JOURNAL OF ALGEBRA 107, 195–216 (1987)

Cyclic Homology, Comodules, and Mixed Complexes

CHRISTIAN KASSEL*

Département de Mathématiques,
7 Rue René Descartes, 67084 Strasbourg, France

Communicated by Richard G. Swan

Received December 11, 1985

In this paper we compute the cyclic homology groups of a tensor product of algebras. From Connes's long exact sequence

$$\cdots \longrightarrow H_n(A, A) \longrightarrow \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \longrightarrow H_{n-1}(A, A) \longrightarrow \cdots, \tag{0.1}$$

relating cyclic and Hochschild homology, one can see that any formula for $\mathrm{HC}_*(A \otimes A')$ will not only involve $\mathrm{HC}_*(A)$ and $\mathrm{HC}_*(A')$, but also the "periodicity" operator S and the Hochschild groups. Fortunately, if the map S brings in some complications, it also endows $\mathrm{HC}_*(A)$ with a comodule structure over the cyclic homology of the ground ring k. Actually, the comodule structure exists already on the complex level: a convenient complex on which S has the form of a canonical surjection is Connes's double complex with differentials b and B.

Now the idea is to view the cyclic homology of an algebra as a composite functor

$$\mathrm{Alg}_k \xrightarrow{C} C_{\mathrm{mix}}(k) \longrightarrow \mathrm{Comod} \xrightarrow{\mathrm{H}_*} \mathrm{Gr}(k).$$

Here Alg_k is the category of k-algebras, $\mathrm{Gr}(k)$ is the category of graded k-modules, Comod is a category of differential graded comodules, H_* stands for homology. The objects of $C_{\mathrm{mix}}(k)$ are what we call *mixed complexes* which are both chain and cochain complexes in a compatible way (also called complexes with an algebraic circle action by Burghelea). Every algebra A gives rise to a mixed complex $C(A)$.

We detail now the contents of the paper. In the first section we define mixed complexes and relate them to classical homological algebra via the

* Research supported in part by NSF Grant 8120790.

0021-8693/87 $3.00

observation that a mixed complex is exactly a differential graded module over the exterior algebra Λ generated by a single element in degree 1. Moreover cyclic homology appears as a Tor-functor over Λ and the cyclic homology $\mathrm{HC}_*(k)$ of the ground ring k identifies with the classifying coalgebra $B(\Lambda)$ as defined by [14].

In this framework, Connes's double complex with differentials b and B can be seen as a bar construction. The latter has classically a comodule structure over $B(\Lambda) = \mathrm{HC}_*(k)$.

Finally we prove that the bar construction corresponding to the tensor product of two mixed complexes is the cotensor product of the corresponding comodules.

Section 2 originates from the fact that given two algebras A and A', $C(A) \otimes C(A')$ and $C(A \otimes A')$ are different mixed complexes; actually, there is not even a morphism between them, because Connes's operator B does not commute with the shuffle map. Here again, a classical notion will solve the problem. We prove that there exists a strongly homotopy linear map (which roughly is a map commuting with B up to higher homotopies) in the sense of Gugenheim–Munkholm [11] between $C(A) \otimes C(A')$ and $C(A \otimes A')$. Hence the corresponding long exact sequences (0.1) are isomorphic.

As an immediate consequence, we get in Section 3 a Künneth-type exact sequence of the form

$$\begin{aligned} 0 &\longrightarrow \mathrm{Cotor}^{\mathrm{HC}(k)}(\mathrm{HC}(A), \mathrm{HC}(A'))[1] \longrightarrow \mathrm{HC}(A \otimes A') \\ &\longrightarrow \mathrm{HC}(A) \,\square_{\mathrm{HC}(k)} \mathrm{HC}(A') \longrightarrow 0 \end{aligned}$$

under mild flatness assumptions. This formula has a lot of applications. Among others, we have (when k contains the field of rational numbers)

$$\mathrm{HC}_n(A[x]) = \mathrm{HC}_n(A) \oplus H_n(A, A)^{(\aleph_0)},$$

$$\mathrm{HC}_n(A[x, x^{-1}]) = \mathrm{HC}_n(A) \oplus \mathrm{HC}_{n-1}(A) \oplus H_n(A, A)^{(\aleph_0)}.$$

In these formulas, there is a "good" part which behaves like algebraic K-theory or Hochschild homology. The "bad" summands represented by the Hochschild groups are unstable, which means that they vanish after inverting the map S. As a matter of fact, we prove that periodic cyclic homology commutes with tensor products

$$\mathrm{HC}_*^{\mathrm{per}}(A \otimes A') = \mathrm{HC}_*^{\mathrm{per}}(A) \otimes \mathrm{HC}_*^{\mathrm{per}}(A').$$

We end the paper with an Appendix in which we set the foundations for a tensor product in the category $\Lambda(k)$ of cyclic k-modules in the sense of Connes [5, 9]. This notion is implicit in [5], but has never appeared with proofs in the literature. We use this construction in Section 1 to prove an isomorphism

$$\mathrm{Tor}_*^{\Lambda(k)}(k, E) \cong \mathrm{Tor}_*^{A}(k, \tilde{E})$$

between the derived functors for a cyclic k-module E and the derived functors for the corresponding mixed complex $\tilde{E}$.

It should be noted that all our constructions and results extend to the cyclic homology of differential graded algebras (replace homology by hyperhomology). We thus recover Theorem B of [2] by purely algebraic methods.

All modules, algebras (always associative and unital) and tensor products are over a fixed commutative ring k. All complexes are nonnegatively differential graded (d.g.) k-modules with degree -1 differential. Graded modules are considered as d.g. modules with zero differential. Given a d.g. module M and an integer p, $M[p]$ denotes the d.g. module with $M[p]_n = M_{n-p}$ and $(-1)^p d_M$ as differential. If an element or a map x has a degree, it will be denoted by $|x|$. Brackets stand for graded commutators: $[x, y] = xy - (-1)^{|x||y|} yx$.

After circulation of a first version of this paper, M. Karoubi informed me that he could prove a Künneth-type formula for cyclic modules and D. Burghelea sent me a preprint, *Künneth formula in cyclic homology*, in which he proves this over a field k of characteristic zero.

I thank the referee for several valuable suggestions and the Institute for Advanced Study in Princeton for their hospitality in 1984–1985 when this paper was written.

1. Mixed Complexes and Classical Homological Algebra

A *mixed complex* (M, b, B) is a nonnegatively graded k-module $(M_n)_n$ endowed with a degree -1 endomorphism b and a degree $+1$ endomorphism B satisfying the following relations:

$$b^2 = B^2 = [B, b] = 0$$

(recall that here $[B, b] = Bb + bB$). Thus a mixed complex is both a chain and a cochain complex. However, for reasons which will appear soon and despite the *a priori* symmetry between b and B, we shall view b as the main differential and consider B merely as a device to perturb the complex (M, b).

Mixed complexes arise from the category $\Lambda(k)$ of contravariant functors from Connes's category Λ to the category of k-modules. An object of $\Lambda(k)$ is called a cyclic k-module and is a simplicial k-module with extra operators which permute cyclically the simplices. To any cyclic k-module E, one can associate naturally a mixed complex $(\tilde{E}, b, B)$ where $\tilde{E}_n = E_n$, b and B are described explicitly in terms of the face, degeneracy and cyclic operators of E (see [5, 9, 15]).

The cyclic homology of a cyclic k-module as defined in [5] and in [9], can be expressed in terms of appropriate Tor-functors in $\Lambda(k)$ (see the Appendix). We shall show in the section that cyclic homology can also be computed as derived functors in the category of mixed complexes.

The way to obtain the cyclic homology of a cyclic k-module E via the corresponding mixed complex $\tilde{E}$ goes through the following construction.

