Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory

Volumes I and II by Donald Yau

Volume III by Niles Johnson and Donald Yau

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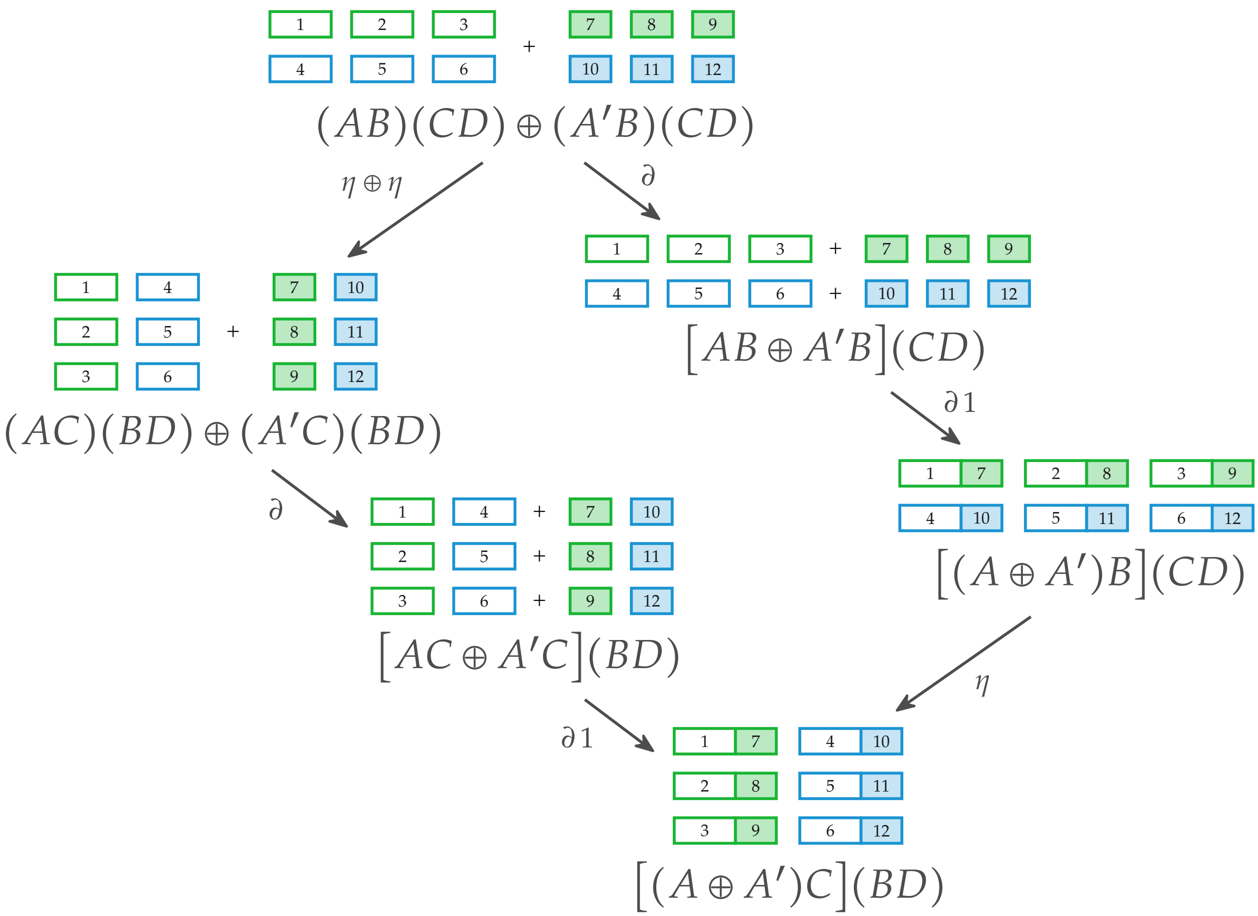

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ABSTRACT. Bimonoidal categories are categorical analogues of rings without additive inverses. They have been actively studied in category theory, homotopy theory, and algebraic K-theory since around 1970. There is an abundance of new applications and questions of bimonoidal categories in mathematics and other sciences. This work provides a unified treatment of bimonoidal and higher ring-like categories, their connection with algebraic K-theory and homotopy theory, and applications to quantum groups and topological quantum computation. With ample background material, extensive coverage, detailed presentation of both well-known and new theorems, and a list of open questions, this work is a user-friendly resource for beginners and experts alike.

Part I.1 proves in detail Laplaza's two coherence theorems and May's strictification theorem of symmetric bimonoidal categories, as well as their bimonoidal analogues. This part includes detailed corrections to several inaccurate statements and proofs found in the literature. Part I.2 proves Baez's Conjecture on the existence of a bi-initial object in a 2-category of symmetric bimonoidal categories. The next main theorem states that a matrix construction, involving the matrix product and the matrix tensor product, sends a symmetric bimonoidal category with invertible distributivity morphisms to a symmetric monoidal bicategory, with no strict structure morphisms in general.

Part II.1 studies braided bimonoidal categories, with applications to quantum groups and topological quantum computation. It is proved that the categories of modules over a braided bialgebra, of Fibonacci anyons, and of Ising anyons form braided bimonoidal categories. Two coherence theorems for braided bimonoidal categories are proved, confirming the Blass-Gurevich conjecture. The rest of this part discusses braided analogues of Baez's Conjecture and the monoidal bicategorical matrix construction in Part I.2. Part II.2 studies ring and bipermutative categories in the sense of Elmendorf-Mandell, braided ring categories, and E_n-monoidal categories, which combine n-fold monoidal categories with ring categories.

Part III.1 is a detailed study of enriched monoidal categories, pointed diagram categories, and enriched multicategories. Using the machinery in Part III.1, Part III.2 discusses the rich interconnection between the higher ring-like categories in Part II.2, homotopy theory, and algebraic K-theory. Starting with a chapter on homotopy theory background, the first half of this part constructs the Segal K-theory functor and the Elmendorf-Mandell K-theory multifunctor from permutative categories to symmetric spectra. For the latter, the detailed treatment here includes identification and correction of some subtle errors concerning its extended domain. The second half applies the K-theory multifunctor to small ring, bipermutative, braided ring, and E_n-monoidal categories to obtain, respectively, strict ring, E_∞-, E_2-, and E_n-symmetric spectra. Appendix III.A discusses open questions related to the topics of this work.

Contents

This is a three volume work, consisting of six parts total. Each volume has identical front and end matter, spanning the three volumes, and its own content in two parts.

Reference Convention. In each of Volumes II and III, the numbering of parts, chapters, and pages restarts at 1. Page numbers are prefixed with a volume number so that, for example, page n of Volume II is printed II.n. References (chapters, definitions, theorems, equations, etc.) to a different volume are preceded by I, II, or III according to the volume. For example, $a.b.c$ in Volume I is denoted by I.$a.b.c$ in Volumes II and III, and Chapter N in Volume III is denoted by Chapter III.N in Volumes I and II.

Preface

Bimonoidal categories are categorical analogues of rings without additive inverses. They have been actively studied in category theory, homotopy theory, and algebraic K-theory since around 1970. There is an abundance of new applications and questions of bimonoidal categories in mathematics and other sciences. This work provides the first unified treatment of bimonoidal and higher ring-like categories, their connection with algebraic K-theory and homotopy theory, and applications to quantum groups and topological quantum computation. With ample background material, extensive coverage, detailed presentation of both well-known and new theorems, and a list of open questions, this work is a user-friendly resource for beginners and experts alike.

Bimonoidal and E_n-Monoidal Categories

A *bimonoidal category* C is a categorical analogue of a rig, which is a ring without additive inverses. In this categorification, the addition, multiplication, 0, and 1 of a rig are replaced by functors and objects in a bimonoidal category. Rig axioms are replaced by natural structure morphisms, along with suitable coherence axioms of their own.

More specifically, in place of the rig addition and multiplication, C has two monoidal structures

$$\big(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}\big) \quad \text{and} \quad \big(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}\big).$$

The first is symmetric monoidal, and called the *additive structure*. The second is plain monoidal, and called the *multiplicative structure*. As with plain monoidal categories, there are variants with braided or symmetric multiplicative structure, and a variety of intermediate multiplicative structures parametrized by E_n-operads.

In place of distributivity relations in a rig, a bimonoidal category has natural *distributivity monomorphisms* for objects A, B, and C:

$$\begin{aligned} A \otimes (B \oplus C) &\xrightarrow{\delta^l_{A,B,C}} (A \otimes B) \oplus (A \otimes C) \\ (A \oplus B) \otimes C &\xrightarrow{\delta^r_{A,B,C}} (A \otimes C) \oplus (B \otimes C). \end{aligned}$$

These data are required to satisfy a finite list of axioms that (i) are checkable in practice and (ii) ensure that (symmetric/braided) bimonoidal categories have good coherence and other categorical properties. An important special case is a *tight* bimonoidal category, in which the distributivity monomorphisms δ^l and δ^r are *isomorphisms*.

A number of examples, arising in both algebraic and homotopical contexts, are discussed throughout the text. Here, we summarize three important ones. More examples are discussed in the next section about quantum science.

(1) The category of finite dimensional complex vector spaces, $\mathsf{Vect}^{\mathbb{C}}$, is a tight symmetric bimonoidal category with its additive and multiplicative structures given by the usual direct sum and tensor product of vector spaces. More generally, each distributive symmetric monoidal category is a tight symmetric bimonoidal category.
(2) The nonnegative integers and permutations form the objects and the morphisms of a tight symmetric bimonoidal category Σ, called the finite ordinal category.
(3) May's bipermutative categories, with the additional axiom $\xi^{\otimes}_{-,\mathbb{0}} = \mathrm{Id}$, are tight symmetric bimonoidal categories.

The definition and coherence theorems for symmetric bimonoidal categories are due to Laplaza [**Lap72a, Lap72b**]. These theorems and their plain bimonoidal analogues are discussed in detail in Part I.1. In addition to providing completely detailed proofs, we also correct some subtle and nontrivial inaccuracies in the original statements and proofs. See Sections I.3.11 and I.4.7 for related discussion. Just as applications of monoidal categories heavily depend on Mac Lane's coherence theorem, Laplaza's two coherence theorems for symmetric bimonoidal categories, as well as their plain and braided analogues, are crucial to their applications.

Part I.2 applies Laplaza's coherence theorems to prove a number of theorems about bimonoidal categories in the context of 2-dimensional categories. These include existence of a bi-initial object, confirming a conjecture of Baez [**Bae18**], and a symmetric monoidal bicategory of matrices, $\mathsf{Mat}^{\mathsf{C}}$, constructed from a tight symmetric bimonoidal category C. (Note the unfortunately subtle confluence of terminology that "symmetric monoidal *bicategory*" and "symmetric *bimonoidal* category" refer to wildly distinct algebraic structures.) In the case $\mathsf{C} = \mathsf{Vect}^{\mathbb{C}}$, $\mathsf{Mat}^{\mathsf{C}}$ is the symmetric monoidal bicategory of *coordinatized 2-vector spaces*, one version of the 2-vector spaces introduced by Kapranov and Voevodsky in [**KV94**].

Braided bimonoidal categories, along with their corresponding coherence and strictification theorems, are discussed in Part II.1. These structures are of interest for applications in quantum science, discussed below. The relevant coherence results are new, confirming a conjecture of Blass and Gurevich [**BG20a**].

Part II.2 introduces a similar but distinct categorification of rigs, called *E_n-monoidal categories*. These have *factorization morphisms* in place of distributivity monomorphisms, and are significant for the K-theoretic applications in Part III.2. The E_n-monoidal structure is a generalization of n-fold monoidal structure due to Balteanu, Fiedorowicz, Schwänzel, and Vogt [**BFSV03**]. Special cases of E_n-monoidal categories include, or are closely related to, the *bipermutative categories* of May [**May77**], the *ring categories* of Elmendorf and Mandell [**EM06**], and the *braided bimonoidal categories* of Richter [**Ric10**].

Applications in Quantum Science

Due to the ubiquity of ring-like structures and categories, bimonoidal categories are increasingly applied in a variety of disciplines in mathematics and other formal sciences related to quantum algebra. To support readers with a variety of interests, this work includes summary and introduction to several such applications.

Quantum Groups. The first part of Chapter II.3 extends a well-known fact in quantum group theory. We observe that the category of left modules over a braided bialgebra, which is also known as a quasitriangular bialgebra in the literature, is a tight braided bimonoidal category.

Topologial Quantum Computation. The second part of Chapter II.3 discusses applications of braided bimonoidal categories to topological quantum computation (TQC). We prove that the Fibonacci anyons and the Ising anyons, which are two of the most important models in TQC, are both tight braided bimonoidal categories.

Centers. Monoidal, braided monoidal, and symmetric monoidal categories are connected by the Drinfeld center and the symmetric center. Kassel [**Kas95**] and Majid [**Maj91**] explain the relationship between the Drinfeld center construction, due to Drinfeld in unpublished work, and modules over the Drinfeld double of a a finite dimensional Hopf algebra. (See Note II.1.7.2 for further explanation and context.) Bimonoidal and ring categorical analogues of these center constructions are discussed in Chapters II.4 and II.9.

Reversible Programming. Section I.2.6 is a brief illustration that symmetric bimonoidal categories naturally arise in reversible programming of finite types. We observe that there is a symmetric bimonoidal groupoid whose objects are syntax of finite types. Note I.2.7.5 directs the reader to further applications in the sheet diagrams of [**CDH∞**] and the work of [**Hin13**] on quantum circuits.

Applications in Algebraic K-Theory

Uses of additive and multiplicative categorical structure in homotopy theory are among the earliest and most well-developed applications of the material from Volumes I and II of this work. Volume III focuses on those applications, beginning with the work of Segal [**Seg74**] that assembles structured ring spectra from permutative categories. Under the Segal K-theory functor K^{Se}, the symmetric monoidal structure of a permutative category C results in the additive structure in the cohomology theory represented by the spectrum $\mathsf{K}^{\mathsf{Se}}(\mathsf{C})$.

The computational importance of multiplicative structure in cohomology motivates significant interest in multiplicative structure for the representing spectra, leading to the highly-structured spectra that are presaged in work of Adams [**Ada95**] and realized in the S-modules of Elmendorf-Kriz-Mandell-May [**EKMM97**], the symmetric spectra of Hovey-Shipley-Smith [**HSS00**], and the orthogonal spectra of Mandell-May-Schwede-Shipley [**MMSS01**], among other equivalent models. Thus, there is corresponding interest in bimonoidal structures for the input categories.

One difficulty, however, is that Segal K-theory does *not* preserve the multiplicative aspect of such structures. As a resolution, the work of Elmendorf and Mandell [**EM06, EM09**] introduces an alternative construction that (a) is suitably equivalent to Segal's construction and (b) preserves multiplicative structure. This is known as Elmendorf-Mandell K-theory, K^{EM}.

The formalism in which these statements can be made precise is that of multicategories, where multilinear functors between permutative categories encode the relevant multiplicative structure. In these terms, the essential difference between the Segal and Elmendorf-Mandell constructions is that the latter is *multifunctorial*, while the former is merely functorial.

Part III.1 develops the necessary supporting theory of enriched monoidal categories and multicategories, but is also of independent interest. Part III.2 contains the applications to algebraic K-theory, including a review of the relevant background in homotopy theory and detailed treatments of both the Segal and Elmendorf-Mandell constructions.

The second half of Part III.2 applies the Elmendorf-Mandell K-theory multifunctor K^{EM} to the E_n-monoidal categories in Part II.2 to produce structured ring spectra. We prove in detail that K^{EM} sends

- small ring categories to strict ring symmetric spectra,
- small bipermutative categories to E_∞-symmetric spectra,
- small braided ring categories to E_2-symmetric spectra, and
- small E_n-monoidal categories to E_n-symmetric spectra for $2 \leq n < \infty$.

The strict ring and E_∞ cases are from [**EM06, EM09**]. The $1 < n < \infty$ cases are new results.

Audience and Features

This work is aimed at graduate students and advanced researchers with an interest in category theory, homotopy theory, algebraic K-theory, and their applications. Below are some features that make this work a unique and user-friendly resource.

Unified Presentation: The literature on bimonoidal categories, higher ring-like categories, enriched monoidal categories, multicategories, and their connection with algebraic K-theory, homotopy theory, and the sciences is scattered across many journal articles over several decades, with varying definitions, notations, and terminology. This work presents these topics in a unified manner, with both well-known and new theorems.

Background Material: To make this work self-contained and to bring the reader quickly up to speed, there is extensive background material on
- basic category theory (Chapter I.1),
- 2-dimensional categories (Chapter I.6),
- braided structures (Chapter II.1),
- abelian categories (Section II.2.3),
- braided, also known as quasitriangular, bialgebras (Section II.3.1),
- enriched monoidal categories (Chapters III.1, III.2, and III.3),
- pointed objects and pointed diagram categories (Chapter III.4),
- enriched multicategories (Chapters III.5 and III.6), and
- homotopy theory (Chapter III.7).

These chapters and sections form a substantial portion of this work.

Open Questions: Appendix III.A discusses open questions related to the topics of this work. The reader is encouraged to take advantage of these open questions and use them as a springboard to read the main text.

Detailed Discussion: This work contains many highly detailed and carefully structured proofs for both known and new theorems. For each major result, our discussion has much more detail than one would normally find in the literature. Our detailed discussion has several purposes.

Exercises with Solutions. Our detailed presentation makes the material accessible to a diverse audience, including those who are new to bimonoidal and higher ring-like categories and algebraic K-theory. Students are encouraged to regard the numerous detailed proofs as exercises with full solutions. Each result whose proof has many different parts has been carefully structured to make it easy for the reader to jump forward and backward.

Axioms. Symmetric bimonoidal categories are defined by 24 axioms, and the list of axioms for (braided) bimonoidal categories is similarly substantial. Our detailed discussion helps the reader see exactly where these axioms are used and why they are needed.

Laplaza's Theorems. The Coherence Theorems I.3.9.1 and I.4.4.3 for symmetric bimonoidal categories are central results in this subject that have been cited and used numerous times in the literature. Their original proofs given by Laplaza in [**Lap72a, Lap72b**] were written in outline form, with much detail and some cases in the proofs completely omitted. Moreover, Laplaza's original proofs and statements of these theorems have several subtle and nontrivial inaccuracies that have never been made explicit before and are not easy to spot. For both archival and educational purposes, we present fully detailed proofs of these theorems and correct the inaccuracies. Sections I.3.11 and I.4.7 have more related discussion.

K-Theory Multifunctors. The K-theory multifunctors in Chapters III.9 and III.10, due to Elmendorf-Mandell [**EM06, EM09**], are fundamental constructions for multiplicative structure of algebraic K-theory spectra. They are essential for our development of E_n-monoidal symmetric spectra from corresponding structure on small permutative categories. We use the theory of enriched monoidal categories and enriched multicategories from Part III.1 to give complete explanations of the constructions and their properties. This treatment corrects an inaccuracy in the statement of [**EM09**, Theorem 1.3] and some other statements about expanding the domain of the K-theory multifunctor. The basic issue has to do with monoidal units and, to the authors' knowledge, has not been previously explained. See Note III.10.8.2 for further discussion.

Reading Guides: In addition to a detailed introduction, almost every chapter has a brief *Reading Guide* that provides an alternative to reading that chapter linearly. Our presentation in the main text follows a straightly logical order and has a lot of detail. By following the reading guide, it is possible to first obtain a bird's-eye view of that chapter before digesting all the detail. The end of this Preface also includes several thematic reading guides for salient topics that span multiple chapters.

Motivation and Explanation: Main definitions and results are often preceded by discussion that motivates the upcoming definitions and proofs. Whenever useful, definitions and results are followed by a detailed explanation that interprets and unpacks the various components. In the text, these are clearly marked as *Motivation* and *Explanation*, respectively. Examples include Motivation I.2.1.1, Explanation I.2.4.7, and Section I.4.1.

Organization: There are extensive cross-references throughout the text. In addition to a detailed index, there are lists of main facts and notations, each organized by chapters. While the text follows a strictly logical order, it is not necessary to read the chapters in a linear order. The reader can jump straight to a section and use the extensive cross-references to fill in the necessary definitions and facts.

Part and Chapter Summaries

Part I.1: Symmetric Bimonoidal Categories

This part studies symmetric bimonoidal categories and bimonoidal categories (Chapter I.2). It presents highly detailed proofs of Laplaza's Coherence Theorems for symmetric bimonoidal categories (Chapters I.3 and I.4), May's Strictification Theorem for tight symmetric bimonoidal categories (Chapter I.5), and their non-symmetric analogues for bimonoidal categories. The only prerequisite for this part is some basic knowledge of category theory, which is summarized in Chapter I.1.

Part I.2: Bicategorical Aspects of Symmetric Bimonoidal Categories

Applying Laplaza's Coherence Theorems, this part proves several new theorems on the connection between symmetric bimonoidal categories and bicategories. All the necessary definitions of 2-dimensional category theory are summarized in Chapter I.6. The first main result is a confirmation of Baez's Conjecture (Chapter I.7) that proves the existence of a bi-initial object in a 2-category of symmetric bimonoidal categories. Chapter I.8 proves that a matrix construction $\mathsf{Mat}^{\mathsf{C}}$ sends each tight symmetric bimonoidal category to a symmetric monoidal bicategory.

Part II.1: Braided Bimonoidal Categories

Starting with a preliminary chapter on the braid groups and braided monoidal categories, this part is a detailed study of braided bimonoidal categories (Chapter II.2), which are strictly more general than Richter's **[Ric10]** and the BD categories of Blass-Gurevich **[BG20a]**. This part discusses applications to quantum groups and topological quantum computation (Chapter II.3), bimonoidal centers (Chapter II.4), coherence and strictification of braided bimonoidal categories (Chapters II.5 and II.6), and the braided versions of Baez's Conjecture and the matrix construction (Chapters II.7 and II.8). Our coherence and strictification theorems confirm the Blass-Gurevich Conjecture. The main theorems in Parts I.1 and I.2 are used in this part.

Part II.2: E_n-Monoidal Categories

This part studies a closely related variant of bimonoidal categories, called ring categories, and their bipermutative, braided, and higher analogues, called E_n-monoidal categories. Ring and bipermutative categories are due to Elmendorf-Mandell **[EM06, EM09]**. An E_n-monoidal category combines n ring categories with a common additive structure and an n-fold monoidal category as in **[BFSV03]**. The categories in this part are applied in Part III.2 to obtain E_n-symmetric spectra via algebraic K-theory. This part is independent of the earlier parts, except for some definitions and statements of theorems.

Part III.1: Enriched Monoidal Categories and Multicategories

To prepare for Part III.2, this part lays the groundwork on enriched monoidal categories (Chapters III.1, III.2, and III.3), smash products (Chapter III.4), and multicategories (Chapters III.5 and III.6). In addition to their roles in the Segal K-theory functor and the Elmendorf-Mandell K-theory multifunctor, the detailed discussion of enriched monoidal categories—including change of enrichment, coherence, self-enrichment, and the Enriched Yoneda Lemma—and multicategories is also of independent interest. These chapters assume only a basic knowledge of monoidal categories, as summarized in Section III.1.1.

Part III.2: Algebraic K-Theory

This part studies the interconnection between E_n-monoidal categories (Part II.2), homotopy theory (Chapter III.7), and algebraic K-theory. The first half discusses in detail the Segal K-theory functor (Chapter III.8) and the Elmendorf-Mandell K-theory multifunctor (Chapters III.9 and III.10) from small permutative categories to symmetric spectra. The second half (Chapters III.11, III.12, and III.13) applies the K-theory multifunctor to small ring, bipermutative, braided ring, and E_n-monoidal categories to obtain, respectively, strict ring, E_∞-, E_2-, and E_n-symmetric spectra. These structured ring spectra are fundamental objects in homotopy theory. Our discussion shows how they arise from E_n-monoidal categories via algebraic K-theory.

In the main text, each chapter starts with a detailed introduction. A summary of each chapter follows.

Part I.1: Symmetric Bimonoidal Categories

Chapter I.1: Basic Category Theory

To make this book self-contained, this chapter reviews the basics of category theory, starting from the definitions of categories, functors, and natural transformations. Then it discusses adjunctions, equivalences of categories, (co)limits, (co)ends, and Kan extensions. The remaining sections review (symmetric) monoidal categories, (symmetric) monoidal functors, monoidal natural transformations, and their coherence theorems.

Chapter I.2: Symmetric Bimonoidal Categories

This chapter introduces symmetric bimonoidal categories and bimonoidal categories. Then we prove Laplaza's Theorem I.2.2.13 that says that half of the 24 symmetric bimonoidal category axioms are formal consequences of the other 12 axioms. The weaker bimonoidal analogue is Proposition I.2.2.14. The remaining sections discuss examples of symmetric bimonoidal categories, including distributive symmetric monoidal categories, the finite ordinal category Σ, a variant Σ', and left and right bipermutative categories. The finite ordinal category Σ is an important part of (i) the distortion category $\mathcal{D}$ (Chapter I.4) used in Laplaza's Second Coherence Theorem I.4.4.3, (ii) Baez's Conjecture (Chapter I.7), and (iii) the braided version of Baez's Conjecture (Chapter II.7). Section I.2.6 contains an application of symmetric bimonoidal categories to reversible programming of finite types.

Chapter I.3: Coherence of Symmetric Bimonoidal Categories

This chapter proves Laplaza's First Coherence Theorem I.3.9.1 for symmetric bimonoidal categories that satisfy a monomorphism assumption. This assumption is automatically satisfied if tightness—that is, the invertibility of the distributivity morphisms δ^l and δ^r—is assumed, but the general form of this theorem only requires that the distributivity morphisms be natural monomorphisms. The analogue of this coherence theorem for bimonoidal categories is Theorem I.3.10.7. Section I.3.11 discusses the main differences between this chapter and Laplaza's original work in [**Lap72a**].

Chapter I.4: Coherence of Symmetric Bimonoidal Categories II

This chapter proves Laplaza's Second Coherence Theorem I.4.4.3 for symmetric bimonoidal categories that satisfy the same monomorphism assumption as in Theorem I.3.9.1. The analogue of this coherence theorem for bimonoidal categories is Theorem I.4.5.8. Section I.4.7 discusses the main differences between this chapter and Laplaza's original work in [**Lap72b**]. Both Coherence Theorems I.3.9.1 and I.4.4.3 say that some formal diagrams in certain symmetric bimonoidal categories commute. The first theorem has an assumption called *regularity* on the common domain of the two paths involved, which is analogous to Mac Lane's Coherence Theorem I.1.3.3 for monoidal categories. The second theorem has an assumption about the two paths themselves, which is reminiscent of the Joyal-Street Coherence Theorem II.1.6.3 for braided monoidal categories. In Chapter II.5, we observe that the second, but not the first, theorem has a braided analogue.

Chapter I.5: Strictification of Tight Symmetric Bimonoidal Categories

This chapter proves May's Strictification Theorem I.5.4.6 of *tight* symmetric bimonoidal categories to right bipermutative categories. The latter are tight symmetric bimonoidal categories whose additive structures and multiplicative structures are both permutative categories, and whose structure morphisms $\lambda^\bullet$, $\rho^\bullet$, δ^r, and $\xi^{\otimes}_{-,\mathbb{0}}$ are identities. Unlike the Coherence Theorems I.3.9.1 and I.4.4.3, the strictification theorem requires the tightness assumption. Our detailed proofs show exactly where the invertibility of δ^l and δ^r is used. Theorem I.5.4.7 is another version of the strictification theorem involving *left* bipermutative categories, in which δ^l, instead of δ^r, is the identity. Theorems I.5.5.11 and I.5.5.12 are the corresponding strictification results for tight bimonoidal categories. Section I.5.6 briefly discusses the history of related strictification theorems and claims. The proofs in this chapter are repurposed in Chapter II.6 to prove the strictification form of the Blass-Gurevich conjecture for braided bimonoidal categories.

Part I.2: Bicategorical Aspects of Symmetric Bimonoidal Categories

Chapter I.6: Definitions from Bicategory Theory

Without assuming any knowledge of 2-dimensional categories, in this chapter we review the basics of 2-/bicategories, pasting diagrams, lax functors, lax transformations, modifications, and adjunctions in bicategories. Then it reviews multiplicative structures, including monoidal bicategories, their braided, sylleptic, and symmetric analogues, the Gray tensor product for 2-categories, (permutative) Gray monoids, and permutative 2-categories. Most of these topics are discussed in detail in the book [**JY21**].

Chapter I.7: Baez's Conjecture

This chapter proves Baez's Conjecture (Theorems I.7.8.1 and I.7.8.3). Section I.7.1 defines a 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ with *flat* small symmetric bimonoidal categories as objects. Flatness (Definition I.3.9.9) is much weaker than tightness, and it guarantees that the Coherence Theorems I.3.9.1 and I.4.4.3 are applicable. The first version of Baez's Conjecture (Theorem I.7.8.1) says that the finite ordinal category Σ is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$. Another version is Theorem I.7.8.3, which says that the variant Σ' of Σ is also such a lax bicolimit. We emphasize that our proof of Baez's Conjecture does *not* use the Strictification Theorems I.5.4.6 and I.5.4.7. This allows us to use flat small symmetric bimonoidal categories in the 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$, instead of the smaller class of tight ones. Section I.7.9 discusses the relationship between our version of Baez's Conjecture and the more restricted version in [**CDH∞, Elg21**] for rig categories, which are multiplicatively nonsymmetric and tight.

Chapter I.8: Symmetric Monoidal Bicategorification

This chapter proves Theorem I.8.15.4. It says that, for each tight symmetric bimonoidal category C, a matrix construction $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory, with no strict structure morphisms in general. Therefore, the construction $\mathsf{Mat}^{\mathsf{C}}$ is a direct connection between tight symmetric bimonoidal categories and symmetric monoidal bicategories. The objects in $\mathsf{Mat}^{\mathsf{C}}$ are nonnegative integers. Its 1-/2-cells are matrices whose entries are objects/morphisms in C. The horizontal composition in the bicategory $\mathsf{Mat}^{\mathsf{C}}$ uses the usual matrix product. The monoidal composition in its monoidal bicategory structure uses the matrix tensor product, which is also known as the Kronecker product. The category of coordinatized 2-vector spaces, which is $\mathsf{Mat}^{\mathsf{C}}$ with $\mathsf{C} = \mathsf{Vect}^{\mathbb{C}}$, is such a symmetric monoidal bicategory. This chapter uses the Coherence Theorems I.3.9.1 and I.3.10.7 and the graph theoretic machinery in Chapter I.3, but neither the Coherence Theorem I.4.4.3 nor the Strictification Theorems I.5.4.6 and I.5.4.7.

Part II.1: Braided Bimonoidal Categories

Chapter II.1: Preliminaries on Braided Structures

To prepare for the rest of Part II.1, this chapter discusses the braid groups and braided monoidal categories. First it defines the braid groups and proves some useful properties for sum braids and block braids. Then it reviews braided monoidal categories and proves some basic properties, including two manifestations of the third Reidemeister move. Next it proves in detail that the Drinfeld center of a monoidal category is a braided monoidal category and that the symmetric center of a braided monoidal category is a symmetric monoidal category. Then it recalls the Joyal-Street Coherence Theorem II.1.6.3 for braided monoidal categories.

Chapter II.2: Braided Bimonoidal Categories

This chapter defines braided bimonoidal categories. They are defined using 12 of the 24 Laplaza axioms of a symmetric bimonoidal category, together with two additional axioms that are variants of the only two Laplaza axioms involving the braiding $\xi^{\otimes}$. In a symmetric bimonoidal category, each of these two variant axioms is equivalent to the original Laplaza axiom. This is reminiscent of the fact that a braided monoidal category has two hexagon axioms, which are equivalent to each other in a symmetric monoidal category. A *tight* braided bimonoidal category—that is, one with invertible distributivity morphisms δ^l and δ^r—is equivalent to a BD category in the sense of Blass and Gurevich [**BG20a**]. The first main observation in this chapter is Theorem II.2.2.1, which says that each braided bimonoidal category satisfies all 24 Laplaza axioms. Therefore, a symmetric bimonoidal category is precisely a braided bimonoidal category whose braiding satisfies the symmetry axiom. The second main result in this chapter says that an abelian category with a compatible (symmetric/braided) monoidal structure is a tight (symmetric/braided) bimonoidal category. The additive structure comes from the abelian structure, and the multiplicative structure comes from the monoidal structure. The braided case of this result is due to Blass and Gurevich [**BG20a**].

Chapter II.3: Applications to Quantum Groups and Topological Quantum Computation

This chapter shows that braided bimonoidal categories arise naturally in quantum groups and topological quantum computation (TQC). The first main observation is Theorem II.3.2.19. It says that for a (symmetric/braided) bialgebra A, the category $\mathsf{Mod}(A)$ of left A-modules, equipped with the usual direct sum and tensor product, is a tight (symmetric/braided) bimonoidal category. This is an extension of the important fact in quantum group theory that, for a braided bialgebra A, $\mathsf{Mod}(A)$ is a braided monoidal category. Next we prove in detail that Fibonacci anyons and Ising anyons, which are two central models in TQC, are both tight braided bimonoidal categories. In each case, the additive structure comes from an abelian category structure, and the multiplicative structure comes from the fusion rules of anyons.

Chapter II.4: Bimonoidal Centers

This chapter generalizes the Drinfeld center of a monoidal category and the symmetric center of a braided monoidal category (Sections II.1.4 and II.1.5) to the bimonoidal setting. Generalizing the Drinfeld center, Theorem II.4.4.3 says that, for each tight bimonoidal category C, the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a tight braided bimonoidal category. Tightness is required for this theorem because the invertibility of δ^l and δ^r is used in the construction of $\overline{\mathsf{C}}^{\mathsf{bi}}$. The proof of this theorem is another good illustration of the axioms of a braided bimonoidal category, since we will use all 24 Laplaza axioms and the two variant axioms in the braided case. Generalizing the symmetric center, Theorem II.4.5.3 says that, for each braided bimonoidal category C, the bimonoidal symmetric center $\mathsf{C}^{\mathsf{sym}}$ is a symmetric bimonoidal category.

Chapter II.5: Coherence of Braided Bimonoidal Categories

This chapter proves the Coherence Theorem II.5.4.4 for braided bimonoidal categories that satisfy a monomorphism assumption. As in the symmetric case (Theorems I.3.9.1 and I.4.4.3), the monomorphism assumption in Theorem II.5.4.4 is automatically satisfied if tightness is assumed. This theorem is the braided analogue of Laplaza's Second Coherence Theorem I.4.4.3. It uses a braided version $\mathcal{D}^{\mathsf{br}}$ of the distortion category that involves the symmetric groups and the braid groups to keep track of, respectively, the additive symmetry $\xi^{\oplus}$ and the braiding $\xi^{\otimes}$. Reminiscent of the Joyal-Street Coherence Theorem II.1.6.3 for braided monoidal categories, Theorem II.5.4.4 says that, if two paths have the same image in the braided distortion category $\mathcal{D}^{\mathsf{br}}$, then they have the same value in the braided bimonoidal category in question. This condition of having the same image in $\mathcal{D}^{\mathsf{br}}$ is very much checkable in practice. In fact, the proofs of the main results in Chapters II.6, II.7, and II.8 all use Theorem II.5.4.4 and involve checking this condition many times. In [**BG20a**], Blass and Gurevich conjectured the existence of a coherence theorem for their BD categories, which are equivalent to our tight braided bimonoidal categories. Theorem II.5.4.4 confirms the Blass-Gurevich Conjecture in the form of commutative formal diagrams.

Chapter II.6: Strictification of Tight Braided Bimonoidal Categories

This chapter proves two Strictification Theorems II.6.3.6 and II.6.3.7 for tight braided bimonoidal categories. As in the symmetric case (Theorems I.5.4.6 and I.5.4.7), strictification requires the tightness assumption because the construction of the strictification uses the invertibility of the distributivity morphisms δ^l and δ^r. A *right permbraided category* is a tight braided bimonoidal category with both the additive and the multiplicative structures strict monoidal, and with identities for the structure morphisms $\lambda^\bullet$, $\rho^\bullet$, δ^r, $\xi^\otimes_{-,\mathbb{0}}$, and $\xi^\otimes_{\mathbb{0},-}$. Theorem II.6.3.6 says that each tight braided bimonoidal category is adjoint equivalent to a right permbraided category via strong braided bimonoidal functors. Theorem II.6.3.7 is the analogue that strictifies each tight braided bimonoidal category to a *left* permbraided category, in which δ^l, instead of δ^r, is the identity. Theorems II.6.3.6 and II.6.3.7 are two further positive answers to the Blass-Gurevich Conjecture [**BG20a**] in the form of strictification.

Chapter II.7: The Braided Baez Conjecture

This chapter proves the braided version of Baez's Conjecture. Section II.7.1 defines the 2-category $\mathsf{Bi}^{\mathsf{fbr}}_{\mathsf{r}}$ with *flat* small braided bimonoidal categories as objects. As in the symmetric case, flatness (Definition II.5.4.5) is much weaker than tightness, and it guarantees that the Braided Bimonoidal Coherence Theorem II.5.4.4 is applicable. The first version of the Braided Baez Conjecture (Theorem II.7.3.4) says that the finite ordinal category Σ is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}^{\mathsf{fbr}}_{\mathsf{r}}$. Another version is Theorem II.7.3.6, which says that the variant Σ' of Σ is also such a lax bicolimit. Also like the symmetric case, the proofs of the Braided Baez Conjecture do *not* use the Strictification Theorems II.6.3.6 and II.6.3.7. This allows us to use flat small braided bimonoidal categories in the 2-category $\mathsf{Bi}^{\mathsf{fbr}}_{\mathsf{r}}$, instead of the smaller class of tight ones. The reader may wonder why the finite ordinal category Σ and its variant Σ' are bi-initial in both the symmetric case (Theorems I.7.8.1 and I.7.8.3) and the braided case. This is analogous to the fact that the ring of integers is initial in both the category of rings and the category of commutative rings.

Chapter II.8: Monoidal Bicategorification

The main Theorem II.8.4.7 in this chapter says that, for each tight braided bimonoidal category C, the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory. While most of the definitions for $\mathsf{Mat}^{\mathsf{C}}$ are the same as in the symmetric case in Chapter I.8, there are two subtleties. First, in the current braided case, the lax functoriality constraint $\boxtimes^2$ of the monoidal composition $\boxtimes$ in $\mathsf{Mat}^{\mathsf{C}}$ has two additional conditions about the braided distortions of the two paths involved; see (II.8.2.15) and (II.8.2.16). These conditions about the braided distortions are necessary because a braid is not determined by its underlying permutation, and the braided distortion category $\mathcal{D}^{\mathsf{br}}$ involves the braid groups. The second subtle point is that, even if C is a tight braided bimonoidal category, the monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$ does *not* seem to have any reasonable braided monoidal bicategory structure in general. We will explain this point in more detail near the end of Section II.8.4. The difficulty once again comes from the fact that the braided distortion category $\mathcal{D}^{\mathsf{br}}$ involves the braid groups, and a braid with an identity underlying permutation is usually not the identity braid.

Part II.2: E_n-Monoidal Categories

Chapter II.9: Ring, Bipermutative, and Braided Ring Categories

This chapter discusses ring and bipermutative categories in the sense of Elmendorf-Mandell and the braided version. The main difference between these categorical notions and their bimonoidal counterparts in Parts I.1 and II.1 is that ring categories have generally non-invertible *factorization morphisms*

$$\begin{aligned}(A \otimes C) \oplus (B \otimes C) &\xrightarrow{\partial^l_{A,B,C}} (A \oplus B) \otimes C\\ (A \otimes B) \oplus (A \otimes C) &\xrightarrow{\partial^r_{A,B,C}} A \otimes (B \oplus C)\end{aligned}$$

that go in the opposite direction as the distributivity morphisms δ^r and δ^l. Ring categories with invertible factorization morphisms are special cases of tight bimonoidal categories, so the latter's strictification theorems in Chapter I.5 also apply to such ring categories. The bipermutative and braided analogues are also true. Similar to the endomorphism rig of a commutative monoid, each small permutative category C yields an endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$. Similar to the reduction of Laplaza's axioms in symmetric bimonoidal categories in Section I.2.2 and the braided version in Theorem II.2.2.1, about half of the ring category axioms are redundant in a bipermutative category and a braided ring category. This is an extension of an observation in [**EM06**, Fig. 1]. Moreover, the Drinfeld center and the symmetric center have natural analogues for these ring-like categories. As we will discuss in Chapters III.11 and III.12, the Elmendorf-Mandell K-theory multifunctor sends small ring, braided ring, and bipermutative categories to, respectively, strict ring, E_2-, and E_∞-symmetric spectra. The strict ring and E_∞ cases are due to Elmendorf-Mandell [**EM06, EM09**], and the E_2 case is new.

Chapter II.10: Iterated and E_n-Monoidal Categories

Keeping in mind that the ring-like categories in Chapter II.9 correspond to E_n-symmetric spectra for $n \in \{1, 2, \infty\}$ via algebriac K-theory, this chapter discusses the categorical structure for the general E_n cases. An n-fold monoidal category in the sense of [**BFSV03**] has n monoidal structures $\otimes_1, \ldots, \otimes_n$ that are strictly associative and unital and interact via the exchange natural transformations

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D)$$

for $1 \leq i < j \leq n$. Monoids in the monoidal category of small n-fold monoidal categories are precisely small $(n+1)$-fold monoidal categories. We introduce the notion of an *E_n-monoidal category* as a permutative category $(\mathsf{C}, \oplus)$ equipped with an n-fold monoidal structure $\{\otimes_i, \eta^{i,j}\}$ and factorization morphisms $\{\partial^{l,i}, \partial^{r,i}\}$ for each monoidal structure $\otimes_i$, such that (i) each $(\oplus, \otimes_i, \partial^{l,i}, \partial^{r,i})$ is a ring category and (ii) several axioms relating $\eta^{i,j}$, $\partial^{l,i}$, and $\partial^{r,i}$ hold. Ring categories are E_1-monoidal categories. Braided ring categories and bipermutative categories are special cases of, respectively, E_2- and E_n-monoidal categories for $n \geq 2$. Moreover, each small category generates a free E_n-monoidal category. In Chapter III.13, we will show that the Elmendorf-Mandell K-theory of a small E_n-monoidal category is an E_n-symmetric spectrum for $n \geq 1$.

Part III.1: Enriched Monoidal Categories and Multicategories

Chapter III.1: Enriched Monoidal Categories

This chapter gives the basic definitions and properties for enriched monoidal categories, including plain, braided, and symmetric variants. Definition III.1.4.25 describes 2-categories of each, with 1- and 2-cells given by appropriately monoidal enriched functors and natural transformations, respectively. For our applications to K-theory in Part III.2, the enriching category V is symmetric monoidal closed. However, our treatment in this chapter addresses the more general case that V is merely monoidal, with additional assumptions about braided or symmetric monoidal structure stated as necessary.

Section III.1.5 discusses the important special case $\mathsf{V} = \mathsf{Cat}$, the category of small categories with its Cartesian product. Explanation III.1.5.3 describes how the monoidal V-categories in this case are strict versions of monoidal bicategories. The braided and symmetric cases are similarly compared.

Chapter III.2: Change of Enrichment

This chapter describes change of enriching category induced by a symmetric monoidal functor, showing that monoidal structures are preserved. Sections III.2.1 through III.2.4 give a thorough treatment of 2-functoriality results. As an application, Corollary III.2.4.17 shows that taking underlying categories gives a 2-functor from small monoidal V-categories, V-functors, and V-natural transformations to ordinary monoidal categories, functors, and natural transformations. Similar statements hold for the braided and symmetric cases.

A partial reverse of Corollary III.2.4.17 is given in Theorem III.2.5.1. The theorem shows that, for given V-enriched data, various enriched monoidal axioms are satisfied if and only if the corresponding monoidal axioms for the underlying data are satisfied. This provides a mechanism to lift ordinary monoidal structures to enriched monoidal structures.

Sections III.2.5 and III.2.6 apply Theorem III.2.5.1 to lift coherence and strictification results for ordinary monoidal, braided, and symmetric monoidal categories to their enriched counterparts. The Enriched Monoidal Coherence Theorem III.2.5.6 and Enriched Epstein's Coherence Theorem III.2.5.8 play a significant role in subsequent chapters.

Chapter III.3: Self-Enrichment and Enriched Yoneda

This chapter restricts to the case that V is a symmetric monoidal closed category. Theorem III.3.3.2 shows, via Theorem III.2.5.1, that the canonical enrichment of V over itself is symmetric monoidal as a V-category. The next several sections develop the theory of V-enriched co/ends followed by the V-Yoneda Lemma (Theorem III.3.6.9) and an equivalent form called the V-Yoneda Density Theorem III.3.7.8. These are applied to develop the Day convolution and internal hom for enriched diagram categories (Theorem III.3.7.22). The remainder of the chapter discusses additional theory of enriched diagram categories and tensor/cotensor structures that will be important for the development of enriched K-theory functors in Part III.2.

Chapter III.4: Pointed Objects, Smash Products, and Pointed Homs

This chapter gives the definitions and properties of smash products and pointed homs. These will be used throughout Part III.2, and the smash product of pointed multicategories, developed in Chapter III.5, will be particularly important.

Section III.4.3 uses the Day convolution and internal hom to develop symmetric monoidal closed structure for pointed diagram categories. The results are summarized in Theorem III.4.3.37. Applications of this material appear in Chapters III.8, III.9, and III.10, where the Segal and Elmendorf-Mandell K-theory constructions are given via certain pointed diagram categories.

Chapter III.5: Multicategories

This chapter gives relevant background on multicategories, multifunctors, and multinatural transformations. Theorem III.5.5.14 shows that the category of small multicategories is complete and cocomplete. The Boardman-Vogt tensor product of multicategories, and the associated smash product for pointed multicategories, are developed in Section III.5.6. The corresponding internal hom and its pointed variant are developed in Section III.5.7.

Chapter III.6: Enriched Multicategories

This chapter develops basic definitions and properties for enriched multicategories. One of our important applications, developed in Section III.6.3, is the enriched multicategory associated to an enriched symmetric monoidal category. Our first use of this is in Section III.6.4 where we describe the Cat-enriched multicategory structure on Multicat, the category of small multicategories. It is induced by showing that the tensor product makes Multicat symmetric monoidal as a Cat-enriched category (Theorem III.6.4.3). The pointed variant, with the smash product of small pointed multicategories, is given in Theorem III.6.4.4 and will be essential for Part III.2.

Sections III.6.5 and III.6.6 cover our second important application of enriched multicategories. The category $\mathsf{PermCat}^{\mathsf{su}}$, consisting of small permutative categories and strictly unital symmetric monoidal functors, has a Cat-enriched multicategory structure given by multilinear functors and multilinear transformations (Definitions III.6.5.4 and III.6.5.11). Propositions III.6.5.10 and III.6.5.13 show that this Cat-enriched multicategory structure is induced from that of small pointed multicategories and their smash product. Section III.6.6 gives a second, direct proof of the Cat-enriched multicategory axioms.

Part III.2: Algebraic *K*-Theory

Chapter III.7: Homotopy Theory Background

This chapter gives relevant background from homotopy theory. Sections III.7.1 and III.7.2 introduce simplicial sets and simplicial homotopy, along with the nerve and geometric realization functors. The category of symmetric spectra, with its symmetric monoidal closed structure, is presented in Sections III.7.3 through III.7.6. Then, Sections III.7.7 and III.7.8 give a short review of Quillen model categories and a number of key examples.

Chapter III.8: Segal K-Theory of Permutative Categories

This chapter presents the K-theory functor K^{Se} due to Segal [**Seg74**]. Its inputs are small permutative categories and its outputs are symmetric spectra. Section III.8.3 describes the key construction as given by Segal. Sections III.8.4

and III.8.5 describe an equivalent construction that compares more readily with the K-theory multifunctor of Elmendorf-Mandell, K^{EM}.

Chapter III.9: Categories of $\mathcal{G}_*$-Objects

This chapter is the first of two that replace the Segal K-theory functor with a simplicially-enriched multifunctor due to Elmendorf-Mandell [**EM06, EM09**]. This chapter focuses on the replacement of Γ-categories and Γ-simplicial sets with pointed diagrams out of a larger indexing category $\mathcal{G}$. The construction of symmetric spectra from such diagram categories is given in Section III.9.3 and is denoted $\mathsf{K}^{\mathcal{G}}$. Sections III.9.2 and III.9.4 use the material from Part III.1 to explain that the new diagram categories and the new functor $\mathsf{K}^{\mathcal{G}}$ are symmetric monoidal, in the enriched sense of Chapter III.1, over the category of pointed simplicial sets.

Chapter III.10: Elmendorf-Mandell K-Theory of Permutative Categories

This chapter is the second of two that replace the Segal K-theory functor with a simplicially enriched multifunctor due to Elmendorf-Mandell [**EM06, EM09**]. This chapter focuses on the construction of $\mathcal{G}_*$-categories from small permutative categories, replacing Segal's construction of Γ-categories from the same. Additional material from Part III.1 is used throughout the chapter to explain that the multi/categories and multi/functors are enriched either in the symmetric monoidal sense of Chapter III.1 or in the multicategorical and multifunctorial sense of Chapter III.6. Section III.10.6 contains the proof that the Segal and Elmendorf-Mandell K-theory symmetric spectra associated to a small permutative category C are level equivalent (Theorem III.10.6.10). Because K^{EM} is an enriched multifunctor, it preserves operad actions. We state this result as Theorem III.10.3.33 and apply it in Chapters III.11, III.12, and III.13.

Chapter III.11: K-Theory of Ring and Bipermutative Categories

This is the first of three chapters that contain algebraic K-theory applications of the ring-like categories in Part II.2. The main K-theory results in this chapter, Corollaries III.11.3.16 and III.11.6.12, are from [**EM06, EM09**], and they are the E_1 and the E_∞ cases. These results state that the Elmendorf-Mandell K-theory multifunctor K^{EM} sends (i) small ring categories to strict ring symmetric spectra and (ii) small bipermutative categories to E_∞-symmetric spectra. They are obtained by combining the multifunctor K^{EM} and the fact that the associative operad and the Barratt-Eccles operad parametrize, respectively, ring and bipermutative category structures on small permutative categories. Since the associative operad has monoids as algebras and the Barratt-Eccles operad is an E_∞-operad, the K-theory results follow.

Chapter III.12: K-Theory of Braided Ring Categories

This chapter contains the E_2 analogues of the results in Chapter III.11. The first part of this chapter discusses the braid operad Br, which generalizes the Barratt-Eccles operad. This is a categorical E_2-operad (Theorem III.12.2.4) whose algebras in Cat are small braided strict monoidal categories (Proposition III.12.3.22). The main categorical input is Theorem III.12.4.5, which says that Br parametrizes braided ring category structures, as in Chapter II.9, on small permutative categories. Applying the K-theory multifunctor K^{EM}, it follows that K^{EM} sends small braided ring categories to E_2-symmetric spectra (Corollary III.12.5.3). The K-theory result, Corollary III.12.5.3, and the main categorical input, Theorem III.12.4.5, are new results.

Chapter III.13: K-Theory of E_n-Monoidal Categories

This chapter contains the general E_n analogues for $n \geq 1$ of the categorical and K-theory results in Chapters III.11 and III.12. The first part of this chapter discusses the n-fold monoidal category operad Mon^n. This is a categorical E_n-operad (Theorem III.13.2.1) whose algebras in Cat are small n-fold monoidal categories (Proposition III.13.3.18) as in Chapter II.10. The main categorical input is Theorem III.13.4.12, which says that Mon^n parametrizes E_n-monoidal category structures on small permutative categories. Applying the K-theory multifunctor K^{EM}, it follows that K^{EM} sends small E_n-monoidal categories to E_n-symmetric spectra for $n \geq 1$ (Corollary III.13.5.2). As in Chapter III.12, the K-theory result, Corollary III.13.5.2, and the main categorical input, Theorem III.13.4.12, are new results.

Appendix III.A: Open Questions

This chapter discusses open questions related to the topics of this work. We encourage the reader to read these open questions at any time and use them as additional motivation for the main text.

Reading Guides

Supplementing the chapter introductions and individual reading guides therein, the following guides describe themes that span multiple chapters. For especially broad topics, we include a selection of general references for background or further reading. The Notes section at the end of each chapter provides additional references relevant to the content of that chapter.

Category Theory. For a refresher of basic category theory, including braided and symmetric monoidal categories, read Chapters I.1 and II.1. For bicategories and 2-categories, read Chapter I.6. For abelian categories, read Section II.2.3. For enriched category theory, read Section III.1.2 and Chapter III.3. Bimonoidal categories are built upon monoidal categories. Thus, a thorough understanding of the definitions and coherence of monoidal categories is necessary to understand bimonoidal categories and their coherence.

References for basic category theory include [**Awo10, BK00, Gra18, Lei14, Rie16, Rom17, Sim11**]. References for more advanced category theory include [**Bor94a, Bor94b, Bor94c, ML98, Mit65, Sch72**]. References for Abelian categories include [**EGNO15, Fre03, Mit65**]. References for enriched categories include [**Bor94b, Cru09, For04, Kel05**]. References for ends and coends include [**Day70, DK69, Lor21**]. For further reference on 2-dimensional categories, we highly recommend [**JY21**].

Symmetric Bimonoidal Categories. To review the axioms of a symmetric bimonoidal category, read Sections I.2.1 and I.2.2. For Laplaza's Coherence Theorems for symmetric bimonoidal categories, read Theorems I.3.9.1 and I.4.4.3. For strictification theorems, read Theorems I.5.4.6, I.5.4.7, I.5.5.11, and I.5.5.12. For their bimonoidal analogues, read Theorems I.3.10.7 and I.4.5.8. For Baez's Conjecture, read Definition I.7.1.8 and Theorem I.7.8.1. The introductions of Chapters I.3, I.4, I.5, and I.7 have more detailed description and reading suggestions. This material on (symmetric) bimonoidal categories is used extensively in Volume II in the discussion of braided bimonoidal categories.

Braided Bimonoidal Categories. To review the axioms of a braided bimonoidal category, read Definition II.2.1.29 and Theorem II.2.2.1. For their coherence and strictification (i.e., the Blass-Gurevich Conjecture), read Theorems II.5.4.4, II.6.3.6, and II.6.3.7. For the braided version of Baez's Conjecture, read Definition II.7.1.5 and Theorem II.7.3.4. The introductions of these chapters have more detailed discussion and reading guides. This material on braided bimonoidal categories is not heavily used in Volume III in our discussion of algebraic K-theory. Instead, these braided structures provide a great illustration of the coherence theory in Volume I and have many applications in the sciences, some of which are discussed in this work.

Applications to Quantum Groups, TQC, and Programming. For applications to quantum group theory, read Sections II.3.1 and II.3.2. For applications to topological quantum computation, read Theorem II.2.4.22 and Sections II.3.3 through II.3.6. For applications to reversible programming, read Section I.2.6. Our treatment is self-contained and assumes no prior knowledge of these topics. These applications are not used in Volume III. Instead, they illustrate the much larger scientific context where categorical structures discussed in this text are applied.

Enriched Monoidal Categories. To review enriched monoidal categories, read Sections III.1.3 through III.1.5. For their coherence theory, read Sections III.2.5 and III.2.6. For change of enrichment, read Proposition III.2.1.2, Theorems III.2.2.7, III.2.3.7, and III.2.4.10, and Corollary III.2.4.17. For symmetric monoidal closed structures in the pointed context, read Theorems III.4.1.8, III.4.2.3, III.4.3.19, and III.4.3.37. Much of this material is known to experts, but it is not easily accessible in the literature in a unified format. Our discussion is self-contained and highly detailed, so the reader can thoroughly learn these topics just from our text. This material is necessary to understand the intricate multicategorical properties of Elmendorf-Mandell K-theory discussed in later chapters.

Multicategories and Operads. To review enriched multicategories, read Section III.6.1. The Boardman-Vogt tensor product of multicategories and its pointed variant are discussed in Sections III.5.6 and III.6.4. For the passage from enriched symmetric monoidal categories to enriched multicategories, read Sections III.5.3 and III.6.3. For the categorically-enriched multicategory of small permutative categories, read Sections III.6.5 and III.6.6. For the associative operad, the Barratt-Eccles operad, the braid operad, and the n-fold monoidal category operad, read Definitions III.11.1.1, III.11.4.10, III.12.1.2, and III.13.1.12. Elmendorf-Mandell K-theory is an enriched multifunctor, which can transport operadic algebras in small permutative categories to the same type of operadic algebras in symmetric spectra. This material is necessary to describe the enriched multifunctoriality of Elmendorf-Mandell K-theory and its applications to highly structured ring spectra. References for multicategories and operads include **[Fre17, MSS02, May72, Yau16]**.

Ring, Bipermutative, Braided Ring, and E_n-Monoidal Categories. For the definitions of ring, bipermutative, braided ring, and E_n-monoidal categories, read Definitions II.9.1.2, II.9.3.2, II.9.5.1, II.10.1.1, and II.10.7.2. For their coherence theory, read Corollaries II.9.1.19, II.9.1.20, II.9.3.12, and II.9.3.13, Theorem II.10.6.8, and Question III.A.2.1. For their description in terms of operads, read Theorems III.11.2.16, III.11.5.5, III.12.4.5, and III.13.4.12. The fact that these monoidal categories with extra structures are algebras over operads is the precise reason why they are sent by Elmendorf-Mandell K-theory to structured ring spectra of the corresponding types.

Basic Homotopy Theory. For a brief review of simplicial objects and the nerve construction, read Sections III.7.1 and III.7.2. To review symmetric spectra, read Sections III.7.3 and III.7.4. For the smash product of symmetric spectra, read Sections III.7.5 and III.7.6. Model category theory is reviewed in Sections III.7.7 and III.7.8, but later chapters do not use model categories in any way. While we assume some basic homotopy theory, we do not assume any prior knowledge of symmetric spectra. Our discussion of symmetric spectra is very gentle and contains a lot of details that are not explicitly available elsewhere in the literature. The main point of both Segal K-theory and Elmendorf-Mandell K-theory is the construction of symmetric spectra from purely categorical data. A detailed understanding of symmetric spectra and their smash product is necessary to fully appreciate the K-theory constructions of Segal and Elmendorf-Mandell.

References for homotopy theory include [**BR20, May99, MP12, Mil20, Ric20, Rie14**]. References for simplicial homotopy theory include [**Cur71, GZ67, GJ09, May92**]. References for further background and a broader perspective on algebraic K-theory include [**Mil71, Qui73, Ros95, Wal85, Wei13**].

Segal K-Theory. To review the passage from Γ-simplicial sets to symmetric spectra, read Definitions III.7.4.5, III.8.1.8, and III.8.2.5. For the passage from small permutative categories to Γ-categories, read Definitions III.8.1.17, III.8.3.1, III.8.3.6, III.8.3.9, and III.8.3.12 and Proposition III.8.4.8. To review Segal's K-theory construction, from small permutative categories to symmetric spectra via Γ-categories, read Definitions III.8.4.10 and III.8.5.1 and Theorem III.8.5.2. Although Segal K-theory does not generally preserve multiplicative structures, it is a fundamental tool in this subject and the main motivation for Elmendorf-Mandell K-theory.

Elmendorf-Mandell K-Theory. To review the passage from $\mathcal{G}_*$-simplicial sets to symmetric spectra, read Sections III.9.1 through III.9.3. Its symmetric monoidality is discussed in Theorem III.9.4.9. For the passage from small permutative categories to $\mathcal{G}_*$-categories, read Section III.10.4. For Elmendorf-Mandell K-theory, from small permutative categories to symmetric spectra via $\mathcal{G}_*$-categories, read Lemma III.10.2.14 and Corollary III.10.3.24. For the comparison between Segal K-theory and Elmendorf-Mandell K-theory, read Theorem III.10.6.10. For the passage from categorical data to highly structured ring spectra via Elmendorf-Mandell K-theory, read Theorem III.10.3.33 and Corollaries III.11.3.16, III.12.5.3, and III.13.5.2. This material is the main point of Volume III; all preceding chapters of Volume III are preparation for it.

Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory

Volume I: Symmetric Bimonoidal Categories and Monoidal Bicategories

Donald Yau

The author dedicates this book to Jacqueline.

Part 1

Symmetric Bimonoidal Categories

CHAPTER 1

Basic Category Theory

In this preliminary chapter, we review basic concepts and results about categories and monoidal categories. This chapter contains only definitions, examples, and statements of facts. For proofs and more detailed discussion of basic category theory, the reader may consult the references in Section 1.4. In Section 1.1, we recall the definitions of categories, functors, natural transformations, adjunctions, equivalences, (co)limits, (co)ends, and Kan extensions.

In Section 1.2, we recall (symmetric) monoidal categories, (symmetric) monoidal functors, and monoidal natural transformations. Monoidal categories are important for this work because essentially all subsequent chapters are based on the concept of a monoidal structure. There will be more discussion of category theory later in this work, including

- Chapter 6 on bicategories and 2-categories,
- Chapter II.1 on braided monoidal categories,
- Section II.2.3 on abelian categories, and
- Part III.1 on enriched monoidal categories and multicategories.

In Section 1.3, we recall several coherence theorems for (symmetric) monoidal categories and functors. The Coherence Theorems 1.3.3 and 1.3.8 for (symmetric) monoidal categories will be used in Chapters 3, 4, II.5, and III.2 to prove

- the Coherence Theorems 3.10.7 and 4.5.8 for bimonoidal categories,
- the Coherence Theorems 3.9.1 and 4.4.3 for symmetric bimonoidal categories,
- the Coherence Theorem II.5.4.4 for braided bimonoidal categories, and
- coherence theorems for enriched monoidal categories.

Epstein's Coherence Theorem 1.3.12 will be used in

- Chapter 7 in the proof of Baez's Conjecture and
- Chapter III.3 in the discussion of the standard enrichment of a symmetric monoidal functor.

An enriched version of this theorem is Theorem III.2.5.8. Sections III.2.2 and III.2.3 contain coherence theorems for change of enrichment.

1.1. Categories

In this section, we fix some notations for categories, functors, natural transformations, adjunctions, equivalences, (co)limits, (co)ends, and Kan extensions.

Categories, Functors, and Natural Transformations.

Definition 1.1.1. A *category* C consists of

- a class of *objects* $\mathsf{Ob}(\mathsf{C})$;

- for each pair of objects X and Y in C, a set of *morphisms* $\mathsf{C}(X, Y)$, which is also denoted by $\mathsf{C}(X; Y)$, with *domain* X and *codomain* Y;
- for each object X in C, an *identity morphism* $1_X \in \mathsf{C}(X, X)$, which is also denoted by id_X; and
- a function

$$\begin{aligned} \mathsf{C}(Y, Z) \times \mathsf{C}(X, Y) &\longrightarrow \mathsf{C}(X, Z) \\ (g, f) &\longmapsto g \circ f = gf, \end{aligned}$$

which is called the *composition*, for objects X, Y, and Z.

These data are required to satisfy the following two conditions.

Unity: $1_Y f = f = f 1_X$ for morphisms $f \in \mathsf{C}(X, Y)$.

Associativity: $(hg)f = h(gf)$ for $(h, g, f) \in \mathsf{C}(Z, W) \times \mathsf{C}(Y, Z) \times \mathsf{C}(X, Y)$.

Moreover, we define the following.

- A category is also called a 1-*category*.
- A category is *small* if it has a set of objects.
- A category is *finite* if it has only finitely many objects and morphisms.
- We usually abbreviate $X \in \mathsf{Ob}(\mathsf{C})$ to $X \in \mathsf{C}$, and denote a morphism $f \in \mathsf{C}(X, Y)$ by $f : X \longrightarrow Y$.
- If gf is defined for two morphisms f and g, then it is called the *composite*.

◇

Definition 1.1.2. For a category C, a *subcategory* D is a category consisting of

- a subcalss $\mathsf{Ob}(\mathsf{D}) \subseteq \mathsf{Ob}(\mathsf{C})$ of objects, and
- a subset $\mathsf{D}(X, Y) \subseteq \mathsf{C}(X, Y)$ of morphisms for objects $X, Y \in \mathsf{D}$

such that the identity morphisms and composition in D are restricted from those in C. A subcategory D of C is *full* if each subset inclusion $\mathsf{D}(X, Y) \subseteq \mathsf{C}(X, Y)$ is an equality. ◇

Definition 1.1.3. Suppose $f : X \longrightarrow Y$ is a morphism in a category C.

- f is called an *isomorphism* if there exists a morphism $g : Y \longrightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$. An isomorphism is sometimes denoted by $X \xrightarrow{\cong} Y$. A category in which every morphism is an isomorphism is called a *groupoid*.
- f is called a *monomorphism* if for any pair of morphisms $g, h : W \longrightarrow X$ with codomain X and a common domain,

$$fg = fh \quad \text{implies} \quad g = h.$$

  In other words, f is cancellable on the left.
- f is called an *epimorphism* if for any pair of morphisms $g', h' : Y \longrightarrow Z$ with domain Y and a common codomain,

$$g'f = h'f \quad \text{implies} \quad g' = h'.$$

  In other words, f is cancellable on the right. ◇

Definition 1.1.4. For categories C and D, a *functor* $F : \mathsf{C} \longrightarrow \mathsf{D}$ consists of

- an assignment on objects

$$\begin{aligned} \mathsf{Ob}(\mathsf{C}) &\longrightarrow \mathsf{Ob}(\mathsf{D}) \\ X &\longmapsto FX \end{aligned}$$

and

- an assignment on morphisms

$$\begin{aligned} \mathsf{C}(X,Y) &\longrightarrow \mathsf{D}(FX,FY) \\ f &\longmapsto Ff. \end{aligned}$$

These data are required to satisfy the following two conditions.

Composition: $F(gf) = (Fg)(Ff)$, provided that the composite gf is defined.
Identities: $F1_X = 1_{FX}$ for each object $X \in \mathsf{C}$.

Moreover, we define the following.

- Functors are composed by composing the assignments on objects and on morphisms.
- The *identity functor* of a category C is determined by the identity assignments on objects and morphisms, and is written as either Id_{C} or 1_{C}. ◇

Example 1.1.5. We let **1** denote the terminal category; it has a unique object $*$ and unique morphism 1_*. ◇

Example 1.1.6. We write Cat for the category with small categories as objects, functors as morphisms, identity functors as identity morphisms, and composition of functors as composition. ◇

Definition 1.1.7. Suppose $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ are functors. A *natural transformation* $\theta : F \longrightarrow G$ consists of a morphism $\theta_X : FX \longrightarrow GX$ in D for each object $X \in \mathsf{C}$ such that the diagram

$$\begin{tikzcd} FX \arrow[r, "\theta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\theta_Y"] & GY \end{tikzcd}$$

in D is commutative for each morphism $f : X \longrightarrow Y$ in C.

Moreover, we define the following.

- Each morphism θ_X is called a *component* of θ.
- The *identity natural transformation* $1_F : F \longrightarrow F$ of a functor F has each component an identity morphism.
- A *natural isomorphism* is a natural transformation in which every component is an isomorphism.
- A *natural monomorphism* is a natural transformation in which every component is a monomorphism. ◇

Definition 1.1.8. Suppose $\theta : F \longrightarrow G$ is a natural transformation for functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$.

- Suppose $\phi : G \longrightarrow H$ is a natural transformation for another functor $H : \mathsf{C} \longrightarrow \mathsf{D}$. The *vertical composition*

$$\phi\theta : F \longrightarrow H$$

is the natural transformation with components

$$(\phi\theta)_X = \phi_X \circ \theta_X : FX \longrightarrow HX \quad \text{for} \quad X \in \mathsf{C}.$$

- Suppose $\theta' : F' \longrightarrow G'$ is a natural transformation for functors $F', G' : \mathsf{D} \longrightarrow \mathsf{E}$. The *horizontal composition*

$$\theta' * \theta : F'F \longrightarrow G'G$$

is the natural transformation whose component $(\theta' * \theta)_X$ for an object $X \in \mathsf{C}$ is defined as either composite in the commutative diagram

$$\begin{tikzcd} F'FX \arrow[r, "\theta'_{FX}"] \arrow[d, "F'\theta_X"'] & G'FX \arrow[d, "G'\theta_X"] \\ F'GX \arrow[r, "\theta'_{GX}"] & G'GX \end{tikzcd}$$

in D. ◇

Adjunctions and Equivalences.

Definition 1.1.9. For categories C and D, an *adjunction* from C to D is a triple (L, R, ϕ) consisting of

- a pair of functors

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "L"] & \mathsf{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

  and
- a bijection

$$\mathsf{D}(LX, Y) \xrightarrow[\cong]{\phi_{X,Y}} \mathsf{C}(X, RY)$$

  that is natural in the objects $X \in \mathsf{C}$ and $Y \in \mathsf{D}$.

Such an adjunction is also called an *adjoint pair*, with L the *left adjoint* and R the *right adjoint*. We also denote such an adjunction by $L \dashv R$. ◇

Unless otherwise specified, we always display the left adjoint on top, pointing to the right. In an adjunction $L \dashv R$ as above, setting $Y = LX$ or $X = RY$, the natural bijection ϕ yields natural transformations

$$\begin{aligned} 1_{\mathsf{C}} &\xrightarrow{\eta} RL \\ LR &\xrightarrow{\varepsilon} 1_{\mathsf{D}}, \end{aligned} \tag{1.1.10}$$

which are called, respectively, the *unit* and the *counit*. The vertically composed natural transformations

$$\begin{aligned} R &\xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R \\ L &\xrightarrow{L\eta} LRL \xrightarrow{\varepsilon L} L \end{aligned} \tag{1.1.11}$$

are equal to 1_R and 1_L, respectively. Here

$$\begin{aligned} \eta R &= \eta * 1_R \\ R\varepsilon &= 1_R * \varepsilon \end{aligned}$$

and similarly for $L\eta$ and εL. The identities in (1.1.11) are called the *triangle identities*. Conversely, an adjunction is completely determined by functors (L, R) and natural transformations (η, ε) that satisfy the triangle identities.

Definition 1.1.12. A functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ is called an *equivalence* if there exist

- a functor $G : \mathsf{D} \longrightarrow \mathsf{C}$ and
- natural isomorphisms

$$1_{\mathsf{C}} \xrightarrow[\cong]{\eta} GF \quad \text{and} \quad FG \xrightarrow[\cong]{\varepsilon} 1_{\mathsf{D}}.$$

If, in addition, F is left adjoint to G with unit η and counit ε, then $(F, G, \eta, \varepsilon)$ is called an *adjoint equivalence*. ◇

A functor F is an equivalence if and only if it is both

- *fully faithful*, which means that each function $\mathsf{C}(X,Y) \longrightarrow \mathsf{D}(FX,FY)$ on morphism sets is a bijection, and
- *essentially surjective*, which means that for each object $Y \in \mathsf{D}$, there exists an isomorphism $FX \xrightarrow{\cong} Y$ for some object $X \in \mathsf{C}$.

Colimits and Limits.

Definition 1.1.13. Suppose $F : \mathsf{D} \longrightarrow \mathsf{C}$ is a functor. A *colimit of* F, if it exists, is a pair $(\mathsf{colim}\, F, \delta)$ consisting of

- an object $\mathsf{colim}\, F \in \mathsf{C}$ and
- a morphism $\delta_d : Fd \longrightarrow \mathsf{colim}\, F$ in C for each object $d \in \mathsf{D}$

such that the following two conditions are satisfied.

Naturality: For each morphism $f : d \longrightarrow d'$ in D, the diagram

$$\begin{tikzcd} Fd \arrow[r, "\delta_d"] \arrow[d, "Ff"'] & \mathsf{colim}\, F \arrow[d, equal] \\ Fd' \arrow[r, "\delta_{d'}"] & \mathsf{colim}\, F \end{tikzcd}$$

in C is commutative. A pair $(\mathsf{colim}\, F, \delta)$ with this property is called a *cocone of* F.

Universality: The pair $(\mathsf{colim}\, F, \delta)$ is *universal* among cocones of F. This means that if (X, δ') is another cocone of F, then there exists a unique morphism $h : \mathsf{colim}\, F \longrightarrow X$ in C such that the diagram

$$\begin{tikzcd} Fd \arrow[r, "\delta_d"] \arrow[d, equal] & \mathsf{colim}\, F \arrow[d, "h"] \\ Fd \arrow[r, "\delta'_d"] & X \end{tikzcd}$$

is commutative for each object $d \in \mathsf{D}$. ◇

A *limit of* F, which is denoted by $(\lim F, \delta)$ if it exists, is defined dually by turning the morphisms δ_d for $d \in \mathsf{D}$ and h backward.

- For a functor $F : \mathsf{D} \longrightarrow \mathsf{C}$, its colimit, if it exists, is also denoted by $\mathsf{colim}_{x \in \mathsf{D}}\, Fx$ and $\mathsf{colim}_{\mathsf{D}}\, F$, and similarly for limits.
- A *small (co)limit* is a (co)limit of a functor whose domain category is a small category.
- A *finite (co)limit* is a (co)limit of a functor whose domain category is a finite category.
- A category C is *(co)complete* if it has all small (co)limits.

A left adjoint $F : \mathsf{C} \longrightarrow \mathsf{D}$ preserves all the colimits that exist in C. In other words, if $H : \mathsf{E} \longrightarrow \mathsf{C}$ has a colimit, then $FH : \mathsf{E} \longrightarrow \mathsf{D}$ also has a colimit, and the natural morphism

$$\underset{e \in E}{\mathsf{colim}}\, FHe \longrightarrow F\Big(\underset{e \in E}{\mathsf{colim}}\, He\Big) \tag{1.1.14}$$

is an isomorphism. Similarly, a right adjoint $G : \mathsf{D} \longrightarrow \mathsf{C}$ preserves all the limits that exist in D.

Example 1.1.15. Here are some special types of colimits in a category C.

(1) An *initial object* $\varnothing^{\mathsf{C}}$ in C is a colimit of the functor $\varnothing \longrightarrow \mathsf{C}$, where $\varnothing$ is the empty category with no objects and no morphisms. It is characterized by the universal property that for each object X in C, there is a unique morphism $\varnothing^{\mathsf{C}} \longrightarrow X$ in C.
(2) A *coproduct* is a colimit of a functor whose domain category is a discrete category, that is, a category with only identity morphisms. We use the symbols $\coprod$ and $\amalg$ to denote coproducts.
(3) A *pushout* is a colimit of a functor whose domain category has the form

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

with three objects and two nonidentity morphisms.
(4) A *coequalizer* is a colimit of a functor whose domain category has the form

$$\bullet \rightrightarrows \bullet$$

with two objects and two nonidentity morphisms.

Terminal objects, products, pullbacks, and equalizers are the corresponding limit concepts. A *zero object* is an object that is both initial and terminal. $\diamond$

Coends and Ends. For a category C, its *opposite category* C^{op} is the category with the same objects as C and with morphism sets

$$\mathsf{C}^{\mathsf{op}}(A, B) = \mathsf{C}(B, A).$$

The identity morphisms and composites in C^{op} are defined by those in C.

Definition 1.1.16. Suppose $F : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \longrightarrow \mathsf{D}$ is a functor.

(1) A *cowedge* of F is a pair (X, ζ) consisting of
  - an object $X \in \mathsf{D}$ and
  - morphisms

$$\zeta_c : F(c, c) \longrightarrow X \quad \text{for} \quad c \in \mathsf{C}$$

such that the diagram

$$\begin{tikzcd} F(d,c) \arrow[r, "F(1_d, g)"] \arrow[d, "F(g, 1_c)"'] & F(d,d) \arrow[d, "\zeta_d"] \\ F(c,c) \arrow[r, "\zeta_c"] & X \end{tikzcd}$$

is commutative for each morphism $g : c \longrightarrow d \in \mathsf{C}$.
(2) A *coend* of F is an initial cowedge $\left(\int^{c \in \mathsf{C}} F(c, c), \omega\right)$. $\diamond$

More explicitly, a coend of F is a cowedge $\left(\int^{c \in \mathsf{C}} F(c, c), \omega\right)$ such that, given any cowedge (X, ζ) of F, there exists a unique morphism

$$\int^{c \in \mathsf{C}} F(c, c) \xrightarrow{h} X \in \mathsf{D}$$

that renders the diagram

$$\begin{tikzcd} F(c,c) \arrow[rr, "\omega_c"] \arrow[dr, "\zeta_c"'] & & \int^{c\in\mathsf{C}} F(c,c) \arrow[dl, "h"] \\ & X & \end{tikzcd}$$

commutative for each object $c \in \mathsf{C}$. Dual to a coend is the notion of an *end*, which is originally due to Yoneda [**Yon60**]. The following observation follows from the definitions of a coend and a coequalizer.

Proposition 1.1.17. *Suppose $F : \mathsf{C}^{\mathsf{op}} \times \mathsf{C} \longrightarrow \mathsf{D}$ is a functor with C small and D cocomplete. Then a coend of F exists and is given by a coequalizer*

$$\int^{c\in\mathsf{C}} F(c,c) = \mathsf{coeq}\left(\coprod_{g\in\mathsf{Mor}(\mathsf{C})} F(d,c) \underset{i_c\circ F(g,1_c)}{\overset{i_d\circ F(1_d,g)}{\rightrightarrows}} \coprod_{c\in\mathsf{C}} F(c,c) \right)$$

in which $g : c \longrightarrow d$ runs through all the morphisms in C. The natural morphism ω_c is the composite

$$\begin{tikzcd} F(c,c) \arrow[rr, "\omega_c"] \arrow[dr, "i_c"'] & & \int^{c\in\mathsf{C}} F(c,c) \\ & \coprod_{c\in\mathsf{C}} F(c,c) \arrow[ur, "\text{natural}"'] & \end{tikzcd}$$

for each object $c \in \mathsf{C}$, with i_c the natural inclusion.

In the above setting, a coend is a colimit.

Kan Extensions.

Definition 1.1.18. Suppose

$$F : \mathsf{C} \longrightarrow \mathsf{D} \quad \text{and} \quad G : \mathsf{C} \longrightarrow \mathsf{E}$$

are functors. A *left Kan extension of F along G* is a pair

$$(\mathsf{Lan}_G F, \theta)$$

consisting of

- a functor
$$\mathsf{Lan}_G F : \mathsf{E} \longrightarrow \mathsf{D}$$
and
- a natural transformation
$$\theta : F \longrightarrow (\mathsf{Lan}_G F)G$$

that is initial among such pairs. ⋄

Explanation 1.1.19. The universal property that defines a left Kan extension in Definition 1.1.18 is that for each

- functor $H : \mathsf{E} \longrightarrow \mathsf{D}$ and
- natural transformation $\varepsilon : F \longrightarrow HG$,

there exists a unique natural transformation

$$\phi : \mathsf{Lan}_G F \longrightarrow H$$

such that ε factors as

$$\varepsilon = (\phi * 1_G)\theta.$$

This equality means that ε is equal to the following pasting of natural transformations, where $L = \mathsf{Lan}_G F$:

$$\begin{tikzcd}
 & & \mathsf{D} \\
\mathsf{C} \arrow[urr, bend left, "F"] \arrow[drr, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & {} & {} \arrow[phantom, "\overset{\exists!\phi}{\Rightarrow}"] \\
 & & \mathsf{E} \arrow[uu, bend left=50, "L"] \arrow[uu, bend right=50, "H"']
\end{tikzcd}$$

A left Kan extension is given by the coend

$$\mathsf{Lan}_G F = \int^{c \in \mathsf{C}} \mathsf{E}(Gc, -) \cdot Fc,$$

assuming that this coend and the coproduct

$$\mathsf{E}(Gc, -) \cdot Fc = \coprod_{\mathsf{E}(Gc,-)} Fc$$

exist in D. This is [**ML98**, X.4 Theorem 1]. $\diamond$

The dual concept of a *right Kan extension of F along G* is a pair $(\mathsf{Ran}_G F, \theta)$ consisting of

- a functor
$$\mathsf{Ran}_G F : \mathsf{E} \longrightarrow \mathsf{D}$$
and
- a natural transformation
$$\theta : (\mathsf{Ran}_G F)G \longrightarrow F$$

that is terminal among such pairs.

1.2. Monoidal Categories

In this section, we recall the definitions of a (symmetric) monoidal category, a (symmetric) monoidal functor, and a monoidal natural transformation.

Monoidal Categories and Functors.

Definition 1.2.1. A *monoidal category* is a tuple

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$$

consisting of

- a category C;
- a functor $\otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$, which is called the *monoidal product*;
- an object $\mathbb{1} \in \mathsf{C}$, which is called the *monoidal unit*;
- a natural isomorphism
$$(X \otimes Y) \otimes Z \xrightarrow[\cong]{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z)$$
for all objects $X, Y, Z \in \mathsf{C}$, which is called the *associativity isomorphism*; and

- natural isomorphisms

$$\mathbb{1} \otimes X \xrightarrow[\cong]{\lambda_X} X \quad \text{and} \quad X \otimes \mathbb{1} \xrightarrow[\cong]{\rho_X} X$$

  for all objects $X \in \mathsf{C}$, which are called the *left unit isomorphism* and the *right unit isomorphism*, respectively.

These data are required to satisfy the following two axioms.

The Unity Axiom: The *middle unity diagram*

(1.2.2)
$$\begin{tikzcd} (X \otimes \mathbb{1}) \otimes Y \arrow[r, "\alpha_{X,\mathbb{1},Y}"] \arrow[d, "\rho_X \otimes 1_Y"'] & X \otimes (\mathbb{1} \otimes Y) \arrow[d, "1_X \otimes \lambda_Y"] \\ X \otimes Y \arrow[r, equal] & X \otimes Y \end{tikzcd}$$

is commutative for all objects $X, Y \in \mathsf{C}$.

The Pentagon Axiom: The pentagon

(1.2.3)
$$\begin{tikzcd}[column sep=small] & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y \otimes Z}"] & \\ ((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W \otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes 1_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\ (W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X \otimes Y,Z}"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "1_W \otimes \alpha_{X,Y,Z}"'] \end{tikzcd}$$

is commutative for all objects $W, X, Y, Z \in \mathsf{C}$.

A *strict monoidal category* is a monoidal category in which the components of α, λ, and ρ are all identity morphisms. ◇

Convention 1.2.4. In a monoidal category, we sometimes use concatenation as an abbreviation for the monoidal product. For example,

$$XY = X \otimes Y, \qquad (XY)Z = (X \otimes Y) \otimes Z,$$

and similarly for morphisms. We sometimes denote a monoidal category by $(\mathsf{C}, \otimes, \mathbb{1})$ or C. On the other hand, to emphasize the monoidal category C, we decorate the monoidal structure accordingly as $\otimes^{\mathsf{C}}$, $\mathbb{1}^{\mathsf{C}}$, α^{C}, λ^{C}, and ρ^{C}. ◇

Remark 1.2.5. By the results in [**Kel64**], in a monoidal category the equality

(1.2.6)
$$\lambda_{\mathbb{1}} = \rho_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1}$$

holds, and the diagrams

(1.2.7)
$$\begin{tikzcd} (\mathbb{1} \otimes X) \otimes Y \arrow[r, "\alpha_{\mathbb{1},X,Y}"] \arrow[d, "\lambda_X \otimes 1_Y"'] & \mathbb{1} \otimes (X \otimes Y) \arrow[d, "\lambda_{X \otimes Y}"] \\ X \otimes Y \arrow[r, equal] & X \otimes Y \end{tikzcd} \qquad \begin{tikzcd} (X \otimes Y) \otimes \mathbb{1} \arrow[r, "\alpha_{X,Y,\mathbb{1}}"] \arrow[d, "\rho_{X \otimes Y}"'] & X \otimes (Y \otimes \mathbb{1}) \arrow[d, "1_X \otimes \rho_Y"] \\ X \otimes Y \arrow[r, equal] & X \otimes Y \end{tikzcd}$$

are commutative. They are called the *left unity diagram* and the *right unity diagram*, respectively. Proofs of these unity properties in the more general setting of bicategories can be found in [**JY21**, Section 2.2]. ◇

Definition 1.2.8. A *monoid* in a monoidal category C is a triple (X, μ, η) with

- X an object in C;

- $\mu : X \otimes X \longrightarrow X$ a morphism, which is called the *multiplication*; and
- $\eta : \mathbb{1} \longrightarrow X$ a morphism, which is called the *unit*.

These data are required to make the following associativity and unity diagrams commutative.

$$\begin{tikzcd}
(X \otimes X) \otimes X \arrow[rr, "\alpha"] \arrow[dd, "\mu\otimes 1_X"'] & & X \otimes (X \otimes X) \arrow[d, "1_X\otimes\mu"] \\
& & X \otimes X \arrow[d, "\mu"] \\
X \otimes X \arrow[rr, "\mu"] & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes X \arrow[r, "\lambda_X"] \arrow[d, "\eta\otimes 1_X"'] & X \arrow[d, equal] \\
X \otimes X \arrow[r, "\mu"] & X \\
X \otimes \mathbb{1} \arrow[r, "\rho_X"] \arrow[u, "1_X\otimes\eta"] & X \arrow[u, equal]
\end{tikzcd}$$

A *morphism* of monoids

$$f : (X, \mu^X, \eta^X) \longrightarrow (Y, \mu^Y, \eta^Y)$$

is a morphism $f : X \longrightarrow Y$ in C that preserves the multiplications and the units in the sense that the diagrams

$$\begin{tikzcd}
X \otimes X \arrow[r, "f\otimes f"] \arrow[d, "\mu^X"'] & Y \otimes Y \arrow[d, "\mu^Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta^X"] \arrow[d, equal] & X \arrow[d, "f"] \\
\mathbb{1} \arrow[r, "\eta^Y"] & Y
\end{tikzcd}$$

are commutative. ◇

Definition 1.2.9. A *comonoid* in a monoidal category C is a triple (Y, Δ, ε) with

- Y an object in C;
- $\Delta : Y \longrightarrow Y \otimes Y$ a morphism, which is called the *comultiplication*; and
- $\varepsilon : Y \longrightarrow \mathbb{1}$ a morphism, which is called the *counit*.

These data are required to make the following coassociativity and counity diagrams commutative.

(1.2.10)
$$\begin{tikzcd}
Y \arrow[r, "\Delta"] \arrow[dd, "\Delta"'] & Y \otimes Y \arrow[d, "\Delta\otimes 1_Y"] \\
& (Y \otimes Y) \otimes Y \arrow[d, "\alpha"'] \\
Y \otimes Y \arrow[r, "1_Y\otimes\Delta"] & Y \otimes (Y \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes Y \arrow[r, "\lambda_Y"] & Y \arrow[d, equal] \\
Y \otimes Y \arrow[u, "\varepsilon\otimes 1_Y"] \arrow[d, "1_Y\otimes\varepsilon"'] & Y \arrow[l, "\Delta"'] \arrow[d, equal] \\
Y \otimes \mathbb{1} \arrow[r, "\rho"] & Y
\end{tikzcd}$$

A *morphism* of comonoids is a morphism of the underlying objects that preserves the comultiplications and the counits. ◇

Definition 1.2.11. For monoidal categories C and D, a *monoidal functor*

$$(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of

- a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$;
- a natural transformation

(1.2.12)
$$FX \otimes FY \xrightarrow{F^2_{X,Y}} F(X \otimes Y) \in \mathsf{D}$$

for objects X and Y in C, which is called the *monoidal constraint*; and

- a morphism

$$\mathbb{1}^{\mathsf{D}} \xrightarrow{F^0} F\mathbb{1}^{\mathsf{C}} \in \mathsf{D}, \tag{1.2.13}$$

which is called the *unit constraint*.

These data are required to satisfy the following associativity and unity axioms.

Associativity: The diagram

$$\begin{tikzcd}
\big(FX \otimes FY\big) \otimes FZ \arrow[r, "\alpha^{\mathsf{D}}"] \arrow[d, "F^2 \otimes 1_{FZ}"'] & FX \otimes \big(FY \otimes FZ\big) \arrow[d, "1_{FX} \otimes F^2"] \\
F(X \otimes Y) \otimes FZ \arrow[d, "F^2"'] & FX \otimes F(Y \otimes Z) \arrow[d, "F^2"] \\
F\big((X \otimes Y) \otimes Z\big) \arrow[r, "F\alpha^{\mathsf{C}}"] & F\big(X \otimes (Y \otimes Z)\big)
\end{tikzcd} \tag{1.2.14}$$

is commutative for all objects $X, Y, Z \in \mathsf{C}$.

