathcal{E}_C \to \mathcal{S}$ is corepresentable by an object $X \in \mathcal{E}_C$, then $\mathscr{F}_{C'}$ is corepresentable by the image $e_!(X) \in \mathcal{C}_{C'}$ (Corollary 8.3.1.9). This proves assertion (2), and assertion (3) is a formal consequence (see Proposition 5.1.4.16). □

Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. To show that the projection map $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ is a cocartesian fibration (at least for $\kappa \gg 0$), we used the fact that the collection of corepresentable functors is closed under the formation of left Kan extensions. To prove Proposition 8.6.5.8, we will need to characterize the collection of π^{corep}-cocartesian edges of the simplicial set $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ more explicitly.

Lemma 8.6.5.13. *Let κ be an uncountable cardinal, let $\mathcal{E}$ be an ∞-category which is locally κ-small, and let $\mathscr{F} : \mathcal{E} \to \mathcal{S}^{<\kappa}$ be a functor. Suppose we are given a full subcategory $\mathcal{E}_0 \subseteq \mathcal{E}$, an object $X \in \mathcal{E}_0$, and a vertex $\eta \in \mathscr{F}(X)$. Then η exhibits the $\mathscr{F}$ as corepresented by the object X if and only if the following conditions are satisfied:* 04E5

(a) *The vertex η exhibits $\mathscr{F}_0 = \mathscr{F}|_{\mathcal{E}_0}$ as corepresented by the object X.*

(b) *The functor $\mathscr{F}$ is left Kan extended from $\mathcal{E}_0$.*

Proof. It follows immediately from the definition that if η exhibits $\mathscr{F}$ as corepresented by X, then it also exhibits $\mathscr{F}_0$ as corepresented by X. We may therefore assume without loss of generality that condition (a) is satisfied. In this case, the desired equivalence follows immediately by combining the criterion of Lemma 8.3.1.7 with the transitivity property of Kan extensions (Proposition 7.3.8.18). □

04E6 **Lemma 8.6.5.14.** *Let κ be an uncountable cardinal, let $\mathcal{E}$ be an ∞-category which is locally κ-small, and let $\mathscr{F} : \mathcal{E} \to \mathcal{S}^{<\kappa}$ be a functor. Suppose we are given a functor $U : \mathcal{E} \to \Delta^1$ and a U-cocartesian morphism $e : X \to Y$ of $\mathcal{E}$ satisfying $U(X) = 0$ and $U(Y) = 1$. Write $\mathcal{E}_0 = \{0\} \times_{\Delta^1} \mathcal{E}$ and $\mathcal{E}_1 = \{1\} \times_{\Delta^1} \mathcal{E}$, and let $\eta \in \mathscr{F}(X)$ be a vertex which exhibits the functor $\mathscr{F}_0 = \mathscr{F}|_{\mathcal{E}_0}$ as corepresented by the object X. The following conditions are equivalent:*

(1) *The functor $\mathscr{F}$ is left Kan extended from $\mathcal{E}_0$.*

(2) *The vertex η exhibits the functor $\mathscr{F}$ as corepresented by X.*

(3) *The vertex $\mathscr{F}(e)(\eta) \in \mathscr{F}(Y)$ exhibits the functor $\mathscr{F}_1 = \mathscr{F}|_{\mathcal{E}_1}$ as corepresented by the object $Y \in \mathcal{E}_1$.*

Proof. The equivalence of (1) and (2) follows from Lemma 8.6.5.13. We will show that (2) and (3) are equivalent. Fix an object $Z \in \mathcal{E}_1$. Then the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}_1}(Y, Z) \arrow[rr, "\circ[e]"] \arrow[dr] & & \mathrm{Hom}_{\mathcal{E}}(X, Z) \arrow[dl] \\
& \mathscr{F}(Z) &
\end{tikzcd}$$

commutes up to homotopy, where the right vertical map is determined by $\eta\mathscr{F}(X)$ and the left vertical map is determined by $\mathscr{F}(e)(\eta) \in \mathscr{F}(Y)$. Our assumption that e is U-cocartesian guarantees that the horizontal map is a homotopy equivalence (Corollary 5.1.2.3). It follows that the left vertical map is a homotopy equivalence if and only if the right vertical map is a homotopy equivalence. The desired result now follows by allowing the object Z to vary. □

04E7 **Lemma 8.6.5.15.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let κ be an uncountable cardinal such that each fiber of U is locally κ-small. Let $\widetilde{e}$ be an edge of the simplicial set $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ corresponding to a pair $(e, \mathscr{F})$, where $e : C \to D$ is an edge of $\mathcal{C}$ and $\mathscr{F} : \Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ is a functor. The following conditions are equivalent:*

(1) *The edge $\widetilde{e}$ is π^{corep}-cocartesian (where $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ denotes the projection map).*

(2) *There exists an object $X \in \mathcal{E}_C$, a vertex $\eta \in \mathscr{F}(X)$ which exhibits $\mathscr{F}|_{\mathcal{E}_C}$ as corepresented by the object X, and a U-cocartesian morphism $\overline{e} : X \to Y$ such that $U(\overline{e}) = e$ and the vertex $\mathscr{F}(\overline{e})(\eta) \in \mathscr{F}(Y)$ exhibits $\mathscr{F}|_{\mathcal{E}_D}$ as corepresented by the object Y.*

(3) *For every object $X \in \mathcal{E}_C$, every vertex $\eta \in \mathscr{F}(X)$ which exhibits $\mathscr{F}|_{\mathcal{E}_C}$ as corepresented by the object X, and every U-cocartesian morphism $\overline{e} : X \to Y$ satisfying $U(\overline{e}) = e$, the vertex $\mathscr{F}(\overline{e})(\eta) \in \mathscr{F}(Y)$ exhibits $\mathscr{F}|_{\mathcal{E}_D}$ as corepresented by the object Y.*

Proof. By virtue of Remark 8.6.5.7, we are free to enlarge the cardinal κ; we may therefore assume without loss of generality that κ is regular and that the ∞-category $\Delta^1 \times_{\mathcal{C}} \mathcal{E}$ is essentially κ-small. In this case, Variant 8.6.5.11 shows that (1) is equivalent to the following:

(1′) The functor $\mathscr{F}$ is left Kan extended from the full subcategory $\mathcal{E}_C \subseteq \Delta^1 \times_{\mathcal{C}} \mathcal{E}$.

The equivalences (1′) ⇔ (2) ⇔ (3) now follow from Lemma 8.6.5.14. □

Proof of Proposition 8.6.5.8. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, and let κ be an uncountable cardinal such that each fiber of U is locally κ-small. It follows from Proposition 8.6.5.12 that the projection map $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ is a cocartesian fibration of simplicial sets. We wish to show that the evaluation map

$$\mathrm{ev} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa} \qquad ((C, \mathscr{F}_C), X) \mapsto \mathscr{F}_C(X)$$

satisfies conditions (a) and (b) of Definition 8.6.4.12. Condition (a) asserts that, for each vertex $C \in \mathcal{C}$, the evaluation map

$$\mathrm{ev}_C : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E}_C, \mathcal{S}^{<\kappa}) \times \mathcal{E}_C \to \mathcal{S}^{<\kappa} \qquad (\mathscr{F}, X) \mapsto \mathscr{F}(X)$$

is a balanced profunctor; this follows from Corollary 8.3.2.21. Assertion (b) is a restatement of the implication (1) ⇒ (3) of Lemma 8.6.5.15. □

Proposition 8.6.5.8 immediately implies the existence assertion of Theorem 8.6.5.1. To establish uniqueness, it will be convenient to introduce some terminology.

Definition 8.6.5.16. Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets and let κ be an uncountable cardinal. We will say that a morphism $\mathscr{K} : \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ is a *weak $\mathcal{C}$-family of corepresentable profunctors* if, for every vertex $C \in \mathcal{C}$, the induced map 04E8

$$\mathscr{K}_C : \mathcal{E}^{\vee}_C \times \mathcal{E}_C \to \mathcal{S}^{<\kappa}$$

is a corepresentable profunctor (Definition 8.3.2.9). We say that $\mathscr{K}$ is a *$\mathcal{C}$-family of corepresentable profunctors* if it is a weak $\mathcal{C}$-family of corepresentable profunctors and satisfies the following additional condition:

$(*)$ Let $f : X \to Y$ be a U-cocartesian edge of $\mathcal{E}$ and let $f^\vee : X^\vee \to Y^\vee$ be a $U^\vee$-cocartesian edge of $\mathcal{E}^\vee$ having the same image $u : C \to D$ in $\mathcal{C}$. Then the map of Kan complexes

$$\mathscr{K}(f^\vee, f) : \mathscr{K}_C(X^\vee, X) \to \mathscr{K}_D(Y^\vee, Y)$$

carries couniversal vertices of $\mathscr{K}_C(X^\vee, X)$ to couniversal vertices of $\mathscr{K}_D(Y^\vee, Y)$.

04E9 **Example 8.6.5.17.** In the situation of Definition 8.6.5.16, the morphism $\mathscr{K}$ exhibits $U^\vee$ as a cocartesian dual of U (in the sense of Variant 8.6.4.13) if and only if it is a $\mathcal{C}$-family of corepresentable profunctors having the further property that each of the profunctors $\mathscr{K}_C$ is *balanced*: that is, it is corepresentable by an equivalence of ∞-categories $(\mathcal{E}_C^\vee)^{\mathrm{op}} \to \mathcal{E}_C$ (see Corollary 8.3.2.20).

Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets, let κ be an uncountable cardinal, and let $\pi : \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ denote the projection map. For any morphism of simplicial sets $U^\vee : \mathcal{E}^\vee \to \mathcal{C}$, we can identify morphisms $\mathscr{K} : \mathcal{E}^\vee \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ with morphisms $F : \mathcal{E}^\vee \to \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ satisfying $\pi \circ F = U$.

04EA **Proposition 8.6.5.18.** *Let $U : \mathcal{E} \to \mathcal{C}$ and $U^\vee : \mathcal{E}^\vee \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. Let κ be an uncountable cardinal such that U is locally κ-small. Fix a morphism $\mathscr{K} : \mathcal{E}^\vee \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$, which we identify with a morphism $F : \mathcal{E}^\vee \to \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$. Then:*

(1) *The morphism $\mathscr{K}$ is a weak $\mathcal{C}$-family of corepresentable profunctors if and only if F factors through the simplicial subset $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \subseteq \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$.*

(2) *The morphism $\mathscr{K}$ is a $\mathcal{C}$-family of corepresentable profunctors if and only if F factors through $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ and carries $U^\vee$-cocartesian edges of $\mathcal{E}^\vee$ to π^{corep}-cocartesian edges of $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$. Here $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ denotes the cocartesian fibration of Proposition 8.6.5.12.*

(3) *The morphism $\mathscr{K}$ exhibits $U^\vee$ as a cocartesian dual of U if and only if $F : \mathcal{E}^\vee \to \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ is an equivalence of cocartesian fibrations over $\mathcal{C}$.*

Proof. Assertion (1) is immediate from the definitions and assertion (2) follows from Lemma 8.6.5.15. Assertion (3) follows by combining (2) with Example 8.6.5.17 (see Proposition 5.1.7.15). □

Proof of Theorem 8.6.5.1. Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Fix an uncountable cardinal κ such that U is locally κ-small. Proposition 8.6.5.8 implies that the projection map $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ is a cocartesian dual of U, and Proposition 8.6.5.18 implies that any other cocartesian dual $U^\vee : \mathcal{E}^\vee \to \mathcal{C}$ is equivalent to π^{corep}. □

Using Proposition 8.6.5.18, we can characterize the dual of a fibration by a universal mapping property.

Corollary 8.6.5.19. *Let $U : \mathcal{E} \to \mathcal{C}$, $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$, and $V : \mathcal{D} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets, let κ be an uncountable cardinal, and let $\mathscr{K} : \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ exhibit $U^{\vee}$ as a cocartesian dual of U. Then:* 04EB

(1) *Composition with $\mathscr{K}$ induces a fully faithful functor*

$$\mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E}^{\vee}) \to \mathrm{Fun}(\mathcal{D} \times_{\mathcal{C}} \mathcal{E}^{\vee}, \mathcal{S}^{<\kappa}).$$

The essential image is spanned by the weak $\mathcal{C}$-families of corepresentable profunctors.

(2) *A morphism $F \in \mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E}^{\vee})$ carries V-cocartesian edges of $\mathcal{D}$ to $U^{\vee}$-cocartesian edges of $\mathcal{E}^{\vee}$ if and only if the composite map*

$$\mathcal{D} \times_{\mathcal{C}} \mathcal{E} \xrightarrow{F \times \mathrm{id}} \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \xrightarrow{\mathscr{K}} \mathcal{S}^{<\kappa}$$

is a $\mathcal{C}$-family of corepresentable profunctors.

(3) *A morphism $F \in \mathrm{Fun}_{/\mathcal{C}}(\mathcal{D}, \mathcal{E}^{\vee})$ is an equivalence of cocartesian fibrations over $\mathcal{C}$ if and only if the composite map*

$$\mathcal{D} \times_{\mathcal{C}} \mathcal{E} \xrightarrow{F \times \mathrm{id}} \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \xrightarrow{\mathscr{K}} \mathcal{S}^{<\kappa}$$

exhibits V as a cocartesian dual of U

Proof. We can identify $\mathscr{K}$ with a morphism $G \in \mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}^{\vee}, \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}))$. It follows from Proposition 8.6.5.18 that G is an equivalence of cocartesian fibrations over $\mathcal{C}$. We can therefore replace $\mathcal{E}^{\vee}$ by $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ and $\mathscr{K}$ by the evaluation map $\mathrm{ev} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$. In this case, assertions (1), (2), and (3) follow immediately from the corresponding assertions of Proposition 8.6.5.18. □

8.6.6 Comparison of Dual and Conjugate Fibrations

In this section, we show that the theory of conjugate fibrations (introduced in §8.6.1) can be regarded as a reformulation of cocartesian duality (introduced in §8.6.4). Our main result can be stated as follows: 04ET

Proposition 8.6.6.1. *Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. Then U is a cocartesian dual of $U^{\vee}$ if and only if the opposite fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a cartesian conjugate of $U^{\vee}$.* 04EU

Before giving the proof of Proposition 8.6.6.1, let us collect some consequences.

Corollary 8.6.6.2 (Symmetry)**.** *Let $\mathcal{C}$ be a simplicial set, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration, and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration. Then $U^{\dagger}$ is a cartesian conjugate of U if and only if U^{op} is a cartesian conjugate of $U^{\dagger,\mathrm{op}}$.* 04EV

Proof. Combine Proposition 8.6.6.1 with Remark 8.6.4.3. □

05JR **Corollary 8.6.6.3.** *For every simplicial set $\mathcal{C}$, the formation of cartesian conjugates induces a bijection*

$$\begin{tikzcd} \{\text{Cocartesian fibrations } U : \mathcal{E} \to \mathcal{C}\}/\text{Equivalence} \arrow[d] \\ \{\text{Cartesian fibrations } U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}\}/\text{Equivalence}. \end{tikzcd}$$

Proof. Combine Proposition 8.6.6.1 with Corollary 8.6.5.2. □

05JS **Example 8.6.6.4.** Let $\mathcal{C}$ be an ∞-category. Then the evaluation functor $\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ is a cocartesian fibration, which is cocartesian dual to the projection map $\lambda_+ : \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}$ of Notation 8.1.1.6. This follows by combining Proposition 8.6.6.1 with Corollary 8.6.3.18 (applied to the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$).

05JT **Corollary 8.6.6.5.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let L be the collection of U-cocartesian edges of $\mathcal{E}$, and let $\mathcal{E}^{\vee}$ be the fiber product $\mathcal{C} \times_{\mathrm{Cospan}(\mathcal{C})} \mathrm{Cospan}^{L,\mathrm{all}}(\mathcal{E})$. Then the projection map $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ is a cocartesian fibration, which is a cocartesian dual of U.*

Proof. Combine Propositions 8.6.6.1 and 8.6.3.5. □

04F7 **Remark 8.6.6.6.** Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, and let $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ be the opposite fibration. By virtue of Theorem 8.6.5.1 and Corollary 8.6.3.14, U admits a cocartesian dual $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ and a cartesian conjugate $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$, which are uniquely determined up to equivalence and opposite to one another (Proposition 8.6.6.1). When $\mathcal{C}$ is an ∞-category, all four of these fibrations can be realized as a suitable restriction of the projection map $\mathrm{Cospan}(U) : \mathrm{Cospan}(\mathcal{E}) \to \mathrm{Cospan}(\mathcal{C})$. Let L denote the collection of all U-cocartesian morphisms of $\mathcal{E}$, and let R denote the collection of all morphisms f of $\mathcal{E}$ such that $U(f)$ is an isomorphism in $\mathcal{C}$. Then:

- Using Proposition 8.1.7.6, we can identify U with the map
$$\mathrm{Cospan}^{\mathrm{all}, L \cap R}(\mathcal{E}) = \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C}).$$
- Using Variant 8.1.7.14, we can identify U^{op} with the map
$$\mathrm{Cospan}^{L \cap R, \mathrm{all}}(\mathcal{E}) = \mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{C}).$$
- Using Proposition 8.6.3.5 (and Remark 8.6.3.7), we can identify $U^{\dagger}$ with the projection map $\mathrm{Cospan}^{R,L}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{iso},\mathrm{all}}(\mathcal{C})$.

- Using Corollary 8.6.6.5 (and Remark 8.6.3.7), we can identify $U^{\vee}$ with the map $\mathrm{Cospan}^{L,R}(\mathcal{E}) \to \mathrm{Cospan}^{\mathrm{all},\mathrm{iso}}(\mathcal{C})$.

Corollary 8.6.6.7. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration of ∞-categories, and suppose we are given a commutative diagram* 04EW

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C} .
\end{tikzcd}$$

The following conditions are equivalent:

(1) *The functor T exhibits $U^{\dagger}$ as a cartesian conjugate of U (in the sense of Definition 8.6.1.1).*

(2) *The functor T exhibits $\mathcal{E}$ as a localization of $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ with respect to W, where W is the collection of all morphisms $w = (w', w'')$ where w' is a U'-cartesian morphism of $\mathcal{E}^{\dagger}$ and w'' is a morphism of $\mathrm{Tw}(\mathcal{C})$ whose image in $\mathcal{C}$ is degenerate.*

Proof. We will show that (2) implies (1); the reverse implication follows from Proposition 8.6.2.12. Using Corollary 8.6.6.3, we can choose a cocartesian fibration $U' : \mathcal{E}' \to \mathcal{C}$ and a commutative diagram

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T'"] \arrow[d] & \mathcal{E}' \arrow[d, "U'"] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

which exhibits $U^{\dagger}$ as a cartesian conjugate of U'. Assume that condition (2) is satisfied, so that we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{E}, \mathcal{E}') \arrow[r, "T\circ"] \arrow[d, "U\circ"] & \mathrm{Fun}((\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}))[W^{-1}], \mathcal{E}') \arrow[d, "U'\circ"] \\
\mathrm{Fun}(\mathcal{E}, \mathcal{C}) \arrow[r, "T\circ"] & \mathrm{Fun}((\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}))[W^{-1}], \mathcal{C}),
\end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories and the vertical maps are isofibrations (Corollary 4.4.5.6). Applying Corollary 4.5.2.32, we deduce that the map

$$(\circ T) : \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}') \to \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}), \mathcal{E}')$$

is fully faithful, and that its essential image consists of those functors $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \to \mathcal{E}'$ which carry each morphism of W to an isomorphism in $\mathcal{E}'$. We may therefore assume without loss of generality that $T' = F \circ T$ for some functor $F \in \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}')$. Proposition 8.6.2.12 implies that T' exhibits $\mathcal{E}'$ as a localization of $\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C})$ with respect to W. It follows that F is an equivalence of ∞-categories (Remark 6.3.1.19), so that T also exhibits $U^{\dagger}$ as a cartesian conjugate of U. □

We now turn to the proof of Proposition 8.6.6.1. Consider first the special case where $\mathcal{C} = \Delta^0$. For any ∞-category $\mathcal{E}$, the projection map $U : \mathcal{E} \to \Delta^0$ can be regarded as a cartesian conjugate of itself (Example 8.6.1.3). Consequently, Proposition 8.6.6.1 reduces to the assertion that U is dual to the opposite fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \Delta^0$: that is, that there exists a balanced profunctor $\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{S}$. This is proved by taking $\mathscr{H}$ to be a Hom-functor for $\mathcal{E}$, in the sense of Definition 8.3.3.1. To prove Proposition 8.6.6.1, we will need a relative variant of this construction.

05JU **Definition 8.6.6.8.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets, so that the induced map

$$\lambda : \operatorname{Tw}(\mathcal{E}) \to \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E}$$

is a left fibration (Proposition 8.1.1.15). Let κ be an uncountable cardinal. We say that a diagram

$$\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$$

is a *relative* Hom-*functor for U* if it is a covariant transport representation for the left fibration λ (Definition 5.6.5.1).

05JV **Remark 8.6.6.9.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets and let

$$\lambda : \operatorname{Tw}(\mathcal{E}) \to \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E}$$

be the left fibration of Proposition 8.1.1.15. If X and Y are vertices of $\mathcal{E}$ and $f : U(X) \to U(Y)$ is an edge of $\mathcal{C}$, then the fiber $\lambda^{-1}\{(X, f, Y)\}$ is a Kan complex, which is homotopy equivalent to the morphism space $\operatorname{Hom}_{\mathcal{E}_f}(X, Y)$, where we abuse notation by identifying X and Y with their preimages in the ∞-category $\mathcal{E}_f = \Delta^1 \times_{\mathcal{C}} \mathcal{E}$. Consequently, if $\mathscr{H}$ is a relative Hom-functor for U, then we have homotopy equivalences $\mathscr{H}(X, f, Y) \simeq \operatorname{Hom}_{\mathcal{E}_f}(X, Y)$.

Remark 8.6.6.10 (Existence and Uniqueness)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets and let κ be an uncountable cardinal. Then U admits a relative Hom-functor 05JW

$$\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$$

if and only if the inner fibration U is locally κ-small (see Proposition 4.7.9.5). Moreover, if this condition is satisfied, then $\mathscr{H}$ is unique up to isomorphism (Corollary 5.6.0.6).

Example 8.6.6.11. Let $\mathcal{E}$ be a locally κ-small ∞-category and let $\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{S}^{<\kappa}$ be a Hom-functor for $\mathcal{E}$. Suppose we are given a functor $U : \mathcal{E} \to \mathcal{C}$, where $\mathcal{C} = \mathrm{N}_{\bullet}(J)$ is the nerve of a partially ordered set J. Then U is automatically an inner fibration (Proposition 4.1.1.10). Moreover, the iterated fiber product $\mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E}$ can be identified with the full subcategory of $\mathcal{E}^{\mathrm{op}} \times \mathcal{E}$ spanned by those objects (X, Y) satisfying $U(X) \leq U(Y)$ (see Example 8.1.0.5). In this case, the restriction of $\mathscr{H}$ to this full subcategory is a relative Hom-functor for the inner fibration U, in the sense of Definition 8.6.6.8. 05JX

Remark 8.6.6.12. Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories and let 05JY

$$\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$$

be a relative Hom-functor for U. Then, for every vertex $C \in \mathcal{C}$, the restriction of $\mathscr{H}$ to

$$\mathcal{E}_C^{\mathrm{op}} \times \mathcal{E}_C \simeq \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \{\mathrm{id}_C\} \times_{\mathcal{C}} \mathcal{E}$$

is Hom-functor for the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$, in the sense of Definition 8.3.3.1.

Notation 8.6.6.13 (Relative Yoneda Embeddings)**.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets, let κ be an uncountable cardinal, and let 05JZ

$$\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$$

be a relative Hom-functor for U. Then $\mathscr{H}$ can be identified with a morphism

$$T : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}),$$

where $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ denotes the relative exponential of Construction 4.5.9.1. In this case, we will refer to T as the *relative Yoneda embedding* of the inner fibration U.

Remark 8.6.6.14. Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. Then the relative Yoneda embedding T of Notation 8.6.6.13 fits into a commutative diagram 05K0

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] & \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \arrow[d] \\
\mathrm{Tw}(\mathcal{C}) \arrow[r] & \mathcal{C} .
\end{tikzcd}$$

Consequently, for each vertex $C \in \mathcal{C}$, the restriction of T to $\mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \{\mathrm{id}_C\}$ can be regarded as a functor

$$T_C : \mathcal{E}_C^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{E}_C, \mathcal{S}^{<\kappa}),$$

which is a contravariant Yoneda embedding for the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$, in the sense of Definition 8.3.3.9. See Remark 8.6.6.12.

We now formulate a more precise version of Proposition 8.6.6.1.

05K1 **Theorem 8.6.6.15.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, let κ be an uncountable cardinal for which U is locally κ-small, and let*

$$T : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$$

be a relative Yoneda embedding for U (see Notation 8.6.6.13). Then:

(a) The morphism T factors through the simplicial subset $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ of Construction 8.6.5.6.

(b) The morphism T exhibits $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ as a cartesian conjugate of the cocartesian fibration $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$.

Proof. We first prove (a). Fix an edge $e : C_0 \to C_1$ of $\mathcal{C}$, let $\mathcal{E}_0$ denote the fiber $\{C_0\} \times_{\mathcal{C}} \mathcal{E}$, and define $\mathcal{E}_1$ similarly. Fix an object $Y \in \mathcal{E}_0$, so that we can identify the pair (Y, e) with a vertex of the fiber product $\mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$. Then $T(Y, e)$ can be regarded as a functor $F : \mathcal{E}_1 \to \mathcal{S}^{<\kappa}$, and we wish to show that F is corepresentable. Replacing U by the projection map $\Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \Delta^1$, we can reduce to the special case where $\mathcal{C} = \Delta^1$ is the standard 1-simplex (with $C_0 = 0$ and $C_1 = 1$). Our assumption that U is a cocartesian fibration guarantees that there exists a U-cocartesian morphism $f : Y \to X$ with $X \in \mathcal{E}_1$. Let $\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{S}^{<\kappa}$ be a Hom-functor for $\mathcal{E}$. Using Example 8.6.6.11, we can identify F with the functor

$$\mathcal{E}_1 \hookrightarrow \mathcal{E} \xrightarrow{\mathscr{H}(Y,-)} \mathcal{S}^{<\kappa},$$

so that f can be identified with a vertex of $\eta \in F(X)$. Since f is U-cocartesian, the vertex η exhibits F as corepresented by the object X (see Lemma 8.6.5.14).

We now prove (b). For each vertex $C \in \mathcal{C}$, the restriction of T to the fiber $\mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \{\mathrm{id}_C\}$ determines a functor $T_C : \mathcal{E}_C^{\mathrm{op}} \to \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E}_C, \mathcal{S}^{<\kappa})$ which is a contravariant Yoneda embedding for the ∞-category $\mathcal{E}_C$ (Remark 8.6.6.14), and is therefore an equivalence of ∞-categories (Theorem 8.3.3.13). We will complete the proof by showing that T satisfies conditions $(2')$ and $(2'')$ of Proposition 8.6.1.13. Both of these conditions make an assertion about an arbitrary edge $e : C_0 \to C_1$ of the simplicial set $\mathcal{C}$; let us denote these assertions by $(2'_e)$ and $(2''_e)$, respectively. To verify them, we can replace U by the projection map $\Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \Delta^1$ as above, and thereby reduce to the special case where $\mathcal{C} = \Delta^1$ is the standard 1-simplex and e is the nondegenerate edge of $\mathcal{C}$. Let $\mathscr{H} : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{S}^{<\kappa}$ be a Hom-functor for $\mathcal{E}$. Using Example 8.6.6.11 again and Variant 8.6.5.11, we can formulate the hypotheses of Proposition 8.6.1.13 more concretely as follows:

($2'_e$) For every object $Y \in \mathcal{E}_0$, the functor $\mathscr{H}(Y, -) : \mathcal{E} \to \mathcal{S}^{<\kappa}$ is left Kan extended from $\mathcal{E}_0$.

($2''_e$) For every object $Y \in \mathcal{E}_0$ and every U-cocartesian morphism $f : Y \to X$ with $X \in \mathcal{E}_1$, the induced natural transformation $\mathscr{H}(X, -) \to \mathscr{H}(Y, -)$ is an isomorphism when restricted to the subcategory $\mathcal{E}_1$. In other words, for every object $W \in \mathcal{E}_1$, precomposition with f induces a homotopy equivalence $\mathscr{H}(X, W) \to \mathscr{H}(Y, W)$.

Assertion ($2'_e$) now follows from Lemma 8.6.5.13, and assertion ($2''_e$) from Corollary 5.1.2.3. □

Proof of Proposition 8.6.6.1. Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets. Fix an uncountable cardinal κ such that U and $U^{\vee}$ are locally κ-small. If U is a cocartesian dual of $U^{\vee}$, then Proposition 8.6.5.18 guarantees that $U^{\vee}$ is equivalent to the cocartesian fibration $\pi^{\mathrm{corep}} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$. Combining this observation with Theorem 8.6.6.15, we conclude that U^{op} is a cartesian conjugate of $U^{\vee}$.

We now prove the converse. Assume that U^{op} is a cartesian conjugate of $U^{\vee}$; we wish to show that U is a cartesian dual of $U^{\vee}$. Applying Theorem 8.6.5.1, we see that $U^{\vee}$ admits a cocartesian dual $U' : \mathcal{E}' \to \mathcal{C}$. The first part of the proof shows that the opposite fibration $U'^{\mathrm{op}} : \mathcal{E}'^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a cartesian conjugate of $U^{\vee}$, and is therefore equivalent to the cartesian fibration U^{op} (Corollary 8.6.3.14). It follows that the cocartesian fibrations U and U' are also equivalent, so that U is also a cocartesian dual of $U^{\vee}$. □

8.6.7 The Opposition Functor

Recall that, for every ∞-category $\mathcal{C}$, the opposite simplicial set $\mathcal{C}^{\mathrm{op}}$ is also an ∞-category 04F8 (Proposition 1.4.2.6). Our goal in this section is to show that the construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ can be promoted to a functor of ∞-categories $\sigma : \mathcal{QC} \to \mathcal{QC}$, where $\mathcal{QC}$ denotes the ∞-category of (small) ∞-categories (Construction 5.5.4.1). Beware that this is not completely obvious from the definition. The ∞-category $\mathcal{QC}$ was obtained as the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat})$, where QCat denotes the simplicial category whose objects are ∞-categories and whose morphism spaces are given by the formula $\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$. The construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ determines an automorphism of QCat as an ordinary category. However, this automorphism is *not* compatible with the simplicial enrichment of QCat: for ∞-categories $\mathcal{C}$ and $\mathcal{D}$, the Kan complex $\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})_{\bullet} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})^{\simeq}$ identifies with the *opposite* of the Kan complex $\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$. To address this point, it is convenient to work with a slight variant of Construction 5.5.4.1.

04F9 **Notation 8.6.7.1.** Let $\mathcal{E}$ be a simplicial category. We define a new simplicial category $\mathcal{E}^{\asymp}$ as follows:

- The objects of $\mathcal{E}^{\asymp}$ are the objects of $\mathcal{E}$.
- For every pair of objects $X, Y \in \mathcal{E}$, the simplicial set $\operatorname{Hom}_{\mathcal{E}^{\asymp}}(X,Y)_\bullet$ is the twisted arrow construction $\operatorname{Tw}(\operatorname{Hom}_{\mathcal{E}}(X,Y)_\bullet)$.
- For every triple of objects $X, Y, Z \in \mathcal{E}$, the composition law
$$\circ : \operatorname{Hom}_{\mathcal{E}^{\asymp}}(Y,Z)_\bullet \times \operatorname{Hom}_{\mathcal{E}^{\asymp}}(X,Y)_\bullet \to \operatorname{Hom}_{\mathcal{E}^{\asymp}}(X,Z)_\bullet$$
is obtained by applying the twisted arrow functor Tw to the composition law for the simplicial category $\mathcal{E}$.

The simplicial category $\mathcal{E}^{\asymp}$ is equipped with a simplicial functor $\pi : \mathcal{E}^{\asymp} \to \mathcal{E}$, which carries each object to itself and is given on morphism spaces by the projection map

$$\operatorname{Hom}_{\mathcal{E}^{\asymp}}(X,Y)_\bullet = \operatorname{Tw}(\operatorname{Hom}_{\mathcal{E}}(X,Y)_\bullet) \to \operatorname{Hom}_{\mathcal{E}}(X,Y)_\bullet$$

described in Notation 8.1.1.6.

04FA **Proposition 8.6.7.2.** *Let $\mathcal{E}$ be a locally Kan simplicial category. Then:*

(1) *The simplicial category $\mathcal{E}^{\asymp}$ of Notation 8.6.7.1 is locally Kan.*

(2) *The forgetful functor $\pi : \mathcal{E}^{\asymp} \to \mathcal{E}$ is a weak equivalence of simplicial categories (see Definition 4.6.8.7).*

(3) *The functor π induces an equivalence of ∞-categories $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{E}^{\asymp}) \to \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{E})$.*

Proof. Assertions (1) and (2) follow immediately from Corollary 8.1.2.3; assertion (3) then follows from Corollary 4.6.8.8. □

04FB **Remark 8.6.7.3** (Comparison with the Conjugate)**.** Let $\mathcal{E}$ be a simplicial category. Recall that the *conjugate* $\mathcal{E}^{\mathrm{c}}$ is a simplicial category having the same objects, with morphism spaces given by $\operatorname{Hom}_{\mathcal{E}^{\mathrm{c}}}(X,Y)_\bullet = \operatorname{Hom}_{\mathcal{E}}(X,Y)^{\mathrm{op}}_\bullet$ (see Example 2.4.2.12). Then there is a canonical isomorphism of simplicial categories $\mathcal{E}^{\asymp} \xrightarrow{\sim} (\mathcal{E}^{\mathrm{c}})^{\asymp}$, which is the identity on objects and given on morphism spaces by the isomorphisms

$$\operatorname{Hom}_{\mathcal{E}^{\asymp}}(X,Y)_\bullet = \operatorname{Tw}(\operatorname{Hom}_{\mathcal{E}}(X,Y)_\bullet) \simeq \operatorname{Tw}(\operatorname{Hom}_{\mathcal{E}}(X,Y)^{\mathrm{op}}_\bullet) = \operatorname{Hom}_{(\mathcal{E}^{\mathrm{c}})^{\asymp}}(X,Y)_\bullet$$

described in Remark 8.1.1.7. Composing this isomorphism with the forgetful functor $(\mathcal{E}^{\mathrm{c}})^{\asymp} \to \mathcal{E}^{\mathrm{c}}$, we obtain a forgetful functor $\pi^{\mathrm{c}} : \mathcal{E}^{\asymp} \to \mathcal{E}^{\mathrm{c}}$. If $\mathcal{E}$ is locally Kan, then Proposition 8.6.7.2 guarantees that π^{c} is a weak equivalence of simplicial categories. We therefore obtain equivalences of ∞-categories

$$\mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{E}) \xleftarrow{\sim} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{E}^{\asymp}) \xrightarrow{\sim} \mathrm{N}^{\mathrm{hc}}_\bullet(\mathcal{E}^{c}).$$

We now specialize to the case of interest to us.

Construction 8.6.7.4. Let QCat denote the simplicial category whose objects are (small) ∞-categories, with morphisms spaces given by $\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}$ (see Construction 5.5.4.1). We let $\mathrm{QCat}^{\asymp}$ denote the simplicial category described in Notation 8.6.7.1, and we let $\mathcal{QC}^{\asymp}$ denote the homotopy coherent nerve $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\mathrm{QCat}^{\asymp})$. 04FC

Proposition 8.6.7.5. *The simplicial set $\mathcal{QC}^{\asymp}$ is an ∞-category. Moreover, the forgetful functor $\mathrm{QCat}^{\asymp} \to \mathrm{QCat}$ of Notation 8.6.7.1 induces an equivalence of ∞-categories $\pi : \mathcal{QC}^{\asymp} \to \mathcal{QC}$.* 04FD

Proof. Apply Proposition 8.6.7.2 to the locally Kan simplicial category $\mathcal{E} = \mathrm{QCat}$. □

Construction 8.6.7.6 (The Opposite Functor)**.** The simplicial category $\mathrm{QCat}^{\asymp}$ is equipped with an automorphism $\widetilde{\sigma}$, given on objects by the construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$ and on morphism spaces by the composition 04FE

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{QCat}^{\asymp}}(\mathcal{C}, \mathcal{D})_{\bullet} &= \mathrm{Tw}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}) \\
&\simeq \mathrm{Tw}((\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq})^{\mathrm{op}}) \\
&\simeq \mathrm{Tw}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})^{\simeq}) \\
&= \mathrm{Hom}_{\mathrm{QCat}^{\asymp}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}),
\end{aligned}$$

where the isomorphism on the second line is supplied by Remark 8.1.1.7. It follows from Proposition 8.6.7.5 that there exists a functor of ∞-categories $\sigma : \mathcal{QC} \to \mathcal{QC}$ for which the diagram

$$\begin{tikzcd}
\mathcal{QC}^{\asymp} \arrow[r, "\mathrm{N}^{\mathrm{hc}}_{\bullet}(\widetilde{\sigma})", "\sim"'] \arrow[d, "\pi"] & \mathcal{QC}^{\asymp} \arrow[d, "\pi"] \\
\mathcal{QC} \arrow[r, "\sigma"] & \mathcal{QC}
\end{tikzcd}$$

commutes up to isomorphism. Moreover, the functor σ is uniquely deterrmined up to isomorphism. We will refer to σ as the *opposition functor* for the ∞-category $\mathcal{QC}$.

The terminology of Construction 8.6.7.6 is justified by the following observation:

Proposition 8.6.7.7. *Let $\sigma : \mathcal{QC} \to \mathcal{QC}$ be the opposition functor of Construction 8.6.7.6. Then the diagram of ∞-categories* 04FF

$$\begin{tikzcd}
\mathrm{N}_{\bullet}(\mathrm{QCat}) \arrow[r, "\sigma_0"] \arrow[d] & \mathrm{N}_{\bullet}(\mathrm{QCat}) \arrow[d] \\
\mathcal{QC} \arrow[r, "\sigma"] & \mathcal{QC}
\end{tikzcd}$$

commutes up to isomorphism, where the functor σ_0 is given by the construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$.

Proof. Let QCat° denote the underlying category of the simplicial category QCat, and let us abuse notation by viewing QCat° as a *constant* simplicial category. We can then identify QCat° with a simplicial subcategory of QCat having the same objects, with morphism spaces given by

$$\mathrm{Hom}_{\mathrm{QCat}^{\circ}}(\mathcal{C}, \mathcal{D})_{\bullet} = \mathrm{sk}_0(\mathrm{Hom}_{\mathrm{QCat}}(\mathcal{C}, \mathcal{D})_{\bullet}).$$

Note that the inclusion map $\mathrm{QCat}^{\circ} \hookrightarrow \mathrm{QCat}$ factors as a composition

$$\mathrm{QCat}^{\circ} \xrightarrow{\iota} \mathrm{QCat}^{\asymp} \xrightarrow{\pi} \mathrm{QCat},$$

where the functor $\iota : \mathrm{QCat}^{\circ} \to \mathrm{QCat}^{\asymp}$ carries each ∞-category $\mathcal{C}$ to itself and each functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ to the vertex

$$\mathrm{id}_F \in \mathrm{Tw}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\simeq}) = \mathrm{Hom}_{\mathrm{QCat}^{\asymp}}(\mathcal{C}, \mathcal{D})_{\bullet}.$$

We then have a diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{N}_{\bullet}(\mathrm{QCat}) \arrow[r, "\sigma_0"] \arrow[d, "\mathrm{N}^{\mathrm{hc}}_{\bullet}(\iota)"] & \mathrm{N}_{\bullet}(\mathrm{QCat}) \arrow[d, "\mathrm{N}^{\mathrm{hc}}_{\bullet}(\iota)"] \\
\mathcal{QC}^{\asymp} \arrow[r, "\mathrm{N}^{\mathrm{hc}}_{\bullet}(\widetilde{\sigma})"] \arrow[d] & \mathcal{QC}^{\asymp} \arrow[d] \\
\mathcal{QC} \arrow[r, "\sigma"] & \mathcal{QC},
\end{tikzcd}$$

where the upper square is strictly commutative and the lower square commutes up to isomorphism. It follows that the outer rectangle also commutes up to isomorphism. □

04FG **Remark 8.6.7.8.** Let $\sigma : \mathcal{QC} \to \mathcal{QC}$ be the opposition functor of Construction 8.6.7.6. Passing to homotopy categories, we obtain a functor $\overline{\sigma} : \mathrm{hQCat} \to \mathrm{hQCat}$. It follows from Proposition 8.6.7.7 that, up to isomorphism, $\overline{\sigma}$ agrees with the automorphism of hQCat which is given on objects by the construction $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$, and on morphisms by the construction $[F] \mapsto [F^{\mathrm{op}}]$; here $[F]$ denotes the isomorphism class of a functor $F : \mathcal{C} \to \mathcal{D}$ and $[F^{\mathrm{op}}]$ the isomorphism class of the opposition functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

04FH **Proposition 8.6.7.9** (Involutivity)**.** *Let $\sigma : \mathcal{QC} \to \mathcal{QC}$ be the opposition functor. Then the composition $\sigma \circ \sigma$ is isomorphic to the identity functor $\mathrm{id}_{\mathcal{QC}}$. In particular, σ is an equivalence of ∞-categories.*

Proof. This follows from Proposition 8.6.7.5, since the composition $\widetilde{\sigma} \circ \widetilde{\sigma}$ is *equal* to the identity functor of the ∞-category $\mathcal{QC}^{\asymp}$. □

Remark 8.6.7.10 (Uniqueness)**.** Let $\operatorname{Aut}(\mathcal{QC})$ denote the full subcategory of $\operatorname{Fun}(\mathcal{QC}, \mathcal{QC})$ spanned by those functors $\mathcal{QC} \to \mathcal{QC}$ which are equivalences of ∞-categories. A theorem of Toën ([55]) guarantees that $\operatorname{Aut}(\mathcal{QC})$ is a Kan complex having exactly two connected components, each of which is contractible (see Corollary [?]). Consequently, the opposition functor $\sigma : \mathcal{QC} \to \mathcal{QC}$ of Construction 8.6.7.4 is characterized (up to a contractible space of choices) by the fact that it is an equivalence of ∞-categories which is not isomorphic to the identity functor $\mathrm{id}_{\mathcal{QC}}$. 04FJ

Recall that every Kan complex X is homotopy equivalent to its opposite X^{op} (Example 8.6.4.18). The following is a more precise statement:

Proposition 8.6.7.11. *Let $\sigma : \mathcal{QC} \to \mathcal{QC}$ be the opposition functor of Construction 8.6.7.6. Then the restriction $\sigma|_{\mathcal{S}}$ is isomorphic to the identity functor from $\mathcal{S} \subset \mathcal{QC}$ to itself.* 04FK

Proof. For every ∞-category $\mathcal{C}$, the image $\sigma(\mathcal{C}) \in \mathcal{QC}$ is equivalent to the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ (Remark 8.6.7.8); in particular, $\mathcal{C}$ is a Kan complex if and only if $\sigma(\mathcal{C})$ is a Kan complex. It follows that σ restrict a functor $\sigma_0 : \mathcal{S} \to \mathcal{S}$, which is also an equivalence of ∞-categories. We wish to show that σ_0 is isomorphic to the identity functor $\mathrm{id}_{\mathcal{S}}$. This follows from Example 8.4.0.4, since $\sigma_0(\Delta^0)$ is homotopy equivalent to the Kan complex $(\Delta^0)^{\mathrm{op}} \simeq \Delta^0$. □

Using the classification of cocartesian fibrations given in §5.6, we can use the opposition functor $\sigma : \mathcal{QC} \to \mathcal{QC}$ to give a reformulation of cocartesian duality.

Proposition 8.6.7.12. *Let $U : \mathcal{E} \to \mathcal{C}$ and $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets having transport representations $\operatorname{Tr}_{\mathcal{E}/\mathcal{C}}, \operatorname{Tr}_{\mathcal{E}^{\vee}/\mathcal{C}} : \mathcal{C} \to \mathcal{QC}$. Then $U^{\vee}$ is a cocartesian dual of U if and only if $\operatorname{Tr}_{\mathcal{E}^{\vee}/\mathcal{C}}$ is isomorphic to $\sigma \circ \operatorname{Tr}_{\mathcal{E}/\mathcal{C}}$, where $\sigma : \mathcal{QC} \to \mathcal{QC}$ denotes the opposition functor of Construction 8.6.7.6.* 04FL

Proof. Assume that $\operatorname{Tr}_{\mathcal{E}^{\vee}/\mathcal{C}}$ is isomorphic to $\sigma \circ \operatorname{Tr}_{\mathcal{E}/\mathcal{C}}$; we will show that $U^{\vee}$ is cocartesian dual to U (the reverse implication then follows formally from the fact that cocartesian duals are unique up to equivalence; see Theorem 8.6.5.1). Let $\pi : \mathrm{QCat}^{\asymp} \to \mathrm{QCat}$ denote the forgetful functor, and let $\pi' : \mathrm{QCat}^{\asymp} \to \mathrm{QCat}$ be the composition of π with the automorphism $\widetilde{\sigma} : \mathrm{QCat}^{\asymp} \simeq \mathrm{QCat}^{\asymp}$ described in Construction 8.6.7.6. By virtue of Proposition 8.6.7.5, we may assume without loss of generality that the covariant transport representation $\mathscr{F}_{+} = \operatorname{Tr}_{\mathcal{E}/\mathcal{C}}$ factors as a composition $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\pi) \circ \mathscr{F}$ for some diagram $\mathscr{F} : \mathcal{C} \to \mathcal{QC}^{\asymp}$. Set $\mathscr{F}_{-} = \mathrm{N}^{\mathrm{hc}}_{\bullet}(\pi') \circ \mathscr{F}$. Our assumption then guarantees that $\mathscr{F}_{-}$ is a covariant transport representation for $U^{\vee}$. We may therefore assume without loss of generality that U and $U^{\vee}$ coincide with the projection maps $\int_{\mathcal{C}} \mathscr{F}_{+} \to \mathcal{C}$ and $\int_{\mathcal{C}} \mathscr{F}_{-} \to \mathcal{C}$, respectively.

We now proceed as in the proof of Proposition 8.6.4.23. Define a simplicial functor $\tau : \mathrm{QCat}^{\simeq} \to \mathrm{QCat}$ as follows:

- On objects, τ is given by the construction $\mathcal{D} \mapsto \mathrm{Tw}(\mathcal{D})$.
- On morphism spaces, τ is given by the morphism of simplicial sets

$$\begin{aligned} \mathrm{Hom}_{\mathrm{QCat}^{\simeq}}(\mathcal{D}, \mathcal{D}')_{\bullet} &= \mathrm{Tw}(\mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq}) \\ &\to \mathrm{Fun}(\mathrm{Tw}(\mathcal{D}), \mathrm{Tw}(\mathcal{D}'))^{\simeq} \\ &= \mathrm{Hom}_{\mathrm{QCat}}(\mathrm{Tw}(\mathcal{D}), \mathrm{Tw}(\mathcal{D}'))_{\bullet}. \end{aligned}$$

which classifies the composition

$$\mathrm{Tw}(\mathcal{D}) \times \mathrm{Tw}(\mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq}) \hookrightarrow \mathrm{Tw}(\mathcal{D} \times \mathrm{Fun}(\mathcal{D}, \mathcal{D}')) \xrightarrow{\mathrm{Tw(ev)}} \mathrm{Tw}(\mathcal{D}'),$$

where $\mathrm{ev} : \mathcal{D} \times \mathrm{Fun}(\mathcal{D}, \mathcal{D}') \to \mathcal{D}'$ is the evaluation map.

Let $\widetilde{\mathscr{F}} : \mathcal{C} \to \mathcal{QC}$ denote the diagram given by the composition $\mathrm{N}^{\mathrm{hc}}_{\bullet}(\tau) \circ \mathscr{F}$, and set $\widetilde{\mathcal{E}} = \int_{\mathcal{C}} \widetilde{\mathscr{F}}$. There is a natural transformation of simplicial functors $\tau \to \pi' \times \pi$, which carries each ∞-category $\mathcal{D}$ to the left fibration $\mathrm{Tw}(\mathcal{D}) \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ of Proposition 8.1.1.11. Applying Corollary [?], see that this natural transformation induces a left fibration

$$\widetilde{\mathcal{E}} = \int_{\mathcal{C}} \widetilde{\mathscr{F}} \xrightarrow{\lambda} \int_{\mathcal{C}} (\mathscr{F}_{-} \times \mathscr{F}_{+}) \simeq \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E}\,.$$

We will complete the proof by showing that λ exhibits $U^{\vee}$ as a cocartesian dual of U: that is, it satisfies conditions (a) and (b) of Definition 8.6.4.1.

(a) Fix a vertex $C \in \mathcal{C}$; we wish to show that the left fibration

$$\lambda_C : \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \widetilde{\mathscr{F}} \to (\{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}_{-}) \times (\{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}_{+})$$

is a balanced coupling. Set $\mathcal{D} = \widetilde{\mathscr{F}}(C)$. Using Example 5.6.2.18, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Tw}(\mathcal{D}) \arrow[r] \arrow[d] & \{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F} \arrow[d, "\lambda_C"] \\ \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[r] & (\{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}_{-}) \times (\{C\} \times_{\mathcal{C}} \int_{\mathcal{C}} \mathscr{F}_{+}), \end{tikzcd}$$

where the horizontal maps are equivalences of ∞-categories. The desired result now follows from the observation that the twisted arrow coupling $\mathrm{Tw}(\mathcal{D}) \to \mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ is balanced (Example 8.2.6.2).

(b) Let $U : \widetilde{\mathcal{E}} \to \mathcal{C}$ denote the projection map, let $f : X \to X'$ be a U-cocartesian edge of $\widetilde{\mathcal{E}}$, and let $\overline{f} : C \to C'$ denote the image of f in the simplicial set $\mathcal{C}$. The functor $\mathscr{F}$ carries the vertex C to an ∞-category $\mathcal{D}$, C' to an ∞-category $\mathcal{D}'$, and $\overline{f}$ to a vertex of the Kan complex $\mathrm{Tw}(\mathrm{Fun}(\mathcal{D}, \mathcal{D}')^{\simeq})$, which we can identify with an isomorphism $u : F_{-} \to F_{+}$ between functors $F_{-}, F_{+} : \mathcal{D} \to \mathcal{D}'$. We can then identify X with a morphism $e : D_{-} \to D_{+}$ in the ∞-category $\mathcal{D}$ and X' with a morphism $e' : D'_{-} \to D'_{+}$ in the ∞-category $\mathcal{D}'$, so that f determines a morphism in the ∞-category $\mathrm{Tw}(\mathcal{D}')$ which we depict informally in the diagram $\mathcal{D}'$ which we depict informally in the diagram

$$\begin{tikzcd}
F_{-}(D_{-}) \arrow[d, "u(e)"] & D'_{-} \arrow[l] \arrow[d, "e'"] \\
F_{+}(D_{+}) \arrow[r] & D'_{+}.
\end{tikzcd}$$

Our assumption that X is universal for the coupling λ_C guarantees that e is an isomorphism in the ∞-category $\mathcal{D}$ (Example 8.2.1.5), so that the left vertical map is an isomorphism in the ∞-category $\mathcal{D}'$. Our assumption that f is U-cocartesian guarantees that the horizontal maps in the diagram are also isomorphisms (Remark 5.6.2.14). It follows that e' is also an isomorphism in $\mathcal{D}$, so that X' is universal for the coupling $\lambda_{C'}$ as desired.

□

Remark 8.6.7.13. Let $\mathcal{QC}_{\mathrm{Obj}}$ denote the ∞-category of pairs $(\mathcal{C}, X)$, where $\mathcal{C}$ is a small 04FM ∞-category and X is an object of $\mathcal{C}$ (see Definition 5.5.6.10). Then the identity functor $\mathrm{id} : \mathcal{QC} \to \mathcal{QC}$ is a covariant transport representation for the universal cocartesian fibration

$$U : \mathcal{QC}_{\mathrm{Obj}} \to \mathcal{QC} \qquad (\mathcal{C}, X) \mapsto \mathcal{C}\,.$$

Applying Proposition 8.6.7.12, we deduce that the opposition functor $\sigma : \mathcal{QC} \to \mathcal{QC}$ is a covariant transport representation for a cocartesian dual of the fibration U. By virtue of Corollary 5.6.5.13, this property characterizes the functor σ up to isomorphism.

Chapter 9

Large ∞-Categories

04KE 9.1 Local Objects and Factorization Systems

04KF 9.1.1 Local Objects

04KG Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to some collection of morphisms W. Recall that F is a *reflective localization* if it admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$. In this case, the functor G is automatically fully faithful (Proposition 6.3.3.6), and its essential image is a reflective subcategory $\mathcal{C}' \subseteq \mathcal{C}$ (Corollary 6.2.2.17). In this situation, we can extract the subcategory $\mathcal{C}'$ directly from W.

02FZ **Definition 9.1.1.1.** Let $\mathcal{C}$ be an ∞-category and let $w : X \to Y$ be a morphism of $\mathcal{C}$. We say that an object $C \in \mathcal{C}$ is *w-local* if precomposition with the homotopy class $[w]$ induces a homotopy equivalence of mapping spaces $\mathrm{Hom}_{\mathcal{C}}(Y, C) \xrightarrow{\circ [w]} \mathrm{Hom}_{\mathcal{C}}(X, C)$. We say that C is *w-colocal* if postcomposition with $[w]$ induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(C, X) \xrightarrow{[w] \circ} \mathrm{Hom}_{\mathcal{C}}(C, Y)$.

If W is a collection of morphisms of $\mathcal{C}$, we say that an object $C \in \mathcal{C}$ is *W-local* if it is w-local for each $w \in W$. Similarly, we say that C is *W-colocal* if it is w-colocal for each $w \in W$.

04KH **Example 9.1.1.2.** Let $\mathcal{C}$ be an ∞-category and let $w : X \to Y$ be an isomorphism in $\mathcal{C}$. Then every object of $\mathcal{C}$ is w-local.

02G1 **Remark 9.1.1.3.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$, which we also view as a collection of morphisms in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. Then an object $Z \in \mathcal{C}$ is W-local (in the sense of Definition 9.1.1.1) if and only if it is W-colocal when viewed as an object of $\mathcal{C}^{\mathrm{op}}$.

04KJ **Remark 9.1.1.4.** Let $\mathcal{C}$ be an ∞-category containing a morphism $w : X \to Y$. Let $\pi : \mathcal{C}_{X/} \to \mathcal{C}$ denote the projection map, so that w can be identified with an object $\widetilde{Y} \in \mathcal{C}_{X/}$

satisfying $\pi(\widetilde{Y}) = Y$. Then an object $C \in \mathcal{C}$ is w-local if and only if, for every object $\widetilde{C} \in \mathcal{C}_{X/}$ satisfying $\pi(\widetilde{C}) = C$, the morphism space $\mathrm{Hom}_{\mathcal{C}_{X/}}(\widetilde{Y}, \widetilde{C})$ is contractible. This follows from the criterion of Remark 3.4.0.6, since $\mathrm{Hom}_{\mathcal{C}_{X/}}(\widetilde{Y}, \widetilde{C})$ can be identified with the homotopy fiber of the composition map $\mathrm{Hom}_{\mathcal{C}}(Y, C) \to \mathrm{Hom}_{\mathcal{C}}(X, C)$ over the vertex corresponding to $\widetilde{C}$ (see Corollary 4.6.9.18).

Remark 9.1.1.5. Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let $w : X \to Y$ be a morphism which belongs to W. Then, for every W-local object C of $\mathcal{C}$, precomposition with the homotopy class $[w]$ induces a bijection $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, C) \xrightarrow{\circ[w]} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, C)$. In particular, if the objects X and Y are W-local, then w is an isomorphism. 02G2

Remark 9.1.1.6. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. If C is a W-local object of $\mathcal{C}$, then any retract of C is also W-local. In particular, the condition that C is W-local depends only on the isomorphism class of C. 04KK

Variant 9.1.1.7. Let $\mathcal{C}$ be an ∞-category, let $w : X \to Y$ be a morphism in $\mathcal{C}$, and let $C \in \mathcal{C}$ be an object which is w-local. Then C is w'-local, for any morphism $w' : X' \to Y'$ which is a retract of w (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$). 04KL

Remark 9.1.1.8. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Then the collection of W-local objects is closed under the formation of all limits which exist in $\mathcal{C}$ (see Corollary 7.4.5.18). Similarly, the collection of W-colocal objects is closed under the formation of all colimits which exist in $\mathcal{C}$. 04KM

Remark 9.1.1.9. Let $\mathcal{C}$ be an ∞-category, and let f be a morphism of $\mathcal{C}$ which is the colimit of a diagram 04KN

$$K \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \qquad v \mapsto f_v$$

which is preserved by the evaluation functors $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$. If an object $C \in \mathcal{C}$ is f_v-local for each vertex $v \in K$, then it is also f-local. This follows from Propositions 7.4.5.17 and 7.1.2.13.

Remark 9.1.1.10. Let $\mathcal{C}$ be an ∞-category containing a pushout diagram 04KP

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$$\begin{tikzcd} X \arrow[r] \arrow[d, "w"] & X' \arrow[d, "w'"] \\ Y \arrow[r] & Y'. \end{tikzcd} \tag{9.1}$$

If an object $C \in \mathcal{C}$ is w-local, then it is also w'-local. This follows immediately from the observation that the representable functor $h_C : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ carries pushout diagrams in $\mathcal{C}$ to pullback diagrams in $\mathcal{S}$ (Corollary 7.4.5.18).

04KR **Remark 9.1.1.11.** Let $\mathcal{C}$ be an ∞-category, let $C \in \mathcal{C}$ be an object, and let W be the collection of all morphisms w in $\mathcal{C}$ such that C is w-local. Then W contains all isomorphisms and has the two-out-of-three property. Moreover, it is also closed under retracts (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$).

04KS **Remark 9.1.1.12.** Let $\mathcal{C}$ be an ∞-category which admits pushouts, let $w : X \to Y$ be a morphism of $\mathcal{C}$, and let $\gamma_{X/Y} : Y \coprod_X Y \to Y$ be the relative codiagonal of w (see Variant 7.6.3.19). If an object $C \in \mathcal{C}$ is w-local, then it is also $\gamma_{X/Y}$-local. This follows by applying Remark 9.1.1.11 to the diagram

$$\begin{tikzcd}
 & Y \coprod_X Y \arrow[dr, "\gamma_{X/Y}"] & \\
Y \arrow[ur, "w'"] \arrow[rr, "\mathrm{id}"] & & Y;
\end{tikzcd}$$

here w' is a pushout of w (so that C is w'-local by virtue of Remark 9.1.1.10). For a partial converse, see Exercise 9.1.3.16.

04KT **Proposition 9.1.1.13.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to some collections of morphisms W, and let C be an object of $\mathcal{C}$. The following conditions are equivalent:*

(1) *The object C is W-local, in the sense of Definition 9.1.1.1.*

(2) *For every object $C' \in \mathcal{C}$, the functor F induces a homotopy equivalence of mapping spaces $\theta_{C',C} : \mathrm{Hom}_{\mathcal{C}}(C', C) \to \mathrm{Hom}_{\mathcal{D}}(F(C'), F(C))$.*

Proof. Fix an uncountable regular cardinal κ for which both $\mathcal{C}$ and $\mathcal{D}$ are essentially κ-small. Precomposition with F determines a functor $F^* : \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$. It follows from Proposition 7.6.7.13 that the functor F^* admits a left adjoint $F_!$ (given by left Kan extension along F^{op}). Let $h^{\mathcal{C}}_{\bullet} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ and $h^{\mathcal{D}}_{\bullet} : \mathcal{D} \to \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ be covariant Yoneda embeddings for $\mathcal{C}$ and $\mathcal{D}$, respectively, so that the diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "h^{\mathcal{C}}_{\bullet}"] \arrow[d, "F"] & \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \arrow[d, "F_!"] \\
\mathcal{D} \arrow[r, "h^{\mathcal{D}}_{\bullet}"] & \mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa})
\end{tikzcd}$$

commutes up to isomorphism (Example 8.4.4.5), where the horizontal maps are fully faithful (Theorem 8.3.3.13). It follows that, for every pair of objects $C', C \in \mathcal{C}$, we can identify $\theta_{C',C}$ with the comparison map

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}^{<\kappa})}(h^{\mathcal{C}}_{C'}, h^{\mathcal{C}}_{C}) &\rightarrow \mathrm{Hom}_{\mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathcal{S}^{<\kappa})}(F_! h^{\mathcal{C}}_{C'}, F_! h^{\mathcal{C}}_{C}) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S}^{<\kappa})}(h^{\mathcal{C}}_{C'}, F^* F_! h^{\mathcal{C}}_{C})\end{aligned}$$

given by precomposition with the unit $u : h^{\mathcal{C}}_{C} \to F^* F_! h^{\mathcal{C}}_{C}$. Combining this observation with Proposition 8.3.1.1, we see that condition (2) can be restated as follows:

(2′) The unit map $u : h^{\mathcal{C}}_{C} \to F^* F_! h^{\mathcal{C}}_{C}$ is an isomorphism in the ∞-category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.

Our assumption that F exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ guarantees that the pullback functor F^* is fully faithful. Using Remark 6.2.2.18, we see that u is an isomorphism if and only the representable functor $h^{\mathcal{C}}_{C}$ belongs to the essential image of F^*: that is, the collection of functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}^{<\kappa}$ which carry every morphism of W to an isomorphism in $\mathcal{S}^{<\kappa}$. This is a reformulation of (1). □

Corollary 9.1.1.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which exhibits $\mathcal{D}$ as a localization of $\mathcal{C}$ with respect to some collection of morphisms W. Suppose that F admits a right adjoint $G : \mathcal{D} \to \mathcal{C}$. Then G is fully faithful, and the essential image of G is spanned by the collection of W-local objects of $\mathcal{C}$.* 04KU

Proof. The assertion that G is fully faithful follows from Proposition 6.3.3.6. Let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be the unit of an adjunction between F and G. Then an object $C \in \mathcal{C}$ belongs to the essential image of G if and only if the morphism $\eta_C : C \to (G \circ F)(C)$ is an isomorphism. This is equivalent to the requirement that, for every object $B \in \mathcal{C}$, composition with η_C induces a homotopy equivalence of mapping spaces $\theta : \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(B, (G \circ F)(C))$. We conclude by observing that θ factors as a composition

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(B), F(C)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(B, (G \circ F)(C))$$

where the second map is the homotopy equivalence of Proposition 6.2.1.17. □

Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by the W-local objects. Beware that, in general, $\mathcal{C}'$ is not a reflective subcategory of $\mathcal{C}$. To ensure this, we need some additional assumptions on W.

Definition 9.1.1.15. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. We say that W is *localizing* if the following conditions are satisfied: 02G0

(1) Every isomorphism of $\mathcal{C}$ is contained in W.

(2) The collection of morphisms W satisfies the two-out-of-three property. That is, for every 2-simplex

$$\begin{tikzcd}
 & Y \arrow[dr, "v"] & \\
X \arrow[ur, "u"] \arrow[rr, "w"] & & Z
\end{tikzcd}$$

in $\mathcal{C}$, if any two of the morphisms u, v, and w belongs to W, then so does the third.

(3) For every object $X \in \mathcal{C}$, there exists a morphism $w : X \to Y$ which belongs to W, where the object Y is W-local.

We will say that W is *colocalizing* if it satisfies conditions (1) and (2), together with the following dual version of (3):

(3′) For every object $X \in \mathcal{C}$, there exists a morphism $w : Y \to X$ which belongs to W, where the object Y is W-colocal.

04KV **Proposition 9.1.1.16.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a reflective localization functor, and let W be the collection of all morphisms w of $\mathcal{C}$ such that $F(w)$ is an isomorphism in $\mathcal{D}$. Then W is localizing.*

Proof. Conditions (1) and (2) of Definition 9.1.1.15 follow immediately from the definitions (and do not require any assumptions on F). We will verify condition (3). Since F is a reflective localization functor, it admits a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$. For every object $Y \in \mathcal{D}$, the image $G(Y) \in \mathcal{C}$ is W-local (Corollary 9.1.1.14). In particular, if X is an object of $\mathcal{C}$, then $(G \circ F)(X)$ is a W-local object of $\mathcal{C}$. Let $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$ be the unit of an adjunction between F and G. To complete the proof, it will suffice to show that the unit map $\eta_X : X \to (G \circ F)(X)$ belongs to W: that is, that $F(\eta_X)$ is an isomorphism in $\mathcal{D}$. Since the functor G is fully faithful, this follows from Remark 6.3.3.5. □

04KW **Corollary 9.1.1.17.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a reflective subcategory, and let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint to the inclusion functor $\iota : \mathcal{C}' \hookrightarrow \mathcal{C}$. Let W be the collection of all morphisms $w : X \to Y$ of $\mathcal{C}$ for which $L(w)$ is an isomorphism in $\mathcal{C}'$. Then:*

(1) *The collection W is localizing (Definition 9.1.1.15).*

(2) *Every object of $\mathcal{C}'$ is W-local (Definition 9.1.1.1).*

(3) *If $\mathcal{C}'$ is replete, then every W-local object of $\mathcal{C}$ belongs to $\mathcal{C}'$.*

Proof. Combine Proposition 9.1.1.16 with Corollary 9.1.1.14. □

We now prove the converse of Corollary 9.1.1.17: every localizing collection of morphisms of an ∞-category $\mathcal{C}$ can be obtained from a reflective localization of $\mathcal{C}$.

Proposition 9.1.1.18. *Let $\mathcal{C}$ be an ∞-category, let W be a localizing collection of morphisms of $\mathcal{C}$, and let $\mathcal{C}'$ denote the full subcategory of $\mathcal{C}$ spanned by the W-local objects. Then:* 02G3

(1) *The full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is reflective (Definition 6.2.2.1).*

(2) *The inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}'$.*

(3) *A morphism w of $\mathcal{C}$ is contained in W if and only if $L(w)$ is an isomorphism in $\mathcal{C}'$.*

(4) *The functor L exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to W.*

Proof. Let X be an object of $\mathcal{C}$. Our assumption that W is localizing guarantees that there exists a morphism $w_X : X \to X'$ which belongs to W, where $X' \in \mathcal{C}'$. By definition, every object $C \in \mathcal{C}'$ is W-local, so composition with w_X induces a homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X', C) \to \mathrm{Hom}_{\mathcal{C}}(X, C)$. It follows that w_X exhibits X' as a $\mathcal{C}'$-reflection of X, in the sense of Definition 6.2.2.1. Assertion (1) follows by allowing the object X to vary. The implication (1) $\Rightarrow$ (2) follows from Proposition 6.2.2.15, and the implication (3) $\Rightarrow$ (4) from Example 6.3.3.7.

It remains to prove (3). Choose a natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \to L$ which exhibits L as a $\mathcal{C}'$-reflection functor (see Definition 6.2.2.12). For each object $X \in \mathcal{C}$, the morphism $\eta_X : X \to L(X)$ exhibits $L(X)$ as a $\mathcal{C}'$-reflection of X, and can therefore be obtained by composing w_X with an isomorphism $X' \xrightarrow{\sim} L(X)$. Since W contains all isomorphisms and is closed under composition, it follows that η_X belongs to W.

For every morphism $w : X \to Y$ in $\mathcal{C}$, the natural transformation η determines a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "w"] \arrow[d, "\eta_X"] & Y \arrow[d, "\eta_Y"] \\
L(X) \arrow[r, "L(w)"] & L(Y)
\end{tikzcd}$$

where η_X and η_Y belong to W. Using the two-out-of-three property, we see that w is contained in W if and only if $L(w)$ is contained in W. Since $L(X)$ and $L(Y)$ are W-local, this is equivalent to the requirement that $L(w)$ is an isomorphism (Remark 9.1.1.5). $\square$

04KX **Corollary 9.1.1.19.** *Let $\mathcal{C}$ be an ∞-category and suppose we are given a pushout diagram*

04KY
$$\begin{tikzcd} X \arrow[r, "w"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "w'"] & Y' \end{tikzcd} \tag{9.2}$$

in $\mathcal{C}$. If W is a localizing collection of morphisms of $\mathcal{C}$ which contains w, then it also contains w'.

Proof. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory of W-local objects and let $L : \mathcal{C} \to \mathcal{C}'$ be a left adjoint to the inclusion. The assumption $w \in W$ guarantees that $L(w)$ is an isomorphism in $\mathcal{C}'$ (Proposition 9.1.1.18). Since L carries the (9.2) to a pushout diagram in the ∞-category $\mathcal{C}'$ (Corollary 7.1.3.21), it follows that $L(w')$ is also an isomorphism (Corollary 7.6.3.24). Applying Proposition 9.1.1.18 again, we conclude that w' belongs to W. □

02G4 **Notation 9.1.1.20.** Let $\mathcal{C}$ be an ∞-category and let W be a localizing collection of morphisms of $\mathcal{C}$. We will often write $\mathcal{C}[W^{-1}]$ for the full subcategory of $\mathcal{C}$ spanned by the W-local objects. By virtue of Proposition 9.1.1.18, this is consistent with Remark 6.3.2.2: that is, we can regard $\mathcal{C}[W^{-1}]$ as a localization of $\mathcal{C}$ with respect to W. This convention is very convenient, since the full subcategory of W-local objects is uniquely determined by $\mathcal{C}$ and W. However, it has the potential to create confusion in some situations: see Warning 9.1.1.23 below.

02G8 **Corollary 9.1.1.21.** *Let $\mathcal{C}$ be an ∞-category. Then the construction $W \mapsto \mathcal{C}[W^{-1}]$ determines a bijection*

$$\begin{tikzcd} \{\text{Localizing collections of morphisms of } \mathcal{C}\} \arrow[d, "\sim"] \\ \{\text{Reflective replete subcategories of } \mathcal{C}\}. \end{tikzcd}$$

Proof. Combine Proposition 9.1.1.18 with Corollary 9.1.1.17. □

Proposition 9.1.1.18 has a counterpart for colocalizing collections of morphisms:

02G5 **Variant 9.1.1.22.** Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$ which is colocalizing, and let $\mathcal{C}'$ denote the full subcategory of $\mathcal{C}$ spanned by the W-colocal objects. Then:

(1) The full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is coreflective.

(2) The inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ admits a right adjoint $L : \mathcal{C} \to \mathcal{C}'$.

(3) A morphism w of $\mathcal{C}$ is contained in W if and only if $L(w)$ is an isomorphism in $\mathcal{C}'$.

(4) The functor L exhibits $\mathcal{C}'$ as a localization of $\mathcal{C}$ with respect to W.

Warning 9.1.1.23. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$ which is both localizing and colocalizing. In this case, Proposition 9.1.1.18 and Variant 9.1.1.22 provide two *different* concrete realizations of the localization $\mathcal{C}[W^{-1}]$, given by the full subcategories $\mathcal{C}' \subseteq \mathcal{C} \supseteq \mathcal{C}''$ spanned by the W-local and W-colocal objects of $\mathcal{C}$, respectively. Note that $\mathcal{C}'$ and $\mathcal{C}''$ are necessarily equivalent as abstract ∞-categories. More precisely, if $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ is a functor which exhibits $\mathcal{C}[W^{-1}]$ as the localization of $\mathcal{C}$ with respect to W, then the restrictions 02G6

$$\mathcal{C}' \xrightarrow{F|_{\mathcal{C}'}} \mathcal{C}[W^{-1}] \xleftarrow{F|_{\mathcal{C}''}} \mathcal{C}''$$

are equivalences of ∞-categories. Beware that $\mathcal{C}'$ and $\mathcal{C}''$ usually do not coincide when regarded *as subcategories of* $\mathcal{C}$. See Warning 6.3.3.12.

9.1.2 Digression: Transfinite Composition

Let $\mathcal{C}$ be an ∞-category and let $X : \mathrm{N}_{\bullet}(\mathbf{Z}_{\geq 0}) \to \mathcal{C}$ be a functor, which we display as a diagram 04KZ

$$X(0) \xrightarrow{f_0} X(1) \xrightarrow{f_1} X(2) \xrightarrow{f_2} X(3) \to \cdots$$

Suppose that X can be extended to a colimit diagram $\overline{X} : \mathrm{N}_{\bullet}(\mathbf{Z}_{\geq 0})^{\triangleright} \to \mathcal{C}$, carrying the cone point to an object $Y = \varinjlim(X)$. In this case, we can evaluate $\overline{X}$ on the edge $\{0\}^{\triangleright} \subseteq \mathrm{N}_{\bullet}(\mathbf{Z}_{\geq 0})^{\triangleright}$ to obtain a morphism $g : X(0) \to Y$. Heuristically, we can think of the morphism g as an "infinite composition" $\cdots f_4 \circ f_3 \circ f_2 \circ f_1 \circ f_0$. Our goal in this section is to extend this heuristic to more general well-ordered diagrams.

In what follows, we assume that the reader is familiar with the theory of ordinals (see §4.7.1 for a review). For every ordinal α, let $\mathrm{Ord}_{\leq \alpha}$ denote the linearly ordered set of ordinals which are less than or equal to α, and let $\mathrm{Ord}_{< \alpha}$ denote the subset consisting of ordinals which are strictly smaller than α.

Definition 9.1.2.1. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. 04L0 We will say that a morphism f of $\mathcal{C}$ is a *transfinite composition of morphisms of* W if there exists an ordinal α and a functor $F : \mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \alpha}) \to \mathcal{C}$ with the following properties:

(a) For every nonzero limit ordinal $\lambda \leq \alpha$, the restriction $F|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \lambda})}$ is a colimit diagram: that is, it exhibits $F(\lambda)$ as a colimit of the restriction $F|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{< \lambda})}$.

(b) For every ordinal $\beta < \alpha$, the morphism $F(\beta) \to F(\beta + 1)$ belongs to W.

(c) The morphism $F(0) \to F(\alpha)$ coincides with f.

In this case, we will say that F *exhibits* f *as a transfinite composition of morphisms of* W.

We say that W is *closed under transfinite composition* if it contains every morphism which is a transfinite composition of morphisms of W.

04L1 **Remark 9.1.2.2.** Let $\mathcal{C}$ be an ordinary category and let W be a collection of morphisms of $\mathcal{C}$. Then a morphism f of $\mathcal{C}$ is a transfinite composition of morphisms belonging to W (in the sense of Definition 1.5.4.10) if and only if the corresponding morphism of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ is a transfinite composition of morphisms belonging to W (in the sense of Definition 9.1.2.1).

04L2 **Variant 9.1.2.3.** Let A be a well-ordered set and let α denote its order type. Then there is a unique order-preserving bijection $\mathrm{Ord}_{<\alpha} \simeq A$, which determines an isomorphism of simplicial sets $u : \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \alpha}) \simeq \mathrm{N}_\bullet(A)^{\triangleright}$. If $\mathcal{C}$ is an ∞-category containing a morphism f and a collection of morphisms W, we will say that a diagram $F : \mathrm{N}_\bullet(A)^{\triangleright} \to \mathcal{C}$ *exhibits* f *as a transfinite composition of morphisms of* W if the composition $\mathrm{N}_\bullet(\mathrm{Ord}_{\leq \alpha}) \simeq \mathrm{N}_\bullet(A)^{\triangleright} \xrightarrow{F} \mathcal{C}$ exhibits f as a transfinite composition of morphisms of W, in the sense of Definition 9.1.2.1.

006N **Example 9.1.2.4.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Then every identity morphism of $\mathcal{C}$ is a transfinite composition of morphisms of W (take $\alpha = 0$ in Definition 9.1.2.1). In particular, if W is closed under transfinite composition, then it contains every identity morphism of $\mathcal{C}$.

006P **Example 9.1.2.5.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Then every morphism of W is a transfinite composition of morphisms of W (take $\alpha = 1$ in Definition 9.1.2.1).

006Q **Example 9.1.2.6.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$ which contains a pair of composable morphisms $f : X \to Y$ and $g : Y \to Z$. Then any composition of f with g is a transfinite composition of morphisms of W (take $\alpha = 2$ in Definition 1.5.4.10). In particular, if W is closed under transfinite composition, then it is closed under composition.

04L3 **Example 9.1.2.7.** Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be an isomorphism in $\mathcal{C}$. Let F denote the composite map $\mathrm{N}_\bullet(\mathbf{Z}_{\geq 0})^{\triangleright} \twoheadrightarrow (\Delta^0)^{\triangleright} \simeq \Delta^1 \xrightarrow{f} \mathcal{C}$, which we display informally as a diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[drr, "f"'] & X \arrow[r, "\mathrm{id}"] \arrow[dr, "f"] & X \arrow[r, "\mathrm{id}"] \arrow[d, "f"] & X \arrow[r, "\mathrm{id}"] \arrow[dl, "f"] & \cdots \\
 & & Y. & &
\end{tikzcd}$$

Since the simplicial set $N_\bullet(\mathbf{Z}_{\geq 0})$ is contractible (Example 3.2.4.2), the functor F is a colimit diagram (Corollary 7.2.3.5), and therefore exhibits f as a transfinite composition of morphisms belonging to the singleton $\{\mathrm{id}_X\}$.

Remark 9.1.2.8. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$ which is closed under transfinite composition. Combining Examples 9.1.2.4 and 9.1.2.7, we deduce that W contains every isomorphism of $\mathcal{C}$. `04L4`

Remark 9.1.2.9. Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$ which is closed under transfinite composition. Then W is closed under isomorphism: that is, if f and g are morphisms of $\mathcal{C}$ which are isomorphic (as objects of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$), then f belongs to W if and only if g belongs to W. This follows by combining Example 9.1.2.6 with Remark 9.1.2.8. In particular, the condition that a morphism f of $\mathcal{C}$ belongs to W depends only on the homotopy class $[f]$. `04L5`

Example 9.1.2.6 admits a partial converse:

Proposition 9.1.2.10. *Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Assume that:* `04L6`

(1) *Every identity morphism of $\mathcal{C}$ belongs to W.*

(2) *The collection W is closed under composition.*

(3) *The collection W is closed under the formation of colimits in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$.*

Then W is closed under transfinite composition.

Proof. Let $f : X \to Y$ be a morphism of $\mathcal{C}$ which is a transfinite composition of morphisms of W; we wish to show that $f \in W$. Choose an ordinal α and a functor $F : N_\bullet(\mathrm{Ord}_{\leq \alpha}) \to \mathcal{C}$ which exhibits f as a transfinite composition of morphisms of W. For each $\beta \leq \alpha$, the functor F carries the ordered pair $(0 \leq \beta)$ to a morphism $f_\beta : F(0) \to F(\beta)$. We will complete the proof by showing that each the morphisms f_β belongs to W. The proof proceeds by transfinite induction on β. If $\beta = 0$, then $f_\beta = \mathrm{id}_X$ and the desired result follows from assumption (1). If β is a nonzero limit ordinal, then the desired result follows from assumption (3). Remark 9.1.1.9. It will therefore suffice to treat the case where $\beta = \gamma + 1$ is a successor ordinal. In this case, the desired result follows by applying assumption (2) to the diagram

$$\begin{tikzcd}
& F(\gamma) \arrow[dr] & \\
X \arrow[ur, "f_\gamma"] \arrow[rr, "f_\beta"] & & F(\beta),
\end{tikzcd}$$

since f_γ belongs to W by virtue of our inductive hypothesis. □

04L7 **Remark 9.1.2.11.** In the statement of Proposition 9.1.2.10, it is not necessary to assume that W is closed under the formation of *all* colimits in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. It suffices to consider colimits of diagrams indexed by $\mathrm{N}_\bullet(\mathrm{Ord}_{<\beta})$ where β is a limit ordinal; moreover, we can further restrict our attention to colimits which are preserved by the evaluation functors $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$.

04L8 **Corollary 9.1.2.12.** *Let $\mathcal{C}$ be an ∞-category and let W be the collection of all isomorphisms in $\mathcal{C}$. Then W is closed under transfinite composition.*

04L9 **Definition 9.1.2.13.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. The *transfinite closure of* W is the smallest collection of morphisms of $\mathcal{C}$ which contains W and is closed under transfinite composition.

04LA **Example 9.1.2.14.** Let $\mathcal{C}$ be an ∞-category and let W be the collection of all isomorphisms in $\mathcal{C}$. It follows from Corollary 9.1.2.12 and Example 9.1.2.7 that W is the smallest collection of morphisms of $\mathcal{C}$ which is closed under transfinite composition: that is, it is the transfinite closure of the empty set.

04LB **Warning 9.1.2.15.** Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let f be a morphism of $\mathcal{C}$. If f is a transfinite composition of morphisms in W, then it belongs to the transfinite closure of W. Beware that, if we strictly adhere to the terminology of Definition 9.1.2.1, then the converse need not be true. For example, if $W = \emptyset$ and f is an isomorphism, then f belongs to the transfinite closure of W (Example 9.1.2.14). However, f is a transfinite composition of morphisms in W if and only if it is an identity morphism (Example 9.1.2.4).

We can rule out the pathological behavior described in Warning 9.1.2.15 adding a mild additional assumption.

04LC **Proposition 9.1.2.16.** *Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let $\overline{W}$ be the collection of all morphisms of $\mathcal{C}$ which are transfinite compositions of morphisms belonging to W. If W contains all identity morphisms, then $\overline{W}$ is closed under transfinite composition (and is therefore the transfinite closure of W).*

Our proof of Proposition 9.1.2.16 will make use of the following:

04LD **Lemma 9.1.2.17.** *Let B be a linearly ordered set and let $A \subseteq B$ be a subset which satisfies the following condition:*

$(*)$ *For every element $b \in B$, the set $\{a \in A : a \leq b\}$ has a largest element b_-, and the set $\{a \in A : b \leq a\}$ has a smallest element b_+.*

Let $K(A,B) \subseteq \mathrm{N}_\bullet(B)$ be the simplicial subset whose n-simplices are given by tuples $(b_0 \leq b_1 \leq b_2 \leq \cdots \leq b_n)$ which satisfy one of the following conditions:

(1) *Each of the elements b_i belongs to A.*

(2) *For every element $a \in A$, either $a \leq b_0$ or $a \geq b_n$.*

Then the inclusion map $\iota : K(A,B) \hookrightarrow \mathrm{N}_\bullet(B)$ is a categorical equivalence of simplicial sets.

Proof. Note that we can identify $K(A,B)$ with the (filtered) colimit of the simplicial subsets $K(A,B')$, where B' ranges over the collection of all subsets of B which are obtained from A by adjoining finitely many elements. Since the collection of categorical equivalences is stable under the formation of filtered colimits (Corollary 4.5.7.2), it will suffice to prove Lemma 9.1.2.17 in the special case where $B \setminus A$ is finite.

Let $A_0 \subseteq A$ be the collection of elements which have the form b_- or b_+, where b is an element of $B \setminus A$. Note that, if $A' \subseteq A$ is a subset which contains A_0 and we set $B' = A' \cup (B \setminus A)$, then the pair (A', B') also satisfies condition $(*)$. Moreover, we have $K(A',B') = K(A,B) \cap \mathrm{N}_\bullet(B')$. It follows that $K(A,B)$ can be written as a filtered colimit of simplicial subsets $K(A',B')$, where A' ranges over finite subsets of A which contain A_0. Applying Corollary 4.5.7.2 again, we are reduced to proving Lemma 9.1.2.17 under the additional assumption that A is finite. We will also assume that A is nonempty (otherwise, B is empty and therefore is nothing to prove).