To any mixed complex (M, b, B), one associates a chain complex $({}_BM, d)$ where

$$ {}_BM_n = M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \cdots $$

and for every $(m_n, m_{n-2}, m_{n-4}, \ldots) \in {}_BM_n$

$$ d(m_n, m_{n-2}, m_{n-4}, \ldots) = (bm_n + Bm_{n-2}, bm_{n-2} + Bm_{n-4}, \ldots). $$

One sees immediately that ${}_BM$ is related to the chain complex (M, b) by the following exact sequence of complexes

$$ 0 \longrightarrow M \longrightarrow {}_BM \xrightarrow{S} {}_BM[2] \longrightarrow 0. \tag{1.1} $$

The map S is obtained by dividing ${}_BM$ by its first factor; it is a canonical surjection.

If we call *homology* (resp. *cyclic homology*) of M and denote by $H_*(M)$ (resp. $\mathrm{HC}_*(M)$) the homology of (M, b) (resp. of $({}_BM, d)$), then the above short exact sequence of complexes gives rise to the long exact sequence

$$ \cdots \longrightarrow H_n(M) \longrightarrow \mathrm{HC}_n(M) \xrightarrow{S} \mathrm{HC}_{n-2}(M) \longrightarrow H_{n-1}(M) \longrightarrow \cdots. \tag{1.2} $$

This terminology is justified by the fact that for a cyclic k-module E, the homology $H_*(\tilde{E})$ is the Hochschild homology of E and $\mathrm{HC}_*(\tilde{E})$ is the cyclic homology of E [9, Proposition II.2.5] or [15, Proposition 1.5].

The previous *adhoc* constructions will become clearer once one makes the simple observation that a mixed complex is nothing but a differential graded module over the graded exterior k-algebra Λ on one generator ε of degree 1 (with zero differential).

As a complex,

$$ \Lambda_0 = k, \qquad \Lambda_1 = k\varepsilon, \qquad \Lambda_i = 0 \quad \text{for } i \geqslant 2. $$

Precisely, a d.g. Λ-module M is a mixed complex (M, b, B), where b is the given differential on M and B corresponds to left multiplication by ε. The defining relations

$$B^2 = [B, b] = 0$$

are equivalent to the fact that $\varepsilon^2 = 0$ and

$$b(\varepsilon m) = (-1)^{|\varepsilon|}\varepsilon bm = -\varepsilon bm.$$

The rest of this section is devoted to classical homological algebra applied to the algebra Λ. We begin with the following

PROPOSITION 1.3. (a) *For every mixed complex* (M, b, B) *or, equivalently, any* d.g. *Λ-module,*

$$\mathrm{HC}_*(M) = \mathrm{Tor}_*^{\Lambda}(k, M),$$

where k is the trivial Λ-module given by the augmentation.

(b) *The functor $E \mapsto \tilde{E}$ from cyclic k-modules to mixed complexes induces the following isomorphism:*

$$\mathrm{Tor}_*^{\Lambda(k)}(k, E) \cong \mathrm{Tor}_*^{\Lambda}(k, \tilde{E}).$$

This means that the graded algebra Λ is a good approximation to the abelian category $\Lambda(k)$.

Proof. Part (b) follows from (a) in view of the Appendix and the above remarks. To prove (a), one uses as a resolution of k, the following exact complex of free Λ-modules

$$L = \{\cdots \longrightarrow \Lambda[2] \xrightarrow{\varepsilon} \Lambda[1] \xrightarrow{\varepsilon} \Lambda\}.$$

It is clear for any Λ-module M that the complex ${}_BM$ defined above with homology $\mathrm{HC}_*(M)$, coincides with the total complex of the bicomplex $L \otimes_\Lambda M$. ∎

The above proposition is also a consequence of the fact that ${}_BM$ is a bar-construction. In order to see this, recall that Husemoller, Moore and Stasheff [14] associated a classifying coalgebra $B(A)$ to any d.g. algebra A. This comes with a canonical "twisting cochain" $\tau\colon B(A) \to A$. In the very simple case of Λ, $B(\Lambda)$ is the graded k-coalgebra $k[u]$ generated by one element u in degree 2, with coproduct Δ, counit ε and "twisting cochain" given by

$$\Delta(u^n) = \sum_{i+j=n} u^i \otimes u^j \qquad \varepsilon(u^n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

and

$$\tau(u^n) = \begin{cases} \varepsilon & \text{if } n = 1 \\ 0 & \text{if } n \neq 1. \end{cases}$$

We follow now [11] and define the two-sided bar construction of two Λ-modules M and N by

$$B(M, \Lambda, N) = M \otimes_\tau B(\Lambda) \otimes_\tau N,$$

the subscript τ meaning that the usual tensor product differential is perturbed by the "twisting cochain" in a canonical way we shall explicit when needed.

In case $N = k$, clearly $B(M, \Lambda, k) = M \otimes_\tau B(\Lambda)$.

PROPOSITION 1.4. *For any mixed complex (M, b, B), the associated complex ${}_BM$ is the same as the bar construction $B(M, \Lambda, k)$*

$${}_BM = B(M, \Lambda, k) = M \otimes_\tau B(\Lambda).$$

Proof. This is clear as graded modules, since

$$({}_BM)_n = M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \cdots = (M \otimes k[u])_n$$

(recall u is of degree 2). The differential on $M \otimes_\tau B(\Lambda)$ is the sum of the tensor product differential and of a degree -1 map d_τ. Let us from now on write any element m of $({}_BM)_n$ as a finite sum

$$m = \sum_{i \geq 0} m_i \otimes u^i,$$

where $m_i \in M_{n-2i}$. According to [11],

$$\begin{aligned} d_\tau(m_p \otimes u^p) &= \sum_{i+j=p} m\tau(u^i) \otimes u^j \\ &= \begin{cases} 0 & \text{if } p = 0 \\ \varepsilon m \otimes u^{p-1} = Bm \otimes u^{p-1} & \text{if } p \geq 1. \end{cases} \end{aligned}$$

Therefore the differential d on $M \otimes_\tau B(\Lambda)$ is given by

$$d\left(\sum_{i \geq 0} m_i \otimes u^i\right) = (bm_0 + Bm_1) \otimes 1 + (bm_1 + Bm_2) \otimes u + \cdots,$$

which is, by definition, the differential on ${}_BM$. ∎

Proposition 1.4 has two consequences. First, it reproves Proposition 1.3 because of the classical fact (see [6] or [11, (2.2)]) that for any d.g. modules M and N over a d.g. algebra A,

$$\operatorname{Tor}_*^A(M, N) \cong H_*(B(M, A, N)). \tag{1.5}$$

Second, for any A-module M, the d.g. k-module $M \otimes_\tau B(A)$ has a natural structure of d.g. $B(A)$-*comodule*. In view of (1.5), this implies that $\operatorname{Tor}_*^A(M, k)$ is a natural $\operatorname{Tor}_*^A(k, k)$-comodule. In case $A = \Lambda$, the *coaction* Δ_M on ${}_BM = M \otimes_\tau k[u]$ is given by

$$\Delta_M(m \otimes u^p) = \sum_{i+j=p} m \otimes u^i \otimes u^j.$$

In this simple case, it is easy to see that a d.g. k-module P is a $k[u]$-comodule if and only if there exists a differential graded k-linear map S: $P \to P[2]$. The coaction Δ_P: $P \otimes k[u]$ is then given by

$$\Delta_P(x) = x \otimes 1 + S(x) \otimes u + S^2(x) \otimes u^2 + \cdots.$$

Applying [10, Sect. 2] or [11, Sect. 1], one sees immediately that the natural $k[u]$-comodule structure on $M \otimes_\tau B(\Lambda) = {}_BM$ yields the map S given by

$$S(m \otimes u^p) = \begin{cases} m \otimes u^{p-1} & \text{if } p \geqslant 1 \\ 0 & \text{if } p = 0 \end{cases}$$

which is the natural projection S in (1.1). Clearly from (1.2) and the isomorphism $\mathrm{HC}_*(k) = k[u]$, for any mixed complex (M, b, B), the cyclic homology $\mathrm{HC}_*(M)$ is a graded $\mathrm{HC}_*(k)$-comodule with respect to Connes's operator S: $\mathrm{HC}_*(M) \to \mathrm{HC}_{*-2}(M)$.

