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A CHROMATIC VANISHING RESULT FOR TR

LIAM KEENAN AND JONAS MCCANDLESS

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Abstract. In this note, we establish a vanishing result for telescopically localized topological restriction homology TR. More precisely, we prove that $T(k)$-local TR vanishes on connective $L_n^{p,f}$-acyclic $\mathbb{E}_1$-rings for every $1 \leq k \leq n$ and deduce consequences for connective Morava K-theory and the Thom spectra $y(n)$. The proof relies on the relationship between TR and the spectrum of curves on K-theory together with fact that algebraic K-theory preserves infinite products of additive ∞-categories which was recently established by Córdova Fedeli [*Topological Hochschild homology of adic rings*, Ph.D. thesis, University of Copenhagen, 2023].

1. Introduction

In this note, we study the telescopic localizations of TR inspired by the work of Land–Mathew–Meier–Tamme [25] and Mathew [29]. Our starting point is the following result which follows from the main result of [25]: If R is an $\mathbb{E}_1$-ring with $L_n^{p,f} R \simeq 0$, then

$$L_{T(k)} \mathrm{K}(R) \simeq 0$$

for every $1 \leq k \leq n$, where $L_n^{p,f}$ denotes the Bousfield localization at $\mathbb{S}[1/p] \oplus T(1) \oplus \cdots \oplus T(n)$ for a fixed prime number p. For instance, if $R = \mathbb{Z}/p^n$ for some integer $n \geq 1$, then $L_{T(1)} \mathrm{K}(\mathbb{Z}/p^n) \simeq 0$. We consider this result as an extension of Quillen's fundamental calculation that $\mathrm{K}(\mathbb{F}_p)_p^\wedge \simeq H\mathbb{Z}_p$ which in particular yields that $L_{T(1)} \mathrm{K}(\mathbb{F}_p) \simeq 0$. This particular consequence was also obtained by Bhatt–Clausen–Mathew [5] by means of a calculation in prismatic cohomology. Additionally, the vanishing result above for $T(k)$-local K-theory can be applied to the Morava K-theories $K(n)$ and to the Thom spectra $y(n)$ considered by Mahowald–Ravenel–Shick in [28].

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1.1. **Results.** We will be interested in similar vanishing results for $T(k)$-local TR[1]. The invariant TR plays an instrumental role in the classical construction of topological cyclic homology in [7, 8, 20], where TC is obtained as the fixedpoints of a Frobenius operator on TR. In §3, we briefly review the construction of TR following [30] which produces TR together with its Frobenius operator entirely in the Borel–equivariant formalism of Nikolaus–Scholze [31]. Even though TR does not feature prominently in the construction of TC given in [31], TR remains an important invariant by virtue of its close relationship to the Witt vectors and the de Rham–Witt complex [19–22]. In [29], Mathew proves that $T(1)$-local TR is truncating on connective $H\mathbb{Z}$-algebras which means that if R is a connective $H\mathbb{Z}$-algebra, then the canonical map of spectra

$$L_{T(1)} \operatorname{TR}(R) \to L_{T(1)} \operatorname{TR}(\pi_0 R)$$

is an equivalence. This property was verified for $T(1)$-local K-theory and $T(1)$-local TC in [5, 25]. Our main result is a version of this at higher chromatic heights:

Theorem 1.1. *Let $n \geq 1$. If R is a connective $\mathbb{E}_1$-ring such that $L_n^{p,f} R \simeq 0$, then*

$$L_{T(k)} \operatorname{TR}(R) \simeq 0$$

for every $1 \leq k \leq n$.