If $B = \emptyset$, there is nothing to prove. We may therefore assume without loss of generality that $B = [n] = \{0 < 1 < \cdots n\}$ for some nonnegative integer n, so that $\mathrm{N}_\bullet(B)$ can be identified with the standard n-simplex Δ^n. Note that the simplicial subset $K(A,B) \subseteq \Delta^n$ contains the spine $\mathrm{Spine}[n]$ of Example 1.5.7.7. The inclusion $\mathrm{Spine}[n] \hookrightarrow \Delta^n$ is inner anodyne (Example 1.5.7.7), and therefore a categorical equivalence. It will therefore suffice to show that the inclusion map $\mathrm{Spine}[n] \hookrightarrow K(A,B)$ is also a categorical equivalence. In fact, we will show that it is inner anodyne.

Write $A = \{a_0 < a_1 < \cdots < a_m\}$, where $a_0 = 0$ and $a_m = n$. Then $\mathrm{N}_\bullet(A)$ is the image of a nondegenerate m-simplex $\sigma : \Delta^m \to \Delta^n$, given by $\sigma(i) = a_i$. Let $K' \subseteq \Delta^n$ denote the simplicial subset consisting of simplices which satisfy condition (1): more concretely, K' is the union of the images of nondegenerate simplices

$$\tau_i : \Delta^{a_i - a_{i-1}} \to \Delta^n \qquad k \mapsto k + a_{i-1}.$$

Note that the inverse image $\sigma^{-1}(K')$ identifies with the spine $\mathrm{Spine}[m]$, so that we have a pushout diagram

$$\begin{tikzcd}
\mathrm{Spine}[m] \arrow[r] \arrow[d] & \Delta^m \arrow[d, "\sigma"] \\
K' \arrow[r] & K(A,B).
\end{tikzcd}$$

Since the inclusion map $\mathrm{Spine}[m] \hookrightarrow \Delta^m$ is inner anodyne (Example 1.5.7.7), it follows that the inclusion $K' \hookrightarrow K(A,B)$ is also inner anodyne. We are therefore reduced to showing that the inclusion map $\mathrm{Spine}[n] \hookrightarrow K'$ is inner anodyne. This follows from the observation that we also have a pushout diagram

$$\begin{tikzcd}
\coprod_{1 \leq i \leq m} \mathrm{Spine}[a_i - a_{i-1}] \arrow[r] \arrow[d] & \mathrm{Spine}[n] \arrow[d] \\
\coprod_{1 \leq i \leq m} \Delta^{a_i - a_{i-1}} \arrow[r, "\{\tau_i\}"] & K',
\end{tikzcd}$$

where the left vertical map is inner anodyne by virtue of Example 1.5.7.7. □

Proof of Proposition 9.1.2.16. Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let $\overline{W}$ be the collection of morphisms which can be written as transfinite compositions of morphisms belonging to W. Suppose we are given a diagram $F : \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \alpha}) \to \mathcal{C}$ which exhibits the underlying map $f : F(0) \to F(\alpha)$ as a transfinite composition of morphisms of $\overline{W}$. We wish to show that f also belongs to $\overline{W}$.

For each ordinal $\beta < \alpha$, let $u_\beta : F(\beta) \to F(\beta + 1)$ denote the morphism of $\mathcal{C}$ obtained by evaluating F on the pair $(\beta, \beta+1)$. By assumption, u_β belongs to $\overline{W}$. We can therefore choose a well-ordered set $(B(\beta), \leq_\beta)$ and a diagram $G_\beta : \mathrm{N}_\bullet(B(\beta))^{\triangleright} \to \mathcal{C}$ which exhibits u_β as a transfinite composition of morphisms of W, in the sense of Variant 9.1.2.3. Since W contains all isomorphisms in $\mathcal{C}$, we can assume without loss of generality that each $B(\beta)$ is nonempty (see Examples 9.1.2.4 and 9.1.2.5), and therefore contains a smallest element a_β. Let a_α be an auxiliary symbol, set $B(\alpha) = \{a_\alpha\}$, and let B denote the disjoint union $\coprod_{\beta \leq \alpha} B(\beta)$. Given elements $b \in B(\beta)$ and $b' \in B(\beta')$, we write $b \leq b'$ if either $\beta < \beta'$, or $\beta = \beta'$ and $b \leq_\beta b'$. Set $A = \{a_\beta\}_{\beta \leq \alpha} \subseteq B$. The construction $\beta \mapsto a_\beta$ determines an order-preserving bijection $\mathrm{Ord}_{\leq \alpha} \xrightarrow{\sim} A$, so that the diagram F can be identified with a functor from $\mathrm{N}_\bullet(A)$ to $\mathcal{C}$. For each $\beta < \alpha$, let us identify G_β with a functor from $\mathrm{N}_\bullet(B(\beta) \cup \{a_{\beta+1}\})$ to $\mathcal{C}$. Then the functors F and $\{G_\beta\}_{\beta < \alpha}$ determine a morphism of simplicial sets $H_0 : K(A,B) \to \mathcal{C}$, where $K(A,B) \subseteq \mathrm{N}_\bullet(B)$ is the simplicial subset appearing in the statement of Lemma 9.1.2.17. Since the inclusion map $K(A,B) \hookrightarrow \mathrm{N}_\bullet(B)$ is a categorical equivalence of simplicial sets, we can extend H_0 to a diagram $H : \mathrm{N}_\bullet(B) \to \mathcal{C}$.

Note that the linear ordering on B is a well-ordering, with largest element a_{α}. We claim that H exhibits f as a transfinite composition of morphisms of W, in the sense of Variant 9.1.2.3. It follows immediately from the construction that H carries the pair $(a_0 \leq a_{\alpha})$ to the morphism f. Moreover, if an element $b \in B$ has an immediate predecessor $b' \in B$, then there is a (unique) ordinal $\beta < \alpha$ such that both b and b' belong to $B(\beta) \cup \{a_{\beta+1}\}$; our assumption on G_{β} then guarantees that the morphism $H(b') \to H(b)$ belongs to W. To complete the proof, it will suffice to show that if $b \neq a_0$ is an element of B which does *not* have an immediate predecessor, then the restriction $H|_{\mathrm{N}_{\bullet}(B_{\leq b})}$ is a colimit diagram in the ∞-category $\mathcal{C}$. Note that b belongs to $B(\beta)$ for some unique ordinal $0 \leq \beta \leq \alpha$. We consider three cases:

- Suppose that b is not equal to a_{β}. In this case, the inclusion map $B(\beta)_{<b} \hookrightarrow B_{<b}$ is cofinal (in the sense of Definition 4.7.1.26), and therefore induces a right cofinal morphism of simplicial sets $\mathrm{N}_{\bullet}(B(\beta)_{<b}) \hookrightarrow \mathrm{N}_{\bullet}(B_{<b})$ (Corollary 7.2.3.4). Using Corollary 7.2.2.2, we are reduced to showing that the diagram $H|_{\mathrm{N}_{\bullet}(B(\beta)_{\leq b})}$ is a colimit diagram in $\mathcal{C}$, which follows from our assumption on G_{β}.

- Suppose that $b = a_{\beta}$ and that $\beta = \gamma + 1$ is a successor ordinal. In this case, the inclusion map $B_{\gamma} \hookrightarrow B_{<b}$ is cofinal, and therefore induces a right cofinal morphism of simplicial sets $\mathrm{N}_{\bullet}(B_{\beta}) \hookrightarrow \mathrm{N}_{\bullet}(B_{<b})$. The desired result now follows again Corollary 7.2.2.2, since the restriction $H|_{\mathrm{N}_{\bullet}(B_{\gamma} \cup \{b\})}$ can be identified with G_{γ} and is therefore a colimit diagram in the ∞-category $\mathcal{C}$.

- Suppose that $b = a_{\beta}$, where β is a nonzero limit ordinal. In this case, the inclusion map $A_{<b} \hookrightarrow B_{<b}$ is cofinal and therefore induces a right cofinal morphism of simplicial sets $\mathrm{N}_{\bullet}(A_{<b}) \hookrightarrow \mathrm{N}_{\bullet}(B_{<b})$. Using Corollary 7.2.2.2, we are reduced to showing that the restriction $H|_{\mathrm{N}_{\bullet}(A_{\leq b})}$ is a colimit diagram in $\mathcal{C}$. We conclude by observing that this restriction identifies with $F|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \beta})}$.

□

Proposition 9.1.2.18. *Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$ which is closed under isomorphism, and let f be a morphism of $\mathcal{C}$ which is a transfinite composition of morphisms of W. If f is not an isomorphism, then it is a transfinite composition of morphisms of W which are not isomorphisms.* 04LE

Proof. Choose a diagram $F : \mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \alpha}) \to \mathcal{C}$ which exhibits f as a transfinite composition of morphisms of W. For every pair of ordinals $0 \leq \gamma \leq \beta \leq \alpha$, let $u_{\beta,\gamma} : F(\beta) \to F(\gamma)$ denote the morphism of $\mathcal{C}$ obtained by evaluating F on the edge $(\gamma \leq \beta)$ of $\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \alpha})$. We write $\beta \sim \gamma$ if, for every ordinal λ satisfying $\gamma \leq \lambda \leq \beta$, the morphisms $u_{\beta,\lambda}$ and $u_{\lambda,\gamma}$ are both isomorphisms. It is not difficult to see that this is an equivalence relation on the set

$\mathrm{Ord}_{\leq \alpha}$. For every ordinal $\beta \leq \alpha$, the equivalence class of β contains a smallest element which we will denote by β_- (since $\mathrm{Ord}_{\leq \alpha}$ is well-ordered), and a largest element which we will denote by β_+ (since the collection of isomorphisms is closed under transfinite composition; see Corollary 9.1.2.12).

Choose a subset $A \subseteq \mathrm{Ord}_{\leq \alpha}$ which contains exactly one representative of each $\sim$-equivalence class. Our assumption that $f = u_{\alpha,0}$ is not an isomorphism guarantees that 0 and α belong to different equivalence classes; we can therefore arrange that both 0 and α are contained in A. We will complete the proof by showing that the diagram $F|_{\mathrm{N}_\bullet(A)}$ exhibits f as a transfinite composition of morphisms of W which are not isomorphisms (in the sense of Variant 9.1.2.3).

For any pair of ordinals $\gamma < \beta$ which belong to A, we have inequalities $\gamma \leq \gamma_+ < \beta_- \leq \beta$. Then $u_{\beta,\gamma}$ factors as a composition

$$F(\gamma) \xrightarrow{u_{\gamma_+,\gamma}} F(\gamma_+) \xrightarrow{u_{\beta_-,\gamma_+}} F(\beta_-) \xrightarrow{u_{\beta,\beta_-}} F(\beta),$$

where the maps on the left and right are isomorphisms. In particular, $u_{\beta,\gamma}$ is isomorphic to u_{β_-,γ_+} as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$. If β is an immediate successor of γ in A, then β_- is an immediate successor of γ_+ in $\mathrm{Ord}_{\leq\alpha}$. Our assumption on F then guarantees that u_{β_-,γ_+} is contained in W. Since W is closed under isomorphism, it follows that $u_{\beta,\gamma}$ is also contained in W. Moreover, $u_{\beta,\gamma}$ cannot be an isomorphism (otherwise we would have $\beta \sim \gamma$, contradicting our assumption that A contains exactly one representative of each equivalence class).

For each element $\beta \in A$, set $A_{\leq \beta} = \{\gamma \in A : \gamma \leq \beta\}$ and $A_{<\beta} = \{\gamma \in A : \gamma < \beta\}$. To complete the proof, it will suffice to show that if $\beta \neq 0$ is not the immediate successor of another element of A, then the restriction $F|_{\mathrm{N}_\bullet(A_{\leq\beta})}$ is a colimit diagram in the ∞-category $\mathcal{C}$. Since u_{β,β_-} is an isomorphism, it will suffice to show that $G = F|_{\mathrm{N}_\bullet(A_{<\beta} \cup \{\beta_-\})}$ is a colimit diagram (Corollary 7.1.2.14). Our assumption that β has no immediate predecessor in A guarantees that β_- is a limit ordinal and that $A_{<\beta}$ is a cofinal subset of in $\mathrm{Ord}_{<\beta_-}$. It follows that the inclusion map $\mathrm{N}_\bullet(A_{<\beta}) \hookrightarrow \mathrm{N}_\bullet(\mathrm{Ord}_{<\beta})$ is right cofinal (Corollary 7.2.3.4). The desired result now follows from Corollary 7.2.2.2, since the restriction $F|_{\mathrm{N}_\bullet(\mathrm{Ord}_{\leq\beta_-})}$ is a colimit diagram in $\mathcal{C}$. □

04LF **Corollary 9.1.2.19.** *Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$ which is closed under isomorphism. Then a morphism f of $\mathcal{C}$ belongs to the transfinite closure of W if and only if f is either an isomorphism or a transfinite composition of morphisms of W.*

Proof. Assume that f belongs to the transfinite closure of W; we will show that f is either an isomorphism or a transfinite composition of morphisms of W (the converse is clear, since the transfinite closure of W contains all isomorphisms: see Remark 9.1.2.8). Let W^+ be the union of W with the collection of all identity morphisms of $\mathcal{C}$. Applying Proposition 9.1.2.16, we see that f is a transfinite composition of morphisms of W^{+}. If f is not an isomorphism, then Proposition 9.1.2.18 guarantees that f is a transfinite composition of morphisms which belong to W^{+} and are not isomorphism, and therefore belong to W. □

9.1.3 Weakly Local Objects

Let $\mathcal{C}$ be a category and let W be a collection of morphisms of $\mathcal{C}$. By definition, an object $C \in \mathcal{C}$ is W-local (in the sense of Definition 9.1.1.1) if, for every morphism $f : X \to C$ in $\mathcal{C}$ and every morphism $w : X \to Y$ which belongs to W, there is a unique morphism $g : Y \to C$ satisfying $g \circ w = f$, as indicated in the diagram 04LG

$$\begin{tikzcd}
 & Y \arrow[dr, dashed, "g"] & \\
X \arrow[ur, "w"] \arrow[rr, "f"] & & C.
\end{tikzcd}$$

It will sometimes be useful to consider the following weaker condition:

Definition 9.1.3.1. Let $\mathcal{C}$ be a category and let $w : X \to Y$ be a morphism of $\mathcal{C}$. We say that an object $C \in \mathcal{C}$ is *weakly w-local* if, for every morphism $f : X \to C$ of $\mathcal{C}$, there exists a morphism $g : Y \to C$ satisfying $g \circ w = f$. If W is a collection of morphisms of $\mathcal{C}$, we say that C is *weakly W-local* if it is weakly w-local for each $w \in W$. 04LH

Example 9.1.3.2 (Kan Complexes)**.** Let $\mathcal{C} = \mathrm{Set}_{\Delta}$ be the category of simplicial sets and let W be the collection of all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, where $n > 0$ and $0 \leq i \leq n$. Then a simplicial set is weakly W-local if and only if it is a Kan complex. 04LJ

Example 9.1.3.3 (∞-Categories)**.** Let $\mathcal{C} = \mathrm{Set}_{\Delta}$ be the category of simplicial sets and let W be the collection of all inner horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, where $0 < i < n$. Then a simplicial set is weakly W-local if and only if it is an ∞-category. 04LK

Example 9.1.3.4 (Contractible Kan Complexes)**.** Let $\mathcal{C} = \mathrm{Set}_{\Delta}$ be the category of simplicial sets and let W be the collection of inclusion maps $\partial \Delta^n \hookrightarrow \Delta^n$. Then a simplicial set is weakly W-local if and only if it is a contractible Kan complex. 04LL

Definition 9.1.3.1 has an obvious counterpart in the setting of ∞-categories:

Definition 9.1.3.5. Let $\mathcal{C}$ be an ∞-category and let $w : X \to Y$ be a morphism of $\mathcal{C}$. We say that an object $C \in \mathcal{C}$ is *weakly w-local* if, for every morphism $f : X \to C$, there exists a 04LM

2-simplex with boundary indicated in the diagram

$$\begin{tikzcd}
 & Y \arrow[dr, dashed] & \\
X \arrow[ur, "w"] \arrow[rr, "f"] & & C.
\end{tikzcd}$$

If W is a collection of morphisms of $\mathcal{C}$, we say that an object $C \in \mathcal{C}$ is *weakly W-local* if it is weakly w-local for each $w \in W$.

04LN **Example 9.1.3.6.** Let $\mathcal{C}$ be a category and let W be a collection of morphisms of $\mathcal{C}$. Then an object $C \in \mathcal{C}$ is weakly W-local (in the sense of Definition 9.1.3.1) if and only if it is weakly W-local when regarded as an object of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ (in the sense of Definition 9.1.3.5).

04LP **Remark 9.1.3.7.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. It follows from Proposition 1.4.4.2 that an object $C \in \mathcal{C}$ is weakly W-local if and only if, for every morphism $w : X \to Y$ which belongs to W, composition with the homotopy class $[w]$ induces a surjection $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, C) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, C)$. In other words, the object C is weakly W-local (in the sense of Definition 9.1.3.5) if and only if it is weakly $[W]$-local when regarded as an object of the homotopy category $\mathrm{h}\mathcal{C}$ (in the sense of Definition 9.1.3.1). Here $[W] = \{[w] : w \in W\}$ denotes the collection of all homotopy classes of morphisms which belong to W.

04LQ **Example 9.1.3.8.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. If an object $C \in \mathcal{C}$ is W-local (in the sense of Definition 9.1.1.1), then it is weakly W-local.

04LR **Example 9.1.3.9.** Let $\mathcal{C}$ be an ∞-category and let $w : X \to Y$ a morphism of $\mathcal{C}$ which admits a left homotopy inverse $r : Y \to X$. Then every object $C \in \mathcal{C}$ is weakly w-local. In particular, if w is an isomorphism, then every object of $\mathcal{C}$ is weakly w-local.

04LS **Remark 9.1.3.10.** Let $\mathcal{C}$ be an ∞-category containing a 2-simplex

$$\begin{tikzcd}
 & Y \arrow[dr, "v"] & \\
X \arrow[ur, "u"] \arrow[rr, "w"] & & Z.
\end{tikzcd}$$

If an object $C \in \mathcal{C}$ is weakly u-local and weakly v-local, then it is weakly w-local. Conversely, if C is weakly w-local, then it is weakly u-local.

Remark 9.1.3.11. Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let $C \in \mathcal{C}$ be an object which factors as a product of some collection of objects $\{C_i\}_{i \in I}$ (see Definition 7.6.1.3). If each C_i is weakly W-local, then C is weakly W-local. In particular, any final object of $\mathcal{C}$ is weakly W-local. 04LT

Remark 9.1.3.12. Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let $C \in \mathcal{C}$ be an object. If C is weakly W-local, then any retract of C is also weakly W-local. In particular, the condition that C is weakly W-local depends only on the isomorphism class of C. 04LU

Variant 9.1.3.13. Let $\mathcal{C}$ be an ∞-category, let $w : X \to Y$ and $w' : X' \to Y'$ be morphisms of $\mathcal{C}$, and suppose that w' is a retract of w (in the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{C})$). If an object $C \in \mathcal{C}$ is weakly w-local, then it is also weakly w'-local. In particular, if we regard the object $C \in \mathcal{C}$ is fixed, then the condition that C is w-local depends only on the isomorphism class of w (as an object of $\operatorname{Fun}(\Delta^1, \mathcal{C})$). 04LV

Proposition 9.1.3.14. *Let $\mathcal{C}$ be an ∞-category containing a pushout diagram* 04LW

04LX
$$\begin{tikzcd}
X \arrow[r] \arrow[d, "w"] & X' \arrow[d, "w'"] \\
Y \arrow[r] & Y'.
\end{tikzcd} \tag{9.3}$$

If an object $C \in \mathcal{C}$ is weakly w-local, then it is also weakly w'-local.

Proof. We have a commutative diagram of sets

$$\begin{tikzcd}
\operatorname{Hom}_{h\mathcal{C}}(Y', C) \arrow[r] & \operatorname{Hom}_{h\mathcal{C}}(X', C) \times_{\operatorname{Hom}_{h\mathcal{C}}(X,C)} \operatorname{Hom}_{h\mathcal{C}}(Y, C) \arrow[r] \arrow[d] & \operatorname{Hom}_{h\mathcal{C}}(X', C) \arrow[d] \\
 & \operatorname{Hom}_{h\mathcal{C}}(Y, C) \arrow[r, "\circ[w]"] & \operatorname{Hom}_{h\mathcal{C}}(X, C),
\end{tikzcd}$$

where the square on the right is a pullback. Our assumption that C is weakly w-local guarantees that the bottom horizontal map is surjective, so that the upper horizontal map on the right is also surjective. Since Proposition 9.3 is a pushout square, the horizontal map on the upper left is also surjective (Warning 7.6.3.3). It follows that the composite map $\operatorname{Hom}_{h\mathcal{C}}(Y', C) \xrightarrow{\circ[w']} \operatorname{Hom}_{h\mathcal{C}}(X', C)$ is also surjective. $\square$

Example 9.1.3.8 admits the following partial converse:

04LY **Proposition 9.1.3.15.** *Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Suppose that every morphism $w : X \to Y$ of W admits a relative codiagonal $\gamma_{X/Y} : Y \coprod_X Y \to Y$ which also belongs to W (Variant 7.6.3.19). Then an object $C \in \mathcal{C}$ is W-local if and only if it is weakly W-local.*

Proof. Fix an object $C \in \mathcal{C}$, and let $h_C : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ denote the functor represented by C. We wish to show that, for every morphism $w : X \to Y$ which belongs to W, the image $h_C(w)$ is a homotopy equivalence of Kan complexes. By virtue of Remark 3.5.1.19, it will suffice to show that $h_C(w)$ is n-connective for every integer $n \geq 0$. The proof proceeds by induction on n. In the case $n = 0$, we wish to show that the composition map $\mathrm{Hom}_{\mathcal{C}}(Y, C) \to \mathrm{Hom}_{\mathcal{C}}(X, C)$ is surjective on connected components, which follows from our assumption that C is weakly W-local. Let us therefore assume that $n > 0$. Using the criterion of Corollary 3.5.1.29 (together with Exercise 7.6.4.13), we are reduced to proving the $(n-1)$-connectivity of the relative diagonal of $h_C(w)$ (formed in the ∞-category $\mathcal{S}$). Since the functor h_C preserves limits (Proposition 7.4.5.17), we can identify the relative diagonal of $h_C(w)$ with $h_C(\gamma_{X/Y})$, where $\gamma_{X/Y} : Y \coprod_X Y \to Y$ denotes a relative codiagonal of w. By assumption, we can arrange that $\gamma_{X/Y}$ is also contained in W, so the desired result follows from our inductive hypothesis. $\square$

04LZ **Exercise 9.1.3.16.** Let $\mathcal{C}$ be an ∞-category and let $w : X \to Y$ be a morphism of $\mathcal{C}$ which admits a relative codiagonal $\gamma_{X/Y} : Y \coprod_X Y \to Y$. Show that an object $C \in \mathcal{C}$ is w-local if and only if it is both $\gamma_{X/Y}$-local and weakly w-local.

We have the following ∞-categorical counterpart of Proposition 1.5.4.11:

04M0 **Proposition 9.1.3.17.** *Let $\mathcal{C}$ be an ∞-category, let $C \in \mathcal{C}$ be an object, and let W be the collection of morphisms $w : X \to Y$ in $\mathcal{C}$ such that C is weakly w-local. Then W is closed under transfinite composition (see Definition 9.1.2.1).*

Proof. Let $U : \mathcal{C}_{/C} \to \mathcal{C}$ be the projection map and let $f : X \to Y$ be a morphism of $\mathcal{C}$ which can be written as a transfinite composition of morphisms of $\mathcal{C}$. Suppose we are given a morphism $X \to C$ in $\mathcal{C}$, which we identify with an object $\widetilde{X} \in \mathcal{C}_{/C}$ satisfying $U(\widetilde{X}) = X$. We wish to show that there is a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ of $\mathcal{C}_{/C}$ satisfying $U(\widetilde{f}) = f$.

Choose an ordinal α and a functor $F : \mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \alpha}) \to \mathcal{C}$ which exhibits f as a transfinite composition of morphisms of W. Let Q be the collection of all ordered pairs $(\beta, \widetilde{F}_{\leq \beta})$, where $\beta \leq \alpha$ is an ordinal and $\widetilde{F}_{\leq \beta} : \mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \beta}) \to \mathcal{C}_{/C}$ is a functor satisfying $\widetilde{F}_{\leq \beta}(0) = \widetilde{X}$ and $U \circ \widetilde{F}_{\leq \beta} = F|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \beta})}$. We regard Q as a partially ordered set, where $(\beta, \widetilde{F}_{\leq \beta}) \leq (\beta', \widetilde{F}'|_{\leq \beta'})$ if $\beta \leq \beta'$ and $\widetilde{F}_{\leq \beta} = \widetilde{F}'_{\leq \beta'}|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \beta})}$.

We first claim that Q satisfies the hypotheses of Zorn's lemma. Let $Q_0 \subseteq Q$ be a linearly ordered subset of Q; we wish to show that Q_0 admits an upper bound. If Q_0 is empty, we can take this upper bound be the pair $(0, \widetilde{F}_{\leq 0})$, where $\widetilde{F}_{\leq 0}$ is the constant functor taking the value $\widetilde{X}$. Without loss of generality, we may assume that Q_0 does not contain a maximal element (otherwise, there is nothing to prove). Write $Q_0 = \{(\beta_i, \widetilde{F}_{\leq \beta_i})\}_{i \in I}$ and let $\beta \leq \alpha$ be the supremum of the set $\{\beta_i\}_{i \in I}$. The functors $\widetilde{F}_{\leq \beta_i}$ can then be amalgamated to a single functor $\widetilde{F}_{<\beta} : \mathrm{N}_{\bullet}(\mathrm{Ord}_{<\beta}) \to \mathcal{C}_{/C}$. To find an upper bound for Q_0, it will suffice to show that the lifting problem

$$\begin{tikzcd}
\mathrm{N}_{\bullet}(\mathrm{Ord}_{<\beta}) \arrow[r, "\widetilde{F}_{<\beta}"] \arrow[d] & \mathcal{C}_{/C} \arrow[d, "U"] \\
\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq\beta}) \arrow[r, "F|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq\beta})}"] & \mathcal{C}
\end{tikzcd}$$

admits a solution. This follows immediately from our assumption that $F|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq\beta})}$ is a colimit diagram in $\mathcal{C}$.

Applying Zorn's lemma, we deduce that Q contains a maximal element $(\beta, \widetilde{F}_{\leq\beta})$. To complete the proof, it will suffice to show that $\beta = \alpha$; we can then take $\widetilde{f}$ to be obtained by applying the functor $\widetilde{F}$ to the edge of $\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq\alpha})$ given by the pair $(0, \alpha)$. Assume otherwise, let $F_{\leq\beta}$ denote the restriction $F|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq\beta})}$, and let $\mathcal{D}$ denote the coslice ∞-category $\mathcal{C}_{F_{\leq\beta}/}$. Then $F_{\leq\beta+1}$ and $\widetilde{F}_{\leq\beta}$ can be identified with objects $D, D' \in \mathcal{D}$, and the maximality of $(\beta, \widetilde{F}_{\leq\beta})$ guarantees that $\mathrm{Hom}_{\mathcal{D}}(D, D') = \emptyset$. Since the inclusion map $\{\beta\} \hookrightarrow \mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq\beta})$ is right anodyne (Corollary 4.6.7.24), the restriction map $V : \mathcal{D} = \mathcal{C}_{F_{\leq\beta}/} \to \mathcal{C}_{F(\beta)/}$ is a trivial Kan fibration (Corollary 4.3.6.13). It follows that the mapping space $\mathrm{Hom}_{\mathcal{C}_{F(\beta)/}}(V(D), V(D'))$ is also empty: that is, there is no 2-simplex of $\mathcal{C}$ with boundary indicated in the diagram

$$\begin{tikzcd}
 & F(\beta+1) \arrow[dr, dashed] & \\
F(\beta) \arrow[ur] \arrow[rr] & & C.
\end{tikzcd}$$

This contradicts our assumption that the morphism $F(\beta) \to F(\beta+1)$ belongs to W. □

Variant 9.1.3.18. Let $\mathcal{C}$ be an ∞-category, let $C \in \mathcal{C}$ be an object, and let W be the collection of morphisms $w : X \to Y$ in $\mathcal{C}$ such that C is w-local. Then W is closed under transfinite composition. 04M1

Proof. Let $\mathcal{C}$ be an ∞-category, let $C \in \mathcal{C}$ be an object, and let W be the collection of morphisms w of $\mathcal{C}$ such that C is w-local. Then W contains all identity morphisms, is closed under composition (Remark 9.1.1.11), and is closed under the formation of colimits in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$ which are preserved by the evaluation functors $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$

(Remark 9.1.1.9). Applying Proposition 9.1.2.10 (and Remark 9.1.2.11), we conclude that W is closed under transfinite composition. □

We now introduce some terminology which is motivated by the preceding discussion.

04M2 **Definition 9.1.3.19.** Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. We will say that W is *weakly saturated* if it satisfies the following conditions:

(1) The collection W is closed under pushouts: that is, for every pushout diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "w"] & X' \arrow[d, "w'"] \\
Y \arrow[r] & Y'
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, if w belongs to W, then w' also belongs to W.

(2) The collection W is closed under the formation of retracts (in the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{C})$).

(3) The collection W is closed under transfinite composition (Definition 9.1.2.1).

04M3 **Remark 9.1.3.20.** Let $\mathcal{C} = \mathrm{N}_{\bullet}(\mathcal{C}_0)$ be the nerve of an ordinary category $\mathcal{C}_0$. Then a collection of morphisms of $\mathcal{C}$ is weakly saturated (in the sense of Definition 9.1.3.19) if and only if is weakly saturated when regarded as a collection of morphisms of $\mathcal{C}_0$ (in the sense of Definition 1.5.4.12).

04M4 **Example 9.1.3.21.** Let $\mathcal{C}$ be an ∞-category, let $C \in \mathcal{C}$ be an object, and let W be the collection of all morphisms $w : X \to Y$ of $\mathcal{C}$ such that C is weakly w-local. Then W is weakly saturated. This follows from Proposition 9.1.3.14, Variant 9.1.3.13, and Proposition 9.1.3.17.

04M5 **Variant 9.1.3.22.** Let $\mathcal{C}$ be an ∞-category, let $C \in \mathcal{C}$ be an object, and let W be the collection of all morphisms $w : X \to Y$ of $\mathcal{C}$ such that C is w-local. Then W is weakly saturated. This follows from Remark 9.1.1.10, Remark 9.1.1.11, and Variant 9.1.3.18.

04M6 **Remark 9.1.3.23.** Let $\mathcal{C}$ be an ∞-category. Any intersection of weakly saturated collections of morphisms of $\mathcal{C}$ is also weakly saturated. In particular, for any collection W of morphisms of $\mathcal{C}$, there is a smallest collection $\overline{W}$ which is weakly saturated and contains W. We will refer to $\overline{W}$ as the *weakly saturated collection generated by* W.

9.1.4 The Small Object Argument

In §3.1.7, we showed that every simplicial set X admits an anodyne morphism $X \hookrightarrow Q$, where Q is a Kan complex (Corollary 3.1.7.2). The proof is easy to describe: if X is not a Kan complex, then there is some horn $\sigma_0 : \Lambda^n_i \to X$ which cannot be extended to an n-simplex of X. This defect can be remedied by replacing X by the pushout $\Delta^n \coprod_{\Lambda^n_i} X$. The desired Kan complex Q is obtained by a (possibly transfinite) iteration of this procedure. A similar strategy can be used to prove many related results (see for example Exercise 3.1.7.11, Proposition 4.1.3.2, and Proposition 4.2.4.7). Following Quillen ([46]), we will refer to this proof strategy as the *small object argument*. Our goal in this section is to formalize a version of this argument in the ∞-categorical setting. First, we need a bit of terminology. 04M7

Definition 9.1.4.1. Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let W' denote the collection of those morphisms $w' : X' \to Y'$ for which there exists a pushout square 04M8

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "w"] & X' \arrow[d, "w'"] \\
Y \arrow[r] & Y',
\end{tikzcd}$$

where $w \in W$. We say that a morphism f of $\mathcal{C}$ is a *transfinite pushout of morphisms of* W if it is a transfinite composition of morphisms of W', in the sense of Definition 9.1.2.1.

Remark 9.1.4.2. Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let $f : X \to Y$ be a morphism of $\mathcal{C}$ which is a transfinite pushout of morphisms which belong to W. Then: 04M9

- If an object $C \in \mathcal{C}$ is weakly W-local, then it is weakly f-local.
- If an object $C \in \mathcal{C}$ is W-local, then it is f-local.

The first assertion follows from Propositions 9.1.3.14 and 9.1.3.17; the second follows from Remark 9.1.1.10 and Variant 9.1.3.18.

We can now formulate the main result of this section:

Theorem 9.1.4.3 (The Small Object Argument)**.** *Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Assume that:* 04MA

- *The ∞-category $\mathcal{C}$ is locally small and admits small colimits.*
- *The collection W is small.*

- *For each morphism $w : X \to Y$ which belongs to W, the object $X \in \mathcal{C}$ is κ-compact for some small cardinal κ (see Definition [?]).*

For every object $C \in \mathcal{C}$, there exists a morphism $f : C \to C'$ where C' is weakly W-local and f is a transfinite pushout of morphisms of W.

04MB **Warning 9.1.4.4.** If $\mathcal{C}$ is (the nerve of) an ordinary category, then the morphism $f : C \to C'$ of Theorem 9.1.4.3 can be chosen to depend functorially on C. Beware that this is generally not possible if $\mathcal{C}$ is an ∞-category (see Example 9.1.4.5).

04MC **Example 9.1.4.5.** Fix an integer $n \geq 0$. Let $\mathcal{C} = \mathcal{S}$ be the ∞-category of spaces and let W be the collection of morphisms of $\mathcal{C}$ given by the inclusion maps $\{\mathrm{Ex}^{\infty}(\partial \Delta^m) \hookrightarrow \mathrm{Ex}^{\infty}(\Delta^m)\}_{0 \leq m \leq n}$. Then an object $X \in \mathcal{C}$ weakly W-local if and only if it is n-connective (see Definition 3.5.1.1). In this case, Theorem 9.1.4.3 asserts that every Kan complex X admits a morphism $f : X \to Y$, where Y is an n-connective Kan complex which can be obtained from X by attaching cells of dimension $\leq n$. Beware that, if $n > 0$, then Y cannot be chosen to depend functorially on X.

04MD **Corollary 9.1.4.6.** *Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, and let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by the W-local objects. Assume that:*

- *The ∞-category $\mathcal{C}$ is locally small and admits small colimits.*
- *The collection W is small.*
- *For each morphism $w : X \to Y$ which belongs to W, the objects X and Y are κ-compact for some small infinite cardinal κ.*

Then $\mathcal{C}'$ is a reflective localization of $\mathcal{C}$.

Proof. For each morphism $w : X \to Y$ of $\mathcal{C}$, choose a morphism $\delta_w : Y \coprod_X Y \to Y$ which is a relative codiagonal of w (see Variant 7.6.3.19). Note that, if X and Y are κ-compact for some infinite cardinal κ, then the pushout $Y \coprod_X Y$ is also κ-compact (Proposition [?]). Let W' be the smallest collection of morphisms of $\mathcal{C}$ which contains W and is closed under the construction $w \mapsto \gamma_w$. By virtue of Remark 9.1.1.12, an object of $\mathcal{C}$ is W-local if and only if it is W'-local. We may therefore replace W by W' and thereby reduce to proving Corollary 9.1.4.6 in the special case where W is closed under the formation of relative codiagonals.

Fix an object $C \in \mathcal{C}$. Using Theorem 9.1.4.3, we see that there exists a morphism $f : C \to C'$, where C' is weakly W-local and f is a transfinite pushout of morphisms which belong to W. Using Proposition 9.1.3.15, we see that C' belongs to the subcategory $\mathcal{C}' \subseteq \mathcal{C}$. To complete the proof, it will suffice to show that f exhibits C' as a $\mathcal{C}'$-reflection of C: that is, every object of $\mathcal{C}'$ is f-local. This follows from Remark 9.1.4.2. □

Our proof of Theorem 9.1.4.3 will require some preliminaries.

Lemma 9.1.4.7. *Let $\mathcal{C}$ be an ∞-category which admits small colimits, let $\{w_s : X_s \to Y_s\}_{s \in S}$ be a small collection of morphisms of $\mathcal{C}$ indexed by a set S, let $\{e_s : X_s \to C\}_{s \in S}$ be another collection of morphisms of $\mathcal{C}$, and let D be a colimit of the diagram* 04ME

$$(S \times \Delta^1) \coprod_{S \times \{0\}} S^{\triangleright} \xrightarrow{(\{w_s\}, \{e_s\})} \mathcal{C}.$$

Then the tautological map $u : C \to D$ is a transfinite pushout of morphisms belonging to $\{w_s\}_{s \in S}$.

Proof. Using the well-ordering theorem (Theorem 4.7.1.34), we can choose an ordinal α and a bijection $\ell : S \to \mathrm{Ord}_{<\alpha}$. Let Q denote the disjoint union $(S \times [1]) \coprod \mathrm{Ord}_{\leq \alpha}$. For elements $q, q' \in Q$, we write $q \leq q'$ if (exactly) one of the following conditions holds:

- There exist an element $s \in S$ such that $q = (s, i)$ and $q' = (s, i')$, where $i \leq i'$.
- We have $q = \beta$ and $q' = \beta'$ for ordinals $\beta, \beta' \in \mathrm{Ord}_{\leq \alpha}$ satisfying $\beta \leq \beta'$ (for the usual ordering of $\mathrm{Ord}_{\leq \alpha}$).
- We have $q = (s, 0)$ for some $s \in S$ and $q' = \beta$ for some $\beta \in \mathrm{Ord}_{\leq \alpha}$.
- We have $q = (s, 1)$ and $q' = \beta$ for some $\beta \in \mathrm{Ord}_{\leq \alpha}$ satisfying $\ell(s) < \beta$.

Let $Q_0 = (S \times [1]) \coprod \{0\}$, which we regard as a partially ordered subset of Q. By construction, the nerve $N_{\bullet}(Q_0)$ can be identified with the pushout $(S \times \Delta^1) \coprod_{S \times \{0\}} S^{\triangleright}$. Consequently, the collections $\{e_s\}_{s \in S}$ and $\{w_s\}_{s \in S}$ determine a diagram $F_0 : N_{\bullet}(Q_{\leq 0}) \to \mathcal{C}$. Since $\mathcal{C}$ admits small colimits, the diagram F_0 admits a left Kan extension $F : N_{\bullet}(Q) \to \mathcal{C}$ (Proposition 7.6.7.13). Then $D = F(\alpha)$ is a colimit of the diagram F_0, and we can identify u with the morphism obtained by evaluating the functor F on the edge of $N_{\bullet}(Q)$ given by the pair $(0 \leq \alpha)$. We will show that the restriction $F|_{N_{\bullet}(\mathrm{Ord}_{\leq \alpha})}$ exhibits u as a transfinite pushout of morphisms belonging to $\{w_s\}_{s \in S}$.

We first claim that that if $\beta \leq \alpha$ is a nonzero limit ordinal, then the restriction $F|_{N_{\bullet}(\mathrm{Ord}_{\leq \beta})}$ is a colimit diagram in $\mathcal{C}$. Set $Q_{\leq \beta} = \{q \in Q : q \leq \beta\}$ and $Q_{<\beta} = \{q \in Q : q < \beta\}$. Since the functor F is left Kan extended from $N_{\bullet}(Q_0)$, it is also left Kan extended from larger ∞-category $N_{\bullet}((S \times [1]) \coprod \mathrm{Ord}_{<\beta})$ (see Corollary 7.3.8.8). It follows that $F|_{N_{\bullet}(Q_{\leq \beta})}$ is a colimit diagram in $\mathcal{C}$. It will therefore suffice to show that $N_{\bullet}(\iota)$ is right cofinal, where ι denotes the inclusion map $\mathrm{Ord}_{<\beta} \hookrightarrow Q_{<\beta}$ (Corollary 7.2.2.2). This is a special case of Corollary 7.2.3.7, since ι admits a left adjoint (given on $S \times [1]$ by the construction $(s, 0) \mapsto 0$ and $(s, 1) \mapsto \ell(s) + 1$).

Now suppose that $\beta = \gamma + 1$ is a successor ordinal. Let $s \in S$ be the unique element satisfying $\ell(s) = \gamma$. We will complete the proof by showing that the morphism $F(\gamma) \to F(\beta)$

in $\mathcal{C}$ can be realized as a pushout of w_s. More precisely, we will show that the functor F carries the diagram

$$\begin{tikzcd}
(s,0) \arrow[r] \arrow[d] & (s,1) \arrow[d] \\
\gamma \arrow[r] & \beta
\end{tikzcd}$$

in $\mathrm{N}_\bullet(Q)$ to a pushout diagram in the ∞-category $\mathcal{C}$. Arguing as above, we see that the restriction $F|_{Q_{\leq \beta}}$ is a colimit diagram. Using Corollary 7.2.2.2 again, we are reduced to showing that $\mathrm{N}_\bullet(\iota)$ is right cofinal, where ι denotes the inclusion of partially ordered sets $\{(s,1) > (s,0) < \beta\} \hookrightarrow Q_{<\beta}$. This again follows from Corollary 7.2.3.7, since ι admits a left adjoint given by the construction

$$(q \in Q_{<\beta}) \mapsto \begin{cases} q & \text{if } q = (s,0) \text{ or } q = (s,1) \\ \beta & \text{otherwise.} \end{cases}$$

□

04MF **Lemma 9.1.4.8.** *Let $\mathcal{C}$ be a locally small ∞-category which admits small colimits, and let W be a small collection of morphisms of $\mathcal{C}$. For every object $C \in \mathcal{C}$, there exists a morphism $u : C \to D$ which is a transfinite pushout of morphisms of W with the following property: for every morphism $w : X \to Y$ which belongs to W and every morphism $e : X \to C$, there exists a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "w"] \arrow[d, "e"] & Y \arrow[d] \\
C \arrow[r, "u"] & D
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$.

Proof. For each morphism $w : X \to Y$ belonging to W, let $\{f_s : X \to C\}_{s \in S_w}$ be a set of representatives for the homotopy classes of morphisms from X to C. Since $\mathcal{C}$ is locally small, the collection S_w is small. Our assumption that W is small then guarantees that the disjoint union $S = \coprod_{w \in W} S_w$ is small. The desired result now follows by applying Lemma 9.1.4.7 to the collection of morphisms $\{f_s\}_{s \in S}$. □

04MG **Lemma 9.1.4.9.** *Let $\mathcal{C}$ be an ∞-category which admits small filtered colimits. Let U be a collection of morphisms with the following property: for every object $D \in \mathcal{C}$, there exists a morphism $u : D \to E$ which belongs to U. Then, for every object $C \in \mathcal{C}$ and every (small) ordinal α, there exists a diagram $F : \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \alpha}) \to \mathcal{C}$ satisfying the following conditions:*

(*a*) *For every nonzero limit ordinal* $\lambda \leq \alpha$*, the restriction* $F|_{\mathrm{N}_\bullet(\mathrm{Ord}_{\leq \lambda})}$ *is a colimit diagram.*

(*b*) *For every ordinal* $\gamma < \alpha$*, the morphism* $F(\gamma) \to F(\gamma+1)$ *belongs to* U*.*

(*c*) *The object* $F(0)$ *coincides with* C*.*

Proof. Let Q denote the collection of all diagrams $F_\beta : \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \beta}) \to \mathcal{C}$ satisfying conditions (*a*), (*b*), and (*c*), where β is an ordinal $\leq \alpha$. We regard Q as a partially ordered set, where $F_\beta \leq F_{\beta'}$ if $\beta \leq \beta'$ and $F_\beta = F_{\beta'}|_{\mathrm{N}_\bullet(\mathrm{Ord}_{\leq \beta})}$. Note that Q is nonempty: it has a least element given by the diagram $\mathrm{N}_\bullet(\mathrm{Ord}_{\leq 0}) \simeq \{C\} \hookrightarrow \mathcal{C}$ taking the value C. We claim that Q satisfies the hypothesis of Zorn's lemma: that is, every linearly ordered set $Q' \subset Q$ admits an upper bound. Without loss of generality, we may assume that Q' is nonempty and has no largest element. In this case, the elements of Q' can be amalgamated to a diagram $F_{<\lambda} : \mathrm{N}_\bullet(\mathrm{Ord}_{<\lambda}) \to \mathcal{C}$, where $\lambda \leq \alpha$ is a nonzero limit ordinal. Our assumption on $\mathcal{C}$ then guarantees that $F_{<\lambda}$ can be extended to a colimit diagram

$$F_\lambda : \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \lambda}) \simeq \mathrm{N}_\bullet(\mathrm{Ord}_{<\lambda})^{\triangleright} \to \mathcal{C}\,.$$

By construction, this diagram satisfies conditions (*a*), (*b*), and (*c*), and is therefore an upper bound for Q'.

Applying Zorn's lemma, we deduce that Q has a maximal element $F_\beta : \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \beta}) \to \mathcal{C}$. We will complete the proof by showing that $\beta < \alpha$. Assume otherwise, and set $X = F_\beta(\beta)$. By assumption, we can choose a morphism $u : X \to Y$ which belongs to U. Let us identify u with an object $\widetilde{Y}$ of the coslice ∞-category $\mathcal{C}_{X/}$. Since the inclusion map $\{\beta+1\} \hookrightarrow \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \beta+1})$ is right anodyne (Example 4.3.7.11), the restriction map $\mathcal{C}_{F_\beta/} \to \mathcal{C}_{X/}$ is a trivial Kan fibration (Corollary 4.3.6.13). We can therefore lift $\widetilde{Y}$ to an object of the ∞-category $\mathcal{C}_{F_\beta/}$, which we can identify with an extension of F_β to a diagram $F_{\beta+1} : \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \beta+1}) \to \mathcal{C}$. By construction, this diagram carries the pair $(\beta, \beta+1)$ to the morphism u of $\mathcal{C}$. It follows that $F_{\beta+1}$ is also an element of Q, contradicting the maximality of F_β. □

Proof of Theorem 9.1.4.3. Let $\mathcal{C}$ be a locally small ∞-category which admits small colimits, let W be a small collection of morphisms of $\mathcal{C}$, and let κ be a small regular cardinal having the property that for each morphism $w : X \to Y$ which belongs to W, the object X is κ-compact. Fix an object $C \in \mathcal{C}$; we wish to show that there exists a morphism $f : C \to C'$ where C' is weakly W-local and f is a transfinite pushout of morphisms belonging to W. Without loss of generality, we may assume that C itself is not weakly W-local (otherwise, we can take $f = \mathrm{id}_C$).

Let W' be the collection of all morphisms of $\mathcal{C}$ which are pushouts of morphisms of W, and let $\overline{W}$ denote the transfinite closure of W' (Definition 9.1.2.13). Let $U \subseteq \overline{W}$ denote the subcollection consisting of those morphisms u which satisfy the requirement of Lemma 9.1.4.8. Using Lemma 9.1.4.9 we deduce that there exists a diagram $F : \mathrm{N}_\bullet(\mathrm{Ord}_{\leq \kappa}) \to \mathcal{C}$

where $F(0) = C$ and F exhibits the induced map $F(0) \xrightarrow{f} F(\kappa)$ as a transfinite composition of morphisms of U.

We first claim that the object $C' = F(\kappa)$ is weakly W-local. Let $w : X \to Y$ be a morphism which belongs to W. We wish to show that every morphism $[\overline{e}] : X \to F(\kappa)$ in the homotopy category $\mathrm{h}\mathcal{C}$ factors through the homotopy class $[w]$. Since F is a colimit diagram and the object X is κ-compact, the morphism $[\overline{e}]$ factors as a composition $X \xrightarrow{[e]} F(\alpha) \to F(\kappa)$ for some ordinal $\alpha < \kappa$ and some morphism $e : X \to F(\alpha)$ in the ∞-category $\mathcal{C}$. Since the transition map $F(\alpha) \to F(\alpha + 1)$ belongs to U, we can choose a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "w"] \arrow[d, "e"] & Y \arrow[d, "e'"] \\
F(\alpha) \arrow[r] & F(\alpha + 1).
\end{tikzcd}$$

It follows that $[\overline{e}]$ factors as a composition $X \xrightarrow{[w]} Y \xrightarrow{[e']} F(\alpha + 1) \to F(\kappa)$.

To complete the proof, it will suffice to show that f is a transfinite composition of morphisms belonging to W'. By construction, f is a transfinite pushout of morphisms of $U \subseteq \overline{W}$, and therefore belongs to $\overline{W}$. This follows from Corollary 9.1.2.19, since f is not an isomorphism (otherwise, the object C would also be weakly W-local, contrary to our initial assumption). □

9.1.5 Lifting Problems in ∞-Categories

04MH Let $\mathcal{C}$ be a category. Recall that a *lifting problem in $\mathcal{C}$* is a commutative diagram

04MJ
$$\begin{tikzcd}
A \arrow[r, "u_0"] \arrow[d, "f"] & X \arrow[d, "g"] \\
B \arrow[r, "\overline{u}"] & Y
\end{tikzcd} \tag{9.4}$$

In this case, a *solution* to the lifting problem (9.4) is a morphism $u : B \to X$ satisfying $u \circ f = u_0$ and $q \circ u = \overline{u}$ (see Definition 1.5.4.1). This definition has an obvious ∞-categorical counterpart:

04MK **Definition 9.1.5.1** (Lifting Problems in ∞-Categories)**.** Let $\mathcal{C}$ be an ∞-category. A *lifting problem in $\mathcal{C}$* is a diagram $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$. In this case, a *solution* to the lifting problem σ is a 3-simplex $\overline{\sigma} : \Delta^3 \to \mathcal{C}$ for which the composition

$$\Delta^1 \times \Delta^1 \xrightarrow{\alpha} \Delta^3 \xrightarrow{\overline{\sigma}} \mathcal{C}$$

coincides with σ, where α denotes the map of simplicial sets given on vertices by $\alpha(i,j) = 2i + j$.

Remark 9.1.5.2. Let us informally display the standard simplex Δ^3 as a diagram 04ML

04MM
$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[ur, dashed] & \bullet .
\end{tikzcd} \tag{9.5}$$

The morphism $\alpha : \Delta^1 \times \Delta^1 \to \Delta^3$ appearing in Definition 9.1.5.1 is a monomorphism of simplicial sets, whose image is the simplicial subset $Q \subseteq \Delta^3$ consisting of those simplices which do not contain the "inner" edge $\mathrm{N}_\bullet(\{1 < 2\})$ which is indicated by the dotted arrow in the diagram (9.5). Stated more informally, Q is the subset of Δ^3 which is "visible from the top" in the diagram (9.5); in particular, Q contains the inner faces $\mathrm{N}_\bullet(\{0 < 1 < 3\})$ and $\mathrm{N}_\bullet(\{0 < 2 < 3\})$, but not the outer faces $\mathrm{N}_\bullet(\{0 < 1 < 2\})$ and $\mathrm{N}_\bullet(\{1 < 2 < 3\})$.

Notation 9.1.5.3. Let $\mathcal{C}$ be an ∞-category. We will often denote a lifting problem σ in $\mathcal{C}$ by a diagram 04MN

04MP
$$\begin{tikzcd}
A \arrow[r, "u_0"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "\overline{u}"] \arrow[ur, dashed] & Y .
\end{tikzcd} \tag{9.6}$$

Here the dotted arrow in the diagram does not indicate part of the data supplied by lifting problem σ; instead, it indicates part of the data of a hypothetical solution.

Stated more concretely, the lifting problem σ is given by the following data:

- Four objects of $\mathcal{C}$, which are indicated by A, B, X, and Y in the diagram (9.6).
- Five morphisms of $\mathcal{C}$, which we will denote by $f : A \to B$, $g : X \to Y$, $u_0 : A \to X$, $\overline{u} : B \to Y$, and $\overline{u}_0 : A \to Y$. Here the first four of these morphisms are indicated as outer edges of the diagram (9.6), while the fifth is left implicit.
- A pair of 2-simplices τ_1 and τ_2 of $\mathcal{C}$, whose boundaries are indicated in the diagrams

$$\begin{tikzcd}
A \arrow[r, "u_0"] \arrow[dr, "\overline{u}_0"'] & X \arrow[d, "g"] \\
& Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, "\overline{u}_0"] & \\
B \arrow[r, "\overline{u}"] & Y .
\end{tikzcd}$$

In other words, τ_1 and τ_2 exhibit the morphism $\overline{u}_0$ as a composition $\overline{u} \circ f$ and a composition $g \circ u_0$, respectively.

A solution to the lifting problem σ is given by the following additional data:

- A morphism $u : B \to X$ (indicated by the dotted arrow in the diagram (9.6).
- A pair of 2-simplices τ_0 and τ_3 of $\mathcal{C}$, whose boundaries are indicated in the diagrams

$$\begin{tikzcd} & X \arrow[d, "g"] \\ B \arrow[ur, "u"] \arrow[r, "\overline{u}"] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u_0"] \arrow[d, "f"] & X \\ B. \arrow[ur, "u"'] & \end{tikzcd}$$

In other words, τ_0 exhibits $\overline{u}$ as a composition $g \circ u$, and τ_3 exhibits u_0 as a composition $u \circ f$.

- A 3-simplex of $\mathcal{C}$ having boundary $(\tau_0, \tau_1, \tau_2, \tau_3)$.

04MQ **Example 9.1.5.4.** Let $\mathcal{C}_0$ be a category and let $\mathcal{C} = \mathrm{N}_\bullet(\mathcal{C}_0)$ denote its nerve. Then lifting problems σ in the ∞-category $\mathcal{C}$ (in the sense of Definition 9.1.5.1) can be identified with lifting problems σ_0 in the ordinary category $\mathcal{C}_0$ (in the sense of Definition 1.5.4.1). In this case, we can also identify solutions to σ with solutions to σ_0.

04MR **Warning 9.1.5.5.** Let $\mathcal{C}$ be an ∞-category. Every lifting problem

04MS
$$\begin{tikzcd} A \arrow[r, "u_0"] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r, "\overline{u}"] \arrow[ur, dashed] & Y \end{tikzcd} \tag{9.7}$$

in $\mathcal{C}$ determines a lifting problem in the homotopy category $\mathrm{h}\mathcal{C}$, given by the diagram

04MT
$$\begin{tikzcd} A \arrow[r, "{[u_0]}"] \arrow[d, "{[f]}"] & X \arrow[d, "{[g]}"] \\ B \arrow[r, "{[\overline{u}]}"] \arrow[ur, dashed] & Y. \end{tikzcd} \tag{9.8}$$

Moreover, every solution to the lifting problem (9.7) determines a solution to the lifting problem (9.8). Beware that the converse is false: it is possible for the lifting problem (9.8) to admit a solution when the lifting problem (9.7) does not (Exercise 9.1.5.6).

Exercise 9.1.5.6. Let $g : X \to S$ be Kan fibration between Kan complexes, where S is connected and X is contractible. Choose a vertex $s \in S$ and let X_s denote the fiber $\{s\} \times_S X$, so that we have a commutative diagram of Kan complexes 04MU

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{s\} \arrow[r] & S \end{tikzcd} \tag{9.9}$$ 04MV

Show that:

- In the ∞-category of spaces $\mathcal{S}$, the lifting problem determined by (9.9) admits a solution only if S is contractible.
- In the homotopy category $\mathrm{h}\mathcal{S}$, the lifting problem determined by (9.9) always has a solution.

Remark 9.1.5.7. Let $\mathcal{C}$ be an ∞-category. Fix a morphism $\overline{u}_0 : A \to Y$ of $\mathcal{C}$, and let $\mathcal{C}_{A//Y}$ denote the ∞-category of Remark 4.6.6.2. The datum of an extension of $\overline{u}_0$ to a lifting problem σ : 04MW

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{9.10}$$ 04MX

can be identified with a pair of objects $\widetilde{B}, \widetilde{X} \in \mathcal{C}_{A//Y}$. In this case, a solution to the lifting problem (9.10) is a morphism from $\widetilde{B}$ to $\widetilde{X}$ in the ∞-category $\mathcal{C}_{A//Y}$.

Definition 9.1.5.8. Let $\mathcal{C}$ be an ∞-category, and let $f : A \to B$ and $g : X \to Y$ be morphisms of $\mathcal{C}$. We will say that f *is weakly left orthogonal to* g if every lifting problem 04MY

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in the ∞-category $\mathcal{C}$ admits a solution. In this case, we will also say that g is *weakly right orthogonal* to f.

04MZ **Example 9.1.5.9.** In the situation of Definition 9.1.5.8, suppose that $\mathcal{C}$ is the nerve of a category $\mathcal{C}_0$. Then f is weakly left orthogonal to g (in the sense of Definition 9.1.5.8) if and only if it is weakly left orthogonal to g when regarded as a morphism of the category $\mathcal{C}_0$ (in the sense of Definition 1.5.4.3).

04N0 **Warning 9.1.5.10.** Let $\mathcal{C}$ be an ∞-category containing a pair of morphisms $f : A \to B$ and $g : X \to Y$. If f is weakly left orthogonal to g in the ∞-category $\mathcal{C}$, then the homotopy class $[f]$ is weakly left orthogonal to $[g]$ in the homotopy category $\mathrm{h}\mathcal{C}$ (see Exercise 1.5.2.10). Beware that the converse is false in general (see Warning 9.1.5.5 and Exercise 9.1.5.6).

04N1 **Variant 9.1.5.11.** Let $\mathcal{C}$ be an ∞-category and let S and T be collections of morphisms of $\mathcal{C}$. We say that *S is weakly left orthogonal to T* if every morphism $f \in S$ is weakly left orthogonal to every morphism $g \in T$. In this case, we also say that *T is weakly right orthogonal to S*. In the special case where $S = \{f\}$ is a singleton, we abbreviate this condition by saying that *f is weakly left orthogonal to T*, or *T is weakly right orthogonal to f*. In the special case $T = \{g\}$ is a singleton, we abbreviate this condition by saying that *g is weakly right orthogonal to S*, or *S is weakly left orthogonal to g*.

04N2 **Remark 9.1.5.12.** Let $\mathcal{C}$ be an ∞-category containing a morphism $g : X \to Y$, which we identify with an object $\widetilde{X}$ of the slice ∞-category $\mathcal{C}_{/Y}$. Let S be a collection of morphisms of $\mathcal{C}$, and let $\widetilde{S}$ denote its inverse image in $\mathcal{C}_{/Y}$. The following conditions are equivalent:

- The morphism g is weakly right orthogonal to S (in the sense of Variant 9.1.5.11).
- The object $\widetilde{X}$ is weakly $\widetilde{S}$-local (in the sense of Definition 9.1.3.5).

04N3 **Example 9.1.5.13.** Let $\mathcal{C}$ be an ∞-category containing a morphism $g : X \to Y$. Then every isomorphism f of $\mathcal{C}$ is weakly left orthogonal to g. This follows from the criterion of Remark 9.1.5.12, since every lift of f to the ∞-category $\mathcal{C}_{/Y}$ is also an isomorphism (Proposition 4.4.2.11).

We now record a few consequences of Remark 9.1.5.12.

04N4 **Proposition 9.1.5.14.** *Let $\mathcal{C}$ be an ∞-category and let $g : X \to Y$ be a morphism of $\mathcal{C}$. If another morphism $f : A \to B$ is weakly left orthogonal to g, then any retract of f (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$) is also weakly left orthogonal to g.*

Proof. Let $f' : A' \to B'$ be a retract of f (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$); we will show that f' is weakly left orthogonal to g. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, and let us identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. By virtue of Remark 9.1.5.12, it will suffice to show that for any morphism $\widetilde{f}'$ of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}') = f'$, the object $\widetilde{X}$ is weakly $\widetilde{f}'$-local. It follows from Corollary 4.2.5.2 that π induces a right fibration $\mathrm{Fun}(\Delta^1, \mathcal{C}_{/Y}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$. Applying Remark 8.5.1.23, we deduce that $\widetilde{f}'$ is a retract of a morphism $\widetilde{f}$ of $\mathcal{C}_{/Y}$ satisfying

$U(\widetilde{f}) = f$. By virtue of Variant 9.1.3.13, it will suffice to show that the object $\widetilde{X}$ is weakly $\widetilde{f}$-local, which follows from our assumption that f is weakly left orthogonal to g (Remark 9.1.5.12). □

Proposition 9.1.5.15. *Let $\mathcal{C}$ be an ∞-category containing a pushout diagram* 04N5

04N6

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r] & B'. \end{tikzcd} \qquad (9.11)$$

If f is weakly left orthogonal to a morphism $g : X \to Y$ of $\mathcal{C}$, then f' is also weakly left orthogonal to g.

Proof. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, and let us identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. By virtue of Remark 9.1.5.12, it will suffice to show that for any morphism $\widetilde{f}'$ of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}') = f'$, the object $\widetilde{X}$ is weakly $\widetilde{f}'$-local. Since π is a right fibration, we can lift (9.11) to a diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d, "\widetilde{f}"] & \widetilde{A}' \arrow[d, "\widetilde{f}'"] \\ \widetilde{B} \arrow[r] & \widetilde{B}' \end{tikzcd}$$

in the ∞-category $\mathcal{C}_{/Y}$, which is also a pushout square (Proposition 7.1.3.19). By virtue of Proposition 9.1.3.14, it will suffice to show that $\widetilde{X}$ is weakly $\widetilde{f}$-local, which follows from our assumption that f is weakly left orthogonal to g (Remark 9.1.5.12). □

Proposition 9.1.5.16. *Let $\mathcal{C}$ be an ∞-category containing a morphism $g : X \to Y$, and let S be the collection of all morphisms of $\mathcal{C}$ which are weakly left orthogonal to g. Then S is closed under transfinite composition (see Definition 9.1.2.1).* 04N7

Proof. Let $f : A \to B$ be a transfinite composition of morphisms of S; we wish to show that f is weakly left orthogonal to g. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, and let us identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. By virtue of Remark 9.1.5.12, it will suffice to show that for every morphism $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}) = f$, the object $\widetilde{X}$ is weakly $\widetilde{f}$-local.

Choose an ordinal α and a diagram $F : \mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \alpha}) \to \mathcal{C}$ which exhibits f as a transfinite composition of morphisms of S (see Definition 9.1.2.1). We will assume that $\alpha > 0$ (otherwise,

f is an identity morphism and the desired result follows from Example 9.1.5.13). In this case, Lemma 4.3.7.8 guarantees that the inclusion map $\mathrm{N}_{\bullet}(\{0 < \alpha\}) \hookrightarrow \mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \alpha})$ is right anodyne. Since π is a right fibration (Proposition 4.3.6.1), we can lift F to a diagram $\widetilde{F} : \mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \alpha}) \to \mathcal{C}$ for which the associated morphism $\widetilde{F}(0) \to \widetilde{F}(\alpha)$ coincides with $\widetilde{f}$. For every nonzero limit ordinal $\lambda \leq \alpha$, Proposition 7.1.3.19 guarantees that the restriction $\widetilde{F}|_{\mathrm{N}_{\bullet}(\mathrm{Ord}_{\leq \lambda})}$ is a colimit diagram in the ∞-category $\mathcal{C}_{/Y}$. Using Remark 9.1.5.12, we see that $\widetilde{F}$ exhibits $\widetilde{f}$ as a transfinite composition of morphisms of $\mathcal{C}_{/Y}$ with respect to which $\widetilde{X}$ is weakly local. Applying Proposition 9.1.3.17, we conclude that $\widetilde{X}$ is weakly $\widetilde{f}$-local, as desired. □

`04N8` **Corollary 9.1.5.17.** *Let $\mathcal{C}$ be an ∞-category, let T be a collection of morphisms of $\mathcal{C}$, and let S be the collection of all morphisms of $\mathcal{C}$ which are weakly left orthogonal to T. Then S is weakly saturated.*

Proof. Combine Propositions 9.1.5.14, 9.1.5.15, and 9.1.5.16 with Remark 9.1.3.23. □

9.1.6 Weak Factorization Systems

`04N9` Throughout this text, we have frequently made use of the fact that every morphism of simplicial sets $h : X \to Z$ admits a factorization $X \xrightarrow{f} Y \xrightarrow{g} Z$, where g is some sort of fibration and the morphism f has innocuous properties. In this section, we develop a general framework for results of this type.

`04NA` **Definition 9.1.6.1.** Let $\mathcal{C}$ be an ∞-category. A *weak factorization system on $\mathcal{C}$* is a pair (S_L, S_R), where S_L and S_R are collections of morphisms of $\mathcal{C}$ which satisfy the following conditions:

(1) For every morphism $h : X \to Z$ of $\mathcal{C}$, there exists a 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

where f belongs to S_L and g belongs to S_R.

(2) Every lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in $\mathcal{C}$ admits a solution, provided that $f \in S_L$ and $g \in S_R$.

(3) The collections S_L and S_R are closed under retracts (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$).

Example 9.1.6.2. Let $\mathcal{C} = \mathrm{Set}_\Delta$ be the category of simplicial sets. We have already encountered several examples of weak factorization systems (S_L, S_R) on $\mathcal{C}$: `04NB`

- We can take S_R to be the collection of Kan fibrations and S_L the collection of anodyne morphisms (Proposition 3.1.7.1).
- We can take S_R to be the collection of inner anodyne morphisms and S_L the collection of inner fibrations (Proposition 4.1.3.2).
- We can take S_R to be the collection of left fibrations and S_L the collection of left anodyne morphisms (Proposition 4.2.4.7).
- We can take S_R to be the collection of right fibrations and S_L the collection of left anodyne morphisms (Variant 4.2.4.8).
- We can take S_R to be the collection of trivial Kan fibrations and S_L the collection of monomorphisms (Exercise 3.1.7.11).

Remark 9.1.6.3 (Symmetry)**.** Let $\mathcal{C}$ be an ∞-category and let (S_L, S_R) be a weak factorization system on $\mathcal{C}$. Then the pair (S_R, S_L) is a weak factorization system on the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. `04NC`

Remark 9.1.6.4. Let $\mathcal{C}$ be an ∞-category. For every collection of morphisms S of $\mathcal{C}$, let $[S]$ be the collection of homotopy classes of morphisms which belong to S. If (S_L, S_R) is a weak factorization system on $\mathcal{C}$, then $([S_L], [S_R])$ is a weak factorization system on (the nerve of) the homotopy category $\mathrm{h}\mathcal{C}$. See Warning 9.1.5.5 and Variant 8.5.1.3. `04ND`

In the situation of Definition 9.1.6.1, the collections S_L and S_R are determined by one another.

Proposition 9.1.6.5. *Let $\mathcal{C}$ be an ∞-category, let (S_L, S_R) be a weak factorization system on $\mathcal{C}$, and let $h : X \to Z$ be a morphism of $\mathcal{C}$. Then h belongs to S_L if and only if it is weakly left orthogonal to S_R, and h belongs to S_R if and only if it is weakly right orthogonal to S_L.* `04NE`

Proof. We will prove the first assertion; the second follows by a similar argument. Assume that h is weakly left orthogonal to S_R; we wish to show that h belongs to S_L (the reverse implication is immediate from the definition). By virtue of Remark 9.1.6.4, we may assume that $\mathcal{C}$ is (the nerve of) an ordinary category. The morphism h admits a factorization

$X \xrightarrow{f} Y \xrightarrow{g} Z$, where $f \in S_L$ and $g \in S_R$. Since h is weakly left orthogonal to g, the lifting problem

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"] & Y \arrow[d, "g"] \\
Z \arrow[r, "\mathrm{id}"] \arrow[ur, dashed, "i"] & Z
\end{tikzcd}$$

admits a solution. We then have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "h"] & X \arrow[r, "\mathrm{id}"] \arrow[d, "f"] & X \arrow[d, "h"] \\
Z \arrow[r, "i"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

which exhibits h as a retract of f, so that h also belongs to S_L. □

04NF **Corollary 9.1.6.6.** *Let $\mathcal{C}$ be an ∞-category and let (S_L, S_R) be a weak factorization system on $\mathcal{C}$. Then S_L is a weakly saturated collection of morphisms of $\mathcal{C}$ (see Definition 9.1.3.19).*

Proof. Combine Proposition 9.1.6.5 with Corollary 9.1.5.17. □

Using the small object argument of §9.1.4, we can produce many examples of weak factorization systems.

04NG **Theorem 9.1.6.7** (Existence of Weak Factorization Systems)**.** *Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Assume that:*

(1) *The ∞-category $\mathcal{C}$ is locally small and admits small colimits.*

(2) *The collection W is small.*

(3) *For every morphism $w : X \to Y$ in W, the object X is κ-compact for some small cardinal κ.*

Then $\mathcal{C}$ admits a weak factorization system (S_L, S_R), where S_L is the weakly saturated collection of morphisms generated by W (Remark 9.1.3.23) and S_R is the collection of morphisms which are weakly right orthogonal to W.

04NH **Remark 9.1.6.8.** Let $\mathcal{C}$ be an ∞-category, let W be a collection of morphisms of $\mathcal{C}$, let $W_{\perp}$ denote the collection of morphisms of $\mathcal{C}$ which are weakly right orthogonal to every morphism of W, and let $\overline{W}$ denote the collection of all morphisms which are weakly left orthogonal to every morphism of $W_{\perp}$. Then $\overline{W}$ is always a weakly saturated collection of morphisms which contains W (Corollary 9.1.5.17). If the hypotheses of Theorem 9.1.6.7 are satisfied, then $\overline{W}$ is the weakly saturated collection generated by W, in the sense of Remark 9.1.3.23.

Proof of Theorem 9.1.6.7. The collection S_L is closed under retracts by construction, and S_R is closed under retracts by virtue of Proposition 9.1.5.14. Corollary 9.1.5.17 guarantees that S_L is weakly left orthogonal to S_R. It will therefore suffice to show that every morphism $h : X \to Z$ of $\mathcal{C}$ factors as a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$, where f belongs to S_L and g belongs to S_R.

Let $\widetilde{\mathcal{C}}$ denote the slice ∞-category $\mathcal{C}_{/Z}$, and let $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ denote the projection map. Our assumption that $\mathcal{C}$ is locally small guarantees that $\widetilde{\mathcal{C}}$ is also locally small (Example 4.7.8.11), and our assumption that $\mathcal{C}$ admits small colimits guarantees that $\widetilde{\mathcal{C}}$ admits small colimits (Corollary 7.1.3.20). Let $\widetilde{W}$ denote a set of representatives for the collection of isomorphism classes of morphisms $\widetilde{w}$ of $\mathcal{C}_{/Z}$ satisfying $\pi(\widetilde{w}) \in W$. Since W is small and $\mathcal{C}$ is locally small, the set $\widetilde{W}$ is also small. For every morphism $\widetilde{w} : \widetilde{A} \to \widetilde{B}$ which belongs to $\widetilde{W}$, the image $A = \pi(\widetilde{A})$ is a κ-compact object of $\widetilde{\mathcal{C}}$ for some small cardinal κ, so that $\widetilde{A}$ is a κ-compact object of $\widetilde{\mathcal{C}}$ (Remark [?]). Let us identify the morphism h with an object $\widetilde{X} \in \widetilde{\mathcal{C}}$ satisfying $\pi(\widetilde{X}) = X$. Applying Theorem 9.1.4.3, we deduce that there is a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in the ∞-category $\widetilde{\mathcal{C}}$ which is a transfinite pushout of morphisms of $\widetilde{W}$, where $\widetilde{Y}$ is $\widetilde{W}$-local. Set $f = \pi(\widetilde{f})$, so that $\widetilde{f}$ can be identified with a diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Since the functor π preserves small colimits (Corollary 7.1.3.20), the morphism f is a transfinite pushout of morphisms belonging to W, and therefore belongs to S_L. Our assumption that $\widetilde{Y}$ is $\widetilde{W}$-local guarantees that g belongs to S_R (Remark 9.1.5.12). □

9.1.7 Orthogonality

In the ∞-categorical setting, it will often be useful to view the collection of solutions to a lifting problem as a space, rather than a set. 04NJ

04NK **Construction 9.1.7.1.** Suppose we are given a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

in an ∞-category $\mathcal{C}$, given by a morphism $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}$. We let $\mathrm{Sol}(\sigma)$ denote the simplicial set $\{\sigma\} \times_{\mathrm{Fun}(Q,\mathcal{C})} \mathrm{Fun}(\Delta^3, \mathcal{C})$, where $Q \subset \Delta^3$ is the simplicial subset described in Remark 9.1.5.2. We will refer to $\mathrm{Sol}(\sigma)$ as the *space of solutions* to the lifting problem σ.

04NL **Remark 9.1.7.2.** In the situation of Construction 9.1.7.1, vertices of the simplicial set $\mathrm{Sol}(\sigma)$ can be identified with solutions to the lifting problem σ (in the sense of Definition 9.1.5.1). In particular, the lifting problem σ admits a solution if and only if $\mathrm{Sol}(\sigma)$ is nonempty.

04NM **Remark 9.1.7.3.** In the situation of Construction 9.1.7.1, the restriction map $\mathrm{Fun}(\Delta^3, \mathcal{C}) \to \mathrm{Fun}(Q, \mathcal{C})$ is an isofibration of ∞-categories (Corollary 4.4.5.3). Moreover, since Q contains every vertex of Δ^3, it is also conservative (Theorem 4.4.4.4). It follows that the solution space $\mathrm{Sol}(\sigma)$ is a Kan complex (Corollary 4.4.3.21).

04NN **Definition 9.1.7.4.** Let $\mathcal{C}$ be an ∞-category, and let $f : A \to B$ and $g : X \to Y$ be morphisms of $\mathcal{C}$. We will say that f *is left orthogonal to* g if, for every lifting problem σ :

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

in the ∞-category $\mathcal{C}$, the solution space $\mathrm{Sol}(\sigma)$ is a contractible Kan complex. In this case, we will also say that g is *right orthogonal to* f.

04NP **Example 9.1.7.5.** Let $\mathcal{C}$ be an ordinary category. Then a morphism $f : A \to B$ is left orthogonal to a morphism $g : X \to Y$ in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ if and only if every lifting problem

$$\begin{tikzcd} A \arrow[r, "u_0"] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r, "\overline{u}"] \arrow[ur, dashed] & Y \end{tikzcd}$$

admits a *unique* solution: that is, there is a unique morphism $u : B \to X$ satisfying $u \circ f = u_0$ and $g \circ u = \overline{u}$.

Remark 9.1.7.6. Let f and g be morphisms in an ∞-category $\mathcal{C}$. Then f is left orthogonal to g in $\mathcal{C}$ if and only if it is right orthogonal to g when regarded as a morphism of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. `04NQ`

Remark 9.1.7.7. Let f and g be morphisms in an ∞-category $\mathcal{C}$. If f is left orthogonal to g (in the sense of Definition 9.1.7.4), then it is weakly left orthogonal to g (in the sense of Definition 9.1.5.8). Beware that the converse is false (Exercise 9.1.7.8). `04NR`

Exercise 9.1.7.8. Let $f : A \twoheadrightarrow B$ be a surjective function between sets, and let $g : X \hookrightarrow Y$ be an injective function between sets. Show that: `04NS`

- The morphism f is left orthogonal to g (in the category of sets).
- The morphism g is weakly left orthogonal to f.
- Unless either f or g is a bijection, the morphism g is not left orthogonal to f.

Variant 9.1.7.9. Let $\mathcal{C}$ be an ∞-category and let S and T be collections of morphisms of $\mathcal{C}$. We say that S *is left orthogonal to* T if every morphism $f \in S$ is weakly left orthogonal to every morphism $g \in T$. In this case, we also say that T *is right orthogonal to* S. In the special case where $S = \{f\}$ is a singleton, we abbreviate this condition by saying that f *is left orthogonal to* T, or T *is right orthogonal to* f. In the special case $T = \{g\}$ is a singleton, we abbreviate this condition by saying that g *is right orthogonal to* S, or S *is left orthogonal to* g. `04NT`

To establish some elementary properties of Definition 9.1.7.4, it will be convenient to give an alternative description of the solution spaces $\mathrm{Sol}(\sigma)$.

Construction 9.1.7.10. Let $\mathcal{C}$ be an ∞-category and let σ : `04NU`

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

be a lifting problem in $\mathcal{C}$. Then σ determines a pair of objects $\widetilde{B}$, $\widetilde{X}$ in the ∞-category $\mathcal{C}_{A//Y}$ (see Remark 9.1.5.7). Let K denote the morphism space $\mathrm{Hom}_{\mathcal{C}_{A//Y}}(\widetilde{B}, \widetilde{X})$. We then have a tautological map $K \times \Delta^1 \to \mathcal{C}_{A//Y}$, which we can identify with a diagram $\{A\} \star (K \times \Delta^1) \star \{Y\} \to \mathcal{C}$. Composing with the quotient map

$$K \times \Delta^3 \simeq K \times (\{A\} \star \Delta^1 \star \{Y\}) \twoheadrightarrow \{A\} \star (K \times \Delta^1) \star \{Y\},$$

we obtain a morphism $K \to \mathrm{Fun}(\Delta^3, \mathcal{C})$, which factors through the simplicial subset $\mathrm{Sol}(\sigma) \subseteq \mathrm{Fun}(\Delta^3, \mathcal{C})$ of Construction 9.1.7.1. We therefore obtain a comparison map $\theta : \mathrm{Hom}_{\mathcal{C}_{A//Y}}(\widetilde{B}, \widetilde{X}) \to \mathrm{Sol}(\sigma)$.

04NV **Proposition 9.1.7.11.** *Let $\mathcal{C}$ be an ∞-category and let σ :*

04NW
$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{9.12}$$

be a lifting problem in $\mathcal{C}$. Then the comparison map

$$\theta : \mathrm{Hom}_{\mathcal{C}_{A//Y}}(\widetilde{B}, \widetilde{X}) \to \mathrm{Sol}(\sigma)$$

of Construction 9.1.7.10 is a homotopy equivalence of Kan complexes.

Proof. Corollary 4.6.6.9 supplies a categorical pullback diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1, \mathcal{C}_{A//Y}) \arrow[r] \arrow[d] & \mathrm{Fun}(\{A\} \star \Delta^1 \star \{Y\}, \mathcal{C}) \arrow[d] \\ \mathrm{Fun}(\partial\Delta^1, \mathcal{C}_{A//Y}) \arrow[r] & \mathrm{Fun}(\{A\} \star \partial\Delta^1 \star \{Y\}, \mathcal{C}), \end{tikzcd}$$

where the vertical maps are isofibrations (Corollary 4.4.5.3). Unwinding the definitions, we see that the comparison map θ is obtained by taking vertical fibers over the vertex corresponding to the pair $(\widetilde{B}, \widetilde{X})$. Corollary 4.5.2.31 guarantees that θ is an equivalence of ∞-categories. Since the source and target of θ are Kan complexes (Remark 9.1.7.3), it is a homotopy equivalence (Example 4.5.1.13). □

04NX **Warning 9.1.7.12.** In the situation of Proposition 9.1.7.11, the comparison map θ need not be an isomorphism of simplicial sets. However, it is always bijective on 0-simplices: vertices of both $\mathrm{Hom}_{\mathcal{C}_{A//Y}}(\widetilde{B}, \widetilde{X})$ and $\mathrm{Sol}(\sigma)$ can be identified with solutions to the lifting problem σ.

04NY **Corollary 9.1.7.13.** *Let $\mathcal{C}$ be an ∞-category containing morphisms $f : A \to B$ and $g : X \to Y$. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ denote the projection map, so that g can be identified with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. The following conditions are equivalent:*

(1) *The morphism g is right orthogonal to f (in the sense of Definition 9.1.7.4).*

(2) *For every morphism $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}) = f$, the object $\widetilde{X} \in \mathcal{C}_{/Y}$ is $\widetilde{f}$-local (in the sense of Definition 9.1.1.1).*

Proof. Combine Proposition 9.1.7.11 with Remark 9.1.1.4. □

Corollary 9.1.7.14. *Let $\mathcal{C}$ be an ∞-category containing morphisms $f : A \to B$ and $g : X \to Y$. If either f or g is an isomorphism, then f is left orthogonal to g.* 04NZ

Proof. Without loss of generality, we may assume that f is an isomorphism. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, so that we can identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. By virtue of Corollary 9.1.7.13, it will suffice to show that $\widetilde{X}$ is $\widetilde{f}$-local for every morphism $\widetilde{f}$ of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}) = f$. This is a special case of Example 9.1.1.2, since $\widetilde{f}$ is an isomorphism (Proposition 4.4.2.11). □

Corollary 9.1.7.15. *Let $\mathcal{C}$ be an ∞-category containing a morphism $g : X \to Y$ and a 2-simplex* 04P0

04P1

$$\begin{tikzcd} & B \arrow[dr, "f''"] & \\ A \arrow[ur, "f'"] \arrow[rr, "f"] & & C. \end{tikzcd} \tag{9.13}$$

Assume that f' is left orthogonal to g. Then f is left orthogonal to g if and only if f'' is left orthogonal to g.

Proof. Assume that f'' is left orthogonal to g; we will show that f is left orthogonal to g (the proof of the converse is similar). Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, so that we can identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. By virtue of Corollary 9.1.7.13, it will suffice to show that the object $\widetilde{X}$ is $\widetilde{f}$-local, for every morphism $\widetilde{f}$ of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}) = f$. Since π is a right fibration (Proposition 4.3.6.1), we can lift (9.13) to a diagram

$$\begin{tikzcd} & \widetilde{B} \arrow[dr, "\widetilde{f}''"] & \\ \widetilde{A} \arrow[ur, "\widetilde{f}'"] \arrow[rr, "\widetilde{f}"] & & \widetilde{C} \end{tikzcd}$$

in the ∞-category $\mathcal{C}_{/Y}$. Corollary 9.1.7.13 guarantees that the object $\widetilde{X}$ is both $\widetilde{f}'$-local and $\widetilde{f}''$-local, so the desired result follows from Remark 9.1.1.11. □

Warning 9.1.7.16. In the situation of Corollary 9.1.7.15, if the morphisms f and f'' are left orthogonal to g, then f' need not be left orthogonal to g. 04P2

Corollary 9.1.7.17. *Let $\mathcal{C}$ be an ∞-category and let $g : X \to Y$ be a morphism of $\mathcal{C}$. If another morphism $f : A \to B$ is left orthogonal to g, then any retract of f (in the ∞-category* $\mathrm{Fun}(\Delta^1, \mathcal{C})$*) is also left orthogonal to g.* 04P3

Proof. We proceed as in the proof of Proposition 9.1.5.14. Let $f' : A' \to B'$ be a retract of f (in the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{C})$); we will show that f' is left orthogonal to g. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, and let us identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. By virtue of Corollary 9.1.7.13, it will suffice to show that for any morphism $\widetilde{f}'$ of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}') = f'$, the object $\widetilde{X}$ is $\widetilde{f}'$-local. It follows from Corollary 4.2.5.2 that π induces a right fibration $\operatorname{Fun}(\Delta^1, \mathcal{C}_{/Y}) \to \operatorname{Fun}(\Delta^1, \mathcal{C})$. Applying Remark 8.5.1.23, we deduce that $\widetilde{f}'$ is a retract of a morphism $\widetilde{f}$ of $\mathcal{C}_{/Y}$ satisfying $U(\widetilde{f}) = f$. By virtue of Variant 9.1.1.7, it will suffice to show that the object $\widetilde{X}$ is $\widetilde{f}$-local, which follows from our assumption that f is left orthogonal to g (Corollary 9.1.7.13). □

04P4 **Corollary 9.1.7.18.** *Let $\mathcal{C}$ be an ∞-category containing a pushout diagram*

04P5
$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\
B \arrow[r] & B'.
\end{tikzcd} \tag{9.14}$$

If f is left orthogonal to a morphism $g : X \to Y$ of $\mathcal{C}$, then f' is also left orthogonal to g.