As a matter of fact, the entire long exact sequence (1.2) is a consequence of the following proposition which follows immediately from [10, Proposition 2.6] and from the vanishing of τ on k.

PROPOSITION 1.6. *For any* d.g. *module M over a* d.g. *algebra A,*

$$M \cong [M \otimes_\tau B(A)] \,\square_{B(A)}\, k.$$

Here as in [6], the symbol $\square_{B(A)}$ stands for the cotensor product of $B(A)$-comodules. The exact sequence (1.2) is then a consequence of Proposition 1.6 applied to $A = \Lambda$ and of the spectral sequence of Theorem 9.2 of [6]. Note that every $k[u]$-comodule has an injective resolution of length 1 so that the higher $\operatorname{Cotor}_*^{k[u]}$ vanish and the spectral sequence degenerates. We are left with $\mathrm{Cotor}_0^{k[u]}$ which is the cotensor product and with $\mathrm{Cotor}_1^{k[u]}$ which we shall denote without subscript.

We end this section by investigating the behaviour of the bar construction with respect to the *tensor product* of modules. We make the following observation: if (M, b, B) and (N, b, B) are mixed complexes, then $(M \otimes N, b, B)$ is also a mixed complex where b and B are extended over the tensor product in the usual way. This is equivalent to the fact that Λ is a Hopf algebra.

THEOREM 1.7. *Let (M, b, B) and (N, b, B) be mixed complexes. The $k[u]$-comodule associated to $(M \otimes N, b, B)$ is the cotensor product of ${}_BM$ and ${}_BN$,*

$$_B(M \otimes N) = {}_BM \,\square_{k[u]}\, {}_BN.$$

Proof. Recall from [6, Proposition 2.1], that if P is a (right) comodule and $k[u] \otimes N$ is a (left) extended comodule, then the map

$$\Delta_P \otimes N\colon \quad P \otimes N \to P \otimes k[u] \otimes N$$

is injective and its image is $P \,\square_{k[u]} (k[u] \otimes N)$. Hence as graded modules, we have

$$\begin{aligned} {}_BM \,\square_{k[u]}\, {}_BN &= (k[u] \otimes M) \,\square_{k[u]} (k[u] \otimes N) \\ &= k[u] \otimes M \otimes N = {}_B(M \otimes N). \end{aligned}$$

To complete the proof, it is enough to check that the map

$$j = (T \otimes k[u] \otimes N) \circ (M \otimes \Delta \otimes N) \circ T \otimes N$$

from ${}_B(M \otimes N) = k[u] \otimes M \otimes N$ into ${}_BM \otimes {}_BN = k[u] \otimes M \otimes k[u] \otimes N$ commutes with the differentials. By T we mean the map $k[u] \otimes M \to M \otimes k[u]$ which interchanges factors and transforms the left comodule ${}_BM$ into a right one.

Let $m \in M_q$, $n \in N_r$ and $u^p \otimes m \otimes n \in {}_B(M \otimes N)_{q+r}$. We adopt the following convention: $u^p = 0$ for $p < 0$. We have to show that $x = y$ with $x = dj(u^p \otimes m \otimes n)$ and $y = jd(u^p \otimes m \otimes n)$,

$$\begin{aligned} x &= d\Big(\sum_{i+j=p} (u^i \otimes m) \otimes (u^j \otimes n)\Big) \\ &= \sum_{i+j=p} (u^i \otimes b(m) \otimes u^j \otimes n + (-1)^{2i+q} u^i \otimes m \otimes u^j \otimes b(n)) \\ &\quad + \sum_{i+j=p} (u^{i-1} \otimes B(m) \otimes u^j \otimes n + (-1)^{2i+q} u^i \otimes m \otimes u^{j-1} \otimes B(n)). \end{aligned}$$

$$
\begin{aligned}
y &= j(u^p \otimes b(m \otimes n) + u^{p-1} \otimes B(m \otimes n)) \\
&= j(u^p \otimes b(m) \otimes n + (-1)^q u^p \otimes m \otimes b(n) \\
&\quad + u^{p-1} \otimes B(m) \otimes n + (-1)^q u^{p-1} \otimes m \otimes B(n)) \\
&= \sum_{i+j=p} (u^i \otimes b(m) \otimes u^j \otimes n + (-1)^q u^i \otimes m \otimes u^j \otimes b(n)) \\
&\quad + \sum_{i+j=p-1} (u^i \otimes B(m) \otimes u^j \otimes n + (-1)^q u^i \otimes m \otimes u^j \otimes B(n)) \\
&= x. \quad \blacksquare
\end{aligned}
$$

Remark. The reader can check that if A is a graded exterior algebra on odd-degree generators, then Theorem 1.7 generalizes to the following isomorphism

$$(M \otimes N) \otimes_\tau B(A) \cong [M \otimes_\tau B(A)] \,\square_{B(A)}\, [B(A) \otimes_\tau N]$$

for any pair of A-modules M and N.

The following is an immediate consequence of Theorem 1.7, [6, Theorem 9.2 and the comments following Proposition 1.6].

COROLLARY 1.8. *Let (M, b, B) and (N, b, B) be mixed complexes. Assume N_n and $\mathrm{HC}_n(N)$ are flat k-modules for all $n \geqslant 0$. Then there exists a functorial Künneth-type short exact sequence*

$$
\begin{aligned}
0 &\longrightarrow \mathrm{Cotor}^{k[u]}(\mathrm{HC}(M), \mathrm{HC}(N))[1] \longrightarrow \mathrm{HC}(M \otimes N) \\
&\longrightarrow \mathrm{HC}(M) \,\square_{k[u]}\, \mathrm{HC}(N) \longrightarrow 0.
\end{aligned}
$$

2. CONNES'S OPERATOR B AND THE SHUFFLE MAP

Any associative unital k-algebra A gives rise to a cyclic k-module as described in [5] and [9], hence to a mixed complex $(C(A), b, B)$. The complex $(C(A), b)$ is the standard Hochschild complex given by $C_n(A) = A^{\otimes(n+1)}$ and

$$
\begin{aligned}
b(a_0 \otimes \cdots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
&\quad + (-1)^n a_n a_0 \oplus \cdots \oplus a_{n-1}.
\end{aligned}
$$

The map B has a more complicated expression. Therefore it is convenient to use the *normalized Hochschild complex* $\bar{C}_*(A) = C_*(A)/D_*$ where D_n is the subcomplex spanned by the elements $a_0 \otimes \cdots \otimes a_n$ such that $a_i = 1$ for some i with $1 \leqslant i \leqslant n$. The differentials b and B pass to the quotient so that $(\bar{C}(A), b, B)$ is a mixed complex. The map B is given on $\bar{C}(A)$ by

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{in}\, 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}$$

and the morphism $(C(A), b, B) \to (\bar{C}(A), b, B)$ is a quasi-isomorphism, in the sense that it induces an isomorphism of the corresponding long exact sequences (1.2) (see [15, Proposition 1.11]). In particular

$$\mathrm{HC}_*(C(A)) = \mathrm{HC}_*(\bar{C}(A)) = \mathrm{HC}_*(A) \tag{2.1}$$

is the *cyclic homology* of the algebra A and

$$H_*(C(A)) = H_*(\bar{C}(A)) = H_*(A, A)$$

is the *Hochschild homology* of A. Note that whereas $\bar{C}(A)$ is a mixed complex, it is not a cyclic module in the sense of Connes, which shows the computational advantage of mixed complexes.