We remark that Theorem 1.1 is a consequence of the work of [25] in the case where R admits a more refined multiplicative structure; If R admits an $\mathbb{E}_m$-ring structure for $m \geq 2$, then the refined cyclotomic trace $\mathrm{K}(R) \to \operatorname{TR}(R)$ is a map of $\mathbb{E}_1$-rings. Consequently, the spectrum $L_{T(k)} \operatorname{TR}(R)$ admits the structure of a $L_{T(k)} \mathrm{K}(R)$-module and $L_{T(k)} \mathrm{K}(R) \simeq 0$ by [25, Theorem 3.8]. A similar sort of reasoning has recently been employed with great success to study redshift phenomena for algebraic K-theory in [10, 12, 16, 33]. We deduce the following results from Theorem 1.1:

Corollary 1.2. *Let $n \geq 1$. Then $L_{T(k)} \operatorname{TR}(\mathbb{Z}/p^n) \simeq 0$ for every $k \geq 1$.*

We stress that Corollary 1.2 is a consequence of the work of [5, 25] by the reasoning above. For $n = 1$, Corollary 1.2 can also be deduced from the work of Mathew [29]. Since $T(1)$-local TR is truncating on connective $H\mathbb{Z}$-algebra it is in particular nilinvariant by [26], so

$$L_{T(1)} \operatorname{TR}(\mathbb{Z}/p^n) \simeq L_{T(1)} \operatorname{TR}(\mathbb{F}_p) \simeq 0,$$

where the final equivalence follows since $\operatorname{TR}(\mathbb{F}_p, p) \simeq H\mathbb{Z}_p$ by Hesselholt–Madsen [20]. As a consquence of Theorem 1.1 we deduce a new chromatic vanishing result for the connective Morava K-theories, which we denote by $k(n)$. While $k(n)$ admits the structure of an $\mathbb{E}_1$-ring, it does not admit the structure of an $\mathbb{E}_2$-ring so we cannot argue using the refined cyclotomic trace above.

Corollary 1.3. *Let $n \geq 2$. Then $L_{T(k)} \operatorname{TR}(k(n)) \simeq 0$ for every $1 \leq k \leq n-1$.*

Similarly, we obtain a chromatic vanishing result for the Thom spectra $y(n)$ considered in [28].

[1]In this note, we will not distinguish between integral $\operatorname{TR}(R)$ and p-typical $\operatorname{TR}(R,p)$, since

$$\operatorname{TR}(R, \mathbb{Z}_p) \simeq \prod_{(k,p)=1} \operatorname{TR}(R, p)$$

and the p-completion map $\operatorname{TR}(R) \to \operatorname{TR}(R, \mathbb{Z}_p)$ is a $T(k)$-local equivalence for every $k \geq 1$ (cf. [25, Lemma 2.2]).

1.2. **Methods.** We end by explaining the strategy of our proof of Theorem 1.1. The key input is the close relationship between TR and the spectrum of curves on K-theory as studied in [4,6,19,30]. For every $\mathbb{E}_1$-ring R, the spectrum of curves on K-theory is defined by

$$\mathrm{C}(R) = \varprojlim_i \Omega\tilde{\mathrm{K}}(R[t]/t^i),$$

where $\tilde{\mathrm{K}}(R[t]/t^i)$ denotes the fiber of the map $\mathrm{K}(R[t]/t^i) \to \mathrm{K}(R)$ induced by the augmentation. If we assume that R is connective, then $\mathrm{TR}(R) \simeq \mathrm{C}(R)$ by [30, Corollary 4.2.5]. This result was preceded by Hesselholt [19] and Betley–Schlichtkrull [4] who established the result for associative rings after profinite completion. Combining the theorem of the weighted heart (cf. [14,17,18]) with the recent result of Córdova Fedeli [13, Corollary 2.11.1] which asserts that algebraic K-theory preserves arbitrary products of additive ∞-categories, we reduce to proving that

$$L_{T(k)}\,\mathrm{K}^{\oplus}\Big(\prod_{i\geq 1}\mathrm{Proj}^{\omega}_{R[t]/t^i}\Big) \simeq 0$$

provided that $L_n^{p,f}R \simeq 0$, where $\mathrm{Proj}^{\omega}_{R[t]/t^i}$ denotes the additive ∞-category of finitely generated projective $R[t]/t^i$-modules and $\mathrm{K}^{\oplus}$ denotes additive algebraic K-theory. This claim can be verified explicitly by using [25, Proposition 3.6].