Proof. We proceed as in the proof of Proposition 9.1.5.15. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, and let us identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. By virtue of Corollary 9.1.7.13, it will suffice to show that for any morphism $\widetilde{f}'$ of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}') = f'$, the object $\widetilde{X}$ is weakly $\widetilde{f}'$-local. Since π is a right fibration, we can lift (9.14) to a diagram

$$\begin{tikzcd}
\widetilde{A} \arrow[r] \arrow[d, "\widetilde{f}"] & \widetilde{A}' \arrow[d, "\widetilde{f}'"] \\
\widetilde{B} \arrow[r] & \widetilde{B}'
\end{tikzcd}$$

in the ∞-category $\mathcal{C}_{/Y}$, which is also a pushout square (Proposition 7.1.3.19). By virtue of Remark 9.1.1.10, it will suffice to show that $\widetilde{X}$ is $\widetilde{f}$-local, which follows from our assumption that f is left orthogonal to g (Corollary 9.1.7.13). □

04P6 **Corollary 9.1.7.19.** *Let $\mathcal{C}$ be an ∞-category, and let f be a morphism of $\mathcal{C}$ which is the colimit of a diagram*

$$Q_0 : K \to \operatorname{Fun}(\Delta^1, \mathcal{C}) \qquad v \mapsto f_v$$

which is preserved by the evaluation functors $\operatorname{ev}_0, \operatorname{ev}_1 : \operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$. Let $g : X \to Y$ be a morphism which is right orthogonal to each of the morphisms f_v. Then g is right orthogonal to f.

Proof. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, and let us identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. By virtue of Corollary 9.1.7.13, it will suffice to show that if $\widetilde{f}$ is a morphism in $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}) = f$, then $\widetilde{X}$ is $\widetilde{f}$-local. Choose a colimit diagram $Q : K^{\triangleright} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ which satisfies $Q|_K = Q_0$ and carries the cone point of $K^{\triangleright}$ to f. Since the inclusion of the cone point into $K^{\triangleright}$ is right anodyne (Example 4.3.7.11) and the projection map $\mathrm{Fun}(\Delta^1, \mathcal{C}_{/Y}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ is a right fibration (Corollary 4.2.5.2) we can lift Q to a diagram $\widetilde{Q} : K^{\triangleright} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ carrying the cone point to $\widetilde{f}$. Using Corollary 7.1.3.20 and Proposition 7.1.6.1, we see that $\widetilde{Q}$ is a colimit diagram which is preserved by the evaluation functors $\mathrm{ev}_0, \mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathcal{C}_{/Y}) \to \mathcal{C}_{/Y}$. For each vertex $v \in K$, let $\widetilde{f}_v = \widetilde{Q}(v)$ is a morphism of $\mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{f}_v) = f_v$. Our assumption that f_v is left orthogonal to g guarantees that $\widetilde{X}$ is a $\widetilde{f}_v$-local object of the ∞-category $\mathcal{C}_{/Y}$ (Corollary 9.1.7.13). Applying Remark 9.1.1.9, we deduce that $\widetilde{X}$ is also $\widetilde{f}$-local. □

Corollary 9.1.7.20. *Let $\mathcal{C}$ be an ∞-category, let $g : X \to Y$ be a morphism of $\mathcal{C}$, and let S be the collection of morphisms of $\mathcal{C}$ which are left orthogonal to g. Then S is weakly saturated.* 04P7

Proof. Combining Corollaries 9.1.7.14, 9.1.7.15, and 9.1.7.19 with Proposition 9.1.2.10 (and Remark 9.1.2.11), we see that S is closed under transfinite composition. Since S is also closed under retracts (Corollary 9.1.7.17) and pushouts (Corollary 9.1.7.18), it is weakly saturated. □

Corollary 9.1.7.21. *Let $\mathcal{C}$ be an ∞-category, let S be a collection of morphisms of $\mathcal{C}$, and let $g : X \to Y$ be a morphism of $\mathcal{C}$ which is weakly right orthogonal to S. Assume that every morphism $f : A \to B$ which belongs to S admits a relative codiagonal $\gamma_{A/B} : B \coprod_A B \to B$ which also belongs to S (see Variant 7.6.3.19). Then g is right orthogonal to S.* 04P8

Proof. Let $\pi : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map, and let us identify g with an object $\widetilde{X} \in \mathcal{C}_{/Y}$ satisfying $\pi(\widetilde{X}) = X$. Let $\widetilde{S}$ denote the collection of those morphisms $\widetilde{f}$ in $\mathcal{C}_{/Y}$ which satisfy $\pi(\widetilde{f}) \in S$. Our assumption that g is weakly right orthogonal to S guarantees that $\widetilde{X}$ is weakly $\widetilde{S}$-local (Remark 9.1.5.12). It follows from Proposition 7.1.3.19 that every morphism of $\widetilde{S}$ admits a relative codiagonal which also belongs to $\widetilde{S}$, so that $\widetilde{X}$ is $\widetilde{S}$-local (Proposition 9.1.3.15). Invoking Corollary 9.1.7.13, we conclude that g is right orthogonal to S. □

Proposition 9.1.7.22. *Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories, let σ :* 04P9

04PA

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{9.15}$$

be a lifting problem in the ∞-category $\mathcal{E}$, and let $\overline{\sigma} = U \circ \sigma$ denote the associated lifting problem in the ∞-category $\mathcal{C}$. If the morphism f is U-cocartesian or the morphism $\widetilde{g}$ is U-cartesian, then U induces a homotopy equivalence of solution spaces $\mathrm{Sol}(\sigma) \to \mathrm{Sol}(\overline{\sigma})$.

Proof. Let us identify the diagram (9.15) with a pair of objects $\widetilde{B}, \widetilde{X} \in \mathcal{E}_{A/\,/Y}$. Note that U induces a functor $\widetilde{U} : \mathcal{E}_{A/\,/Y} \to \mathcal{C}_{U(A)/\,/U(Y)}$, and we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}_{A/\,/Y}}(\widetilde{B}, \widetilde{X}) \arrow[r] \arrow[d] & \mathrm{Sol}(\sigma) \arrow[d] \\
\mathrm{Hom}_{\mathcal{C}_{U(A)/\,/U(Y)}}(\widetilde{U}(\widetilde{B}), \widetilde{U}(\widetilde{X})) \arrow[r] & \mathrm{Sol}(\overline{\sigma}),
\end{tikzcd}$$

where the horizontal maps are the homotopy equivalences supplied by Proposition 9.1.7.11. It will therefore suffice to show that the left vertical map is a homotopy equivalence. Without loss of generality, we may assume that the morphism f is U-cocartesian. In this case, we will complete the proof by showing that the object $\widetilde{B} \in \mathcal{E}_{A/\,/Y}$ is $\widetilde{U}$-initial. Let $U_{/Y} : \mathcal{E}_{/Y} \to \mathcal{C}_{/U(Y)}$ be the inner fibration induced by U; by virtue of Example 7.1.5.9, it will suffice to show that the lower left of the diagram (9.15) is $U_{/Y}$-cocartesian when viewed as a morphism in $\mathcal{E}_{/Y}$. This follows from our assumption that f is U-cocartesian (Corollary 5.1.1.14). □

04PB **Corollary 9.1.7.23.** *Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories and let f and g be morphisms of $\mathcal{E}$. Assume either that f is U-cocartesian or that g is U-cartesian. Then:*

- *If $U(f)$ is left orthogonal to $U(g)$ in the ∞-category $\mathcal{C}$, then f is left orthogonal to g in the ∞-category $\mathcal{E}$.*
- *If $U(f)$ is weakly left orthogonal to $U(g)$ in the ∞-category $\mathcal{C}$, then f is weakly left orthogonal to g in the ∞-category $\mathcal{E}$.*

9.1.8 Uniqueness of Factorizations

04PQ Let $\mathcal{C}$ be an ∞-category, let S_L and S_R be collections of morphisms of $\mathcal{C}$, and let $f : X \to Z$ be a morphism which factors as a composition

$$X \xrightarrow{f_L} Y \xrightarrow{f_R} Z$$

where f_L belongs to S_L and f_R belongs to S_R. Our goal in this section is to show that, if S_L is left orthogonal to S_R, then this factorization is essentially unique (Theorem 9.1.8.2).

Notation 9.1.8.1. Let $\mathcal{C}$ be an ∞-category and let S_L and S_R be collections of morphisms of $\mathcal{C}$. We let $\mathrm{Fun}_L(\Delta^2, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ spanned by those diagrams 04PS

$$\begin{tikzcd}
& Y \arrow[dr, "f_R"] & \\
X \arrow[ur, "f_L"] \arrow[rr, "f"] & & Z
\end{tikzcd}$$

where f belongs to S_L, and $\mathrm{Fun}_R(\Delta^2, \mathcal{C})$ the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ spanned by those diagrams where g belongs to S_R. We let $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C})$ denote the intersection $\mathrm{Fun}_L(\Delta^2, \mathcal{C}) \cap \mathrm{Fun}_R(\Delta^2, \mathcal{C})$.

We can now formulate our main result.

Theorem 9.1.8.2. *Let $\mathcal{C}$ be an ∞-category and let S_L and S_R be collections of morphisms of $\mathcal{C}$. If S_L is left orthogonal to S_R, then the restriction functor* 04QC

$$D : \mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \qquad \sigma \mapsto d_1^2(\sigma)$$

is fully faithful. The converse holds if S_L and S_R contain all identity morphisms of $\mathcal{C}$.

The proof of Theorem 9.1.8.2 will require some preliminaries. We begin by giving another description of the space of solutions to a lifting problem.

Notation 9.1.8.3. Let $\mathcal{C}$ be an ∞-category containing morphisms $f : A \to B$ and $g : X \to Y$. We let ${}_f \mathrm{Fun}(\Delta^3, \mathcal{C})_g$ denote the the iterated fiber product 04PW

$$\{f\} \times_{\mathrm{Fun}(\mathrm{N}_\bullet(\{0<1\})} \mathrm{Fun}(\Delta^3, \mathcal{C}) \times_{\mathrm{Fun}_{\mathrm{N}_\bullet(\{2<3\})}} \{g\},$$

whose objects can be identified with diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & X \arrow[d, "g"] \\
B \arrow[ur] \arrow[r] & Y
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$.

Lemma 9.1.8.4. *Let $\mathcal{C}$ be an ∞-category containing morphisms $f : A \to B$ and $g : X \to Y$. Then precomposition with the inclusion map $\mathrm{N}_\bullet(\{1 < 2\}) \hookrightarrow \Delta^3$ induces a trivial Kan fibration of simplicial sets ${}_f \mathrm{Fun}(\Delta^3, \mathcal{C})_g \to \mathrm{Hom}_{\mathcal{C}}(B, X)$. In particular, the simplicial set ${}_f \mathrm{Fun}(\Delta^3, \mathcal{C})_g$ is a Kan complex.* 04PX

Proof. By construction, we have a pullback diagram of simplicial sets

$$\begin{tikzcd}
{}_f \operatorname{Fun}(\Delta^3, \mathcal{C})_g \arrow[r] \arrow[d] & \operatorname{Fun}(\Delta^3, \mathcal{C}) \arrow[d] \\
\operatorname{Hom}_{\mathcal{C}}(B, X) \arrow[r] & \operatorname{Fun}(\operatorname{Spine}[3], \mathcal{C}),
\end{tikzcd}$$

where the right vertical map is a trivial Kan fibration (see Example 1.5.7.7). □

04PY **Notation 9.1.8.5.** Let $\mathcal{C}$ be an ∞-category containing morphisms $f : A \to B$ and $g : X \to Y$. We let $\widetilde{f} = s_1^1(f)$ and $\widetilde{g} = s_0^1(g)$ denote the degenerate 2-simplices of $\mathcal{C}$ depicted in the diagram

$$A \xrightarrow{f} B \xrightarrow{\mathrm{id}} B \qquad X \xrightarrow{\mathrm{id}} X \xrightarrow{g} Y.$$

Let $\beta : \Delta^2 \times \Delta^1 \to \Delta^3$ denote the morphism of simplicial sets given on vertices by the formulae

$$\beta(0,0) = 0 \qquad \beta(1,0) = 1 = \beta(2,0) \qquad \beta(0,1) = 2 = \beta(1,1) \qquad \beta(2,1) = 3.$$

Then precomposition with β determines a functor $\operatorname{Fun}(\Delta^3, \mathcal{C}) \to \operatorname{Fun}(\Delta^2 \times \Delta^1, \mathcal{C})$, which restricts to a map of Kan complexes $T : {}_f \operatorname{Fun}(\Delta^3, \mathcal{C})_g \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^2, \mathcal{C})}(\widetilde{f}, \widetilde{g})$. Concretely, T carries a 3-simplex

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"] & X \arrow[d, "g"] \\
B \arrow[r, "w"] \arrow[ur, "v"] & Y
\end{tikzcd}$$

to the morphism in $\operatorname{Fun}(\Delta^2, \mathcal{C})$ depicted in the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"] & B \arrow[r, "\mathrm{id}_B"] \arrow[d, "v"] & B \arrow[d, "w"] \\
X \arrow[r, "\mathrm{id}_X"] & X \arrow[r, "g"] & Y.
\end{tikzcd}$$

04PZ **Lemma 9.1.8.6.** *Let $\mathcal{C}$ be an ∞-category containing morphisms $f : A \to B$ and $g : X \to Y$. Then the comparison map*

$$T : {}_f \operatorname{Fun}(\Delta^3, \mathcal{C})_g \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^2, \mathcal{C})}(\widetilde{f}, \widetilde{g})$$

of Notation 9.1.8.5 is a homotopy equivalence.

Proof. By construction, the diagram $\widetilde{f} : \Delta^2 \to \mathcal{C}$ is left Kan extended from the simplicial subset $\Delta^1 \subset \Delta^2$. Applying Corollary 7.3.6.9, we deduce that the restriction functor $\operatorname{Fun}(\Delta^2, \mathcal{C}) \to \operatorname{Fun}(\Delta^1, \mathcal{C})$ determines a trivial Kan fibration $R : \operatorname{Hom}_{\operatorname{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f}, \widetilde{g}) \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f, \mathrm{id}_X)$. Similarly, the 1-simplex id_X can be viewed as a diagram $\Delta^1 \to \mathcal{C}$ which is right Kan extended from the vertex $\{1\} \subset \Delta^2$, so the evaluation functor $\mathrm{ev}_1 : \operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ induces a trivial Kan fibration $Q : \operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f, \mathrm{id}_X) \to \operatorname{Hom}_{\mathcal{C}}(B, X)$. We are therefore reduced to showing that the composite map

$$_f\operatorname{Fun}(\Delta^3, \mathcal{C})_g \xrightarrow{T} \operatorname{Hom}_{\operatorname{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f}, \widetilde{g}) \xrightarrow{R} \operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f, \mathrm{id}_X) \xrightarrow{Q} \operatorname{Hom}_{\mathcal{C}}(B, X)$$

is a homotopy equivalence, which follows from Lemma 9.1.8.4. □

In the situation of Lemma 9.1.8.6, restriction to the "long edge" of Δ^2 determines an inner fibration of ∞-categories $\operatorname{Fun}(\Delta^2, \mathcal{C}) \to \operatorname{Fun}(\Delta^1, \mathcal{C})$ (Corollary 4.1.4.2), and therefore induces a Kan fibration of mapping spaces

$$\operatorname{Hom}_{\operatorname{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f}, \widetilde{g}) \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f, g).$$

Lemma 9.1.8.7. *Let $\mathcal{C}$ be an ∞-category containing a lifting problem σ :* 04Q0

04Q1

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "s"] & Y, \end{tikzcd} \qquad (9.16)$$

which we identify with a morphism from f to g in the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{C})$. *Then the comparison map $T : {}_f\operatorname{Fun}(\Delta^3, \mathcal{C})_g \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f}, \widetilde{g})$ of Notation 9.1.8.5 restricts to a homotopy equivalence of Kan complexes*

$$T_0 : \operatorname{Sol}(\sigma) \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f}, \widetilde{g}) \times_{\operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f,g)} \{\sigma\}.$$

Proof. Let $\alpha : \Delta^1 \times \Delta^1 \hookrightarrow \Delta^3$ be the morphism of simplicial sets given on vertices by the formula $\alpha(i, j) = 2i + j$ (see Definition 9.1.5.1). Then precomposition with α induces an isofibration of ∞-categories $U : \operatorname{Fun}(\Delta^3, \mathcal{C}) \to \operatorname{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$, which restricts to an isofibration $U_0 : {}_f\operatorname{Fun}(\Delta^3, \mathcal{C})_g \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f, g)$. Since the source and target of U_0 are Kan complexes, it is a Kan fibration (Corollary 4.4.3.10). The desired result now follows by applying Corollary 3.3.7.5 to the diagram of Kan complexes

$$\begin{tikzcd}
{}_f \operatorname{Fun}(\Delta^3, \mathcal{C})_g \arrow[rr, "T"] \arrow[dr, "U_0"] & & \operatorname{Hom}_{\operatorname{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f}, \widetilde{g}) \arrow[dl] \\
& \operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f,g), &
\end{tikzcd}$$

since T is a homotopy equivalence (Lemma 9.1.8.6). □

04Q2 **Corollary 9.1.8.8.** *Let $\mathcal{C}$ be an ∞-category containing 2-simplices σ and τ. Suppose that the initial edge $f = d_2^2(\sigma)$ is left orthogonal to the final edge $g = d_0^2(\tau)$. Then the restriction map $\theta : \operatorname{Hom}_{\operatorname{Fun}(\Delta^2,\mathcal{C})}(\sigma, \tau) \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f,g)$ is a trivial Kan fibration.*

Proof. By virtue of Proposition 3.3.7.6, it will suffice to show that every fiber of θ is a contractible Kan complex. Let $D : \operatorname{Fun}(\Delta^2, \mathcal{C}) \to \operatorname{Fun}(\Delta^1, \mathcal{C})$ denote the functor given by precomposition with the inclusion map $\Delta^1 \simeq \operatorname{N}_\bullet(\{0 < 2\}) \hookrightarrow \Delta^2$, and let $E : \operatorname{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$ be given by evaluation at the final vertex $1 \in \Delta^1$. Let $\gamma : \Delta^2 \times \Delta^1 \to \Delta^2$ be the morphism of simplicial sets given on vertices by the formula $\gamma(i,j) = \begin{cases} 1 & \text{if } (i,j) = (0,2) \\ i & \text{otherwise.} \end{cases}$. Then the composition

$$\Delta^2 \times \Delta^1 \xrightarrow{\gamma} \Delta^2 \xrightarrow{\sigma} \mathcal{C}$$

can be regarded as a morphism $e : \widetilde{f} \to \sigma$ in the ∞-category $\operatorname{Fun}(\Delta^2, \mathcal{C})$. It follows from Corollary 5.3.7.5 that the morphism e is $(E \circ D)$-cocartesian and that $D(e)$ is E-cocartesian. Consequently, the morphism e is D-cocartesian (Corollary 5.1.2.6). Using Proposition 5.1.2.1, we deduce that every fiber of θ is homotopy equivalent to a fiber of the restriction map $\theta' : \operatorname{Hom}_{\operatorname{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f}, \tau) \to \operatorname{Hom}_{\operatorname{Fun}(\Delta^1,\mathcal{C})}(f,g)$. It will therefore suffice to prove Corollary 9.1.8.8 in the special case where $\sigma = \widetilde{f}$. By a similar argument, we may also assume that τ is the degenerate 2-simplex $\widetilde{g} = s_0^1(g)$. In this case, Lemma 9.1.8.7 guarantees that every fiber of θ is homotopy equivalent to the space of solutions to some lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r] \arrow[ur, dashed, "s"] & Y,
\end{tikzcd}$$

which is contractible by virtue of our assumption that f is left orthogonal to g. □

Proof of Theorem 9.1.8.2. Let $\mathcal{C}$ be an ∞-category and let S_L and S_R be collections of morphisms of $\mathcal{C}$. Suppose first that S_L is left orthogonal to S_R. In this case Corollary 9.1.8.8 guarantees that the restriction map

$$\theta : \mathrm{Hom}_{\mathrm{Fun}(\Delta^2,\mathcal{C})}(\sigma,\sigma') \to \mathrm{Hom}_{\mathrm{Fun}(\Delta^1,\mathcal{C})}(d_1^2(\sigma), d_1^2(\sigma'))$$

is a homotopy equivalence whenever σ belongs to $\mathrm{Fun}_L(\Delta^2,\mathcal{C})$ and σ' belongs to $\mathrm{Fun}_R(\Delta^2,\mathcal{C})$. It follows that the functor

$$D : \mathrm{Fun}_{LR}(\Delta^2,\mathcal{C}) \to \mathrm{Fun}(\Delta^1,\mathcal{C}) \qquad \sigma \mapsto d_1^2(\sigma)$$

is fully faithful.

We now prove the converse. Assume that D is fully faithful and that S_L and S_R contain all identity morphisms of $\mathcal{C}$; we wish to show that S_L is left orthogonal to S_R. Suppose we are given a lifting problem τ :

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, where f belongs to S_L and g belongs to S_R. We wish to show that the solution space $\mathrm{Sol}(\tau)$ is contractible. Let $\widetilde{f} = s_1^1(f)$ and $\widetilde{g} = s_0^1(g)$ denote the degenerate 2-simplices of $\mathcal{C}$ defined in Notation 9.1.8.3. Since S_L and S_R contain all identity morphisms, we can view $\widetilde{f}$ and $\widetilde{g}$ as objects of the ∞-category $\mathrm{Fun}_{LR}(\Delta^2,\mathcal{C})$. Assumption (1) guarantees that the Kan fibration $\mathrm{Hom}_{\mathrm{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f},\widetilde{g}) \to \mathrm{Hom}_{\mathrm{Fun}(\Delta^1,\mathcal{C})}(f,g)$ is a homotopy equivalence. Lemma 9.1.8.7 supplies a homotopy equivalence of $\mathrm{Sol}(\tau)$ with the fiber $\mathrm{Hom}_{\mathrm{Fun}(\Delta^2,\mathcal{C})}(\widetilde{f},\widetilde{g})_\tau$, which is contractible by virtue of Proposition 3.3.7.6. □

9.1.9 Factorization Systems

Motivated by Theorem 9.1.8.2, we introduce the following variant of Definition 9.1.6.1: 04PC

Definition 9.1.9.1. Let $\mathcal{C}$ be an ∞-category. A *factorization system on* $\mathcal{C}$ is a pair (S_L, S_R), where S_L and S_R are collections of morphisms of $\mathcal{C}$ which satisfy the following conditions: 04PD

(1) For every morphism $f : X \to Z$ of $\mathcal{C}$, there exists a 2-simplex

$$\begin{tikzcd}
& Y \arrow[dr, "f_R"] & \\
X \arrow[ur, "f_L"] \arrow[rr, "f"] & & Z
\end{tikzcd}$$

where f_L belongs to S_L and f_R belongs to S_R.

(2) Every morphism of S_L is left orthogonal to every morphism of S_R (Definition 9.1.7.4).

(3) The collections S_L and S_R are closed under isomorphism (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$).

04PE **Remark 9.1.9.2** (Symmetry)**.** Let $\mathcal{C}$ be an ∞-category and let (S_L, S_R) be a factorization system on $\mathcal{C}$. Then the pair (S_R, S_L) is a weak factorization system on the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$.

04PF **Example 9.1.9.3** (Trivial Factorization Systems)**.** Let $\mathcal{C}$ be an ∞-category, let W be the collection of all isomorphisms in $\mathcal{C}$, and let A denote the collection of all morphisms in $\mathcal{C}$. Then the pairs (W, A) and (A, W) are factorization systems on $\mathcal{C}$ (see Corollary 9.1.7.14).

We now give some more interesting examples of factorization systems. Recall that a functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$ is *categorically n-connective* if it is m-full for every nonnegative integer $m \leq n$ (Definition 4.8.7.1), and *essentially $(n-1)$-categorical* if it is m-full for $m > n$ (Definition 4.8.6.1). Let $\mathcal{QC}$ denote the ∞-category of (small) ∞-categories (Construction 5.5.4.1).

05E7 **Proposition 9.1.9.4.** *Let n be an integer, let S_L denote the collection of all categorically n-connective functors, and let S_R denote the collection of all essentially $(n-1)$-categorical functors. Then the pair (S_L, S_R) is a factorization system on the ∞-category $\mathcal{QC}$.*

Proof. We first observe that S_R is closed under the formation of relative diagonals: that is, if a functor $G : \mathcal{C} \to \mathcal{D}$ is essentially $(n-1)$-categorical, then the relative diagonal of G (formed in the ∞-category $\mathcal{QC}$) has the same property. Using Exercise 7.6.4.13, we can identify the relative diagonal of G with the inclusion map $\iota : \mathcal{C} \hookrightarrow \mathcal{C} \times^{\mathrm{h}}_{\mathcal{D}} \mathcal{C}$. For $n \geq 1$, it follows from Variant 4.8.6.15 that ι is essentially $(n-2)$-categorical, and therefore also essentially $(n-1)$-categorical (Remark 4.8.6.6). If $n \leq 0$, then the functor G is fully faithful, so ι is an equivalence of ∞-categories.

It follows from Remarks 4.8.5.16, 4.8.5.17, and 4.8.5.18 that S_L and S_R are invariant under isomorphism. Theorem 4.8.8.3 asserts that every functor $F : \mathcal{C} \to \mathcal{E}$ admits a factorization $\mathcal{C} \xrightarrow{F_L} \mathcal{D} \xrightarrow{F_R} \mathcal{E}$, where F_L belongs to S_L and F_R belongs to S_R. We will complete the proof by showing that S_L is left orthogonal to S_R. By virtue of (the dual of) Corollary 9.1.7.21, it will suffice to show that S_L is weakly left orthogonal to S_R: that is, every lifting problem

05E8
$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d, "F"] & \mathcal{C} \arrow[d, "G"] \\
\mathcal{B} \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd} \tag{9.17}$$

in the ∞-category $\mathcal{QC}$ admits a solution, provided that F is categorically n-connective and G is essentially $(n-1)$-categorical. By virtue of Corollary 5.6.5.16, we may assume that (9.17) arises from a commutative diagram in the category of simplicial sets. Using Corollary

4.5.2.23, we can further assume that F is a monomorphism of simplicial sets and that G is an isofibration. In this case, the lifting problem (9.17) already admits a solution in the category of simplicial sets: see Corollary 4.8.7.18 and Remark 4.8.7.19. □

Corollary 9.1.9.5. *Let n be an integer, let S_L denote the collection of all n-connective morphisms between Kan complexes, and let S_R denote the collection of all $(n-1)$-truncated morphisms between Kan complexes. Then the pair (S_L, S_R) determines a factorization system on the ∞-category $\mathcal{S}$.* 05E9

Proof. Recall that a morphism of Kan complexes is n-connective if and only if it is categorically n-connective (Example 4.8.7.3), and $(n-1)$-truncated if and only if it is essentially $(n-1)$-categorical (Example 4.8.6.3). It follows immediately from Proposition 9.1.9.4 that S_L and S_R are closed under isomorphism, and that S_L is left orthogonal to S_R. To complete the proof, it suffices to show that every morphism of Kan complexes $f : X \to Z$ admits a factorization $X \xrightarrow{f_L} Y \xrightarrow{f_R} Z$, where f_L is n-connective and f_R is $(n-1)$-truncated. This is the content of Corollary 4.8.8.9. □

Proposition 9.1.9.6. *Let $\mathcal{C}$ be an ∞-category, let (S_L, S_R) be a factorization system on $\mathcal{C}$, and let $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \subseteq \mathrm{Fun}(\Delta^2, \mathcal{C})$ be the full subcategory of Notation 9.1.8.1. Then the restriction map* 04QD

$$D : \mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \qquad \sigma \mapsto d_1^2(\sigma)$$

is a trivial Kan fibration.

Proof. Condition (1) of Definition 9.1.9.1 guarantees that D is surjective on objects, and condition (2) guarantees that D is fully faithful (Theorem 9.1.8.2). Applying the criterion Theorem 4.6.2.20, we deduce that D is an equivalence of ∞-categories. Condition (3) of Definition 9.1.9.1 guarantees that the full subcategory $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \subseteq \mathrm{Fun}(\Delta^2, \mathcal{C})$ is replete, so that D is an isofibration of ∞-categories (see Corollary 4.4.5.3). Applying Proposition 4.5.5.20, we conclude that D is a trivial Kan fibration. □

Corollary 9.1.9.7. *Let $\mathcal{C}$ be an ∞-category and let (S_L, S_R) be a factorization system on $\mathcal{C}$. Then every isomorphism in $\mathcal{C}$ is contained in both S_L and S_R.* 04QE

Proof. Let W be the collection of all isomorphisms in $\mathcal{C}$, and set $S_L^+ = S_L \cup W$ and $S_R^+ = S_R \cup W$. Using Corollary 9.1.7.14, we deduce that S_L^+ is left orthogonal to S_R^+, so that (S_L^+, S_R^+) is also a factorization system on $\mathcal{C}$. Let $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C})$ be as in Notation 9.1.8.1 and define $\mathrm{Fun}^{+}_{LR}(\Delta^2, \mathcal{C})$ similarly. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \arrow[rr] \arrow[dr, "D"'] & & \mathrm{Fun}^{+}_{LR}(\Delta^2, \mathcal{C}) \arrow[dl, "D^{+}"] \\
& \mathrm{Fun}(\Delta^1, \mathcal{C}) &
\end{tikzcd}$$

where both of the vertical maps are trivial Kan fibrations (Proposition 9.1.9.6). It follows that the inclusion map $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \hookrightarrow \mathrm{Fun}^{+}_{LR}(\Delta^2, \mathcal{C})$ is an equivalence of ∞-categories. Since $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C})$ is a replete full subcategory of $\mathrm{Fun}^{+}_{LR}(\Delta^2, \mathcal{C})$, we must have $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) = \mathrm{Fun}^{+}_{LR}(\Delta^2, \mathcal{C})$. In particular, if $f : X \to Y$ is an isomorphism in $\mathcal{C}$, then the degenerate 2-simplices $s^1_0(f)$ and $s^1_1(f)$ are both contained in $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C})$, so that f is contained in both S_L and S_R. □

04QF **Corollary 9.1.9.8.** *Let $\mathcal{C}$ be an ∞-category and let (S_L, S_R) be a factorization system on $\mathcal{C}$. Then S_L and S_R are closed under retracts (in the ∞-category* $\mathrm{Fun}(\Delta^1, \mathcal{C})$*).*

Proof. We will show that S_L is closed under retracts; the analogous statement for S_R follows by a similar argument. By virtue of Proposition 9.1.9.6, the restriction map

$$D : \mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \qquad \sigma \mapsto d^2_1(\sigma)$$

is a trivial Kan fibration. It therefore admits a section $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}_{LR}(\Delta^2, \mathcal{C})$, which carries each morphism $f : X \to Z$ of $\mathcal{C}$ to a 2-simplex σ_f :

$$\begin{tikzcd}
& Y \arrow[dr, "f_R"] & \\
X \arrow[ur, "f_L"] \arrow[rr, "f"] & & Z,
\end{tikzcd}$$

where $f_L \in S_L$ and $f_R \in S_R$. We will complete the proof by showing that f belongs to S_L if and only if f_R is an isomorphism in $\mathcal{C}$. One direction is clear: if f_R is an isomorphism, then f is isomorphic to f_L in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$, and therefore belongs to S_L by virtue of our assumption that S_L is closed under isomorphism. For the converse, assume that f belongs to S_L. Since id_Z belongs to S_R (Corollary 9.1.9.7), the degenerate 2-simplex $\widetilde{f} = s^1_1(f)$ can be regarded as an object of $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C})$ satisfying $D(\widetilde{f}) = f = D(\sigma_f)$. Since D is an equivalence of ∞-categories, the 2-simplex σ_f is isomorphic to $\widetilde{f}$ as an object of the ∞-category $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C})$. It follows that $f_R = d^2_0(\sigma_f)$ is isomorphic to $\mathrm{id}_Z = d^2_0(\widetilde{f})$ as an object of the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$, so that f_R is an isomorphism (Example 4.4.1.14). □

Corollary 9.1.9.9. *Let $\mathcal{C}$ be an ∞-category and let (S_L, S_R) be a factorization system on $\mathcal{C}$. Then (S_L, S_R) is a weak factorization system on $\mathcal{C}$.* 04QG

Proof. The only nontrivial point is to verify that S_L and S_R are closed under retracts, which follows from Corollary 9.1.9.8. □

Beware that the converse of Corollary 9.1.9.9 is false in general:

Exercise 9.1.9.10. Let Set denote the category of sets and let $\mathcal{C} = \mathrm{N}_{\bullet}(\mathrm{Set})$ be the associated ∞-category. Let S be the collection of surjective functions, and let I be the collection of injective functions. Show that: 04PH

- The pair (S, I) is a factorization system on $\mathcal{C}$.
- The pair (I, S) is a weak factorization system on $\mathcal{C}$.
- The pair (I, S) is not a factorization system on $\mathcal{C}$.

In the situation of Definition 9.1.9.1, either of the collections S_L and S_R can be recovered from the other.

Proposition 9.1.9.11. *Let $\mathcal{C}$ be an ∞-category, let (S_L, S_R) be a factorization system on $\mathcal{C}$, and let f be a morphism of $\mathcal{C}$. The following conditions are equivalent:* 04PJ

(1) *The morphism f belongs to S_L.*

(2) *The morphism f is left orthogonal to S_R.*

(3) *The morphism f is weakly left orthogonal to S_R.*

Proof. The implication (1) ⇒ (2) is immediate from the definition, the implication (2) ⇒ (3) follows from Remark 9.1.7.7, and the implication (3) ⇒ (1) follows from Proposition 9.1.6.5 (together with Corollary 9.1.9.9). □

Corollary 9.1.9.12. *Let $\mathcal{C}$ be an ∞-category which admits pushouts and let (S_L, S_R) be a weak factorization system on $\mathcal{C}$. The following conditions are equivalent:* 04PK

(1) *The pair (S_L, S_R) is a factorization system on $\mathcal{C}$.*

(2) *For every 2-simplex*

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

of $\mathcal{C}$, if f and h belong to S_L, then g also belongs to S_L.

(3) *For every morphism $f : X \to Y$ which belongs to S_L, the relative codiagonal $\gamma_{X/Y} : Y \coprod_X Y \to Y$ also belongs to S_L.*

Proof. We first show that (1) ⇒ (2). Assume that (S_L, S_R) is a factorization system and consider a 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

of $\mathcal{C}$. If f and h belong to S_L, then they are left orthogonal to S_R. Applying Corollary 9.1.7.15, we deduce that g is also left orthogonal to S_R, so that $g \in S_L$ by virtue of Proposition 9.1.9.11.

We now show that (2) implies (3). Let $f : X \to Y$ be a morphism which belongs to S_L. Then the relative codiagonal $\gamma_{X/Y}$ fits into a commutative diagram

$$\begin{tikzcd} & Y \coprod_X Y \arrow[dr, "\gamma_{X/Y}"] & \\ Y \arrow[ur, "f'"] \arrow[rr, "\mathrm{id}_Y"] & & Y, \end{tikzcd}$$

where f' is a pushout of f. Since S_L is weakly saturated (Corollary 9.1.6.6), it contains the morphisms f' and id_Y. If condition (2) is satisfied, then $\gamma_{X/Y}$ also contains $\gamma_{X/Y}$.

We now complete the proof by showing that (3) implies (1). Let g be a morphism of $\mathcal{C}$ which belongs to S_R. Then g is weakly right orthogonal to S_L, and we wish to show that g is right orthogonal to S_L. This follows by combining assumption (3) with Corollary 9.1.7.21. □

04PL **Proposition 9.1.9.13** (Lifting Factorization Systems)**.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories and let (S_L, S_R) be a weak factorization system on $\mathcal{C}$. Let $\widetilde{S}_L$ denote the collection of all U-cocartesian morphisms $\widetilde{f}$ in $\mathcal{E}$ satisfying $U(\widetilde{f}) \in S_L$, and let $\widetilde{S}_R$ be the collection of all morphisms $\widetilde{g}$ in $\mathcal{E}$ satisfying $U(\widetilde{g}) \in S_R$. Then the pair $(\widetilde{S}_L, \widetilde{S}_R)$ is a weak factorization system on $\mathcal{E}$. If (S_L, S_R) is a factorization system on $\mathcal{C}$, then $(\widetilde{S}_L, \widetilde{S}_R)$ is a factorization system on $\mathcal{E}$.*

Proof. By assumption, the collection S_L is weakly left orthogonal to S_R. Applying Corollary 9.1.7.23, we see that $\widetilde{S}_L$ is weakly left orthogonal to $\widetilde{S}_R$ (and left orthogonal if the pair (S_L, S_R) is a factorization system on $\mathcal{C}$). Since S_L and S_R are closed under isomorphism, the collections $\widetilde{S}_L$ and $\widetilde{S}_R$ have the same property (see Corollary 8.5.1.13). We will complete the proof by showing that the pair $(\widetilde{S}_L, \widetilde{S}_R)$ satisfies condition (1) of Definition 9.1.6.1. Let $\widetilde{h} : \widetilde{X} \to \widetilde{Z}$ be a morphism in the ∞-category $\mathcal{E}$, and let $h : X \to Z$ denote its image in the ∞-category $\mathcal{C}$. Since (S_L, S_R) is a weak factorization system, we can choose a 2-simplex σ :

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

of $\mathcal{C}$, where f belongs to S_L and g belongs to S_R. Our assumption that U is a cocartesian fibration guarantees that we can lift f to a U-cocartesian morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in the ∞-category $\mathcal{E}$. Since $\widetilde{f}$ is U-cocartesian, we can lift σ to a 2-simplex $\widetilde{\sigma}$:

$$\begin{tikzcd}
& \widetilde{Y} \arrow[dr, "\widetilde{g}"] & \\
\widetilde{X} \arrow[ur, "\widetilde{f}"] \arrow[rr, "\widetilde{h}"] & & \widetilde{Z}
\end{tikzcd}$$

in the ∞-category $\mathcal{E}$. By construction, we have $\widetilde{f} \in \widetilde{S}_L$ and $\widetilde{g} \in \widetilde{S}_R$. □

Corollary 9.1.9.14. *Let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of ∞-categories, let S be the collection of all U-cocartesian morphisms of $\mathcal{E}$, and let T be the collection of all morphisms f of $\mathcal{E}$ such that $U(f)$ is an isomorphism in $\mathcal{C}$. Then the pair (S, T) is a factorization system on $\mathcal{E}$.* 04PM

Proof. Combine Proposition 9.1.9.13 with Example 9.1.9.3. □

One can produce many examples of factorization systems using the small object argument of §9.1.4.

Theorem 9.1.9.15 (Existence of Factorization Systems)**.** *Let $\mathcal{C}$ be an ∞-category and let W be a collection of morphisms of $\mathcal{C}$. Assume that:* 04PN

(1) *The ∞-category $\mathcal{C}$ is locally small and admits small colimits.*

(2) *The collection W is small.*

(3) *For every morphism $w : X \to Y$ in W, the objects X and Y are κ-compact for some small cardinal κ.*

Then $\mathcal{C}$ admits a factorization system (S_L, S_R), where S_R is the collection of morphisms of $\mathcal{C}$ which are right orthogonal to W.

Proof. By virtue of Corollary 9.1.7.21 (and Proposition [?]), we can enlarge W to arrange that every morphism $w : A \to B$ which belongs to W admits a relative codiagonal $\gamma_{A/B} : B \coprod_A B \to B$ which also belongs to W. Applying Theorem 9.1.6.7, we conclude that $\mathcal{C}$ admits a weak factorization system (S_L, S_R), where S_L is the weakly saturated class of morphisms generated by W and S_R is the collection of morphisms which are weakly right orthogonal to W. Using Corollary 9.1.7.21, we see that a morphism $g : X \to Y$ of $\mathcal{C}$ belongs to S_R if and only if it is right orthogonal to W. If this condition is satisfied, then Corollary 9.1.7.20 guarantees that g is right orthogonal to W. Allowing g to vary, we conclude that (S_L, S_R) is a factorization system. □

04PP **Remark 9.1.9.16.** In the situation of Theorem 9.1.9.15, the collection S_L is characterized by the fact that it is the smallest weakly saturated collection of morphisms which contains W and also satisfies the equivalent conditions of Corollary 9.1.9.12. Beware that it is generally larger than the weakly saturated collection of morphisms generated by W.

We close this section by recording a converse to Proposition 9.1.9.6:

04PT **Theorem 9.1.9.17.** *Let $\mathcal{C}$ be an ∞-category, let S_L and S_R be collections of morphisms of $\mathcal{C}$, and let $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \subseteq \mathrm{Fun}(\Delta^2, \mathcal{C})$ be the full subcategory of Notation 9.1.8.1. Then (S_L, S_R) is a factorization system on $\mathcal{C}$ if and only if it satisfies the following conditions:*

(1) *The restriction map*

$$D : \mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \qquad \sigma \mapsto d_1^2(\sigma)$$

is an equivalence of ∞-categories.

(2) *Every identity morphism of $\mathcal{C}$ is contained in both S_L and S_R.*

(3) *The collections S_L and S_R are closed under isomorphism (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$).*

Proof. The necessity of (3) is immediate from the definitions, and the necessity of (1) and (2) follow from Proposition 9.1.9.6 and Corollary 9.1.9.7, respectively. For the converse, assume that conditions (1), (2), and (3) are satisfied. Combining (1) and (3) with Theorem 9.1.8.2, we deduce that S_L is left orthogonal to S_R. We are therefore reduced to proving that the functor D is surjective on objects. Assumption (3) guarantees that the full subcategory $\mathrm{Fun}_{LR}(\Delta^2, \mathcal{C}) \subseteq \mathrm{Fun}(\Delta^2, \mathcal{C})$ is replete, so that D is an isofibration (see Corollary 4.4.5.3). It will therefore suffice to show that D is essentially surjective, which follows from assumption (1). □

Corollary 9.1.9.18 (Exponentiation of Factorization Systems)**.** *Let $\mathcal{C}$ be an ∞-category equipped with a factorization system (S_L, S_R) and let K be a simplicial set. Then the ∞-category $\operatorname{Fun}(K, \mathcal{C})$ admits a factorization system (S_L^K, S_R^K), where S_L^K denotes the collection of all morphisms f in $\operatorname{Fun}(K, \mathcal{C})$ such that $f(v) \in S_L$ for each vertex v of K, and S_R^K is defined similarly.* 04PV

Proof. Since S_L and S_R contain identity morphisms and are closed under isomorphism, the collections S_L^K and S_R^K have the same properties. By virtue of Theorem 9.1.9.17, it will suffice to show that the restriction map

$$D_K : \operatorname{Fun}_{LR}(\Delta^2, \operatorname{Fun}(K, \mathcal{C})) \to \operatorname{Fun}(\Delta^1, \operatorname{Fun}(K, \mathcal{C})) \qquad \sigma \mapsto d_1^2(\sigma)$$

is an equivalence of ∞-categories. This follows from Remark 4.5.1.16, since D_K is obtained by applying the functor $\operatorname{Fun}(K, \bullet)$ to the restriction map $D : \operatorname{Fun}_{LR}(\Delta^2, \mathcal{C}) \to \operatorname{Fun}(\Delta^1, \mathcal{C})$. □

9.2 Truncated Objects of ∞-Categories

9.2.1 Truncated Objects 05EA

Let n be an integer. Recall that a Kan complex X is *n-truncated* if, for every integer $m \geq n+2$, every morphism $\partial \Delta^m \to X$ can be extended to an m-simplex of X. We now introduce a counterpart of this condition for objects of an arbitrary ∞-category. 05EB

Definition 9.2.1.1. Let $\mathcal{C}$ be an ∞-category and let n be an integer. We say that an object $X \in \mathcal{C}$ is *n-truncated* if, for every object $Y \in \mathcal{C}$, the morphism space $\operatorname{Hom}_{\mathcal{C}}(Y, X)$ is an n-truncated Kan complex. 05EC

Remark 9.2.1.2. In the formulation of Definition 9.2.1.1, we can replace $M = \operatorname{Hom}_{\mathcal{C}}(Y, X)$ by any Kan complex which is homotopy equivalent M. For example, we can replace M by the pinched morphism spaces $\operatorname{Hom}^{\mathrm{L}}_{\mathcal{C}}(Y, X)$ and $\operatorname{Hom}^{\mathrm{R}}_{\mathcal{C}}(Y, X)$ (see Proposition 4.6.5.10). 05ED

Example 9.2.1.3. Let $\mathcal{C}$ be an ∞-category. For $n \leq -2$, an object $X \in \mathcal{C}$ is n-truncated if and only if it is a final object of $\mathcal{C}$ (Definition 4.6.7.1). In particular, this condition is independent of n, so long as $n \leq -2$. Consequently, in the setting of Definition 9.2.1.1, there is no loss of generality in assuming that $n \geq -2$. 05EE

Example 9.2.1.4. Let X be a Kan complex and let n be an integer. The following conditions are equivalent: 05EF

(1) The Kan complex X is n-truncated, in the sense of Definition 3.5.7.1.

(2) For every Kan complex Y, the Kan complex $\operatorname{Fun}(Y, X)$ is n-truncated.

(3) For every simplicial set Y, the Kan complex $\mathrm{Fun}(Y, X)$ is n-truncated.

(4) The Kan complex X is n-truncated when regarded as an object of the ∞-category $\mathcal{S}$ (in the sense of Definition 9.2.1.1).

The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are immediate, the implication (1) $\Rightarrow$ (3) follows from Corollary 3.5.9.27, and the equivalence (2) $\Leftrightarrow$ (4) follows from the homotopy equivalence $\mathrm{Fun}(Y, X) \to \mathrm{Hom}_{\mathcal{S}}(Y, X)$ of Remark 5.5.1.5.

05EG **Remark 9.2.1.5.** Let $\mathcal{C}$ be an ∞-category, and let X and Y be objects of $\mathcal{C}$. If X is n-truncated and Y is a retract of X, then Y is also n-truncated. In particular, if X and Y are isomorphic, then X is n-truncated if and only if Y is n-truncated.

05EH **Remark 9.2.1.6.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let n be an integer, and let $X \in \mathcal{C}$ be an object whose image $F(X)$ is an n-truncated object of $\mathcal{D}$. If the functor F is essentially $(n+1)$-categorical (Definition 4.8.6.1), then X is an n-truncated object of $\mathcal{C}$ (see Proposition 3.5.9.13). In particular, if F is fully faithful, then X is an n-truncated object of $\mathcal{C}$.

05EJ **Remark 9.2.1.7.** Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories. Then an object $X \in \mathcal{C}$ is n-truncated if and only if the image $Y = F(X)$ is an n-truncated object of $\mathcal{D}$. The "if" direction follows from Remark 9.2.1.6. For the converse, suppose that X is n-truncated and let $G : \mathcal{D} \to \mathcal{C}$ be a homotopy inverse to F. Then $G(Y) \in \mathcal{C}$ is isomorphic to X, and is therefore an n-truncated object of $\mathcal{C}$ (Remark 9.2.1.5). Since G is fully faithful, Remark 9.2.1.6 guarantees that Y is an n-truncated object of $\mathcal{D}$.

05EK **Remark 9.2.1.8.** Let $\mathcal{C}$ be an ∞-category and let n be an integer. Then $\mathcal{C}$ is locally n-truncated (in the sense of Definition 4.8.2.1) if and only if every object $X \in \mathcal{C}$ is n-truncated (in the sense of Definition 9.2.1.1).

05EL **Remark 9.2.1.9** (Monotonicity)**.** Let $\mathcal{C}$ be an ∞-category and let $m \leq n$ be integer. If an object $X \in \mathcal{C}$ is m-truncated, then it is also n-truncated (see Remark 3.5.9.6).

05EM **Remark 9.2.1.10.** Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and let $n \geq -2$ be an integer. The following conditions are equivalent:

(1) The object $X \in \mathcal{C}$ is n-truncated, in the sense of Definition 9.2.1.1.

(2) The constant map $\partial \Delta^{n+2} \to \{\mathrm{id}_X\} \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, X)$ exhibits X as a power of itself by $\partial \Delta^{n+2}$, in the sense of Definition 7.6.2.1.

(3) The constant map
$$(\partial \Delta^{n+2})^{\triangleright} \simeq \Lambda^{n+3}_{n+3} \to \{X\} \hookrightarrow \mathcal{C}$$
is a limit diagram in $\mathcal{C}$, in the sense of Definition 7.1.2.4.

The equivalence (1) $\Leftrightarrow$ (2) follows from Corollary 3.5.9.22, and the equivalence (2) $\Leftrightarrow$ (3) from Remark 7.6.2.6.

Remark 9.2.1.11. In the formulation of Remark 9.2.1.10, we can replace $\partial \Delta^{n+2}$ by any simplicial set K of the same weak homotopy type (that is, any simplicial set K for which the geometric realization $|K|$ is homotopy equivalent to a sphere of dimension $n+1$). For example, we can take K to be the subdivision $\mathrm{Sd}(\Delta^{n+1})$ (see Proposition 3.3.4.8). 05EN

Remark 9.2.1.12. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves finite limits. Then, for every n-truncated object $X \in \mathcal{C}$, the image $F(X)$ is an n-truncated object of $\mathcal{D}$. This follows from the criterion of Remark 9.2.1.10. 05EP

Remark 9.2.1.13. Let $\mathcal{C}$ be an ∞-category and let n be an integer. Then an object $X \in \mathcal{C}$ is n-truncated if and only if the right fibration $\mathcal{C}_{/X} \to \mathcal{C}$ is essentially n-categorical. This follows from the criterion of Corollary 5.1.5.18 (together with Remark 9.2.1.2). 05EQ

Proposition 9.2.1.14. *Let $\mathcal{C}$ be an ∞-category and let n be an integer. Let X be an object of $\mathcal{C}$, and let us abuse notation by identifying X with its image in the homotopy n-category $\mathrm{h}_{\leq n}(\mathcal{C})$ (Notation 4.8.4.2). Then X is $(n-1)$-truncated if and only if the diagram of ∞-categories* 05K2

$$\begin{tikzcd}
\mathcal{C}_{/X} \arrow[r] \arrow[d] & \mathrm{h}_{\leq n}(\mathcal{C})_{/X} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathrm{h}_{\leq n}(\mathcal{C})
\end{tikzcd} \tag{9.18}$$
05K3

is a categorical pullback square.

Proof. Since the vertical maps in the diagram (9.18) are right fibrations (Proposition 4.3.6.1), it is a categorical pullback square if and only if, for every object $Y \in \mathcal{C}$, the induced map of right-pinched morphism spaces

$$\theta : \mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(Y, X) = \{Y\} \times_{\mathcal{C}} \mathcal{C}_{/Y} \to \{Y\} \times_{\mathrm{h}_{\leq n}(\mathcal{C})} (\mathrm{h}_{\leq n}(\mathcal{C}))_{/X} = \mathrm{Hom}^{\mathrm{R}}_{\mathrm{h}_{\leq n(()\mathcal{C})}}(Y, X)$$

is a homotopy equivalence (Corollary 5.1.6.4). It follows from Corollary 4.8.4.8 that θ exhibits $\mathrm{Hom}^{\mathrm{R}}_{\mathrm{h}_{\leq n(()\mathcal{C})}}(Y, X)$ as an $(n-1)$-truncation of $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(Y, X)$. In particular, it is a homotopy equivalence if and only if $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(Y, X)$ is $(n-1)$-truncated. This condition is satisfied for every object $Y \in \mathcal{C}$ if and only if the object X is $(n-1)$-truncated (Remark 9.2.1.2). □

Corollary 9.2.1.15. *Let $\mathcal{C}$ be an ∞-category, let n be an integer, and let X be an $(n-1)$-truncated object of $\mathcal{C}$. Then the canonical map $\mathrm{h}_{\leq n}(\mathcal{C}_{/X}) \to \mathrm{h}_{\leq n}(\mathcal{C})_{/X}$ is an equivalence of ∞-categories.* 05K4

Proof. Combine Propositions 9.2.1.14 and 4.8.4.20. □

05ER **Proposition 9.2.1.16** (Limits of Truncated Objects)**.** *Let $\mathcal{C}$ be an ∞-category and let n be an integer. Then the collection of n-truncated objects of $\mathcal{C}$ is closed under limits. That is, if $F : A^{\triangleleft} \to \mathcal{C}$ is a limit diagram in $\mathcal{C}$ having the property that $F(a)$ is n-truncated for each vertex $a \in A$, then F carries the cone point of $A^{\triangleleft}$ to an n-truncated object of $\mathcal{C}$.*

Proof. Fix an object $C \in \mathcal{C}$ and let $h^C : \mathcal{C} \to \mathcal{S}$ denote the functor corepresented by C. We wish to show that the composition

$$A^{\triangleleft} \xrightarrow{F} \mathcal{C} \xrightarrow{h^C} \mathcal{S}$$

carries the cone point of $A^{\triangleleft}$ to an n-truncated Kan complex. This follows from Remark 7.4.5.9, since $h^C \circ F$ is a limit diagram in the ∞-category $\mathcal{S}$ (Corollary 7.4.5.18). □

05ES **Remark 9.2.1.17.** Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by those objects $Y \in \mathcal{C}$ for which the morphism space $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is n-truncated. Since the representable functor

$$h_X : \mathcal{C}^{\mathrm{op}} \to \mathcal{S} \qquad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

carries colimits in the ∞-category $\mathcal{C}$ to limits in the ∞-category of spaces $\mathcal{S}$, (Corollary 7.4.5.18), Remark 7.4.5.9 guarantees that the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is closed under the formation of colimits. Consequently, if $\mathcal{C}$ is generated (under the formation of small colimits) by some full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$, then the object X is n-truncated if and only if the morphism space $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is n-truncated for each object $Y \in \mathcal{C}_0$.

Definition 9.2.1.1 can be reformulated as a filling condition:

05ET **Proposition 9.2.1.18.** *Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and let $n \geq -2$ be an integer. Then X is n-truncated if and only if it satisfies the following condition for each $m \geq n + 3$:*

$(*_m)$ *Every morphism $\sigma : \partial \Delta^m \to \mathcal{C}$ satisfying $\sigma(m) = X$ can be extended to an m-simplex of $\mathcal{C}$.*

Proof. This is a special case of Proposition 4.8.6.20, since the object X is n-truncated if and only if the right fibration $\mathcal{C}_{/X} \to \mathcal{C}$ is essentially $(n + 1)$-categorical (Remark 9.2.1.13). □

We close this section by classifying the truncated objects of the ∞-category $\mathcal{QC}$ of (small) ∞-categories (Construction 5.5.4.1).

05EU **Proposition 9.2.1.19.** *Let $\mathcal{C}$ be an ∞-category and let n be an integer. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is n-truncated when viewed as an object of $\mathcal{QC}$, in the sense of Definition 9.2.1.1.*

(2) *The Kan complex $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}$ is n-truncated.*

(3) *The core $\mathcal{C}^{\simeq}$ is an n-truncated Kan complex. Moreover, for every pair of objects $X, Y \in \mathcal{C}$, the morphism space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is also an n-truncated Kan complex.*

Proof. By virtue of Proposition [?], the ∞-category $\mathcal{QC}$ is generated under colimits by the object $\Delta^1 \in \mathcal{QC}$. The equivalence (1) $\Leftrightarrow$ (2) now follows by combining Remarks 9.2.1.17 and 5.5.4.5. We next show that, if condition (2) is satisfied, then the core $\mathcal{C}^{\simeq}$ is an n-truncated Kan complex. Let $\mathrm{Isom}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by the isomorphisms of $\mathcal{C}$. Then the diagonal map

$$\mathcal{C} \hookrightarrow \mathrm{Isom}(\mathcal{C}) \qquad X \mapsto \mathrm{id}_X$$

is an equivalence of ∞-categories (Corollary 4.5.3.13), and therefore restricts to a homotopy equivalence of Kan complexes $\mathcal{C}^{\simeq} \hookrightarrow \mathrm{Isom}(\mathcal{C})^{\simeq}$. We are therefore reduced to showing that the Kan complex $\mathrm{Isom}(\mathcal{C})^{\simeq}$ is n-truncated. Since $\mathrm{Isom}(\mathcal{C})^{\simeq}$ is a summand of the Kan complex $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}$, this follows immediately from assumption (2) if $n \geq -1$. The case $n \leq -2$ then follows from the additional observation that if $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq}$ is nonempty, then the ∞-category $\mathcal{C}$ is nonempty, so $\mathrm{Isom}(\mathcal{C})^{\simeq}$ is nonempty.

We now complete the proof by showing that (2) and (3) are equivalent. By virtue of the preceding argument, we may assume that the core $\mathcal{C}^{\simeq}$ is n-truncated, so the product $\mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$ is n-truncated (Remark 3.5.7.6). Using Proposition 3.5.9.13, we see that condition (2) is satisfied if and only if the map of Kan complexes

$$U : \mathrm{Fun}(\Delta^1, \mathcal{C})^{\simeq} \to \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq} \qquad (f : X \to Y) \mapsto (X, Y)$$

is n-truncated. Since U is a Kan fibration (Corollary 4.4.5.4), this is equivalent to the requirement that each fiber of U is an n-truncated Kan complex (Proposition 3.5.9.8), which is a restatement of (3). □

Remark 9.2.1.20. If $n \geq -1$, we can reformulate condition (3) of Proposition 9.2.1.19 as follows: 05EV

(3′) For every pair of objects $X, Y \in \mathcal{C}$, the morphism space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is n-truncated. Moreover, the summand $\mathrm{Isom}_{\mathcal{C}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ spanned by the isomorphisms from X to Y is $(n-1)$-truncated.

See Example 3.5.9.18.

Corollary 9.2.1.21. *Let $\mathcal{C}$ be an ∞-category and let n be an integer. Then:* 05EW

- *If $\mathcal{C}$ is an n-truncated object of $\mathcal{QC}$ (in the sense of Definition 9.2.1.1), then it is locally n-truncated (in the sense of Definition 4.8.2.1).*

- *If $n \geq -1$ and $\mathcal{C}$ is locally $(n-1)$-truncated, then it is an n-truncated object of $\mathcal{QC}$.*

05EX **Warning 9.2.1.22.** In general, neither implication of Corollary 9.2.1.21 is reversible. See Example 9.2.2.11.

9.2.2 Example: Discrete and Subterminal Objects

04UX We now consider some important special cases of Definition 9.2.1.1.

05EY **Definition 9.2.2.1.** Let $\mathcal{C}$ be an ∞-category. We will say that an object $X \in \mathcal{C}$ is *discrete* if, for every object $C \in \mathcal{C}$, every connected component of the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, X)$ is contractible.

04UY **Definition 9.2.2.2.** Let $\mathcal{C}$ be an ∞-category. We will say that an object $X \in \mathcal{C}$ is *subterminal* if, for every object $C \in \mathcal{C}$, the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, X)$ is either empty or contractible.

05EZ **Remark 9.2.2.3.** Let $\mathcal{C}$ be an ∞-category. Then:

- An object $X \in \mathcal{C}$ is discrete (in the sense of Definition 9.2.2.1) if and only if is 0-truncated (in the sense of Definition 9.2.2.1).

- An object $X \in \mathcal{C}$ is subterminal (in the sense of Definition 9.2.2.2) if and only if it is (-1)-truncated.

See Examples 3.5.7.4 and 3.5.7.5.

04V0 **Example 9.2.2.4.** Let $\mathcal{C}$ be an ∞-category. Then every final object of $\mathcal{C}$ is subterminal, and every subterminal object of $\mathcal{C}$ is discrete.

04V1 **Example 9.2.2.5.** Let X be a Kan complex, which we regard as an object of the ∞-category of spaces $\mathcal{S}$ (Construction 5.5.1.1). Then:

- The Kan complex X is a discrete object of the ∞-category $\mathcal{S}$ (in the sense of Definition 9.2.2.1) if and only if every connected component of X is contractible: that is, the projection map $X \to \pi_0(X)$ is a homotopy equivalence.

- The Kan complex X is a subterminal object of the ∞-category $\mathcal{S}$ (in the sense of Definition 9.2.2.2) if and only if X is either empty or contractible.

See Example 9.2.1.4.

05F0 **Example 9.2.2.6.** Let $\mathcal{C} = \mathrm{N}_{\bullet}(\mathcal{C}_0)$ be the nerve of an ordinary category $\mathcal{C}_0$. Then:

- Every object of $\mathcal{C}$ is discrete.
- An object $X \in \mathcal{C}$ is subterminal (in the sense of Definition 9.2.2.2) if and only if it is subterminal in the sense of classical category theory: that is, for every object $Y \in \mathcal{C}$, there is at most one morphism from Y to X.

Remark 9.2.2.7. Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. If X is discrete, then the canonical map $\mathrm{h}(\mathcal{C}_{/X}) \to (\mathrm{h}\mathcal{C})_{/X}$ is an equivalence of categories. This is a special case of Corollary 9.2.1.15. 05K5

Proposition 9.2.2.8. *Let $\mathcal{C}$ be an ∞-category and let* 05K6

$$\begin{tikzcd} X_{01} \arrow[r] \arrow[d] & X_0 \arrow[d] \\ X_1 \arrow[r] & X \end{tikzcd} \qquad (9.19)$$ 05K7

be a pullback diagram in $\mathcal{C}$. If the object X is discrete, then (9.19) determines a pullback diagram in the homotopy category $\mathrm{h}\mathcal{C}$.

Beware that the conclusion of Proposition 9.2.2.8 is generally false if we do not assume that the object X is discrete (see Warning 7.6.3.3).

Proof of Proposition 9.2.2.8. By virtue of Proposition 7.6.3.14, it will suffice to show that the the tautological map $F : \mathcal{C}_{/X} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})_{/X}$ preserves products. Since X is discrete, we can use Remark 9.2.2.7 to identify F with the canonical map from $\mathcal{C}_{/X}$ to (the nerve of) its homotopy category, which always preserves products (see Warning 7.6.1.11). □

We now record a partial converse to Example 9.2.2.6.

Definition 9.2.2.9. Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *locally discrete* if every object $X \in \mathcal{C}$ is discrete. 05F1

Note that an ∞-category $\mathcal{C}$ is locally discrete if and only if it is locally 0-truncated, in the sense of Definition 4.8.2.1. Invoking Corollary 4.8.2.15, we obtain the following:

Remark 9.2.2.10. Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent: 05F2

- The ∞-category $\mathcal{C}$ is locally discrete.
- The comparison map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ is a trivial Kan fibration.
- There exists an ordinary category $\mathcal{C}_0$ and an equivalence of ∞-categories $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathcal{C}_0)$.

05F3 **Example 9.2.2.11.** Let $\mathcal{QC}$ be the ∞-category of (small) ∞-categories (Construction 5.5.4.1). Then an object $\mathcal{C} \in \mathcal{QC}$ is discrete (in the sense of Definition 9.2.2.1) if and only if it satisfies the following pair of conditions:

- The ∞-category $\mathcal{C}$ is locally discrete: that is, there exists an equivalence $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathcal{C}_0)$, where $\mathcal{C}_0$ is an ordinary category (Remark 9.2.2.10).
- For every object $X \in \mathcal{C}_0$, the automorphism group $\mathrm{Aut}(X)$ is trivial.

See Proposition 9.2.1.19. Beware that the second condition cannot be omitted.

04V2 **Remark 9.2.2.12.** Let $\mathcal{C}$ be an ∞-category, let $X \in \mathcal{C}$ be an object, and let $Y \in \mathcal{C}$ be a retract of X. If X is discrete, then Y is also discrete. If X is subterminal, then Y is also subterminal. See Remark 9.2.1.5.

04V3 **Remark 9.2.2.13.** Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories, and let X be an object of $\mathcal{C}$. Then:

- If $F(X)$ is a discrete object of $\mathcal{D}$, then X is a discrete object of $\mathcal{C}$.
- If $F(X)$ is a subterminal object of $\mathcal{D}$, then X is a subterminal object of $\mathcal{C}$.

In both cases, the converse holds if F is an equivalence of ∞-categories.

04V4 **Remark 9.2.2.14.** Let $\mathcal{C}$ be an ∞-category. Then an object $X \in \mathcal{C}$ is discrete if and only if it satisfies the following condition for every integer $m \geq 3$:

$(*_m)$ Every morphism $\sigma : \partial\Delta^m \to \mathcal{C}$ satisfying $\sigma(m) = X$ can be extended to an m-simplex of $\mathcal{C}$.

In this case, X is subterminal if and only if it also satisfies condition $(*_2)$. See Proposition 9.2.1.18.

05F4 **Remark 9.2.2.15.** Let $\mathcal{C}$ be an ∞-category. An object $X \in \mathcal{C}$ is subterminal if and only if the diagram

$$X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$$

exhibits X as a product of X with itself. See Remark 9.2.1.10.

05F5 **Remark 9.2.2.16.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then:

- If F preserves finite limits, then it carries discrete objects of $\mathcal{C}$ to discrete objects of $\mathcal{D}$ (see Remark 9.2.1.12).
- If F preserves pairwise products, then it carries subterminal objects of $\mathcal{C}$ to subterminal objects of $\mathcal{D}$ (see Remark 9.2.2.15).

Notation 9.2.2.17 (The Heart of an ∞-Category)**.** Let $\mathcal{C}$ be an ∞-category. We let $\mathcal{C}^{\heartsuit}$ denote the full subcategory of $\mathcal{C}$ spanned by the discrete objects of $\mathcal{C}$. We will refer to $\mathcal{C}^{\heartsuit}$ as the *heart* of the ∞-category $\mathcal{C}$. 05F6

Let $\mathrm{Disc}(\mathcal{C})$ denote the homotopy category of $\mathcal{C}^{\heartsuit}$. By construction, the ∞-category $\mathcal{C}^{\heartsuit}$ is locally discrete, so Remark 9.2.2.10 guarantees that the comparison map $\mathcal{C} \to \mathrm{N}_{\bullet}(\mathrm{h}\mathcal{C})$ restricts to a trivial Kan fibration

$$\mathcal{C}^{\heartsuit} \to \mathrm{N}_{\bullet}(\mathrm{Disc}(\mathcal{C})).$$

For this reason, we will often abuse terminology by identifying the heart $\mathcal{C}^{\heartsuit}$ with the ordinary category $\mathrm{Disc}(\mathcal{C})$, which we also refer to as the *heart* of $\mathcal{C}$.

Notation 9.2.2.18. Let $\mathcal{C}$ be an ∞-category. We let $\mathrm{Sub}(\mathcal{C})$ denote the collection of isomorphism classes of subterminal objects of $\mathcal{C}$. If X is a subterminal object of $\mathcal{C}$, we let $[X] \in \mathrm{Sub}(\mathcal{C})$ denote its isomorphism class. Given a pair of subterminal objects X and X', we write $[X] \subseteq [X']$ if there exists a morphism $f : X \to X'$ in the ∞-category $\mathcal{C}$. Note that the relation $\subseteq$ is a partial ordering on the set $\mathrm{Sub}(\mathcal{C})$. 04V9

Remark 9.2.2.19. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by the subterminal objects of $\mathcal{C}$. Then the construction $X \mapsto [X]$ induces a trivial Kan fibration of ∞-categories $\mathcal{C}' \to \mathrm{N}_{\bullet}(\mathrm{Sub}(\mathcal{C}))$. Stated more informally, the partially ordered set $\mathrm{Sub}(\mathcal{C})$ can be identified with the full subcategory $\mathcal{C}' \subseteq \mathcal{C}$. 05F7

Remark 9.2.2.20. Let $\mathcal{C}$ be an ∞-category. 04VA

- If $\mathcal{C}$ has a final object, then the partially ordered set $\mathrm{Sub}(\mathcal{C})$ has a largest element: namely, the isomorphism class $[X]$, where X is any final object of $\mathcal{C}$.
- If $\mathcal{C}$ admits finite products, then $\mathrm{Sub}(\mathcal{C})$ is a lower semilattice: that is, every finite subset of $\mathrm{Sub}(\mathcal{C})$ has a greatest lower bound. In particular, every pair of elements $[X], [Y] \in \mathrm{Sub}(\mathcal{C})$ have a greatest lower bound which we will denote by $[X] \cap [Y]$, given by the isomorphism class of the product $X \times Y$.

9.2.3 Truncated Morphisms

We now introduce a relative version of Definition 9.2.1.1. 05F8

Definition 9.2.3.1. Let $\mathcal{C}$ be an ∞-category and let n be an integer. We say that a morphism $f : X \to Y$ of $\mathcal{C}$ is *n-truncated* if, for every object $C \in \mathcal{C}$, composition with the homotopy class $[f]$ induces an n-truncated morphism of Kan complexes $\mathrm{Hom}_{\mathcal{C}}(C, X) \xrightarrow{[f]\circ} \mathrm{Hom}_{\mathcal{C}}(C, Y)$. 05F9

05FA **Remark 9.2.3.2.** In the situation of Definition 9.2.3.1, the composition map

$$\theta : \operatorname{Hom}_{\mathcal{C}}(C, X) \xrightarrow{[f]\circ} \operatorname{Hom}_{\mathcal{C}}(C, Y)$$

is only well-defined up to homotopy (see Notation 4.6.9.15). However, the condition that θ is n-truncated depends only on its homotopy class (Remark 3.5.9.5).

05FB **Remark 9.2.3.3.** Let $f : X \to Y$ be a morphism in an ∞-category $\mathcal{C}$. The condition that f is n-truncated depends only on the homotopy class $[f]$, regarded as a morphism in the homotopy category $\mathrm{h}\mathcal{C}$.

05FC **Remark 9.2.3.4** (Monotonicity)**.** Let $\mathcal{C}$ be an ∞-category and let $m \leq n$ be integers. If $f : X \to Y$ is an m-truncated morphism of $\mathcal{C}$, then it is also n-truncated.

05FD **Example 9.2.3.5.** Let $\mathcal{C}$ be an ∞-category. For $n \leq -2$, a morphism $f : X \to Y$ of $\mathcal{C}$ is n-truncated if and only if it is an isomorphism. See Example 3.5.9.2.

05FE **Example 9.2.3.6.** Let $f : X \to Y$ be a morphism of Kan complexes and let n be an integer. The following conditions are equivalent:

(1) The morphism f is n-truncated, in the sense of Definition 3.5.9.1.

(2) For every Kan complex K, composition with f induces an n-truncated morphism $\operatorname{Fun}(K, X) \to \operatorname{Fun}(K, Y)$.

(3) For every simplicial set K, composition with f induces an n-truncated morphism $\operatorname{Fun}(K, X) \to \operatorname{Fun}(K, Y)$.

(4) The morphism f is n-truncated when regarded as a morphism in the ∞-category $\mathcal{S}$ of spaces, in the sense of Definition 9.2.3.1.

The implications (3) ⇒ (2) ⇒ (1) are immediate, the implication (1) ⇒ (3) follows from Corollary 3.5.9.26, and the equivalence (2) ⇔ (4) follows from Remark 5.5.1.5.

05FF **Proposition 9.2.3.7.** *Let $\mathcal{C}$ be an ∞-category, let n be an integer, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then the morphism f is n-truncated (in the sense of Definition 9.2.3.1) if and only if it is n-truncated when regarded as an object of the slice ∞-category $\mathcal{C}_{/Y}$ (in the sense of Definition 9.2.1.1).*

Proof. By definition, f is n-truncated as an object of $\mathcal{C}_{/Y}$ if and only if, for every morphism $g : C \to Y$ of $\mathcal{C}$, the morphism space $K = \operatorname{Hom}_{\mathcal{C}_{/Y}}(g, f)$ is n-truncated. Using Corollary 4.6.9.18, we can identify K with the homotopy fiber of the composition map $\operatorname{Hom}_{\mathcal{C}}(C, X) \xrightarrow{[f]\circ} \operatorname{Hom}_{\mathcal{C}}(C, Y)$ over the vertex $g \in \operatorname{Hom}_{\mathcal{C}}(C, Y)$. The desired result now follows from Corollary 3.5.9.12. □

Corollary 9.2.3.8 (Homotopy Invariance)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then f is n-truncated if and only if the image $F(f) : F(X) \to F(Y)$ is n-truncated.* 05FG

Proof. Using Corollary 4.6.4.19, we see that F induces an equivalence of ∞-categories $\mathcal{C}_{/Y} \to \mathcal{D}_{/F(Y)}$. The desired result now follows by combining Proposition 9.2.3.7 with Remark 9.2.1.7. □

Corollary 9.2.3.9. *Let $\mathcal{C}$ be an ∞-category, let $n \geq 0$ be an integer, let $\square^{n+1}$ denote the simplicial cube of dimension $n+1$ (Notation 2.4.5.2), and let $y \in \square^{n+1}$ be the final vertex. Let $Q : \square^{n+1} \to \Delta^1$ be the morphism given on vertices by* 05FH

$$Q(v) = \begin{cases} 1 & \text{if } v = y \\ 0 & \text{otherwise.} \end{cases}$$

Then a morphism $f : X \to Y$ of $\mathcal{C}$ is $(n-2)$-truncated if and only if the composite map

$$\square^{n+1} \xrightarrow{Q} \Delta^1 \xrightarrow{f} \mathcal{C}$$

is a limit diagram in $\mathcal{C}$.

Proof. Let us identify $\square^{n+1}$ with the iterated join $\{x\} \star \mathrm{Sd}(\partial\Delta^n) \star \{y\}$, where $\mathrm{Sd}(\partial\Delta^n)$ denotes the subdivision of $\partial\Delta^n$ (see Proposition 3.3.3.16). Using Remark 7.1.2.11, we see that $f \circ Q$ is a limit diagram in $\mathcal{C}$ if and only if the constant map

$$\{x\} \star \mathrm{Sd}(\partial\Delta^n) \to \{f\} \hookrightarrow \mathcal{C}_{/Y}$$

is a limit diagram in the slice ∞-category $\mathcal{C}_{/Y}$. The desired result now follows by combining Proposition 9.2.3.7 with Remark 9.2.1.11. □

Corollary 9.2.3.10. *Let $U : \mathcal{C} \to \mathcal{D}$ be a right fibration of ∞-categories, let n be an integer, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then f is n-truncated if and only if $U(f)$ is an n-truncated morphism of $\mathcal{D}$.* 05FJ

Proof. Combine Corollaries 9.2.3.9 and 7.1.5.17. □

Corollary 9.2.3.11. *Let $\mathcal{C}$ be an ∞-category and let n be an integer. Then the collection of n-truncated morphisms of $\mathcal{C}$ is closed under retracts (in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$).* 05FK

Proof. Combine Corollaries 9.2.3.9 and 8.5.1.12. □

Corollary 9.2.3.12. *Let $\mathcal{C}$ be an ∞-category, let $f : X \to Y$ be a morphism of $\mathcal{C}$, and let $n \geq -2$ be an integer. Then f is n-truncated if and only if it satisfies the following condition for every positive integer $m \geq n+4$:* 05FL

$(*_m)$ *If $\sigma : \Lambda^m_m \to \mathcal{C}$ is a diagram having the property that the composite map*

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{m-1 < m\}) \hookrightarrow \Lambda^m_m \xrightarrow{\sigma} \mathcal{C}$$

is equal to f, then σ can be extended to an m-simplex of $\mathcal{C}$.

Proof. Combine Propositions 9.2.3.7 and 9.2.1.18. □

05FM **Proposition 9.2.3.13.** *Let $\mathcal{C}$ be an ∞-category, let n be an integer, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then:*

(1) *If Y is an n-truncated morphism and f is an n-truncated morphism, then X is an n-truncated object.*

(2) *If X is an n-truncated object and Y is an $(n+1)$-truncated object, then f is an n-truncated morphism.*

Proof. Let $C \in \mathcal{C}$ be an object and let $\theta : \mathrm{Hom}_{\mathcal{C}}(C, X) \to \mathrm{Hom}_{\mathcal{C}}(C, Y)$ be given by composition with the homotopy class $[f]$. Invoking Proposition 3.5.9.13, we obtain:

(1_C) If the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, Y)$ is n-truncated and θ is n-truncated, then the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, X)$ is n-truncated.

(2_C) If the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, X)$ is n-truncated and the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, Y)$ is $(n+1)$-truncated, then θ is n-truncated.

Proposition 9.2.3.13 follows by allowing the object C to vary. □

05FN **Corollary 9.2.3.14.** *Let $\mathcal{C}$ be an ∞-category, let n be an integer, and let Y be an n-truncated object of $\mathcal{C}$. Then a morphism $f : X \to Y$ is n-truncated if and only if the object X is n-truncated.*

Proof. Combine Proposition 9.2.3.13 with Remark 9.2.3.4. □

05FP **Example 9.2.3.15.** Let $\mathcal{C}$ be an ∞-category which contains a final object Y. Then every object $X \in \mathcal{C}$ admits a morphism $f : X \to Y$ which is uniquely determined up to homotopy. In this case, the object X is n-truncated (in the sense of Definition 9.2.1.1) if and only if the morphism f is n-truncated (in the sense of Definition 9.2.3.1).

05FQ **Corollary 9.2.3.16** (Composition)**.** *Let $\mathcal{C}$ be an ∞-category containing a 2-simplex*

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z,
\end{tikzcd}$$

and let n be an integer. Then:

(1) *If the morphisms f and g are n-truncated, then the morphism h is n-truncated.*

(2) *If the morphism h is n-truncated and the morphism g is $(n+1)$-truncated, then the morphism f is n-truncated.*

Proof. Apply Proposition 9.2.3.13 to the slice ∞-category $\mathcal{C}_{/Z}$ (see Proposition 9.2.3.7). ☐

Proposition 9.2.3.17 (Pullbacks of Truncated Morphisms)**.** *Let $\mathcal{C}$ be an ∞-category containing a pullback diagram* 05FR

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

and let n be an integer. If f is n-truncated, then f' is also n-truncated.

Proof. Let $C \in \mathcal{C}$ be an object. Applying Proposition 7.4.5.17, we obtain a pullback diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C, X') \arrow[r] \arrow[d, "\theta'"] & \mathrm{Hom}_{\mathcal{C}}(C, X) \arrow[d, "\theta"] \\
\mathrm{Hom}_{\mathcal{C}}(C, Y') \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(C, Y)
\end{tikzcd}$$

in the ∞-category of spaces. Corollary 7.6.4.11 guarantees that if θ is n-truncated, then θ' is also n-truncated. Proposition 9.2.3.17 now follows by allowing the object C to vary. ☐

Proposition 9.2.3.18. *Let $\mathcal{C}$ be an ∞-category, let $n \geq -1$ be an integer, and let X be an object of $\mathcal{C}$ for which there exists a product $X \times X$. Then X is n-truncated if and only if the diagonal map $\delta_X : X \to X \times X$ is $(n-1)$-truncated.* 05FS

Proof. For each object $C \in \mathcal{C}$, Example 3.5.9.18 shows that the mapping space $\mathrm{Hom}_{\mathcal{C}}(C, X)$ is n-truncated if and only if the diagonal map

$$\mathrm{Hom}_{\mathcal{C}}(C, X) \to \mathrm{Hom}_{\mathcal{C}}(C, X) \times \mathrm{Hom}_{\mathcal{C}}(C, X)$$

is $(n-1)$-truncated. The desired result now follows by allowing the object C to vary. ☐

Corollary 9.2.3.19. *Let $\mathcal{C}$ be an ∞-category, let $n \geq -1$ be an integer, and let $f : X \to Y$ be a morphism of $\mathcal{C}$ for which there exists a fiber product $X \times_Y X$. Then f is n-truncated if if and only if the relative diagonal $\delta_{X/Y} : X \to X \times_Y X$ is $(n-1)$-truncated (see Notation 7.6.3.18).* 05FT

Proof. Let us identify the morphism f with an object $\overline{X}$ of the slice ∞-category $\mathcal{C}_{/Y}$. By virtue of Proposition 7.6.3.14, there exists a product $\overline{X} \times \overline{X}$ in the ∞-category $\mathcal{C}_{/Y}$, whose image in $\mathcal{C}$ is the fiber product $X \times_Y X$. Moreover, the relative diagonal $\delta_{X/Y}$ can be identified with the image of the diagonal map $\delta_{\overline{X}} : \overline{X} \to \overline{X} \times \overline{X}$ under the forgetful functor $\mathcal{C}_{/Y} \to \mathcal{C}$. Applying Corollary 9.2.3.10, we see that $\delta_{X/Y}$ is an $(n-1)$-truncated morphism of $\mathcal{C}$ if and only if $\delta_{\overline{X}}$ is an $(n-1)$-truncated morphism of $\mathcal{C}_{/Y}$. By virtue of Proposition 9.2.3.18, this is equivalent to the requirement that $\overline{X}$ is n-truncated as an object of $\mathcal{C}_{/Y}$. The desired result now follows from the criterion of Proposition 9.2.3.7. □

05FU **Corollary 9.2.3.20.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Suppose that $\mathcal{C}$ admits pullbacks and that the functor F preserves pullbacks. Then, for every integer n, the functor F carries n-truncated morphisms of $\mathcal{C}$ to n-truncated morphisms of $\mathcal{D}$.*

Proof. For $n \leq -2$, a morphism is n-truncated if and only if it is an isomorphism (Example 9.2.3.5), so the desired result follows from Remark 1.5.1.6. The general case follows by induction on n, using Corollary 9.2.3.19. □

9.2.4 Monomorphisms

04VD Let $\mathcal{C}$ be a category. Recall that a morphism $f : X_0 \to X$ of $\mathcal{C}$ is a *monomorphism* if, for every object C of $\mathcal{C}$, the composition map

$$\mathrm{Hom}_{\mathcal{C}}(C, X_0) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(C, X)$$

is injective. This notion has an obvious counterpart in the setting of ∞-categories.

04VE **Definition 9.2.4.1.** Let $\mathcal{C}$ be an ∞-category and let $f : X_0 \to X$ be a morphism of $\mathcal{C}$. We say that f is a *monomorphism* if, for every object $C \in \mathcal{C}$, the composition map

$$\mathrm{Hom}_{\mathcal{C}}(C, X_0) \xrightarrow{[f]\circ} \mathrm{Hom}_{\mathcal{C}}(C, X)$$

induces a homotopy equivalence of $\mathrm{Hom}_{\mathcal{C}}(C, X_0)$ with a summand of $\mathrm{Hom}_{\mathcal{C}}(C, X)$.