The aim of this section is to study the relationship between $C(A) \otimes C(A')$ and $C(A \otimes A')$ for two algebras A and A'. Classically, there exist two graded k-linear maps, the Alexander–Whitney map f and the shuffle map g,

$$C(A) \otimes C(A') \underset{g}{\overset{f}{\rightleftarrows}} C(A \otimes A')$$

both commuting with the differential b and passing to the normalized complexes; moreover they are quasi-isomorphisms (see [3, IX]). Unfortunately they do not commute with the operator B, which means that f and g are *not* morphisms of d.g. Λ-modules. Nevertheless, the classical notion of *strongly homotopy linear map* (see [11, 3.3]) will enable us to show that the shuffle map induces a natural isomorphism between $\mathrm{HC}_*(C(A) \otimes C(A'))$ and $\mathrm{HC}_*(C(A \otimes A'))$.

DEFINITION 2.2. Given two d.g. Λ-modules M and N, a strongly homotopy Λ-map from M to N is a collection of graded maps $G^{(i)}: M \to N$ of degree $2i$ for all $i \geqslant 0$ such that

$$[G^{(0)}, b] = 0$$

and

$$[G^{(i+1)}, b] = -[G^{(i)}, B] \qquad \text{for all} \quad i \geqslant 0.$$

The reader can easily check that this definition is exactly the one given in [11, Sect. 3] when specialized to the case of the graded algebra Λ. Note that the degree 0 map $G^{(0)}$ is a morphism of complexes. It commutes with B not strictly, but up to higher homotopies. The interest of strongly homotopy linear maps for our purposes lies in the following result which is a precise reformulation of Theorem 3.5 of [11] for Λ.

PROPOSITION 2.3. *Let (M, b, B) and (N, b, B) be two mixed complexes. Assume there exists a strongly homotopy Λ-map $(G^{(i)})_{i\geq 0}$ from M to N. Then there exists a map of complexes G: ${}_BM \to {}_BN$ such that the following diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & {}_BM & \xrightarrow{S} & {}_BM[2] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle G^{(0)}} & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle G[2]} & & \\ 0 & \longrightarrow & N & \longrightarrow & {}_BN & \xrightarrow{S} & {}_BN[2] & \longrightarrow & 0 \end{array}$$

is commutative.

Proof. Recall $({}_BM)_n = M_n \oplus M_{n-2} \oplus \cdots$. In matrix notation, the differential d of ${}_BM$ and ${}_BN$ can be written as follows:

$$d = \begin{pmatrix} b & B & 0 & 0 & \cdots \\ 0 & b & B & 0 & \\ 0 & 0 & b & B & \\ \vdots & & & & \ddots \end{pmatrix}.$$

Define G by

$$G = \begin{pmatrix} G^{(0)} & G^{(1)} & G^{(2)} & \cdots \\ 0 & G^{(0)} & G^{(1)} & \\ 0 & 0 & G^{(0)} & \\ \vdots & & & \ddots \end{pmatrix}.$$

The hypotheses verified by the maps $G^{(i)}$ imply immediately $[G, d] = 0$. The rest of the proof is straightforward. ∎

We come now to the main result of this section.

THEOREM 2.4. *Given two unital associative k-algebras A and A', there exists a strongly homotopy Λ-map from $\bar{C}(A) \otimes \bar{C}(A')$ to $\bar{C}(A \otimes A')$ such that $G^{(0)}$ is the shuffle map g.*

The theorem has two important consequences. To state the first one, define for any k-algebra A, $H^*_{\mathrm{DR}}(A)$ as the cohomology of the complex

$$\cdots \xrightarrow{B} H_i(A, A) \xrightarrow{B} H_{i+1}(A, A) \longrightarrow \cdots.$$

This notation is justified by the fact that this complex is isomorphic to the de Rham complex $(\Omega^*_{A/k}, d)$ when A is a smooth commutative algebra over a field of characteristic 0 [15].

COROLLARY 2.5. *Suppose A' and $H_*(A', A')$ are flat k-modules. Then for any algebra A, there is a spectral sequence converging to $H^*_{\mathrm{DR}}(A \otimes A')$ with* $\mathrm{Tor}^k(H^*_{\mathrm{DR}}(A), H^*_{\mathrm{DR}}(A'))$ *as E^2-term. In particular, if k is a field*

$$H^*_{\mathrm{DR}}(A \otimes A') \cong H^*_{\mathrm{DR}}(A) \otimes H^*_{\mathrm{DR}}(A').$$

Proof. Under our assumptions,

$$\begin{aligned} H_*(A, A) \otimes H_*(A', A') &\cong H_*(\bar{C}(A) \otimes \bar{C}(A')) \cong H_*(\bar{C}(A \otimes A')) \\ &\cong H_*(A \otimes A', A \otimes A'). \end{aligned}$$

Now Theorem 2.4 shows that g commutes with B on the level of Hochschild homology. Hence, the above string of isomorphisms are chain isomorphisms for B. We conclude by applying the Künneth spectral sequence. ▌

COROLLARY 2.6. *For any pair of algebras A and A', one has*

$$\mathrm{HC}_*(C(A) \otimes C(A')) \cong \mathrm{HC}_*(C(A \otimes A')).$$

Proof. It is an immediate consequence of Theorem 2.4, Proposition 2.3, (2.1), the fact that g is a quasi-isomorphism and the five lemma. ▌

Proof of Theorem 2.4. We set $M = \bar{C}(A) \otimes \bar{C}(A')$ and $N = \bar{C}(A \otimes A')$. We have to construct a family of maps $(G^{(i)})_{i \geqslant 0}$ of degree $2i$ from M to N satisfying the conditions of Definition 2.2; $G^{(0)}$ will be the shuffle map. We shall construct the maps $G^{(i)}$ such that

$$G^{(i)} = F^{(i)} g$$

with $F^{(i)}: N \to N$ of degree $2i$, satisfying the relations

$$[F^{(i+1)}, b] = -[G^{(i)}, B] f \qquad \text{for all} \quad i \geqslant 0. \tag{2.7}$$

Then clearly

$$\begin{aligned} [G^{(i+1)}, b] = [F^{(i+1)} g, b] = [F^{(i+1)}, b]\, g &= -[G^{(i)}, B] fg \\ &= -[G^{(i)}, B], \end{aligned}$$

since the shuffle map is a strict right inverse to the Alexander–Whitney map on the normalized complexes.

Since a given N_n is the target of only finitely many nonzero $F^{(i)}$ or $G^{(i)}$ maps, we shall construct the maps $F_p^{(i)}: N_p \to N_{p+2i}$ (resp. $G_p^{(i)}: M_p \to N_{p+2i}$) by induction on $p+2i=n$. The method we use was inspired to us by Goodwillie's proof of [9, II.4.2].

For $n=0$, we take $F_0^{(0)} = \text{identity}$ and $G_0^{(0)} = g_0$. Let us fix now an integer $n \geqslant 1$ and suppose we have constructed maps $F_p^{(i)}$ and the corresponding $G_p^{(i)} = F_p^{(i)} g_p$ such that $p+2i<n$ and such that the relations (2.7) are satisfied.