2. Preliminaries on weight structures and K-theory

The main technical apparatus for deducing our chromatic vanishing result for TR is the notion of a weight structure on a stable ∞-category in conjunction with the closely related theorem of the weighted heart (cf. [14,17]). This will help us reduce to studying additive algebraic K-theory of additive ∞-categories.

Definition 2.1. A weight structure on a stable ∞-category $\mathcal{C}$ consists of a pair of full subcategories $\mathcal{C}_{[0,\infty]}$ and $\mathcal{C}_{[-\infty,0]}$ of $\mathcal{C}$ such that the following conditions are satisfied:

(1) The full subcategories $\mathcal{C}_{[0,\infty]}$ and $\mathcal{C}_{[-\infty,0]}$ are closed under retracts in $\mathcal{C}$.
(2) For $X \in \mathcal{C}_{[-\infty,0]}$ and $Y \in \mathcal{C}_{[0,\infty]}$, the mapping spectrum $\mathrm{map}_{\mathcal{C}}(X,Y)$ is connective.
(3) For every $X \in \mathcal{C}$, there is a fiber sequence

$$X' \to X \to X''$$

with $X' \in \mathcal{C}_{[-\infty,0]}$ and $X''[-1] \in \mathcal{C}_{[0,\infty]}$.

Let $\mathcal{C}_{[a,\infty]}$ denote the full subcategory of $\mathcal{C}$ spanned by those objects X of $\mathcal{C}$ with $X[-a] \in \mathcal{C}_{[0,\infty]}$, and let $\mathcal{C}_{[-\infty,b]}$ denote the full subcategory of $\mathcal{C}$ spanned by those objects X with $X[-b] \in \mathcal{C}_{[-\infty,0]}$. The heart of the weight structure is the subcategory $\mathcal{C}^{\mathrm{ht}} = \mathcal{C}_{[0,0]}$, where $\mathcal{C}_{[a,b]} = \mathcal{C}_{[a,\infty]} \cap \mathcal{C}_{[-\infty,b]}$. The weight structure is said to be exhaustive if every object is bounded, in the sense that

$$\mathcal{C} = \bigcup_{n\in\mathbb{Z}} \mathcal{C}_{[-n,n]}.$$

A weighted ∞-category is a stable ∞-category equipped with a weight structure.

Remark 2.2. The heart of a weighted ∞-category is an additive ∞-category ([17, Lemma 3.1.2]).

We recall the following terminology which will play an important role throughout this note. For every connective $\mathbb{E}_1$-ring R, let Proj_R^{ω} denote the full subcategory of the ∞-category $\mathrm{LMod}_R^{\geq 0}$ spanned by those connective left R-modules which are finitely generated and projective. Recall that an object of Proj_R^{ω} can be written as a retract of a finitely generated free R-module (cf. [27, Proposition 7.2.2.7]). For any not necessarily connective $\mathbb{E}_1$-ring, let Perf_R denote the ∞-category of perfect R-modules defined as the smallest stable subcategory of LMod_R which contains R and is closed under retracts. The following is our main example of interest:

Example 2.3. For a connective $\mathbb{E}_1$-ring R, let $\mathrm{Perf}_{R,\geq 0}$ be the full subcategory of Perf_R spanned by those perfect R-modules which are connective, and let $\mathrm{Perf}_{R,\leq 0}$ denote the full subcategory of Perf_R spanned by those perfect R-modules M which have projective amplitude ≤ 0. This means that every R-linear map $M \to N$ is nullhomotopic provided that N is 1-connective. The pair $(\mathrm{Perf}_{R,\geq 0}, \mathrm{Perf}_{R,\leq 0})$ defines an exhaustive weight structure on Perf_R whose heart is equivalent to the additive ∞-category Proj_R^{ω} of finitely generated projective R-modules (cf. [18, 1.38 & 1.39]); while the proofs therein are stated for connective $\mathbb{E}_\infty$-rings, the same arguments work in the $\mathbb{E}_1$ case.