04VF **Warning 9.2.4.2.** Let $f : X_0 \to X$ be a morphism of Kan complexes. The assertion that f is a monomorphism can be given two different interpretations:

(1) The map f is a monomorphism in the ordinary category of Set_{Δ} of simplicial sets.

(2) The map f is a monomorphism in the ∞-category $\mathcal{S}$ of spaces.

Beware that these conditions are unrelated to one another. Condition (2) is homotopy invariant: it is the requirement that f restricts to a homotopy equivalence of X_0 with a summand of X (Example 9.2.4.10). Condition (1) is very far from being homotopy invariant: we can always arrange that it is satisfied by replacing X by a homotopy equivalent Kan complex (see Exercise 3.1.7.11).

Notation 9.2.4.3. Let $\mathcal{C}$ be an ∞-category and let f be a morphism of $\mathcal{C}$ having source X_0 and target X. If f is a monomorphism, we will sometimes visually emphasize this by denoting f with a hooked arrow (that is, we will write $f : X_0 \hookrightarrow X$ in place of $f : X_0 \to X$). Beware that this convention can be ambiguous in some situations (for example if $\mathcal{C} = \mathcal{S}$ is the ∞-category of spaces; see Warning 9.2.4.2). 04VG

Variant 9.2.4.4. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. We will say that f is an *epimorphism* if it is a monomorphism when viewed as a morphism of the ∞-category $\mathcal{C}^{\mathrm{op}}$: that is, if the induced map 04VS

$$\mathrm{Hom}_{\mathcal{C}}(Y, C) \xrightarrow{\circ[f]} \mathrm{Hom}_{\mathcal{C}}(X, C)$$

induces a homotopy equivalence of $\mathrm{Hom}_{\mathcal{C}}(Y, C)$ with a summand of $\mathrm{Hom}_{\mathcal{C}}(X, C)$, for each object $C \in \mathcal{C}$. We will generally avoid this terminology, to avoid confusion with the notion of *quotient morphism* which we introduce in §10.2.2 (see Warning 10.2.2.10).

Remark 9.2.4.5. Let $\mathcal{C}$ be an ∞-category. Then a morphism $f : X_0 \to X$ is a monomorphism (in the sense of Definition 9.2.4.1) if and only if it is (-1)-truncated (in the sense of Definition 9.2.3.1). See Example 3.5.9.3. 05FV

Example 9.2.4.6. Let $\mathcal{C}$ be a category and let $f : X_0 \to X$ be a morphism in $\mathcal{C}$. Then f is a monomorphism in the ∞-category $\mathrm{N}_{\bullet}(\mathcal{C})$ (in the sense of Definition 9.2.4.1) if and only if it is a monomorphism in the usual category-theoretic sense. 04VH

Example 9.2.4.7. Let $\mathcal{C}$ be an ∞-category and let $f : X_0 \to X$ be a morphism of $\mathcal{C}$. Then: 04VJ

- If the object X is subterminal and f is a monomorphism, then the object X_0 is also subterminal.
- If the object X_0 is subterminal and the object X is discrete, then f is a monomorphism.

In particular, if X is subterminal, then f is a monomorphism if and only if X_0 is subterminal. See Proposition 9.2.3.13.

Example 9.2.4.8. Let $\mathcal{C}$ be an ∞-category containing a final object $\mathbf{1}$, and let X be an object of $\mathcal{C}$. Then there is a morphism $f : X \to \mathbf{1}$, which is uniquely determined up to homotopy. It follows from Example 9.2.4.7 that f is a monomorphism if and only if X is subterminal. 04VK

Example 9.2.4.9. Let $\mathcal{C}$ be an ∞-category. Then every isomorphism in $\mathcal{C}$ is a monomorphism. 04VL

Example 9.2.4.10. Let $f : X_0 \to X$ be a map of Kan complexes. Then f is a monomorphism in the ∞-category of spaces $\mathcal{S}$ if and only if it induces a homotopy equivalence of X_0 with a summand of X. See Example 9.2.1.4. 05FW

04VM **Warning 9.2.4.11.** Let $\mathcal{C}$ be an ∞-category and let $i : X_0 \to X$ be a morphism of $\mathcal{C}$ which admits a left homotopy inverse $r : X \to X_0$. If $\mathcal{C}$ is (the nerve of) an ordinary category, then i is automatically a monomorphism. In general, this is not necessarily true. For example, let (X, x) be a pointed Kan complex, and regard the inclusion map $i : \{x\} \to X$ as a morphism in the ∞-category $\mathcal{S}$ of spaces. Then i has a left homotopy inverse (given by the constant map $X \to \{x\}$). However, i is a monomorphism in the ∞-category $\mathcal{S}$ only if x belongs to a contractible connected component of X (Example 9.2.4.10).

04VN **Remark 9.2.4.12.** Let $\mathcal{C}$ be an ∞-category and let $f : X_0 \to X$ be a morphism in $\mathcal{C}$. If f is a monomorphism, then the homotopy class $[f] : X_0 \to X$ is a monomorphism in the ordinary category $\mathrm{h}\mathcal{C}$. Beware that the converse is false in general.

04VQ **Remark 9.2.4.13.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $f : X_0 \to X$ be a morphism of $\mathcal{C}$.

- If F is fully faithful and $F(f)$ is a monomorphism in $\mathcal{D}$, then f is a monomorphism in $\mathcal{C}$.
- If F is an equivalence of ∞-categories, then $F(f)$ is a monomorphism in $\mathcal{D}$ if and only if f is a monomorphism in $\mathcal{C}$.

04VP **Remark 9.2.4.14.** Let $\mathcal{C}$ be an ∞-category and let $f : X_0 \to X$ be a morphism of $\mathcal{C}$. The condition that f is a monomorphism depends only on the homotopy class $[f] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X_0, X)$.

04VR **Remark 9.2.4.15.** Let $\mathcal{C}$ be an ∞-category, and suppose that we are given a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

in $\mathcal{C}$, where g is a monomorphism. Then f is a monomorphism if and only if h is a monomorphism. In particular, the collection of monomorphisms is closed under composition. See Corollary 9.2.3.16.

05FX **Remark 9.2.4.16.** Let $\mathcal{C}$ be an ∞-category and let $f : X_0 \to X$ be a morphism of $\mathcal{C}$. Then f is a monomorphism if and only if it is subterminal when viewed as an object of the ∞-category $\mathcal{C}_{/X}$. See Proposition 9.2.3.7.

05FY **Remark 9.2.4.17.** Let $\mathcal{C}$ be an ∞-category and let $f : X_0 \to X$ be a morphism of $\mathcal{C}$. Then f is a monomorphism if and only if it satisfies the following condition for each $m \geq 3$:

$(*_m)$ Let $\sigma : \Lambda^m_m \to \mathcal{C}$ be a morphism of simplicial sets for which the composition

$$\Delta^1 \simeq \mathrm{N}_\bullet(\{m-1 < m\}) \subset \Lambda^m_m \xrightarrow{\sigma} \mathcal{C}$$

coincides with f. Then σ can be extended to an m-simplex of $\mathcal{C}$.

This follows by combining Remarks 9.2.2.14 and 9.2.4.16.

Remark 9.2.4.18. Let $\mathcal{C}$ be an ∞-category, let $f : X_0 \to X$ be a morphism of $\mathcal{C}$, and let σ denote the composite map 05FZ

$$\Delta^1 \times \Delta^1 \xrightarrow{(i,j)\mapsto ij} \Delta^1 \xrightarrow{u} \mathcal{C},$$

which we depict as a diagram

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X_0 \arrow[d, "f"] \\
X_0 \arrow[r, "f"] & X.
\end{tikzcd}$$

Then f is a monomorphism if and only if σ is a pullback square in $\mathcal{C}$. This follows by combining Remarks 9.2.4.16 and 9.2.2.15 (see Proposition 7.6.3.14).

Remark 9.2.4.19. Let $\mathcal{C}$ be an ∞-category which admits pullbacks. Stated more informally, Remark 9.2.4.18 asserts that a morphism $f : X_0 \to X$ of $\mathcal{C}$ is a monomorphism if and only if the relative diagonal $\delta_{X_0/X} : X_0 \to X_0 \times_X X_0$ is an isomorphism. 04VX

From the criterion of Remark 9.2.4.18, we immediately obtain the following:

Proposition 9.2.4.20. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves pullbacks. Then F carries monomorphisms in $\mathcal{C}$ to monomorphisms in $\mathcal{D}$.* 05G0

Remark 9.2.4.21. In the statement of Proposition 9.2.4.20, it is not necessary to assume that the ∞-categories $\mathcal{C}$ and $\mathcal{D}$ admit pullbacks (we only need to know that F preserves those pullback squares which exist in $\mathcal{C}$). 04W1

Example 9.2.4.22. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which admits a left adjoint. Then F carries subterminal objects of $\mathcal{C}$ to subterminal objects of $\mathcal{D}$, and carries monomorphisms in $\mathcal{C}$ to monomorphisms in $\mathcal{D}$. This follows from Proposition 9.2.4.20 and Remark 9.2.2.16, since F preserves limit diagrams (Corollary 7.1.3.21). 04W2

Remark 9.2.4.23. Let $F : \mathcal{C} \to \mathcal{D}$ be a right fibration of ∞-categories and let f be a morphism in $\mathcal{C}$. Then f is a monomorphism if and only if $F(f)$ is a monomorphism in the ∞-category $\mathcal{D}$. See Corollary 9.2.3.10. 05G1

05G2 **Remark 9.2.4.24.** Let $\mathcal{C}$ be an ∞-category and let $f : X \hookrightarrow Y$ be a monomorphism in $\mathcal{C}$. If $f' : X' \to Y'$ is a retract of f (in the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{C})$), then f' is also a monomorphism. See Corollary 9.2.3.11.

04W8 **Definition 9.2.4.25.** Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. A *subobject of* X is a subterminal object of the slice ∞-category $\mathcal{C}_{/X}$: that is, an object which is given by a monomorphism $f : X_0 \hookrightarrow X$ in the ∞-category $\mathcal{C}$ (see Remark 9.2.4.16). In this situation, we will sometimes abuse terminology by referring to X_0 as a *subobject of* X and writing $X_0 \subseteq X$; in this case, we implicitly assume that a monomorphism $X_0 \hookrightarrow X$ has been specified.

04W9 **Notation 9.2.4.26.** Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. We let $\operatorname{Sub}(X)$ denote the set $\operatorname{Sub}(\mathcal{C}_{/X})$ of isomorphism classes of subterminal objects of $\mathcal{C}_{/X}$ (see Notation 9.2.2.18). If $f : X_0 \hookrightarrow X$ is a monomorphism, we write $[X_0] \in \operatorname{Sub}(X)$ for the isomorphism class of f. We will sometimes abuse notation by identifying the isomorphism class $[X_0]$ with the object X_0 itself: by virtue of Remark 9.2.2.19, this identification is essentially harmless provided that X_0 is understood as an object of the slice ∞-category $\mathcal{C}_{/X}$ (that is, provided that we remember the data of the monomorphism f). We will refer to $\operatorname{Sub}(X)$ as the *set of subobjects of* X, and we endow it with the partial ordering described in Notation 9.2.2.18: that is, if $f_0 : X_0 \hookrightarrow X$ and $f_1 : X_1 \hookrightarrow X$ are monomorphisms, we write $[X_0] \subseteq [X_1]$ if there exists a 2-simplex

$$\begin{tikzcd}
X_0 \arrow[rr, "g"] \arrow[dr, "f_0"'] & & X_1 \arrow[dl, "f_1"] \\
& Y &
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. In this case, g is automatically a monomorphism (see Remark 9.2.4.15).

04WA **Example 9.2.4.27.** Let X be a Kan complex, which we regard as an object of the ∞-category $\mathcal{S}$. Using Example 9.2.4.10, we can identify $\operatorname{Sub}(X)$ with the partially ordered collection of all summands of X. Alternatively, we can identify $\operatorname{Sub}(X)$ with the collection of all subsets of the set $\pi_0(X)$ (see Exercise 1.2.1.16).

04WB **Remark 9.2.4.28.** Let $\mathcal{C}$ be an ∞-category. For every object $X \in \mathcal{C}$, the identity morphism $\operatorname{id}_X : X \to X$ is a monomorphism (Example 9.2.4.9), so we can regard X as a subobject of itself. Moreover, the isomorphism class $[X]$ is a largest element of the partially ordered set $\operatorname{Sub}(X)$ (see Remark 9.2.2.20).

04WC **Remark 9.2.4.29.** Let $\mathcal{C}$ be an ∞-category which admits fiber products. Then, for every object $X \in \mathcal{C}$, the slice ∞-category $\mathcal{C}_{/X}$ admits finite products (Corollary 7.6.3.20). It follows that the partially ordered set $\operatorname{Sub}(X)$ is a lower semilattice (see Remark 9.2.2.20). In particular, every pair of objects $[X_0], [X_1] \in \mathrm{Sub}(X)$ have a greatest lower bound $[X_0] \cap [X_1]$ in $\mathrm{Sub}(X)$, given concretely by the isomorphism class of the fiber product $[X_0 \times_X X_1]$.

Remark 9.2.4.30 (Pullbacks of Monomorphisms)**.** Let $\mathcal{C}$ be an ∞-category containing a commutative diagram 04WD

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "i"] & Y_0 \arrow[d, "j"] \\ X \arrow[r, "f"] & Y, \end{tikzcd} \tag{9.20}$$

04WE

where $j : Y_0 \to Y$ is a monomorphism. Then (9.20) is a pullback square if and only if the following conditions are satisfied:

- The morphism $i : X_0 \to X$ is also a monomorphism.
- The diagram (9.20) determines a pullback square in the homotopy category $\mathrm{h}\mathcal{C}$. That is, a morphism $g : C \to X_0$ factors (up to homotopy) through i if and only if the $f \circ g$ factors (up to homotopy) through j.

In particular, the collection of monomorphisms in $\mathcal{C}$ is closed under pullbacks.

Construction 9.2.4.31 (Inverse Images)**.** Let $\mathcal{C}$ be an ∞-category which admits fiber products. Then every morphism $f : X \to Y$ in $\mathcal{C}$ determines a pullback functor 04WF

$$f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X} \qquad Y' \mapsto X \times_Y Y'$$

(see Proposition 7.6.3.16). The functor f^* has a left adjoint, and therefore carries subterminal objects of $\mathcal{C}_{/X}$ to subterminal objects of $\mathcal{C}_{/Y}$. Passing to isomorphism classes, we obtain a map of partially ordered sets $f^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$, given concretely by the formula $f^{-1}[Y_0] = [Y_0 \times_Y X]$. Since the functor f^* preserves products, f^{-1} is a homomorphism of lower semilattices: that is, it satisfies the identities

$$f^{-1}([Y_0] \cap [Y_1]) = f^{-1}([Y_0]) \cap f^{-1}([Y_1]) \qquad f^{-1}([Y]) = [X].$$

We close this section with a discussion of monomorphisms in the ∞-category $\mathcal{QC}$ of (small) ∞-categories.

Proposition 9.2.4.32. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The following conditions are equivalent:* 04W5

(1) *The functor F is a monomorphism in the ∞-category $\mathcal{QC}$.*

(2) *For every pair of objects $X, Y \in \mathcal{C}$, the functor F induces a homotopy equivalence from* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *to a summand of* $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ *which contains every isomorphism from $F(X)$ to $F(Y)$.*

(3) *The functor F induces an equivalence from $\mathcal{C}$ to a replete subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$.*

Proof. We first show that (1) implies (2). By virtue of Corollary 4.5.2.23, we may assume without loss of generality that F is an isofibration of ∞-categories. In this case, it follows from Exercise 7.6.4.13 that the diagonal inclusion $\delta : \mathcal{C} \hookrightarrow \mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ (formed in the ordinary category of simplicial sets) can be identified with the relative diagonal of F in the ∞-category $\mathcal{QC}$. Combining this observation with Remark 9.2.4.18, we deduce that F is a monomorphism (in the ∞-category $\mathcal{QC}$) if and only if δ is an equivalence of ∞-categories. In particular, if F is a monomorphism, then δ is fully faithful: that is, for every pair of objects $X, Y \in \mathcal{C}$, the induced map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C} \times_{\mathcal{D}} \mathcal{C}}(\delta(X), \delta(Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(X,Y) \times_{\mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))} \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

is a homotopy equivalence. Our assumption that F is an isofibration guarantees that the map $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ is Kan fibration (Proposition 4.6.1.21). Applying Corollary 3.5.1.31, we deduce that $F_{X,Y}$ restricts to a homotopy equivalence of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ with a summand of $\mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$. To complete the proof, it will suffice to show that this summand contains every isomorphism from $F(X)$ to $F(Y)$. In fact, we will prove something more precise: the induced map of cores $F^{\simeq} : \mathcal{C}^{\simeq} \to \mathcal{D}^{\simeq}$ is a trivial Kan fibration from $\mathcal{C}^{\simeq}$ to a summand of $\mathcal{D}^{\simeq}$. This follows again from Corollary 3.5.1.31, since $F^{\simeq}$ is a Kan fibration (Proposition 4.4.3.7).

We now show that (2) implies (3). As above, we may assume that F is an isofibration. Let $\mathrm{h}\mathcal{C}$ and $\mathrm{h}\mathcal{D}$ denote the homotopy categories of $\mathcal{C}$ and $\mathcal{D}$, respectively. We define a subcategory $\mathrm{h}\mathcal{D}_0 \subseteq \mathrm{h}\mathcal{D}$ as follows:

- An object $\overline{X}$ of $\mathrm{h}\mathcal{D}$ belongs to the subcategory $\mathrm{h}\mathcal{D}_0$ if and only if it is the image of an object X of $\mathrm{h}\mathcal{C}$.
- A morphism $\overline{u} : \overline{X} \to \overline{Y}$ of $\mathrm{h}\mathcal{D}$ belongs to the subcategory $\mathrm{h}\mathcal{D}_0$ if and only if it is the image of a morphism u of $\mathrm{h}\mathcal{C}$.

We first claim that the subcategory $\mathrm{h}\mathcal{D}_0$ is well-defined: that is, if $\overline{u} : \overline{X} \to \overline{Y}$ and $\overline{v} : \overline{Y} \to \overline{Z}$ are composable morphisms of $\mathrm{h}\mathcal{D}$ which can be lifted to morphisms $u : X \to Y$ and $v : Y' \to Z$ of $\mathrm{h}\mathcal{C}$, then the composite morphism $\overline{v} \circ \overline{u}$ has the same property. Assumption (2) guarantees that the identity morphism $\mathrm{id}_{\overline{Y}}$ belongs to the image of the map

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(Y, Y') = \pi_0(\mathrm{Hom}_{\mathcal{C}}(Y, Y')) \to \pi_0(\mathrm{Hom}_{\mathcal{D}}(\overline{Y}, \overline{Y})) \simeq \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(\overline{Y}, \overline{Y}).$$

That is, there exists a morphism $e : Y \to Y'$ in $\mathrm{h}\mathcal{C}$ satisfying $F(e) = \mathrm{id}_{\overline{Y}}$. Replacing v by the composition $v \circ e$, we can arrange that $Y = Y'$: that is, that u and v are composable morphisms in the category $\mathrm{h}\mathcal{C}$. It then follows that $\overline{v} \circ \overline{u} = F(v \circ u)$ is also morphism of $\mathrm{h}\mathcal{D}_0$, as desired.

By virtue of Proposition 4.1.2.10, the subcategory $\mathrm{h}\mathcal{D}_0 \subseteq \mathrm{h}\mathcal{D}$ is the homotopy category of a (unique) subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$. Using condition (2), we see that the subcategory $\mathcal{D}_0$ is replete. By construction, the functor F factors as a composition $\mathcal{C} \xrightarrow{F_0} \mathcal{D}_0 \hookrightarrow \mathcal{D}$. For every pair of objects $X, Y \in \mathcal{C}$, we can identify $\mathrm{Hom}_{\mathcal{D}_0}(F(X), F(Y))$ with the summand of $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ given by the essential image of $F_{X,Y}$. Invoking assumption (2), we see that the functor F_0 is fully faithful. By construction, F_0 is also surjective on objects, and is therefore an equivalence of ∞-categories (Theorem 4.6.2.20). This completes the proof of the implication (2) $\Rightarrow$ (3).

We now show that (3) implies (1). Assume that F induces an equivalence from $\mathcal{C}$ to a replete subcategory $\mathcal{D}_0 \subseteq \mathcal{C}$; we wish to show that F is a monomorphism. By virtue of Remark 9.2.4.15 (and Example 9.2.4.9), it will suffice to show that the inclusion map $\iota : \mathcal{D}_0 \hookrightarrow \mathcal{D}$ is a monomorphism in $\mathcal{QC}$. Fix an ∞-category $\mathcal{B}$, so that composition with the homotopy class $[\iota]$ induces a map of Kan complexes $\theta : \mathrm{Hom}_{\mathcal{QC}}(\mathcal{B}, \mathcal{D}_0) \to \mathrm{Hom}_{\mathcal{QC}}(\mathcal{B}, \mathcal{D})$. We wish to show that θ induces a homotopy equivalence from $\mathrm{Hom}_{\mathcal{QC}}(\mathcal{B}, \mathcal{D}_0)$ to a summand of $\mathrm{Hom}_{\mathcal{QC}}(\mathcal{B}, \mathcal{D})$. By virtue of Remark 5.5.4.5, it will suffice to prove the analogous assertion for the inclusion map $\mathrm{Fun}(\mathcal{B}, \mathcal{D}_0)^{\simeq} \hookrightarrow \mathrm{Fun}(\mathcal{B}, \mathcal{D})^{\simeq}$, which follows immediately from Corollary 4.4.3.13. □

Corollary 9.2.4.33. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor of ∞-categories. Then F is a monomorphism in the ∞-category $\mathcal{QC}$.* 04W6

Proof. Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be the essential image of F. By virtue of Proposition 9.2.4.32, it will suffice to show that F induces an equivalence from $\mathcal{C}$ to $\mathcal{D}_0$, which is a reformulation of the requirement that F is fully faithful (Corollary 4.6.2.22). □

Warning 9.2.4.34. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a subcategory. Beware that, if we do not assume that $\mathcal{C}_0$ is replete (or full), then the inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ need not be a monomorphism in $\mathcal{QC}$. For example, suppose that $\mathcal{C} = \mathrm{N}_{\bullet}(\mathcal{D})$ is the nerve of a category $\mathcal{D}$. Then the 0-skeleton $\mathcal{C}_0 = \mathrm{sk}_0(\mathcal{C})$ is always subcategory of $\mathcal{C}$ (namely, the subcategory spanned by the identity morphisms of $\mathcal{C}$). However, the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is a monomorphism in $\mathcal{QC}$ if and only if every isomorphism in $\mathcal{D}$ is an identity morphism. 04W7

9.3 Fiberwise Cocompletions

Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories. For each vertex $C \in \mathcal{C}$, the fiber $\mathcal{E}_C$ is an ∞-category. In §8.4, we proved that $\mathcal{E}_C$ admits a *cocompletion*: that is, an ∞-category 05K8

$\widehat{\mathcal{E}}_C$ which is universal among cocomplete ∞-categories with a functor from $\mathcal{E}_C$ (Proposition 8.4.5.3). Our goal in this section is to show that the collection of ∞-categories $\{\widehat{\mathcal{E}}_C\}_{C \in \mathcal{C}}$ can be regarded as the fibers of another inner fibration $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$, where $\widehat{\mathcal{E}}$ is an ∞-category we refer to as the *fiberwise cocompletion of* $\mathcal{E}$. We begin by articulating the relationship between the fibrations U and $\widehat{U}$. To simplify the discussion, let us assume that the inner fibration U is essentially small (see Definition 9.3.1.12 for a more general discussion).

05K9 **Definition 9.3.0.1.** Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially small inner fibration of ∞-categories. We say that a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

exhibits $\widehat{\mathcal{E}}$ *as a fiberwise cocompletion of* $\mathcal{E}$ if the following conditions are satisfied:

(1) For every object $C \in \mathcal{C}$, the map of fibers $H_C : \mathcal{E}_C \to \widehat{\mathcal{E}}_C$ exhibits $\widehat{\mathcal{E}}_C$ as a cocompletion of $\mathcal{E}_C$ (see Definition 8.4.0.1).

(2) The functor H is fully faithful.

(3) The functor $\widehat{U}$ is a locally cartesian fibration.

(4) For every morphism $f : C \to D$ of $\mathcal{C}$, the contravariant transport functor $f^* : \widehat{\mathcal{E}}_D \to \widehat{\mathcal{E}}_C$ preserves small colimits.

One of our principal goals in this section is to prove the following:

05KA **Theorem 9.3.0.2.** *Let* $U : \mathcal{E} \to \mathcal{C}$ *be an essentially small inner fibration of* ∞*-categories. Then there exists a commutative diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ *as a fiberwise cocompletion of* $\mathcal{E}$*. Moreover, the inner fibration* $\widetilde{U}$ *is unique up to equivalence.*

We begin by establishing the uniqueness assertion of Theorem 9.3.0.2. In the situation of Definition 9.3.0.1, the inner fibration $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$ satisfies the following condition:

$(*)$ For each object $C \in \mathcal{C}$, the ∞-category $\widehat{\mathcal{E}}_C = \{C\} \times_{\mathcal{C}} \widehat{\mathcal{E}}$ is cocomplete and the inclusion functor $\widehat{\mathcal{E}}_C \hookrightarrow \widehat{\mathcal{E}}$ carries (small) colimit diagrams in $\mathcal{E}_C$ to U-colimit diagrams in $\mathcal{E}$.

More generally, we say that an inner fibration of ∞-categories $U' : \mathcal{E}' \to \mathcal{C}$ is *cocomplete* if it satisfies condition $(*)$ (Definition 9.3.1.1). In §9.3.1, we show that if a commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

exhibits $\widehat{\mathcal{E}}$ as a fiberwise cocompletion of $\mathcal{E}$, then it is universal in the following sense: every functor $F : \mathcal{E} \to \mathcal{E}'$ satisfying $U' \circ F = U$ admits an essentially unique extension $\widehat{F} : \widehat{\mathcal{E}} \to \mathcal{E}'$ having the property that, for each object $C \in \mathcal{C}$, the map of fibers $\widehat{F}_C : \widehat{\mathcal{E}}_C \to \mathcal{E}'_C$ preserves small colimits (Theorem 9.3.1.20). It follows that $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$ is determined (up to equivalence) by the inner fibration U (Remark 9.3.1.21).

To complete the proof of Theorem 9.3.0.2, we must show that every (essentially small) inner fibration $U : \mathcal{E} \to \mathcal{C}$ admits a fiberwise cocompletion $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$. We begin by treating some special cases:

- Assume that U is a cocartesian fibration. In §9.3.2 we show that there exists a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

  which exhibits $\widehat{\mathcal{E}}$ as a fiberwise cocompletion of $\mathcal{E}$. Moreover, $\widehat{U}$ is also a cocartesian fibration and the functor H carries U-cocartesian morphisms of $\mathcal{E}$ to $\widehat{U}$-cocartesian morphisms of $\widehat{\mathcal{E}}$ (Theorem 9.3.2.1). Our proof uses the theory of dual fibrations developed in §8.6. More precisely, we show that we can take $\widehat{\mathcal{E}}$ to be the relative exponential $\operatorname{Fun}(\mathcal{E}^{\vee} / \mathcal{C}, \mathcal{S})$, where $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ is a cocartesian dual of U (Lemma 9.3.2.4).

- Assume that U is a cartesian fibration. In §9.3.3 we show that there exists a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

  which exhibits $\widehat{\mathcal{E}}$ as a fiberwise cocompletion of $\mathcal{E}$. Moreover, $\widehat{U}$ is also a cartesian fibration and the functor H carries U-cartesian morphisms of $\mathcal{E}$ to $\widehat{U}$-cartesian morphisms of $\widehat{\mathcal{E}}$ (Theorem 9.3.3.1). Our strategy is to reduce to the cocartesian case, using the compatibility of fiberwise cocompletion with the formation of conjugate fibrations (Proposition 9.3.3.3).

In §9.3.4, we complete the proof of Theorem 9.3.0.2 by proving the existence of a fiberwise cocompletion for a general inner fibration $U : \mathcal{E} \to \mathcal{C}$. Our strategy is to embed $\mathcal{E}$ into the oriented fiber product $\mathcal{E} \tilde{\times}_{\mathcal{C}} \mathcal{C}$, and thereby reduce to the case where U is a cocartesian fibration (using the oriented fiber product $\mathcal{C} \tilde{\times}_{\mathcal{C}} \mathcal{E}$, we could instead reduce to the case where U is a cartesian fibration).

Recall that an inner fibration of simplicial sets $U : \mathcal{E} \to \mathcal{C}$ is *exponentiable* if the pullback functor

$$(\mathrm{Set}_{\Delta})_{/\mathcal{C}} \to (\mathrm{Set}_{\Delta})_{/\mathcal{E}} \qquad \mathcal{C}' \mapsto \mathcal{C}' \times_{\mathcal{C}} \mathcal{E}$$

preserves categorical equivalences (Definition 4.5.9.10). In §9.3.6, we introduce a closely related condition which has somewhat better formal properties. We say that an inner fibration $U : \mathcal{E} \to \mathcal{C}$ is *flat* if, for every 2-simplex σ of $\mathcal{C}$, the inclusion map

$$\Lambda^2_1 \times_{\mathcal{C}} \mathcal{E} \hookrightarrow \Delta^2 \times_{\mathcal{C}} \mathcal{E}$$

is a categorical equivalence of simplicial sets (Definition 9.3.6.1). It follows immediately from the definitions that if U is an exponentiable inner fibration, then it is flat. In §9.3.6, we show that the converse holds if U is an isofibration (Corollary 9.3.6.30). The key observation is that that an inner fibration $U : \mathcal{E} \to \mathcal{C}$ is flat if and only if the fiberwise cocompletion $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$ is a cartesian fibration (Lemma 9.3.6.13). In §9.3.8 we show that, under this assumption, we can give a concrete description of $\widehat{\mathcal{E}}$ in terms of the relative Yoneda embedding of U (Theorem 9.3.8.4).

9.3.1 Uniqueness of Fiberwise Cocompletions

05KB

Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories. Our goal in this section is to show that if there exists a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise cocompletion of $\mathcal{E}$ (in the sense of Definition 9.3.0.1), then the inner fibration $\widetilde{U}$ is uniquely determined up to equivalence. To prove this, we will show that the ∞-category $\widehat{\mathcal{E}}$ can be characterized by a universal mapping property (Theorem 9.3.1.20). To formulate it, we will need some terminology.

Definition 9.3.1.1. Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories and let $\mathbb{K}$ be a collection of simplicial sets. We say that U is $\mathbb{K}$-*cocomplete* if it satisfies the following conditions: 05KC

(1) For every object $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is $\mathbb{K}$-cocomplete (Definition 8.4.5.1). That is, it admits K-indexed colimits, for every simplicial set $K \in \mathbb{K}$.

(2) Let C be an object of $\mathcal{C}$ and let $K \in \mathbb{K}$. Then every colimit diagram $K^{\triangleright} \to \mathcal{E}_C$ is a U-colimit diagram in $\mathcal{E}$.

If κ is a regular cardinal, we say that the inner fibration $U : \mathcal{E} \to \mathcal{C}$ is κ-*cocomplete* if it $\mathbb{K}$-cocomplete, where $\mathbb{K}$ denotes the collection of all κ-small simplicial sets.

Example 9.3.1.2. Let $\mathcal{E}$ be an ∞-category and let $\mathbb{K}$ be a collection of simplicial sets. Then $\mathcal{E}$ is $\mathbb{K}$-cocomplete (in the sense of Definition 8.4.5.1) if and only if the inner fibration $\mathcal{E} \to \Delta^0$ is $\mathbb{K}$-cocomplete (in the sense of Definition 9.3.1.1). 05KD

Proposition 9.3.1.3. *Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories and let $\mathbb{K}$ be a collection of simplicial sets. Then U is $\mathbb{K}$-cocomplete if and only if, for every morphism $\Delta^1 \to \mathcal{C}$, the inner fibration $\Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \Delta^1$ is $\mathbb{K}$-cocomplete.* 05KE

Proof. This follows immediately from the characterization of U-colimit diagrams supplied by Proposition 7.1.5.22. □

Using Proposition 9.3.1.3, we can formulate a slightly more general version of Definition 9.3.1.1:

05KF **Variant 9.3.1.4.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories and let $\mathbb{K}$ be a collection of simplicial sets. We say that U is $\mathbb{K}$-*cocomplete* if, for every edge $\Delta^1 \to \mathcal{C}$, the projection map $\Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \Delta^1$ is a $\mathbb{K}$-cocomplete inner fibration of ∞-categories, in the sense of Definition 9.3.1.1. By virtue of Proposition 9.3.1.3, this agrees with Definition 9.3.1.1 in the special case where $\mathcal{C}$ is an ∞-category.

05KG **Remark 9.3.1.5.** Let $\mathbb{K}$ be a collection of simplicial sets and suppose we are given a pullback diagram

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r] & \mathcal{C},
\end{tikzcd}$$

where U and U' are inner fibrations. If U is $\mathbb{K}$-cocomplete, then U' is also $\mathbb{K}$-cocomplete.

05KH **Example 9.3.1.6.** Let $U : \mathcal{E} \to \mathcal{C}$ be a locally cartesian fibration of simplicial sets, and let $\mathbb{K}$ be a collection of simplicial sets. Then U is $\mathbb{K}$-cocomplete if and only if, for each vertex $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is $\mathbb{K}$-cocomplete. To prove this, we can assume without loss of generality that $\mathcal{C} = \Delta^1$, so that U is a cartesian fibration. In this case, the desired result follows from Corollary 7.1.5.20.

05KJ **Example 9.3.1.7.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. We say that U is *idempotent complete* if it satisfies either of the following equivalent conditions:

(a) For each vertex $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is idempotent complete.

(b) The inner fibration U is $\{\mathrm{N}_\bullet(\mathrm{Idem})\}$-cocomplete, where Idem is the category introduced in Construction 8.5.2.7.

The implication (b) $\Rightarrow$ (a) is immediate (see the proof of Proposition 8.5.5.2). To prove the converse, we may assume without loss of generality that $\mathcal{C} = \Delta^1$; in this case, the result follows from the observation that the inclusion map $\{0\} \times_{\mathcal{C}} \mathcal{E} \hookrightarrow \mathcal{E}$ preserves $\mathrm{N}_\bullet(\mathrm{Idem})$-indexed colimiits (Corollary 8.5.3.12).

We will soon show that if $U : \mathcal{E} \to \mathcal{C}$ is an arbitrary inner fibration, then there is a universal way to enlarge $\mathcal{E}$ to obtain a $\mathbb{K}$-cocomplete inner fibration $\widetilde{U} : \widetilde{\mathcal{E}} \to \mathcal{C}$. We begin by characterizing this enlargement in the special case $\mathcal{C} = \Delta^1$.

05KK **Definition 9.3.1.8.** Let $\mathbb{K}$ be a collection of simplicial sets and suppose that we are given a commutative diagram of ∞-categories

05KL
$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \Delta^1. &
\end{tikzcd} \tag{9.21}$$

We say that the diagram (9.21) *exhibits* $\widehat{\mathcal{E}}$ *as a fiberwise* $\mathbb{K}$*-cocompletion of* $\mathcal{E}$ if the following conditions are satisfied:

(1) For $i \in \{0,1\}$, the map of fibers $H_i : \mathcal{E}_i \to \widehat{\mathcal{E}}_i$ exhibits $\widehat{\mathcal{E}}_i$ as a $\mathbb{K}$-cocompletion of $\mathcal{E}_i$ (Definition 8.4.5.1).

(2) The functor H is fully faithful.

(3) The inner fibration $\widehat{U}$ is $\mathbb{K}$-cocomplete (Definition 9.3.1.1).

(4) For every object $X \in \mathcal{E}_0$, the functor

$$\widehat{\mathcal{E}}_1 \to \mathcal{S} \qquad Y \mapsto \operatorname{Hom}_{\widehat{\mathcal{E}}}(H(X), Y)$$

preserves $\mathbb{K}$-indexed colimits.

If κ is a regular cardinal, we say that (9.21) *exhibits* $\widehat{\mathcal{E}}$ *as a fiberwise* κ*-cocompletion of* $\mathcal{E}$ if it exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$, where $\mathbb{K}$ is the collection of all κ-small simplicial sets.

Remark 9.3.1.9. In condition (4) of Definition 9.3.1.8, we have implicitly assumed that the ∞-category $\widehat{\mathcal{E}}$ is locally small and that each of the simplicial sets $K \in \mathbb{K}$ is (essentially) small. If these conditions are not satisfied, then we should replace the ∞-category $\mathcal{S}$ by $\mathcal{S}^{<\lambda}$, where λ is some regular cardinal having the property that $\widehat{\mathcal{E}}$ is locally τ-small and each $K \in \mathbb{K}$ is essentially τ-small. 05KM

Remark 9.3.1.10. Let $\mathbb{K}$ be a collection of simplicial sets and let $H : \mathcal{C} \to \widehat{\mathcal{C}}$ be a functor of ∞-categories which exhibits $\widehat{\mathcal{C}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{C}$. Then the diagram 05KN

$$\begin{tikzcd}
\mathcal{C} \times \Delta^1 \arrow[rr, "H \times \mathrm{id}"] & & \widehat{\mathcal{C}} \times \Delta^1 \arrow[dl] \\
& \Delta^1 &
\end{tikzcd}$$

exhibits $\widehat{\mathcal{C}} \times \Delta^1$ as a fiberwise cocompletion of $\mathcal{C} \times \Delta^1$. Conditions (1) and (3) of Definition 9.3.1.8 are immediate, and conditions (2) and (4) follow from Variant 8.4.6.9.

05KP **Proposition 9.3.1.11.** *Let κ be an uncountable regular cardinal and suppose we are given a commutative diagram of ∞-categories*

05KQ
$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \Delta^1, &
\end{tikzcd} \tag{9.22}$$

where $\mathcal{E}$ is essentially κ-small. Then (9.22) exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$ if and only if it satisfies conditions (1) *and* (2) *of Definition 9.3.1.8, together with the following:*

($3'$) *The functor $\widehat{U}$ is a cartesian fibration.*

($4'$) *The contravariant transport functor $\widehat{\mathcal{E}}_1 \to \widehat{\mathcal{E}}_0$ preserves κ-small colimits.*

Proof. Without loss of genearlity, we may assume that H satisfies conditions (1) and (2) of Definition 9.3.1.8. The implication ($3'$) $\Rightarrow$ (3) is a special case of Example 9.3.1.6. Moreover, if ($3'$) is satisfied, then we can use condition (1) (together with Proposition 8.4.2.5) to identify the contravariant transport functor $\widehat{\mathcal{E}}_1 \to \widehat{\mathcal{E}}_0$ with the functor

$$\widehat{\mathcal{E}}_1 \to \operatorname{Fun}(\mathcal{E}_0^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \qquad \widehat{Y} \mapsto \operatorname{Hom}_{\widehat{\mathcal{E}}}(H(-), \widehat{Y}).$$

In this case, the equivalence (4) $\Leftrightarrow$ ($4'$) follows from the observation that κ-small colimits in the ∞-category $\operatorname{Fun}(\mathcal{E}_0^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ are computed levelwise (Corollary 7.1.6.2).

We will complete the proof by showing that if H satisfies conditions (3) and (4), then U is a cartesian fibration. Fix a regular cardinal λ having exponential cofinality $\geq \kappa$ such that $\widehat{\mathcal{E}}$ is locally λ-small. By virtue of Corollary 6.2.3.2, it will suffice to show that for each object $\widehat{Y} \in \widehat{\mathcal{E}}_1$, the functor

$$\mathscr{F} : \widehat{\mathcal{E}}_0^{\mathrm{op}} \to \mathcal{S}^{<\lambda} \qquad \widehat{X} \mapsto \operatorname{Hom}_{\widehat{\mathcal{E}}}(\widehat{X}, \widehat{Y})$$

is representable by an object of $\widehat{\mathcal{E}}_0$. Condition (3) guarantees that $\mathscr{F}$ preserves κ-small limits. Using condition (1) and Corollary 8.4.3.10, we are reduced to showing that for each object $X \in \mathcal{E}_0$, the Kan complex $\mathscr{F}(H(X)) = \operatorname{Hom}_{\widehat{\mathcal{E}}}(H(X), \widehat{Y})$ is κ-small.

Let now us regard the object X as fixed. Condition (4) guarantees that the functor

$$\widehat{\mathcal{E}}_0 \to \mathcal{S}^{<\lambda} \qquad \widehat{Y} \mapsto \operatorname{Hom}_{\widehat{\mathcal{E}}}(H(X), \widehat{Y})$$

preserves κ-small colimits. In particular, the collection of objects $\widehat{Y}$ for which $\operatorname{Hom}_{\widehat{\mathcal{E}}}(H(X), \widehat{Y})$ is essentially κ-small is closed under κ-small colimits (Example 7.6.7.8). Since $\widehat{\mathcal{E}}_1$ is generated under κ-small colimits by the image of H, we can assume that $\widehat{Y} = H(Y)$ for some object $Y \in \mathcal{E}_1$. In this case, condition (2) supplies a homotopy equivalence

$$\mathrm{Hom}_{\mathcal{E}}(X, Y) \to \mathrm{Hom}_{\widehat{\mathcal{E}}}(H(X), \widehat{Y}),$$

so the desired result follows from our assumption that $\mathcal{E}$ is essentially κ-small. □

We now extend the scope of Definition 9.3.1.8.

Definition 9.3.1.12. Let $\mathbb{K}$ be a collection of simplicial sets and suppose we are given a commutative diagram of simplicial sets 05KR

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\ & \mathcal{C}, & \end{tikzcd} \tag{9.23}$$

05KS

where U and $\widehat{U}$ are inner fibrations. We will say that the diagram (9.23) *exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$* if, for every edge $e : \Delta^1 \to \mathcal{C}$, the induced map

$$H_e : \Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \Delta^1 \times_{\mathcal{C}} \widehat{\mathcal{E}}$$

exhibits $\Delta^1 \times_{\mathcal{C}} \widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\Delta^1 \times_{\mathcal{C}} \mathcal{E}$ (in the sense of Definition 9.3.1.8).

If κ is a regular cardinal, we say that the diagram (9.23) *exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$* if it exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$, where $\mathbb{K}$ is the collection of all κ-small simplicial sets.

Remark 9.3.1.13. In the special case where $\mathcal{C} = \Delta^1$, Definition 9.3.1.12 reduces to Definition 9.3.1.8: see Remark 9.3.1.10. 05KT

Remark 9.3.1.14. In the situation of Definition 9.3.1.12, suppose that $\mathcal{C}$ is an ∞-category. Then H is fully faithful if and only if, for every morphism $\Delta^1 \to \mathcal{C}$, the induced map 05KU

$$\Delta^1 \times_{\mathcal{C}} \mathcal{E} \to \Delta^1 \times_{\mathcal{C}} \widehat{\mathcal{E}}$$

is fully faithful (see Variant 4.8.6.19). In particular, if H exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$, then H is fully faithful.

Remark 9.3.1.15. Let $\mathbb{K}$ be a collection of simplicial sets and suppose we are given a commutative diagram of simplicial sets 05KV

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\ & \mathcal{C} & \end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. Let $\mathcal{E}' \subseteq \widehat{\mathcal{E}}$ be the full simplicial subset spanned by those vertices which belong to the image of H. It follows from Remark 9.3.1.14 and Proposition 5.1.7.9 that the induced map $\mathcal{E} \to \mathcal{E}'$ is an equivalence of inner fibrations over $\mathcal{C}$, so that the inclusion map $\mathcal{E}' \hookrightarrow \widehat{\mathcal{E}}$ exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}'$.

We now show that, in the special case where $\mathbb{K}$ is the collection of small simplicial sets and U is an essentially small inner fibration of ∞-categories, Definition 9.3.1.12 reduces to Definition 9.3.0.1. Following the convention of Remark 4.7.0.5, this can be regarded as a special case of the following:

05KW **Proposition 9.3.1.16.** *Let κ be an uncountable regular cardinal and suppose we are given a commutative diagram of ∞-categories*

05KX
$$\begin{tikzcd} \mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\ & \mathcal{C}, & \end{tikzcd} \tag{9.24}$$

where U and $\widehat{U}$ are inner fibrations and U is essentially κ-small. Then (9.24) exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$ (in the sense of Definition 9.3.1.12) if and only if the following conditions are satisfied:

(1) *For each object $C \in \mathcal{C}$, the induced map of fibers $H_C : \mathcal{E}_C \to \widehat{\mathcal{E}}_C$ exhibits $\widehat{\mathcal{E}}_C$ as a κ-cocompletion of $\mathcal{E}_C$.*

(2) *The functor H is fully faithful.*

(3) *The inner fibration $\widehat{U}$ is locally cartesian.*

(4) *For each morphism $f : C \to C'$ of $\mathcal{C}$, the contravariant transport functor $f^* : \widehat{\mathcal{E}}_{C'} \to \widehat{\mathcal{E}}_C$ preserves κ-small colimits.*

Proof. Using Remark 9.3.1.14, we can reduce to the special case $\mathcal{C} = \Delta^1$, which follows from Proposition 9.3.1.11. □

05KY **Example 9.3.1.17** (Fiberwise Idempotent Completion)**.** Suppose we are given a commutative diagram of simplicial sets

05KZ
$$\begin{tikzcd} \mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\ & \mathcal{C}, & \end{tikzcd} \tag{9.25}$$

where U and $\widehat{U}$ are inner fibrations. We say that (9.25) *exhibits $\widehat{\mathcal{E}}$ as a fiberwise idempotent completion of $\mathcal{E}$* if it exhibits $\widehat{\mathcal{E}}$ as a $\{\mathrm{N}_{\bullet}(\mathrm{Idem})\}$-cocompletion of $\mathcal{E}$, in the sense of Definition 9.3.1.12; here Idem is the category introduced in Construction 8.5.2.7.

Proposition 9.3.1.18. *Suppose we are given a commutative diagram of ∞-categories* 05L0

05L1

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C}, &
\end{tikzcd} \tag{9.26}$$

where U and $\widehat{U}$ are inner fibrations. Then (9.26) exhibits $\widehat{\mathcal{E}}$ as a fiberwise idempotent completion of $\mathcal{E}$ if and only if it satisfies the following pair of conditions:

(1) *For each object $C \in \mathcal{C}$, the ∞-category $\widehat{\mathcal{E}}_C$ is idempotent complete.*

(2) *The functor H is fully faithful.*

Proof. As in Proposition 9.3.1.16, we can use Remark 9.3.1.14 to reduce to the case $\mathcal{C} = \Delta^1$; in this case, the result follows from Example 9.3.1.7. □

We can now formulate our main result.

Notation 9.3.1.19. Let $\mathbb{K}$ be a collection of simplicial sets, and let $U : \mathcal{E} \to \mathcal{C}$ and $U' : \mathcal{E}' \to \mathcal{C}$ be $\mathbb{K}$-cocomplete inner fibrations of simplicial sets. We let $\mathrm{Fun}^{\mathbb{K}}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}')$ denote the full subcategory of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}')$ spanned by those commutative diagrams 05L2

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathcal{C} &
\end{tikzcd}$$

having the property that, for every vertex $C \in \mathcal{C}$, the induced map of fibers $F_C : \mathcal{E}_C \to \mathcal{E}'_C$ preserves K-indexed colimits, for each $K \in \mathbb{K}$.

Theorem 9.3.1.20. *Let $\mathbb{K}$ be a collection of simplicial sets. Suppose we are given a diagram of simplicial sets* 05L3

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C}, &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. Then, for every $\mathbb{K}$-cocomplete inner fibration $U' : \mathcal{E}' \to \mathcal{C}$, precomposition with H induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathbb{K}}_{/\mathcal{C}}(\widehat{\mathcal{E}}, \mathcal{E}') \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}').$$

05L4 **Remark 9.3.1.21** (Uniqueness)**.** Let $\mathbb{K}$ be a collection of simplicial sets and let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. It follows from Theorem 9.3.1.20 that if there exists a diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\ & \mathcal{C} & \end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$, then the inner fibration $\widehat{U}$ is unique up to equivalence (in the sense of Definition 5.1.7.1). More precisely, suppose we are given another commutative diagram

05L5
$$\begin{tikzcd} \mathcal{E} \arrow[rr, "H'"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}}' \arrow[dl, "\widehat{U}'"] \\ & \mathcal{C} & \end{tikzcd} \qquad (9.27)$$

where $\widehat{U}'$ is a $\mathbb{K}$-cocomplete inner fibration. Theorem 9.3.1.20 then guarantees the existence (and essential uniqueness) of a morphism $F \in \mathrm{Fun}^{\mathbb{K}}_{/\mathcal{C}}(\widehat{\mathcal{E}}, \widehat{\mathcal{E}}')$ such that H' is isomorphic to $F \circ H$ (as an object of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \widehat{\mathcal{E}}')$). In this case, the diagram (9.27) exhibits $\widehat{\mathcal{E}}'$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$ if and only if F is an equivalence of inner fibrations over $\mathcal{C}$.

05L6 **Example 9.3.1.22.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\ & \mathcal{C}, & \end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise idempotent completion of $\mathcal{E}$. Then, for any idempotent complete inner fibration $U' : \mathcal{E}' \to \mathcal{C}$, precomposition with H induces an equivalence of ∞-categories

$$\mathrm{Fun}_{/\mathcal{C}}(\widehat{\mathcal{E}}, \mathcal{E}') \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}').$$

This follows by combining Theorem 9.3.1.20 with Corollary 8.5.3.12.

Our proof of Theorem 9.3.1.20 will require some preliminaries.

Lemma 9.3.1.23. *Let $\widehat{\mathcal{D}}$ be an ∞-category and let $\mathcal{D} \subseteq \widehat{\mathcal{D}}$ be a full subcategory. Suppose that the inclusion functor $\mathcal{D} \hookrightarrow \widehat{\mathcal{D}}$ exhibits $\widehat{\mathcal{D}}$ as a $\mathbb{K}$-cocompletion of $\mathcal{D}$, for some collection of simplicial sets $\mathbb{K}$. Let $U : \mathcal{E} \to \mathcal{C}$ be a $\mathbb{K}$-cocomplete inner fibration of ∞-categories, let $C \in \mathcal{C}$ be an object, and let $F : \widehat{\mathcal{D}} \to \mathcal{E}_C$ be a functor. The following conditions are equivalent:* 05L7

(1) *The functor F preserves K-indexed colimits, for each $K \in \mathbb{K}$.*

(2) *The functor F is left Kan extended from $\mathcal{D}$.*

(3) *When regarded as a functor from $\widehat{\mathcal{D}}$ to $\mathcal{E}$, the functor F is U-left Kan extended from $\mathcal{D}$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Lemma 8.4.5.9, and the implication (3) $\Rightarrow$ (2) is a special case of Corollary 7.3.3.23. We will complete the proof by showing that (1) implies (3). By virtue of Remark 7.3.3.24, we may assume that $\mathcal{C} = \Delta^1$ and that C is the initial vertex; in this case, we wish to show that F is left Kan extended from $\mathcal{D}$ (when regarded as a functor from $\widehat{\mathcal{D}}$ to $\mathcal{E}$). Fix an uncountable regular cardinal κ such that every simplicial set $K \in \mathbb{K}$ is essentially κ-small. Let λ be a cardinal of exponential cofinality $\geq \kappa$ such that $\mathcal{E}$ is locally λ-small. By virtue of Proposition 7.4.5.17, it will suffice to show that for every object $E \in \mathcal{E}$, the functor

$$\mathscr{F} : \widehat{\mathcal{D}} \to (\mathcal{S}^{<\lambda})^{\mathrm{op}} \qquad D \mapsto \mathrm{Hom}_{\mathcal{E}}(F(D), E)$$

is left Kan extended from $\mathcal{D}$. By virtue of Lemma 8.4.5.9, this is equivalent to the requirement that $\mathscr{F}$ preserves K-indexed colimits for each $K \in \mathbb{K}$. This follows from (1) together with our assumption that U is $\mathbb{K}$-cocomplete. □

Lemma 9.3.1.24. *Let $\mathbb{K}$ be a collection of simplicial sets and suppose we are given a commutative diagram of ∞-categories* 05L8

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C}, &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. Let X be an object of $\widehat{\mathcal{E}}$ having image $C = \widehat{U}(X)$ and set $\mathcal{E}_{/X} = \mathcal{E} \times_{\widehat{\mathcal{E}}} \widehat{\mathcal{E}}_{/X}$. If U is an isofibration, then the inclusion map

$$\{\mathrm{id}_C\} \times_{\mathcal{C}_{/C}} \mathcal{E}_{/X} \hookrightarrow \mathcal{E}_{/X}$$

is right cofinal.

Proof. Fix an object $\widetilde{Y} \in \mathcal{E}_{/X}$, which we identify with a pair (Y, v) where Y is an object of $\mathcal{E}$ and $v : H(Y) \to X$ is a morphism in $\widehat{\mathcal{E}}$. By virtue of Theorem 7.2.3.1, it will suffice to show that the ∞-category $\mathcal{A} = \{\mathrm{id}_C\} \times_{\mathcal{C}_{/C}} (\mathcal{E}_{/C})_{\widetilde{Y}/}$ is weakly contractible. Since $\widehat{U}$ is an inner fibration, the map

$$\mathcal{E}_{/X} = \mathcal{E} \times_{\widehat{\mathcal{E}}} \widehat{\mathcal{E}}_{/X} \to \mathcal{E} \times_{\mathcal{C}} \mathcal{C}_{/C}$$

is a right fibration (Proposition 4.3.6.8). Using our assumption that U is an isofibration, we deduce that the map $\mathcal{E}_{/X} \to \mathcal{C}_{/C}$ is also an isofibration. Let $\widetilde{D}$ denote the image of $\widetilde{Y}$ under this isofibration, which we identify with a morphism $f : D \to C$ in the ∞-category $\mathcal{C}$ (so that $D = U(Y)$ and $f = \widehat{U}(v)$). Set $\mathcal{B} = \{\mathrm{id}_C\} \times_{\mathcal{C}_{/C}} (\mathcal{C}_{/C})_{\widetilde{D}/}$, so that U induces an isofibration

$$\mathcal{A} = \{\mathrm{id}_C\} \times_{\mathcal{C}_{/C}} (\mathcal{E}_{/X})_{\widetilde{Y}/}) \to \{\mathrm{id}_C\} \times_{\mathcal{C}_{/C}} (\mathcal{C}_{/C})_{\widetilde{D}/} = \mathcal{B}.$$

Since id_C is final when viewed as an object of $\mathcal{C}_{/C}$ (Proposition 4.6.7.22), the simplicial set $\mathcal{B}$ is a contractible Kan complex. Let $B \in \mathcal{B}$ be the vertex which corresponds to the degenerate 2-simplex of $\mathcal{C}$ depicted in the diagram

$$\begin{tikzcd}
 & C \arrow[dr, "\mathrm{id}_C"] & \\
D \arrow[ur, "f"] \arrow[rr, "f"] & & C,
\end{tikzcd}$$

so that the inclusion map $\{B\} \hookrightarrow \mathcal{B}$ is an equivalence of ∞-categories. Applying Corollary 4.5.2.30, we deduce that the inclusion map $\{B\} \times_{\mathcal{B}} \mathcal{A} \hookrightarrow \mathcal{A}$ is an equivalence of ∞-categories. In particular, it is a weak homotopy equivalence. Consequently, to show that $\mathcal{A}$ is weakly contractible, it will suffice to show that the fiber $\{B\} \times_{\mathcal{B}} \mathcal{A}$ is weakly contractible. Replacing $\mathcal{E}$ and $\widehat{\mathcal{E}}$ by the fiber products $\Delta^1 \times_{\mathcal{C}} \mathcal{E}$ and $\Delta^1 \times_{\mathcal{C}} \widehat{\mathcal{E}}$, respectively, we are reduced to proving that $\mathcal{A}$ is weakly contractible under the additional assumption that $\mathcal{C} = \Delta^1$, where $U(Y) = 0$ and $\widehat{U}(X) = 1$.

For $i \in \{0, 1\}$, set $\mathcal{E}_i = \{i\} \times_{\mathcal{C}} \mathcal{E}$ and $\widehat{\mathcal{E}}_i = \{i\} \times_{\mathcal{C}} \widehat{\mathcal{E}}$. Fix an uncountable regular cardinal κ such that $\widehat{\mathcal{E}}$ is locally κ-small and every simplicial set $K \in \mathbb{K}$ is essentially κ-small. Then the projection map $\widehat{\mathcal{E}}_{H(Y)/} \to \widehat{\mathcal{E}}$ admits a covariant transport representation $\mathscr{F} : \widehat{\mathcal{E}} \to \mathcal{S}^{<\kappa}$, given informally by the formula $\mathscr{F}(E) = \mathrm{Hom}_{\widehat{\mathcal{E}}}(H(Y), E)$. Our assumption that H exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$ guarantees that the restriction $\mathscr{F}|_{\widehat{\mathcal{E}}_1}$ preserves K-indexed colimits, for each $K \in \mathbb{K}$. Applying Corollary 8.4.5.10 (and our assumption that H is fully faithful), we conclude that H induces a left cofinal functor

$$\mathcal{E}_1 \times_{\mathcal{E}} \mathcal{E}_{Y/} \xrightarrow{\sim} \mathcal{E}_1 \times_{\widehat{\mathcal{E}}} \widehat{\mathcal{E}}_{H(Y)/} \to \widehat{\mathcal{E}}_1 \times_{\widehat{\mathcal{E}}} \widehat{\mathcal{E}}_{H(Y)/}.$$

By virtue of Theorem 7.2.3.1, this is a reformulation of the assertion that the ∞-category $\mathcal{A}$ is weakly contractible (for every choice of object $X \in \widehat{\mathcal{E}}_1$). $\square$

Proposition 9.3.1.25. *Let $\mathbb{K}$ be a collection of simplicial sets. Suppose we are given a commutative diagram of ∞-categories* 05L9

$$\begin{tikzcd}
\widehat{\mathcal{E}} \arrow[rr, "F"] \arrow[dr, "\widehat{U}"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathcal{C} &
\end{tikzcd}$$

and let $\mathcal{E} \subseteq \widehat{\mathcal{E}}$ be a full subcategory. Assume that $U = \widehat{U}|_{\mathcal{E}}$ is an isofibration, that the inclusion map $\mathcal{E} \hookrightarrow \widehat{\mathcal{E}}$ exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$, and that U' is a $\mathbb{K}$-cocomplete inner fibration. The following conditions are equivalent:

(1) *The functor F is U'-left Kan extended from $\mathcal{E}$.*

(2) *For each object $C \in \mathcal{C}$ and each $K \in \mathbb{K}$, the functor $F_C : \widehat{\mathcal{E}}_C \to \mathcal{E}'_C$ preserves K-indexed colimits.*

Proof. Fix an object $C \in \mathcal{C}$, and let X be an object of $\widehat{\mathcal{E}}$ satisfying $\widehat{U}(X) = C$. Combining Lemma 9.3.1.24 with Corollary 7.2.2.2, we see that the following conditions are equivalent:

(1_X) The functor F is U'-left Kan extended from $\mathcal{E}$ at the object $X \in \widehat{\mathcal{E}}$.

(2_X) The functor F_C is U'-left Kan extended from $\mathcal{E}_C$ at the object $X \in \widehat{\mathcal{E}}_C$.

Allowing the object X to vary (with C fixed) and applying Lemma 9.3.1.23, we see that the following are equivalent:

(1_C) For each object $X \in \widehat{\mathcal{E}}$ satisfying $\widehat{U}(X) = C$, the functor F is U'-left Kan extended from $\mathcal{E}$ at the object X.

(2_C) For each $K \in \mathbb{K}$, the functor F_C preserves K-indexed colimits.

The equivalence (1) $\Leftrightarrow$ (2) now follows by allowing the object $C \in \mathcal{C}$ to vary. □

Variant 9.3.1.26. Let $\mathbb{K}$ be a collection of simplicial sets, let $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$ be an inner fibration of ∞-categories, and let $\mathcal{E} \subseteq \widehat{\mathcal{E}}$ be a full subcategory. Assume that $U = \widehat{U}|_{\mathcal{E}}$ is an isofibration and that the inclusion functor $\mathcal{E} \hookrightarrow \widehat{\mathcal{E}}$ exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. Let $U' : \mathcal{E}' \to \mathcal{C}$ be a $\mathbb{K}$-cocomplete isofibration. Then every lifting problem 05LA

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F^0"] \arrow[d] & \mathcal{E}' \arrow[d, "U'"] \\
\widehat{\mathcal{E}} \arrow[r] \arrow[ur, dashed, "F"] & \mathcal{C}
\end{tikzcd}$$

admits a solution, where the functor F is U'-left Kan extended from $\mathcal{E}$.

Proof. Fix an object $C \in \mathcal{C}$, so that F^0 restricts to a functor $F^0_C : \mathcal{E}_C \to \mathcal{E}'_C$. Since the inclusion functor $\mathcal{E}_C \hookrightarrow \widehat{\mathcal{E}}_C$ exhibits $\widehat{\mathcal{E}}_C$ as a $\mathbb{K}$-cocompletion of $\mathcal{E}_C$, the functor F^0_C admits an (essentially unique) extension $F_C : \widehat{\mathcal{E}}_C \to \mathcal{E}'_C$ which preserves K-indexed colimits for each $K \in \mathbb{K}$. It follows from Lemma 9.3.1.23 that F_C is U'-left Kan extended from $\mathcal{E}_C$. In particular, for each object $X \in \widehat{\mathcal{E}}_C$, the composition

$$(\mathcal{E}_C \times_{\widehat{\mathcal{E}}_C} (\widehat{\mathcal{E}}_C)_{/X})^{\triangleright} \widehat{\mathcal{E}}_C \xrightarrow{F_C} \mathcal{E}'$$

is a U'-colimit diagram. Since the inclusion map

$$\mathcal{E}_C \times_{\widehat{\mathcal{E}}_C} (\widehat{\mathcal{E}}_C)_{/X} \hookrightarrow \mathcal{E} \times_{\widehat{\mathcal{E}}} \widehat{\mathcal{E}}_{/X}$$

is right cofinal (Lemma 9.3.1.24), Proposition 7.2.2.9 guarantees that the lifting problem

$$\begin{tikzcd}
\mathcal{E} \times_{\widehat{\mathcal{E}}} \widehat{\mathcal{E}}_{/X} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r, "F"] & \mathcal{E}' \arrow[d, "U'"] \\
(\mathcal{E} \times_{\widehat{\mathcal{E}}} \widehat{\mathcal{E}}_{/X})^{\triangleright} \arrow[r] \arrow[urr, dashed] & \widehat{\mathcal{E}} \arrow[r, "\widehat{U}"] & \mathcal{C}
\end{tikzcd}$$

admits a solution, where the dotted arrow is a U'-colimit diagram. The desired result now follows by allowing the object X to vary and applying the criterion of Proposition 7.3.5.5. □

We now prove a special case of Theorem 9.3.1.20 (which is sufficient for most applications).

05LB **Lemma 9.3.1.27.** *Let $\mathbb{K}$ be a collection of simplicial sets. Suppose we are given a diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where U and $\widehat{U}$ are isofibrations and H exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. Then, for every $\mathbb{K}$-cocomplete isofibration $U' : \mathcal{E}' \to \mathcal{C}$, precomposition with H induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathbb{K}}_{/\mathcal{C}}(\widehat{\mathcal{E}}, \mathcal{E}') \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}').$$

Proof. Since the functor H is fully faithful (see Variant 4.8.6.19), we can assume without loss of generality that $\mathcal{E}$ is a replete full subcategory of $\widehat{\mathcal{E}}$ (and that H is the inclusion functor). In this case, Proposition 9.3.1.25 identifies $\operatorname{Fun}^{\mathbb{K}}_{/\mathcal{C}}(\widehat{\mathcal{E}}, \mathcal{E}')$ with the full subcategory of $\operatorname{Fun}_{/\mathcal{C}}(\widehat{\mathcal{E}}, \mathcal{E}')$ spanned by those functors $F : \widehat{\mathcal{E}} \to \mathcal{E}'$ which are U'-left Kan extended from $\mathcal{E}$. Combining this observation with Theorem 7.3.6.14 and Variant 9.3.1.26, we see that the restriction functor $\operatorname{Fun}^{\mathbb{K}}_{/\mathcal{C}}(\widehat{\mathcal{E}}, \mathcal{E}') \to \operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}, \mathcal{E}')$ is a trivial Kan fibration. □

Proof of Theorem 9.3.1.20. For every morphism of simplicial sets $S \to \mathcal{C}$, we let $\operatorname{Fun}^{\mathbb{K}}_{/\mathcal{C}}(S \times_{\mathcal{C}} \widehat{\mathcal{E}}, \mathcal{E}')$ denote the (replete) full subcategory of $\operatorname{Fun}_{/\mathcal{C}}(S \times_{\mathcal{C}} \widehat{\mathcal{E}}, \mathcal{E}')$ spanned by those morphisms $F : S \times_{\mathcal{C}} \widehat{\mathcal{E}} \to \mathcal{E}'$ having the property that, for each vertex $s \in S$, the functor $F_s : \{s\} \times_{\mathcal{C}} \widehat{\mathcal{E}} \to \{s\} \times_{\mathcal{C}} \mathcal{E}'$ preserves K-indexed colimits for each $K \in \mathbb{K}$. We will prove the following:

$(*)$ For every morphism of simplicial sets $S \to \mathcal{C}$, precomposition with H induces an equivalence of ∞-categories

$$\theta_S : \operatorname{Fun}^{\mathbb{K}}_{/\mathcal{C}}(S \times_{\mathcal{C}} \widehat{\mathcal{E}}, \mathcal{E}') \to \operatorname{Fun}_{/\mathcal{C}}(S \times_{\mathcal{C}} \mathcal{E}, \mathcal{E}').$$

Writing S as the union of its skeleta $\operatorname{sk}_n(S)$, we can realize θ_S as the limit of a tower of comparison maps

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{Fun}^{\mathbb{K}}_{/\mathcal{C}}(\operatorname{sk}_1(S) \times_{\mathcal{C}} \widehat{\mathcal{E}}, \mathcal{E}') \arrow[r] \arrow[d, "\theta_{\operatorname{sk}_1(S)}"] & \operatorname{Fun}^{\mathbb{K}}_{/\mathcal{C}}(\operatorname{sk}_0(S) \times_{\mathcal{C}} \widehat{\mathcal{E}}, \mathcal{E}') \arrow[d, "\theta_{\operatorname{sk}_0(S)}"] \\
\cdots \arrow[r] & \operatorname{Fun}_{/\mathcal{C}}(\operatorname{sk}_1(S) \times_{\mathcal{C}} \mathcal{E}, \mathcal{E}') \arrow[r] & \operatorname{Fun}_{/\mathcal{C}}(\operatorname{sk}_0(S) \times_{\mathcal{C}} \mathcal{E}, \mathcal{E}'),
\end{tikzcd}$$

where the horizontal maps are isofibrations (Variant 4.4.5.11). Consequently, to show that θ_S is an equivalence of ∞-categories, it will suffice to show that $\theta_{\operatorname{sk}_n(S)}$ is an equivalence of ∞-categories for each $n \geq -1$. We may therefore assume without loss of generality that S has dimension $\leq n$, for some integer $n \geq -1$.

We now proceed by induction on n. Assume that $n \geq 0$ (otherwise, S is empty and there is nothing to prove). Let S' be the $(n-1)$-skeleton of S, so that Proposition 1.1.4.12 supplies a pushout diagram

$$\begin{tikzcd}
T' \arrow[r] \arrow[d] & T \arrow[d] \\
S' \arrow[r] & S
\end{tikzcd}$$

where T is a coproduct of standard n-simplices (indexed by the nondegenerate n-simplices of S) and T' is the coproduct of their boundaries. It follows that θ_S can be realized as the horizontal limit of a diagram of comparison maps

$$\begin{tikzcd}
\mathrm{Fun}^{\mathbb{K}}_{/\mathcal{C}}(S' \times_{\mathcal{C}} \widehat{\mathcal{E}}, \mathcal{E}') \arrow[r] \arrow[d, "\theta_{S'}"] & \mathrm{Fun}^{\mathbb{K}}_{/\mathcal{C}}(T' \times_{\mathcal{C}} \widehat{\mathcal{E}}, \mathcal{E}') \arrow[r] \arrow[d, "\theta_{T'}"] & \mathrm{Fun}^{\mathbb{K}}_{/\mathcal{C}}(T \times_{\mathcal{C}} \widehat{\mathcal{E}}, \mathcal{E}') \arrow[d, "\theta_{T}"] \\
\mathrm{Fun}_{/\mathcal{C}}(S' \times_{\mathcal{C}} \mathcal{E}, \mathcal{E}') \arrow[r] & \mathrm{Fun}_{/\mathcal{C}}(T' \times_{\mathcal{C}} \mathcal{E}, \mathcal{E}') & \mathrm{Fun}_{/\mathcal{C}}(T \times_{\mathcal{C}} \mathcal{E}, \mathcal{E}'), \arrow[l]
\end{tikzcd}$$

where the horizontal maps on the right are isofibrations (Variant 4.4.5.11). Since the simplicial sets S' and T' have dimension $< n$, our inductive hypothesis guarantees that $\theta_{S'}$ and $\theta_{T'}$ are equivalences of ∞-categories. Consequently, to show that θ_S is an equivalence of ∞-categories, it will suffice to show that θ_T is an equivalence of ∞-categories (Corollary 4.5.2.30). Replacing $\mathcal{C}$ by T, we are reduced to proving Theorem 9.3.1.20 in the special case where $\mathcal{C}$ is a coproduct of standard simplices. In this case, $\mathcal{C}$ is an ∞-category and the inner fibrations $U : \mathcal{E} \to \mathcal{C}$, $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$, and $U' : \mathcal{E}' \to \mathcal{C}$ are automatically isofibrations (Example 4.4.1.6), so the desired result follows from Lemma 9.3.1.27. □

9.3.2 Fiberwise Cocompletions of Cocartesian Fibrations

05LC We now specialize our theory of fiberwise cocompletions to the setting of cocartesian fibrations. In this case, Definition 9.3.1.12 can be reformulated:

05LD **Theorem 9.3.2.1.** *Let $\mathbb{K}$ be a collection of simplicial sets and let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then a commutative diagram*

05LE
$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd} \qquad (9.28)$$

exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$ if and only if the following conditions are satisfied:

(a) For each vertex $C \in \mathcal{C}$, the functor $H_C : \mathcal{E}_C \to \widehat{\mathcal{E}}_C$ exhibits the ∞-category $\widehat{\mathcal{E}}_C$ as a $\mathbb{K}$-cocompletion of the ∞-category $\mathcal{E}_C$ (Definition 8.4.5.1).

(b) The morphism $\widehat{U}$ is a cocartesian fibration of simplicial sets.

(c) *For every edge $f : C \to C'$ of $\mathcal{C}$, the covariant transport functor $f_! : \widehat{\mathcal{E}}_C \to \widehat{\mathcal{E}}_{C'}$ preserves K-indexed colimits for each $K \in \mathbb{K}$.*

(d) *The morphism H carries U-cocartesian edges of $\mathcal{E}$ to $\widehat{U}$-cocartesian edges of $\widehat{\mathcal{E}}$.*

Moreover, there exists a diagram (9.28) which satisfies these conditions.

Our proof of Theorem 9.3.2.1 will require some preliminaries.

Lemma 9.3.2.2. *Let $\mathbb{K}$ be a collection of simplicial sets and let $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$ be a cocartesian fibration. Then $\widehat{U}$ is $\mathbb{K}$-cocomplete (in the sense of Variant 9.3.1.4) if and only if it satisfies the following conditions:* 05LF

- *For each object $C \in \mathcal{C}$, the ∞-category $\widehat{\mathcal{E}}_C = \{C\} \times_{\mathcal{C}} \widehat{\mathcal{E}}$ is $\mathbb{K}$-cocomplete.*
- *For each edge $f : C \to C'$ of $\mathcal{C}$, the covariant transport functor $f_! : \mathcal{E}_C \to \mathcal{E}_{C'}$ preserves K-indexed colimits for each $K \in \mathbb{K}$.*

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is an ∞-category (or even that $\mathcal{C} = \Delta^1$). In this case, the desired result follows from the characterization of $\widehat{U}$-colimit diagrams supplied by Proposition 7.3.9.2. □

Lemma 9.3.2.3. *Let $\mathbb{K}$ be a collection of simplicial sets, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets, and suppose we are given a commutative diagram* 05LG

05LH
$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd} \qquad (9.29)$$

which satisfies the conditions of Theorem 9.3.2.1. Then H exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$.

Proof. Without loss of generality, we may assume that $\mathcal{C} = \Delta^1$. For $i \in \{0, 1\}$, let $\mathcal{E}_i$ denote the fiber $\{i\} \times_{\Delta^1} \mathcal{E}$ and define $\widehat{\mathcal{E}}_i$ similarly, so that H restricts to a functor $H_i : \mathcal{E}_i \to \widehat{\mathcal{E}}_i$. By assumption, the functor H_i exhibits $\widehat{\mathcal{E}}_i$ as a $\mathbb{K}$-cocompletion of $\mathcal{E}_i$, and is therefore fully faithful (Proposition 8.4.5.3). Since H carries U-cocartesian morphisms of $\mathcal{E}$ to $\widehat{U}$-cocartesian morphisms of $\widehat{\mathcal{E}}$, it follows that H is fully faithful (Proposition 5.1.6.7). It follows from Lemma 9.3.2.2 that the cocartesian fibration $\widehat{U}$ is $\mathbb{K}$-cocomplete.

Fix an uncountable regular cardinal κ such that $\widehat{\mathcal{E}}$ is locally κ-small and every simplicial set $K \in \mathbb{K}$ is essentially κ-small. To complete the proof, it will suffice to show that for every object $X \in \mathcal{E}_0$, the functor

$$\mathscr{F} : \widehat{\mathcal{E}}_1 \to \mathcal{S}^{<\kappa} \qquad Y \mapsto \operatorname{Hom}_{\widehat{\mathcal{E}}}(H(X), Y)$$

commutes with K-indexed colimits, for each $K \in \mathbb{K}$. Since U is a cocartesian fibration, we can choose a U-cocartesian morphism $u : X \to X'$ of $\mathcal{E}$ with $X' \in \mathcal{E}_1$. By assumption, $H(u) : H(X) \to H(X')$ is a $\widehat{U}$-cocartesian morphism in the ∞-category $\widehat{\mathcal{E}}$, so the functor $\mathscr{F}$ is corepresented by the object $H(X')$. The desired result now follows from the characterization of $\mathbb{K}$-cocompletions given in Variant 8.4.6.9. □

Our next goal is to prove the existence assertion of Theorem 9.3.2.1. We begin by treating the case where $\mathbb{K}$ is the collection of κ-small simplicial sets, for some uncountable regular cardinal κ, and the cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ is essentially κ-small. In the special case $\mathcal{C} = \Delta^0$, Theorem 9.3.2.1 reduces to the assertion that there exists a functor $H : \mathcal{E} \to \widehat{\mathcal{E}}$ which exhibits $\widehat{\mathcal{E}}$ as a cocompletion of $\mathcal{E}$. In §8.4, we proved this using an explicit construction: we can take $\widehat{\mathcal{E}}$ to be the ∞-category $\mathrm{Fun}(\mathcal{E}^{\mathrm{op}}, \mathcal{S})$ and H to be the covariant Yoneda embedding (Theorem 8.4.0.3). We now extend this construction to the relative setting, using the duality theory developed in §8.6.

05LJ **Lemma 9.3.2.4.** *Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \to \mathcal{C}$ and $U^\vee : \mathcal{E}^\vee \to \mathcal{C}$ be essentially κ-small cocartesian fibrations of simplicial sets, and let $\mathrm{Fun}(\mathcal{E}^\vee / \mathcal{C}, \mathcal{S}^{<\kappa})$ be the relative exponential of Construction 4.5.9.1. Suppose we are given a morphism*

$$\mathscr{K} : \mathcal{E}^\vee \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa},$$

which exhibits $U^\vee$ as a cocartesian dual of U (see Variant 8.6.4.13). Then $\mathscr{K}$ is classified by a commutative diagram with a diagram

05LK
$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \mathrm{Fun}(\mathcal{E}^\vee / \mathcal{C}, \mathcal{S}^{<\kappa}) \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} . &
\end{tikzcd} \tag{9.30}$$

which satisfies the conditions of Theorem 9.3.2.1.

Proof. It follows from Proposition 8.6.5.12 that $\widehat{U}$ is both a cartesian and cocartesian fibration, so that condition (b) of Theorem 9.3.2.1 is satisfied and condition (c) is automatic (Corollary 7.1.3.21). Condition (d) follows from Proposition 8.6.5.18, and condition (a) from Theorem 8.4.3.3. □

05LL **Example 9.3.2.5.** Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets which is essentially κ-small, and let $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$ be as in Construction 8.6.5.6. It follows from Proposition 8.6.5.8 that the evaluation functor

$$\mathrm{ev} : \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa} \qquad ((C, \mathscr{F}_C), X) \mapsto \mathscr{F}_C(X)$$

exhibits U as a cocartesian dual of the projection map $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$. Applying Lemma 9.3.2.4 (with the roles of $\mathcal{E}$ and $\mathcal{E}^{\vee}$ switched) and Lemma 9.3.2.3, we conclude that the diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

exhibits $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ as a fiberwise κ-cocompletion of $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$.

Lemma 9.3.2.6. *Let $\mathbb{K}$ be a collection of simplicial sets and let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Then there exists a commutative diagram* 05LM

05LN

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C}. &
\end{tikzcd} \tag{9.31}$$

which satisfies the conditions of Theorem 9.3.2.1.

Proof. Choose an uncountable regular cardinal κ such that U is essentially κ-small and each $K \in \mathbb{K}$ is κ-small. Using Theorem 8.6.5.1 (and Proposition 8.6.4.15), we can choose a cocartesian fibration $U^{\vee} : \mathcal{E}^{\vee} \to \mathcal{C}$ and a diagram $\mathscr{K} : \mathcal{E}^{\vee} \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ which exhibits $U^{\vee}$ as a cocartesian dual of U. Let $\mathbb{K}'$ be the collection of all κ-small simplicial sets and set $\widehat{\mathcal{E}}' = \mathrm{Fun}(\mathcal{E}^{\vee} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$, so that Lemma 9.3.2.4 supplies a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H'"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}}' \arrow[dl, "\widehat{U}'"] \\
& \mathcal{C} &
\end{tikzcd}$$

which satisfies the conditions of Theorem 9.3.2.1 with respect to $\mathbb{K}'$.

For each object $C \in \mathcal{C}$, let $\widehat{\mathcal{E}}_C$ denote smallest full subcategory of $\widehat{\mathcal{E}}$ which contains the essential image of the functor $H'_C : \mathcal{E}_C \to \widehat{\mathcal{E}}'_C$ and is closed under K-indexed colimits for each $K \in \mathbb{K}$. For each edge $f : C \to D$ of the simplicial set $\mathcal{C}$, Remark 5.2.2.14 guarantees that the diagram

$$\begin{tikzcd}
\mathcal{E}_C \arrow[r] \arrow[d, "H'_C"] & \mathcal{E}_D \arrow[d, "H'_D"] \\
\widehat{\mathcal{E}}'_C \arrow[r, "f_!"] & \widehat{\mathcal{E}}'_D
\end{tikzcd}$$

commutes up to isomorphism, where the horizontal maps are given by covariant transport along f. In particular, the covariant transport functor $f_! : \widehat{\mathcal{E}}'_C \to \widehat{\mathcal{E}}'_D$ carries the essential image of H'_C into the essential image of H'_D. Since $f_!$ preserves κ-small colimits, it carries $\widehat{\mathcal{E}}_C$ into $\widehat{\mathcal{E}}_D$.

Let $\widehat{\mathcal{E}}$ denote the full simplicial subset of $\widehat{\mathcal{E}}'$ spanned by those vertices X which belong to $\widehat{\mathcal{E}}_C$, where $C = \widehat{U}'(X)$. Applying Proposition 5.1.4.16, we see that $\widehat{U} = \widehat{U}'|_{\widehat{\mathcal{E}}}$ is a cocartesian fibration from $\widehat{\mathcal{E}}$ to $\mathcal{C}$, and that an edge of $\widehat{\mathcal{E}}$ is $\widehat{U}$-cocartesian if and only if it is $\widehat{U}'$-cocartesian when regarded as an edge of $\widehat{\mathcal{E}}'$. It follows that the commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

automatically satisfies conditions (b) and (d) of Theorem 9.3.2.1. Moreover, for each edge $f : C \to D$ of $\mathcal{C}$, the associated covariant transport functor $\widehat{\mathcal{E}}_C \to \widehat{\mathcal{E}}_D$ is can be identified with the restriction of $f_!$ to $\widehat{\mathcal{E}}_C$, and therefore preserves K-indexed colimits for each $K \in \mathbb{K}$. To complete the proof, it will suffice to show that for every vertex $C \in \mathcal{C}$, the induced map $H_C : \mathcal{E}_C \to \widehat{\mathcal{E}}_C$ exhibits $\widehat{\mathcal{E}}_C$ as a $\mathbb{K}$-cocompletion of $\mathcal{E}_C$. This follows from Proposition 8.4.5.7. □

Proof of Theorem 9.3.2.1. Let $\mathbb{K}$ be a collection of simplicial sets and let $U : \mathcal{E} \to \mathcal{C}$ be a cocartesian fibration of simplicial sets. Using Lemma 9.3.2.6, we can choose a commutative diagram

05LP

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd} \tag{9.32}$$

which satisfies conditions (a), (b), (c), and (d) of Theorem 9.3.2.1. Lemma 9.3.2.3 guarantees any such diagram (9.32) exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. Conversely, if we are given another diagram

05LQ

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}}' \arrow[dl, "\widehat{U}'"] \\
& \mathcal{C} &
\end{tikzcd} \tag{9.33}$$

which exhibits $\widehat{\mathcal{E}}'$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$, then Remark 9.3.1.21 guarantees that there is an equivalence $F : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}'$ of inner fibrations over $\mathcal{C}$ such that H' is isomorphic to $F \circ H$ as an object of $\operatorname{Fun}_{/\mathcal{C}}(\mathcal{E}, \widehat{\mathcal{E}}')$. It then follows that (9.33) also satisfies conditions (a), (b), (c), and (d) of Theorem 9.3.2.1. □

9.3.3 Fiberwise Cocompletion of Cartesian Fibrations

We now study fiberwise cocompletion in the setting of cartesian fibrations. Here the main result is the following: 05LR

Theorem 9.3.3.1. *Let $\mathbb{K}$ be a collection of simplicial sets and let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of simplicial sets. Then a commutative diagram* 05LS

05LT

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd} \tag{9.34}$$

exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$ if and only if the following conditions are satisfied:

(a) *For each vertex $C \in \mathcal{C}$, the functor $H_C : \mathcal{E}_C \to \widehat{\mathcal{E}}_C$ exhibits the ∞-category $\widehat{\mathcal{E}}_C$ as a $\mathbb{K}$-cocompletion of the ∞-category $\mathcal{E}_C$ (Definition 8.4.5.1).*

(b) *The morphism $\widehat{U}$ is a cartesian fibration of simplicial sets.*

(c) *For every edge $f : C \to C'$ of $\mathcal{C}$, the contravariant transport functor $f^* : \widehat{\mathcal{E}}_{C'} \to \widehat{\mathcal{E}}_C$ preserves K-indexed colimits for each $K \in \mathbb{K}$.*

(d) *The morphism H carries U-cartesian edges of $\mathcal{E}$ to $\widehat{U}$-cartesian edges of $\widehat{\mathcal{E}}$.*

Moreover, there exists a diagram (9.34) which satisfies these conditions.