Unity: The diagrams

$$\begin{tikzcd}
\mathbb{1}^{\mathsf{D}} \otimes FX \arrow[r, "\lambda^{\mathsf{D}}_{FX}"] \arrow[d, "F^0 \otimes 1_{FX}"'] & FX \\
F\mathbb{1}^{\mathsf{C}} \otimes FX \arrow[r, "F^2"] & F(\mathbb{1}^{\mathsf{C}} \otimes X) \arrow[u, "F\lambda^{\mathsf{C}}_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes \mathbb{1}^{\mathsf{D}} \arrow[r, "\rho^{\mathsf{D}}_{FX}"] \arrow[d, "1_{FX} \otimes F^0"'] & FX \\
FX \otimes F\mathbb{1}^{\mathsf{C}} \arrow[r, "F^2"] & F(X \otimes \mathbb{1}^{\mathsf{C}}) \arrow[u, "F\rho^{\mathsf{C}}_X"']
\end{tikzcd} \tag{1.2.15}$$

are commutative for all objects $X \in \mathsf{C}$. They are called the *left unity diagram* and the *right unity diagram*, respectively.

A monoidal functor (F, F^2, F^0) is sometimes abbreviated to F.

Moreover, a monoidal functor (F, F^2, F^0) is said to be

- *unital* if F^0 is an isomorphism;
- *strictly unital* if F^0 is the identity morphism;
- *strong* if F^0 and the components of F^2 are isomorphisms; and
- *strict* if F^0 and the components of F^2 are identity morphisms. $\diamond$

Definition 1.2.16. For monoidal functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, a *monoidal natural transformation* $\theta : F \longrightarrow G$ is a natural transformation between the underlying functors such that the diagrams

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\theta_X \otimes \theta_Y"] \arrow[d, "F^2"'] & GX \otimes GY \arrow[d, "G^2"] \\
F(X \otimes Y) \arrow[r, "\theta_{X \otimes Y}"] & G(X \otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1}^{\mathsf{D}} \arrow[r, "F^0"] \arrow[d, equal] & F\mathbb{1}^{\mathsf{C}} \arrow[d, "\theta_{\mathbb{1}^{\mathsf{C}}}"] \\
\mathbb{1}^{\mathsf{D}} \arrow[r, "G^0"] & G\mathbb{1}^{\mathsf{C}}
\end{tikzcd} \tag{1.2.17}$$

are commutative for all objects $X, Y \in \mathsf{C}$. $\diamond$

Symmetric Monoidal Categories and Functors. Next we consider symmetric analogues of the above concepts.

Definition 1.2.18. A *symmetric monoidal category* is a pair (C, ξ) consisting of the following data.

- $\mathsf{C} = (\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category as in Definition 1.2.1.

- ξ is a natural isomorphism

$$X \otimes Y \xrightarrow[\cong]{\xi_{X,Y}} Y \otimes X \tag{1.2.19}$$

  for objects $X, Y \in \mathsf{C}$, which is called the *symmetry isomorphism* or the *braiding*.

These data are required to satisfy the following three axioms.

The Symmetry Axiom: The diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, "\xi_{X,Y}"] \arrow[dr, equal] & Y \otimes X \arrow[d, "\xi_{Y,X}"] \\ & X \otimes Y \end{tikzcd} \tag{1.2.20}$$

is commutative for all objects $X, Y \in \mathsf{C}$.

The Unit Axiom: The diagram

$$\begin{tikzcd} X \otimes \mathbb{1} \arrow[r, "\xi_{X,\mathbb{1}}"] \arrow[d, "\rho_X"'] & \mathbb{1} \otimes X \arrow[d, "\lambda_X"] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{1.2.21}$$

is commutative for all objects $X \in \mathsf{C}$.

The Hexagon Axiom: The diagram

$$\begin{tikzcd} & X \otimes (Z \otimes Y) \arrow[r, "1_X \otimes \xi_{Z,Y}"] & X \otimes (Y \otimes Z) \arrow[dr, "\alpha^{-1}"] & \\ (X \otimes Z) \otimes Y \arrow[ur, "\alpha"] \arrow[dr, "\xi_{X\otimes Z,Y}"'] & & & (X \otimes Y) \otimes Z \\ & Y \otimes (X \otimes Z) \arrow[r, "\alpha^{-1}"] & (Y \otimes X) \otimes Z \arrow[ur, "\xi_{Y,X} \otimes 1_Z"'] & \end{tikzcd} \tag{1.2.22}$$

is commutative for all objects $X, Y, Z \in \mathsf{C}$.

A symmetric monoidal category is said to be *strict* if the underlying monoidal category is strict. A symmetric strict monoidal category is also called a *permutative category*. ◇

Definition 1.2.23. Suppose (C, ξ) is a symmetric monoidal category.

- A *commutative monoid* is a monoid (X, μ, η) such that the diagram

$$\begin{tikzcd} X \otimes X \arrow[r, "\xi_{X,X}"] \arrow[d, "\mu"'] & X \otimes X \arrow[d, "\mu"] \\ X \arrow[r, equal] & X \end{tikzcd}$$

  is commutative.

- A *cocommutative comonoid* is a comonoid (Y, Δ, ε) such that the diagram

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d, "\Delta"'] & Y \arrow[d, "\Delta"] \\ Y \otimes Y \arrow[r, "\xi_{Y,Y}"] & Y \otimes Y \end{tikzcd} \tag{1.2.24}$$

  is commutative.

A morphism of (co)commutative (co)monoids is a morphism of the underlying (co)monoids. ◇

Definition 1.2.25. For symmetric monoidal categories C and D, a *symmetric monoidal functor* $(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$ is a monoidal functor between the underlying monoidal categories such that the diagram

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\xi_{FX,FY}", "\cong"'] \arrow[d, "F^2"'] & FY \otimes FX \arrow[d, "F^2"] \\ F(X \otimes Y) \arrow[r, "F\xi_{X,Y}", "\cong"'] & F(Y \otimes X) \end{tikzcd} \tag{1.2.26}$$

is commutative for all objects $X, Y \in \mathsf{C}$. A symmetric monoidal functor is said to be *strong* (respectively, *strict*, *unital*, or *strictly unital*) if the underlying monoidal functor is so. ◇

Example 1.2.27. Here are some examples of symmetric monoidal categories.

- $(\mathsf{Set}, \times, *)$ is the category of sets and functions. A monoid in Set is a monoid in the usual sense.
- $(\mathsf{Cat}, \times, \mathbf{1})$ is the category of small categories and functors.
- $(\mathsf{Vect}^{\Bbbk}, \otimes, \Bbbk)$ is the category of $\Bbbk$-vector spaces and $\Bbbk$-linear maps over a field $\Bbbk$, with $\otimes$ the tensor product of $\Bbbk$-vector spaces. ◇

Definition 1.2.28. A symmetric monoidal category C is *closed* if, for each object X, the functor

$$- \otimes X : \mathsf{C} \longrightarrow \mathsf{C}$$

admits a right adjoint, which is denoted by $[X, -]$ and is called the *internal hom*. ◇

1.3. Coherence

In this section, we recall several coherence results for monoidal categories, symmetric monoidal categories, and symmetric monoidal functors. Each of them states either that some formal diagrams commute, or that a monoidal structure can be strictified. See the references in Notes 1.4.4 and 1.4.5 for detailed discussion of these topics.

Monoidal Categories. To state the first type of coherence result, we use the next two definitions.

Definition 1.3.1. A *word* of length $n \geq 0$ is defined inductively as follows.

- The only word of length 0 is the symbol e.
- The only word of length 1 is the symbol $-$.
- If u and v are words of lengths m and n, respectively, then $u \square v$ is a word of length $m + n$.

Moreover, we define the following.

- A *left normalized word* is the word e, $-$, $u \,\square\, e$, or $u \,\square\, -$, with u a left normalized word.
- A *right normalized word* is the word e, $-$, $e \,\square\, u$, or $- \,\square\, u$, with u a right normalized word.
- For a monoidal category C, each word w of length n determines a functor $w : \mathsf{C}^n \longrightarrow \mathsf{C}$ by interpreting
  - the length 0 word e as the constant functor at $\mathbb{1}$;
  - the length 1 word $-$ as the identity functor 1_{C}; and
  - $\square$ as the monoidal product in C.

  We also call this functor a *word*. ◇

Definition 1.3.2. For a monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$, *canonical maps* are natural isomorphisms between words of the same length, defined inductively by the following four conditions.

- The identity morphism of $\mathbb{1}$ is a canonical map.
- The identity natural transformation of 1_{C} is a canonical map.
- α, λ, ρ, and their inverses are canonical maps.
- Canonical maps are closed under $\otimes$ and vertical composites. ◇

Theorem 1.3.3 (Mac Lane's Coherence)**.** *Suppose u and v are words $\mathsf{C}^n \longrightarrow \mathsf{C}$ of the same length in a monoidal category C. Then there exists a unique canonical map $u \longrightarrow v$.*

Example 1.3.4. The word

$$u = \big((- \,\square\, -) \,\square\, -\big) \,\square\, -$$

is left normalized, and the word

$$v = - \,\square\, \big(- \,\square (- \,\square\, -)\big)$$

is right normalized, both of length 4. Regarding the words u and v as functors $\mathsf{C}^4 \longrightarrow \mathsf{C}$, they are given by

$$\begin{aligned} u(x_1, x_2, x_3, x_4) &= \big((x_1 \otimes x_2) \otimes x_3\big) \otimes x_4 \quad \text{and} \\ v(x_1, x_2, x_3, x_4) &= x_1 \otimes \big(x_2 \otimes (x_3 \otimes x_4)\big). \end{aligned}$$

Each of the two paths in the pentagon axiom (1.2.3) is a canonical map from u to v. Mac Lane's Coherence Theorem 1.3.3 extends the unity axiom (1.2.2) and the pentagon axiom (1.2.3). ◇

Theorem 1.3.5 (Mac Lane's Strictification)**.** *For each monoidal category C, there exist a strict monoidal category C_{st} and an adjoint equivalence*

$$\begin{tikzcd} \mathsf{C}_{\mathsf{st}} \arrow[r, shift left, "L"] & \mathsf{C} \arrow[l, shift left, "R"] \end{tikzcd}$$

with (i) both L and R strong monoidal functors, and (ii) $LR = 1_{\mathsf{C}}$.

Symmetric Monoidal Categories. The symmetric versions of the above coherence results involve the following concepts.

Definition 1.3.6. The symmetric group on n letters is denoted by Σ_n. Suppose C is a monoidal category.

- For a word w of length n and a permutation $\sigma \in \Sigma_n$, the *permuted word* $w\sigma : \mathsf{C}^n \longrightarrow \mathsf{C}$ is the composite functor $w \circ \sigma$, where $\sigma : \mathsf{C}^n \longrightarrow \mathsf{C}^n$ is given by

$$\sigma(x_1, \ldots, x_n) = \big(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\big)$$

with the x_i's all objects, or all morphisms, in C.

- For a symmetric monoidal category (C, ξ), a *permuted canonical map* is a natural isomorphism between permuted words of the same length, defined as in Definition 1.3.2 by also allowing the symmetry isomorphism ξ. ◇

Example 1.3.7. For the left normalized word

$$u = \big((- \,\square\, -) \,\square\, -\big) \,\square\, -$$

and the cycle

$$\sigma = (1,4,2,3) \in \Sigma_4,$$

the permuted word $u\sigma : \mathsf{C}^4 \longrightarrow \mathsf{C}$ is the functor given by

$$(u\sigma)(x_1, x_2, x_3, x_4) = \big((x_3 \otimes x_4) \otimes x_2\big) \otimes x_1. \quad ◇$$

Theorem 1.3.8 (Symmetric Coherence)**.** *Suppose $u\sigma$ and $v\tau$ are two permuted words of the same length in a symmetric monoidal category C. Then there exists a unique permuted canonical map $u\sigma \longrightarrow v\tau$.*

Example 1.3.9. The Symmetric Coherence Theorem 1.3.8 extends the symmetry axiom (1.2.20), the unit axiom (1.2.21), and the hexagon axiom (1.2.22). ◇

Theorem 1.3.10 (Symmetric Strictification)**.** *For each symmetric monoidal category C, there exist a permutative category C_{st} and an adjoint equivalence*

$$\begin{tikzcd} \mathsf{C}_{\mathsf{st}} \arrow[r, shift left, "L"] & \mathsf{C} \arrow[l, shift left, "R"] \end{tikzcd}$$

with (i) both L and R strong symmetric monoidal functors, and (ii) $LR = 1_{\mathsf{C}}$.

Monoidal Functors. Next we recall a coherence theorem for (symmetric) monoidal functors. The following definition and theorem have a plain monoidal version and a symmetric monoidal version. We will state them together using Definitions 1.3.1, 1.3.2, and 1.3.6.

Definition 1.3.11. Suppose $(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$ is a (symmetric) monoidal functor between (symmetric) monoidal categories.

(1) The set of *F-iterates* is the set of functors $\mathsf{C}^n \longrightarrow \mathsf{D}$ for $n \geq 1$ defined inductively by the following two conditions.
   - $Fw : \mathsf{C}^n \longrightarrow \mathsf{D}$ is an F-iterate for each (permuted) word $w : \mathsf{C}^n \longrightarrow \mathsf{C}$ of length n.
   - If $G : \mathsf{C}^m \longrightarrow \mathsf{D}$ and $H : \mathsf{C}^n \longrightarrow \mathsf{D}$ are F-iterates, then so is the composite
   $$\mathsf{C}^{m+n} = \mathsf{C}^m \times \mathsf{C}^n \xrightarrow{G \times H} \mathsf{D} \times \mathsf{D} \xrightarrow{\otimes} \mathsf{D}.$$

(2) The set of *F-coherent maps* is the set of natural transformations between F-iterates defined inductively as follows.
   - Suppose θ is a (permuted) canonical map in C that does not involve $\mathbb{1}^{\mathsf{C}}$, λ^{C}, or ρ^{C}. Then $1_F * \theta$ is an F-coherent map.
   - The identity natural transformation, α^{D}, their inverses, and ξ^{D} in the symmetric case, applied to F-iterates are F-coherent maps.
   - F^2 is an F-coherent map.
   - F-coherent maps are closed under vertical composites and $\otimes^{\mathsf{D}}$. ◇

Theorem 1.3.12 (Epstein's Coherence)**.** *Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a (symmetric) monoidal functor between (symmetric) monoidal categories, and $G, H : \mathsf{C}^n \longrightarrow \mathsf{D}$ are F-iterates. Then there exists at most one F-coherent map from G to H.*

Example 1.3.13. Consider the associativity axiom (1.2.14) for a monoidal functor $F : \mathsf{C} \longrightarrow \mathsf{D}$. All six functors $\mathsf{C}^3 \longrightarrow \mathsf{D}$ defined by the vertices in (1.2.14) are F-iterates. Each path along the boundary of that diagram is a component of an F-coherent map. Epstein's Coherence Theorem 1.3.12 extends the associativity axiom (1.2.14) and the compatibility axiom (1.2.26) in the symmetric case. ⋄

1.4. Notes

1.4.1 (Categories)**.** For more detailed discussion of basic category theory, we refer the reader to the books [**Awo10, BK00, Gra18, Lei14, Rie16, Rom17, Sim11**]. For higher level discussion of categories, the reader may consult the books [**Bor94a, Bor94b, Bor94c, Ehr65, ML98, Mit65, Sch72**]. The founding paper [**EML45**] of category theory is very readable and contains applications to topology. ⋄

1.4.2 (Ends and Coends)**.** More discussion about (co)ends and enriched diagram categories is in [**Day70, DK69**] and Chapter III.3. A systematic discussion about (co)ends is in [**Lor21**] and [**ML98**, IX.5–8]. ⋄

1.4.3 (Terminology)**.** What we call a (symmetric) monoidal category is what Joyal and Street [**JS93**] called a *(symmetric) tensor category*. A monoidal functor is sometimes called a *lax monoidal functor* in the literature to emphasize the fact that the structure morphisms F^2 and F^0 are not necessarily invertible. A strong monoidal functor is also known as a *tensor functor*. ⋄

1.4.4 (Coherence)**.** Mac Lane's Coherence Theorem 1.3.3 is [**ML63**, Th. 5.2]; see also [**ML98**, VII.2 Cor.]. Mac Lane's Strictification Theorem 1.3.5 is [**ML98**, XI.3 Th. 1], which is a consequence of the coherence theorem. The Symmetric Coherence Theorem 1.3.8 is [**ML98**, XI.1 Th. 1]. The Symmetric Strictification Theorem 1.3.10 is proved by adapting the proof of Theorem 1.3.5 in the nonsymmetric case, by incorporating the symmetry and using Theorem 1.3.8. ⋄

1.4.5 (Coherence of Monoidal Functors)**.** Theorem 1.3.12 is [**Eps66**, Th. 1.6]. The proof of the plain monoidal version of Theorem 1.3.12 is obtained from Epstein's proof by systematically removing all instances of the symmetry isomorphism. Another coherence result for monoidal functors is [**JS93**, Th. 1.7]. ⋄

CHAPTER 2

Symmetric Bimonoidal Categories

In this chapter, we introduce symmetric bimonoidal categories in the sense of Laplaza and discuss some examples and one application to computer science. The definition is given in Section 2.1. A symmetric bimonoidal category is a categorical analogue of a commutative rig, in which one can add, multiply, and distribute products over sums. A symmetric bimonoidal category is a category equipped with

- two symmetric monoidal structures, one additive and one multiplicative,
- two multiplicative zero natural isomorphisms, and
- two distributivity natural monomorphisms,

along with a list of carefully chosen axioms. A symmetric bimonoidal category is called *tight* if the distributivity morphisms are natural isomorphisms. We also define bimonoidal categories, which do not have a multiplicative symmetry.

In Section 2.2, we prove Laplaza's theorem, with one improvement as discussed in Note 2.7.1, that half of the 24 axioms of a symmetric bimonoidal category are formal consequences of the others. Therefore, in practice, to check that one has a symmetric bimonoidal category, one has to check 12 axioms instead of 24. This reduction Theorem 2.2.13 does not apply to bimonoidal categories because its proof heavily uses the multiplicative symmetry. The only reduction of the list of axioms in the nonsymmetric case is in Proposition 2.2.14.

The next few sections contain examples of symmetric bimonoidal categories. In Section 2.3, we observe that distributive symmetric monoidal categories are symmetric bimonoidal categories whose sums are coproducts. In particular, distributive categories and the category of modules over a commutative ring are symmetric bimonoidal categories.

Section 2.4 contains an important example of a symmetric bimonoidal category Σ with finite ordinals as objects and their bijections as morphisms.

- In Section 4.2, we will construct a symmetric bimonoidal category, called the distortion category, that extends Σ. The distortion category is an essential ingredient in the formulation of the Coherence Theorem 4.4.3.
- In Chapter 7, we will show that Σ is bicategorically an initial object in a 2-category of small symmetric bimonoidal categories.
- In Chapter II.7, we will show that Σ is bicategorically an initial object in a 2-category of small braided bimonoidal categories.

This section also contains a variant of Σ denoted by Σ'.

In Section 2.5, we observe that right bipermutative categories, with Σ' being one example, are tight symmetric bimonoidal categories. A right bipermutative category has two permutative structures, one additive $\oplus$ and one multiplicative $\otimes$, with identities for both multiplicative zeros, the right distributivity, and $\xi^{\otimes}_{-,\mathbb{0}}$, along with several carefully chosen axioms. In Section 5.4, we will show that *tight* symmetric bimonoidal categories can be strictified to equivalent right bipermutative categories. In the second half of this section, we discuss left bipermutative categories, in which the left distributivity is the identity. The symmetric bimonoidal category Σ is an example of a left bipermutative category.

In Section 2.6, we discuss an application of symmetric bimonoidal categories to computer science from [**CS16**]. The main observation is that there is a symmetric bimonoidal groupoid Π with syntax of finite types as objects. The symmetric bimonoidal groupoid Π provides a variant of the Curry-Howard-Lambek correspondence for reversible programming of finite types with sums and products. Applications of *braided* bimonoidal categories to quantum group theory and topological quantum computation are discussed in Chapter II.3.

The following table summarizes basic properties of (symmetric) bimonoidal categories, along with their coherence theorems, strictification theorems, and bicategorical properties.

	bimonoidal categories	symmetric bimonoidal categories
axioms	22	24
reduced axioms	21 (2.2.14)	12 (2.2.13)
coherence	3.10.7, 4.5.8	3.9.1, 4.4.3
strictification	5.5.11, 5.5.12	5.4.6, 5.4.7
Baez's Conjecture		7.8.1, 7.8.3
bicategorification	8.4.12	8.15.4

Braided bimonoidal categories will be discussed in Part II.1. For open questions related to bimonoidal categories, see Appendix III.A.1.

2.1. Definitions

In this section, we define symmetric bimonoidal categories and bimonoidal categories, and provide some basic examples. Many more examples will be given in the coming sections.

Motivation 2.1.1. To motivate the definition of a symmetric bimonoidal category, consider a commutative rig $(R,+,0,\times,1)$, which is a commutative ring without additive inverses, such as the ring $\mathbb{Z}$ of integers. Both the addition $+$ and the multiplication $\times$ are associative and commutative. They are 2-sided unital with respect to 0 and 1, respectively. For elements $x,y,z \in R$, the following multiplicative zero and distributivity properties hold:

$$0x = 0 = x0$$
$$x(y+z) = xy + xz$$
$$(x+y)z = xz + yz$$

A symmetric bimonoidal category is a categorified version of a commutative rig, with the commutative monoids $(R,+,0)$ and $(R,\times,1)$ replaced by two symmetric monoidal structures. The above four rig axioms are replaced by natural transformations that satisfy a list of coherence axioms. In the next several chapters, we will see that these coherence axioms yield good coherence properties of symmetric bimonoidal categories. $\diamond$

Definition 1.2.18 of a symmetric monoidal category is used in the next definition. In Theorem 2.2.13, we show that in each of the following twelve groups of axioms, only the first axiom is needed. However, for later reference and usage, it is more convenient to have all of them in one place.

Definition 2.1.2. A *symmetric bimonoidal category* is a tuple

$$\big(\mathsf{C},(\oplus,\mathbb{0},\alpha^{\oplus},\lambda^{\oplus},\rho^{\oplus},\xi^{\oplus}),(\otimes,\mathbb{1},\alpha^{\otimes},\lambda^{\otimes},\rho^{\otimes},\xi^{\otimes}),(\lambda^{\bullet},\rho^{\bullet}),(\delta^{l},\delta^{r})\big)$$

consisting of the following data.

- $(\mathsf{C},\oplus,\mathbb{0},\alpha^{\oplus},\lambda^{\oplus},\rho^{\oplus},\xi^{\oplus})$ is a symmetric monoidal category, which is called the *additive structure*.
- $(\mathsf{C},\otimes,\mathbb{1},\alpha^{\otimes},\lambda^{\otimes},\rho^{\otimes},\xi^{\otimes})$ is a symmetric monoidal category, which is called the *multiplicative structure*.
- $\lambda^{\bullet}$ and $\rho^{\bullet}$ are natural isomorphisms

$$\mathbb{0}\otimes A \xrightarrow[\cong]{\lambda^{\bullet}_A} \mathbb{0} \xleftarrow[\cong]{\rho^{\bullet}_A} A\otimes\mathbb{0} \quad \text{for} \quad A\in\mathsf{C}, \tag{2.1.3}$$

  which are called the *left multiplicative zero* and the *right multiplicative zero*, respectively.
- δ^l and δ^r are natural monomorphisms

$$\begin{gathered} A\otimes(B\oplus C) \xrightarrow{\delta^l_{A,B,C}} (A\otimes B)\oplus(A\otimes C)\\ (A\oplus B)\otimes C \xrightarrow{\delta^r_{A,B,C}} (A\otimes C)\oplus(B\otimes C) \end{gathered} \tag{2.1.4}$$

  for objects $A,B,C\in\mathsf{C}$, which are called the *left distributivity morphism* and the *right distributivity morphism*, respectively.

To simplify the presentation, we often abbreviate $\otimes$ using concatenation. In the absence of parentheses, $\otimes$ always takes precedence over $\oplus$. For example, the left distributivity morphism is abbreviated to $A(B\oplus C)\longrightarrow AB\oplus AC$.

The above data are required to make the following 24 diagrams in C commutative for all objects A,B,C, and D in C. They are collectively known as *Laplaza's Axioms*.

Distributivity and Multiplicative Symmetry:

$$\begin{tikzcd} (A\oplus B)C \arrow[r, "\delta^r_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A\oplus B,C}"'] & AC\oplus BC \arrow[d, "\xi^{\otimes}_{A,C}\oplus\xi^{\otimes}_{B,C}"] \\ C(A\oplus B) \arrow[r, "\delta^l_{C,A,B}"] & CA\oplus CB \end{tikzcd} \tag{2.1.5}$$

Distributivity and Additive Symmetry:

$$\begin{tikzcd} A(B\oplus C) \arrow[r, "\delta^l_{A,B,C}"] \arrow[d, "1_A\xi^{\oplus}_{B,C}"'] & AB\oplus AC \arrow[d, "\xi^{\oplus}_{AB,AC}"] \\ A(C\oplus B) \arrow[r, "\delta^l_{A,C,B}"] & AC\oplus AB \end{tikzcd} \tag{2.1.6}$$

$$\begin{tikzcd}
(A \oplus B)C \arrow[r, "\delta^r_{A,B,C}"] \arrow[d, "\xi^{\oplus}_{A,B} 1_C"'] & AC \oplus BC \arrow[d, "\xi^{\oplus}_{AC,BC}"] \\
(B \oplus A)C \arrow[r, "\delta^r_{B,A,C}"] & BC \oplus AC
\end{tikzcd} \tag{2.1.7}$$

Distributivity and Additive Associativity:

$$\begin{tikzcd}
[(A \oplus B) \oplus C]D \arrow[r, "\delta^r_{A\oplus B,C,D}"] \arrow[d, "\alpha^{\oplus}_{A,B,C} 1_D"'] & (A \oplus B)D \oplus CD \arrow[r, "\delta^r_{A,B,D} \oplus 1_{CD}"] & (AD \oplus BD) \oplus CD \arrow[d, "\alpha^{\oplus}_{AD,BD,CD}"] \\
[A \oplus (B \oplus C)]\, D \arrow[r, "\delta^r_{A,B\oplus C,D}"] & AD \oplus (B \oplus C)D \arrow[r, "1_{AD} \oplus \delta^r_{B,C,D}"] & AD \oplus (BD \oplus CD)
\end{tikzcd} \tag{2.1.8}$$

$$\begin{tikzcd}
A[(B \oplus C) \oplus D] \arrow[r, "\delta^l_{A,B\oplus C,D}"] \arrow[d, "1_A \alpha^{\oplus}_{B,C,D}"'] & A(B \oplus C) \oplus AD \arrow[r, "\delta^l_{A,B,C} \oplus 1_{AD}"] & (AB \oplus AC) \oplus AD \arrow[d, "\alpha^{\oplus}_{AB,AC,AD}"] \\
A[B \oplus (C \oplus D)] \arrow[r, "\delta^l_{A,B,C\oplus D}"] & AB \oplus A(C \oplus D) \arrow[r, "1_{AB} \oplus \delta^l_{A,C,D}"] & AB \oplus (AC \oplus AD)
\end{tikzcd} \tag{2.1.9}$$

Distributivity and Multiplicative Associativity:

$$\begin{tikzcd}
(AB)(C \oplus D) \arrow[rr, "\delta^l_{AB,C,D}"] \arrow[d, "\alpha^{\otimes}_{A,B,C\oplus D}"'] & & (AB)C \oplus (AB)D \arrow[d, "\alpha^{\otimes}_{A,B,C} \oplus \alpha^{\otimes}_{A,B,D}"] \\
A[B(C \oplus D)] \arrow[r, "1_A \delta^l_{B,C,D}"] & A(BC \oplus BD) \arrow[r, "\delta^l_{A,BC,BD}"] & A(BC) \oplus A(BD)
\end{tikzcd} \tag{2.1.10}$$

$$\begin{tikzcd}
[(A \oplus B)C]D \arrow[r, "\delta^r_{A,B,C} 1_D"] \arrow[d, "\alpha^{\otimes}_{A\oplus B,C,D}"'] & (AC \oplus BC)D \arrow[r, "\delta^r_{AC,BC,D}"] & (AC)D \oplus (BC)D \arrow[d, "\alpha^{\otimes}_{A,C,D} \oplus \alpha^{\otimes}_{B,C,D}"] \\
(A \oplus B)(CD) \arrow[rr, "\delta^r_{A,B,CD}"] & & A(CD) \oplus B(CD)
\end{tikzcd} \tag{2.1.11}$$

$$\begin{tikzcd}
[A(B \oplus C)]D \arrow[r, "\delta^l_{A,B,C} 1_D"] \arrow[d, "\alpha^{\otimes}_{A,B\oplus C,D}"'] & (AB \oplus AC)D \arrow[r, "\delta^r_{AB,AC,D}"] & (AB)D \oplus (AC)D \arrow[d, "\alpha^{\otimes}_{A,B,D} \oplus \alpha^{\otimes}_{A,C,D}"] \\
A[(B \oplus C)D] \arrow[r, "1_A \delta^r_{B,C,D}"] & A(BD \oplus CD) \arrow[r, "\delta^l_{A,BD,CD}"] & A(BD) \oplus A(CD)
\end{tikzcd} \tag{2.1.12}$$

2-By-2 Distributivity:

$$\begin{tikzcd}
(A \oplus B)(C \oplus D) \arrow[r, "\delta^r_{A,B,C\oplus D}"] \arrow[d, "\delta^l_{A\oplus B,C,D}"'] & A(C \oplus D) \oplus B(C \oplus D) \arrow[d, "\delta^l_{A,C,D} \oplus \delta^l_{B,C,D}"] \\
(A \oplus B)C \oplus (A \oplus B)D \arrow[d, "\delta^r_{A,B,C} \oplus \delta^r_{A,B,D}"'] & (AC \oplus AD) \oplus (BC \oplus BD) \arrow[d, "\alpha^{\oplus}_{AC,AD,BC\oplus BD}"] \\
(AC \oplus BC) \oplus (AD \oplus BD) \arrow[d, "\alpha^{\oplus}_{AC,BC,AD\oplus BD}"'] & AC \oplus [AD \oplus (BC \oplus BD)] \arrow[d, "1_{AC} \oplus (\alpha^{\oplus})^{-1}"] \\
AC \oplus [BC \oplus (AD \oplus BD)] \arrow[d, "1_{AC} \oplus (\alpha^{\oplus})^{-1}"'] & AC \oplus [(AD \oplus BC) \oplus BD] \arrow[d, "1_{AC} \oplus (\xi^{\oplus}_{AD,BC} \oplus 1_{BD})"] \\
AC \oplus [(BC \oplus AD) \oplus BD] \arrow[r, equal] & AC \oplus [(BC \oplus AD) \oplus BD]
\end{tikzcd} \tag{2.1.13}$$

Multiplicative Zero of 0:

$$\mathbb{0} \otimes \mathbb{0} \underset{\rho^{\bullet}_{\mathbb{0}}}{\overset{\lambda^{\bullet}_{\mathbb{0}}}{\rightrightarrows}} \mathbb{0} \tag{2.1.14}$$

Multiplicative Zero of a Sum:

$$\begin{tikzcd}
\mathbb{0}(A \oplus B) \arrow[r, "\delta^{l}_{\mathbb{0},A,B}"] \arrow[d, "\lambda^{\bullet}_{A\oplus B}"'] & \mathbb{0}A \oplus \mathbb{0}B \arrow[d, "\lambda^{\bullet}_{A}\oplus\lambda^{\bullet}_{B}"] \\
\mathbb{0} & \mathbb{0} \oplus \mathbb{0} \arrow[l, "\lambda^{\oplus}_{\mathbb{0}}"']
\end{tikzcd} \tag{2.1.15}$$

$$\begin{tikzcd}
(A \oplus B)\mathbb{0} \arrow[r, "\delta^{r}_{A,B,\mathbb{0}}"] \arrow[d, "\rho^{\bullet}_{A\oplus B}"'] & A\mathbb{0} \oplus B\mathbb{0} \arrow[d, "\rho^{\bullet}_{A}\oplus\rho^{\bullet}_{B}"] \\
\mathbb{0} & \mathbb{0} \oplus \mathbb{0} \arrow[l, "\lambda^{\oplus}_{\mathbb{0}}"']
\end{tikzcd} \tag{2.1.16}$$

Multiplicative Zero and Multiplicative Unit:

$$\mathbb{0} \otimes \mathbb{1} \underset{\rho^{\otimes}_{\mathbb{0}}}{\overset{\lambda^{\bullet}_{\mathbb{1}}}{\rightrightarrows}} \mathbb{0} \tag{2.1.17}$$

$$\mathbb{1} \otimes \mathbb{0} \underset{\lambda^{\otimes}_{\mathbb{0}}}{\overset{\rho^{\bullet}_{\mathbb{1}}}{\rightrightarrows}} \mathbb{0} \tag{2.1.18}$$

Symmetry of Multiplicative Zero:

$$\begin{tikzcd}
A \otimes \mathbb{0} \arrow[rr, "\xi^{\otimes}_{A,\mathbb{0}}"] \arrow[dr, "\rho^{\bullet}_{A}"'] & & \mathbb{0} \otimes A \arrow[dl, "\lambda^{\bullet}_{A}"] \\
& \mathbb{0} &
\end{tikzcd} \tag{2.1.19}$$

Multiplicative Zero and Multiplicative Associativity:

$$\begin{tikzcd}
(AB)\mathbb{0} \arrow[r, "\alpha^{\otimes}_{A,B,\mathbb{0}}"] \arrow[d, "\rho^{\bullet}_{AB}"'] & A(B\mathbb{0}) \arrow[d, "1_A\rho^{\bullet}_{B}"] \\
\mathbb{0} & A\mathbb{0} \arrow[l, "\rho^{\bullet}_{A}"']
\end{tikzcd} \tag{2.1.20}$$

$$\begin{tikzcd}
(A\mathbb{0})B \arrow[rr, "\alpha^{\otimes}_{A,\mathbb{0},B}"] \arrow[d, "\rho^{\bullet}_{A}1_B"'] & & A(\mathbb{0}B) \arrow[d, "1_A\lambda^{\bullet}_{B}"] \\
\mathbb{0}B \arrow[dr, "\lambda^{\bullet}_{B}"'] & & A\mathbb{0} \arrow[dl, "\rho^{\bullet}_{A}"] \\
& \mathbb{0} &
\end{tikzcd} \tag{2.1.21}$$

$$\begin{tikzcd}
(\mathbb{0}A)B \arrow[r, "\lambda^{\bullet}_A 1_B"] \arrow[d, "\alpha^{\otimes}_{\mathbb{0},A,B}"'] & \mathbb{0}B \arrow[d, "\lambda^{\bullet}_B"] \\
\mathbb{0}(AB) \arrow[r, "\lambda^{\bullet}_{AB}"] & \mathbb{0}
\end{tikzcd} \tag{2.1.22}$$

Additive and Multiplicative Zero:

$$\begin{tikzcd}
A(\mathbb{0} \oplus B) \arrow[r, "\delta^l_{A,\mathbb{0},B}"] \arrow[d, "1_A \lambda^{\oplus}_B"'] & A\mathbb{0} \oplus AB \arrow[d, "\rho^{\bullet}_A \oplus 1_{AB}"] \\
AB & \mathbb{0} \oplus AB \arrow[l, "\lambda^{\oplus}_{AB}"']
\end{tikzcd} \tag{2.1.23}$$

$$\begin{tikzcd}
(\mathbb{0} \oplus B)A \arrow[r, "\delta^r_{\mathbb{0},B,A}"] \arrow[d, "\lambda^{\oplus}_B 1_A"'] & \mathbb{0}A \oplus BA \arrow[d, "\lambda^{\bullet}_A \oplus 1_{BA}"] \\
BA & \mathbb{0} \oplus BA \arrow[l, "\lambda^{\oplus}_{BA}"']
\end{tikzcd} \tag{2.1.24}$$

$$\begin{tikzcd}
A(B \oplus \mathbb{0}) \arrow[r, "\delta^l_{A,B,\mathbb{0}}"] \arrow[d, "1_A \rho^{\oplus}_B"'] & AB \oplus A\mathbb{0} \arrow[d, "1_{AB} \oplus \rho^{\bullet}_A"] \\
AB & AB \oplus \mathbb{0} \arrow[l, "\rho^{\oplus}_{AB}"']
\end{tikzcd} \tag{2.1.25}$$

$$\begin{tikzcd}
(B \oplus \mathbb{0})A \arrow[r, "\delta^r_{B,\mathbb{0},A}"] \arrow[d, "\rho^{\oplus}_B 1_A"'] & BA \oplus \mathbb{0}A \arrow[d, "1_{BA} \oplus \lambda^{\bullet}_A"] \\
BA & BA \oplus \mathbb{0} \arrow[l, "\rho^{\oplus}_{BA}"']
\end{tikzcd} \tag{2.1.26}$$

Distributivity and Multiplicative Unit:

$$\begin{tikzcd}
\mathbb{1}(A \oplus B) \arrow[rr, "\delta^l_{\mathbb{1},A,B}"] \arrow[dr, "\lambda^{\otimes}_{A \oplus B}"'] & & \mathbb{1}A \oplus \mathbb{1}B \arrow[dl, "\lambda^{\otimes}_A \oplus \lambda^{\otimes}_B"] \\
& A \oplus B &
\end{tikzcd} \tag{2.1.27}$$

$$\begin{tikzcd}
(A \oplus B)\mathbb{1} \arrow[rr, "\delta^r_{A,B,\mathbb{1}}"] \arrow[dr, "\rho^{\otimes}_{A \oplus B}"'] & & A\mathbb{1} \oplus B\mathbb{1} \arrow[dl, "\rho^{\otimes}_A \oplus \rho^{\otimes}_B"] \\
& A \oplus B &
\end{tikzcd} \tag{2.1.28}$$

This finishes the definition of a symmetric bimonoidal category.

A *bimonoidal category* has the same definition as a symmetric bimonoidal category except for the following two conditions.

- The multiplicative symmetry $\xi^{\otimes}$ is omitted, and $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes})$ is a monoidal category.
- The axioms (2.1.5) and (2.1.19) are omitted.

Moreover, we define the following.

- A (symmetric) bimonoidal category is *small* if its class of objects is a set.
- A (symmetric) bimonoidal category is *tight* if both δ^l and δ^r are natural isomorphisms.
- A *(symmetric) bimonoidal groupoid* is a (symmetric) bimonoidal category in which each morphism is an isomorphism.
- The objects $\mathbb{0}$ and $\mathbb{1}$ are called the *additive zero* and the *multiplicative unit*, respectively.
- $\oplus$ and $\otimes$ are called the *sum* and the *product*, respectively.
- $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, and $\xi^{\oplus}$ are called the *additive associativity isomorphism*, the *left additive zero*, the *right additive zero*, and the *additive symmetry isomorphism*, respectively.
- $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, and $\xi^{\otimes}$ are called the *multiplicative associativity isomorphism*, the *left multiplicative unit*, the *right multiplicative unit*, and the *multiplicative symmetry isomorphism*, respectively. ⋄

Explanation 2.1.29. Consider the definition of a symmetric bimonoidal category.

(1) The 24 axioms are purposely divided into 12 groups. We will see in Section 2.2 that in each group, only the first axiom is actually needed. Therefore, a symmetric bimonoidal category is defined by 12 axioms. See Theorem 2.2.13.
(2) The left distributivity and the right distributivity (2.1.4) are only required to be natural monomorphisms. Laplaza's coherence results in Chapters 3 and 4 do not need the invertibility of the distributivity morphisms.
(3) The 2-by-2 distributivity axiom (2.1.13) is the only axiom that involves the inverse of a structure morphism, namely $(\alpha^{\oplus})^{-1}$. ⋄

Explanation 2.1.30. Consider the definition of a bimonoidal category.

(1) The additive structure $(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$ is still a symmetric monoidal category.
(2) The axioms
    - (2.1.5) relating δ^l and δ^r, and
    - (2.1.19) relating $\lambda^{\bullet}$ and $\rho^{\bullet}$

    are omitted because they are the only ones among Laplaza's 24 axioms that involve the multiplicative symmetry $\xi^{\otimes}$.

Due to the absence of $\xi^{\otimes}$, the reduction of Laplaza's axioms in Theorem 2.2.13 from 24 to 12 does *not* apply to bimonoidal categories. The only exception is in Proposition 2.2.14. Therefore, a bimonoidal category is defined by 21 Laplaza axioms. ⋄

Example 2.1.31 (Commutative Rigs)**.** Suppose

$$(R, +, \times, 0_R, 1_R)$$

is a *commutative rig*, that is, a commutative ring without additive inverses. Then R becomes a small and tight symmetric bimonoidal category with only identity morphisms and

$$(+, \times, 0_R, 1_R) = (\oplus, \otimes, \mathbb{0}, \mathbb{1}).$$

In particular, the structure morphisms $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\xi^{\otimes}$, $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^l, and δ^r are all identities.

Similarly, if R is a *rig*, that is, a ring without additive inverses, then R becomes a small and tight bimonoidal category as above with only identity morphisms.

Therefore, a (symmetric) bimonoidal category is a categorification of a (commutative) rig. ⋄

Example 2.1.32 (Vector Spaces)**.** The category $\mathsf{Vect}^{\mathbb{C}}$ of finite dimensional complex vector spaces is a tight symmetric bimonoidal category with

- the usual direct sum $\oplus$ and tensor product $\otimes$ of finite dimensional complex vector spaces;
- $\mathbb{0}$ the 0 vector space; and
- $\mathbb{1} = \mathbb{C}$ as a 1-dimensional complex vector space.

The structure morphisms $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\xi^{\otimes}$, $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^{l}, and δ^{r} are the canonical isomorphisms. The axioms in Definition 2.1.2 are satisfied because, in each case, the two composites are both given by the canonical isomorphism. ⋄

2.2. Reduction of Axioms

In this section, we observe that in a symmetric bimonoidal category, only 12 of the 24 axioms are necessary; the other 12 axioms are formal consequences. We emphasize that this reduction of the list of axioms applies only to symmetric bimonoidal categories and *not* bimonoidal categories. The reason is that the proofs in this section use the multiplicative symmetry $\xi^{\otimes}$, with only one exception as in Proposition 2.2.14. The braided analogue of the reduction Theorem 2.2.13 is Theorem II.2.2.1.

Motivation 2.2.1. To motivate these reductions, recall that there are 12 groups of axioms in Definition 2.1.2, four of which have only one axiom each. In the other eight groups of axioms, we will show that only the first axiom is necessary. The axiom (2.1.5) describes how the left distributivity δ^{l} and the right distributivity δ^{r} determine each other. Once this axiom is assumed, axioms involving δ^{l} have analogues for δ^{r}. Moreover, the axiom (2.1.19) states that the left multiplicative zero $\lambda^{\bullet}$ and the right multiplicative zero $\rho^{\bullet}$ determine each other. Once this axiom is assumed, axioms involving $\lambda^{\bullet}$ have analogues for $\rho^{\bullet}$. ⋄

Convention 2.2.2. The following list of conventions is used throughout the rest of this book.

(1) The subscripts of a natural transformation are omitted if there is no danger of confusion. For example, we often write $\delta^{l}_{A,B,C}$ as δ^{l}.
(2) The inverse of $\alpha^{\oplus}$ is abbreviated to $\alpha^{-\oplus}$, and similarly for other invertible structure morphisms.
(3) In each diagram, if an isomorphism, such as $\xi^{\otimes}$ or $\alpha^{\otimes}$, is replaced by its inverse, then we denote the resulting equivalent version by the same reference without further comment.
(4) If a diagram is commutative because of the naturality of some structure, such as δ^{l}, $\xi^{\oplus}$, $\xi^{\otimes}$, or $\oplus$, then it is denoted by nat.
(5) If a diagram is commutative because of naturality and some other property P, then only P is displayed. ⋄

To simplify the presentation, we adopt the following in most of this section.

Convention 2.2.3. Suppose

$$\big(\mathsf{C},(\oplus,\mathbb{0},\alpha^{\oplus},\lambda^{\oplus},\rho^{\oplus},\xi^{\oplus}),(\otimes,\mathbb{1},\alpha^{\otimes},\lambda^{\otimes},\rho^{\otimes},\xi^{\otimes}),(\lambda^{\bullet},\rho^{\bullet}),(\delta^{l},\delta^{r})\big)$$

consists of the same data as in Definition 2.1.2, in which Laplaza's axioms are *not* assumed, with A, B, C, and D objects in C. $\diamond$

Lemma 2.2.4. *Under Convention 2.2.3, if (2.1.5) is satisfied, then (2.1.6) is equivalent to (2.1.7).*

Proof. The outer diagram below is (2.1.6), and the inner rectangle is (2.1.7).

$$\begin{tikzcd}
A(B\oplus C) \arrow[rrr, "\delta^l"] \arrow[ddd, "1\xi^{\oplus}"'] \arrow[dr, "\xi^{\otimes}"] & & & AB\oplus AC \arrow[dl, "\xi^{\otimes}\oplus\xi^{\otimes}"'] \arrow[ddd, "\xi^{\oplus}"] \\
& (B\oplus C)A \arrow[r, "\delta^r"] \arrow[d, "\xi^{\oplus}1"] \arrow[u, phantom, "(2.1.5)"] \arrow[l, phantom, "\mathsf{nat}"] & BA\oplus CA \arrow[d, "\xi^{\oplus}"'] \arrow[r, phantom, "\mathsf{nat}"] & \\
& (C\oplus B)A \arrow[r, "\delta^r"] \arrow[ur, phantom, "(2.1.7)"] & CA\oplus BA & \\
A(C\oplus B) \arrow[rrr, "\delta^l"] \arrow[ur, "\xi^{\otimes}"] & & & AC\oplus AB \arrow[ul, "\xi^{\otimes}\oplus\xi^{\otimes}"] \arrow[ulll, phantom, "(2.1.5)"]
\end{tikzcd}$$

Since $\xi^{\otimes}$ is a natural isomorphism, the outer diagram is commutative if and only if the inner rectangle is commutative. $\square$

Lemma 2.2.5. *Under Convention 2.2.3, if (2.1.5) is satisfied, then (2.1.8) is equivalent to (2.1.9).*

Proof. The outer diagram below is (2.1.9), and the middle rectangle is (2.1.8).

$$\begin{tikzcd}
A[(B\oplus C)\oplus D] \arrow[rr, "\delta^l"] \arrow[ddddd, "1\alpha^{\oplus}"'] \arrow[ddr, "\xi^{\otimes}"] & & A(B\oplus C)\oplus AD \arrow[rrr, "\delta^l\oplus 1"] \arrow[dr, "1\oplus\xi^{\otimes}"] \arrow[dd, "\xi^{\otimes}\oplus\xi^{\otimes}"'] & & & (AB\oplus AC)\oplus AD \arrow[dl, "1\oplus\xi^{\otimes}"'] \arrow[ddddd, "\alpha^{\oplus}"] \\
& & & A(B\oplus C)\oplus DA \arrow[r, "\delta^l\oplus 1"] \arrow[dl, "\xi^{\otimes}\oplus 1"] & (AB\oplus AC)\oplus DA \arrow[d, "(\xi^{\otimes}\oplus\xi^{\otimes})\oplus 1"'] & \\
& {[(B\oplus C)\oplus D]A} \arrow[r, "\delta^r"] \arrow[d, "\alpha^{\oplus}1"] & (B\oplus C)A\oplus DA \arrow[rr, "\delta^r\oplus 1"] & & (BA\oplus CA)\oplus DA \arrow[d, "\alpha^{\oplus}"'] & \\
& {[B\oplus(C\oplus D)]A} \arrow[r, "\delta^r"] & BA\oplus(C\oplus D)A \arrow[rr, "1\oplus\delta^r"] & & BA\oplus(CA\oplus DA) & \\
& & & AB\oplus(C\oplus D)A \arrow[ul, "\xi^{\otimes}\oplus 1"'] \arrow[r, "1\oplus\delta^r"] & AB\oplus(CA\oplus DA) \arrow[u, "\xi^{\otimes}\oplus 1"] & \\
A[B\oplus(C\oplus D)] \arrow[rr, "\delta^l"'] \arrow[uur, "\xi^{\otimes}"] & & AB\oplus A(C\oplus D) \arrow[rrr, "1\oplus\delta^l"'] \arrow[uu, "\xi^{\otimes}\oplus\xi^{\otimes}"] \arrow[ur, "1\oplus\xi^{\otimes}"] & & & AB\oplus(AC\oplus AD) \arrow[ul, "1\oplus(\xi^{\otimes}\oplus\xi^{\otimes})"']
\end{tikzcd}$$

Since $\xi^{\otimes}$ is a natural isomorphism, the outer diagram is commutative if and only if the middle rectangle is commutative. $\square$

Lemma 2.2.6. *Under Convention 2.2.3, if (2.1.5) is satisfied, then (2.1.10) is equivalent to (2.1.11).*

Proof. The outer diagram below is (2.1.10), and the middle rectangle is (2.1.11).

$$\begin{tikzcd}
(AB)(C\oplus D) \arrow[rrrr, "\delta^l"] \arrow[dr, "\xi^{\otimes}"] \arrow[ddddd, "\alpha^{\otimes}"'] \arrow[drrr, phantom, "(2.1.5)"] & & & & (AB)C\oplus(AB)D \arrow[dl, "\xi^{\otimes}\oplus\xi^{\otimes}"'] \arrow[ddddd, "\alpha^{\otimes}\oplus\alpha^{\otimes}"] \\
& (C\oplus D)(AB) \arrow[rr, "\delta^r"] \arrow[d, "1\xi^{\otimes}"] \arrow[drr, phantom, "\mathrm{nat}"] & & C(AB)\oplus D(AB) \arrow[d, "1\xi^{\otimes}\oplus1\xi^{\otimes}"'] & \\
& (C\oplus D)(BA) \arrow[rr, "\delta^r"] \arrow[l, phantom, "1.3.8"] \arrow[drr, phantom, "(2.1.11)"] & & C(BA)\oplus D(BA) \arrow[r, phantom, "1.3.8"] & \\
& {[(C\oplus D)B]A} \arrow[r, "\delta^r 1"] \arrow[u, "\alpha^{\otimes}"'] \arrow[dr, phantom, "(2.1.5)"] & (CB\oplus DB)A \arrow[r, "\delta^r"] \arrow[dr, phantom, "\mathrm{nat}"] & (CB)A\oplus(DB)A \arrow[u, "\alpha^{\otimes}\oplus\alpha^{\otimes}"] & \\
& {[B(C\oplus D)]A} \arrow[r, "\delta^l 1"] \arrow[u, "\xi^{\otimes}1"'] \arrow[dr, phantom, "\mathrm{nat}"] & (BC\oplus BD)A \arrow[r, "\delta^r"] \arrow[u, "(\xi^{\otimes}\oplus\xi^{\otimes})1"'] \arrow[dr, phantom, "(2.1.5)"] & (BC)A\oplus(BD)A \arrow[u, "\xi^{\otimes}1\oplus\xi^{\otimes}1"] & \\
A[B(C\oplus D)] \arrow[ur, "\xi^{\otimes}"'] \arrow[rr, "1\delta^l"'] & & A(BC\oplus BD) \arrow[rr, "\delta^l"'] \arrow[u, "\xi^{\otimes}"'] & & A(BC)\oplus A(BD) \arrow[ul, "\xi^{\otimes}\oplus\xi^{\otimes}"]
\end{tikzcd}$$

The left and the right trapezoids are commutative by Theorem 1.3.8. Since $\alpha^{\otimes}$ and $\xi^{\otimes}$ are natural isomorphisms, the outer diagram (2.1.10) is commutative if and only if the middle rectangle (2.1.11) is commutative. $\square$

Lemma 2.2.7. *Under Convention 2.2.3, if (2.1.5) and (2.1.10) are satisfied, then so is (2.1.12).*

Proof. The outer diagram below is (2.1.12).

$$
\begin{tikzcd}
{[A(B\oplus C)]D} \arrow[rr, "\delta^l 1"] \arrow[ddddd, "\alpha^{\otimes}"'] & & (AB\oplus AC)D \arrow[rr, "\delta^r"] \arrow[d, phantom, "\text{nat}"', shift right=8ex] \arrow[drr, phantom, "(2.1.5)"] & & (AB)D\oplus(AC)D \arrow[ddddd, "\alpha^{\otimes}\oplus\alpha^{\otimes}"] \\
& D[A(B\oplus C)] \arrow[ul, "\xi^{\otimes}"'] \arrow[r, "1\delta^l"] & D[AB\oplus AC] \arrow[u, "\xi^{\otimes}"] \arrow[r, "\delta^l"] \arrow[d, phantom, "(2.1.10)"] & D(AB)\oplus D(AC) \arrow[ur, "\xi^{\otimes}\oplus\xi^{\otimes}"] & \\
\arrow[r, phantom, "1.3.8"] & (DA)(B\oplus C) \arrow[u, "\alpha^{\otimes}"'] \arrow[rr, "\delta^l"] \arrow[d, "\xi^{\otimes}1"'] & \arrow[d, phantom, "\text{nat}"] & (DA)B\oplus(DA)C \arrow[u, "\alpha^{\otimes}\oplus\alpha^{\otimes}"'] \arrow[d, "\xi^{\otimes}1\oplus\xi^{\otimes}1"] & \arrow[l, phantom, "1.3.8"'] \\
& (AD)(B\oplus C) \arrow[rr, "\delta^l"] \arrow[d, "\alpha^{\otimes}"'] & \arrow[d, phantom, "(2.1.10)"] & (AD)B\oplus(AD)C \arrow[d, "\alpha^{\otimes}\oplus\alpha^{\otimes}"'] & \\
& A[D(B\oplus C)] \arrow[dl, "1\xi^{\otimes}"] \arrow[r, "1\delta^l"] \arrow[d, phantom, "(2.1.5)"] & A(DB\oplus DC) \arrow[r, "\delta^l"] \arrow[d, "1(\xi^{\otimes}\oplus\xi^{\otimes})"'] \arrow[dr, phantom, "\text{nat}"] & A(DB)\oplus A(DC) \arrow[dr, "1\xi^{\otimes}\oplus 1\xi^{\otimes}"'] & \\
A[(B\oplus C)D] \arrow[rr, "1\delta^r"'] & & A(BD\oplus CD) \arrow[rr, "\delta^l"'] & & A(BD)\oplus A(CD)
\end{tikzcd}
$$

The left and the right trapezoids are commutative by Theorem 1.3.8. Since $\alpha^{\otimes}$ and $\xi^{\otimes}$ are invertible, if (2.1.10) is commutative, then so is the outer diagram (2.1.12). □

Lemma 2.2.8. *Under Convention 2.2.3, if (2.1.5) and (2.1.19) are satisfied, then (2.1.15) is equivalent to (2.1.16).*

Proof. Consider the following diagram.

$$
\begin{tikzcd}
\mathbb{0}(A\oplus B) \arrow[rrr, "\delta^l"] \arrow[dd, "\xi^{\otimes}"'] \arrow[dr, "\lambda^{\bullet}"] & \arrow[d, phantom, "(2.1.19)"'] & \arrow[d, phantom, "(2.1.15)"] & \mathbb{0}A\oplus\mathbb{0}B \arrow[dd, "\xi^{\otimes}\oplus\xi^{\otimes}"] \arrow[dl, "\lambda^{\bullet}\oplus\lambda^{\bullet}"'] \\
& \mathbb{0} & \mathbb{0}\oplus\mathbb{0} \arrow[l, "\lambda^{\oplus}"'] \arrow[d, phantom, "(2.1.16)"] \arrow[r, phantom, "(2.1.19)"] & \\
(A\oplus B)\mathbb{0} \arrow[rrr, "\delta^r"] \arrow[ur, "\rho^{\bullet}"'] & & & A\mathbb{0}\oplus B\mathbb{0} \arrow[ul, "\rho^{\bullet}\oplus\rho^{\bullet}"]
\end{tikzcd}
$$

The outer diagram is commutative by (2.1.5). Since $\xi^{\otimes}$, $\lambda^{\oplus}$, and $\rho^{\bullet}$ are natural isomorphisms, the top trapezoid is commutative if and only if the bottom trapezoid is commutative. □

Lemma 2.2.9. *Under Convention 2.2.3, if (2.1.19) is satisfied, then (2.1.17) is equivalent to (2.1.18).*

Proof. Consider the following diagram.

$$\begin{tikzcd}
\mathbb{1}\mathbb{0} \arrow[rr, bend left, "\xi^{\otimes}_{\mathbb{1},\mathbb{0}}"] \arrow[dr, "\rho^{\bullet}_{\mathbb{1}}"] \arrow[dr, bend right=60, "\lambda^{\otimes}_{\mathbb{0}}"'] \arrow[rr, phantom, "(2.1.19)"', near start, shift right=2] & & \mathbb{0}\mathbb{1} \arrow[dl, "\lambda^{\bullet}_{\mathbb{1}}"'] \arrow[dl, bend left=60, "\rho^{\otimes}_{\mathbb{0}}"] \\
 & \mathbb{0} &
\end{tikzcd}$$

$$\text{left triangle: }(2.1.18) \qquad \text{right triangle: }(2.1.17)$$

The outer diagram is commutative by the axioms (1.2.20) and (1.2.21) in the symmetric monoidal category $(\mathsf{C}, \xi^{\otimes})$. Since $\xi^{\otimes}$ is invertible, the left triangle is commutative if and only if the right triangle is commutative. $\square$

Lemma 2.2.10. *Under Convention 2.2.3, if (2.1.19) is satisfied, then (2.1.20) implies both (2.1.21) and (2.1.22).*

Proof. The outer diagram below is (2.1.21).

$$\begin{tikzcd}
(A\mathbb{0})B \arrow[rrrrrr, "\alpha^{\otimes}"] \arrow[ddd, "\rho^{\bullet}1"'] \arrow[dr, "\xi^{\otimes}"] & & & & & & A(\mathbb{0}B) \arrow[ddd, "1\lambda^{\bullet}"] \arrow[dl, "1\xi^{\otimes}"'] \\
 & B(A\mathbb{0}) \arrow[r, "\alpha^{-\otimes}"] \arrow[d, "1\rho^{\bullet}"] & (BA)\mathbb{0} \arrow[ddr, "\rho^{\bullet}"'] & & (AB)\mathbb{0} \arrow[ll, "\xi^{\otimes}1"'] \arrow[ddl, "\rho^{\bullet}"] & A(B\mathbb{0}) \arrow[l, "\alpha^{-\otimes}"'] \arrow[ddr, "1\rho^{\bullet}"'] & \\
 & B\mathbb{0} \arrow[drr, "\rho^{\bullet}"] & & & & & \\
\mathbb{0}B \arrow[ur, "\xi^{\otimes}"] \arrow[rrr, "\lambda^{\bullet}"] & & & \mathbb{0} & & & A\mathbb{0} \arrow[lll, "\rho^{\bullet}"']
\end{tikzcd}$$

The top trapezoid is commutative by Theorem 1.3.8. Therefore, assuming (2.1.19), (2.1.20) implies (2.1.21).

For the other assertion, consider the following diagram.

$$\begin{tikzcd}
(\mathbb{0}A)B \arrow[rr, "\alpha^{\otimes}"] \arrow[dr, "\lambda^{\bullet}1"] \arrow[ddd, "\xi^{\otimes}1"'] & & \mathbb{0}(AB) \arrow[r, "\xi^{\otimes}"] \arrow[d, "\lambda^{\bullet}"'] & (AB)\mathbb{0} \arrow[dl, "\rho^{\bullet}"] \arrow[ddd, "\alpha^{\otimes}"] \\
 & \mathbb{0}B \arrow[r, "\lambda^{\bullet}"] & \mathbb{0} & \\
 & & A\mathbb{0} \arrow[u, "\rho^{\bullet}"] & \\
(A\mathbb{0})B \arrow[uur, "\rho^{\bullet}1"] \arrow[rr, "\alpha^{\otimes}"'] & & A(\mathbb{0}B) \arrow[u, "1\lambda^{\bullet}"] \arrow[r, "1\xi^{\otimes}"'] & A(B\mathbb{0}) \arrow[ul, "1\rho^{\bullet}"']
\end{tikzcd}$$

The outer diagram is commutative by Theorem 1.3.8. In the previous paragraph, we showed that assuming (2.1.19), (2.1.20) implies (2.1.21). Since $\lambda^{\bullet}$, $\rho^{\bullet}$, and $\xi^{\otimes}$

are natural isomorphisms, we infer that (2.1.20) also implies (2.1.22), which is the upper left trapezoid. $\square$

Lemma 2.2.11. *Under Convention 2.2.3, if (2.1.5), (2.1.6), and (2.1.19) are satisfied, then the four axioms (2.1.23), (2.1.24), (2.1.25), and (2.1.26) are equivalent to each other.*

Proof. The outer diagram below is commutative by (2.1.5).

$$\begin{tikzcd}
A(\mathbb{0}\oplus B) \arrow[rrr, "\delta^l"] \arrow[dddd, "\xi^{\otimes}"'] \arrow[dr, "1\lambda^{\oplus}"] & & & A\mathbb{0}\oplus AB \arrow[dl, "\rho^{\bullet}\oplus 1"'] \arrow[dddd, "\xi^{\otimes}\oplus\xi^{\otimes}"] \\
& AB \arrow[dd, "\xi^{\otimes}"] \arrow[r, phantom, "(2.1.23)"] & \mathbb{0}\oplus AB \arrow[l, "\lambda^{\oplus}"'] \arrow[dd, "1\oplus\xi^{\otimes}"'] & \\
\arrow[r, phantom, "\text{nat}"] & \arrow[r, phantom, "\text{nat}"] & \arrow[r, phantom, "(2.1.19)"] & \\
& BA \arrow[r, phantom, "(2.1.24)"] & \mathbb{0}\oplus BA \arrow[l, "\lambda^{\oplus}"'] & \\
(\mathbb{0}\oplus B)A \arrow[rrr, "\delta^r"] \arrow[ur, "\lambda^{\oplus}1"] & & & \mathbb{0}A\oplus BA \arrow[ul, "\lambda^{\bullet}\oplus 1"']
\end{tikzcd}$$

Since $\lambda^{\oplus}$, $\lambda^{\bullet}$, and $\xi^{\otimes}$ are natural isomorphisms, the above diagram shows the equivalence between (2.1.23) and (2.1.24).

The outer diagram below is commutative by (2.1.6).

$$\begin{tikzcd}
A(\mathbb{0}\oplus B) \arrow[rrr, "\delta^l"] \arrow[dddd, "1\xi^{\oplus}"'] \arrow[dr, "1\lambda^{\oplus}"] & & & A\mathbb{0}\oplus AB \arrow[dl, "\rho^{\bullet}\oplus 1"'] \arrow[dddd, "\xi^{\oplus}"] \\
& AB \arrow[dd, "1"] \arrow[r, phantom, "(2.1.23)"] & \mathbb{0}\oplus AB \arrow[l, "\lambda^{\oplus}"'] \arrow[dd, "\xi^{\oplus}"'] & \\
\arrow[r, phantom, "(1.2.20)\ (1.2.21)"] & \arrow[r, phantom, "(1.2.20)+(1.2.21)"] & \arrow[r, phantom, "\text{nat}"] & \\
& AB \arrow[r, phantom, "(2.1.25)"] & AB\oplus\mathbb{0} \arrow[l, "\rho^{\oplus}"'] & \\
A(B\oplus\mathbb{0}) \arrow[rrr, "\delta^l"] \arrow[ur, "1\rho^{\oplus}"] & & & AB\oplus A\mathbb{0} \arrow[ul, "1\oplus\rho^{\bullet}"']
\end{tikzcd}$$

Since $\lambda^{\oplus}$, $\rho^{\oplus}$, $\xi^{\oplus}$, and $\rho^{\bullet}$ are natural isomorphisms, the above diagram shows the equivalence between (2.1.23) and (2.1.25). Note that this diagram is valid even without assuming (2.1.5), (2.1.19), and the existence of $\xi^{\otimes}$.

The outer diagram below is commutative by (2.1.5).

$$
\begin{tikzcd}
A(B\oplus\mathbb{0}) \arrow[rrr, "\delta^l"] \arrow[dr, "1\rho^{\oplus}"] \arrow[ddd, "\xi^{\otimes}"'] \arrow[drr, phantom, "(2.1.25)"] & & & AB\oplus A\mathbb{0} \arrow[dl, "1\oplus\rho^{\bullet}"'] \arrow[ddd, "\xi^{\otimes}\oplus\xi^{\otimes}"] \\
\arrow[r, phantom, "\text{nat}"] & AB \arrow[d, "\xi^{\otimes}"] \arrow[r, phantom, "\text{nat}" near end, shift right=4ex] & AB\oplus\mathbb{0} \arrow[l, "\rho^{\oplus}"'] \arrow[d, "\xi^{\otimes}\oplus 1"'] \arrow[r, phantom, "(2.1.19)"] & \\
& BA & BA\oplus\mathbb{0} \arrow[l, "\rho^{\oplus}"'] & \\
(B\oplus\mathbb{0})A \arrow[rrr, "\delta^r"'] \arrow[ur, "\rho^{\oplus}1"'] \arrow[urr, phantom, "(2.1.26)"] & & & BA\oplus\mathbb{0}A \arrow[ul, "1\oplus\lambda^{\bullet}"]
\end{tikzcd}
$$

Since $\rho^{\oplus}$, $\lambda^{\bullet}$, and $\xi^{\otimes}$ are natural isomorphisms, the above diagram shows the equivalence between (2.1.25) and (2.1.26). $\square$

Lemma 2.2.12. *Under Convention 2.2.3, if (2.1.5) is satisfied, then (2.1.27) is equivalent to (2.1.28).*

Proof. The outer diagram below is commutative by (2.1.5).

$$
\begin{tikzcd}
\mathbb{1}(A\oplus B) \arrow[rr, "\delta^l"] \arrow[dr, "\lambda^{\otimes}"'] \arrow[dd, "\xi^{\otimes}"'] & (2.1.27) & \mathbb{1}A\oplus\mathbb{1}B \arrow[dl, "\lambda^{\otimes}\oplus\lambda^{\otimes}"] \arrow[dd, "\xi^{\otimes}\oplus\xi^{\otimes}"] \\
& A\oplus B & \\
(A\oplus B)\mathbb{1} \arrow[rr, "\delta^r"'] \arrow[ur, "\rho^{\otimes}"] & (2.1.28) & A\mathbb{1}\oplus B\mathbb{1} \arrow[ul, "\rho^{\otimes}\oplus\rho^{\otimes}"']
\end{tikzcd}
$$

The left and right triangles are commutative by the axioms (1.2.20) and (1.2.21) in the symmetric monoidal category $(\mathsf{C},\xi^{\otimes})$. Since $\lambda^{\otimes}$ and $\rho^{\otimes}$ are natural isomorphisms, the above diagram shows the equivalence between (2.1.27) and (2.1.28). $\square$

Combining Lemmas 2.2.4 through 2.2.12, we obtain the following main observation of this section.

Theorem 2.2.13. *In Definition 2.1.2 of a symmetric bimonoidal category, it is sufficient to assume the first axiom in each of the twelve groups of axioms, namely, (2.1.5), (2.1.6), (2.1.8), (2.1.10), (2.1.13), (2.1.14), (2.1.15), (2.1.17), (2.1.19), (2.1.20), (2.1.23), and (2.1.27).*

As noted at the beginning of this section, Theorem 2.2.13 does not apply to bimonoidal categories because its proof uses the multiplicative symmetry $\xi^{\otimes}$. There is only one exception as follows.

Proposition 2.2.14. *In a bimonoidal category, the two axioms (2.1.23) and (2.1.25) are equivalent, so only one of them is necessary.*

Proof. The second diagram in the proof of Lemma 2.2.11, which establishes the equivalence between (2.1.23) and (2.1.25), is valid in a bimonoidal category because it does not involve the multiplicative symmetry. □

2.3. Distributive Symmetric Monoidal Categories

In this section, we discuss a class of examples of symmetric bimonoidal categories whose sums are coproducts.

Definition 2.3.1. A symmetric monoidal category C is said to be *distributive* if the following two conditions hold.

(1) C has finite coproducts. Fix an initial object $\varnothing \in \mathsf{C}$.
(2) The natural morphisms

$$\coprod_{i=1}^n (A \otimes B_i) \xrightarrow{d_l} A \otimes \left(\coprod_{i=1}^n B_i\right)$$

$$\coprod_{i=1}^n (B_i \otimes A) \xrightarrow{d_r} \left(\coprod_{i=1}^n B_i\right) \otimes A$$

are isomorphisms for $n = 0, 2$ and for objects $A, B_i \in \mathsf{C}$, with $\coprod$ denoting coproduct, and with an empty coproduct denoting $\varnothing$. ⋄

Recall that a symmetric bimonoidal category is *tight* if both distributivity δ^l and δ^r are natural isomorphisms.

Proposition 2.3.2. *Suppose C is a distributive symmetric monoidal category. Then C yields a tight symmetric bimonoidal category with the following data.*

The Multiplicative Structure: *It is the given symmetric monoidal structure on C.*
The Additive Structure: *It is $(\mathsf{C}, \amalg, \varnothing)$, with $\alpha^\oplus$, $\lambda^\oplus$, $\rho^\oplus$, and $\xi^\oplus$ given by the universal properties of coproducts and $\varnothing$.*
The Multiplicative Zeros: *$\lambda^\bullet = d_r^{-1}$ and $\rho^\bullet = d_l^{-1}$ in the case $n = 0$.*
The Distributivity Morphisms: *$\delta^l = d_l^{-1}$ and $\delta^r = d_r^{-1}$ in the case $n = 2$.*

Proof. The universal properties of coproducts and $\varnothing$ imply that $(\mathsf{C}, \amalg, \varnothing)$ is a symmetric monoidal category. By definition, δ^l and δ^r are natural isomorphisms. The twelve axioms in Theorem 2.2.13 all follow from the university properties of coproducts and $\varnothing$, and the naturality of the multiplicative structure. □

Below are some examples of distributive symmetric monoidal categories, to which Proposition 2.3.2 applies to yield tight symmetric bimonoidal categories.

Example 2.3.3 (Symmetric Monoidal Closed Categories)**.** If C is a symmetric monoidal closed category as in Definition 1.2.28 with finite coproducts, then the natural morphisms d_l and d_r are isomorphisms. Therefore, C is a distributive symmetric monoidal category. ⋄

Example 2.3.4 (Modules)**.** For each commutative ring R, the category $\mathsf{Mod}(R)$ of R-modules is symmetric monoidal with

- the usual tensor product of R-modules;
- coproducts given by direct sums $\oplus$ of R-modules; and
- internal hom given by R-linear maps.

As a special case of Example 2.3.3, $\mathsf{Mod}(R)$ is a distributive symmetric monoidal category, hence also a symmetric bimonoidal category with $(\amalg, \varnothing) = (\oplus, 0)$. ⋄

Example 2.3.5 (Distributive Categories)**.** A *distributive category* is a category with finite products $\times$ and finite coproducts such that the natural morphism d_l is an isomorphism for $n = 0, 2$ and $\otimes = \times$. The symmetry isomorphism of $\times$ implies that d_r is also an isomorphism for $n = 0, 2$. Therefore, a distributive category is a distributive symmetric monoidal category, hence also a symmetric bimonoidal category with $\otimes = \times$, $\mathbb{1} = *$ (a chosen terminal object), $\oplus = \amalg$, and $\mathbb{0} = \varnothing$ (a chosen initial object). $\diamond$

The symmetric bimonoidal categories in Proposition 2.3.2 and Examples 2.3.3 through 2.3.5 all have $(\oplus, \mathbb{0}) = (\amalg, \varnothing)$. In other words, the additive structure is given by coproducts. For the symmetric bimonoidal categories Σ and Σ' in Section 2.4, their sums are *not* given by coproducts.

2.4. Finite Ordinal Category

In this section, we discuss a symmetric bimonoidal category Σ that will play an important role in Chapters 4, 7, and II.7.

(1) In Chapter 4, Σ is extended to the distortion category $\mathcal{D}$ that is a crucial ingredient in formulating and proving the Coherence Theorem 4.4.3.
(2) In Chapter 7, we show that Σ is bicategorically an initial object in a suitable 2-category of small symmetric bimonoidal categories.
(3) In Chapter II.7, we show that Σ is bicategorically an initial object in a suitable 2-category of small braided bimonoidal categories.

A variation of Σ is discussed in the second half of this section. Further explanation of the formulas in the next definition is given in Explanation 2.4.7.

Definition 2.4.1. Define Σ by the following data.

Category: As a category, Σ is defined as follows.

Objects: They are nonnegative integers $n \geq 0$.

Morphisms: For objects m and n, the morphism set is

$$\Sigma(m,n) = \begin{cases} \Sigma_n & \text{if } m = n \text{, and} \\ \varnothing & \text{if } m \neq n. \end{cases}$$

Here Σ_n is the symmetric group on n letters.

Composition is that of bijections, and $\mathrm{id}_n \in \Sigma_n$ is the identity morphism on n.

The Additive Structure: The functor

$$\Sigma \times \Sigma \xrightarrow{\oplus} \Sigma$$

is defined on objects by

$$m \oplus n = m + n,$$

and on morphisms

$$\Sigma_m \times \Sigma_n \xrightarrow{\oplus} \Sigma_{m+n}$$

by the *block sum*

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq m \text{, and} \\ \tau(i-m) + m & \text{if } m+1 \leq i \leq m+n \end{cases} \tag{2.4.2}$$

for $(\sigma, \tau) \in \Sigma_m \times \Sigma_n$.

The additive zero $\mathbb{0}$ is the object 0, which is a strict two-sided unit for $\oplus$. The additive structure isomorphisms $\alpha^{\oplus}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$ are identities. The additive symmetry isomorphism

$$m \oplus n = m + n \xrightarrow{\xi^{\oplus}_{m,n}} n + m = n \oplus m$$

is the bijection defined by

$$\xi^{\oplus}_{m,n}(j) = \begin{cases} j + n & \text{if } 1 \leq j \leq m, \text{ and} \\ j - m & \text{if } m + 1 \leq j \leq m + n. \end{cases} \tag{2.4.3}$$

The Multiplicative Structure: The functor

$$\Sigma \times \Sigma \xrightarrow{\otimes} \Sigma$$

is defined on objects by

$$m \otimes n = mn,$$

and on morphisms

$$\Sigma_m \times \Sigma_n \xrightarrow{\otimes} \Sigma_{mn}$$

by

$$(\sigma \otimes \tau)\big(i + (j-1)m\big) = \sigma(i) + \big(\tau(j) - 1\big)m \tag{2.4.4}$$

for $(\sigma, \tau) \in \Sigma_m \times \Sigma_n$, $1 \leq i \leq m$, and $1 \leq j \leq n$.

The multiplicative unit $\mathbb{1}$ is the object 1, which is a strict two-sided unit for $\otimes$. The multiplicative structure isomorphisms $\alpha^{\otimes}$, $\lambda^{\otimes}$, and $\rho^{\otimes}$ are identities.

The multiplicative symmetry isomorphism

$$m \otimes n = mn \xrightarrow{\xi^{\otimes}_{m,n}} nm = n \otimes m$$

is the bijection defined by

$$\xi^{\otimes}_{m,n}\big(i + (j-1)m\big) = j + (i-1)n \tag{2.4.5}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

The Multiplicative Zeros: The left multiplicative zero $\lambda^{\bullet}$ and the right multiplicative zero $\rho^{\bullet}$ are both identity natural transformations. This is well defined because

$$m \otimes 0 = 0 = 0 \otimes m.$$

Distributivity: The left distributivity morphism

$$m \otimes \big(n \oplus p\big) \xrightarrow{\delta^{l}_{m,n,p}} \big(m \otimes n\big) \oplus \big(m \otimes p\big)$$

is the identity permutation in $\Sigma_{m(n+p)}$. The right distributivity morphism

$$\big(m \oplus n\big) \otimes p \xrightarrow{\delta^{r}_{m,n,p}} \big(m \otimes p\big) \oplus \big(n \otimes p\big)$$

is the permutation in $\Sigma_{(m+n)p}$ defined by

$$\begin{aligned} \delta^r\big(i + (k-1)(m+n)\big) &= i + (k-1)m \\ \delta^r\big(j + m + (k-1)(m+n)\big) &= j + (k-1)n + pm \end{aligned} \tag{2.4.6}$$

for $1 \le i \le m$, $1 \le j \le n$, and $1 \le k \le p$.

This finishes the definition of Σ. $\diamond$

Explanation 2.4.7. Here we provide geometric intuition of the constructions in Definition 2.4.1.

Objects and morphisms: The object $m \in \Sigma$ is regarded as an interval with m objects, arranged from left to right. A morphism in Σ permutes the objects within such an interval.

Sums: The object $m \oplus n = m + n$ is the combined interval with m on the left and n on the right.

Block sums: In the block sum $\sigma \oplus \tau$ in (2.4.2), σ acts on the interval on the left, and τ acts on the interval on the right.

Additive symmetry: $\xi^{\oplus}_{m,n}$ in (2.4.3) interchanges the two intervals and leaves the order within each interval unchanged.

Products: The object $m \otimes n = mn$ consists of n intervals, each with m objects. The elements in n are now used as indices for these n intervals. Alternatively, it is an $n \times m$ matrix with each row a copy of m and with the rows indexed by n.

Product of morphisms: In the bijection $\sigma \otimes \tau \in \Sigma_{mn}$ in (2.4.4), $\tau \in \Sigma_n$ permutes the n intervals, and $\sigma \in \Sigma_m$ permutes within each interval. In the matrix description, τ permutes the n rows, and σ permutes the m columns.

Multiplicative symmetry: $\xi^{\otimes}_{m,n}$ in (2.4.5) redistributes n intervals of m objects each to m intervals of n objects each. Specifically, it sends the ith object in the jth interval in the domain to the jth object in the ith interval in the codomain. In the matrix description, this corresponds to taking the transpose of an $n \times m$ matrix.

Left distributivity: $\delta^l_{m,n,p}$ is the identity because $(n+p)$ intervals of m objects each is already equal to n intervals of m objects each followed by p intervals of m objects each.

Right distributivity: $\delta^r_{m,n,p}$ in (2.4.6) redistributes p intervals of $(m+n)$ objects each to p intervals of m objects each followed by p intervals of n objects each. Alternatively, in the matrix description, we first arrange $(m+n)p$ objects into a $p \times (m+n)$ matrix of the form $[A|B]$ with
- A the $p \times m$ matrix consisting of the first m columns and
- B the $p \times n$ matrix consisting of the last n columns.

Then this component of δ^r rearranges the matrix $[A|B]$ to the array $\left[\frac{A}{B}\right]$, leaving the order of the entries in each of A and B unchanged. The two rows in (2.4.6) correspond to the action of δ^r on, respectively, the (k,i)-entry in A and the (k,j)-entry in B. $\diamond$

Recall from Definitions 1.2.1 and 1.2.18 that a *permutative category* is a symmetric strict monoidal category.

Proposition 2.4.8. *Σ in Definition 2.4.1 is a small and tight symmetric bimonoidal category whose additive structure and multiplicative structure are both permutative categories.*

Proof. That the additive structure is a strict monoidal category follows from the fact that $\alpha^{\oplus}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$ are identities. The unit axiom (1.2.21) holds because $\xi^{\oplus}_{m,0}$ is the identity map. The symmetry axiom (1.2.20) and the hexagon axiom (1.2.22) both follow from the description in Explanation 2.4.7 of $\xi^{\oplus}_{m,n}$ as interchanging two consecutive intervals. So the additive structure is a symmetric strict monoidal category.

To see that the multiplicative associativity can be defined as the identity, suppose $\sigma \in \Sigma_m$, $\tau \in \Sigma_n$, and $\pi \in \Sigma_p$. Then both $(\sigma \otimes \tau) \otimes \pi$ and $\sigma \otimes (\tau \otimes \pi)$ in Σ_{mnp} are given by the bijection

$$i + (j-1)m + (k-1)(mn) \longmapsto \sigma(i) + \big(\tau(j)-1\big)m + \big(\pi(k)-1\big)(mn) \tag{2.4.9}$$

for $1 \le i \le m$, $1 \le j \le n$, and $1 \le k \le p$. See Explanation 2.4.14. Therefore, the identity natural transformation $\alpha^{\otimes}$ is well defined. Both identity natural transformations $\lambda^{\otimes}$ and $\rho^{\otimes}$ are well defined by (2.4.4) and that $\mathbb{1} = 1$ is a strict two-sided unit for $\otimes$. That the multiplicative structure is a strict monoidal category follows from the fact that $\alpha^{\otimes}$, $\lambda^{\otimes}$, and $\rho^{\otimes}$ are identities.

Next we check the symmetric monoidal category axioms.

- The unit axiom (1.2.21) holds because $\xi^{\otimes}_{m,1}$ is the identity map.
- The symmetry axiom (1.2.20) holds by the description of $\xi^{\otimes}_{m,n}$ as redistributing n intervals of m objects each to m intervals of n objects each. Alternatively, regarding $\xi^{\otimes}_{m,n}$ as taking the transpose of an $n \times m$ matrix, its inverse is the transpose given by $\xi^{\otimes}_{n,m}$.
- The hexagon axiom (1.2.22) is equivalent to the commutativity of the following diagram for $m, n, p \ge 0$.

$$\begin{tikzcd} m \otimes n \otimes p \arrow[rr, "1_m \otimes \xi^{\otimes}_{n,p}"] \arrow[dr, "\xi^{\otimes}_{mn,p}"'] & & m \otimes p \otimes n \arrow[dl, "\xi^{\otimes}_{m,p} \otimes 1_n"] \\ & p \otimes m \otimes n & \end{tikzcd}$$

  This diagram is commutative because both composites are given by the bijection

$$i + (j-1)m + (k-1)(mn) \longmapsto k + [i - 1 + (j-1)m]\, p \tag{2.4.10}$$

  for $1 \le i \le m$, $1 \le j \le n$, and $1 \le k \le p$. See Explanation 2.4.14.

Therefore, the multiplicative structure is a symmetric strict monoidal category.

For each of the twelve axioms listed in Theorem 2.2.13 for Σ, we check that the two relevant permutations are equal. First, for the axiom (2.1.6) with $A = m$, $B = n$, and $C = p$, each of the two composites is the permutation in $\Sigma_{m(n+p)}$ given by

$$\begin{cases} i + (j-1)m \longmapsto i + (p+j-1)m \\ i + (n+k-1)m \longmapsto i + (k-1)m \end{cases} \tag{2.4.11}$$

for $1 \le i \le m$, $1 \le j \le n$, and $1 \le k \le p$. See Explanation 2.4.15.

For the axiom (2.1.5), with the same notation as above, each of the two composites is the permutation in $\Sigma_{(m+n)p}$ given by

$$\begin{cases} i+(k-1)(m+n) \longmapsto k+(i-1)p \\ j+m+(k-1)(m+n) \longmapsto k+(m+j-1)p \end{cases} \tag{2.4.12}$$

for $1 \le i \le m$, $1 \le j \le n$, and $1 \le k \le p$. See Explanation 2.4.16.

For the 2-by-2 distributivity axiom (2.1.13) with $D = q$, each of the two composites is the permutation in $\Sigma_{(m+n)(p+q)}$ given by

$$\begin{cases} i+(k-1)(m+n) \longmapsto i+(k-1)m \\ j+m+(k-1)(m+n) \longmapsto j+(k-1)n+mp \\ i+(m+n)(p+l-1) \longmapsto i+(l-1)m+(m+n)p \\ j+m+(m+n)(p+l-1) \longmapsto j+(l-1)n+(m+n)p+mq \end{cases} \tag{2.4.13}$$

for $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$, and $1 \le l \le q$. See Explanation 2.4.17.

The other nine axioms—(2.1.9), which is equivalent to (2.1.8) by Lemma 2.2.5, (2.1.10), (2.1.14), (2.1.15), (2.1.17), (2.1.19), (2.1.20), (2.1.23), and (2.1.27)—hold because each permutation involved is the identity. □

We now provide geometric description of the bijections that appeared in the proof of Proposition 2.4.8.

Explanation 2.4.14. For the bijection in (2.4.10), suppose

$$A = [\bullet, \ldots, \bullet]$$

is a $1 \times m$ matrix with m objects, and M is the $p \times mn$ matrix

$$M = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pn} \end{bmatrix}$$

with each A_{kj} a copy of A for $1 \le j \le n$ and $1 \le k \le p$. Then the bijection in (2.4.10), namely $\xi^{\otimes}_{mn,p}$, sends M to its transpose

$$M^T = \begin{bmatrix} A^T_{11} & \cdots & A^T_{p1} \\ \vdots & \ddots & \vdots \\ A^T_{1n} & \cdots & A^T_{pn} \end{bmatrix}$$

with the superscript T denoting transpose. The formula (2.4.10) is the action of this bijection on the ith object in A_{kj}.

For the bijection in (2.4.9), namely

$$(\sigma \otimes \tau) \otimes \pi = \sigma \otimes (\tau \otimes \pi) \in \Sigma_{mnp},$$

regard M as a $p \times n$ matrix with (k, j)-entry A_{kj}. Then each of these two bijections permutes

- the p rows of M via $\pi \in \Sigma_p$;
- the n columns of M via $\tau \in \Sigma_n$; and
- the objects in each $1 \times m$ matrix $A_{kj} = A$ via $\sigma \in \Sigma_m$. ◇

Explanation 2.4.15. For the bijection in (2.4.11), first arrange $m(n+p)$ objects into an $(n+p) \times m$ matrix of the form $M = \left[\frac{A}{B}\right]$ with

- A the $n \times m$ matrix consisting of the top n rows and

- B the $p \times m$ matrix consisting of the bottom p rows.

Then the bijection in (2.4.11) rearranges the matrix $\left[\frac{A}{B}\right]$ to $\left[\frac{B}{A}\right]$ by swapping the blocks A and B. The two rows in (2.4.11) correspond to the action of this bijection on, respectively,

- the (j,i)-entry in A in M and
- the (k,i)-entry in B in M. $\diamond$

Explanation 2.4.16. For the bijection in (2.4.12), first arrange $(m+n)p$ objects into a $p \times (m+n)$ matrix of the form $M = [A|B]$ with

- A the $p \times m$ matrix consisting of the first m columns and
- B the $p \times n$ matrix consisting of the last n columns.

Then the bijection in (2.4.12) sends the matrix $[A|B]$ to its transpose $\left[\frac{A^T}{B^T}\right]$, with A^T and B^T denoting the transposes of, respectively, A and B. The two rows in (2.4.12) correspond to the action of this bijection on, respectively,

- the (k,i)-entry in A in M and
- the (k,j)-entry in B in M. $\diamond$

Explanation 2.4.17. For the bijection in (2.4.13), first arrange $(m+n)(p+q)$ objects into a $(p+q) \times (m+n)$ matrix M of the form

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

with

- A a $p \times m$ matrix,
- B a $p \times n$ matrix,
- C a $q \times m$ matrix, and
- D a $q \times n$ matrix.

Then the bijection in (2.4.13) rearranges M to the array

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}.$$

The four rows in (2.4.13) correspond to the action of this bijection on, respectively,

- the (k,i)-entry in A in M,
- the (k,j)-entry in B in M,
- the (l,i)-entry in C in M, and
- the (l,j)-entry in D in M. $\diamond$

Next we describe a variant of Σ in which the right distributivity is an identity.

Definition 2.4.18. Define Σ' by the following data.

The Additive Structure: The additive structure

$$(\Sigma', \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$$

is the same as that of Σ in Definition 2.4.1.

The Multiplicative Structure: The functor

$$\Sigma' \times \Sigma' \xrightarrow{\otimes} \Sigma'$$

is again defined by $m \otimes n = mn$ on objects, and by

$$(\sigma \otimes \tau)\big(j + (i-1)n\big) = \tau(j) + \big(\sigma(i) - 1\big)n \tag{2.4.19}$$

on morphisms for $(\sigma, \tau) \in \Sigma_m \times \Sigma_n$, $1 \le i \le m$, and $1 \le j \le n$.

The multiplicative unit $\mathbb{1}$ is the object 1, which is a strict two-sided unit for $\otimes$. The multiplicative structure isomorphisms $\alpha^{\otimes}$, $\lambda^{\otimes}$, and $\rho^{\otimes}$ are identities.

The multiplicative symmetry isomorphism

$$mn \xrightarrow{\xi'^{\otimes}_{m,n}} mn$$

is defined by

$$\xi'^{\otimes}_{m,n}\big(j + (i-1)n\big) = i + (j-1)m \tag{2.4.20}$$

for $1 \le i \le m$ and $1 \le j \le n$.