We now want to construct

$$F_{p+1}^{(i)}: \quad N_{p+1} = \bar{C}_{p+1}(A \otimes A') \to N_{p+1+2i} = \bar{C}_{p+1+2i}(A \otimes A')$$

such that

$$bF_{p+1}^{(i)} = G_{p+2}^{(i-1)} Bf_{p+1} - BG_{p+1}^{(i-1)} f_{p+1} + F_p^{(i)} b$$

and such that $p+1+2i=n$.

We can use an auxiliary induction on i, which starts with $i=0$ $F_{p+2i}^{(0)} = \text{identity}$.

Now as in [9, II.4.2], we note that $F_{p+1}^{(i)}$ which can be viewed as a multilinear map from $(A \otimes A')^{p+2}$ into $\bar{C}_{p+1+2i}(A \otimes A')$, corresponds naturally to an element $x \in \bar{C}_{p+1+2i}(T)$ where T is the tensor algebra generated over k by variables $a_0, a_0', a_1, a_1', \ldots, a_{p+1}, a_{p+1}'$.

The problem is now to find $x \in \bar{C}_{p+1+2i}(T)$ such that $bx = y$, where

$$y = G_{p+2}^{(i-1)} Bf_{p+1} - BG_{p+1}^{(i-1)} f_{p+1} + F_p^{(i)} b.$$

Let us prove that y is a cycle in $\bar{C}_{p+2i}(T)$.

$$\begin{aligned}
by &= bG_{p+2}^{(i-1)} Bf_{p+1} + BbG_{p+1}^{(i-1)} f_{p+1} + bF_p^{(i)} b \\
&= (G_{p+3}^{(i-2)} B - BG_{p+2}^{(i-2)} + G_{p+1}^{(i)} b)\, Bf_{p+1} \\
&\quad + B(G_{p+2}^{(i-2)} B - BG_{p+1}^{(i-2)} + G_p^{(i-1)} b)\, f_{p+1} + bF_p^{(i)} b \\
&= G_{p+1}^{(i)} bBf_{p+1} + BG_p^{(i-1)} bf_{p+1} + bF_p^{(i)} b \\
&= -(G_{p+1}^{(i)} B - BG_p^{(i-1)})\, f_p b + bF_p^{(i)} b \\
&= -(bF_p^{(i)} - F_{p-1}^{(i)} b) b + bF_p^{(i)} b \\
&= 0.
\end{aligned}$$

By [15, Lemma 5.2], all cycles in $\bar{C}_q(T)$ are boundaries provided $q \geqslant 2$.

So there exists x such that $bx = y$ when $p+2i \geqslant 2$. And the corresponding representative $F_{p+1}^{(i)}$ can be chosen to be multilinear by the same argument as in [9]. The above method does not apply for $(p, i) = (-1, 1)$, so that we have to construct $F_0^{(1)}$ by hand.

Identifying $a \in A$ with $a \otimes 1 \in A \otimes A'$ and $a' \in A'$ with $1 \otimes a' \in A \otimes A'$, we denote an element $a \otimes a'$ in $A \otimes A'$ by aa'.

We set

$$F_0^{(1)}(a_0 a_0') = 1 \otimes a_0 \otimes a_0'.$$

The reader will easily check the desired relation for $F_0^{(1)}$, namely

$$bF_0^{(1)} = g_1 Bf_0 - Bg_0 f_0,$$

by making use of the above expressions for b and B and of the formulas given for f and g in [3, XI]. ▮

3. Application to Cyclic Homology

As an immediate consequence of Corollaries 1.8 and 2.6, we have

THEOREM 3.1. *Let A and A' be unital associative algebras over the commutative ring k. Assume that A' and* $\mathrm{HC}_*(A')$ *are flat k-modules. Then the following sequence*

$$\begin{aligned} 0 &\longrightarrow \mathrm{Cotor}^{k[u]}(\mathrm{HC}_*(A), \mathrm{HC}_*(A'))[1] \longrightarrow \mathrm{HC}_*(A \otimes A') \\ &\longrightarrow \mathrm{HC}_*(A) \,\square_{k[u]}\, \mathrm{HC}_*(A') \longrightarrow 0 \end{aligned}$$

is exact.

We list now a series of applications of 3.1 in which A' is a flat k-algebra such that $\mathrm{HC}_*(A') = k[u] \otimes U \oplus V$, where U and V are flat k-modules, $k[u] \otimes U$ is an *extended comodule* and V is a *trivial comodule* in the sense of [6] (this means that for $k[u] \otimes U$ the coaction is given by $\Delta \otimes U$ and the map S corresponding to V is zero). Then using Theorem 1.6 and classical properties of Cotor (see also the proof of [2, Theorem B(b)]), one concludes immediately that for any k-algebra A,

$$\mathrm{HC}_*(A \otimes A') = \mathrm{HC}_*(A) \otimes U \oplus H_*(A, A) \otimes V. \tag{3.2}$$

In the following six examples for which we use (3.2), k is a commutative ring containing a field of characteristic zero, A is any unital associative k-algebra.

Examples

(3.3) In the case of polynomials, $\mathrm{HC}_*(k[x]) = k[u] \oplus I$. Here I is the augmentation ideal of $k[u]$; it is a trivial comodule concentrated in degree 0. Therefore

$$\mathrm{HC}_n(A[x]) = \mathrm{HC}_n(A) \oplus H_n(A, A) \otimes I.$$

(3.4) For Laurent polynomials, we have $\mathrm{HC}_*(k[x, x^{-1}]) = k[u] \otimes (k \oplus kv) \oplus I'$. The module $k \oplus kv$ is generated by 1 in degree 0 and v in degree 1; I' is the augmentation ideal of $k[x, x^{-1}]$ and a trivial comodule. Hence

$$\mathrm{HC}_n(A[x, x^{-1}]) = \mathrm{HC}_n(A) \oplus \mathrm{HC}_{n-1}(A) \oplus H_n(A, A) \otimes I'.$$

This result can be found in [1]. The above two examples are special cases of the following one.

(3.5) Let A' be a smooth commutative algebra over a field k of characteristic 0; then by [15, Theorem 2.9], $\mathrm{HC}_*(A') = k[u] \otimes H^*_{\mathrm{DR}}(A') \oplus d\Omega^*_{A'/k}$ where the second summand is a trivial comodule. Then

$$\mathrm{HC}_n(A \otimes A') = \left(\bigoplus_{p+q=n} \mathrm{HC}_p(A) \otimes H^q_{\mathrm{DR}}(A') \right) \oplus \left(\bigoplus_{p+q=n} H_p(A, A) \otimes d\Omega^q_{A'/k} \right).$$

(3.6) Concerning other affine schemes, a computation by Goodwillie (unpublished) and by Masuda–Natsume [18] shows that if k is a field of characteristic zero and $P(x)$ is a degree m polynomial with coefficients in k and with r distinct roots in an extension of k, then $\mathrm{HC}_*(k[x]/P(x)) = k[u]^r \oplus k[u]^{m-r}$, where the second summand is a trivial comodule. It follows immediately that

$$\mathrm{HC}_n(A[x]/P(x)) = \mathrm{HC}_n(A)^r \oplus H_n(A, A)^{m-r} \oplus H_{n-2}(A, A)^{m-r} \oplus \cdots.$$

(3.7) Let J be the algebra of generalized Jacobi matrices consisting of matrices $(a_{ij})_{i,j \in \mathbf{Z}}$ with only finitely many nonzero diagonals. Feigin and Tsygan [7] proved that $\mathrm{HC}_*(J) = k[u][1]$. Consequently,

$$\mathrm{HC}_n(A \otimes J) = \mathrm{HC}_{n-1}(A).$$

Note that the obvious embedding of $k[x, x^{-1}]$ into J (given by "counting the diagonals") detects the interesting factor $\mathrm{HC}_{n-1}(A)$ in $\mathrm{HC}_n(A[x, x^{-1}])$.