Our strategy is to reduce Theorem 9.3.3.1 from its counterpart for cocartesian fibrations (Theorem 9.3.2.1), which was proved in §9.3.2. To carry out the reduction, we show that fiberwise cocompletion is compatible with the formation of conjugate fibrations (Proposition 9.3.3.3). As a first step, we record the easy direction of Theorem 9.3.3.1:

05LU **Lemma 9.3.3.2.** *Let $\mathbb{K}$ be a collection of simplicial sets, let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of simplicial sets, and suppose we are given a commutative diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which satisfies the conditions of Theorem 9.3.3.1. Then H exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$.

Proof. Without loss of generality, we may assume that $\mathcal{C} = \Delta^1$. For $i \in \{0, 1\}$, let $\mathcal{E}_i$ denote the fiber $\{i\} \times_{\Delta^1} \mathcal{E}$ and define $\widehat{\mathcal{E}}_i$ similarly, so that H restricts to a functor $H_i : \mathcal{E}_i \to \widehat{\mathcal{E}}_i$. By assumption, the functor H_i exhibits $\widehat{\mathcal{E}}_i$ as a $\mathbb{K}$-cocompletion of $\mathcal{E}_i$, and is therefore fully faithful (Proposition 8.4.5.3). Since H carries U-cartesian morphisms of $\mathcal{E}$ to $\widehat{U}$-cartesian morphisms of $\widehat{\mathcal{E}}$, it follows that H is fully faithful (Proposition 5.1.6.7). It follows from Example 9.3.1.6 that the fibration $\widehat{U}$ is $\mathbb{K}$-cocomplete.

Fix an uncountable regular cardinal κ such that $\widehat{\mathcal{E}}$ is locally κ-small and every simplicial set $K \in \mathbb{K}$ is essentially κ-small. To complete the proof, it will suffice to show that for every object $X \in \mathcal{E}_0$, the functor

$$\mathscr{F} : \widehat{\mathcal{E}}_1 \to \mathcal{S}^{<\kappa} \qquad Y \mapsto \operatorname{Hom}_{\widehat{\mathcal{E}}}(H(X), Y)$$

commutes with K-indexed colimits, for each $K \in \mathbb{K}$. Since $\widehat{U}$ is a cartesian fibration, this functor factors as a composition $\widehat{\mathcal{E}}_1 \xrightarrow{f^*} \widehat{\mathcal{E}}_0 \xrightarrow{\mathscr{G}} \mathcal{S}^{<\kappa}$, where f^* is given by contravariant transport along the nondegenerate edge of $\mathcal{C}$ and the functor $\mathscr{G}$ is corepresented by $H(X)$. We are therefore reduced to showing that the functor $\mathscr{G}$ preserves K-indexed colimits, which follows from the recognition principle of Variant 8.4.6.9. □

05LV **Proposition 9.3.3.3.** *Let $\mathbb{K}$ be a collection of simplicial sets, let $U : \mathcal{E} \to \mathcal{C}$ and $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$ be cocartesian fibrations of simplicial sets, and suppose we are given a commutative diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. Let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ and $\widehat{U}^{\dagger} : \widehat{\mathcal{E}}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be cartesian fibrations, and let

$$T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E} \qquad \widehat{T} : \widehat{\mathcal{E}}^{\dagger} \times^{\mathrm{op}}_{\mathcal{C}} \mathrm{Tw}(\mathcal{C}) \to \widehat{\mathcal{E}}$$

be morphisms which exhibit $U^{\dagger}$ and $\widehat{U}^{\dagger}$ as cartesian conjugates of U and $\widehat{U}$, respectively. Then there exists a morphism $H^{\dagger} : \mathcal{E}^{\dagger} \to \widehat{\mathcal{E}}^{\dagger}$ satisfying $\widehat{U}^{\dagger} \circ H^{\dagger} = U^{\dagger}$, and for which the diagram

05LW

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "T"] \arrow[d, "H^{\dagger} \times \mathrm{id}"] & \mathcal{E} \arrow[d, "H"] \\
\widehat{\mathcal{E}}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "\widehat{T}"] & \widehat{\mathcal{E}}
\end{tikzcd} \tag{9.35}$$

commutes up to isomorphism (in the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}), \widehat{\mathcal{E}})$). In this case, the diagram

05LX

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \arrow[rr, "H^{\dagger}"] \arrow[dr, "U^{\dagger}"'] & & \widehat{\mathcal{E}}^{\dagger} \arrow[dl, "\widehat{U}^{\dagger}"] \\
& \mathcal{C}^{\mathrm{op}} &
\end{tikzcd} \tag{9.36}$$

satisfies the hypotheses of Theorem 9.3.3.1, and therefore exhibits $\widehat{\mathcal{E}}^{\dagger}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}^{\dagger}$.

Proof. By virtue of Corollary 5.6.7.3, we may assume that $\mathcal{C}$ is an ∞-category. It follows from the criterion of Theorem 9.3.2.1 that the functor H carries U-cocartesian morphisms of $\mathcal{E}$ to $\widehat{U}$-cocartesian morphisms of $\widehat{\mathcal{E}}$. Invoking Example 8.6.2.10, we deduce that there is an object $H^{\dagger} \in \mathrm{Fun}_{/\mathcal{C}^{\mathrm{op}}}(\mathcal{E}^{\dagger}, \widehat{\mathcal{E}}^{\dagger})$ for which the diagram (9.35) commutes up to isomorphism (moreover, $H^{\dagger}$ is unique up to isomorphism). We complete the proof by showing that the diagram (9.36) satisfies the conditions of Theorem 9.3.3.1:

(*a*) Fix an object $C \in \mathcal{C}$; we wish to show that the functor $H^{\dagger}_{C}$ exhibits the ∞-category $\widehat{\mathcal{E}}^{\dagger}_{C}$ as a $\mathbb{K}$-cocompletion of the ∞-category $\mathcal{E}^{\dagger}_{C}$. Using (9.35), we obtain a diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E}^{\dagger}_{C} \arrow[r, "T_C"] \arrow[d, "H^{\dagger}_{C}"] & \mathcal{E}_C \arrow[d, "H_C"] \\
\widehat{\mathcal{E}}^{\dagger}_{C} \arrow[r, "\widehat{T}_C"] & \widehat{\mathcal{E}}_C
\end{tikzcd}$$

which commutes up to isomorphism, where the horizontal maps are equivalences. The desired result now follows from our assumption that H_C exhibits the ∞-category $\widehat{\mathcal{E}}_C$ as a $\mathbb{K}$-cocompletion of $\mathcal{E}_C$.

(*b*) The morphism $\widehat{U}^{\dagger}$ is a cartesian fibration by assumption.

(*c*) Fix a morphism $f : C \to C'$ of $\mathcal{C}$ and a simplicial set $K \in \mathbb{K}$; we wish to show that the contravariant transport functor $f^* : \widehat{\mathcal{E}}^{\dagger}_C \to \widehat{\mathcal{E}}^{\dagger}_{C'}$ preserves K-indexed colimits. Proposition 8.6.1.5 then guarantees that the diagram of ∞-categories

$$\begin{tikzcd}
\widehat{\mathcal{E}}^{\dagger}_C \arrow[r, "f^*"] \arrow[d, "\widehat{T}_C"] & \widehat{\mathcal{E}}^{\dagger}_{C'} \arrow[d, "\widehat{T}_{C'}"] \\
\widehat{\mathcal{E}}_C \arrow[r, "f_!"] & \widehat{\mathcal{E}}_{C'}
\end{tikzcd}$$

commutes up to isomorphism, where $f_!$ is given by covariant transport along f for the cocartesian fibration $\widehat{U}$. Since the vertical maps are equivalences of ∞-categories, we are reduced to proving that the functor $f_!$ preserves K-indexed colimits, which follows from Theorem 9.3.2.1.

(*d*) The morphism $H^{\dagger}$ carries $U^{\dagger}$-cartesian edges of $\mathcal{E}^{\dagger}$ to $\widehat{U}^{\dagger}$-cartesian edges of $\widehat{\mathcal{E}}^{\dagger}$: this follows from Remark 8.6.2.11.

□

05LY **Corollary 9.3.3.4.** *Let $\mathbb{K}$ be a collection of simplicial sets and let $U^{\dagger} : \mathcal{E}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ be a cartesian fibration of simplicial sets. Then there exists a diagram*

05LZ

$$\begin{tikzcd}
\mathcal{E}^{\dagger} \arrow[rr, "H^{\dagger}"] \arrow[dr, "U^{\dagger}"'] & & \widehat{\mathcal{E}}^{\dagger} \arrow[dl, "\widehat{U}^{\dagger}"] \\
& \mathcal{C}^{\mathrm{op}} &
\end{tikzcd} \tag{9.37}$$

which satisfies the conditions of Theorem 9.3.3.1.

Proof. Using Corollary 8.6.6.3, we can choose a cocartesian fibration $U : \mathcal{E} \to \mathcal{C}$ and a morphism $T : \mathcal{E}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathcal{E}$ which exhibits $U^{\dagger}$ as a cartesian conjugate of U. Applying

Theorem 9.3.2.1, we can choose a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$, where $\widehat{U}$ is a cocartesian fibration. Using Proposition 8.6.2.3, we can choose a cartesian fibration $\widehat{U}^{\dagger} : \widehat{\mathcal{E}}^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ and a morphism $\widehat{T} : \widehat{\mathcal{E}}^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \widehat{\mathcal{E}}$ which exhibits $\widehat{U}^{\dagger}$ as a cartesian conjugate of $\widehat{U}$. The desired result now follows from Proposition 9.3.3.3. □

Proof of Theorem 9.3.3.1. Let $\mathbb{K}$ be a collection of simplicial sets and let $U : \mathcal{E} \to \mathcal{C}$ be a cartesian fibration of simplicial sets. Using Corollary 9.3.3.4, we can choose a commutative diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which satisfies the conditions of Theorem 9.3.3.1, and therefore exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$ (Lemma 9.3.3.2). Conversely, if we are given another diagram

05M0

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}}' \arrow[dl, "\widehat{U}'"] \\
& \mathcal{C} &
\end{tikzcd} \tag{9.38}$$

which exhibits $\widehat{\mathcal{E}}'$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$, then Remark 9.3.1.21 guarantees that there is an equivalence $F : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}'$ of inner fibrations over $\mathcal{C}$ such that H' is isomorphic to $F \circ H$ as an object of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \widehat{\mathcal{E}}')$. It then follows that (9.38) the conditions of Theorem 9.3.2.1. □

9.3.4 Existence of Fiberwise Cocompletions

Our goal in this section is to prove the following existence result for fiberwise cocompletions: 05M1

05M2 **Theorem 9.3.4.1.** *Let $\mathbb{K}$ be a collection of simplicial sets and let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. Then there exists a commutative diagram*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$ (in the sense of Definition 9.3.1.12).

Our strategy for proving Theorem 9.3.4.1 is to reduce to the special case where U is a cocartesian fibration, which was treated in §9.3.2 (see Theorem 9.3.2.1). To carry out the reduction, we will need the following relative version of Corollary 8.4.6.10:

05M3 **Lemma 9.3.4.2.** *Let $\mathbb{K}$ be a collection of simplicial sets, let $U' : \mathcal{E}' \to \mathcal{C}$ be an inner fibration of simplicial sets, and and suppose we are given a commutative diagram*

05M4
$$\begin{tikzcd}
\mathcal{E}' \arrow[rr, "H'"] \arrow[dr, "U'"'] & & \widehat{\mathcal{E}}' \arrow[dl, "\widehat{U}'"] \\
& \mathcal{C} &
\end{tikzcd} \tag{9.39}$$

which exhibits $\widehat{\mathcal{E}}'$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}'$. Let $\mathcal{E}$ be a full simplicial subset of $\mathcal{E}'$ and let $\widehat{\mathcal{E}} \subseteq \widehat{\mathcal{E}}'$ be the full simplicial subset spanned by those vertices Y with the following property:

- *If $C = \widehat{U}'(Y)$, then Y belongs to the smallest full subcategory of $\widehat{\mathcal{E}}'_C$ which contains the essential image of $H|_{\mathcal{E}_C}$ and is closed under K-indexed colimits, for each $K \in \mathbb{K}$.*

Set $H = H'|_{\mathcal{E}}$, $U = U'|_{\mathcal{E}}$, and $\widehat{U} = \widehat{U}'|_{\widehat{\mathcal{E}}}$. Then the diagram

05M5
$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd} \tag{9.40}$$

exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$.

Proof. Without loss of generality, we may assume that $\mathcal{C} = \Delta^1$; in this case, we must show that the diagram (9.40) satisfies conditions (1) through (4) of Definition 9.3.1.8. Condition (1) follows from Corollary 8.4.6.10, and the remaining conditions follow immediately from the corresponding conditions on the diagram (9.39) $\square$

We now prove a special case of Theorem 9.3.4.1, which is already sufficient for most applications:

Lemma 9.3.4.3. *Let $\mathbb{K}$ be a collection of simplicial sets and let $U : \mathcal{E} \to \mathcal{C}$ be an isofibration of ∞-categories Then there exists a commutative diagram of ∞-categories* `05M6`

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$ (in the sense of Definition 9.3.1.12). Moreover, $\widehat{U}$ is also an isofibration.

Proof. Choose a factorization of U as a composition $\mathcal{E} \xrightarrow{F} \mathcal{E}' \xrightarrow{U'} \mathcal{C}$, where U' is a cocartesian fibration of ∞-categories and F is fully faithful. For example, we can take $\mathcal{E}'$ to be the oriented fiber product $\mathcal{E} \tilde{\times}_{\mathcal{C}} \mathcal{C}$, $U' : \mathcal{E}' \to \mathcal{C}$ be the projection onto the second factor (which is a cocartesian fibration by virtue of Corollary 5.3.7.3), and F to be the inclusion map

$$\mathcal{E} \simeq \mathcal{E} \times_{\mathcal{C}} \mathcal{C} \hookrightarrow \mathcal{E} \tilde{\times}_{\mathcal{C}} \mathcal{C}$$

(which is fully faithful by virtue of Corollary 4.5.2.22). Let $\mathcal{E}'_0 \subseteq \mathcal{E}'$ be the essential image of F. Since U' is an isofibration (Proposition 5.1.4.8), it restricts to an isofibration $\mathcal{E}'_0 \to \mathcal{C}$. Applying Proposition 5.1.7.5, we see that the functor $F : \mathcal{E} \to \mathcal{E}'_0$ is an equivalence of inner fibrations over $\mathcal{C}$. We may therefore replace $\mathcal{E}$ by $\mathcal{E}'_0$ and thereby reduce to the special case where $\mathcal{E}$ is a replete full subcategory of $\mathcal{E}'$.

Applying Theorem 9.3.2.1, we deduce that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{E}' \arrow[rr, "H'"] \arrow[dr, "U'"'] & & \widehat{\mathcal{E}}' \arrow[dl, "\widehat{U}'"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}'$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}'$. Moreover, the morphism $\widehat{U}'$ is a cocartesian fibration (and therefore an isofibration; see Proposition 5.1.4.8). Define $\widehat{\mathcal{E}} \subseteq \widehat{\mathcal{E}}'$

as in the statement of Lemma 9.3.4.2. Then $\widehat{\mathcal{E}}$ is a replete full subcategory of $\widehat{\mathcal{E}}'$, so the restriction map $\widehat{U} = \widehat{U}'|_{\widehat{\mathcal{E}}}$ is an isofibration. Moreover, Lemma 9.3.4.2 guarantees that the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H'|_{\mathcal{E}}"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. □

We now prove a relative version of Theorem 9.3.4.1.

`05M7` **Lemma 9.3.4.4.** *Let $\mathbb{K}$ be a collection of simplicial sets, let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a simplicial subset. Set $\mathcal{E}_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{C}$, let $U_0 : \mathcal{E}_0 \to \mathcal{C}_0$ be the projection onto the first factor, and suppose we are given a diagram*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[rr, "H_0"] \arrow[dr, "U_0"'] & & \widehat{\mathcal{E}}_0 \arrow[dl, "\widehat{U}_0"] \\
& \mathcal{C}_0 &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}_0$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}_0$. Then there exists a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise cocompletion of $\mathcal{E}$ and an isomorphism $\widehat{\mathcal{E}}_0 \simeq \mathcal{C}_0 \times_{\mathcal{C}} \widehat{\mathcal{E}}$ which carries H_0 to the map $(U_0, H) : \mathcal{E}_0 \to \mathcal{C}_0 \times_{\mathcal{C}} \widehat{\mathcal{E}}$.

Proof. Using Proposition 1.1.4.12, we can reduce to the case where $\mathcal{C} = \Delta^n$ is a standard simplex and $\mathcal{C}_0 = \partial \Delta^n$ is its boundary. In this case, U is an isofibration of ∞-categories

(Example 4.4.1.6). Applying Lemma 9.3.4.3, we deduce that there exists a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H'"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}}' \arrow[dl, "\widehat{U}'"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}'$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. Set $\widehat{\mathcal{E}}'_0 = \mathcal{C}_0 \times_{\mathcal{C}} \widehat{\mathcal{E}}$, so that $H'_0 = H'|_{\mathcal{E}_0}$ exhibits $\widehat{\mathcal{E}}'_0$ as a fiberwise $\mathbb{K}$-completion of $\mathcal{E}_0$. Applying Remark 9.3.1.21, we deduce that there exists an isomorphism $H'_0 \simeq F_0 \circ H_0$ in the ∞-category $\mathrm{Fun}_{/\mathcal{C}_0}(\mathcal{E}_0, \widehat{\mathcal{E}}'_0)$, where $F_0 : \widehat{\mathcal{E}}_0 \to \widehat{\mathcal{E}}'_0$ is an equivalence of inner fibrations over $\mathcal{C}_0$. Using Lemma 5.6.7.1, we can choose an inner fibration $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$ and an equivalence $F : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}'$ satisfying $F_0 = F|_{\widehat{\mathcal{E}}_0}$. Composition with F induces an equivalence of ∞-categories $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \widehat{\mathcal{E}}) \to \mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \widehat{\mathcal{E}}')$. We can therefore choose a functor $H : \mathcal{E} \to \widehat{\mathcal{E}}$ such that $F \circ H$ is isomorphic to H' in the ∞-category $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \widehat{\mathcal{E}}')$. It follows that $F_0 \circ H|_{\mathcal{E}_0}$ is isomorphic to $F_0 \circ H_0$ as objects of the ∞-category $\mathrm{Fun}_{/\mathcal{C}_0}(\mathcal{E}_0, \widehat{\mathcal{E}}'_0)$. Since F_0 is an equivalence of inner fibrations over $\mathcal{C}_0$, it follows that H_0 and $H|_{\mathcal{E}_0}$ are isomorphic as objects of the ∞-category $\mathrm{Fun}_{/\mathcal{C}_0}(\mathcal{E}_0, \widehat{\mathcal{E}}_0)$. Since the restriction functor $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{E}, \widehat{\mathcal{E}}) \to \mathrm{Fun}_{/\mathcal{C}_0}(\mathcal{E}_0, \widehat{\mathcal{E}}_0)$ is an isofibration (Proposition 4.1.4.1), we can arrange (after replacing H by an isomorphic functor if necessary) that $H|_{\mathcal{E}_0} = H_0$. We conclude by observing that the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

exhibits $\widehat{\mathcal{E}}$ as a fiberwise $\mathbb{K}$-cocompletion of $\mathcal{E}$. □

Proof of Theorem 9.3.4.1. Apply Lemma 9.3.4.4 in the special case $\mathcal{C}_0 = \emptyset$. □

9.3.5 Digression: Morita Equivalence

Recall that a morphism of simplicial sets $F : \mathcal{C} \to \mathcal{D}$ is a *categorical equivalence* if, for every ∞-category $\mathcal{E}$, precomposition with F induces an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ (Definition 4.5.3.1). We now consider a slightly weaker version of this condition. 05M8

05M9 **Definition 9.3.5.1.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. We say that F is a *Morita equivalence* if, for every idempotent complete ∞-category $\mathcal{E}$, precomposition with F induces an equivalence of ∞-categories $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$.

05MA **Example 9.3.5.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. If F is a categorical equivalence, then it is a Morita equivalence. Beware that the converse is false in general.

05MB **Example 9.3.5.3.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which exhibits $\mathcal{D}$ as an idempotent completion of $\mathcal{C}$. Then F is a Morita equivalence (Proposition 8.5.5.2).

05MC **Remark 9.3.5.4.** Let $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ be a monomorphism of simplicial sets and let $\mathcal{E}$ be an idempotent complete ∞-category. If ι is a Morita equivalence, then the functor $\operatorname{Fun}(\mathcal{C}, \mathcal{E}) \xrightarrow{\circ \iota} \operatorname{Fun}(\mathcal{C}_0, \mathcal{E})$ is both an isofibration (Corollary 4.4.5.3) and an equivalence of ∞-categories, and therefore a trivial Kan fibration (Proposition 4.5.5.20). In particular, every diagram $\mathcal{C}_0 \to \mathcal{E}$ can be extended to $\mathcal{C}$.

05MD **Remark 9.3.5.5.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be morphisms of simplicial sets. If two of the morphisms F, G, and $G \circ F$ are Morita equivalences, then so is the third. In particular, the collection of Morita equivalences is closed under composition.

05ME **Remark 9.3.5.6** (Isomorphism Invariance)**.** Let $\mathcal{C}$ be a simplicial set, let $\mathcal{D}$ be an ∞-category, and suppose we are given a pair of diagrams $F, F' : \mathcal{C} \to \mathcal{D}$ which are isomorphic (as objects of the ∞-category $\operatorname{Fun}(\mathcal{C}, \mathcal{D})$). Then F is a Morita equivalence if and only if F' is a Morita equivalence.

05MF **Remark 9.3.5.7.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then F is a Morita equivalence if and only if it satisfies the following condition:

$(*)$ For every idempotent complete ∞-category $\mathcal{E}$, precomposition with F induces a bijection of sets
$$\pi_0(\operatorname{Fun}(\mathcal{D}, \mathcal{E})^{\simeq}) \to \pi_0(\operatorname{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}).$$

The necessity of condition $(*)$ is immediate. Conversely, suppose that condition $(*)$ is satisfied, and let $\mathcal{E}$ be an idempotent complete ∞-category; we wish to show that the functor $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \xrightarrow{\circ F} \operatorname{Fun}(\mathcal{C}, \mathcal{E})$ is an equivalence of ∞-categories. By virtue of Proposition 4.5.1.22, it will suffice to show that for every simplicial set K, the induced map
$$\pi_0(\operatorname{Fun}(K, \operatorname{Fun}(\mathcal{D}, \mathcal{E}))^{\simeq}) \to \pi_0(\operatorname{Fun}(K, \operatorname{Fun}(\mathcal{C}, \mathcal{E})^{\simeq})$$
is a bijection. This follows by applying condition $(*)$ to the ∞-category $\operatorname{Fun}(K, \mathcal{E})$ (which is idempotent complete by virtue of Corollary 8.5.4.10).

05MG **Proposition 9.3.5.8.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then F is a Morita equivalence if and only if it satisfies the following pair of conditions:*

(a) *The functor F is fully faithful.*

(b) *For every object $Y \in \mathcal{D}$, there exists an object $X \in \mathcal{C}$ such that Y is a retract of $F(X)$.*

Proof. Using Corollary 8.5.5.3, we can choose a functor $H : \mathcal{D} \to \widehat{\mathcal{D}}$ which exhibits $\widehat{\mathcal{D}}$ as an idempotent completion of $\mathcal{D}$. Then H is a Morita equivalence (Example 9.3.5.3). By virtue of Remark 9.3.5.5, we can replace F by the composite functor $H \circ F$ and thereby reduce to proving Proposition 9.3.5.8 in the special case where $\mathcal{D}$ is idempotent complete. In this case, the desired result is a reformulation of Proposition 8.5.5.2. $\square$

Corollary 9.3.5.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. Then F is a categorical equivalence if and only if it is a Morita equivalence and the induced map of homotopy categories $\mathrm{h}F : \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is essentially surjective.* `05MH`

Proof. Using Remark 9.3.5.5 (and Proposition 4.1.3.2), we can reduce to the case where $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, in which case the desired result follows from the criterion of Proposition 9.3.5.8. $\square$

Proposition 9.3.5.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets. The following conditions are equivalent:* `05MJ`

(1) *The morphism F is a Morita equivalence.*

(2) *The morphism F^{op} is a Morita equivalence.*

(3) *For every uncountable cardinal κ, precomposition with F induces an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.*

(4) *There exists an uncountable regular cardinal κ such that $\mathcal{C}$ and $\mathcal{D}$ are essentially κ-small and precomposition with F induces an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are immediate. We complete the proof by showing that (4) implies (1). Without loss of generality, we may assume that $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories. Fix a regular cardinal κ such that $\mathcal{C}$ and $\mathcal{D}$ are essentially κ-small and assume that precomposition with F induces an equivalence of ∞-categories

$$G : \mathrm{Fun}(\mathcal{D}, \mathcal{S}^{<\kappa}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{S}^{<\kappa}).$$

Let $\widehat{\mathcal{C}} \subseteq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa})$ be the full subcategory spanned by the atomic object (see Proposition 8.5.5.5) and define $\widehat{\mathcal{D}} \subseteq \mathrm{Fun}(\mathcal{D}, \mathcal{S}^{<\kappa})$ similarly. Then G induces an equivalence of ∞-categories $\widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$, which admits a homotopy inverse $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$. It follows from Example

8.4.4.5 that the diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "h^{\bullet}_{\mathcal{C}}"] & \mathcal{D} \arrow[d, "h^{\bullet}_{\mathcal{D}}"] \\
\widehat{\mathcal{C}} \arrow[r, "\widehat{F}", "\sim"'] & \widehat{\mathcal{D}}
\end{tikzcd}$$

commutes up to isomorphism. The vertical maps exhibit $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ as idempotent completions of $\mathcal{C}$ and $\mathcal{D}$, respectively (Proposition 8.5.5.5), and are therefore Morita equivalences (Example 9.3.5.3). Combining this observation with Remarks 9.3.5.6 and 9.3.5.5, we conclude that F is a Morita equivalence. □

05MK **Proposition 9.3.5.11.** *Suppose we are given a commutative diagram of simplicial sets*

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "F"] \arrow[d, "U"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C} \arrow[r, "\overline{F}"] & \mathcal{C}'
\end{tikzcd}$$

with the following properties:

(1) *The morphisms U and U' are cartesian fibrations.*

(2) *The morphism F carries U-cartesian edges of $\mathcal{E}$ to U'-cartesian edges of $\mathcal{E}'$.*

(3) *The morphism $\overline{F}$ is a Morita equivalence of simplicial sets*

(4) *For each vertex $C \in \mathcal{C}$ having image $C' = \overline{F}(C)$, the functor*

$$F_C : \mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E} \to \{C'\} \times_{\mathcal{C}'} \mathcal{E}' = \mathcal{E}'_{C'}$$

is a Morita equivalence of ∞-categories.

Then F is a Morita equivalence of simplicial sets.

Proof. Using Corollary 5.6.7.3, we can choose a pullback diagram

$$\begin{tikzcd}
\mathcal{E}' \arrow[r, "G"] \arrow[d, "U'"] & \mathcal{E}'' \arrow[d, "U''"] \\
\mathcal{C}' \arrow[r, "\overline{G}"] & \mathcal{C}'',
\end{tikzcd}$$

where U'' is a cartesian fibration, $\overline{G}$ is inner anodyne, and $\mathcal{C}''$ is an ∞-category. It follows from Corollary 5.6.7.6 that G is a categorical equivalence of simplicial sets; in particular, it is a Morita equivalence (Example 9.3.5.2). Consequently, to show that F is a Morita equivalence, it will suffice to show that the composite map $G \circ F$ is a Morita equivalence (Remark 9.3.5.5). We may therefore replace U' by U'', and thereby reduce to proving Proposition 9.3.5.11 in the special case where $\mathcal{C}'$ is an ∞-category. Similarly, we may assume that $\mathcal{C}$ is an ∞-category. To complete the proof, it will suffice to show that the functor $F : \mathcal{E} \to \mathcal{E}'$ satisfies the conditions of Proposition 9.3.5.8:

(a) Since $\overline{F}$ is a Morita equivalence of ∞-categories, it is fully faithful. Similarly, for each object $C \in \mathcal{C}$ having image $C' = \overline{F}(C)$, condition (4) guarantees that the functor $F_C : \mathcal{E}_C \to \mathcal{E}'_{C'}$ is fully faithful. Using condition (2) and Proposition 5.1.6.7, we conclude that F is fully faithful.

(b) Let Y be an object of $\mathcal{E}'$; we wish to show that Y is a retract of $F(X)$, for some object $X \in \mathcal{E}$. Set $\overline{Y} = U'(Y)$. Since $\overline{F}$ is a Morita equivalence, $\overline{Y}$ is a retract of $C' = \overline{F}(C)$, for some object $C \in \mathcal{C}$. Choose a retraction diagram $\overline{\sigma}$:

$$\begin{tikzcd}
 & C' \arrow[dr, "\overline{r}"] & \\
\overline{Y} \arrow[ur, "\overline{i}"] \arrow[rr, "\mathrm{id}"] & & \overline{Y}
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Our assumption that U' is a cartesian fibration guarantees that we can lift $\overline{\sigma}$ to a retraction diagram

$$\begin{tikzcd}
 & X' \arrow[dr, "r"] & \\
Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}"] & & Y
\end{tikzcd}$$

in the ∞-category $\mathcal{E}'$. Since the functor F_C is a Morita equivalence, there exists an object $X \in \mathcal{E}_C$ such that X' is a retract of $F(X)$ in the ∞-category $\mathcal{E}'_{C'}$, and therefore also in the ∞-category $\mathcal{E}'$. Applying Remark 8.5.1.6, we conclude that Y is a retract of $F(X)$.

$\square$

9.3.6 Application: Flat Inner Fibrations

05ML Recall that an inner fibration of simplicial sets $U : \mathcal{E} \to \mathcal{C}$ is *exponentiable* if the pullback functor

$$(\mathrm{Set}_{\Delta})_{/\mathcal{C}} \to (\mathrm{Set}_{\Delta})_{/\mathcal{E}} \qquad \mathcal{C}' \mapsto \mathcal{C}' \times_{\mathcal{C}} \mathcal{E}$$

preserves categorical equivalences of simplicial sets (Definition 4.5.9.10). In this section, we study a weaker version of this condition.

05MM **Definition 9.3.6.1.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. We say that U is *flat* if, for every 2-simplex σ of $\mathcal{C}$, the inclusion map

$$\Lambda^2_1 \times_{\mathcal{C}} \mathcal{E} \hookrightarrow \Delta^2 \times_{\mathcal{C}} \mathcal{E}$$

is a categorical equivalence of simplicial sets.

05MN **Example 9.3.6.2.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. If U is exponentiable, then it is flat.

05MP **Remark 9.3.6.3** (Homotopy Invariance)**.** Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where the vertical maps are inner fibrations and F is an equivalence of inner fibrations over $\mathcal{C}$ (see Definition 5.1.7.1). Then U is flat if and only if U' is flat.

05MQ **Remark 9.3.6.4.** Suppose we are given a pullback diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "U'"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}' \arrow[r, "F"] & \mathcal{C}.
\end{tikzcd}$$

If U is a flat inner fibration, then U' is a flat inner fibration. The converse holds if F is surjective on 2-simplices.

05MR **Remark 9.3.6.5.** Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibraiton of simplicial sets. Then U is flat if and only if the opposite inner fibration $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is flat.

Our first goal in this section is to establish a weak converse to Example 9.3.6.2.

Theorem 9.3.6.6. *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of simplicial sets. Then, for every Morita equivalence of simplicial sets $\mathcal{C}' \to \mathcal{C}$, the projection map $\mathcal{C}' \times_{\mathcal{C}} \mathcal{E} \to \mathcal{E}$ is also a Morita equivalence.* 05MS

Corollary 9.3.6.7. *Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. The following conditions are equivalent:* 05MT

(1) *The pullback functor*

$$(\mathrm{Set}_{\Delta})_{/\mathcal{C}} \to (\mathrm{Set}_{\Delta})_{/\mathcal{E}} \qquad \mathcal{C}' \mapsto \mathcal{C}' \times_{\mathcal{C}} \mathcal{E}$$

preserves Morita equivalences of simplicial sets. That is, if $F : \mathcal{C}'' \to \mathcal{C}'$ is a Morita equivalence in $(\mathrm{Set}_{\Delta})_{/\mathcal{C}}$, then the induced map $F_{\mathcal{E}} : \mathcal{C}'' \times_{\mathcal{C}} \mathcal{E} \to \mathcal{C}' \times_{\mathcal{C}} \mathcal{E}$ is also a Morita equivalence.

(2) *If $F : \mathcal{C}'' \to \mathcal{C}'$ is a categorical equivalence in $(\mathrm{Set}_{\Delta})_{/\mathcal{C}}$, then $F_{\mathcal{E}}$ is a Morita equivalence.*

(3) *If $F : \mathcal{C}'' \to \mathcal{C}'$ is a categorical equivalence in $(\mathrm{Set}_{\Delta})_{/\mathcal{C}}$ which is surjective on vertices, then $F_{\mathcal{E}}$ is a categorical equivalence.*

(4) *The inner fibration U is flat.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are immediate from the definitions (see Corollary 9.3.5.9), and the implication (4) ⇒ (1) is a reformulation of Theorem 9.3.6.6. □

Corollary 9.3.6.8. *Let $V : \mathcal{E} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ be flat inner fibrations of simplicial sets. Then the composite map $(U \circ V) : \mathcal{E} \to \mathcal{C}$ is also a flat inner fibration.* 05MU

Our proof of Theorem 9.3.6.6 will make use of the following flatness criterion:

Proposition 9.3.6.9. *Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets which is essentially κ-small, and suppose we are given a commutative diagram* 05MV

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$. Then U is flat if and only if $\widehat{U}$ is a cartesian fibration.

05MW **Example 9.3.6.10.** Suppose we are given a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where F is fully faithful and induces a Morita equivalence $F_C : \mathcal{E}_C \to \mathcal{E}'_C$, for each object $C \in \mathcal{C}$. If κ is an uncountable regular cardinal such that U' is essentially κ-small, then any fiberwise κ-cocompletion of $\mathcal{E}'$ can also be regarded as a fiberwise κ-cocompletion of $\mathcal{E}$ (see Proposition 9.3.5.10). Applying the criterion of Proposition 9.3.6.9, we conclude that U is flat if and only if U' is flat. In particular, an inner fibration $U : \mathcal{E} \to \mathcal{C}$ is flat if and only if its fiberwise idempotent completion is flat.

The proof of Proposition 9.3.6.9 will require some preliminaries.

05MX **Lemma 9.3.6.11.** *Let K be a simplicial set and let $\widehat{U} : \widehat{\mathcal{E}} \to \mathcal{C}$ be a cartesian fibration of simplicial sets having the property that, for each vertex $C \in \mathcal{C}$, the ∞-category $\widehat{\mathcal{E}}_C = \{C\} \times_{\mathcal{C}} \widehat{\mathcal{E}}$ admits K-indexed colimits. Let $\mathcal{D}$ be an ∞-category which admits K-indexed colimits, let $F : \mathcal{E} \to \mathcal{D}$ be a diagram, and let S be the collection of vertices $C \in \mathcal{C}$ for which the functor $F_C = F|_{\mathcal{E}_C}$ preserves K-indexed colimits. Then S is closed under retracts in the homotopy category $\mathrm{h}\mathcal{C}$.*

Proof. Using Corollary 5.6.7.3, we can reduce to the case where $\mathcal{C}$ is an ∞-category. Fix objects $C, C' \in \mathcal{C}$ such that C is a retract of C', and assume that $F_{C'}$ preserves K-indexed colimits; we wish to show that F_C also preserves K-indexed colimits. Choose a colimit diagram $\overline{u} : K^{\triangleright} \to \widehat{\mathcal{E}}_C$; we will show that $F_C \circ \overline{u}$ is a colimit diagram in the ∞-category $\mathcal{D}$. Set $u = \overline{u}|_K$. It follows from Theorem 5.2.1.1 that $\widehat{U}$ induces a cartesian fibration $\mathrm{Fun}(K, \widehat{\mathcal{E}}) \to \mathrm{Fun}(K, \mathcal{C})$. Applying Remark 8.5.1.23, we deduce that there is a diagram $u' : K \to \widehat{\mathcal{E}}_{C'}$ having the property that u is a retract of u' in the ∞-category $\mathrm{Fun}(K, \widehat{\mathcal{E}})$. Since $\widehat{\mathcal{E}}_{C'}$ admits K-indexed colimits, we can extend u' to a colimit diagram $\overline{u}' : K^{\triangleright} \to \widehat{\mathcal{E}}_{C'}$. Since $\widehat{U}$ is a cartesian fibration, $\overline{u}$ and $\overline{u}'$ are $\widehat{U}$-colimit diagrams in the ∞-category $\widehat{\mathcal{E}}$ (Corollary 7.3.3.23). Using Theorem 7.3.6.14, we see that any diagram which exhibits u as a retract of u' can be extended to a diagram which exhibits $\overline{u}$ as a retract of $\overline{u}'$. It follows that $F_C \circ \overline{u}$ is a retract of $F_{C'} \circ \overline{u}'$ (in the ∞-category $\mathrm{Fun}(K^{\triangleright}, \mathcal{D})$). Consequently, to show that $F_C \circ \overline{u}$ is a colimit diagram in $\mathcal{D}$, it will suffice to show that $F_{C'} \circ \overline{u}'$ is a colimit diagram in $\mathcal{D}$ (Corollary 8.5.1.12). This follows from our assumption that $F_{C'}$ preserves K-indexed colimits. □

Lemma 9.3.6.12. *Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets which is essentially κ-small, and suppose we are given a commutative diagram* 05MY

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$. Let $\overline{F} : \mathcal{C}' \to \mathcal{C}$ be a Morita equivalence of simplicial sets. If $\widehat{U}$ is a cartesian fibration, then the projection map $\mathcal{C}' \times_{\mathcal{C}} \mathcal{E} \to \mathcal{E}$ is also a Morita equivalence of simplicial sets.

Proof. Set $\mathcal{E}' = \mathcal{C}' \times_{\mathcal{C}} \mathcal{E}$ and $\widehat{\mathcal{E}}' = \mathcal{C}' \times_{\mathcal{C}} \widehat{\mathcal{E}}$, so that $\overline{F}$ induces projection maps $F : \mathcal{E}' \to \mathcal{E}$ and $\widehat{F} : \widehat{\mathcal{E}}' \to \widehat{\mathcal{E}}$. We wish to show that, for every idempotent complete ∞-category $\mathcal{D}$, precomposition with F induces an equivalence of ∞-categories $\mathrm{Fun}(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}', \mathcal{D})$. By virtue of Proposition 9.3.5.10, we may assume without loss of generality that $\mathcal{D}$ is κ-cocomplete (in fact, we can assume that $\mathcal{D} = \mathcal{S}^{<\lambda}$ for some uncountable regular cardinal $\lambda \geq \kappa$). Let $\mathrm{Fun}^{\kappa}(\widehat{\mathcal{E}}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\widehat{\mathcal{E}}, \mathcal{D})$ spanned by those diagrams $G : \widehat{\mathcal{E}} \to \mathcal{D}$ having the property that, for each vertex $C \in \mathcal{C}$, the functor $G_C = G|_{\widehat{\mathcal{E}}_C}$ preserves κ-small colimits, and define $\mathrm{Fun}^{\kappa}(\widehat{\mathcal{E}}', \mathcal{D}) \subseteq \mathrm{Fun}(\widehat{\mathcal{E}}, \mathcal{D})$ similarly. We then have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}^{\kappa}(\widehat{\mathcal{E}}, \mathcal{D}) \arrow[r, "\circ \widehat{F}"] \arrow[d] & \mathrm{Fun}^{\kappa}(\widehat{\mathcal{E}}', \mathcal{D}) \arrow[d] \\
\mathrm{Fun}(\mathcal{E}, \mathcal{D}) \arrow[r, "\circ F"] & \mathrm{Fun}(\mathcal{E}', \mathcal{D}),
\end{tikzcd}$$

where the vertical maps are equivalences by virtue of the universal property of fiberwise κ-cocompletions (Theorem 9.3.1.20). It will therefore suffice to show that precomposition with $\widehat{F}$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\kappa}(\widehat{\mathcal{E}}, \mathcal{D}) \to \mathrm{Fun}^{\kappa}(\widehat{\mathcal{E}}', \mathcal{D})$. Since $\overline{F}$ is a Morita equivalence and $\widehat{U}$ is a cartesian fibration, the morphism $\widehat{F}$ is also a Morita equivalence (Proposition 9.3.5.11). In particular, precomposition with $\widehat{F}$ induces an equivalence of ∞-categories $\mathrm{Fun}(\widehat{\mathcal{E}}, \mathcal{D}) \to \mathrm{Fun}(\widehat{\mathcal{E}}', \mathcal{D})$. To complete the proof, it will suffice to show that an object $G \in \mathrm{Fun}(\widehat{\mathcal{E}}, \mathcal{D})$ belongs to the subcategory $\mathrm{Fun}^{\kappa}(\widehat{\mathcal{E}}, \mathcal{D})$ if and only if $G \circ \widehat{F}$ belongs to $\mathrm{Fun}^{\kappa}(\widehat{\mathcal{E}}', \mathcal{D})$. This follows from Lemma 9.3.6.11. □

Lemma 9.3.6.13. *Let κ be an uncountable regular cardinal, let $\mathcal{E}$ be an ∞-category which* 05MZ *is essentially κ-small, and suppose we are given a commutative diagram of ∞-categories*

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \Delta^2 &
\end{tikzcd}$$

which exhibits $\widehat{E}$ as a fiberwise κ-cocompletion of $\mathcal{E}$. The following conditions are equivalent:

(1) *The inclusion map $\Lambda^2_1 \times_{\Delta^2} \mathcal{E} \hookrightarrow \mathcal{E}$ is a categorical equivalence of simplicial sets.*

(2) *The functor $\widehat{U}$ is a cartesian fibration.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Lemma 9.3.6.12 and Corollary 9.3.5.9. Conversely, suppose that condition (1) is satisfied. Set $\mathcal{E}_0 = \Lambda^2_1 \times_{\Delta^2} \mathcal{E}$ and $\widehat{\mathcal{E}}_0 = \Lambda^2_1 \times_{\Delta^2} \widehat{\mathcal{E}}$. Using Proposition 5.6.7.2, we can choose a pullback diagram

$$\begin{tikzcd}
\widehat{\mathcal{E}}_0 \arrow[r] \arrow[d, "\widehat{U}_0"] & \widehat{\mathcal{E}}' \arrow[d, "\widehat{U}'"] \\
\Lambda^2_1 \arrow[r] & \Delta^2,
\end{tikzcd}$$

where $\widehat{U}'$ is a cartesian fibration. Let us abuse notation by identifying $\widehat{\mathcal{E}}_0$ with its image in $\widehat{\mathcal{E}}'$. It follows from Proposition 5.3.6.1 that the inclusion map $\widehat{\mathcal{E}}_0 \hookrightarrow \widehat{\mathcal{E}}'$ is a categorical equivalence of simplicial sets, so there exists a functor $F : \widehat{\mathcal{E}}' \to \widehat{\mathcal{E}}$ which is the identity on $\widehat{\mathcal{E}}_0$. We will complete the proof by showing that F is an equivalence of inner fibrations over Δ^2, so that $\widehat{U}$ is also a cartesian fibration (Proposition 5.1.7.14).

Note that the inner $\widehat{U}$ and $\widehat{U}'$ are κ-cocomplete. It will therefore suffice to show that, for every κ-cocomplete inner fibration $V : \mathcal{D} \to \Delta^2$, precomposition with F induces an equivalence of ∞-categories

$$\mathrm{Fun}^{\kappa}_{/\Delta^2}(\widehat{\mathcal{E}}, \mathcal{D}) \to \mathrm{Fun}^{\kappa}_{/\Delta^2}(\widehat{\mathcal{E}}', \mathcal{D}).$$

Since the inclusion $\widehat{\mathcal{E}}_0 \hookrightarrow \widehat{\mathcal{E}}'$ is a categorical equivalence, this is equivalent to the statement that the restriction map

$$\mathrm{Fun}^{\kappa}_{/\Delta^2}(\widehat{\mathcal{E}}, \mathcal{D}) \to \mathrm{Fun}^{\kappa}_{/\Lambda^2_1}(\widehat{\mathcal{E}}_0, \mathcal{D}_0)$$

is an equivalence, where $\mathcal{D}_0 = \Lambda^2_1 \times_{\Delta^2} \mathcal{D}$. Invoking the universal property of fiberwise cocompletion (Theorem 9.3.1.20), we are reduced to showing that the restriction map $\mathrm{Fun}_{/\Delta^2}(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}_{/\Delta^2}(\Lambda^2_1 \times_{\Delta^2}, \mathcal{E}, \mathcal{D})$ is an equivalence of ∞-categories, which follows immediately from assumption (1). $\square$

Proof of Proposition 9.3.6.9. Let $U : \mathcal{E} \to \mathcal{C}$ be an essentially κ-small inner fibration and let

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

be a diagram which exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$. It follows from Corollary 5.1.5.11 that $\widehat{U}$ is a cartesian fibration if and only if, for every 2-simplex σ of $\mathcal{C}$, the projection map $\widehat{U}_{\sigma} : \Delta^2 \times_{\mathcal{C}} \widehat{\mathcal{E}} \to \Delta^2$ is a cartesian fibration. By virtue of Lemma 9.3.6.13, this is equivalent to the requirement that U is flat. □

Proof of Theorem 9.3.6.6. Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of simplicial sets and let $\mathcal{C}' \to \mathcal{C}$ be a Morita equivalence; we wish to show that the projection map $\mathcal{C}' \times_{\mathcal{C}} \mathcal{E} \to \mathcal{E}$ is also a Morita equivalence. Choose an uncountable regular cardinal κ such that U is essentially κ-small. Using Theorem 9.3.4.1, we can choose a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \Delta^1 &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$. By virtue of Lemma 9.3.6.12, it will suffice to show that $\widehat{U}$ is a cartesian fibration, which is a reformulation of our assumption that U is flat (Proposition 9.3.6.9). □

We record another consequence of Proposition 9.3.6.9.

Corollary 9.3.6.14. *Every functor of ∞-categories $U : \mathcal{E} \to \Delta^1$ is a flat inner fibration.* 05N0

Proof. Choose an uncountable regular cardinal κ for which $\mathcal{E}$ is essentially κ-small. Using Theorem 9.3.4.1, we can choose a diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \Delta^1 &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$. It follows from Proposition 9.3.1.11 that $\widehat{U}$ is a cartesian fibration, so that U is flat by virtue of Proposition 9.3.6.9. □

05N1 **Remark 9.3.6.15.** By definition, an inner fibration of simplicial sets $U : \mathcal{E} \to \mathcal{C}$ is flat if, for every 2-simplex σ of $\mathcal{C}$, the inclusion map

$$\Lambda^2_1 \times_{\mathcal{C}} \mathcal{E} \hookrightarrow \Delta^2 \times_{\mathcal{C}} \mathcal{E}$$

is a categorical equivalence of simplicial sets. It follows from Corollary 9.3.6.14 that it suffices to check this condition in the case where σ is nondegenerate.

05N2 **Corollary 9.3.6.16.** *A functor of ∞-categories $U : \mathcal{E} \to \Delta^2$ is a flat inner fibration if and only if the inclusion map $\Lambda^2_1 \times_{\Delta^2} \mathcal{E} \hookrightarrow \mathcal{E}$ is a categorical equivalence of simplicial sets.*

05N3 **Proposition 9.3.6.17.** *Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of ∞-categories. Then U is exponentiable if and only if it is a flat isofibration.*

Proof. Assume that U is a flat isofibration; we will show that U is exponentiable (the converse follows from Remark 4.5.9.12 and Example 9.3.6.2). Suppose we are given a commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathcal{E}'' \arrow[r, "F"] \arrow[d] & \mathcal{E}_0 \arrow[r, "G"] \arrow[d] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}_1 \arrow[r, "\overline{F}"] & \mathcal{C}_0 \arrow[r, "\overline{G}"] & \mathcal{C}
\end{tikzcd}$$

in which both squares are pullbacks, where $\overline{F}$ is a categorical equivalence; we wish to show that F is also a categorical equivalence. Using Proposition 4.1.3.2, we can factor $\overline{G}$ as a composition $\mathcal{C}_0 \xrightarrow{\iota_0} \mathcal{C}_0^+ \xrightarrow{V_0} \mathcal{C}$, where ι_0 is inner anodyne and V_0 is an inner fibration. In particular, $\mathcal{C}_0^+$ is an ∞-category. Replacing $\mathcal{C}$ by $\mathcal{C}_0^+$ (and U by the projection map $\mathcal{C}_0^+ \times_{\mathcal{C}} \mathcal{E} \to \mathcal{C}_0^+$), we can assume that $\overline{G}$ is inner anodyne. In this case, Corollary 9.3.6.7 guarantees that G is a categorical equivalence of simplicial sets. It will therefore suffice to show that the composition $(G \circ F) : \mathcal{E}_1 \to \mathcal{E}$ is a categorical equivalence. Applying Proposition 4.1.3.2 again, we can factor $\overline{G} \circ \overline{F}$ as a composition $\mathcal{C}_1 \xrightarrow{\iota_1} \mathcal{C}_1^+ \xrightarrow{V_1} \mathcal{C}$, where ι_1 is inner anodyne and V_1 is an inner fibration. Applying Corollary 9.3.6.7 again, we conclude that the inclusion map $\mathcal{E}_1 \hookrightarrow \mathcal{C}_1^+ \times_{\mathcal{C}} \mathcal{E}$ is a categorical equivalence of simplicial sets. We are therefore reduced to showing that the projection map $\mathcal{C}_1^+ \times_{\mathcal{C}} \mathcal{E} \to \mathcal{E}$ is an equivalence of ∞-categories. Note that $\overline{F}$, $\overline{G}$, and ι_1 are categorical equivalences of simplicial sets, so the inner fibration V_1 is an equivalence of ∞-categories. The desired result now follows from Corollary 4.5.2.29 (since U is an isofibration). □

05N4 **Warning 9.3.6.18.** In the formulation of Proposition 9.3.6.17, the assumption that U is an isofibration cannot be omitted. For example, let $\mathcal{C}$ be a contractible Kan complex containing two vertices C_0 and C_1, let $\mathcal{D}_1$ be an ∞-category which is the idempotent completion of a full subcategory $\mathcal{D}_0 \subseteq \mathcal{D}_1$, and let $\mathcal{E} \subseteq \mathcal{C} \times \mathcal{D}_1$ be the full subcategory spanned by those objects (C_i, D) where D is contained in $\mathcal{D}_i$. It follows from Example 9.3.6.10 that the projection map $U : \mathcal{E} \to \mathcal{C}$ is a flat inner fibration. However, if $\mathcal{D}_0$ is not idempotent complete, then U is not an isofibration.

Corollary 9.3.6.19. *Let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets. Then U is flat if and only if, for every n-simplex $\sigma : \Delta^n \to \mathcal{C}$, the projection map $\Delta^n \times_{\mathcal{C}} \mathcal{E} \to \Delta^n$ is exponentiable. Moreover, it suffices to verify this condition in the case $n = 2$.* 05N5

Proof. Combine Proposition 9.3.6.17 with Example 4.4.1.6. □

We now show that the counterexample of Warning 9.3.6.18 is essentially the only way that a flat inner fibration can fail to be exponentiable.

Lemma 9.3.6.20. *Suppose we are given a commutative diagram of ∞-categories* 05N6

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where U is a flat inner fibration, U' is an inner fibration, and F is an equivalence of ∞-categories. Then:

(1) *For every object $C \in \mathcal{C}$, the induced map of fibers $F_C : \mathcal{E}_C \to \mathcal{E}'_C$ is a Morita equivalence.*

(2) *The inner fibration U' is flat.*

Note that, in the statement of Lemma 9.3.6.20, we do *not* assume that F is an equivalence of inner fibrations over $\mathcal{C}$ (otherwise, there would be nothing to prove).

Proof of Lemma 9.3.6.20. We will prove (1); assertion (2) then follows from Example 9.3.6.10. Using Corollary 4.5.2.23, we can factor the inclusion map $\{C\} \hookrightarrow \mathcal{C}$ as a composition $\{C\} \hookrightarrow \widetilde{\mathcal{C}} \xrightarrow{V} \mathcal{C}$, where V is an isofibration and $\widetilde{\mathcal{C}}$ is a contractible Kan complex. Set $\widetilde{\mathcal{E}} = \widetilde{\mathcal{C}} \times_{\mathcal{C}} \mathcal{E}$ and $\widetilde{\mathcal{E}}' = \widetilde{\mathcal{C}} \times_{\mathcal{C}} \mathcal{E}'$, so that we have a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E}_C \arrow[r, "F_C"] \arrow[d] & \mathcal{E}'_C \arrow[d] \\
\widetilde{\mathcal{E}} \arrow[r, "\widetilde{F}"] & \widetilde{\mathcal{E}}'.
\end{tikzcd}$$

It follows from Theorem 9.3.6.6 that the vertical maps are Morita equivalences, and from Corollary 4.5.2.29 that $\widetilde{F}$ is an equivalence of ∞-categories. Applying Remark 9.3.5.5, we conclude that F_C is a Morita equivalence. □

05N7 **Warning 9.3.6.21.** Suppose we are given a commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{E}' \arrow[dl, "U'"] \\
& \mathcal{C}, &
\end{tikzcd}$$

where U and U' are inner fibrations and F is an equivalence of ∞-categories (but not necessarily an equivalence of inner fibrations over $\mathcal{C}$). Lemma 9.3.6.20 asserts that if U is flat, then U' is also flat. Beware that the converse generally does not hold: if U' is flat, then U need not be flat. For example, suppose that U' is an isomorphism and that $F : \mathcal{E} \hookrightarrow \mathcal{E}'$ is an isomorphism from $\mathcal{E}$ to a full subcategory of $\mathcal{E}'$. If F is essentially surjective, then it is an equivalence of ∞-categories. In this case, the inner fibration $U = U' \circ F$ is flat if and only if F is an isomorphism.

05N8 **Proposition 9.3.6.22.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of ∞-categories. Then U factors as a composition $\mathcal{E} \xrightarrow{F} \mathcal{E}' \xrightarrow{U'} \mathcal{C}$, where F is an equivalence of ∞-categories and U' is a flat isofibration. Moreover, for each object $C \in \mathcal{C}$, the induced map $F_C : \mathcal{E}_C \to \mathcal{E}'_C$ is a Morita equivalence.*

Proof. Combine Corollary 4.5.2.23 with Lemma 9.3.6.20. □

05N9 **Corollary 9.3.6.23.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of ∞-categories. Assume that, for each vertex $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C = \{C\} \times_{\mathcal{C}} \mathcal{E}$ is idempotent complete. Then U is an isofibration. In particular, U is exponentiable.*

Proof. Using Proposition 9.3.6.22, we can factor U as a composition $\mathcal{E} \xrightarrow{F} \mathcal{E}' \xrightarrow{U'} \mathcal{C}$, where F is an equivalence of ∞-categories and U' is an isofibration. For each object $C \in \mathcal{C}$, Lemma 9.3.6.20 guarantees that the induced map $F_C : \mathcal{E}_C \to \mathcal{E}'_C$ is a Morita equivalence; in particular, every object of $\mathcal{E}'_C$ is a retract of $F(X)$, for some object $X \in \mathcal{E}_C$ (see Proposition 9.3.5.8). Since $\mathcal{E}_C$ is idempotent complete, it follows that F_C is an equivalence of ∞-categories. Applying Corollary 5.1.7.10, we conclude that F is an equivalence of inner fibrations over $\mathcal{C}$, so that U is also an isofibration (Proposition 5.1.7.14). □

We now extend some of the preceding results to the case of inner fibrations between simplicial sets.

Proposition 9.3.6.24. *Let $\mathcal{C}$ be a simplicial set, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a simplicial subset which contains every vertex of $\mathcal{C}$, and let $U_0 : \mathcal{E}_0 \to \mathcal{C}_0$ be a flat inner fibration. If the inclusion map $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ is inner anodyne, then there exists a pullback diagram* 05NA

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d, "U_0"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}_0 \arrow[r] & \mathcal{C},
\end{tikzcd}$$

where U is a flat inner fibration.

Proof. Choose an uncountable regular cardinal κ for which the inner fibration U_0 is essentially κ-small. Using Theorem 9.3.4.1, we can choose a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[rr, "H_0"] \arrow[dr, "U_0"'] & & \widehat{\mathcal{E}}_0 \arrow[dl, "\widehat{U}_0"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}_0$ as a fiberwise κ-cocompletion of $\mathcal{E}_0$. Let $\mathcal{E}'_0$ be the full simplicial subset of $\widehat{\mathcal{E}}_0$ spanned by those vertices which belong to the image of H_0. It follows from Remark 9.3.1.15 that H_0 induces an equivalence $\mathcal{E}_0 \to \mathcal{E}'_0$ of inner fibrations over $\mathcal{C}_0$. Using Lemma 5.6.7.1 (and Remark 9.3.6.3), we can replace $\mathcal{E}_0$ by $\mathcal{E}'_0$ and thereby reduce to the case where H_0 is an isomorphism from $\mathcal{E}_0$ onto a full simplicial subset of $\widehat{\mathcal{E}}_0$.

Since U_0 is flat, the morphism $\widehat{U}_0$ is a cartesian fibration (Proposition 9.3.6.9). Using Proposition 5.6.7.2, we can choose a pullback diagram

$$\begin{tikzcd}
\widehat{\mathcal{E}}_0 \arrow[r] \arrow[d, "\widehat{U}_0"] & \widehat{\mathcal{E}} \arrow[d, "\widehat{U}"] \\
\mathcal{C}_0 \arrow[r, "\iota"] & \mathcal{C}
\end{tikzcd}$$

where $\widehat{U}$ is a cartesian fibration. Let $\mathrm{Tr}_{\widehat{\mathcal{E}}/\mathcal{C}} : \mathrm{h}\mathcal{C}^{\mathrm{op}} \to \mathrm{h}\mathcal{QC}$ be the homotopy transport representation of $\widehat{U}$. Since ι induces an isomorphism of homotopy categories $\mathrm{h}\mathcal{C}_0 \xrightarrow{\sim} \mathrm{h}\mathcal{C}$, Proposition 9.3.1.16 guarantees that the functor $\mathrm{Tr}_{\widehat{\mathcal{E}}/\mathcal{C}}$ carries each object of $\mathrm{h}\mathcal{C}$ to a κ-cocomplete ∞-category and each morphism of $\mathrm{h}\mathcal{C}$ to a functor which preserves κ-small colimits.

Let $\mathcal{E} \subseteq \widehat{\mathcal{E}}$ be the full simplicial subset spanned by those vertices which belong to the image of $\mathcal{E}_0$, and set $U = \widehat{U}|_{\mathcal{E}}$. Applying the criterion of Proposition 9.3.1.16, we see that the diagram

$$\begin{tikzcd}
\mathcal{E} \arrow[rr] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

exhibits $\widehat{\mathcal{E}}$ as a fiberwise κ-cocompletion of $\mathcal{E}$. Since $\widehat{U}$ is a cartesian fibration, Proposition 9.3.6.9 guarantees that the inner fibration U is flat. By construction, the isomorphism $\widehat{\mathcal{E}}_0 \xrightarrow{\sim} \mathcal{C}_0 \times_{\mathcal{C}} \widehat{\mathcal{E}}$ restricts to an isomorphism of $\mathcal{E}_0$ with the fiber product $\mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}$. □

05NB **Exercise 9.3.6.25.** In the special case $\mathcal{C} = \Delta^2$ and $\mathcal{C}_0 = \Lambda^2_1$, Proposition 9.3.6.24 reduces to the assertion that every inner fibration $U_0 : \mathcal{E}_0 \to \Lambda^2_1$ fits into a pullback diagram

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r, "F"] \arrow[d, "U_0"] & \mathcal{E} \arrow[d, "U"] \\
\Lambda^2_1 \arrow[r] & \Delta^2,
\end{tikzcd}$$

where $\mathcal{E}$ is an ∞-category and F is a categorical equivalence of simplicial sets (see Corollary 9.3.6.16). Use the small object argument to give a more direct proof of this statement.

05NC **Corollary 9.3.6.26.** *Let $U_0 : \mathcal{E}_0 \to \mathcal{C}_0$ be a flat inner fibration of simplicial sets. Then there exists a pullback diagram*

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d, "U_0"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}_0 \arrow[r, "\iota"] & \mathcal{C},
\end{tikzcd}$$

where U is a flat inner fibration of ∞-categories. Moreover, we may assume that ι is inner anodyne.

05ND **Corollary 9.3.6.27.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of simplicial sets. Assume that, for each vertex $C \in \mathcal{C}$, the ∞-category $\mathcal{E}_C$ is idempotent complete. Then U is an isofibration.*

Proof. Combine Corollaries 9.3.6.26 and 9.3.6.23. □

Corollary 9.3.6.28. *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of simplicial sets and suppose we are given a digram* 05NE

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "U"'] & & \widehat{\mathcal{E}} \arrow[dl, "\widehat{U}"] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}$ as a fiberwise idempotent completion of $\mathcal{E}$. Then $\widehat{U}$ is a flat isofibration.

Proof. It follows from Example 9.3.6.10 that $\widehat{U}$ is a flat inner fibration. Since the fibers of $\widehat{U}$ are idempotent complete, it is an isofibration (Corollary 9.3.6.27). □

Proposition 9.3.6.29. *Let $U_0 : \mathcal{E}_0 \to \mathcal{C}_0$ be a flat isofibration of simplicial sets. Then there exists a pullback diagram* 05NF

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d, "U_0"] & \mathcal{E} \arrow[d, "U"] \\
\mathcal{C}_0 \arrow[r, "\iota"] & \mathcal{C},
\end{tikzcd}$$

where U is a flat isofibration of ∞-categories. Moreover, we may assume that ι is inner anodyne.

Proof. Using Corollary 9.3.6.26, we can choose an inner anodyne map $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$ and a pullback diagram

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d, "U_0"] & \mathcal{E}' \arrow[d, "U'"] \\
\mathcal{C}_0 \arrow[r, "\iota"] & \mathcal{C},
\end{tikzcd}$$

where U' is a flat isofibration. Using Proposition 9.3.6.22, we can factor U' as a composition $\mathcal{E}' \xrightarrow{F} \mathcal{E}'' \xrightarrow{U''} \mathcal{C}$, where F is an equivalence of ∞-categories and U'' is a flat isofibration. Set $\mathcal{E}''_0 = \mathcal{C}_0 \times_{\mathcal{C}} \mathcal{E}''$, so that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d, "F_0"] & \mathcal{E}' \arrow[d, "F"] \\
\mathcal{E}''_0 \arrow[r] & \mathcal{E}''.
\end{tikzcd}$$

Since U' and U'' are flat, Proposition 9.3.6.6 guarantees that the horizontal maps are categorical equivalences. Since F is an equivalence of ∞-categories, it follows that F_0 is a categorical equivalence of simplicial sets (Remark 4.5.3.5), and therefore an equivalence of isofibrations over $\mathcal{C}_0$ (Proposition 5.1.7.5). Applying Lemma 5.6.7.1, we conclude that there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{E}_0 \arrow[r] \arrow[d] & \mathcal{E} \arrow[r, "G"] \arrow[d, "U"] & \mathcal{E}'' \arrow[d, "U''"] \\
\mathcal{C}_0 \arrow[r, "\iota"] & \mathcal{C} \arrow[r, "\mathrm{id}"] & \mathcal{C},
\end{tikzcd}$$

where the left square is a pullback, the upper horizontal composition is F_0, and G is an equivalence of inner fibrations over $\mathcal{C}$. Since U'' is a flat isofibration, it follows that U is also a flat isofibration (Remark 9.3.6.3 and Proposition 5.1.7.14). □

05NG **Corollary 9.3.6.30.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat isofibration of simplicial sets. Then U is exponentiable.*

Proof. By virtue of Proposition 9.3.6.29, we may assume that $\mathcal{C}$ is an ∞-category. In this case, the result follows from Proposition 9.3.6.17. □

9.3.7 Flatness and Morphism Spaces

05NH Let $\mathcal{E}$ be an ∞-category. For each morphism $f : X \to Z$ of $\mathcal{E}$, we let $\mathcal{E}_{X/\,/Z}$ denote the ∞-category whose objects are diagrams of the form

$$\begin{tikzcd}
 & Y \arrow[dr] & \\
X \arrow[ur] \arrow[rr, "f"] & & Z
\end{tikzcd}$$

(see Notation 4.6.6.1). Our goal in this section is to prove the following:

05NJ **Theorem 9.3.7.1.** *Let $U : \mathcal{E} \to \Delta^2$ be a functor of ∞-categories. Then U is flat if and only if, for every morphism $f : X \to Z$ in $\mathcal{E}$ satisfying $U(X) = 0$ and $U(Z) = 2$, the ∞-category $\{1\} \times_{\Delta^2} \mathcal{E}_{X/\,/Z}$ is weakly contractible.*

Stated more informally, Theorem 9.3.7.1 asserts that a functor $U : \mathcal{E} \to \Delta^2$ is flat if every morphism from an object $X \in \mathcal{E}_0$ to an object $Z \in \mathcal{E}_2$ admits a factorization through an object $Y \in \mathcal{E}_1$, which is unique up to (weakly) contractible choice. To carry out the proof, it will be useful to work with another formulation of this condition.

Lemma 9.3.7.2. *Let κ be an uncountable regular cardinal, let $\mathcal{E}$ be an ∞-category which is essentially κ-small, let X be an object of $\mathcal{E}$, and let $h_X : \mathcal{E} \to \mathcal{S}^{<\kappa}$ be a functor corepresented by X. Let $U : \mathcal{E} \to \Delta^2$ be an inner fibration of ∞-categories, let $\mathcal{E}_{\geq 1}$ be the full subcategory of $\mathcal{E}$ spanned by those objects E satisfying $U(E) \geq 1$, and let $\mathcal{E}_1$ denote the full subcategory spanned by those objects satisfying $U(E) = 1$. If U is flat, then the functor $h_X|_{\mathcal{E}_{\geq 1}}$ is left Kan extended from $\mathcal{E}_1$.* 05NK

Proof. Choose a functor $\mathscr{F}_{\geq 1} : \mathcal{E}_{\geq 1} \to \mathcal{S}$, and set $\mathscr{F}_1 = \mathscr{F}_{\geq 1}|_{\mathcal{E}_1}$. By virtue of Corollary 7.3.6.13, it will suffice to show that the restriction map

$$\theta : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{E}_{\geq 1}, \mathcal{S}^{<\kappa})}(h_X|_{\mathcal{E}_{\geq 1}}, \mathscr{F}_{\geq 1}) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{E}_1, \mathcal{S}^{<\kappa})}(h_X|_{\mathcal{E}_1}, \mathscr{F}_1)$$

is a homotopy equivalence of Kan complexes. Let $\mathcal{E}_{\leq 1}$ be the full subcategory of $\mathcal{E}$ spanned by those objects E satisfying $U(E) \leq 1$, and choose a regular cardinal λ of exponential cofinality $\geq \kappa$. Then the ∞-category $\mathcal{S}^{<\lambda}$ is κ-complete (Example 7.6.7.4). Since $\mathcal{E}_{\leq 1}$ is essentially κ-small, the functor $\mathscr{F}_1$ admits a right Kan extension $\mathscr{F}_{\leq 1} : \mathcal{E}_{\leq 1} \to \mathcal{S}^{<\lambda}$. Our assumption that U is flat guarantees that the restriction functor

$$T : \mathrm{Fun}(\mathcal{E}, \mathcal{S}^{<\lambda}) \to \mathrm{Fun}(\mathcal{E}_{\leq 1}, \mathcal{S}^{<\lambda}) \times_{\mathrm{Fun}(\mathcal{E}_1, \mathcal{S}^{<\lambda})} \mathrm{Fun}(\mathcal{E}_{\geq 1}, \mathcal{S}^{<\lambda})$$

is a trivial Kan fibration, so that we can choose a functor $\mathscr{F} : \mathcal{E} \to \mathcal{S}^{<\lambda}$ satisfying $\mathscr{F}|_{\mathcal{E}_{\leq 1}} = \mathscr{F}_{\leq 1}$ and $\mathscr{F}|_{\mathcal{E}_{\geq 1}} = \mathscr{F}_{\geq 1}$. Since T is a trivial Kan fibration, the diagram of Kan complexes

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{E}, \mathcal{S}^{<\lambda})}(h_X, \mathscr{F}) \arrow[r, "\theta'"] \arrow[d] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{E}_{\leq 1}, \mathcal{S}^{<\lambda})}(h_X|_{\mathcal{E}_{\leq 1}}, \mathscr{F}_{\leq 1}) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{E}_{\geq 1}, \mathcal{S}^{<\lambda})}(h_X|_{\mathcal{E}_{\geq 1}}, \mathscr{F}_{\geq 1}) \arrow[r, "\theta"] & \mathrm{Hom}_{\mathrm{Fun}(\mathcal{E}_1, \mathcal{S}^{<\lambda})}(h_X|_{\mathcal{E}_1}, \mathscr{F}_1).
\end{tikzcd}$$

Our assumption that $\mathscr{F}_{\leq 1}$ is right Kan extended from $\mathscr{F}_1$ guarantees that the right vertical map is a homotopy equivalence (Corollary 7.3.6.13). Consequently, to show that θ is a homotopy equivalence, it will suffice to show that θ' is a homotopy equivalence. This follows from the ∞-categorical version of Yoneda's lemma (Proposition 8.3.1.3): the source and target of θ' can both be identified with the Kan complex $\mathscr{F}(X)$. □

The converse of Lemma 9.3.7.2 is also true:

Lemma 9.3.7.3. *Let $U : \mathcal{E} \to \Delta^2$ and κ be as in Lemma 9.3.7.2. Suppose that, for every object $X \in \mathcal{E}$ satisfying $U(X) = 0$, the functor $h_X|_{\mathcal{E}_{\geq 1}}$ is left Kan extended from $\mathcal{E}_1$. Then U is flat.* 05NL

Proof. Using Proposition 9.3.6.24 (or Exercise 9.3.6.25), we can choose a flat inner fibration $U' : \mathcal{E}' \to \Delta^2$ and an isomorphism $F_0 : \Lambda^2_1 \times_{\Delta^2} \mathcal{E}' \xrightarrow{\sim} \Lambda^2_1 \times_{\Delta^2} \mathcal{E}$ of simplicial sets over Λ^2_1. Since the inclusion map $\Lambda^2_1 \times_{\Delta^2} \mathcal{E}' \hookrightarrow \mathcal{E}'$ is a categorical equivalence of simplicial sets, we can extend F_0 to a functor of ∞-categories $F : \mathcal{E}' \to \mathcal{E}$ (which automatically satisfies $U \circ F = U'$). To prove that U is flat, it will suffice to show that F is an equivalence of inner fibrations over Δ^2 (Remark 9.3.6.3).

By construction, F is bijective on objects. By virtue of Corollary 5.1.7.10, we are reduced to showing that F is fully faithful: that is, for every pair of objects $X', Y' \in \mathcal{E}'$ having images $X = F(X')$ and $Y = F(Y')$, the morphism $\overline{F}$ induces a homotopy equivalence of Kan complexes $\mathrm{Hom}_{\mathcal{E}'}(X', Y') \to \mathrm{Hom}_{\mathcal{E}}(X, Y)$. We may assume that $U(X) = 0$ (otherwise, the result follows immediately from the fact that F is an isomorphism). Choose an uncountable regular cardinal κ such that $\mathcal{E}$ and $\mathcal{E}'$ are essentially κ-small, let $h_X : \mathcal{E} \to \mathcal{S}^{<\kappa}$ be the functor represented by X, and define $h_{X'} : \mathcal{E}' \to \mathcal{S}^{<\kappa}$ similarly. The functor F then induces a natural transformation $\alpha : h_{X'} \to h_X \circ F$, and we wish to show that α is an isomorphism. Let $\mathcal{E}'_{\leq 1}$ denote the full subcategory of $\mathcal{E}'$ spanned by those objects Y satisfying $U(Y) \leq 1$, and define $\mathcal{E}'_1$ and $\mathcal{E}'_{\geq 1}$ similarly. Since F_0 is an isomorphism, the natural transformation α is an isomorphism when restricted to $\mathcal{E}'_{\leq 1}$. It will therefore suffice to show that α is also an isomorphism when restricted $\mathcal{E}'_{\geq 1}$. Our assumption (and the fact that F_0 is an isomorphism) guarantees that $(h_X \circ F)|_{\mathcal{E}'_{\geq 1}}$ is left Kan extended from $\mathcal{E}'_1$. Since U' is flat, Lemma 9.3.7.2 guarantees that the functor $h_{X'}|_{\mathcal{E}'_{\geq 1}}$ is also left Kan extended from $\mathcal{E}'_1$. The desired result now follows from the fact that α is an isomorphism when restricted to $\mathcal{E}'_1$. □

05NM **Proposition 9.3.7.4.** *Let $U : \mathcal{E} \to \Delta^2$ be a functor of ∞-categories. Then U is flat if and only if, for every object $X \in \mathcal{E}$ satisfying $U(X) = 0$, the inclusion functor*

$$\iota_X : \{1\} \times_{\Delta^2} \mathcal{E}_{X/} \hookrightarrow \mathrm{N}_\bullet(\{1 < 2\}) \times_{\Delta^2} \mathcal{E}_{X/}$$

is left cofinal.

Proof. Choose an uncountable regular cardinal κ such that $\mathcal{E}$ is essentially κ-small. Using Lemmas 9.3.7.2 and 9.3.7.3, we see that U is flat if and only if, for every object $X \in \mathcal{E}$ satisfying $U(X) = 0$, the corepresentable functor $h_X : \mathcal{E} \to \mathcal{S}^{<\kappa}$ has the property that $h_X|_{\mathcal{E}_{\geq 1}}$ is left Kan extended from $\mathcal{E}_1$. Since h_X is a covariant transport representation for the left fibration $\mathcal{E}_{X/} \to \mathcal{E}$, this is equivalent to the requirement that ι_X is left cofinal (see Corollary 7.4.5.16). □

Proof of Theorem 9.3.7.1. Let $U : \mathcal{E} \to \Delta^2$ be a functor of ∞-categories and let X be an object of $\mathcal{E}$ satisfying $U(X) = 0$. By virtue of Proposition 9.3.7.4, it will suffice to show that the following conditions are equivalent:

(1) The inclusion functor

$$\iota_X : \{1\} \times_{\Delta^2} \mathcal{E}_{X/} \hookrightarrow \mathrm{N}_\bullet(\{1 < 2\}) \times_{\Delta^2} \mathcal{E}_{X/}$$

is left cofinal.

(2) For every morphism $f : X \to Z$ of $\mathcal{E}$ where $U(Z) = 2$, the ∞-category $\{1\} \times_{\Delta^2} \mathcal{E}_{X//Z}$ is weakly contractible.

This follows from the cofinality criterion of Theorem 7.2.3.1. □

9.3.8 Fiberwise Cocompletion via the Yoneda Embedding

Recall that, if $\mathcal{E}$ is an essentially small ∞-category, then the contravariant Yoneda embedding $h^\bullet : \mathcal{E}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{E}, \mathcal{S})$ exhibits the ∞-category $\mathrm{Fun}(\mathcal{E}, \mathcal{S})$ as a cocompletion of $\mathcal{E}$ (Theorem 8.4.0.3). Our goal in this section is to prove a relative version of this statement. 05NN

Proposition 9.3.8.1. *Let κ be an uncountable regular cardinal and let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of simplicial sets which is essentially κ-small. Then:* 05NP

(1) *The projection map $V : \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ is a cocartesian fibration of simplicial sets (see Construction 4.5.9.1).*

(2) *Let $V^\dagger : \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})^\dagger \to \mathcal{C}^{\mathrm{op}}$ be a cartesian conjugate of V (Definition 8.6.1.1). Then $V^\dagger$ is a fiberwise κ-cocompletion of $U^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$.*

The first assertion of Proposition 9.3.8.1 is a special case of the following:

Lemma 9.3.8.2. *Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \to \mathcal{C}$ be an inner fibration of simplicial sets which is essentially κ-small, and let $\mathcal{D}$ be an ∞-category which is κ-cocomplete. Then the projection map $V : \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{D}) \to \mathcal{C}$ is a cocartesian fibration. Moreover, for every edge $e : C \to C'$ of $\mathcal{C}$, the covariant transport functor* 05NQ

$$\mathrm{Fun}(\mathcal{E}_C, \mathcal{D}) = \{C\} \times_{\mathcal{C}} \mathrm{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{D}) \to \{C'\} \times_{\mathcal{C}} \mathrm{Fun}(\mathcal{E} / \mathcal{C}) = \mathrm{Fun}(\mathcal{E}_{C'}, \mathcal{D})$$

preserves κ-small colimits.

Proof. To show that V is a cocartesian fibration, we may assume without loss of generality that $\mathcal{C} = \Delta^n$ is a standard simplex (Proposition 5.1.4.7). In this case, U is exponentiable (Corollary 9.3.6.19), so the desired result follows from Variant 8.6.5.11. To prove the second assertion, we may assume that $\mathcal{C} = \Delta^1$ with $C = 0$ and $C' = 1$. In this case, $\mathcal{E}_C$ and $\mathcal{E}_{C'}$ are full subcategories of $\mathcal{E}$, and determine restriction functors $R : \mathrm{Fun}(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}_C, \mathcal{D})$ and $R' : \mathrm{Fun}(\mathcal{E}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{E}_{C'}, \mathcal{D})$. The covariant transport functor $e_!$ can then be identified with the composition $R' \circ R^L$, where R^L is left adjoint to R (given by left Kan extension along the inclusion $\mathcal{E}_C \hookrightarrow \mathcal{E}$). Since the functor R^L preserves all colimits (Corollary 7.1.3.21), we are reduced to showing that the functor R' preserves κ-small colimits, which follows from the criterion of Proposition 7.1.6.1. □

To prove Proposition 9.3.8.1, we may assume without loss of generality that $U : \mathcal{E} \to \mathcal{C}$ is a flat inner fibration of ∞-categories (Corollary 9.3.6.26). In this case, the results of §8.6.2 provide an explicit example of a cartesian dual of the inner fibration V.

`05NR` **Construction 9.3.8.3.** Let κ be an uncountable regular cardinal, let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of ∞-categories which is essentially κ-small, and let $V : \operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ be the cocartesian fibration of Lemma 9.3.8.2. We let $\operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$ denote the full subcategory

$$\operatorname{Fun}^{\mathrm{CCart}}_{/\mathcal{C}}(\operatorname{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}) \subseteq \operatorname{Fun}_{/\mathcal{C}}(\operatorname{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}))$$

introduced in Construction 8.6.2.2, and let $V^{\dagger} : \operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ denote the projection map. It follows from Proposition 8.6.2.3 that the evaluation map

$$\mathrm{ev} : \operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \to \operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})$$

exhibits $V^{\dagger}$ as a cartesian conjugate of V.

By virtue of Lemma 9.3.8.2 (and Corollary 9.3.6.26), Proposition 9.3.8.1 is a consequence of the following more precise assertion:

`05NS` **Theorem 9.3.8.4.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of ∞-categories, let κ be an uncountable regular cardinal for which U is essentially κ-small, and let*

$$H : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \operatorname{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$$

be a relative Hom*-functor for U (see Definition 8.6.6.8), so that H is classified by a map*

$$h : \mathcal{E}^{\mathrm{op}} \to \operatorname{Fun}(\operatorname{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} / \mathcal{C}^{\mathrm{op}}, \mathcal{S}^{<\kappa}) \simeq \operatorname{Fun}_{/\mathcal{C}}(\operatorname{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})).$$

Then:

(1) *The functor h factors through the full subcategory*

$$\operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \subseteq \operatorname{Fun}_{/\mathcal{C}}(\operatorname{Tw}(\mathcal{C}) / \mathcal{C}^{\mathrm{op}}, \operatorname{Fun}(\mathcal{E} / \mathcal{C}, \mathcal{S}^{<\kappa}))$$

of Construction 9.3.8.3.

(2) *The diagram*

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{op}} \arrow[rr, "h"] \arrow[dr, "U^{\mathrm{op}}"'] & & \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \arrow[dl, "V^{\dagger}"] \\
& \mathcal{C}^{\mathrm{op}} &
\end{tikzcd}$$

exhibits $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$ *as a fiberwise* κ*-cocompletion of* $\mathcal{E}^{\mathrm{op}}$*.*

We will carry out the proof of Theorem 9.3.8.4 in several steps.

Remark 9.3.8.5. Let κ be an uncountable regular cardinal and let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of ∞-categories which is essentially κ-small. Suppose we are given a commutative diagram of simplicial sets 05NT

$$\begin{tikzcd}
\mathcal{D}^{\mathrm{op}} \arrow[rr, "h"] \arrow[dr] & & \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \arrow[dl] \\
& \mathcal{C}^{\mathrm{op}}, &
\end{tikzcd}$$

which we identify with a morphism $H : \mathcal{D}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$. Using the criterion of Variant 8.6.5.11, we see that h factors through $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$ if and only if H satisfies the following condition:

($*$) Let X be a vertex of $\mathcal{D}$ having image $C \in \mathcal{C}$, let e be an edge of $\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$, and let H_e denote the composition

$$\Delta^1 \times_{\mathcal{C}} \mathcal{E} \xrightarrow{e \times \mathrm{id}} \{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \hookrightarrow \mathcal{D}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \xrightarrow{H} \mathcal{S}^{<\kappa}.$$

Then the functor H_e is left Kan extended from $\{0\} \times_{\mathcal{C}} \mathcal{E}$.

Moreover, it suffice to verify condition ($*$) under the additional assumption that e belongs to the image of the equivalence $\mathcal{C}_{C/} \to \{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C})$ (see Proposition 8.1.2.9).