The Multiplicative Zeros: Both $\lambda^{\bullet}$ and $\rho^{\bullet}$ are identity natural transformations.

Distributivity: The right distributivity morphism δ^r is the identity natural transformation. The left distributivity morphism

$$m \otimes \big(n \oplus p\big) \xrightarrow{\delta^l_{m,n,p}} \big(m \otimes n\big) \oplus \big(m \otimes p\big)$$

is the permutation in $\Sigma_{m(n+p)}$ defined by

$$\begin{aligned} \delta^l\big(j + (i-1)(n+p)\big) &= j + (i-1)n \\ \delta^l\big(k + n + (i-1)(n+p)\big) &= k + (i-1)p + mn \end{aligned} \tag{2.4.21}$$

for $1 \le i \le m$, $1 \le j \le n$, and $1 \le k \le p$.

This finishes the definition of Σ'. $\diamond$

Explanation 2.4.22. The additive structure of Σ' has the same geometric interpretation as that of Σ in Explanation 2.4.7. Consider the rest of Σ' in Definition 2.4.18.

Products: The object $m \otimes n = mn$ is now regarded as consisting of m intervals, each with n objects. Alternatively, it is an $m \times n$ matrix with each row a copy of n and with the rows indexed by m.

Product of morphisms: In the bijection $\sigma \otimes \tau \in \Sigma_{mn}$ in (2.4.19), $\sigma \in \Sigma_m$ permutes the m intervals, and $\tau \in \Sigma_n$ permutes within each interval. In the matrix description, σ permutes the m rows, and τ permutes the n columns.

Multiplicative symmetry: $\xi'^{\otimes}_{m,n}$ in (2.4.20) redistributes m intervals of n objects each to n intervals of m objects each by sending the jth object in the ith interval in the domain to the ith object in the jth interval in the codomain. In the matrix description, this corresponds to taking the transpose of an $m \times n$ matrix.

Right distributivity: $\delta^r_{m,n,p}$ is the identity because $(m+n)$ intervals of p objects each is already equal to m intervals of p objects each followed by n intervals of p objects each.

Left distributivity: $\delta^l_{m,n,p}$ in (2.4.21) redistributes m intervals of $(n+p)$ objects each to m intervals of n objects each followed by m intervals of p objects each. Alternatively, in the matrix description, we first arrange $m(n+p)$ objects into an $m \times (n+p)$ matrix of the form $[A|B]$ with

- A the $m \times n$ matrix consisting of the first n columns and

- B the $m \times p$ matrix consisting of the last p columns.

Then this component of δ^l rearranges the matrix $[A|B]$ to the array $\left[\frac{A}{B}\right]$, leaving the order of the entries in each of A and B unchanged. The two rows in (2.4.21) correspond to the action of δ^l on, respectively, the (i,j)-entry in A and the (i,k)-entry in B. $\diamond$

A minor modification of the proof of Proposition 2.4.8 yields the following.

Proposition 2.4.23. *Σ' in Definition 2.4.18 is a small and tight symmetric bimonoidal category whose additive structure and multiplicative structure are both permutative categories.*

2.5. Bipermutative Categories

In this section, we introduce strict analogues of a symmetric bimonoidal category, which are called a right, respectively left, bipermutative category. We observe that right and left bipermutative categories are tight symmetric bimonoidal categories. In Chapter 5, we will show that tight symmetric bimonoidal categories can be strictified to right and left bipermutative categories. First we define right bipermutative categories.

Right Bipermutative Categories.

Motivation 2.5.1. To motivate the definition of a right bipermutative category, recall from Section 2.4 the two symmetric bimonoidal categories, Σ and Σ', whose additive and multiplicative structures are both permutative categories, that is, symmetric strict monoidal categories. Their left and right multiplicative zeros, $\lambda^\bullet$ and $\rho^\bullet$, are both identities. However, only one of the two distributivity morphisms is an identity: δ^l in Σ and δ^r in Σ'. These two examples suggest that, in defining a strict analogue of a symmetric bimonoidal category, we should demand that the multiplicative zeros and only one of the two distributivity morphisms be the identities. In a right bipermutative category, the right distributivity morphism δ^r is the identity. $\diamond$

Definition 2.5.2. A *right bipermutative category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\lambda^\bullet, \rho^\bullet), (\delta^l, \delta^r)\big)$$

consisting of the following data.

- $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ and $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ are permutative categories.
- $\lambda^\bullet$, $\rho^\bullet$, δ^l, and δ^r are natural transformations as in (2.1.3) and (2.1.4).

The above data are required to satisfy the following four conditions.

(1) $\lambda^\bullet$ and $\rho^\bullet$ are both equal to the identity natural transformation of the constant functor $\mathsf{C} \longrightarrow \mathsf{C}$ at $\mathbb{0}$.
(2) δ^r is an identity natural transformation.
(3) $\xi^{\otimes}_{A,\mathbb{0}} : A \otimes \mathbb{0} \longrightarrow \mathbb{0} \otimes A$ is the identity morphism of $\mathbb{0}$ for each object A.
(4) The axioms (2.1.5), (2.1.7), and (2.1.13) are satisfied.

This finishes the definition of a right bipermutative category. $\diamond$

Explanation 2.5.3. Consider Definition 2.5.2.

- The axiom (2.1.5) is equivalent to the following commutative diagram.

$$\begin{tikzcd} A(B\oplus C) \arrow[r, "\delta^l"] \arrow[d, "\xi^{\otimes}"'] & AB\oplus AC \\ (B\oplus C)A \arrow[r, "\delta^r", "="'] & BA\oplus CA \arrow[u, "\xi^{\otimes}\oplus\xi^{\otimes}"'] \end{tikzcd} \tag{2.5.4}$$

In particular, the left distributivity morphism δ^l is a natural isomorphism.

- The axiom (2.1.7) means the commutativity of the following diagram.

$$\begin{tikzcd} (A\oplus B)C \arrow[r, "\delta^r", "="'] \arrow[d, "\xi^{\oplus}1"'] & AC\oplus BC \arrow[d, "\xi^{\oplus}"] \\ (B\oplus A)C \arrow[r, "\delta^r", "="'] & BC\oplus AC \end{tikzcd} \tag{2.5.5}$$

- The axiom (2.1.13) means the commutativity of the following diagram.

$$\begin{tikzcd} (A\oplus B)(C\oplus D) \arrow[rr, "\delta^r", "="'] \arrow[d, "\delta^l"'] & & A(C\oplus D)\oplus B(C\oplus D) \arrow[d, "\delta^l\oplus\delta^l"] \\ (A\oplus B)C\oplus(A\oplus B)D \arrow[dr, "="', "\delta^r\oplus\delta^r"] & & AC\oplus AD\oplus BC\oplus BD \arrow[dl, "1\oplus\xi^{\oplus}\oplus 1"] \\ & AC\oplus BC\oplus AD\oplus BD & \end{tikzcd} \tag{2.5.6}$$

$\diamond$

Recall from Definition 2.1.2 that a symmetric bimonoidal category is *tight* if both δ^l and δ^r are natural isomorphisms.

Proposition 2.5.7. *Each right bipermutative category is a tight symmetric bimonoidal category.*

Proof. The right distributivity morphism δ^r is the identity by definition. The left distributivity morphism δ^l is a natural isomorphism by (2.5.4). Consider the 24 axioms in Definition 2.1.2 for a right bipermutative category.

- The axioms (2.1.5), (2.1.7), and (2.1.13) hold by assumption.
- The axiom (2.1.6) holds by Lemma 2.2.4.
- The axiom (2.1.19) holds by the identity assumptions on $\lambda^{\bullet}$, $\rho^{\bullet}$, and $\xi^{\otimes}_{-,\mathbb{0}}$.
- The axioms (2.1.8), (2.1.11), (2.1.14), (2.1.16), (2.1.17), (2.1.20), (2.1.24), and (2.1.28) hold by the assumption that $\lambda^{\bullet}$, $\rho^{\bullet}$, and δ^r are identities.

We finish the proof by applying Lemmas 2.2.5 through 2.2.12 to obtain all 24 axioms of a symmetric bimonoidal category. $\square$

Example 2.5.8. The symmetric bimonoidal category Σ' in Proposition 2.4.23 is a right bipermutative category. $\diamond$

Example 2.5.9 (Coordinatized Vector Spaces)**.** A variation of the tight symmetric bimonoidal category $\mathsf{Vect}^{\mathbb{C}}$ in Example 2.1.32 is the right bipermutative category $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$ of *coordinatized* finite dimensional complex vector spaces defined as follows.

- Its objects are nonnegative integers.

- A morphism $m \longrightarrow n$ is an $n \times m$ complex matrix.
- Categorical composition is given by matrix multiplication.
- The identity morphism $1_n : n \longrightarrow n$ is the $n \times n$ identity matrix.

It becomes a right bipermutative category with the following data.

- $m \oplus n = m + n$ on objects, with 0 as the additive zero.
- For complex matrices A and B, $A \oplus B$ is the matrix direct sum
$$A \oplus B = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$
with each 0 representing a 0 matrix of appropriate size.
- $\xi^{\oplus}_{m,n} : m + n \longrightarrow n + m$ is the permutation matrix
$$\xi^{\oplus}_{m,n} = \left[\begin{array}{c|c} 0 & 1_n \\ \hline 1_m & 0 \end{array}\right]$$
This permutation matrix is obtained from the identity matrix 1_{m+n} by permuting its $m + n$ rows using the block permutation in (2.4.3).
  - For a matrix C with $n + m$ columns, $C\xi^{\oplus}_{m,n}$ is obtained from C by swapping the first n columns with the last m columns.
  - For a matrix D with $m + n$ rows, $\xi^{\oplus}_{m,n}D$ is obtained from D by swapping the first m rows with the last n rows.
- $m \otimes n = mn$ on objects, with 1 as the multiplicative unit.
- For complex matrices
  - $A = (A_{ji}) : m \longrightarrow n$ and
  - $B = (B_{lk}) : p \longrightarrow q$,

  $A \otimes B : mp \longrightarrow nq$ is the following matrix tensor product.
$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nm}B \end{bmatrix}$$
Each $A_{ji}B = (A_{ji}B_{lk})_{l,k}$ is the scalar product.
- $\xi^{\otimes}_{m,n} : mn \longrightarrow nm$ is the permutation matrix obtained from the identity matrix 1_{mn} by permuting its mn rows using the permutation in (2.4.20).
  - For a matrix C with mn columns, $C\xi^{\otimes}_{m,n}$ is obtained from C by permuting its columns using the permutation in (2.4.5).
  - For a matrix D with mn rows, $\xi^{\otimes}_{m,n}D$ is obtained from D by permuting its rows using the permutation in (2.4.20).
- The structure morphisms $\lambda^{\bullet}$, $\rho^{\bullet}$, and δ^{r} are identities.
- δ^{l} is defined as the composite in (2.5.4).

This finishes the right bipermutative categorical data of $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$.

The naturality of $\xi^{\oplus}$ follows from the matrix equalities
$$\begin{bmatrix} B & 0 \\ 0 & A \end{bmatrix}\begin{bmatrix} 0 & 1_n \\ 1_m & 0 \end{bmatrix} = \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1_{n'} \\ 1_{m'} & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$
for $A : m \longrightarrow m'$ and $B : n \longrightarrow n'$. The naturality of $\xi^{\otimes}$ means the equality
$$\xi^{\otimes}_{m',n'}(A \otimes B) = (B \otimes A)\xi^{\otimes}_{m,n}$$

that expresses $B \otimes A$ as a row and column permutation of $A \otimes B$. The naturality of δ^r follows from the following matrix equalities.

$$(A \oplus B) \otimes C = \left[\begin{array}{c|c} A \otimes C & 0 \\ \hline 0 & B \otimes C \end{array}\right] = (A \otimes C) \oplus (B \otimes C)$$

On the other hand, δ^l is not the identity because

$$C \otimes (A \oplus B) \neq (C \otimes A) \oplus (C \otimes B)$$

for general matrices A, B, and C. For example, if

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A = B = (1), \tag{2.5.10}$$

then

$$C \otimes \big((1) \oplus (1)\big) = C \otimes 1_2 = \begin{bmatrix} 0 & 1_2 \\ 1_2 & 0 \end{bmatrix},$$

while

$$\big(C \otimes (1)\big) \oplus \big(C \otimes (1)\big) = C \oplus C = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}.$$

Next we check the right bipermutative category axioms.

- The diagram (2.5.4) is commutative by the definition of δ^l in terms of δ^r and $\xi^{\otimes}$.
- The diagram (2.5.5) is commutative because

$$\xi^{\oplus}_{m,n} \otimes 1_p = \begin{bmatrix} 0 & 1_{np} \\ 1_{mp} & 0 \end{bmatrix} = \xi^{\oplus}_{mp,np}$$

  for $m, n, p \geq 0$.
- The diagram (2.5.6) is commutative for the following reasons:
  - The axiom (2.1.13) holds in Σ', which is equivalent to (2.5.6).
  - The two composites in (2.5.6) for $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$ are the two permutation matrices obtained from $1_{(m+n)(p+q)}$ by permuting its rows using the two corresponding permutations in (2.5.6) for Σ'.

A more conceptual way to think about $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$ is to regard each object m as the complex vector space $\mathbb{C}^m$ equipped with the standard Kronecker basis. A morphism $m \longrightarrow n$ is then a $\mathbb{C}$-linear map $\mathbb{C}^m \longrightarrow \mathbb{C}^n$, regarded as a complex $n \times m$ matrix via the standard bases. Composition of $\mathbb{C}$-linear maps

$$\mathbb{C}^m \longrightarrow \mathbb{C}^n \longrightarrow \mathbb{C}^p$$

corresponds to matrix multiplication. The identity matrix 1_n corresponds to the identity map on $\mathbb{C}^n$. The rest of the structures in $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$ are similarly interpreted as in $\mathsf{Vect}^{\mathbb{C}}$ for the vector spaces $\mathbb{C}^n$ with the standard bases.

We remark that there is an incorrect claim in [**KV94**, Ex. 5.6] that says that in $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$ both distributivity morphisms are the identities. As we illustrated with a simple example (2.5.10) above, δ^l in $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$ is not the identity. $\diamond$

Left Bipermutative Categories. Next we define left bipermutative categories, in which the left distributivity morphism δ^l is the identity.

Definition 2.5.11. A *left bipermutative category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^l, \delta^r)\big)$$

consisting of the following data.

- $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ and $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ are permutative categories.
- $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^l, and δ^r are natural transformations as in (2.1.3) and (2.1.4).

The above data are required to satisfy the following four conditions.

(1) $\lambda^{\bullet}$ and $\rho^{\bullet}$ are both equal to the identity natural transformation of the constant functor $\mathsf{C} \longrightarrow \mathsf{C}$ at $\mathbb{0}$.
(2) δ^l is an identity natural transformation.
(3) $\xi^{\otimes}_{A,\mathbb{0}} : A \otimes \mathbb{0} \longrightarrow \mathbb{0} \otimes A$ is the identity morphism of $\mathbb{0}$ for each object A.
(4) The axioms (2.1.5), (2.1.6), and (2.1.13) are satisfied.

This finishes the definition of a left bipermutative category. ◇

Explanation 2.5.12. Consider Definition 2.5.11.

- The axiom (2.1.5) is equivalent to the following commutative diagram.

$$\begin{tikzcd}
(A \oplus B)C \arrow[r, "\delta^r"] \arrow[d, "\xi^{\otimes}"'] & AC \oplus BC \\
C(A \oplus B) \arrow[r, "\delta^l", "="'] & CA \oplus CB \arrow[u, "\xi^{\otimes} \oplus \xi^{\otimes}"']
\end{tikzcd} \tag{2.5.13}$$

  In particular, the right distributivity morphism δ^r is a natural isomorphism.
- The axiom (2.1.6) means the commutativity of the following diagram.

$$\begin{tikzcd}
A(B \oplus C) \arrow[r, "\delta^l", "="'] \arrow[d, "1\xi^{\oplus}"'] & AB \oplus AC \arrow[d, "\xi^{\oplus}"] \\
A(C \oplus B) \arrow[r, "\delta^l", "="'] & AC \oplus AB
\end{tikzcd} \tag{2.5.14}$$

- The axiom (2.1.13) means the commutativity of the following diagram.

$$\begin{tikzcd}
(A \oplus B)(C \oplus D) \arrow[rr, "\delta^r"] \arrow[d, "\delta^l"', "="] & & A(C \oplus D) \oplus B(C \oplus D) \arrow[d, "="', "\delta^l \oplus \delta^l"] \\
(A \oplus B)C \oplus (A \oplus B)D \arrow[dr, "\delta^r \oplus \delta^r"'] & & AC \oplus AD \oplus BC \oplus BD \arrow[dl, "1 \oplus \xi^{\oplus} \oplus 1"] \\
& AC \oplus BC \oplus AD \oplus BD &
\end{tikzcd} \tag{2.5.15}$$

◇

Proposition 2.5.16. *Each left bipermutative category is a tight symmetric bimonoidal category.*

Proof. The left distributivity morphism δ^l is the identity by definition. The right distributivity morphism δ^r is a natural isomorphism by (2.5.13). Consider the 24 axioms in Definition 2.1.2 for a left bipermutative category.

- The axioms (2.1.5), (2.1.6), and (2.1.13) hold by assumption.
- The axiom (2.1.19) holds by the identity assumptions on $\lambda^\bullet$, $\rho^\bullet$, and $\xi^\otimes_{-,\mathbb{0}}$.
- The axioms (2.1.9), (2.1.10), (2.1.14), (2.1.15), (2.1.17), (2.1.20), (2.1.23), and (2.1.27) hold by the assumption that $\lambda^\bullet$, $\rho^\bullet$, and δ^r are identities.

We finish the proof by applying Lemmas 2.2.4 through 2.2.12 to obtain all 24 axioms of a symmetric bimonoidal category. □

Example 2.5.17. The symmetric bimonoidal category Σ in Proposition 2.4.8 is a left bipermutative category. ⋄

2.6. Application: Reversible Programming of Finite Types

In this section, we discuss an application of symmetric bimonoidal categories to reversible programming of finite types with sums and products. This section is based on the paper [**CS16**]. The main observation [**CS16**, Th. 3] is Theorem 2.6.2, which asserts the existence of a symmetric bimonoidal groupoid Π with syntax of finite types as objects. This is actually true by the construction of Π. Theorem 2.6.2 is a bimonoidal manifestation of the theme of the paper [**BS11**], in which the close connection between category theory, physics, and computation is described. Here the categorical concept is a symmetric bimonoidal groupoid. Sums and products of objects in such a category model syntax of finite types. Near the end of this section, we point out a slight improvement of Π. More discussion about [**CS16**] and Π is in Note 2.7.3 and Examples 3.9.10 and 4.4.5. In particular, Laplaza's Coherence Theorems 3.9.1 and 4.4.3 apply to Π.

Motivation 2.6.1. The Curry-Howard-Lambek correspondence relates the following concepts:

Type theory: types, programs (or type equivalences), and program transformations (or equivalences of type equivalences);

Propositional logic: propositions, proofs, and transformations of proofs; and

Cartesian closed categories: objects, morphisms, and commutative diagrams.

Extending the work of [**BJS11, JS12**], in [**CS16**] a variant of the Curry-Howard-Lambek correspondence is proposed for finite type reversible programming. In categorical language, *finite type* means that there are functors representing sums and products, along with appropriate distributivity and units. Therefore, the concept of finite type is naturally associated with bimonoidal categories.

Reversibility can be interpreted in several ways.

Category theory: In a category, reversibility means that each morphism is invertible, so we are dealing with groupoids.

Physics: The conservation of information in physics is another motivation for reversibility. The No-Hiding Theorem [**BP07, KGB⁺19, SPK11**] states essentially that no physical processes can destroy quantum information. If computation is regarded as a physical process, then it makes sense that this process can be reversed.

Homotopy type theory: A *type isomorphism* is a fundamental concept in homotopy type theory [**Pro13**], where a proof of an equality between two terms is thought of as a path between two points in a topological space. Since each path has an inverse, so should a proof.

In summary, bimonoidal groupoids form the appropriate framework for finite type reversible programming. ⋄

Recall from Definition 2.1.2 that a *symmetric bimonoidal groupoid* is a symmetric bimonoidal category in which each morphism is invertible.

Theorem 2.6.2. *There is a symmetric bimonoidal groupoid* Π *with*

- *syntax of finite types as objects and*
- Π*-terms and* Π*-combinators as morphisms.*

In the rest of this section, we explain the category Π as defined in [**CS16**] and point out one improvement.

Syntax of Finite Types as Objects. The objects in the groupoid Π are defined inductively as follows.

- Each Agda type is an object in Π. Agda is the dependently typed functional programming language available at `https://wiki.portal.chalmers.se/agda/pmwiki.php`. It is an extension of Martin-Löf type theory.
- Π is equipped with two distinguished objects $\mathbb{0}$ and $\mathbb{1}$.
- Inductively, if A and B are objects in Π, then so are their *sum* $A \oplus B$ and *product* $A \otimes B$.

In the language of type theory and propositional logic:

- $\mathbb{0}$ is the empty type $\perp$, which corresponds to inconsistency.
- $\mathbb{1}$ is the unit type $\top$, which corresponds to the trivially true proposition.
- $\oplus$ is the sum type $\uplus$, which corresponds to the disjunction of propositions.
- $\otimes$ is the product type $\times$, which corresponds to the conjunction of propositions.

In [**CS16**], the objects $A \oplus B$ and $A \otimes B$ are denoted by, respectively, $A + B$ and $A * B$. Instead of starting with Agda types, we can, in fact, start with any class of objects in the above definition, and the construction below is still valid.

Π**-Terms.** There are two kinds of morphisms in Π. Morphisms of the first kind are called Π-*terms* in [**CS16**, Fig. 1 and Section 6.1]. Using the notation in Definition 2.1.2, Π-terms include the following isomorphisms for objects A, B, and C in Π.

Identity Morphisms:

$$A \xrightarrow{1_A} A$$

The Additive Structure:

$$(A \oplus B) \oplus C \underset{\alpha^{-\oplus}_{A,B,C}}{\overset{\alpha^{\oplus}_{A,B,C}}{\rightleftarrows}} A \oplus (B \oplus C) \qquad\qquad A \oplus B \underset{\xi^{-\oplus}_{A,B}}{\overset{\xi^{\oplus}_{A,B}}{\rightleftarrows}} B \oplus A$$

$$\mathbb{0} \oplus A \underset{\lambda^{-\oplus}_{A}}{\overset{\lambda^{\oplus}_{A}}{\rightleftarrows}} A \underset{\rho^{-\oplus}_{A}}{\overset{\rho^{\oplus}_{A}}{\leftrightarrows}} A \oplus \mathbb{0}$$

The Multiplicative Structure:

$$(A \otimes B) \otimes C \underset{\alpha^{-\otimes}_{A,B,C}}{\overset{\alpha^{\otimes}_{A,B,C}}{\rightleftarrows}} A \otimes (B \otimes C) \qquad\qquad A \otimes B \underset{\xi^{-\otimes}_{A,B}}{\overset{\xi^{\otimes}_{A,B}}{\rightleftarrows}} B \otimes A$$

$$\mathbb{1} \otimes A \underset{\lambda^{-\otimes}_{A}}{\overset{\lambda^{\otimes}_{A}}{\rightleftarrows}} A \underset{\rho^{-\otimes}_{A}}{\overset{\rho^{\otimes}_{A}}{\leftrightarrows}} A \otimes \mathbb{1}$$

The Multiplicative Zeros:

$$\mathbb{0} \otimes A \underset{\lambda^{-\bullet}_{A}}{\overset{\lambda^{\bullet}_{A}}{\rightleftarrows}} \mathbb{0} \underset{\rho^{-\bullet}_{A}}{\overset{\rho^{\bullet}_{A}}{\leftrightarrows}} A \otimes \mathbb{0}$$

Distributivity:

$$A \otimes (B \oplus C) \underset{\delta^{-l}_{A,B,C}}{\overset{\delta^{l}_{A,B,C}}{\rightleftarrows}} (A \otimes B) \oplus (A \otimes C)$$

$$(A \oplus B) \otimes C \underset{\delta^{-r}_{A,B,C}}{\overset{\delta^{r}_{A,B,C}}{\rightleftarrows}} (A \otimes C) \oplus (B \otimes C)$$

The distributivity morphisms δ^l and δ^r are also equipped with inverses because Π is a groupoid. These isomorphisms are denoted by different symbols in [**CS16**]. For example, $\lambda^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\bullet}$, and δ^r are denoted by, respectively, $unite_{+}l$, $assocr_{*}$, *absorbr*, and *dist* in [**CS16**, Fig. 1].

Π**-Combinators.** Suppose $f : A \longrightarrow B$, $g : B \longrightarrow C$, and $h : A' \longrightarrow B'$ are morphisms in Π. Then, inductively, so are the following.

Composition: $gf : A \longrightarrow C$, whose inverse is $f^{-1}g^{-1}$.
Sum: $f \oplus h : A \oplus A' \longrightarrow B \oplus B'$, whose inverse is $f^{-1} \oplus h^{-1}$.
Product: $f \otimes h : A \otimes A' \longrightarrow B \otimes B'$, whose inverse is $f^{-1} \otimes h^{-1}$.

In [**CS16**, Fig. 2], composition, sum, and product are denoted by, respectively, $\odot$, $\oplus$, and $\otimes$, and are called Π-*combinators*. The above data are subject to the relations that Π is a symmetric bimonoidal groupoid as in Definition 2.1.2. To be more explicit, the relations ensure the following.

- Π with its identity morphisms, composition, and inverses is a groupoid.
- The additive structure $(\Pi, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$ is a symmetric monoidal category.
- The multiplicative structure $(\Pi, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$ is a symmetric monoidal category.
- $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^l, and δ^r are natural isomorphisms.
- The Laplaza axioms (2.1.5)–(2.1.28) are satisfied.

This finishes the construction of Π.

An Improvement of Π. The relations above are stated in [**CS16**, Fig. 3–12] with the following 13 Laplaza axioms, where Fig. N refers to that figure in [**CS16**].

- (2.1.5) and (2.1.6) in Fig. 12;
- (2.1.8), (2.1.10), and (2.1.13) in Fig. 11;
- (2.1.14), (2.1.15), (2.1.17), (2.1.19), (2.1.21), (2.1.22), and (2.1.23) in Fig. 10; and
- (2.1.27) in Fig. 6.

It is possible to shorten the above list as follows. By Lemma 2.2.10, the axiom (2.1.20) implies both (2.1.21) and (2.1.22), as long as (2.1.19) is assumed. Therefore, in [**CS16**, Fig. 10], the two relations (2.1.21) and (2.1.22) may be replaced by the single relation (2.1.20). Using the notation in [**CS16**], the relation (2.1.20) is stated as follows.

$$absorbl \quad \Leftrightarrow \quad (assocr_*) \odot (id\!\leftrightarrow \ \otimes absorbl) \odot absorbl$$

By Theorem 2.2.13, this shortened list of relations defines a symmetric bimonoidal groupoid.

2.7. Notes

2.7.1 (Symmetric Bimonoidal Categories)**.** The 24 axioms in Definition 2.1.2 of a symmetric bimonoidal category are due to Laplaza [**Lap72a**]. The only differences between our definition and Laplaza's is that his associativity isomorphisms $\alpha^{\oplus}$ and $\alpha^{\otimes}$, which he denoted by α' and α, move parentheses from right to left, instead of left to right. Since these are natural isomorphisms, these differences are only cosmetic. All the lemmas in Section 2.2 are also due to Laplaza, with one exception. In Lemma 2.2.10, the assertion that (2.1.20) implies (2.1.21) is not included in [**Lap72a**]. $\diamond$

2.7.2 (Terminology)**.** The name *symmetric bimonoidal category* goes back to at least [**May77**], but the literature contains several different names for this concept.

- In [**Lap72a, Lap72b**] Laplaza called such a category *coherent*.
- They are also called *symmetric ring categories* or *symmetric rig categories* elsewhere, including [**CS16, Elg21**].
- In other places, including [**BG20a, Hin13**], the word *distributive* is used instead of bimonoidal, ring, or rig.

One main reason we use the word *bimonoidal* is that it aligns better with bipermutative categories, including those in Section 2.5 and those of Elmendorf-Mandell in Chapters II.9 and III.11.

The word *bimonoidal* in categorical probability theory [**FP18**] means something different. In that paper, a *bimonoidal monad* is a monad on a monoidal category equipped with both a monoidal functor structure and an opmonoidal functor structure that are compatible in an appropriate sense. $\diamond$

2.7.3 (Symmetric Rig Categories)**.** There are two remarks concerning some definitions in [**CS16**], on which Section 2.6 is based.

- [**CS16**, Def. 8] of a braided monoidal category includes only the left hexagon axiom and omits the right hexagon axiom in (II.1.3.17). In a symmetric monoidal category, these two hexagon axioms are equivalent to each other by the symmetry axiom (1.2.20), so only one of them is needed.

However, in a general braided monoidal category, the two hexagon axioms are not equivalent to each other, so both of them are needed.

- In [**CS16**, Def. 10], a symmetric rig (= tight bimonoidal) category is defined as a rig (= tight bimonoidal) category in which the multiplicative structure is symmetric. This is not correct because a bimonoidal category is not equipped with a multiplicative symmetry $\xi^{\otimes}$, so the Laplaza axioms (2.1.5) and (2.1.19) are not included. A symmetric bimonoidal category is a bimonoidal category in which
  - the multiplicative structure is symmetric, *and*
  - the axioms (2.1.5) and (2.1.19) are satisfied. ◇

2.7.4 (Bipermutative Categories)**.** Our right bipermutative category is almost the same as May's bipermutative category [**May77**], except that May did not include the axiom that $\xi^{\otimes}_{-,\mathbb{0}}$ be the identity. The material in Section 2.5 on right bipermutative categories is due to May [**May77**, Section 6.3]. The symmetric bimonoidal category Σ' in Proposition 2.4.23 appeared in [**May77**, Example 6.5.1]. In Chapter 5 we will show that tight symmetric bimonoidal categories can be strictified to right bipermutative categories. In the literature, the name *bipermutative category* refers to either May's version or a different version due to Elmendorf-Mandell [**EM06**], which we will discuss in Chapters II.9 and III.11. See also Note 5.6.3. ◇

2.7.5 (Applications to Diagrammatic Calculus)**.** In addition to Section 2.6 on reversible programming of finite types, bimonoidal categories have other applications in computer science. Sheet diagrams for tight bimonoidal categories that generalize string diagrams for monoidal categories [**JS91a, Sel11**] are discussed in [**CDH∞**]; see also Example 3.10.9 and Note 7.9.2. Sheet diagrams are three-dimensional, two for the additive structure $\oplus$ and an additional one for the multiplicative structure $\otimes$.

- Precursors of sheet diagrams appeared in [**Sta15**] to diagrammatically separate the quantum parts of quantum circuits from the classical wires.
- Sheet diagrams in [**Del20**] provide a diagrammatic calculus for faceted dataflow in OpenRefine (`https://openrefine.org`), which is a popular data wrangling software.

For open questions related to sheet diagrams, see Questions III.A.1.6 and III.A.2.8. Moreover, in [**Hin13**], tight symmetric bimonoidal categories are used to provide a categorical proof of the operational equivalence of two quantum circuits. Chapter II.3 discusses applications of braided bimonoidal categories to quantum groups and topological quantum computation. ◇

CHAPTER 3

Coherence of Symmetric Bimonoidal Categories

The purpose of this chapter is to prove the Coherence Theorem 3.9.1 for symmetric bimonoidal categories, which is due to Laplaza [**Lap72a**]. This coherence result says that certain formal diagrams in symmetric bimonoidal categories commute, provided that a condition about monomorphisms is satisfied. We emphasize that this coherence theorem, like Definition 2.1.2, only requires the distributivity morphisms δ^l and δ^r in (2.1.4) to be natural monomorphisms, instead of natural isomorphisms. The end of Section 3.9 contains examples where Theorem 3.9.1 is applicable. Theorem 3.10.7 is the multiplicatively nonsymmetric analogue of Theorem 3.9.1 for bimonoidal categories. For open questions related to coherence of symmetric bimonoidal categories, see Question III.A.1.6.

Outline of the Coherence Theorem. The proof of Theorem 3.9.1 involves a series of reductions until the Symmetric Coherence Theorem 1.3.8, for both the additive structure $\oplus$ and the multiplicative structure $\otimes$, can be applied to finish the proof. The reduction steps for Theorem 3.9.1 are roughly analogous to the proof of Mac Lane's Coherence Theorem 1.3.3 as given in [**ML63**, Theorem 5.2] and [**ML98**, VII.2 Corollary]. For coherence of monoidal categories, one first reduces to the situation where the monoidal unit, the left unit isomorphism, and the right unit isomorphism are not involved. One is then left with the associativity isomorphisms. A further argument reduces the proof to applying the monoidal category axioms and the naturality of some structure morphisms.

Since a symmetric bimonoidal category has more structure and many more axioms than a monoidal category, the proof of Theorem 3.9.1 has many more steps and cases than that of Theorem 1.3.3. Here is a brief outline of the reduction steps for the Coherence Theorem 3.9.1 for symmetric bimonoidal categories.

(1) Reduce away the additive zeros $\lambda^{\oplus}$ and $\rho^{\oplus}$, and the multiplicative zeros $\lambda^{\bullet}$ and $\rho^{\bullet}$.
   - In Section 3.3, we first prove the existence and the uniqueness of such reductions for objects.
   - In Sections 3.4 and 3.5, we show the existence of such reductions for paths.

   Section 3.5 ends with a special case of the Coherence Theorem 3.9.1; see Proposition 3.5.33.
(2) Reduce away the distributivity morphisms δ^l and δ^r.
   - The existence of such reductions for objects is proved in Section 3.6.
   - The existence of such reductions for paths is proved in Lemma 3.6.12 and Section 3.7.
(3) Reduce away the multiplicative units $\lambda^{\otimes}$ and $\rho^{\otimes}$ in Section 3.8.

(4) In Section 3.9, we assemble the results in the previous sections to reduce the proof of Theorem 3.9.1 to the situation where only the structure morphisms $\alpha^{\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, $\xi^{\otimes}$, and their inverses (that is, the additive and the multiplicative associativity and symmetry) are left. Then we apply the Symmetric Coherence Theorem 1.3.8 both additively and multiplicatively to finish the proof of Theorem 3.9.1.

The proof of the Coherence Theorem 3.10.7 for bimonoidal categories is obtained from that of Theorem 3.9.1 by removing $\xi^{\pm\otimes}$, and by using Theorem 1.3.3 instead of Theorem 1.3.8 for the multiplicative structure.

Mac Lane's Coherence Theorems 1.3.3 and 1.3.8 are stated in terms of words, canonical maps, and their symmetric variants. Section 3.1 contains the symmetric bimonoidal analogues of these concepts. Many proofs in this chapter involve inductions on formal words. Section 3.2 contains several concepts on formal words that will be used in later sections to perform inductions.

As we mentioned above, the Coherence Theorem 3.9.1 is due to Laplaza, and we generally follow the broad outline of his original proof. However, there are some nontrivial differences between this chapter and Laplaza's original proof, which will be discussed in Section 3.11. The relation to a 2-monad approach is discussed in Note 3.11.7.

Reading Guide. The proof of Theorem 3.9.1 is presented straightly linearly in this chapter. Since this proof has many steps and cases, as a possible alternative to reading this chapter linearly, we suggest the following.

- First read Theorem 3.9.1, whose proof is only about three pages long by that point, without worrying about the terms and results from earlier sections. The point is to first obtain a bird's-eye view of the structure of the proof.
- Then read the earlier sections, but skip all the proofs, or at least the longer ones.
- With all the necessary concepts and statements of preliminary results in mind, read Theorem 3.9.1 again. The outlined proof above should now make sense.
- After enough rest and some mental preparation, read the proofs in the earlier sections.

We deliberately divided the proof of Theorem 3.9.1 into many lemmas and cases to clarify the overall structure of the proof, and to make jumping forward and backward easier. Students are encouraged to regard the lemmas, the cases within each lemma, and their detailed proofs as exercises with full solutions.

Detail. In addition to proving Theorem 3.9.1, the many detailed proofs in this chapter have several additional purposes.

Axioms: Some of the proofs in this chapter are where the axioms of a symmetric bimonoidal category in Definition 2.1.2 are used. For example, Lemmas 3.4.12, 3.5.9, and 3.6.12 use, respectively, 2, 3, and 11 symmetric bimonoidal category axioms. Seeing these axioms arising naturally in these proofs helps demystify them.

Corrections: Laplaza's original proof contains some inaccuracies that have never been made explicit before. For both educational and archival purposes, it is important to rectify these issues. However, without going into detail, it would be difficult to (i) pinpoint these subtle issues, and (ii) discuss the corrections and where they fit in the big picture. Section 3.11 contains more detailed discussion of the differences between this chapter and Laplaza's original proof, and the necessary corrections for the latter.

Other Theorems: Several other main theorems in this book use the preliminary results in this chapter and Theorem 3.9.1.

- The proof of the Coherence Theorem 4.4.3 uses the same reduction steps as in the proof of Theorem 3.9.1. So it uses essentially all the preliminary results in this chapter before Section 3.9.
- The proof of the Strictification Theorem 5.4.6 of tight symmetric bimonoidal categories to equivalent right bipermutative categories involves Theorem 3.9.1 many times. In fact, both the definition of the associated right bipermutative category and the proofs of its properties use coherence. See Explanation 5.2.31, Lemmas 5.3.1, 5.3.4, 5.3.8, and 5.4.4, and Note 5.6.2.
- The proofs of Baez's Conjecture (Theorem 7.8.1) and the Bicategorification Theorem 8.15.4 use Theorem 3.9.1.
- The proof of the Coherence Theorem II.5.4.4 of braided bimonoidal categories uses many of the proofs in this chapter. A detailed treatment here will allow us to be both precise and concise at the same time in the braided case.

Throughout this chapter, as in Definition 2.1.2, we often abbreviate $\otimes$ using concatenation, with $\otimes$ taking precedence over $\oplus$ in the absence of parentheses. For example,

$$(ab)c \oplus a'(b'c') = \big((a \otimes b) \otimes c\big) \oplus \big(a' \otimes (b' \otimes c')\big).$$

3.1. Regularity

The main objective of this section is to introduce the language and notation that are needed to state and prove the first coherence theorem for symmetric bimonoidal categories.

Motivation 3.1.1. In the Coherence Theorems 1.3.3 and 1.3.8 for (symmetric) monoidal categories, the assertions are stated in terms of

- formal variables in the forms of (permuted) words in Definitions 1.3.1 and 1.3.6; and
- natural isomorphisms in the forms of (permuted) canonical maps in Definitions 1.3.2 and 1.3.6.

To state and prove the first Coherence Theorem 3.9.1 for symmetric bimonoidal categories, we first need to develop symmetric bimonoidal analogues of words and canonical maps.

The formal alphabets are taken from a set X with two distinguished elements 0^X and 1^X. The analogues of words are elements in the free $\{\oplus,\otimes\}$-algebra X^{fr} in Definition 3.1.2. The analogues of morphisms between words are paths consisting of prime edges in Definition 3.1.8. These two concepts together constitute the graph of X, which is denoted by $\mathsf{Gr}(X)$, in Definition 3.1.9. The first half of this section contains all the definitions necessary to define the graph of X.

To apply these concepts to symmetric bimonoidal categories C, we use a certain graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ in Definition 3.1.14. This is analogous to interpreting a word as a functor in Definition 1.3.1. With this graph morphism, we can define the *value* of a path in $\mathsf{Gr}(X)$ as the corresponding composite morphism in C. The first coherence theorem of symmetric bimonoidal categories then takes on a familiar form: two paths from a to b in $\mathsf{Gr}(X)$ have the same value in C, provided a satisfies a regularity condition.

The regularity condition ensures that we do *not* assert generally false statements such as this: $1_{x\oplus x}$ and $\xi^{\oplus}_{x,x} : x \oplus x \longrightarrow x \oplus x$ are equal for all objects x. The second half of this section is devoted to the concept of regularity. $\diamond$

Elementary Graph. To define the graph of X in Definition 3.1.9, we first define its vertices in the next definition.

Definition 3.1.2. Suppose S is a set. The *free* $\{\oplus, \otimes\}$*-algebra* of S is the set S^{fr} defined inductively by the following two conditions.

- $S \subset S^{\mathsf{fr}}$.
- If $a, b \in S^{\mathsf{fr}}$, then the symbols
$$a \oplus b \quad \text{and} \quad a \otimes b$$
also belong to S^{fr}. They are called, respectively, the *sum* and the *product* of a and b.

To simplify the presentation, we sometimes abbreviate $a \otimes b$ to ab. In the absence of clarifying parentheses, $\otimes$ takes precedence over $\oplus$. $\diamond$

Example 3.1.3. For $a, \ldots, f \in S$,
$$(ab \oplus cd)e \oplus f = \Big[\big((a \otimes b) \oplus (c \otimes d)\big) \otimes e\Big] \oplus f$$
in S^{fr}. $\diamond$

We also need the following concept of a graph.

Definition 3.1.4. A *graph* $G = (V, E)$ is a pair consisting of the following data.

- V is a class. An element in V is called a *vertex* in G.
- E assigns to each ordered pair (u, v), with $u, v \in V$, a set $E(u, v)$, an element of which is called an *edge* with *domain* u and *codomain* v. We also denote such an edge by
$$u \longrightarrow v, \quad e : u \longrightarrow v, \quad \text{or} \quad u \xrightarrow{\;e\;} v$$
if e is the name of the edge.

A *path* in such a graph is a nonempty finite sequence of edges $(e_n, \ldots, e_1)$ as in
$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_n.$$
Such a path is denoted by $v_0 \longrightarrow\!\!\!\!\rightarrow v_n$ and is said to have *length* n, domain v_0, and codomain v_n. $\diamond$

Example 3.1.5. Each category C has an associated graph (V, E), with V the class of objects in C, and $E(u, v) = \mathsf{C}(u, v)$ for objects $u, v \in \mathsf{C}$. A nonempty finite sequence of composable morphisms in C yields a path in the associated graph. $\diamond$

The edges in the graph of X in Definition 3.1.9 are built from the elementary edges in the next definition.

Definition 3.1.6. Suppose X is a set with two distinguished elements 0^X and 1^X, called the *additive zero* and the *multiplicative unit*, respectively. The *elementary graph of* X, denoted by $\mathsf{Gr}^{\mathsf{el}}(X)$, is the graph defined as follows.

Vertices: The set of vertices in $\mathsf{Gr}^{\mathsf{el}}(X)$ is the free $\{\oplus, \otimes\}$-algebra X^{fr} of X.

Edges: Edges in $\mathsf{Gr}^{\mathsf{el}}(X)$ are of the following types for all $x, y, z \in X^{\mathsf{fr}}$.

The Additive Structure:

$$(x \oplus y) \oplus z \underset{\alpha^{-\oplus}_{x,y,z}}{\overset{\alpha^{\oplus}_{x,y,z}}{\rightleftarrows}} x \oplus (y \oplus z) \qquad\qquad x \oplus y \underset{\xi^{-\oplus}_{x,y}}{\overset{\xi^{\oplus}_{x,y}}{\rightleftarrows}} y \oplus x$$

$$0^X \oplus x \underset{\lambda^{-\oplus}_{x}}{\overset{\lambda^{\oplus}_{x}}{\rightleftarrows}} x \underset{\rho^{-\oplus}_{x}}{\overset{\rho^{\oplus}_{x}}{\leftrightarrows}} x \oplus 0^X$$

The Multiplicative Structure:

$$(x \otimes y) \otimes z \underset{\alpha^{-\otimes}_{x,y,z}}{\overset{\alpha^{\otimes}_{x,y,z}}{\rightleftarrows}} x \otimes (y \otimes z) \qquad\qquad x \otimes y \underset{\xi^{-\otimes}_{x,y}}{\overset{\xi^{\otimes}_{x,y}}{\rightleftarrows}} y \otimes x$$

$$1^X \otimes x \underset{\lambda^{-\otimes}_{x}}{\overset{\lambda^{\otimes}_{x}}{\rightleftarrows}} x \underset{\rho^{-\otimes}_{x}}{\overset{\rho^{\otimes}_{x}}{\leftrightarrows}} x \otimes 1^X$$

The Multiplicative Zeros:

$$0^X \otimes x \underset{\lambda^{-\bullet}_{x}}{\overset{\lambda^{\bullet}_{x}}{\rightleftarrows}} 0^X \underset{\rho^{-\bullet}_{x}}{\overset{\rho^{\bullet}_{x}}{\leftrightarrows}} x \otimes 0^X$$

Distributivity:

$$x \otimes (y \oplus z) \xrightarrow{\delta^l_{x,y,z}} (x \otimes y) \oplus (x \otimes z)$$
$$(x \oplus y) \otimes z \xrightarrow{\delta^r_{x,y,z}} (x \otimes z) \oplus (y \otimes z)$$

Identity:

$$x \xrightarrow{1_x} x$$

This finishes the definition of $\mathsf{Gr}^{\mathsf{el}}(X)$.

Moreover, we define the following.

- The set of edges in $\mathsf{Gr}^{\mathsf{el}}(X)$ is denoted by $\mathsf{E}_{\mathsf{el}}(X)$, the elements of which are called *elementary edges*.
- $\alpha^{\oplus}$ and $\alpha^{-\oplus}$ are *formal inverses* of each other, and similarly for the other 9 pairs of elementary edges in the first three groups above.
- 1_x is called the *identity* of x.
- The names in Definition 2.1.2 are reused for elementary edges. For example, $\lambda^{\bullet}$ is called the left multiplicative zero, and δ^r is called the right distributivity. ◇

Graph. The next two definitions will be combined to yield the graph of X in Definition 3.1.9.

Definition 3.1.7. With $(X, 0^X, 1^X)$ as in Definition 3.1.6, consider the free $\{\oplus, \otimes\}$-algebra $\mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ of the set $\mathsf{E}_{\mathsf{el}}(X)$ of elementary edges. The *domain* and *codomain* of an element $f \in \mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ are elements in X^{fr} defined inductively as follows.

- For an elementary edge $f \in \mathsf{E}_{\mathsf{el}}(X)$, its (co)domain are those of f in the elementary graph $\mathsf{Gr}^{\mathsf{el}}(X)$.
- Suppose $f_1, f_2 \in \mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ with
  - $u_i \in X^{\mathsf{fr}}$ the domain of f_i and
  - $v_i \in X^{\mathsf{fr}}$ the codomain of f_i

  already defined for $i = 1, 2$. Then:
  - $f_1 \oplus f_2$ has domain $u_1 \oplus u_2$ and codomain $v_1 \oplus v_2$.
  - $f_1 \otimes f_2$ has domain $u_1 \otimes u_2$ and codomain $v_1 \otimes v_2$. ◇

Definition 3.1.8. Continuing Definition 3.1.7, *identity prime edges* and *nonidentity prime edges* are elements in $\mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ defined inductively by the following four conditions.

- Elementary edges of the type 1_x for $x \in X^{\mathsf{fr}}$ are identity prime edges.
- Elementary edges not of the type 1_x for $x \in X^{\mathsf{fr}}$ are nonidentity prime edges.
- If $e_1, e_2 \in \mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ are identity prime edges, then so are $e_1 \oplus e_2$ and $e_1 \otimes e_2$.
- If f is a nonidentity prime edge, and if e is an identity prime edge, then

$$f \oplus e, \quad e \oplus f, \quad f \otimes e, \quad \text{and} \quad e \otimes f$$

  are nonidentity prime edges.

A *prime edge* means either an identity prime edge or a nonidentity prime edge. The set of prime edges is denoted by $\mathsf{E}^{\mathsf{pr}}(X)$. An identity prime edge is also called an *identity*. ◇

Definition 3.1.9. With $(X, 0^X, 1^X)$ as in Definition 3.1.6, the *graph of* X, which is denoted by $\mathsf{Gr}(X)$, is the graph defined as follows.

Vertices: The set of vertices in $\mathsf{Gr}(X)$ is the free $\{\oplus, \otimes\}$-algebra X^{fr} of X.
Edges: The set of edges in $\mathsf{Gr}(X)$ is the set $\mathsf{E}^{\mathsf{pr}}(X)$ of prime edges as in Definition 3.1.8, with (co)domain as in Definition 3.1.7. ◇

Definition 3.1.10. Consider $\mathsf{Gr}(X)$.

- Suppose $f : a \longrightarrow b$ is a prime edge that does not involve δ^l and δ^r. Its *formal inverse* $g : b \longrightarrow a$ is the prime edge obtained from f as follows.
  - Each identity 1_x for $x \in X^{\mathsf{fr}}$ in f is replaced by 1_x in the opposite direction.
  - If f involves an elementary edge ϵ that is not an identity, then replace ϵ by its formal inverse.
- Suppose $P : a \longrightarrow\!\!\!\!\!\!\to b$ is a path in which each prime edge does not involve δ^l and δ^r. Its *formal inverse* $Q : b \longrightarrow\!\!\!\!\!\!\to a$ is the path obtained from P by replacing each of its prime edges by its formal inverse. ◇

Example 3.1.11. Suppose $x, y, z \in X^{\mathsf{fr}}$. With $\otimes$ abbreviated to concatenation, below are two paths in $\mathsf{Gr}(X)$.

$$\begin{tikzcd}
& x(y\oplus z) \arrow[rr, "1_x(\lambda_y^{-\otimes}\oplus 1_z)"] & & x(1^X y\oplus z) \arrow[dr, "\delta^l_{x,1^X y,z}"] & \\
x(y\oplus z)\oplus 0^X \arrow[ur, "\rho^\oplus_{x(y\oplus z)}"] \arrow[d, "\delta^l_{x,y,z}\oplus 1_{0^X}"'] & & & & x(1^X y)\oplus xz \\
(xy\oplus xz)\oplus 0^X \arrow[r, "\alpha^\oplus_{xy,xz,0^X}"] & xy\oplus(xz\oplus 0^X) \arrow[rr, "1_{xy}\oplus\rho^\oplus_{xz}"] & & xy\oplus xz \arrow[ur, "(1_x\lambda_y^{-\otimes})\oplus 1_{xz}"'] &
\end{tikzcd}$$

The top path has length 3, with the outer edges elementary edges, and the middle one a nonidentity prime edge. The bottom path has length 4, with $\alpha^\oplus$ an elementary edge, and the other three edges nonidentity prime edges. $\diamond$

Example 3.1.12. For elements $w, x, y, z \in X^{\mathsf{fr}}$,

$$w \oplus x(y\oplus z) \xrightarrow{1_w \oplus 1_x 1_{y\oplus z}} w\oplus x(y\oplus z)$$

is an identity. The formal inverse of the prime edge

$$w \oplus x(y\oplus z) \xrightarrow{1_w \oplus \lambda_x^{-\otimes} 1_{y\oplus z}} w\oplus (1^X x)(y\oplus z)$$

is the prime edge

$$w \oplus (1^X x)(y\oplus z) \xrightarrow{1_w \oplus \lambda_x^{\otimes} 1_{y\oplus z}} w\oplus x(y\oplus z),$$

and vice versa. $\diamond$

The Value of a Path. The graph of X is interpreted in a symmetric bimonoidal category via the following concept.

Definition 3.1.13. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs. A *graph morphism* $f : G_1 \longrightarrow G_2$ consists of functions

- $f_V : V_1 \longrightarrow V_2$ and
- $f_E : E_1(u,v) \longrightarrow E_2\big(f_V u, f_V v\big)$ for $u, v \in V_1$.

To simplify the notation, both f_V and f_E are denoted by f below. $\diamond$

Definition 3.1.14. Suppose given the data (X, C, φ) as follows.

- X is a set with two distinguished elements 0^X and 1^X as in Definition 3.1.6.
- C is an arbitrary symmetric bimonoidal category as in Definition 2.1.2, equipped with the graph structure in Example 3.1.5.
- $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is any function such that

$$\varphi(0^X) = \mathbb{0} \quad \text{and} \quad \varphi(1^X) = \mathbb{1}. \tag{3.1.15}$$

Extend φ to a graph morphism

$$\mathsf{Gr}(X) \xrightarrow{\varphi} \mathsf{C}$$

as follows.

Vertices: For $x, y \in X^{\mathsf{fr}}$ such that $\varphi x, \varphi y \in \mathsf{Ob}(\mathsf{C})$ are already defined, we define

$$\begin{aligned} \varphi(x \oplus y) &= \varphi x \oplus \varphi y \quad \text{and} \\ \varphi(x \otimes y) &= \varphi x \otimes \varphi y. \end{aligned} \tag{3.1.16}$$

Elementary Edges: φ sends each elementary edge to the structure morphism in C with the same name, and with the subscripts replaced by their images under φ.

Prime Edges: If $f_1, f_2 \in \mathsf{E}^{\mathsf{pr}}(X)$ are prime edges with at most one of them nonidentity, and with $\varphi(f_1)$ and $\varphi(f_2)$ already defined, then we define

$$\begin{aligned} \varphi(f_1 \oplus f_2) &= \varphi(f_1) \oplus \varphi(f_2) \quad \text{and} \\ \varphi(f_1 \otimes f_2) &= \varphi(f_1) \otimes \varphi(f_2). \end{aligned} \tag{3.1.17}$$

This finishes the definition of the graph morphism φ.

Moreover, we define the following.

- For a path $P = (f_n, \ldots, f_1)$ in $\mathsf{Gr}(X)$ with domain u and codomain v, its *value* in C is the composite

$$\varphi P = \varphi(f_n) \circ \cdots \circ \varphi(f_1) \in \mathsf{C}(\varphi u; \varphi v). \tag{3.1.18}$$

- A diagram with vertices and edges in $\mathsf{Gr}(X)$ is *commutative in* C if its image under the graph morphism φ is a commutative diagram in C.
- A diagram with vertices and edges in $\mathsf{Gr}(X)$ is *commutative* if it is commutative in each symmetric bimonoidal category C and for each function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ satisfying (3.1.15). ◇

Remark 3.1.19. In Definition 3.1.14, the 24 symmetric bimonoidal category axioms are not needed. Therefore, the definition still makes sense if C only has the data portion of a symmetric bimonoidal category. ◇

Example 3.1.20. The graph morphism φ sends elementary edges to structure morphisms in C. Below are two examples.

$$\begin{aligned} \varphi(\alpha^{\oplus}_{x,y,z}) &= \alpha^{\oplus}_{\varphi x, \varphi y, \varphi z} : (\varphi x \oplus \varphi y) \oplus (\varphi z) \longrightarrow \varphi x \oplus (\varphi y \oplus \varphi z) \\ \varphi(\lambda^{\bullet}_x) &= \lambda^{\bullet}_{\varphi x} : \mathbb{0} \otimes \varphi x \longrightarrow \mathbb{0}. \end{aligned}$$

◇

Example 3.1.21. The image under φ of an identity prime edge is an identity morphism by the functoriality of $\oplus$ and $\otimes$ in C. If f is a prime edge with formal inverse g, then

$$\varphi(f)^{-1} = \varphi(g),$$

and similarly for a path with a formal inverse. ◇

Example 3.1.22. For elements $x, y, z \in X^{\mathsf{fr}}$, the diagram

$$\begin{tikzcd} (x \oplus y)z \arrow[r, "\delta^r_{x,y,z}"] \arrow[d, "\xi^{\otimes}_{x \oplus y, z}"'] & xz \oplus yz \arrow[r, "1_{xz} \oplus \xi^{\otimes}_{y,z}"] & xz \oplus zy \arrow[d, "\xi^{\otimes}_{x,z} \oplus 1_{zy}"] \\ z(x \oplus y) \arrow[rr, "\delta^l_{z,x,y}"] & & zx \oplus zy \end{tikzcd}$$

in $\mathsf{Gr}(X)$ is commutative. This is true because its image under the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ is commutative by

- the axiom (2.1.5) in C applied to the objects φx, φy, and φz; and

- the functoriality of $\oplus$ in C, which implies
$$\big(\xi^{\otimes}_{x,z} \oplus 1_{zy}\big)\big(1_{xz} \oplus \xi^{\otimes}_{y,z}\big) = \xi^{\otimes}_{x,z} \oplus \xi^{\otimes}_{y,z}.$$

Similarly, each of the other 23 symmetric bimonoidal category axioms in C gives a commutative diagram in $\mathsf{Gr}(X)$. ◇

Regularity. Next we introduce a restriction, called regularity, on the elements in X^{fr} for the first coherence theorem of symmetric bimonoidal categories. As discussed in Motivation 3.1.1, this restriction is needed to avoid incoherent situations. Regularity is defined in terms of the following concept.

Definition 3.1.23. Suppose $(X, 0^X, 1^X)$ is as in Definition 3.1.6. Define its *strict* $\{\oplus,\otimes\}$*-algebra* X^{st} as the quotient set of X^{fr} in Definition 3.1.2 by the smallest relation that contains the following identifications for all elements $x, y, z \in X^{\mathsf{fr}}$.

The Additive Structure:
$$(x \oplus y) \oplus z = x \oplus (y \oplus z)$$
$$0^X \oplus x = x = x \oplus 0^X$$
$$x \oplus y = y \oplus x$$

The Multiplicative Structure:
$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$
$$1^X \otimes x = x = x \otimes 1^X$$
$$x \otimes y = y \otimes x$$

The Multiplicative Zeros:
$$0^X \otimes x = 0^X = x \otimes 0^X$$

Distributivity:
$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$$
$$(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$$

This finishes the definition of X^{st}.

Moreover, denote by

$$X^{\mathsf{fr}} \xrightarrow{\mathsf{supp}} X^{\mathsf{st}} \tag{3.1.24}$$

the quotient map, called the *support*. ◇

Definition 3.1.25. An element $x \in X^{\mathsf{fr}}$ is *regular* if there exist elements $x^i_j \in X$ for $1 \le i \le m$ and $1 \le j \le k_i$ for each i with $m, k_1, \ldots, k_m > 0$, such that the following three conditions hold.

(i) The equality
$$\mathsf{supp}(x) = \mathsf{supp}\Bigg(\bigoplus_{i=1}^{m}\big(x^i_1 \otimes \cdots \otimes x^i_{k_i}\big)\Bigg) \tag{3.1.26}$$
holds in X^{st}. The iterated sum $\bigoplus_{i=1}^m$ and each of the m iterated products $x^i_1 \otimes \cdots \otimes x^i_{k_i}$ have some bracketings. By the definition of X^{st}, different bracketings yield the same support.

(ii) For each $1 \le i \le m$, the elements $x^i_1, \ldots, x^i_{k_i} \in X$ are distinct.

(iii) The m elements

$$\mathsf{supp}\big(x_1^i \otimes \cdots \otimes x_{k_i}^i\big) \in X^{\mathsf{st}}$$

for $1 \leq i \leq m$ are distinct. $\diamond$

The rest of this section contains basic observations and examples about support and regular elements.

Example 3.1.27. For elements $a, \ldots, f \in X$,

$$\begin{aligned}\mathsf{supp}\big((ab \oplus cd)e \oplus f\big) &= \mathsf{supp}\big(\,[(ab)e \oplus (cd)e] \oplus f\big)\\ &= \mathsf{supp}\big(a(be) \oplus [c(de) \oplus f]\,\big).\end{aligned}$$

In general, because of the relations that define X^{st}, when computing the support of an element in X^{fr}, we may

- distributive $\otimes$ over $\oplus$ as much as possible and
- ignore the additive bracketings of the summands and the multiplicative bracketings within each summand. $\diamond$

Example 3.1.28. Elements in X are regular. If $x_1, \ldots, x_m$ are distinct elements in X, then the elements

$$x_1 \oplus \cdots \oplus x_m \quad \text{and} \quad x_1 \otimes \cdots \otimes x_m \in X^{\mathsf{fr}}$$

with any bracketings are regular. On the other hand, $x_1 \oplus x_1$ and $x_1 \otimes x_1$ are not regular. $\diamond$

Lemma 3.1.29. *The following statements hold.*

(1) If two elements in X^{fr} have the same support, then one of them is regular if and only if the other one is regular.

(2) If $x \twoheadrightarrow y$ is a path in $\mathsf{Gr}(X)$ as in Definition 3.1.9, then

$$\mathsf{supp}(x) = \mathsf{supp}(y).$$

Proof. The first assertion holds because regularity depends only on the support of an element in X^{fr}.

If $x \longrightarrow y$ is an elementary edge as in Definition 3.1.6, then x and y are connected by one of the relations that define X^{st} as a quotient set of X^{fr}. So they have the same support. This implies that if $x \longrightarrow y$ is a prime edge as in Definition 3.1.8, then x and y have the same support. The second assertion now follows from an induction on the length of a path in $\mathsf{Gr}(X)$, with the initial case being a single prime edge. $\square$

For the next observation, consider the 24 axioms (2.1.5)–(2.1.28) of a symmetric bimonoidal category with

- A, B, C, and D taken to be distinct elements in X;
- $\mathbb{0}$ and $\mathbb{1}$ interpreted as 0^X and 1^X, respectively; and
- each vertex in each diagram regarded as an element in X^{fr}.

The next observation provides many examples of regular elements.

Proposition 3.1.30. *In each of the 24 axioms in the setting of the previous paragraph:*

(1) All the vertices have the same support.

(2) All the vertices are regular.

Proof. All the vertices in each axiom have the same support because they are all connected by the relations that define X^{st} as a quotient set of X^{fr}.

For the second assertion, it suffices to show that in each axiom, one vertex is regular. Together with the first assertion and Lemma 3.1.29(1), one vertex being regular would imply that all the vertices are regular.

For the first axiom (2.1.5) consider the vertex $AC \oplus BC$ and the following four cases.

(1) If $A, B, C \in X \setminus \{0^X\}$ then
$$\mathsf{supp}(AC) \neq \mathsf{supp}(BC).$$

(2) If $A = 0^X$, then $B, C \neq 0^X$, and
$$\mathsf{supp}(AC \oplus BC) = \mathsf{supp}(BC).$$

(3) If $B = 0^X$, then $A, C \neq 0^X$, and
$$\mathsf{supp}(AC \oplus BC) = \mathsf{supp}(AC).$$

(4) If $C = 0^X$, then
$$\mathsf{supp}(AC \oplus BC) = \mathsf{supp}(0^X \oplus 0^X) = \mathsf{supp}(0^X).$$

In each of the four cases, we conclude that $AC \oplus BC$ is regular because A, B, and C are distinct in X. The same kind of case-by-case analysis applies to the next 8 axioms (2.1.6)–(2.1.13).

Each of the axioms (2.1.14)–(2.1.22) has a vertex equal to 0^X, which is regular. Each of the axioms (2.1.23)–(2.1.26) has a vertex equal to AB or BA, which are both regular. Each of the axioms (2.1.27) and (2.1.28) has a vertex equal to $A \oplus B$, which is regular. □

Example 3.1.31. In Example 3.1.11, the seven vertices in X^{fr} have the same support, which is that of $xy \oplus xz$. If x, y, and z are distinct elements in X, then $xy \oplus xz$ is regular by the same kind of case-by-case proof in Proposition 3.1.30. ◇

Example 3.1.32. The three elements in X^{fr} appearing in Example 3.1.27 have the same support. If $a, \ldots, f \in X$ are distinct, then these three elements in X^{fr} are all regular by the same kind of case-by-case proof in Proposition 3.1.30. ◇

3.2. Induction Devices

The purpose of this section is to introduce some tools for performing induction arguments in the graph of X as in Definition 3.1.9. These concepts will be used in subsequent sections to prove the Coherence Theorem 3.9.1 for symmetric bimonoidal categories. We will need the following three concepts for induction arguments.

Definition 3.2.1. Denote by $\mathbb{Z}_+$ the set of positive integers. Suppose $(X, 0^X, 1^X)$ is a set with two distinguished elements as in Definition 3.1.6, and with X^{fr} as in Definition 3.1.2.

(1) The *norm*

$$X^{\mathsf{fr}} \xrightarrow{\mathsf{norm}} \mathbb{Z}_+ \tag{3.2.2}$$

is the function defined inductively by the following two conditions:
- $\mathsf{norm}(x) = 1$ for $x \in X$.

- For $a, b \in X^{\mathsf{fr}}$ with $\mathsf{norm}(a)$ and $\mathsf{norm}(b)$ already defined, define

$$\mathsf{norm}(a \oplus b) = \mathsf{norm}(a \otimes b) = \mathsf{norm}(a) + \mathsf{norm}(b).$$

(2) The *rank*

$$X^{\mathsf{fr}} \xrightarrow{\mathsf{rank}} \mathbb{Z}_+ \tag{3.2.3}$$

is the function defined by

$$\mathsf{rank}(a) = 2 \cdot \mathsf{norm}(a) \quad \text{for} \quad a \in X^{\mathsf{fr}}.$$

(3) The *size*

$$X^{\mathsf{fr}} \xrightarrow{\mathsf{size}} \mathbb{Z}_+ \tag{3.2.4}$$

is the function defined inductively by the following two conditions:

- $\mathsf{size}(x) = 2$ for $x \in X$.
- For $a, b \in X^{\mathsf{fr}}$ with $\mathsf{size}(a)$ and $\mathsf{size}(b)$ already defined, define

$$\begin{aligned} &\mathsf{size}(a \oplus b) = \mathsf{size}(a) + \mathsf{size}(b) \quad \text{and} \\ &\mathsf{size}(a \otimes b) = \mathsf{size}(a) \cdot \mathsf{size}(b). \end{aligned}$$

We call $\mathsf{norm}(a)$, $\mathsf{rank}(a)$, and $\mathsf{size}(a)$, respectively, the norm of a, the rank of a, and the size of a. ◇

Explanation 3.2.5. The norm of $a \in X^{\mathsf{fr}}$ is computed by counting the number of elements in X appearing in the expression of a, counting multiplicity. So if an element $x \in X$ appears n times in a, then it is counted as n in the norm of a.

The size of a is computed by replacing

- each element of X appearing in a by the number 2 and
- $\{\oplus, \otimes\}$ in X^{fr} by $\{+, \times\}$ of integers. ◇

Example 3.2.6. Suppose $x^i_j \in X$ for $1 \le i \le m$ and $1 \le j \le k_i$ for each i with $m, k_1, \ldots, k_m > 0$. Consider the element

$$a = \bigoplus_{i=1}^{m} \big(x^i_1 \cdots x^i_{k_i}\big) \in X^{\mathsf{fr}} \tag{3.2.7}$$

appearing in (3.1.26), in which the iterated sum $\oplus_{i=1}^m$ and each of the m iterated products $x^i_1 \cdots x^i_{k_i}$ have some bracketings. Then its norm, rank, and size are as follows.

$$\begin{aligned} \mathsf{norm}(a) &= k_1 + \cdots + k_m \\ \mathsf{rank}(a) &= 2(k_1 + \cdots + k_m) \\ \mathsf{size}(a) &= 2^{k_1} + \cdots + 2^{k_m} \end{aligned}$$

Furthermore, the equality

$$\mathsf{size}(a) = \mathsf{rank}(a)$$

holds if and only if $k_i \le 2$ for each $1 \le i \le m$. ◇

Example 3.2.8. For the element

$$b = (x_1 \oplus x_2)(x_3 \oplus x_4) \in X^{\mathsf{fr}}$$

with $x_1, x_2, x_3, x_4 \in X$, we have

$$\mathsf{norm}(b) = 4, \quad \mathsf{rank}(b) = 8, \quad \text{and} \quad \mathsf{size}(b) = 16.$$ ◇

Below are some basic observations that we will later use for induction arguments.

Lemma 3.2.9. *The following statements hold for elements $a \in X^{\mathsf{fr}}$.*

(1) $\mathsf{norm}(a) = 1$ *if and only if* $a \in X$.
(2) $\mathsf{size}(a) - \mathsf{rank}(a) \geq 0$.

Proof. For assertion (1), if $a \in X$, then $\mathsf{norm}(a) = 1$ by definition. If a does not lie in X, then a has the form $a_1 \oplus a_2$ or $a_1 \otimes a_2$ for some $a_1, a_2 \in X^{\mathsf{fr}}$. So $\mathsf{norm}(a) \geq 2$.

For assertion (2), first observe that the rank can be equivalently defined as follows.

- $\mathsf{rank}(x) = 2$ for $x \in X$.
- For $a, b \in X^{\mathsf{fr}}$ with $\mathsf{rank}(a)$ and $\mathsf{rank}(b)$ already defined, define

$$\mathsf{rank}(a \oplus b) = \mathsf{rank}(a \otimes b) = \mathsf{rank}(a) + \mathsf{rank}(b).$$

Now we prove the second assertion by induction on the norm. For norm 1, that is, for an element $x \in X$, there are equalities

$$\mathsf{size}(x) = \mathsf{rank}(x) = 2.$$

The induction step follows from the inequalities for $a, b \in X^{\mathsf{fr}}$:

$$\begin{aligned}
\mathsf{rank}(a \oplus b) &= \mathsf{rank}(a) + \mathsf{rank}(b) \\
&\leq \mathsf{size}(a) + \mathsf{size}(b) \qquad &&(\text{since } \mathsf{norm}(a), \mathsf{norm}(b) < \mathsf{norm}(a \oplus b)) \\
&= \mathsf{size}(a \oplus b); \\
\mathsf{rank}(a \otimes b) &= \mathsf{rank}(a) + \mathsf{rank}(b) \\
&\leq \mathsf{rank}(a) \cdot \mathsf{rank}(b) &&(\text{since } \mathsf{rank}(a), \mathsf{rank}(b) \geq 2) \\
&\leq \mathsf{size}(a) \cdot \mathsf{size}(b) &&(\text{since } \mathsf{norm}(a), \mathsf{norm}(b) < \mathsf{norm}(a \oplus b)) \\
&= \mathsf{size}(a \otimes b).
\end{aligned}$$

This proves assertion (2). □

Lemma 3.2.10. *Suppose $f : a \longrightarrow b$ is a prime edge that is either an identity or involves a single instance of $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{\pm\otimes}$. Then the following statements hold.*

(1) $\mathsf{norm}(a) = \mathsf{norm}(b)$.
(2) $\mathsf{rank}(a) = \mathsf{rank}(b)$.
(3) $\mathsf{size}(a) = \mathsf{size}(b)$.

Proof. If f is an identity, then $a = b$, which implies the desired equalities.

For $x,y,z \in X^{\mathsf{fr}}$ and each $\odot \in \{\oplus,\otimes\}$, there are equalities as follows.

$$\begin{aligned}
\mathsf{norm}((x\odot y)\odot z) &= \mathsf{norm}(x)+\mathsf{norm}(y)+\mathsf{norm}(z)\\
&= \mathsf{norm}(x\odot(y\odot z))\\
\mathsf{norm}(x\odot y) &= \mathsf{norm}(x)+\mathsf{norm}(y)\\
&= \mathsf{norm}(y\odot x)\\
\mathsf{size}((x\oplus y)\oplus z) &= \mathsf{size}(x)+\mathsf{size}(y)+\mathsf{size}(z)\\
&= \mathsf{size}(x\oplus(y\oplus z))\\
\mathsf{size}((x\otimes y)\otimes z) &= \mathsf{size}(x)\cdot\mathsf{size}(y)\cdot\mathsf{size}(z)\\
&= \mathsf{size}(x\otimes(y\otimes z))\\
\mathsf{size}(x\oplus y) &= \mathsf{size}(x)+\mathsf{size}(y)\\
&= \mathsf{size}(y\oplus x)\\
\mathsf{size}(x\otimes y) &= \mathsf{size}(x)\cdot\mathsf{size}(y)\\
&= \mathsf{size}(y\otimes x)
\end{aligned}$$

They imply the assertions for norm and size. The assertion for rank follows from the definition $\mathsf{rank} = 2\cdot\mathsf{norm}$. □

The next few assertions are about the quantity $\mathsf{size}-\mathsf{rank}$.

Lemma 3.2.11. *For elements $a,b \in X^{\mathsf{fr}}$ and each $\odot \in \{\oplus,\otimes\}$, the inequality*

$$\mathsf{size}(a\odot b)-\mathsf{rank}(a\odot b) \geq \mathsf{size}(c)-\mathsf{rank}(c)$$

holds for each $c \in \{a,b\}$.

Proof. The inequality for $\oplus$ follows from Lemma 3.2.9 (2) and the following computation.

$$\begin{aligned}
&\mathsf{size}(a\oplus b)-\mathsf{rank}(a\oplus b)\\
&= \mathsf{size}(a)+\mathsf{size}(b)-\mathsf{rank}(a)-\mathsf{rank}(b)\\
&= \big(\mathsf{size}(a)-\mathsf{rank}(a)\big)+\big(\mathsf{size}(b)-\mathsf{rank}(b)\big)
\end{aligned}$$

Similarly, the inequality for $\otimes$ follows from the last equality above and the following computation.

$$\begin{aligned}
&\mathsf{size}(a\otimes b)-\mathsf{rank}(a\otimes b)\\
&= \mathsf{size}(a)\cdot\mathsf{size}(b)-\mathsf{rank}(a)-\mathsf{rank}(b)\\
&\geq \mathsf{size}(a)+\mathsf{size}(b)-\mathsf{rank}(a)-\mathsf{rank}(b)
\end{aligned}$$

The inequality uses the fact that $\mathsf{size}(a),\mathsf{size}(b) \geq 2$. □

Next we want to describe elements in X^{fr} that satisfy $\mathsf{size}(a) = \mathsf{rank}(a)$. The following are two preliminary observations in this direction, and use the concept of a prime edge in Definition 3.1.8.

Lemma 3.2.12. *Suppose $a \longrightarrow b$ is a prime edge that involves either δ^l or δ^r. Then the inequality*

$$\mathsf{size}(a)-\mathsf{rank}(a) > \mathsf{size}(b)-\mathsf{rank}(b)$$

holds.

Proof. For $x, y, z \in X^{\mathsf{fr}}$, there are (in)equalities as follows.

$$\begin{aligned}\mathsf{size}(x(y \oplus z)) &= \mathsf{size}(x) \cdot \big(\mathsf{size}(y) + \mathsf{size}(z)\big)\\ &= \mathsf{size}(xy \oplus xz)\\ \mathsf{rank}(x(y \oplus z)) &= \mathsf{rank}(x) + \mathsf{rank}(y) + \mathsf{rank}(z)\\ &< \mathsf{rank}(x) + \mathsf{rank}(y) + \mathsf{rank}(x) + \mathsf{rank}(z)\\ &= \mathsf{rank}(xy \oplus xz)\end{aligned}$$

These (in)equalities imply that in going from the domain to the codomain of the elementary edge

$$x(y \oplus z) \xrightarrow{\delta^l_{x,y,z}} xy \oplus xz,$$

and similarly any prime edge involving δ^l, the quantity $(\mathsf{size} - \mathsf{rank})$ strictly decreases. An almost identical computation proves the case for δ^r. $\square$

Lemma 3.2.13. *If an element $a \in X^{\mathsf{fr}}$ satisfies*

$$\mathsf{size}(a) = \mathsf{rank}(a),$$

then a is not the domain of any prime edge that involves either δ^l or δ^r.

Proof. Suppose to the contrary that there exists a prime edge $a \longrightarrow b$ involving δ^l or δ^r. Then the hypothesis on a and Lemma 3.2.12 imply that

$$0 = \mathsf{size}(a) - \mathsf{rank}(a) > \mathsf{size}(b) - \mathsf{rank}(b).$$

This cannot happen by Lemma 3.2.9(2). $\square$

Example 3.2.14. The converse of Lemma 3.2.13 is *not* true. For example, for elements $x, y, z \in X$, the product $(x \otimes y) \otimes z$ is not the domain of any prime edge that involves either δ^l or δ^r. However, it has rank 6 and size 8. In other words, in order for an element $a \in X^{\mathsf{fr}}$ to not be the domain of any prime edge involving either δ^l or δ^r, the property

$$\mathsf{size}(a) = \mathsf{rank}(a)$$

is sufficient but *not* necessary. $\diamond$

The following observation characterizes elements in X^{fr} with size equal to rank.

Proposition 3.2.15. *For elements $a \in X^{\mathsf{fr}}$, the equality*

$$\mathsf{size}(a) = \mathsf{rank}(a)$$

holds if and only if a has the form

$$a = a_1 \oplus \cdots \oplus a_m$$

for some additive bracketing and some $m \geq 1$ such that for each $1 \leq i \leq m$, either

- *$a_i \in X$, or*
- *$a_i = a_i^1 \otimes a_i^2$ for some elements $a_i^1, a_i^2 \in X$.*

Proof. First suppose $a = a_1 \oplus \cdots \oplus a_m$ with some additive bracketing and with each a_i either in X or the product of two elements in X. Then $\mathsf{size}(a) = \mathsf{rank}(a)$ by Example 3.2.6.

Conversely, suppose $\mathsf{size}(a) = \mathsf{rank}(a)$. To show that a has the desired form, consider the three possible cases.

(i) If $a \in X$ then we are done.

(ii) Next suppose

$$a = a_1 \otimes \cdots \otimes a_n$$

for some $n \geq 2$ and some multiplicative bracketing, and with each $a_i \in X^{\mathsf{fr}}$ not of the form $a_i^1 \otimes a_i^2$ for any $a_i^1, a_i^2 \in X^{\mathsf{fr}}$. So each a_i is either in X or has the form $a_i^1 \oplus a_i^2$. Lemma 3.2.13 implies that each $a_i \in X$. Since

$$\mathsf{rank}(a) = 2n \leq 2^n = \mathsf{size}(a),$$

the assumption $\mathsf{size}(a) = \mathsf{rank}(a)$ implies $n = 2$. So a is a product of two elements in X, which is of the desired form.

(iii) For the remaining case, we may assume that a has the form

$$a = a_1 \oplus \cdots \oplus a_m$$

for some $m \geq 2$ and some additive bracketing, and with each $a_i \in X^{\mathsf{fr}}$ not of the form $a_i^1 \oplus a_i^2$ for any $a_i^1, a_i^2 \in X^{\mathsf{fr}}$. For each $1 \leq i \leq m$, either

- $a_i \in X$, or
- by Lemma 3.2.13 a_i is a finite product of elements in X.

Therefore, a has the form (3.2.7), which is a finite sum of finite products of elements in X. By Example 3.2.6, there are equalities

$$\begin{aligned}\mathsf{rank}(a) &= 2(k_1 + \cdots + k_m) \quad \text{and}\\ \mathsf{size}(a) &= 2^{k_1} + \cdots + 2^{k_m}.\end{aligned}$$

The assumption $\mathsf{size}(a) = \mathsf{rank}(a)$ implies $k_i \leq 2$ for $1 \leq i \leq m$. So a is a finite sum with each summand either in X, or is a product of two elements in X.

This finishes the proof of the other direction. $\square$

Lemma 3.2.16. *Suppose $f : a \longrightarrow b$ is a prime edge that involves either $\lambda^{\otimes}$ or $\rho^{\otimes}$.*

(1) The following inequality holds.

$$\mathsf{size}(a) - \mathsf{rank}(a) \geq \mathsf{size}(b) - \mathsf{rank}(b). \tag{3.2.17}$$

(2) (3.2.17) is an equality if and only if f is a sum of identities and an elementary edge $\lambda_x^{\otimes}$ or $\rho_x^{\otimes}$ with $x \in X$.

Proof. Suppose f involves $\lambda^{\otimes}$; the proof for $\rho^{\otimes}$ is almost identical.

For assertion (1), first consider the following (in)equalities for $c \in X^{\mathsf{fr}}$.

$$\begin{aligned}&\mathsf{size}(1^X \otimes c) - \mathsf{rank}(1^X \otimes c)\\ &= \mathsf{size}(1^X) \cdot \mathsf{size}(c) - \mathsf{rank}(1^X) - \mathsf{rank}(c)\\ &= \big(\mathsf{size}(c) - \mathsf{rank}(c)\big) + \big(\mathsf{size}(c) - 2\big)\\ &\geq \mathsf{size}(c) - \mathsf{rank}(c)\end{aligned} \tag{3.2.18}$$

This is an equality if and only if $\mathsf{size}(c) = 2$, that is, if $c \in X$. Since the prime edge f must involve one elementary edge of the form $\lambda_c^{\otimes}$ for some $c \in X^{\mathsf{fr}}$ and identities, a computation similar to (3.2.18) proves the inequality (3.2.17).

For assertion (2), (3.2.18) with $c \in X$ proves the "if" direction. Conversely, if (3.2.17) is an equality, then the elementary edge in f has the form $\lambda_x^{\otimes}$ for some $x \in X$ by (3.2.18). Moreover, this $\lambda_x^{\otimes}$ is a summand of f, with all other summands identities. This is true because, by a computation similar to (3.2.18), for a product of $1^X \otimes x$ with some other elements in X^{fr}, the quantity $\mathsf{size} - \mathsf{rank}$ strictly decreases when 1^X is removed. □

3.3. Reduction of Additive and Multiplicative Zeros

As the first reduction step for the Coherence Theorem 3.9.1, in this section we show that each element in the free $\{\oplus, \otimes\}$-algebra X^{fr} has a unique 0^X-reduction, in which all instances of 0^X that can be added or multiplied have been eliminated. Although a path from an element to its 0^X-reduction is not unique, we observe that its value in C is unique.

Convention 3.3.1. For the rest of this chapter, the following conventions are in effect.

(1) $(X, 0^X, 1^X)$ is as in Definition 3.1.6, so X is a set with two distinguished elements 0^X and 1^X. The free $\{\oplus, \otimes\}$-algebra X^{fr} of X is as in Definition 3.1.2.
(2) C is an arbitrary symmetric bimonoidal category as in Definition 2.1.2.
(3) $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is an arbitrary function such that

$$\varphi(0^X) = \mathbb{0} \quad \text{and} \quad \varphi(1^X) = \mathbb{1},$$

and

$$\mathsf{Gr}(X) \xrightarrow{\ \varphi\ } \mathsf{C}$$

is the associated graph morphism in Definition 3.1.14. The *value* in C of a path (3.1.18) and the *commutativity* of a diagram in $\mathsf{Gr}(X)$ are defined in that definition. ◇

Definition 3.3.2. A 0^X*-prime edge* is a nonidentity prime edge in the sense of Definition 3.1.8 that involves $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\bullet}$, or $\rho^{\bullet}$. ◇

Example 3.3.3. In Example 3.1.11, there are two 0^X-prime edges, namely, $\rho^{\oplus}_{x(y \oplus z)}$ and $1_{xy} \oplus \rho^{\oplus}_{xz}$. ◇

Definition 3.3.4. Suppose $a \in X^{\mathsf{fr}}$.

(1) The element a is 0^X*-reduced* if either
  - $a = 0^X$, or
  - the expression of a in terms of elements of X does not involve 0^X.
(2) A 0^X*-reduction* of a is a path $P : a \twoheadrightarrow b$ in $\mathsf{Gr}(X)$ such that the following two statements hold.
  - b is 0^X-reduced.
  - Each edge in P is either an identity or a 0^X-prime edge. ◇

The following is basically the definition, but it is useful to state it precisely for later usage.

Lemma 3.3.5. *The following statements hold.*

(1) Each element in X is 0^X-reduced.
(2) An element $a \in X^{\mathsf{fr}}$ is 0^X-reduced if and only if a is not the domain of any 0^X-prime edges.

Proof. An element in X is either 0^X or in $X \setminus \{0^X\}$. In either case, it is 0^X-reduced.

For the second assertion, the domain of each 0^X-prime edge contains 0^X and at least one other element in X^{fr}, which may also be 0^X. Such an element is not 0^X-reduced.

Conversely, if $a \in X^{\mathsf{fr}}$ is not 0^X-reduced, then, by assertion (1), its expression in terms of elements of X involves at least two elements, and 0^X is among them. So a is the domain of some 0^X-prime edge. $\square$

Next we show the existence of 0^X-reductions.

Lemma 3.3.6. *Each element in X^{fr} has a 0^X-reduction.*

Proof. Suppose $a \in X^{\mathsf{fr}}$. The proof is by induction on the number n of instances of 0^X in the expression of a in terms of elements of X. If $n = 0$, then a is 0^X-reduced, and the identity 1_a is a 0^X-reduction of a.

Inductively, suppose $n > 0$. If $a = 0^X$, then it is 0^X-reduced, and 1_a is again a 0^X-reduction.

If $a \neq 0^X$, then a is not 0^X-reduced. By Lemma 3.3.5 (2), a is the domain of some 0^X-prime edge $f : a \longrightarrow a'$. The number of 0^X in the expression of a' in terms of elements of X is $n - 1$. By the induction hypothesis, a' has a 0^X-reduction $P' : a' \twoheadrightarrow b$. The path

$$a \xrightarrow{\;f\;} a' \xrightarrow{\;P'\;} b$$

in $\mathsf{Gr}(X)$ is a 0^X-reduction of a. $\square$

Lemma 3.3.7. *Suppose $P : a \twoheadrightarrow b$ is a 0^X-reduction of an element $a \in X^{\mathsf{fr}}$. Then*

$$\mathsf{supp}(a) = \mathsf{supp}(0^X) \quad \text{if and only if} \quad b = 0^X.$$

Proof. By Lemma 3.1.29 (2), there is an equality

$$\mathsf{supp}(a) = \mathsf{supp}(b).$$

So if $b = 0^X$, then

$$\mathsf{supp}(a) = \mathsf{supp}(b) = \mathsf{supp}(0^X). \tag{3.3.8}$$

Conversely, since b is 0^X-reduced, it is either 0^X or contains no 0^X. Therefore, if (3.3.8) holds, then $b = 0^X$, since otherwise its support cannot be the support of 0^X. $\square$

Motivation 3.3.9. Uniqueness of 0^X-reductions requires more care than existence. For example, the diagram

$$\begin{tikzcd}
& x \otimes 0^X \arrow[dr, bend left, "\rho^{\bullet}_x"] & \\
0^X \oplus (x \otimes 0^X) \arrow[ur, bend left, "\lambda^{\oplus}_{x \otimes 0^X}"] \arrow[dr, bend right, "1_{0^X} \oplus \rho^{\bullet}_x"'] & & 0^X \\
& 0^X \oplus 0^X \arrow[ur, bend left, "\lambda^{\oplus}_{0^X}"] \arrow[ur, bend right, "\rho^{\oplus}_{0^X}"'] &
\end{tikzcd}$$

contains three different 0^X-reductions of $0^X \oplus (x \otimes 0^X) \in X^{\mathsf{fr}}$. Therefore, uniqueness should *not* refer to the paths themselves, but rather (i) the codomain and (ii) the values of the paths in the symmetric bimonoidal category C. Uniqueness of 0^X-reductions in these two senses are proved in the next two results. $\diamond$

Notation 3.3.10. For a path $P : a \twoheadrightarrow b$ in $\mathsf{Gr}(X)$ and an element $c \in X^{\mathsf{fr}}$, the path

$$a \oplus c \xrightarrow{P \oplus 1_c} b \oplus c$$

in $\mathsf{Gr}(X)$ is obtained from P by replacing each of its edges h by $h \oplus 1_c$. The paths

- $1_c \oplus P : c \oplus a \twoheadrightarrow c \oplus b$,
- $P \otimes 1_c : a \otimes c \twoheadrightarrow b \otimes c$, and
- $1_c \otimes P : c \otimes a \twoheadrightarrow c \otimes b$

are defined analogously by replacing each edge h in P by $1_c \oplus h$, $h \otimes 1_c$, and $1_c \otimes h$, respectively. ◇

Lemma 3.3.11. *For any two 0^X-reductions*

$$\begin{tikzcd} & b_1 \\ a \arrow[ur, bend left, two heads, "P_1"] \arrow[dr, bend right, two heads, "P_2"'] & \\ & b_2 \end{tikzcd}$$

of an element $a \in X^{\mathsf{fr}}$, the equality $b_1 = b_2$ holds.

Proof. The proof is by induction on the norm of a. If $\mathsf{norm}(a) = 1$, then $a \in X$. By Lemma 3.3.5, the only possible 0^X-reductions of a are paths consisting of identity prime edges. They all have codomains equal to a.

For the induction step, suppose $\mathsf{norm}(a) > 1$. So a has the form $a_1 \oplus a_2$ or $a_1 \otimes a_2$ for some $a_1, a_2 \in X^{\mathsf{fr}}$ with strictly lower norms than a.

If a contains no 0^X in its expression, then the only possible 0^X-reductions of a are paths consisting of identity prime edges. They all have codomains equal to a.