The algebraic K-theory of a weighted ∞-category is often determined by the additive algebraic K-theory of its heart by virtue of the theorem of the weighted heart first established by Fontes [14] but we also refer the reader to [17, Corollary 8.1.3, Remark 8.1.4]. Let $\mathcal{A}$ denote an additive ∞-category regarded as a symmetric monoidal ∞-category with the cocartesian symmetric monoidal structure, so that the core $\mathcal{A}^{\simeq}$ inherits the structure of an $\mathbb{E}_\infty$-monoid. Recall that the additive algebraic K-theory of $\mathcal{A}$ is defined by

$$\mathrm{K}^{\oplus}(\mathcal{A}) = (\mathcal{A}^{\simeq})^{\mathrm{grp}},$$

where $(\mathcal{A}^{\simeq})^{\mathrm{grp}}$ denotes the group completion of the $\mathbb{E}_\infty$-monoid $\mathcal{A}^{\simeq}$. We have the following result which will play an instrumental role below (cf. [14, Theorem 5.1] and [17, Corollary 8.1.3]):

Theorem 2.4. *The canonical map of spectra*

$$\mathrm{K}^{\oplus}(\mathcal{C}^{\mathrm{ht}}) \to \mathrm{K}(\mathcal{C})$$

is an equivalence for every stable ∞-category $\mathcal{C}$ equipped with an exhaustive weight structure.

3. Chromatic vanishing results

The main goal of this section is to prove Theorem 1.1 from §1 and discuss various consequences. As explained, our proof of this result relies on the close relationship between TR and the spectrum of curves in K-theory (cf. [4, 19, 30]). We will regard TR as a functor $\mathrm{TR} : \mathrm{Alg}_{\mathbb{E}_1}^{\mathrm{cn}} \to \mathrm{Sp}$ given by

$$\mathrm{TR}(R) \simeq \mathrm{map}_{\mathrm{CycSp}}(\widetilde{\mathrm{THH}}(\mathbb{S}[t]), \mathrm{THH}(R))$$

following [30] and this agrees with the classical construction of TR by [30, Theorem 3.3.12]. By virtue of our assumption that R is connective, there is an equivalence of spectra

$$\mathrm{TR}(R) \simeq \varprojlim \Omega \widetilde{\mathrm{K}}(R[t]/t^i),$$

where $\tilde{\mathrm{K}}(R[t]/t^i)$ denotes the fiber of the map $\mathrm{K}(R[t]/t^i) \to \mathrm{K}(R)$ induced by the augmentation. In this generality, the result was obtained by the second author in [30] preceded by Hesselholt [19] and Betley–Schlichtkrull [4] who proved the result for associative rings after profinite completion. Recall that $L_n^{p,f}$ denotes the Bousfield localization at the spectrum $\mathbb{S}[1/p] \oplus T(1) \oplus \cdots \oplus T(n)$. With this equivalence at our disposal, we prove the following result:

Theorem 3.1. *Let $n \geq 1$. If R is a connective $\mathbb{E}_1$-ring such that $L_n^{p,f} R \simeq 0$, then*

$$L_{T(k)} \mathrm{TR}(R) \simeq 0$$

for every $1 \leq k \leq n$.

The limit in the definition of the spectrum of curves on K-theory above does not commute with $T(k)$-localization. Instead, the proof of Theorem 3.1 relies on the following result, which is proved by combining the theorem of the weighted heart and a recent result which asserts that additive algebraic K-theory preserves infinite products of additive ∞-categories, due to Córdova Fedeli [13].

Proposition 3.2. *Let R be a connective $\mathbb{E}_1$-ring such that $L_n^{p,f} R \simeq 0$. If $\{S_i\}_{i \in I}$ is collection of connective $\mathbb{E}_1$-rings with a map of $\mathbb{E}_1$-rings $R \to S_i$ for every $i \in I$, then*

$$L_{T(k)}\big(\prod_{i \in I} \mathrm{K}(S_i)\big) \simeq 0$$

for every $1 \leq k \leq n$.