We now prove the first assertion of Theorem 9.3.8.4:

Lemma 9.3.8.6. *Let* $U : \mathcal{E} \to \mathcal{C}$ *be a flat inner fibration of* ∞*-categories, let* κ *be an uncountable regular cardinal for which* U *is essentially* κ*-small, and let* $H : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ *be a relative* Hom*-functor for* U*. Then* H *is classified by a functor* $h : \mathcal{E}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$*.* 05NU

Proof. Fix an object $X \in \mathcal{E}$ having image $C = U(X)$ and an edge e of the ∞-category $\mathcal{C}_{C/}$, which we identify with a 2-simplex σ :

$$\begin{tikzcd}
& C' \arrow[dr] & \\
C \arrow[ur] \arrow[rr] & & C''
\end{tikzcd}$$

of $\mathcal{C}$, and let $H_e : \mathrm{N}_\bullet(\{C' < C''\}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$ be as in condition $(*)$ of Remark 9.3.8.5; we wish to show that the functor F_e is left Kan extended from the subcategory $\{C'\} \times_{\mathcal{C}} \mathcal{E}$. Without loss of generality, we may assume that $\mathcal{C} = \Delta^2$ and that σ is the identity morphism. In this case, we can identify F_e with the composition

$$\mathrm{N}_\bullet(\{1 < 2\}) \times_{\Delta^2} \mathcal{E} \hookrightarrow \mathcal{E} \xrightarrow{\mathscr{F}} \mathcal{S}^{<\kappa},$$

where $\mathscr{F}$ is corepresented by the object $X \in \mathcal{E}$. Since U is flat, the desired result is a reformulation of Lemma 9.3.7.2. □

We now prove a special case of Theorem 9.3.8.4.

05NV **Lemma 9.3.8.7.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of ∞-categories, let κ be an uncountable regular cardinal for which U is essentially κ-small, and let*

$$H : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \mathcal{E} \to \mathcal{S}^{<\kappa}$$

be a relative Hom*-functor for U. Then the induced map $h : \mathcal{E}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$ exhibits* $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ *as a fiberwise κ-cocompletion of $\mathcal{E}^{\mathrm{op}}$ (relative to $\mathcal{C}^{\mathrm{op}}$).*

Proof. Let us identify H with a functor $F : \mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$. If X an object of $\mathcal{E}$ having image $C = U(X)$ and $u : C \to D$ is a morphism of $\mathcal{C}$, our assumption that U is a cocartesian fibration guarantees that we can lift u to a U-cocartesian morphism $\widetilde{u} : X \to Y$ of $\mathcal{E}$. In this case, we can identify $F(X, u)$ with a functor $\mathcal{E}_D \to \mathcal{S}^{<\kappa}$ which is corepresented by the object Y. It follows that F factors through the full subcategory $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \subseteq \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ of Construction 8.6.5.6. Unwinding the definitions, we have a commutative diagram

05NW
$$\begin{tikzcd}
\mathcal{E}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "F"] \arrow[d, "h \times \mathrm{id}"] & \mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \arrow[d] \\
\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \arrow[r, "\mathrm{ev}"] & \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}).
\end{tikzcd} \tag{9.41}$$

Let $V : \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$ and $V^{\dagger} : \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \to \mathcal{C}^{\mathrm{op}}$ denote the projection maps, and set $V^{\mathrm{corep}} = V|_{\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})}$. Then the lower horizontal map exhibits $V^{\dagger}$ as a cartesian conjugate of V (Proposition 8.6.2.3), and the upper horizontal map exhibits U^{op} as a cartesian conjugate of V^{corep} (Theorem 8.6.6.15). Moreover, the inclusion map $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa}) \subseteq \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ exhibits $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ as a fiberwise κ-cocompletion of $\mathrm{Fun}^{\mathrm{corep}}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$ (Example 9.3.2.5). Applying Proposition 9.3.3.3, we conclude that h exhibits $\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$ as a fiberwise κ-cocompletion of $\mathcal{E}^{\mathrm{op}}$. □

To prove Theorem 9.3.8.4, we need the following relative version of Yoneda's lemma:

Lemma 9.3.8.8. *Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of ∞-categories which is essentially κ-small. Then the functor $h : \mathcal{E}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$ of Lemma 9.3.8.6 is fully faithful.* 05NX

Proof. Using Theorem 9.3.4.1, we can choose a diagram

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{op}} \arrow[rr, "F^{\mathrm{op}}"] \arrow[dr, "U^{\mathrm{op}}"'] & & \widehat{\mathcal{E}}^{\mathrm{op}} \arrow[dl, "\widehat{U}^{\mathrm{op}}"] \\
& \mathcal{C}^{\mathrm{op}} &
\end{tikzcd}$$

which exhibits $\widehat{\mathcal{E}}^{\mathrm{op}}$ as a fiberwise κ-cocompletion of $\mathcal{E}^{\mathrm{op}}$. Since U is flat, the inner fibration U^{op} is also flat (Remark 9.3.6.5). Applying the criterion of Proposition 9.3.6.9, we see that $\widehat{U}^{\mathrm{op}}$ is a cartesian fibration. Let λ be a regular cardinal of exponential cofinality $\geq \kappa$ such that $\widehat{U}$ is essentially λ-small. Let h^{+} denote the composition of h with the inclusion functor

$$\begin{aligned}
\mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} &\hookrightarrow \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\,\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})) \\
&\hookrightarrow \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\,\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\lambda})).
\end{aligned}$$

We will complete the proof by showing that h^{+} is fully faithful.

Since λ has exponential cofinality $\geq \kappa$, the ∞-category $\mathcal{S}^{<\lambda}$ admits κ-small limits (Example 7.6.7.4). Let $\mathrm{Fun}(\widehat{\mathcal{E}}/\,\mathcal{C}, \mathcal{S}^{<\lambda})$ denote the relative exponential of Construction 4.5.9.1. By definition, objects of $\mathrm{Fun}(\widehat{\mathcal{E}}/\,\mathcal{C}, \mathcal{S}^{<\lambda})$ can be identified with pairs $(C, \mathscr{G})$, where C is an object of $\mathcal{C}$ and $\mathscr{G} : \widehat{\mathcal{E}}_{C} \to \mathcal{S}^{<\lambda}$ is a functor. Let $\mathrm{Fun}'(\widehat{\mathcal{E}}/\,\mathcal{C}, \mathcal{S}^{<\lambda})$ be the full subcategory spanned by those pairs $(C, \mathscr{F})$, where the functor $\mathscr{G}$ preserves κ-small limits. Let $\widehat{H} : \widehat{\mathcal{E}}^{\mathrm{op}} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Tw}(\mathcal{C}) \times_{\mathcal{C}} \widehat{\mathcal{E}} \to \mathcal{S}^{<\lambda}$ be a relative Hom-functor for $\widehat{U}$, which we identify with a map

$$\widehat{h} : \widehat{\mathcal{E}}^{\mathrm{op}} \to \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\,\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\widehat{\mathcal{E}}/\,\mathcal{C}, \mathcal{S}^{<\lambda}).$$

Since corepresentable functors preserve limits (Corollary 7.4.5.18), the functor $\widehat{h}$ factors through the full subcategory $\mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\,\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}'(\widehat{\mathcal{E}}/\,\mathcal{C}, \mathcal{S}^{<\lambda}))$. Since F is fully faithful

(Remark 9.3.1.14), the functor h^{+} is isomorphic to the composition

$$\begin{aligned} \mathcal{E}^{\mathrm{op}} & \xrightarrow{F^{\mathrm{op}}} \widehat{\mathcal{E}}^{\mathrm{op}} \\ & \xrightarrow{\widehat{h}} \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}'(\widehat{\mathcal{E}}/\mathcal{C}, \mathcal{S}^{<\lambda})) \\ & \xrightarrow{T} \mathrm{Fun}_{/\mathcal{C}}(\mathrm{Tw}(\mathcal{C})/\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\lambda})), \end{aligned}$$

where T is given by precomposition with F. It follows from the universal property of Theorem 9.3.1.20 that precomposition with F induces an equivalence $\mathrm{Fun}'(\widehat{\mathcal{E}}/\mathcal{C}, \mathcal{S}^{<\lambda}) \to \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\lambda})$ of inner fibrations over $\mathcal{C}$, so that T is an equivalence of ∞-categories. To complete the proof, it will suffice to show that the functor $\widehat{h}$ is fully faithful. In other words, we can replace U by $\widehat{U}$ (and κ by the cardinal λ) and thereby reduce to proving Lemma 9.3.8.8 under the assumption that U is a cocartesian fibration. In this case, the desired result follows from Lemma 9.3.8.7 (and Remark 9.3.1.14). □

Proof of Theorem 9.3.8.4. Let $U : \mathcal{E} \to \mathcal{C}$ be a flat inner fibration of ∞-categories which is essentially κ-small. It follows from Lemma 9.3.8.8 that a relative Hom-functor for U determines a fully faithful functor $h : \mathcal{E}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$. We wish to show that the diagram

$$\begin{tikzcd} \mathcal{E}^{\mathrm{op}} \arrow[rr, "h"] \arrow[dr, "U^{\mathrm{op}}"'] & & \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \arrow[dl, "V^{\dagger}"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

exhibits $\mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$ as a fiberwise κ-cocompletion of $\mathcal{E}^{\mathrm{op}}$. By construction, $V^{\dagger}$ is a cartesian fibration which is conjugate to $V : \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa}) \to \mathcal{C}$. In particular, for every morphism $f : C \to C'$ of $\mathcal{C}$, the contravariant transport functor

$$\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \to \{C'\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$$

for the cartesian fibration $V^{\dagger}$ can be identified with the covariant transport functor

$$\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa}) \to \{C'\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa})$$

for the cocartesian fibration V (see Proposition 8.6.1.5), and therefore preserves κ-small colimits (Lemma 9.3.8.2). By virtue of Proposition 9.3.1.16, it will suffice to show that for each object $C \in \mathcal{C}$, the map of fibers

$$h_C : \mathcal{E}^{\mathrm{op}}_C \to \{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Fun}(\mathcal{E}/\mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$$

exhibits $\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger}$ as a κ-cocompletion of $\mathcal{E}_C^{\mathrm{op}}$. Unwinding the definitions, we see that the composition of h_C with the equivalence

$$\{C\} \times_{\mathcal{C}^{\mathrm{op}}} \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})^{\dagger} \simeq \{C\} \times_{\mathcal{C}} \mathrm{Fun}(\mathcal{E} \,/\, \mathcal{C}, \mathcal{S}^{<\kappa})$$

identifies with the contravariant Yoneda embedding for the ∞-category $\mathcal{E}_C$. The desired result now follows from Theorem 8.4.3.3. $\square$

Chapter 10

Exactness and Animation

04QH 10.1 Simplicial Objects of ∞-Categories

04QJ Let $\mathcal{C}$ be a category. Recall that a simplicial object of $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, where $\mathbf{\Delta}$ is the simplex category introduced in Definition 1.1.0.2. This notion has an obvious counterpart in the setting of ∞-categories:

04QK **Definition 10.1.0.1** (Simplicial Objects)**.** Let $\mathcal{C}$ be an ∞-category. A *simplicial object of* $\mathcal{C}$ is a functor from the ∞-category $\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}})$ to $\mathcal{C}$. A *cosimplicial object of* $\mathcal{C}$ is a functor from $\mathrm{N}_\bullet(\mathbf{\Delta})$ to $\mathcal{C}$.

04QL **Notation 10.1.0.2.** Let $\mathcal{C}$ be an ∞-category. We will often use the notation $X_\bullet$ to indicate a simplicial object of $\mathcal{C}$. In this case, we write X_n for the value of the functor $X_\bullet$ on the object $[n] \in \mathbf{\Delta}^{\mathrm{op}}$. Similarly, we often use an expression like $X^\bullet$ to indicate a cosimplicial object of $\mathcal{C}$, and X^n for its value on the object $[n] \in \mathbf{\Delta}$.

04QM **Example 10.1.0.3.** Let $\mathcal{C}$ be a category. Then (co)simplicial objects of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ (in the sense of Definition 10.1.0.1) can be identified with (co)simplicial objects of $\mathcal{C}$ (in the sense of Definition 1.1.0.4).

04QN **Notation 10.1.0.4** (Face and Degeneracy Operators)**.** Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$. For every pair of integers $0 \leq i \leq n$, we let $s^n_i : X_n \to X_{n+1}$ denote the morphism induced by the surjection $\sigma^i_n : [n+1] \twoheadrightarrow [n]$ of Construction 1.1.2.1; we will refer to s^n_i as the *ith degeneracy operator* for the simplicial object $X_\bullet$. If $n > 0$, we let $d^n_i : X_n \to X_{n-1}$ denote the morphism induced by the inclusion of linearly ordered sets $\delta^i_n : [n-1] \hookrightarrow [n]$ introduced in Construction 1.1.1.4. We will refer to d^n_i as the *ith face operator* of $X_\bullet$.

04QP **Warning 10.1.0.5.** If $\mathcal{C}$ is an ordinary category, then a simplicial object $X_\bullet$ of $\mathcal{C}$ is completely determined by the collection of objects $\{X_n\}_{n \geq 0}$, together with the face and degeneracy operators

$$d_i^n : X_n \to X_{n-1} \qquad s_i^n : X_n \to X_{n+1}$$

(see Proposition 1.1.2.14). In the setting of ∞-categories, this is no longer true.

10.1.1 Geometric Realization

Let S be a simplicial set. Recall that the *geometric realization* of S is a topological space $|S|$ which corepresents the functor 04QQ

$$(X \in \mathrm{Top}) \mapsto \mathrm{Hom}_{\mathrm{Set}_\Delta}(S, \mathrm{Sing}_\bullet(X));$$

here Top denotes the category whose objects are topological spaces and whose morphisms are continuous functions (Definition 1.2.3.1). This property determines the topological space $|S|$ up to homeomorphism: that is, up to isomorphism in the category Top. We now formulate a homotopy-invariant counterpart of this universal property, which determines the topological space $|S|$ up to homotopy equivalence (rather than homeomorphism). In what follows, we regard Top as a simplicially enriched category (see Example 2.4.1.5), and we let $\mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Top})$ denote its homotopy coherent nerve.

Proposition 10.1.1.1. *Let S be a simplicial set. Then the geometric realization $|S|$ is a colimit of the diagram* 04QR

$$\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}) \xrightarrow{S} \mathrm{N}_\bullet(\mathrm{Set}) \subset \mathrm{N}_\bullet^{\mathrm{hc}}(\mathrm{Top}).$$

Proposition 10.1.1.1 admits a more combinatorial formulation:

Variant 10.1.1.2. Let $S = S_\bullet$ be a simplicial set. Then the Kan complex $\mathrm{Sing}_\bullet(|S|)$ is a colimit of the diagram 04QS

$$\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}) \xrightarrow{S} \mathrm{N}_\bullet(\mathrm{Set}) \subset \mathcal{S}.$$

Proof of Variant 10.1.1.2. Let Kan denote the ordinary category of Kan complexes, and let $\mathscr{F} : \boldsymbol{\Delta}^{\mathrm{op}} \to \mathrm{Kan}$ be the functor which carries each object $[n] \in \boldsymbol{\Delta}^{\mathrm{op}}$ to the set of n-simplices S_n (regarded as a constant simplicial set). Let $\underrightarrow{\mathrm{holim}}(\mathscr{F})$ denote the homotopy colimit of the diagram $\mathscr{F}$ (Construction 5.3.2.1). By virtue of Proposition 7.5.7.1 (and Example 1.4.2.5), it will suffice to show that there is a weak homotopy equivalence of simplicial sets $v : \underrightarrow{\mathrm{holim}}(\mathscr{F})^{\mathrm{op}} \to \mathrm{Sing}_\bullet(|S|)$. Using Example 5.3.2.5 (and Example 5.2.6.4), we can identify $\underrightarrow{\mathrm{holim}}(\mathscr{F})^{\mathrm{op}}$ with the nerve of the category of simplices $\boldsymbol{\Delta}_S$ (see Construction 1.1.3.9). We complete the proof by taking v to be the composition

$$\mathrm{N}_\bullet(\boldsymbol{\Delta}_S) \xrightarrow{\psi} S \xrightarrow{u} \mathrm{Sing}_\bullet(|S|),$$

where u is the weak homotopy equivalence of Theorem 3.6.4.1 and $\psi : \mathrm{N}_\bullet(\boldsymbol{\Delta}_S) \to S$ is the comparison map of Construction 3.3.3.9. By virtue of Variant 6.3.7.4, the morphism

ψ is universally localizing, and is therefore also a weak homotopy equivalence (Remark 6.3.6.5). □

Proof of Proposition 10.1.1.1. Let $\mathcal{T}$ be the ∞-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Top})$, and let $\mathcal{T}_0 \subseteq \mathcal{T}$ be the full subcategory spanned by those topological spaces which have the homotopy type of a CW complex. It follows from Example 6.2.2.7 that $\mathcal{T}_0$ is a coreflective subcategory of $\mathcal{T}$; in particular, the inclusion map $\mathcal{T}_0 \hookrightarrow \mathcal{T}$ preserves colimits (Variant 7.1.3.24). It will therefore suffice to show that for every simplicial set S, the geometric realization $|S|$ is a colimit of the diagram $\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}) \xrightarrow{S} \mathrm{N}_\bullet(\mathrm{Set}) \subset \mathcal{T}_0$. This is a reformulation of Variant 10.1.1.2, since the functor $X \mapsto \mathrm{Sing}_\bullet(X)$ determines an equivalence of ∞-categories $\mathcal{T}_0 \to \mathcal{S}$ (Remark 5.5.1.9). □

Motivated by Proposition 10.1.1.1, we introduce the following terminology:

04QT **Definition 10.1.1.3** (Geometric Realization)**.** Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$. We will say that an object $X \in \mathcal{C}$ is a *geometric realization of* $X_\bullet$ if it is a colimit of the diagram $X_\bullet : \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}) \to \mathcal{C}$.

04QU **Warning 10.1.1.4.** Let $S = S_\bullet$ be a simplicial set. Proposition 10.1.1.1 asserts that the topological space $|S|$ introduced in §1.2.3 is a geometric realization of S (in the sense of Definition 10.1.1.3), provided that we regard S as a simplicial object of the ∞-category $\mathrm{N}^{\mathrm{hc}}_\bullet(\mathrm{Top})$ (by equipping each of the sets S_n with the discrete topology). Beware that $|S_\bullet|$ is usually *not* a geometric realization of S (in the sense of Definition 10.1.1.3) if we regard S as a simplicial object of the ∞-category $\mathrm{N}_\bullet(\mathrm{Set})$. The latter is a colimit of the diagram $\boldsymbol{\Delta}^{\mathrm{op}} \xrightarrow{S} \mathrm{Set}$, which identifies with the set of connected components $\pi_0(S)$ (see Remark 1.2.1.20), or equivalently with the set of path components of the topological space $|S|$ (see Corollary 1.2.3.19).

04QV **Notation 10.1.1.5.** Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$. It follows from Proposition 7.1.1.12 that, if $X_\bullet$ admits a geometric realization X, then the isomorphism class of X is uniquely determined. To emphasize this, we will often denote X by $|X_\bullet|$ and refer to it as *the* geometric realization of $X_\bullet$. Beware that, in the case where $\mathcal{C}$ is (the nerve of) the category of sets, this is incompatible with the convention of Notation 1.2.3.3 (see Warning 10.1.1.4).

04QW **Exercise 10.1.1.6.** Let $X_\bullet$ be a simplicial object of an ordinary category $\mathcal{C}$. Show that an object $X \in \mathcal{C}$ is a geometric realization of $X_\bullet$ (in the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$) if and only if it is a coequalizer of the face operators $d^1_0, d^1_1 : X_1 \rightrightarrows X_0$. For a slightly more general statement, see Corollary 10.1.2.12.

04QX **Example 10.1.1.7** (Simplicial Abelian Groups)**.** Let Ab denote the category of abelian groups. By virtue of the Dold-Kan correspondence (Theorem 2.5.6.1), there is an equivalence of categories $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Ch}(\mathbf{Z})_{\geq 0}$, which carries each simplicial abelian group $A_\bullet$ to its normalized Moore complex

$$\mathrm{N}_*(A) = (\cdots \to \mathrm{N}_2(A) \xrightarrow{\partial} \mathrm{N}_1(A) \xrightarrow{\partial} \mathrm{N}_0(A)).$$

Under this equivalence, the coequalizer of the pair of face operators $d^1_0, d^1_1 : A_1 \rightrightarrows A_0$ can be identified with the 0th homology group $\mathrm{H}_0(\mathrm{N}_*(A)) = \mathrm{coker}(\partial : \mathrm{N}_1(A) \to \mathrm{N}_0(A))$, or alternatively with the homotopy group $\pi_0(A_\bullet)$ (see Exercise 3.2.2.22). Using Exercise 10.1.1.6 we see that the $\pi_0(A)$ can be regarded as a geometric realization of $A_\bullet$ in the category of abelian groups. In particular, the forgetful functor Ab → Set commutes with the formation of geometric realizations (this is a special case of a more general phenomenon, which we will return to in §[?]).

Remark 10.1.1.8. Let $X_\bullet$ be a simplicial object of a category $\mathcal{C}$. It follows from Exercise 10.1.1.6 that a geometric realization of $X_\bullet$ (if it exists) depends only on the pair of face operators $d^1_0, d^1_1 : X_1 \rightrightarrows X_0$. Beware that, in the ∞-categorical setting, this is generally not true: the geometric realization $|X_\bullet|$ is sensitive to information about the *entire* simplicial object $X_\bullet$. `04QY`

Variant 10.1.1.9. Let $X^\bullet$ be a cosimplicial object of an ∞-category $\mathcal{C}$. We will say that an object $X \in \mathcal{C}$ is a *totalization of* $X^\bullet$ if it is a limit of the diagram $X^\bullet : \mathrm{N}_\bullet(\mathbf{\Delta}) \to \mathcal{C}$. If this condition is satisfied, then X is uniquely determined up to isomorphism. To emphasize this, we will often denote X by $\mathrm{Tot}(X^\bullet)$ and refer to it as *the* totalization of $X^\bullet$. `04QZ`

For many applications, the language of Definition 10.1.1.3 is insufficiently precise. Given a simplicial object $X_\bullet$ of an ∞-category $\mathcal{C}$, we would like to view its geometric realization $|X_\bullet|$ not abstractly as an object of $\mathcal{C}$, but as an object of the coslice ∞-category $\mathcal{C}_{X_\bullet/}$. For this purpose, it will be convenient to introduce some additional terminology.

Definition 10.1.1.10 (The Augmented Simplex Category)**.** For each integer $n \geq -1$, let $[n]$ denote the linearly ordered set $\{0 < 1 < \cdots < n\}$, so that $[-1]$ is the empty set. We let $\mathbf{\Delta}_+$ denote the category whose objects are the linearly ordered sets $\{[n]\}_{n \geq -1}$, and whose morphisms are nondecreasing functions. We will refer to $\mathbf{\Delta}_+$ as the *augmented simplex category.* `04R0`

Remark 10.1.1.11. The augmented simplex category $\mathbf{\Delta}_+$ of Definition 10.1.1.10 contains the simplex category $\mathbf{\Delta}$ of Definition 1.1.0.2 as a full subcategory (spanned by the objects $[n]$ for $n \geq 0$). Moreover, $\mathbf{\Delta}_+$ can be obtained from $\mathbf{\Delta}$ by adjoining a single object $[-1]$, which is an initial object satisfying $\mathrm{Hom}_{\mathbf{\Delta}_+}([n], [-1]) = \emptyset$ for $n \geq 0$. In other words, $\mathbf{\Delta}_+$ can be identified with the left cone $\mathbf{\Delta}^{\triangleleft}$ (see Example 4.3.2.5). `04R1`

Definition 10.1.1.12 (Augmented Simplicial Objects)**.** Let $\mathcal{C}$ be an ∞-category. An *augmented simplicial object of* $\mathcal{C}$ is a functor from the ∞-category $\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}_+)$ to $\mathcal{C}$. An *augmented cosimplicial object* is a functor from the ∞-category $\mathrm{N}_\bullet(\mathbf{\Delta}_+)$ to $\mathcal{C}$. `04R2`

04R3 **Notation 10.1.1.13.** Let $\mathcal{C}$ be an ∞-category. We will often use the notation $X_{\bullet}$ to indicate an augmented simplicial object of $\mathcal{C}$. In this case, we write X_n for the value of the functor $X_{\bullet}$ on the object $[n] \in \mathbf{\Delta}_{+}^{\mathrm{op}}$. Similarly, we often use the expression $X^{\bullet}$ to indicate an augmented cosimplicial object of $\mathcal{C}$, and X^n for its value on the object $[n] \in \mathbf{\Delta}_{+}$.

04R4 **Remark 10.1.1.14.** Let $\mathcal{C}$ be an ∞-category. Every augmented simplicial object of $\mathcal{C}$ determines a simplicial object of $\mathcal{C}$, by restriction along the inclusion of full subcategories $\mathbf{\Delta}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_{+}^{\mathrm{op}}$. For this reason, we will sometimes use the notation $\overline{X}_{\bullet}$ to indicate an augmented simplicial object of $\mathcal{C}$, to distinguish it from the underlying simplicial object $X_{\bullet} = \overline{X}_{\bullet}|_{\mathrm{N}_{\bullet}(\mathbf{\Delta}^{\mathrm{op}})}$.

04R5 **Remark 10.1.1.15.** Let $\mathcal{C}$ be an ∞-category containing an object X. By virtue of Remark 10.1.1.11, the following data are equivalent:

- Augmented simplicial objects $\mathrm{N}_{\bullet}(\mathbf{\Delta}_{+}^{\mathrm{op}}) \to \mathcal{C}$ carrying the object $[-1]$ to X.
- Simplicial objects of the slice ∞-category $\mathcal{C}_{/X}$.

We will often invoke this equivalence implicitly, using the notation $X_{\bullet}$ to indicate both an augmented simplicial object of $\mathcal{C}$ (satisfying $X_{-1} = X$) and the associated simplicial object of $\mathcal{C}_{/X}$.

04R6 **Definition 10.1.1.16.** Let $\mathcal{C}$ be an ∞-category containing an object X, let $\overline{X}_{\bullet}$ be an augmented simplicial object of $\mathcal{C}$ satisfying $\overline{X}_{-1} = X$, and let $X_{\bullet} = \underline{X}_{\bullet}|_{\mathrm{N}_{\bullet}(\mathbf{\Delta}^{\mathrm{op}})}$ denote its underlying simplicial object. We will say that $\overline{X}_{\bullet}$ *exhibits X as a geometric realization of* $X_{\bullet}$ if it is a colimit diagram in the ∞-category $\mathcal{C}$, in the sense of Variant 7.1.2.5.

Similarly, if $\overline{X}^{\bullet}$ is an augmented cosimplicial object of $\mathcal{C}$ satisfying $\overline{X}^{-1} = X$ and $X^{\bullet} = \overline{X}^{\bullet}|_{\mathrm{N}_{\bullet}(\mathbf{\Delta})}$ is the underlying cosimplicial object, we say that $\overline{X}^{\bullet}$ *exhibits X as a totalization of* $X^{\bullet}$ if it is a limit diagram in the ∞-category $\mathcal{C}$, in the sense of Definition 7.1.2.4.

04R7 **Remark 10.1.1.17.** Let $\mathcal{C}$ be an ∞-category, let $X_{\bullet}$ be a simplicial object of $\mathcal{C}$, and let X be an object of $\mathcal{C}$. Then X is a geometric realization of $X_{\bullet}$ (in the sense of Definition 10.1.1.3) if and only if there exists an augmented simplicial object $\overline{X}_{\bullet}$ which exhibits X as a geometric realization of $X_{\bullet}$ (in the sense of Definition 10.1.1.16). See Remark 7.1.2.7.

10.1.2 Semisimplicial Objects

04R8 Let $\mathbf{\Delta}$ denote the simplex category (Definition 1.1.0.2), and let $\mathbf{\Delta}_{\mathrm{inj}}$ denote the subcategory of $\mathbf{\Delta}$ whose morphisms are strictly increasing functions $[m] \hookrightarrow [n]$ (Definition 1.1.1.2). It will often be useful to consider the following variant of Definition 10.1.0.1:

Definition 10.1.2.1 (Semisimplicial Objects)**.** Let $\mathcal{C}$ be an ∞-category. A *semisimplicial object of* $\mathcal{C}$ is a functor $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}) \to \mathcal{C}$. A *cosemisimplicial object of* $\mathcal{C}$ is a functor $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}) \to \mathcal{C}$. `04R9`

Notation 10.1.2.2. Let $\mathcal{C}$ be an ∞-category. We will often use the notation $X_\bullet$ to indicate a semisimplicial object of $\mathcal{C}$. In this case, we write X_n for the value of the functor $X_\bullet$ on the object $[n] \in \mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}$. Similarly, we often use an expression like $X^\bullet$ to indicate a cosemisimplicial object of $\mathcal{C}$, and X^n for its value on the object $[n] \in \mathbf{\Delta}_{\mathrm{inj}}$. `04RA`

Example 10.1.2.3. Let $\mathcal{C}$ be a category. Then (co)semisimplicial objects of the ∞-category $\mathrm{N}_\bullet(\mathcal{C})$ (in the sense of Definition 10.1.2.1) can be identified with (co)semisimplicial objects of $\mathcal{C}$ (in the sense of Definition 1.1.1.2). `04RB`

Example 10.1.2.4. Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. The composite functor `04RC`

$$\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}) \subset \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \xrightarrow{X_\bullet} \mathcal{C}$$

is a semisimplicial object of $\mathcal{C}$, which we will refer to as the *underlying semisimplicial object of* $X_\bullet$. We will often abuse notation by identifying $X_\bullet$ with its underlying semisimplicial object. Similarly, every cosimplicial object $X^\bullet$ of $\mathcal{C}$ has an underlying cosemisimplicial object, given by the composition

$$\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}) \subset \mathrm{N}_\bullet(\mathbf{\Delta}) \xrightarrow{X^\bullet} \mathcal{C}\,.$$

Remark 10.1.2.5 (Face Operators)**.** Let $X_\bullet$ be a semisimplicial object of an ∞-category. For every pair of integers $0 \leq i \leq n$ with $n > 0$, we let $d_i^n : X_n \to X_{n-1}$ denote the morphism induced by the inclusion of linearly ordered sets $\delta_n^i : [n-1] \hookrightarrow [n]$ introduced in Construction 1.1.1.4. We will refer to d_i^n as the *ith face operator* of the semisimplicial object $X_\bullet$. `04RD`

The utility of Definition 10.1.2.1 stems in part from the fact that, for many purposes, passage from simplicial to semisimplicial objects does not lose very much information.

Proposition 10.1.2.6. *The inclusion $\mathbf{\Delta}_{\mathrm{inj}} \subset \mathbf{\Delta}$ determines a left cofinal functor of ∞-categories $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}) \hookrightarrow \mathrm{N}_\bullet(\mathbf{\Delta})$.* `04RE`

Proof. By virtue of Theorem 7.2.3.1, it will suffice to show that for every integer $n \geq 0$, the category $\mathcal{C} = \mathbf{\Delta}_{\mathrm{inj}} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]}$ has weakly contractible nerve. Let $C_0 \in \mathcal{C}$ denote the object corresponding to the inclusion map $[0] \simeq \{n\} \hookrightarrow [n]$. For every object $C \in \mathcal{C}$, given by a nondecreasing function $\alpha : [m] \to [n]$, we let $F(C) \in \mathcal{C}$ denote the object given by the nondecreasing function $\alpha^+ : [m+1] \to [n]$ given by the formula

$$\alpha^+(i) = \begin{cases} \alpha(i) & \text{if } 0 \leq i \leq m \\ n & \text{if } i = m+1. \end{cases}$$

Note that we have canonical maps $C \xrightarrow{\beta_-} F(C) \xleftarrow{\beta_+} C_0$, given by the inclusions

$$\{0 < 1 < \cdots < m\} \hookrightarrow \{0 < 1 < \cdots < m+1\} \hookleftarrow \{m+1\}.$$

These morphisms depend functorially on C, and therefore furnish natural transformations of functors $\mathrm{id}_{\mathcal{C}} \to F \leftarrow \underline{C_0}$, where $\underline{C_0} : \mathcal{C} \to \mathcal{C}$ denotes the constant functor taking the value C_0. It follows that the identity morphism of the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ is homotopic to the constant morphism $\mathrm{N}_\bullet(\mathcal{C}) \twoheadrightarrow \{C_0\} \hookrightarrow \mathrm{N}_\bullet(\mathcal{C})$, so that the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ is contractible (and, in particular, it is weakly contractible). □

04RF **Corollary 10.1.2.7.** *Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. Then an object $X \in \mathcal{C}$ is a geometric realization of $X_\bullet$ (in the sense of Definition 10.1.1.3) if and only if it is a colimit of the underlying diagram $\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}) \subset \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \xrightarrow{X_\bullet} \mathcal{C}$.*

Proof. Combine Proposition 10.1.2.6 with Corollary 7.2.2.11. □

04RG **Example 10.1.2.8.** Let S be a simplicial set and let $|S|$ denote its geometric realization as a topological space (Definition 1.2.3.1). Combining Proposition 10.1.1.1 with Corollary 10.1.2.7, we deduce that the homotopy type of the topological space $|S|$ depends only on the underlying semisimplicial set of S. Compare with Corollary 3.4.5.5.

Motivated by Corollary 10.1.2.7, we introduce the following terminology:

04RH **Definition 10.1.2.9.** Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be a semisimplicial object of $\mathcal{C}$. We say that an object $X \in \mathcal{C}$ is a *geometric realization of* $X_\bullet$ if it is a colimit of the diagram $X_\bullet : \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}) \to \mathcal{C}$. If $X^\bullet$ is a cosemisimplicial object of $\mathcal{C}$, we say that an object $X \in \mathcal{C}$ is a *totalization of* $X^\bullet$ if it is a limit of the diagram $X^\bullet : \mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}) \to \mathcal{C}$.

04RJ **Remark 10.1.2.10.** Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$. Corollary 10.1.2.7 asserts that an object $X \in \mathcal{C}$ is a geometric realization of $X_\bullet$ (in the sense of Definition 10.1.1.3) if and only if it is a geometric realization of the underlying semisimplicial object of $X_\bullet$ (in the sense of Definition 10.1.2.9). In particular, $X_\bullet$ admits a geometric realization if and only if its underlying semisimplicial object admits a geometric realization.

In the setting of classical category theory, the notion of geometric realization can be made more concrete.

04RK **Proposition 10.1.2.11.** *Let $\mathcal{C}$ be a category, let Y be an object of $\mathcal{C}$, and let $\underline{Y}$ denote the constant semisimplicial object of $\mathcal{C}$ taking the value Y. For every semisimplicial object $X_\bullet$ of $\mathcal{C}$, the evaluation map*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}, \mathcal{C})}(X_\bullet, \underline{Y}) \to \mathrm{Hom}_{\mathcal{C}}(X_0, Y)$$

is a monomorphism, whose image is the set of morphisms $\epsilon : X_0 \to Y$ which satisfy the following condition:

$(*)$ *The face operators* $d^1_0, d^1_1 : X_1 \to X_0$ *of the simplicial object* $X_\bullet$ *satisfy* $\epsilon \circ d^1_0 = \epsilon \circ d^1_1$.

Proof. For every integer $n \geq 0$, let ι_n denote the inclusion map $[0] = \{0\} \hookrightarrow \{0 < 1 < \cdots < n\} = [n]$ and write $\iota^*_n : X_n \to X_0$ for the associated morphism of $\mathcal{C}$. If $f_\bullet : X_\bullet \to \underline{Y}$ is a morphism of semisimplicial objects, then we must have $f_n = f_0 \circ \iota^*_n$ for each $n \geq 0$; in particular, $f_\bullet$ is uniquely determined by the morphism $\epsilon = f_0$. To complete the proof, it will suffice to show that if a morphism $\epsilon : X_0 \to Y$ satisfies condition $(*)$, then the collection $\{(\epsilon \circ \iota^*_n) : X_n \to Y\}_{n \geq 0}$ determines a morphism of semisimplicial objects from $X_\bullet$ to $\underline{Y}$ (the converse follows immediately from the definitions). Fix a strictly increasing function $\alpha : [m] \hookrightarrow [n]$; we wish to show that the diagram

$$\begin{tikzcd}
X_n \arrow[r, "\iota^*_n"] \arrow[d, "\alpha^*"'] & X_0 \arrow[d, "\epsilon"] \\
X_m \arrow[r, "\epsilon \circ \iota^*_m"] & Y
\end{tikzcd} \tag{10.1}$$ 04RL

commutes. If $\alpha(0) = 0$, then $\iota_n = \alpha \circ \iota_m$. It follows that $\iota^*_n = \iota^*_m \circ \alpha^*$, and the desired result follows by composing with ϵ on both sides. We may therefore assume without loss of generality that $\alpha(0) > 0$. Let $\beta : [1] \hookrightarrow [n]$ be the strictly increasing function given by $\beta(0) = 0$ and $\beta(1) = \alpha(0)$. Then (10.1) can be identified with the outer rectangle of the diagram

$$\begin{tikzcd}
X_n \arrow[r, "\beta^*"] \arrow[d, "\alpha^*"'] & X_1 \arrow[r, "d^1_1"] \arrow[d, "d^1_0"] & X_0 \arrow[d, "\epsilon"] \\
X_m \arrow[r, "\iota^*_m"] & X_0 \arrow[r, "\epsilon"] & Y,
\end{tikzcd}$$

where the left square commutes by the naturality of the construction $[k] \mapsto X_k$, and the right square commutes by virtue of assumption $(*)$. □

Corollary 10.1.2.12. *Let* $X_\bullet$ *be a semisimplicial object of a category* $\mathcal{C}$. *Then an object* $X \in \mathcal{C}$ *is a geometric realization of* $X_\bullet$ *(in the* ∞*-category* $\mathrm{N}_\bullet(\mathcal{C})$*) if and only if it is a coequalizer of the face operators* $d^1_0, d^1_1 : X_1 \rightrightarrows X_0$. 04RM

For some applications, we will need more precise language for discussing geometric realizations of semisimplicial objects.

Notation 10.1.2.13. Let $\mathbf{\Delta}_+$ be the augmented simplex category (Definition 10.1.1.10). We let $\mathbf{\Delta}_{\mathrm{inj},+}$ denote the (non-full) subcategory of $\mathbf{\Delta}_+$ whose morphisms are strictly increasing 04RN functions $[m] \hookrightarrow [n]$. Note that $\mathbf{\Delta}_{\mathrm{inj},+}$ can be obtained from the category $\mathbf{\Delta}_{\mathrm{inj}}$ by adjoining an initial object $[-1]$ satisfying $\mathrm{Hom}_{\mathbf{\Delta}_{+,\mathrm{inj}}}([n],[-1]) = \emptyset$ for $n \geq 0$. Consequently, $\mathbf{\Delta}_{\mathrm{inj},+}$ can be identified with the left cone $\mathbf{\Delta}_{\mathrm{inj}}^{\triangleleft}$ (see Example 4.3.2.5).

04RP **Proposition 10.1.2.14.** *The diagram of ∞-categories*

$$\begin{tikzcd}
\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}) \arrow[r] \arrow[d] & \mathrm{N}_\bullet(\mathbf{\Delta}) \arrow[d] \\
\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj},+}) \arrow[r] & \mathrm{N}_\bullet(\mathbf{\Delta}_{+})
\end{tikzcd}$$

is a categorical pushout square.

Proof. By virtue of Proposition 7.2.2.1, this is a reformulation of Proposition 10.1.2.6. □

04RQ **Definition 10.1.2.15** (Augmented Semisimplicial Objects)**.** Let $\mathcal{C}$ be an ∞-category. An *augmented semisimplicial object of* $\mathcal{C}$ is a functor from the ∞-category $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj},+}^{\mathrm{op}})$ to $\mathcal{C}$. An *augmented cosemisimplicial object* is a functor from the ∞-category $\mathrm{N}_\bullet(\mathbf{\Delta}_{+})$ to $\mathcal{C}$.

04RR **Notation 10.1.2.16.** Let $\mathcal{C}$ be an ∞-category. We will often use the notation $X_\bullet$ to indicate an augmented semisimplicial object of $\mathcal{C}$. In this case, we write X_n for the value of the functor $X_\bullet$ on the object $[n] \in \mathbf{\Delta}_{\mathrm{inj},+}^{\mathrm{op}}$. Similarly, we often use the expression $X^\bullet$ to indicate an augmented cosimplicial object of $\mathcal{C}$, and X^n for its value on the object $[n] \in \mathbf{\Delta}_{\mathrm{inj},+}$.

04RS **Remark 10.1.2.17.** Let $\mathcal{C}$ be an ∞-category. Every augmented semisimplicial object of $\mathcal{C}$ determines a semisimplicial object of $\mathcal{C}$, by restriction along the inclusion of full subcategories $\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}} \hookrightarrow \mathbf{\Delta}_{\mathrm{inj},+}^{\mathrm{op}}$. For this reason, we will sometimes use the notation $\overline{X}_\bullet$ to indicate an augmented semisimplicial object of $\mathcal{C}$, to distinguish it from the underlying simplicial object $X_\bullet = \overline{X}_\bullet|_{\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}})}$.

04RT **Remark 10.1.2.18.** Let $\mathcal{C}$ be an ∞-category. It follows from Proposition 10.1.2.14 that the diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_{+}^{\mathrm{op}}),\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}),\mathcal{C}) \arrow[d] \\
\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj},+}^{\mathrm{op}}),\mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}),\mathcal{C})
\end{tikzcd}$$

is a categorical pullback square. In particular, if $X_\bullet$ is a simplicial object of $\mathcal{C}$, then the datum of an augmentation of $X_\bullet$ is equivalent to the datum of an augmentation on the underlying semisimplicial object of $X_\bullet$.

Remark 10.1.2.19 (Face Operators)**.** For every pair of integers $0 \leq i \leq n$, there is a unique increasing function $\delta^i_n : [n-1] \hookrightarrow [n]$ whose image is the set $[n] \setminus \{i\}$, given concretely by the formula 04RU

$$\delta^i_n(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i. \end{cases}$$

If $X_\bullet$ is an augmented semisimplicial object of an ∞-category $\mathcal{C}$, then evaluation on the morphism δ^i_n determines a map $d^n_i : X_n \to X_{n-1}$, which we will refer to as the *ith face operator* for the augmented semisimplicial object $X_\bullet$. If $n > 0$, this recover the face operators for the underlying semisimplicial object of $X_\bullet$ (Remark 10.1.2.5). In the case $n = 0$, we obtain a new operator $d^0_0 : X_0 \to X_{-1}$.

Remark 10.1.2.20. Let $\mathcal{C}$ be an ∞-category containing an object X. By virtue of Remark 10.1.1.11, the following data are equivalent: 04RV

- Augmented semisimplicial objects $\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}_+) \to \mathcal{C}$ carrying the object $[-1]$ to X.
- Semisimplicial objects of the slice ∞-category $\mathcal{C}_{/X}$.

We will often invoke this equivalence implicitly, using the notation $X_\bullet$ to indicate both an augmented simplicial object of $\mathcal{C}$ (satisfying $X_{-1} = X$) and the associated simplicial object of $\mathcal{C}_{/X}$.

Remark 10.1.2.21 (Augmented Moore Complexes)**.** Let $A_\bullet$ be an augmented semisimplicial object of the category of abelian groups. For each $n \geq 0$, let $\partial : A_n \to A_{n-1}$ denote the group homomorphism given by the alternating sum 04RW

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i d^n_i(\sigma).$$

The diagram

$$\cdots \to A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0 \xrightarrow{\partial} A_{-1}$$

is a chain complex of abelian groups which we will denote by $\mathrm{C}^{\mathrm{aug}}_*(A)$ and refer to as the *augmented Moore complex of* $A_\bullet$. Note that, when restricted to nonnegative degrees, this recovers the Moore complex of the underlying semisimplicial abelian group (see Construction 2.5.5.1).

Variant 10.1.2.22. Let $A_\bullet$ be an augmented simplicial object of the category of abelian groups. Let us abuse notation by identifying $A_\bullet$ with the underlying simplicial abelian group, and let 04RX

$$\mathrm{D}_*(A) \subseteq \mathrm{C}_*(A) \subseteq \mathrm{C}^{\mathrm{aug}}_*(A)$$

be the subcomplex generated by the images of the degeneracy operators (see Proposition 2.5.5.6). We let $\mathrm{N}^{\mathrm{aug}}_*(A)$ denote the quotient complex $\mathrm{C}^{\mathrm{aug}}_*(A)/\mathrm{D}_*(A)$, which we will refer to as the *normalized augmented Moore complex* of $A_\bullet$. Note that, when restricted to nonnegative degrees, this recovers the normalized Moore complex of the underlying simplicial abelian group (Construction 2.5.5.7).

04RY **Definition 10.1.2.23.** Let $\mathcal{C}$ be an ∞-category containing an object X, let $\overline{X}_\bullet$ be an augmented semisimplicial object of $\mathcal{C}$ satisfying $\overline{X}_{-1} = X$, and let $X_\bullet = \underline{X}_\bullet|_{\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}})}$ denote its underlying semisimplicial object. We will say that $\overline{X}_\bullet$ *exhibits X as a geometric realization of $X_\bullet$* if it is a colimit diagram in the ∞-category $\mathcal{C}$, in the sense of Variant 7.1.2.5.

Similarly, if $\overline{X}^\bullet$ is an augmented cosemisimplicial object of $\mathcal{C}$ satisfying $\overline{X}^{-1} = X$ and $X^\bullet = \overline{X}^\bullet|_{\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}})}$ is the underlying cosemisimplicial object, we say that $\overline{X}^\bullet$ *exhibits X as a totalization of $X^\bullet$* if it is a limit diagram in the ∞-category $\mathcal{C}$, in the sense of Definition 7.1.2.4.

04RZ **Remark 10.1.2.24.** Let $\mathcal{C}$ be an ∞-category, let $X_\bullet$ be a semisimplicial object of $\mathcal{C}$, and let X be an object of $\mathcal{C}$. Then X is a geometric realization of $X_\bullet$ (in the sense of Definition 10.1.2.9) if and only if there exists an augmented semisimplicial object $\overline{X}_\bullet$ which exhibits X as a geometric realization of $X_\bullet$ (in the sense of Definition 10.1.2.23). See Remark 7.1.2.7.

By virtue of Example 4.3.2.15, we can formulate Proposition 10.1.2.11 as follows:

04S0 **Proposition 10.1.2.25.** *Let $\mathcal{C}$ be a category, let X be an object of $\mathcal{C}$, and let $X_\bullet$ be a semisimplicial object of $\mathcal{C}$. The following data are equivalent:*

- *Extensions of $X_\bullet$ to an augmented semisimplicial object $\overline{X}_\bullet$ satisfying $\overline{X}_{-1} = X$.*
- *Morphisms $\epsilon : X_0 \to X$ satisfying $\epsilon \circ d^1_0 = \epsilon \circ d^1_1$, where $d^1_0, d^1_1 : X_1 \rightrightarrows X_0$ are the face operators of the semisimplicial object $X_\bullet$.*

Here the equivalence is implemented by taking ϵ to be the face operator $d^0_0 : X_0 \to X_{-1}$ of Remark 10.1.2.19.

04S1 **Remark 10.1.2.26.** In the situation of Proposition 10.1.2.25, the augmented semisimplicial object $\overline{X}_\bullet$ exhibits X as a geometric realization of $X_\bullet$ (in the sense of Definition 10.1.1.16) if and only if it the morphism ϵ exhibits X as a coequalizer of the face operators $d^1_0, d^1_1 : X_1 \rightrightarrows X_0$.

Combining Propositions 10.1.2.25 and 1.1.1.9, we obtain an explicit characterization of augmented semisimplicial objects of ordinary categories:

04S2 **Corollary 10.1.2.27.** *Let $\mathcal{C}$ be a category and let $\{X_n\}_{n \geq -1}$ be a sequence of objects of $\mathcal{C}$. Then a system of morphisms $\{d^n_i : X_n \to X_{n-1}\}_{0 \leq i \leq n}$ arise as the face operators of an augmented semisimplicial object $X_\bullet$ of $\mathcal{C}$ if and only if they satisfy the following condition:*

($*$) *For all integers $n > 0$ and $0 \leq i < j \leq n$, we have an equality $d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$ (as morphisms from X_n to X_{n-2}).*

If this condition is satisfied, then the augmented semisimplicial object $X_\bullet$ is uniquely determined.

Variant 10.1.2.28. Let $\mathcal{C}$ be a category and let $\{X_n\}_{n \geq -1}$ be a sequence of objects of $\mathcal{C}$. Then morphisms 04S3

$$\{d_i^n : X_n \to X_{n-1}\}_{0 \leq i \leq n} \qquad \{s_i^n : X_n \to X_{n+1}\}_{0 \leq i \leq n}$$

are the face and degeneracy operators for an augmented simplicial object $X_\bullet$ of $\mathcal{C}$ if and only if they satisfy the following conditions:

(1) For all integers $n > 0$ and $0 \leq i < j \leq n$, we have an equality $d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$ (as morphisms from X_n to X_{n-2}).

(2) For all integers $0 \leq i \leq j \leq n$, we have an equality $s_i^{n+1} \circ s_j^n = s_{j+1}^{n+1} \circ s_i^n$ (as morphisms from X_n to X_{n+2}).

(3) For all integers $0 \leq i, j \leq n$, we have an equality

$$d_i^{n+1} \circ s_j^n = \begin{cases} s_{j-1}^{n-1} \circ d_i^n & \text{if } i < j \\ \mathrm{id}_{X_n} & \text{if } i = j \text{ or } i = j+1 \\ s_j^{n-1} \circ d_{i-1}^n & \text{if } i > j+1 \end{cases}$$

(as morphisms from X_n to X_n).

If these conditions are satisfied, then the augmented simplicial object $X_\bullet$ is uniquely determined.

Proof. Combine Proposition 1.1.2.14, Remark 10.1.2.18, and Corollary 10.1.2.27. □

We close this section with a few remarks concerning the relationship between simplicial and semisimplicial objects.

Proposition 10.1.2.29. *Let $\mathcal{C}$ be an ∞-category which admits finite coproducts and let $X_\bullet$ be a semisimplicial object of $\mathcal{C}$. Then there exists a simplicial object $X_\bullet^+$ of $\mathcal{C}$ and a natural transformation of semisimplicial objects $f_\bullet : X_\bullet \to X_\bullet^+$ which exhibits $X_\bullet^+$ as a left Kan extension of $X_\bullet$ along the inclusion map $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}) \subseteq \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}})$.* 04S4

Proof. We will show that $X_\bullet$ satisfies the criterion of Proposition 7.3.5.1. Fix an object $[n] \in \mathbf{\Delta}$, let $\mathcal{E}$ denote the fiber product $\mathbf{\Delta}_{\mathrm{inj}} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{[n]/}$, and let F denote the composite map

$$\mathrm{N}_\bullet(\mathcal{E}^{\mathrm{op}}) \to \mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}) \xrightarrow{X_\bullet} \mathcal{C}\,.$$

We wish to show that F admits a colimit in $\mathcal{C}$.

By definition, objects of the category $\mathcal{E}$ can be identified with pairs $([m], \alpha)$, where $[m]$ is an object of $\mathbf{\Delta}_{\mathrm{inj}}$ is an integer and $\alpha : [n] \to [m]$ is a nondecreasing function. Let $\mathcal{E}_0 \subseteq \mathcal{E}$ denote the full subcategory spanned by those objects $([m], \alpha)$ where α is a surjection. Note that any morphism $\alpha : [n] \to [m]$ in $\mathbf{\Delta}$ factors uniquely as a composition $[n] \xrightarrow{\alpha'} [m'] \xrightarrow{\beta} [m]$, where α' is a surjection and β is an injection. The pair $([m'], \alpha')$ is then an object of the subcategory $\mathcal{E}_0$, and the morphism $\beta : ([m'], \alpha') \to ([m], \alpha)$ exhibits exhibits $([m'], \alpha')$ as a $\mathcal{E}_0$-coreflection of $([m], \alpha)$ (see Definition 6.2.2.1). It follows that the inclusion map $\mathrm{N}_\bullet(\mathcal{E}_0^{\mathrm{op}}) \hookrightarrow \mathrm{N}_\bullet(\mathcal{E}^{\mathrm{op}})$ is right cofinal (Corollary 7.2.2.7). Consequently, to show that F admits a colimit in $\mathcal{C}$, it will suffice to show that the restriction $F|_{\mathrm{N}_\bullet(\mathcal{E}_0^{\mathrm{op}})}$ admits a colimit in $\mathcal{C}$ (Corollary 7.2.2.10). Since the category $\mathcal{E}_0$ has finitely many objects and only identity morphisms, this follows from our assumption that $\mathcal{C}$ admits finite coproducts. □

04S5 **Remark 10.1.2.30.** Let $\mathcal{C}$ be an ∞-category, let $Y_\bullet$ be a simplicial object of $\mathcal{C}$, and suppose we are given a morphism $f_\bullet : X_\bullet \to Y_\bullet$ of semisimplicial objects of $\mathcal{C}$. It follows from the proof of Proposition 10.1.2.29 that $f_\bullet$ exhibits $Y_\bullet$ as a left Kan extension of $X_\bullet$ along the inclusion map $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}) \subseteq \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}})$ if and only if, for every integer $n \geq 0$, the following condition is satisfied:

$(*_n)$ Let E be the collection of all surjections $\alpha : [n] \twoheadrightarrow [m]$ in the category $\mathbf{\Delta}$. For each $\alpha \in E$, let $g_\alpha : X_m \to Y_n$ be a composition of $f_m : X_m \to Y_m$ with the morphism $\alpha^* : Y_m \to Y_n$. Then the collection $\{g_\alpha\}_{\alpha \in E}$ exhibit Y_n as a coproduct of the collection of objects $\{X_m\}_{(\alpha:[n] \twoheadrightarrow [m]) \in E}$.

Compare with Construction 3.3.1.6.

10.1.3 Skeletal Simplicial Objects

05G3 Recall that every set S can be regarded as a simplicial set, by identifying it with the constant functor

$$\underline{S} : \mathbf{\Delta}^{\mathrm{op}} \to \{S\} \hookrightarrow \mathrm{Set}.$$

This construction has a counterpart in an ∞-category:

05G4 **Definition 10.1.3.1.** Let $\mathcal{C}$ be an ∞-category. For each object $C \in \mathcal{C}$, we let $\underline{C}$ denote the simplicial object of $\mathcal{C}$ given by the constant functor

$$\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \to \{C\} \hookrightarrow \mathcal{C}.$$

We say that a simplicial object $X_\bullet$ of $\mathcal{C}$ is *constant* if it is equal to $\underline{C}$, for some object $C \in \mathcal{C}$ (in this case, we must have $C = X_0$). We say that $X_\bullet$ is *essentially constant* it is isomorphic to a constant simplicial object $\underline{C}$, for some $C \in \mathcal{C}$.

Proposition 10.1.3.2. *Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. The following conditions are equivalent:* 05G5

(1) *The simplicial object $X_\bullet$ is essentially constant: that is, there exists an isomorphism of simplicial objects $\alpha : \underline{C} \to X_\bullet$ for some object $C \in \mathcal{C}$.*

(2) *The functor $X_\bullet : \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \to \mathcal{C}$ carries each morphism in the category $\mathbf{\Delta}^{\mathrm{op}}$ to an isomorphism in the ∞-category $\mathcal{C}$.*

(3) *The functor $X_\bullet$ is left Kan extended from the full subcategory $\{[0]\} \subseteq \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}})$.*

(4) *For every simplicial object $Y_\bullet$ of $\mathcal{C}$, the restriction map*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}),\mathcal{C})}(X_\bullet, Y_\bullet) \to \mathrm{Hom}_{\mathcal{C}}(X_0, Y_0)$$

is a homotopy equivalence.

Proof. The equivalences (1) ⇔ (2) ⇔ (3) are special cases of Corollary 7.3.3.14, since $[0]$ is a final object of the category $\mathbf{\Delta}$. The equivalence (3) ⇔ (4) follows from Corollary 7.3.6.13. □

We now introduce a generalization of Definition 10.1.3.1.

Notation 10.1.3.3. Let $\mathbf{\Delta}$ denote the simplex category (Definition 1.1.0.2). For every integer n, we let $\mathbf{\Delta}^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}$ spanned by those objects $[m] = \{0 < 1 < \cdots < m\}$ where $0 \leq m \leq n$ (see Construction 1.1.3.9). 05G6

Definition 10.1.3.4. Let $\mathcal{C}$ be an ∞-category and let n be an integer. We say that a simplicial object $X_\bullet$ of $\mathcal{C}$ is *n-skeletal* if the functor 05G7

$$\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}$$

is left Kan extended from the full subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}^{\leq n})^{\mathrm{op}} \subseteq \mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}$.

Example 10.1.3.5. Let $\mathcal{C}$ be an ∞-category. A simplicial object $X_\bullet$ of $\mathcal{C}$ is 0-skeletal (in the sense of Definition 10.1.3.4) if and only if it is essentially constant (in the sense of Definition 10.1.3.1). See Proposition 10.1.3.2. 05G8

Example 10.1.3.6. Let $X_\bullet$ be a simplicial set and let n be an integer. Then $X_\bullet$ is n-skeletal (in the sense of Definition 10.1.3.4) if and only if it has dimension $\leq n$ (in the sense of Definition 1.1.3.1) . This is a reformulation of Proposition 1.1.3.11 (see Remark 1.1.3.12). 05G9

Exercise 10.1.3.7. Let n be an integer. Show that a simplicial abelian group $A_\bullet$ is n-skeletal (when regarded as a simplicial object of the ∞-category $\mathrm{N}_\bullet(\mathrm{Ab})$) if and only if the normalized Moore complex 05GA

$$\cdots \to \mathrm{N}_2(A) \xrightarrow{\partial} \mathrm{N}_1(A) \xrightarrow{\partial} \mathrm{N}_0(A)$$

is concentrated in degrees $\leq n$: that is, the abelian group $\mathrm{N}_k(A)$ vanishes for $k > n$. Beware that this condition does not guarantee that $A_\bullet$ is n-skeletal when regarded as a simplicial set.

05GB **Remark 10.1.3.8** (Monotonicity)**.** Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$ and let $m \leq n$ be integers. If $X_\bullet$ is m-skeletal, then it is also n-skeletal. See Corollary 7.3.8.8.

05GC **Remark 10.1.3.9.** In the formulation of Definition 10.1.3.4, we allow n to be an arbitrary integer. However, for $n < 0$, the notion becomes degenerate: a simplicial object $X_\bullet$ is n-skeletal if and only if each X_m is an initial object of $\mathcal{C}$. In this case, $X_\bullet$ is an initial object in the ∞-category of simplicial objects $\mathrm{Fun}(\mathrm{N}_\bullet(\boldsymbol{\Delta})^{\mathrm{op}}, \mathcal{C})$.

To make Definition 10.1.3.4 more explicit, it will be convenient to introduce an auxiliary construction.

05GD **Construction 10.1.3.10** (Degeneracy Cubes)**.** Fix an integer $k \geq 0$, set $K = \{1, 2, \cdots, k\}$, and let $\square^k = \square^K$ denote the simplicial cube of dimension k (Notation 2.4.5.2). Recall that $\square^k$ can be identified with the nerve of the set $P(K)$ of subsets of K, partially ordered with respect to inclusion.

For every subset $J \subseteq K$ having cardinality j, let $\alpha_J : [k] \twoheadrightarrow [j]$ denote the nondecreasing function function which carries an element $k' \in [k]$ to the cardinality of the intersection $J \cap \{1, 2, \cdots, k'\}$. The construction $J \mapsto \alpha_J$ then determines a functor from $P(K)^{\mathrm{op}}$ to the coslice category $\boldsymbol{\Delta}_{[k]/}$.

If $\mathcal{C}$ is an ∞-category and $X_\bullet$ is a simplicial object of $\mathcal{C}$, we let $\sigma_k : \square^k \to \mathcal{C}$ denote the map given by the composition

$$\square^k \simeq \mathrm{N}_\bullet(P(K)) \xrightarrow{J \mapsto \alpha_J} \mathrm{N}_\bullet(\boldsymbol{\Delta}_{[k]/})^{\mathrm{op}} \to \mathrm{N}_\bullet(\boldsymbol{\Delta})^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}\,.$$

We will refer to σ_k as the *kth degeneracy cube* of the simplicial object $X_\bullet$.

05GE **Example 10.1.3.11.** Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$. For small values of k, the degeneracy cube $\sigma_k : \square^k \to \mathcal{C}$ of Construction 10.1.3.10 can be described more concretely:

- The degeneracy cube σ_0 can be identified with the object X_0 of $\mathcal{C}$.
- The degeneracy cube σ_1 can be identified with the degeneracy operator $s_0^0 : X_0 \to X_1$ of Notation 10.1.0.4.

- The degeneracy cube σ_2 is a square diagram

$$\begin{tikzcd}
X_0 \arrow[r, "s_0^0"] \arrow[d, "s_0^0"] & X_1 \arrow[d, "s_0^1"] \\
X_1 \arrow[r, "s_1^1"] & X_2
\end{tikzcd}$$

  which witnesses the identity $[s_0^1] \circ [s_0^0] = [s_1^1] \circ [s_0^0]$ in the homotopy category $\mathrm{h}\mathcal{C}$.

Notation 10.1.3.12. Let k be a nonnegative integer. For every integer n, we let $\mathbf{\Delta}_{[k]/}^{\leq n}$ denote the full subcategory of the coslice category $\mathbf{\Delta}_{[k]/}$ spanned by objects which correspond to nondecreasing functions $[k] \to [m]$, where $m \leq n$. `05GF`

Lemma 10.1.3.13. *Let $k \geq 0$ and n be integers, let $K = \{1, \cdots, k\}$, and let $P^{\leq n}(K)$ denote the partially ordered collection of all subsets $J \subseteq K$ which have cardinality $\leq n$. Then the assignment $J \mapsto \alpha_J$ of Construction 10.1.3.10 determines a right cofinal functor of ∞-categories* `05GG`

$$\alpha : \mathrm{N}_\bullet(P^{\leq n}(K)) \to \mathrm{N}_\bullet(\mathbf{\Delta}_{[k]/}^{\leq n})^{\mathrm{op}}.$$

Proof. This is a special case of Corollary 7.2.3.7, since the functor α has a left adjoint (which carries a morphism $f : [k] \to [m]$ to the subset $J = \{j \in K : f(j-1) < f(j)\} \in P_{\leq n}(K)$). □

Proposition 10.1.3.14. *Let $\mathcal{C}$ be an ∞-category, let $X_\bullet$ be a simplicial object of $\mathcal{C}$, and let n be an integer. The following conditions are equivalent:* `05GH`

(1) *The simplicial object $X_\bullet$ is n-skeletal, in the sense of Definition 10.1.3.4.*

(2) *Let $k \geq 0$ be a nonnegative integer and set $K = \{1, 2, \cdots, k\}$. Then the kth degeneracy cube*

$$\sigma_k : \square^k = \mathrm{N}_\bullet(P(K)) \to \mathcal{C}$$

exhibits X_k as a colimit of the diagram $\sigma_k|_{\mathrm{N}_\bullet(P^{\leq n}(K))}$.

(3) *For every nonnegative integer $k > n$, the k-degeneracy cube σ_k is a colimit diagram in $\mathcal{C}$.*

Proof. The equivalence (1) ⇔ (2) follows immediately from Lemma 10.1.3.13 (together with Corollary 7.2.2.3). For each integer $k \geq 0$, set $K = \{1, 2, \cdots, k\}$ and consider the following conditions:

(2_k) The degeneracy cube $\sigma_k : \mathrm{N}_\bullet(P(K)) \to \mathcal{C}$ exhibits X_k as a colimit of the diagram $\sigma_k|_{\mathrm{N}_\bullet(P^{\leq n}(K))}$.

(3_k) The degeneracy cube σ_k is a colimit diagram in $\mathcal{C}$.

Note that condition (2_k) is automatic for $k \leq n$. We will complete the proof by showing that if $k > n$ and condition (2_ℓ) is satisfied for every integer $0 \leq \ell < k$, then conditions (2_k) and (3_k) are equivalent. Our hypothesis that condition (2_ℓ) is satisfied for $\ell < k$ guarantees that the functor $\sigma_k|_{\mathrm{N}_\bullet(P^{\leq k-1}(K))}$ is left Kan extended from the full subcategory $\mathrm{N}_\bullet(P^{\leq n}(K)) \subseteq \mathrm{N}_\bullet(P^{\leq k-1}(K))$. The equivalence of (2_k) and (3_k) is therefore a special case of Corollary 7.3.8.2. □

05GJ **Remark 10.1.3.15.** Let $\mathcal{C}$ be an ∞-category and let n be an integer. Using Proposition 10.1.3.14, we see that the condition that a simplicial object $X_\bullet$ of $\mathcal{C}$ is n-skeletal depends only on the restriction of $X_\bullet$ to the (non-full) subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{surj}})^{\mathrm{op}} \subset \mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}$ (see Notation 1.1.2.12). Stated more informally (and slightly incorrectly), the condition of n-skeletality depends only on the degeneracy operators of $X_\bullet$, and not on its face operators.

05GK **Corollary 10.1.3.16.** *Let $\mathcal{C}$ be an ∞-category and let $n \geq 0$ be an integer. Then a simplicial object $X_\bullet$ of $\mathcal{C}$ is $(n-1)$-skeletal if and only if it is n-skeletal and the degeneracy cube $\sigma_n : \square^n \to \mathcal{C}$ is a colimit diagram in $\mathcal{C}$.*

05GL **Corollary 10.1.3.17.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $n \geq 0$ be an integer. Assume that $\mathcal{C}$ admits pushouts and that F preserves pushouts. If $X_\bullet$ is an n-skeletal simplicial object of $\mathcal{C}$, then $F(X_\bullet)$ is an n-skeletal simplicial object of $\mathcal{D}$.*

Proof. Fix an integer $k > n$, and let $\sigma_k : \square^k \to \mathcal{C}$ be the kth degeneracy cube of the simplicial object $X_\bullet$. Set $K = \{1, 2, \cdots, k\}$, and let $P^{>0}(K)$ denote the collection of all nonempty subsets of K. Then σ_k can be identified with a functor σ_k° from $\mathrm{N}_\bullet(P^{>0}(K))$ to the coslice ∞-category $\mathcal{C}_{X_0/}$. Our assumption that $X_\bullet$ is n-skeletal guarantees that σ_k is a colimit diagram in $\mathcal{C}$ (Proposition 10.1.3.14), or equivalently that σ_k° is a colimit diagram in the ∞-category $\mathcal{C}_{X_0/}$ (Remark 7.1.2.11). Since the functor F preserves pushouts, the induced functor of coslice ∞-categories $F_{X_0/} : \mathcal{C}_{X_0/} \to \mathcal{D}_{F(X_0)/}$ preserves finite colimits (Example 7.6.3.28). In particular, $F_{X_0/} \circ \sigma_k^\circ$ is a colimit diagram in the ∞-category $\mathcal{D}_{F(X_0)/}$, so that $F \circ \sigma_k$ is a colimit diagram in $\mathcal{D}$ (Remark 7.1.2.11). Allowing k to vary, we conclude that $F(X_\bullet)$ is an n-skeletal simplicial object of $\mathcal{D}$ (Proposition 10.1.3.14). □

Let $X = X_\bullet$ be a simplicial set. Recall that the *n-skeleton of X* is the largest simplicial subset $\mathrm{sk}_n(X) \subseteq X$ of dimension $\leq n$ (see Construction 1.1.4.1 and Corollary 1.1.4.7). This construction has a counterpart for more general simplicial objects.

05GM **Definition 10.1.3.18.** Let $\mathcal{C}$ be an ∞-category, let $u : Y_\bullet \to X_\bullet$ be a morphism between simplicial objects of $\mathcal{C}$, and let n be an integer. We will say that u *exhibits $Y_\bullet$ as an n-skeleton of $X_\bullet$* if the following conditions are satisfied:

- The simplicial object $Y_\bullet$ is n-skeletal.

- For $0 \leq m \leq n$, the induced map $Y_m \to X_m$ is an isomorphism in the ∞-category $\mathcal{C}$.

Example 10.1.3.19. Let $X = X_\bullet$ be a simplicial set. For every integer n, the inclusion of simplicial sets $\mathrm{sk}_n(X) \hookrightarrow X$ exhibits $\mathrm{sk}_n(X)$ as an n-skeleton of X, in the sense of Definition 10.1.3.18: see Proposition 1.1.4.6. `05GN`

Remark 10.1.3.20 (Uniqueness)**.** Let $\mathcal{C}$ be an ∞-category, let $X_\bullet$ be a simplicial object of $\mathcal{C}$, and let n be an integer. If there exists a morphism of simplicial objects $u : Y_\bullet \to X_\bullet$ which exhibits $Y_\bullet$ as an n-skeleton of $X_\bullet$, then $Y_\bullet$ is uniquely determined up to isomorphism and depends functorially on $X_\bullet$. To emphasize this dependence, we will denote the object $Y_\bullet$ by $\mathrm{sk}_n(X)$ and refer to to it as the *n-skeleton of $X_\bullet$*. By virtue of Example 10.1.3.19, this reduces to the standard definition in the special case where $\mathcal{C}$ is (the nerve of) the category of sets. `05GP`

Using Corollary 7.3.6.13, we see that the n-skeleton of a simplicial object $X_\bullet$ is characterized by the following universal mapping property:

- If $Z_\bullet$ is any n-skeletal simplicial object of $\mathcal{C}$, then composition with u induces a homotopy equivalence of mapping spaces

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C})}(Z_\bullet, \mathrm{sk}_n(X_\bullet)) \to \mathrm{Hom}_{\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C})}(Z_\bullet, X_\bullet).$$

Example 10.1.3.21. Let $A_\bullet$ be a simplicial abelian group and let $\mathrm{N}_*(A)$ denote the normalized Moore complex of $A_\bullet$ (Construction 2.5.5.7). For every integer $n \geq 0$, let $\mathrm{N}_{\leq n}(A)$ denote the subcomplex of $\mathrm{N}_*(A)$ depicted in the diagram `05GQ`

$$\cdots \to 0 \to \mathrm{N}_n(A) \xrightarrow{\partial} \mathrm{N}_{n-1}(A) \to \cdots \to \mathrm{N}_1(A) \xrightarrow{\partial} \mathrm{N}_0(A) \to 0 \to \cdots$$

Then the inclusion map

$$\mathrm{K}(\mathrm{N}_{\leq n}(A)) \hookrightarrow \mathrm{K}(\mathrm{N}_*(A)) \simeq A_\bullet$$

exhibits the Eilenberg-MacLane space $\mathrm{K}(\mathrm{N}_{\leq n}(A))$ as an n-skeleton of $A_\bullet$ in the category of simplicial abelian groups. Beware that the image of this inclusion is usually larger than the n-skeleton of $A_\bullet$ as a simplicial set (see Exercise 10.1.3.7).

Example 10.1.3.22 (0-Skeleta)**.** Let $\mathcal{C}$ be an ∞-category, let $X_\bullet$ be a simplicial object of $\mathcal{C}$, and set $C = X_0$. Then the constant simplicial object $\underline{C}$ is an n-skeleton of $X_\bullet$. More precisely, the identity morphism $\mathrm{id} : C \xrightarrow{\sim} X_0$ admits an (essentially unique) extension to a morphism of simplicial objects $\underline{C} \to X_\bullet$ which exhibits $\underline{C}$ as a 0-skeleton of $X_\bullet$. `05GR`

Proposition 10.1.3.23 (Existence of Skeleta)**.** *Let $\mathcal{C}$ be an ∞-category and let $n \geq 0$ be an integer. If $\mathcal{C}$ admits pushouts, then every simplicial object $X_\bullet$ of $\mathcal{C}$ admits an n-skeleton.* `05GS`

Proof. We will show that the functor

$$\mathrm{N}_\bullet(\mathbf{\Delta}^{\leq n})^{\mathrm{op}} \hookrightarrow \mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}$$

admits a left Kan extension $Y_\bullet : \mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{C}$; Corollary 7.3.6.13 then guarantees that there is an (essentially unique) morphism of simplicial objects $u : Y_\bullet \to X_\bullet$ which is the identity when restricted to $\mathrm{N}_\bullet(\mathbf{\Delta}^{\leq n})^{\mathrm{op}}$. By virtue of Corollary 7.3.5.8, it will suffice to show that for every integer $k > n$, the diagram

$$F : \mathrm{N}_\bullet(\mathbf{\Delta}^{\leq n}_{[k]/})^{\mathrm{op}} \to \mathrm{N}_\bullet(\mathbf{\Delta}^{\leq n})^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}$$

admits a colimit. Set $K = \{1, 2, \cdots, k\}$, let $P^{\leq n}(K)$ denote the collection of all subsets of K having cardinality $\leq n$, and let F_0 denote the composition of F with the right cofinal functor

$$\mathrm{N}_\bullet(P^{\leq n}(K)) \to \mathrm{N}_\bullet(\mathbf{\Delta}^{\leq n}_{[k]/})^{\mathrm{op}}$$

supplied by Lemma 10.1.3.13. Let $Q \subseteq P^{\leq n}(K)$ denote the collection of nonempty subsets of K of cardinality $\leq n$, so that F_0 can be identified with a functor $G : \mathrm{N}_\bullet(Q) \to \mathcal{C}_{X_0/}$. Since $\mathcal{C}$ admits pushouts, the coslice ∞-category $\mathcal{C}_{X_0/}$ admits finite limits (Example 7.6.3.28). In particular, the functor G admits a colimit in $\mathcal{C}_{X_0/}$, which we can identify with a colimit of F_0 in the ∞-category $\mathcal{C}$ (Remark 7.1.2.11). □

`05GT` **Warning 10.1.3.24.** If $X = X_\bullet$ is a simplicial set, then the comparison map $\mathrm{sk}_n(X) \to X$ is a monomorphism of simplicial sets. Beware that the analogous statement is generally false for simplicial objects of more general ∞-categories.

10.1.4 Coskeletal Simplicial Objects

`05GU` Let S be a set. For each $n \geq 0$, we let $\check{\mathrm{C}}_n(S) = \mathrm{Hom}([n], S)$ denote the collection of functions from the set $[n] = \{0 < 1 < \cdots < n\}$ into S. The construction $[n] \mapsto \check{\mathrm{C}}_n(S)$ determines a simplicial set $\check{\mathrm{C}}_\bullet(S)$, which we will refer to as the *Čech nerve* of S. In this section, we study an ∞-categorical counterpart of this construction.

`04SW` **Definition 10.1.4.1.** Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. We will say that $X_\bullet$ is a *Čech nerve* if, for every integer $n \geq 0$, the following condition is satisfied:

$(*_n)$ For $0 \leq i \leq n$, let $\nu_i : X_n \to X_0$ be the morphism of $\mathcal{C}$ induced by the inclusion $[0] \simeq \{i\} \subseteq [n]$. Then the morphisms $\{\nu_i\}_{0 \leq i \leq n}$ exhibit X_n as a product of $(n+1)$-copies of X_0.

`04SX` **Remark 10.1.4.2.** Let $\mathcal{C}$ be an ∞-category and let $C_\bullet : \mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object of $\mathcal{C}$. Then $X_\bullet$ is a Čech nerve if and only if it is right Kan extended from the full subcategory of $\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}$ spanned by the object $[0]$.

Definition 10.1.4.3. Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. We will say that a simplicial object $X_\bullet$ of $\mathcal{C}$ is a *Čech nerve of* X if $X_\bullet$ is a Čech nerve (in the sense of Definition 10.1.4.1) and $X_0 = X$. 04SY

Notation 10.1.4.4. Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$ which admits a Čech nerve $X_\bullet$. It follows from Corollary 7.3.6.13 that, for every simplicial object $Y_\bullet$ of $\mathcal{C}$, the restriction map 04SZ

$$\mathrm{Hom}_{\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}},\mathcal{C})}(Y_\bullet, X_\bullet) \to \mathrm{Hom}_{\mathcal{C}}(Y_0, X_0) = \mathrm{Hom}_{\mathcal{C}}(Y_0, X)$$

is a homotopy equivalence. In particular, the simplicial object $X_\bullet$ is unique up to isomorphism and depends functorially on X. To emphasize this dependence, we will denote $X_\bullet$ by $\check{\mathrm{C}}_\bullet(X)$ and refer to it as *the* Čech nerve of the object X.

Proposition 10.1.4.5 (Existence)**.** *Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. Then X admits a Čech nerve $\check{\mathrm{C}}_\bullet(X)$ if and only if, for every nonempty finite set J, there exists a product of J copies of X in the ∞-category $\mathcal{C}$.* 04T0

Proof. For every integer $n \geq 0$, the category $\{[0]\} \times_{\mathbf{\Delta}} \mathbf{\Delta}_{/[n]}$ is isomorphic to the finite set $\{0, 1, \cdots, n\}$, regarded as a category having only identity morphisms. By virtue of Remark 10.1.4.2, the desired result is a special case of the existence criterion for Kan extensions (Corollary 7.3.5.8). □

Corollary 10.1.4.6. *Let $\mathcal{C}$ be an ∞-category which admits finite products. Then every object $X \in \mathcal{C}$ admits a Čech nerve $\check{\mathrm{C}}_\bullet(X)$.* 04T1

Remark 10.1.4.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves finite products. Then the induced functor $\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{D})$ carries Čech nerves to Čech nerves. In particular, if X is an object of $\mathcal{C}$ which admits a Čech nerve $\check{\mathrm{C}}_\bullet(X)$, then the image $Y = F(X)$ also admits a Čech nerve, given by $\check{\mathrm{C}}_\bullet(Y) = F(\check{\mathrm{C}}_\bullet(X))$. 04T2

Corollary 10.1.4.8. *Let $\mathcal{C}$ be an ∞-category which admits finite products. Then the evaluation functor* 04T3

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C} \qquad X_\bullet \mapsto X_0$$

admits a right adjoint, given on objects by the Čech nerve $X \mapsto \check{\mathrm{C}}_\bullet(X)$.

Proof. Combine Corollaries 10.1.4.6 and 7.3.6.4. □

We now introduce a generalization of Definition 10.1.4.1.

Definition 10.1.4.9. Let $\mathcal{C}$ be an ∞-category and let n be an integer. We say that a simplicial object $X_\bullet$ of $\mathcal{C}$ is *n-coskeletal* if the functor 05GV

$$\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}$$

is right Kan extended from the full subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}^{\leq n})^{\mathrm{op}} \subseteq \mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}$.

05GW **Example 10.1.4.10.** Let $\mathcal{C}$ be an ∞-category. A simplicial object $X_\bullet$ of $\mathcal{C}$ is 0-coskeletal if and only if it is a Čech nerve (Remark 10.1.4.2).

05GX **Example 10.1.4.11.** Let $X_\bullet$ be a simplicial set and let n be an integer. The following conditions are equivalent:

- The simplicial set $X_\bullet$ is n-coskeletal in the sense of Definition 3.5.3.1: that is, the restriction map $\operatorname{Hom}_{\mathrm{Set}_\Delta}(\Delta^m, X) \to \operatorname{Hom}_{\mathrm{Set}_\Delta}(\partial\Delta^m, X)$ is bijective for $m > n$.
- The simplicial set $X_\bullet$ is n-coskeletal in the sense of Definition 10.1.4.9: that is, it is a right Kan extension of its restriction to (the opposite of) the subcategory $\mathbf{\Delta}^{\leq n} \subset \mathbf{\Delta}$.

This is a restatement of Corollary 3.5.3.13 (see Remark 3.5.3.14).

05GY **Remark 10.1.4.12** (Monotonicity)**.** Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$ and let $m \leq n$ be integers. If $X_\bullet$ is m-coskeletal, then it is also n-coskeletal. See Corollary 7.3.8.8.

05GZ **Remark 10.1.4.13.** In the formulation of Definition 10.1.4.9, we allow n to be an arbitrary integer. However, for $n < 0$, the notion becomes degenerate: a simplicial object $X_\bullet$ is n-coskeletal if and only if each X_m is a final object of $\mathcal{C}$. In this case, $X_\bullet$ is a final object of the ∞-category of simplicial objects $\operatorname{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta})^{\mathrm{op}}, \mathcal{C})$.

Definition 10.1.4.9 has a counterpart for semisimplicial objects.

05H0 **Variant 10.1.4.14.** For every integer n, we let $\mathbf{\Delta}_{\mathrm{inj}}^{\leq n} = \mathbf{\Delta}_{\mathrm{inj}} \cap \mathbf{\Delta}^{\leq n}$ denote the category whose objects are linearly ordered sets $[m] = \{0 < 1 < \cdots < m\}$ for $0 \leq m \leq n$, and whose morphisms are strictly increasing functions. If $\mathcal{C}$ is an ∞-category, we say that a semisimplicial object $X_\bullet : \mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}})^{\mathrm{op}} \to \mathcal{C}$ is *n-coskeletal* if it is right Kan extended from the full subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{inj}}^{\leq n})^{\mathrm{op}}$.

05H1 **Proposition 10.1.4.15.** *Let $\mathcal{C}$ be an ∞-category, let n be an integer, and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. Then $X_\bullet$ is n-coskeletal (in the sense of Definition 10.1.4.9) if and only if its underlying semisimplicial object is n-coskeletal (in the sense of Variant 10.1.4.14).*

Proof. Fix an integer $k \geq 0$, $\mathbf{\Delta}_{/[k]}^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}_{/[k]}$ spanned by those objects which correspond to nondecreasing functions $\alpha : [m] \to [k]$ where $m \leq n$, and let $\mathcal{J}$ be the full subcategory of $\mathbf{\Delta}_{/[k]}^{\leq n}$ spanned by those objects where α is strictly increasing. Unwinding the definitions, we see that $X_\bullet$ is n-coskeletal if and only if, for every integer $k \geq 0$, the composite functor

$$F : \mathrm{N}_\bullet((\mathbf{\Delta}_{/[k]}^{\leq n}))^{\triangleright} \hookrightarrow \mathrm{N}_\bullet(\mathbf{\Delta}_{/[k]}^{\triangleright}) \to \mathrm{N}_\bullet(\mathbf{\Delta}) \xrightarrow{X_\bullet} \mathcal{C}^{\mathrm{op}}$$

is a colimit diagram in the ∞-category $\mathcal{C}^{\mathrm{op}}$. Similarly, the underlying semisimplicial object of $X_\bullet$ is n-coskeletal if and only if, for every integer $k \geq 0$, the restriction $F|_{\mathrm{N}_\bullet(\mathcal{J})^{\triangleright}}$ is a colimit diagram in $\mathcal{C}^{\mathrm{op}}$. Consequently, to show that these conditions are equivalent, it will suffice to prove that the inclusion functor $\mathrm{N}_\bullet(\mathcal{J}) \hookrightarrow \mathrm{N}_\bullet(\mathbf{\Delta}^{\leq n}_{/[k]})$ is right cofinal (Corollary 7.2.2.3). This is a special case of Corollary 7.2.3.7, since the inclusion functor $\mathcal{J} \hookrightarrow \mathbf{\Delta}^{\leq n}_{/[k]}$ has a left adjoint (which carries a nondecreasing function $\alpha : [m] \to [k]$ to the inclusion map $\mathrm{im}(\alpha) \hookrightarrow [k]$; here we abuse notation by identifying $\mathrm{im}(\alpha)$ with the corresponding object of $\mathbf{\Delta}$). □

Corollary 10.1.4.16. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves finite limits and let n be an integer. Then:* 05H2

(1) *If $X_\bullet$ is an n-coskeletal simplicial object of $\mathcal{C}$, then $F(X_\bullet)$ is an n-coskeletal simplicial object of $\mathcal{D}$.*

(2) *If $X_\bullet$ is an n-coskeletal semisimplicial object of $\mathcal{C}$, then $F(X_\bullet)$ is an n-coskeletal semisimplicial object of $\mathcal{D}$.*

Proof. Assertion (2) is immediate from the definitions (since the category $\mathcal{J}$ appearing in the proof of Proposition 10.1.4.15 is a finite partially ordered set). Assertion (1) follows by combining (2) with Proposition 10.1.4.15. □

Recall that a morphism of simplicial sets $f : X_\bullet \to Y_\bullet$ exhibits $Y_\bullet$ as an n-coskeleton of $X_\bullet$ if $Y_\bullet$ is n-coskeletal and f is bijective on m-simplices for $m \leq n$. This notion has an obvious counterpart for simplicial objects in general:

Definition 10.1.4.17. Let $\mathcal{C}$ be an ∞-category, let $u : X_\bullet \to Y_\bullet$ be a morphism between simplicial objects of $\mathcal{C}$, and let n be an integer. We will say that u *exhibits $Y_\bullet$ as an n-coskeleton of $X_\bullet$* if the following conditions are satisfied: 05H3

- The simplicial object $Y_\bullet$ is n-coskeletal.
- For $0 \leq m \leq n$, the induced map $X_m \to Y_m$ is an isomorphism in the ∞-category $\mathcal{C}$.