We now assume that a contains at least one 0^X in its expression in terms of elements of X. Adding or removing an identity in a path do not change the (co)domain. Therefore, to show that $b_1 = b_2$, we may assume that for $i = 1, 2$, P_i has the form

$$a \xrightarrow{f_i} c_i \overset{P_i'}{\twoheadrightarrow} b_i$$

with

- the first edge f_i a 0^X-prime edge and
- $P_i' : c_i \twoheadrightarrow b_i$ a 0^X-reduction of c_i, whose norm is strictly less than that of a.

The induction hypothesis applied to c_i implies that the codomains of any two 0^X-reductions of c_i are equal. To show that $b_1 = b_2$, we consider the two possible cases.

First, if $f_1 = f_2$, then $c_1 = c_2$. This implies that $b_1 = b_2$.

If $f_1 \neq f_2$, then up to renaming them, they fall into one of the following cases.

(1) f_1 and f_2 act on different summands of a, as in the following diagram.

$$\begin{tikzcd}
& c_1 = a_1' \oplus a_2 \arrow[dr, "1_{a_1'} \oplus f_2'"] \arrow[drr, bend left, two heads, "Q_1"] & & \\
a = a_1 \oplus a_2 \arrow[ur, "f_1 = f_1' \oplus e_{a_2}"] \arrow[dr, "f_2 = e_{a_1} \oplus f_2'"'] & & d = a_1' \oplus a_2' \arrow[r, two heads, "Q"] & d' \\
& c_2 = a_1 \oplus a_2' \arrow[ur, "f_1' \oplus 1_{a_2'}"'] \arrow[urr, bend right, two heads, "Q_2"'] & &
\end{tikzcd}$$

Here:

- $e_{a_i} : a_i \longrightarrow a_i$ is an identity prime edge for $i = 1, 2$.
- $f_1 = f_1' \oplus e_{a_2}$ for some 0^X-prime edge $f_1' : a_1 \longrightarrow a_1'$.
- $f_2 = e_{a_1} \oplus f_2'$ for some 0^X-prime edge $f_2' : a_2 \longrightarrow a_2'$.
- By Lemma 3.3.6, $d = a_1' \oplus a_2'$ has a 0^X-reduction $Q : d \longrightarrow\!\!\!\!\rightarrow d'$.

Define the paths

$$\begin{aligned}
Q_1 &= \big(Q, 1_{a_1'} \oplus f_2'\big) : c_1 \longrightarrow\!\!\!\!\rightarrow d' \\
Q_2 &= \big(Q, f_1' \oplus 1_{a_2'}\big) : c_2 \longrightarrow\!\!\!\!\rightarrow d'
\end{aligned}$$

in $\mathsf{Gr}(X)$ as displayed above. Since these are 0^X-reductions of c_1 and c_2, respectively, the induction hypothesis implies that $b_1 = d' = b_2$.

(2) Case (1) has an analogue involving $\otimes$ instead of $\oplus$ in the diamond in the previous diagram. The proof for $b_1 = b_2$ in this case is the same as above after replacing $\oplus$ with $\otimes$.

(3) f_1 and f_2 act on the same summand of a, as in the following diagram.

$$\begin{tikzcd}
& c_1 = a_1' \oplus a_2 \arrow[dr, two heads, "R_1 \oplus 1_{a_2}"] \arrow[drr, bend left, two heads, "Q_1"] & & \\
a = a_1 \oplus a_2 \arrow[ur, "f_1 = f_1' \oplus e_{a_2}"] \arrow[dr, "f_2 = f_2' \oplus e_{a_2}'"'] & & d = b' \oplus a_2 \arrow[r, two heads, "Q"] & d' \\
& c_2 = a_1'' \oplus a_2 \arrow[ur, two heads, "R_2 \oplus 1_{a_2}"'] \arrow[urr, bend right, two heads, "Q_2"'] & &
\end{tikzcd}$$

Here:

- f_1 is as in case (1).
- $e_{a_2}' : a_2 \longrightarrow a_2$ is an identity prime edge.
- $f_2 = f_2' \oplus e_{a_2}'$ for some 0^X-prime edge $f_2' : a_1 \longrightarrow a_1''$.
- Since
$$\mathsf{norm}(a_1) < \mathsf{norm}(a),$$
the induction hypothesis applied to a_1, and Lemma 3.3.6 applied to a_1' and a_1'', together imply that there exist 0^X-reductions
$$\begin{tikzcd}
a_1' \arrow[dr, two heads, "R_1"] & \\
& b' \\
a_1'' \arrow[ur, two heads, "R_2"'] &
\end{tikzcd}$$

with a common codomain b'.
- By Lemma 3.3.6, $d = b' \oplus a_2$ has a 0^X-reduction $Q : d \twoheadrightarrow d'$.
- Each path $R_i \oplus 1_{a_2} : c_i \twoheadrightarrow d$ is defined in Notation 3.3.10.

Define the paths

$$\begin{aligned} Q_1 &= \big(Q, R_1 \oplus 1_{a_2}\big) : c_1 \twoheadrightarrow d' \\ Q_2 &= \big(Q, R_2 \oplus 1_{a_2}\big) : c_2 \twoheadrightarrow d' \end{aligned}$$

in $\mathsf{Gr}(X)$ as displayed above. Since these are 0^X-reductions of c_1 and c_2, respectively, the induction hypothesis implies that $b_1 = d' = b_2$.

(4) There are three variants of case (3) obtained by the following modifications in the diamond in the previous diagram.
- Permute the two summands in each vertex and each edge or path.
- Replace $\oplus$ with $\otimes$.
- Permute the two summands in each vertex and each edge or path, and replace $\oplus$ with $\otimes$.

In each of these three cases, after the modifications, the proof is the same as in case (3).

(5) f_1 acts on a summand, and f_2 acts on all of a, as in the following diagram.

$$\begin{tikzcd}
 & c_1 = d \oplus 0^X \arrow[dr, "\rho^{\oplus}_d"] \arrow[rr, bend left, two heads, "Q_1"] & & \\
a = a_1 \oplus 0^X \arrow[ur, "f_1 \,=\, f_1' \oplus 1_{0^X}"] \arrow[dr, "f_2 \,=\, \rho^{\oplus}_{a_1}"'] & & d \arrow[r, two heads, "Q"] & d' \\
 & c_2 = a_1 \arrow[ur, "f_1'"'] \arrow[urr, bend right, two heads, "Q_2"'] & &
\end{tikzcd}$$

Here:
- $f_1 = f_1' \oplus 1_{0^X}$ for some 0^X-prime edge $f_1' : a_1 \longrightarrow d$.
- By Lemma 3.3.6, there exists a 0^X-reduction $Q : d \twoheadrightarrow d'$.

Define the paths

$$\begin{aligned} Q_1 &= \big(Q, \rho^{\oplus}_d\big) : c_1 \twoheadrightarrow d' \\ Q_2 &= \big(Q, f_1'\big) : c_2 \twoheadrightarrow d' \end{aligned}$$

in $\mathsf{Gr}(X)$ as displayed above. Since these are 0^X-reductions of c_1 and c_2, respectively, the induction hypothesis implies that $b_1 = d' = b_2$.

(6) Case (5) has a variant with $a = 0^X \oplus a_1$. The proof for this case is obtained from that of case (5) by replacing $(\rho^{\oplus}, f_1' \oplus 1_{0^X})$ with $(\lambda^{\oplus}, 1_{0^X} \oplus f_1')$.

(7) f_1 acts on a factor, and f_2 acts on all of a, as in the following diagram.

$$\begin{tikzcd}[column sep=small] & c_1 = a_1' \otimes 0^X \arrow[dr, "\rho^\bullet_{a_1'}"] & \\ a = a_1 \otimes 0^X \arrow[ur, "f_1 \,=\, f_1' \otimes 1_{0^X}"] \arrow[dr, "f_2 \,=\, \rho^\bullet_{a_1}"'] & & 0^X \\ & c_2 = 0^X \arrow[ur, "1_{0^X}"'] & \end{tikzcd}$$

Since $\rho^\bullet_{a_1'}$ and 1_{0^X} are 0^X-reductions of c_1 and c_2, respectively, the induction hypothesis implies that $b_1 = 0^X = b_2$.

(8) Case (7) has a variant with $a = 0^X \otimes a_1$. The proof for this case is obtained from that of case (7) by replacing $(\rho^\bullet, f_1' \otimes 1_{0^X})$ with $(\lambda^\bullet, 1_{0^X} \otimes f_1')$.

(9) Both f_1 and f_2 act on all of a, as in the following diagram.

$$\begin{tikzcd}[column sep=small] & c_1 = 0^X \arrow[dr, "1_{0^X}"] & \\ a = 0^X \oplus 0^X \arrow[ur, "f_1 \,=\, \lambda^\oplus_{0^X}"] \arrow[dr, "f_2 \,=\, \rho^\oplus_{0^X}"'] & & 0^X \\ & c_2 = 0^X \arrow[ur, "1_{0^X}"'] & \end{tikzcd}$$

Since 1_{0^X} is a 0^X-reduction of each of c_1 and c_2, the induction hypothesis implies that $b_1 = 0^X = b_2$.

(10) Case (9) has a variant with $(\oplus, \lambda^\oplus_{0^X}, \rho^\oplus_{0^X})$ replaced by $(\otimes, \lambda^\bullet_{0^X}, \rho^\bullet_{0^X})$. Again the induction hypothesis implies that $b_1 = 0^X = b_2$.

This finishes the proof of the induction step. □

Lemma 3.3.12. *Any two 0^X-reductions*

$$\begin{tikzcd} a \arrow[r, two heads, bend left, "P_1"] \arrow[r, two heads, bend right, "P_2"'] & b \end{tikzcd}$$

of an element $a \in \mathsf{X}^{\mathsf{fr}}$ have the same value in C.

Proof. We use Lemma 3.3.11 and its inductive proof. If either $\mathsf{norm}(a) = 1$, or inductively $\mathsf{norm}(a) > 1$ with a containing no 0^X, then a 0^X-reduction of a consists of identities. So its value in C must be the identity morphism.

Following the proof of Lemma 3.3.11, we may assume that each P_i takes the form

$$\begin{tikzcd} a \arrow[r, "f_i"] & c_i \arrow[r, two heads, "P_i'"] & b \end{tikzcd}$$

with

- f_i a 0^X-prime edge and
- $P_i' : c_i \twoheadrightarrow b$ a 0^X-reduction of c_i.

The induction hypothesis applied to c_i says that every 0^X-reduction of c_i has the same value in C as P_i'.

If $f_1 = f_2$, then $c_1 = c_2$. The induction hypothesis implies that P_1' and P_2', and therefore also P_1 and P_2, have the same value in C.

If $f_1 \neq f_2$, then we follow the proof of Lemma 3.3.11 and reuse the diagrams there with $d' = b$.

- In cases (1) and (2), the diamond is commutative by, respectively, the functoriality of $\oplus$ and $\otimes$ in C.
- In cases (3) and (4), the diamond is commutative by functoriality and the induction hypothesis applied to a_1.
- In cases (5)–(8), the diamond is commutative by, respectively, the naturality of $\rho^\oplus$, $\lambda^\oplus$, $\rho^\bullet$, and $\lambda^\bullet$.
- In case (9), the diamond is commutative by the unity property (1.2.6), that is, $\lambda^\oplus_{\mathbb{0}} = \rho^\oplus_{\mathbb{0}}$ in the symmetric monoidal category $(\mathsf{C}, \oplus, \mathbb{0})$.
- In case (10), the diamond is commutative by the axiom (2.1.14), that is, $\lambda^\bullet_{\mathbb{0}} = \rho^\bullet_{\mathbb{0}}$ in C.

Now we observe that P_1 and P_2 have the same value in C in each case. In each of cases (1)–(6), the other two subdiagrams are commutative by the definitions of the paths Q_1 and Q_2. The induction hypothesis applied to each c_i implies that P_i' and Q_i have the same value in C. The commutativity of the diagram in C then implies that P_1 and P_2 have the same value in C.

In each of cases (7)–(10), the diagram is the diamond with $b = 0^X$.

- In case (7), the induction hypothesis implies that the values of P_1' and P_2' in C are, respectively, $\rho^\bullet_{ga_1'}$ and $1_{\mathbb{0}}$. The commutative diamond in C implies that P_1 and P_2 have the same value in C.
- The proof for case (8) is obtained from that of case (7) by replacing $\rho^\bullet_{ga_1'}$ with $\lambda^\bullet_{ga_1'}$.
- In cases (9) and (10), the value in C of any 0^X-reduction of $c_1 = c_2 = 0^X$ is $1_{\mathbb{0}}$. Therefore, P_1 and P_2 have values, respectively,
  - $\lambda^\oplus_{\mathbb{0}}$ and $\rho^\oplus_{\mathbb{0}}$ in case (9) and
  - $\lambda^\bullet_{\mathbb{0}}$ and $\rho^\bullet_{\mathbb{0}}$ in case (10).

  They are equal by, respectively, (1.2.6) and (2.1.14), as explained above.

This finishes the induction step. □

Example 3.3.13. In Motivation 3.3.9, the three 0^X-reductions of $0^X \oplus (x \otimes 0^X) \in X^{\mathsf{fr}}$ have the same value in C for the following reasons.

- The upper left sub-diagram is commutative by the naturality of $\lambda^\oplus$ in C.
- The lower right sub-diagram is commutative by axiom (1.2.6) in the symmetric monoidal category $(\mathsf{C}, \oplus, \mathbb{0})$, which says that $\lambda^\oplus_{\mathbb{0}} = \rho^\oplus_{\mathbb{0}}$. ◇

3.4. Zero Reduction of Paths

A reduction step in the proof of the Coherence Theorem 3.9.1 involves eliminating additive zeros, multiplicative zeros, and their inverses in paths. In this section, we first give a precise definition of such a reduction; see Definition 3.4.5. In such a reduction, the (co)domain is a 0^X-reduction of the (co)domain of the original path. Then we prove several basic properties about them. The proof of the existence of such a reduction is given in Section 3.5. Recall elementary edges, prime edges, and 0^X-prime edges in Definitions 3.1.6, 3.1.8, and 3.3.2. Recall that Convention 3.3.1 is in effect.

Definition 3.4.1. Consider the graph $\mathsf{Gr}(X)$ of X in Definition 3.1.9.

- An *inverse* 0^X*-prime edge* is a nonidentity prime edge that involves $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\lambda^{-\bullet}$, or $\rho^{-\bullet}$.
- A 0^X*-free path* is a path that does not contain any 0^X-prime edges and inverse 0^X-prime edges.
- A 0^X*-free edge* is a 0^X-free path of length 1. $\diamond$

Explanation 3.4.2. In a 0^X-free path, every prime edge is either an identity or involves a single instance of $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, $\xi^{\pm\otimes}$, $\lambda^{\pm\otimes}$, $\rho^{\pm\otimes}$, δ^l, or δ^r. $\diamond$

Example 3.4.3. For elements $a, b \in X^{\mathsf{fr}}$, the prime edges

$$a \xrightarrow{\lambda_a^{-\oplus}} 0^X \oplus a \xrightarrow{\rho_b^{-\bullet} \oplus 1_a} (b \otimes 0^X) \oplus a$$

are inverse 0^X-prime edges. $\diamond$

Example 3.4.4. Suppose $P : a \twoheadrightarrow b$ is a 0^X-free path. Then the paths

- $P \oplus 1_c : a \oplus c \twoheadrightarrow b \oplus c$,
- $1_c \oplus P : c \oplus a \twoheadrightarrow c \oplus b$,
- $P \otimes 1_c : a \otimes c \twoheadrightarrow b \otimes c$, and
- $1_c \otimes P : c \otimes a \twoheadrightarrow c \otimes b$

in Notation 3.3.10 are 0^X-free paths for each element $c \in X^{\mathsf{fr}}$. $\diamond$

The next definition uses the concepts of commutativity near the end of Definition 3.1.14, and 0^X-reductions of an element in X^{fr} in Definition 3.3.4.

Definition 3.4.5. Suppose $P : a \twoheadrightarrow b$ is a path in $\mathsf{Gr}(X)$. A 0^X*-reduction* of P is a 0^X-free path $R : a' \twoheadrightarrow b'$ such that for any 0^X-reductions

- $Q_a : a \twoheadrightarrow a'$ of a and
- $Q_b : b \twoheadrightarrow b'$ of b,

the diagram

$$\begin{tikzcd} a \arrow[r, two heads, "P"] \arrow[d, two heads, "Q_a"'] & b \arrow[d, two heads, "Q_b"] \\ a' \arrow[r, two heads, "R"] & b' \end{tikzcd} \tag{3.4.6}$$

is commutative in the sense of Definition 3.1.14. $\diamond$

The goal of this section and Section 3.5 is to show that each path in $\mathsf{Gr}(X)$ has a 0^X-reduction; see Proposition 3.5.32.

Explanation 3.4.7. Consider Definition 3.4.5.

(1) The domain a' of R must be the codomain of a 0^X-reduction of a, and the codomain b' of R must be the codomain of a 0^X-reduction of b. The elements a' and b' are well defined, that is, uniquely determined by a and b, by Lemmas 3.3.6 and 3.3.11. In particular, R is a 0^X-free path whose domain and codomain are both 0^X-reduced in the sense of Definition 3.3.4.

(2) For the commutativity of the diagram (3.4.6), the equality

$$(R, Q_a) = (Q_b, P)$$

of paths in $\mathsf{Gr}(X)$ is sufficient but *not* necessary. The definition only requires that the image of the diagram under the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ in Definition 3.1.14 be a commutative diagram in C for arbitrary C and φ.

The following observation says that for a 0^X-free path to be a 0^X-reduction of a given path, it suffices to check the commutativity of the diagram (3.4.6) for one pair of 0^X-reductions (Q_a, Q_b). $\diamond$

Lemma 3.4.8. *Suppose the following statements hold:*

- *$P : a \twoheadrightarrow b$ is a path in $\mathsf{Gr}(X)$.*
- *$Q_a : a \twoheadrightarrow a'$ is a 0^X-reduction of a.*
- *$Q_b : b \twoheadrightarrow b'$ is a 0^X-reduction of b.*
- *$R : a' \twoheadrightarrow b'$ is a 0^X-free path.*
- *The diagram*

$$\begin{tikzcd} a \arrow[r, two heads, "P"] \arrow[d, two heads, "Q_a"'] & b \arrow[d, two heads, "Q_b"] \\ a' \arrow[r, two heads, "R"] & b' \end{tikzcd}$$

is commutative.

Then R is a 0^X-reduction of P.

Proof. By Lemmas 3.3.11 and 3.3.12, for each element in X^{fr}, the codomain and the value in C of a 0^X-reduction are unique. Therefore, in the given diagram, if Q_a and Q_b are replaced by any other 0^X-reductions of a and b, respectively, then the resulting diagram is still commutative. $\square$

Lemma 3.4.8 will be used below to check that a given path has a 0^X-reduction. Recall from Definition 3.3.4 the concept of an element in X^{fr} being 0^X-reduced.

Lemma 3.4.9. *Suppose $P : a \twoheadrightarrow b$ is a 0^X-free path.*

(1) *If a contains no 0^X, then every vertex in P contains no 0^X and is 0^X-reduced.*
(2) *If $a = 0^X$, then the following statements hold.*
  (i) *Each edge in P is either an identity or a prime edge involving a single instance of $\alpha^{\pm\otimes}$, $\xi^{\pm\otimes}$, $\lambda^{\pm\otimes}$, or $\rho^{\pm\otimes}$.*
  (ii) *Each vertex in P is a finite product with precisely one 0^X and all other factors 1^X.*

Proof. If the expression of a in terms of elements of X contains no 0^X, then the same is true for all other vertices in P because edges in P, as described in Explanation 3.4.2, do not introduce any 0^X from the domain to the codomain.

For assertion (2), consider the subset $S \subset X^{\mathsf{fr}}$ in which each element is a finite product with precisely one 0^X and all other factors 1^X. Then $a = 0^X$ is in S as the special case with no factors of 1^X. Consider an edge

$$c \xrightarrow{f} b$$

in $\mathsf{Gr}(X)$ with

- $c \in S$ and
- f a prime edge that is also a 0^X-free path.

See Explanation 3.4.2 for the possibilities of f. We consider all possible cases below.

- If f is an identity, then $b = c \in S$.
- Suppose f is not an identity, and $c = 0^X$. Then f must be either

$$0^X \xrightarrow{\lambda^{-\otimes}_{0^X}} 1^X \otimes 0^X \quad \text{or} \quad 0^X \xrightarrow{\rho^{-\otimes}_{0^X}} 0^X \otimes 1^X.$$

  Moreover, its codomain is also in S.
- Suppose f is not an identity, and $c \neq 0^X$. Since $c \in S$, the prime edge f can only involve a single instance of $\alpha^{\pm\otimes}$ (if $\mathsf{norm}(c) \geq 3$), $\xi^{\pm\otimes}$, $\lambda^{\pm\otimes}$, or $\rho^{\pm\otimes}$. Moreover, its codomain is also in S.

Therefore, an induction on the length of P proves assertion (2), with the previous paragraph proving both the initial case and the first edge of the induction step. □

Example 3.4.10. In the context of Lemma 3.4.9 (2), for a 0^X-free path $P : 0^X \twoheadrightarrow b$, it does *not* follow that the vertices in P other than the domain are 0^X-reduced. For example, in the 0^X-free path

$$\begin{tikzcd} 0^X \arrow[d, "\lambda^{-\otimes}_{0^X}"'] & & & (1^X \otimes 1^X) \otimes (0^X \otimes 1^X) \\ 1^X \otimes 0^X \arrow[r, "\rho^{-\otimes}_{1^X \otimes 0^X}"] & (1^X \otimes 0^X) \otimes 1^X \arrow[rr, "\alpha^{\otimes}_{1^X,0^X,1^X}"] & & 1^X \otimes (0^X \otimes 1^X) \arrow[u, "\lambda^{-\otimes}_{1^X} \otimes 1_{0^X \otimes 1^X}"'] \end{tikzcd}$$

all the vertices except 0^X are *not* 0^X-reduced. ⋄

Recall from (3.1.18) the value in C of a path in $\mathsf{Gr}(X)$. The next two observations are about the values in C of paths with 0^X as the domain. They provide further information in the setting of Lemma 3.4.9 (2).

Lemma 3.4.11. *Suppose $P : 0^X \twoheadrightarrow 0^X$ is a 0^X-free path. Then the value of P in C is $1_{\mathbb{0}}$.*

Proof. This follows from Lemma 3.4.9 (2) and the uniqueness part of Theorem 1.3.8 for the symmetric monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \xi^\otimes)$. □

The next observation will be needed in the proof of Lemma 3.4.14.

Lemma 3.4.12. *Suppose*

- *$P : 0^X \twoheadrightarrow b$ is a 0^X-free path, and*
- *$Q_b : b \twoheadrightarrow 0^X$ is a 0^X-reduction of b.*

Then the value in C of the path

$$0^X \xrightarrow{P} b \xrightarrow{Q_b} 0^X \tag{3.4.13}$$

is the identity morphism $1_{\mathbb{0}}$.

Proof. By Lemma 3.4.9 (2), b is a finite product with precisely one 0^X and all other factors 1^X. An induction on $\mathsf{norm}(b)$, starting with $\mathsf{norm}(b) = 1$ and $b = 0^X$, implies that there is a 0^X-reduction

$$b \xrightarrow{Q_b} 0^X$$

of b such that the following statements hold.

- $Q_b = 1_{0^X}$ if $b = 0^X$.
- If $b \neq 0^X$, then Q_b consists of prime edges, each involving $\lambda^\bullet$ or $\rho^\bullet$ with subscript a finite product of copies of 1^X.

One example of Q_b is the path below.

$$\begin{tikzcd} b = 1^X \otimes \big(0^X \otimes (1^X \otimes 1^X)\big) \arrow[r, "1_{1^X} \otimes \lambda^\bullet_{1^X \otimes 1^X}"] & 1^X \otimes 0^X \arrow[d, "\rho^\bullet_{1^X}"] \\ & 0^X \end{tikzcd}$$

In general, the value in C of each prime edge in Q_b is one of the following:

- $1_{\mathbb{0}}$ if $Q_b = 1_{0^X}$;
- a canonical map as in Definition 1.3.2 involving a single instance of $\lambda^\bullet_{\mathbb{1}} = \rho^\otimes_{\mathbb{0}}$ (2.1.17) and identities, if f has a copy of $\lambda^\bullet_{1^X}$;
- a canonical map involving a single instance of $\rho^\bullet_{\mathbb{1}} = \lambda^\otimes_{\mathbb{0}}$ (2.1.18) and identities, if f has a copy of $\rho^\bullet_{1^X}$; or
- a canonical map involving identities, $\lambda^\otimes$, and $\rho^\otimes$, if f has a copy of $\lambda^\bullet$ or $\rho^\bullet$ whose subscript is a product of at least two copies of 1^X.

For an example of the last case, consider the prime edge

$$1^X \otimes \big(0^X \otimes (1^X \otimes 1^X)\big) \xrightarrow{1_{1^X} \otimes \lambda^\bullet_{1^X \otimes 1^X}} 1^X \otimes 0^X$$

with an instance of $\lambda^\bullet_{1^X \otimes 1^X}$. Its value in C factors as the composite below by the naturality of $\lambda^\bullet$ in C.

$$\begin{tikzcd} \mathbb{1} \otimes \big(\mathbb{0} \otimes (\mathbb{1} \otimes \mathbb{1})\big) \arrow[rr, "1_{\mathbb{1}} \otimes \lambda^\bullet_{\mathbb{1} \otimes \mathbb{1}}"] \arrow[dr, "1_{\mathbb{1}} \otimes (1_{\mathbb{0}} \otimes \lambda^\otimes_{\mathbb{1}})"'] & & \mathbb{1} \otimes \mathbb{0} \\ & \mathbb{1} \otimes (\mathbb{0} \otimes \mathbb{1}) \arrow[ur, "1_{\mathbb{1}} \otimes \lambda^\bullet_{\mathbb{1}} = 1_{\mathbb{1}} \otimes \rho^\otimes_{\mathbb{0}}"'] & \end{tikzcd}$$

The axiom (2.1.17) in C, which says that $\lambda^\bullet_{\mathbb{1}} = \rho^\otimes_{\mathbb{0}}$, implies that the above composite is a canonical map involving identities, $\lambda^\otimes$, and $\rho^\otimes$.

For a general finite product I of at least two copies of $\mathbb{1}$, there is a finite composite $I \longrightarrow \mathbb{1}$ consisting of identities, $\lambda^\otimes$, and $\rho^\otimes$. In this case:

- The left slanted morphism in the previous diagram is a canonical map involving identities, $\lambda^\otimes$, and $\rho^\otimes$.
- The right slanted morphism uses either
  - $\lambda^\bullet_{\mathbb{1}} = \rho^\otimes_{\mathbb{0}}$ as above, if the prime edge involves $\lambda^\bullet$; or
  - the axiom (2.1.18), which says that $\rho^\bullet_{\mathbb{1}} = \lambda^\otimes_{\mathbb{0}}$, if the prime edge involves $\rho^\bullet$.

Therefore, in general the value of Q_b in C is the $\mathbb{0}$-component of a canonical map involving identities, $\lambda^\otimes$, and $\rho^\otimes$.

By Lemma 3.4.9 (2), every edge in P is either an identity or a prime edge involving a single instance of $\alpha^{\pm\otimes}$, $\xi^{\pm\otimes}$, $\lambda^{\pm\otimes}$, or $\rho^{\pm\otimes}$. Therefore, the value of the path (3.4.13) in C is the $\mathbb{0}$-component of a permuted canonical map $- \longrightarrow -$. It is the identity morphism $1_{\mathbb{0}}$ by the uniqueness part of the Symmetric Coherence Theorem 1.3.8 for the symmetric monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \xi^\otimes)$.

We have proved the assertion for one choice of a 0^X-reduction of b. Therefore, by Lemma 3.3.12, the assertion remains true for any other 0^X-reduction of b. □

The next observation will be used in the proofs of several lemmas in Section 3.5.

Lemma 3.4.14. *Suppose*

- $P : a \twoheadrightarrow b$ *is a* 0^X*-free path with* a *and* b *both* 0^X*-reduced, and*
- $c \in X^{\mathsf{fr}}$ *is* 0^X*-reduced.*

Then the paths

$$P \oplus 1_c, \quad 1_c \oplus P, \quad P \otimes 1_c, \quad \text{and} \quad 1_c \otimes P$$

in Notation 3.3.10 have 0^X*-reductions.*

Proof. Each of a and c has two possibilities. It is either 0^X, or its expression in terms of elements of X contains no 0^X. We consider the possible cases separately.

First, if a contains no 0^X, then b also contains no 0^X by Lemma 3.4.9 (1). There are now two sub-cases depending on the form of c.

(1) If c also contains no 0^X, then $a \oplus c$ and $b \oplus c$ contain no 0^X, and are 0^X-reduced. Since $P \oplus 1_c$ is a 0^X-free path, it is a 0^X-reduction of itself. Essentially the same argument works for the other three types of paths.

(2) If $c = 0^X$, then the diagram

$$\begin{tikzcd} a \oplus 0^X \arrow[r, two heads, "P\oplus 1_{0^X}"] \arrow[d, "\rho^{\oplus}_a"'] & b \oplus 0^X \arrow[d, "\rho^{\oplus}_b"] \\ a \arrow[r, two heads, "P"] & b \end{tikzcd}$$

is commutative by the naturality of $\rho^{\oplus}$ in C. The left and the right vertical edges are 0^X-reductions of $a \oplus 0^X$ and $b \oplus 0^X$, respectively. Since P is a 0^X-free path, it is a 0^X-reduction of $P \oplus 1_c$.

Similarly, for the paths $1_c \oplus P$, $P \otimes 1_c$, and $1_c \otimes P$, the diagrams below are commutative by the naturality of, respectively, $\lambda^{\oplus}$, $\rho^{\bullet}$, and $\lambda^{\bullet}$ in C.

$$\begin{tikzcd} 0^X \oplus a \arrow[r, two heads, "1_{0^X}\oplus P"] \arrow[d, "\lambda^{\oplus}_a"'] & 0^X \oplus b \arrow[d, "\lambda^{\oplus}_b"] \\ a \arrow[r, two heads, "P"] & b \end{tikzcd} \qquad \begin{tikzcd} a \otimes 0^X \arrow[r, two heads, "P\otimes 1_{0^X}"] \arrow[d, "\rho^{\bullet}_a"'] & b \otimes 0^X \arrow[d, "\rho^{\bullet}_b"] \\ 0^X \arrow[r, two heads, "1_{0^X}"] & 0^X \end{tikzcd} \qquad \begin{tikzcd} 0^X \otimes a \arrow[r, two heads, "1_{0^X}\otimes P"] \arrow[d, "\lambda^{\bullet}_a"'] & 0^X \otimes b \arrow[d, "\lambda^{\bullet}_b"] \\ 0^X \arrow[r, two heads, "1_{0^X}"] & 0^X \end{tikzcd}$$

They show that P, 1_{0^X}, and 1_{0^X}, respectively, are the corresponding 0^X-reductions.

Next if $a = 0^X$, then b is a finite product with precisely one 0^X and all other factors 1^X by Lemma 3.4.9 (2). We consider the following diagram with $Q_b : b \twoheadrightarrow 0^X$ any 0^X-reduction of b.

$$\begin{tikzcd} 0^X \oplus c \arrow[r, two heads, "P\oplus 1_c"] \arrow[d, "1_{0^X}\oplus 1_c"'] & b \oplus c \arrow[d, "Q_b\oplus 1_c"] \\ 0^X \oplus c \arrow[r, two heads, "1_{0^X}\oplus 1_c"] \arrow[d, "\lambda^{\oplus}_c"'] & 0^X \oplus c \arrow[d, "\lambda^{\oplus}_c"] \\ c \arrow[r, "1_c"] & c \end{tikzcd}$$

The bottom rectangle is commutative by the functoriality of $\oplus$ in C. The top rectangle is commutative by the functoriality of $\oplus$ in C and Lemma 3.4.12. Since c

is 0^X-reduced, the left and the right vertical paths are 0^X-reductions of $0^X \oplus c$ and $b \oplus c$, respectively. Therefore, 1_c is a 0^X-reduction of $P \oplus 1_c$.

For $1_c \oplus P$, we modify the previous diagram by (i) switching the two summands in each vertex, each edge, and each path in the top rectangle, and (ii) replacing $\lambda^{\oplus}_c$ with $\rho^{\oplus}_c$.

For $P \otimes 1_c$, we modify the previous diagram by replacing (i) $\oplus$ with $\otimes$, and (ii) the bottom rectangle with the rectangle below.

$$\begin{tikzcd} 0^X \otimes c \arrow[r, two heads, "1_{0^X} \otimes 1_c"] \arrow[d, "\lambda^{\bullet}_c"'] & 0^X \otimes c \arrow[d, "\lambda^{\bullet}_c"] \\ 0^X \arrow[r, "1_{0^X}"] & 0^X \end{tikzcd}$$

This is commutative by the functoriality of $\otimes$ in C. In this case, 1_{0^X} is a 0^X-reduction of $P \otimes 1_c$.

For $1_c \otimes P$, we modify the proof for $P \otimes 1_c$ by (i) switching the two factors in each vertex, each edge, and each path in the top rectangle, and (ii) replacing $\lambda^{\bullet}_c$ with $\rho^{\bullet}_c$. □

3.5. Existence of Zero Reduction of Paths

In this section, we give a detailed proof that every path in $\mathsf{Gr}(X)$ as in Definition 3.1.9 has a 0^X-reduction in the sense of Definition 3.4.5; see Proposition 3.5.32. A path in $\mathsf{Gr}(X)$ consists of prime edges as in Definition 3.1.8, and prime edges are built from elementary edges as in Definition 3.1.6. The proof of Proposition 3.5.32 is broken down into a series of lemmas covering the various types of prime edges. This section ends with a preliminary version of the Coherence Theorem 3.9.1 for paths whose domains have the same support as 0^X; see Proposition 3.5.33. Recall that Convention 3.3.1 is in effect.

Lemma 3.5.1. *Suppose $f : a \longrightarrow b$ is a prime edge of one of the following types:*

- *an identity,*
- *a 0^X-prime edge as in Definition 3.3.2, or*
- *an inverse 0^X-prime edge as in Definition 3.4.1.*

Then f has a 0^X-reduction.

Proof. Recall from Lemma 3.3.6 that each element in X^{fr} has a 0^X-reduction.

- If f is an identity, then its value in C is an identity morphism. If $Q_a : a \twoheadrightarrow a'$ is a 0^X-reduction of a, then the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, two heads, "Q_a"'] & a \arrow[d, two heads, "Q_a"] \\ a' \arrow[r, two heads, "1_{a'}"] & a' \end{tikzcd}$$

is commutative. So the identity $1_{a'}$ is a 0^X-reduction of f.

- If f is a 0^X-prime edge, and if $Q_b : b \twoheadrightarrow b'$ is a 0^X-reduction of b, then $(Q_b, f) : a \twoheadrightarrow b'$ is a 0^X-reduction of a. Since the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "(Q_b,f)"', two heads] & b \arrow[d, "Q_b", two heads] \\ b' \arrow[r, "1_{b'}", two heads] & b' \end{tikzcd}$$

  is commutative, the identity $1_{b'}$ is a 0^X-reduction of f.
- If f is an inverse 0^X-prime edge, denote by $f^{-1} : b \longrightarrow a$ the 0^X-prime edge obtained from f by replacing its unique copy of $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\lambda^{-\bullet}$, or $\rho^{-\bullet}$ with its formal inverse. If $Q_a : a \twoheadrightarrow a'$ is a 0^X-reduction of a, then $(Q_a, f^{-1}) : b \twoheadrightarrow a'$ is a 0^X-reduction of b. Since the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "Q_a"', two heads] & b \arrow[d, "(Q_a,f^{-1})", two heads] \\ a' \arrow[r, "1_{a'}", two heads] & a' \end{tikzcd}$$

  is commutative, the identity $1_{a'}$ is a 0^X-reduction of f.

Therefore, by Lemma 3.4.8, f has a 0^X-reduction. $\square$

Lemma 3.5.2. *Suppose $f : a \longrightarrow b$ is a prime edge whose domain is 0^X-reduced in the sense of Definition 3.3.4. Then f has a 0^X-reduction.*

Proof. If f satisfies the hypothesis of Lemma 3.5.1, then we are done. Otherwise, f is a 0^X-free path. There are two cases.

First suppose that the expression of a in terms of elements of X contains no 0^X. By Lemma 3.4.9(1), the codomain b is also 0^X-reduced. The identities $1_a : a \longrightarrow a$ and $1_b : b \longrightarrow b$ are 0^X-reductions of a and b, respectively, and the diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "1_a"'] & b \arrow[d, "1_b"] \\ a \arrow[r, "f"] & b \end{tikzcd}$$

is commutative. Therefore, f is a 0^X-free path, and is a 0^X-reduction of f.

Next suppose $a = 0^X$. This implies that 1_a is a 0^X-reduction of a, and that f is either

$$a = 0^X \xrightarrow{\lambda^{-\otimes}_{0^X}} 1^X \otimes 0^X = b \quad \text{or} \quad a = 0^X \xrightarrow{\rho^{-\otimes}_{0^X}} 0^X \otimes 1^X = b.$$

In these two cases, a 0^X-reduction of b is $\rho^{\bullet}_{1^X}$ on the left and $\lambda^{\bullet}_{1^X}$ on the right. The diagrams

$$\begin{tikzcd} 0^X \arrow[r, "\lambda^{-\otimes}_{0^X}"] \arrow[d, "1_{0^X}"'] & 1^X \otimes 0^X \arrow[d, "\rho^{\bullet}_{1^X}"] \\ 0^X \arrow[r, "1_{0^X}"] & 0^X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0^X \arrow[r, "\rho^{-\otimes}_{0^X}"] \arrow[d, "1_{0^X}"'] & 0^X \otimes 1^X \arrow[d, "\lambda^{\bullet}_{1^X}"] \\ 0^X \arrow[r, "1_{0^X}"] & 0^X \end{tikzcd}$$

are commutative by the axioms (2.1.18) and (2.1.17), which state that $\rho^{\bullet}_{\mathbb{1}} = \lambda^{\otimes}_{\mathbb{0}}$ and $\lambda^{\bullet}_{\mathbb{1}} = \rho^{\otimes}_{\mathbb{0}}$, respectively. Therefore, 1_{0^X} is a 0^X-reduction of f. $\square$

Lemma 3.5.3. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a is not 0^X-reduced, and*
- *f involves an instance of $\alpha^{\pm\oplus}$.*

Then f has a 0^X-reduction.

Proof. By assumption, $\mathsf{norm}(a) \geq 3$, and a contains at least one 0^X in its expression in terms of elements of X. First assume that f involves an instance of $\alpha^{\oplus}$. We proceed by induction on $\mathsf{norm}(a)$.

If $\mathsf{norm}(a) = 3$, then

$$a = (x \oplus y) \oplus z \xrightarrow{f = \alpha^{\oplus}_{x,y,z}} x \oplus (y \oplus z) = b \tag{3.5.4}$$

with $x, y, z \in X$, at least one of which is 0^X. Depending on whether each of x, y, and z is 0^X, there are seven cases. Consider, for example, the case $x = z = 0^X \neq y$, as in the following diagram.

$$\begin{tikzcd}
a = (0^X \oplus y) \oplus 0^X \arrow[r, "\alpha^{\oplus}_{0^X,y,0^X}"] \arrow[d, "\lambda^{\oplus}_y \oplus 1_{0^X}"'] & 0^X \oplus (y \oplus 0^X) = b \arrow[d, "\lambda^{\oplus}_{y \oplus 0^X}"] \\
y \oplus 0^X \arrow[r, "1_{y \oplus 0^X}"] \arrow[d, "\rho^{\oplus}_y"'] & y \oplus 0^X \arrow[d, "\rho^{\oplus}_y"] \\
y \arrow[r, "1_y"] & y
\end{tikzcd} \tag{3.5.5}$$

- The top rectangle is commutative by the left unity property (1.2.7) in the monoidal category $(\mathsf{C}, \oplus, 0^X, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus})$.
- The bottom rectangle is commutative by definition.
- The left and the right vertical paths are, respectively, 0^X-reductions of a and b.
- 1_y is a 0^X-free path with $y \in X$ 0^X-reduced by Lemma 3.3.5 (1).

Therefore, 1_y is a 0^X-reduction of f in this case. The other six cases are proved in the same way, using either the unity axiom (1.2.2) or the unity properties (1.2.7) in the monoidal category $(\mathsf{C}, \oplus)$. In each case, an identity is a 0^X-reduction of f.

For the induction step, suppose $\mathsf{norm}(a) > 3$. Either

$$a = a_1 \oplus a_2 \quad \text{or} \quad a = a_1 \otimes a_2$$

for some $a_1, a_2 \in X^{\mathsf{fr}}$. We consider these two cases separately below.

First suppose a is a sum $a = a_1 \oplus a_2$, and $f = f_1 \oplus e$ splits as a sum for some

- prime edge $f_1 : a_1 \longrightarrow b_1$ involving an instance of $\alpha^{\oplus}$ and
- identity $e : a_2 \longrightarrow a_2$.

There are two sub-cases depending on a_1.

- If a_1 is 0^X-reduced, then by Lemma 3.5.2, f_1 has a 0^X-reduction R_1.
- If a_1 is not 0^X-reduced, then, since $\mathsf{norm}(a_1) < \mathsf{norm}(a)$, the induction hypothesis implies that f_1 has a 0^X-reduction R_1.

In either case, there is a diagram

$$\begin{tikzcd} a_1 \arrow[r, "f_1"] \arrow[d, "Q_{a_1}"'] & b_1 \arrow[d, "Q_{b_1}"] \\ a_1' \arrow[r, two heads, "R_1"] & b_1' \end{tikzcd} \tag{3.5.6}$$

that is commutative, with Q_{a_1} and Q_{b_1} any 0^X-reductions of a_1 and b_1, respectively, and with R_1 a 0^X-reduction of f_1, hence in particular a 0^X-free path.

Consider the following diagram.

$$\begin{tikzcd} a = a_1 \oplus a_2 \arrow[r, "f = f_1 \oplus e"] \arrow[d, "Q_{a_1} \oplus 1_{a_2}"'] & b_1 \oplus a_2 = b \arrow[d, "Q_{b_1} \oplus 1_{a_2}"] \\ a_1' \oplus a_2 \arrow[r, two heads, "R_1 \oplus 1_{a_2}"] \arrow[d, "1_{a_1'} \oplus Q_{a_2}"'] & b_1' \oplus a_2 \arrow[d, "1_{b_1'} \oplus Q_{a_2}"] \\ a_1' \oplus a_2' \arrow[r, two heads, "R_1 \oplus 1_{a_2'}"] \arrow[d, "Q_1"'] & b_1' \oplus a_2' \arrow[d, "Q_2"] \\ a' \arrow[r, two heads, "R"] & b' \end{tikzcd} \tag{3.5.7}$$

- $Q_{a_2} : a_2 \twoheadrightarrow a_2'$ is any 0^X-reduction of a_2.
- The top two rectangles are commutative by the functoriality of $\oplus$ in C.
- $Q_1 : a_1' \oplus a_2' \twoheadrightarrow a'$ is any 0^X-reduction of $a_1' \oplus a_2'$.
- $Q_2 : b_1' \oplus a_2' \twoheadrightarrow b'$ is any 0^X-reduction of $b_1' \oplus a_2'$.
- R is a 0^X-reduction of $R_1 \oplus 1_{a_2'}$, which exists by Lemma 3.4.14. So R is a 0^X-free path, and the bottom rectangle is commutative.
- The left and the right vertical paths are 0^X-reductions of a and b, respectively.

Therefore, by Lemma 3.4.8, R is a 0^X-reduction of f.

If f has the form $e \oplus f_1$, then we slightly modify the argument in the previous two paragraphs by switching the two summands in the top two rectangles in (3.5.7).

If f does not split as a sum, then $f = \alpha^{\oplus}_{x,y,z}$ as in (3.5.4) for some $x, y, z \in X^{\mathsf{fr}}$. For each $w \in \{x, y, z\}$, suppose $Q_w : w \twoheadrightarrow w'$ is a 0^X-reduction of w. The diagram

$$\begin{tikzcd} a = (x \oplus y) \oplus z \arrow[r, "f = \alpha^{\oplus}_{x,y,z}"] \arrow[d, "(Q_x \oplus 1_y) \oplus 1_z"'] & x \oplus (y \oplus z) = b \arrow[d, "Q_x \oplus (1_y \oplus 1_z)"] \\ (x' \oplus y) \oplus z \arrow[r, "\alpha^{\oplus}_{x',y,z}"] \arrow[d, "(1_{x'} \oplus Q_y) \oplus 1_z"'] & x' \oplus (y \oplus z) \arrow[d, "1_{x'} \oplus (Q_y \oplus 1_z)"] \\ (x' \oplus y') \oplus z \arrow[r, "\alpha^{\oplus}_{x',y',z}"] \arrow[d, "(1_{x'} \oplus 1_{y'}) \oplus Q_z"'] & x' \oplus (y' \oplus z) \arrow[d, "1_{x'} \oplus (1_{y'} \oplus Q_z)"] \\ a' = (x' \oplus y') \oplus z' \arrow[r, "\alpha^{\oplus}_{x',y',z'}"] & x' \oplus (y' \oplus z') = b' \end{tikzcd} \tag{3.5.8}$$

is commutative by the naturality of $\alpha^{\oplus}$ in C. Each w' is 0^X-reduced, so it is either 0^X or contains no 0^X.

- If x', y', and z' all contain no 0^X, then a' and b' also contain no 0^X, and are 0^X-reduced. The left and the right vertical paths in (3.5.8) are 0^X-reductions of a and b, respectively. Therefore, $\alpha^{\oplus}_{x',y',z'}$ is a 0^X-reduction of f in this case.
- Otherwise, at least one of x', y', and z' is 0^X. We reuse the argument in (3.5.5) applied to $\alpha^{\oplus}_{x',y',z'}$, and append the resulting diagram to the bottom of (3.5.8), and similarly for the other six cases. For each $w \in \{x, y, z\}$, the condition "$w \neq 0^X$" is replaced by "w' contains no 0^X." In each case, the combined diagram is commutative, and shows that an identity is a 0^X-reduction of f.

So far we have proved the induction step when a is a sum.

Next suppose $a = a_1 \otimes a_2$. Then f, being a prime edge involving $\alpha^{\oplus}$, must split as a product, say $f_1 \otimes e$ with

- $f_1 : a_1 \longrightarrow b_1$ a prime edge involving $\alpha^{\oplus}$ and
- $e : a_2 \longrightarrow a_2$ an identity.

The diagram (3.5.6) still applies. We reuse the argument in (3.5.7) by replacing $\oplus$ with $\otimes$ to conclude that f has a 0^X-reduction. If f has the form $e \otimes f_1$, then we modify the argument further by switching the two factors in the top two rectangles. This finishes the induction and the case with f involving an instance of $\alpha^{\oplus}$.

If f involves an instance of $\alpha^{-\oplus}$, then we reuse the above argument by replacing $\alpha^{\oplus}$ with $\alpha^{-\oplus}$. $\square$

Lemma 3.5.9. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a is not 0^X-reduced, and*
- *f involves an instance of $\alpha^{\pm\otimes}$.*

Then f has a 0^X-reduction.

Proof. First suppose that f involves an instance of $\alpha^{\otimes}$. We proceed by induction on $\mathsf{norm}(a) \geq 3$ as in the proof of Lemma 3.5.3, most of which is reused here by replacing $\oplus$ with $\otimes$.

In the initial case of the induction with $\mathsf{norm}(a) = 3$ and $f = \alpha^{\otimes}_{x,y,z}$, the diagram (3.5.5) becomes the following diagram.

$$\begin{tikzcd}
a = (0^X \otimes y) \otimes 0^X \arrow[rr, "f = \alpha^{\otimes}_{0^X,y,0^X}"] \arrow[d, "\lambda^{\bullet}_y \otimes 1_{0^X}"'] & & 0^X \otimes (y \otimes 0^X) = b \arrow[dd, "\lambda^{\bullet}_{y \otimes 0^X}"] \\
0^X \otimes 0^X \arrow[d, "\lambda^{\bullet}_{0^X}"'] & & \\
0^X \arrow[rr, "1_{0^X}"] & & 0^X
\end{tikzcd} \tag{3.5.10}$$

This diagram is commutative by the axiom (2.1.22), so 1_{0^X} is a 0^X-reduction of f. The other six cases follow similarly by the axioms (2.1.20)–(2.1.22).

For the induction step, we first consider the case with $a = a_1 \otimes a_2$ and $f = f_1 \otimes e$ for some

- prime edge $f_1 : a_1 \longrightarrow b_1$ involving an instance of $\alpha^{\otimes}$ and

- identity $e : a_2 \longrightarrow a_2$.

We reuse the argument in the two paragraphs containing (3.5.6) and (3.5.7) by changing $\oplus$ to $\otimes$. The case with $f = e \otimes f_1$ is similar, with the two factors switched.

If f does not split as a product, then $f = \alpha^{\otimes}_{x,y,z}$ for some $x, y, z \in X^{\mathsf{fr}}$. We reuse the paragraph containing (3.5.8) by changing $\oplus$ to $\otimes$, (3.5.5) to (3.5.10), and similarly for the other six cases. So far we have proved the induction step when a is a product.

Next suppose $a = a_1 \oplus a_2$. Then f, being a prime edge involving $\alpha^{\otimes}$, must split as a sum, say $f_1 \oplus e$ with f_1 and e as above. We reuse the argument in (3.5.7), and conclude that f has a 0^X-reduction. If f has the form $e \oplus f_1$, then we modify the argument by switching the two summands in the top two rectangles. This finishes the induction and the case with f involving an instance of $\alpha^{\otimes}$.

If f involves an instance of $\alpha^{-\otimes}$, then we reuse the above argument by replacing $\alpha^{\otimes}$ with $\alpha^{-\otimes}$. □

Lemma 3.5.11. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a is not 0^X-reduced, and*
- *f involves an instance of $\xi^{\pm\oplus}$.*

Then f has a 0^X-reduction.

Proof. By assumption, $\mathsf{norm}(a) \geq 2$, and a contains at least one 0^X in its expression in terms of elements of X. First assume that f involves an instance of $\xi^{\oplus}$. We proceed by induction on $\mathsf{norm}(a)$.

If $\mathsf{norm}(a) = 2$, then

$$a = x \oplus y \xrightarrow{f = \xi^{\oplus}_{x,y}} y \oplus x = b$$

with $x, y \in X$, at least one of which is 0^X. Depending on whether each of x and y is 0^X, there are three cases corresponding to the following three diagrams.

(3.5.12)
$$\begin{tikzcd} 0^X \oplus y \arrow[r, "\xi^{\oplus}_{0^X,y}"] \arrow[d, "\lambda^{\oplus}_y"'] & y \oplus 0^X \arrow[d, "\rho^{\oplus}_y"] \\ y \arrow[r, "1_y"] & y \end{tikzcd} \qquad \begin{tikzcd} x \oplus 0^X \arrow[r, "\xi^{\oplus}_{x,0^X}"] \arrow[d, "\rho^{\oplus}_x"'] & 0^X \oplus x \arrow[d, "\lambda^{\oplus}_x"] \\ x \arrow[r, "1_x"] & x \end{tikzcd} \qquad \begin{tikzcd} 0^X \oplus 0^X \arrow[r, "\xi^{\oplus}_{0^X,0^X}"] \arrow[d, "\rho^{\oplus}_{0^X}"'] & 0^X \oplus 0^X \arrow[d, "\lambda^{\oplus}_{0^X}"] \\ 0^X \arrow[r, "1_{0^X}"] & 0^X \end{tikzcd}$$

Each of these diagrams is commutative by the symmetry axiom (1.2.20) and the unit axiom (1.2.21) in the symmetric monoidal category $(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$. In each case, an identity is a 0^X-reduction of f.

For the induction step, suppose $\mathsf{norm}(a) > 2$. Either

$$a = a_1 \oplus a_2 \quad \text{or} \quad a = a_1 \otimes a_2$$

for some $a_1, a_2 \in X^{\mathsf{fr}}$. We consider these two cases separately below.

First suppose a is a sum $a = a_1 \oplus a_2$, and $f = f_1 \oplus e$ splits as a sum for some

- prime edge $f_1 : a_1 \longrightarrow b_1$ involving an instance of $\xi^{\oplus}$ and
- identity $e : a_2 \longrightarrow a_2$.

We obtain a 0^X-reduction of f by reusing the two paragraphs containing (3.5.6) and (3.5.7). If f has the form $e \oplus f_1$, then we slightly modify the argument by switching the two summands in the top two rectangles in (3.5.7).

If f does not split as a sum, then $f = \xi^{\oplus}_{a_1,a_2}$. For each $w \in \{a_1, a_2\}$, suppose $Q_w : w \twoheadrightarrow w'$ is a 0^X-reduction of w. The diagram

$$\begin{tikzcd}
a = a_1 \oplus a_2 \arrow[r, "f = \xi^{\oplus}_{a_1,a_2}"] \arrow[d, "Q_{a_1} \oplus 1_{a_2}"'] & a_2 \oplus a_1 = b \arrow[d, "1_{a_2} \oplus Q_{a_1}"] \\
a_1' \oplus a_2 \arrow[r, "\xi^{\oplus}_{a_1',a_2}"] \arrow[d, "1_{a_1'} \oplus Q_{a_2}"'] & a_2 \oplus a_1' \arrow[d, "Q_{a_2} \oplus 1_{a_1'}"] \\
a' = a_1' \oplus a_2' \arrow[r, "\xi^{\oplus}_{a_1',a_2'}"] & a_2' \oplus a_1' = b'
\end{tikzcd} \tag{3.5.13}$$

is commutative by the naturality of $\xi^{\oplus}$ in C. Each w' is 0^X-reduced, so it is either 0^X or contains no 0^X.

- If a_1' and a_2' contain no 0^X, then a' and b' also contain no 0^X, and are 0^X-reduced. The left and the right vertical paths are 0^X-reductions of a and b, respectively. Therefore, $\xi^{\oplus}_{a_1',a_2'}$ is a 0^X-reduction of f in this case.
- Otherwise, at least one of a_1' and a_2' is 0^X. We reuse the argument in (3.5.12) applied to $\xi^{\oplus}_{a_1',a_2'}$, and append the resulting diagram to the bottom of (3.5.13). In each case, the combined diagram is commutative, and shows that an identity is a 0^X-reduction of f.

So far we have proved the induction step when a is a sum.

Next suppose $a = a_1 \otimes a_2$. Then f, being a prime edge involving $\xi^{\oplus}$, must split as a product, say $f_1 \otimes e$ with

- $f_1 : a_1 \longrightarrow b_1$ a prime edge involving $\xi^{\oplus}$ and
- $e : a_2 \longrightarrow a_2$ an identity.

We reuse (3.5.6) and the argument in (3.5.7) by replacing $\oplus$ with $\otimes$ to conclude that f has a 0^X-reduction. If f has the form $e \otimes f_1$, then we modify the argument further by switching the two factors in the top two rectangles. This finishes the induction and the case with f involving an instance of $\xi^{\oplus}$.

If f involves an instance of $\xi^{-\oplus}$, then we reuse the above argument by replacing $\xi^{\oplus}$ with $\xi^{-\oplus}$. $\square$

Lemma 3.5.14. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a is not 0^X-reduced, and*
- *f involves an instance of $\xi^{\pm\otimes}$.*

Then f has a 0^X-reduction.

Proof. First suppose that f involves an instance of $\xi^{\otimes}$. We proceed by induction on $\mathsf{norm}(a) \geq 2$ as in the proof of Lemma 3.5.11, most of which is reused here by replacing $\oplus$ with $\otimes$.

In the initial case of the induction with $\mathsf{norm}(a) = 2$ and $f = \xi^{\otimes}_{x,y}$, the diagrams in (3.5.12) become the following diagrams.

$$\begin{tikzcd}
0^X \otimes y \arrow[r, "\xi^{\otimes}_{0^X,y}"] \arrow[d, "\lambda^{\bullet}_y"'] & y \otimes 0^X \arrow[d, "\rho^{\bullet}_y"] \\
0^X \arrow[r, "1_{0^X}"] & 0^X
\end{tikzcd}
\qquad
\begin{tikzcd}
x \otimes 0^X \arrow[r, "\xi^{\otimes}_{x,0^X}"] \arrow[d, "\rho^{\bullet}_x"'] & 0^X \otimes x \arrow[d, "\lambda^{\bullet}_x"] \\
0^X \arrow[r, "1_{0^X}"] & 0^X
\end{tikzcd}
\qquad
\begin{tikzcd}
0^X \otimes 0^X \arrow[r, "\xi^{\otimes}_{0^X,0^X}"] \arrow[d, "\rho^{\bullet}_{0^X}"'] & 0^X \otimes 0^X \arrow[d, "\lambda^{\bullet}_{0^X}"] \\
0^X \arrow[r, "1_{0^X}"] & 0^X
\end{tikzcd} \tag{3.5.15}$$

Each of these diagrams is commutative by the axiom (2.1.19), and also the symmetry axiom (1.2.20) in $(\mathsf{C}, \otimes)$ for the leftmost diagram. In each case, 1_{0^X} is a 0^X-reduction of f.

For the induction step, we first consider the case with $a = a_1 \otimes a_2$, and $f = f_1 \otimes e$ for some

- prime edge $f_1 : a_1 \longrightarrow b_1$ involving an instance of $\xi^{\otimes}$ and
- identity $e : a_2 \longrightarrow a_2$.

We reuse the argument in the two paragraphs containing (3.5.6) and (3.5.7) by changing $\oplus$ to $\otimes$. The case with $f = e \otimes f_1$ is similar, with the two factors switched.

If f does not split as a product, then $f = \xi^{\otimes}_{a_1,a_2}$ for some $a_1, a_2 \in X^{\mathsf{fr}}$. We reuse the paragraph containing (3.5.13) by changing $\oplus$ to $\otimes$ and (3.5.12) to (3.5.15). So far we have proved the induction step when a is a product.

Next suppose $a = a_1 \oplus a_2$. Then f, being a prime edge involving $\xi^{\otimes}$, must split as a sum, say $f_1 \oplus e$ with f_1 and e as above. We reuse the argument in (3.5.7), and conclude that f has a 0^X-reduction. If f has the form $e \oplus f_1$, then we modify the argument by switching the two summands in the top two rectangles. This finishes the induction and the case with f involving an instance of $\xi^{\otimes}$.

If f involves an instance of $\xi^{-\otimes}$, then we reuse the above argument by replacing $\xi^{\otimes}$ with $\xi^{-\otimes}$. □

Lemma 3.5.16. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a is not 0^X-reduced, and*
- *f involves an instance of $\lambda^{\otimes}$.*

Then f has a 0^X-reduction.

Proof. By assumption, $\mathsf{norm}(a) \geq 2$, and a contains at least one of each of 0^X and 1^X in its expression in terms of elements of X. We proceed by induction on $\mathsf{norm}(a)$.

If $\mathsf{norm}(a) = 2$, then

$$a = 1^X \otimes 0^X \xrightarrow{f = \lambda^{\otimes}_{0^X}} 0^X = b.$$

The diagram

$$\begin{tikzcd} 1^X \otimes 0^X \arrow[r, "\lambda^{\otimes}_{0^X}"] \arrow[d, "\rho^{\bullet}_{1^X}"'] & 0^X \arrow[d, "1_{0^X}"] \\ 0^X \arrow[r, "1_{0^X}"] & 0^X \end{tikzcd} \tag{3.5.17}$$

is commutative by the axiom (2.1.18), which says that $\lambda^{\otimes}_{\mathbb{0}} = \rho^{\bullet}_{\mathbb{1}}$. Therefore, 1_{0^X} is a 0^X-reduction of f.

For the induction step, suppose $\mathsf{norm}(a) > 2$. Either

$$a = a_1 \oplus a_2 \quad \text{or} \quad a = a_1 \otimes a_2$$

for some $a_1, a_2 \in X^{\mathsf{fr}}$. We consider these two cases separately below.

First suppose $a = a_1 \oplus a_2$. Then f, being a prime edge with an instance of $\lambda^{\otimes}$, must split as a sum, say $f = f_1 \oplus e$ for some

- prime edge $f_1 : a_1 \longrightarrow b_1$ involving an instance of $\lambda^{\otimes}$ and
- identity $e : a_2 \longrightarrow a_2$.

We reuse the argument in the two paragraphs containing (3.5.6) and (3.5.7) to conclude that f has a 0^X-reduction. If f has the form $e \oplus f_1$, then we slightly modify the argument by switching the two summands in the top two rectangles in (3.5.7).

Next suppose $a = a_1 \otimes a_2$. If f splits as a product $f_1 \otimes e$ with f_1 and e as above, then we reuse the argument in the two paragraphs containing (3.5.6) and (3.5.7) by changing $\oplus$ to $\otimes$. If $f = e \otimes f_1$, then we modify the argument further by switching the two factors in the top two rectangles in (3.5.7).

If f does not split as a product, then

$$a = 1^X \otimes x \xrightarrow{f = \lambda^{\otimes}_x} x = b$$

with $x = a_2 \in X^{\mathsf{fr}}$ having at least one 0^X. Suppose $Q_x : x \twoheadrightarrow x'$ is any 0^X-reduction of x. The diagram

$$\begin{tikzcd}
a = 1^X \otimes x \arrow[r, "f = \lambda^{\otimes}_x"] \arrow[d, twoheadrightarrow, "1_{1^X} \otimes Q_x"'] & x = b \arrow[d, twoheadrightarrow, "Q_x"] \\
1^X \otimes x' \arrow[r, "\lambda^{\otimes}_{x'}"] & x'
\end{tikzcd}$$

is commutative by the naturality of $\lambda^{\otimes}$ in C. The element x' is 0^X-reduced, so it is either 0^X or contains no 0^X.

- If x' contains no 0^X, then $1^X \otimes x'$ contains no 0^X, and is 0^X-reduced. Since the left vertical path is a 0^X-reduction of a, the 0^X-free path $\lambda^{\otimes}_{x'}$ is a 0^X-reduction of f in this case.
- If $x' = 0^X$, then we append the diagram (3.5.17) to the bottom of the previous diagram to conclude that 1_{0^X} is a 0^X-reduction of f.

This finishes the induction. □

Lemma 3.5.18. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a is not 0^X-reduced, and*
- *f involves an instance of $\lambda^{-\otimes}$.*

Then f has a 0^X-reduction.

Proof. By assumption, $\mathsf{norm}(a) \geq 2$, and a contains at least one 0^X in its expression in terms of elements of X. We proceed by induction on $\mathsf{norm}(a)$.

If $\mathsf{norm}(a) = 2$, then $a = a_1 \oplus a_2$ or $a_1 \otimes a_2$ for some $a_1, a_2 \in X$, at least one of which is 0^X. The prime edge f has one of the following six forms.

$$\begin{array}{ll}
a_1 \oplus a_2 \xrightarrow{\lambda^{-\otimes}_{a_1} \oplus 1_{a_2}} (1^X \otimes a_1) \oplus a_2 & a_1 \otimes a_2 \xrightarrow{\lambda^{-\otimes}_{a_1} \otimes 1_{a_2}} (1^X \otimes a_1) \otimes a_2 \\
a_1 \oplus a_2 \xrightarrow{1_{a_1} \oplus \lambda^{-\otimes}_{a_2}} a_1 \oplus (1^X \otimes a_2) & a_1 \otimes a_2 \xrightarrow{1_{a_1} \otimes \lambda^{-\otimes}_{a_2}} a_1 \otimes (1^X \otimes a_2) \\
a_1 \oplus a_2 \xrightarrow{\lambda^{-\otimes}_{a_1 \oplus a_2}} 1^X \otimes (a_1 \oplus a_2) & a_1 \otimes a_2 \xrightarrow{\lambda^{-\otimes}_{a_1 \otimes a_2}} 1^X \otimes (a_1 \otimes a_2)
\end{array} \tag{3.5.19}$$

First consider the upper left case in (3.5.19). There are two subcases as follows.

- Suppose $a_1 = 0^X$. The diagram

$$\begin{tikzcd}
a = 0^X \oplus a_2 \arrow[r, "\lambda^{-\otimes}_{0^X} \oplus 1_{a_2}"] \arrow[d, "1_{0^X \oplus a_2}"'] & (1^X \otimes 0^X) \oplus a_2 = b \arrow[d, "\rho^{\bullet}_{1^X} \oplus 1_{a_2}"] \\
0^X \oplus a_2 \arrow[r, "1_{0^X \oplus a_2}"] \arrow[d, "\lambda^{\oplus}_{a_2}"'] & 0^X \oplus a_2 \arrow[d, "\lambda^{\oplus}_{a_2}"] \\
a_2 \arrow[r, "1_{a_2}"] & a_2
\end{tikzcd} \tag{3.5.20}$$

is commutative by the axiom (2.1.18), which says that $\lambda^{\otimes}_{\mathbb{0}} = \rho^{\bullet}_{\mathbb{1}}$. The left and the right vertical paths are 0^X-reductions of a and b, respectively. Therefore, 1_{a_2} is a 0^X-reduction of f.
- Suppose $a_2 = 0^X \neq a_1$. The diagram

$$\begin{tikzcd}
a = a_1 \oplus 0^X \arrow[r, "\lambda^{-\otimes}_{a_1} \oplus 1_{0^X}"] \arrow[d, "\rho^{\oplus}_{a_1}"'] & (1^X \otimes a_1) \oplus 0^X = b \arrow[d, "\rho^{\oplus}_{1^X \otimes a_1}"] \\
a_1 \arrow[r, "\lambda^{-\otimes}_{a_1}"] & 1^X \otimes a_1
\end{tikzcd}$$

is commutative by the naturality of $\rho^{\oplus}$ in C. The left and the right vertical edges are 0^X-reductions of a and b, respectively. Therefore, $\lambda^{-\otimes}_{a_1}$ is a 0^X-reduction of f.

The left middle case in (3.5.19) is proved by almost the same argument.

Consider the left bottom case in (3.5.19). There are three sub-cases as follows.

- If $a_1 = 0^X \neq a_2$, then the diagram

$$\begin{tikzcd}
0^X \oplus a_2 \arrow[r, "\lambda^{-\otimes}_{0^X \oplus a_2}"] \arrow[d, "\lambda^{\oplus}_{a_2}"'] & 1^X \otimes (0^X \oplus a_2) \arrow[d, "1_{1^X} \otimes \lambda^{\oplus}_{a_2}"] \\
a_2 \arrow[r, "\lambda^{-\otimes}_{a_2}"] & 1^X \otimes a_2
\end{tikzcd} \tag{3.5.21}$$

is commutative by the naturality of $\lambda^{\otimes}$ in C. Therefore, $\lambda^{-\otimes}_{a_2}$ is a 0^X-reduction of f.
- Similarly, if $a_1 \neq 0^X = a_2$, then the diagram

$$\begin{tikzcd}
a_1 \oplus 0^X \arrow[r, "\lambda^{-\otimes}_{a_1 \oplus 0^X}"] \arrow[d, "\rho^{\oplus}_{a_1}"'] & 1^X \otimes (a_1 \oplus 0^X) \arrow[d, "1_{1^X} \otimes \rho^{\oplus}_{a_1}"] \\
a_1 \arrow[r, "\lambda^{-\otimes}_{a_1}"] & 1^X \otimes a_1
\end{tikzcd} \tag{3.5.22}$$

is commutative by the naturality of $\lambda^{\otimes}$ in C. It shows that $\lambda^{-\otimes}_{a_1}$ is a 0^X-reduction of f.

- If $a_1 = a_2 = 0^X$, then the diagram

(3.5.23)
$$\begin{tikzcd}
0^X \oplus 0^X \arrow[r, "\lambda^{-\otimes}_{0^X \oplus 0^X}"] \arrow[d, "\lambda^{\oplus}_{0^X}"'] & 1^X \otimes (0^X \oplus 0^X) \arrow[d, "1_{1^X} \otimes \lambda^{\oplus}_{0^X}"] \\
0^X \arrow[r, "\lambda^{-\otimes}_{0^X}"] \arrow[d, "1_{0^X}"'] & 1^X \otimes 0^X \arrow[d, "\rho^{\bullet}_{1^X}"] \\
0^X \arrow[r, "1_{0^X}"] & 0^X
\end{tikzcd}$$

is commutative by the naturality of $\lambda^{\otimes}$ for the top rectangle, and the axiom (2.1.18) in C for the bottom rectangle. Therefore, 1_{0^X} is a 0^X-reduction of f.

The three right cases in (3.5.19) are proved by almost the same argument. This finishes the proof for the initial case.

For the induction step, suppose $\mathsf{norm}(a) > 2$. Then $a = a_1 \oplus a_2$ or $a_1 \otimes a_2$ for some $a_1, a_2 \in X^{\mathsf{fr}}$, at least one of which contains 0^X. The prime edge f has one of the six forms in (3.5.19).

First consider the upper left case in (3.5.19). There are two subcases as follows.

- Suppose a_1 contains 0^X. If, furthermore, a_1 is 0^X-reduced, then $a_1 = 0^X$. The diagram

$$\begin{tikzcd}
a = 0^X \oplus a_2 \arrow[r, "\lambda^{-\otimes}_{0^X} \oplus 1_{a_2}"] \arrow[d, "1_{0^X \oplus a_2}"'] & (1^X \otimes 0^X) \oplus a_2 = b \arrow[d, "\rho^{\bullet}_{1^X} \oplus 1_{a_2}"] \\
0^X \oplus a_2 \arrow[r, "1_{0^X \oplus a_2}"] \arrow[d, two heads, "Q"'] & 0^X \oplus a_2 \arrow[d, two heads, "Q"] \\
a' \arrow[r, "1_{a'}"] & a'
\end{tikzcd}$$

is commutative by the axiom (2.1.18), where Q is any 0^X-reduction of $0^X \oplus a_2$. Since the left and the right vertical paths are 0^X-reductions of a and b, respectively, $1_{a'}$ is a 0^X-reduction of f.

Next suppose a_1 is not 0^X-reduced. Since $\mathsf{norm}(a_1) < \mathsf{norm}(a)$, the induction hypothesis implies that the prime edge $\lambda^{-\otimes}_{a_1}$ has a 0^X-reduction $R_1 : a'_1 \twoheadrightarrow b'_1$. The diagram

(3.5.24)
$$\begin{tikzcd}
a_1 \arrow[r, "\lambda^{-\otimes}_{a_1}"] \arrow[d, two heads, "Q_{a_1}"'] & 1^X \otimes a_1 \arrow[d, two heads, "Q_{b_1}"] \\
a'_1 \arrow[r, two heads, "R_1"] & b'_1
\end{tikzcd}$$

is commutative, with Q_{a_1} and Q_{b_1} any 0^X-reductions of, respectively, a_1 and $b_1 = 1^X \otimes a_1$. With $f = \lambda^{-\otimes}_{a_1} \oplus 1_{a_2}$, we now reuse the argument in the paragraph containing (3.5.7) to conclude that f has a 0^X-reduction.

- Suppose a_1 does not contain 0^X. Since a_1 is 0^X-reduced, the prime edge $\lambda^{-\otimes}_{a_1}$ has a 0^X-reduction $R_1 : a'_1 \twoheadrightarrow b'_1$ by Lemma 3.5.2 (whose proof shows that $R_1 = \lambda^{-\otimes}_{a_1}$). The diagram (3.5.24) exists in this case, so again the argument in the paragraph containing (3.5.7) shows that f has a 0^X-reduction.

The left middle case in (3.5.19) is proved by almost the same argument.

For the left bottom case in (3.5.19) consider the following diagram, where $Q_i : a_i \twoheadrightarrow a_i'$ is a 0^X-reduction of a_i for $i = 1, 2$.

$$\begin{tikzcd}
a = a_1 \oplus a_2 \arrow[r, "\lambda^{-\otimes}_{a_1 \oplus a_2}"] \arrow[d, twoheadrightarrow, "Q_1 \oplus 1_{a_2}"'] & 1^X \otimes (a_1 \oplus a_2) = b \arrow[d, twoheadrightarrow, "1_{1^X} \otimes (Q_1 \oplus 1_{a_2})"] \\
a_1' \oplus a_2 \arrow[r, "\lambda^{-\otimes}_{a_1' \oplus a_2}"] \arrow[d, twoheadrightarrow, "1_{a_1'} \oplus Q_2"'] & 1^X \otimes (a_1' \oplus a_2) \arrow[d, twoheadrightarrow, "1_{1^X} \otimes (1_{a_1'} \oplus Q_2)"] \\
a_1' \oplus a_2' \arrow[r, "\lambda^{-\otimes}_{a_1' \oplus a_2'}"] & 1^X \otimes (a_1' \oplus a_2')
\end{tikzcd} \tag{3.5.25}$$

This diagram is commutative by the naturality of $\lambda^{\otimes}$ in C. Each a_i' is 0^X-reduced, so it is either 0^X or contains no 0^X. There are four subcases as follows.

- If both a_1' and a_2' contain no 0^X, then $a_1' \oplus a_2'$ and $1^X \otimes (a_1' \oplus a_2')$ contain no 0^X, and are 0^X-reduced. Since the left and the right vertical paths are 0^X-reductions of a and b, respectively, the 0^X-free path $\lambda^{-\otimes}_{a_1' \oplus a_2'}$ is a 0^X-reduction of f.
- If $a_1' = 0^X$, and if a_2' contains no 0^X, then we append the diagram (3.5.21) with a_2' in place of a_2 to the bottom of the diagram (3.5.25). The combined diagram shows that $\lambda^{-\otimes}_{a_2'}$ is a 0^X-reduction of f.
- If a_1' contains no 0^X, and if $a_2' = 0^X$, then we append the diagram (3.5.22) with a_1' in place of a_1 to the bottom of the diagram (3.5.25). The combined diagram shows that $\lambda^{-\otimes}_{a_1'}$ is a 0^X-reduction of f.
- If $a_1' = a_2' = 0^X$, then we append the diagram (3.5.23) to the bottom of the diagram (3.5.25). The combined diagram shows that 1_{0^X} is a 0^X-reduction of f.

The three right cases in (3.5.19) are proved by almost the same argument.

This finishes the induction. $\square$

Lemma 3.5.26. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a is not 0^X-reduced, and*
- *f involves an instance of $\rho^{\pm\otimes}$.*

Then f has a 0^X-reduction.

Proof. The proofs are obtained from those of Lemmas 3.5.16 and 3.5.18 by replacing $(\lambda^{\otimes}, \lambda^{-\otimes})$ with $(\rho^{\otimes}, \rho^{-\otimes})$, and the axiom (2.1.18) by the axiom (2.1.17), which states that $\rho^{\otimes}_{\mathbb{0}} = \lambda^{\bullet}_{\mathbb{1}}$. $\square$

Lemma 3.5.27. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a is not 0^X-reduced, and*
- *f involves an instance of δ^l or δ^r.*

Then f has a 0^X-reduction.

Proof. By assumption, $\mathsf{norm}(a) \geq 3$, and a contains at least one 0^X in its expression in terms of elements of X. First assume that f involves an instance of δ^l. We proceed by induction on $\mathsf{norm}(a)$. For the rest of this proof, we abbreviate $\otimes$ to concatenation.

If $\mathsf{norm}(a) = 3$, then

$$a = x(y \oplus z) \xrightarrow{f = \delta^l_{x,y,z}} xy \oplus xz = b \tag{3.5.28}$$

for some $x, y, z \in X$, at least one of which is 0^X. Depending on whether each of x, y, and z is 0^X, there are seven cases as follows.

(1) If $y = 0^X$, then the diagram

$$\begin{tikzcd} a = x(0^X \oplus z) \arrow[r, "f = \delta^l_{x,0^X,z}"] \arrow[dd, "1_x\lambda^{\oplus}_z"'] & x0^X \oplus xz = b \arrow[d, "\rho^{\bullet}_x \oplus 1_{xz}"] \\ & 0^X \oplus xz \arrow[d, "\lambda^{\oplus}_{xz}"] \\ xz \arrow[r, "1_{xz}"] & xz \end{tikzcd} \tag{3.5.29}$$

is commutative by the axiom (2.1.23).

(i) If $x, z \neq 0^X$, then xz is 0^X-reduced. Therefore, 1_{xz} is a 0^X-reduction of f in this case.

(ii) If either $x = 0^X$ or $z = 0^X$, then we append the diagram

$$\begin{tikzcd} 0^X z \arrow[r, "1_{0^X z}"] \arrow[d, "\lambda^{\bullet}_z"'] & 0^X z \arrow[d, "\lambda^{\bullet}_z"] \\ 0^X \arrow[r, "1_{0^X}"] & 0^X \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} x0^X \arrow[r, "1_{x0^X}"] \arrow[d, "\rho^{\bullet}_x"'] & x0^X \arrow[d, "\rho^{\bullet}_x"] \\ 0^X \arrow[r, "1_{0^X}"] & 0^X \end{tikzcd}$$

to the bottom of the diagram (3.5.29). In each case, the combined diagram is commutative, and shows that 1_{0^X} is a 0^X-reduction of f. If $x = z = 0^X$, then we can use either one of these two diagrams.

(2) If $y \neq 0^X$, then at least one of x and z is 0^X.

(i) If $z = 0^X$, then the diagram

$$\begin{tikzcd} a = x(y \oplus 0^X) \arrow[r, "f = \delta^l_{x,y,0^X}"] \arrow[dd, "1_x\rho^{\oplus}_y"'] & xy \oplus x0^X = b \arrow[d, "1_{xy} \oplus \rho^{\bullet}_x"] \\ & xy \oplus 0^X \arrow[d, "\rho^{\oplus}_{xy}"] \\ xy \arrow[r, "1_{xy}"] & xy \end{tikzcd} \tag{3.5.30}$$

is commutative by the axiom (2.1.25).

- If $x \neq 0^X$, then xy is 0^X-reduced. The diagram (3.5.30) shows that 1_{xy} is a 0^X-reduction of f.

- If $x = 0^X$, then we append the diagram

$$\begin{tikzcd}
0^X y \arrow[r, "1_{0^X y}"] \arrow[d, "\lambda^\bullet_y"'] & 0^X y \arrow[d, "\lambda^\bullet_y"] \\
0^X \arrow[r, "1_{0^X}"] & 0^X
\end{tikzcd}$$

to the bottom of the diagram (3.5.30). The combined diagram is commutative, and shows that 1_{0^X} is a 0^X-reduction of f.

(ii) If $z \neq 0^X$, then $x = 0^X$. The diagram

$$\begin{tikzcd}
0^X(y \oplus z) \arrow[r, "f = \delta^l_{0^X,y,z}"] \arrow[ddd, "\lambda^\bullet_{y\oplus z}"'] & 0^X y \oplus 0^X z \arrow[d, "\lambda^\bullet_y \oplus 1_{0^X z}"] \\
& 0^X \oplus 0^X z \arrow[d, "1_{0^X} \oplus \lambda^\bullet_z"] \\
& 0^X \oplus 0^X \arrow[d, "\lambda^\oplus_{0^X}"] \\
0^X \arrow[r, "1_{0^X}"] & 0^X
\end{tikzcd}$$

is commutative by the axiom (2.1.15) and the functoriality of $\oplus$ in C. Therefore, 1_{0^X} is a 0^X-reduction of f.

This finishes the proof of the initial case.

For the induction step, suppose $\mathsf{norm}(a) > 3$. Either $a = a_1 \oplus a_2$ or $a = a_1 \otimes a_2$ for some $a_1, a_2 \in X^{\mathsf{fr}}$, at least one of which contains 0^X. We consider these cases separately below.

If $a = a_1 \oplus a_2$, then f, being a prime edge involving an instance of δ^l, must split as a sum, say $f = f_1 \oplus e$ for some

- prime edge $f_1 : a_1 \longrightarrow b_1$ involving an instance of δ^l and
- identity $e : a_2 \longrightarrow a_2$.

We reuse the argument in the two paragraphs containing (3.5.6) and (3.5.7) to conclude that f has a 0^X-reduction. If f has the form $e \oplus f_1$, then we slightly modify the argument by switching the two summands in the top two rectangles in (3.5.7).

Next suppose $a = a_1 \otimes a_2$. If $f = f_1 \otimes e$ with f_1 and e as above, then we reuse the argument in the two paragraphs containing (3.5.6) and (3.5.7) by changing $\oplus$ to $\otimes$. If $f = e \otimes f_1$, then we modify the argument further by switching the two factors in the top two rectangles in (3.5.7).

If f does not split as a product, then $f = \delta^l_{x,y,z}$ as in (3.5.28) for some $x, y, z \in X^{\mathsf{fr}}$, at least one of which contains 0^X. For each $w \in \{x, y, z\}$, suppose $Q_w : w \longrightarrow\!\!\!\!\rightarrow w'$

is a 0^X-reduction of w. Consider the following diagram.

$$\begin{tikzcd}
a = x(y \oplus z) \arrow[r, "f = \delta^l_{x,y,z}"] \arrow[dd, "Q_x 1_{y\oplus z}"'] & xy \oplus xz = b \arrow[d, "Q_x 1_y \oplus 1_{xz}"] \\
 & x'y \oplus xz \arrow[d, "1_{x'y} \oplus Q_x 1_z"] \\
x'(y \oplus z) \arrow[r, "\delta^l_{x',y,z}"] \arrow[d, "1'_x(Q_y \oplus 1_z)"'] & x'y \oplus x'z \arrow[d, "1_{x'} Q_y \oplus 1_{x'z}"] \\
x'(y' \oplus z) \arrow[r, "\delta^l_{x',y',z}"] \arrow[d, "1_{x'}(1_{y'} \oplus Q_z)"'] & x'y' \oplus x'z \arrow[d, "1_{x'y'} \oplus 1_{x'} Q_z"] \\
x'(y' \oplus z') \arrow[r, "\delta^l_{x',y',z'}"] & x'y' \oplus x'z'
\end{tikzcd} \tag{3.5.31}$$

This diagram is commutative by the naturality of δ^l in C. Each w' is 0^X-reduced, so it is either 0^X or contains no 0^X.

- If x', y', and z' all contain no 0^X, then $x'(y' \oplus z')$ and $x'y' \oplus x'z'$ also contain no 0^X, and are 0^X-reduced. Since the left and the right vertical paths in (3.5.31) are 0^X-reductions of a and b, respectively, the 0^X-free path $\delta^l_{x',y',z'}$ is a 0^X-reduction of f in this case.
- Otherwise, at least one of x', y', and z' is 0^X. In each of the seven subcases, we (i) reuse the argument for the initial case above applied to $\delta^l_{x',y',z'}$, and (ii) append the resulting diagram to the bottom of (3.5.31). For each $w \in \{x, y, z\}$, the condition "$w \neq 0^X$" is replaced by "w' contains no 0^X." In each case, we conclude that an identity is a 0^X-reduction of f.

This finishes the induction and the case with f containing an instance of δ^l.

The proof for the case with f containing an instance of δ^r is obtained from the above proof by replacing

- $\delta^l_{x,y,z}$ with $\delta^r_{y,z,x} : (y \oplus z)x \longrightarrow yx \oplus zx$ and
- the axioms (2.1.23), (2.1.25), and (2.1.15) in C with, respectively, the axioms (2.1.24), (2.1.26), and (2.1.16).

This finishes the proof. $\square$

Now we combine the preliminary lemmas above to prove the following main result of this section.

Proposition 3.5.32. *Each path in* $\mathsf{Gr}(X)$ *has a* 0^X*-reduction.*

Proof. Suppose $P = (f_n, \ldots, f_1)$ is a path in $\mathsf{Gr}(X)$ for some $n \geq 1$ as displayed below.

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n$$

- For each $0 \leq j \leq n$, suppose $Q_j : a_j \twoheadrightarrow a'_j$ is any 0^X-reduction of a_j, which exists by Lemma 3.3.6. Moreover, a'_j is uniquely determined by a_j by Lemma 3.3.11.

- For each $1 \leq i \leq n$, the prime edge $f_i : a_{i-1} \longrightarrow a_i$ has a 0^X-reduction $R_i : a'_{i-1} \twoheadrightarrow a'_i$ by one of Lemmas 3.5.1 through 3.5.3, 3.5.9, 3.5.11, 3.5.14, 3.5.16, 3.5.18, 3.5.26, and 3.5.27.

Each square in the diagram

$$\begin{tikzcd}
a_0 \arrow[r, "f_1"] \arrow[d, twoheadrightarrow, "Q_0"'] & a_1 \arrow[r, "f_2"] \arrow[d, twoheadrightarrow, "Q_1"] & \cdots \arrow[r, "f_n"] & a_n \arrow[d, twoheadrightarrow, "Q_n"] \\
a'_0 \arrow[r, twoheadrightarrow, "R_1"] & a'_1 \arrow[r, twoheadrightarrow, "R_2"] & \cdots \arrow[r, twoheadrightarrow, "R_n"] & a'_n
\end{tikzcd}$$

is commutative by Definition 3.4.5. Therefore, the 0^X-free path

$$a'_0 \xrightarrow{(R_n,\ldots,R_1)} a'_n$$

is a 0^X-reduction of P by Lemma 3.4.8. □

Recall from (3.1.24) the support of an element in X^{fr}. Next is a preliminary form of the Coherence Theorem 3.9.1.

Proposition 3.5.33. *Suppose*

$$\begin{tikzcd}
a \arrow[r, bend left, twoheadrightarrow, "P_1"] \arrow[r, bend right, twoheadrightarrow, "P_2"'] & b
\end{tikzcd}$$

are two paths in $\mathsf{Gr}(X)$ *such that*

$$\mathsf{supp}(a) = \mathsf{supp}(0^X).$$

Then the values of P_1 *and* P_2 *in* C *are equal.*

Proof. By Lemma 3.1.29 (2),

$$\mathsf{supp}(a) = \mathsf{supp}(b) = \mathsf{supp}(0^X).$$

By Lemmas 3.3.6 and 3.3.7 and Proposition 3.5.32, there exist 0^X-reductions

- $Q_a : a \twoheadrightarrow 0^X$ of a,
- $Q_b : b \twoheadrightarrow 0^X$ of b, and
- $R_i : 0^X \twoheadrightarrow 0^X$ of P_i for each $i = 1, 2$.

For each $i = 1, 2$, the diagram

$$\begin{tikzcd}
a \arrow[r, twoheadrightarrow, "P_i"] \arrow[d, twoheadrightarrow, "Q_a"'] & b \arrow[d, twoheadrightarrow, "Q_b"] \\
0^X \arrow[r, twoheadrightarrow, "R_i"] & 0^X
\end{tikzcd}$$

is commutative by Definition 3.4.5. By Lemma 3.4.11, the value of R_i in C is the identity morphism $1_{\mathbb{0}}$. Therefore, the value of P_i in C is the value of Q_a in C followed by the inverse of the value of Q_b in C. □

3.6. Reduction of Distributivity

The main observation in this and the next sections is Proposition 3.7.19. It says that each path whose vertices do not involve 0^X can be replaced by a 0^X-free path that does not involve the distributivity edges δ^l and δ^r. This will constitute another reduction step in the Coherence Theorem 3.9.1, where we eliminate δ^l and δ^r. In this section, we first define the preliminary concept of a δ-reduction of an element in X^{fr}, and show its existence in Lemma 3.6.9. The main technical result of this section is Lemma 3.6.12, which says that, under some conditions, a 0^X-free edge may be replaced by a 0^X-free path that contains no δ^l and δ^r. This observation is used repeatedly in the proof of Proposition 3.7.19.

Some of the definitions and results in this section are distributivity analogues of those in Sections 3.3 through 3.5. Recall that Convention 3.3.1 is in effect.

Motivation 3.6.1. The next definition contains the distributivity analogues of 0^X-prime edges, 0^X-reduced, and 0^X-reductions in Definitions 3.3.2 and 3.3.4. We observed in Lemma 3.3.5 that an element in X^{fr} is 0^X-reduced if and only if it is not the domain of any 0^X-prime edges. The distributivity analogue of a 0^X-reduced element in X^{fr}, which is called δ-reduced, is modeled after this lemma. In Lemma 3.6.5, we will provide a more intrinsic characterization of δ-reduced elements. ⋄

Definition 3.6.2. Consider the graph $\mathsf{Gr}(X)$ of X in Definition 3.1.9.