(3.8) Let A_m be the Weyl algebra of polynomial differential operators on the mth dimensional affine space. Then $\mathrm{HC}_*(A_m) = k[u][2m]$ (see [7]). Hence, for any k-algebra A,

$$\mathrm{HC}_n(A \otimes A_m) = \mathrm{HC}_{n-2m}(A).$$

We give also an application which is valid without any restriction on the ground ring k. Let us recall (see [21]) that a k-algebra A is *separable* if it is projective as a module over $A \otimes A^0$. The following algebras are separable: (i) the algebra of $n \times n$-matrices with entries in k, (ii) $k[G]$ where G is a finite group with order invertible in k, (iii) a finite product of finite dimensional simple algebras over a field k whose centres are separable extensions of k, (iv) a finitely generated commutative unramified algebra, (v) an étale algebra (see [20, Chaps. I and III]).

PROPOSITION 3.9. *Let A' be a separable flat algebra over a commutative ring k. Then for any associative k-algebra A,*

$$\mathrm{HC}_*(A \otimes A') = \mathrm{HC}_*(A) \otimes A'/[A', A'].$$

Proof. Since A' is $A' \otimes A'^0$-projective, its Hochschild homology is trivial, i.e., $H_n(A', A') = 0$ for $n \neq 0$. By Connes's long exact sequence, $\mathrm{HC}_*(A') = k[u] \otimes A'/[A', A']$. We apply (3.2) to conclude. ∎

We turn now to *periodic cyclic homology*, which is obtained from cyclic homology by inverting the map S. More precisely, given a mixed complex (M, b, B), the inverse limit $\varprojlim_{m}\, {}_BM[2m]$ with respect to the natural surjection $S\colon M \to M[2]$ (cf. Sect. 1) is the $\mathbf{Z}/2$-graded complex

$$\hat{M} = \Big(\prod_{n \text{ even}} M_n \underset{b+B}{\overset{b+B}{\rightleftarrows}} \prod_{n \text{ odd}} M_n \Big).$$

We denote by $\mathrm{HC}_*^{\mathrm{per}}(M)$ the $\mathbf{Z}/2$-graded k-module $H_*(\hat{M})$ and call it the *periodic cyclic homology* of M. It is related to cyclic homology by a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 \mathrm{HC}_{*+2m+1}(M) \longrightarrow \mathrm{HC}_*^{\mathrm{per}}(M) \longrightarrow \varprojlim \mathrm{HC}_{*+2m}(M) \longrightarrow 0.$$

When M is the mixed complex $C(A)$ associated to an associative algebra A (see Sect. 2), then we write $\mathrm{HC}_*^{\mathrm{per}}(A)$ for $\mathrm{HC}_*^{\mathrm{per}}(C(A))$. Our definition coincides with the one given in [9, II.3]. Periodic cyclic homology is of great geometric significance, as illustrated in [4, 8, 9] and in [1] for group rings. For instance, when A is the coordinate ring of an affine algebraic variety V over a field k of characteristic zero, Feigin and Tsygan [8] proved that $\mathrm{HC}_*^{\mathrm{per}}(A)$ is isomorphic to the crystalline cohomology of V defined by Grothendieck [19] as a generalization of de Rham cohomology for singular varieties.

Given two mixed complexes M and N, there exists a natural chain map $\hat{M} \otimes \hat{N} \to \widehat{M \otimes N}$; but it is not necessarily an isomorphism since infinite direct products do not commute with the tensor product. However, we shall prove this map is a quasi-isomorphism if M (or N) has the following Property (P):

(P) M is flat over k and $\mathrm{HC}_*(M)$ is the direct sum of an extended comodule $k[u] \otimes U$ and of a trivial comodule V where U is a finitely generated projective k-module and V is a flat k-module.

If M satisfies (P), the inverse system $\{\cdots \to \mathrm{HC}_*(M) \xrightarrow{S} \mathrm{HC}_{*-2}(M) \to \cdots\}$ is Mittag–Leffler and $\mathrm{HC}_*^{\mathrm{per}}(M) = U$. By extension, we say that a k-algebra A has Property (P) if the associated mixed complex $C(A)$ has the same property. Notice that all algebras in Examples (3.3)–(3.9) above have this property (provided $\mathrm{HC}_0(A')$ is finitely generated and projective over k in (3.9)).

THEOREM 3.10. *Let A and A' be associative algebras over a commutative ring k. Assume A' has Property* (P). *Then one has an isomorphism*

$$\mathrm{HC}_*^{\mathrm{per}}(A \otimes A') = \mathrm{HC}_*^{\mathrm{per}}(A) \otimes \mathrm{HC}_*^{\mathrm{per}}(A')$$

of $\mathbf{Z}/2$*-graded vector spaces.*

Proof. The theorem is a consequence of the isomorphisms

$$\mathrm{HC}_*^{\mathrm{per}}(\bar{C}(A) \otimes \bar{C}(A')) = \mathrm{HC}_*^{\mathrm{per}}(\bar{C}(A \otimes A')) = \mathrm{HC}_*^{\mathrm{per}}(A \otimes A')$$

which follow from Proposition 2.3 and Theorem 2.4 and of the following fact: if M and N are mixed complexes, N has Property (P) and $\mathrm{HC}_*(N) = k[u] \otimes U \oplus V$ as above, then $\mathrm{HC}_*(M \otimes N) = \mathrm{HC}_*^{\mathrm{per}}(M) \otimes U$.

In order to prove this assertion, we explicit the map S on ${}_B(M \otimes N)$ in terms of the corresponding maps on ${}_BM$ and ${}_BN$. Let us resume the notations of Section 1 and point out that Theorem 1.7 can be reformulated under the following exact sequence of complexes

$$0 \longrightarrow {}_B(M \otimes N) \xrightarrow{j} {}_BM \otimes_B N \xrightarrow{S \otimes \mathrm{id} - \mathrm{id} \otimes S} {}_BM \otimes_B N[2] \longrightarrow 0.$$

When one identifies ${}_BM$ with $k[u] \otimes M$ (and similarly for ${}_BN$ and ${}_B(M \otimes N)$), the map j is given by: $j(u^i \otimes m \otimes n) = \sum_{p+q=i} u^p \otimes m \otimes u^q \otimes n$. Let us consider the endomorphism $S' = \mathrm{id} \otimes S$ of ${}_BM \otimes_B N$. It commutes with $S \otimes \mathrm{id} - \mathrm{id} \otimes S$ and one sees immediately that $S' \circ j = j \circ S$. Therefore in order to understand the inverse system $\{\cdots \to {}_B(M \otimes N) \xrightarrow{S} {}_B(M \otimes N)[2] \to \cdots\}$, we look at the inverse system $\{\cdots \to {}_BM \otimes {}_BN \xrightarrow{S'} {}_BM \otimes {}_BN[2] \to \cdots\}$.