Remark 10.1.4.18. Definition 10.1.4.17 has an obvious counterpart for semisimplicial objects. If $u : X_\bullet \to Y_\bullet$ is a morphism between semisimplicial objects of an ∞-category $\mathcal{C}$, we say that u *exhibits $Y_\bullet$ as an n-coskeleton of $X_\bullet$* if $Y_\bullet$ is n-coskeletal and the morphism u induces an isomorphism $X_m \to Y_m$ for $0 \leq m \leq n$. By virtue of Proposition 10.1.4.15, this recovers Definition 10.1.4.17 in the case where u arises from a morphism between simplicial objects of $\mathcal{C}$. 05H4

05H5 **Remark 10.1.4.19** (Uniqueness). Let $\mathcal{C}$ be an ∞-category, let $X_\bullet$ be a simplicial object of $\mathcal{C}$, and let n be an integer. If there exists a morphism of simplicial objects $u : X_\bullet \to Y_\bullet$ which exhibits $Y_\bullet$ as an n-coskeleton of $X_\bullet$, then $Y_\bullet$ is uniquely determined up to isomorphism and depends functorially on $X_\bullet$. To emphasize this dependence, we will denote the object $Y_\bullet$ by $\mathrm{cosk}_n(X)_\bullet$ and refer to to it as the *n-skeleton of $X_\bullet$*. In the special case where $\mathcal{C}$ is (the nerve of) the category of sets, this recovers the convention of Notation 3.5.3.18.

Using Corollary 7.3.6.13, we see that the n-skeleton of a simplicial object $X_\bullet$ is characterized by the following universal mapping property:

- If $Z_\bullet$ is any n-coskeletal simplicial object of $\mathcal{C}$, then composition with u induces a homotopy equivalence of mapping spaces

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{N}_\bullet(\boldsymbol{\Delta})^{\mathrm{op}},\mathcal{C})}(\mathrm{cosk}_n(X)_\bullet, Z_\bullet) \to \mathrm{Hom}_{\mathrm{Fun}(\mathrm{N}_\bullet(\boldsymbol{\Delta})^{\mathrm{op}},\mathcal{C})}(X_\bullet, Z_\bullet).$$

05H6 **Example 10.1.4.20.** Let $\mathcal{C}$ be an ∞-category and let $C_\bullet$ be a simplicial object of $\mathcal{C}$. If the object $X = C_0$ admits a Čech nerve $\check{\mathrm{C}}_\bullet(X)$ (Definition 10.1.4.3), then the identity map $C_0 \to X$ can be promoted to a morphism of simplicial objects $C_\bullet \to \check{\mathrm{C}}_\bullet(X)$ (see Notation 10.1.4.4) which exhibits $\check{\mathrm{C}}_\bullet(X)$ as a 0-coskeleton of $C_\bullet$.

05H7 **Proposition 10.1.4.21** (Existence of Coskeleta). *Let $\mathcal{C}$ be an ∞-category which admits finite limits and let n be an integer. Then every (semi)simplicial object $X_\bullet$ of $\mathcal{C}$ admits an n-coskeleton $\mathrm{cosk}_n(X)_\bullet$.*

Proof. We will prove the assertion for simplicial objects; the analogous statement for semisimplicial objects is similar (but easier). It will suffice to show that the functor

$$\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\leq n})^{\mathrm{op}} \hookrightarrow \mathrm{N}_\bullet(\boldsymbol{\Delta})^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}$$

admits a right Kan extension $Y_\bullet : \mathrm{N}_\bullet(\boldsymbol{\Delta})^{\mathrm{op}} \to \mathcal{C}$; Corollary 7.3.6.13 then guarantees that there is an (essentially unique) morphism of simplicial objects $u : X_\bullet \to Y_\bullet$ which is the identity when restricted to $\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\leq n})^{\mathrm{op}}$. By virtue of Corollary 7.3.5.8, it will suffice to show that for every integer k, the diagram

$$G : \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\leq n}_{/[k]})^{\mathrm{op}} \to \mathrm{N}_\bullet(\boldsymbol{\Delta})^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}$$

admits a limit. As in the proof of Proposition 10.1.4.15, we observe that the inclusion map $\mathrm{N}_\bullet(\mathcal{J}) \hookrightarrow \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\leq n}_{/[k]})$ is right cofinal, where $\mathcal{J} \subseteq \boldsymbol{\Delta}^{\leq n}_{/[k]}$ is the full subcategory spanned by the injective maps $[m] \hookrightarrow [k]$. We are therefore reduced to showing that $G|_{\mathrm{N}_\bullet(\mathcal{J})^{\mathrm{op}}}$ has a limit in $\mathcal{C}$ (Corollary 7.2.2.10), which follows from our assumption that $\mathcal{C}$ admits finite limits (since $\mathcal{J}$ is the category associated to a finite partially ordered set). $\square$

10.1.5 The Čech Nerve of a Morphism

We now consider a relative version of Definition 10.1.4.1. 04SV

Definition 10.1.5.1. Let $\mathcal{C}$ be an ∞-category and let $C_\bullet$ be an augmented simplicial object of $\mathcal{C}$, which we identify with a simplicial object $C'_\bullet$ of the ∞-category $\mathcal{C}_{/C_{-1}}$ (see Remark 10.1.1.15). We will say that $C_\bullet$ is a *Čech nerve* if the simplicial object $C'_\bullet$ is a Čech nerve in the ∞-category $\mathcal{C}_{/C_{-1}}$ (see Definition 10.1.4.1). 04T4

Remark 10.1.5.2. Let $\mathcal{C}$ be an ∞-category. Stated more informally, an augmented simplicial object $C_\bullet$ of $\mathcal{C}$ is a Čech nerve if, for every integer $n \geq 0$, it exhibits C_n as an iterated fiber product 04T5

$$C_0 \times_{C_{-1}} C_0 \times_{C_{-1}} \cdots \times_{C_{-1}} C_0$$

(where the factor C_0 appears $n+1$ times).

Remark 10.1.5.3. In the augmented simplex category $\mathbf{\Delta}_+$, there is unique morphism $\delta^0_0 : [-1] \to [0]$. This morphism determines a fully faithful functor $[1] \to \mathbf{\Delta}_+$, whose image is the full subcategory $\mathbf{\Delta}_+^{\leq 0} \subseteq \mathbf{\Delta}_+$ spanned by the objects $[0]$ and $[-1]$. Combining Remarks 10.1.4.2 and 7.3.2.4, we see that an augmented simplicial object $X_\bullet$ of an ∞-category $\mathcal{C}$ is a Čech nerve if and only if it is right Kan extended from the subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}_+^{\leq 0})^{\mathrm{op}} \subset \mathrm{N}_\bullet(\mathbf{\Delta}_+)^{\mathrm{op}}$. 04T6

Definition 10.1.5.4. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. We will say that an augmented simplicial object $C_\bullet$ of $\mathcal{C}$ is a *Čech nerve of f* if $C_\bullet$ is a Čech nerve (in the sense of Definition 10.1.5.1) and the face operator $d^0_0 : C_0 \to C_{-1}$ coincides with the morphism f (so that $C_0 = X$ and $C_{-1} = Y$). 04T7

Notation 10.1.5.5. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. It follows from Remarks 10.1.5.3 and 7.3.6.6 that if f admits a Čech nerve $C_\bullet$, then the augmented simplicial object $C_\bullet$ is determined up to isomorphism and depends functorially on f. To emphasize this dependence, we will denote $C_\bullet$ by $\check{\mathrm{C}}_\bullet(X/Y)$ and refer to it as *the* Čech nerve of the morphism $f : X \to Y$. Alternatively, we can identify $\check{\mathrm{C}}_\bullet(X/Y)$ with the simplicial object of $\mathcal{C}_{/Y}$ given by the Čech nerve of f (in the sense of Notation 10.1.4.4). 04T8

Proposition 10.1.5.6. *Let $\mathcal{C}$ be an ∞-category which admits pullbacks. Then every morphism $f : X \to Y$ in $\mathcal{C}$ admits a Čech nerve $\check{\mathrm{C}}_\bullet(X/Y)$.* 04T9

Proof. Apply Corollary 10.1.4.6 to the ∞-category $\mathcal{C}_{/Y}$, which admits finite products by virtue of our assumption that $\mathcal{C}$ admits pullbacks (Corollary 7.6.3.20). □

Remark 10.1.5.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves fiber products. Then the induced functor of augmented simplicial objects 04TA

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}), \mathcal{C}) \to \mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}), \mathcal{D})$$

carries Čech nerves to Čech nerves (see Remark 10.1.4.7). In particular, if $u : X \to Y$ is a morphism of $\mathcal{C}$ which admits a Čech nerve $\check{C}_\bullet(X/Y)$, then the morphism $F(u) : F(X) \to F(Y)$ admits a Čech nerve in the ∞-category $\mathcal{D}$, given by $F(\check{C}_\bullet(X/Y))$.

04TB **Corollary 10.1.5.8.** *Let $\mathcal{C}$ be an ∞-category which admits pullbacks. Then the forgetful functor*

$$\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}), \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \qquad C_\bullet \mapsto (d_0^0 : C_0 \to C_{-1})$$

admits a right adjoint, given on objects by the construction $(f : X \to Y) \mapsto \check{C}_\bullet(X/Y)$.

Proof. Combine Proposition 10.1.5.6 with Corollary 7.3.6.4. □

It will sometimes be useful to consider a generalization of Definition 10.1.5.1.

05H8 **Notation 10.1.5.9.** Let $\mathbf{\Delta}_+$ be the augmented simplex category (Definition 10.1.1.10). For every integer n, we let $\mathbf{\Delta}_+^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}_+$ spanned by the collection of objects $\{[m]\}_{-1 \leq m \leq n}$.

05H9 **Definition 10.1.5.10.** Let $\mathcal{C}$ be an ∞-category and let n be an integer. We say that an augmented simplicial object $C_\bullet$ of $\mathcal{C}$ is *n-coskeletal* if the functor

$$C_\bullet : \mathrm{N}_\bullet(\mathbf{\Delta}_+)^{\mathrm{op}} \to \mathcal{C}$$

is right Kan extended from the full subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}_+^{\leq n})^{\mathrm{op}}$.

05HA **Example 10.1.5.11.** Let $\mathcal{C}$ be an ∞-category and let $C_\bullet$ be an augmented simpicial object of $\mathcal{C}$. Then $C_\bullet$ is 0-coskeletal (in the sense of Definition 10.1.5.10) if and only if it is a Čech nerve (in the sense of Definition 10.1.5.1). See Remark 10.1.5.3.

05HB **Example 10.1.5.12.** Let $\mathcal{C}$ be an ∞-category. For $n \leq -2$, an augmented simplicial object $C_\bullet$ of $\mathcal{C}$ is n-coskeletal if and only if each C_m is a final object of $\mathcal{C}$.

05HC **Example 10.1.5.13.** Let $\mathcal{C}$ be an ∞-category and let $C_\bullet$ be an augmented simplicial object of $\mathcal{C}$. The following conditions are equivalent:

(1) The augmented simplicial object $C_\bullet$ is (-1)-coskeletal, in the sense of Definition 10.1.5.10.

(2) The augmented simplicial object $C_\bullet$ is essentially constant: that is, it is isomorphic to a constant functor from $\mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}})$ to $\mathcal{C}$.

(3) The functor $C_\bullet : \mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}) \to \mathcal{C}$ carries each morphism in the category $\mathbf{\Delta}_+^{\mathrm{op}}$ to an isomorphism in the ∞-category $\mathcal{C}$.

(4) For every augmented simplicial object $X_\bullet$ of $\mathcal{C}$, the restriction map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}), \mathcal{C})}(X_\bullet, C_\bullet) \to \mathrm{Hom}_{\mathcal{C}}(X_{-1}, C_{-1})$$

is a homotopy equivalence.

The equivalences (1) ⇔ (2) ⇔ (3) are special cases of Corollary 7.3.3.14, since $[-1]$ is an initial object of the category $\mathbf{\Delta}_{+}$. The equivalence (1) ⇔ (4) follows from Corollary 7.3.6.13.

Variant 10.1.5.14. For every integer n, we let $\mathbf{\Delta}_{+,\mathrm{inj}}^{\leq n}$ denote the category whose objects are linearly ordered sets $[m] = \{0 < 1 < \cdots < n\}$ for $-1 \leq m \leq n$, and whose morphisms are strictly increasing functions. We say that an augmented semisimplicial object $C_\bullet$ of an ∞-category $\mathcal{C}$ is *n-coskeletal* if the functor 05HD

$$C_\bullet : \mathrm{N}_\bullet(\mathbf{\Delta}_{+,\mathrm{inj}})^{\mathrm{op}} \to \mathcal{C}$$

is a right Kan extension of its restriction to $\mathrm{N}_\bullet(\mathbf{\Delta}_{+,\mathrm{inj}}^{\leq n})^{\mathrm{op}}$.

Remark 10.1.5.15. Let $\mathcal{C}$ be an ∞-category and let $C_\bullet$ be an augmented simplicial object of $\mathcal{C}$, which we identify with a (semi)simplicial object $C'_\bullet$ of the slice ∞-category $\mathcal{C}_{/C_{-1}}$ (see Remark 10.1.1.15). Then, for $n \geq -1$, the augmented simplicial object $C_\bullet$ is n-coskeletal (in the sense of Definition 10.1.5.10) if and only if the simplicial object $C'_\bullet$ is n-coskeletal (in the sense of Definition 10.1.4.9). Moreover, the analogous statement holds for semisimplicial objects. See Remark 7.3.2.4. 05HE

Warning 10.1.5.16. Let $\mathcal{C}$ be an ∞-category and let $C_\bullet$ be an augmented simplicial object of $\mathcal{C}$. It follow from Remark 10.1.5.15 that if C_{-1} is a final object of $\mathcal{C}$, then $C_\bullet$ is n-coskeletal if and only if its underlying simplicial object is n-coskeletal. Beware that neither implication holds in general if we do not assume that C_{-1} is final. 05HF

Remark 10.1.5.17. Let $\mathcal{C}$ be an ∞-category and let $C_\bullet$ be an augmented simplicial object of $\mathcal{C}$. For every integer n, the augmented simplicial object $C_\bullet$ is n-coskeletal (in the sense of Definition 10.1.5.10) if and only if its underlying augmented semisimplicial object is n-coskeletal (in the sense of Variant 10.1.5.14). For $n \leq -2$, this is trivial (see Example 10.1.5.12). For $n \geq -1$, it follows by combining Remark 10.1.5.15 with Proposition 10.1.4.15 (applied to the slice ∞-category $\mathcal{C}_{/C_{-1}}$). 05HG

To make Definition 10.1.5.10 more concrete, it will be convenient to introduce a dual version of Construction 10.1.3.10.

Construction 10.1.5.18 (Face Cubes)**.** Fix an integer $k \geq -1$, and let $\square^{k+1}$ be the simplicial cube of dimension $k+1$ (Notation 2.4.5.2). In what follows, we will identify $\square^{k+1}$ with the opposite of the nerve of the partially ordered set $P([k])$ of all subsets of $[k] = \{0 < 1 < \cdots < k\}$. Note that there is an isomorphism of categories $P([k]) \to (\mathbf{\Delta}_{+,\mathrm{inj}})_{/[k]}$, which carries each subset $J \subseteq [k]$ of cardinality $j+1$ to the unique strictly increasing function $[j] \hookrightarrow [k]$ having image J. If $C_\bullet$ is an augmented semisimplicial object of an ∞-category $\mathcal{C}$, we let $\tau_k : \square^{k+1} \to \mathcal{C}$ denote the composite functor 05HH

$$\square^{k+1} \simeq \mathrm{N}_\bullet((\mathbf{\Delta}_{+,\mathrm{inj}})_{/[k]})^{\mathrm{op}} \to \mathrm{N}_\bullet(\mathbf{\Delta}_{+,\mathrm{inj}}^{\mathrm{op}}) \xrightarrow{C_\bullet} \mathcal{C}\,.$$

We will refer to τ_k as the *kth face cube* of the augmented semisimplicial object $C_\bullet$.

05HJ **Example 10.1.5.19.** Let $C_\bullet$ be an augmented semisimplicial object of an ∞-category $\mathcal{C}$. For small values of k, the face cube $\tau_k : \square^{k+1} \to \mathcal{C}$ of Construction 10.1.5.18 can be described more explicitly:

- The face cube τ_{-1} can be identified with the object C_{-1} of $\mathcal{C}$.
- The face cube τ_0 can be identified with the face operator $d^0_0 : C_0 \to C_{-1}$.
- The face cube τ_1 is a square diagram

$$\begin{tikzcd}
C_1 \arrow[r, "d^1_0"] \arrow[d, "d^1_1"] & C_0 \arrow[d, "d^0_0"] \\
C_0 \arrow[r, "d^0_0"] & C_{-1}
\end{tikzcd}$$

which witnesses the identity $[d^0_0] \circ [d^1_0] = [d^0_0] \circ [d^1_1]$ in the homotopy category $\mathrm{h}\mathcal{C}$.

05HK **Proposition 10.1.5.20.** *Let $\mathcal{C}$ be an ∞-category, let $C_\bullet$ be an augmented semisimplicial object of $\mathcal{C}$, and let n be an integer. The following conditions are equivalent:*

(1) *The augmented semisimplicial object $C_\bullet$ is $(n-1)$-coskeletal, in the sense of Variant 10.1.5.14.*

(2) *For each $k \geq n$, the face cube $\tau_k : \square^{k+1} \to \mathcal{C}$ of Construction 10.1.5.18 is a limit diagram in $\mathcal{C}$.*

Proof. We proceed as in the proof of Proposition 10.1.3.14. Let us identify each τ_k with a functor $\mathrm{N}_\bullet(P([k]))^{\mathrm{op}} \to \mathcal{C}$, where $P([k])$ denotes the collection of all subsets of $\{0 < 1 < \cdots < k\}$. Let $P^{\leq n}([k])$ denote the subset of $P([k])$ consisting of subsets of cardinality $\leq n$. Unwinding the definitions, we see that $C_\bullet$ is $(n-1)$-coskeletal if and only if the following condition is satisfied for each $k \geq n$:

(1_k) The functor τ_k exhibits C_k as a limit of its restriction to $\mathrm{N}_\bullet(P^{\leq n}([k]))^{\mathrm{op}}$.

Similarly, (2) asserts that the following condition is satisfied for each $k \geq n$:

(2_k) The face cube $\tau_k : \square^{k+1} \to \mathcal{C}$ of Construction 10.1.5.18 is a limit diagram in $\mathcal{C}$.

To complete the proof, it will suffice to show that if condition (1_ℓ) is satisfied for $n \leq \ell < k$, then conditions (1_k) and (2_k) are equivalent. Our hypothesis that condition (2_ℓ) is satisfied for $\ell < k$ guarantees that the functor $\tau_k|_{\mathrm{N}_\bullet(P^{\leq k}([k]))^{\mathrm{op}}}$ is right Kan extended from the full subcategory $\mathrm{N}_\bullet(P^{\leq n}([k]))^{\mathrm{op}} \subseteq \mathrm{N}_\bullet(P^{\leq k}([k]))^{\mathrm{op}}$. The equivalence of (1_k) and (2_k) is therefore a special case of Corollary 7.3.8.2. □

Corollary 10.1.5.21. *Let $\mathcal{C}$ be an ∞-category and let $n \geq -1$ be an integer. Then an augmented (semi)simplicial object $C_\bullet$ of $\mathcal{C}$ is $(n-1)$-coskeletal if and only if it is n-coskeletal and the face cube $\tau_n : \square^{n+1} \to \mathcal{C}$ is a limit diagram in $\mathcal{C}$.* 05HL

Corollary 10.1.5.22. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, let $C_\bullet$ be an augmented (semi)simplicial object of $\mathcal{C}$, and let $n \geq -1$ be an integer. Assume that $\mathcal{C}$ admits pullbacks and that F preserves pullbacks. If $C_\bullet$ is n-coskeletal, then the image $F(C_\bullet)$ is n-coskeletal.* 05HM

10.1.6 Split Simplicial Objects

We now introduce a tool which is often useful for computing geometric realizations of simplicial objects. 04S6

Notation 10.1.6.1. We define a category $\mathbf{\Delta}_{\mathrm{min}}$ as follows: 04S7

- The objects of $\mathbf{\Delta}_{\mathrm{min}}$ are linearly ordered sets $[n] = \{0 < 1 < \cdots < n\}$, where n is a nonnegative integer.
- A morphism from $[m]$ to $[n]$ in the category $\mathbf{\Delta}_{\mathrm{min}}$ is a nondecreasing function $\alpha : [m] \to [n]$ satisfying $\alpha(0) = 0$.

Remark 10.1.6.2. By construction, the category $\mathbf{\Delta}_{\mathrm{min}}$ is a (non-full) subcategory of the simplex category $\mathbf{\Delta}$ of Definition 1.1.0.2. It can therefore also be regarded as a subcategory of the augmented simplex category $\mathbf{\Delta}_{+}$ of Definition 10.1.1.10. The inclusion functor $\mathbf{\Delta}_{\mathrm{min}} \hookrightarrow \mathbf{\Delta}_{+}$ admits a left adjoint $C_{+} : \mathbf{\Delta}_{+} \to \mathbf{\Delta}_{\mathrm{min}}$, given concretely by the construction $C_{+}([n]) = [0] \star [n] \simeq [n+1]$. We will refer to C_{+} as the *concatenation functor*. We let $C : \mathbf{\Delta} \to \mathbf{\Delta}_{\mathrm{min}}$ denote the restriction of C_{+} to the simplex category $\mathbf{\Delta}$, which we will also refer to as the *concatenation functor*. 04S8

Definition 10.1.6.3. Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$ (Definition 10.1.1.12). A *splitting* of $X_\bullet$ is a functor $\overline{X} : \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{min}}) \to \mathcal{C}$ for which the composition 04S9

$$\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}_{+}) \xrightarrow{C_{+}^{\mathrm{op}}} \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{min}}) \xrightarrow{\overline{X}} \mathcal{C}$$

is equal to $X_\bullet$; here C_{+} denotes the concatenation functor $[n] \mapsto [n] \star [0]$ of Remark 10.1.6.2. We will say that the augmented simplicial object $X_\bullet$ is *split* if there exists a splitting of $X_\bullet$.

Remark 10.1.6.4 (Extra Degeneracies)**.** Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$. For every integer $n \geq -1$, the function 04SA

$$\sigma^0_{n+1} : [n+2] \to [n+1] \qquad i \mapsto \begin{cases} 0 & \text{if } i = 0 \\ i-1 & \text{if } i > 0 \end{cases}$$

belongs to the subcategory $\mathbf{\Delta}_{\mathrm{min}} \subseteq \mathbf{\Delta}$. If $\overline{X}$ is a splitting of $X_\bullet$, then evaluation on σ^0_{n+1} determines a morphism

$$h_n : X_n = \overline{X}([n+1]) \to \overline{X}([n+2]) = X_{n+1}.$$

Heuristically, one can think of the morphisms $\{h_n\}_{n \geq -1}$ as "extra" degeneracy operators on the augmented simplicial object $X_\bullet$. In the homotopy category $\mathrm{h}\mathcal{C}$, these operators satisfy the identities

04SB
$$d_i^{n+1} \circ h_n \sim \begin{cases} \mathrm{id}_{X_n} & \text{if } i = 0 \\ h_{n-1} \circ d_{i-1}^n & \text{otherwise} \end{cases} \tag{10.2}$$

04SC
$$s_i^{n+1} \circ h_n \sim \begin{cases} h_{n+1} \circ h_n & \text{if } i = 0 \\ h_{n+1} \circ s_{i-1}^n & \text{otherwise.} \end{cases} \tag{10.3}$$

04SD **Exercise 10.1.6.5.** Let $\mathcal{C}$ be an ordinary category and let $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$. Show that the construction of Remark 10.1.6.4 determines a bijection from the set of splittings of $X_\bullet$ (in the sense of Definition 10.1.6.3) to the collection of systems $\{h_n : X_n \to X_{n+1}\}_{n \geq -1}$ satisfying the identities (10.2) and (10.3).

04SE **Example 10.1.6.6.** Let $A_\bullet$ be an augmented simplicial abelian group, and let

$$\mathrm{C}^{\mathrm{aug}}_*(A) = (\cdots \to A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0 \xrightarrow{\partial} A_{-1})$$

denote its augmented Moore complex (Remark 10.1.2.21). Suppose we are given a splitting of $A_\bullet$, and let $\{h_n : A_n \to A_{n+1}\}_{n \geq -1}$ be the extra degeneracy operators described in Remark 10.1.6.4. Then the collection $\{h_n\}$ is a contracting homotopy for $\mathrm{C}^{\mathrm{aug}}_*(A)$, in the sense of Definition 2.5.0.5: that is, the homomorphism

$$(h_{n-1} \circ \partial + \partial \circ h_n) : A_n \to A_n$$

is equal to the identity for each $n \geq -1$ (where we adopt the convention that $h_n \circ \partial = 0$ for $n = -1$). This follows from the calculation

$$\begin{aligned} h_{n-1} \circ \partial + \partial \circ h_n &= (\sum_{i=0}^{n} (-1)^i h_{n-1} \circ d_i^n) + (\sum_{j=0}^{n+1} (-1)^j d_j^{n+1} \circ h_n) \\ &= (\sum_{i=0}^{n} (-1)^i (h_{n-1} \circ d_i^n - d_{i+1}^{n+1} \circ h_n)) + d_0^{n+1} \circ h_n \\ &= \mathrm{id}_{A_n}. \end{aligned}$$

where the final equality follows from the identities (10.2).

Variant 10.1.6.7. In the situation of Example 10.1.6.6, let $\mathrm{N}_*^{\mathrm{aug}}(A)$ denote the augmented normalized Moore complex of $A_\bullet$ (Variant 10.1.2.22). It follows from (10.3) that, for every integer $n \geq 0$, the operator 04SF

$$\mathrm{C}_n^{\mathrm{aug}}(A) = A_n \xrightarrow{h_n} A_{n+1} = \mathrm{C}_{n+1}^{\mathrm{aug}}(A)$$

carries degenerate n-simplices of $A_\bullet$ to degenerate $(n+1)$-simplices of $A_\bullet$, and therefore descends to an operator $\overline{h}_n : \mathrm{N}_n^{\mathrm{aug}}(A) \to \mathrm{N}_{n+1}^{\mathrm{aug}}(A)$. The collection of homomorphisms $\{\overline{h}_n\}$ then determine a contracting homotopy for the chain complex $\mathrm{N}_*^{\mathrm{aug}}(A)$.

Warning 10.1.6.8. Let $A_\bullet$ be an augmented simplicial abelian group. In general, not every contracting homotopy for the chain complex $\mathrm{N}_*^{\mathrm{aug}}(A)$ can be obtained from the construction of Variant 10.1.6.7. A splitting of $A_\bullet$ determines a system of homomorphisms $\{h_n : A_n \to A_{n+1}\}_{n \geq 0}$ which satisfy the identity $h_{n+1} \circ h_n = s_0^{n+1} \circ h_n$ (Remark 10.1.6.4). In particular, the composition $h_{n+1} \circ h_n$ carries every n-simplex of $A_\bullet$ to a degenerate $(n+2)$-simplex of $A_\bullet$. It follows that the composite map 04SG

$$\mathrm{N}_n^{\mathrm{aug}}(A) \xrightarrow{\overline{h}_n} \mathrm{N}_{n+1}^{\mathrm{aug}}(A) \xrightarrow{\overline{h}_{n+1}} \mathrm{N}_{n+2}^{\mathrm{aug}}(A)$$

vanishes; for a general contracting homotopy, the analogous statement need not be true.

The utility of Definition 10.1.6.3 stems from the following:

Proposition 10.1.6.9. *Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$. If $X_\bullet$ is split, then it is a colimit diagram in $\mathcal{C}$.* 04SH

Proof. Let $\overline{X} : \mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}}) \to \mathcal{C}$ be a splitting of $X_\bullet$, and let $C_+ : \mathbf{\Delta}_+ \to \mathbf{\Delta}_{\min}$ denote the concatenation functor of Remark 10.1.6.2. Let us abuse notation by identifying $\mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}})$ with the cone $\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright}$. We wish to show that the augmented simplicial object

$$(X_\bullet = \overline{X} \circ \mathrm{N}_\bullet(C_+^{\mathrm{op}})) : \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright} \to \mathcal{C}$$

is a colimit diagram in $\mathcal{C}$.

Note that $[0]$ is initial when viewed as an object of the category $\mathbf{\Delta}_{\min}$, and therefore final when viewed as an object of the ∞-category $\mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}})$. Unwinding the definitions, we see that the functor $\mathrm{N}_\bullet(C_+^{\mathrm{op}})$ factors as a composition

$$\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright} \xrightarrow{\mathrm{N}_\bullet(C)^{\triangleright}} \mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}})^{\triangleright} \xrightarrow{R} \mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}}),$$

where R is the identity when restricted to $\mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}})$ and carries the cone point of $\mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}})^{\triangleright}$ to $[0]$. Applying Corollary 7.2.2.6, we deduce that $(\overline{X} \circ R) : \mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}})^{\triangleright} \to \mathcal{C}$ is a colimit diagram. Consequently, to show that $X_\bullet$ is a colimit diagram, it will suffice to show that the functor $\mathrm{N}_\bullet(C^{\mathrm{op}}) : \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \to \mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}})$ is right cofinal (Corollary 7.2.2.3). This is a special case of Corollary 7.2.3.7, since the concatenation functor C is left adjoint to the inclusion $\mathbf{\Delta}_{\min} \hookrightarrow \mathbf{\Delta}$ (Remark 10.1.6.2). □

04SJ **Remark 10.1.6.10.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories and let $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$, so that $F(X_\bullet)$ is an augmented cosimplicial object of $\mathcal{D}$. Composition with the functor F carries splittings of $X_\bullet$ to splittings of $F(X_\bullet)$. Consequently, if $X_\bullet$ is split, then $F(X_\bullet)$ is also split. In particular, if $X_\bullet$ is split, then $F(X_\bullet)$ is a colimit diagram in $\mathcal{D}$ (Proposition 10.1.6.9).

04SK **Variant 10.1.6.11.** Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. A *splitting* of $X_\bullet$ is a functor $\overline{X} : \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}_{\mathrm{min}}) \to \mathcal{C}$ for which the composition

$$\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}_{+}) \xrightarrow{C^{\mathrm{op}}} \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}_{\mathrm{min}}) \xrightarrow{\overline{X}} \mathcal{C}$$

is equal to $X_\bullet$; here C denotes the concatenation functor $[n] \mapsto [n] \star [0]$ of Remark 10.1.6.2. We will say that the simplicial object $X_\bullet$ is *split* if there exists a splitting of $X_\bullet$.

04SL **Warning 10.1.6.12.** The terminology of Variant 10.1.6.11 (and Definition 10.1.6.3) is potentially confusing. We will use the term *split simplicial object* to refer to a simplicial object $X_\bullet$ of an ∞-category $\mathcal{C}$ for which there *exists* a splitting $\overline{X} : \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}_{\mathrm{min}}) \to \mathcal{C}$. Unless otherwise specified, we do not assume that a particular splitting has been chosen. Beware that $\overline{X}$ is not uniquely determined by $X_\bullet$. However, the underlying augmented simplicial object

$$\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}_{+}) \xrightarrow{\mathrm{N}_\bullet(C^{\mathrm{op}}_{+})} \mathrm{N}_\bullet(\boldsymbol{\Delta}_{\mathrm{min}}) \xrightarrow{\overline{X}} \mathcal{C}$$

is determined up to isomorphism by $X_\bullet$: by virtue of Proposition 10.1.6.9, it is an extension of $X_\bullet$ to a colimit diagram in $\mathcal{C}$.

04SM **Corollary 10.1.6.13.** *Let $X_\bullet$ be a split simplicial object of an ∞-category $\mathcal{C}$. Then $X_\bullet$ admits a geometric realization $|X_\bullet|$. Moreover, the geometric realization of $X_\bullet$ is preserved by any functor of ∞-categories $F : \mathcal{C} \to \mathcal{D}$.*

Proof. The first assertion follows from Proposition 10.1.6.9, and the second from Remark 10.1.6.10. □

The Čech nerve construction of §10.1.5 provides an abundant supply of split simplicial objects.

05HN **Proposition 10.1.6.14.** *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$ which admits a Čech nerve $\check{\mathrm{C}}_\bullet(X/Y)$. Then the augmented simplicial object $\check{\mathrm{C}}_\bullet(X/Y)$ splits if and only if f admits a right homotopy inverse.*

04TL **Corollary 10.1.6.15.** *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$ which admits a Čech nerve $\check{\mathrm{C}}_\bullet(X/Y)$. If f admits a right homotopy inverse, then $\check{\mathrm{C}}_\bullet(X/Y)$ is a colimit diagram: that is, it exhibits Y as a geometric realization of its underlying simplicial object.*

Proof. Combine Propositions 10.1.6.14 and 10.1.6.9. □

Our proof of Proposition 10.1.6.14 will require some preliminaries.

Notation 10.1.6.16. Let $\mathbf{\Delta}_{\mathrm{min}}$ be the category introduced in Notation 10.1.6.1. For every integer n, we let $\mathbf{\Delta}_{\mathrm{min}}^{\leq n}$ denote the full subcategory of $\mathbf{\Delta}_{\mathrm{min}}$ spanned by the collection of objects $\{[m]\}_{0 \leq m \leq n}$. `05HP`

Example 10.1.6.17. For $n < 0$, the category $\mathbf{\Delta}_{\mathrm{min}}^{\leq n}$ is empty. For $n = 0$, it contains a single object $[0]$, and no morphisms other than the identity morphism. `05HQ`

Example 10.1.6.18. Let Ret be the category introduced in Construction 8.5.0.2: that is, the category which is freely generated by a pair of morphisms $i : Y \to X$ and $r : X \to Y$ satisfying the identity $r \circ i = \mathrm{id}_Y$. By virtue of Exercise 8.5.0.3, there is a unique functor $\mathrm{Ret} \to \mathbf{\Delta}$ which carries i to the inclusion map $[0] \hookrightarrow [1]$ and r to the constant function $[1] \twoheadrightarrow [0]$. This functor induces an isomorphism from Ret onto the subcategory $\mathbf{\Delta}_{\mathrm{min}}^{\leq 1} \subset \mathbf{\Delta}$ of Notation 10.1.6.16. `05HR`

Lemma 10.1.6.19. *Let $\mathcal{C}$ be an ∞-category, let $X_\bullet$ be an augmented simplicial object of $\mathcal{C}$, and let $\overline{X} : \mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{min}}^{\mathrm{op}}) \to \mathcal{C}$ be a splitting of $X_\bullet$ (in the sense of Definition 10.1.6.3). For every integer n, the following conditions are equivalent:* `04TE`

(1) *The functor $\overline{X}$ is right Kan extended from the full subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{min}}^{\leq n+1})$ of Notation 10.1.6.16.*

(2) *The augmented simplicial object $X_\bullet$ is n-coskeletal: that is, it is right Kan extended from the subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}_{+}^{\leq n})^{\mathrm{op}}$ (Definition 10.1.5.10).*

Proof. For each integer $k \geq -1$, we will show that the following conditions are equivalent:

(1_k) The functor $\overline{X}$ is right Kan extended from the subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}_{\mathrm{min}}^{\leq n+1})^{\mathrm{op}}$ at the object $[k+1]$.

(2_k) The functor $X_\bullet$ is right Kan extended from the subcategory $\mathrm{N}_\bullet(\mathbf{\Delta}_{+}^{\leq n})^{\mathrm{op}}$ at the object $[k]$.

Let $(\mathbf{\Delta}_{+}^{\leq n})_{/[k]}$ denote the fiber product $(\mathbf{\Delta}_{+})_{/[k]} \times_{\mathbf{\Delta}_{+}} \mathbf{\Delta}_{+}^{\leq n}$, and define $(\mathbf{\Delta}_{\mathrm{min}}^{\leq n+1})_{/[k+1]}$ similarly. By virtue of Corollary 7.2.2.3, it will suffice to show that the concatentation functor C_+ induces a right cofinal functor

$$G : \mathrm{N}_\bullet((\mathbf{\Delta}_{+}^{\leq n})_{/[k]}) \to \mathrm{N}_\bullet((\mathbf{\Delta}_{\mathrm{min}}^{\leq n+1})_{/[k+1]}).$$

This follows from Corollary 7.2.3.7, since the functor G admits a left adjoint F (which carries a morphism $\alpha : [m] \to [k+1]$ of $\mathbf{\Delta}_{\mathrm{min}}$ to the nondecreasing function

$$\{k \in [m] : \alpha(k) > 0\} \xrightarrow{\alpha} \{1 < 2 < \cdots < k\} \simeq [k].$$

□

04TF **Variant 10.1.6.20.** Let n be an integer and let $\mathcal{C}$ be an ∞-category which is equipped with a functor $T : \mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\leq n+1})^{\mathrm{op}} \to \mathcal{C}$. The following conditions are equivalent:

(1) The functor T admits a right Kan extension $\overline{F} : \mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\mathrm{op}}) \to \mathcal{C}$.

(2) The composite functor

$$\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{+}^{\leq n})^{\mathrm{op}} \xrightarrow{C_{+}} \mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\leq n+1})^{\mathrm{op}} \xrightarrow{T} \mathcal{C}$$

can be extended to an n-coskeletal augmented simplicial object of $\mathcal{C}$.

Proof. We maintain the notations from the proof of Lemma 10.1.6.19. By virtue of Corollary 7.3.5.8, it will suffice to show that for every integer $k \geq -1$, the following conditions are equivalent:

(1_k) The diagram

$$\mathrm{N}_{\bullet}((\boldsymbol{\Delta}_{\mathrm{min}}^{\leq n+1})_{/[k+1]}) \to \mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\leq n+1}) \xrightarrow{T} \mathcal{C}^{\mathrm{op}}$$

admits a colimit in the ∞-category $\mathcal{C}^{\mathrm{op}}$.

(2_k) The diagram

$$\mathrm{N}_{\bullet}((\boldsymbol{\Delta}_{+}^{\leq n})_{/[k]}) \xrightarrow{G} \mathrm{N}_{\bullet}((\boldsymbol{\Delta}_{\mathrm{min}}^{\leq n+1})_{/[k+1]}) \to \mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\leq n+1}) \xrightarrow{T} \mathcal{C}^{\mathrm{op}}$$

admits a colimit in the ∞-category $\mathcal{C}^{\mathrm{op}}$.

As in the proof of Lemma 10.1.6.19, the functor G is right cofinal, so the equivalence of (1_k) and (2_k) is a special case of Corollary 7.2.2.10. □

04TG **Proposition 10.1.6.21.** *Let $\mathcal{C}$ be an ∞-category, let n be an integer, let $\mathrm{Fun}'(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{+}^{\mathrm{op}}), \mathcal{C})$ be the full subcategory of $\mathrm{Fun}(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{+}^{\mathrm{op}}), \mathcal{C})$ spanned by the n-coskeletal augmented simplicial objects of $\mathcal{C}$, and let $\mathrm{Fun}'(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\mathrm{op}}), \mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\mathrm{op}}), \mathcal{C})$ be its inverse image. Then precomposition with the concatenation functor C_{+} induces a trivial Kan fibration*

$$\theta : \mathrm{Fun}'(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\mathrm{op}}), \mathcal{C}) \to \mathrm{Fun}'(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{\mathrm{min}}^{\leq n+1})^{\mathrm{op}}, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{+}^{\leq n})^{\mathrm{op}}, \mathcal{C})} \mathrm{Fun}'(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{+}^{\mathrm{op}}), \mathcal{C}).$$

Proof. Let $\mathrm{Fun}'(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{+}^{\leq n})^{\mathrm{op}}, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{N}_{\bullet}(\boldsymbol{\Delta}_{+}^{\leq n})^{\mathrm{op}}, \mathcal{C})$ spanned by those functors which can be extended to n-coskeletal augmented simplicial objects of $\mathcal{C}$, and define $\mathrm{Fun}'(\mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\leq n+1})^{\mathrm{op}}, \mathcal{C})$ similarly. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}'(\mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}}), \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}'(\mathrm{N}_\bullet(\mathbf{\Delta}_{+}^{\mathrm{op}}), \mathcal{C}) \arrow[d] \\
\mathrm{Fun}'(\mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\leq n+1})^{\mathrm{op}}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}'(\mathrm{N}_\bullet(\mathbf{\Delta}_{+}^{\leq n})^{\mathrm{op}}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\leq n+1})^{\mathrm{op}}, \mathcal{C}) \arrow[r] & \mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_{+}^{\leq n})^{\mathrm{op}}, \mathcal{C}).
\end{tikzcd}$$

Combining Lemma 10.1.6.19 with Corollary 7.3.6.15, we deduce that the upper vertical maps are trivial Kan fibrations; in particular, the upper half of the diagram is a categorical pullback square (Proposition 4.5.2.21). Variant 10.1.6.20 guarantees that the lower half of the square is a pullback diagram. Since bottom horizontal map is an isofibration of ∞-categories (Corollary 4.4.5.3), it is a categorical pullback square (Corollary 4.5.2.27). It follows that the outer rectangle is a categorical pullback square (Proposition 4.5.2.18), so that θ is an equivalence of ∞-categories (Proposition 4.5.2.26). Corollary 4.4.5.3 guarantees that θ is also an isofibration, so it is a trivial Kan fibration (Proposition 4.5.5.20). □

Proof of Proposition 10.1.6.14. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$ which admits a Čech nerve $\check{\mathrm{C}}_\bullet(X/Y)$. Applying Proposition 10.1.6.21 (in the case $n = 0$), we see that precomposition with the inclusion map

$$\mathrm{Ret}^{\mathrm{op}} \simeq \mathrm{Ret} \simeq \mathbf{\Delta}_{\min}^{\leq 1} \hookrightarrow \mathbf{\Delta}_{\min}$$

of Example 10.1.6.18 induces a trivial Kan fibration

$$\{\check{\mathrm{C}}_\bullet(X/Y)\} \times_{\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_{+}^{\mathrm{op}}), \mathcal{C})} \mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}}), \mathcal{C}) \to \{f\} \times_{\mathrm{Fun}(\Delta^1, \mathcal{C})} \mathrm{Fun}(\mathrm{N}_\bullet(\mathrm{Ret}), \mathcal{C}).$$

In particular, the left hand side is nonempty if and only if the right hand side is nonempty: that is, the Čech nerve $\check{\mathrm{C}}_\bullet(X/Y)$ splits if and only if f has a right homotopy inverse. □

We close this section by describing an important special class of split simplicial objects.

Construction 10.1.6.22 (Decalage)**.** Let $\mathcal{C}$ be an ∞-category and let $X_\bullet$ be a simplicial object of $\mathcal{C}$. We let $\mathrm{Dec}_+(X)_\bullet$ denote the augmented simplicial object of $\mathcal{C}$ given by the composition 04SN

$$\mathrm{N}_\bullet(\mathbf{\Delta}_{+}^{\mathrm{op}}) \xrightarrow{\mathrm{N}_\bullet(C_{+}^{\mathrm{op}})} \mathrm{N}_\bullet(\mathbf{\Delta}_{\min}^{\mathrm{op}}) \subset \mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \xrightarrow{X_\bullet} \mathcal{C},$$

where C_+ denote the concatenation functor of Remark 10.1.6.2. We will refer to $\mathrm{Dec}_+(X)_\bullet$ as the *augmented decalage* of $X_\bullet$. We let $\mathrm{Dec}(X)_\bullet$ denote the underlying simplicial object of $\mathrm{Dec}_+(X)_\bullet$, given by the composition

$$\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}) \xrightarrow{\mathrm{N}_\bullet(C^{\mathrm{op}})} \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}_{\mathrm{min}}) \subset \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}) \xrightarrow{X_\bullet} \mathcal{C}\,.$$

We will defer to $\mathrm{Dec}(X)_\bullet$ as the *decalage* of $\mathrm{Dec}(X)_\bullet$.

04SP **Remark 10.1.6.23.** More informally, the augmented decalage of a simplicial object $X_\bullet$ is given by the formula $\mathrm{Dec}_+(X)_n = X_{n+1}$. Moreover, for every pair of integers $0 \leq i \leq n$, the face and degeneracy operators

$$d^n_i : \mathrm{Dec}_+(X)_n \to \mathrm{Dec}_+(X)_{n-1} \qquad s^n_i : \mathrm{Dec}_+(X)_n \to \mathrm{Dec}_+(X)_{n+1}$$

coincide with the face and degeneracy operators

$$d^{n+1}_{i+1} : X_{n+1} \to X_n \qquad s^{n+1}_{i+1} : X_{n+1} \to X_{n+2}.$$

04SQ **Example 10.1.6.24.** Let X be a simplicial set. Then the decalage $\mathrm{Dec}(X)_\bullet$ can be identified with the disjoint union of coslice constructions $\coprod_x X_{x/}$, where the coproduct is indexed by the collection of all vertices $x \in X$.

04SR **Remark 10.1.6.25.** Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$. Then the augmented simplicial object $\mathrm{Dec}_+(X)_\bullet$ is split: it admits a splitting given by the diagram

$$\mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}_{\mathrm{min}}) \subset \mathrm{N}_\bullet(\boldsymbol{\Delta}^{\mathrm{op}}) \xrightarrow{X_\bullet} \mathcal{C}\,.$$

In particular, Proposition 10.1.6.9 guarantees that $\mathrm{Dec}_+(X)_\bullet$ is a colimit diagram in $\mathcal{C}$: that is, it exhibits the object $X_0 \in \mathcal{C}$ as a geometric realization of the decalage $\mathrm{Dec}(X)_\bullet$.

04SS **Remark 10.1.6.26.** Let $\iota : \boldsymbol{\Delta}_{\mathrm{min}} \hookrightarrow \boldsymbol{\Delta}$ denote the inclusion functor, and let

$$C : \boldsymbol{\Delta} \to \boldsymbol{\Delta}_{\mathrm{min}} \qquad [n] \mapsto [0] \star [n] = [n+1]$$

denote the concatenation functor of Remark 10.1.6.2. There is a natural transformation $\eta : \mathrm{id}_{\boldsymbol{\Delta}} \to \iota \circ C$, which carries each object $[n] \in \boldsymbol{\Delta}$ to the inclusion map

$$[n] \hookrightarrow [n+1] \qquad i \mapsto i+1.$$

If $X_\bullet$ is a simplicial object of an ∞-category $\mathcal{C}$, then composition with η determines a natural transformation of simplicial objects $T_\bullet : \mathrm{Dec}(X)_\bullet \to X_\bullet$, given termwise by the face operator $\mathrm{Dec}(X)_n = X_{n+1} \xrightarrow{d^{n+1}_0} X_n$.

The natural transformation η is the unit of an adjunction between ι and C; it admits a compatible counit $\epsilon : C \circ \iota \to \mathrm{id}_{\mathbf{\Delta}_{\min}}$, which carries each object $[n]$ to the quotient map

$$[n+1] \twoheadrightarrow [n] \qquad i \mapsto \max(0, i-1).$$

We therefore have a commutative diagram

04ST

$$\begin{tikzcd}
 & C \circ \iota \circ C \arrow[dr, "\mathrm{id}_C \circ \epsilon"] & \\
C \arrow[ur, "\eta \circ \mathrm{id}_C"] \arrow[rr, "\mathrm{id}_C"] & & C
\end{tikzcd} \tag{10.4}$$

in the functor category $\mathrm{Fun}(\mathbf{\Delta}, \mathbf{\Delta}_{\min})$. If $\overline{X}$ is a splitting of the simplicial object $X_\bullet$, then precomposition with (10.4) determines a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Dec}(X)_\bullet \arrow[dr, "T_\bullet"] & \\
X_\bullet \arrow[ur, "h_\bullet"] \arrow[rr, "\mathrm{id}"] & & X_\bullet
\end{tikzcd}$$

in the ∞-category of simplicial objects $\mathrm{Fun}(\mathrm{N}_\bullet(\mathbf{\Delta}^{\mathrm{op}}), \mathcal{C})$. Here $h_\bullet$ is given termwise by the extra degeneracy map $h_n : X_n \to X_{n+1} = \mathrm{Dec}(X)_n$ appearing in Remark 10.1.6.4. In particular, if $X_\bullet$ is a split simplicial object of $\mathcal{C}$, then it is a retract of the decalage $\mathrm{Dec}(X)_\bullet$.

Warning 10.1.6.27. Let $X_\bullet$ be a simplicial object of an ∞-category $\mathcal{C}$. It follows from Remark 10.1.6.26 that every splitting of $X_\bullet$ determines a right homotopy inverse to the comparison map $T_\bullet : \mathrm{Dec}(X)_\bullet \to X_\bullet$. Beware that, in general, not every right homotopy inverse can be obtained in this way. For example, suppose that $\mathcal{C}$ is (the nerve of) an ordinary category. Unwinding the definitions, we see that a morphism of simplicial objects from $X_\bullet$ to $\mathrm{Dec}(X)_\bullet$ is given by a collection of morphisms $h_n : X_n \to \mathrm{Dec}(X)_n = X_{n+1}$ which satisfy the identities 04SU

$$d_i^{n+1} \circ h_n = h_{n-1} \circ d_{i-1}^n \qquad s_i^{n+1} \circ h_n = h_{n+1} \circ s_{i-1}^n$$

for $0 < i \leq n+1$. Moreover, $h_\bullet$ is a right inverse of $T_\bullet$ if and only if it satisfies the further identity $d_0^{n+1} \circ h_n = \mathrm{id}_{X_n}$ for each $n \geq 0$. However, $h_\bullet$ arises from a splitting of the simplicial object $X_\bullet$ only if it also satisfies the identities $s_0^{n+1} \circ h_n = h_{n+1} \circ h_n$; see Exercise 10.1.6.5 (compare with Warning 10.1.6.8).

10.2 Regular ∞-Categories

04TM Let X and Y be sets and let $f : X \to Y$ be a function. Recall that the *image* of f is the subset $\mathrm{im}(f) \subseteq Y$ consisting of those elements $y \in Y$ satisfying $y = f(x)$ for some element $x \in X$. Writing i for the inclusion of $\mathrm{im}(f)$ into Y, the function f then factors as a composition

$$X \overset{f_0}{\twoheadrightarrow} \mathrm{im}(f) \overset{i}{\hookrightarrow} Y.$$

This factorization admits a more abstract characterization: it is determined (up to unique isomorphism) by the requirements that i is injective (that is, it is a monomorphism in the category of sets) and that f_0 is surjective (that is, it is an epimorphism in the category of sets).

The construction $f \mapsto \mathrm{im}(f)$ has counterparts in many other categories. For example, every homomorphism of commutative rings $f : R \to S$ has a tautological factorization

$$R \overset{f_0}{\twoheadrightarrow} \mathrm{im}(f) \overset{i}{\hookrightarrow} S,$$

which is again characterized (up to unique isomorphism) by the requirements that that i is injective and that f_0 is surjective. Here the first demand is equivalent to the condition that i is a monomorphism in the category of commutative rings. However, the second demand is more subtle. Every surjective ring homomorphism is an epimorphism in the category of commutative rings, but the converse is false in general.

04TN **Example 10.2.0.1.** Let $\mathbf{Q}$ denote the field of rational numbers, let $\mathbf{Z} \subseteq \mathbf{Q}$ denote the ring of integers, and let $f : \mathbf{Z} \hookrightarrow \mathbf{Q}$ denote the inclusion map. Then f is both a monomorphism and an epimorphism in the category of commutative rings. Consequently, the ring homomorphism f admits (at least) two factorizations as an epimorphism followed by a monomorphism, given by the diagrams

$$\mathbf{Z} \xrightarrow{\mathrm{id}} \mathbf{Z} \xrightarrow{f} \mathbf{Q} \qquad \mathbf{Z} \xrightarrow{f} \mathbf{Q} \xrightarrow{\mathrm{id}} \mathbf{Q}.$$

To address the phenomenon described in Example 10.2.0.1, it is convenient to modify the definition of epimorphism.

04TP **Definition 10.2.0.2.** Let $\mathcal{C}$ be a category which admits fiber products. We will say that a morphism $f : X \to Y$ of $\mathcal{C}$ is a *regular epimorphism* if it exhibits Y as a coequalizer of the pair of projection maps $\pi_0, \pi_1 : X \times_Y X \to X$.

04TQ **Remark 10.2.0.3.** Let $\mathcal{C}$ be a category which admits fiber products and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then f is an epimorphism if and only if, for every object $Z \in \mathcal{C}$, the function

$$\theta_Z : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z) \qquad g \mapsto g \circ f$$

is injective. The condition that f is a regular epimorphism is (in general) stronger: it requires also that the image of θ_Z is the collection of morphisms $h : X \to Z$ which satisfy the identity $h \circ \pi_0 = h \circ \pi_1$; here π_0 and π_1 denote the projection maps from $X \times_Y X$ to X.

Example 10.2.0.4. Let $\mathcal{C} = \mathrm{Set}$ be the category of sets and let $f : X \twoheadrightarrow Y$ be an epimorphism in $\mathcal{C}$: that is, a surjective function. Then f is a regular epimorphism: that is, it exhibits Y as a quotient of the equivalence relation $\equiv_f$, defined by the requirement 04TR

$$(x \equiv_f x') \Leftrightarrow (f(x) = f(x')).$$

Exercise 10.2.0.5. Let $f : R \to S$ be a homomorphism of commutative rings. Show that f is a regular epimorphism (in the category of commutative rings) if and only if it is surjective (as a map of sets). In particular, the inclusion map $\mathbf{Z} \hookrightarrow \mathbf{Q}$ of Example 10.2.0.1 is an epimorphism in the category of commutative rings which is not regular. 04TS

Let $\mathcal{C}$ be a category which admits fiber products and let $f : X \to Y$ be a morphism in $\mathcal{C}$. We will say that an object $Y_0 \in \mathcal{C}$ is an *image of f* if the morphism f factors as a composition

$$X \overset{f_0}{\twoheadrightarrow} Y_0 \overset{i}{\hookrightarrow} Y,$$

where i is a monomorphism and f_0 is a regular epimorphism. It is not difficult to show that if such a factorization exists, then it is uniquely determined up to (canonical) isomorphism: for example, the object Y_0 can be recovered as the coequalizer of the pair of projection maps $X \times_Y X \rightrightarrows X$. To emphasize the uniqueness, we will typically denote the object Y_0 by $\mathrm{im}(f)$ and refer to it as *the* image of f. This motivates the following:

Definition 10.2.0.6. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is *regular* if it satisfies the following conditions: 04TT

(1) The category $\mathcal{C}$ admits finite limits (in particular, it admits fiber products).

(2) Every morphism $f : X \to Y$ of $\mathcal{C}$ has an image: that is, we can write f as a composition

$$X \overset{f_0}{\twoheadrightarrow} Y_0 \overset{i}{\hookrightarrow} Y$$

where i is a monomorphism and f_0 is a regular epimorphism.

(3) The collection of regular epimorphisms is stable under the formation of pullbacks. That is, for every pullback diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

in the category $\mathcal{C}$, if f is a regular epimorphism, then f' is also a regular epimorphism.

04TU **Example 10.2.0.7.** The axioms of Definition 10.2.0.6 tend to be satisfied by any category $\mathcal{C}$ whose objects can be described as "sets with algebraic structure." For example:

- The category of sets is regular.
- The category of groups is regular.
- The category of abelian groups is regular.
- The category of associative rings is regular.
- The category of commutative rings is regular.

04TV **Example 10.2.0.8.** Let $\mathcal{C}$ be the category of partially ordered sets (with morphisms given by nondecreasing functions). Then $\mathcal{C}$ is not regular: it satisfies conditions (1) and (2) of Definition 10.2.0.6, but does not satisfy condition (3) (see Exercise 10.2.2.16).

Our goal in this section is to extend Definition 10.2.0.6 to the setting of ∞-categories. The first step is to find an appropriate ∞-categorical counterpart for the notion of regular epimorphism. Let $\mathcal{C}$ be an ∞-category which admits fiber products. Then, to every morphism $f : X \to Y$ of $\mathcal{C}$, one can associate a diagram

04TW
$$\begin{tikzcd} X \times_Y X \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & X \arrow[r, "f"] & Y. \end{tikzcd} \tag{10.5}$$

If $\mathcal{C}$ is (the nerve of) an ordinary category, then f is a regular epimorphism if and only if (10.5) is a coequalizer diagram. In the ∞-categorical setting, this condition is almost never satisfied (even if f is an isomorphism). To guarantee that a morphism $g : X \to Z$ factors (up to homotopy) through f, it is typically not enough to know that $g \circ \pi_0$ is homotopic to $g \circ \pi_1$: one needs a homotopy satisfying further coherence conditions, whose formalization involves iterated fiber products $X \times_Y X \times_Y \cdots \times_Y X$. Recall that the collection of all such fiber products can be organized into an augmented simplicial object $\check{\mathrm{C}}_\bullet(X/Y)$ called the *Čech nerve* of f (Definition 10.1.5.4), which we display informally as

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X \times_Y X \times_Y X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \times_Y X \arrow[r, shift left, "\pi_0"] \arrow[r, shift right, "\pi_1"'] & X \arrow[r] & Y. \end{tikzcd}$$

We will say that f is a *quotient morphism* if $\check{\mathrm{C}}_\bullet(X/Y)$ is a colimit diagram in the ∞-category $\mathcal{C}$.

04TX **Remark 10.2.0.9.** In §10.2.2, we adopt a slightly different definition of quotient morphism (Definition 10.2.2.1), which makes sense in any ∞-category $\mathcal{C}$ (that is, we do not need to assume that $\mathcal{C}$ admits fiber products). Our definition is formulated using the language of *sieves*, which we review in §10.2.1. When $\mathcal{C}$ admits fiber products, the sieve-theoretic definition reduces to the requirement that $\check{\mathrm{C}}_\bullet(X/Y)$ is a colimit diagram (see Proposition 10.2.2.4).

Warning 10.2.0.10. Let $\mathcal{C} = \mathrm{N}_\bullet(\mathcal{C}_0)$ be the nerve of an ordinary category $\mathcal{C}_0$, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then f is a quotient morphism if and only if it is a regular epimorphism in $\mathcal{C}_0$, in the sense of Definition 10.2.0.2 (see Corollary 10.2.2.7). In particular, every quotient morphism in $\mathcal{C}$ is an epimorphism. Beware that, if $\mathcal{C}$ is not assumed to be the nerve of an ordinary category, then the analogous statement is false: quotient morphisms in $\mathcal{C}$ are usually not epimorphisms (that is, they are not monomorphisms when viewed as morphisms in the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$). See Warning 10.2.2.10). 04TY

Let $\mathcal{C}$ be an ∞-category containing a morphism $f : X \to Y$. We will say that an object $Y_0 \in \mathcal{C}$ is an *image of f* if there exists a diagram

$$\begin{tikzcd}
& Y_0 \arrow[dr, "i"] & \\
X \arrow[ur, "f_0"] \arrow[rr, "f"] & & Y,
\end{tikzcd}$$

where f_0 is a quotient morphism and i is a monomorphism (Definition 10.2.3.1). In §10.2.3, we will show that if such a diagram exists, then it is unique up to isomorphism (in fact, up to a contractible space of choices: see Proposition 10.2.3.14). Following our discussion of the classical case, we will typically denote the object Y_0 by $\mathrm{im}(f)$ and refer to it as the *image* of the morphism f (Notation 10.2.3.12).

Let $\mathcal{C}$ be an ∞-category which admits fiber products. Beware that, in general, the collection of quotient morphisms in $\mathcal{C}$ is not closed under pullback (see Exercise 10.2.2.16). We say that a morphism $f : X \to Y$ if $\mathcal{C}$ is a *universal quotient morphism* if, for every pullback diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y,
\end{tikzcd}$$

the morphism f' is a quotient morphism. In §10.2.4, we extend this definition to the setting of an ∞-category $\mathcal{C}$ which does not necessarily admit pullbacks (see Definition 10.2.4.1 and Corollary 10.2.4.7) and study its properties. The notion of universal quotient morphism is in some respects better behaved than the notion of quotient morphism: for example, the collection of universal quotient morphisms is always closed under composition (Proposition 10.2.4.12), while the collection of quotient morphisms need not be (Exercise 10.2.2.15).

Armed with a good theory of quotient morphisms, we can formulate an ∞-categorical analogue of Definition 10.2.0.6. We say that an ∞-category $\mathcal{C}$ is *regular* if it satisfies the following axioms (Definition 10.2.5.1):

(1) The ∞-category $\mathcal{C}$ admits finite limits.

(2) Every morphism $f : X \to Y$ of $\mathcal{C}$ has an image: that is, we can write f as a composition of a quotient morphism $X \twoheadrightarrow Y_0$ with a monomorphism $Y_0 \hookrightarrow Y$.

(3) The collection of quotient morphisms in $\mathcal{C}$ is stable under pullback: that is, every quotient morphism is a universal quotient morphism.

In §10.2.5, we discuss various formulations of this definition and give some examples of regular ∞-categories. In particular, we show that the ∞-category of spaces $\mathcal{S}$ is regular (Corollary 10.2.5.6) and that the collection of regular ∞-categories is closed under the formation of slice constructions (Proposition 10.2.5.9) and left exact localization (Proposition 10.2.5.19).

10.2.1 Sieves

04TZ Let $\mathcal{C}$ be a category. Recall that a *sieve on* $\mathcal{C}$ is a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ satisfying the following condition:

- If $f : X \to Y$ is a morphism of $\mathcal{C}$ and Y belongs to the subcategory $\mathcal{C}^0$, then X also belongs to the subcategory $\mathcal{C}^0$.

This condition has a counterpart in the setting of ∞-categories.

04U0 **Definition 10.2.1.1.** Let $\mathcal{C}$ be a simplicial set. A *sieve on* $\mathcal{C}$ is a simplicial subset $\mathcal{C}^0 \subseteq \mathcal{C}$ for which the inclusion map $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ is a right fibration.

04U1 **Example 10.2.1.2.** Let $\mathcal{C}$ be a simplicial set. Then the simplicial subsets $\emptyset, \mathcal{C} \subseteq \mathcal{C}$ are sieves on $\mathcal{C}$.

04U2 **Remark 10.2.1.3** (Base Change)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of simplicial sets, and let $\mathcal{D}^0 \subseteq \mathcal{D}$ be a sieve on $\mathcal{D}$. Then the inverse image $\mathcal{C}^0 = F^{-1}(\mathcal{D}^0)$ is a sieve on $\mathcal{C}$.

04U3 **Remark 10.2.1.4** (Transitivity)**.** Let $\mathcal{C}$ be a simplicial set containing simplicial subsets $\mathcal{C}^1 \subseteq \mathcal{C}^0 \subseteq \mathcal{C}$, where $\mathcal{C}^0$ is a sieve on $\mathcal{C}$. Then $\mathcal{C}^1$ is a sieve on $\mathcal{C}^0$ if and only if it is a sieve on $\mathcal{C}$.

04U4 **Proposition 10.2.1.5.** *Let $\mathcal{C}$ be a simplicial set. Then a simplicial subset $\mathcal{C}^0 \subseteq \mathcal{C}$ is a sieve if and only if it satisfies the following condition:*

(∗) *Let $\sigma : \Delta^n \to \mathcal{C}$ be an n-simplex of $\mathcal{C}$. If the final vertex $\sigma(n)$ is contained in $\mathcal{C}^0$, then σ is contained in $\mathcal{C}^0$.*

Proof. For every integer $n \geq 0$, the inclusion map $\{n\} \hookrightarrow \Delta^n$ is right anodyne (Example 4.3.7.11). If the inclusion map $\iota : \mathcal{C}^0 \hookrightarrow \mathcal{C}$ is a right fibration, then condition $(\ast)$ is a special case of Proposition 4.2.4.5. Conversely, suppose that condition $(\ast)$ is satisfied, and let $\sigma : \Delta^n \to \mathcal{C}$ be an n-simplex of $\mathcal{C}$. For every integer $0 < i \leq n$, the horn Λ^n_i contains the final vertex $\{n\} \subseteq \Delta^n$. Consequently, if the restriction $\sigma|_{\Lambda^n_i}$ factors (uniquely) through ι, then condition $(\ast)$ guarantees that σ factors (uniquely) through ι. Allowing n and i to vary, we conclude that ι is a right fibration. □

Corollary 10.2.1.6. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a simplicial subset. Then $\mathcal{C}^0$ is a sieve on $\mathcal{C}$ if and only if it is a full subcategory of $\mathcal{C}$ which satisfies the following condition:* 04U5

(∗) *If $f : X \to Y$ is a morphism of $\mathcal{C}$ and Y belongs to the subcategory $\mathcal{C}^0$, then X also belongs to the subcategory $\mathcal{C}^0$.*

Proof. By definition, $\mathcal{C}^0$ is a sieve on $\mathcal{C}$ if and only if the inclusion map $\iota : \mathcal{C}^0 \hookrightarrow \mathcal{C}$ is a right fibration. In particular, this guarantees that ι is an inner fibration, so that $\mathcal{C}^0$ is a subcategory of $\mathcal{C}$. It also guarantees that a morphism $f : X \to Y$ is contained in $\mathcal{C}^0$ if and only if the object Y is contained in $\mathcal{C}^0$, so that the subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ is full and satisfies $(\ast)$. Conversely, if $\mathcal{C}^0 \subseteq \mathcal{C}$ is a full subcategory satisfying condition $(\ast)$, then ι satisfies the criterion of Proposition 10.2.1.5 and is therefore a right fibration. □

Example 10.2.1.7. Let $\mathcal{C}$ be a category and let S be a simplicial subset of $\mathrm{N}_\bullet(\mathcal{C})$. Then S is a sieve on $\mathrm{N}_\bullet(\mathcal{C})$ (in the sense of Definition 10.2.1.1) if and only if it has the form $\mathrm{N}_\bullet(\mathcal{C}^0)$, where $\mathcal{C}^0$ is sieve on $\mathcal{C}$ (in the usual category-theoretic sense). 04U6

Corollary 10.2.1.8. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a sieve. Then $\mathcal{C}^0$ is a replete full subcategory of $\mathcal{C}$. In particular, $\mathcal{C}^0$ is an ∞-category.* 04U7

Remark 10.2.1.9. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a sieve, and let $f : K \to \mathcal{C}^0$ be a diagram. If the simplicial set K is nonempty, then the inclusion map $\mathcal{C}^0_{/f} \hookrightarrow \mathcal{C}_{/f}$ is an isomorphism. In particular, an extension $\overline{f} : K^{\triangleleft} \to \mathcal{C}^0$ is a limit diagram in the ∞-category $\mathcal{C}^0$ if and only if it is a limit diagram in $\mathcal{C}$. 04U8

Example 10.2.1.10. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a sieve, and let X be an object of $\mathcal{C}^0$. If $C_\bullet$ is any simplicial object of $\mathcal{C}$ satisfying $C_0 = X$, then $C_\bullet$ can also be regarded as a simplicial object of $\mathcal{C}^0$. Applying Remark 10.2.1.9, we deduce that $C_\bullet$ is a Čech nerve of X in the ∞-category $\mathcal{C}^0$ if and only if it is a Čech nerve of X in the ∞-category $\mathcal{C}$ (see Definition 10.1.4.1). 04U9

04UA **Proposition 10.2.1.11.** *Let $U : \mathcal{E} \to \mathcal{C}$ be a morphism of simplicial sets. The following conditions are equivalent:*

(1) *The morphism U restricts to an isomorphism of $\mathcal{E}$ with a sieve $\mathcal{C}^0 \subseteq \mathcal{C}$.*

(2) *The morphism U is a right covering map (Definition 4.2.3.8) and, for every vertex $C \in \mathcal{C}$, the fiber $U^{-1}\{C\}$ has at most one element.*

Proof. The implication (1) ⇒ (2) follows from Example 4.2.3.12. Conversely, suppose that condition (2) is satisfied, and let $\mathrm{h}\mathcal{C}$ denote the homotopy category of $\mathcal{C}$. Our assumption that U is a right covering map guarantees the existence of a pullback square

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "U"] \arrow[d] & \mathrm{N}_\bullet(\int^{\mathrm{h}\mathcal{C}} \mathscr{F}) \arrow[d] \\
\mathcal{C} \arrow[r] & \mathrm{N}_\bullet(\mathrm{h}\mathcal{C})
\end{tikzcd}$$

where $\mathscr{F} : \mathrm{h}\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is the contravariant homotopy transport representation of $\mathcal{C}$, given concretely by the formula $\mathscr{F}(C) = U^{-1}\{C\}$ (see Corollary 5.2.7.4). Our assumption that each of the sets $\mathscr{F}(C)$ has at most one element guarantees that the right vertical map induces an isomorphism from $\int^{\mathrm{h}\mathcal{C}} \mathscr{F}$ to the sieve $\mathcal{D} \subseteq \mathrm{h}\mathcal{C}$ spanned by those objects C for which the fiber $U^{-1}\{C\}$ is nonempty. Condition (1) now follows from Example 10.2.1.7 and Remark 10.2.1.3. □

04UB **Corollary 10.2.1.12.** *Let $\mathcal{C}$ be a simplicial set and let $\mathcal{D} = \mathrm{h}\mathcal{C}$ denote its homotopy category. Then the construction $(\mathcal{D}^0 \subseteq \mathcal{D}) \mapsto \mathrm{N}_\bullet(\mathcal{D}^0) \times_{\mathrm{N}_\bullet(\mathcal{D})} \mathcal{C}$ induces a bijection*

$$\{\text{Sieves } \mathcal{D}^0 \subseteq \mathcal{D}\} \to \{\text{Sieves } \mathcal{C}^0 \subseteq \mathcal{C}\}.$$

04UC **Remark 10.2.1.13.** Let $\mathcal{C}$ be a simplicial set. Then the collection of sieves on $\mathcal{C}$ is closed under the formation of intersections. In particular, for every vertex $X \in \mathcal{C}$, there is a smallest sieve $\mathcal{C}^0 \subseteq \mathcal{C}$ containing X. We will refer to $\mathcal{C}^0$ as the *sieve generated by* X. If $\mathcal{C}$ is an ∞-category, then $\mathcal{C}^0$ admits a more explicit description: it is the full subcategory of $\mathcal{C}$ spanned by those objects C for which there exists a morphism $f : C \to X$.

04UZ **Remark 10.2.1.14.** Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the sieve generated by X (Remark 10.2.1.13): that is, the full subcategory of $\mathcal{C}$ spanned by those objects C for which the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, X)$ is nonempty. Then X is a subterminal object of $\mathcal{C}$ (in the sense of Definition 9.2.2.2) if and only if it is a final object of $\mathcal{C}^0$ (in the sense of Definition 4.6.7.1).

Remark 10.2.1.15. Let $\mathcal{C}$ be an ∞-category, let X be an object of $\mathcal{C}$, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the sieve generated by X (Remark 10.2.1.13). Then $\mathcal{C}^0$ is the essential image of the forgetful functor $U : \mathcal{C}_{/X} \to \mathcal{C}$. In particular, U determines a functor $U^0 : \mathcal{C}_{/X} \to \mathcal{C}^0$. Moreover, the following conditions are equivalent: 04V5

(1) The object $X \in \mathcal{C}$ is subterminal (see Definition 9.2.2.2).

(2) The functor U^0 is a trivial Kan fibration.

(3) The functor U^0 is an equivalence of ∞-categories.

The equivalence (1) $\Leftrightarrow$ (2) is a reformulation of Remark 9.2.2.14. Note that U^0 is a pullback of U, and therefore a right fibration (Proposition 4.3.6.1). In particular, U^0 is an isofibration (Example 4.4.1.11), so the equivalence (2) $\Leftrightarrow$ (3) follows from Proposition 4.5.5.20.

Remark 10.2.1.16. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Suppose that $\mathcal{C}$ admits finite limits, so that f admits a Čech nerve $\check{\mathrm{C}}(X/Y)_\bullet$ (Proposition 10.1.5.6). If Y is a subterminal object of $\mathcal{C}$, then the underlying simplicial object of $\check{\mathrm{C}}(X/Y)_\bullet$ is also a Čech nerve of the object X. This follows by combining Remark 10.2.1.15 with Example 10.2.1.10. 04V6

Remark 10.2.1.17. Let $\mathcal{C}$ be an ∞-category containing a 2-simplex 04W3

04W4

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z, \end{tikzcd} \tag{10.6}$$

where g is a monomorphism. Suppose that $\mathcal{C}$ admits fiber products, so that the morphisms f and h admit Čech nerves $\check{\mathrm{C}}_\bullet(X/Y)$ and $\check{\mathrm{C}}_\bullet(X/Z)$ (Proposition 10.1.5.6). Then the underlying simplicial objects of $\check{\mathrm{C}}_\bullet(X/Y)$ and $\check{\mathrm{C}}_\bullet(X/Z)$ are canonically isomorphic. To see this, let us regard (10.6) as morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in the slice ∞-category $\mathcal{C}_{/Z}$. Since the forgetful functor $\mathcal{C}_{/Z} \to \mathcal{C}$ preserves pullbacks (Corollary 7.1.5.18), we can identify $\check{\mathrm{C}}_\bullet(X/Y)$ with the image of $\check{\mathrm{C}}_\bullet(\widetilde{X}/\widetilde{Y})$. The desired result now follows by applying Example 10.2.1.16 to the object $\widetilde{Y} \in \mathcal{C}_{/Z}$ (which is subterminal by virtue of Remark 9.2.4.16).

It will often be convenient to work with a variant Definition 10.2.1.1.

Definition 10.2.1.18. Let $\mathcal{C}$ be an ∞-category and let Y be an object of $\mathcal{C}$. A *sieve on* Y is a sieve on the slice ∞-category $\mathcal{C}_{/Y}$; that is, a full subcategory $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ satisfying the following condition: 04UD

$(*)$ For every 2-simplex

$$\begin{tikzcd}
X' \arrow[rr] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$, if f is contained in the subcategory $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$, then f' is also contained in $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$.

`04UE` **Example 10.2.1.19.** Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. By virtue of Remark 10.2.1.13, there is a smallest sieve $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ on the object Y which contains the morphism f. We will refer to $\mathcal{C}^0_{/Y}$ as the *sieve generated by* f. Concretely, a morphism $e : C \to Y$ belongs to the sieve $\mathcal{C}^0_{/Y}$ if and only if there exists a commutative diagram

$$\begin{tikzcd}
C \arrow[rr] \arrow[dr, "e"'] & & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Stated more informally, a morphism e belongs to the sieve $\mathcal{C}^0_{/Y}$ if and only if it factors through f.

`04UF` **Remark 10.2.1.20.** Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be a sieve on an object Y. Then $\mathcal{C}^0_{/Y}$ coincides with $\mathcal{C}_{/Y}$ if and only if it contains the identity morphism $\mathrm{id}_Y : Y \to Y$. In particular, if $\mathcal{C}^0_{/Y}$ is the sieve generated by a morphism $f : X \to Y$ (Example 10.2.1.19), then $\mathcal{C}^0_{/Y} = \mathcal{C}_{/Y}$ if and only if the morphism f admits a right homotopy inverse $s : Y \to X$.

`04UG` **Remark 10.2.1.21.** Let $\mathcal{C}$ be an ∞-category, let Y be an object of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be a sieve on Y. The condition that a morphism $f : X \to Y$ belongs to $\mathcal{C}^0_{/Y}$ depends only on the isomorphism class of f as an object of the ∞-category $\mathcal{C}_{/Y}$. In particular, if X is fixed, then the condition that f belongs to $\mathcal{C}^0_{/Y}$ depends only on the homotopy class $[f] \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$.

We have the following variant of Corollary 10.2.1.12:

`04UH` **Proposition 10.2.1.22.** *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{D} = \mathrm{h}\mathcal{C}$ be its homotopy category, and let Y be an object of $\mathcal{C}$ (which we also regard as an object of $\mathcal{D}$). Then the construction $(\mathcal{D}^0_{/Y} \subseteq \mathcal{D}_{/Y}) \mapsto \mathrm{N}_\bullet(\mathcal{D}^0_{/Y}) \times_{\mathrm{N}_\bullet(\mathcal{D}_{/Y})} \mathcal{C}_{/Y}$ induces a bijection*

$$\{\text{Sieves } \mathcal{D}^0_{/Y} \subseteq \mathcal{D}_{/Y}\} \xrightarrow{\sim} \{\text{Sieves } \mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}\}.$$

Warning 10.2.1.23. Proposition 10.2.1.22 is not a special case of Corollary 10.2.1.12, because the slice category $\mathcal{D}_{/Y}$ is usually not equivalent to the homotopy category of $\mathcal{C}_{/Y}$. 04UJ

Proof of Proposition 10.2.1.22. Let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be a sieve on the ∞-category $\mathcal{C}_{/Y}$. We wish to show that there is a unique sieve $\mathcal{D}^0_{/Y} \subseteq \mathcal{D}_{/Y}$ with the following property: a morphism $f : X \to Y$ belongs to the sieve $\mathcal{C}^0_{/Y}$ if and only if the homotopy class $[f]$ belongs to the sieve $\mathcal{D}^0_{/Y}$. The uniqueness assertion is immediate. To prove existence, we define $\mathcal{D}^0_{/Y}$ to be the full subcategory of $\mathcal{D}_{/Y}$ spanned by those homotopy classes $[f] : X \to Y$ such that f belongs to $\mathcal{C}^0_{/Y}$; by virtue of Remark 10.2.1.21, this condition depends only on the homotopy class $[f]$ and not on the choice of representative f. To complete the proof, it will suffice to show that the subcategory $\mathcal{D}^0_{/Y} \subseteq \mathcal{D}_{/Y}$ is a sieve. Suppose we are given a commutative diagram

04UK

$$\begin{tikzcd}
X' \arrow[rr] \arrow[dr, "{[f']}"'] & & X \arrow[dl, "{[f]}"] \\
& Y &
\end{tikzcd} \tag{10.7}$$

in the homotopy category $\mathcal{D} = \mathrm{h}\mathcal{C}$. We wish to show that if $[f]$ belongs to $\mathcal{D}^0_{/Y}$, then $[f']$ also belongs to $\mathcal{D}^0_{/Y}$. This follows from our assumption that $\mathcal{C}^0_{/Y}$ is a sieve on Y, since (10.7) can be lifted to a 2-simplex in the ∞-category $\mathcal{C}$. □

Notation 10.2.1.24 (Pullback Sieves)**.** Let $\mathcal{C}$ be an ∞-category, let $f : X \to Y$ be a morphism of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be a sieve on the object Y. We let $f^*(\mathcal{C}^0_{/Y})$ denote the full subcategory of $\mathcal{C}_{/X}$ spanned by those objects $e : C \to X$ for which the composition $(f \circ e) : C \to Y$ belongs to $\mathcal{C}^0_{/Y}$. By virtue of Remark 10.2.1.21, this condition is independent of the choice of composition $f \circ e$. The subcategory $f^*(\mathcal{C}^0_{/Y})$ is a sieve on the object X, which we will refer to as the *pullback of* $\mathcal{C}^0_{/Y}$ *along the morphism* f. 04UL

Example 10.2.1.25. Let $\mathcal{C}$ be an ∞-category containing a pullback diagram 04UM

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"] & Y,
\end{tikzcd}$$

and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be the sieve on Y generated by the morphism f. Then the pullback $u^*(\mathcal{C}^0_{/Y})$ is the sieve on Y' generated by the morphism f'. In other words, a morphism $[v] : C \to X'$ in the homotopy category $\mathrm{h}\mathcal{C}$ factors through $[f']$ if and only if the composite morphism $[u] \circ [v]$ factors through $[f]$ (see Warning 7.6.3.3).

Let $\mathcal{C}$ be an ∞-category. Recall that a full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ is *dense* if the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is left Kan extended from $\mathcal{C}^0$ (Definition 8.4.1.5). We now consider a slight variant of this condition.

04UN **Definition 10.2.1.26.** Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$. We say that a sieve $\mathcal{C}^0_{/X} \subseteq \mathcal{C}_{/X}$ on X is *dense* if the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ is left Kan extended from $\mathcal{C}^0_{/X}$.

04UP **Warning 10.2.1.27.** The terminology of Definition 10.2.1.26 has the potential to create confusion. Since the forgetful functor $\mathcal{C}_{/X} \to \mathcal{C}$ creates colimits (Proposition 7.1.3.19), a sieve $\mathcal{C}^0_{/X} \subseteq \mathcal{C}_{/X}$ which is dense in the sense of Definition 10.2.1.26 is also dense when regarded as a full subcategory of $\mathcal{C}_{/X}$ (in the sense of Definition 8.4.1.5). Beware that the converse is false in general (Example 10.2.1.28). However, it is true if $\mathcal{C}$ admits finite products (Proposition 10.2.1.29).

04UQ **Example 10.2.1.28.** Let $\mathcal{C}$ be the 1-dimensional simplicial set associated to the directed graph depicted in the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \\
Y & B, \arrow[l] \arrow[u]
\end{tikzcd}$$

and let $\mathcal{C}^0_{/X} \subseteq \mathcal{C}_{/X}$ be the sieve spanned by the objects A and B. Then $\mathcal{C}^0_{/X}$ is dense when regarded as a full category of $\mathcal{C}_{/X}$ (in the sense of Definition 8.4.1.5), but not when regarded as a sieve on X (in the sense of Definition 10.2.1.26).

04UR **Proposition 10.2.1.29.** *Let $\mathcal{C}$ be an ∞-category which admits pairwise products and let $X \in \mathcal{C}$. Then a sieve $\mathcal{C}^0_{/X} \subseteq \mathcal{C}_{/X}$ is dense (in the sense of Definition 10.2.1.26) if and only if it is dense when regarded as a subcategory of $\mathcal{C}_{/X}$ (in the sense of Definition 8.4.1.5).*

Proof. Assume that $\mathcal{C}^0_{/X}$ is a dense subcategory of $\mathcal{C}_{/X}$; we will show that it is dense when regarded as a sieve (for the reverse implication, see Warning 10.2.1.27). By assumption, the identity functor $\mathrm{id} : \mathcal{C}_{/X} \to \mathcal{C}_{/X}$ is left Kan extended from $\mathcal{C}^0_{/X}$. We wish to show that the forgetful functor $U : \mathcal{C}_{/X} \to \mathcal{C}$ is also left Kan extended from $\mathcal{C}^0_{/X}$. To prove this, it suffices to show that the functor U preserves colimits. This is a special case of Corollary 7.1.3.21, since the functor U admits a right adjoint (given on objects by the construction $Y \mapsto X \times Y$; see Proposition 7.6.1.12). $\square$

04US **Example 10.2.1.30.** Let $\mathcal{C}$ be an ∞-category. For every object $X \in \mathcal{C}$, the ∞-category $\mathcal{C}_{/X}$ is a dense sieve on X (see Example 7.3.3.8).