- A *δ-prime edge* is a prime edge that involves either δ^l or δ^r.
- An element $a \in X^{\mathsf{fr}}$ is *δ-reduced* if it is not the domain of any δ-prime edges.
- A *δ-reduction* of $a \in X^{\mathsf{fr}}$ is a path $P : a \twoheadrightarrow b$ such that the following two statements hold.
  - b is δ-reduced.
  - Each edge in P is either an identity or a δ-prime edge. ⋄

To characterize δ-reduced elements, we use the following concepts.

Definition 3.6.3. Suppose $a \in X^{\mathsf{fr}}$.

- The element a is a *monomial* if either
  - $a \in X$, or
  - $a = a_1 \otimes \cdots \otimes a_k$ for some multiplicative bracketing and some elements $a_j \in X$ for $1 \leq j \leq k$ with $k \geq 2$. Each a_j is called a *factor* of the monomial a.
- The element a is a *polynomial* if either
  - a is a monomial, or
  - $a = a^1 \oplus \cdots \oplus a^m$ for some additive bracketing and some monomials $a^i \in X^{\mathsf{fr}}$ for $1 \leq i \leq m$ with $m \geq 2$. Each a^i is called a monomial in the polynomial a. ⋄

Explanation 3.6.4. A polynomial is a nonempty sum in which each summand is either an element in X or a product of several elements in X. ⋄

Lemma 3.6.5. *For an element $a \in X^{\mathsf{fr}}$, the following statements are equivalent.*

(1) a is a polynomial
(2) $1_a : a \longrightarrow a$ is a δ-reduction of a.
(3) a is δ-reduced.

Proof. First observe that the domain of a δ-prime edge must involve an expression of the form

- $x \otimes (y \oplus z)$ for δ^l or
- $(y \oplus z) \otimes x$ for δ^r

for some $x, y, z \in X^{\mathsf{fr}}$. A polynomial does not contain these expressions, so it is δ-reduced. Therefore, if a is a polynomial, then the identity 1_a is a δ-reduction of a, proving (1) $\Rightarrow$ (2).

If 1_a is a δ-reduction of a, then its codomain a is δ-reduced by definition, proving (2) $\Rightarrow$ (3).

For (3) $\Rightarrow$ (1), suppose a is δ-reduced. A general element $a \in X^{\mathsf{fr}}$ can always be expressed in the form

$$a = a^1 \oplus \cdots \oplus a^m$$

for some additive bracketing and $m \geq 1$ such that each a^i is not a sum. In other words, each a^i is either in X or has the form $a^i_1 \otimes a^i_2$. Since a is δ-reduced, expressions of the forms $x \otimes (y \oplus z)$ and $(y \oplus z) \otimes x$ do not appear in a. So each a^i is a monomial, and a is a polynomial. $\square$

Example 3.6.6. For elements $a_1, \ldots, a_n \in X$ for $n \geq 1$, the elements

$$a_1 \oplus \cdots \oplus a_n \quad \text{and} \quad a_1 \otimes \cdots \otimes a_n \in X^{\mathsf{fr}}$$

with any additive, respectively multiplicative, bracketing are δ-reduced. $\diamond$

Example 3.6.7. In Definition 2.1.2 of a symmetric bimonoidal category, consider the 24 axioms, and interpret A, B, C, and D as elements in X, $\mathbb{0}$ as 0^X, and $\mathbb{1}$ as 1^X.

- In each of (2.1.5)–(2.1.13), (2.1.15)–(2.1.16), and (2.1.23)–(2.1.28), the common domain is *not* δ-reduced, and the common codomain is δ-reduced.
- In each of (2.1.14) and (2.1.17)–(2.1.22), each vertex is δ-reduced. $\diamond$

Example 3.6.8. Consider elements $a, b, c, d \in X$.

- $\delta^l_{a,b,c} : a(b \oplus c) \longrightarrow ab \oplus ac$ is a δ-reduction of $a(b \oplus c)$.
- $\delta^r_{a,b,c} : (a \oplus b)c \longrightarrow ac \oplus bc$ is a δ-reduction of $(a \oplus b)c$.
- The following two paths are two δ-reductions of $(a \oplus b)(c \oplus d)$.

$$\begin{tikzcd}
& (a \oplus b)(c \oplus d) \arrow[dl, "\delta^l_{a \oplus b, c, d}"'] \arrow[dr, "\delta^r_{a,b,c \oplus d}"] & \\
(a \oplus b)c \oplus (a \oplus b)d \arrow[d, "\delta^r_{a,b,c} \oplus 1_{(a \oplus b)d}"'] & & a(c \oplus d) \oplus b(c \oplus d) \arrow[d, "\delta^l_{a,c,d} \oplus 1_{b(c \oplus d)}"] \\
(ac \oplus bc) \oplus (a \oplus b)d \arrow[d, "1_{ac \oplus bc} \oplus \delta^r_{a,b,d}"'] & & (ac \oplus ad) \oplus b(c \oplus d) \arrow[d, "1_{ac \oplus ad} \oplus \delta^l_{b,c,d}"] \\
(ac \oplus bc) \oplus (ad \oplus bd) & & (ac \oplus ad) \oplus (bc \oplus bd)
\end{tikzcd}$$

This example shows that the codomain of a δ-reduction of a given element in X^{fr} is *not* unique in general. This is different from the situation for 0^X-reductions, where the codomain is uniquely determined, as shown in Lemma 3.3.11. $\diamond$

In Lemma 3.3.6, we observed that each element in X^{fr} has a 0^X-reduction. Next is the distributivity analogue of that assertion. Recall rank and size in Definition 3.2.1.

Lemma 3.6.9. *Each element in X^{fr} has a δ-reduction.*

Proof. We proceed by induction on $\mathsf{size}(a) - \mathsf{rank}(a)$ for elements $a \in X^{\mathsf{fr}}$. If

$$\mathsf{size}(a) - \mathsf{rank}(a) = 0,$$

then by Proposition 3.2.15, a is a polynomial in the sense of Definition 3.6.3, in which each monomial is either an element in X or a product of two elements in X. Therefore, by Lemma 3.6.5, the identity 1_a is a δ-reduction of a.

For the induction step, suppose $\mathsf{size}(a) - \mathsf{rank}(a) > 0$. If a is δ-reduced, then by Lemma 3.6.5 again, 1_a is a δ-reduction of a.

If a is not δ-reduced, then there exists a δ-prime edge $f : a \longrightarrow b$ with domain a. The induction hypothesis applies to b by Lemma 3.2.12, so there exists a δ-reduction $P : b \longrightarrow\!\!\!\!\rightarrow b'$ of b. The combined path

$$a \xrightarrow{\;f\;} b \xrightarrow{\;P\;}\!\!\!\!\rightarrow b'$$

is a δ-reduction of a. This finishes the induction. $\square$

The next definition contains the distributivity analogue of a 0^X-free path in Definition 3.4.1.

Definition 3.6.10. Consider $\mathsf{Gr}(X)$.

- A *δ-free path* is a path that does not contain any δ-prime edges.
- A *δ-free edge* is a δ-free path of length 1.
- A *$(0^X, \delta)$-free path* is a path that is both a 0^X-free path and a δ-free path.
- A *$(0^X, \delta)$-free edge* is a $(0^X, \delta)$-free path of length 1. $\diamond$

Explanation 3.6.11. In a δ-free path, every prime edge is either an identity or involves a single instance of $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\alpha^{\pm\otimes}$, $\xi^{\pm\otimes}$, $\lambda^{\pm\otimes}$, $\rho^{\pm\otimes}$, $\lambda^{\pm\bullet}$, or $\rho^{\pm\bullet}$.

In a $(0^X, \delta)$-free path, every prime edge is either an identity or involves a single instance of $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, $\xi^{\pm\otimes}$, $\lambda^{\pm\otimes}$, or $\rho^{\pm\otimes}$. $\diamond$

The goal for the rest of this section is to prove the following preliminary version of Proposition 3.7.19. Recall from Definition 3.4.1 that a 0^X-free edge is a prime edge that does not involve $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\bullet}$, $\rho^{\bullet}$, and their formal inverses.

Lemma 3.6.12. *In $\mathsf{Gr}(X)$, suppose*

- *$f : a \longrightarrow b$ is a 0^X-free edge, and*
- *$g : a \longrightarrow c$ is a δ-prime edge.*

Then there exists a diagram

$$\begin{tikzcd}[column sep=small] & a \arrow[dl, "g"'] \arrow[dr, "f"] & \\ c \arrow[d, "D_2"', two heads] & & b \arrow[d, "D_1", two heads] \\ c' \arrow[rr, "R", two heads] & & b' \end{tikzcd} \tag{3.6.13}$$

such that the following statements hold.

1. *The above diagram is commutative.*
2. *Each path D_i is either a single identity or consists of δ-prime edges.*

(3) *R is a $(0^X, \delta)$-free path. Moreover, R involves only instances of $\lambda^{\pm\otimes}$ or $\rho^{\pm\otimes}$ if and only if f involves an elementary edge of the same type.*

Proof. Recall the possibilities of a 0^X-free edge in Explanation 3.4.2.

- If f is an identity, then $D_1 = g$, and each of D_2 and R is a single identity.
- If $f = g$, then each of D_1, D_2, and R is a single identity.

In the rest of this proof, we may assume that f is not an identity, and that $f \neq g$.

The proof proceeds by induction on $\mathsf{norm}(a) \geq 3$. If $\mathsf{norm}(a) = 3$, then g is either

$$a = x(y \oplus z) \xrightarrow{\delta^l_{x,y,z}} xy \oplus xz = c \quad \text{or} \quad a = (y \oplus z)x \xrightarrow{\delta^r_{y,z,x}} yx \oplus zx = c$$

for some $x, y, z \in X$. First suppose $g = \delta^l_{x,y,z}$. Depending on what f is, there are the following subcases. In what follows, *functoriality* means the functoriality of $\oplus$, $\otimes$, or both in C.

(1) If $f = \lambda^{\otimes}_{y \oplus z}$ with $x = 1^X$, then D_1, D_2, and R in the diagram (3.6.13) are defined as follows.

(3.6.14)
$$\begin{tikzcd}
& a = 1^X(y \oplus z) \arrow[dl, "g = \delta^l_{1^X,y,z}"'] \arrow[dr, "\lambda^{\otimes}_{y\oplus z} = f"] & \\
c = 1^X y \oplus 1^X z \arrow[d, "D_2 = 1_c"'] & & y \oplus z = b \arrow[d, "1_b = D_1"] \\
c' = 1^X y \oplus 1^X z \arrow[r, "\lambda^{\otimes}_y \oplus 1_{1^X z}"] \arrow[rr, bend right, "R"'] & y \oplus 1^X z \arrow[r, "1_y \oplus \lambda^{\otimes}_z"] & y \oplus z = b'
\end{tikzcd}$$

This diagram is commutative by the axiom (2.1.27) and functoriality.

(2) If f involves $\lambda^{-\otimes}$, then there are five subcases depending on whether $\lambda^{-\otimes}$ is applied to a, x, y, z, or $y \oplus z$.

(i) If $f = \lambda^{-\otimes}_a$, then D_1, D_2, and R in (3.6.13) are defined as follows.

(3.6.15)
$$\begin{tikzcd}
& a = x(y \oplus z) \arrow[dl, "g = \delta^l_{x,y,z}"'] \arrow[dr, "\lambda^{-\otimes}_a = f"] & \\
c = xy \oplus xz \arrow[d, "D_2 = 1_c"'] & & 1^X[x(y \oplus z)] = b \arrow[d, "1_{1^X}\delta^l_{x,y,z} = D_1"] \\
c' = xy \oplus xz \arrow[rr, "R = \lambda^{-\otimes}_{xy\oplus xz}"] & & 1^X(xy \oplus xz) = b'
\end{tikzcd}$$

This diagram is commutative by the naturality of $\lambda^{\otimes}$ in C.

(ii) If f involves $\lambda^{-\otimes}_x$, then we consider the following diagram.

(3.6.16)
$$\begin{tikzcd}
& a = x(y \oplus z) \arrow[dl, "g = \delta^l_{x,y,z}"'] \arrow[dr, "\lambda^{-\otimes}_x 1_{y\oplus z} = f"] & \\
c = xy \oplus xz \arrow[d, "D_2 = 1_c"'] & & (1^X x)(y \oplus z) = b \arrow[d, "\delta^l_{1^X x,y,z} = D_1"] \\
c' = xy \oplus xz \arrow[r, "\lambda^{-\otimes}_x 1_y \oplus 1_{xz}"] \arrow[rr, bend right, "R"'] & (1^X x)y \oplus xz \arrow[r, "1 \oplus \lambda^{-\otimes}_x 1_z"] & (1^X x)y \oplus (1^X x)z = b'
\end{tikzcd}$$

In the lower right horizontal edge, 1 means $1_{(1^X x)y}$. This diagram is commutative by the naturality of δ^l and functoriality.

(iii) If f involves $\lambda_y^{-\otimes}$ or $\lambda_z^{-\otimes}$, then there are two diagrams similar to the one in case (ii) that are commutative for the same reasons.

(iv) If f involves $\lambda_{y\oplus z}^{-\otimes}$, then we consider the following diagram.

$$\begin{tikzcd}
& a = x(y\oplus z) \arrow[dl, "g = \delta^l_{x,y,z}"'] \arrow[dr, "1_x\lambda^{-\otimes}_{y\oplus z} = f"] \arrow[dd, "1_x(\lambda^{-\otimes}_y \oplus 1_z)"'] & \\
c = xy\oplus xz \arrow[dd, "D_2 = 1_c"] \arrow[drr, bend right=70, twoheadrightarrow, "R"'] & & x\left[1^X(y\oplus z)\right] = b \arrow[d, "1_x\delta^l_{1^X,y,z}"] \arrow[dd, bend left=60, twoheadrightarrow, "D_1"] \\
& x(1^X y\oplus z) \arrow[r, "1_x(1_{1^X y}\oplus\lambda^{-\otimes}_z)"] & x(1^X y\oplus 1^X z) \arrow[d, "\delta^l_{x,1^X y,1^X z}"] \\
c' = xy\oplus xz \arrow[r, "1_x\lambda^{-\otimes}_y\oplus 1_{xz}"] \arrow[rr, bend right, twoheadrightarrow, "R"'] & x(1^X y)\oplus xz \arrow[r, "1\oplus 1_x\lambda^{-\otimes}_z"] & x(1^X y)\oplus x(1^X z) = b'
\end{tikzcd} \tag{3.6.17}$$

In the lower right horizontal edge, 1 means $1_{x(1^X y)}$. The upper right trapezoid is commutative by the axiom (2.1.27) and functoriality. The left subdiagram is commutative by the naturality of δ^l and functoriality.

(3) If f involves $\rho^{-\otimes}$, then as in case (2), there are five further subcases depending on whether $\rho^{-\otimes}$ is applied to a, x, y, z, or $y\oplus z$. The proofs for these five subcases are obtained from those in case (2) by
- replacing $\lambda^{-\otimes}$ with $\rho^{-\otimes}$ throughout and
- replacing $\delta^l_{1^X,y,z}$ and (2.1.27) with, respectively, $\delta^r_{y,z,1^X}$ and (2.1.28) in the upper right trapezoid in the diagram (3.6.17).

(4) If $f = \xi^{\otimes}_{x,y\oplus z}$, then we consider the following diagram.

$$\begin{tikzcd}
& a = x(y\oplus z) \arrow[dl, "g = \delta^l_{x,y,z}"'] \arrow[dr, "\xi^{\otimes}_{x,y\oplus z} = f"] & \\
c = xy\oplus xz \arrow[d, "D_2 = 1_c"'] & & (y\oplus z)x = b \arrow[d, "\delta^r_{y,z,x} = D_1"] \\
c' = xy\oplus xz \arrow[r, "\xi^{\otimes}_{x,y}\oplus 1_{xz}"] \arrow[rr, bend right, twoheadrightarrow, "R"'] & yx\oplus xz \arrow[r, "1_{yx}\oplus\xi^{\otimes}_{x,z}"] & yx\oplus zx = b'
\end{tikzcd} \tag{3.6.18}$$

This diagram is commutative by functoriality, the axiom (2.1.5), and the symmetry axiom (1.2.20) in $(\mathsf{C},\otimes)$.

(5) If $f = \xi^{-\otimes}_{y\oplus z,x}$, then we reuse the diagram in case (4) by replacing each $\xi^{\otimes}_{x,?}$ with $\xi^{-\otimes}_{?,x}$. The resulting diagram is commutative by functoriality and the axiom (2.1.5).

(6) If $f = 1_x\xi^{\oplus}_{y,z}$, then we consider the following diagram.

$$\begin{tikzcd}[column sep=small] & a = x(y\oplus z) \arrow[dl, "g = \delta^l_{x,y,z}"'] \arrow[dr, "1_x\xi^{\oplus}_{y,z} = f"] & \\ c = xy \oplus xz \arrow[d, "D_2 = ", "1_c"'] & & x(z\oplus y) = b \arrow[d, "\delta^l_{x,z,y}", "= D_1"'] \\ c' = xy \oplus xz \arrow[rr, "R = \xi^{\oplus}_{xy,xz}"] & & xz \oplus xy \end{tikzcd} \tag{3.6.19}$$

This diagram is commutative by the axiom (2.1.6).

(7) If $f = 1_x\xi^{-\oplus}_{z,y}$, then we reuse the diagram in case (6) by replacing each $\xi^{\oplus}_{*,?}$ with $\xi^{-\oplus}_{?,*}$. The resulting diagram is commutative by the axiom (2.1.6).

This finishes the initial case when $g = \delta^l_{x,y,z}$.

The proof for the case $g = \delta^r_{y,z,x}$ is obtained by slightly modifying the diagrams in cases (1)–(7) above, and using the symmetric bimonoidal category axioms

- (2.1.5), (2.1.27), (2.1.28), and
- (2.1.7) instead of (2.1.6) in the analogues of cases (6) and (7) if $f = \xi^{\oplus}_{y,z}1_x$ or $\xi^{-\oplus}_{z,y}1_x$.

This finishes the initial case of the induction.

For the induction step, suppose $\mathsf{norm}(a) > 3$. Either

$$a = a_1 \oplus a_2 \quad \text{or} \quad a = a_1 \otimes a_2$$

for some $a_1, a_2 \in X^{\mathsf{fr}}$ with $\mathsf{norm}(a_i) < \mathsf{norm}(a)$ for $i = 1,2$. There are cases corresponding to the following situations.

(1) f and g act on the same summand or factor of a. To say that f acts on a summand, say a_1 of $a = a_1 \oplus a_2$, means that $f = f_1 \oplus e_2$ for some prime edge $f_1 : a_1 \longrightarrow b_1$ and some identity $e_2 : a_2 \longrightarrow a_2$. The meaning of f acting on a factor of a is similar with $\otimes$ instead of $\oplus$.

(2) f and g act on different summands or factors of a.

(3) f acts on all of a, and g acts on a summand or factor of a. To say that f acts on all of a means that f does not split as a sum or a product.

(4) f acts on a summand or factor of a, and g acts on all of a.

(5) f and g both act on all of a.

For the rest of this proof $\odot$ denotes either $\oplus$ or $\otimes$, but not both in the same diagram. We automatically adjust the symbols to avoid too many subscripts and superscripts.

(1) f and g act on the same $\odot$-factor of a, as in the following diagram.

$$\begin{tikzcd}[column sep=small] & a = a_1 \odot a_2 \arrow[dl, "g = g_1 \odot e_2'"'] \arrow[dr, "f_1 \odot e_2 = f"] & \\ c = c_1 \odot a_2 \arrow[d, "D_2 = ", "D_2^1 \odot 1_{a_2}"'] & & b_1 \odot a_2 = b \arrow[d, "D_1^1 \odot 1_{a_2}", "= D_1"'] \\ c' = c_1' \odot a_2 \arrow[rr, "R = R^1 \odot 1_{a_2}"] & & b_1' \odot a_2 = b' \end{tikzcd}$$

- $e_2, e_2' : a_2 \longrightarrow a_2$ are identities.
- $f_1 : a_1 \longrightarrow b_1$ is a 0^X-free edge.
- $g_1 : a_1 \longrightarrow c_1$ is a δ-prime edge.

- The paths

$$\begin{tikzcd} c_1 \arrow[d, "D_2^1"'] & b_1 \arrow[d, "D_1^1"] \\ c_1' \arrow[r, "R^1"] & b_1' \end{tikzcd}$$

as in (3.6.13) are obtained from the induction hypothesis applied to a_1.
- The above diagram is commutative by functoriality.

If f and g both act on a_2 instead of a_1, then there is a similar diagram obtained from the induction hypothesis applied to a_2 and functoriality.

(2) f and g act on different $\odot$-factors of a.

(i) If f is a δ-prime edge, then we consider the following diagram.

$$\begin{tikzcd} & a = a_1 \odot a_2 \arrow[dl, "g = e_1 \odot g_2"'] \arrow[dr, "f_1 \odot e_2 = f"] & \\ c = a_1 \odot c_2 \arrow[d, "D_2 = f_1 \odot 1_{c_2}"'] & & b_1 \odot a_2 = b \arrow[d, "1_{b_1} \odot g_2 = D_1"] \\ c' = b_1 \odot c_2 \arrow[rr, "R = 1_{c'}"] & & b_1 \odot c_2 = b' \end{tikzcd}$$

- $e_1 : a_1 \longrightarrow a_1$ and $e_2 : a_2 \longrightarrow a_2$ are identities.
- $f_1 : a_1 \longrightarrow b_1$ and $g_2 : a_2 \longrightarrow c_2$ are δ-prime edges.

This diagram is commutative by functoriality.

(ii) If f is not a δ-prime edge, then we consider the following diagram.

$$\begin{tikzcd} & a = a_1 \odot a_2 \arrow[dl, "g = e_1 \odot g_2"'] \arrow[dr, "f_1 \odot e_2 = f"] & \\ c = a_1 \odot c_2 \arrow[d, "D_2 = 1_c"'] & & b_1 \odot a_2 = b \arrow[d, "1_{b_1} \odot g_2 = D_1"] \\ c' = a_1 \odot c_2 \arrow[rr, "R = f_1 \odot 1_{c_2}"] & & b_1 \odot c_2 = b' \end{tikzcd}$$

- e_1, e_2, and g_2 are as in case (i).
- $f_1 : a_1 \longrightarrow b_1$ is a $(0^X, \delta)$-free edge.

This diagram is commutative by functoriality.

(iii) If f and g act on a_2 and a_1, respectively, then there are two similar diagrams obtained from those in cases (i) and (ii) by switching the two $\odot$-factors.

(3) f acts on all of a, and g acts on a $\odot$-factor of a.

(i) If $f = \xi^{\odot}_{a_1,a_2}$, then we consider the following diagram.

$$\begin{tikzcd} & a = a_1 \odot a_2 \arrow[dl, "g = g_1 \odot e_2"'] \arrow[dr, "\xi^{\odot}_{a_1,a_2} = f"] & \\ c = c_1 \odot a_2 \arrow[d, "D_2 = 1_c"'] & & a_2 \odot a_1 = b \arrow[d, "1_{a_2} \odot g_1 = D_1"] \\ c' = c_1 \odot a_2 \arrow[rr, "R = \xi^{\odot}_{c_1,a_2}"] & & a_2 \odot c_1 = b' \end{tikzcd}$$

- $e_2 : a_2 \longrightarrow a_2$ is an identity.
- $g_1 : a_1 \longrightarrow c_1$ is a δ-prime edge.

This diagram is commutative by the naturality of $\xi^{\odot}$ in C. There is a similar diagram if g acts on the $\odot$-factor a_2 instead of a_1.

(ii) If $f = \xi^{-\odot}_{a_2,a_1}$, then we reuse the two diagrams in case (i) by replacing $\xi^{\odot}_{?,a_2}$ with $\xi^{-\odot}_{a_2,?}$. These diagrams are commutative by the naturality of $\xi^{\odot}$ in C.

(iii) If $f = \alpha^{\odot}_{a_1,a_2,a_3}$, and if g acts on one of the three $\odot$-factors of a, then we consider the following diagram.

$$
\begin{tikzcd}
& a = (a_1 \odot a_2) \odot a_3 \arrow[dl, "g \,=\, (g_1 \odot e_2) \odot e_3"'] \arrow[dr, "\alpha^{\odot}_{a_1,a_2,a_3} \,=\, f"] & \\
c = (c_1 \odot a_2) \odot a_3 \arrow[d, "D_2 \,=\, 1_c"'] & & a_1 \odot (a_2 \odot a_3) = b \arrow[d, "g_1 \odot 1_{a_2 \odot a_3} \,=\, D_1"] \\
c' = (c_1 \odot a_2) \odot a_3 \arrow[rr, "R \,=\, \alpha^{\odot}_{c_1,a_2,a_3}"] & & c_1 \odot (a_2 \odot a_3) = b'
\end{tikzcd}
$$

- $e_2 : a_2 \longrightarrow a_2$ and $e_3 : a_3 \longrightarrow a_3$ are identities.
- $g_1 : a_1 \longrightarrow c_1$ is a δ-prime edge.

This diagram is commutative by the naturality of $\alpha^{\odot}$ in C. There are two similar diagrams if g acts on a_2 or a_3 instead of a_1.

(iv) If $f = \alpha^{-\odot}_{a_1,a_2,a_3}$, and if g acts on one of the three $\odot$-factors of a, then we reuse the diagrams in case (iii) by replacing $\alpha^{\odot}$ with $\alpha^{-\odot}$. These diagrams are commutative by the naturality of $\alpha^{\odot}$ in C.

(v) If $f = \alpha^{\otimes}_{a_1,a_2,a_3}$, and if g involves δ^l on the first two $\otimes$-factors of a with $a_2 = x \oplus y$, then we consider the following diagram.

$$
\begin{tikzcd}
& a = [a_1(x \oplus y)]a_3 \arrow[dl, "g \,=\, \delta^l_{a_1,x,y} e_3"'] \arrow[dr, "\alpha^{\otimes}_{a_1,x \oplus y,a_3} \,=\, f"] & \\
c = (a_1 x \oplus a_1 y)a_3 \arrow[dd, "D_2 \,=\, \delta^r_{a_1 x, a_1 y, a_3}"'] \arrow[ddrr, bend right=40, "R"'] & & a_1[(x \oplus y)a_3] = b \arrow[d, "1_{a_1}\delta^r_{x,y,a_3}"'] \arrow[dd, bend left=60, "D_1"] \\
& & a_1(xa_3 \oplus ya_3) \arrow[d, "\delta^l_{a_1,xa_3,ya_3}"'] \\
c' = (a_1 x)a_3 \oplus (a_1 y)a_3 \arrow[r, "\alpha^{\otimes}_{a_1,x,a_3} \oplus 1_{(a_1 y)a_3}"] & a_1(xa_3) \oplus (a_1 y)a_3 \arrow[r, "1_{a_1(xa_3)} \oplus \alpha^{\otimes}_{a_1,y,a_3}"] & a_1(xa_3) \oplus a_1(ya_3) = b'
\end{tikzcd}
$$

Here $e_3 : a_3 \longrightarrow a_3$ is an identity. This diagram is commutative by the axiom (2.1.12).

(vi) If $f = \alpha^{\otimes}_{a_1,a_2,a_3}$, and if g involves δ^r on the first two $\otimes$-factors of a with $a_1 = x \oplus y$, then there is a diagram similar to the one in case (v). This diagram is commutative by the axiom (2.1.11).

(vii) If $f = \alpha^{-\otimes}_{a_1,a_2,a_3}$, and if g involves δ^l on the last two $\otimes$-factors of a with $a_3 = y \oplus z$, then we consider the following diagram.

$$\begin{tikzcd}[column sep=small]
& & a = a_1\left[a_2(y \oplus z)\right] \arrow[dll, "g \,=\, e_1\delta^l_{a_2,y,z}"'] \arrow[drr, "\alpha^{-\otimes}_{a_1,a_2,y\oplus z} \,=\, f"] & & \\
c = a_1(a_2y \oplus a_2z) \arrow[d, "D_2 \,=\, \delta^l_{a_1,a_2y,a_2z}"'] & & & & (a_1a_2)(y \oplus z) = b \arrow[d, "\delta^l_{a_1a_2,y,z} \,=\, D_1"] \\
c' = a_1(a_2y) \oplus a_1(a_2z) \arrow[rr, "\alpha^{-\otimes}_{a_1,a_2,y} \oplus 1_{a_1(a_2z)}"] \arrow[rrrr, bend right, "R"'] & & (a_1a_2)y \oplus a_1(a_2z) \arrow[rr, "1_{(a_1a_2)y} \oplus \alpha^{-\otimes}_{a_1,a_2,z}"] & & (a_1a_2)y \oplus (a_1a_2)z = b'
\end{tikzcd}$$

Here $e_1 : a_1 \longrightarrow a_1$ is an identity. This diagram is commutative by the axiom (2.1.10).

(viii) If $f = \alpha^{-\otimes}_{a_1,a_2,a_3}$, and if g involves δ^r on the last two $\otimes$-factors of a with $a_2 = x \oplus y$, then there is a diagram similar to the one in case (vii). This diagram is commutative by the axiom (2.1.12).

(ix) If $f = \lambda^{\otimes}_{a_2} : 1^X \otimes a_2 \longrightarrow a_2$, then we consider the following diagram.

$$\begin{tikzcd}
& a = 1^X \otimes a_2 \arrow[dl, "g \,=\, 1_{1^X} \otimes g_2"'] \arrow[dr, "\lambda^{\otimes}_{a_2} \,=\, f"] & \\
c = 1^X \otimes c_2 \arrow[d, "D_2 \,=\, 1_{1^X \otimes c_2}"'] & & a_2 = b \arrow[d, "g_2 \,=\, D_1"] \\
c' = 1^X \otimes c_2 \arrow[rr, "R \,=\, \lambda^{\otimes}_{c_2}"] & & c_2 = b'
\end{tikzcd}$$

Here $g_2 : a_2 \longrightarrow c_2$ is a δ-prime edge. This diagram is commutative by the naturality of $\lambda^{\otimes}$ in C.

(x) If $f = \lambda^{-\otimes}_a$, and if g acts on the $\odot$-factor a_1, then we consider the following diagram.

$$\begin{tikzcd}
& a = a_1 \odot a_2 \arrow[dl, "g \,=\, g_1 \odot e_2"'] \arrow[dr, "\lambda^{-\otimes}_a \,=\, f"] & \\
c = c_1 \odot a_2 \arrow[d, "D_2 \,=\, 1_{c_1 \odot a_2}"'] & & 1^X(a_1 \odot a_2) = b \arrow[d, "1_{1^X}(g \odot 1_{a_2}) \,=\, D_1"] \\
c' = c_1 \odot a_2 \arrow[rr, "R \,=\, \lambda^{-\otimes}_{c_1 \odot a_2}"] & & 1^X(c_1 \odot a_2) = b'
\end{tikzcd}$$

- $e_2 : a_2 \longrightarrow a_2$ is an identity.
- $g_1 : a_1 \longrightarrow c_1$ is a δ-prime edge.

This diagram is commutative by the naturality of $\lambda^{\otimes}$ in C.
If g acts on the $\odot$-factor a_2, then we slightly modify the above diagram by switching the two $\odot$-factors of a.

(xi) If

$$a_1 \otimes 1^X \xrightarrow{f = \rho^{\otimes}_{a_1}} a_1 \quad \text{or} \quad a \xrightarrow{f = \rho^{-\otimes}_a} a \otimes 1^X,$$

then we slightly modify the diagrams in cases (ix) and (x). The resulting diagrams are commutative by the naturality of $\rho^{\otimes}$ in C.

(xii) If $f = \delta^{l}_{a_1,x,y}$ with $a_2 = x \oplus y$, and if g acts on a_1, then we consider the following diagram.

$$\begin{tikzcd}
& a = a_1(x \oplus y) \arrow[dl, "g \,=\, g_1 e_2"'] \arrow[dr, "\delta^{l}_{a_1,x,y} \,=\, f"] & \\
c = c_1(x \oplus y) \arrow[ddd, "D_2 \,=\, \delta^{l}_{c_1,x,y}"'] & & a_1 x \oplus a_1 y = b \arrow[d, "g_1 1_x \oplus 1_{a_1 y}"'] \arrow[ddd, bend left=60, two heads, "D_1"] \\
& & c_1 x \oplus a_1 y \arrow[dd, "1_{c_1 x} \oplus g_1 1_y"'] \\
& & \\
c' = c_1 x \oplus c_1 y \arrow[rr, "R \,=\, 1_{c'}"] & & c_1 x \oplus c_1 y = b'
\end{tikzcd}$$

- $e_2 : a_2 \longrightarrow a_2$ is an identity.
- $g_1 : a_1 \longrightarrow c_1$ is a δ-prime edge.

This diagram is commutative by functoriality and the naturality of δ^{l} in C.

If g acts on x or y, then there are two diagrams similar to the one above that are commutative for the same reasons.

(xiii) If

$$(w \oplus x)a_2 \xrightarrow{f = \delta^{r}_{w,x,a_2}} wa_2 \oplus xa_2$$

with $a_1 = w \oplus x$, and if g acts on one of w, x, and a_2, then we slightly modify the diagrams in case (xii). The resulting diagrams are commutative by functoriality and the naturality of δ^{r} in C.

(4) f acts on a $\odot$-factor of a, and g acts on all of a.

First suppose $g = \delta^{l}_{a_1,y,z}$ with $a_2 = y \oplus z$. Then f may act on a_1, y, z, or a_2, leading to the following cases.

(i) If f acts on a_1, and if f is not a δ-prime edge, then we consider the following diagram.

$$\begin{tikzcd}
& a = a_1(y \oplus z) \arrow[dl, "g \,=\, \delta^{l}_{a_1,y,z}"'] \arrow[dr, "f_1 \oplus e_2 \,=\, f"] & \\
c = a_1 y \oplus a_1 z \arrow[d, "D_2 \,=\, 1_c"'] & & b_1(y \oplus z) = b \arrow[d, "\delta^{l}_{b_1,y,z} \,=\, D_1"'] \\
c' = a_1 y \oplus a_1 z \arrow[r, "f_1 1_y \oplus 1_{a_1 z}"] \arrow[rr, bend right=30, two heads, "R"'] & b_1 y \oplus a_1 z \arrow[r, "1_{b_1 y} \oplus f_1 1_z"] & b_1 y \oplus b_1 z = b'
\end{tikzcd}$$

- $e_2 : a_2 \longrightarrow a_2$ is an identity.
- $f_1 : a_1 \longrightarrow b_1$ is a $(0^{\mathsf{X}}, \delta)$-free edge.

This diagram is commutative by functoriality and the naturality of δ^{l} in C.

If f acts on a_1, and if f is a δ-prime edge, then we modify the previous diagram by defining

- $c' = b'$,
- $D_2 : c \twoheadrightarrow c' = b'$ as the bottom composite in the previous diagram, and
- $R = 1_{c'}$.

The resulting diagram is commutative for the same reasons.

(ii) If f acts on y or z, then there are diagrams similar to those in case (i) that are commutative for the same reasons.

(iii) If

$$a_1(y \oplus z) \xrightarrow{f = e_1 \xi^{\oplus}_{y,z}} a_1(z \oplus y)$$

for some identity $e_1 : a_1 \longrightarrow a_1$, then we reuse the argument in the paragraph containing (3.6.19) and the axiom (2.1.6).
Similarly, if $f = e_1 \xi^{-\oplus}_{z,y}$, then we reuse the argument in case (7) in the initial case.

(iv) If
- $a = w[(x \oplus y) \oplus z]$ with $a_1 = w$ and $a_2 = (x \oplus y) \oplus z$, and
- $f = e\alpha^{\oplus}_{x,y,z}$ for some identity $e : w \longrightarrow w$,

then we consider the following diagram.

$$\begin{tikzcd}
 & a = w[(x \oplus y) \oplus z] \arrow[dl, bend right, "g = \delta^l_{w,x\oplus y,z}"'] \arrow[dr, bend left, "e\alpha^{\oplus}_{x,y,z} = f"] & \\
c = w(x \oplus y) \oplus wz \arrow[dd, "D_2 = \delta^l_{w,x,y} \oplus 1_{wz}"'] & & w[x \oplus (y \oplus z)] = b \arrow[d, "\delta^l_{w,x,y\oplus z}"] \arrow[dd, bend left=60, two heads, "D_1"] \\
 & & wx \oplus w(y \oplus z) \arrow[d, "1_{wx} \oplus \delta^l_{w,y,z}"] \\
c' = (wx \oplus wy) \oplus wz \arrow[rr, "R = \alpha^{\oplus}_{wx,wy,wz}"] & & wx \oplus (wy \oplus wz) = b'
\end{tikzcd}$$

This diagram is commutative by the axiom (2.1.9).

(v) If
- $a = w[x \oplus (y \oplus z)]$ with $a_1 = w$ and $a_2 = x \oplus (y \oplus z)$, and
- $f = e\alpha^{-\oplus}_{x,y,z}$ for some identity $e : w \longrightarrow w$,

then we consider the following diagram.

$$\begin{tikzcd}
 & a = w[x \oplus (y \oplus z)] \arrow[dl, bend right, "g = \delta^l_{w,x,y\oplus z}"'] \arrow[dr, bend left, "e\alpha^{-\oplus}_{x,y,z} = f"] & \\
c = wx \oplus w(y \oplus z) \arrow[dd, "D_2 = 1_{wx} \oplus \delta^l_{w,y,z}"'] & & w[(x \oplus y) \oplus z] = b \arrow[d, "\delta^l_{w,x\oplus y,z}"] \arrow[dd, bend left=60, two heads, "D_1"] \\
 & & w(x \oplus y) \oplus wz \arrow[d, "\delta^l_{w,x,y} \oplus 1_{wz}"] \\
c' = wx \oplus (wy \oplus wz) \arrow[rr, "R = \alpha^{-\oplus}_{wx,wy,wz}"] & & (wx \oplus wy) \oplus wz = b'
\end{tikzcd}$$

This diagram is commutative by the axiom (2.1.9).

Cases (i)–(v) account for all the cases when $g = \delta^l$.

If $a_1 = x \oplus y$, and if

$$(x \oplus y)a_2 \xrightarrow{g = \delta^r_{x,y,a_2}} xa_2 \oplus ya_2,$$

then f may act on a_2, x, y, or a_1, leading to analogues of cases (i)–(v) above. We slightly modify the diagrams above using

- functoriality and the naturality of δ^r in the analogues of cases (i) and (ii) when f acts on one of a_2, x, and y;
- (2.1.7) instead of (2.1.6) in the analogue of case (iii) when $f = \xi^{\oplus}_{x,y}e_2$ or $\xi^{-\oplus}_{y,x}e_2$ for some identity $e_2 : a_2 \longrightarrow a_2$; and
- (2.1.8) instead of (2.1.9) in the analogues of cases (iv) and (v) when $f = \alpha^{\oplus}_{w,x,y}e$ or $f = \alpha^{-\oplus}_{w,x,y}e$ for some identity $e : z \longrightarrow z$.

(5) f and g both act on all of a.

First suppose $g = \delta^l$. Depending on what f is, there are the following cases.

(i) If $f = \alpha^{\otimes}$, then we consider the following diagram.

$$\begin{tikzcd}
& a = (wx)(y \oplus z) \arrow[dl, "g = \delta^l_{wx,y,z}"'] \arrow[dr, "\alpha^{\otimes}_{w,x,y\oplus z} = f"] & \\
c = (wx)y \oplus (wx)z \arrow[ddd, "D_2 = 1_c"'] & & w[x(y \oplus z)] = b \arrow[d, "1_w\delta^l_{x,y,z}"'] \arrow[ddd, bend left, "D_1"] \\
& & w(xy \oplus xz) \arrow[dd, "\delta^l_{w,xy,xz}"'] \\
& & \\
c' = (wx)y \oplus (wx)z \arrow[r, "\alpha^{\otimes}_{w,x,y} \oplus 1_{(wx)z}"] \arrow[rr, bend right, "R"'] & w(xy) \oplus (wx)z \arrow[r, "1_{w(xy)} \oplus \alpha^{\otimes}_{w,x,z}"] & w(xy) \oplus w(xz) = b'
\end{tikzcd}$$

This diagram is commutative by functoriality and (2.1.10).

(ii) If $f = \xi^{\otimes}_{x,y\oplus z}$ and $g = \delta^l_{x,y,z}$, then we reuse the diagram (3.6.18).

(iii) If $f = \xi^{-\otimes}_{y\oplus z,x}$ and $g = \delta^l_{x,y,z}$, then we reuse case (5) in the initial case.

(iv) If $f = \lambda^{\otimes}_{y\oplus z}$ and $g = \delta^l_{1^X,y,z}$, then we reuse the diagram (3.6.14).

(v) If $f = \lambda^{-\otimes}_a$, then we reuse the diagram (3.6.15).

(vi) If $f = \rho^{-\otimes}_a$, then we reuse the diagram (3.6.15) by replacing $\lambda^{-\otimes}$ with $\rho^{-\otimes}$, and using the naturality of $\rho^{\otimes}$ in C.

(vii) If $f = \delta^r$, then we consider the following diagram, where some subscripts are omitted to save space.

$$
\begin{tikzcd}
 & a = (w \oplus x)(y \oplus z) \arrow[dl, bend right, "g = \delta^l_{w\oplus x,y,z}"'] \arrow[dr, bend left, "\delta^r_{w,x,y\oplus z} = f"] & \\
c = (w \oplus x)y \oplus (w \oplus x)z \arrow[d, "\delta^r_{w,x,y} \oplus 1"'] \arrow[ddd, bend right=60, "D_2"'] & & w(y \oplus z) \oplus x(y \oplus z) = b \arrow[d, "\delta^l_{w,y,z} \oplus 1"] \arrow[ddd, bend left=60, "D_1"] \\
(wy \oplus xy) \oplus (w \oplus x)z \arrow[dd, "1 \oplus \delta^r_{w,x,z}"'] & & (wy \oplus wz) \oplus x(y \oplus z) \arrow[dd, "1 \oplus \delta^l_{x,y,z}"] \\
 & & \\
c' = (wy \oplus xy) \oplus (wz \oplus xz) \arrow[rr, "R"] \arrow[d, "\alpha^{\oplus}"'] & & (wy \oplus wz) \oplus (xy \oplus xz) = b' \\
wy \oplus [xy \oplus (wz \oplus xz)] \arrow[d, "1 \oplus \alpha^{-\oplus}"'] & & wy \oplus [wz \oplus (xy \oplus xz)] \arrow[u, "\alpha^{-\oplus}"'] \\
wy \oplus [(xy \oplus wz) \oplus xz] \arrow[rr, "1 \oplus (\xi^{-\oplus}_{wz,xy} \oplus 1)"] & & wy \oplus [(wz \oplus xy) \oplus xz] \arrow[u, "1 \oplus \alpha^{\oplus}"']
\end{tikzcd}
\tag{3.6.20}
$$

The bottom half of this diagram is the definition of the $(0^X, \delta)$-free path R, which consists of five prime edges. This diagram is commutative by the axiom (2.1.13).

Cases (i)–(vii) above account for all the cases when $g = \delta^l$.

Similarly, if $g = \delta^r$, then there are the following cases.

(i) If $f = \alpha^{-\otimes}$, then there is a diagram similar to the one in case (i) above, which is commutative by functoriality and (2.1.11).

(ii) If $f = \xi^{\otimes}$, then we use functoriality and (2.1.5).

(iii) If $f = \xi^{-\otimes}$, then we use functoriality, (2.1.5), and the symmetry axiom (1.2.20) in $(\mathsf{C}, \otimes)$.

(iv) If $f = \rho^{\otimes}$, then we use functoriality and (2.1.28).

(v) If $f = \lambda^{-\otimes}$, then we use the δ^r analogue of the diagram (3.6.15) and the naturality of $\lambda^{\otimes}$ in C.

(vi) If $f = \rho^{-\otimes}$, then we use the $(\delta^r, \rho^{-\otimes})$ analogue of the diagram (3.6.15) and the naturality of $\rho^{\otimes}$ in C.

(vii) If $f = \delta^l$, then we reuse the diagram in case (vii) above by
- switching (f, D_1) with (g, D_2) and
- reversing R and every edge in it and replacing $(\alpha^{\oplus}, \alpha^{-\oplus}, \xi^{-\oplus})$ with $(\alpha^{-\oplus}, \alpha^{\oplus}, \xi^{\oplus})$.

This finishes case (5) and the induction. The proof of Lemma 3.6.12 is now complete. □

3.7. Zero and Delta Reduction of Paths

Recall from Definition 3.6.2 that a δ-reduction of an element $a \in X^{\mathsf{fr}}$ is a path with domain a and codomain a δ-reduced element in which each edge is either an identity or a δ-prime edge. The next definition is the path analogue of a δ-reduction. In Proposition 3.5.32, we showed that each path in $\mathsf{Gr}(X)$ has a 0^X-reduction. The goal of this section is to extend that observation by also eliminating the distributivity morphisms; see Proposition 3.7.19. This will constitute another reduction step in the proof of the Coherence Theorem 3.9.1. Recall that Convention 3.3.1 is in effect.

Definition 3.7.1. Suppose

- $P : a \twoheadrightarrow b$ is a path in $\mathsf{Gr}(X)$ whose vertices do not contain 0^X, and
- for each $x \in \{a, b\}$, $D_x : x \twoheadrightarrow x'$ is a δ-reduction of x.

A $(0^X, \delta)$-*reduction* of (P, D_a, D_b) is a $(0^X, \delta)$-free path $R : a' \twoheadrightarrow b'$ such that the diagram

$$\begin{tikzcd} a \arrow[r, two heads, "P"] \arrow[d, two heads, "D_a"'] & b \arrow[d, two heads, "D_b"] \\ a' \arrow[r, two heads, "R"] & b' \end{tikzcd} \tag{3.7.2}$$

is commutative in the sense of Definition 3.1.14. We also call R a $(0^X, \delta)$-reduction of P, suppressing D_a and D_b from the notation. ⋄

The main result of this section, Proposition 3.7.19, asserts the existence of a $(0^X, \delta)$-reduction.

Explanation 3.7.3. Consider Definition 3.7.1.

(1) Since the vertices in P contain no 0^X, P is a 0^X-free path as in Definition 3.4.1. Its edges do not contain $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\lambda^{\pm\bullet}$, and $\rho^{\pm\bullet}$. In other words, as described in Explanation 3.4.2, each prime edge in P is either an identity or contains a single instance of $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, $\xi^{\pm\otimes}$, $\lambda^{\pm\otimes}$, $\rho^{\pm\otimes}$, δ^l, or δ^r.

(2) The (co)domain of R is a δ-reduction of the (co)domain of the original path P. Each element in X^{fr} has a δ-reduction by Lemma 3.6.9. However, neither its codomain nor its value in C is unique, as we explained in Example 3.6.8.

(3) For the commutativity of the diagram (3.7.2), the equality

$$(R, D_a) = (D_b, P)$$

of paths in $\mathsf{Gr}(X)$ is sufficient but *not* necessary. The definition only requires that the image of the diagram under the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ in Definition 3.1.14 be a commutative diagram in C for arbitrary C and φ. ⋄

Example 3.7.4. Suppose P is the path

$$(w \oplus x)(y \oplus z) \xrightarrow{P = \delta^r_{w,x,y\oplus z}} w(y \oplus z) \oplus x(y \oplus z)$$

for some elements $w, x, y, z \in X \smallsetminus \{0^X\}$. Reusing the diagram (3.6.20):

- $D_a = (D_2, g) : a \twoheadrightarrow c'$ is a δ-reduction of a.
- $D_b = D_1 : b \twoheadrightarrow b'$ is a δ-reduction of b.
- $R : c' \twoheadrightarrow b'$ is a $(0^X, \delta)$-reduction of (P, D_a, D_b).

The commutativity of the diagram (3.7.2) follows from the axiom (2.1.13). ⋄

Lemma 3.7.5. *Suppose $P : a \twoheadrightarrow b$ is a $(0^X, \delta)$-free path with a and b both δ-reduced. Then P is a $(0^X, \delta)$-reduction of P for any choices of δ-reductions of a and b.*

Proof. Since a and b are not the domains of any δ-prime edges, any δ-reduction $D_x : x \twoheadrightarrow x'$ of $x \in \{a, b\}$, which exists by Lemma 3.6.9, is a sequence of identities with $x' = x$. Therefore, the diagram (3.7.2) is commutative if $R = P$. □

The proof of the existence of a $(0^X, \delta)$-reduction is split into several cases, the first of which uses the following two facts about symmetric bimonoidal categories.

Lemma 3.7.6. *For objects $Y_1, \ldots, Y_n \in \mathsf{C}$ with $n \geq 2$, the diagram*

$$\begin{tikzcd}[column sep=large]
 & \mathbb{1}(Y_1 \oplus \cdots \oplus Y_n) = W \arrow[d, "D"] \\
Y = Y_1 \oplus \cdots \oplus Y_n \arrow[ur, bend left, "\lambda_Y^{-\otimes}"] \arrow[dr, bend right, "L"'] & \\
 & \mathbb{1}Y_1 \oplus \cdots \oplus \mathbb{1}Y_n = Z
\end{tikzcd} \tag{3.7.7}$$

defined as follows is commutative:

- *Y has some additive bracketing, and $W = \mathbb{1}Y$.*
- *Z is obtained from Y by replacing each summand Y_k with $\mathbb{1}Y_k$.*
- *$L = \lambda_{Y_1}^{-\otimes} \oplus \cdots \oplus \lambda_{Y_n}^{-\otimes}$ with the same additive bracketing as Y.*
- *D is a composite of morphisms, each being a sum of*
  - *identity morphisms, for which there are zero of them for the first morphism in D; and*
  - *one $\delta^l_{\mathbb{1},Y',Y''}$ with each of Y' and Y'' a sum of consecutive Y_k's with bracketings inherited from Y.*

Proof. The proof is by induction on n. The $n = 2$ case of (3.7.7) is commutative by the axiom (2.1.27).

For the induction step, suppose $n > 2$, and $Y = Y^1 \oplus Y^2$ with

- $Y^1 = Y_1 \oplus \cdots \oplus Y_k$ and
- $Y^2 = Y_{k+1} \oplus \cdots \oplus Y_n$

with bracketings inherited from Y and some $1 \leq k \leq n-1$. Moreover, $L = L^1 \oplus L^2$ with

- $L^1 = \lambda_{Y_1}^{-\otimes} \oplus \cdots \oplus \lambda_{Y_k}^{-\otimes}$ and
- $L^2 = \lambda_{Y_{k+1}}^{-\otimes} \oplus \cdots \oplus \lambda_{Y_n}^{-\otimes}$

with bracketings inherited from Y.

The diagram (3.7.7) factors as follows.

$$\begin{tikzcd}[column sep=huge]
 & \mathbb{1}(Y^1 \oplus Y^2) = W \arrow[d, "\delta^l_{\mathbb{1},Y^1,Y^2}"'] \arrow[dd, bend left=60, "D"] \\
Y = Y^1 \oplus Y^2 \arrow[ur, bend left, "\lambda_Y^{-\otimes}"] \arrow[r, "\lambda_{Y^1}^{-\otimes} \oplus \lambda_{Y^2}^{-\otimes}"] \arrow[dr, bend right, "L^1 \oplus L^2"'] & \mathbb{1}Y^1 \oplus \mathbb{1}Y^2 \arrow[d, "D^1 \oplus D^2"'] \\
 & \big(\mathbb{1}Y_1 \oplus \cdots \oplus \mathbb{1}Y_k\big) \oplus \big(\mathbb{1}Y_{k+1} \oplus \cdots \oplus \mathbb{1}Y_n\big) = Z
\end{tikzcd}$$

The top half is commutative by (2.1.27). Consider the bottom half.

- If $k = 1$, then $Y^1 = Y_1$, and $D^1 = 1_{\mathbb{1}Y_1}$. The induction hypothesis is applied to Y^2.
- If $k = n-1$, then $Y^2 = Y_n$, and $D^2 = 1_{\mathbb{1}Y_n}$. The induction hypothesis is applied to Y^1.
- If $1 < k < n-1$, then the induction hypothesis is applied to Y^1 and Y^2.

In each case, the bottom half of the previous diagram is commutative. $\square$

Next is the $\rho^{\otimes}$ analogue of Lemma 3.7.6.

Lemma 3.7.8. *For objects $Y_1, \ldots, Y_n \in \mathsf{C}$ with $n \geq 2$, the diagram*

$$\begin{tikzcd} & (Y_1 \oplus \cdots \oplus Y_n)\mathbb{1} = W \arrow[d, "D"] \\ Y = Y_1 \oplus \cdots \oplus Y_n \arrow[ur, bend left, "\rho_Y^{-\otimes}"] \arrow[r, bend right, "L"'] & Y_1\mathbb{1} \oplus \cdots \oplus Y_n\mathbb{1} = Z \end{tikzcd}$$

defined as follows is commutative:

- *Y has some additive bracketing, and $W = Y\mathbb{1}$.*
- *Z is obtained from Y by replacing each summand Y_k with $Y_k\mathbb{1}$.*
- *$L = \rho_{Y_1}^{-\otimes} \oplus \cdots \oplus \rho_{Y_n}^{-\otimes}$ with the same additive bracketing as Y.*
- *D is a composite of morphisms, each being a sum of*
  - *identity morphisms, for which there are zero of them for the first morphism in D; and*
  - *one $\delta^r_{Y',Y'',\mathbb{1}}$ with each of Y' and Y'' a sum of consecutive Y_k's with bracketings inherited from Y.*

Proof. This is a slight modification of the proof of Lemma 3.7.6, using the axiom (2.1.28) instead of (2.1.27). $\square$

Next is the first preliminary case of Proposition 3.7.19.

Lemma 3.7.9. *Suppose $f : a \longrightarrow b$ is a prime edge such that*

- *a and b do not contain 0^X, and*
- *a is δ-reduced.*

Then f has a $(0^X, \delta)$-reduction for any choices of δ-reductions of a and b.

Proof. For each $x \in \{a, b\}$, suppose $D_x : x \twoheadrightarrow x'$ is a δ-reduction of x, which exists by Lemma 3.6.9. First we make a few observations.

- By Lemma 3.6.5, a being δ-reduced means that it is a polynomial in the sense of Definition 3.6.3.
- Since a, being δ-reduced, is not the domain of any δ-prime edge, the δ-reduction D_a is a sequence of identities of a, and $a' = a$.
- The absence of 0^X in a and b, and that a is δ-reduced, imply that f is a $(0^X, \delta)$-free edge as in Definition 3.6.10.

The possibilities of the $(0^X, \delta)$-free edge f are listed in Explanation 3.6.11.

First suppose f is either an identity or involves an instance of

- $\alpha^{\pm\oplus}, \xi^{\pm\oplus}, \alpha^{\pm\otimes}, \xi^{\pm\otimes}, \lambda^{\otimes}, \rho^{\otimes}$, or

- $\lambda^{-\otimes}$ or $\rho^{-\otimes}$ applied to either a monomial in a or some $\otimes$-factor of a monomial in a.

Then b is also a polynomial, which means that it is δ-reduced. The δ-reduction D_b is a sequence of identities of b, and $b' = b$. Therefore, $R = f$ is a $(0^X, \delta)$-reduction of f.

If f is not as in the previous paragraph, then f involves $\lambda^{-\otimes}$ or $\rho^{-\otimes}$ applied to a sum of at least two monomials in a. Consider the $\lambda^{-\otimes}$ case first. In other words:

- $a \in X^{\mathsf{fr}}$ has the form
$$a = \underbrace{a^1 \oplus \cdots \oplus a^{i-1}}_{\text{empty if } i=1} \oplus A \oplus \underbrace{a^{j+1} \oplus \cdots \oplus a^m}_{\text{empty if } j=m}$$
for some additive bracketing and some
$$A = a^i \oplus \cdots \oplus a^j \in X^{\mathsf{fr}}$$
with some additive bracketing, $1 \leq i < j \leq m$ with $m \geq 2$, and each a^k a monomial as in Definition 3.6.3.
- $b \in X^{\mathsf{fr}}$ has the form
$$b = a^1 \oplus \cdots \oplus a^{i-1} \oplus 1^X A \oplus a^{j+1} \oplus \cdots \oplus a^m$$
with the same additive bracketing as a.
- The prime edge f involves the elementary edge $\lambda_A^{-\otimes} : A \longrightarrow 1^X A$.

Note that b is not δ-reduced because δ^l can be applied to $1^X A$. Since $D_b : b \twoheadrightarrow b'$ is a δ-reduction of b, the codomain b' has the form
$$b' = a^1 \oplus \cdots \oplus a^{i-1} \oplus A' \oplus a^{j+1} \oplus \cdots \oplus a^m$$
with the same additive bracketing as a, where
$$A' = 1^X a^i \oplus \cdots \oplus 1^X a^j$$
with the same additive bracketing as A. Moreover, each prime edge in D_b is either an identity or involves δ^l.

The desired path $R : a = a' \twoheadrightarrow b'$ is defined using the following objects and edges.

- For $0 \leq l \leq j-i+1$, define $A_l \in X^{\mathsf{fr}}$ as the element obtained from A by replacing each of the l leftmost summands $a^?$ by $1^X a^?$. Here are some examples.
$$\begin{aligned} A_0 &= a^i \oplus \cdots \oplus a^j = A \\ A_1 &= 1^X a^i \oplus a^{i+1} \oplus \cdots \oplus a^j \\ A_{j-i} &= 1^X a^i \oplus \cdots \oplus 1^X a^{j-1} \oplus a^j \\ A_{j-i+1} &= 1^X a^i \oplus \cdots \oplus 1^X a^j = A' \end{aligned}$$
- For $0 \leq l \leq j-i+1$, define $a_l \in X^{\mathsf{fr}}$ as the element obtained from a by replacing A with A_l. Here are some examples.
$$\begin{aligned} a_0 &= a^1 \oplus \cdots \oplus a^{i-1} \oplus A \oplus a^{j+1} \oplus \cdots \oplus a^m = a \\ a_1 &= a^1 \oplus \cdots \oplus a^{i-1} \oplus A_1 \oplus a^{j+1} \oplus \cdots \oplus a^m \\ a_{j-i+1} &= a^1 \oplus \cdots \oplus a^{i-1} \oplus A' \oplus a^{j+1} \oplus \cdots \oplus a^m = b' \end{aligned}$$

- For $1 \leq l \leq j-i+1$, define $r_l : a_{l-1} \longrightarrow a_l$ as the prime edge involving the elementary edge
$$a^{i+l-1} \xrightarrow{\lambda^{-\otimes}_{a^{i+l-1}}} 1^X a^{i+l-1}$$
and the identity elementary edge of every other summand in a_{l-1}. For example, the prime edge
  - $r_1 : a = a_0 \longrightarrow a_1$ involves $\lambda^{-\otimes}_{a^i} : a^i \longrightarrow 1^X a^i$, and
  - $r_{j-i+1} : a_{j-i} \longrightarrow a_{j-i+1} = b'$ involves $\lambda^{-\otimes}_{a^j} : a^j \longrightarrow 1^X a^j$.

Define the $(0^X, \delta)$-free path
$$R = \big(r_{j-i+1}, \ldots, r_1\big)$$
in $\mathsf{Gr}(X)$, that is,
$$a = a_0 \xrightarrow{r_1} a_1 \xrightarrow{r_2} \cdots \xrightarrow{r_{j-i+1}} a_{j-i+1} = b'.$$
We aim to show that R is a δ-reduction of f.

To show that the diagram (3.7.2) is commutative, it suffices to concentrate on the summands A in a, $1^X A$ in b, and A' in b', since f, D_b, and R in other summands are identities. The relevant diagram is commutative by Lemma 3.7.6 applied as follows:

- $Y_l = \varphi a^{i+l-1}$ for $1 \leq l \leq j-i+1 = n$, with φ the graph morphism in Definition 3.1.14. For example, $Y_1 = \varphi a^i$, and $Y_n = \varphi a^j$.
- $Y = Y_1 \oplus \cdots \oplus Y_n$ has the same additive bracketing as A, and $Y = \varphi A$.
- $W = \mathbb{1} Y = \varphi(1^X A)$.
- $Z = \mathbb{1} Y_1 \oplus \cdots \oplus \mathbb{1} Y_n = \varphi A'$.
- $\lambda^{-\otimes}_Y : Y \longrightarrow \mathbb{1} Y$ is the nonidentity summand of φf.
- $L = \lambda^{-\otimes}_{Y_1} \oplus \cdots \oplus \lambda^{-\otimes}_{Y_n}$ is the nonidentity summand of the composite of φR.
- D is the nonidentity summand of the composite of φD_b.

This shows that R is a δ-reduction of f when f involves $\lambda^{-\otimes}$.

If f involves $\rho^{-\otimes}$, then we slightly modify the argument above by replacing

- $\big(\lambda^{-\otimes}, \delta^l, 1^X A, 1^X a^k\big)$ with $\big(\rho^{-\otimes}, \delta^r, A 1^X, a^k 1^X\big)$ and
- Lemma 3.7.6 with Lemma 3.7.8.

All the possible cases of f, as in Explanation 3.6.11, have been considered. □

Next is the second preliminary case of Proposition 3.7.19.

Lemma 3.7.10. *Under the hypotheses of Definition 3.7.1, suppose that each edge in the path $P : a \twoheadrightarrow b$ is of one of the following types:*

- *an identity or*
- *a prime edge involving $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, $\xi^{\pm\otimes}$, δ^l, or δ^r.*

Then P has a $(0^X, \delta)$-reduction.

Proof. The proof is by induction on $\mathsf{size}(a) - \mathsf{rank}(a)$, which is always nonnegative by Lemma 3.2.9 (2). First suppose
$$\mathsf{size}(a) - \mathsf{rank}(a) = 0.$$
By Lemmas 3.2.9, 3.2.10, and 3.2.12, P does not contain δ^l and δ^r, and each vertex in P satisfies $\mathsf{size} = \mathsf{rank}$. By Proposition 3.2.15, each vertex in P is a polynomial in the sense of Definition 3.6.3, which is δ-reduced by Lemma 3.6.5. Since P is

a $(0^X,\delta)$-free path with a and b both δ-reduced, P is a $(0^X,\delta)$-reduction of P by Lemma 3.7.5.

For the induction step, suppose

$$\mathsf{size}(a) - \mathsf{rank}(a) > 0$$

and that the path P consists of the prime edges $(f_n,\ldots,f_1)$ as in

$$a = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n = b \tag{3.7.11}$$

for some $n \geq 1$. The δ-reductions $D_x : x \twoheadrightarrow x'$ of $x \in \{a,b\}$ are already chosen in Definition 3.7.1. If $n > 1$, then for each $1 \leq i \leq n-1$, choose a δ-reduction $D_i : a_i \twoheadrightarrow a_i'$ of a_i, which exists by Lemma 3.6.9. If for each $1 \leq i \leq n$, the prime edge f_i has a $(0^X,\delta)$-reduction $R_i : a_{i-1}' \twoheadrightarrow a_i'$ with respect to these δ-reductions, then the concatenated path

$$R = (R_n,\ldots,R_1)$$

is a $(0^X,\delta)$-reduction of P, as in the following diagram.

$$\begin{tikzcd}
a = a_0 \arrow[r, "f_1"] \arrow[d, "D_a"'] & a_1 \arrow[r, "f_2"] \arrow[d, "D_1"] & \cdots \arrow[r, "f_n"] & a_n = b \arrow[d, "D_b"] \\
a' = a_0' \arrow[r, two heads, "R_1"] & a_1' \arrow[r, two heads, "R_2"] & \cdots \arrow[r, two heads, "R_n"] & a_n' = b'
\end{tikzcd}$$

Therefore, to show that P has a $(0^X,\delta)$-reduction, it suffices to show that each prime edge $f : a \longrightarrow b$ of the form stated in the lemma, with a and b containing no 0^X, has a $(0^X,\delta)$-reduction.

Moreover, if a is δ-reduced, then f has a $(0^X,\delta)$-reduction by Lemma 3.7.9. Therefore, we may assume that a is not δ-reduced, which by Lemma 3.6.5 means that a is not a polynomial. The rest of the induction step is divided into two cases depending on f.

First suppose that the prime edge $f : a \longrightarrow b$ does not involve δ^l and δ^r. In other words, either

- f is an identity, or
- f involves $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{\pm\otimes}$.

Lemma 3.2.10 implies that there is an equality

$$\mathsf{size}(a) - \mathsf{rank}(a) = \mathsf{size}(b) - \mathsf{rank}(b). \tag{3.7.12}$$

Moreover, that a is not a polynomial and the current hypothesis on f imply that b is not a polynomial, so b is also not δ-reduced by Lemma 3.6.5. Since a and b are not δ-reduced, each δ-reduction $D_x : x \twoheadrightarrow x'$ of $x \in \{a,b\}$ has at least one δ-prime edge. We are concerned about constructing a diagram (3.7.2) that is commutative, and identity prime edges are sent by the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ in Definition 3.1.14 to identity morphisms. Therefore, we may assume that the first edge in each δ-reduction D_x is a δ-prime edge. In the diagram (3.7.13) below, these δ-prime edges are denoted by $a \longrightarrow a_1$ and $b \longrightarrow b_1$.

Using Lemma 3.6.12 and the induction hypothesis (IH), a $(0^X,\delta)$-reduction $R : a' \twoheadrightarrow b'$ of f is obtained in the following diagram, which we will explain in detail.

$$
\begin{tikzcd}
a \arrow[rr, "f"] \arrow[dd, "\delta"'] \arrow[ddd, bend right=60, two heads, "D_a"'] \arrow[drr, phantom, "3.6.12"] & & b \arrow[rr, two heads, "\delta"] \arrow[d, "\delta"] \arrow[rrr, bend left=40, two heads, "D_b"] \arrow[drr, phantom, "3.6.12"] & & b_1 \arrow[r, two heads, "\overline{\delta}"] \arrow[d, two heads, "{(1,\delta)}"] & b' \arrow[ddd, "1_{b'}"] \\
& & b_1' \arrow[r, two heads, "{(1,\delta)}"] \arrow[d, two heads, "{(1,\delta)}"] & e \arrow[r, two heads, "{(\alpha,\xi)}"] \arrow[dd, two heads, "\overline{\delta}"] & h \arrow[dd, two heads, "\overline{\delta}"] \arrow[dr, phantom, "\mathrm{IH}"] & \\
a_1 \arrow[r, two heads, "{(1,\delta)}"] \arrow[d, two heads, "\overline{\delta}"] \arrow[dr, phantom, "\mathrm{IH}"] & c \arrow[r, two heads, "{(\alpha,\xi)}"] \arrow[d, two heads, "\overline{\delta}"] \arrow[dr, phantom, "\mathrm{IH}"] & d \arrow[d, two heads, "\overline{\delta}"] \arrow[r, phantom, "\mathrm{IH}"] & \arrow[r, phantom, "\mathrm{IH}"] & & \\
a' \arrow[r, two heads, "R_1"] \arrow[rrrrr, bend right=30, two heads, "R"'] & c' \arrow[r, two heads, "R_2"] & d' \arrow[r, two heads, "R_3"] & e' \arrow[r, two heads, "R_4"] & h' \arrow[r, two heads, "R_5"] & b'
\end{tikzcd}
\tag{3.7.13}
$$

The following statements are about the diagram (3.7.13):

- Each subdiagram labeled by 3.6.12 is obtained by applying that lemma.
- Each subdiagram labeled by IH is obtained by applying the induction hypothesis.
- Each subdiagram is commutative.
- Each edge labeled by δ is a δ-prime edge.
- Each path labeled by:
  - $\overline{\delta}$ is a δ-reduction.
  - $(1,\delta)$ is either a single identity or consists of δ-prime edges.
  - (α,ξ) consists of identities and prime edges involving $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{\pm\otimes}$.
- The δ-reductions D_a and D_b are as displayed. The respective first edges $a \longrightarrow a_1$ and $b \longrightarrow b_1$ are both δ-prime edges by the last sentence of the previous paragraph.
- Each R_i is a $(0^X,\delta)$-reduction of the horizontal path above it. Their concatenation R is a $(0^X,\delta)$-reduction of f.

With more detail, we begin with the prime edge f, and the δ-reductions D_a and D_b, and perform the following steps.

(1) Using the current hypothesis on f and that $a \longrightarrow a_1$ is a δ-prime edge, we first apply Lemma 3.6.12 to obtain the upper left square in (3.7.13). The resulting path $b \longrightarrow\!\!\!\!\to d$ has at least one δ-prime edge $b \longrightarrow b_1'$ by the following (in)equalities.

$$
\begin{aligned}
&\mathsf{size}(d) - \mathsf{rank}(d) \\
&= \mathsf{size}(c) - \mathsf{rank}(c) && \text{(by Lemma 3.2.10)} \\
&\leq \mathsf{size}(a_1) - \mathsf{rank}(a_1) && \text{(by Lemma 3.2.12)} \\
&< \mathsf{size}(a) - \mathsf{rank}(a) && \text{(by Lemma 3.2.12)} \\
&= \mathsf{size}(b) - \mathsf{rank}(b) && \text{(by (3.7.12))}
\end{aligned}
$$

Therefore, $b \longrightarrow\!\!\!\!\to d$ cannot be a single identity, and must contain a δ-prime edge $b \longrightarrow b_1'$.

(2) Next we apply Lemma 3.6.12 to the δ-prime edges $b \longrightarrow b_1$ and $b \longrightarrow b_1'$ to obtain the top middle rectangle in (3.7.13). Since a and b contain no 0^X, all other vertices involved so far—namely, a_1, a', b_1, b', b_1', c, d, e, and h—also contain no 0^X.

(3) For each $y \in \{c, d, e, h\}$, choose a δ-reduction $y \longrightarrow\!\!\!\!\rightarrow y'$, which exists by Lemma 3.6.9. The concatenated paths

$$b_1' \twoheadrightarrow d \twoheadrightarrow d'$$
$$b_1 \twoheadrightarrow h \twoheadrightarrow h'$$

are δ-reductions of b_1' and b_1, respectively. Moreover, since b' is δ-reduced, the identity $1_{b'}$ is a δ-reduction.

(4) We observed in step (1) that a_1 and c both have $\mathsf{size} - \mathsf{rank}$ less than that of a. Furthermore, by Lemma 3.2.12 and (3.7.12), b_1, b_1', and e also have $\mathsf{size} - \mathsf{rank}$ less than that of a. The induction hypothesis is applied five times as indicated by IH in (3.7.13) to obtain $(0^X, \delta)$-reductions
- $R_1 : a' \twoheadrightarrow c'$ of $a_1 \twoheadrightarrow c$;
- $R_2 : c' \twoheadrightarrow d'$ of $c \twoheadrightarrow d$;
- $R_3 : d' \twoheadrightarrow e'$ of $b_1' \twoheadrightarrow e$;
- $R_4 : e' \twoheadrightarrow h'$ of $e \twoheadrightarrow h$; and
- $R_5 : h' \twoheadrightarrow b'$ of $b_1 \twoheadrightarrow b'$.

The concatenated path

$$R = (R_5, R_4, R_3, R_2, R_1) : a' \twoheadrightarrow b'$$

is $(0^X, \delta)$-free, and the diagram (3.7.2) with $P = f$ is commutative by (3.7.13). Therefore, R is a $(0^X, \delta)$-reduction of f.

Next we consider the case when the prime edge f involves a single instance of δ^l or δ^r. Using Lemma 3.6.12 and the induction hypothesis (IH), a $(0^X, \delta)$-reduction $R : a' \twoheadrightarrow b'$ of f is obtained in the following diagram.

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[d, "\delta"] \arrow[dd, bend right=60, two heads, "D_a"'] \arrow[drr, phantom, "3.6.12"] & & b \arrow[r, two heads, "D_b"] \arrow[d, two heads, "{(1,\delta)}"] & b' \arrow[dd, "1_{b'}"] \\
a_1 \arrow[r, two heads, "{(1,\delta)}"] \arrow[d, two heads, "\overline{\delta}"] \arrow[dr, phantom, "\mathrm{IH}"] & c \arrow[r, two heads, "{(\alpha,\xi)}"] \arrow[d, two heads, "\overline{\delta}"] \arrow[dr, phantom, "\mathrm{IH}"] & d \arrow[d, two heads, "\overline{\delta}"] \arrow[r, phantom, "\mathrm{IH}"] & {} \\
a' \arrow[r, two heads, "R_1"] \arrow[rrr, bend right, two heads, "R"'] & c' \arrow[r, two heads, "R_2"] & d' \arrow[r, two heads, "R_3"] & b'
\end{tikzcd} \tag{3.7.14}$$

The steps for obtaining this diagram are essentially those for (3.7.13). In more detail:

(1) Using the fact that f and $a \longrightarrow a_1$ are δ-prime edges, we apply Lemma 3.6.12 to obtain the upper left rectangle in (3.7.14).

(2) For each $y \in \{c, d\}$, choose a δ-reduction $y \twoheadrightarrow y'$ using Lemma 3.6.9. The path

$$b \twoheadrightarrow d \twoheadrightarrow d'$$

is a δ-reduction of b, and $1_{b'}$ is a δ-reduction of b', since b' is δ-reduced.

(3) By Lemma 3.2.12, the elements a_1, b, and c all have $\mathsf{size} - \mathsf{rank}$ less than that of a. The induction hypothesis is applied three times as indicated by IH. Each R_i is a $(0^X, \delta)$-reduction of the horizontal path above it.

The bottom concatenated path

$$R = (R_3, R_2, R_1) : a' \twoheadrightarrow b'$$

is a $(0^X, \delta)$-reduction of f. This finishes the induction. $\square$

Next is the preliminary case of Proposition 3.7.19 involving $\lambda^{\otimes}$ and $\rho^{\otimes}$.

Lemma 3.7.15. *Under the hypotheses of Definition 3.7.1, suppose that each prime edge in $P : a \twoheadrightarrow b$ involves a single instance of $\lambda^{\otimes}$ or $\rho^{\otimes}$. Then P has a $(0^X, \delta)$-reduction.*

Proof. The proof is by induction on $\mathsf{size}(a) - \mathsf{rank}(a)$, which is always nonnegative by Lemma 3.2.9 (2). First suppose

$$\mathsf{size}(a) - \mathsf{rank}(a) = 0.$$

By Lemma 3.2.16 (1), each vertex in P satisfies $\mathsf{size} = \mathsf{rank}$. By Proposition 3.2.15, each vertex in P is a polynomial in the sense of Definition 3.6.3, which is δ-reduced by Lemma 3.6.5. Since P is a $(0^X, \delta)$-free path with a and b both δ-reduced, P is a $(0^X, \delta)$-reduction of P by Lemma 3.7.5.

For the induction step, suppose

$$\mathsf{size}(a) - \mathsf{rank}(a) > 0.$$

Reusing the reasoning in the second and the third paragraphs in the proof of Lemma 3.7.10, it suffices to show that each prime edge $f : a \longrightarrow b$ involving $\lambda^{\otimes}$ or $\rho^{\otimes}$, with a and b containing no 0^X and with a not δ-reduced, has a $(0^X, \delta)$-reduction. Using Lemmas 3.6.12 and 3.7.10 and the induction hypothesis (IH), we modify the diagram (3.7.14) as follows to obtain a $(0^X, \delta)$-reduction of f.

$$\begin{tikzcd}
a \arrow[rr, "f"] \arrow[d, "\delta"] \arrow[dd, bend right=60, two heads, "D_a"'] \arrow[drr, phantom, "3.6.12"] & & b \arrow[r, two heads, "D_b"] \arrow[d, two heads, "(1,\delta)"] \arrow[ddr, phantom, "3.7.10"] & b' \arrow[dd, "1_{b'}"] \\
a_1 \arrow[r, two heads, "(1,\delta)"] \arrow[d, two heads, "\overline{\delta}"] \arrow[dr, phantom, "3.7.10"] & c \arrow[r, two heads, "(\lambda^{\otimes},\rho^{\otimes})"] \arrow[d, two heads, "\overline{\delta}"] \arrow[dr, phantom, "\mathrm{IH}"] & d \arrow[d, two heads, "\overline{\delta}"] & \\
a' \arrow[r, two heads, "R_1"] \arrow[rrr, bend right, two heads, "R"'] & c' \arrow[r, two heads, "R_2"] & d' \arrow[r, two heads, "R_3"] & b'
\end{tikzcd} \tag{3.7.16}$$

The steps for obtaining this diagram are as follows.

(1) Since a is not δ-reduced, the δ-reduction D_a has at least one δ-prime edge. We may assume that it is the first edge in D_a, as explained in the paragraph containing (3.7.12). This δ-prime edge is $a \longrightarrow a_1$ in (3.7.16).

(2) Starting with the prime edge f, which involves either $\lambda^{\otimes}$ or $\rho^{\otimes}$, and the δ-prime edge $a \longrightarrow a_1$, apply Lemma 3.6.12 to obtain the upper left rectangle in (3.7.16). By that lemma, the path $c \twoheadrightarrow d$ consists of only prime edges involving either $\lambda^{\otimes}$ or $\rho^{\otimes}$, according to f.

(3) For each $y \in \{c, d\}$, choose a δ-reduction $y \twoheadrightarrow y'$ using Lemma 3.6.9. The path

$$b \twoheadrightarrow d \twoheadrightarrow d'$$

is a δ-reduction of b, and $1_{b'}$ is a δ-reduction of b', since b' is δ-reduced.

(4) Since the paths

$$\begin{array}{c} a_1 \longrightarrow\!\!\!\!\rightarrow c \\ b \xrightarrow{D_b}\!\!\!\!\rightarrow b' \end{array}$$

consist of identities and δ-prime edges, we can apply Lemma 3.7.10 to each of them to obtain the $(0^X, \delta)$-reductions R_1 and R_3, respectively.

(5) By Lemma 3.2.12, c has $\mathsf{size} - \mathsf{rank}$ less than that of a. The induction hypothesis applies to the path $c \longrightarrow\!\!\!\!\rightarrow d$ to yield the $(0^X, \delta)$-reduction R_2.

The bottom concatenated path

$$R = (R_3, R_2, R_1) : a' \longrightarrow\!\!\!\!\rightarrow b'$$

is a $(0^X, \delta)$-reduction of f. This finishes the induction. $\square$

Remark 3.7.17. The reason that Lemmas 3.7.10 and 3.7.15 are not proved by one overall induction is that in the diagram (3.7.16), b may have the same $\mathsf{size} - \mathsf{rank}$ as a by Lemma 3.2.16. If the two inductions are combined into one, then we would have to separately consider the case

$$\mathsf{size}(a) - \mathsf{rank}(a) = \mathsf{size}(b) - \mathsf{rank}(b)$$

with f as in Lemma 3.2.16 (2). The argument involved would be no easier than the proof of Lemma 3.7.15. $\diamond$

Next is the preliminary case of Proposition 3.7.19 involving $\lambda^{-\otimes}$ and $\rho^{-\otimes}$.

Lemma 3.7.18. *Under the hypotheses of Definition 3.7.1, suppose that each prime edge in $P : a \longrightarrow\!\!\!\!\rightarrow b$ involves a single instance of $\lambda^{-\otimes}$ or $\rho^{-\otimes}$. Then P has a $(0^X, \delta)$-reduction.*

Proof. Suppose $Q : b \longrightarrow\!\!\!\!\rightarrow a$ is the formal inverse of P as in Definition 3.1.10. Each prime edge in Q involves either $\lambda^{\otimes}$ or $\rho^{\otimes}$. By Lemma 3.7.15, Q has a $(0^X, \delta)$-reduction $S : b' \longrightarrow\!\!\!\!\rightarrow a'$, making the diagram

$$\begin{tikzcd} a \arrow[d, "D_a"'] & b \arrow[l, "Q"'] \arrow[d, "D_b"] \\ a' & b' \arrow[l, "S"] \end{tikzcd}$$

commutative. The formal inverse $R : a' \longrightarrow\!\!\!\!\rightarrow b'$ of S is a $(0^X, \delta)$-reduction of P because

$$\varphi(P) = \varphi(Q)^{-1} \quad \text{and} \quad \varphi(R) = \varphi(S)^{-1}$$

for the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ in Definition 3.1.14. $\square$

Next is the main result of this section.

Proposition 3.7.19. *Under the hypotheses of Definition 3.7.1, the path $P : a \longrightarrow\!\!\!\!\rightarrow b$ has a $(0^X, \delta)$-reduction.*

Proof. By the reasoning in the paragraph containing (3.7.11), it suffices to show that each prime edge in P has a $(0^X, \delta)$-reduction. By assumption, each prime edge $f : c \longrightarrow d$ in P has the property that c and d contain no 0^X. The possibilities of these 0^X-free edges are listed in Explanation 3.4.2. By Lemmas 3.7.10, 3.7.15, and 3.7.18, each prime edge in P has a $(0^X, \delta)$-reduction. $\square$

3.8. Reduction of Multiplicative Units

In Proposition 3.7.19, we showed that each path whose vertices contain no 0^X has a $(0^X, \delta)$-reduction. In this section, we go one step further and eliminate the multiplicative units; see Proposition 3.8.14. In the first half of this section, we define a 1^X-reduction of an element in X^{fr}, and prove its existence and uniqueness in a suitable sense. In the second half of this section, we define a 1^X-reduction of a path and establish its existence under suitable conditions. Recall that Convention 3.3.1 is in effect.

Next is the multiplicative unit analogue of a 0^X-reduction and a δ-reduction in Definitions 3.3.4 and 3.6.2.

Definition 3.8.1. Consider the graph $\mathsf{Gr}(X)$ of X in Definition 3.1.9.

- A 1^X*-prime edge* is a prime edge that involves either $\lambda^{\otimes}$ or $\rho^{\otimes}$.
- An element $a \in X^{\mathsf{fr}}$ is 1^X*-reduced* if it is not the domain of any 1^X-prime edge.
- A 1^X*-reduction* of $a \in X^{\mathsf{fr}}$ is a path $P : a \twoheadrightarrow b$ such that the following two statements hold.
  - b is 1^X-reduced.
  - Each edge in P is either an identity or a 1^X-prime edge. ◇

Example 3.8.2. Here are some examples of 1^X-prime edges.

- In Example 3.1.12, $1_w \oplus \lambda^{\otimes}_x 1_{y\oplus z}$ is a 1^X-prime edge.
- In Lemma 3.6.12, if f is a 1^X-prime edge, then every edge in the path R is a 1^X-prime edge.
- In the diagram (3.7.16), every edge in the path $c \twoheadrightarrow d$ is a 1^X-prime edge. ◇

Example 3.8.3. The domain of a 1^X-prime edge must contain at least one instance of 1^X. Therefore, each element in X^{fr} that contains no 1^X is 1^X-reduced. However, the converse is not true. For example, if $x, y \in X$ with $y \neq 1^X$, then the elements

$$1^X \oplus x \quad \text{and} \quad y \otimes (x \oplus 1^X)$$

contain 1^X and are 1^X-reduced. ◇

Recall the concept of a polynomial in Definition 3.6.3. The following result provides a characterization of a δ-reduced element in X^{fr} that is also 1^X-reduced.

Lemma 3.8.4. *Suppose $a \in X^{\mathsf{fr}}$ is δ-reduced. Then a is 1^X-reduced if and only if it is a polynomial with each monomial either equal to 1^X or containing no instances of 1^X.*

Proof. By Lemma 3.6.5, being δ-reduced is equivalent to being a polynomial. If each monomial in a is either 1^X or contains no 1^X, then a is 1^X-reduced by the following two facts:

- The domain of $\lambda^{\otimes}$ contains an expression of the form $1^X \otimes ?$.
- The domain of $\rho^{\otimes}$ contains an expression of the form $? \otimes 1^X$.

Conversely, suppose it is not the case that each monomial in a is either 1^X or contains no 1^X. Then a contains a monomial with at least two elements in X that contains 1^X. Therefore, a is the domain of some 1^X-prime edge. □

Next we establish the existence of 1^X-reductions.

Lemma 3.8.5. *Each element in X^{fr} has a 1^X-reduction.*

Proof. We proceed by induction on $\mathsf{norm}(a)$ for elements $a \in X^{\mathsf{fr}}$, where norm is as in Definition 3.2.1. If $\mathsf{norm}(a) = 1$, then $a \in X$. So a is 1^X-reduced, and the identity 1_a is a 1^X-reduction of a.

For the induction step, suppose $\mathsf{norm}(a) > 0$. If a is 1^X-reduced, then the identity 1_a is a 1^X-reduction of a. If a is not 1^X-reduced, then there exists a 1^X-prime edge $f : a \longrightarrow b$ with domain a. Since an instance of 1^X is removed when going from a to b, the inequality

$$\mathsf{norm}(a) > \mathsf{norm}(b)$$

holds. So the induction hypothesis applies to b to yield a 1^X-reduction $P : b \twoheadrightarrow b'$ of b. The combined path

$$a \xrightarrow{f} b \overset{P}{\twoheadrightarrow} b'$$

is a 1^X-reduction of a. This finishes the induction. $\square$

Next is the 1^X-reduction analogue of Lemma 3.3.11. It says that for a δ-reduced element in X^{fr}, the codomain of a 1^X-reduction is uniquely determined.

Lemma 3.8.6. *For any two 1^X-reductions*

$$\begin{tikzcd} & b_1 \\ a \arrow[ur, bend left, two heads, "P_1"] \arrow[dr, bend right, two heads, "P_2"'] & \\ & b_2 \end{tikzcd}$$

of a δ-reduced element $a \in X^{\mathsf{fr}}$, the equality $b_1 = b_2$ holds.

Proof. The element a, being δ-reduced, is a polynomial by Lemma 3.6.5. Each 1^X-prime edge in each P_i eliminates one copy of 1^X in a monomial that contains at least one other element in X. So both b_1 and b_2 are still polynomials, hence δ-reduced, and 1^X-reduced. By Lemma 3.8.4, the following statements hold.

- If a monomial in a is either a single copy of 1^X or contains no 1^X, then it is still in b_i.
- If a monomial in a contains at least two copies of 1^X and no other elements of X, then it becomes 1^X in b_i.
- If a monomial in a contains at least one copy of 1^X and at least one element in $X \setminus \{1^X\}$, then the corresponding monomial in b_i has all the copies of 1^X removed.

Therefore, $b_1 = b_2$. $\square$

Next is the 1^X-reduction analogue of Lemma 3.3.12. Recall the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ in Definition 3.1.14, with respect to which values in C are defined.

Lemma 3.8.7. *Any two 1^X-reductions*

$$\begin{tikzcd} a \arrow[r, bend left, two heads, "P_1"] \arrow[r, bend right, two heads, "P_2"'] & b \end{tikzcd}$$

of a δ-reduced element $a \in X^{\mathsf{fr}}$ have the same value in C.

Proof. The element a, being δ-reduced, is a polynomial by Lemma 3.6.5. Suppose the polynomial a has the form

$$a = a^1 \oplus \cdots \oplus a^m$$

for some $m \geq 1$, with each a^j a monomial, and for some additive bracketing. The polynomial b has the form

$$b = b^1 \oplus \cdots \oplus b^m$$

with the same additive bracketing as a and with each monomial b^j obtained from a^j as described in the proof of Lemma 3.8.6. In particular, each monomial b^j is 1^X or contains no 1^X.

By the functoriality of $\oplus$ in C, for each $i = 1, 2$, the 1^X-reduction P_i has value in C a sum

$$\varphi P_i = P_i^1 \oplus \cdots \oplus P_i^m$$

with the same additive bracketing as a and with

$$\varphi a^j \xrightarrow{P_i^j} \varphi b^j$$

for $1 \leq j \leq m$. To show that P_1 and P_2 have the same values in C, it suffices to show the equality

$$P_1^j = P_2^j : \varphi a^j \longrightarrow \varphi b^j.$$

By the proof of Lemma 3.8.6, P_i^j is a composite of morphisms, each being a $\otimes$ of identity morphisms and a component of either $\lambda^{\otimes}$ or $\rho^{\otimes}$. Mac Lane's Coherence Theorem 1.3.3 in the monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes})$ implies that $P_1^j = P_2^j$ for each $1 \leq j \leq n$. □

We now define the 1^X analogue of a $(0^X, \delta)$-free path in Definition 3.6.10.

Definition 3.8.8. A 1^X-*free path* is a $(0^X, \delta)$-free path that contains no $\lambda^{\pm\otimes}$ and $\rho^{\pm\otimes}$. ◇

Explanation 3.8.9. In a 1^X-free path, every prime edge is either an identity or involves a single instance of $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{\pm\otimes}$. ◇

Motivation 3.8.10. In the next definition, the $(0^X, \delta)$-free path P should be thought of as a $(0^X, \delta)$-reduction of some other path as in Definition 3.7.1. The aim is to further eliminate multiplicative units, thereby replacing P by a 1^X-free path. ◇

Definition 3.8.11. Suppose

- $P : a \twoheadrightarrow b$ is a $(0^X, \delta)$-free path, and
- for each $x \in \{a, b\}$, $L_x : x \twoheadrightarrow x'$ is a 1^X-reduction of x.