Proof. For $i \in I$, the stable ∞-category Perf_{S_i} admits an exhaustive weight structure whose heart is equivalent to the additive ∞-category $\mathrm{Proj}^\omega_{S_i}$ by Example 2.3. The canonical composite

$$\mathrm{K}^{\oplus}\Big(\prod_{i \in I} \mathrm{Proj}^\omega_{S_i}\Big) \to \prod_{i \in I} \mathrm{K}^{\oplus}(\mathrm{Proj}^\omega_{S_i}) \to \prod_{i \in I} \mathrm{K}(\mathrm{Perf}_{S_i})$$

is an equivalence by [13, Corollary 2.11.1] and Theorem 2.4, so we have reduced to proving that

$$L_{T(k)} \mathrm{K}^{\oplus}\Big(\prod_{i \in I} \mathrm{Proj}^\omega_{S_i}\Big) \simeq 0$$

for $1 \leq k \leq n$. By [25, Proposition 3.6], it suffices to prove that the endomorphism $\mathbb{E}_1$-rings of

$$\mathcal{A} = \prod_{i \in I} \mathrm{Proj}^\omega_{S_i}$$

vanish after $L_n^{p,f}$-localization. If $P \in \mathcal{A}$, then the endomorphism $\mathbb{E}_1$-ring of P is given by

$$\mathrm{End}_{\mathcal{A}}(P) \simeq \prod_{i \in I} \mathrm{map}_{S_i}(P_i, P_i),$$

where $\mathrm{map}_{S_i}(P_i, P_i)$ denotes the mapping spectrum in LMod_{S_i}. For each $i \in I$, we may choose a positive integer $n_i \geq 1$ such that P_i is a retract of $S_i^{\oplus n_i}$ by virtue of our assumption that P_i is a finitely generated projective S_i-module. Consequently, we obtain a retract diagram of spectra

$$\mathrm{End}_{\mathcal{A}}(P) \to \prod_{i \in I} S_i^{\oplus n_i^2} \to \mathrm{End}_{\mathcal{A}}(P)$$

which proves the desired statement since the middle term is a left R-module, hence vanishes after $L_n^{p,f}$-localization by virtue of our assumption that R is $L_n^{p,f}$-acyclic. $\square$

Remark 3.3. In general, E-acyclic spectra are not closed under infinite products; for each $n \geq 0$, the nth Postnikov truncation $\tau_{\leq n}\mathbb{S}$ is $K(1)$-acyclic, whereas $\prod_{n\geq 0} \tau_{\leq n}\mathbb{S}$ is not, else $L_{K(1)}\mathbb{S} \simeq 0$. The assumptions of Proposition 3.2 should be viewed as a uniformity condition on the spectra $\mathrm{K}(S_i)$, forcing their product to become acyclic.

Proof of Theorem 3.1. Since R is a connective $\mathbb{E}_1$-ring, there is an equivalence of spectra $\mathrm{TR}(R) \simeq \mathrm{C}(R)$ by [30, Corollary 4.2.5]. Thus, the spectrum $\Sigma\,\mathrm{TR}(R)$ is the fiber of a suitable map

$$\prod_{i\geq 1} \widetilde{\mathrm{K}}(R[t]/t^i) \to \prod_{i\geq 1} \widetilde{\mathrm{K}}(R[t]/t^i)$$

which proves the desired statement as these products vanish after $T(k)$-localization for $1 \leq k \leq n$ by virtue of Proposition 3.2. $\square$

Remark 3.4. As remarked above, we have used work by Córdova Fedeli [13] in a crucial way. This result on K-theory of additive ∞-categories is part of a long tradition of examining the interaction of algebraic K-theory and infinite products of categories. One of the first results of this kind is due to Carlsson, who showed that K-theory preserves infinite products of exact 1-categories with a cylinder functor [11]. In [24], Kasprowski–Winges proved that K-theory preserves infinite products of additive categories. Furthermore, Kasprowski–Winges [23] used a characterization of Grayson [15] to prove that non-connective algebraic K-theory preserves infinite products of stable ∞-categories and this was used in [9] with Bunke to prove the analogous statement of prestable ∞-categories.