Taking the homology of the above exact sequence yields an exact sequence as in Corollary 1.8 and passing to the inverse limit gives the following six-term exact sequence

$$0 \longrightarrow \varprojlim \operatorname{Cotor}(\mathrm{HC}(M), \mathrm{HC}(N))[1] \longrightarrow \varprojlim \mathrm{HC}(M \otimes N)$$
$$\longrightarrow \varprojlim \mathrm{HC}(M) \mathbin{\square} \mathrm{HC}(N) \longrightarrow \varprojlim{}^1 \operatorname{Cotor}(\mathrm{HC}(M), \mathrm{HC}(N))[2]$$
$$\longrightarrow \varprojlim{}^1 \mathrm{HC}(M \otimes N)[1] \longrightarrow \varprojlim{}^1 \mathrm{HC}(M) \mathbin{\square} \mathrm{HC}(N)[1] \longrightarrow 0.$$

Now under the hypotheses,

$$\operatorname{Cotor}(\mathrm{HC}(M), \mathrm{HC}(\mathrm{N})) = \operatorname{Cotor}(\mathrm{HC}(M), V)$$
$$= \operatorname{Coker}(\mathrm{HC}(M) \otimes V[-2] \xrightarrow{S \otimes \mathrm{id}} \mathrm{HC}(M) \otimes V)$$

on which $S' = 0$. Therefore

$$\varprojlim \operatorname{Cotor}(\mathrm{HC}(M), \mathrm{HC}(N)) = \varprojlim{}^1 \operatorname{Cotor}(\mathrm{HC}(M), \mathrm{HC}(N)) = 0.$$

Concerning the cotensor product, we have $\mathrm{HC}(M) \mathbin{\square} \mathrm{HC}(N) = \mathrm{HC}(N) \mathbin{\square} (k[u] \otimes U) \oplus \mathrm{HC}(M) \mathbin{\square} V$. Now $\mathrm{HC}(M) \mathbin{\square} V = \operatorname{Ker}(\mathrm{HC}(M) \otimes V \xrightarrow{S \otimes \mathrm{id}} \mathrm{HC}(M) \otimes V[2])$ on which $S' = 0$.

We are left with $\mathrm{HC}(M) \mathbin{\square} (k[u] \otimes U)$. We know that $i\colon \mathrm{HC}(M) \otimes U \to \mathrm{HC}(M) \mathbin{\square} (k[u] \otimes U)$ given by $i(m \otimes x) = \sum_{i \geq 0} S^i m \otimes u^i \otimes x$ is an isomorphism. Let us show that S' corresponds to $S \otimes \mathrm{id}$ under this isomorphism.

$$S' \circ i(m \otimes x) = S'\left(\sum_{i \geq 0} S^i m \otimes u^i \otimes x\right) = (\mathrm{id} \otimes S)\left(\sum_{i \geq 0} S^i m \otimes u^i \otimes x\right)$$
$$= \sum_{i \geq 1} S^i m \otimes u^{i-1} \otimes x = \sum_{i \geq 0} S^{i+1} m \otimes u^i \otimes x = i(Sm \otimes x).$$

One concludes that

$$\varprojlim \mathrm{HC}(M \otimes N) = \varprojlim (\mathrm{HC}(M) \otimes U) = (\varprojlim \mathrm{HC}(M)) \otimes U$$

because of the finiteness condition on U. Similarly, $\varprojlim{}^1 \mathrm{HC}(M \otimes N) = (\varprojlim{}^1 \mathrm{HC}(M)) \otimes U$. By the existence of a natural map $\mathrm{HC}_*^{\mathrm{per}}(M) \otimes U = \mathrm{HC}_*^{\mathrm{per}}(M) \otimes \mathrm{HC}_*^{\mathrm{per}}(N) \to \mathrm{HC}_*^{\mathrm{per}}(M \otimes N)$ and the exact sequence relating periodic cyclic homology to the $\varprojlim$- and $\varprojlim{}^1$-terms, one has $\mathrm{HC}_*^{\mathrm{per}}(M \otimes N) = \mathrm{HC}_*^{\mathrm{per}}(M) \otimes U$. ∎

We list a few immediate applications.

COROLLARY 3.11. *Let k be a field. Then for any pair of groups G and H such that $k[G]$ or $k[H]$ has Property* (P),

$$\mathrm{HC}_*^{\mathrm{per}}(k[G \times H]) = \mathrm{HC}_*^{\mathrm{per}}(k[G]) \otimes \mathrm{HC}_*^{\mathrm{per}}(k[H]).$$

This result generalizes Corollary IV_p in [1]. The following one is of geometric nature.

COROLLARY 3.12. *Let $O[V]$ be the coordinate ring of an affine algebraic variety over a field k of characteristic zero. Then for any associative algebra A,*

(a) *if V is smooth,*

$$\mathrm{HC}_*^{\mathrm{per}}(A \otimes O[V]) = \mathrm{HC}_*^{\mathrm{per}}(A) \otimes H_{\mathrm{DR}}^*(V)$$

(where the de Rham cohomology of V is given the usual $\mathbf{Z}/2$-grading).

(b) *if V is arbitrary,*

$$\mathrm{HC}_*^{\mathrm{per}}(A \otimes O[V]) = \mathrm{HC}_*^{\mathrm{per}}(A) \otimes H_{\mathrm{cris}}^*(V)$$

where $H_{\mathrm{cris}}^*(V)$ is the crystalline cohomology of V (see [19]).

Proof. It results from Theorem 3.10 and from [15, Theorem 2.9] (for part (a)) and from [8, Theorem 5] (for part (b)). By [19], (a) is a special case of (b). ∎

Corollary 3.12 implies the following isomorphisms (k containing the field of rational numbers)

$$\mathrm{HC}_*^{\mathrm{per}}(A[x]) = \mathrm{HC}_*^{\mathrm{per}}(A). \tag{3.13}$$

$$\mathrm{HC}_*^{\mathrm{per}}(A[x, x^{-1}]) = \mathrm{HC}_*^{\mathrm{per}}(A) \oplus \mathrm{HC}_{*-1}^{\mathrm{per}}(A). \tag{3.14}$$

If the polynomial $P(x)$ is as in (3.6), then $\mathrm{HC}_*^{\mathrm{per}}(k[x]/P(x)) = k^r$ counts the roots of P. Hence

$$\mathrm{HC}_*^{\mathrm{per}}(A[x]/P(x)) = \mathrm{HC}_*^{\mathrm{per}}(A)^r. \tag{3.15}$$

Finally, corresponding to (3.7) and (3.8), one has (with the same notations as above)

$$\mathrm{HC}_*^{\mathrm{per}}(A \otimes J) = \mathrm{HC}_{*-1}^{\mathrm{per}}(A). \tag{3.16}$$

$$\mathrm{HC}_*^{\mathrm{per}}(A \otimes A_m) = \mathrm{HC}_*^{\mathrm{per}}(A). \tag{3.17}$$

This last isomorphism was also proved by Masuda [17].

Remark. Theorem 3.10 is related to the following classical result of equivariant homotopy theory (see [13]). Let G be the circle group. If G acts on a space X, let X^G be the set of all points of X with finite isotropy subgroup and let $H_G^*(X)$ denote the rational G-equivariant cohomology of X. If u is a generator of $H^2(BG)$, let us denote $H_G^*(X)[u^{-1}]$ by $\hat{H}_G^*(X)$. Then if X is a finite G-space, there is a natural isomorphism

$$\hat{H}_G^*(X) \cong \hat{H}_G^*(X^G) \cong H^*(X^G) \otimes \hat{H}_G^*(pt).$$

Consequently, $\hat{H}_G^*(X \times Y)$ is the tensor product of $\hat{H}_G^*(X)$ and $\hat{H}_G^*(Y)$ over $\hat{H}_G^*(pt)$.

APPENDIX: Derived Functors in the Category $\Lambda(k)$

This section is devoted to some homological algebra on the abelian category $\Lambda(\mathfrak{A})$ of contravariant functors from Connes's category Λ (see [5]) into a given abelian category $\mathfrak{A}$. When $\mathfrak{A}$ is the category of modules over a commutative ring k, we denote the above functor category by $\Lambda(k)$ and the opposite category by $\Lambda(k)^0$.

Our aim is to define cyclic homology as derived functor of an appropriate tensor product in $\Lambda(k)$. This is implicit in [5, IV] and was used in [2] and [9], but has never appeared explicitly in the literature.