A 1^X-*reduction* of (P, L_a, L_b) is a 1^X-free path $R : a' \twoheadrightarrow b'$ such that the diagram

$$\begin{tikzcd} a \arrow[r, two heads, "P"] \arrow[d, two heads, "L_a"'] & b \arrow[d, two heads, "L_b"] \\ a' \arrow[r, two heads, "R"] & b' \end{tikzcd} \tag{3.8.12}$$

is commutative in the sense of Definition 3.1.14. We also call R a 1^X-reduction of P, suppressing L_a and L_b from the notation. ◇

The main result of this section, Proposition 3.8.14, asserts the existence of a 1^X-reduction when a and b are δ-reduced. First we need the following preliminary observation.

Lemma 3.8.13. *Suppose:*

- *$f : a \longrightarrow b$ is a $(0^X, \delta)$-free edge whose domain a is δ-reduced.*
- *For each $x \in \{a, b\}$, $L_x : x \twoheadrightarrow x'$ is a 1^X-reduction of x.*
- *If f contains $\lambda^{-\otimes}$ or $\rho^{-\otimes}$, then its subscript is a monomial or a factor of a monomial in a.*

Then a 1^X-reduction of f exists.

Proof. By Lemma 3.6.5, a is a polynomial as in Definition 3.6.3. The possibilities of the $(0^X, \delta)$-free edge f in Explanation 3.6.11 and the last assumption on f imply that b is also a polynomial and δ-reduced. By the proof of Lemma 3.8.6, the following statements hold:

- There is a canonical bijection between the set of monomials in a and the set of monomials in a'. Each monomial in a' is either equal to one in a, or is obtained from one in a by removing copies of 1^X. If the monomial started with some element in $X \setminus \{1^X\}$, then all the copies of 1^X are removed. If the monomial only has copies of 1^X, then all but one copies are removed.
- The additive bracketing of a' is equal to that of a.
- The analogous statements also hold for b and b'.

We now consider the possible cases of f.

(1) Suppose f is an identity or involves an instance of $\alpha^{\pm\oplus}$ or $\xi^{\pm\oplus}$. By the remarks above, the monomials in a and b are the same, and the monomials in a' and b' are the same. There is a prime edge $f' : a' \longrightarrow b'$ of the same type as f whose subscripts are the corresponding monomials in a'. The path $R = f'$ is a 1^X-reduction of f because the diagram (3.8.12) is commutative by
   - the naturality of $\alpha^{\oplus}$ or $\xi^{\oplus}$ and
   - Lemma 3.8.7 applied to each monomial in a.

(2) Suppose f involves an instance of $\alpha^{\pm\otimes}$. Then a and b differ only by the multiplicative bracketing within a single monomial, say m_a in a and m_b in b. Denote by $m_{a'}$ and $m_{b'}$ the corresponding monomials in a' and b', respectively. Then a' and b' differ by at most the multiplicative bracketings in $m_{a'}$ and $m_{b'}$. Define $R : a' \longrightarrow b'$ as consisting of a single prime edge that is either an identity or contains an instance of $\alpha^{\pm\otimes}$ that moves the multiplicative brackets from $m_{a'}$ to $m_{b'}$ in precisely the same way as f with copies of 1^X removed, as described earlier in this proof. The diagram (3.8.12) is commutative by
   - the functoriality of $\oplus$ and
   - the uniqueness in Mac Lane's Coherence Theorem 1.3.3 in $(\mathsf{C}, \otimes)$ applied to each monomial in a.

(3) Suppose f involves an instance of $\xi^{\pm\otimes}$. Then a and b differ only by the multiplicative bracketing, and the order of the factors within a single monomial. We slightly modify the argument in the previous case, using instead the uniqueness in the Symmetric Coherence Theorem 1.3.8 in the symmetric monoidal category $(\mathsf{C}, \otimes)$. The path $R : a' \longrightarrow b'$ consists

of a single prime edge that is either an identity or contains an instance of $\xi^{\pm\otimes}$.

(4) If f involves an instance of $\lambda^{\otimes}$ or $\rho^{\otimes}$, then the concatenated path

$$a \xrightarrow{f} b \xrightarrow{L_b} b'$$

is a 1^X-reduction of a. There is an equality $a' = b'$, and the paths L_a and (L_b, f) have the same values in C by Lemmas 3.8.6 and 3.8.7. Therefore, with $R = 1_{a'}$ the diagram (3.8.12) is commutative.

(5) Suppose f involves an instance of $\lambda^{-\otimes}$ or $\rho^{-\otimes}$ with subscript a monomial or a factor of a monomial in a. Then a and b only differ by an extra copy of 1^X in a monomial in b. This implies that $a' = b'$. With $R = 1_{a'}$ the diagram (3.8.12) is commutative by
- the functoriality of $\oplus$ and
- the uniqueness in Mac Lane's Coherence Theorem 1.3.3 in $(\mathsf{C}, \otimes)$ applied to each monomial in a.

The above cases account for all the possibilities of f. □

The following main result of this section is a 1^X-reduction analogue of Propositions 3.5.32 and 3.7.19, which assert the existence of a 0^X-reduction and a $(0^X, \delta)$-reduction, respectively, under suitable conditions. Recall from Lemma 3.6.5 that an element in X^{fr} is δ-reduced if and only if it is a polynomial.

Proposition 3.8.14. *Suppose*
- *$P : a \twoheadrightarrow b$ is a $(0^X, \delta)$-free path with a and b both δ-reduced, and*
- *for each $x \in \{a, b\}$, $L_x : x \twoheadrightarrow x'$ is a 1^X-reduction of x.*

Then a 1^X-reduction of P exists.

Proof. Since P contains no δ-prime edges, and since a and b are δ-reduced,
- the naturality of $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, and $\xi^{\otimes}$, and
- the Symmetric Coherence Theorem 1.3.8 for $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$

imply the following statement:

> P has the same value in C as a $(0^X, \delta)$-free path $Q : a \twoheadrightarrow b$ that does *not* contain $\lambda^{-\otimes}$ and $\rho^{-\otimes}$ with subscript a sum.

For example, for elements $x, y \in X^{\mathsf{fr}}$, the two paths below have the same value in C by naturality and coherence as stated above.

$$\begin{tikzcd}
x \oplus y \arrow[rr, "\xi^{\oplus}_{x,y}"] \arrow[d, "\lambda^{-\otimes}_{x\oplus y}"'] & & y \oplus x \\
1^X(x \oplus y) \arrow[d, "1_{1^X}\xi^{\oplus}_{x,y}"'] & & 1^X(y \oplus x) \arrow[u, "\lambda^{\otimes}_{y\oplus x}"'] \\
1^X(y \oplus x) \arrow[d, "\rho^{-\otimes}_{1^X(y\oplus x)}"'] & & (1^X 1^X)(y \oplus x) \arrow[u, "\lambda^{\otimes}_{1^X} 1_{y\oplus x}"'] \\
{[1^X(y \oplus x)]\, 1^X} \arrow[rr, "\xi^{\otimes}_{1^X(y\oplus x),1^X}"] & & 1^X\, [1^X(y \oplus x)] \arrow[u, "\alpha^{-\otimes}_{1^X,1^X,y\oplus x}"']
\end{tikzcd}$$

Therefore, we may assume that P does not contain $\lambda^{-\otimes}$ and $\rho^{-\otimes}$ with subscript a sum.

Suppose the path P consists of the prime edges $(f_n, \ldots, f_1)$ as in

$$a = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} a_n = b$$

for some $n \geq 1$. Observe the following:

- Since a is a polynomial, the reduction in the previous paragraph implies that each vertex in P is a polynomial, that is, δ-reduced.
- If $n > 1$, then for each $1 \leq i \leq n-1$, choose a 1^X-reduction $L_i : a_i \twoheadrightarrow a_i'$ of a_i, which exists by Lemma 3.8.5. Then for each $1 \leq i \leq n$, the prime edge f_i has a 1^X-reduction $R_i : a_{i-1}' \twoheadrightarrow a_i'$ by the reduction in the previous paragraph and Lemma 3.8.13.

The concatenated path

$$R = (R_n, \ldots, R_1)$$

is a 1^X-reduction of P because the diagram

$$\begin{tikzcd}
a = a_0 \arrow[r, "f_1"] \arrow[d, two heads, "L_a"'] & a_1 \arrow[r, "f_2"] \arrow[d, two heads, "L_1"] & \cdots \arrow[r, "f_n"] & a_n = b \arrow[d, two heads, "L_b"] \\
a' = a_0' \arrow[r, two heads, "R_1"] & a_1' \arrow[r, two heads, "R_2"] & \cdots \arrow[r, two heads, "R_n"] & a_n' = b'
\end{tikzcd}$$

is commutative in the sense of Definition 3.1.14. □

3.9. The First Coherence Theorem

In this section, we use the results in the previous sections to prove the following coherence theorem for symmetric bimonoidal categories. Some examples of symmetric bimonoidal categories where this theorem can be applied are given after the proof. See Section 3.11 for further discussion about this theorem. The statement of the following theorem uses Definitions 2.1.2, 3.1.9, 3.1.14, 3.1.25, and 3.6.2 and Convention 3.3.1.

Theorem 3.9.1 (Laplaza's First Coherence)**.** *Suppose* C *is a symmetric bimonoidal category in which the value of each* δ*-prime edge is a monomorphism. If*

$$\begin{tikzcd}
a \arrow[r, bend left, "P_1"] \arrow[r, bend right, "P_2"'] & b
\end{tikzcd}$$

are two paths in $\mathsf{Gr}(X)$ *with* $a \in X^{\mathsf{fr}}$ *regular, then the values of* P_1 *and* P_2 *in* C *are equal.*

Proof. This proof involves a series of reductions. Since a is regular, the existence of a path from a to b implies that b is also regular by Lemma 3.1.29.

Reduction step 1

Choose 0^X-reductions $Q_x : x \twoheadrightarrow x'$ of $x \in \{a, b\}$, which exist by Lemma 3.3.6. The codomain x' and the value of the path Q_x in C are both uniquely determined by x by Lemmas 3.3.11 and 3.3.12. Since each $x \in \{a, b\}$ is regular, by Lemma 3.1.29

each x' is regular. By Proposition 3.5.32, each path P_i for $i = 1, 2$ has a 0^X-reduction R_i such that the diagram

$$\begin{tikzcd} a \arrow[r, twoheadrightarrow, "P_i"] \arrow[d, "Q_a"'] & b \arrow[d, "Q_b"] \\ a' \arrow[r, twoheadrightarrow, "R_i"] & b' \end{tikzcd}$$

is commutative. Since $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\bullet}$, and $\rho^{\bullet}$ are natural isomorphisms in C, the values of Q_a and Q_b in C are isomorphisms. Therefore, to show that P_1 and P_2 have the same value in C, it suffices to show that R_1 and R_2 have the same value in C. In other words:

> *We may assume that each $P_i : a \twoheadrightarrow b$ is a 0^X-free path with a and b both 0^X-reduced and regular.*

Reduction step 2

If

$$\mathsf{supp}(a) = \mathsf{supp}(0^X),$$

then Proposition 3.5.33 implies that P_1 and P_2 have the same value in C. If

$$\mathsf{supp}(a) \neq \mathsf{supp}(0^X),$$

then a being 0^X-reduced as in Definition 3.3.4 implies that a contains no 0^X. That b is 0^X-reduced and Lemma 3.1.29(2) imply that b also contains no 0^X. In other words:

> *We may assume that each $P_i : a \twoheadrightarrow b$ is a 0^X-free path with a and b containing no 0^X and regular.*

Reduction step 3

Next suppose $D_b : b \twoheadrightarrow b'$ is a δ-reduction of b, which exists by Lemma 3.6.9. By Lemma 3.1.29, b' is regular. Observe the following:

- In each category, monomorphisms are closed under composition.
- Each edge in the δ-reduction D_b is either an identity or a δ-prime edge.
- The value of each δ-prime edge is a monomorphism in C by assumption.

These remarks imply that the value of D_b in C is a monomorphism. Therefore, it suffices to show that the two concatenated paths

$$\begin{tikzcd} a \arrow[r, twoheadrightarrow, bend left, "P_1"] \arrow[r, twoheadrightarrow, bend right, "P_2"'] & b \arrow[r, twoheadrightarrow, "D_b"] & b' \end{tikzcd} \tag{3.9.2}$$

have the same value in C. In other words:

> *We may assume that each $P_i : a \twoheadrightarrow b$ is a 0^X-free path with a and b containing no 0^X and regular, and with b δ-reduced.*

In particular, the identity 1_b is a δ-reduction of b.

Reduction step 4

Suppose $D_a : a \twoheadrightarrow a'$ is a δ-reduction of a, which exists by Lemma 3.6.9. By Lemma 3.1.29, a' is regular. By Proposition 3.7.19, each P_i has a $(0^X, \delta)$-reduction R_i such that the diagram

$$\begin{tikzcd} a \arrow[r, two heads, "P_i"] \arrow[d, two heads, "D_a"'] & b \arrow[d, two heads, "1_b"] \\ a' \arrow[r, two heads, "R_i"] & b \end{tikzcd}$$

is commutative in the sense of Definition 3.1.14. To show that P_1 and P_2 have the same value in C, it suffices to show that R_1 and R_2 have the same value in C. In other words:

> *We may assume that each $P_i : a \twoheadrightarrow b$ is a $(0^X, \delta)$-free path with a and b containing no 0^X, δ-reduced, and regular.*

Reduction step 5

Choose 1^X-reductions $L_x : x \twoheadrightarrow x'$ of $x \in \{a, b\}$, which exist by Lemma 3.8.5. By Lemma 3.1.29, each x' is regular. Since each $x \in \{a, b\}$ is δ-reduced, by Lemmas 3.8.6 and 3.8.7 the codomain x' and the value of L_x in C are uniquely determined by x. By Proposition 3.8.14, each P_i has a 1^X-reduction R_i such that the diagram

$$\begin{tikzcd} a \arrow[r, two heads, "P_i"] \arrow[d, two heads, "L_a"'] & b \arrow[d, two heads, "L_b"] \\ a' \arrow[r, two heads, "R_i"] & b' \end{tikzcd}$$

is commutative in the sense of Definition 3.1.14. Since $\lambda^{\otimes}$ and $\rho^{\otimes}$ are natural isomorphisms in C, the values of L_a and L_b in C are isomorphisms. So it suffices to show that R_1 and R_2 have the same value in C. In other words:

> *We may assume that each $P_i : a \twoheadrightarrow b$ is a 1^X-free path with a and b containing no 0^X, δ-reduced, 1^X-reduced, and regular.*

Let us explain this reduction step more explicitly.

(1) By Definition 3.1.25 and Lemmas 3.6.5 and 3.8.4, the conditions on $x \in \{a, b\}$—namely, containing no 0^X, δ-reduced, 1^X-reduced, and regular—mean that x is a polynomial as in Definition 3.6.3 such that the following statements hold.
   - (i) Each monomial in x either
     - is equal to 1^X, or
     - contains no 0^X and 1^X.
   - (ii) Different monomials in x have different supports as in (3.1.24).
   - (iii) Within each monomial in x, the factors are all distinct elements in X.

   For example, for distinct elements $p, q, r, s \in X \setminus \{0^X, 1^X\}$, the polynomial

   $$\left[1^X \oplus (pq)r\right] \oplus p(qs)$$

   satisfies all three conditions stated above.

(2) As stated in Explanation 3.8.9, that each P_i is a 1^X-free path means that each of its prime edges is either an identity or involves an instance of $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{\pm\otimes}$. It follows that each vertex in each P_i is a polynomial that satisfies (i)–(iii) in (1) above.

When applied to a polynomial, prime edges involving:

- $\alpha^{\pm\oplus}$ move additive brackets, but do not change the monomials.
- $\xi^{\pm\oplus}$ permute monomials, but do not change the monomials.
- $\alpha^{\pm\otimes}$ move multiplicative brackets within a monomial, but do not change the additive bracketing and the order of the monomials.
- $\xi^{\pm\otimes}$ permute factors within a monomial, but do not change the additive bracketing and the order of the monomials.

These remarks and the naturality of $\alpha^{\oplus}$ and $\xi^{\oplus}$ in C imply that for each $i = 1,2$, there is a diagram in $\mathsf{Gr}(X)$

(3.9.3)
$$\begin{tikzcd} a \arrow[rr, two heads, "P_i"] \arrow[dr, two heads, "P_i'"'] & & b \\ & c_i \arrow[ur, two heads, "P_i''"'] & \end{tikzcd}$$

that is commutative in the sense of Definition 3.1.14 such that the following two statements hold.

- $P_i' : a \twoheadrightarrow c_i$ consists of identities and prime edges involving $\alpha^{\pm\oplus}$ or $\xi^{\pm\oplus}$.
- $P_i'' : c_i \twoheadrightarrow b$ consists of identities and prime edges involving $\alpha^{\pm\otimes}$ or $\xi^{\pm\otimes}$.

Moreover, there is an equality

$$c_1 = c_2$$

in X^{fr} because of the following facts.

- P_1' and P_2' only move additive brackets and permute monomials.
- P_1'' and P_2'' only move multiplicative brackets and permute factors within each monomial.
- Different monomials in a have different supports by condition (1)(ii).
- If there are paths $p \twoheadrightarrow r$ and $q \twoheadrightarrow r$ for elements $p,q,r \in X^{\mathsf{fr}}$, then

  $$\mathsf{supp}(p) = \mathsf{supp}(r) = \mathsf{supp}(q)$$

  by Lemma 3.1.29.

Finally, with c denoting $c_1 = c_2$, the following two statements hold.

- The Symmetric Coherence Theorem 1.3.8 for the additive structure

  $$(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$$

  and condition (1)(ii) imply that the paths P_1' and $P_2' : a \twoheadrightarrow c$ have the same value in C.
- The Symmetric Coherence Theorem 1.3.8 for the multiplicative structure

  $$(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$$

  and condition (1)(iii), when applied to each monomial in c, imply that the paths P_1'' and $P_2'' : c \twoheadrightarrow b$ have the same value in C.

Therefore, $P_1 = (P_1'', P_1')$ and $P_2 = (P_2'', P_2')$ have the same value in C. □

Explanation 3.9.4. In Theorem 3.9.1, there are two assumptions:

(i) The value of each δ-prime edge is a monomorphism in C.
(ii) $a \in X^{\mathsf{fr}}$ is regular.

These assumptions are only used in the following ways.

(1) Assumption (i) is only used in the reduction step (3.9.2) to make sure that the δ-reduction $D_b : b \twoheadrightarrow b'$, which consists of identities and δ-prime edges, has value in C a monomorphism. By Definition 3.6.2, a δ-prime edge is an iterated $\oplus$ and $\otimes$ of identities and one elementary edge δ^l or δ^r as in Definition 3.1.6. Its value in C has the same description, with identity morphisms instead of identities. However, even though $\delta^l_{x,y,z}$ and $\delta^r_{x,y,z}$ are monomorphisms in C, without further assumptions on C, it does *not* automatically follow that morphisms such as

$$\delta^l_{x,y,z} \oplus 1_w, \qquad \delta^l_{x,y,z} \otimes 1_w, \quad \text{and} \quad (\delta^l_{x,y,z} \oplus 1_w) \otimes 1_v \tag{3.9.5}$$

are monomorphisms. Therefore, assumption (i) is imposed to ensure that the value of the δ-reduction D_b is a monomorphism in C.

(2) Even without the regularity assumption (ii), the diagrams (3.9.3) still exist, and each vertex in each path P_i still satisfies (1)(i) in the proof of Theorem 3.9.1. The regularity assumption is used to obtain conditions (1)(ii) and (1)(iii) in that proof. As explained there, these conditions then imply:
   - $c_1 = c_2$.
   - The paths P_1' and P_2' have the same value in C.
   - The paths P_1'' and P_2'' have the same value in C. ⋄

Example 3.9.6 (Tight Symmetric Bimonoidal Categories)**.** If C is a tight symmetric bimonoidal category, then Theorem 3.9.1 applies to C for each choice of a graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ as in Definition 3.1.14. Recall that *tight* means that the distributivity morphisms δ^l and δ^r are natural isomorphisms, not just natural monomorphisms. Therefore, the value of each δ-prime edge is an isomorphism, hence a monomorphism, in C. In particular, Theorem 3.9.1 applies to the tight symmetric bimonoidal categories Σ and Σ' in Propositions 2.4.8 and 2.4.23. ⋄

Example 3.9.7 (Distributive Symmetric Monoidal Categories)**.** These are also tight symmetric bimonoidal categories by Proposition 2.3.2. As explained in Example 3.9.6, Theorem 3.9.1 applies to each distributive symmetric monoidal category. In particular, it applies to

- symmetric monoidal closed categories with finite coproducts by Example 2.3.3;
- the category of modules over a commutative ring by Example 2.3.4; and
- distributive categories by Example 2.3.5. ⋄

Example 3.9.8 (Bipermutative Categories)**.** Right and left bipermutative categories in Definitions 2.5.2 and 2.5.11 are tight symmetric bimonoidal categories by Propositions 2.5.7 and 2.5.16, respectively. By Example 3.9.6, Theorem 3.9.1 applies to each right, respectively left, bipermutative category. ⋄

Definition 3.9.9. A (symmetric) bimonoidal category C is called *flat* if each iterated sum and product of a component of δ^l or δ^r with a finite number of identity morphisms is a monomorphism. ⋄

Example 3.9.10. Tight (symmetric) bimonoidal categories—that is, those with δ^l and δ^r natural isomorphisms—are flat. In particular, all the symmetric bimonoidal categories in Sections 2.3 through 2.6 are flat. Suppose C is a flat symmetric bimonoidal category. Then the morphisms in (3.9.5) and the values of δ-prime edges are monomorphisms. Therefore, Theorem 3.9.1 applies to C. ⋄

Example 3.9.11. Suppose $x, y, z \in X$ are distinct elements, and that the value of each δ-prime edge is a monomorphism in C. Then the two paths in Example 3.1.11 have the same value in C by Theorem 3.9.1 because the domain $x(y \oplus z) \oplus 0^X$ is regular, as we mentioned in Example 3.1.31. $\diamond$

Example 3.9.12. The proof of Theorem 3.9.1 will be reused in Section 4.4 to prove the second coherence theorem for symmetric bimonoidal categories. $\diamond$

Example 3.9.13. Theorem 3.9.1 will be applied many times in Chapter 5 in the construction of a right bipermutative category that is equivalent in a suitable sense to a given tight symmetric bimonoidal category. See Explanation 5.2.31, Example 5.2.32, and Lemmas 5.3.4, 5.3.7, 5.3.8, and 5.4.4. $\diamond$

3.10. Coherence of Bimonoidal Categories

In this section, we discuss the multiplicatively nonsymmetric analogue of the Coherence Theorem 3.9.1 that applies to bimonoidal categories instead of symmetric bimonoidal categories. It is also the bimonoidal analogue of Mac Lane's Coherence Theorem 1.3.3. As in the symmetric case, the Bimonoidal Coherence Theorem 3.10.7 does *not* require the invertibility of the distributivity morphisms δ^l and δ^r. Instead, it assumes a much weaker monomorphism condition.

The statement and the proof of Theorem 3.10.7 are obtained from those of Theorem 3.9.1 by

- omitting the multiplicative symmetry $\xi^{\otimes}$ and its inverse and
- using Mac Lane's Coherence Theorem 1.3.3 instead of Theorem 1.3.8 for the multiplicative structure $(\mathsf{C}, \otimes)$.

Below we will state the key definitions in the nonsymmetric context.

Convention 3.10.1. Throughout this section, suppose the triple (X, C, φ) consists of the following data.

- X is a set of formal variables with two distinguished elements $\{0^X, 1^X\}$ as in Definition 3.1.6.
- C is a bimonoidal category as in Definition 2.1.2, equipped with the graph structure in Example 3.1.5. In particular, its multiplicative structure

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes})$$

  is a monoidal category, and C satisfies the 22 Laplaza axioms excluding (2.1.5) and (2.1.19).
- $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is a function that satisfies $\varphi(0^X) = \mathbb{0}$ and $\varphi(1^X) = \mathbb{1}$. $\diamond$

Definition 3.10.2. Under Convention 3.10.1, define the following.

- In Definition 3.1.6, an elementary edge in $\mathsf{Gr}^{\mathsf{el}}(X)$ is *nonsymmetric* if it is not of the form $\xi^{\otimes}_{x,y}$ and $\xi^{-\otimes}_{x,y}$ for any $x, y \in X^{\mathsf{fr}}$.
- In Definition 3.1.8, a prime edge is *nonsymmetric* if it only involves nonsymmetric elementary edges.
- The *nonsymmetric graph of* X, which is denoted by $\mathsf{Gr}^{\mathsf{ns}}(X)$, is obtained from Definition 3.1.9 by restricting to nonsymmetric prime edges. An edge or a path in $\mathsf{Gr}^{\mathsf{ns}}(X)$ is said to be *nonsymmetric*.
- The notions of
  - the graph morphism $\varphi : \mathsf{Gr}^{\mathsf{ns}}(X) \longrightarrow \mathsf{C}$,
  - the *value* in C of a path in $\mathsf{Gr}^{\mathsf{ns}}(X)$, and

  - *commutativity* in C

  are obtained from those in Definition 3.1.14 by restricting to $\mathsf{Gr}^{\mathsf{ns}}(X)$.
- The *nonsymmetric strict* $\{\oplus,\otimes\}$*-algebra* X^{ns} is obtained from X^{st} in Definition 3.1.23 by omitting the relation

$$x \otimes y = y \otimes x \quad \text{for} \quad x, y \in X^{\mathsf{fr}}.$$

- The quotient map

$$X^{\mathsf{fr}} \xrightarrow{\mathsf{nsupp}} X^{\mathsf{ns}}$$

  is called the *nonsymmetric support*.
- An element $x \in X^{\mathsf{fr}}$ is *nonsymmetric regular* if it satisfies the conditions (i) and (iii) in Definition 3.1.25, but with
  - the nonsymmetric support instead of the support and
  - X^{ns} instead of X^{st}. ◇

Lemma 3.10.3. *The following statements hold.*

(1) *If two elements in X^{fr} have the same nonsymmetric support, then one of them is nonsymmetric regular if and only if the other one is nonsymmetric regular.*

(2) *If $x \twoheadrightarrow y$ is a nonsymmetric path in $\mathsf{Gr}^{\mathsf{ns}}(X)$, then*

$$\mathsf{nsupp}(x) = \mathsf{nsupp}(y).$$

Proof. Reuse the proof for the symmetric case in Lemma 3.1.29 by replacing support, regularity, paths, and prime edges by their nonsymmetric counterparts in Definition 3.10.2. □

Example 3.10.4.

- A path in $\mathsf{Gr}(X)$ that is
  - a 0^X-reduction as in Definition 3.3.4,
  - a δ-reduction as in Definition 3.6.2, or
  - a 1^X-reduction as in Definition 3.8.1,

  is nonsymmetric. In other words, it belongs to $\mathsf{Gr}^{\mathsf{ns}}(X)$.
- Each path in Examples 3.1.11, 3.1.12, and 3.6.7 and Motivation 3.3.9 is nonsymmetric.
- Each path in (3.6.14)–(3.6.17), (3.6.19), and (3.6.20) is nonsymmetric.

On the other hand, each of the two paths in Example 3.1.22 is *not* nonsymmetric because it involves $\xi^{\otimes}$. ◇

Example 3.10.5. Regularity implies nonsymmetric regularity. Indeed, the support $\mathsf{supp} : X^{\mathsf{fr}} \longrightarrow X^{\mathsf{st}}$ in (3.1.24) factors into two projection maps

$$X^{\mathsf{fr}} \xrightarrow{\mathsf{nsupp}} X^{\mathsf{ns}} \longrightarrow X^{\mathsf{st}}.$$

Therefore, if an element $w \in X^{\mathsf{fr}}$ is regular as in Definition 3.1.25, then it is also nonsymmetric regular as in Definition 3.10.2. The next example demonstrates the difference between regularity and its nonsymmetric analogue. ◇

Example 3.10.6. Nonsymmetric regularity does *not* imply regularity. For example, suppose $x, y \in X \smallsetminus \{0^X, 1^X\}$ are distinct elements, and

$$z = (x \otimes y) \oplus (y \otimes x) \in X^{\mathsf{fr}}.$$

Then the equality

$$\mathsf{supp}(x \otimes y) = \mathsf{supp}(y \otimes x) \in X^{\mathsf{st}}$$

implies that z is *not* regular as in Definition 3.1.25. However, since

$$\mathsf{nsupp}(x \otimes y) \neq \mathsf{nsupp}(y \otimes x) \in X^{\mathsf{ns}},$$

$z \in X^{\mathsf{fr}}$ is nonsymmetric regular in the sense of Definition 3.10.2. ⋄

For the statement of the following coherence theorem for bimonoidal categories, recall from Definition 3.6.2 that a *δ-prime edge* is a prime edge in $\mathsf{Gr}(X)$ that involves either δ^l or δ^r. Since a prime edge has at most one nonidentity elementary edge, a δ-prime edge is nonsymmetric in the sense of Definition 3.10.2.

Theorem 3.10.7 (Bimonoidal Coherence)**.** *Under Convention 3.10.1, suppose that the value of each δ-prime edge is a monomorphism. If*

$$\begin{tikzcd} a \arrow[r, bend left, "P_1"] \arrow[r, bend right, "P_2"'] & b \end{tikzcd}$$

are two paths in $\mathsf{Gr}^{\mathsf{ns}}(X)$ with $a \in X^{\mathsf{fr}}$ nonsymmetric regular, then the values of P_1 and P_2 in C are equal.

Proof. The proof of the current nonsymmetric case is obtained from the proofs of Theorem 3.9.1 and the preliminary results in the earlier sections of this chapter by the following procedure:

- We systematically remove all $\xi^{\pm\otimes}$ in Section 3.1–Section 3.9.
- We use the notions in Definition 3.10.2 in place of their counterparts in Section 3.1.
- We use Mac Lane's Coherence Theorem 1.3.3 instead of the Symmetric Coherence Theorem 1.3.8 for the monoidal category $(\mathsf{C}, \otimes)$ in
  - Lemmas 3.4.11 and 3.4.12 and Proposition 3.8.14 and
  - the second bullet item in the last paragraph in the proof of Theorem 3.9.1.

In particular, in the diagram (3.9.3), each path $P_i'' : c \twoheadrightarrow b$ consists of identities and $\alpha^{\pm\otimes}$, so it only moves multiplicative brackets within each monomial in c. This is the reason why condition (ii) in Definition 3.1.25 is not included in the definition of nonsymmetric regularity. □

Example 3.10.8. Recall from Definition 3.9.9 that a bimonoidal category C is *flat* if each iterated sum and product of a component of δ^l or δ^r with a finite number of identity morphisms is a monomorphism. Tight bimonoidal categories—that is, those with δ^l and δ^r natural isomorphisms—are flat. If C is flat, then the value of each δ-prime edge is a monomorphism, so Theorem 3.10.7 applies to C. ⋄

Example 3.10.9 (Sheet Diagrams)**.** We mentioned in Note 2.7.5 that sheet diagrams for tight bimonoidal categories are discussed in [**CDH∞**]. Just as string diagrams for monoidal categories [**JS91a**] use Mac Lane's Coherence Theorem 1.3.3, sheet diagrams in [**CDH∞**] use the Bimonoidal Coherence Theorem 3.10.7. In particular, regularity in [**CDH∞**, Def. 13] and the coherence theorem [**CDH∞**, Th. 4] may be replaced by, respectively, nonsymmetric regularity in Definition 3.10.2 and Theorem 3.10.7. See Note 7.9.2 for the use of strictification in [**CDH∞**]. ⋄

3.11. Notes

As we mentioned in the introduction of this chapter, the Coherence Theorem 3.9.1 for symmetric bimonoidal categories is due to Laplaza [**Lap72a**, Proposition 10], and the proof presented in this chapter follows the general outline of Laplaza's original proof. Here we point out some of the main differences between Laplaza's proof and this chapter, and explain how some inaccuracies in Laplaza's proof are fixed. In Note 3.11.7, we briefly discuss a possible 2-monad approach.

3.11.1 (Terminology and Notation)**.** The table below shows the correspondence of terminology and notation between this chapter and [**Lap72a**].

Definition	Concept	Laplaza [**Lap72a**]
3.1.6	$0^X, 1^X, X^{\mathsf{fr}}, \mathsf{Gr}^{\mathsf{el}}(X)$	$n, u, \underline{A}, \underline{G}$
3.1.7	$\mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$	$\underline{H}$
3.1.9	$\mathsf{Gr}(X)$	$\underline{T}$
3.1.23	X^{st}	$\underline{A}^*$
3.1.6	elementary edge $\longrightarrow$	$\rightharpoondown$
3.1.9	path $\twoheadrightarrow$ in $\mathsf{Gr}(X)$	$\multimap$
3.3.4	0^X-reduction	reduction
3.6.2	δ-reduction	rappel
3.8.1	1^X-reduction	normalization

Moreover, the concept of a graph in Definition 3.1.4 is not explicitly defined in [**Lap72a**]. ◇

3.11.2 (Level of Detail)**.** Laplaza's original proof is given partly in outline form, and some of the cases and proofs are omitted entirely. In this chapter, we provided all the detail and explained all the cases in full. For example:

- Section 3.5 and the main result Proposition 3.5.32, with about 15 pages total, correspond to [**Lap72a**, Prop. 5], which is condensed to about 2 pages there. Some cases, such as the one in Lemma 3.5.11, are not explicitly mentioned there.
- Lemma 3.6.12 corresponds to [**Lap72a**, Prop. 6], where the initial case of the induction with $\mathsf{norm}(a) = 3$ and some cases in the induction step are not explicitly mentioned.
- The proofs of the assertions in Section 3.8 are omitted in [**Lap72a**]. ◇

In this chapter, we corrected a few inaccurate statements in [**Lap72a**]. These are discussed in Notes 3.11.3 through 3.11.6 below.

3.11.3 (Zero-Reduction of Paths)**.** In the proof of [**Lap72a**, Prop. 5, page 50, last paragraph], it is stated that, in our terminology,

> if $P : a \twoheadrightarrow b$ is a 0^X-free path with $a \in X^{\mathsf{fr}}$ 0^X-reduced, then each vertex in P is 0^X-reduced.

This statement is only partially correct because it does not account for the case $a = 0^X$, which is 0^X-reduced. For example, the elementary edge

$$0^X \xrightarrow{\lambda^{-\otimes}_{0^X}} 1^X \otimes 0^X$$

is a 0^X-free path, since it does not involve $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\lambda^{\pm\bullet}$, and $\rho^{\pm\bullet}$. However, the codomain $1^X \otimes 0^X$ is not 0^X-reduced. Example 3.4.10 has another example of a 0^X-free path whose vertices other than the domain are not 0^X-reduced.

These nontrivial cases involving 0^X-free paths with domain 0^X must be taken into account in the proofs of Propositions 3.5.32 and 3.5.33. An explicit characterization of 0^X-free paths with domain 0^X is given in Lemma 3.4.9(2). Subsequently, Lemmas 3.4.11 and 3.4.12 deal with the relevant cases, which are used in the proofs of Proposition 3.5.33 and Lemma 3.4.14, respectively. ⋄

3.11.4 (Size and Rank)**.** Just before [**Lap72a**, Prop. 6], it is stated that, in our terminology, for an element $a \in X^{\mathsf{fr}}$ the following are equivalent.

(i) a is a polynomial.
(ii) The identity 1_a is a δ-reduction of a.
(iii) $\mathsf{rank}(a) = \mathsf{size}(a)$.

This is only partially correct. While (i) and (ii) are equivalent by Lemma 3.6.5, (iii) is strictly stronger than (i) and (ii). Indeed, by Proposition 3.2.15, if $\mathsf{rank}(a) = \mathsf{size}(a)$, then a is a special kind of polynomial in which each monomial is either an element in X or a product of two elements in X. On the other hand, for elements $x, y, z \in X$, the monomial

$$a = (xy)z \in X^{\mathsf{fr}}$$

has $\mathsf{rank}(a) = 6$ and $\mathsf{size}(a) = 8$. In other words, (iii) implies (i), but (i) does not imply (iii). ⋄

3.11.5 (Zero and Delta Reduction of Paths)**.** The proof of [**Lap72a**, Prop. 7], which corresponds to our Proposition 3.7.19, requires the following corrections.

(1) The incorrect identification of (ii) and (iii) discussed in Note 3.11.4 implies that the proof of [**Lap72a**, Prop. 7] should have contained the case in Lemma 3.7.9, which in turn uses Lemmas 3.7.6 and 3.7.8. As we explained with the monomial $a = (xy)z$, even if $\mathsf{size}(a) > \mathsf{rank}(a)$, the identity 1_a may still be a δ-reduction of a. In this case, there are no δ-prime edges with domain a. The induction diagram in [**Lap72a**, page 58] relies on the existence of such a δ-prime edge with domain a, which is denoted by $a \longrightarrow a_1$ there.
(2) In the proof of [**Lap72a**, Prop. 7], Laplaza first reduced to the case where the path P is a single prime edge, and then considered three separate cases, each proved by an induction. However, in the diagrams in [**Lap72a**, pages 58-59], the induction steps apply the induction hypothesis for a general path. The correct proof uses Lemmas 3.7.10, 3.7.15, and 3.7.18, each considering a path with some restrictions on its prime edges. The induction diagrams are (3.7.13), (3.7.14), and (3.7.16). ⋄

3.11.6 (Monomorphisms in Theorem 3.9.1)**.** The statement of [**Lap72a**, Prop. 10], which corresponds to Theorem 3.9.1, does not include the hypothesis that the value of each δ-prime edge is a monomorphism in C. However, this condition is actually used implicitly in the first sentence of the second paragraph in the proof of [**Lap72a**, Prop. 10]. In the proof of Theorem 3.9.1, this hypothesis is used in the reduction step in (3.9.2). This point was elaborated in Explanation 3.9.4(1). ⋄

3.11.7 (2-Monads)**.** For symmetric bimonoidal categories with δ^l and δ^r natural isomorphisms (not just monomorphisms), that is, *tight* symmetric bimonoidal categories, we mentioned in Example 3.9.6 that Theorem 3.9.1 applies. This special case of Theorem 3.9.1 is also claimed to follow from the 2-monad coherence result in [**Kel74**, pages 371–373]. This 2-monad approach has the advantage of being more conceptual in the sense that it is part of a more general 2-monad coherence result. On the other hand, there are additional issues to take into account.

First, this 2-monad approach relies on a substantial body of 2-category theory, including [**Kel74**], a number of papers in its references, and the references in those papers. Some basic 2-dimensional category theory is discussed at an elementary level in the book [**JY21**]. Coherence theory of 2-monad algebras goes beyond that book, and a systematic and highly detailed treatment comparable to that book does not exist.

Furthermore, to show that the results in [**Kel74**] are applicable to tight symmetric bimonoidal categories is itself a coherence theorem that requires a careful proof. Among other things, one has to (i) construct the relevant 2-monad and (ii) show that its pseudo algebras are precisely tight symmetric bimonoidal categories. Detailed proofs of these statements would be long and involved. An example of such a detailed proof is in [**JY21**, Ch. 9], where the authors constructed a 2-monad whose pseudo algebras are precisely cloven fibrations.

There is also a set-theoretic issue. In Theorem 3.9.1, there is no restriction on the size of the underlying category C, as long as it is a category. On the other hand, the 2-monad approach would involve the category Cat of all small categories, thereby restricting C to a small category. One may deal with this issue by assuming Grothendieck's Axiom of Universes, as in [**JY21**, Section 1.1]. Nevertheless, this is an extra set-theoretic axiom to deal with a nontrivial issue. $\diamond$

CHAPTER 4

Coherence of Symmetric Bimonoidal Categories II

This chapter contains the second coherence theorem for symmetric bimonoidal categories, which is also due to Laplaza [**Lap72b**]. As in Theorem 3.9.1, the main Coherence Theorem 4.4.3 in this chapter states that some formal diagrams commute in symmetric bimonoidal categories that satisfy a monomorphism condition. In Theorem 3.9.1, the regularity condition is imposed on the common domain of the two paths. In Theorem 4.4.3, the regularity hypothesis is replaced by an assumption about the paths themselves. Roughly speaking, this new hypothesis says that the two paths permute the formal variables involved in the same way. Theorem 4.5.8 is the multiplicatively nonsymmetric analogue of Theorem 4.4.3 for bimonoidal categories. Moreover, Theorem II.5.4.4 is a braided version of this coherence result. For open questions related to this chapter, see Question III.A.5.6.

Organization. In Section 4.1, we explain the intuition and motivate the constructions and proofs in later sections.

The new hypothesis that replaces regularity involves a certain sequence of permutations, called the *distortion* of a path. This notion is defined in two stages. In Section 4.2, we define the distortion category $\mathcal{D}$ whose morphisms encode the sequences of permutations. In Theorem 4.2.29, we carefully prove that the distortion category is a left bipermutative category in the sense of Definition 2.5.11. By Proposition 2.5.16, the distortion category is a tight symmetric bimonoidal category. The distortion category extends the left bipermutative category Σ in Definition 2.4.1. Parts of the proof of Proposition 2.4.8 are reused in the proof of Theorem 4.2.29.

In Section 4.3, we associate to each path in $\mathsf{Gr}(X)$ in Definition 3.1.9 a morphism in the distortion category. By definition this morphism is the distortion of the path; see Definition 4.3.1. The distortion is the value of a path in the sense of (3.1.18), applied to the distortion category. The rest of this section contains observations and examples that illustrate the concept of distortion.

In Section 4.4, we prove the Coherence Theorem 4.4.3. This proof reuses much of the proof of Theorem 3.9.1, in particular its five reduction steps without the regularity condition. In the proof of Theorem 3.9.1, the last part after the five reduction steps was dealt with using the regularity assumption. In the proof of Theorem 4.4.3, the last part after the reduction steps is dealt with using the distortion assumption on the paths; see Lemma 4.4.1. This section ends with examples where Theorem 4.4.3 may be applied.

In Section 4.5, we discuss the multiplicatively nonsymmetric version of Theorem 4.4.3. We first define the multiplicatively nonsymmetric analogue of the distortion category, which is called the additive distortion category and is denoted by

$\mathcal{D}^{\mathrm{ad}}$. Using $\mathcal{D}^{\mathrm{ad}}$, we define the additive distortion of a nonsymmetric path in Definition 3.10.2. The main Coherence Theorem 4.5.8 of this section states that, if two parallel nonsymmetric paths have the same additive distortion, then they have the same value in the bimonoidal category under consideration. Its proof is obtained from that of Theorem 4.4.3 by removing $\xi^{\pm\otimes}$ and using Theorem 1.3.3 instead of Theorem 1.3.8 for the multiplicative structure.

Section 4.6 provides a conceptual description of the distortion category and the additive distortion category as Grothendieck constructions over the finite ordinal category Σ. This observation is a repackaging of the definitions of $\mathcal{D}$ and $\mathcal{D}^{\mathrm{ad}}$ that provides a better perspective about the relationship between $\mathcal{D}$, $\mathcal{D}^{\mathrm{ad}}$, and Σ. The braided version for the braided distortion category is Proposition II.5.5.3.

Although we partly follow the outline of Laplaza's original proof in Theorem 4.4.3, there are some significant differences between this chapter and Laplaza's original proof. These issues are discussed in Section 4.7. In particular, we correct some inaccuracies in Laplaza's original proof; see Notes 4.7.3 and 4.7.4.

Reading Guide. As a possible alternative to reading this chapter linearly, we offer the following suggestion.

- First read Section 4.1, which motivates the distortion category, the distortion of a path, and their roles in the proof of Theorem 4.4.3.
- In Section 4.2, read Definitions 4.2.1, 4.2.7, 4.2.14, and 4.2.23, the explanation after them, and the statement of Theorem 4.2.29. Together they define the distortion category $\mathcal{D}$. Save the proofs for a second reading.
- In Section 4.3, first read Definition 4.3.1, which defines the distortion of a path. Then read Lemmas 4.3.5 and 4.3.6, which describe the images of polynomials in the distortion category, and Examples 4.3.7 and 4.3.8. Save the rest of this section for a second reading.
- In Section 4.4, first read Theorem 4.4.3, whose proof is only about half a page long. Then read Lemma 4.4.1, which is the key step where distortion replaces regularity.
- With a good grasp of the structure of the proof of Theorem 4.4.3, go back and read the proofs in Section 4.2, which mainly consist of computation involving permutations, and the second half of Section 4.3 starting with Convention 4.3.9.

As in Chapter 3, we divided the proof of Theorem 4.4.3 into a number of lemmas to clarify the overall structure of the proof and to make jumping forward and backward easier. Students are encouraged to regard the detailed proofs in Section 4.2 as exercises with full solutions.

Detail. In addition to proving Theorem 4.4.3, the detailed proofs in this chapter have the following additional purposes.

- As in Chapter 3, Laplaza's original proof of Theorem 4.4.3 contains some inaccuracies that have never been made explicit before. To pinpoint these subtle issues and discuss the corrections, it is necessary to dive into the detail. Section 4.7 contains more detailed discussion of the differences between this chapter and Laplaza's original proof and necessary corrections for the latter.
- The proof of Baez's Conjecture (Theorem 7.8.1) uses Theorem 4.4.3 a number of times.

- The proof of the Coherence Theorem II.5.4.4 of braided bimonoidal categories uses modified versions of the proofs in this chapter. A detailed treatment here will allow us to be both precise and concise at the same time in the braided case.

Concepts in Chapter 3 are also used in this chapter. Before Section 4.5, Convention 3.3.1 is in effect. In particular, C is a symmetric bimonoidal category, and X is a set with two distinguished elements 0^X and 1^X, with $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ the graph morphism in Definition 3.1.14.

4.1. Motivation

In this section, we motivate

(i) the distortion category to be defined in Section 4.2,
(ii) the distortion of a path to be defined in Section 4.3, and
(iii) the proof of Theorem 4.4.3 in Section 4.4.

The distortion category is a crucial component in the formulation and proof of Theorem 4.4.3, the main coherence result of this chapter.

The Coherence Theorem 3.9.1 says that two parallel paths in $\mathsf{Gr}(X)$ whose domain is regular have the same value in the symmetric bimonoidal category C, in which the values of δ-prime edges are assumed to be monomorphisms. The assumption is imposed on the domain and not on the two paths. The regularity assumption excludes domains such as $x \oplus x$ and $x \otimes x$, which are not regular, for any $x \in X$. However, there should be some coherence properties even for parallel paths whose domains are not regular. For example, the composite

$$x \otimes x \xrightarrow{\xi^{\otimes}_{x,x}} x \otimes x \xrightarrow{\xi^{\otimes}_{x,x}} x \otimes x$$

is equal to $1_{x \otimes x}$. We seek an alternative coherence result that imposes a suitable condition on the paths and not on the domain.

The Coherence Theorem 4.4.3 is a variant of Theorem 3.9.1 that replaces the regularity assumption on the domain with an assumption about the two paths in question. As we pointed out in Explanation 3.9.4 (2), even without the regularity assumption, the diagrams (3.9.3) still exist along with condition (1)(i) in that proof for each vertex in each path P_i. The regularity assumption is used to obtain conditions (1)(ii) and (1)(iii) there. These conditions are used to conclude that the component paths P_1' and P_2' have the same value in C, and similarly for P_1'' and P_2''.

An alternative to regularity should be a concept such that the following statements hold.

(1) The reduction steps, without the regularity condition, leading up to the diagrams (3.9.3) are still valid.
(2) From the diagram (3.9.3), the new assumption on the paths P_1 and P_2 implies that they have the same value in C.

In the diagram (3.9.3), for either path $P_1 = (P_1'', P_1')$ or $P_2 = (P_2'', P_2')$, the following statements hold.

- The elements a, b, and c_i are polynomials as in Definition 3.6.3, with each monomial either equal to 1^X or containing no 0^X and 1^X.
- The first path P_i' only moves additive brackets and permutes the set of monomials.

- The second path P_i'' only moves multiplicative brackets and permutes the factors within each monomial.

Assuming step (1) above is possible, to achieve step (2), we would want the paths P_1 and P_2 to

(i) permute the set of monomials in the same way and
(ii) permute the factors in each monomial in the same way.

Suppose

$$a = a^1 \oplus \cdots \oplus a^m \in X^{\mathsf{fr}}$$

is a polynomial with some additive bracketing and with $m \geq 1$ monomials, and each monomial

$$a^j = a_1^j \otimes \cdots \otimes a_{r_j}^j \in X^{\mathsf{fr}}$$

for $1 \leq j \leq m$ has some multiplicative bracketing and $r_j \geq 1$ factors in X.

- To record the number of monomials in a and the number of factors in each monomial, we use the finite sequence of nonnegative integers
$$\underline{r} = (r_1, \ldots, r_m).$$
- To achieve (i), we need to associate to each path P_i a permutation
$$\sigma_i \in \Sigma_m \quad \text{such that} \quad \sigma_1 = \sigma_2.$$
Here Σ_m denotes the symmetric group on m letters as in Definition 2.4.1. The permutation σ_i permutes the m monomials $a^1, \ldots, a^m$ in a, ignoring its additive bracketing.
- To achieve (ii), we need to associate to each path P_i a permutation
$$\tau_i^j \in \Sigma_{r_j} \quad \text{such that} \quad \tau_1^j = \tau_2^j$$
for each $1 \leq j \leq m$. The permutation τ_i^j permutes the r_j factors in the monomial a^j, ignoring its multiplicative bracketing.

Therefore:

- To each object $a \in X^{\mathsf{fr}}$, we want to associate a finite sequence $\underline{r}$ of integers that, if a is a polynomial, records the number of monomials and the number of factors in each monomial in a.
- To each path $P : a \twoheadrightarrow b$ in $\mathsf{Gr}(X)$, we want to associate a finite sequence of permutations
$$\big(\sigma; \tau^1, \ldots, \tau^m\big)$$
that, if a and b are polynomials with the same number of monomials, records
  - how P permutes the monomials in a to those in b via σ and
  - how P permutes the factors in each monomial in a to those in b via the τ^j.

This discussion suggests a category with

- finite sequences of nonnegative integers $\underline{r} = (r_1, \ldots, r_m)$ as objects and
- finite sequences of permutations $(\sigma; \tau^1, \ldots, \tau^m)$ as morphisms.

Furthermore, this should be a symmetric bimonoidal category whose sum $\oplus$ and product $\otimes$ correspond to those in X^{fr}. This is the distortion category $\mathcal{D}$ to be defined in Section 4.2 below. In Section 2.4, we already constructed a symmetric bimonoidal category Σ that corresponds to m and $\sigma \in \Sigma_m$ above. By Definition 2.4.1 and Proposition 2.4.8, Σ is a left bipermutative category as in Definition 2.5.11. In particular, its structure morphisms $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\xi^{\otimes}_{-,\mathbb{0}}$, $\lambda^{\bullet}$, $\rho^{\bullet}$, and δ^{l} are identities. We will show that the distortion category is a left bipermutative category that extends Σ in Theorem 4.2.29.

The Coherence Theorem 4.4.3 states that if the paths P_1 and P_2 have the same value in the distortion category, then they have the same value in the symmetric bimonoidal category C, in which the value of each δ-prime edge is a monomorphism. The proof is a precise version of the discussion above involving steps (1) and (2). Step (1) is in the proof of Theorem 4.4.3, and step (2) is in Lemma 4.4.1.

4.2. The Distortion Category

In this section, we construct the distortion category as motivated in Section 4.1, and observe that it is a left bipermutative category as in Definition 2.5.11. In Section 4.3, we will use the distortion category to define the distortion of a path in $\mathsf{Gr}(X)$ in Definition 3.1.9. The notion of the distortion of a path is part of the statement of the Coherence Theorem 4.4.3. Here is an outline of this section.

- The underlying category of the distortion category is defined in Definition 4.2.1 and verified in Lemma 4.2.5.
- The additive structure of the distortion category is defined in Definition 4.2.7 and verified in Lemma 4.2.12.
- The multiplicative structure of the distortion category is defined in Definition 4.2.14 and verified in Lemma 4.2.19.
- The rest of the left bipermutative category structure $(\lambda^{\bullet}, \rho^{\bullet}, \delta^{l}, \delta^{r})$ for the distortion category is given in Definition 4.2.23 and shown to be well defined in Lemma 4.2.25.
- Theorem 4.2.29 proves that the distortion category is a left bipermutative category.

The Underlying Category. First we define the underlying category of the distortion category. Recall that Σ_m denotes the symmetric group on m letters. Its identity permutation is denoted by id_m.

Definition 4.2.1. Define the *distortion category* $\mathcal{D}$ as follows.

Objects: An object in $\mathcal{D}$ is a finite sequence

$$\underline{r} = (r_1, \ldots, r_m)$$

with $m \geq 0$ and with each r_j for $1 \leq j \leq m$ a nonnegative integer. We call m the *length* of $\underline{r}$, which is denoted by $|\underline{r}|$. The unique sequence with length 0 is denoted by $\varnothing$.

Morphisms: Suppose $\underline{s} = (s_1, \ldots, s_n)$ is an object in $\mathcal{D}$. With $\underline{r}$ as above, the morphism set $\mathcal{D}(\underline{r}; \underline{s})$ is defined as follows.

- If $m \neq n$, then $\mathcal{D}(\underline{r}; \underline{s})$ is empty.
- If $m = n$, then $\mathcal{D}(\underline{r}; \underline{s})$ is the set of finite sequences of permutations

$$\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) \in \Sigma_m \times \Sigma_{r_1} \times \cdots \times \Sigma_{r_m}$$

such that

$$\sigma\underline{r} = \big(r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(m)}\big) = \underline{s}. \tag{4.2.2}$$

The last equality means that

$$r_{\sigma^{-1}(j)} = s_j \quad \text{for} \quad 1 \le j \le m.$$

Identities: The identity morphism of an object $\underline{r}$ as above is the sequence

$$1_{\underline{r}} = \big(\mathrm{id}_m; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}\big) \tag{4.2.3}$$

of identity permutations, with $1_\varnothing = (\mathrm{id}_0;)$.

Composition: Suppose given morphisms

$$\underline{r} \xrightarrow{\underline{\sigma}} \underline{s} \xrightarrow{\underline{\tau}} \underline{t}$$

in $\mathcal{D}$ with $\underline{\sigma}$ as above and with

$$\underline{\tau} = (\tau; \tau_1, \ldots, \tau_m) \in \Sigma_m \times \Sigma_{r_{\sigma^{-1}(1)}} \times \cdots \times \Sigma_{r_{\sigma^{-1}(m)}}.$$

Their composite

$$\Big(\ \underline{r} \xrightarrow{\underline{\tau\sigma}} \underline{t}\ \Big) \in \mathcal{D}(\underline{r}; \underline{t})$$

is defined as

$$\underline{\tau\sigma} = \big(\tau\sigma; \tau_{\sigma(1)}\sigma_1, \ldots, \tau_{\sigma(m)}\sigma_m\big) \in \Sigma_m \times \Sigma_{r_1} \times \cdots \times \Sigma_{r_m}. \tag{4.2.4}$$

This finishes the definition of the distortion category. ◇

Lemma 4.2.5. *The distortion category $\mathcal{D}$ in Definition 4.2.1 is a groupoid.*

Proof. The unity axiom of a category in Definition 1.1.1 holds because $1_{\underline{r}}$ is a sequence of identity permutations. For the associativity axiom, suppose $\underline{\sigma}$ and $\underline{\tau}$ are as in Definition 4.2.1, and $\underline{\pi} \in \mathcal{D}(\underline{t}; \underline{u})$ is given by

$$\underline{\pi} = (\pi; \pi_1, \ldots, \pi_m) \in \Sigma_m \times \Sigma_{t_1} \times \cdots \times \Sigma_{t_m}$$

with each $t_j = r_{(\tau\sigma)^{-1}(j)}$. Then

$$\begin{aligned}
\underline{\pi}(\underline{\tau\sigma}) &= \big(\pi\tau\sigma; \pi_{(\tau\sigma)(1)}\tau_{\sigma(1)}\sigma_1, \ldots, \pi_{(\tau\sigma)(m)}\tau_{\sigma(m)}\sigma_m\big)\\
&= \big(\pi\tau\sigma; \pi_{\tau(\sigma(1))}\tau_{\sigma(1)}\sigma_1, \ldots, \pi_{\tau(\sigma(m))}\tau_{\sigma(m)}\sigma_m\big)\\
&= (\underline{\pi\tau})\underline{\sigma}.
\end{aligned}$$

Therefore, $\mathcal{D}$ is a category.

To see that $\mathcal{D}$ is a groupoid, suppose

$$\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) \in \mathcal{D}(\underline{r}; \underline{s})$$

is a morphism. Then the morphism

$$\underline{\sigma}^{-1} = \Big(\sigma^{-1}; (\sigma_{\sigma^{-1}(1)})^{-1}, \ldots, (\sigma_{\sigma^{-1}(m)})^{-1}\Big) \in \mathcal{D}(\underline{s}; \underline{r})$$

is the inverse of $\underline{\sigma}$. □

Explanation 4.2.6. Consider the distortion category $\mathcal{D}$.

(1) An object $\underline{r} = (r_1, \ldots, r_m)$ may be thought of as consisting of m consecutive intervals, with the jth interval having r_j objects.

(2) A morphism $\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) \in \mathcal{D}(\underline{r}; \underline{s})$ consists of

- a permutation $\sigma \in \Sigma_m$ that permutes the m intervals and leaves the order within each interval unchanged; and
- for each $1 \le j \le m$, a permutation $\sigma_j \in \Sigma_{r_j}$ that permutes the r_j objects in the jth interval in $\underline{r}$.

We think of $\underline{s}$ as the result of first applying the permutation σ_j to the jth interval in $\underline{r}$ for each $1 \le j \le m$, followed by the permutation σ that permutes the m resulting intervals.

The identity morphisms and composition are then what one would expect using the above geometric interpretation of objects and morphisms. $\diamond$

The Additive Structure. Next we define the additive structure in the distortion category. We continue to use the notations in Definition 4.2.1. In the next two definitions, we reuse some of the structure in the left bipermutative category Σ in Definition 2.4.1 and Explanation 2.4.7.

Definition 4.2.7. Define the additive structure

$$\big(\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}\big)$$

in the distortion category $\mathcal{D}$ as follows.

The Sum: The functor

$$\mathcal{D} \times \mathcal{D} \xrightarrow{\oplus} \mathcal{D}$$

is defined as follows.

Objects: For objects $\underline{r} = (r_1, \ldots, r_m)$ and $\underline{r}' = (r_1', \ldots, r_k')$ in $\mathcal{D}$, define the object

$$\underline{r} \oplus \underline{r}' = \big(r_1, \ldots, r_m, r_1', \ldots, r_k'\big) \tag{4.2.8}$$

with length $|\underline{r}| + |\underline{r}'|$.

Morphisms: For morphisms $\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) \in \mathcal{D}(\underline{r}; \underline{s})$ and

$$\underline{\sigma}' = (\sigma'; \sigma_1', \ldots, \sigma_k') \in \Sigma_k \times \Sigma_{r_1'} \times \cdots \times \Sigma_{r_k'}$$

in $\mathcal{D}(\underline{r}'; \underline{s}')$ with $|\underline{r}'| = |\underline{s}'| = k$, define the morphism

$$\underline{\sigma} \oplus \underline{\sigma}' = \Big(\sigma \oplus \sigma'; \sigma_1, \ldots, \sigma_m, \sigma_1', \ldots, \sigma_k'\Big) \in \mathcal{D}\big(\underline{r} \oplus \underline{r}'; \underline{s} \oplus \underline{s}'\big). \tag{4.2.9}$$

Here $\sigma \oplus \sigma' \in \Sigma_{m+k}$ is their block sum in (2.4.2).

The Additive Zero: The object $\mathbb{0}$ is defined as the empty sequence $\varnothing \in \mathcal{D}$.

Associativity and Unity: The natural transformations $\alpha^{\oplus}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$, with components, respectively,

$$(\underline{r} \oplus \underline{r}') \oplus \underline{r}'' \xrightarrow{\alpha^{\oplus}_{\underline{r},\underline{r}',\underline{r}''}} \underline{r} \oplus (\underline{r}' \oplus \underline{r}'')$$
$$\varnothing \oplus \underline{r} \xrightarrow{\lambda^{\oplus}_{\underline{r}}} \underline{r}$$
$$\underline{r} \oplus \varnothing \xrightarrow{\rho^{\oplus}_{\underline{r}}} \underline{r}$$

are defined as the identities.

The Additive Symmetry: The natural transformation $\xi^{\oplus}$ has components

$$\underline{r} \oplus \underline{r}' \xrightarrow{\xi^{\oplus}_{\underline{r},\underline{r}'}} \underline{r}' \oplus \underline{r}$$

defined as

$$\xi^{\oplus}_{\underline{r},\underline{r}'} = \big(\xi^{\oplus}_{m,k}; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}, \mathrm{id}_{r'_1}, \ldots, \mathrm{id}_{r'_k}\big) \tag{4.2.10}$$

with $\xi^{\oplus}_{m,k} \in \Sigma_{m+k}$ the block permutation defined in (2.4.3).

This finishes the definition of $\big(\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}\big)$ in $\mathcal{D}$. ⋄

Explanation 4.2.11. We continue Explanation 4.2.6.

(1) The object $\underline{r} \oplus \underline{r}'$ in (4.2.8) is the concatenation of the m intervals in $\underline{r}$ with the k intervals in $\underline{r}'$.

(2) Consider the morphism $\underline{\sigma} \oplus \underline{\sigma}'$ in (4.2.9).
   - The block sum $\sigma \oplus \sigma'$ permutes
     - the m intervals in $\underline{r}$ via σ and
     - the k intervals in $\underline{r}'$ via σ'.
   - The permutations σ_j for $1 \le j \le m$ and σ'_l for $1 \le l \le k$ permute, respectively,
     - the r_j objects in the jth interval in $\underline{r}$ and
     - the r'_l objects in the lth interval in $\underline{r}'$.

   Starting with $\underline{r} \oplus \underline{r}'$, applying all the permutations σ_j and σ'_l, followed by the block sum $\sigma \oplus \sigma'$, the result is $\underline{s} \oplus \underline{s}'$.

(3) Regarding each of $\underline{r}$ and $\underline{r}'$ as a single block in $\underline{r} \oplus \underline{r}'$, the additive symmetry $\xi^{\oplus}_{\underline{r},\underline{r}'}$ in (4.2.10) swaps these two blocks and leaves the order in each block unchanged. ⋄

Recall from Definitions 1.2.1 and 1.2.18 that a permutative category means a symmetric strict monoidal category.

Lemma 4.2.12. *With the additive structure in Definition 4.2.7, $\mathcal{D}$ is a permutative category.*

Proof. We already observed in Lemma 4.2.5 that $\mathcal{D}$ is a groupoid. To show that $\oplus$ is a functor, first observe that the sum in (4.2.9) of two identity morphisms as in (4.2.3) is another identity morphism because the block sum preserves identity morphisms.

The sum in (4.2.9) preserves composition as in (4.2.4) because for morphisms

$$\begin{aligned} \underline{\tau} &= (\tau; \tau_1, \ldots, \tau_m) \in \mathcal{D}(\underline{s}; \underline{t}) \quad \text{and} \\ \underline{\tau}' &= (\tau'; \tau'_1, \ldots, \tau'_k) \in \mathcal{D}(\underline{s}'; \underline{t}'), \end{aligned}$$

there are equalities in $\mathcal{D}\big(\underline{r} \oplus \underline{r}'; \underline{t} \oplus \underline{t}'\big)$ as follows.

$$\begin{aligned} &(\underline{\tau} \oplus \underline{\tau}')(\underline{\sigma} \oplus \underline{\sigma}') \\ &= \Big(\tau \oplus \tau'; \tau_1, \ldots, \tau_m, \tau'_1, \ldots, \tau'_k\Big)\Big(\sigma \oplus \sigma'; \sigma_1, \ldots, \sigma_m, \sigma'_1, \ldots, \sigma'_k\Big) \\ &= \Big((\tau \oplus \tau')(\sigma \oplus \sigma'); \tau_{\sigma(1)}\sigma_1, \ldots, \tau_{\sigma(m)}\sigma_m, \tau'_{\sigma'(1)}\sigma'_1, \ldots, \tau'_{\sigma'(k)}\sigma'_k\Big) \\ &= \Big((\tau\sigma) \oplus (\tau'\sigma'); \tau_{\sigma(1)}\sigma_1, \ldots, \tau_{\sigma(m)}\sigma_m, \tau'_{\sigma'(1)}\sigma'_1, \ldots, \tau'_{\sigma'(k)}\sigma'_k\Big) \\ &= \big(\tau\sigma; \tau_{\sigma(1)}\sigma_1, \ldots, \tau_{\sigma(m)}\sigma_m\big) \oplus \big(\tau'\sigma'; \tau'_{\sigma'(1)}\sigma'_1, \ldots, \tau'_{\sigma'(k)}\sigma'_k\big) \\ &= (\underline{\tau\sigma}) \oplus (\underline{\tau'\sigma'}) \end{aligned}$$

Therefore, $\oplus : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$ is a functor.

The identity natural transformations $\alpha^{\oplus}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$ are well defined, with the last two by the definitions of the empty sequence $\varnothing$ and its identity morphism $1_{\varnothing} = (\mathrm{id}_0;)$. The unity axiom (1.2.2) and the pentagon axiom (1.2.3) are satisfied because each morphism involved is an identity. Therefore, $\mathcal{D}$ is a strict monoidal category.

The naturality of $\xi^{\oplus}$ in (4.2.10) means the commutativity of the following diagram for morphisms $\underline{\sigma} \in \mathcal{D}(\underline{r};\underline{s})$ and $\underline{\sigma}' \in \mathcal{D}(\underline{r}';\underline{s}')$.

$$\begin{tikzcd}
\underline{r} \oplus \underline{r}' \arrow[r, "\xi^{\oplus}_{\underline{r},\underline{r}'}"] \arrow[d, "\underline{\sigma}\oplus\underline{\sigma}'"'] & \underline{r}' \oplus \underline{r} \arrow[d, "\underline{\sigma}'\oplus\underline{\sigma}"] \\
\underline{s} \oplus \underline{s}' \arrow[r, "\xi^{\oplus}_{\underline{s},\underline{s}'}"] & \underline{s}' \oplus \underline{s}
\end{tikzcd}$$

- The respective first component of the two composites, namely
$$\xi^{\oplus}_{m,k}(\sigma \oplus \sigma') \quad \text{and} \quad (\sigma' \oplus \sigma)\xi^{\oplus}_{m,k} \in \Sigma_{m+k},$$
are both given by the permutation below for $1 \le j \le m$ and $1 \le l \le k$.
$$\begin{cases} j \longmapsto k + \sigma(j) \\ m + l \longmapsto \sigma'(l) \end{cases}$$
- In each of the two composites, the remaining components are given by the permutations
$$(\sigma_1, \ldots, \sigma_m, \sigma'_1, \ldots, \sigma'_k).$$

Therefore, $\xi^{\oplus}$ is a natural transformation.

Next we check the symmetric monoidal category axioms for $\mathcal{D}$.

- The symmetry axiom (1.2.20) holds because the block permutation $\xi^{\oplus}_{m,k} \in \Sigma_{m+k}$ has inverse $\xi^{\oplus}_{k,m}$.
- The unit axiom (1.2.21) holds because $\xi^{\oplus}_{\underline{r},\varnothing}$ is the identity morphism $1_{\underline{r}}$.
- The hexagon axiom (1.2.22) is satisfied because, with $\alpha^{\oplus}$ being the identity, block permutations (2.4.3) satisfy the same hexagon axiom, that is,

$$\xi^{\oplus}_{m+k,p} = \big(\xi^{\oplus}_{m,p} \oplus \mathrm{id}_k\big)\big(\mathrm{id}_m \oplus \xi^{\oplus}_{k,p}\big) \in \Sigma_{m+k+p}. \tag{4.2.13}$$

Therefore, $\mathcal{D}$ is a permutative category with the additive structure. □

The Multiplicative Structure. Next we define the multiplicative structure in the distortion category. We continue to reuse the structure in the left bipermutative category Σ in Definition 2.4.1 and Explanation 2.4.7.

Definition 4.2.14. Define the multiplicative structure

$$\big(\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}\big)$$

in the distortion category $\mathcal{D}$ as follows.

The Product: The functor

$$\mathcal{D} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D}$$

is defined as follows.

Objects: For objects $\underline{r} = (r_1, \ldots, r_m)$ and $\underline{r}' = (r'_1, \ldots, r'_k)$ in $\mathcal{D}$, define the object

$$\underline{r} \otimes \underline{r}' = \Big(\big\{r_j + r'_1\big\}_{j=1}^m, \ldots, \big\{r_j + r'_k\big\}_{j=1}^m\Big) \tag{4.2.15}$$

with length $|\underline{r}| \cdot |\underline{r}'|$. In this object, the $\big(j + (l-1)m\big)$th entry is $r_j + r'_l$ for each $1 \le j \le m$ and $1 \le l \le k$.

Morphisms: For morphisms $\underline{\sigma} \in \mathcal{D}(\underline{r}; \underline{s})$ and $\underline{\sigma}' \in \mathcal{D}(\underline{r}'; \underline{s}')$ as in Definition 4.2.7, define the morphism

$$\underline{\sigma} \otimes \underline{\sigma}' = \Big(\sigma \otimes \sigma'; \big\{\sigma_j \oplus \sigma'_1\big\}_{j=1}^m, \ldots, \big\{\sigma_j \oplus \sigma'_k\big\}_{j=1}^m\Big) \in \mathcal{D}\big(\underline{r} \otimes \underline{r}'; \underline{s} \otimes \underline{s}'\big) \tag{4.2.16}$$

with
- $\sigma \otimes \sigma' \in \Sigma_{mk}$ the permutation defined in (2.4.4) and
- each $\sigma_j \oplus \sigma'_l \in \Sigma_{r_j + r'_l}$ a block sum (2.4.2).

In (4.2.16), after $\sigma \otimes \sigma'$, the $\big(j + (l-1)m\big)$th entry is $\sigma_j \oplus \sigma'_l$ for each $1 \le j \le m$ and $1 \le l \le k$.

The Multiplicative Unit: The object $\mathbb{1}$ is the sequence $(0) \in \mathcal{D}$ with length 1 and entry 0.

Associativity and Unity: The natural transformations $\alpha^\otimes$, $\lambda^\otimes$, and $\rho^\otimes$, with components, respectively,

$$(\underline{r} \otimes \underline{r}') \otimes \underline{r}'' \xrightarrow{\alpha^\otimes_{\underline{r}, \underline{r}', \underline{r}''}} \underline{r} \otimes (\underline{r}' \otimes \underline{r}'')$$
$$(0) \otimes \underline{r} \xrightarrow{\lambda^\otimes_{\underline{r}}} \underline{r}$$
$$\underline{r} \otimes (0) \xrightarrow{\rho^\otimes_{\underline{r}}} \underline{r}$$

are defined as the identities.

The Multiplicative Symmetry: The natural transformation $\xi^\otimes$ has components

$$\underline{r} \otimes \underline{r}' \xrightarrow{\xi^\otimes_{\underline{r}, \underline{r}'}} \underline{r}' \otimes \underline{r}$$

defined as

$$\xi^\otimes_{\underline{r}, \underline{r}'} = \Big(\xi^\otimes_{m,k}; \big\{\xi^\oplus_{r_j, r'_1}\big\}_{j=1}^m, \ldots, \big\{\xi^\oplus_{r_j, r'_k}\big\}_{j=1}^m\Big) \tag{4.2.17}$$

with
- $\xi^\otimes_{m,k} \in \Sigma_{mk}$ the permutation defined in (2.4.5) and
- each $\xi^\oplus_{r_j, r'_l} \in \Sigma_{r_j + r'_l}$ a block permutation (2.4.3).

After $\xi^\otimes_{m,k}$, the $\big(j + (l-1)m\big)$th entry is $\xi^\oplus_{r_j, r'_l}$ for each $1 \le j \le m$ and $1 \le l \le k$.

This finishes the definition of $\big(\otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \xi^\otimes\big)$ in $\mathcal{D}$. ◇

Explanation 4.2.18. We continue Explanations 4.2.6 and 4.2.11.

(1) The object $\underline{r} \otimes \underline{r}'$ in (4.2.15) is a $k \times m$ matrix, whose rows are indexed by the k intervals in $\underline{r}'$. For each $1 \le l \le k$, the lth row $\{r_j + r'_l\}_{j=1}^m$ is obtained from $\underline{r}$ by concatenating each of its m intervals with the lth interval in $\underline{r}'$. Therefore, for each $1 \le j \le m$, the (l, j)-entry of this matrix is an interval with $r_j + r'_l$ objects.

(2) Consider the morphism $\underline{\sigma} \otimes \underline{\sigma}' \in \mathcal{D}(\underline{r} \otimes \underline{r}'; \underline{s} \otimes \underline{s}')$ in (4.2.16).

- The first entry $\sigma \otimes \sigma' \in \Sigma_{mk}$, which is defined in (2.4.4),
  - permutes the k rows of the matrix $\underline{r} \otimes \underline{r}'$ via $\sigma' \in \Sigma_k$ and
  - permutes the m columns of the matrix via $\sigma \in \Sigma_m$.
- The other entries form a $k \times m$ matrix of permutations. Its (l, j)-entry is the block sum $\sigma_j \oplus \sigma'_l \in \Sigma_{r_j + r'_l}$.

Starting with the $k \times m$ matrix $\underline{r} \otimes \underline{r}'$, first applying the block sums $\sigma_j \oplus \sigma'_l$ to its entries, and then permuting the rows and columns via $\sigma \otimes \sigma'$, the resulting $k \times m$ matrix is $\underline{s} \otimes \underline{s}'$.

(3) Consider the multiplicative symmetry $\xi^{\otimes}_{\underline{r},\underline{r}'} \in \mathcal{D}(\underline{r} \otimes \underline{r}'; \underline{r}' \otimes \underline{r})$ in (4.2.17).
- The first entry $\xi^{\otimes}_{m,k} \in \Sigma_{mk}$, which is defined in (2.4.5), transposes the $k \times m$ matrix $\underline{r} \otimes \underline{r}'$.
- The other entries form a $k \times m$ matrix of permutations. Its (l, j)-entry is the block permutation $\xi^{\oplus}_{r_j, r'_l} \in \Sigma_{r_j + r'_l}$ in (2.4.3).

Starting with the $k \times m$ matrix $\underline{r} \otimes \underline{r}'$, first applying the block permutations $\xi^{\oplus}_{r_j, r'_l}$ to its entries, and then taking the transpose $\xi^{\otimes}_{m,k}$, the resulting $m \times k$ matrix is $\underline{r}' \otimes \underline{r}$. ◇

Lemma 4.2.19. *With the multiplicative structure in Definition 4.2.14, $\mathcal{D}$ is a permutative category.*

Proof. We use the notations in Definitions 4.2.1, 4.2.7, and 4.2.14. We already observed in Lemma 4.2.5 that $\mathcal{D}$ is a groupoid.

To show that $\otimes$ is a functor, first observe that the product in (4.2.16) of two identity morphisms as in (4.2.3) is another identity morphism because

- the block sum preserves identity morphisms and
- $\mathrm{id}_m \otimes \mathrm{id}_k = \mathrm{id}_{mk}$.

The product in (4.2.16) preserves composition as in (4.2.4) because for morphisms

$$\begin{aligned} \underline{\tau} &= (\tau; \tau_1, \ldots, \tau_m) \in \mathcal{D}(\underline{s}; \underline{t}) \quad \text{and} \\ \underline{\tau}' &= (\tau'; \tau'_1, \ldots, \tau'_k) \in \mathcal{D}(\underline{s}'; \underline{t}'), \end{aligned}$$

there are equalities in $\mathcal{D}\big(\underline{r} \otimes \underline{r}'; \underline{t} \otimes \underline{t}'\big)$ as follows.

$$\begin{aligned}
&(\underline{\tau} \otimes \underline{\tau}')(\underline{\sigma} \otimes \underline{\sigma}') \\
&= \Big(\tau \otimes \tau'; \{\tau_j \oplus \tau'_1\}_{j=1}^m, \ldots, \{\tau_j \oplus \tau'_k\}_{j=1}^m\Big)\Big(\sigma \otimes \sigma'; \{\sigma_j \oplus \sigma'_1\}_{j=1}^m, \ldots, \{\sigma_j \oplus \sigma'_k\}_{j=1}^m\Big) \\
&= \Big((\tau \otimes \tau')(\sigma \otimes \sigma'); \big\{(\tau_{\sigma(j)} \oplus \tau'_{\sigma'(1)})(\sigma_j \oplus \sigma'_1)\big\}_{j=1}^m, \ldots, \big\{(\tau_{\sigma(j)} \oplus \tau'_{\sigma'(k)})(\sigma_j \oplus \sigma'_k)\big\}_{j=1}^m\Big) \\
&= \Big((\tau\sigma) \otimes (\tau'\sigma'); \big\{\tau_{\sigma(j)}\sigma_j \oplus \tau'_{\sigma'(1)}\sigma'_1\big\}_{j=1}^m, \ldots, \big\{\tau_{\sigma(j)}\sigma_j \oplus \tau'_{\sigma'(k)}\sigma'_k\big\}_{j=1}^m\Big) \\
&= \big(\tau\sigma; \tau_{\sigma(1)}\sigma_1, \ldots, \tau_{\sigma(m)}\sigma_m\big) \otimes \big(\tau'\sigma'; \tau'_{\sigma'(1)}\sigma'_1, \ldots, \tau'_{\sigma'(k)}\sigma'_k\big) \\
&= (\underline{\tau\sigma}) \otimes (\underline{\tau'\sigma'})
\end{aligned}$$

Therefore, $\otimes : \mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}$ is a functor.

To see that $\alpha^{\otimes}$ is well defined, suppose $\underline{r}'' = (r''_1, \ldots, r''_p)$ is an object in $\mathcal{D}$. Then both $(\underline{r} \otimes \underline{r}') \otimes \underline{r}''$ and $\underline{r} \otimes (\underline{r}' \otimes \underline{r}'')$ are given by the sequence

$$\begin{aligned} &\Big(\big\{r_j + r'_1 + r''_1\big\}_{j=1}^m, \ldots, \big\{r_j + r'_k + r''_1\big\}_{j=1}^m, \\ &\ldots, \big\{r_j + r'_1 + r''_p\big\}_{j=1}^m, \ldots, \big\{r_j + r'_k + r''_p\big\}_{j=1}^m\Big). \end{aligned} \tag{4.2.20}$$

In this sequence, the $\big(j+(l-1)m+(t-1)mk\big)$th entry is $r_j + r'_l + r''_t$ for $1 \le j \le m$, $1 \le l \le k$, and $1 \le t \le p$.

Next suppose

$$\underline{\sigma}'' = (\sigma''; \sigma''_1, \ldots, \sigma''_p) \in \mathcal{D}(\underline{r}''; \underline{s}'')$$

is a morphism. Then there is an equality

$$(\sigma \otimes \sigma') \otimes \sigma'' = \sigma \otimes (\sigma' \otimes \sigma'') \in \Sigma_{mkp}$$

by (2.4.9). So the first entry in $(\underline{\sigma} \otimes \underline{\sigma}') \otimes \underline{\sigma}''$ is equal to the first entry in $\underline{\sigma} \otimes (\underline{\sigma}' \otimes \underline{\sigma}'')$. For the equality of their remaining entries, we

- reuse (4.2.20) by replacing $(+, r_j, r'_l, r''_t)$ with $(\oplus, \sigma_j, \sigma'_l, \sigma''_t)$ for $1 \le j \le m$, $1 \le l \le k$, and $1 \le t \le p$; and
- use the fact that block sums of permutations is strictly associative.

Therefore, we can define $\alpha^{\otimes}$ as the identity.

The identity natural transformations $\lambda^{\otimes}$ and $\rho^{\otimes}$ are well defined by the definitions of the sequence $\mathbb{1} = (0)$ and its identity morphism

$$1_{(0)} = (\mathrm{id}_1; \mathrm{id}_0).$$

The unity axiom (1.2.2) and the pentagon axiom (1.2.3) are satisfied because each morphism involved is an identity. Therefore, $\mathcal{D}$ is a strict monoidal category with the multiplicative structure.

The naturality of $\xi^{\otimes}$ in (4.2.17) means the commutativity of the following diagram for morphisms $\sigma \in \mathcal{D}(\underline{r}; \underline{s})$ and $\sigma' \in \mathcal{D}(\underline{r}'; \underline{s}')$.

$$\begin{tikzcd}
\underline{r} \otimes \underline{r}' \arrow[r, "\xi^{\otimes}_{\underline{r},\underline{r}'}"] \arrow[d, "\underline{\sigma}\otimes\underline{\sigma}'"'] & \underline{r}' \otimes \underline{r} \arrow[d, "\underline{\sigma}'\otimes\underline{\sigma}"] \\
\underline{s} \otimes \underline{s}' \arrow[r, "\xi^{\otimes}_{\underline{s},\underline{s}'}"] & \underline{s}' \otimes \underline{s}
\end{tikzcd}$$

By (4.2.2), (4.2.16), and (4.2.17), the above diagram involves the following four morphisms in $\mathcal{D}$.

$$\begin{aligned}
\underline{\sigma} \otimes \underline{\sigma}' &= \Big(\sigma \otimes \sigma'; \{\sigma_j \oplus \sigma'_1\}_{j=1}^m, \ldots, \{\sigma_j \oplus \sigma'_k\}_{j=1}^m\Big) \\
\underline{\sigma}' \otimes \underline{\sigma} &= \Big(\sigma' \otimes \sigma; \{\sigma'_l \oplus \sigma_1\}_{l=1}^k, \ldots, \{\sigma'_l \oplus \sigma_m\}_{l=1}^k\Big) \\
\xi^{\otimes}_{\underline{r},\underline{r}'} &= \Big(\xi^{\otimes}_{m,k}; \big\{\xi^{\oplus}_{r_j, r'_1}\big\}_{j=1}^m, \ldots, \big\{\xi^{\oplus}_{r_j, r'_k}\big\}_{j=1}^m\Big) \\
\xi^{\otimes}_{\underline{s},\underline{s}'} &= \Big(\xi^{\otimes}_{m,k}; \big\{\xi^{\oplus}_{r_{\sigma^{-1}(j)}, r'_{\sigma'^{-1}(1)}}\big\}_{j=1}^m, \ldots, \big\{\xi^{\oplus}_{r_{\sigma^{-1}(j)}, r'_{\sigma'^{-1}(k)}}\big\}_{j=1}^m\Big)
\end{aligned}$$

Using (4.2.4) and interpreting $\xi^{\otimes}_{m,k}$ as taking the transpose of a matrix as in Explanation 4.2.18, there are equalities as follows.

$$\begin{aligned}
&\xi^{\otimes}_{\underline{s},\underline{s}'}(\underline{\sigma} \otimes \underline{\sigma}') \\
&= \Big(\xi^{\otimes}_{m,k}(\sigma \otimes \sigma'); \big\{\xi^{\oplus}_{r_j, r'_1}(\sigma_j \oplus \sigma'_1)\big\}_{j=1}^m, \ldots, \big\{\xi^{\oplus}_{r_j, r'_k}(\sigma_j \oplus \sigma'_k)\big\}_{j=1}^m\Big) \\
&= \Big((\sigma' \otimes \sigma)\xi^{\otimes}_{m,k}; \big\{(\sigma'_1 \oplus \sigma_j)\xi^{\oplus}_{r_j, r'_1}\big\}_{j=1}^m, \ldots, \big\{(\sigma'_k \oplus \sigma_j)\xi^{\oplus}_{r_j, r'_k}\big\}_{j=1}^m\Big) \\
&= (\underline{\sigma}' \otimes \underline{\sigma})\xi^{\otimes}_{\underline{r},\underline{r}'}
\end{aligned}$$

Therefore, $\xi^{\otimes}$ is a natural transformation.

Next we check the symmetric monoidal category axioms for $\mathcal{D}$ with the multiplicative structure.

- The symmetry axiom (1.2.20) holds because
  - the permutation $\xi^{\otimes}_{m,k} \in \Sigma_{mk}$ in (2.4.5) has inverse $\xi^{\otimes}_{k,m}$ and
  - the block permutation $\xi^{\oplus}_{r_j,r'_l} \in \Sigma_{r_j+r'_l}$ in (2.4.3) has inverse $\xi^{\oplus}_{r'_l,r_j}$.
- The unit axiom (1.2.21) holds because $\xi^{\otimes}_{\underline{r},(0)}$ is the identity morphism $1_{\underline{r}}$.

It remains to prove the hexagon axiom (1.2.22) for $\mathcal{D}$ with the multiplicative structure. Since $\alpha^{\otimes}$ is the identity, the hexagon axiom means the commutativity of the following diagram for objects $\underline{r}, \underline{r}', \underline{r}'' \in \mathcal{D}$.

(4.2.21)
$$\begin{tikzcd}
\underline{r} \otimes \underline{r}' \otimes \underline{r}'' \arrow[rr, "1_{\underline{r}} \otimes \xi^{\otimes}_{\underline{r}',\underline{r}''}"] \arrow[dr, "\xi^{\otimes}_{\underline{r}\otimes\underline{r}',\underline{r}''}"'] & & \underline{r} \otimes \underline{r}'' \otimes \underline{r}' \arrow[dl, "\xi^{\otimes}_{\underline{r},\underline{r}''} \otimes 1_{\underline{r}'}"] \\
& \underline{r}'' \otimes \underline{r} \otimes \underline{r}' &
\end{tikzcd}$$

To check that this diagram is commutative, we first compute its three morphisms using (4.2.3), (4.2.16), and (4.2.17) as follows, assuming $|\underline{r}| = m$, $|\underline{r}'| = k$, and $|\underline{r}''| = p$.