Remark 10.2.1.31. Let $\mathcal{C}$ be an ∞-category and let $\mathcal{C}^0_{/X} \subseteq \mathcal{C}^1_{/X} \subseteq \mathcal{C}_{/X}$ be sieves on an object X. If $\mathcal{C}^0_{/X}$ is dense, then $\mathcal{C}^1_{/X}$ is also dense. See Proposition 7.3.8.6. `04UT`

Remark 10.2.1.32. Let $\mathcal{C}$ be an ∞-category, let Y be an object of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be a sieve on X. Let $f : X \to Y$ be a morphism of $\mathcal{C}$, which we regard as an object of $\mathcal{C}_{/Y}$, and let $\mathcal{C}^0_{/X} = f^*(\mathcal{C}^0_{/Y})$ be the pullback sieve (Notation 10.2.1.24). Then the forgetful functor $\mathcal{C}_{/Y} \to \mathcal{C}$ is left Kan extended from $\mathcal{C}^0_{/Y}$ at f if and only if the following condition is satisfied: `04UU`

$(*_f)$ The composite map

$$(\mathcal{C}^0_{/X})^{\triangleright} \hookrightarrow \mathcal{C}^{\triangleright}_{/X} \to \mathcal{C}$$

is a colimit diagram in the ∞-category $\mathcal{C}$.

In particular, the sieve $\mathcal{C}^0_{/Y}$ is dense if and only if it satisfies condition $(*_f)$ for *every* morphism $f : X \to Y$ of $\mathcal{C}$.

Proposition 10.2.1.33. *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. For every dense sieve $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$, the pullback sieve $f^* \mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/X}$ is also dense.* `04UV`

Proof. This is an immediate consequence of the criterion of Remark 10.2.1.32. □

Proposition 10.2.1.34 (Transitivity)**.** *Let $\mathcal{C}$ be an ∞-category, let Y be an object of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y}, \mathcal{C}^1_{/Y} \subseteq \mathcal{C}_{/Y}$ be sieves on Y. Assume that:* `04UW`

(1) *The sieve $\mathcal{C}^0_{/Y}$ is dense.*

(2) *For each morphism $f : X \to Y$ which belongs to $\mathcal{C}^0_{/Y}$, the pullback sieve $f^*(\mathcal{C}^1_{/Y}) \subseteq \mathcal{C}_{/X}$ is dense.*

Then $\mathcal{C}^1_{/Y}$ is also a dense sieve.

Proof. Let $U : \mathcal{C}_{/Y} \to \mathcal{C}$ denote the projection map; we wish to show that U is left Kan extended from $\mathcal{C}^1_{/Y}$. Assumption (1) guarantees that U is left Kan extended from $\mathcal{C}^0_{/Y}$. By virtue of Corollary 7.3.8.8, it will suffice to show that $U|_{\mathcal{C}^0_{/Y}}$ is left Kan extended from the intersection $\mathcal{C}^{01}_{/Y} = \mathcal{C}^0_{/Y} \cap \mathcal{C}^1_{/Y}$. Fix a morphism $f : X \to Y$ which belongs to the sieve $\mathcal{C}^0_{/Y}$; we wish to show that U is left Kan extended from $\mathcal{C}^{01}_{/Y}$ at f. This follows from our assumption that $f^* \mathcal{C}^{01}_{/Y} = f^* \mathcal{C}^1_{/Y}$ is a dense sieve on X. □

10.2.2 Quotient Morphisms

Let X and Y be sets, and let $f : X \to Y$ be a function. The function f determines an equivalence relation $\equiv_f$ on X, defined by the requirement `04WG`

$$(x \equiv_f x') \Leftrightarrow (f(x) = f(x')).$$

If f is surjective, then it induces an bijection from $X/\equiv_f$ to Y. Stated in more categorical terms, this means that for every set S, composition with f induces a bijection

$$\begin{tikzcd}
\{\text{Functions } Y \to S\} \arrow[d] \\
\{\text{Functions } g : X \to S \text{ satisfying } g(x) = g(x') \text{ when } f(x) = f(x')\}.
\end{tikzcd}$$

Our goal in this section is to study an ∞-categorical counterpart of this condition.

04WH **Definition 10.2.2.1.** Let $\mathcal{C}$ be an ∞-category, let $f : X \to Y$ be a morphism of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be the sieve generated by f (see Example 10.2.1.19). We will say that f is a *quotient morphism* if the composite map

$$(\mathcal{C}^0_{/Y})^{\triangleright} \hookrightarrow (\mathcal{C}_{/Y})^{\triangleright} \to \mathcal{C}$$

is a colimit diagram in the ∞-category $\mathcal{C}$.

04WJ **Notation 10.2.2.2.** Let $\mathcal{C}$ be an ∞-category and let f be a morphism of $\mathcal{C}$ having source X and target Y. If f is a quotient morphism, we will often visually emphasize this by denoting f with a double-headed arrow (that is, we will write $f : X \twoheadrightarrow Y$ in place of $f : X \to Y$). Beware that this notation does *not* indicate that f is an epimorphism (see Warning 10.2.2.10).

04WK **Exercise 10.2.2.3.** Let $\mathcal{C}$ be an ∞-category. Show that every isomorphism in $\mathcal{C}$ is a quotient morphism (see Example 10.2.4.3 for a stronger statement).

Stated more informally, a morphism $f : X \to Y$ is a quotient morphism if the object Y can be recovered as the colimit $\varinjlim_{C \to Y} C$, indexed by the ∞-category of morphisms $g : C \to Y$ which factor through f. If the ∞-category $\mathcal{C}$ admits fiber products, this condition admits a more concrete formulation.

04WL **Proposition 10.2.2.4.** *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$ which admits a Čech nerve $\check{\mathrm{C}}(X/Y)_{\bullet} : \mathrm{N}_{\bullet}(\mathbf{\Delta}^{\mathrm{op}}_{+}) \to \mathcal{C}$ (see Definition 10.1.5.4). Then f is a quotient morphism if and only if $\check{\mathrm{C}}_{\bullet}(X/Y)$ is a colimit diagram in $\mathcal{C}$.*

Stated more informally, Proposition 10.2.2.4 asserts that $f : X \to Y$ is a quotient morphism if and only if it exhibits Y as a geometric realization of the simplicial object depicted in the diagram

$$\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} X \times_Y X \times_Y X \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} X \times_Y X \rightrightarrows X.$$

The proof will require some preliminaries.

Lemma 10.2.2.5. *Let $\mathcal{C}$ be an ∞-category and let X be an object of $\mathcal{C}$ which admits a Čech nerve $\check{C}(X)_\bullet : N_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \to \mathcal{C}$ (see Definition 10.1.4.3). Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the sieve generated by X (Example 10.2.1.19). Then the functor $\check{C}(X)_\bullet : N_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \to \mathcal{C}^0$ is right cofinal.* 04WM

Proof. Let C be an object of $\mathcal{C}$ and let $h^C : \mathcal{C} \to \mathcal{S}$ be a functor corepresented by C. Since h^C preserves finite products (Proposition 7.4.5.17), the composition

$$N_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \xrightarrow{\check{C}(X)_\bullet} \mathcal{C} \xrightarrow{h^C} \mathcal{S}$$

is a simplicial object of $\mathcal{S}$ which can be identified with the Čech nerve of the Kan complex $h^C(\check{C}(X)_0) \simeq \mathrm{Hom}_{\mathcal{C}}(C, X)$ (Remark 10.1.4.7). If C belongs to the sieve $\mathcal{C}^0$, then the morphism space $\mathrm{Hom}_{\mathcal{C}}(C, X)$ is nonempty. Applying Corollary 10.1.6.15, we conclude that the geometric realization $|h^C(\check{C}(X)_\bullet)|$ is contractible. The desired result now follows by allowing the object C to vary and applying the criterion of Proposition 7.4.5.11. □

Variant 10.2.2.6. Let $\mathcal{C}$ be an ∞-category, let $f : X \to Y$ be a morphism of $\mathcal{C}$ which admits a Čech nerve $\check{C}(X/Y)_\bullet : N_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}) \to \mathcal{C}$, and let $\mathcal{C}^0_{/Y}$ denote the sieve generated by f. Then $\check{C}(X/Y)_\bullet$ determines a right cofinal functor $N_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \to \mathcal{C}^0_{/Y}$. 04WN

Proof. Apply Lemma 10.2.2.5 to the slice ∞-category $\mathcal{C}_{/Y}$. □

Proof of Proposition 10.2.2.4. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$ which admits a Čech nerve $\check{C}(X/Y)_\bullet : N_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}) \to \mathcal{C}$. We wish to show that f is a quotient morphism if and only if $\check{C}(X/Y)_\bullet$ is a colimit diagram in $\mathcal{C}$. Let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ denote the sieve generated by X and let Q denote the composite map

$$(\mathcal{C}^0_{/Y})^{\triangleright} \hookrightarrow (\mathcal{C}_{/Y})^{\triangleright} \to \mathcal{C}.$$

Let us identify $\check{C}(X/Y)_\bullet$ with a functor $F : N_\bullet(\mathbf{\Delta}^{\mathrm{op}}) \to \mathcal{C}^{\mathrm{op}}_{/Y}$. Unwinding the definitions, we wish to show that Q is a colimit diagram if and only if the composite functor

$$N_\bullet(\mathbf{\Delta}^{\mathrm{op}})^{\triangleright} \xrightarrow{F^{\triangleright}} (\mathcal{C}^0_{/Y})^{\triangleright} \xrightarrow{Q} \mathcal{C}$$

is a colimit diagram. This is a special case of Corollary 7.2.2.3, since the functor F is right cofinal (Variant 10.2.2.6). □

Corollary 10.2.2.7. *Let $\mathcal{C}$ be a category which admits fiber products and let $f : X \to Y$ be a morphism in $\mathcal{C}$. The following conditions are equivalent:* 04WP

(1) *The morphism f is a quotient morphism in the ∞-category $N_\bullet(\mathcal{C})$ (in the sense of Definition 10.2.2.1).*

(2) *The morphism f is a regular epimorphism: that is, it exhibits Y as a coequalizer of the projection maps $X \times_Y X \rightrightarrows X$ (Definition 10.2.0.2).*

Proof. Combine Proposition 10.2.2.4 with Corollary 10.1.2.12. □

`04WQ` **Example 10.2.2.8.** Let X and Y be sets. Then a function $f : X \to Y$ is a quotient morphism (in the category of sets) if and only if it is surjective (see Example 10.2.0.4).

`04WR` **Variant 10.2.2.9.** Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. If f is a quotient morphism (in the sense of Definition 10.2.2.1), then it is an epimorphism.

Proof. Suppose that we are given a pair of morphisms $e_0, e_1 : Y \to Z$ in $\mathcal{C}$ satisfying $e_0 \circ f = e_1 \circ f$; we wish to show that $e_0 = e_1$. By virtue of our assumption that f is a quotient morphism, it will suffice to show that $e_0 \circ h = e_1 \circ h$ for every morphism $h : C \to Y$ which belongs to the sieve $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ generated by f. In this case, we can write $h = f \circ g$ for some morphism $g : C \to X$; the desired result then follows from the calculation

$$e_0 \circ h = e_0 \circ (f \circ g) = (e_0 \circ f) \circ g = (e_1 \circ f) \circ g = e_1 \circ (f \circ g) = e_1 \circ h.$$

□

`04WS` **Warning 10.2.2.10.** Let $f : X \to Y$ be a quotient morphism in an ∞-category $\mathcal{C}$. If $\mathcal{C}$ is not (the nerve of) an ordinary category, then f need not be an epimorphism. For example, let (X, x) be a pointed Kan complex, and let $\iota : \{x\} \to X$ denote the inclusion map, which we regard as a morphism in the ∞-category $\mathcal{S}$ of spaces. Then:

- The morphism ι is an epimorphism (in the ∞-category $\mathcal{S}$) if and only if X is contractible. To see this, we observe that the identity map id_X and the constant map $c : X \to \{x\} \xrightarrow{\iota} X$ become homotopic after precomposition with ι; if ι is an epimorphism, it follows that id_X is homotopic to c.

- The morphism ι is a quotient morphism if and only if X is connected (see Proposition 10.2.4.17).

`04WT` **Proposition 10.2.2.11** (Homotopy Invariance)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of ∞-categories and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then f is a quotient morphism if and only if $F(f)$ is a quotient morphism in the ∞-category $\mathcal{D}$.*

Proof. Let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be the sieve generated by f, and let $\mathcal{D}^0_{/F(Y)} \subseteq \mathcal{D}^0_{/F(Y)}$ be the sieve generated by $F(f)$. Since F is fully faithful, $\mathcal{C}^0_{/Y}$ is the inverse image of $\mathcal{D}^0_{/F(Y)}$ under the functor $F_{/Y} : \mathcal{C}_{/Y} \to \mathcal{D}_{/F(Y)}$ induced by F. Corollary 4.6.4.19 guarantees that $F_{/Y}$ is an equivalence of ∞-categories, and therefore induces an equivalence $F^0_{/Y} : \mathcal{C}^0_{/Y} \to \mathcal{D}^0_{/F(Y)}$ (Corollary 4.5.2.29). In particular, $F^0_{/Y}$ is right cofinal (Corollary 7.2.1.13). Applying Corollary 7.2.2.3, we deduce that $F(f)$ is a quotient morphism if and only if the composite functor

$$(\mathcal{C}^0_{/Y})^{\triangleright} \hookrightarrow (\mathcal{C}_{/Y})^{\triangleright} \to \mathcal{C} \xrightarrow{F} \mathcal{D}$$

is a colimit diagram in $\mathcal{D}$. By virtue of Proposition 7.1.3.9, this is equivalent to the requirement that f is a quotient morphism in $\mathcal{C}$. □

Corollary 10.2.2.12. *Let $\mathcal{C}$ be an ∞-category and let f_0 and f_1 be morphisms of $\mathcal{C}$ which are isomorphic (when viewed as objects of the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{C})$). Then f_0 is a quotient morphism if and only if f_1 is a quotient morphism.* `04WU`

Proof. Let $\operatorname{Isom}(\mathcal{C})$ denote the full subcategory of $\operatorname{Fun}(\Delta^1, \mathcal{C})$ spanned by the isomorphisms. By virtue of Corollary 4.4.5.10, the evaluation functors $\mathrm{ev}_0, \mathrm{ev}_1 : \operatorname{Isom}(\mathcal{C}) \to \mathcal{C}$ are equivalences of ∞-categories. Our assumption that f_0 is isomorphic to f_1 guarantees that there exists a morphism $\widetilde{f}$ of $\operatorname{Isom}(\mathcal{C})$ satisfying $\mathrm{ev}_0(\widetilde{f}) = f_0$ and $\mathrm{ev}_1(\widetilde{f}) = f_1$. Using Proposition 10.2.2.11, we see that the condition that f_0 is a quotient morphism in $\mathcal{C}$ is equivalent to the condition that $\widetilde{f}$ is a quotient morphism in $\operatorname{Isom}(\mathcal{C})$, which is also equivalent to the condition that f_1 is a quotient morphism in $\mathcal{C}$. □

Example 10.2.2.13. Let $\mathcal{C}$ be an ∞-category containing a pair of morphisms $f_0, f_1 : X \to Y$ which are homotopic. Then f_0 is a quotient morphism if and only if f_1 is a quotient morphism. This is a special case of Corollary 10.2.2.12, but can also be deduced immediately from the definition (since f_0 and f_1 generate the same sieve on Y). `04WV`

Proposition 10.2.2.14. *Let $\mathcal{C}$ be an ∞-category, let $q : K \to \mathcal{C}$ be a diagram, and let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be a morphism in the ∞-category $\mathcal{C}_{/q}$ having image $f : X \to Y$ in $\mathcal{C}$. If f is a quotient morphism in $\mathcal{C}$, then $\widetilde{f}$ is a quotient morphism in $\mathcal{C}_{/q}$.* `04WW`

Proof. Set $\widetilde{\mathcal{C}} = \mathcal{C}_{/q}$, so that we have a commutative diagram of forgetful functors

$$\begin{tikzcd}
\widetilde{\mathcal{C}}_{/\widetilde{Y}} \arrow[r, "V'"] \arrow[d, "\widetilde{U}"] & \mathcal{C}_{/Y} \arrow[d, "U"] \\
\widetilde{\mathcal{C}} \arrow[r, "V"] & \mathcal{C} .
\end{tikzcd}$$

Let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ denote the sieve generated by f, so that $\widetilde{\mathcal{C}}^0_{/\widetilde{Y}} = V'^{-1} \mathcal{C}^0_{/Y}$ is the sieve generated by $\widetilde{f}$. Note that V is a right fibration (Proposition 4.3.6.1), so that V' is a trivial Kan fibration (Corollary 4.3.7.13). In particular, the induced map $\widetilde{\mathcal{C}}^0_{/\widetilde{Y}} \to \mathcal{C}^0_{/Y}$ is a trivial Kan fibration, and therefore right cofinal (Corollary 7.2.1.13). Combining our assumption that f is a quotient morphism with Corollary 7.2.2.3, we deduce that the composite functor

$$(\widetilde{\mathcal{C}}^0_{/\widetilde{Y}})^{\triangleright} \hookrightarrow (\widetilde{\mathcal{C}}_{/\widetilde{Y}})^{\triangleright} \to \widetilde{\mathcal{C}} \xrightarrow{V} \mathcal{C}$$

is a colimit diagram in the ∞-category $\mathcal{C}$. Since the functor V is conservative and creates colimits (Proposition 7.1.3.19), we conclude that $\widetilde{f}$ is a quotient morphism. □

We close this section by recording two negative results, highlighting that the collection of quotient morphisms in an ∞-category $\mathcal{C}$ has poor closure properties in general:

- The collection of quotient morphisms need not be closed under composition (Exercise 10.2.2.15).
- The collection of quotient morphisms need not be closed under the formation of pullbacks (Exercise 10.2.2.16).

Both of these defects can be remedied by working instead with the class of *universal quotient morphisms*, which we study in §10.2.4 (see Definition 10.2.4.1).

04WX **Exercise 10.2.2.15.** Let $\mathcal{C}$ be the (nerve of the) ordinary category depicted informally by the diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[d, shift right, "e_0"'] \arrow[d, shift left, "e_1"] \arrow[dr] & \widetilde{Y} \arrow[d, shift right, "g_0"'] \arrow[d, shift left, "g_1"] \arrow[dr] & \\
X \arrow[r, "f"] & Y \arrow[r, "h"] & Z,
\end{tikzcd}$$

so that $f \circ e_0 = f \circ e_1$ and $h \circ g_0 = h \circ g_1$. Show that f and h are quotient morphisms in $\mathcal{C}$, but the composition $(h \circ f) : X \to Z$ is not a quotient morphism.

04WY **Exercise 10.2.2.16.** Let $\mathcal{C}$ be the category of partially ordered sets (where morphisms are nondecreasing functions). Let $Q = \{a, b, c, d\}$ be a set with four elements, endowed with the partial ordering indicated in the diagram

$$\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \arrow[r] & d.
\end{tikzcd}$$

Let $f : Q \to [2] = \{0 < 1 < 2\}$ be the nondecreasing function given by

$$f(a) = 0 \qquad f(b) = 1 = f(c) \qquad f(d) = 2,$$

so that we have a pullback diagram of partially ordered sets

04WZ

$$\begin{tikzcd}
\{a, d\} \arrow[r] \arrow[d, "f_0"] & Q \arrow[d, "f"] \\
\{0 < 2\} \arrow[r] & {[2]}.
\end{tikzcd} \tag{10.8}$$

Show that f is a quotient morphism in (the nerve of) the category $\mathcal{C}$, but that f_0 is not.

Variant 10.2.2.17. In the situation of Exercise 10.2.2.16, we can apply the nerve functor to (10.8) and obtain a commutative diagram of ∞-categories 04X0

$$\begin{tikzcd}
\{a,d\} \arrow[r] \arrow[d, "F_0"] & \mathrm{N}_\bullet(Q) \arrow[d, "F"] \\
\mathrm{N}_\bullet(\{0<2\}) \arrow[r] & \Delta^2,
\end{tikzcd}$$

which we can regard as a pullback square in the ∞-category $\mathcal{QC}$. Show that F is a quotient morphism in $\mathcal{QC}$, but that F_0 is not (beware that this is not a formal consequence of Exercise 10.2.2.16: the construction $P \mapsto \mathrm{N}_\bullet(P)$ does not preserve quotient morphisms in general).

10.2.3 Images

Let X and Y be sets. Recall that the *image* of a function $f : X \to Y$ is defined to be the subset $\mathrm{im}(f) = \{y \in Y : f^{-1}\{y\} \neq \emptyset\}$. More abstractly, the set $\mathrm{im}(f)$ is characterized (up to isomorphism) by the requirement that f factors as a composition 04X1

$$X \overset{q}{\twoheadrightarrow} \mathrm{im}(f) \overset{i}{\hookrightarrow} Y,$$

where q is surjective and i is injective. This motivates the following:

Definition 10.2.3.1. Let $\mathcal{C}$ be an ∞-category, let Y be an object of $\mathcal{C}$, and let $Y_0 \subseteq Y$ be a subobject: that is, an object of $\mathcal{C}$ equipped with a (specified) monomorphism $i : Y_0 \hookrightarrow Y$ (see Definition 9.2.4.25). We will say that Y_0 is an *image* of a morphism $f : X \to Y$ if the homotopy class $[f]$ factors as a composition $[i] \circ [q]$, where $q : X \twoheadrightarrow Y_0$ is a quotient morphism in $\mathcal{C}$. 04X2

Remark 10.2.3.2. In the situation of Definition 10.2.3.1, our assumption that i is a monomorphism guarantees that the composition map $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y_0) \xrightarrow{[i]\circ} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(X, Y)$ is injective. It follows that if there exists a morphism $q : X \to Y_0$ satisfying $[f] = [i] \circ [q]$, then q is uniquely determined up to homotopy. In particular, the condition that q is a quotient morphism is independent of the choice of q (see Example 10.2.2.13). 04X3

Remark 10.2.3.3. In the situation of Definition 10.2.3.1, Y_0 is an image of f if and only if there exists a 2-simplex 04X4

04X5

$$\begin{tikzcd}
& Y_0 \arrow[dr, "i"] & \\
X \arrow[ur, "q"] \arrow[rr, "f"] & & Y
\end{tikzcd} \tag{10.9}$$

in the ∞-category $\mathcal{C}$, where q is a quotient morphism. If this condition is satisfied, we say that the 2-simplex (10.9) *exhibits Y_0 as an image of f.*

04X6 **Example 10.2.3.4** (Images of Sets)**.** Let $f : X \to Y$ be a function between sets, and set $Y_0 = \{y \in Y : f^{-1}\{y\} \neq \emptyset\}$. Then f determines a surjection from X to Y_0, which is a quotient morphism in the category of sets (Example 10.2.2.8). It follows that the commutative diagram

$$\begin{tikzcd}
& Y_0 \arrow[dr, "i"] & \\
X \arrow[ur, "f"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

exhibits Y_0 as an image of f.

04X7 **Example 10.2.3.5** (Images of Monomorphisms)**.** Let $\mathcal{C}$ be an ∞-category and let $f : X \hookrightarrow Y$ be a monomorphism in $\mathcal{C}$. Since the identity map id_X is a quotient morphism (Exercise 10.2.2.3), the left-degenerate 2-simplex

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

exhibits X as an image of f.

04X8 **Proposition 10.2.3.6** (Images of Quotient Morphisms)**.** *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then f is a quotient morphism if and only if Y is an image of f (when regarded as a subobject of itself).*

Proof. Assume first that f is a quotient morphism. Since the identity map id_Y is a monomorphism (Example 9.2.4.9), the right-degenerate 2-simplex

$$\begin{tikzcd}
& Y \arrow[dr, "\mathrm{id}_Y"] & \\
X \arrow[ur, "f"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

exhibits Y as an image of f. Conversely, if Y is an image of f, then f factors as the composition of a quotient morphism and an isomorphism, and is therefore a quotient morphism by virtue of Corollary 10.2.2.12. $\square$

Warning 10.2.3.7. The terminology of Definition 10.2.3.1 is not entirely standard. In the setting of additive categories, many authors refer to an object Y as the *image* of a morphism $f : X \to Y$ if it is a kernel of the tautological map $Y \twoheadrightarrow \operatorname{coker}(f)$. This agrees with Definition 10.2.3.1 when $\mathcal{C}$ is an abelian category (Proposition [?]), but not in general. 04X9

Warning 10.2.3.8 (Essential Images)**.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Recall that the *essential image* of F is the full subcategory $\mathcal{D}_0 \subseteq \mathcal{D}$ spanned by objects $D \in \mathcal{D}$ which are isomorphic to $F(C)$, for some object $C \in \mathcal{C}$ (Definition 4.6.2.11). In this case, the inclusion map $\iota : \mathcal{D}_0 \hookrightarrow \mathcal{D}$ is always a monomorphism in $\mathcal{QC}$ (Corollary 9.2.4.33), and F factors (uniquely) as the composition of ι with a functor $F_0 : \mathcal{C} \to \mathcal{D}_0$. Beware that this factorization generally does *not* exhibit $\mathcal{D}_0$ as an image of F in the ∞-category $\mathcal{QC}$, in the sense of Definition 10.2.3.1: that is, the functor F_0 need not be a quotient morphism in $\mathcal{QC}$. This fails, for example, if F is the inclusion functor $\partial\Delta^1 \hookrightarrow \Delta^1$. 04XA

We now show that if $f : X \to Y$ is a morphism in $\mathcal{C}$ which admits an image Y_0, then the subobject $Y_0 \subseteq Y$ is uniquely determined up to isomorphism.

Lemma 10.2.3.9. *Let $q : X \twoheadrightarrow Y$ be a quotient morphism in an ∞-category $\mathcal{C}$. Then every subterminal object $C \in \mathcal{C}$ is q-local.* 04XB

Proof. We wish to show that the composition map $\operatorname{Hom}_{\mathcal{C}}(Y, C) \xrightarrow{\circ[q]} \operatorname{Hom}_{\mathcal{C}}(X, C)$ is a homotopy equivalence of Kan complexes. Since C is subterminal, both mapping spaces are either empty or contractible. It will therefore suffice to show that if $\operatorname{Hom}_{\mathcal{C}}(X, C)$ is nonempty, then $\operatorname{Hom}_{\mathcal{C}}(Y, C)$ is also nonempty.

Let $\mathcal{C}^0_{/Y}$ be the sieve generated by q. Since q is a quotient morphism, Y is a colimit of the diagram

$$F : \mathcal{C}^0_{/Y} \hookrightarrow \mathcal{C}_{/Y} \to \mathcal{C};$$

that is, it can be lifted to an initial object $\widetilde{Y}$ of the coslice ∞-category $\mathcal{C}_{F/}$. Since $C \in \mathcal{C}$ is subterminal, the projection map $U : \mathcal{C}_{/C} \to \mathcal{C}$ restricts to a trivial Kan fibration from $\mathcal{C}_{/C}$ to a sieve $\mathcal{C}^1 \subseteq \mathcal{C}$. The assumption that $\operatorname{Hom}_{\mathcal{C}}(X, C)$ is nonempty guarantees that F takes values in $\mathcal{C}^1$ and therefore factors through $\mathcal{C}_{/C}$. A choice of factorization determines a lift of C to an object $\widetilde{C} \in \mathcal{C}_{F/}$. Since $\widetilde{Y}$ is an initial object of $\mathcal{C}_{F/}$, we can choose a morphism $\widetilde{u} : \widetilde{Y} \to \widetilde{C}$ in the ∞-category $\mathcal{C}_{F/}$. Applying the forgetful functor $\mathcal{C}_{F/} \to \mathcal{C}$, we obtain a morphism $u : Y \to C$ in $\mathcal{C}$. □

Lemma 10.2.3.10. *Let $\mathcal{C}$ be an ∞-category and let $q : X \twoheadrightarrow Y$ be a quotient morphism in $\mathcal{C}$. Then q is left orthogonal to every monomorphism $i : C \hookrightarrow D$ of $\mathcal{C}$.* 04XC

Proof. Let $U : \mathcal{C}_{/D} \to \mathcal{C}$ denote the projection map, so that the monomorphism i can be identified with a subterminal object $\widetilde{C} \in \mathcal{C}_{/D}$ satisfying $U(\widetilde{C}) = C$ (Remark 9.2.4.16). By virtue of Corollary 9.1.7.13, it will suffice to show that the object $\widetilde{C}$ is $\widetilde{q}$-local for every morphism $\widetilde{q}$ of $\mathcal{C}_{/D}$ satisfying $U(\widetilde{q}) = q$. This follows from Lemma 10.2.3.9, since $\widetilde{q}$ is a quotient morphism in the ∞-category $\mathcal{C}_{/D}$ (Proposition 10.2.2.14). □

`04XD` **Proposition 10.2.3.11.** *Let $\mathcal{C}$ be an ∞-category containing a 2-simplex*

`04XE`

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "f"] & \\ Y_0 \arrow[r, "i_0"] & Y \end{tikzcd} \tag{10.10}$$

which exhibits Y_0 as an image of f. Then, for any monomorphism $i_1 : Y_1 \hookrightarrow Y$ of $\mathcal{C}$, the following conditions are equivalent:

(1) *The morphism f factors (up to homotopy) through i_1. That is, there exists a 2-simplex*

`04XF`

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & Y_1 \arrow[d, "i_1"] \\ & Y \end{tikzcd} \tag{10.11}$$

in the ∞-category $\mathcal{C}$.

(2) *The containment $[Y_0] \subseteq [Y_1]$ holds (where we regard the isomorphism classes $[Y_0]$ and $[Y_1]$ as elements of the partiallly ordered set $\operatorname{Sub}(Y)$; see Notation 9.2.4.26).*

Proof. The implication (2) ⇒ (1) follows immediately from the definitions. To prove the converse, we note that in the situation of (1), we can amalgamate the diagrams (10.10) and (10.11) to obtain a lifting problem

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q"] & Y_1 \arrow[d, "i_1"] \\ Y_0 \arrow[r, "i_0"] \arrow[ur, dashed] & Y \end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Since q is a quotient morphism and i_1 is a monomorphism, Lemma 10.2.3.10 guarantees that this lifting problem admits an (essentially unique) solution, which proves (2). □

Notation 10.2.3.12. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism in $\mathcal{C}$ which admits an image $Y_0 \subseteq Y$. It follows from Proposition 10.2.3.11 that the isomorphism class $[Y_0]$ is uniquely determined by f (as an object of the partially ordered set $\mathrm{Sub}(Y)$; see Notation 9.2.4.26). To emphasize this, we will denote the isomorphism class $[Y_0]$ by $\mathrm{im}(f)$ and refer to it as *the* image of f. We will sometimes abuse notation by identifying $\mathrm{im}(f)$ with the object Y_0, viewed either as an object of the slice ∞-category $\mathcal{C}_{/Y}$ or as an object of the ∞-category $\mathcal{C}$. `04XG`

Corollary 10.2.3.13. *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Then f is an isomorphism if and only if it is both a monomorphism and a quotient morphism.* `05HS`

Proof. Without loss of generality, we may assume that f is a monomorphism. Applying Example 10.2.3.5, we see that the isomorphism class $[X] \in \mathrm{Sub}(Y)$ is an image of f. It follows that f is an isomorphism if and only if $\mathrm{im}(f) = [Y]$. By virtue of Proposition 10.2.3.6, this is equivalent to the requirement that f is a quotient morphism. □

Proposition 10.2.3.14 (Uniqueness of Images). *Let $\mathcal{C}$ be an ∞-category and let $\mathrm{Fun}'(\Delta^2, \mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ spanned by those 2-simplices* `04XH`

$$\begin{tikzcd}
 & Y_0 \arrow[dr, "i"] & \\
X \arrow[ur, "q"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

which exhibit Y_0 as an image of f (see Remark 10.2.3.3). Then the restriction functor

$$D : \mathrm{Fun}'(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \qquad \sigma \mapsto d_1^2(\sigma)$$

is a trivial Kan fibration from $\mathrm{Fun}'(\Delta^2, \mathcal{C})$ to the full subcategory $\mathrm{Fun}'(\Delta^1, \mathcal{C}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ spanned by those morphisms $f : X \to Y$ which admit an image in $\mathcal{C}$.

Proof. Combining Lemma 10.2.3.10 with Theorem 9.1.8.2, we deduce that the functor D is fully faithful, and therefore induces an equivalence from $\mathrm{Fun}'(\Delta^2, \mathcal{C})$ to the full subcategory $\mathrm{Fun}'(\Delta^1, \mathcal{C}) \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$. To complete the proof, it will suffice to show that D is an isofibration (Proposition 4.5.5.20). This follows from Corollary 4.4.5.3, since $\mathrm{Fun}'(\Delta^2, \mathcal{C})$ is a replete subcategory of $\mathrm{Fun}(\Delta^2, \mathcal{C})$ (see Corollary 10.2.2.12 and Remark 9.2.4.24). □

Remark 10.2.3.15. Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Suppose that $\mathcal{C}$ admits fiber products, so that f admits a Čech nerve $\check{C}_\bullet(X/Y)$ (Proposition `04XJ`

10.1.5.6). If f has an image, then $\mathrm{im}(f)$ can be identified with the geometric realization of the underlying simplicial object of $\check{\mathrm{C}}_\bullet(X/Y)$. To see this, choose a 2-simplex

$$\begin{tikzcd}
 & Y_0 \arrow[dr, "i"] & \\
X \arrow[ur, "q"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

which exhibits Y_0 as an image of f. Since i is a monomorphism, Remark 10.2.1.17 supplies an isomorphism between the underlying simplicial objects of $\check{\mathrm{C}}_\bullet(X/Y)$ and $\check{\mathrm{C}}_\bullet(X/Y_0)$. It will therefore suffice to show that $\check{\mathrm{C}}_\bullet(X/Y_0)$ is a colimit diagram in $\mathcal{C}$, which is a reformulation of our assumption that q is a quotient morphism (Proposition 10.2.2.4).

04XK **Warning 10.2.3.16.** Let $\mathcal{C}$ be an ∞-category which admits fiber products, let $f : X \to Y$ be a morphism of $\mathcal{C}$, and let $C_\bullet$ denote the underlying simplicial object of the Čech nerve $\check{\mathrm{C}}_\bullet(X/Y)$. Remark 10.2.3.15 asserts that if f has an image, then that image can be identified with a geometric realization of $C_\bullet$. Beware that the converse is false in general. Suppose that $C_\bullet$ admits a geometric realization $|C_\bullet|$, given by the image on an initial object E of the coslice ∞-category $\mathcal{C}_{C_\bullet/}$. The augmented simplicial object $\check{\mathrm{C}}_\bullet(X/Y)$ determines another object $\widetilde{Y} \in \mathcal{C}_{C_\bullet/}$, so there is an (essentially unique) morphism from E to $\widetilde{Y}$. The forgetful functor $\mathcal{C}_{C_\bullet/} \to \mathcal{C}_{X/}$ carries this morphism to a 2-simplex

04XL
$$\begin{tikzcd}
 & {|C_\bullet|} \arrow[dr, "i"] & \\
X \arrow[ur, "q"] \arrow[rr, "f"] & & Y
\end{tikzcd} \tag{10.12}$$

in the ∞-category $\mathcal{C}$. In this situation, the following conditions are equivalent:

- The morphism i is a monomorphism.
- The diagram (10.12) exhibits $|C_\bullet|$ as an image of f.
- The morphism f has an image in $\mathcal{C}$.

04XM **Definition 10.2.3.17.** Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ *has images* if every morphism $f : X \to Y$ of $\mathcal{C}$ has an image $\mathrm{im}(f) \in \mathrm{Sub}(Y)$.

04XN **Remark 10.2.3.18** (Functoriality of Images)**.** Let $\mathcal{C}$ be an ∞-category and let $\mathrm{Fun}'(\Delta^2, \mathcal{C}) \subseteq \mathrm{Fun}(\Delta^2, \mathcal{C})$ be the full subcategory described in Proposition 10.2.3.14. Then $\mathcal{C}$ has images if and only if the restriction functor

$$D : \mathrm{Fun}'(\Delta^2, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C}) \qquad \sigma \mapsto d^2_1(\sigma)$$

is a trivial Kan fibration. If this condition is satisfied, then D admits a section which carries each morphism $f : X \to Y$ of $\mathcal{C}$ to a 2-simplex

$$\begin{tikzcd}
 & \mathrm{im}(f) \arrow[dr] & \\
X \arrow[ur] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

which exhibits $\mathrm{im}(f)$ as an image of f. In particular, we can promote the construction $f \mapsto \mathrm{im}(f)$ as a functor of ∞-categories $\mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathcal{C}$.

Remark 10.2.3.19. Let $\mathcal{C}$ be an ∞-category, let Q denote the collection of all quotient morphisms in $\mathcal{C}$, and let M denote the collection of all monomorphisms in $\mathcal{C}$. Then Q and M are closed under isomorphism (see Corollary 10.2.2.12 and Remark 9.2.4.24), and Q is left orthogonal to M (Lemma 10.2.3.10). It follows that $\mathcal{C}$ has images if and only if the pair (Q, M) is a factorization system on $\mathcal{C}$ (Definition 9.1.9.1). 04XP

Proposition 10.2.3.20. *Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$. If $\mathcal{C}$ has images, the following conditions are equivalent:* 04XQ

(1) *The morphism f is a quotient morphism.*

(2) *The morphism f is left orthogonal to every monomorphism in $\mathcal{C}$.*

(3) *The morphism f is weakly left orthogonal to every monomorphism in $\mathcal{C}$.*

Proof. Combine Remark 10.2.3.19 with Proposition 9.1.9.11. □

Corollary 10.2.3.21. *Let $\mathcal{C}$ be an ∞-category which has images, and let* 04XR

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z
\end{tikzcd}$$

be a 2-simplex of $\mathcal{C}$, where f is a quotient morphism. Then g is a quotient morphism if and only if h is a quotient morphism. In particular, the collection of quotient morphisms is closed under composition.

Proof. Combine Proposition 10.2.3.20 with Corollary 9.1.7.15. □

04XS **Corollary 10.2.3.22.** *Let $\mathcal{C}$ be an ∞-category containing a pushout diagram*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y \arrow[r] & Y'.
\end{tikzcd}$$

If $\mathcal{C}$ is has images and f is a quotient morphism, then f' is also a quotient morphism.

Proof. Combine Proposition 10.2.3.20 with Corollary 9.1.7.18. □

04XT **Corollary 10.2.3.23.** *Let $\mathcal{C}$ be an ∞-category with images. Then the collection of quotient morphisms in $\mathcal{C}$ is closed under retracts (in the ∞-category* $\mathrm{Fun}(\Delta^1, \mathcal{C})$*).*

Proof. Combine Proposition 10.2.3.20 with Corollary 9.1.7.17. □

04XU **Exercise 10.2.3.24.** Show that the conclusion of Corollary 10.2.3.23 holds for every ∞-category $\mathcal{C}$: that is, it is not necessary to assume that $\mathcal{C}$ has images.

10.2.4 Universal Quotient Morphisms

04XV Let $\mathcal{C}$ be an ∞-category. In §10.2.2, we observed that the collection of quotient morphisms in $\mathcal{C}$ can exhibit some bad behavior: they need not be closed under composition (Exercise 10.2.2.15) or under the formation of pullbacks (Exercise 10.2.2.16). These deficiencies can be remedied by adopting a more restrictive definition.

04XW **Definition 10.2.4.1** (Universal Quotient Morphisms)**.** Let $\mathcal{C}$ be an ∞-category, let $f : X \to Y$ be a morphism of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be the sieve generated by f (see Example 10.2.1.19). We say that f is a *universal quotient morphism* if the sieve $\mathcal{C}^0_{/X}$ is dense (in the sense of Definition 10.2.1.26).

04XX **Remark 10.2.4.2.** Let $\mathcal{C}$ be an ∞-category and let $f : X \to Y$ be a morphism of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be the sieve generated by f. Then f is a quotient morphism if and only if the forgetful functor $\mathcal{C}_{/Y} \to \mathcal{C}$ is left Kan extended from $\mathcal{C}^0_{/Y}$ at the object $\mathrm{id}_Y : Y \to Y$ (see Remark 10.2.1.32). In particular, if f is a universal quotient morphism, then f is a quotient morphism. Beware that the converse is false in general (see Example 10.2.4.11).

04XY **Example 10.2.4.3.** Let $\mathcal{C}$ be an ∞-category, let $f : X \to Y$ be a morphism of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be the sieve generated by f. If f admits a right homotopy inverse $s : Y \to X$, then the sieve $\mathcal{C}^0_{/Y}$ coincides with $\mathcal{C}_{/Y}$, and is therefore dense. It follows that f is a universal quotient morphism. In particular, every isomorphism is a universal quotient morphism.

Remark 10.2.4.4. Let $\mathcal{C}$ be an ∞-category, let Y be an object of $\mathcal{C}$, and let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be a sieve on Y. If $\mathcal{C}^0_{/Y}$ contains a universal quotient morphism $f : X \twoheadrightarrow Y$, then it is dense. See Remark 10.2.1.31. `04XZ`

Remark 10.2.4.5. Let $\mathcal{C}$ be an ∞-category containing a 2-simplex `04Y0`

$$\begin{tikzcd}
 & Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z.
\end{tikzcd}$$

If h is a universal quotient morphism, then g is also a universal quotient morphism. This is a special case of Remark 10.2.4.4.

Proposition 10.2.4.6. *Let $\mathcal{C}$ be an ∞-category containing a pullback diagram* `04Y1`

`04Y2`

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "u"] & Y.
\end{tikzcd} \tag{10.13}$$

If f is a universal quotient morphism, then f' is also a universal quotient morphism.

Proof. Let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ be the sieve generated by f. Our assumption that f is a universal quotient morphism guarantees that $\mathcal{C}^0_{/Y}$ is a dense sieve on Y. Applying Proposition 10.2.1.33, we deduce that the pullback $u^* \mathcal{C}^0_{/Y}$ is a dense sieve on Y'. Since (10.13) is a pullback square, the sieve $u^* \mathcal{C}^0_{/Y'}$ is generated by f', so that f' is also a universal quotient morphism. □

The terminology of Definition 10.2.4.1 is motivated by the following result:

Corollary 10.2.4.7. *Let $\mathcal{C}$ be an ∞-category which admits fiber products and let $f : X \to Y$ be a morphism of $\mathcal{C}$. The following conditions are equivalent:* `04Y3`

(1) *The morphism f is a universal quotient morphism.*

(2) *For every pullback diagram*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

of $\mathcal{C}$, the morphism f' is a universal quotient morphism.

(3) *For every pullback diagram*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

of $\mathcal{C}$, the morphism f' is a quotient morphism.

Proof. The implication (1) ⇒ (2) follows from Proposition 10.2.4.6, the implication (2) ⇒ (3) from Remark 10.2.4.2, and the implication (3) ⇒ (1) from the criterion of Remark 10.2.1.32 (together with Example 10.2.1.25). □

04Y4 **Corollary 10.2.4.8.** *Let $\mathcal{C}$ be an ∞-category which admits fiber products. The following conditions are equivalent:*

(1) *Every quotient morphism in $\mathcal{C}$ is a universal quotient morphism.*

(2) *The collection of quotient morphisms in $\mathcal{C}$ is closed under pullbacks. That is, for every pullback diagram*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r] & Y
\end{tikzcd}$$

where f is a quotient morphism, f' is also a quotient morphism.

04Y5 **Corollary 10.2.4.9.** *Let X and Y be sets, and let $f : X \to Y$ be a function. The following conditions are equivalent:*

(1) *The function f is a universal quotient morphism in the category of sets.*

(2) *The function f is a quotient morphism in the category of sets.*

(3) *The function f is surjective.*

Proof. The implication (1) ⇒ (2) follows from Remark 10.2.4.2 and the equivalence (2) ⇔ (3) follows from Example 10.2.2.8. Since the collection of surjections is closed under pullbacks, Corollary 10.2.4.7 guarantees that (3) ⇒ (1). □

04Y6 **Corollary 10.2.4.10.** *Let $\mathcal{C}$ be an ∞-category which admits pullbacks, and suppose that geometric realizations in $\mathcal{C}$ are universal (see Definition [?]). Then every quotient morphism in $\mathcal{C}$ is a universal quotient morphism.*

Proof. Combine Corollary 10.2.4.7 and Proposition 10.2.2.4 (together with Remark 10.1.5.7). □

Example 10.2.4.11. Let $\mathcal{C}$ be (the nerve of) the category of partially ordered sets. Then Exercise 10.2.2.16 supplies an example of a quotient morphism $f : Q \twoheadrightarrow [2]$ in $\mathcal{C}$ which is not a universal quotient morphism. 04Y7

Proposition 10.2.4.12. *Let $\mathcal{C}$ be an ∞-category containing a 2-simplex* 04Y8

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z.
\end{tikzcd}$$

If f and g are universal quotient morphisms, then h is also a universal quotient morphism.

Proof. Let $\mathcal{C}^0_{/Z}$ and $\mathcal{C}^1_{/Z}$ be the sieves generated by g and h, respectively. By assumption, the sieve $\mathcal{C}^0_{/Z}$ is dense, and we wish to show that $\mathcal{C}^1_{/Z}$ is also dense. By virtue of Proposition 10.2.1.34, it will suffice to show that for every morphism $u : Z' \to Z$ which belongs to $\mathcal{C}^0_{/Z}$, the pullback $u^* \mathcal{C}^1_{/Z'}$ is a dense sieve on Z'. Using Proposition 10.2.1.33, we can reduce to the special case where u is the morphism $g : Y \twoheadrightarrow Z$. In this case, the pullback sieve $u^*(\mathcal{C}^1_{/Y}) \subseteq \mathcal{C}_{/Y}$ contains the universal quotient morphism $f : X \twoheadrightarrow Y$, and is therefore dense (Remark 10.2.4.4). □

Variant 10.2.4.13. Let $\mathcal{C}$ be an ∞-category containing a 2-simplex 04Y9

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] \arrow[rr, "h"] & & Z.
\end{tikzcd}$$

If f is a universal quotient morphism and g is a quotient morphism, then h is a quotient morphism.

Proof. Let $\mathcal{C}^0_{/Z}$ and $\mathcal{C}^1_{/Z}$ be the sieves on X generated by g and h, respectively. Our assumption that g is a quotient morphism guarantees that the functor

$$\overline{Q} : (\mathcal{C}^0_{/Z})^{\triangleright} \to (\mathcal{C}_{/Z})^{\triangleright} \to \mathcal{C}$$

is a colimit diagram in the ∞-category $\mathcal{C}$, and we wish to show that the restriction $\overline{Q}|_{(\mathcal{C}^1_{/Z})^{\triangleright}}$ is also a colimit diagram. By virtue of Corollary 7.3.8.2, it will suffice to show that the restriction $Q = \overline{Q}|_{\mathcal{C}^0_{/Z}}$ is left Kan extended from the full subcategory $\mathcal{C}^1_{/Z}$. Fix a morphism $u : Z' \to Z$ which belongs to the sieve $\mathcal{C}^0_{/Z}$; we wish to show that Q is left Kan extended from $\mathcal{C}^1_{/Z}$ at u. In fact, we will prove a slightly stronger assertion: the pullback $u^*(\mathcal{C}^1_{/Z})$ is a dense sieve on Z'. Using Proposition 10.2.1.33, we are reduced to proving this in the special case where u is the morphism $g : Y \to Z$. In this case, the sieve $u^*(\mathcal{C}^1_{/Z})$ contains the quotient morphism f, and is therefore dense by virtue of Remark 10.2.4.4. □

04YA **Corollary 10.2.4.14.** *Let $\mathcal{C}$ be an ∞-category. Then the collection of universal quotient morphisms of $\mathcal{C}$ is closed under retracts (in the ∞-category $\operatorname{Fun}(\Delta^1, \mathcal{C})$).*

Proof. Let $f : X \twoheadrightarrow Y$ be a universal quotient morphism in $\mathcal{C}$ and let $f' : X' \to Y'$ be a retract of f, so that we have a commutative diagram

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "f'"] & X \arrow[r, "r_X"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
Y' \arrow[r] & Y \arrow[r, "r_Y"] & Y'
\end{tikzcd}$$

where the vertical compositions are homotopic to the identity. We wish to show that f' is also a universal quotient morphism. By virtue of Remark 10.2.4.5, it will suffice to show that the composition $(f' \circ r_X) : X \to Y'$ is a universal quotient morphism. Using the commutativity of the diagram, we can write $f' \circ r_X$ as a composition of r_Y with f. Since f is a universal quotient morphism by assumption and r_Y is a universal quotient morphism by virtue of Example 10.2.4.3, the desired result follows from Proposition 10.2.4.12. □

04YB **Proposition 10.2.4.15.** *Let $\mathcal{C}$ be an ∞-category, let $q : K \to \mathcal{C}$ be a diagram, and let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be a morphism in the ∞-category $\mathcal{C}_{/q}$ having image $f : X \to Y$ in $\mathcal{C}$. If f is a universal quotient morphism in $\mathcal{C}$, then $\widetilde{f}$ is a universal quotient morphism in $\mathcal{C}_{/q}$.*

Proof. Set $\widetilde{\mathcal{C}} = \mathcal{C}_{/q}$, so that we have a commutative diagram of forgetful functors

$$\begin{tikzcd}
\widetilde{\mathcal{C}}_{/\widetilde{Y}} \arrow[r, "V'"] \arrow[d, "\widetilde{U}"] & \mathcal{C}_{/Y} \arrow[d, "U"] \\
\widetilde{\mathcal{C}} \arrow[r, "V"] & \mathcal{C} .
\end{tikzcd}$$

Let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ denote the sieve generated by f. Since f is a universal quotient morphism, the functor U is left Kan extended from $\mathcal{C}^0_{/Y}$. Note that V is a right fibration (Proposition

4.3.6.1), so that V' is a trivial Kan fibration (Corollary 4.3.7.13). In particular, V' is a right fibration, so that the functor $U \circ V' = V \circ \widetilde{U}$ is left Kan extended from the subcategory $\widetilde{\mathcal{C}}^0_{/\widetilde{Y}} = V'^{-1}\,\mathcal{C}^0_{/Y}$ (Corollary 7.3.8.5). Since the functor V is conservative and creates colimits (Proposition 7.1.3.19), it follows that $\widetilde{U}$ is also left Kan extended from $\widetilde{\mathcal{C}}^0_{/\widetilde{Y}}$. We conclude by observing that $\widetilde{\mathcal{C}}^0_{/\widetilde{Y}}$ is the sieve generated by $\widetilde{f}$, so that $\widetilde{f}$ is a universal quotient morphism in $\widetilde{\mathcal{C}}$. □

We close this section by characterizing (universal) quotient morphisms in the ∞-category $\mathcal{S}$ of spaces.

Lemma 10.2.4.16. *Let $\mathcal{C}$ be an ∞-category, let $\mathcal{C}' \subseteq \mathcal{C}$ be a dense full subcategory, and let $f : X \to Y$ be a morphism of $\mathcal{C}$. Suppose that, for every object $C \in \mathcal{C}'$, postcomposition with $[f]$ induces a surjection $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, X) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, Y)$. Then f is a universal quotient morphism.* `04YC`

Proof. Our assumption that $\mathcal{C}'$ is dense guarantees that the identity functor $\mathcal{C} \to \mathcal{C}$ is left Kan extended from $\mathcal{C}'$. Let $U : \mathcal{C}_{/Y} \to \mathcal{C}$ be the projection map and let $\mathcal{C}'_{/Y} \subseteq \mathcal{C}_{/Y}$ denote the inverse image of $\mathcal{C}'$. Since U is a right fibration (Proposition 4.3.6.1), the functor U is left Kan extended from $\mathcal{C}'_{/Y}$. Let $\mathcal{C}^0_{/Y} \subseteq \mathcal{C}_{/Y}$ denote the sieve generated by f. Our hypothesis guarantees that $\mathcal{C}^0_{/Y}$ contains $\mathcal{C}'_{/Y}$. Applying Corollary 7.3.8.8, we conclude that U is left Kan extended from $\mathcal{C}^0_{/Y}$: that is, f is a universal quotient morphism. □

Proposition 10.2.4.17. *Let $f : X \to Y$ be a map of Kan complexes. The following conditions are equivalent:* `04YD`

(1) *The map f is a universal quotient morphism in the ∞-category $\mathcal{S}$ (Definition 10.2.4.1).*

(2) *The map f is a quotient morphism in the ∞-category $\mathcal{S}$ (Definition 10.2.2.1).*

(3) *The map f is 0-connective: that is, it induces a surjection $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$.*

Proof. The implication (1) ⇒ (2) is a special case of Corollary 10.2.4.10. We next show that (2) implies (3). Assume that f is a quotient morphism, so that Y can be identified with the geometric realization of the Čech nerve $\check{\mathrm{C}}(X/Y)_\bullet$ in the ∞-category $\mathcal{S}$ (Proposition 10.2.2.4). Note that the functor

$$\mathcal{S} \to \mathrm{N}_\bullet(\mathrm{Set}) \qquad S \mapsto \pi_0(S)$$

preserves the formation of geometric realizations (since it is left adjoint to the inclusion functor). It follows that $\pi_0(Y)$ can be identified with the geometric realization of $\pi_0(\check{\mathrm{C}}(X/Y)_\bullet)$ in the category of sets: that is, with the coequalizer of the projection maps $\pi_0(X \times_Y X) \rightrightarrows \pi_0(X)$ (see Corollary 10.1.2.12). In particular, the tautological map $\pi_0(X) \to \pi_0(Y)$ is surjective.

We now show that (3) implies (1). Assume that condition (3) is satisfied. For every contractible Kan complex C, the composition map $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, X) \xrightarrow{[f]\circ} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, Y)$ can be identified with $\pi_0(f)$, and is therefore surjective. Since the contractible Kan complexes span a dense subcategory of $\mathcal{S}$ (Example 8.4.2.3), Lemma 10.2.4.16 implies that f is a universal quotient morphism. □

10.2.5 Regular ∞-Categories

04YE We now formulate an ∞-categorical counterpart of Definition 10.2.0.6.

04YF **Definition 10.2.5.1.** Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is *regular* if it satisfies the following conditions:

(1) The ∞-category $\mathcal{C}$ admits finite limits.

(2) The ∞-category $\mathcal{C}$ has images. That is, every morphism $f : X \to Y$ of $\mathcal{C}$ can be extended to a 2-simplex

$$\begin{tikzcd} & Y_0 \arrow[dr, "i"] & \\ X \arrow[ur, "q"] \arrow[rr, "f"] & & Y, \end{tikzcd}$$

where q is a quotient morphism and i is a monomorphism.

(3) The collection of quotient morphisms in $\mathcal{C}$ is closed under pullback. That is, for every pullback diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

of $\mathcal{C}$, if f is a quotient morphism, then f' is also a quotient morphism.

04YG **Example 10.2.5.2.** Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is regular (in the sense of Definition 10.2.0.6) if and only if $\mathrm{N}_\bullet(\mathcal{C})$ is a regular ∞-category (in the sense of Definition 10.2.5.1). See Corollary 10.2.2.7.

Definition 10.2.5.1 admits a number of reformulations.

04YH **Proposition 10.2.5.3.** *Let $\mathcal{C}$ be an ∞-category which has images and admits finite limits. The following conditions are equivalent:*

(1) *The ∞-category $\mathcal{C}$ is regular: that is, the collection of quotient morphisms in $\mathcal{C}$ is closed under pullbacks.*

(2) *Every quotient morphism in $\mathcal{C}$ is a universal quotient morphism.*

(3) *Every morphism $f : X \to Y$ of $\mathcal{C}$ can be realized as the composition of a universal quotient morphism $q : X \twoheadrightarrow Y_0$ and a monomorphism $i : Y_0 \hookrightarrow Y$.*

(4) *For every pullback diagram*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "u"] \\
X \arrow[r, "f"] & Y,
\end{tikzcd} \tag{10.14}$$

04YJ

the image $\mathrm{im}(f')$ *coincides with* $u^{-1}(\mathrm{im}(f))$ *(as an element of the set* $\mathrm{Sub}(Y')$*).*

Proof. The equivalence of (1) $\Leftrightarrow$ (2) is a special case of Corollary 10.2.4.8, and the implication (2) $\Rightarrow$ (3) is immediate from the definitions. We next show that (3) implies (4). Fix a diagram of the form (10.14). If condition (3) is satisfied, then we can choose a diagram

$$\begin{tikzcd}
 & & Y' \arrow[d, "u"] \\
X \arrow[r, "q"] & Y_0 \arrow[r, "i"] & Y
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$ where q is a universal quotient morphism, i is a monomorphism, and the lower vertical composition coincides with f. Since $\mathcal{C}$ admits finite limits, this diagram admits a right Kan extension

$$\begin{tikzcd}
Y' \times_Y X \arrow[r, "q'"] \arrow[d] & Y' \times_Y Y_0 \arrow[r, "i'"] \arrow[d] & Y' \arrow[d, "u"] \\
X \arrow[r, "q"] & Y_0 \arrow[r, "i"] & Y,
\end{tikzcd} \tag{10.15}$$

04YK

so that the right square and outer rectangle are pullback diagrams. By construction, the inverse image $u^{-1}(\mathrm{im}(f))$ is the isomorphism class of the fiber product $Y' \times_Y Y_0$ (regarded as an object of the ∞-category $\mathcal{C}_{/Y'}$ via the morphism i'). On the other hand, the uniqueness of limits guarantees that $Y' \times_Y X$ is isomorphic to X' as an object of $\mathcal{C}_{/Y'}$, so the image of f' coincides with the image of the composite morphism $i' \circ q'$. To prove that this image coincides with $[Y' \times_Y Y_0]$, it suffices to show that q' is a quotient morphism in $\mathcal{C}$. This follows from Corollary 10.2.4.7, since the left half of (10.15) is also a pullback square (Proposition 7.6.3.25).

We now complete the proof by showing that (4) implies (1). Suppose we are given a pullback square (10.14), where f is a quotient morphism; we wish to show that f' is also a quotient morphism. By virtue of Proposition 10.2.3.6, the assumption that f is a quotient morphism guarantees that $\mathrm{im}(f) = [Y]$ is the largest element of $\mathrm{Sub}(Y)$, and we wish to show that $\mathrm{im}(f') = [Y']$ is the largest element of $\mathrm{Sub}(Y')$. This follows immediately from (4), since the inverse image construction $u^{-1} : \mathrm{Sub}(Y) \to \mathrm{Sub}(Y')$ preserves largest elements (see Construction 9.2.4.31). □

04YL **Remark 10.2.5.4.** Let $\mathcal{C}$ be an ∞-category which has images and admits finite limits. Then, for every pullback diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "u"] \\
X \arrow[r, "f"] & Y,
\end{tikzcd}$$

we always have a containment $\mathrm{im}(f') \subseteq u^{-1}(\mathrm{im}(f))$ in $\mathrm{Sub}(Y')$; this follows from the characterization of $\mathrm{im}(f)$ supplied by Proposition 10.2.3.11.

04YM **Corollary 10.2.5.5.** *Let $\mathcal{C}$ be an ∞-category which admits finite limits. Then $\mathcal{C}$ is regular if and only if every morphism $f : X \to Y$ can be obtained by composing a universal quotient morphism $q : X \twoheadrightarrow Y_0$ with a monomorphism $i : Y_0 \hookrightarrow Y$.*

04YN **Corollary 10.2.5.6.** *Let $\mathcal{S}$ denote the ∞-category of spaces. Then $\mathcal{S}$ is a regular ∞-category.*

Proof. Corollary 7.4.5.6 guarantees that $\mathcal{S}$ admits finite limits. By virtue of Corollary 10.2.5.5, it will suffice to show that every map of Kan complexes $f : X \to Y$ factors as a composition $i \circ q$, where $q : X \twoheadrightarrow Y_0$ is a universal quotient morphism in $\mathcal{S}$ and $i : Y_0 \hookrightarrow Y$ is a monomorphism in $\mathcal{S}$. For this, we can take $i : Y_0 \hookrightarrow Y$ to be the inclusion of the essential image of f (which is a monomorphism by Example 9.2.4.10), and $q : X \to Y_0$ to be the restriction of f (which is a universal quotient morphism by Proposition 10.2.4.17. □

04YP **Remark 10.2.5.7.** Let $f : X \to Y$ be a morphism of Kan complexes and let $Y_0 \subseteq Y$ be its essential image. The proof of Corollary 10.2.5.6 shows that Y_0 is an image of f in the ∞-category $\mathcal{S}$.

Warning 10.2.5.8. Let $\mathcal{QC}$ denote the ∞-category of (small) ∞-categories. Then $\mathcal{QC}$ contains quotient morphisms which are not universal quotient morphisms (see Variant 10.2.2.17). In particular, $\mathcal{QC}$ is not regular. `04YQ`

Proposition 10.2.5.9. *Let $\mathcal{C}$ be a regular ∞-category. Then, for every object $Z \in \mathcal{C}$, the slice ∞-category $\mathcal{C}_{/Z}$ is regular.* `04YR`

Proof. It follows from Remark 7.1.2.11 that the ∞-category $\mathcal{C}_{/Z}$ admits finite limits. By virtue of Corollary 10.2.5.5, it will suffice to show that every morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ can be realized as the composition of a universal quotient morphism $\widetilde{X} \twoheadrightarrow \widetilde{Y}_0$ with a monomorphism $\widetilde{Y}_0 \hookrightarrow \widetilde{Y}$. Let $f : X \to Y$ denote the image of $\widetilde{f}$ in the ∞-category $\mathcal{C}$. Since $\mathcal{C}$ is regular, we can choose a 2-simplex σ :

$$\begin{tikzcd} & Y_0 \arrow[dr, "i"] & \\ X \arrow[ur, "q"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

of $\mathcal{C}$, where q is a universal quotient morphism and i is a monomorphism. The inclusion map $\mathrm{N}_{\bullet}(\{0 < 2\}) \hookrightarrow \Delta^2$ is right anodyne (Lemma 4.3.7.8), so we can lift σ to a 2-simplex

$$\begin{tikzcd} & \widetilde{Y}_0 \arrow[dr, "\widetilde{i}"] & \\ \widetilde{X} \arrow[ur, "\widetilde{q}"] \arrow[rr, "\widetilde{f}"] & & \widetilde{Y} \end{tikzcd}$$

in the ∞-category $\mathcal{C}_{/Z}$. We conclude by observing that $\widetilde{i}$ is a monomorphism (Remark 9.2.4.23) and $\widetilde{q}$ is a universal quotient morphism (Proposition 10.2.4.15). □

We now study functors between regular ∞-categories.

Definition 10.2.5.10. Let $\mathcal{C}$ and $\mathcal{D}$ be regular ∞-categories. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is *regular* if it preserves finite limits and carries quotient morphisms of $\mathcal{C}$ to quotient morphisms of $\mathcal{D}$. `04YS`

Remark 10.2.5.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves pullbacks. Then F carries monomorphisms in $\mathcal{C}$ to monomorphisms in $\mathcal{D}$ (Proposition 9.2.4.20). In particular, for every object $Y \in \mathcal{C}$, the functor F carries subobjects of Y to subobjects of $F(Y)$, and therefore induces a map of partially ordered sets $\mathrm{Sub}(Y) \to \mathrm{Sub}(F(Y))$. `04YT`

04YU **Proposition 10.2.5.12.** *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which have images, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves pullback squares. The following conditions are equivalent:*

(1) *The functor F carries quotient morphisms in $\mathcal{C}$ to quotient morphisms in $\mathcal{D}$.*

(2) *For every 2-simplex σ :*

$$\begin{tikzcd} & Y_0 \arrow[dr, "i"] & \\ X \arrow[ur, "q"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

in the ∞-category $\mathcal{C}$ which exhibit Y_0 as an image of f, the 2-simplex $F(\sigma)$ exhibits $F(Y_0)$ as an image of $F(f)$ in the ∞-category $\mathcal{D}$.

(3) *For every morphism $f : X \to Y$ in $\mathcal{C}$, the map* $\mathrm{Sub}(Y) \to \mathrm{Sub}(F(Y))$ *of Remark 10.2.5.11 carries* $\mathrm{im}(f)$ *to* $\mathrm{im}(F(f))$.

Proof. The implication (1) ⇒ (2) follows from the observation that F preserves monomorphisms (Proposition 9.2.4.20), and the implication (2) ⇒ (3) is immediate from the definitions. We will complete the proof by showing that (3) implies (1). Let $f : X \twoheadrightarrow Y$ be a quotient morphism in $\mathcal{C}$; we wish to show that $F(f)$ is a quotient morphism in $\mathcal{D}$. By virtue of Proposition 10.2.3.6, our hypothesis can be reformulated as an equality $\mathrm{im}(f) = [Y]$ in the partially ordered set $\mathrm{Sub}(Y)$, and we wish to prove an equality $\mathrm{im}(F(f)) = [F(Y)]$ in the partially ordered set $\mathrm{Sub}(F(Y))$. This is clear, since the map $\mathrm{Sub}(Y) \to \mathrm{Sub}(F(Y))$ preserves largest elements (see Remark 10.2.5.11). □

04YV **Remark 10.2.5.13.** Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories with images, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves pullback diagrams. For any morphism $f : X \to Y$ of $\mathcal{C}$, we always have an inclusion $F(\mathrm{im}(f)) \subseteq \mathrm{im}(F(f))$ in the partially ordered set $\mathrm{Sub}(F(Y))$.

04YW **Corollary 10.2.5.14.** *Let $\mathcal{C}$ and $\mathcal{D}$ be regular ∞-categories, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits. Then F is regular if and only if, for every morphism $f : X \to Y$ in the ∞-category $\mathcal{C}$, the map* $\mathrm{Sub}(Y) \to \mathrm{Sub}(F(Y))$ *of Remark 10.2.5.11 carries* $\mathrm{im}(f)$ *to* $\mathrm{im}(F(f))$.

04YX **Example 10.2.5.15.** Let $\mathcal{C}$ be a regular ∞-category. Then, for every object $X \in \mathcal{C}$, the slice ∞-category $\mathcal{C}_{/X}$ is also regular (Proposition 10.2.5.9). Moreover, for every morphism $f : X \to Y$ of $\mathcal{C}$, Proposition 7.6.3.16 guarantees that there exists a functor

$$f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X} \qquad Z \mapsto X \times_Y Z$$

given by pullback along f. The functor f^* is regular: it preserves finite limits since it right adjoint to the postcomposition functor $(f \circ \bullet) : \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$, and preserves quotient morphisms by virtue of Proposition 10.2.5.3.

Proposition 10.2.5.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which admit finite limits and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits and geometric realizations of simplicial objects. Then F carries quotient morphisms of $\mathcal{C}$ to quotient morphisms of $\mathcal{D}$. In particular, if the ∞-categories $\mathcal{C}$ and $\mathcal{D}$ are regular, then the functor F is regular.* `04YY`

Proof. Let $f : X \twoheadrightarrow Y$ be a quotient morphism in $\mathcal{C}$; we wish to that $F(f)$ is quotient morphism in $\mathcal{D}$. Let $\check{\mathrm{C}}_\bullet(X/Y) : \mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}) \to \mathcal{C}$ be a Čech nerve of f (Notation 10.1.5.5. Since f is a quotient morphism (Remark 10.2.4.2), $\check{\mathrm{C}}_\bullet(X/Y)$ is a colimit diagram in $\mathcal{C}$ (Proposition 10.2.2.4). Our assumption that F preserves geometric realizations guarantees that $F \circ \check{\mathrm{C}}_\bullet(X/Y)$ is a colimit diagram in the ∞-category $\mathcal{D}$. Since F preserves finite limits, $F \circ \check{\mathrm{C}}_\bullet(X/Y)$ is a Čech nerve of the morphism $F(f) : F(X) \to F(Y)$. Applying Proposition 10.2.2.4 again, we deduce that $F(f)$ is a quotient morphism in $\mathcal{D}$. □

We now record some closure properties for the collection of regular ∞-categories.

Proposition 10.2.5.17. *Let $\mathcal{C}$ be an ∞-category which admits pullbacks, let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under the formation of pullbacks, and let $q : X \to Y$ be a morphism in $\mathcal{C}_0$. If q is a quotient morphism in $\mathcal{C}$, then it is also a quotient morphism in $\mathcal{C}_0$. If q is a universal quotient morphism in $\mathcal{C}$, then it is also a universal quotient morphism in $\mathcal{C}_0$.* `04YZ`

Proof. Assume that q is a quotient morphism in $\mathcal{C}$; we will show that it is also a quotient morphism in $\mathcal{C}_0$ (the analogous assertion for universal quotient morphisms then follows from the criterion of Corollary 10.2.4.7). Since $\mathcal{C}$ admits pullbacks and $\mathcal{C}_0$ is stable under the formation of pullbacks, it follows that $\mathcal{C}_0$ also admits pullbacks. Applying Proposition 10.1.5.6, we deduce that q admits a Čech nerve $\check{\mathrm{C}}_\bullet(X/Y) : \mathrm{N}_\bullet(\mathbf{\Delta}_+^{\mathrm{op}}) \to \mathcal{C}_0$, which is also a Čech nerve of q in the ∞-category $\mathcal{C}$. Since q is a quotient morphism, $\check{\mathrm{C}}_\bullet(Z/Y)$ is a colimit diagram in $\mathcal{C}$ (Proposition 10.2.2.4). It follows that $\check{\mathrm{C}}_\bullet(X/Y)$ is also a colimit diagram in $\mathcal{C}_0$, so that q is a quotient morphism in $\mathcal{C}_0$. □

Corollary 10.2.5.18. *Let $\mathcal{C}$ be a regular ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a full subcategory which is closed under finite limits. Assume that, for every morphism f of $\mathcal{C}_0$, the image $\mathrm{im}(f)$ (formed in the ∞-category $\mathcal{C}$) can be chosen to belong to $\mathcal{C}_0$. Then $\mathcal{C}_0$ is also regular.* `04Z0`

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Since $\mathcal{C}$ is regular, f can be factored as the composition of a universal quotient morphism $q : X \twoheadrightarrow Y_0$ with a monomorphism $i : Y_0 \hookrightarrow Y$. If X and Y belong to $\mathcal{C}_0$, then our assumption guarantees that we can arrange that Y_0 is also contained in $\mathcal{C}_0$. In this case, i is also a monomorphism in the ∞-category $\mathcal{C}_0$, and

Proposition 10.2.5.17 guarantees that q is a universal quotient morphism in the subcategory $\mathcal{C}_0$. Allowing f to vary and invoking Corollary 10.2.5.5, we conclude that $\mathcal{C}_0$ is regular. $\square$

04Z1 **Proposition 10.2.5.19.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which admits a fully faithful right adjoint $G : \mathcal{D} \to \mathcal{C}$. Suppose that the ∞-category $\mathcal{C}$ is regular and that F preserves finite limits. Then the ∞-category $\mathcal{D}$ is also regular, and F is a regular functor.*

Proof. Since the functor F has a right adjoint, it preserves geometric realizations of simplicial objects (Corollary 7.1.3.21). Applying Proposition 10.2.5.16, we deduce that the functor F carries quotient morphisms in $\mathcal{C}$ to quotient morphisms in $\mathcal{D}$. It will therefore suffice to show that $\mathcal{D}$ is regular.

It follows from Corollary 7.1.3.27 (together with Corollary 6.2.2.17) that the ∞-category $\mathcal{D}$ admits finite limits. We next show that every morphism $v : D \to D'$ in $\mathcal{D}$ has an image. Let $\epsilon : (F \circ G) \to \mathrm{id}_{\mathcal{D}}$ be the counit of an adjunction between F and G. Since G is fully faithful, the natural transformation ϵ is an isomorphism. We can therefore replace v by the morphism $(F \circ G)(v)$, and thereby reduce to the case where $v = F(u)$ for some morphism $u : C \to C'$ in $\mathcal{C}$. In this case, our assumption that $\mathcal{C}$ has images guarantees that we can factor u as a composition $C \overset{q}{\twoheadrightarrow} C'_0 \overset{i}{\hookrightarrow} C'$, where q is a quotient morphism and i is a monomorphism (in the ∞-category $\mathcal{C}$). It follows that v can be written as the composition of $F(q)$ (which is a quotient morphism in $\mathcal{D}$, as noted above) with $F(i)$ (which is a monomorphism in $\mathcal{D}$ by virtue of Proposition 9.2.4.20). In particular, the object $F'(C_0)$ is an image of v.

We now complete the proof by showing that if

04Z2
$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d] & Z' \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd} \qquad (10.16)$$

is a pullback diagram in $\mathcal{C}$ where f is a quotient morphism, then f' is also a quotient morphism. Since $\mathcal{C}$ is regular, we can choose a 2-simplex

$$\begin{tikzcd}
& Y \arrow[dr] & \\
G(X) \arrow[ur, "q"] \arrow[rr, "G(f)"] & & G(Z),
\end{tikzcd}$$

which exhibits Y as an image of $G(f)$. It follows that $F(\sigma)$ exhibits $F(Y)$ as an image of the morphism $(F \circ G)(f)$ in the ∞-category $\mathcal{D}$. Note that $(F \circ G)(f)$ is isomorphic to f

(as an object of $\mathrm{Fun}(\Delta^1, \mathcal{D})$), and is therefore a quotient morphism (Corollary 10.2.2.12). Applying Proposition 10.2.3.6, we conclude that $F(i)$ is an isomorphism in $\mathcal{D}$.

Amalgamating the 2-simplex σ with $G(g)$, we obtain a diagram

$$\begin{tikzcd}
 & & G(Z') \arrow[d, "G(g)"] \\
G(X) \arrow[r, "q"] & Y \arrow[r, "i"] & G(Z)
\end{tikzcd}$$

in the ∞-category $\mathcal{C}$. Since $\mathcal{C}$ admits finite limits, this diagram admits a right Kan extension

04Z3

$$\begin{tikzcd}
G(X) \times_{G(Z)} G(Z') \arrow[r, "q'"] \arrow[d] & Y \times_{G(Z)} G(Z') \arrow[r, "i'"] \arrow[d] & G(Z') \arrow[d, "g"] \\
G(X) \arrow[r, "q"] & Y \arrow[r, "i"] & G(Z),
\end{tikzcd} \tag{10.17}$$

so that the square on the right and the outer rectangle are pullback squares. Note that, after applying the functor F, the outer rectangle of this diagram is isomorphic to (10.16). We are therefore reduced to showing that the functor F carries the upper horizontal composition in (10.17) to a quotient morphism in $\mathcal{D}$. Since F preserves pullback squares, $F(i')$ is a pullback of $F(i)$ and is therefore an isomorphism (Corollary 7.6.3.24). Using Corollary 10.2.2.12, we are reduced to showing that $F(q')$ is a quotient morphism in $\mathcal{D}$. In fact, we claim that q' is a pullback morphism of $\mathcal{C}$. This follows from our assumption that $\mathcal{C}$ is regular, since q is a quotient morphism by construction and the left half the diagram (10.17) is a pullback square (Proposition 7.6.3.25). □

Corollary 10.2.5.20. *Let $\mathcal{C}$ be a regular ∞-category and let $\mathcal{C}_0 \subseteq \mathcal{C}$ be a reflective subcategory, so that the inclusion functor $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{C}_0$. If the functor L preserves finite limits, then $\mathcal{C}_0$ is a regular ∞-category and L is a regular functor.* 04Z4

Bibliography

[1] J. Adámek and J. Rosický. On sifted colimits and generalized varieties. *Theory Appl. Categ.*, 8:33–53, 2001.

[2] M. G. Barratt, V. K. A. M. Gugenheim, and J. C. Moore. On semisimplicial fibre-bundles. *American Journal of Mathematics*, 81:639, 1959.

[3] C. Barwick, S. Glasman, and D. Nardin. Dualizing cartesian and cocartesian fibrations. *Theory and Application of Categories*, 33(4):67–94, 2018.

[4] J. Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[5] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[6] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

[7] R. Brown. Groupoids and van Kampen's theorem. *Proc. London Math. Soc. (3)*, 17:385–401, 1967.

[8] A. Campbell. A counterexample in quasi-category theory. *Proc. Amer. Math. Soc.*, 148(1):37–40, 2020.

[9] P. Cohen. The independence of the continuum hypothesis. *Proc. Nat. Acad. Sci. U.S.A.*, 50:1143–1148, 1963.

[10] P. J. Cohen. The independence of the continuum hypothesis. II. *Proc. Nat. Acad. Sci. U.S.A.*, 51:105–110, 1964.

[11] J.-M. Cordier. Sur la notion de diagramme homotopiquement cohérent. *Cahiers Topologie Géom. Différentielle*, 23(1):93–112, 1982. Third Colloquium on Categories, Part VI (Amiens, 1980).

[12] J.-M. Cordier and T. Porter. Vogt's theorem on categories of homotopy coherent diagrams. *Math. Proc. Cambridge Philos. Soc.*, 100(1):65–90, 1986.

[13] P. Deligne. Action du groupe des tresses sur une catégorie. *Invent. Math.*, 128(1):159–175, 1997.

[14] A. Dold. Homology of symmetric products and other functors of complexes. *Ann. of Math. (2)*, 68:54–80, 1958.

[15] C. H. Dowker. Topology of metric complexes. *Amer. J. Math.*, 74:555–577, 1952.

[16] V. Drinfeld. On the notion of geometric realization. *Mosc. Math. J.*, 4(3):619–626, 782, 2004.

[17] J. W. Duskin. Simplicial matrices and the nerves of weak n-categories. I. Nerves of bicategories. *Theory Appl. Categ.*, 9:198–308, 2001/02. CT2000 Conference (Como).

[18] S. Eilenberg and S. Mac Lane. On the groups $H(\Pi, n)$. I. *Ann. of Math. (2)*, 58:55–106, 1953.

[19] S. Eilenberg and J. A. Zilber. On products of complexes. *Amer. J. Math.*, 75:200–204, 1953.

[20] G. Faonte. Simplicial nerve of an $\mathcal{A}_\infty$-category. *Theory Appl. Categ.*, 32:Paper No. 2, 31–52, 2017.

[21] P. Freyd and A. Heller. Splitting homotopy idempotents ii. *Journal of Pure and Applied Algebra*, 89(1):93–106, 1993.

[22] K. Fukaya. Floer homology and mirror symmetry. II. In *Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999)*, volume 34 of *Adv. Stud. Pure Math.*, pages 31–127. Math. Soc. Japan, Tokyo, 2002.

[23] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[24] A. Gagna, Y. Harpaz, and E. Lanari. On the equivalence of all models for $(\infty, 2)$-categories, 2020.

[25] J. B. Giever. On the equivalence of two singular homology theories. *Ann. of Math. (2)*, 51:178–191, 1950.

[26] K. Gödel. *The Consistency of the Continuum Hypothesis.* Annals of Mathematics Studies, No. 3. Princeton University Press, Princeton, N. J., 1940.

[27] A. Grothendieck. *Revêtements étales et groupe fondamental. Fasc. II: Exposés 6, 8 à 11*, volume 1960/61 of *Séminaire de Géométrie Algébrique*. Institut des Hautes Études Scientifiques, Paris, 1963.

[28] G. Heuts and I. Moerdijk. Left fibrations and homotopy colimits. *Math. Z.*, 279(3-4):723–744, 2015.

[29] J. R. Isbell. Adequate subcategories. *Illinois J. Math.*, 4:541–552, 1960.

[30] A. Joyal. Notes on quasi-categories.

[31] A. Joyal. The theory of quasi-categories and its applications.

[32] A. Joyal. Quasi-categories and Kan complexes. *J. Pure Appl. Algebra*, 175(1-3):207–222, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[33] D. M. Kan. On c. s. s. complexes. *Amer. J. Math.*, 79:449–476, 1957.

[34] D. M. Kan. Adjoint functors. *Trans. Amer. Math. Soc.*, 87:294–329, 1958.

[35] D. M. Kan. A combinatorial definition of homotopy groups. *Ann. of Math. (2)*, 67:282–312, 1958.

[36] D. M. Kan. Functors involving c.s.s. complexes. *Trans. Amer. Math. Soc.*, 87:330–346, 1958.

[37] C. Kapulkin, P. L. Lumsdaine, and V. Voevodsky. Univalence in simplicial sets.

[38] J. Kock. Elementary remarks on units in monoidal categories. *Math. Proc. Cambridge Philos. Soc.*, 144(1):53–76, 2008.

[39] R. D. Leitch. The homotopy commutative cube. *J. London Math. Soc. (2)*, 9:23–29, 1974/75.

[40] S. Mac Lane. Natural associativity and commutativity. *Rice Univ. Studies*, 49(4):28–46, 1963.

[41] M. Mather. Pull-backs in homotopy theory. *Canadian J. Math.*, 28(2):225–263, 1976.

[42] M. C. McCord. Classifying spaces and infinite symmetric products. *Trans. Amer. Math. Soc.*, 146:273–298, 1969.

[43] J. Milnor. Construction of universal bundles. I. *Ann. of Math. (2)*, 63:272–284, 1956.

[44] J. Milnor. The geometric realization of a semi-simplicial complex. *Ann. of Math. (2)*, 65:357–362, 1957.

[45] D. Quillen. Higher algebraic K-theory. I. pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[46] D. G. Quillen. *Homotopical algebra.* Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[47] C. Rezk. Free colimit completion in ∞-categories, 2022.

[48] J. Rosický. On homotopy varieties. *Adv. Math.*, 214(2):525–550, 2007.

[49] N. Saavedra Rivano. *Catégories Tannakiennes.* Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin-New York, 1972.

[50] C. Sattler. The equivalence extension property and model structures.

[51] G. Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[52] J. Shah. Parametrized higher category theory, 2021.

[53] M. Shulman. All $(\infty, 1)$-toposes have strict univalent universes.

[54] R. Street. The algebra of oriented simplexes. *J. Pure Appl. Algebra*, 49(3):283–335, 1987.

[55] B. Toën. Vers une axiomatisation de la théorie des catégories supérieures. *K-Theory*, 34(3):233–263, 2005.

[56] D. Verity. Weak complicial sets. II. Nerves of complicial Gray-categories. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 441–467. Amer. Math. Soc., Providence, RI, 2007.

[57] D. Verity. Complicial sets characterising the simplicial nerves of strict ω-categories. *Mem. Amer. Math. Soc.*, 193(905):xvi+184, 2008.

[58] D. R. B. Verity. Weak complicial sets. I. Basic homotopy theory. *Adv. Math.*, 219(4):1081–1149, 2008.

[59] R. M. Vogt. Homotopy limits and colimits. *Math. Z.*, 134:11–52, 1973.

[60] J. H. C. Whitehead. Combinatorial homotopy. I. *Bull. Amer. Math. Soc.*, 55:213–245, 1949.