Lemma A.1. *If $\mathfrak{A}$ has enough projective objects, then so has $\Lambda(\mathfrak{A})$.*

Proof. This is a general statement resulting from the theory of Kan extensions (see for instance [12], Chap. 9). We recall that the functor $I\colon \Lambda(\mathfrak{A}) \to \mathfrak{A}^{\Lambda}$ defined by $I(E)_\lambda = E_\lambda$ has a left adjoint given by

$$J((M_\lambda)_\lambda)_\mu = \bigoplus_{\lambda \in \Lambda} J_\lambda(M_\lambda)_\mu,$$

where for any object λ in Λ, the functor $J_\lambda\colon \mathfrak{A} \to \Lambda(\mathfrak{A})$ is given by

$$J_\lambda(M)_\mu = \bigoplus_{\alpha : \mu \to \lambda} M_\alpha \qquad \text{where} \quad M_\alpha = M.$$

The adjointness relation is given by

$$\mathrm{Hom}_{\Lambda(\mathfrak{A})}(J((M_\lambda)_\lambda), E) \cong \prod_{\lambda \in \Lambda} \mathrm{Hom}_{\mathfrak{A}}(M_\lambda, E_\lambda).$$

This proves immediately that if $(P_\lambda)_\lambda$ is a family of projective objects in $\mathfrak{A}$, then $J((P_\lambda)_\lambda)$ is projective in $\Lambda(\mathfrak{A})$.

To prove the lemma, we have only to show that for any object E in $\Lambda(\mathfrak{A})$, there exists an epimorphism $J((P_\lambda)_\lambda) \to E$. For each $\lambda \in \Lambda$, we choose a projective P_λ in $\mathfrak{A}$ mapping epimorphically onto E_λ. The map $J((P_\lambda)_\lambda)_\mu \to E_\mu$ given for each μ by adjointness factors obviously by two epimorphisms through the summand P_μ corresponding to the identity of μ. This finishes the proof. ∎

It results also from this proof that any projective object in $\Lambda(\mathfrak{A})$ is a retract of a direct sum of objects of the form $J_\lambda(P_\lambda)$ where P_λ is projective in $\mathfrak{A}$.

We define now the desired tensor product of cyclic k-modules as a functor

$$\Lambda(k)^0 \times \Lambda(k) \to (k).$$

DEFINITION A.2. Given E in $\Lambda(K)^0$ and F in $\Lambda(k)$, their tensor product is the k-module

$$E \underset{\Lambda}{\otimes} F = \left(\bigoplus_{\lambda \in \Lambda} E_\lambda \otimes F_\lambda \right) \Big/ V,$$

where V is the sub-k-module generated by all elements of the form $\alpha_*(e) \otimes f - e \otimes \alpha^*(f)$ where $e \in E_\lambda$, $f \in F_\mu$ and α is a map from λ to μ in Λ.

Notice that in the category $\Lambda(k)^0$, one has

$$J_\lambda(M)_\mu = M \otimes k[\mathrm{Hom}_\Lambda(\lambda, \mu)].$$

Then one has immediately

$$J_\lambda(M) \underset{\Lambda}{\otimes} F \cong M \otimes F_\lambda. \tag{A.3}$$

LEMMA A.4. *The functor $\otimes_\Lambda$ is right exact and is left balanced in the sense of Cartan–Eilenberg* [3].

Proof. The right-exactness comes from the fact that $\otimes_\Lambda$ is a quotient of a direct sum and the ordinary tensor product is right exact. We have now to prove that if E is projective in $\Lambda(k)^0$, then $F \mapsto E \otimes_\Lambda F$ is *left* exact. By previous remarks, it is enough to prove it for $E = J_\lambda(P)$ and a projective module P. Which is true because of (A.3) and of the flatness of P. ∎

It follows from [3, Chap. V, Sect. 8] that $\otimes_\Lambda$ has derived functors $\mathrm{Tor}_*^{\Lambda(k)}(-, -)$ which can be computed by resolutions of either variable.

PROPOSITION A.5. *Let E be a cyclic k-module, i.e., an object of $\Lambda(k)$, then its cyclic homology is isomorphic to*

$$\mathrm{HC}_*(E) \cong \mathrm{Tor}_*^{\Lambda(k)}(k, E),$$

where k is the constant functor in $\Lambda(k)^0$.

Proof. In [5, IV], Connes constructs a projective resolution of k in $\Lambda^0(k)$ with the objects $J_\lambda(k)$. Using (A.3), one sees immediatement that the tensor product of Connes's resolution with E over Λ gives the bicomplex defining cyclic homology in [9] or in [15]. ∎

REFERENCES

1. D. BURGHELEA, Cyclic homology of group rings, *Comment. Math. Helv.* **60** (1985), 354–365.
2. D. BURGHELEA AND Z. FIEDOROWICZ, Cyclic homology and algebraic K-theory of spaces, II, *Topology* **25** (1986), 303–317.

3. H. Cartan and S. Eilenberg, "Homological Algebra," Princeton Univ. Press, Princeton, N.J., 1956.
4. A. Connes, Non-commutative differential geometry, Chap. II. De Rham homology and non-commutative algebra, *Publ. Math. IHES* **62** (1985), 41–144.
5. A. Connes, Cohomologie cyclique et foncteurs Ext^n, *C. R. Acad. Sci. Paris* **296** (1983), 953–958.
6. S. Eilenberg and J. C. Moore, Homology and fibrations. I. Coalgebras, cotensor product and its derived functors, *Comment. Math. Helv.* **40** (1966), 199–236.
7. B. L. Feigin and B. L. Tsygan, Cohomology of Lie algebras of generalized Jacobi matrices, *Funct. Anal. Appl.* **17** (1983), 153–155.
8. B. L. Feigin and B. L. Tsygan, Additive K-theory and crystalline cohomology, *Funct. Anal. Appl.* **19** (1985), 124–132.
9. T. G. Goodwillie, Cyclic homology and the free loopspace, *Topology* **24** (1985), 187–215.
10. V. K. A. M. Gugenheim, On the chain complex of a fibration, *Illinois J. Math.* **16** (1972), 398–414.
11. V. K. A. M. Gugenheim and H. J. Munkholm, On the extended functoriality of Tor and Cotor, *J. Pure Appl. Algebra* **4** (1974), 9–29.
12. P. J. Hilton and U. Stammbach, "A Course in Homological Algebra," Springer-Verlag, Berlin, 1971.
13. W. Y. Hsiang, "Cohomology Theory of Topological Transformation Groups," Ergeb. Math., Vol. 85, Springer-Verlag, Berlin, 1975.
14. D. Husemoller, J. C. Moore, and J. Stasheff, Differential homological algebra and homogeneous spaces, *J. Pure Appl. Algebra* **5** (1974), 113–185.
15. J. L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, *Comment. Math. Helv.* **59** (1984), 565–591.
16. T. Masuda, Cyclic cohomology of $A[X]$ and $A[X, X^{-1}]$, *Publ. Res. Inst. Math. Sci.* **22** (1986), 365–393.
17. T. Masuda, Dualité pour le produit croisé différentiel et sa cohomologie cyclique périodique, *C. R. Acad. Sci. Paris* **301** (1985), 551–553.
18. T. Masuda and T. Natsume, Cyclic cohomology of certain affine schemes, *Publ. Res. Inst. Math. Sci.* **21** (1985), 1261–1279.
19. A. Grothendieck, Crystals and the Rham cohomology of schemes, notes by J. Coates and D. Jussila, in Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam, 1968.
20. M. Raynaud, "Anneaux locaux henséliens," Lecture Notes in Math., Vol. 169, Springer-Verlag, Berlin, 1970.
21. M. Auslander and O. Goldman, The Brauer group of a commutative ring, *Trans. Amer. Math. Soc.* **97** (1960), 367–409.