(4.2.22)
$$\begin{aligned}
\xi^{\otimes}_{\underline{r}\otimes\underline{r}',\underline{r}''} &= \Big(\xi^{\otimes}_{mk,p}; \big\{\xi^{\oplus}_{r_j+r'_1,r''_1}\big\}_{j=1}^{m}, \ldots, \big\{\xi^{\oplus}_{r_j+r'_k,r''_1}\big\}_{j=1}^{m}, \ldots, \\
&\qquad \big\{\xi^{\oplus}_{r_j+r'_1,r''_p}\big\}_{j=1}^{m}, \ldots, \big\{\xi^{\oplus}_{r_j+r'_k,r''_p}\big\}_{j=1}^{m}\Big) \\
1_{\underline{r}} \otimes \xi^{\otimes}_{\underline{r}',\underline{r}''} &= \big(\mathrm{id}_m; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}\big) \otimes \Big(\xi^{\otimes}_{k,p}; \big\{\xi^{\oplus}_{r'_l,r''_1}\big\}_{l=1}^{k}, \ldots, \big\{\xi^{\oplus}_{r'_l,r''_p}\big\}_{l=1}^{k}\Big) \\
&= \Big(\mathrm{id}_m \otimes \xi^{\otimes}_{k,p}; \big\{\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_1,r''_1}\big\}_{j=1}^{m}, \ldots, \big\{\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_k,r''_1}\big\}_{j=1}^{m}, \ldots, \\
&\qquad \big\{\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_1,r''_p}\big\}_{j=1}^{m}, \ldots, \big\{\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_k,r''_p}\big\}_{j=1}^{m}\Big) \\
\xi^{\otimes}_{\underline{r},\underline{r}''} \otimes 1_{\underline{r}'} &= \Big(\xi^{\otimes}_{m,p}; \big\{\xi^{\oplus}_{r_j,r''_1}\big\}_{j=1}^{m}, \ldots, \big\{\xi^{\oplus}_{r_j,r''_p}\big\}_{j=1}^{m}\Big) \otimes \big(\mathrm{id}_k; \mathrm{id}_{r'_1}, \ldots, \mathrm{id}_{r'_k}\big) \\
&= \Big(\xi^{\otimes}_{m,p} \otimes \mathrm{id}_k; \big\{\xi^{\oplus}_{r_j,r''_1} \oplus \mathrm{id}_{r'_1}\big\}_{j=1}^{m}, \ldots, \big\{\xi^{\oplus}_{r_j,r''_p} \oplus \mathrm{id}_{r'_1}\big\}_{j=1}^{m}, \ldots, \\
&\qquad \big\{\xi^{\oplus}_{r_j,r''_1} \oplus \mathrm{id}_{r'_k}\big\}_{j=1}^{m}, \ldots, \big\{\xi^{\oplus}_{r_j,r''_p} \oplus \mathrm{id}_{r'_k}\big\}_{j=1}^{m}\Big)
\end{aligned}$$

Using (4.2.4) and (4.2.22), the composite in the diagram (4.2.21) is as follows.

$$\begin{aligned}
&\big(\xi^{\otimes}_{\underline{r},\underline{r}''} \otimes 1_{\underline{r}'}\big)\big(1_{\underline{r}} \otimes \xi^{\otimes}_{\underline{r}',\underline{r}''}\big) \\
&= \Big(\big(\xi^{\otimes}_{m,p} \otimes \mathrm{id}_k\big)\big(\mathrm{id}_m \otimes \xi^{\otimes}_{k,p}\big); \\
&\quad \big\{\big(\xi^{\oplus}_{r_j,r''_1} \oplus \mathrm{id}_{r'_1}\big)\big(\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_1,r''_1}\big)\big\}_{j=1}^{m}, \ldots, \big\{\big(\xi^{\oplus}_{r_j,r''_1} \oplus \mathrm{id}_{r'_k}\big)\big(\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_k,r''_1}\big)\big\}_{j=1}^{m}, \ldots \\
&\quad \big\{\big(\xi^{\oplus}_{r_j,r''_p} \oplus \mathrm{id}_{r'_1}\big)\big(\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_1,r''_p}\big)\big\}_{j=1}^{m}, \ldots, \big\{\big(\xi^{\oplus}_{r_j,r''_p} \oplus \mathrm{id}_{r'_k}\big)\big(\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_k,r''_p}\big)\big\}_{j=1}^{m}\Big)
\end{aligned}$$

The hexagon axiom for

- the permutation $\xi^{\otimes}_{mk,p}$ in (2.4.10) and
- the block permutation $\xi^{\oplus}_{m+k,p}$ in (4.2.13)

implies the following equalities for $1 \le j \le m$, $1 \le l \le k$, and $1 \le t \le p$.

$$\begin{aligned}
\xi^{\otimes}_{mk,p} &= \big(\xi^{\otimes}_{m,p} \otimes \mathrm{id}_k\big)\big(\mathrm{id}_m \otimes \xi^{\otimes}_{k,p}\big)\\
\xi^{\oplus}_{r_j+r'_l,r''_t} &= \big(\xi^{\oplus}_{r_j,r''_t} \oplus \mathrm{id}_{r'_l}\big)\big(\mathrm{id}_{r_j} \oplus \xi^{\oplus}_{r'_l,r''_t}\big)
\end{aligned}$$

It follows that the composite in the diagram (4.2.21) is equal to $\xi^{\otimes}_{\underline{r}\otimes\underline{r}',\underline{r}''}$ as computed in (4.2.22). This proves the hexagon axiom for $\mathcal{D}$. Therefore, $\mathcal{D}$ is a permutative category with the multiplicative structure. $\square$

The Multiplicative Zeros and Distributivity. Next we define the rest of the left bipermutative category structure for the distortion category. The following definition and lemma use Definitions 4.2.1, 4.2.7, and 4.2.14.

Definition 4.2.23. Define the natural transformations

$$\big(\lambda^{\bullet}, \rho^{\bullet}, \delta^{l}, \delta^{r}\big)$$

for the distortion category $\mathcal{D}$ as follows.

The Multiplicative Zeros: $\lambda^{\bullet}$ and $\rho^{\bullet}$ are the identity natural transformations

$$\mathbb{0} \otimes \underline{r} \xrightarrow[=]{\lambda^{\bullet}_{\underline{r}}} \mathbb{0} \xleftarrow[=]{\rho^{\bullet}_{\underline{r}}} \underline{r} \otimes \mathbb{0}$$

for objects $\underline{r} \in \mathcal{D}$.

The Left Distributivity: δ^{l} is the identity natural transformation

$$\underline{r} \otimes (\underline{r}' \oplus \underline{r}'') \xrightarrow[=]{\delta^{l}_{\underline{r},\underline{r}',\underline{r}''}} (\underline{r} \otimes \underline{r}') \oplus (\underline{r} \otimes \underline{r}'')$$

for objects $\underline{r}, \underline{r}', \underline{r}'' \in \mathcal{D}$.

The Right Distributivity: δ^{r} is the natural transformation with components the composites

(4.2.24)
$$\begin{tikzcd}
(\underline{r} \oplus \underline{r}') \otimes \underline{r}'' \arrow[r, "\delta^{r}_{\underline{r},\underline{r}',\underline{r}''}"] \arrow[d, "\xi^{\otimes}_{\underline{r}\oplus\underline{r}',\underline{r}''}"'] & (\underline{r} \otimes \underline{r}'') \oplus (\underline{r}' \otimes \underline{r}'') \\
\underline{r}'' \otimes (\underline{r} \oplus \underline{r}') \arrow[r, "\delta^{l}_{\underline{r}'',\underline{r},\underline{r}'}", "="'] & (\underline{r}'' \otimes \underline{r}) \oplus (\underline{r}'' \otimes \underline{r}') \arrow[u, "\xi^{\otimes}_{\underline{r}'',\underline{r}} \oplus \xi^{\otimes}_{\underline{r}'',\underline{r}'}"']
\end{tikzcd}$$

for objects $\underline{r}, \underline{r}', \underline{r}'' \in \mathcal{D}$.

This finishes the definition of $(\lambda^{\bullet}, \rho^{\bullet}, \delta^{l}, \delta^{r})$ for $\mathcal{D}$. $\diamond$

Lemma 4.2.25. *The natural transformations $(\lambda^{\bullet}, \rho^{\bullet}, \delta^{l}, \delta^{r})$ in Definition 4.2.23 are well defined.*

Proof. Since $\mathbb{0}$ is the empty sequence $\varnothing \in \mathcal{D}$, which has length 0, by (4.2.15) both $\mathbb{0} \otimes \underline{r}$ and $\underline{r} \otimes \mathbb{0}$ are equal to $\varnothing$. The naturality of $\lambda^{\bullet}$ and $\rho^{\bullet}$ follows from (4.2.16) and the fact that the identity morphism $1_{\varnothing}$ is $(\mathrm{id}_0;)$.

To check that the left distributivity δ^l is well defined, first observe that (4.2.8) and (4.2.15) imply the following equalities.

$$\begin{aligned}&\underline{r}\otimes(\underline{r}'\oplus\underline{r}'')\\ &=(r_1,\ldots,r_m)\otimes\big(r_1',\ldots,r_k',r_1'',\ldots,r_p''\big)\\ &=\Big(\big\{r_j+r_1'\big\}_{j=1}^m,\ldots,\big\{r_j+r_k'\big\}_{j=1}^m,\big\{r_j+r_1''\big\}_{j=1}^m,\ldots,\big\{r_j+r_p''\big\}_{j=1}^m\Big)\\ &=(\underline{r}\otimes\underline{r}')\oplus(\underline{r}\otimes\underline{r}'')\end{aligned}\tag{4.2.26}$$

To prove the naturality of δ^l, consider morphisms $\sigma\in\mathcal{D}(\underline{r};\underline{s})$, $\sigma'\in\mathcal{D}(\underline{r}';\underline{s}')$, and $\sigma''\in\mathcal{D}(\underline{r}'';\underline{s}'')$. There is an equality of permutations

$$\sigma\otimes(\sigma'\oplus\sigma'')=(\sigma\otimes\sigma')\oplus(\sigma\otimes\sigma'')\in\Sigma_{m(k+p)}$$

because, by (2.4.2) and (2.4.4), both sides are given by the following bijection for $1\le j\le m$, $1\le l\le k$, and $1\le t\le p$.

$$\begin{cases} j+(l-1)m \longmapsto \sigma(j)+\big(\sigma'(l)-1\big)m\\ j+\big[k+(t-1)\big]m \longmapsto \sigma(j)+\big[k+(\sigma''(t)-1)\big]m\end{cases}$$

So the first entry in $\underline{\sigma}\otimes(\underline{\sigma}'\oplus\underline{\sigma}'')$ is equal to that in $(\underline{\sigma}\otimes\underline{\sigma}')\oplus(\underline{\sigma}\otimes\underline{\sigma}'')$. To see that their remaining entries are also equal, we use

- (4.2.9), (4.2.16), and
- (4.2.26) by replacing $(+,r_j,r_l',r_t'')$ with $(\oplus,\sigma_j,\sigma_l',\sigma_t'')$ for $1\le j\le m$, $1\le l\le k$, and $1\le t\le p$.

This proves that δ^l is a well-defined natural transformation.

Finally, the right distributivity δ^r in (4.2.24) is a well-defined natural transformation by

- the functoriality of $\oplus$ in Lemma 4.2.12,
- the fact that $\xi^{\otimes}$ is a natural transformation in Lemma 4.2.19, and
- the fact that $\delta^l=\mathrm{Id}$ is a natural transformation.

Therefore, $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^l, and δ^r in $\mathcal{D}$ are all well defined. □

Next we provide an explicit description of the right distributivity morphism in the distortion category.

Lemma 4.2.27. *For objects $\underline{r},\underline{r}',\underline{r}''\in\mathcal{D}$ with lengths $|\underline{r}|=m$, $|\underline{r}'|=k$, and $|\underline{r}''|=p$, the right distributivity morphism*

$$\begin{array}{c}(\underline{r}\oplus\underline{r}')\otimes\underline{r}''=\Big(\big\{r_j+r_1''\big\}_{j=1}^m,\big\{r_l'+r_1''\big\}_{l=1}^k,\ldots,\big\{r_j+r_p''\big\}_{j=1}^m,\big\{r_l'+r_p''\big\}_{l=1}^k\Big)\\ \Big\downarrow \delta^r_{\underline{r},\underline{r}',\underline{r}''}\\ (\underline{r}\otimes\underline{r}'')\oplus(\underline{r}'\otimes\underline{r}'')=\Big(\big\{r_j+r_1''\big\}_{j=1}^m,\ldots,\big\{r_j+r_p''\big\}_{j=1}^m,\big\{r_l'+r_1''\big\}_{l=1}^k,\ldots,\big\{r_l'+r_p''\big\}_{l=1}^k\Big)\end{array}$$

in (4.2.24) is given by

$$\Big((\xi^{\otimes}_{p,m}\oplus\xi^{\otimes}_{p,k})\xi^{\otimes}_{m+k,p};\big\{\mathrm{id}_{r_j+r_1''}\big\}_{j=1}^m,\big\{\mathrm{id}_{r_l'+r_1''}\big\}_{l=1}^k,\ldots,\big\{\mathrm{id}_{r_j+r_p''}\big\}_{j=1}^m,\big\{\mathrm{id}_{r_l'+r_p''}\big\}_{l=1}^k\Big).$$

Proof. The domain and the codomain of $\delta^r_{\underline{r},\underline{r}',\underline{r}''}$ are as stated by (4.2.8) and (4.2.15). Using (4.2.8), (4.2.9), and (4.2.17), we compute the two vertical morphisms in (4.2.24) as follows.

$$\begin{aligned} \xi^{\otimes}_{\underline{r}\oplus\underline{r}',\underline{r}''} &= \Big(\xi^{\otimes}_{m+k,p}; \{\xi^{\oplus}_{r_j,r''_1}\}_{j=1}^{m}, \{\xi^{\oplus}_{r'_l,r''_1}\}_{l=1}^{k}, \ldots, \{\xi^{\oplus}_{r_j,r''_p}\}_{j=1}^{m}, \{\xi^{\oplus}_{r'_l,r''_p}\}_{l=1}^{k}\Big) \\ \xi^{\otimes}_{\underline{r}'',\underline{r}} \oplus \xi^{\otimes}_{\underline{r}'',\underline{r}'} &= \big(\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,k}; \{\xi^{\oplus}_{r''_t,r_1}\}_{t=1}^{p}, \ldots, \{\xi^{\oplus}_{r''_t,r_m}\}_{t=1}^{p}, \{\xi^{\oplus}_{r''_t,r'_1}\}_{t=1}^{p}, \ldots, \{\xi^{\oplus}_{r''_t,r'_k}\}_{t=1}^{p}\big) \end{aligned}$$

The stated formula for $\delta^r_{\underline{r},\underline{r}',\underline{r}''}$ now follows from (4.2.4) and the fact that the inverse of the block permutation $\xi^{\oplus}_{u,v}$ in (2.4.3) is $\xi^{\oplus}_{v,u}$. □

Explanation 4.2.28. The permutation

$$(\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,k})\xi^{\otimes}_{m+k,p} \in \Sigma_{(m+k)p}$$

in δ^r in Lemma 4.2.27 is given by the following bijection for $1 \le j \le m$, $1 \le l \le k$, and $1 \le t \le p$.

$$\begin{cases} j + (t-1)(m+k) \longmapsto j + (t-1)m \\ l + m + (t-1)(m+k) \longmapsto l + (t-1)k + mp \end{cases}$$

This permutation rearranges p intervals of $m+k$ objects each to p intervals of m objects each followed by p intervals of k objects each. Except for a small notational difference (with n instead of k), this is the bijection in (2.4.6). A geometric description was given in Explanation 2.4.7. ◇

The Main Result. Recall the notion of a left bipermutative category in Definition 2.5.11. Proposition 2.5.16 states that each left bipermutative category is a tight symmetric bimonoidal category. By Proposition 2.4.8 and Example 2.5.17, Σ is a left bipermutative category. The following result extends that observation to the distortion category.

Theorem 4.2.29. *With the data in Definitions 4.2.7, 4.2.14, and 4.2.23, the distortion category $\mathcal{D}$ is a left bipermutative category.*

Proof. By Lemmas 4.2.12 and 4.2.19, both the additive structure and the multiplicative structure of $\mathcal{D}$ are permutative categories. By Lemma 4.2.25, the natural transformation δ^r and the identity natural transformations $\lambda^{\bullet}$, $\rho^{\bullet}$, and δ^l are well defined.

Next, for an object $\underline{r} \in \mathcal{D}$, both $\varnothing \otimes \underline{r}$ and $\underline{r} \otimes \varnothing$ are equal to $\varnothing$. It follows that

$$\xi^{\otimes}_{\underline{r},\varnothing} = (\mathrm{id}_0;) = 1_{\varnothing} : \varnothing \longrightarrow \varnothing.$$

The axiom (2.1.5) holds by the definition of δ^r in (4.2.24). It remains to check the axioms (2.1.6) and (2.1.13).

The axiom (2.1.6) states that the following diagram is commutative for objects $\underline{r}, \underline{r}', \underline{r}'' \in \mathcal{D}$.

$$\begin{tikzcd} \underline{r} \otimes (\underline{r}' \oplus \underline{r}'') \arrow[r, "\delta^l_{\underline{r},\underline{r}',\underline{r}''}", "="'] \arrow[d, "1_{\underline{r}}\otimes\xi^{\oplus}_{\underline{r}',\underline{r}''}"'] & (\underline{r} \otimes \underline{r}') \oplus (\underline{r} \otimes \underline{r}'') \arrow[d, "\xi^{\oplus}_{\underline{r}\otimes\underline{r}',\underline{r}\otimes\underline{r}''}"] \\ \underline{r} \otimes (\underline{r}'' \oplus \underline{r}') \arrow[r, "\delta^l_{\underline{r},\underline{r}'',\underline{r}'}", "="'] & (\underline{r} \otimes \underline{r}'') \oplus (\underline{r} \otimes \underline{r}') \end{tikzcd}$$

Using (2.4.3), (4.2.3), (4.2.10), and (4.2.15), the previous diagram is commutative by the following equalities.

$$\begin{aligned}
&1_{\underline{r}} \otimes \xi^{\oplus}_{\underline{r}',\underline{r}''}\\
&= \big(\mathrm{id}_m; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}\big) \otimes \big(\xi^{\oplus}_{k,p}; \mathrm{id}_{r'_1}, \ldots, \mathrm{id}_{r'_k}, \mathrm{id}_{r''_1}, \ldots, \mathrm{id}_{r''_p}\big)\\
&= \Big(\mathrm{id}_m \otimes \xi^{\oplus}_{k,p}; \big\{\mathrm{id}_{r_j} \oplus \mathrm{id}_{r'_1}\big\}_{j=1}^m, \ldots, \big\{\mathrm{id}_{r_j} \oplus \mathrm{id}_{r'_k}\big\}_{j=1}^m, \big\{\mathrm{id}_{r_j} \oplus \mathrm{id}_{r''_1}\big\}_{j=1}^m, \ldots, \big\{\mathrm{id}_{r_j} \oplus \mathrm{id}_{r''_p}\big\}_{j=1}^m\Big)\\
&= \Big(\xi^{\oplus}_{mk,mp}; \big\{\mathrm{id}_{r_j+r'_1}\big\}_{j=1}^m, \ldots, \big\{\mathrm{id}_{r_j+r'_k}\big\}_{j=1}^m, \big\{\mathrm{id}_{r_j+r''_1}\big\}_{j=1}^m, \ldots, \big\{\mathrm{id}_{r_j+r''_p}\big\}_{j=1}^m\Big)\\
&= \xi^{\oplus}_{\underline{r}\otimes\underline{r}',\underline{r}\otimes\underline{r}''}
\end{aligned}$$

In the first component, the equality of permutations

$$\mathrm{id}_m \otimes \xi^{\oplus}_{k,p} = \xi^{\oplus}_{mk,mp} \in \Sigma_{m(k+p)}$$

is the axiom (2.1.6) in Σ. We already checked this in (2.4.11) in the proof of Proposition 2.4.8. Moreover, a geometric description was given in Explanation 2.4.15.

The axiom (2.1.13) states that the following diagram is commutative for objects $\underline{a}, \underline{b}, \underline{c}, \underline{d} \in \mathcal{D}$.

$$\begin{tikzcd}
(\underline{a} \oplus \underline{b}) \otimes (\underline{c} \oplus \underline{d}) \arrow[rr, "\delta^r_{\underline{a},\underline{b},\underline{c}\oplus\underline{d}}"] \arrow[d, "\delta^l_{\underline{a}\oplus\underline{b},\underline{c},\underline{d}}"', "="] & & \underline{a} \otimes (\underline{c} \oplus \underline{d}) \oplus \underline{b} \otimes (\underline{c} \oplus \underline{d}) \arrow[d, "="', "\delta^l_{\underline{a},\underline{c},\underline{d}} \oplus \delta^l_{\underline{b},\underline{c},\underline{d}}"] \\
(\underline{a} \oplus \underline{b}) \otimes \underline{c} \oplus (\underline{a} \oplus \underline{b}) \otimes \underline{d} \arrow[dr, "\delta^r_{\underline{a},\underline{b},\underline{c}} \oplus \delta^r_{\underline{a},\underline{b},\underline{d}}"'] & & (\underline{a} \otimes \underline{c}) \oplus (\underline{a} \otimes \underline{d}) \oplus (\underline{b} \otimes \underline{c}) \oplus (\underline{b} \otimes \underline{d}) \arrow[dl, "1_{\underline{a}\otimes\underline{c}} \oplus \xi^{\oplus}_{\underline{a}\otimes\underline{d},\underline{b}\otimes\underline{c}} \oplus 1_{\underline{b}\otimes\underline{d}}"] \\
& (\underline{a} \otimes \underline{c}) \oplus (\underline{b} \otimes \underline{c}) \oplus (\underline{a} \otimes \underline{d}) \oplus (\underline{b} \otimes \underline{d}) &
\end{tikzcd} \tag{4.2.30}$$

By (4.2.3), (4.2.4), (4.2.9), (4.2.10), and Lemma 4.2.27, in each of the two composites in (4.2.30), each entry after the first one is an identity permutation. Therefore, it suffices to check the commutativity of the diagram (4.2.30) in the first entry only. In other words, by Lemma 4.2.27, it suffices to check the equality of permutations

$$\begin{aligned}
&\big[(\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,n})\xi^{\otimes}_{m+n,p}\big] \oplus \big[(\xi^{\otimes}_{q,m} \oplus \xi^{\otimes}_{q,n})\xi^{\otimes}_{m+n,q}\big]\\
&= \big[\mathrm{id}_{mp} \oplus \xi^{\oplus}_{mq,np} \oplus \mathrm{id}_{nq}\big]\big[(\xi^{\otimes}_{p+q,m} \oplus \xi^{\otimes}_{p+q,n})\xi^{\otimes}_{m+n,p+q}\big]
\end{aligned}$$

in $\Sigma_{m+n,p+q}$ if $|\underline{a}| = m$, $|\underline{b}| = n$, $|\underline{c}| = p$, and $|\underline{d}| = q$. This equality is the axiom (2.1.13) in Σ. We already checked this in (2.4.13) in the proof of Proposition 2.4.8. Moreover, a geometric description was given in Explanation 2.4.17. □

Corollary 4.2.31. *The Coherence Theorem 3.9.1 applies to the distortion category.*

Proof. The distortion category is a left bipermutative category, hence in particular a tight symmetric bimonoidal category, by Theorem 4.2.29 and Proposition 2.5.16. As explained in Example 3.9.6, Theorem 3.9.1 applies to $\mathcal{D}$. □

4.3. The Distortion of a Path

In this section, we define the graph morphism from $\mathsf{Gr}(X)$ in Definition 3.1.9 to the distortion category $\mathcal{D}$ that defines the distortion of a path in $\mathsf{Gr}(X)$. Since the distortion category $\mathcal{D}$ is a left bipermutative category by Theorem 4.2.29, it is in particular a tight symmetric bimonoidal category by Proposition 2.5.16. Recall that Convention 3.3.1 still stands in this chapter. In particular, we assume that X is a set with two distinguished elements 0^X and 1^X as in Definition 3.1.6, and X^{fr} is the free $\{\oplus,\otimes\}$-algebra of X in Definition 3.1.2. After the definition of the distortion of a path in $\mathsf{Gr}(X)$, we illustrate it with a series of observations and examples. An important special case of Corollary 4.2.31 is Corollary 4.3.12. It says that any two parallel paths in $\mathsf{Gr}(X)$ whose common domain is regular have the same distortion.

The next definition uses the left bipermutative category structure of the distortion category $\mathcal{D}$ in Definitions 4.2.1, 4.2.7, 4.2.14, and 4.2.23.

Definition 4.3.1. Consider the distortion category $\mathcal{D}$.

- Define the function $\vartheta : X \longrightarrow \mathsf{Ob}(\mathcal{D})$ as follows.

$$\vartheta(x) = \begin{cases} (1) & \text{if } x \in X \smallsetminus \{0^X, 1^X\}, \\ \varnothing & \text{if } x = 0^X, \text{ and} \\ (0) & \text{if } x = 1^X. \end{cases} \tag{4.3.2}$$

- Using the same symbol, define the associated graph morphism

$$\mathsf{Gr}(X) \xrightarrow{\ \vartheta\ } \mathcal{D} \tag{4.3.3}$$

  as in Definition 3.1.14, applied to the function in (4.3.2) and the symmetric bimonoidal category $\mathcal{D}$.
- For a path P in $\mathsf{Gr}(X)$, its value $\vartheta P \in \mathcal{D}$ in the sense of (3.1.18) is called the *distortion of P*.

This finishes the definition. $\diamond$

Explanation 4.3.4. By Definitions 4.2.7 and 4.2.14, in the distortion category $\mathcal{D}$, the empty sequence $\varnothing$ is the additive zero $\mathbb{0}$, and the sequence (0) of length 1 is the multiplicative unit $\mathbb{1}$. Therefore, Definition 3.1.14 is indeed applicable to the function ϑ in (4.3.2). Each object $a \in X^{\mathsf{fr}}$ is sent by ϑ to an object in $\mathcal{D}$, which is a finite sequence of nonnegative integers. The distortion of each path in $\mathsf{Gr}(X)$ is a morphism in $\mathcal{D}$ as in Definition 4.2.1, which is in particular a nonempty finite sequence of permutations. $\diamond$

To understand the distortion of a path, first we describe the action of ϑ on monomials and polynomials. Recall from Definition 3.6.3 that a *monomial* in X^{fr} is either an element in X, or a finite product of at least two elements in X with some multiplicative bracketing. A *polynomial* is either a monomial, or a finite sum of at least two monomials with some additive bracketing.

Lemma 4.3.5. *Suppose*

$$a = a_1 \otimes \cdots \otimes a_p \in X^{\mathsf{fr}}$$

for some multiplicative bracketing and some elements $a_j \in X^{\mathsf{fr}}$ for $1 \le j \le p$ with $p \ge 1$.

(1) $\vartheta(a) = \vartheta(a_1) \otimes \cdots \otimes \vartheta(a_p) \in \mathcal{D}$.
(2) If $a_j = 0^X$ for some $1 \le j \le p$, then

$$\vartheta(a) = \varnothing \in \mathcal{D}.$$

(3) If $a_j \in X \smallsetminus \{0^X\}$ for each $1 \le j \le p$, then

$$\vartheta(a) = (n) \in \mathcal{D},$$

where $n \ge 0$ is the number of a_j's that belong to $X \smallsetminus \{0^X, 1^X\}$.

(4) *The image of a monomial under ϑ has length either 0 or 1, depending on whether it contains 0^X or not.*

Proof. The multiplicative associativity $\alpha^{\otimes}$ in the distortion category $\mathcal{D}$ is the identity by Definition 4.2.14. Therefore, $\vartheta(a) \in \mathcal{D}$ is independent of the choice of a multiplicative bracketing of a. Assertion (1) follows from the definition (3.1.16) of the graph morphism ϑ on objects in X^{fr}.

For assertion (2), by definition $\vartheta(0^X) = \varnothing$. Moreover, $\underline{r} \otimes \varnothing$ and $\varnothing \otimes \underline{r}$ are both $\varnothing \in \mathcal{D}$ for each object $\underline{r} \in \mathcal{D}$. It follows from assertion (1) that $\vartheta(a) = \varnothing$ if $a_j = 0^X$ for some $1 \leq j \leq p$.

For assertion (3), by definition each

$$\vartheta(a_j) = (\epsilon_j) \in \mathcal{D} \quad \text{with} \quad \epsilon_j = \begin{cases} 0 & \text{if } a_j = 1^X \text{, and} \\ 1 & \text{if } a_j \in X \smallsetminus \{0^X, 1^X\}. \end{cases}$$

By assertion (1) and (4.2.15), it follows that $\vartheta(a)$ has length 1, and is given by

$$\begin{aligned} \vartheta(a_1) \otimes \cdots \otimes \vartheta(a_p) &= (\epsilon_1) \otimes \cdots \otimes (\epsilon_p) \\ &= \big(\epsilon_1 + \cdots + \epsilon_p\big). \end{aligned}$$

In other words, its only entry is the number of a_j's that belong to $X \smallsetminus \{0^X, 1^X\}$.

Assertion (4) follows from assertions (2) and (3). □

Lemma 4.3.6. *Suppose*

$$a = a^1 \oplus \cdots \oplus a^m \in X^{\mathsf{fr}}$$

for some additive bracketing and some elements $a^i \in X^{\mathsf{fr}}$ for $1 \leq i \leq m$ with $m \geq 1$. Then:

(1) $\vartheta(a) = \big(\vartheta(a^1), \ldots, \vartheta(a^m)\big) \in \mathcal{D}$.

(2) *If a^i is a monomial for each $1 \leq i \leq m$, then $\vartheta(a)$ has length $k \geq 0$, where k is the number of a^i's that contain no 0^X.*

Proof. The additive associativity $\alpha^{\oplus}$ in the distortion category $\mathcal{D}$ is the identity by Definition 4.2.7. Therefore, $\vartheta(a) \in \mathcal{D}$ is independent of the choice of an additive bracketing of a. Assertion (1) follows from (3.1.16) and (4.2.8). Assertion (2) follows from assertion (1) and Lemma 4.3.5 (4). □

In the following examples, the symbol $\otimes$ in X^{fr} is omitted.

Example 4.3.7. For objects $u, v, w, x, y, z \in X \smallsetminus \{0^X, 1^X\}$, consider the polynomial

$$a = (uv1^X) \oplus (1^X 1^X 1^X) \oplus (w1^X x 1^X y) \oplus (z0^X) \in X^{\mathsf{fr}}$$

with any additive bracketing and any multiplicative bracketing within each monomial. By Lemma 4.3.5 (2) and (3) and Lemma 4.3.6 (1),

$$\begin{aligned} \vartheta(a) &= \big(\vartheta(uv1^X), \vartheta(1^X 1^X 1^X), \vartheta(w1^X x 1^X y), \vartheta(z0^X)\big) \\ &= (2, 0, 3) \in \mathcal{D}, \end{aligned}$$

which has length 3. ◇

Example 4.3.8. For objects $r, s, t \in X \smallsetminus \{0^X, 1^X\}$, consider the polynomial

$$b = r \oplus (0^X 0^X) \oplus (st) \in X^{\mathsf{fr}}$$

with any additive bracketing. With $a \in X^{\mathsf{fr}}$ the polynomial in Example 4.3.7, there are the following objects in $\mathcal{D}$.

$$\begin{aligned}
\vartheta(b) &= \big(\vartheta(r), \vartheta(0^X 0^X), \vartheta(st)\big)\\
&= (1,2)\\
\vartheta(a \oplus b) &= \big(\vartheta(a), \vartheta(b)\big)\\
&= (2,0,3,1,2)\\
\vartheta(a \otimes b) &= \vartheta(a) \otimes \vartheta(b)\\
&= (2,0,3) \otimes (1,2)\\
&= \big(2+1, 0+1, 3+1, 2+2, 0+2, 3+2\big)\\
&= (3,1,4,4,2,5)
\end{aligned}$$

The second-to-last equality follows from the definition (4.2.15) of $\otimes$ in $\mathcal{D}$. ◇

Next we provide examples of distortions of paths.

Convention 4.3.9. For Lemma 4.3.11, below we interpret the 24 diagrams (2.1.5)–(2.1.28) in Definition 2.1.2 in $\mathsf{Gr}(X)$ as follows.

- We interpret each object there as an element in X^{fr} with $A, B, C, D \in X^{\mathsf{fr}}$, and with $\mathbb{0}$ and $\mathbb{1}$ interpreted as 0^X and 1^X, respectively.
- We interpret each morphism there as the corresponding prime edge in $\mathsf{Gr}(X)$ as in Definition 3.1.8, with one kind of exceptions as stated next.
- If a morphism is the sum of two nonidentity structure morphisms, then we interpret it as a path of length 2 in $\mathsf{Gr}(X)$ consisting of the two corresponding prime edges.

In this way, we interpret each of those 24 diagrams as consisting of two paths in $\mathsf{Gr}(X)$ with a common domain and a common codomain. ◇

Example 4.3.10. The morphism

$$AC \oplus BC \xrightarrow{\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}} CA \oplus CB$$

in (2.1.5) is interpreted as the path

$$AC \oplus BC \xrightarrow{\xi^{\otimes}_{A,C} \oplus 1_{BC}} CA \oplus BC \xrightarrow{1_{CA} \oplus \xi^{\otimes}_{B,C}} CA \oplus CB$$

of length 2 in $\mathsf{Gr}(X)$. Besides (2.1.5), such a sum of two nonidentity structure morphisms happens in (2.1.10)–(2.1.13), (2.1.15), (2.1.16), (2.1.27), and (2.1.28). ◇

Lemma 4.3.11. *Under Convention 4.3.9, in each diagram in (2.1.5)–(2.1.28), the two paths in $\mathsf{Gr}(X)$ have the same distortion.*

Proof. The assertion means that if we apply the graph morphism $\vartheta : \mathsf{Gr}(X) \longrightarrow \mathcal{D}$ in (4.3.3) to each of those 24 diagrams, then the result is a commutative diagram in $\mathcal{D}$. Therefore, the assertion follows from the fact that the distortion category $\mathcal{D}$ is a left bipermutative category, hence in particular a symmetric bimonoidal category, by Theorem 4.2.29 and Proposition 2.5.16. □

Next is Corollary 4.2.31 for the graph morphism $\vartheta : \mathsf{Gr}(X) \longrightarrow \mathcal{D}$ in (4.3.3) that defines distortion. In other words, it is the special case of the Coherence Theorem 3.9.1 for the distortion category $\mathcal{D}$ and ϑ. The following result is *not* used in the proof of the Coherence Theorem 4.4.3. Recall the notion of regularity from Definition 3.1.25.

Corollary 4.3.12. *If*

$$\begin{tikzcd} a \arrow[r, bend left, "P_1"] \arrow[r, bend right, "P_2"'] & b \end{tikzcd}$$

are two paths in $\mathsf{Gr}(X)$ *with* $a \in X^{\mathsf{fr}}$ *regular, then* P_1 *and* P_2 *have the same distortion.*

The next example is an illustration of Corollary 4.3.12.

Example 4.3.13. Consider

- *distinct* elements $a_1, a_2, a_3, b_1, b_2 \in X$;
- polynomials

$$a = (a_1 \oplus a_2) \oplus a_3 \quad \text{and} \quad b = b_1 \oplus b_2 \in X^{\mathsf{fr}};$$

  and
- the following two paths in $\mathsf{Gr}(X)$ in which most of the $\otimes$ symbols are omitted.

$$\begin{tikzcd}[column sep=small]
a \otimes b = \big((a_1 \oplus a_2) \oplus a_3\big)(b_1 \oplus b_2) \arrow[dd, "\delta^l_{a,b_1,b_2}"'] \arrow[dr, bend left, "\delta^r_{a_1 \oplus a_2, a_3, b}"] & \\
& (a_1 \oplus a_2)(b_1 \oplus b_2) \oplus a_3(b_1 \oplus b_2) \arrow[dd, "\delta^r_{a_1,a_2,b} \oplus 1_{a_3(b_1 \oplus b_2)}"] \\
\big((a_1 \oplus a_2) \oplus a_3\big)b_1 \oplus \big((a_1 \oplus a_2) \oplus a_3\big)b_2 \arrow[dd, "D^r"'] & \\
& \big(a_1(b_1 \oplus b_2) \oplus a_2(b_1 \oplus b_2)\big) \oplus a_3(b_1 \oplus b_2) \arrow[ddl, bend left, "D^l"] \\
\big((a_1b_1 \oplus a_2b_1) \oplus a_3b_1\big) \oplus \big((a_1b_2 \oplus a_2b_2) \oplus a_3b_2\big) \arrow[d, "Z"'] & \\
\big((a_1b_1 \oplus a_1b_2) \oplus (a_2b_1 \oplus a_2b_2)\big) \oplus (a_3b_1 \oplus a_3b_2) &
\end{tikzcd} \tag{4.3.14}$$

Consider the diagram (4.3.14).

- The path D^l consists of three δ-prime edges as in Definition 3.6.2, each containing one instance of $\delta^l_{a_i,b_1,b_2}$ for $1 \leq i \leq 3$. There are six possibilities of D^l depending on the order of these δ-prime edges. The path D^l may be chosen as any one of them.
- The path D^r consists of four δ-prime edges, each containing one instance of $\delta^r_{a_1 \oplus a_2, a_3, b_j}$ or $\delta^r_{a_1,a_2,b_j}$ for $j = 1, 2$. There are five possibilities of D^r depending on the order of these δ-prime edges, where $\delta^r_{a_1 \oplus a_2, a_3, b_j}$ must occur before $\delta^r_{a_1,a_2,b_j}$ for each $j = 1, 2$. The path D^r may be chosen as any one of them.
- Each prime edge in the path Z contains an instance of $\alpha^{\oplus}$ or $\xi^{\oplus}$. These prime edges move the additive brackets (for $\alpha^{\oplus}$) and permute the six monomials (for $\xi^{\oplus}$). There are infinitely many such paths with the prescribed domain and codomain, and Z may be chosen as any one of them.

Suppose

$$P_1 = \big(Z, D^r, \delta^l_{a,b_1,b_2}\big) \quad \text{and} \qquad (4.3.15)$$
$$P_2 = \big(D^l, \delta^r_{a_1,a_2,b} \oplus 1_{a_3(b_1 \oplus b_2)}, \delta^r_{a_1 \oplus a_2, a_3, b}\big)$$

are, respectively, the left path and the right path in (4.3.14). Since the elements a_1, a_2, a_3, b_1, and b_2 are distinct, the common domain $a \otimes b \in X^{\mathsf{fr}}$ of P_1 and P_2 is regular as in Definition 3.1.25. Corollary 4.3.12 implies that the paths P_1 and P_2 have the same distortion. In other words,

$$\vartheta P_1 = \vartheta P_2$$

as morphisms in the distortion category $\mathcal{D}$. ◇

The next two examples are variations of Example 4.3.13.

Example 4.3.16. In Example 4.3.13, suppose instead that the elements

$$a_1, a_2, a_3, b_1, b_2 \in X \smallsetminus \{0^X, 1^X\}$$

are *not* all distinct. This implies that $a \otimes b \in X^{\mathsf{fr}}$ is not regular, so Corollary 4.3.12 does not apply to the two paths in (4.3.14).

However, Corollary 4.3.12 will apply if we first replace the elements as follows.

- Define the set
$$X' = X \amalg \{a'_1, a'_2, a'_3, b'_1, b'_2\}.$$
It is obtained from X by
  - adjoining five distinct symbols a'_1, a'_2, a'_3, b'_1, and b'_2 not in X and
  - keeping 0^X and 1^X as the additive zero and the multiplicative unit.
- Next we extend the function $\vartheta : X \longrightarrow \mathsf{Ob}(\mathcal{D})$ in (4.3.2) to the function $\vartheta' : X' \longrightarrow \mathsf{Ob}(\mathcal{D})$ by defining
$$\vartheta'(a'_i) = \vartheta'(b'_j) = (1) \quad \text{for} \quad i = 1,2,3 \quad \text{and} \quad j = 1,2.$$
This is well defined because $a'_i, b'_j \in X' \smallsetminus \{0^X, 1^X\}$.
- Using the associated graph morphism $\vartheta' : \mathsf{Gr}(X') \longrightarrow \mathcal{D}$ as in (4.3.3), we consider the prime version of the diagram (4.3.14) in $\mathsf{Gr}(X')$, in which a_i and b_j are replaced by, respectively, a'_i and b'_j for $1 \le i \le 3$ and $1 \le j \le 2$.

The resulting left path P'_1 and right path P'_2 in $\mathsf{Gr}(X')$ satisfy

$$\vartheta P_j = \vartheta' P'_j \in \mathcal{D} \quad \text{for} \quad j = 1,2 \qquad (4.3.17)$$

because $a_i, b_j \in X \smallsetminus \{0^X, 1^X\}$ implies

$$\vartheta(a_i) = (1) = \vartheta'(a'_i)$$
$$\vartheta(b_j) = (1) = \vartheta'(b'_j).$$

Moreover, the element

$$a' \otimes b' = \big((a'_1 \oplus a'_2) \oplus a'_3\big) \otimes (b'_1 \oplus b'_2) \in X'^{\mathsf{fr}}$$

is regular because it has the same support (3.1.24) as

$$\big((a'_1 b'_1 \oplus a'_1 b'_2) \oplus (a'_2 b'_1 \oplus a'_2 b'_2)\big) \oplus (a'_3 b'_1 \oplus a'_3 b'_2) \in X'^{\mathsf{fr}},$$

and the elements in $\{a'_1, a'_2, a'_3, b'_1, b'_2\}$ are all distinct. The paths P'_1 and P'_2 have the same distortion by Corollary 4.3.12. By (4.3.17), the paths P_1 and P_2 also have the same distortion. ◇

Example 4.3.18. In Example 4.3.13, as an elementary alternative to applying Corollary 4.3.12, in this example we explicitly compute the distortions of the two paths in the diagram (4.3.14), starting with P_1 in (4.3.15). For the following computation, we assume that a_1, a_2, a_3, b_1, and b_2 belong to $X \smallsetminus \{0^X, 1^X\}$. Lemmas 4.3.5 and 4.3.6 imply that

$$\begin{aligned}\vartheta(a \otimes b) &= \vartheta(a) \otimes \vartheta(b)\\ &= (1,1,1) \otimes (1,1)\\ &= (2,2,2,2,2,2) \in \mathcal{D}.\end{aligned}$$

The common codomain of the two paths is also sent by ϑ to this object in $\mathcal{D}$. If some of the elements in $\{a_1, a_2, a_3, b_1, b_2\}$ are 0^X or 1^X, then the computation below simplifies accordingly, since $\vartheta(0^X) = \varnothing$ and $\vartheta(1^X) = (0)$.

The distortion category $\mathcal{D}$ is a left bipermutative category by Theorem 4.2.29. Its only nonidentity structure isomorphisms are δ^r in (4.2.24), $\xi^\oplus$ in (4.2.10), and $\xi^\otimes$ in (4.2.17), the last of which is not involved in the paths P_1 and P_2. Therefore, we may ignore the additive bracketing, and each prime edge involving either $\alpha^\oplus$ or δ^l has an identity morphism as its distortion.

Lemma 4.2.27 and Explanation 4.2.28 imply that each of the four δ-prime edges in D^r has an identity morphism as its distortion, since $\xi^\otimes_{?,1}$ and $\xi^\otimes_{1,?}$ are both identity permutations. Therefore, the distortion of the path D^r is also an identity morphism.

For the path Z, we simply need to additively permute the middle four monomials to the order in the codomain. We can accomplish this by additively permuting two adjacent monomials at a time. For $1 \le i \le 3$ and $1 \le j \le 2$,

$$\vartheta(a_i b_j) = (1) \otimes (1) = (2) \in \mathcal{D},$$

which has length 1, by Lemma 4.3.5(3). By definition (4.2.10),

$$\xi^\oplus_{(2),(2)} = \big(\xi^\oplus_{1,1}; \mathrm{id}_2, \mathrm{id}_2\big) : (2,2) \longrightarrow (2,2) \in \mathcal{D},$$

and $\xi^\oplus_{1,1} \in \Sigma_2$ is the unique nonidentity permutation. We conclude that the distortion of P_1 is the morphism

$$\vartheta P_1 = \vartheta Z = \big(\sigma; \mathrm{id}_2, \ldots, \mathrm{id}_2\big) : (2,2,2,2,2,2) \longrightarrow (2,2,2,2,2,2) \in \mathcal{D} \tag{4.3.19}$$

with $\sigma \in \Sigma_6$ the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6\\ 1 & 3 & 5 & 2 & 4 & 6\end{pmatrix}. \tag{4.3.20}$$

Next, to compute the distortion of the path P_2 in (4.3.15), we apply (4.2.4), Lemma 4.2.27, and Explanation 4.2.28 to its first two prime edges. In the morphism

$$\begin{aligned}&\vartheta\big(\delta^r_{a_1,a_2,b} \oplus 1_{a_3(b_1 \oplus b_2)}\big)\vartheta\big(\delta^r_{a_1 \oplus a_2, a_3, b}\big)\\ &= \big(\delta^r_{(1),(1),(1,1)} \oplus 1_{(2,2)}\big)\delta^r_{(1,1),(1),(1,1)} \in \mathcal{D},\end{aligned} \tag{4.3.21}$$

the first entry is the product permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6\\ 1 & 3 & 2 & 4 & 5 & 6\end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6\\ 1 & 2 & 5 & 3 & 4 & 6\end{pmatrix} \in \Sigma_6.$$

This is equal to σ in (4.3.20). The other entries of the morphism in (4.3.21) are all identity permutations. So the distortion of the first two prime edges in P_2 is equal to ϑP_1 in (4.3.19). Since the path D^l consists of only prime edges involving δ^l, whose distortions are identities, ϑD^l is an identity morphism. Therefore, the distortion of the path P_2 is also given by (4.3.19). $\diamond$

4.4. The Second Coherence Theorem

In this section, we prove the main coherence result of this chapter. See Section 4.7 for more discussion of this theorem. We remind the reader that Convention 3.3.1 is still in effect, with C an arbitrary symmetric bimonoidal category. In particular, for a path in $\mathsf{Gr}(X)$:

- Its *value* in C as in (3.1.18) is defined using the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ in Definition 3.1.14.
- Its *distortion* is defined in Definition 4.3.1 using the graph morphism $\vartheta : \mathsf{Gr}(X) \longrightarrow \mathcal{D}$. The distortion of a path is a morphism in the distortion category $\mathcal{D}$.

Before we prove the Coherence Theorem 4.4.3, we consider the following preliminary version, in which Definitions 3.6.2, 3.8.1, and 3.8.8 are used.

Lemma 4.4.1. *Suppose*

$$\begin{tikzcd} a \arrow[r, bend left, "P_1"] \arrow[r, bend right, "P_2"'] & b \end{tikzcd}$$

are two paths in $\mathsf{Gr}(X)$ *such that the following two statements hold.*

- *P_1 and P_2 have the same distortion and are 1^X-free paths.*
- *a and b contain no 0^X and are δ-reduced and 1^X-reduced.*

Then P_1 and P_2 have the same value in C.

Proof. We reuse parts of the proof of Theorem 3.9.1. First we observe the following.

- By Lemmas 3.6.5 and 3.8.4, the assumptions on a and b mean that each of them is a polynomial as in Definition 3.6.3, in which each monomial is equal to 1^X, or contains no 0^X and 1^X.
- By Explanation 3.8.9, the assumption that each path P_i for $i = 1, 2$ is a 1^X-free path means that each of its prime edges is either an identity, or involves an instance of
  - $\alpha^{\pm\oplus}$, which moves the additive brackets;
  - $\xi^{\pm\oplus}$, which permutes the set of monomials;
  - $\alpha^{\pm\otimes}$, which moves the multiplicative brackets in a monomial; or
  - $\xi^{\pm\otimes}$, which permutes the factors in a monomial.

  Together with the previous remark, it follows that each vertex in each path P_i is a polynomial in which each monomial is equal to 1^X, or contains no 0^X and 1^X.

These remarks, and the naturality of $\alpha^{\oplus}$ and $\xi^{\oplus}$ in each symmetric bimonoidal category, imply that for each $i = 1, 2$, there is a diagram in $\mathsf{Gr}(X)$

$$\begin{tikzcd} a \arrow[rr, "P_i"] \arrow[dr, "P_i'"'] & & b \\ & c_i \arrow[ur, "P_i''"'] & \end{tikzcd}$$

that is commutative in the sense of Definition 3.1.14, in particular in C and $\mathcal{D}$, such that the following two statements hold.

- $P_i' : a \twoheadrightarrow c_i$ consists of identities and prime edges involving $\alpha^{\pm\oplus}$ or $\xi^{\pm\oplus}$. In particular, P_i' only moves additive brackets and permutes the set of monomials.
- $P_i'' : c_i \twoheadrightarrow b$ consists of identities and prime edges involving $\alpha^{\pm\otimes}$ or $\xi^{\pm\otimes}$. In particular, P_i'' only moves multiplicative brackets and permutes the factors in each monomial.

Our next objective is to prove the following three equalities.

(i) $\vartheta(P_1') = \vartheta(P_2')$.
(ii) $\vartheta(P_1'') = \vartheta(P_2'')$.
(iii) $c_1 = c_2$.

Recall that the only nonidentity structure isomorphisms in the distortion category $\mathcal{D}$ are δ^r in (4.2.24), $\xi^\oplus$ in (4.2.10), and $\xi^\otimes$ in (4.2.17). Suppose that P_1 has distortion

$$\vartheta(P_1) = \big(\sigma; \sigma_1, \ldots, \sigma_m\big) \in \mathcal{D}(\vartheta a; \vartheta b).$$

The above description of P_1' and P_1'', (4.2.4), Lemma 4.3.5 (3), and Lemma 4.3.6 imply the following equalities.

$$\begin{aligned}\vartheta(P_1') &= \big(\sigma; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}\big) \in \mathcal{D}(\vartheta a; \vartheta c_1)\\ \vartheta(P_1'') &= \big(\mathrm{id}_m; \sigma_{\sigma^{-1}(1)}, \ldots, \sigma_{\sigma^{-1}(m)}\big) \in \mathcal{D}(\vartheta c_1; \vartheta b)\end{aligned}$$

- m is the common number of monomials in each vertex in P_i' and P_i''.
- For each $1 \le i \le m$, r_i is the number of factors in $X \setminus \{1^X\}$ in the ith monomial in $a \in X^{\mathsf{fr}}$.
- $\sigma \in \Sigma_m$, and each $\sigma_i \in \Sigma_{r_i}$.

The same analysis also applies to P_2. Since $\vartheta(P_1) = \vartheta(P_2)$ by assumption, it follows that there are equalities as follows.

$$\begin{aligned}\vartheta(P_1') = \vartheta(P_2') &= \big(\sigma; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}\big)\\ \vartheta(P_1'') = \vartheta(P_2'') &= \big(\mathrm{id}_m; \sigma_{\sigma^{-1}(1)}, \ldots, \sigma_{\sigma^{-1}(m)}\big)\end{aligned} \tag{4.4.2}$$

It follows from (4.2.2) and the first line in (4.4.2) that, ignoring additive brackets, both $P_1' : a \twoheadrightarrow c_1$ and $P_2' : a \twoheadrightarrow c_2$ permute the set of monomials in a via $\sigma \in \Sigma_m$. Moreover, since $P_1'' : c_1 \twoheadrightarrow b$ and $P_2'' : c_2 \twoheadrightarrow b$ neither move the additive brackets nor permute the set of monomials, both c_1 and c_2 have the same additive bracketing as b. It follows that

$$c_1 = c_2 \in X^{\mathsf{fr}}.$$

Our final objective is to show that P_1' and P_2' have the same value in C, and similarly for P_1'' and P_2''.

Write c for the element $c_1 = c_2$. Since the paths P_1' and $P_2' : a \twoheadrightarrow c$ both permute the set of monomials in a via $\sigma \in \Sigma_m$, the Symmetric Coherence Theorem 1.3.8 for the additive structure

$$(\mathsf{C}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus)$$

implies that P_1' and P_2' have the same value in C.

It follows from the second line in (4.4.2) that, ignoring multiplicative brackets, the paths P_1'' and $P_2'' : c \twoheadrightarrow b$ both permute the factors in the ith monomial in c via the permutation $\sigma_{\sigma^{-1}(i)}$ for each $1 \leq i \leq m$. The Symmetric Coherence Theorem 1.3.8 for the multiplicative structure

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}),$$

when applied to each monomial in c, implies that the paths P_1'' and P_2'' have the same value in C. Therefore, $P_1 = (P_1'', P_1')$ and $P_2 = (P_2'', P_2')$ have the same value in C. □

The following coherence theorem is the main result of this chapter. Recall the concept of a δ-prime edge in Definition 3.6.2.

Theorem 4.4.3 (Laplaza's Second Coherence)**.** *Suppose* C *is a symmetric bimonoidal category in which the value of each* δ*-prime edge is a monomorphism. If*

$$\begin{tikzcd} a \arrow[r, bend left, "P_1"] \arrow[r, bend right, "P_2"'] & b \end{tikzcd}$$

are two paths in $\mathsf{Gr}(X)$ *with the same distortion, then their values in* C *are equal.*

Proof. We reuse the reduction steps in the proof of Theorem 3.9.1, along with the following remarks.

(1) The distortion category $\mathcal{D}$ is a groupoid by Lemma 4.2.5. Therefore, if two diagrams in $\mathsf{Gr}(X)$

$$\begin{tikzcd} a \arrow[r, two heads, "P_1"] \arrow[d, "Q_a"'] & b \arrow[d, "Q_b"] \\ a' \arrow[r, two heads, "R_1"] & b' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, two heads, "P_2"] \arrow[d, "Q_a"'] & b \arrow[d, "Q_b"] \\ a' \arrow[r, two heads, "R_2"] & b' \end{tikzcd}$$

are commutative in the sense of Definition 3.1.14, in particular in C and $\mathcal{D}$, such that P_1 and P_2 have the same distortion (that is, $\vartheta P_1 = \vartheta P_2$ in $\mathcal{D}$), then R_1 and R_2 have the same distortion.

(2) As discussed in Explanation 3.9.4 (2), the five reduction steps in the proof of Theorem 3.9.1 can be performed with the regularity condition omitted everywhere. By the previous remark, if the two paths have the same distortion to begin with, then after each of these reduction steps, the two new paths also have the same distortion.

Therefore, by the five reduction steps in the proof of Theorem 3.9.1 with regularity omitted, we reduce to the setting of Lemma 4.4.1, which finishes the proof. □

Remark 4.4.4. In Theorem 4.4.3, the monomorphism assumption on C is needed because, as discussed in Explanation 3.9.4 (1), this condition is used in the reduction step involving the diagram (3.9.2). ◇

Example 4.4.5. As in Examples 3.9.6 through 3.9.8 and 3.9.10, Theorem 4.4.3 is applicable in the following types of categories.

- Tight symmetric bimonoidal categories, that is, those with δ^l and δ^r (2.1.4) natural isomorphisms, not just natural monomorphisms.
- Distributive symmetric monoidal categories in Definition 2.3.1.
- Symmetric monoidal closed categories with finite coproducts in Example 2.3.3.
- The category of modules over a commutative ring in Example 2.3.4.

- Distributive categories in Example 2.3.5.
- Right and left bipermutative categories in Definitions 2.5.2 and 2.5.11.
- The symmetric bimonoidal groupoid Π in Theorem 2.6.2.
- Flat symmetric bimonoidal categories in Definition 3.9.9. $\diamond$

Example 4.4.6. Consider the diagram in Example 3.1.11 with $x, y, z \in X^{\mathsf{fr}}$ arbitrary. Each of the two paths has the identity morphism as its distortion because $\alpha^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, and δ^{l} are identities in the distortion category $\mathcal{D}$ by Definitions 4.2.7, 4.2.14, and 4.2.23. By Theorem 4.4.3, these two paths have the same value in C, provided that the value of each δ-prime edge is a monomorphism. This is an example where Theorem 4.4.3 applies, but Theorem 3.9.1 does not apply immediately unless the domain $x(y \oplus z) \oplus 0^{X}$ is regular. $\diamond$

4.5. Coherence of Bimonoidal Categories II

Convention 3.10.1 is in effect throughout this section, so C is a bimonoidal category as in Definition 2.1.2. In this section, we discuss the multiplicatively nonsymmetric analogue of the Coherence Theorem 4.4.3 that applies to bimonoidal categories instead of symmetric bimonoidal categories. As in the symmetric case, the main Coherence Theorem 4.5.8 in this section does *not* require the invertibility of the distributivity morphisms δ^{l} and δ^{r}. Instead, it assumes a much weaker monomorphism condition.

Motivation 4.5.1. Just as Theorem 4.4.3 is phrased in terms of the distortion category $\mathcal{D}$ in Definition 4.2.1, Theorem 4.5.8 is phrased in terms of a multiplicatively nonsymmetric analogue of $\mathcal{D}$, which we call the additive distortion category and is denoted by $\mathcal{D}^{\mathsf{ad}}$. As discussed in Section 4.1, in a morphism $\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m)$ in the distortion category $\mathcal{D}$, we think of each $\sigma_j \in \Sigma_{r_j}$ as permuting the factors in an r_j-fold product

$$a_1^{j} \otimes \cdots \otimes a_{r_j}^{j}$$

with some multiplicative bracketing. A bimonoidal category does not have a multiplicative symmetry, so permutations of $\otimes$-factors are no longer allowed. Therefore, in the additive distortion category, each morphism is a permutation that takes one sequence of nonnegative integers to another; see (4.5.3).

We observe in Lemma 4.5.6 that $\mathcal{D}^{\mathsf{ad}}$ is a tight bimonoidal category, and it embeds into the distortion category $\mathcal{D}$. The additive distortion of a path in the nonsymmetric graph $\mathsf{Gr}^{\mathsf{ns}}(X)$ is defined as its value in $\mathcal{D}^{\mathsf{ad}}$. Theorem 4.5.8 states that any two parallel nonsymmetric paths with the same additive distortion also have the same value in C. $\diamond$

Now we define the multiplicatively nonsymmetric analogue of the distortion category.

Definition 4.5.2. Define the *additive distortion category*

$$\big(\mathcal{D}^{\mathsf{ad}}, (\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}), (\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}), (\lambda^{\bullet}, \rho^{\bullet}, \delta^{l}, \delta^{r})\big)$$

as follows.

Objects: An object in $\mathcal{D}^{\mathsf{ad}}$ is a finite sequence

$$\underline{r} = (r_1, \ldots, r_m)$$

with $m \geq 0$ and with each r_j for $1 \leq j \leq m$ a nonnegative integer. We call m the *length* of $\underline{r}$, which is denoted by $|\underline{r}|$. The unique sequence with length 0 is denoted by $\varnothing$.

Morphisms: Suppose $\underline{s} = (s_1, \ldots, s_n)$ is an object in $\mathcal{D}^{\mathrm{ad}}$. With $\underline{r}$ as above, the morphism set $\mathcal{D}^{\mathrm{ad}}(\underline{r};\underline{s})$ consists of permutations

$$\sigma \in \Sigma_m \quad \text{such that} \quad \sigma\underline{r} = \big(r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(m)}\big) = \underline{s}. \tag{4.5.3}$$

Other Structures: Except for δ^r, the other structures in $\mathcal{D}^{\mathrm{ad}}$ are defined as in Definitions 4.2.1, 4.2.7, 4.2.14, and 4.2.23 for objects, and by restricting to the first entry for morphisms.

Right Distributivity: δ^r is the natural transformation

$$(\underline{r} \oplus \underline{r}') \otimes \underline{r}'' \xrightarrow{\delta^r_{\underline{r},\underline{r}',\underline{r}''}} (\underline{r} \otimes \underline{r}'') \oplus (\underline{r}' \otimes \underline{r}'')$$

for objects $\underline{r}, \underline{r}', \underline{r}'' \in \mathcal{D}^{\mathrm{ad}}$, that is given by the permutation

$$\delta^r_{\underline{r},\underline{r}',\underline{r}''} = (\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,k})\xi^{\otimes}_{m+k,p} \in \Sigma_{(m+k)p} \tag{4.5.4}$$

with $|\underline{r}| = m$, $|\underline{r}'| = k$, and $|\underline{r}''| = p$.

This finishes the definition of the additive distortion category. ◇

Explanation 4.5.5. Consider Definition 4.5.2.

- If $\mathcal{D}^{\mathrm{ad}}(\underline{r};\underline{s}) \neq \varnothing$, then $|\underline{r}| = |\underline{s}|$.
- Identity morphisms and composition are those of the symmetric groups.
- $\mathbb{0} = \varnothing$ and $\mathbb{1} = (0)$.
- Consider objects $\underline{r}, \underline{r}' \in \mathcal{D}^{\mathrm{ad}}$.
  - $\underline{r} \oplus \underline{r}'$ is their concatenation as in (4.2.8).
  - $\underline{r} \otimes \underline{r}'$ is as in (4.2.15). Its $\big(j + (l-1)m\big)$th entry is $r_j + r'_l$ for $1 \leq j \leq |\underline{r}|$ and $1 \leq l \leq |\underline{r}'|$.
- Consider morphisms $\sigma \in \mathcal{D}^{\mathrm{ad}}(\underline{r};\underline{s})$ and $\sigma' \in \mathcal{D}^{\mathrm{ad}}(\underline{r}';\underline{s}')$.
  - $\sigma \oplus \sigma' \in \mathcal{D}^{\mathrm{ad}}\big(\underline{r} \oplus \underline{r}';\underline{s} \oplus \underline{s}'\big)$ is their block sum in (2.4.2).
  - $\sigma \otimes \sigma' \in \mathcal{D}^{\mathrm{ad}}\big(\underline{r} \otimes \underline{r}';\underline{s} \otimes \underline{s}'\big)$ is the permutation in (2.4.4).
- The additive symmetry is given by
$$\xi^{\oplus}_{\underline{r},\underline{r}'} = \xi^{\oplus}_{|\underline{r}|,|\underline{r}'|} \in \Sigma_{|\underline{r}|+|\underline{r}'|},$$
which is the block permutation in (2.4.3).
- $\alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \lambda^{\bullet}, \rho^{\bullet}$, and δ^l are identity natural transformations.
- The permutation in (4.5.4) that defines δ^r in $\mathcal{D}^{\mathrm{ad}}$ is the first entry of the corresponding component of δ^r in $\mathcal{D}$. See Lemma 4.2.27 and Explanation 4.2.28.

Note that $\mathcal{D}^{\mathrm{ad}}$ does *not* have a multiplicative symmetry. Instead of using (4.2.24) as in $\mathcal{D}$, we define δ^r in $\mathcal{D}^{\mathrm{ad}}$ by directly specifying the permutation (4.5.4). ◇

Lemma 4.5.6. *The following statements hold.*

(1) The underlying category of $\mathcal{D}^{\mathrm{ad}}$ is a groupoid.

(2) There is a faithful embedding

$$\mathcal{D}^{\mathrm{ad}} \xrightarrow{\iota} \mathcal{D}$$

that is defined as follows.

- *It is the identity function on objects.*

- *It sends each morphism σ as in (4.5.3) to* $\big(\sigma; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}\big)$.

(3) *$\mathcal{D}^{\mathrm{ad}}$ is a tight bimonoidal category as in Definition 2.1.2.*

Proof. For assertion (1), $\mathcal{D}^{\mathrm{ad}}$ is a groupoid because its identity morphisms and composition are defined using, respectively, identity permutations and composition of permutations.

The functoriality of $\iota : \mathcal{D}^{\mathrm{ad}} \longrightarrow \mathcal{D}$ follows from (4.2.3) and (4.2.4). It is faithful—that is, injective on morphism sets—by the definition of morphisms in Definition 4.2.1. This proves assertion (2).

For assertion (3), note that we have the following facts:

- $\mathcal{D}^{\mathrm{ad}}$ and $\mathcal{D}$ have the same objects. Structure morphisms in $\mathcal{D}^{\mathrm{ad}}$ are those in $\mathcal{D}$ restricted to the first entry. For δ^r, this follows from Lemma 4.2.27.
- The distortion category $\mathcal{D}$ is a left bipermutative category and, therefore, a tight symmetric bimonoidal category by Proposition 2.5.16 and Theorem 4.2.29. In particular, $\mathcal{D}$ is a tight bimonoidal category.

The faithful embedding ι and these facts imply that $\mathcal{D}^{\mathrm{ad}}$ is also a tight bimonoidal category. □

Recall from Definition 3.10.2 the nonsymmetric graph $\mathsf{Gr}^{\mathsf{ns}}(X)$.

- Its vertex set is the free $\{\oplus, \otimes\}$-algebra X^{fr}.
- Its edges are nonsymmetric prime edges, that is, prime edges that do not involve $\xi^{\pm\otimes}$.

Next we define the multiplicatively nonsymmetric analogue of the distortion of a path.

Definition 4.5.7. Consider the additive distortion category $\mathcal{D}^{\mathrm{ad}}$ in Definition 4.5.2.

- Define the function $\vartheta : X \longrightarrow \mathsf{Ob}(\mathcal{D}^{\mathrm{ad}}) = \mathsf{Ob}(\mathcal{D})$ as in (4.3.2), that is,

$$\vartheta(x) = \begin{cases} (1) & \text{if } x \in X \smallsetminus \{0^X, 1^X\}, \\ \varnothing & \text{if } x = 0^X, \text{ and} \\ (0) & \text{if } x = 1^X. \end{cases}$$

- Using the same symbol, define the associated graph morphism

$$\mathsf{Gr}^{\mathsf{ns}}(X) \xrightarrow{\ \vartheta\ } \mathcal{D}^{\mathrm{ad}}$$

  as in Definition 3.10.2, applied to the function ϑ and the bimonoidal category $\mathcal{D}^{\mathrm{ad}}$.
- For a path P in $\mathsf{Gr}^{\mathsf{ns}}(X)$, its value $\vartheta P \in \mathcal{D}^{\mathrm{ad}}$ in the sense of Definition 3.10.2 is called the *additive distortion of P*.

This finishes the definition. ◇

We are now ready for the second coherence theorem for bimonoidal categories. It is the multiplicatively nonsymmetric analogue of Theorem 4.4.3.

Theorem 4.5.8 (Bimonoidal Coherence II)**.** *Under Convention 3.10.1, suppose that the value of each δ-prime edge is a monomorphism. If*

$$\begin{tikzcd} a \arrow[r, bend left, "P_1"] \arrow[r, bend right, "P_2"'] & b \end{tikzcd}$$

are two paths in $\mathsf{Gr}^{\mathsf{ns}}(X)$ *with the same additive distortion, then their values in* C *are equal.*

Proof. This proof is obtained from that of Theorem 4.4.3 by the following procedure:

- As in the proof of Theorem 3.10.7, we first adapt the five reduction steps in the proof of Theorem 3.9.1 by
  - removing all instances of $\xi^{\pm\otimes}$;
  - using Definition 3.10.2 instead of their counterparts in Section 3.1; and
  - using Theorem 1.3.3 instead of Theorem 1.3.8 for the monoidal category $(\mathsf{C}, \otimes)$.

  After these reduction steps, we are reduced to the setting of Lemma 4.4.1 with
  - $\xi^{\pm\otimes}$ removed and
  - $\mathsf{Gr}^{\mathsf{ns}}(X)$ and additive distortion instead of, respectively, $\mathsf{Gr}(X)$ and distortion.
- Following the proof of Lemma 4.4.1 with $\xi^{\pm\otimes}$ removed, in place of (4.4.2), we have the equalities

$$\begin{aligned}\vartheta(P_1') &= \vartheta(P_2') = \sigma\\ \vartheta(P_1'') &= \vartheta(P_2'') = \mathrm{id}_m.\end{aligned}$$

  The nonsymmetric paths P_1'' and P_2'' involve only identities and $\alpha^{\pm\otimes}$, so they only move multiplicative brackets within each monomial in c. In the next-to-the-last paragraph, we use Theorem 1.3.3 instead of Theorem 1.3.8 in $(\mathsf{C}, \otimes)$.

With these adjustments, the proof of Theorem 4.4.3 is applicable in the current nonsymmetric setting. □

Example 4.5.9. Theorem 4.5.8 applies to flat, in particular, tight, bimonoidal categories as in Definition 3.9.9. ◇

4.6. Distortion Categories as Grothendieck Constructions

In this section, we describe the distortion category $\mathcal{D}$ and the additive distortion category $\mathcal{D}^{\mathrm{ad}}$ as Grothendieck constructions over the finite ordinal category Σ. We first recall from **[JY21**, 10.1.1] the relevant construction.

Definition 4.6.1. For a category C and a functor $F : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$, the *Grothendieck construction* $\int_{\mathsf{C}} F$ is the category defined as follows.

- An object in $\int_{\mathsf{C}} F$ is a pair (A, X) consisting of objects $A \in \mathsf{C}$ and $X \in FA$.
- A morphism

$$(f, p) : (A, X) \longrightarrow (B, Y) \in \int_{\mathsf{C}} F$$

  consists of
  - a morphism $f : A \longrightarrow B$ in C and
  - a morphism $p : X \longrightarrow (Ff)(Y)$ in FA.
- The identity morphism of an object (A, X) is the pair $(1_A, 1_X)$ of identity morphisms.

- For another morphism
$$(g,q) : (B,Y) \longrightarrow (C,Z),$$
the composition with (f,p) is defined as
$$(g,q) \circ (f,p) = \big(gf, (Ff)(q) \circ p\big) : (A,X) \longrightarrow (C,Z).$$

This finishes the definition of $\int_{\mathsf{C}} F$. $\diamond$

Explanation 4.6.2. The notation $\int_{\mathsf{C}} F$ is meant to suggest that it is the integration of the categories FA for $A \in \mathsf{C}$. The composite $(g,q) \circ (f,p)$ may be visualized as follows, where we write f^*, g^*, and $(gf)^*$ for, respectively, Ff, Fg, and $F(gf)$.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C \\
FA & FB \arrow[l, "f^*"'] & FC \arrow[l, "g^*"'] \\
X \arrow[d, "p"'] & & \\
f^*Y \arrow[d, "f^*q"'] & Y \arrow[l, mapsto] \arrow[d, "q"'] & \\
(gf)^*Z = f^*g^*Z & g^*Z \arrow[l, mapsto] & Z \arrow[l, mapsto]
\end{tikzcd}$$

Since $f^*g^* = (gf)^*$ by the functoriality of F, it follows that
$$(f^*q)p : X \longrightarrow (gf)^*Z$$
is a well-defined morphism in FA. $\diamond$

The Distortion Category. Recall from Definition 2.4.1 the finite ordinal category Σ, with objects $n \geq 0$ and morphism sets
$$\Sigma(m,n) = \begin{cases} \Sigma_m & \text{if } m = n \text{ and} \\ \varnothing & \text{if } m \neq n. \end{cases}$$

The n-fold Cartesian product of Σ is denoted by $\Sigma^{\times n}$, with $\Sigma^{\times 0} = *$. We will use the following functor to relate the finite ordinal category and the distortion category.

Definition 4.6.3. Define a functor
$$F : \Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$$
by the following assignments on objects and morphisms.
$$\begin{aligned} n &\longmapsto \Sigma^{\times n} \\ \big(\Sigma(n,n) = \Sigma_n \ni \sigma\big) &\longmapsto \big(\sigma^{-1} \in \mathsf{Cat}(\Sigma^{\times n}, \Sigma^{\times n})\big) \end{aligned}$$

Here σ^{-1} is the functor that permutes the factors via $\sigma^{-1} \in \Sigma_n$, that is,
$$\sigma^{-1}(x_1, \ldots, x_n) = \big(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\big). \tag{4.6.4}$$

This finishes the definition of the functor F. $\diamond$

The following observation provides a conceptual explanation of the distortion category $\mathcal{D}$ in terms of the finite ordinal category Σ.

Proposition 4.6.5. *There is a canonical isomorphism of categories*

$$\mathcal{D} \cong \int_\Sigma F$$

with $F : \Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ *the functor in Definition 4.6.3.*

Proof. This follows from unpacking Definition 4.6.1 of the Grothendieck construction for the functor F. In more detail, an object in $\int_\Sigma F$ is a pair $(m, \underline{r})$ consisting of

- an object $m \geq 0$ in Σ and
- an object $\underline{r} \in F(m) = \Sigma^{\times m}$, that is, a sequence $(r_1, \ldots, r_m)$ with each $r_j \geq 0$.

A morphism

$$\underline{\sigma} : (m, \underline{r}) \longrightarrow (n, \underline{s}) \in \int_\Sigma F$$

consists of

- a morphism $\sigma \in \Sigma(m,n)$, that is, a permutation $\sigma \in \Sigma_m$ with $m = n$, and
- a morphism

  $$(\sigma_1, \ldots, \sigma_m) : \underline{r} \longrightarrow \sigma^{-1}\underline{s} = \big(s_{\sigma(1)}, \ldots, s_{\sigma(m)}\big) \in \Sigma^{\times m},$$

  that is, a permutation

  $$\sigma_j \in \Sigma(r_j, r_j) = \Sigma_{r_j} \quad \text{for each} \quad 1 \leq j \leq m$$

  with $r_j = s_{\sigma(j)}$.

So objects and morphisms in the Grothendieck construction $\int_\Sigma F$ are the same as those in the distortion category $\mathcal{D}$ (Definition 4.2.1). Similarly, the identity morphisms and composition in $\int_\Sigma F$ are the same as those in $\mathcal{D}$. □

The Additive Distortion Category. There is a similar description for the additive distortion category $\mathcal{D}^{\mathsf{ad}}$ (Definition 4.5.2) that involves the following functor. In the following definition, we regard the set $\mathbb{N}$ of nonnegative integers as a discrete category with only identity morphisms.

Definition 4.6.6. Define a functor

$$F^{\mathsf{ad}} : \Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$$

by the following assignments on objects and morphisms.

$$\begin{gathered} n \longmapsto \mathbb{N}^{\times n} \\ \big(\Sigma(n,n) = \Sigma_n \ni \sigma\big) \longmapsto \big(\sigma^{-1} \in \mathsf{Cat}(\mathbb{N}^{\times n}, \mathbb{N}^{\times n})\big) \end{gathered}$$

Here σ^{-1} is the functor that permutes the factors via $\sigma^{-1} \in \Sigma_n$ as in (4.6.4). This finishes the definition of the functor F^{ad}. ◇

Proposition 4.6.7. *There is a canonical isomorphism of categories*

$$\mathcal{D}^{\mathsf{ad}} \cong \int_\Sigma F^{\mathsf{ad}}$$

with $F^{\mathsf{ad}} : \Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ *the functor in Definition 4.6.6.*

Proof. Reuse the proof of Proposition 4.6.5 by replacing the category

$$F(m) = \Sigma^{\times m} \quad \text{with} \quad F^{\mathrm{ad}}(m) = \mathbb{N}^{\times m}.$$

Since $\mathbb{N}$ is a discrete category, the categorical structure of $\int_\Sigma F^{\mathrm{ad}}$ agrees with that of $\mathcal{D}^{\mathrm{ad}}$ (Definition 4.5.2). □

The braided version for the braided distortion category is Proposition II.5.5.3. See also Question III.A.5.6.

4.7. Notes

As we mentioned in the introduction of this chapter, the Coherence Theorem 4.4.3 is due to Laplaza **[Lap72b]**, and the proof presented in this chapter partially follows the general outline of Laplaza's original proof. Here we point out the main differences between Laplaza's proof and the one in this chapter. Besides some cosmetic differences in notation and additional detail in this chapter, Laplaza's original proof contains some inaccuracies that we have corrected in this chapter. See Notes 4.7.3 and 4.7.4. In Note 4.7.5, we briefly discuss a possible 2-monad approach.

4.7.1 (Notation)**.** In addition to the differences in notation in Note 3.11.1, the table below shows the correspondence between our notation and the ones in **[Lap72b]**.

Definition	Concept	Laplaza **[Lap72b]**
4.2.1	distortion category $\mathcal{D}$	$\underline{D}$
4.2.1	symmetric group Σ_m	P_m
(4.2.10)	block permutation $\xi^{\oplus}_{m,k}$	$t_{m,k}$
(4.2.17)	permutation $\xi^{\otimes}_{m,k}$	$\tau_{m,k}$
4.3.1	distortion ϑP	$\operatorname{dist}(P)$

◇

4.7.2 (Level of Detail)**.** In **[Lap72b]**, Laplaza omitted most of the proofs of Lemmas 4.2.5, 4.2.12, 4.2.19, and 4.2.25 and Theorem 4.2.29, which together show that the distortion category is a left bipermutative category. ◇

4.7.3 (Corrections for Laplaza's Parts I–III and Monomorphisms)**.** Laplaza's original proof of Theorem 4.4.3 in **[Lap72b]** is divided into what he called Parts I–V. Parts I–III there correspond to the five reduction steps in the proof of Theorem 3.9.1 without regularity, as used in the proof of Theorem 4.4.3. As we saw in the proof of Theorem 3.9.1, these five reduction steps use essentially all the results in Chapter 3 before Section 3.9. In particular, these include

- Proposition 3.5.32 in reduction step 1 and
- Proposition 3.7.19 in reduction step 4.

These preliminary results correspond to **[Lap72a**, Prop. 5 and 7**]**. Moreover, the monomorphism assumption in the statement of Theorem 4.4.3, which is used in reduction step 3, is not included in **[Lap72b]**. Therefore, the necessary corrections discussed in Notes 3.11.3 through 3.11.6 still apply. ◇

4.7.4 (Corrections for Laplaza's Parts IV and V)**.** In the proof of Theorem 4.4.3, after the five reduction steps, we finished the proof using Lemma 4.4.1. This is significantly different from the way Laplaza finished his Parts IV and V in **[Lap72b]**, which depend on [**Lap72b**, Lemma 3.1]. As we now explain, that lemma is incorrect.

The setting for that lemma is that, in addition to the function $\vartheta : X \longrightarrow \mathsf{Ob}(\mathcal{D})$ as in Definition 4.3.1, we are given the following data.

- Another set X' equipped with two distinguished elements $0^{X'}$ and $1^{X'}$.
- A function $f : X' \longrightarrow X$ such that $f(0^{X'}) = 0^X$ and $f(1^{X'}) = 1^X$.

The assertion of [**Lap72b**, Lemma 3.1] is that for each path Q in $\mathsf{Gr}(X')$, the distortion of Q is equal to the distortion of $f(Q)$. Here $f(Q)$ is the path in $\mathsf{Gr}(X)$ obtained from Q by applying f to each element in X' that appears in the vertices in Q and in the subscripts of the prime edges. For example, if $a, b, c, d \in X'$, then

$$f\big(\delta^l_{a,b,c} \oplus 1_d\big) = \delta^l_{f(a),f(b),f(c)} \oplus 1_{f(d)}.$$

To see that [**Lap72b**, Lemma 3.1] is incorrect, consider the following data.

(i) The set $X' = X \amalg \{x'\}$ obtained from X by adjoining a new element x', with $0^{X'} = 0^X$ and $1^{X'} = 1^X$.

(ii) The function $f : X' \longrightarrow X$ that extends the identity function on X by $f(x') = 1^X$.

(iii) The identity path

$$x' \xrightarrow{Q = 1_{x'}} x' \quad \text{in} \quad \mathsf{Gr}(X').$$

By Definition 4.3.1, since $x' \in X' \setminus \{0^{X'}, 1^{X'}\}$, the distortion of Q is the identity morphism

$$\vartheta'(1_{x'}) = 1_{\vartheta'(x')} = 1_{(1)} = (\mathrm{id}_1; \mathrm{id}_1) \in \mathcal{D}\big((1);(1)\big).$$

On the other hand, if we first apply f, and then take the distortion, we obtain the identity morphism

$$\vartheta f(1_{x'}) = 1_{\vartheta f(x')} = 1_{\vartheta(1^X)} = 1_{(0)} = (\mathrm{id}_1; \mathrm{id}_0) \in \mathcal{D}\big((0);(0)\big).$$

This is *not* equal to $1_{(1)}$, since even their domains are different objects.

Another simple counterexample is the additive symmetry

$$x' \oplus x' \xrightarrow{\xi^{\oplus}_{x',x'}} x' \oplus x' \quad \text{in} \quad \mathsf{Gr}(X').$$

Using (4.2.10), its images under ϑ' and ϑf are as follows.

$$\begin{aligned} \vartheta'(\xi^{\oplus}_{x',x'}) &= \xi^{\oplus}_{(1),(1)} = \big(\xi^{\oplus}_{1,1}; \mathrm{id}_1, \mathrm{id}_1\big) \in \mathcal{D}\big((1) \oplus (1); (1) \oplus (1)\big) \\ \vartheta f(\xi^{\oplus}_{x',x'}) &= \xi^{\oplus}_{(0),(0)} = \big(\xi^{\oplus}_{1,1}; \mathrm{id}_0, \mathrm{id}_0\big) \in \mathcal{D}\big((0) \oplus (0); (0) \oplus (0)\big) \end{aligned}$$

These are different morphisms in $\mathcal{D}$.

The above setting of (i) and (ii), that is,

$$X' = X \amalg \{x'\}, \qquad f|_X = \mathrm{Id}_X, \quad \text{and} \quad f(x') = 1^X,$$

appeared in the proof of [**Lap72b**, Lemma 3.2], which is used in [**Lap72b**, Part IV, page 233]. Therefore, the setting of (i) and (ii) cannot be ignored. The invalidity of [**Lap72b**, Lemma 3.1] implies that Laplaza's proof of [**Lap72b**, Parts IV and V], which uses that lemma, is also invalid. In our proof of Theorem 4.4.3, we substituted Laplaza's Parts IV and V with Lemma 4.4.1, which completely bypasses all the lemmas in [**Lap72b**, Section 3].

In Example 4.3.16, the assumption that $a_i, b_j \notin \{0^X, 1^X\}$ excludes the situations similar to the counterexamples above. $\diamond$

4.7.5 (2-Monads)**.** By Example 4.4.5, Theorem 4.4.3 applies to *tight* symmetric bimonoidal categories, that is, those with δ^l and δ^r (2.1.4) natural isomorphisms. This special case of Theorem 4.4.3 is also claimed to follow from the 2-monad coherence result in [**Kel74**, pages 371-373]. The discussion in Note 3.11.7 still applies here. $\diamond$

CHAPTER 5

Strictification of Tight Symmetric Bimonoidal Categories

A right bipermutative category C as in Definition 2.5.2 has

- two permutative structures $(\oplus, \mathbb{0}, \xi^{\oplus})$ and $(\otimes, \mathbb{1}, \xi^{\otimes})$ and
- identities for the right distributivity δ^r, the multiplicative zeros $\lambda^{\bullet}$ and $\rho^{\bullet}$, and $\xi^{\otimes}_{-,\mathbb{0}}$,

along with three carefully chosen symmetric bimonoidal category axioms. We observed in Proposition 2.5.7 that each right bipermutative category is a tight symmetric bimonoidal category, where *tight* means that δ^l and δ^r are natural isomorphisms. In this chapter, we show that each tight symmetric bimonoidal category is, in a suitable sense to be defined below, adjoint equivalent to a right bipermutative category. This theorem is originally due to May **[May77]**. The Strictification Theorem 5.4.6 is the bimonoidal analogue of Theorem 1.3.10. Moreover, a multiplicatively nonsymmetric analogue is the Strictification Theorem 5.5.11 for tight bimonoidal categories. A braided version that strictifies tight braided bimonoidal categories is Theorem II.6.3.6. Question III.A.1.6 is an open question related to strictification of tight symmetric bimonoidal categories.

Organization. To discuss equivalences between symmetric bimonoidal categories, in Section 5.1, we define symmetric bimonoidal functors. Each symmetric bimonoidal functor is a functor with two symmetric monoidal structures, one additive and one multiplicative, that satisfy two compatibility axioms for the multiplicative zeros and the distributivity morphisms. While the axioms are stated for the right multiplicative zeros and the right distributivity morphisms, they are actually equivalent to the left variants. Symmetric bimonoidal functors compose as expected, and they constitute the morphisms in a category $\mathsf{Bi}^{\mathsf{sy}}$ with small symmetric bimonoidal categories as objects. The second half of this section contains examples of symmetric bimonoidal functors and equivalences. This section does not rely on the Coherence Theorems 3.9.1 and 4.4.3, and may be read immediately after Chapter 2.

The proof of the Strictification Theorem 5.4.6 begins in Section 5.2. For a tight symmetric bimonoidal category C, in this section, we construct an associated right bipermutative category A whose objects are formal polynomials in the objects in C. This section contains most of the data of the associated right bipermutative category.

The proof that A is a right bipermutative category is given in Section 5.3. The tightness assumption on C, that is, the invertibility of its distributivity morphisms, is used to define the products of morphisms and the multiplicative symmetry in A. See (5.2.29), (5.2.36), and the explanation following them.

Section 5.4 contains the other half of the proof of the Strictification Theorem 5.4.6. We define two functors between C and A in opposite directions and observe that they (i) constitute an adjoint equivalence and (ii) are both symmetric bimonoidal equivalences. It is in this sense that a given tight symmetric bimonoidal category C is equivalent to a right bipermutative category A. At the end of this section, we record a variant of this strictification result that involves a left bipermutative category; see Theorem 5.4.7.

For monoidal categories, the proof of the Strictification Theorem 1.3.5 relies on the Coherence Theorem 1.3.3. The reason is that the construction of an associated strict monoidal category requires the commutativity of some formally defined diagrams, which are guaranteed by coherence. In a similar manner, for tight symmetric bimonoidal categories, the proof of the Strictification Theorem 5.4.6 also relies heavily on the Coherence Theorem 3.9.1. See, for example, Explanation 5.2.31 and Lemmas 5.3.1, 5.3.4, 5.3.8, and 5.4.4. Each time Theorem 3.9.1 is used to ensure that a certain diagram in C is commutative.

Section 5.5 contains the Strictification Theorems 5.5.11 and 5.5.12 for tight bimonoidal categories. They are the multiplicatively nonsymmetric analogues of Theorems 5.4.6 and 5.4.7. We define the nonsymmetric analogues of symmetric bimonoidal functors and right (left) bipermutative categories, which are called, respectively, bimonoidal functors and right (left) rigid bimonoidal categories. In the nonsymmetric analogue of A in Definition 5.5.5, the left distributivity morphism is defined in terms of δ^l in C, instead of using the diagram (5.3.6) and the multiplicative symmetry. Therefore, the proof of Proposition 5.5.10 that the nonsymmetric analogue of A satisfies the 22 Laplaza axioms for a bimonoidal category is also somewhat different from the symmetric case in Section 5.3.

For a brief history of strictification for tight symmetric bimonoidal categories, the reader is referred to Section 5.6.

Reading Guide. The proof of the Strictification Theorem 5.4.6 is presented with full detail straightly linearly in this chapter. As a possible alternative to reading this chapter linearly, we offer the following suggestion.

(1) In Section 5.1, read Definition 5.1.1 of a symmetric bimonoidal functor.
(2) In Section 5.2, pay special attention to Explanations 5.2.25, 5.2.30, 5.2.31, and 5.2.37. Skip Example 5.2.32 during the first reading.
(3) In Section 5.3, skip all the proofs during the first reading.
(4) In Section 5.4, first read Definitions 5.4.1 and 5.4.2 and the statements of Lemmas 5.4.3 through 5.4.5 and Theorems 5.4.6 and 5.4.7.
(5) With a clear understanding of the structure of the proof of Theorem 5.4.6, go back and read the parts skipped earlier.

As in Chapters 3 and 4, in this chapter we deliberately divided the proof of Theorem 5.4.6 into many lemmas to clarify the overall structure of the proof and to make jumping forward and backward easier. Students are encouraged to regard the many lemmas and their detailed proofs as exercises with full solutions.

Detail. The detailed proof of Theorem 5.4.6 does more than proving this theorem.

(1) We emphasize that Theorem 5.4.6 only applies to *tight* symmetric bimonoidal categories, that is, those with δ^l and δ^r natural isomorphisms. The detailed constructions and proofs in this chapter show precisely

how the invertibility of the distributivity morphisms is used. See Explanations 5.2.25, 5.2.31, and 5.2.37. This is related to an incorrect claim in [**EM06**] that their bipermutative categories, whose factorization morphisms are not invertible in general, are equivalent to right bipermutative categories. The reader is referred to Note 5.6.3 for more discussion related to this issue.

(2) May's proof of Theorem 5.4.6 is given as an outline in [**May77**, 6.3.5], where the Coherence Theorem 3.9.1 is not explicitly mentioned. Our detailed constructions and proofs in this chapter show precisely where Theorem 3.9.1 is used. The reader is referred to Note 5.6.2 for more discussion related to this issue.

(3) The proof of the Strictification Theorem II.6.3.6 of tight braided bimonoidal categories uses modified versions of many of the proofs in this chapter. A detailed treatment here will allow us to be both precise and concise at the same time in the braided case.

5.1. Symmetric Bimonoidal Functors

In this section, we define functors between symmetric bimonoidal categories and observe that there is a category $\mathsf{Bi}^{\mathsf{sy}}$ of small symmetric bimonoidal categories and symmetric bimonoidal functors. Examples of symmetric bimonoidal functors are discussed in the second half of this section.

Recall from Definitions 1.2.11 and 1.2.25 the concept of a symmetric monoidal functor, and from Definition 2.1.2 the concept of a symmetric bimonoidal category.

Definition 5.1.1. Suppose C and D are symmetric bimonoidal categories. A *symmetric bimonoidal functor* from C to D is a tuple

$$\left(F, F^2_\oplus, F^0_\oplus, F^2_\otimes, F^0_\otimes\right) : \mathsf{C} \longrightarrow \mathsf{D}$$

consisting of the following data.

- $(F, F^2_\oplus, F^0_\oplus) : \mathsf{C} \longrightarrow \mathsf{D}$ is a symmetric monoidal functor from the additive structure of C to the additive structure of D.
- $(F, F^2_\otimes, F^0_\otimes) : \mathsf{C} \longrightarrow \mathsf{D}$ is a symmetric monoidal functor from the multiplicative structure of C to the multiplicative structure of D.

These data are required to make the following two diagrams in D commutative for all objects $A, B, C \in \mathsf{C}$.