Another attempt to prove Proposition 3.2 proceeds by invoking a recent result of Kasprowski–Winges [23], which asserts that the canonical map of spectra

$$\mathrm{K}\left(\prod_{i\in I} \mathrm{Perf}_{S_i}\right) \to \prod_{i\in I} \mathrm{K}(S_i)$$

is an equivalence. Proceeding as in the proof of Proposition 3.2, it would suffice to prove that the endomorphism $\mathbb{E}_1$-rings of the product of the stable ∞-categories Perf_{S_i} vanish after $L_n^{p,f}$-localization. Previous (incorrect) attempts by the authors to prove Theorem 3.1 involved showing that a v_n-self map $v \in \pi_*(\mathrm{End}(V))$ induced a null-homotopic endomorphism of $\mathrm{End}_{S_i}(M_i) \otimes \mathrm{End}(V)$ as an $\mathrm{End}(V)$-module. This is closely related to the following assertion:

- $(*)$ Let E denote the endomorphism $\mathbb{E}_1$-ring of a finite spectrum V of type n. If $v : \Sigma^k E \to E$ is the associated v_n self-map of E, then there is a canonical lift of v to a map of E-E-bimodules.

By the description of the $\mathbb{E}_1$-center as Hochschild cohomology, the statement $(*)$ is equivalent to asking for a lift of the class $v \in \pi_*(E)$ to a class $\tilde{v} \in \pi_*\mathcal{Z}_{\mathbb{E}_1}(E)$ along the $\mathbb{E}_1$-map $\mathcal{Z}_{\mathbb{E}_1}(E) \to E$. Classes which do lift in this way can be viewed as "homotopically central" elements of E, and we remark that such lifts exist for all $\mathbb{E}_2$-rings, by the universal property of the $\mathbb{E}_1$-center.[2]

[2]Additionally, we note that for R an $\mathbb{E}_1$-ring with a central nilpotent element $x \in \pi_*(R)$, iterates of the R-module map $\Sigma^{-k}x : M \to \Sigma^{-k}M$ will typically fail to be nullhomotopic as a map of R-modules unless x lifts to the $\mathbb{E}_1$-center.

However, the assertion $(*)$ is false as we learned from Maxime Ramzi, and we thank him for help with the following argument. In what follows, let $\mathrm{Sp}_{K(n)}$ denote the category of $K(n)$-local spectra, where $K(n)$ denotes the Morava K-theory spectrum for some prime p and natural number n. Similarly, we let $\mathrm{map}_{K(n)}(-,-)$ denote the internal hom in this symmetric monoidal category. If such a lift of a v_n-self map exists, then we obtain an equivalence of $L_{K(n)}E$-$L_{K(n)}E$-bimodules

$$\varphi : \Sigma^k L_{K(n)}E \to L_{K(n)}E,$$

and there is an equivalence of $\mathbb{E}_1$-rings $\mathrm{End}_{K(n)}(L_{K(n)}V) \simeq L_{K(n)}E$ since V is a finite spectrum. The ∞-category of $K(n)$-local spectra is equivalent to the ∞-category $\mathrm{Mod}_{L_{K(n)}E}(\mathrm{Sp}_{K(n)})$ since $L_{K(n)}V$ is a compact generator of $\mathrm{Sp}_{K(n)}$. As a consequence, for every $K(n)$-local spectrum X, we obtain an equivalence $\Sigma^k X \to X$ by base-changing along φ. This is a contradiction since the homotopy groups of a $K(n)$-local spectrum are in general not periodic, as illustrated for instance by the $K(n)$-local sphere. The homotopy groups of $L_{K(1)}\mathbb{S}$ for $p > 2$ were completely calculated in unpublished work by Adams–Baird and independently by Ravenel in [32]. More recently, the homotopy of $(L_{K(n)}\mathbb{S})_{\mathbb{Q}}$ was completely calculated by Barthel–Schlank–Stapleton–Weinstein in [2] for all primes p and all natural numbers n. These computations show that $L_{K(n)}\mathbb{S}$ is not periodic.

Finally, we explore some immediate consequences of Theorem 3.1.

Corollary 3.5. *Let R be a connective $\mathbb{E}_1$-algebra over $\mathbb{Z}/p^j$. If $n \geq 1$, then $L_{T(n)}\,\mathrm{TR}(R) \simeq 0$.*

Proof. Note that $L_n^{p,f} R$ is a module over $L_n^{p,f}\mathbb{Z}/p^j \simeq 0$, so the assertion follows from Theorem 3.1. □

Recall that Corollary 3.5 above also follows from [5, 25, 29] as discussed in the introduction. We deduce some consequence for connective Morava K-theory. Let $k(n)$ denote the connective cover of the nth Morava K-theory $K(n)$. The spectrum $k(n)$ carries the structure of an $\mathbb{E}_1$-ring but not the structure of an $\mathbb{E}_2$-ring. We have the following:

Corollary 3.6. *If $n \geq 2$, then $L_{T(k)}\,\mathrm{TR}(k(n)) \simeq 0$ for every $1 \leq k \leq n-1$.*

Proof. For $n \geq 2$, the canonical map $k(n) \to \mathbb{F}_p$ is a $L_{n-1}^{p,f}$-local equivalence by [25, Lemma 2.2], so the assertion follows from Theorem 3.1. □

Remark 3.7. There is a fiber sequence of spectra

$$\mathrm{K}(\mathbb{F}_p) \to \mathrm{K}(k(n)) \to \mathrm{K}(K(n)),$$

by [1, Proposition 4.4] preceded by [3]. We consider this as an analogue of Quillen's dévissage theorem for algebraic K-theory of ring spectra. One might ask whether we can establish a similar fiber sequence for TR. In particular, this would allow us to deduce an analogue of Corollary 3.6 for the non-connective Morava K-theory.

Let $y(n)$ denote the Thom spectrum considered in [28, Section 3]. This is the Thom spectrum associated to the map of $\mathbb{E}_1$-spaces

$$\Omega J_{p^{n-1}} S^2 \hookrightarrow \Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge),$$

where $J_{p^{n-1}}S^2$ is the $2(p^{n-1})$-skeleton of ΩS^3, which has a single cell in each even dimension. The map $\Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$ is the spherical fibration constructed by

Mahowald (for $p = 2$) and Hopkins (for p odd) whose Thom spectrum is $H\mathbb{F}_p$. We have the following:

Corollary 3.8. *If $n \geq 2$, then $L_{T(k)}\,\mathrm{TR}(y(n)) \simeq 0$ for every $1 \leq k \leq n-1$.*

Proof. This follows immediately by combining Theorem 3.1 with [25, Lemma 4.14]. □

Remark 3.9. If R is a connective $H\mathbb{Z}$-algebra, then the canonical map

$$L_{T(1)}\,\mathrm{K}(R) \to L_{T(1)}\,\mathrm{K}(R[1/p])$$

is an equivalence by [5, 25]. The analogue of this result does not hold for TC as explained in [25, Remark 4.28], which in particular means that the result also does not prolong to TR. However, at chromatic heights $n \geq 2$, TC does satisfy a version of chromatic purity (cf. [25, Corollary 4.5]). In particular, if $A \to B$ is an $L_n^{p,f}$-local equivalence of $\mathbb{E}_1$-rings, then the induced map

$$L_{T(n)}\,\mathrm{TC}(\tau_{\geq 0}A) \xrightarrow{\simeq} L_{T(n)}\,\mathrm{TC}(\tau_{\geq 0}B)$$

is an equivalence. One can wonder whether such a statement is true of $T(n)$-local TR, but our methods here do not seem to shed light on this problem.

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Department of Mathematics, University of Minnesota, 206 Church Street SE, Minneapolis, Minnesota 55455

Email address: keena169@umn.edu

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany

Email address: mccandless@mpim-bonn.mpg.de