Multiplicative Zero:

$$\begin{tikzcd}
(FA)\mathbb{0} \arrow[r, "1_{FA}F^0_\oplus"] \arrow[d, "\rho^\bullet_{FA}"'] & (FA)(F\mathbb{0}) \arrow[d, "F^2_\otimes"] \\
\mathbb{0} \arrow[dr, "F^0_\oplus"'] & F(A\mathbb{0}) \arrow[d, "F\rho^\bullet_A"] \\
& F\mathbb{0}
\end{tikzcd} \tag{5.1.2}$$

Distributivity:

$$\begin{tikzcd}
(FA \oplus FB)(FC) \arrow[r, "\delta^r"] \arrow[d, "F^2_\oplus 1_{FC}"'] & (FA)(FC) \oplus (FB)(FC) \arrow[d, "F^2_\otimes \oplus F^2_\otimes"] \\
F(A \oplus B)(FC) \arrow[d, "F^2_\otimes"'] & F(AC) \oplus F(BC) \arrow[d, "F^2_\oplus"] \\
F\big((A \oplus B)C\big) \arrow[r, "F\delta^r"] & F(AC \oplus BC)
\end{tikzcd} \tag{5.1.3}$$

This finishes the definition of a symmetric bimonoidal functor.

Moreover:

- A symmetric bimonoidal functor as above is sometimes abbreviated to F.
- We write
$$F_\oplus = (F, F^2_\oplus, F^0_\oplus) \quad \text{and} \quad F_\otimes = (F, F^2_\otimes, F^0_\otimes)$$
for the two symmetric monoidal functors, which are called the *additive structure* and the *multiplicative structure*, respectively.
- A symmetric bimonoidal functor is
  - *robust* if $F^2_\oplus$, $F^0_\oplus$, and $F^0_\otimes$ are isomorphisms;
  - *strong* if both $F_\oplus$ and $F_\otimes$ are strong symmetric monoidal functors, that is, if $F^2_\oplus$, $F^0_\oplus$, $F^2_\otimes$, and $F^0_\otimes$ are isomorphisms;
  - *unitary* if it is strong, and if $F^0_\oplus$ and $F^0_\otimes$ are identities;
  - *strict* if both $F_\oplus$ and $F_\otimes$ are strict symmetric monoidal functors, that is, if $F^2_\oplus$, $F^0_\oplus$, $F^2_\otimes$, and $F^0_\otimes$ are identities; and
  - a *symmetric bimonoidal equivalence* if it is also an equivalence of categories. ◇

The axioms (5.1.2) and (5.1.3) of a symmetric bimonoidal functor are stated in terms of the right multiplicative zeros and the right distributivity morphisms. The following observation says that they can also be stated in terms of the left multiplicative zeros and the left distributivity morphisms.

Proposition 5.1.4. *Suppose*

$$\big(F, F^2_\oplus, F^0_\oplus, F^2_\otimes, F^0_\otimes\big) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of the same data as in Definition 5.1.1.

(1) The multiplicative zero axiom (5.1.2) is equivalent to the commutativity of the following diagram for all objects $A \in \mathsf{C}$.

$$\begin{tikzcd}
\mathbb{0}(FA) \arrow[r, "F^0_\oplus 1_{FA}"] \arrow[d, "\lambda^\bullet_{FA}"'] & (F\mathbb{0})(FA) \arrow[d, "F^2_\otimes"] \\
\mathbb{0} \arrow[dr, "F^0_\oplus"'] & F(\mathbb{0}A) \arrow[d, "F\lambda^\bullet_A"] \\
& F\mathbb{0}
\end{tikzcd} \tag{5.1.5}$$

(2) *The distributivity axiom (5.1.3) is equivalent to the commutativity of the following diagram for all objects $A, B, C \in \mathsf{C}$.*

$$\begin{tikzcd}
(FA)(FB \oplus FC) \arrow[r, "\delta^l"] \arrow[d, "1_{FA}F^2_\oplus"'] & (FA)(FB) \oplus (FA)(FC) \arrow[d, "F^2_\otimes \oplus F^2_\otimes"] \\
(FA)F(B \oplus C) \arrow[d, "F^2_\otimes"'] & F(AB) \oplus F(AC) \arrow[d, "F^2_\oplus"] \\
F\big(A(B \oplus C)\big) \arrow[r, "F\delta^l"] & F(AB \oplus AC)
\end{tikzcd} \tag{5.1.6}$$

Proof. For the first assertion, consider the following diagram.

$$\begin{tikzcd}
\mathbb{0}(FA) \arrow[rrrr, "F^0_\oplus 1_{FA}"] \arrow[dr, "\xi^\otimes"] \arrow[ddd, "\lambda^\bullet"'] & & \text{nat} & & (F\mathbb{0})(FA) \arrow[dl, "\xi^\otimes"'] \arrow[ddd, "F^2_\otimes"] \\
& (FA)\mathbb{0} \arrow[rr, "1_{FA}F^0_\oplus"] \arrow[d, "\rho^\bullet"] & & (FA)(F\mathbb{0}) \arrow[d, "F^2_\otimes"'] & \\
\substack{(2.1.19)\\(1.2.20)} & \mathbb{0} \arrow[dr, "F^0_\oplus"] & (5.1.2) & F(A\mathbb{0}) \arrow[dl, "F\rho^\bullet"'] & (1.2.26) \\
\mathbb{0} \arrow[ur, "="] \arrow[rr, "F^0_\oplus"'] & & F\mathbb{0} & \substack{(2.1.19)\\(1.2.20)} & F(\mathbb{0}A) \arrow[ul, "F\xi^\otimes"'] \arrow[ll, "F\lambda^\bullet"]
\end{tikzcd}$$

The outer diagram is (5.1.5). Since $\xi^\otimes$ is a natural isomorphism, the outer diagram is commutative if and only if the inside pentagon (5.1.2) is commutative.

For the second assertion, consider the following diagram.

$$\begin{tikzcd}
(FA)(FB \oplus FC) \arrow[rrr, "\delta^l"] \arrow[dr, "\xi^\otimes"] \arrow[dd, "1F^2_\oplus"'] & & & (FA)(FB) \oplus (FA)(FC) \arrow[dl, "\xi^\otimes \oplus \xi^\otimes"'] \arrow[dd, "F^2_\otimes \oplus F^2_\otimes"] \\
\text{nat} & (FB \oplus FC)(FA) \arrow[r, "\delta^r"] \arrow[d, "F^2_\oplus 1"] & (FB)(FA) \oplus (FC)(FA) \arrow[d, "F^2_\otimes \oplus F^2_\otimes"'] & (1.2.26) \\
(FA)F(B \oplus C) \arrow[r, "\xi^\otimes"] \arrow[dd, "F^2_\otimes"'] & F(B \oplus C)(FA) \arrow[d, "F^2_\otimes"] \arrow[r, phantom, "(5.1.3)"] & F(BA) \oplus F(CA) \arrow[d, "F^2_\oplus"'] & F(AB) \oplus F(AC) \arrow[l, "F\xi^\otimes \oplus F\xi^\otimes"'] \arrow[dd, "F^2_\oplus"] \\
(1.2.26) & F\big((B \oplus C)A\big) \arrow[r, "F\delta^r"] & F(BA \oplus CA) & \text{nat} \\
F\big(A(B \oplus C)\big) \arrow[rrr, "F\delta^l"] \arrow[ur, "F\xi^\otimes"] & & & F(AB \oplus AC) \arrow[ul, "F(\xi^\otimes \oplus \xi^\otimes)"']
\end{tikzcd}$$

The outer diagram is (5.1.6). Since $\xi^\otimes$ is a natural isomorphism, the outer diagram is commutative if and only if the inside rectangle (5.1.3) is commutative. $\square$

Composition.

Definition 5.1.7. Suppose

$$\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$$

are symmetric bimonoidal functors. The *composite*

$$\big(GF, (GF)^2_\oplus, (GF)^0_\oplus, (GF)^2_\otimes, (GF)^0_\otimes\big) : \mathsf{C} \longrightarrow \mathsf{E}$$

is defined by the composites

$$\begin{aligned}\big(GF, (GF)^2_\oplus, (GF)^0_\oplus\big) &= G_\oplus \circ F_\oplus \quad \text{and}\\ \big(GF, (GF)^2_\otimes, (GF)^0_\otimes\big) &= G_\otimes \circ F_\otimes\end{aligned}$$

of symmetric monoidal functors. $\diamond$

Explanation 5.1.8. In Definition 5.1.7, the underlying functor of the composite is the composite functor $GF : \mathsf{C} \longrightarrow \mathsf{E}$. The additive structure $(GF)_\oplus = G_\oplus \circ F_\oplus$ of the composite is defined by the following composites.

$$\begin{tikzcd} \mathbb{0} \arrow[r, "G^0_\oplus"] \arrow[rr, bend left, "(GF)^0_\oplus"] & G\mathbb{0} \arrow[r, "G(F^0_\oplus)"] & GF\mathbb{0} \end{tikzcd}$$

$$\begin{tikzcd} GFA \oplus GFB \arrow[r, "G^2_\oplus"] \arrow[rr, bend left, "(GF)^2_\oplus"] & G(FA \oplus FB) \arrow[r, "G(F^2_\oplus)"] & GF(A \oplus B) \end{tikzcd}$$

Two similar composites, with $(\oplus, \mathbb{0})$ replaced by $(\otimes, \mathbb{1})$, define the multiplicative structure $(GF)_\otimes = G_\otimes \circ F_\otimes$ of the composite. $\diamond$

Lemma 5.1.9. *In Definition 5.1.7, $GF : \mathsf{C} \longrightarrow \mathsf{E}$ is a symmetric bimonoidal functor. Moreover, if both F and G are robust (respectively, strong, unitary, or strict), then so is GF.*

Proof. Since the composite of two symmetric monoidal functors is a symmetric monoidal functor, it remains to check the axioms (5.1.2) and (5.1.3) for GF. For (5.1.2), consider the following diagram.

$$\begin{tikzcd} (GFA)\mathbb{0} \arrow[r, "1G^0_\oplus"] \arrow[d, "\rho^\bullet"'] & (GFA)(G\mathbb{0}) \arrow[r, "1G(F^0_\oplus)"] \arrow[d, "G^2_\otimes"] & (GFA)(GF\mathbb{0}) \arrow[d, "G^2_\otimes"] \arrow[dd, bend left, "(GF)^2_\otimes"] \\ \mathbb{0} \arrow[d, "G^0_\oplus"'] & G((FA)\mathbb{0}) \arrow[r, "G(1F^0_\oplus)"] \arrow[dl, "G\rho^\bullet"'] & G((FA)(F\mathbb{0})) \arrow[d, "G(F^2_\otimes)"] \\ G\mathbb{0} \arrow[r, "G(F^0_\oplus)"] & GF\mathbb{0} & GF(A\mathbb{0}) \arrow[l, "GF\rho^\bullet"'] \end{tikzcd}$$

By the definitions of $(GF)^0_\oplus$ and $(GF)^2_\otimes$, the outer diagram is (5.1.2) for GF. The top right rectangle is commutative by the naturality of $G^2_\otimes$. The left and the bottom trapezoids are commutative by the axiom (5.1.2) for F and G.

A similar diagram using

- (5.1.3) for F and G,

- the naturality of $G^2_\oplus$ and $G^2_\otimes$, and
- the definitions of $(GF)^2_\oplus$ and $(GF)^2_\otimes$,

proves the commutativity of (5.1.3) for GF. This shows that GF is a symmetric bimonoidal functor.

The other assertion follows from the following facts:

- If both $F^0_\oplus$ and $G^0_\oplus$ are isomorphisms (respectively, identities), then so is $(GF)^0_\oplus$.
- If both $F^2_\oplus$ and $G^2_\oplus$ are isomorphisms (respectively, identities), then so is $(GF)^2_\oplus$.
- Similar statements for $(F^0_\otimes, G^0_\otimes, (GF)^0_\otimes)$ and $(F^2_\otimes, G^2_\otimes, (GF)^2_\otimes)$ hold.

These statements hold by the definitions of the additive and the multiplicative structures of GF in Explanation 5.1.8. □

A subcategory is called *wide* if it contains all the objects in the larger category.

Proposition 5.1.10. *There is a category* $\mathsf{Bi}^{\mathsf{sy}}$ *defined by the following data.*

- *The objects are small symmetric bimonoidal categories as in Definition 2.1.2.*
- *The morphisms are symmetric bimonoidal functors as in Definition 5.1.1.*
- *Identity morphisms are identity functors with identity monoidal structures.*
- *Composition is as in Definition 5.1.7.*

Moreover, $\mathsf{Bi}^{\mathsf{sy}}$ *has the following wide subcategories:*

- $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{r}}$ *with robust symmetric bimonoidal functors as morphisms.*
- $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{sg}}$ *with strong symmetric bimonoidal functors as morphisms.*
- $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{u}}$ *with unitary symmetric bimonoidal functors as morphisms.*
- $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{st}}$ *with strict symmetric bimonoidal functors as morphisms.*

Proof. Lemma 5.1.9 shows that composition of symmetric bimonoidal functors is well defined. Associativity and unity of composition follow from those of symmetric monoidal functors. Therefore, $\mathsf{Bi}^{\mathsf{sy}}$ is a category.

The other assertion about wide subcategories holds because each of the four kinds of symmetric bimonoidal functors—robust, strong, unitary, and strict—is closed under composition by Lemma 5.1.9. □

Examples. The rest of this section contains examples of symmetric bimonoidal functors. Recall from Definition 2.3.1 that a distributive symmetric monoidal category is a symmetric monoidal category with finite coproducts over which the monoidal product distributes up to natural isomorphisms. By Proposition 2.3.2, each distributive symmetric monoidal category is a tight symmetric bimonoidal category with $(\oplus, \mathbb{0}) = (\amalg, \varnothing)$.

Proposition 5.1.11. *Suppose* C *and* D *are distributive symmetric monoidal categories, and*

$$(F, F^2_\otimes, F^0_\otimes) : (\mathsf{C}, \otimes^{\mathsf{C}}, \mathbb{1}^{\mathsf{C}}) \longrightarrow (\mathsf{D}, \otimes^{\mathsf{D}}, \mathbb{1}^{\mathsf{D}})$$

is a symmetric monoidal functor. Then F *induces a symmetric bimonoidal functor if* C *and* D *are regarded as symmetric bimonoidal categories with additive structures given by coproducts.*

Proof. The additive structure of F is defined by the morphisms

$$\begin{aligned} \varnothing &\xrightarrow{F^0_\oplus} F\varnothing \\ FA \amalg FB &\xrightarrow{F^2_\oplus} F(A \amalg B) \end{aligned} \tag{5.1.12}$$

for objects $A, B \in \mathsf{C}$. The first morphism is the unique morphism from the chosen initial object in D. The second morphism is determined by F applied to the natural morphisms

$$\begin{aligned} A &\longrightarrow A \amalg B \\ B &\longrightarrow A \amalg B. \end{aligned}$$

The universal properties of coproducts and initial objects imply that

$$(F, F^2_\oplus, F^0_\oplus) : (\mathsf{C}, \amalg, \varnothing) \longrightarrow (\mathsf{D}, \amalg, \varnothing)$$

is a symmetric monoidal functor. Moreover,

- the multiplicative zero axiom (5.1.2) holds by the universal property of the initial object in D, and
- the distributivity axiom (5.1.3) holds by the naturality of $F^2_\otimes$ and the universal property of coproducts.

Therefore, F is a symmetric bimonoidal functor. □

Example 5.1.13 (Ring Morphisms)**.** Suppose $f : R \longrightarrow S$ is a morphism of commutative rings. Then restriction of scalars via f defines a symmetric monoidal functor

$$f^* : \mathsf{Mod}(S) \longrightarrow \mathsf{Mod}(R)$$

from the category of S-modules to the category of R-modules. By Example 2.3.4 and Proposition 5.1.11, f^* is a symmetric bimonoidal functor when $\mathsf{Mod}(S)$ and $\mathsf{Mod}(R)$ are regarded as symmetric bimonoidal categories with additive structures given by direct sums. ⋄

Example 5.1.14 (Distributive Categories)**.** Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor between distributive categories as in Example 2.3.5, such that the natural morphisms

$$\begin{aligned} F* &\xrightarrow{t} * \\ F(A \times B) &\xrightarrow{p} FA \times FB \end{aligned}$$

for objects $A, B \in \mathsf{C}$, which are dual to those in (5.1.12), are isomorphisms. The universal properties of products and terminal objects imply that

$$\left(F, p^{-1}, t^{-1}\right) : (\mathsf{C}, \times, *) \longrightarrow (\mathsf{D}, \times, *)$$

is a symmetric monoidal functor. By Example 2.3.5 and Proposition 5.1.11, F is a symmetric bimonoidal functor when C and D are regarded as symmetric bimonoidal categories with $(\oplus, \mathbb{0}) = (\amalg, \varnothing)$. ⋄

Recall the small and tight symmetric bimonoidal categories Σ and Σ' in Section 2.4. Their underlying categories are given by finite ordinals and bijections, and they have the same additive structure. In Σ the left distributivity δ^l is the identity, and in Σ' the right distributivity δ^r is the identity. Moreover, Σ' is a right bipermutative category as in Definition 2.5.2, and Σ is a left bipermutative category as in Definition 2.5.11.

Proposition 5.1.15. *There is a symmetric bimonoidal equivalence*

$$\Sigma \xrightarrow{F} \Sigma'$$

that is defined by the following data.

- *The additive structure $(F, F^2_\oplus, F^0_\oplus)$ is the identity symmetric monoidal functor.*
- *$F^0_\otimes : 0 \longrightarrow 0$ is the identity permutation in Σ_0.*
- *$F^2_\otimes : mn \longrightarrow mn$ is the multiplicative symmetry isomorphism $\xi'^{\otimes}_{m,n}$ in Σ' in (2.4.20).*

Proof. To check that the multiplicative structure $F_\otimes = (F, F^2_\otimes, F^0_\otimes)$ is a symmetric monoidal functor, first observe that the unity axioms (1.2.15) hold because each morphism involved is an identity morphism.

Since the multiplicative associativity isomorphisms in both Σ and Σ' are identities, the associativity axiom (1.2.14) becomes the following diagram in Σ' for $m, n, p \geq 0$.

$$\begin{tikzcd} mnp \arrow[r, "1_m \xi'^{\otimes}_{n,p}"] \arrow[d, "\xi'^{\otimes}_{m,n} 1_p"'] & mpn \arrow[d, "\xi'^{\otimes}_{m,pn}"] \\ nmp \arrow[r, "\xi'^{\otimes}_{nm,p}"] & pnm \end{tikzcd}$$

This diagram commutes because both composites are given by the bijection

$$k + (j-1)p + (i-1)np \longmapsto i + (j-1)m + (k-1)mn$$

for $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq p$.

The compatibility of $F^2_\otimes$ with the multiplicative symmetries (1.2.26) holds because each composite is the identity. This follows from the fact that the multiplicative symmetry isomorphism $\xi^{\otimes}_{m,n}$ in Σ in (2.4.5) is the inverse of $\xi'^{\otimes}_{m,n}$ in Σ'. So the multiplicative structure $F_\otimes$ is a symmetric monoidal functor.

The multiplicative zero axiom (5.1.2) holds because $\Sigma'(0,0) = \Sigma_0$ contains only the identity morphism of 0.

Since $F^2_\oplus$ and δ^l in Σ are identities, the distributivity axiom, in the equivalent form (5.1.6), is the following diagram in Σ' for $m, n, p \geq 0$.

$$\begin{tikzcd} m(n+p) \arrow[r, "\delta^l_{m,n,p}"] \arrow[d, "\xi'^{\otimes}_{m,n+p}"'] & mn + mp \arrow[d, "\xi'^{\otimes}_{m,n} \oplus \xi'^{\otimes}_{m,p}"] \\ (n+p)m \arrow[r, "="] & nm + pm \end{tikzcd}$$

This is equal to axiom (2.1.5) in the symmetric bimonoidal category Σ' because δ^r in Σ' is the identity. Therefore, F is a symmetric bimonoidal functor.

Finally, F is a symmetric bimonoidal equivalence because its underlying functor is the identity functor. □

Proposition 5.1.16. *There is a symmetric bimonoidal equivalence*

$$\Sigma' \xrightarrow{G} \Sigma$$

that is defined by the following data.

- *The additive structure $(G, G^2_\oplus, G^0_\oplus)$ is the identity symmetric monoidal functor.*

- $G^0_\otimes : 0 \longrightarrow 0$ *is the identity permutation in* Σ_0.
- $G^2_\otimes : mn \longrightarrow mn$ *is the multiplicative symmetry isomorphism* $\xi^\otimes_{m,n}$ *in* Σ *in (2.4.5).*

Furthermore, G and F in Proposition 5.1.15 are inverse isomorphisms in $\mathsf{Bi}^{\mathsf{sy}}$.

Proof. The first assertion, that G is a symmetric bimonoidal equivalence, is proved by a slight modification of the proof of Proposition 5.1.15. The second assertion, that $GF = 1_\Sigma$ and $FG = 1_{\Sigma'}$, follows from the equality

$$\xi^\otimes_{m,n} = \left(\xi'^{\otimes}_{m,n}\right)^{-1}$$

for all m and n. □

5.2. Associated Right Bipermutative Category: Definitions

As the first step of the strictification theorem, in this section, we give a detailed construction of a right bipermutative category associated to a tight symmetric bimonoidal category. Recall that *tight* means that the distributivity morphisms δ^l and δ^r are natural isomorphisms, not just natural monomorphisms. Most of the data of the associated right bipermutative category are defined in this section. The proof that it actually satisfies the axioms of a right bipermutative category is given in Section 5.3. In Section 5.4, we will show that the associated right bipermutative category is equivalent to the given tight symmetric bimonoidal category.

Convention 5.2.1. For the rest of this section, unless specified otherwise, assume that C is a tight symmetric bimonoidal category as in Definition 2.1.2. ◇

Motivation 5.2.2. The objects of the right bipermutative category A associated to C are formal polynomials as in

$$X = \sum_{i=1}^{r} X^i_1 \cdots X^i_{k_i}$$

with each X^i_j an object in C. In the actual construction, this is defined in (5.2.4), with the ith monomial in (5.2.5). With a suitably defined product, which is (5.2.7) below, the right distributivity is the identity. Similar to the proofs of the Strictification Theorems 1.3.5 and 1.3.10, the proof that A is a right bipermutative category involves using Laplaza's First Coherence Theorem 3.9.1 multiple times. Choosing a suitable convention for additive bracketing and multiplicative bracketing, the object X can be interpreted in C using its tight symmetric bimonoidal structure. This association is then extended to the desired equivalence of symmetric bimonoidal categories. The tightness assumption on C is essential to the constructions of the product of morphisms (5.2.29) and the multiplicative symmetry (5.2.36) in A. ◇

Since the definition of A involves many components, to improve readability we split it into several parts. First we define its objects, the additive zero, the multiplicative unit, the sum, and the product on objects. The following definition only requires a class $\mathsf{Ob}(\mathsf{C})$.

Definition 5.2.3. Define the tuple

$$\left(\mathsf{Ob}(\mathsf{A}), \oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}\right)$$

as follows.

Objects: $\mathsf{Ob}(\mathsf{A})$ is the class of finite sequences

$$\underline{a} = \{a^1, \ldots, a^r\} \tag{5.2.4}$$

with *additive length* $r \geq 0$, such that for each $1 \leq i \leq r$, a^i is a finite sequence

$$a^i = \left(a^i_1, \ldots, a^i_{m_i}\right) \tag{5.2.5}$$

with *multiplicative length* $m_i \geq 0$ and each $a^i_j \in \mathsf{Ob}(\mathsf{C})$. We call

- a^i the *ith monomial* in $\underline{a}$ and
- a^i_j the *jth alphabet* in a^i.

The Additive Zero: Define

$$\mathbb{0}^{\mathsf{A}} = \varnothing \in \mathsf{Ob}(\mathsf{A}),$$

which is the unique object with additive length 0.

The Multiplicative Unit: Define

$$\mathbb{1}^{\mathsf{A}} = \{\varnothing\} \in \mathsf{Ob}(\mathsf{A}),$$

which has additive length 1 and whose only monomial has multiplicative length 0.

The Sum: With $\underline{a}$ as above and $\underline{b} = \{b^1, \ldots, b^s\}$, define their *sum* by concatenation,

$$\underline{a} \oplus^{\mathsf{A}} \underline{b} = \left\{a^1, \ldots, a^r, b^1, \ldots, b^s\right\} \in \mathsf{Ob}(\mathsf{A}), \tag{5.2.6}$$

which has additive length $r + s$.

The Product: With $\underline{a}$ and $\underline{b}$ as above, define their *product* by

$$\underline{a} \otimes^{\mathsf{A}} \underline{b} = \left\{(a^1, b^1), \ldots, (a^1, b^s), \ldots, (a^r, b^1), \ldots, (a^r, b^s)\right\} \in \mathsf{Ob}(\mathsf{A}), \tag{5.2.7}$$

which has additive length rs. For $1 \leq i \leq r$ and $1 \leq j \leq s$, the $(j + (i-1)s)$th monomial in $\underline{a} \otimes^{\mathsf{A}} \underline{b}$ is the concatenation

$$(a^i, b^j) = \left(a^i_1, \ldots, a^i_{m_i}, b^j_1, \ldots, b^j_{n_j}\right)$$

if a^i is as in (5.2.5) and

$$b^j = \left(b^j_1, \ldots, b^j_{n_j}\right).$$

Its multiplicative length is $m_i + n_j$. ◇

Lemma 5.2.8. *In the context of Definition 5.2.3, the following statements hold.*

1. $\oplus^{\mathsf{A}}$ *is strictly associative, and* $\mathbb{0}^{\mathsf{A}}$ *is a strict two-sided unit for* $\oplus^{\mathsf{A}}$.
2. $\otimes^{\mathsf{A}}$ *is strictly associative, and* $\mathbb{1}^{\mathsf{A}}$ *is a strict two-sided unit for* $\otimes^{\mathsf{A}}$.
3. *The following equalities hold for all objects* $\underline{a}$.

$$\underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}} = \mathbb{0}^{\mathsf{A}} = \mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a}. \tag{5.2.9}$$

4. *The right distributive law*

$$\left(\underline{a} \oplus^{\mathsf{A}} \underline{b}\right) \otimes^{\mathsf{A}} \underline{c} = \left(\underline{a} \otimes^{\mathsf{A}} \underline{c}\right) \oplus^{\mathsf{A}} \left(\underline{b} \otimes^{\mathsf{A}} \underline{c}\right) \tag{5.2.10}$$

holds for $\underline{a}, \underline{b}, \underline{c} \in \mathsf{Ob}(\mathsf{A})$.

Proof. The first assertion follows from the definitions of $\oplus^{\mathsf{A}}$ as concatenation and $\mathbb{0}^{\mathsf{A}}$ as the object with additive length 0.

For the second assertion, suppose $\underline{c} = \{c^1, \ldots, c^t\}$. Then both $(\underline{a} \otimes^{\mathsf{A}} \underline{b}) \otimes^{\mathsf{A}} \underline{c}$ and $\underline{a} \otimes^{\mathsf{A}} (\underline{b} \otimes^{\mathsf{A}} \underline{c})$ are equal to the object

$$\begin{gathered}\big\{(a^1,b^1,c^1),\ldots,(a^1,b^1,c^t),\ldots,(a^1,b^s,c^1),\ldots,(a^1,b^s,c^t),\\ \ldots,(a^r,b^1,c^1),\ldots,(a^r,b^1,c^t),\ldots,(a^r,b^s,c^1),\ldots,(a^r,b^s,c^t)\big\}.\end{gathered}$$

The object $\mathbb{1}^{\mathsf{A}}$ is a strict two-sided unit for $\otimes^{\mathsf{A}}$ because

$$(a^i, \varnothing) = a^i = (\varnothing, a^i).$$

The third assertion holds because the objects $\underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}}$ and $\mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a}$ both have additive length $r \cdot 0 = 0 \cdot r = 0$.

The right distributive law holds because both sides are equal to the object

$$\begin{gathered}\big\{(a^1,c^1),\ldots,(a^1,c^t),\ldots,(a^r,c^1),\ldots,(a^r,c^t),\\ (b^1,c^1),\ldots,(b^1,c^t),\ldots,(b^s,c^1),\ldots,(b^s,c^t)\big\}\end{gathered}$$

in A. □

The morphisms in A are defined by first interpreting the objects in A as objects in C using the following bracketing convention.

Definition 5.2.11. Suppose $(\mathsf{D}, \odot)$ is a monoidal category, and $x_1, x_2, \ldots$ are objects in D.

- Define the *right normalized bracketing* inductively by

$$(5.2.12)\qquad \big(x_1 \odot \cdots \odot x_k\big)_{\mathsf{rt}} = \begin{cases} x_1 & \text{if } k = 1 \text{ and}\\ x_1 \odot (x_2 \odot \cdots \odot x_k)_{\mathsf{rt}} & \text{if } k > 1.\end{cases}$$

- Define the *left normalized bracketing* inductively by

$$(5.2.13)\qquad \big(x_1 \odot \cdots \odot x_k\big)_{\mathsf{lt}} = \begin{cases} x_1 & \text{if } k = 1 \text{ and}\\ (x_1 \odot \cdots \odot x_{k-1})_{\mathsf{lt}} \odot x_k & \text{if } k > 1.\end{cases}$$

The subscripts rt and lt stand for *right* and *left*, respectively. ◇

In this chapter, we only use the right normalized bracketing. Left normalized bracketing will be used in Chapter 7; see Definition 7.2.2.

Remark 5.2.14. Definition 5.2.11 requires a lot less than a monoidal category. In particular, with $\odot \in \{\oplus, \otimes\}$, right and left normalized bracketings also make sense for elements in the free $\{\oplus, \otimes\}$-algebra X^{fr} in Definition 3.1.2. ◇

Example 5.2.15. The first few right normalized bracketings are

$$x_1, \quad x_1 \odot x_2, \quad x_1 \odot (x_2 \odot x_3), \quad \text{and} \quad x_1 \odot (x_2 \odot (x_3 \odot x_4)).$$

The first few left normalized bracketings are

$$x_1, \quad x_1 \odot x_2, \quad (x_1 \odot x_2) \odot x_3, \quad \text{and} \quad ((x_1 \odot x_2) \odot x_3) \odot x_4. \qquad ◇$$

We now interpret objects in A as objects in C. The following definition uses the structure $(\oplus, \mathbb{0}, \otimes, \mathbb{1})$ in the assumed symmetric bimonoidal category C.

Definition 5.2.16. Continuing Definition 5.2.3, define a function

$$(5.2.17)\qquad \mathsf{Ob}(\mathsf{A}) \xrightarrow{\ \pi\ } \mathsf{Ob}(\mathsf{C})$$

as follows.

- For a finite sequence $(a_1, \ldots, a_m)$ of objects in C, first define the object

$$\pi(a_1, \ldots, a_m) = \begin{cases} \mathbb{1} \in \mathsf{C} & \text{if } m = 0 \text{ and} \\ (a_1 \otimes \cdots \otimes a_m)_{\mathsf{rt}} \in \mathsf{C} & \text{if } m > 0, \end{cases} \tag{5.2.18}$$

  with the right normalized bracketing (5.2.12) defined in $(\mathsf{C}, \otimes)$.
- For an object $\underline{a} = \{a^1, \ldots, a^r\} \in \mathsf{Ob}(\mathsf{A})$ as in (5.2.4), define the object

$$\pi\underline{a} = \begin{cases} \mathbb{0} \in \mathsf{C} & \text{if } r = 0 \text{ and} \\ \left(\pi a^1 \oplus \cdots \oplus \pi a^r\right)_{\mathsf{rt}} \in \mathsf{C} & \text{if } r > 0, \end{cases} \tag{5.2.19}$$

  with the right normalized bracketing defined in $(\mathsf{C}, \oplus)$ and each πa^i as in (5.2.18).

This finishes the definition of the function π. ◇

Example 5.2.20. There are objects

$$\begin{aligned} \pi\mathbb{0}^{\mathsf{A}} &= \mathbb{0} \\ \pi\mathbb{1}^{\mathsf{A}} &= \mathbb{1} \\ \pi\left\{(a_1^1, a_2^1), (\mathbb{1}^{\mathsf{A}}), (a_1^3, a_2^3, a_3^3)\right\} &= (a_1^1 \otimes a_2^1) \oplus \left[\mathbb{1} \oplus \left(a_1^3 \otimes (a_2^3 \otimes a_3^3)\right)\right] \\ \pi\left(\underline{a} \oplus^{\mathsf{A}} \underline{b}\right) &= \left(\pi a^1 \oplus \cdots \oplus \pi a^r \oplus \pi b^1 \oplus \cdots \oplus \pi b^s\right)_{\mathsf{rt}} \end{aligned}$$

in C for objects $\underline{a}, \underline{b} \in \mathsf{Ob}(\mathsf{A})$ with additive lengths $r, s > 0$. ◇

Definition 5.2.21. Continuing Definition 5.2.16, define the following.

- For objects $\underline{a}, \underline{b} \in \mathsf{Ob}(\mathsf{A})$, define the *morphism* set

$$\mathsf{A}(\underline{a}; \underline{b}) = \mathsf{C}(\pi\underline{a}; \pi\underline{b}) \tag{5.2.22}$$

  with π the function in (5.2.17).
- Define the *identity morphism* $1_{\underline{a}} = 1_{\pi\underline{a}} \in \mathsf{A}(\underline{a}; \underline{a})$.
- Define the *composition* in A

$$\mathsf{A}(\underline{b}; \underline{c}) \times \mathsf{A}(\underline{a}; \underline{b}) = \mathsf{C}(\pi\underline{b}; \pi\underline{c}) \times \mathsf{C}(\pi\underline{a}; \pi\underline{b}) \longrightarrow \mathsf{C}(\pi\underline{a}; \pi\underline{c}) = \mathsf{A}(\underline{a}; \underline{c})$$

  as the one in C. ◇

Since C is a category, we obtain the following.

Lemma 5.2.23. *Under Definitions 5.2.3 and 5.2.21,* A *is a category.*

To define the sum and the product of morphisms in A, the following notations are used. Recall from Definitions 3.1.9 and 3.1.14 the graph of X and the value in C of a path in $\mathsf{Gr}(X)$.

Definition 5.2.24.

(1) A *Mac Lane coherence isomorphism* in C, which is denoted by $\cong^{\oplus}_{\mathsf{ML}}$, is the value $\varphi P : \varphi u \longrightarrow \varphi v$ in C of a path $P : u \longrightarrow v$ in $\mathsf{Gr}(X)$ that satisfies the following three conditions.
   (i) P only involves identities, $\alpha^{\pm\oplus}$, $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, and $\xi^{\pm\oplus}$.
   (ii) In addition to the distinguished elements $\{0^X, 1^X\}$, the set X contains a specific element x_m for each monomial m in each object in A that appears in φu.

(iii) For each monomial m in (ii), the equality

$$\varphi(x_m) = \pi(m) \in \mathsf{C}$$

holds, with $\pi(m)$ as in (5.2.18).

(2) A *Laplaza coherence isomorphism* in C, which is denoted by $\cong_{\mathsf{Lap}}$, is the value $\varphi P : \varphi v \longrightarrow \varphi w$ in C of a path $P : v \longrightarrow w$ in $\mathsf{Gr}(X)$ that satisfies the following three conditions.

(i) P does *not* involve $\xi^{\pm\otimes}$.

(ii) In addition to the distinguished elements $\{0^X, 1^X\}$, the set X contains a specific element x_a for each alphabet a in each object in A that appears in φv.

(iii) For each alphabet a in (ii), the equality

$$\varphi(x_a) = a \in \mathsf{C}$$

holds.

(3) The inverse of a Laplaza coherence isomorphism is denoted by $\cong_{\mathsf{Lap}}^{-1}$. ◇

Explanation 5.2.25. Consider Definition 5.2.24.

(1) A Mac Lane coherence isomorphism is in particular a composite of isomorphisms, each being a $\oplus$ and $\otimes$ of identity morphisms and at most one component of $\alpha^{\pm\oplus}$, $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, and $\xi^{\pm\oplus}$.

(2) A Laplaza coherence isomorphism is in particular a composite of isomorphisms, each being a $\oplus$ and $\otimes$ of identity morphisms and at most one component of $\alpha^{\pm\oplus}$, $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, $\lambda^{\pm\otimes}$, $\rho^{\pm\otimes}$, $\lambda^{\pm\bullet}$, $\rho^{\pm\bullet}$, δ^l, and δ^r. Laplaza coherence isomorphisms exclude $\xi^{\pm\otimes}$ because in each construction below involving $\cong_{\mathsf{Lap}}$, $\xi^{\pm\otimes}$ are never used.

(3) The inverse of a Laplaza coherence isomorphism has the same description as a Laplaza coherence isomorphism, but with δ^l and δ^r replaced by their inverses. This only makes sense if the symmetric bimonoidal category is tight. Moreover, a Mac Lane coherence isomorphism is also a Laplaza coherence isomorphism. ◇

Next we extend the sum and the product in A, defined for objects in (5.2.6) and (5.2.7), to morphisms. The product of two morphisms involves the inverse of a Laplaza coherence isomorphism. This is where we first use the tightness assumption on the given symmetric bimonoidal category C. Recall from Example 3.9.6 that Theorem 3.9.1 applies to each tight symmetric bimonoidal category.

Definition 5.2.26. Continuing Definition 5.2.21, suppose given two morphisms

$$\begin{aligned} &\underline{a} \xrightarrow{f} \underline{b} \in \mathsf{A}(\underline{a};\underline{b}) = \mathsf{C}(\pi\underline{a};\pi\underline{b}) \\ &\underline{c} \xrightarrow{g} \underline{d} \in \mathsf{A}(\underline{c};\underline{d}) = \mathsf{C}(\pi\underline{c};\pi\underline{d}) \end{aligned} \tag{5.2.27}$$

with $\underline{a}, \underline{b}, \underline{c}, \underline{d} \in \mathsf{Ob}(\mathsf{A})$. Their *sum* and *product* are, respectively, the morphisms

$$\begin{aligned} &\underline{a} \oplus^{\mathsf{A}} \underline{c} \xrightarrow{f \oplus^{\mathsf{A}} g} \underline{b} \oplus^{\mathsf{A}} \underline{d} \in \mathsf{A}\big(\underline{a} \oplus^{\mathsf{A}} \underline{c}; \underline{b} \oplus^{\mathsf{A}} \underline{d}\big) = \mathsf{C}\big(\pi(\underline{a} \oplus^{\mathsf{A}} \underline{c}); \pi(\underline{b} \oplus^{\mathsf{A}} \underline{d})\big) \\ &\underline{a} \otimes^{\mathsf{A}} \underline{c} \xrightarrow{f \otimes^{\mathsf{A}} g} \underline{b} \otimes^{\mathsf{A}} \underline{d} \in \mathsf{A}\big(\underline{a} \otimes^{\mathsf{A}} \underline{c}; \underline{b} \otimes^{\mathsf{A}} \underline{d}\big) = \mathsf{C}\big(\pi(\underline{a} \otimes^{\mathsf{A}} \underline{c}); \pi(\underline{b} \otimes^{\mathsf{A}} \underline{d})\big) \end{aligned} \tag{5.2.28}$$

defined as the following composites in C.

$$\begin{tikzcd}
\pi(\underline{a} \oplus^{\mathsf{A}} \underline{c}) \arrow[r, "f \oplus^{\mathsf{A}} g"] \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"'] & \pi(\underline{b} \oplus^{\mathsf{A}} \underline{d}) \\
\pi\underline{a} \oplus \pi\underline{c} \arrow[r, "f \oplus g"] & \pi\underline{b} \oplus \pi\underline{d} \arrow[u, "\cong^{\oplus}_{\mathsf{ML}}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi(\underline{a} \otimes^{\mathsf{A}} \underline{c}) \arrow[r, "f \otimes^{\mathsf{A}} g"] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{d}) \\
\pi\underline{a} \otimes \pi\underline{c} \arrow[r, "f \otimes g"] & \pi\underline{b} \otimes \pi\underline{d} \arrow[u, "\cong_{\mathsf{Lap}}"']
\end{tikzcd} \tag{5.2.29}$$

More detail of these two diagrams are given in Explanations 5.2.30 and 5.2.31 below. ◇

Explanation 5.2.30. Consider the left diagram in (5.2.29) that defines $f \oplus^{\mathsf{A}} g$.

- If $\underline{a} = \mathbb{0}^{\mathsf{A}}$, then the left vertical isomorphism is $\lambda^{-\oplus}_{\pi\underline{c}}$.
- If $\underline{c} = \mathbb{0}^{\mathsf{A}}$, then the left vertical isomorphism is $\rho^{-\oplus}_{\pi\underline{a}}$.
- If $\underline{b} = \mathbb{0}^{\mathsf{A}}$, then the right vertical isomorphism is $\lambda^{\oplus}_{\pi\underline{d}}$.
- If $\underline{d} = \mathbb{0}^{\mathsf{A}}$, then the right vertical isomorphism is $\rho^{\oplus}_{\pi\underline{b}}$.
- If none of $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$ is $\mathbb{0}^{\mathsf{A}}$, then the vertical isomorphisms involve only identity morphisms and $\alpha^{\pm\oplus}$.

The existence and the uniqueness of the vertical isomorphisms are guaranteed by Mac Lane's Coherence Theorem 1.3.3. ◇

Explanation 5.2.31. Consider the right diagram in (5.2.29) that defines $f \otimes^{\mathsf{A}} g$.

- If $\underline{a} = \mathbb{0}^{\mathsf{A}}$ or $\mathbb{1}^{\mathsf{A}}$, then the left vertical isomorphism is $\lambda^{-\bullet}_{\pi\underline{c}}$ or $\lambda^{-\otimes}_{\pi\underline{c}}$, respectively.
- If $\underline{c} = \mathbb{0}^{\mathsf{A}}$ or $\mathbb{1}^{\mathsf{A}}$, then the left vertical isomorphism is $\rho^{-\bullet}_{\pi\underline{a}}$ or $\rho^{-\otimes}_{\pi\underline{a}}$, respectively.
- If $\underline{b} = \mathbb{0}^{\mathsf{A}}$ or $\mathbb{1}^{\mathsf{A}}$, then the right vertical isomorphism is $\lambda^{\bullet}_{\pi\underline{d}}$ or $\lambda^{\otimes}_{\pi\underline{d}}$, respectively.
- If $\underline{d} = \mathbb{0}^{\mathsf{A}}$ or $\mathbb{1}^{\mathsf{A}}$, then the right vertical isomorphism is $\rho^{\bullet}_{\pi\underline{b}}$ or $\rho^{\otimes}_{\pi\underline{b}}$, respectively.
- If none of $\underline{a}$, $\underline{b}$, $\underline{c}$, and $\underline{d}$ is $\mathbb{0}^{\mathsf{A}}$ or $\mathbb{1}^{\mathsf{A}}$, then the following statements hold.
  - $\cong^{-1}_{\mathsf{Lap}}$ involves only identity morphisms, $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\alpha^{-\otimes}$, $\lambda^{-\otimes}$, $\rho^{-\otimes}$, δ^{-l}, and δ^{-r}.
  - $\cong_{\mathsf{Lap}}$ involves only identity morphisms, $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, δ^{l}, and δ^{r}.

  In $\cong_{\mathsf{Lap}}$, $\lambda^{\otimes}$ and $\rho^{\otimes}$ are involved if and only if, respectively, $\underline{b}$ and $\underline{d}$ contain monomials with multiplicative length 0. Similarly, in $\cong^{-1}_{\mathsf{Lap}}$, $\lambda^{-\otimes}$ and $\rho^{-\otimes}$ are involved if and only if, respectively, $\underline{a}$ and $\underline{c}$ contain monomials with multiplicative length 0.

The existence of the left vertical isomorphism follows from an induction on the sum of the additive lengths of $\underline{a}$ and $\underline{c}$. The existence of the right vertical isomorphism follows similarly from an induction on the sum of the additive lengths of $\underline{b}$ and $\underline{d}$. The uniqueness of the vertical isomorphisms is guaranteed by Theorem 3.9.1. ◇

Example 5.2.32. Here is an example of $\cong_{\mathsf{Lap}}$ in the right diagram in (5.2.29). Suppose

- $\underline{b} = \{(b), (b_1, b_2)\}$ and
- $\underline{d} = \{(d_1, d_2), (d)\}$

are objects in A with b, b_1, b_2, d_1, d_2, $d \in \mathsf{C}$. To save space, we will abbreviate $\otimes$ in C using concatenation, with $\otimes$ taking precedence over $\oplus$. Then there are the following objects.

$$\begin{aligned}
\pi\underline{b} &= b \oplus b_1 b_2 \in \mathsf{C} \\
\pi\underline{d} &= d_1 d_2 \oplus d \in \mathsf{C} \\
\pi\underline{b} \otimes \pi\underline{d} &= \big(b \oplus b_1 b_2\big)\big(d_1 d_2 \oplus d\big) \\
\underline{b} \otimes^{\mathsf{A}} \underline{d} &= \big\{(b, d_1, d_2), (b, d), (b_1, b_2, d_1, d_2), (b_1, b_2, d)\big\} \in \mathsf{A} \\
\pi(\underline{b} \otimes^{\mathsf{A}} \underline{d}) &= b(d_1 d_2) \oplus \big\{bd \oplus \big[b_1(b_2(d_1 d_2)) \oplus b_1(b_2 d)\big]\big\} \in \mathsf{C}
\end{aligned}$$

To go from $\pi\underline{b} \otimes \pi\underline{d}$ to $\pi(\underline{b} \otimes^{\mathsf{A}} \underline{d})$, we regard b, b_1, b_2, d_1, d_2, and d as formal distinct symbols, even if some of them may be the same object in C. Starting from $\pi\underline{b} \otimes \pi\underline{d}$, we distribute using δ^l and δ^r as far as possible. By the constructions of $\otimes^{\mathsf{A}}$ and π, the result matches $\pi(\underline{b} \otimes^{\mathsf{A}} \underline{d})$, except possibly for

- the additive bracketing,
- the order of the monomials, and
- the multiplicative bracketing of each monomial.

To correct these discrepancies, we apply $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, and $\alpha^{\pm\otimes}$.

The following composite in C is one example of a Laplaza coherence isomorphism $\cong_{\mathsf{Lap}}$. Subscripts in the morphisms are omitted to save space.

$$\begin{tikzcd}
\pi\underline{b} \otimes \pi\underline{d} \arrow[rr, "\cong_{\mathsf{Lap}}"] \arrow[d, "\delta^r"'] & & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{d}) \\
b(d_1 d_2 \oplus d) \oplus (b_1 b_2)(d_1 d_2 \oplus d) \arrow[d, "\delta^l \oplus 1"'] & & b(d_1 d_2) \oplus \big\{bd \oplus \big[b_1(b_2(d_1 d_2)) \oplus (b_1 b_2)d\big]\big\} \arrow[u, "1 \oplus \{1 \oplus [1 \oplus \alpha^{\otimes}]\}"'] \\
\big[b(d_1 d_2) \oplus bd\big] \oplus (b_1 b_2)(d_1 d_2 \oplus d) \arrow[dr, "1 \oplus \delta^l"'] & & b(d_1 d_2) \oplus \big\{bd \oplus \big[(b_1 b_2)(d_1 d_2) \oplus (b_1 b_2)d\big]\big\} \arrow[u, "1 \oplus \{1 \oplus [\alpha^{\otimes} \oplus 1]\}"'] \\
& \big[b(d_1 d_2) \oplus bd\big] \oplus \big[(b_1 b_2)(d_1 d_2) \oplus (b_1 b_2)d\big] \arrow[ur, "\alpha^{\oplus}"'] &
\end{tikzcd}$$

Another Laplaza coherence isomorphism

$$\pi\underline{b} \otimes \pi\underline{d} \xrightarrow{\cong_{\mathsf{Lap}}} \pi(\underline{b} \otimes^{\mathsf{A}} \underline{d})$$

starts with δ^l, followed by two morphisms involving δ^r, then followed by morphisms involving $\xi^{\oplus}$, $\alpha^{\pm\oplus}$, and $\alpha^{\otimes}$. This composite involves an instance of $\xi^{\oplus}$ to swap the middle two terms to match those in $\pi(\underline{b} \otimes^{\mathsf{A}} \underline{d})$. In each case, $\xi^{\otimes}$ is not needed.

To see that these two Laplaza coherence isomorphisms are equal in C, we apply Theorem 3.9.1 as follows. Consider the set

$$X = \big\{0^X, 1^X, b, b_1, b_2, d_1, d_2, d\big\}$$

as in Definition 3.1.6, with b, b_1, b_2, d_1, d_2, and d regarded as formal distinct symbols different from 0^X and 1^X. The element

$$\big(b \oplus (b_1 \otimes b_2)\big) \otimes \big((d_1 \otimes d_2) \oplus d\big) \in X^{\mathsf{fr}}$$

is regular as in Definition 3.1.25 because its support (3.1.24) is

$$bd_1d_2 \oplus bd \oplus b_1b_2d_1d_2 \oplus b_1b_2d \in X^{\mathsf{st}}.$$

Here $\otimes$ is abbreviated to concatenation as above, and the support of an element is denoted by the same symbol.

The graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ in Definition 3.1.14 is defined as follows:

- $\varphi(0^X) = \mathbb{0}$ and $\varphi(1^X) = \mathbb{1}$.
- φ sends each of the formal symbols b, b_1, b_2, d_1, d_2, and d in X to the corresponding object in C.

Each of the two Laplaza coherence isomorphisms above is the value in C, in the sense of (3.1.18), of a path in $\mathsf{Gr}(X)$ that does not involve $\xi^{\pm\otimes}$. Theorem 3.9.1 implies that these two Laplaza coherence isomorphisms are equal. $\diamond$

The following observation allows us to define the associativity isomorphisms and the unit isomorphisms in A to be identities.

Lemma 5.2.33. *For morphisms $f_i \in \mathsf{A}(\underline{a}_i;\underline{b}_i)$ for $1 \le i \le 3$, the following equalities hold.*

$$\begin{gathered}
(f_1 \oplus^{\mathsf{A}} f_2) \oplus^{\mathsf{A}} f_3 = f_1 \oplus^{\mathsf{A}} (f_2 \oplus^{\mathsf{A}} f_3)\\
(f_1 \otimes^{\mathsf{A}} f_2) \otimes^{\mathsf{A}} f_3 = f_1 \otimes^{\mathsf{A}} (f_2 \otimes^{\mathsf{A}} f_3)\\
1_{\mathbb{0}^{\mathsf{A}}} \oplus^{\mathsf{A}} f_1 = f_1 = f_1 \oplus^{\mathsf{A}} 1_{\mathbb{0}^{\mathsf{A}}}\\
1_{\mathbb{1}^{\mathsf{A}}} \otimes^{\mathsf{A}} f_1 = f_1 = f_1 \otimes^{\mathsf{A}} 1_{\mathbb{1}^{\mathsf{A}}}
\end{gathered}$$

Proof. In Lemma 5.2.8, we observed that $\oplus^{\mathsf{A}}$ is strictly associative on objects with $\mathbb{0}^{\mathsf{A}}$ as a strict two-sided unit, and similarly for $(\otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}})$.

For the first equality, consider the following diagram in C.

$$\begin{tikzcd}
\pi\big((\underline{a}_1 \oplus^{\mathsf{A}} \underline{a}_2) \oplus^{\mathsf{A}} \underline{a}_3\big) \arrow[r, "="] \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"'] & \pi\big(\underline{a}_1 \oplus^{\mathsf{A}} (\underline{a}_2 \oplus^{\mathsf{A}} \underline{a}_3)\big) \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"] \\
(\pi\underline{a}_1 \oplus \pi\underline{a}_2) \oplus \pi\underline{a}_3 \arrow[r, "\alpha^{\oplus}"] \arrow[d, "(f_1 \oplus f_2) \oplus f_3"'] & \pi\underline{a}_1 \oplus (\pi\underline{a}_2 \oplus \pi\underline{a}_3) \arrow[d, "f_1 \oplus (f_2 \oplus f_3)"] \\
(\pi\underline{b}_1 \oplus \pi\underline{b}_2) \oplus \pi\underline{b}_3 \arrow[r, "\alpha^{\oplus}"] \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"'] & \pi\underline{b}_1 \oplus (\pi\underline{b}_2 \oplus \pi\underline{b}_3) \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"] \\
\pi\big((\underline{b}_1 \oplus^{\mathsf{A}} \underline{b}_2) \oplus^{\mathsf{A}} \underline{b}_3\big) \arrow[r, "="] & \pi\big(\underline{b}_1 \oplus^{\mathsf{A}} (\underline{b}_2 \oplus^{\mathsf{A}} \underline{b}_3)\big)
\end{tikzcd}$$

The left vertical composite defines $(f_1 \oplus^{\mathsf{A}} f_2) \oplus^{\mathsf{A}} f_3$, and the right vertical composite defines $f_1 \oplus^{\mathsf{A}} (f_2 \oplus^{\mathsf{A}} f_3)$. The middle rectangle is commutative by the naturality of $\alpha^{\oplus}$ in C. The top and the bottom rectangles are commutative by Mac Lane's Coherence Theorem 1.3.3. This proves the first equality.

The second equality is proved by modifying the above diagram by replacing

- $(\oplus^{\mathsf{A}}, \oplus, \alpha^{\oplus})$ with $(\otimes^{\mathsf{A}}, \otimes, \alpha^{\otimes})$,
- the top two $\cong^{\oplus}_{\mathsf{ML}}$ with $\cong^{-1}_{\mathsf{Lap}}$, and
- the bottom two $\cong^{\oplus}_{\mathsf{ML}}$ with $\cong_{\mathsf{Lap}}$.

The resulting diagram commutes by the naturality of $\alpha^{\otimes}$ in C and Theorem 3.9.1.

Using Explanation 5.2.30, the equalities

$$1_{\mathbb{0}^{\mathsf{A}}} \oplus^{\mathsf{A}} f_1 = f_1 = f_1 \oplus^{\mathsf{A}} 1_{\mathbb{0}^{\mathsf{A}}}$$

follow from the naturality of $\lambda^{\oplus}$ and $\rho^{\oplus}$ in C.

Using Explanation 5.2.31, the equalities

$$1_{\mathbb{1}^{\mathsf{A}}} \otimes^{\mathsf{A}} f_1 = f_1 = f_1 \otimes^{\mathsf{A}} 1_{\mathbb{1}^{\mathsf{A}}}$$

follow from the naturality of $\lambda^{\otimes}$ and $\rho^{\otimes}$ in C. □

Using the symmetry isomorphisms $\xi^{\oplus}$ and $\xi^{\otimes}$ in C, next we define the symmetry isomorphisms in A. The multiplicative symmetry isomorphism in A requires the inverse of a Laplaza coherence isomorphism and, therefore, uses the tightness assumption on C.

Definition 5.2.34. Continuing Definition 5.2.26, for objects $\underline{a}, \underline{b} \in \mathsf{Ob}(\mathsf{A})$, define the morphisms

$$\begin{aligned} \underline{a} \oplus^{\mathsf{A}} \underline{b} &\xrightarrow{\xi^{\oplus\mathsf{A}}_{\underline{a};\underline{b}}} \underline{b} \oplus^{\mathsf{A}} \underline{a} \in \mathsf{A}\big(\underline{a} \oplus^{\mathsf{A}} \underline{b}; \underline{b} \oplus^{\mathsf{A}} \underline{a}\big) = \mathsf{C}\big(\pi(\underline{a} \oplus^{\mathsf{A}} \underline{b}); \pi(\underline{b} \oplus^{\mathsf{A}} \underline{a})\big) \\ \underline{a} \otimes^{\mathsf{A}} \underline{b} &\xrightarrow{\xi^{\otimes\mathsf{A}}_{\underline{a};\underline{b}}} \underline{b} \otimes^{\mathsf{A}} \underline{a} \in \mathsf{A}\big(\underline{a} \otimes^{\mathsf{A}} \underline{b}; \underline{b} \otimes^{\mathsf{A}} \underline{a}\big) = \mathsf{C}\big(\pi(\underline{a} \otimes^{\mathsf{A}} \underline{b}); \pi(\underline{b} \otimes^{\mathsf{A}} \underline{a})\big) \end{aligned} \tag{5.2.35}$$

as the following composites in C.

$$\begin{tikzcd} \pi(\underline{a} \oplus^{\mathsf{A}} \underline{b}) \arrow[r, "\xi^{\oplus\mathsf{A}}_{\underline{a};\underline{b}}"] \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"'] & \pi(\underline{b} \oplus^{\mathsf{A}} \underline{a}) \\ \pi\underline{a} \oplus \pi\underline{b} \arrow[r, "\xi^{\oplus}_{\pi\underline{a};\pi\underline{b}}"] & \pi\underline{b} \oplus \pi\underline{a} \arrow[u, "\cong^{\oplus}_{\mathsf{ML}}"'] \end{tikzcd} \qquad \begin{tikzcd} \pi(\underline{a} \otimes^{\mathsf{A}} \underline{b}) \arrow[r, "\xi^{\otimes\mathsf{A}}_{\underline{a};\underline{b}}"] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{a}) \\ \pi\underline{a} \otimes \pi\underline{b} \arrow[r, "\xi^{\otimes}_{\pi\underline{a};\pi\underline{b}}"] & \pi\underline{b} \otimes \pi\underline{a} \arrow[u, "\cong_{\mathsf{Lap}}"'] \end{tikzcd} \tag{5.2.36}$$

◇

Explanation 5.2.37. The vertical isomorphisms in (5.2.36) are as in (5.2.29), which are described explicitly in Explanations 5.2.30 and 5.2.31. Moreover, $\xi^{\oplus\mathsf{A}}$ is a Mac Lane coherence isomorphism. ◇

5.3. Associated Right Bipermutative Category: Proofs

We now show in multiple steps that A is a right bipermutative category as in Definition 2.5.2.

Lemma 5.3.1. *Under Definitions 5.2.3, 5.2.21, 5.2.26, and 5.2.34,*

- $(\mathsf{A}, \oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \xi^{\oplus\mathsf{A}})$ *and*
- $(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}, \xi^{\otimes\mathsf{A}})$

are both permutative categories.

Proof. The functoriality of $\oplus^{\mathsf{A}} : \mathsf{A} \times \mathsf{A} \longrightarrow \mathsf{A}$ follows from

- its construction in (5.2.29),
- the functoriality of $\oplus$ in C, and
- the uniqueness in Theorem 1.3.8.

We observed in Lemma 5.2.8 that $\oplus^{\mathsf{A}}$ is strictly associative on objects, and $\mathbb{0}^{\mathsf{A}}$ is a strict two-sided unit. Using Lemma 5.2.33, we define

- the associativity isomorphism $\alpha^{\oplus\mathsf{A}}$,
- the left unit isomorphism $\lambda^{\oplus\mathsf{A}}$, and
- the right unit isomorphism $\rho^{\oplus\mathsf{A}}$

in A to be the identity natural transformations. Then

$$(\mathsf{A}, \oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \alpha^{\oplus\mathsf{A}}, \lambda^{\oplus\mathsf{A}}, \rho^{\oplus\mathsf{A}})$$

is a strict monoidal category because the unity axiom (1.2.2) and the pentagon axiom (1.2.3) are satisfied when each edge involved is an identity morphism.

Each component of $\xi^{\oplus\mathsf{A}}$ is by definition an isomorphism. Its naturality follows from the naturality of $\xi^{\oplus}$ in C, the definition of $f \oplus^{\mathsf{A}} g$ in (5.2.29), and the uniqueness in Theorem 1.3.8. The symmetric monoidal category axioms (1.2.20), (1.2.21), and (1.2.22) in $(\mathsf{A}, \oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \xi^{\oplus\mathsf{A}})$ follow from those in $(\mathsf{C}, \oplus)$, the naturality of $\xi^{\oplus}$, and the uniqueness in Theorem 1.3.8. Therefore, $(\mathsf{A}, \oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \xi^{\oplus\mathsf{A}})$ is a permutative category.

The proof for $(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}, \xi^{\otimes\mathsf{A}})$ is essentially the same as above, with $\alpha^{\otimes\mathsf{A}}$, $\lambda^{\otimes\mathsf{A}}$, and $\rho^{\otimes\mathsf{A}}$ defined as the identity natural transformations using Lemma 5.2.33. For the naturality of $\xi^{\otimes\mathsf{A}}$ and the symmetric monoidal category axioms in A, we use the naturality of $\xi^{\otimes}$ in C, the definition of $f \otimes^{\mathsf{A}} g$ in (5.2.29), Theorem 3.9.1, and the symmetric monoidal category axioms in $(\mathsf{C}, \otimes)$. □

Lemma 5.3.2. *Suppose $\underline{a} \in \mathsf{A}$ is an object.*

(1) The morphisms

$$\underline{a} \oplus^{\mathsf{A}} \mathbb{0}^{\mathsf{A}} \underset{\xi^{\oplus\mathsf{A}}_{\mathbb{0}^{\mathsf{A}};\underline{a}}}{\overset{\xi^{\oplus\mathsf{A}}_{\underline{a};\mathbb{0}^{\mathsf{A}}}}{\rightleftarrows}} \mathbb{0}^{\mathsf{A}} \oplus^{\mathsf{A}} \underline{a}$$

are both equal to $1_{\underline{a}}$.

(2) The morphisms

$$\underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}} \underset{\xi^{\otimes\mathsf{A}}_{\mathbb{0}^{\mathsf{A}};\underline{a}}}{\overset{\xi^{\otimes\mathsf{A}}_{\underline{a};\mathbb{0}^{\mathsf{A}}}}{\rightleftarrows}} \mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a}$$

are both equal to $1_{\mathbb{0}^{\mathsf{A}}}$.

Proof. For the first assertion, by the left diagram in (5.2.36) with $\underline{b} = \mathbb{0}^{\mathsf{A}}$ and Explanation 5.2.30, $\xi^{\oplus\mathsf{A}}_{\underline{a};\mathbb{0}^{\mathsf{A}}}$ is the following composite in C.

$$\pi\underline{a} \xrightarrow{\rho^{-\oplus}_{\pi\underline{a}}} \pi\underline{a} \oplus \mathbb{0} \xrightarrow{\xi^{\oplus}_{\pi\underline{a};\mathbb{0}}} \mathbb{0} \oplus \pi\underline{a} \xrightarrow{\lambda^{\oplus}_{\pi\underline{a}}} \pi\underline{a}$$

This composite is equal to $1_{\pi\underline{a}} = 1_{\underline{a}}$ by the unit axiom (1.2.21) in the additive structure in C. The other morphism $\xi^{\oplus\mathsf{A}}_{\mathbb{0}^{\mathsf{A}};\underline{a}}$ is also equal to $1_{\underline{a}}$ by the symmetry axiom (1.2.20) in the permutative category $(\mathsf{A}, \oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \xi^{\oplus\mathsf{A}})$.

For the second assertion, by the right diagram in (5.2.36) with $\underline{b} = \mathbb{0}^{\mathsf{A}}$ and Explanation 5.2.31, $\xi^{\otimes\mathsf{A}}_{\underline{a};\mathbb{0}^{\mathsf{A}}}$ is the following composite in C.

$$\mathbb{0} \xrightarrow{\rho^{-\bullet}_{\pi\underline{a}}} \pi\underline{a} \otimes \mathbb{0} \xrightarrow{\xi^{\otimes}_{\pi\underline{a};\mathbb{0}}} \mathbb{0} \otimes \pi\underline{a} \xrightarrow{\lambda^{\bullet}_{\pi\underline{a}}} \mathbb{0}$$

This composite is equal to $1_{\mathbb{0}} = 1_{\mathbb{0}^{\mathsf{A}}}$ by axiom (2.1.19) in C. The other morphism $\xi^{\otimes\mathsf{A}}_{\mathbb{0}^{\mathsf{A}};\underline{a}}$ is also equal to $1_{\mathbb{0}^{\mathsf{A}}}$ by the symmetry axiom (1.2.20) in the permutative category $(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}, \xi^{\otimes\mathsf{A}})$. □

The next observation allows us to define the multiplicative zeros in A to be the identities.

Lemma 5.3.3. *For each morphism $f \in \mathsf{A}(\underline{a};\underline{b})$, the equalities*

$$f \otimes^{\mathsf{A}} 1_{\mathbb{0}^{\mathsf{A}}} = 1_{\mathbb{0}^{\mathsf{A}}} = 1_{\mathbb{0}^{\mathsf{A}}} \otimes^{\mathsf{A}} f$$

hold in $\mathsf{A}(\mathbb{0}^{\mathsf{A}};\mathbb{0}^{\mathsf{A}})$.

Proof. We observed in Lemma 5.2.8 that

$$\underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}} = \mathbb{0}^{\mathsf{A}} = \mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a}$$

for each object $\underline{a}$ in A. By the right diagram in (5.2.29) with

$$g = 1_{\mathbb{0}^{\mathsf{A}}} = 1_{\pi\mathbb{0}^{\mathsf{A}}} = 1_{\mathbb{0}}$$

and Explanation 5.2.31, $f \otimes^{\mathsf{A}} 1_{\mathbb{0}^{\mathsf{A}}}$ is the following composite in C.

$$\mathbb{0} \xrightarrow{\rho^{-\bullet}_{\pi\underline{a}}} \pi\underline{a} \otimes \mathbb{0} \xrightarrow{f \otimes 1_{\mathbb{0}}} \pi\underline{b} \otimes \mathbb{0} \xrightarrow{\rho^{\bullet}_{\pi\underline{b}}} \mathbb{0}$$

This composite is equal to $1_{\mathbb{0}}$ by the naturality of $\rho^{\bullet}$ in C. The equality

$$1_{\mathbb{0}^{\mathsf{A}}} = 1_{\mathbb{0}^{\mathsf{A}}} \otimes^{\mathsf{A}} f$$

follows similarly from the naturality of $\lambda^{\bullet}$ in C. □

The next observation allows us to define the right distributivity in A to be the identity.

Lemma 5.3.4. *For morphisms $f_i \in \mathsf{A}(\underline{a}_i;\underline{b}_i)$ for $1 \le i \le 3$, the equality*

$$(f_1 \oplus^{\mathsf{A}} f_2) \otimes^{\mathsf{A}} f_3 = (f_1 \otimes^{\mathsf{A}} f_3) \oplus^{\mathsf{A}} (f_2 \otimes^{\mathsf{A}} f_3)$$

holds in $\mathsf{A}\big((\underline{a}_1 \oplus^{\mathsf{A}} \underline{a}_2) \otimes^{\mathsf{A}} \underline{a}_3; (\underline{b}_1 \oplus^{\mathsf{A}} \underline{b}_2) \otimes^{\mathsf{A}} \underline{b}_3\big)$.

Proof. By (5.2.10), the right distributive law holds strictly for objects in A. For the desired equality, consider the following diagram in C, with $\otimes$ abbreviated to concatenation.

$$\begin{tikzcd}
\pi\big((\underline{a}_1 \oplus^{\mathsf{A}} \underline{a}_2) \otimes^{\mathsf{A}} \underline{a}_3\big) \arrow[r, "="] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] & \pi\big((\underline{a}_1 \otimes^{\mathsf{A}} \underline{a}_3) \oplus^{\mathsf{A}} (\underline{a}_2 \otimes^{\mathsf{A}} \underline{a}_3)\big) \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"] \\
(\pi\underline{a}_1 \oplus \pi\underline{a}_2)(\pi\underline{a}_3) \arrow[r, "\delta^r"] \arrow[d, "(f_1 \oplus f_2) f_3"'] & (\pi\underline{a}_1)(\pi\underline{a}_3) \oplus (\pi\underline{a}_2)(\pi\underline{a}_3) \arrow[d, "f_1 f_3 \oplus f_2 f_3"] \\
(\pi\underline{b}_1 \oplus \pi\underline{b}_2)(\pi\underline{b}_3) \arrow[r, "\delta^r"] \arrow[d, "\cong_{\mathsf{Lap}}"'] & (\pi\underline{b}_1)(\pi\underline{b}_3) \oplus (\pi\underline{b}_2)(\pi\underline{b}_3) \arrow[d, "\cong_{\mathsf{Lap}}"] \\
\pi\big((\underline{b}_1 \oplus^{\mathsf{A}} \underline{b}_2) \otimes^{\mathsf{A}} \underline{b}_3\big) \arrow[r, "="] & \pi\big((\underline{b}_1 \otimes^{\mathsf{A}} \underline{b}_3) \oplus^{\mathsf{A}} (\underline{b}_2 \otimes^{\mathsf{A}} \underline{b}_3)\big)
\end{tikzcd}$$

By definition (5.2.29), the left vertical composite is

$$(f_1 \oplus^{\mathsf{A}} f_2) \otimes^{\mathsf{A}} f_3,$$

and the right vertical composite is

$$(f_1 \otimes^{\mathsf{A}} f_3) \oplus^{\mathsf{A}} (f_2 \otimes^{\mathsf{A}} f_3).$$

The middle rectangle is commutative by the naturality of δ^r in C. The top and the bottom rectangles are commutative by Theorem 3.9.1 in C. □

Definition 5.3.5. Consider the category A.

- Define the right distributivity morphism

$$(\underline{a} \oplus^{\mathsf{A}} \underline{b}) \otimes^{\mathsf{A}} \underline{c} \xrightarrow{\delta^{r\mathsf{A}}_{\underline{a};\underline{b};\underline{c}}} (\underline{a} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{b} \otimes^{\mathsf{A}} \underline{c})$$

for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$ as the identity natural transformation. This is well defined by Lemma 5.3.4.

- Define the left distributivity $\delta^{l\mathsf{A}}$ as the composite

$$\begin{tikzcd}
\underline{a} \otimes^{\mathsf{A}} (\underline{b} \oplus^{\mathsf{A}} \underline{c}) \arrow[r, "\delta^{l\mathsf{A}}_{\underline{a};\underline{b};\underline{c}}"] \arrow[d, "\xi^{\otimes\mathsf{A}}_{\underline{a};\underline{b}\oplus^{\mathsf{A}}\underline{c}}"'] & (\underline{a} \otimes^{\mathsf{A}} \underline{b}) \oplus^{\mathsf{A}} (\underline{a} \otimes^{\mathsf{A}} \underline{c}) \\
(\underline{b} \oplus^{\mathsf{A}} \underline{c}) \otimes^{\mathsf{A}} \underline{a} \arrow[r, "\delta^{r\mathsf{A}}_{\underline{b};\underline{c};\underline{a}}", "="'] & (\underline{b} \otimes^{\mathsf{A}} \underline{a}) \oplus^{\mathsf{A}} (\underline{c} \otimes^{\mathsf{A}} \underline{a}) \arrow[u, "\xi^{\otimes\mathsf{A}}_{\underline{b};\underline{a}} \oplus^{\mathsf{A}} \xi^{\otimes\mathsf{A}}_{\underline{c};\underline{a}}"']
\end{tikzcd} \tag{5.3.6}$$

for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$. Its naturality follows from the naturality of $\xi^{\otimes\mathsf{A}}$ and the functoriality of $\oplus^{\mathsf{A}}$. ◇

Next we check axioms (2.1.7) and (2.1.13) in A.

Lemma 5.3.7. *Axiom (2.1.7) holds in* A.

Proof. Axiom (2.1.7) is the following diagram for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$.

$$\begin{tikzcd}
(\underline{a} \oplus^{\mathsf{A}} \underline{b}) \otimes^{\mathsf{A}} \underline{c} \arrow[r, "\delta^{r\mathsf{A}}", "="'] \arrow[d, "\xi^{\oplus\mathsf{A}}_{\underline{a};\underline{b}} \otimes^{\mathsf{A}} 1_{\underline{c}}"'] & (\underline{a} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{b} \otimes^{\mathsf{A}} \underline{c}) \arrow[d, "\xi^{\oplus\mathsf{A}}_{\underline{a}\otimes^{\mathsf{A}}\underline{c};\underline{b}\otimes^{\mathsf{A}}\underline{c}}"] \\
(\underline{b} \oplus^{\mathsf{A}} \underline{a}) \otimes^{\mathsf{A}} \underline{c} \arrow[r, "\delta^{r\mathsf{A}}", "="'] & (\underline{b} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{a} \otimes^{\mathsf{A}} \underline{c})
\end{tikzcd}$$

By (5.2.29) and (5.2.36), the above diagram is the outer diagram in C below.

$$\begin{tikzcd}
\pi\big((\underline{a} \oplus^{\mathsf{A}} \underline{b}) \otimes^{\mathsf{A}} \underline{c}\big) \arrow[rr, "="] \arrow[dd, "\cong_{\mathsf{Lap}}^{-1}"'] & & \pi\big((\underline{a} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{b} \otimes^{\mathsf{A}} \underline{c})\big) \arrow[dd, "\cong^{\oplus}_{\mathsf{ML}}"] \\
& (\pi\underline{a})(\pi\underline{c}) \oplus (\pi\underline{b})(\pi\underline{c}) \arrow[dr, "\cong_{\mathsf{Lap}} \oplus \cong_{\mathsf{Lap}}"] \arrow[ddd, "\xi^{\oplus}"] & \\
(\pi\underline{a} \oplus \pi\underline{b})(\pi\underline{c}) \arrow[ur, "\delta^r"'] \arrow[d, "\xi^{\oplus}1"'] & & \pi(\underline{a} \otimes^{\mathsf{A}} \underline{c}) \oplus \pi(\underline{b} \otimes^{\mathsf{A}} \underline{c}) \arrow[d, "\xi^{\oplus}"] \\
(\pi\underline{b} \oplus \pi\underline{a})(\pi\underline{c}) \arrow[dr, "\delta^r"] \arrow[dd, "\cong_{\mathsf{Lap}}"'] & & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{c}) \oplus \pi(\underline{a} \otimes^{\mathsf{A}} \underline{c}) \arrow[dd, "\cong^{\oplus}_{\mathsf{ML}}"] \\
& (\pi\underline{b})(\pi\underline{c}) \oplus (\pi\underline{a})(\pi\underline{c}) \arrow[ur, "\cong_{\mathsf{Lap}} \oplus \cong_{\mathsf{Lap}}"] & \\
\pi\big((\underline{b} \oplus^{\mathsf{A}} \underline{a}) \otimes^{\mathsf{A}} \underline{c}\big) \arrow[rr, "="] & & \pi\big((\underline{b} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{a} \otimes^{\mathsf{A}} \underline{c})\big)
\end{tikzcd}$$

The top and the bottom subdiagrams are commutative by Theorem 3.9.1. The middle left trapezoid is commutative by axiom (2.1.7) in C. The middle right trapezoid is commutative by the naturality of $\xi^{\oplus}$. □

Lemma 5.3.8. *Axiom (2.1.13) holds in* A.

Proof. By Lemma 5.3.1 and Definition 5.3.5, the additive associativity $\alpha^{\oplus\mathsf{A}}$ and the right distributivity $\delta^{r\mathsf{A}}$ in A are both identity natural transformations, and the left distributivity $\delta^{l\mathsf{A}}$ is the composite in (5.3.6). By the definitions of $\oplus^{\mathsf{A}}$ in (5.2.29) and $\xi^{\otimes\mathsf{A}}$ in (5.2.36), axiom (2.1.13) in A is the outer diagram in C below for objects

$\underline{a},\underline{b},\underline{c},\underline{d} \in \mathsf{A}$. To save space, we abbreviate $\oplus^{\mathsf{A}}$, $\otimes^{\mathsf{A}}$, and $\pi(?)$ to, respectively, $\oplus$, $\otimes$, and $?'$.

$$\begin{tikzcd}
\big((\underline{a}\oplus\underline{b})\otimes(\underline{c}\oplus\underline{d})\big)' \arrow[rr, "="] \arrow[dd, "\cong_{\mathsf{Lap}}^{-1}"'] & & \big(\underline{a}\otimes(\underline{c}\oplus\underline{d})\oplus\underline{b}\otimes(\underline{c}\oplus\underline{d})\big)' \arrow[d, "\cong_{\mathsf{ML}}^{\oplus}"] \\
& & \big(\underline{a}\otimes(\underline{c}\oplus\underline{d})\big)'\oplus\big(\underline{b}\otimes(\underline{c}\oplus\underline{d})\big)' \arrow[d, "\cong_{\mathsf{Lap}}^{-1}\oplus\cong_{\mathsf{Lap}}^{-1}"] \\
(\underline{a}\oplus\underline{b})'(\underline{c}\oplus\underline{d})' \arrow[r, "\cong_{\mathsf{ML}}^{\oplus}"] \arrow[d, "\xi^{\otimes}"'] & (\underline{a}'\oplus\underline{b}')(\underline{c}\oplus\underline{d})' \arrow[r, "\delta^r"] & \underline{a}'(\underline{c}\oplus\underline{d})'\oplus\underline{b}'(\underline{c}\oplus\underline{d})' \arrow[d, "\xi^{\otimes}\oplus\xi^{\otimes}"] \\
(\underline{c}\oplus\underline{d})'(\underline{a}\oplus\underline{b})' \arrow[d, "\cong_{\mathsf{Lap}}"'] & & (\underline{c}\oplus\underline{d})'\underline{a}'\oplus(\underline{c}\oplus\underline{d})'\underline{b}' \arrow[d, "\cong_{\mathsf{Lap}}\oplus\cong_{\mathsf{Lap}}"] \\
\big((\underline{c}\oplus\underline{d})\otimes(\underline{a}\oplus\underline{b})\big)' \arrow[rr, "\cong_{\mathsf{ML}}^{\oplus}"] \arrow[d, "="'] & & \big((\underline{c}\oplus\underline{d})\otimes\underline{a}\big)'\oplus\big((\underline{c}\oplus\underline{d})\otimes\underline{b}\big)' \arrow[d, "="] \\
\big(\underline{c}\otimes(\underline{a}\oplus\underline{b})\oplus\underline{d}\otimes(\underline{a}\oplus\underline{b})\big)' \arrow[d, "\cong_{\mathsf{ML}}^{\oplus}"'] & & \big((\underline{c}\otimes\underline{a})\oplus(\underline{d}\otimes\underline{a})\big)'\oplus\big((\underline{c}\otimes\underline{b})\oplus(\underline{d}\otimes\underline{b})\big)' \arrow[d, "\cong_{\mathsf{ML}}^{\oplus}\oplus\cong_{\mathsf{ML}}^{\oplus}"] \\
\big(\underline{c}\otimes(\underline{a}\oplus\underline{b})\big)'\oplus\big(\underline{d}\otimes(\underline{a}\oplus\underline{b})\big)' \arrow[d, "\cong_{\mathsf{Lap}}^{-1}\oplus\cong_{\mathsf{Lap}}^{-1}"'] & & \big((\underline{c}\otimes\underline{a})'\oplus(\underline{d}\otimes\underline{a})'\big)\oplus\big((\underline{c}\otimes\underline{b})'\oplus(\underline{d}\otimes\underline{b})'\big) \arrow[d, "(\cong_{\mathsf{Lap}}^{-1}\oplus\cong_{\mathsf{Lap}}^{-1})\oplus(\cong_{\mathsf{Lap}}^{-1}\oplus\cong_{\mathsf{Lap}}^{-1})"] \\
\underline{c}'(\underline{a}\oplus\underline{b})'\oplus\underline{d}'(\underline{a}\oplus\underline{b})' \arrow[r, "(\delta^l\,\cong_{\mathsf{ML}}^{\oplus})\oplus(\delta^l\,\cong_{\mathsf{ML}}^{\oplus})"] \arrow[d, "\xi^{\otimes}\oplus\xi^{\otimes}"'] & \mathsf{Z} \arrow[r, "\cong_{\mathsf{ML}}^{\oplus}"] & (\underline{c}'\underline{a}'\oplus\underline{d}'\underline{a}')\oplus(\underline{c}'\underline{b}'\oplus\underline{d}'\underline{b}') \arrow[d, "(\xi^{\otimes}\oplus\xi^{\otimes})\oplus(\xi^{\otimes}\oplus\xi^{\otimes})"] \\
(\underline{a}\oplus\underline{b})'\underline{c}'\oplus(\underline{a}\oplus\underline{b})'\underline{d}' \arrow[d, "\cong_{\mathsf{Lap}}"'] & & (\underline{a}'\underline{c}'\oplus\underline{a}'\underline{d}')\oplus(\underline{b}'\underline{c}'\oplus\underline{b}'\underline{d}') \arrow[d, "\cong_{\mathsf{Lap}}"] \\
\big((\underline{a}\oplus\underline{b})\otimes\underline{c}\oplus(\underline{a}\oplus\underline{b})\otimes\underline{d}\big)' \arrow[dr, "="'] & & \big((\underline{a}\otimes\underline{c})\oplus(\underline{a}\otimes\underline{d})\oplus(\underline{b}\otimes\underline{c})\oplus(\underline{b}\otimes\underline{d})\big)' \arrow[dl, "\cong_{\mathsf{ML}}^{\oplus}"] \\
& \big((\underline{a}\otimes\underline{c})\oplus(\underline{b}\otimes\underline{c})\oplus(\underline{a}\otimes\underline{d})\oplus(\underline{b}\otimes\underline{d})\big)' &
\end{tikzcd}$$

In the above diagram, the object Z is

$$\begin{aligned} Z &= [(\pi\underline{c})(\pi\underline{a})\oplus(\pi\underline{c})(\pi\underline{b})]\oplus[(\pi\underline{d})(\pi\underline{a})\oplus(\pi\underline{d})(\pi\underline{b})] \\ &= (\underline{c}'\underline{a}'\oplus\underline{c}'\underline{b}')\oplus(\underline{d}'\underline{a}'\oplus\underline{d}'\underline{b}'). \end{aligned}$$

Each of the four subdiagrams is commutative by Theorem 3.9.1. □

Proposition 5.3.9. *Associated to each tight symmetric bimonoidal category* C, *the tuple*

$$\big(\mathsf{A},(\oplus^{\mathsf{A}},\mathbb{0}^{\mathsf{A}},\xi^{\oplus\mathsf{A}}),(\otimes^{\mathsf{A}},\mathbb{1}^{\mathsf{A}},\xi^{\otimes\mathsf{A}}),(\lambda^{\bullet\mathsf{A}}=1,\rho^{\bullet\mathsf{A}}=1),(\delta^{r\mathsf{A}}=1,\delta^{l\mathsf{A}})\big)$$

given by Lemmas 5.3.1 and 5.3.3 and Definition 5.3.5 is a right bipermutative category.

Proof. We check the four conditions in Definition 2.5.2.

- The first two conditions say that $\lambda^{\bullet\mathsf{A}}$, $\rho^{\bullet\mathsf{A}}$, and $\delta^{r\mathsf{A}}$ are identity natural transformations. They are true by definition.
- The third condition says that $\xi^{\otimes\mathsf{A}}_{-,\mathbb{0}^{\mathsf{A}}}$ is $1_{\mathbb{0}^{\mathsf{A}}}$. This holds by Lemma 5.3.2 (2).

- Axioms (2.1.7) and (2.1.13) hold by Lemmas 5.3.7 and 5.3.8, respectively. Axiom (2.1.5) holds by the definition (5.3.6) of $\delta^{l\mathsf{A}}$.

Therefore, A is a right bipermutative category. $\square$

5.4. Strictification

In this section, we finish the proof that each tight symmetric bimonoidal category can be strictified to an equivalent right bipermutative category. There is also a variant involving a left bipermutative category. Convention 5.2.1 is still in effect, so C is a tight symmetric bimonoidal category. Moreover, A denotes the associated right bipermutative category in Proposition 5.3.9. First we define the functors that constitute an equivalence between them.

Definition 5.4.1. Define the functor $\pi : \mathsf{A} \longrightarrow \mathsf{C}$ as follows.

- Its assignment on objects is the function in (5.2.17).
- For objects $\underline{a}, \underline{b} \in \mathsf{A}$, its assignment on morphisms
$$\mathsf{A}(\underline{a};\underline{b}) \xrightarrow{\ \pi\ } \mathsf{C}(\pi\underline{a};\pi\underline{b})$$
is the identity function using (5.2.22).

That π is a functor is part of Definition 5.2.21. $\diamond$

Definition 5.4.2. Define the functor $\iota : \mathsf{C} \longrightarrow \mathsf{A}$ as follows.

- Using the notations in (5.2.4) and (5.2.5), for each object $X \in \mathsf{C}$, define the object
$$\iota X = \{(X)\} \in \mathsf{A}.$$
It has additive length 1, and its only monomial has multiplicative length 1 consisting of X.
- On morphism sets, it is the identity function
$$\mathsf{C}(X;Y) = \mathsf{C}(\pi\iota X;\pi\iota Y) = \mathsf{A}(\iota X;\iota Y).$$

That ι is a functor is part of Definition 5.2.21. $\diamond$

Lemma 5.4.3. *There is an adjoint equivalence*
$$\mathsf{A} \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathsf{C}$$
with counit $\varepsilon : \pi\iota \longrightarrow 1_{\mathsf{C}}$ *the identity natural transformation.*

Proof. By construction, $\pi\iota = 1_{\mathsf{C}}$. For objects $\underline{a} \in \mathsf{A}$ and $Y \in \mathsf{C}$, the adjunction $\pi \dashv \iota$ is defined by the natural equalities
$$\mathsf{C}(\pi\underline{a};Y) = \mathsf{C}(\pi\underline{a};\pi\iota Y) = \mathsf{A}(\underline{a};\iota Y).$$
The counit of the adjunction is the identity by construction. The unit of the adjunction associates to each object $\underline{a} \in \mathsf{A}$ the isomorphism
$$\big(\underline{a} \xrightarrow{\eta_{\underline{a}}} \iota\pi\underline{a}\big) = 1_{\pi\underline{a}} \in \mathsf{A}(\underline{a};\iota\pi\underline{a}) = \mathsf{C}(\pi\underline{a};\pi\iota\pi\underline{a}) = \mathsf{C}(\pi\underline{a};\pi\underline{a}).$$
Therefore, the unit is a natural isomorphism. $\square$

Next we observe that π and ι are symmetric bimonoidal functors. Recall from Definition 5.1.1 that a symmetric bimonoidal functor F is *unitary* if $F^0_\oplus$ and $F^0_\otimes$ are identities, and if $F^2_\oplus$ and $F^2_\otimes$ are natural isomorphisms.

Lemma 5.4.4. *There is a unitary symmetric bimonoidal equivalence* $\mathsf{A} \longrightarrow \mathsf{C}$ *with underlying functor* π.

Proof. Since $\pi\mathbb{0}^{\mathsf{A}} = \mathbb{0}$ and $\pi\mathbb{1}^{\mathsf{A}} = \mathbb{1}$, we define

$$\mathbb{0} \xrightarrow{\pi^0_\oplus} \pi\mathbb{0}^{\mathsf{A}}$$
$$\mathbb{1} \xrightarrow{\pi^0_\otimes} \pi\mathbb{1}^{\mathsf{A}}$$

as the identity morphisms of $\mathbb{0}$ and $\mathbb{1}$ in C, respectively. Next we define

$$\pi\underline{a} \oplus \pi\underline{b} \xrightarrow{\pi^2_\oplus} \pi(\underline{a} \oplus^{\mathsf{A}} \underline{b})$$
$$\pi\underline{a} \otimes \pi\underline{b} \xrightarrow{\pi^2_\otimes} \pi(\underline{a} \otimes^{\mathsf{A}} \underline{b})$$

for objects $\underline{a}, \underline{b} \in \mathsf{A}$ as, respectively, the Mac Lane coherence isomorphism $\cong^{\oplus}_{\mathsf{ML}}$ and the Laplaza coherence isomorphism $\cong_{\mathsf{Lap}}$ in (5.2.29) as the right vertical isomorphisms. Their naturality follows from the definitions of $\oplus^{\mathsf{A}}$ and $\otimes^{\mathsf{A}}$ on morphisms in (5.2.29), and the definition of π as the identity assignment on morphisms.

The triple $(\pi, \pi^2_\oplus, \pi^0_\oplus)$ is a symmetric monoidal functor because each diagram in Definitions 1.2.11 and 1.2.25 is commutative by Theorem 1.3.8. Similarly, the triple $(\pi, \pi^2_\otimes, \pi^0_\otimes)$ is a symmetric monoidal functor because each necessary diagram is commutative by Theorem 3.9.1.

The multiplicative zero axiom (5.1.2) holds for π because

- $\pi^0_\oplus$ and $\rho^{\bullet\mathsf{A}}$ are identity natural transformations, and
- there is an equality

$$\pi^2_\otimes = \rho^\bullet_{\pi\underline{a}} : (\pi\underline{a}) \otimes (\pi\mathbb{0}^{\mathsf{A}}) = (\pi\underline{a}) \otimes \mathbb{0} \longrightarrow \mathbb{0} = \pi(\underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}}).$$

The distributivity axiom (5.1.3) holds for π by Theorem 3.9.1. Therefore, the tuple

$$(\pi, \pi^2_\oplus, \pi^0_\oplus, \pi^2_\otimes, \pi^0_\otimes)$$

is a unitary symmetric bimonoidal functor. Finally, π is an equivalence of categories by Lemma 5.4.3. $\square$

Recall from Definition 5.1.1 that a symmetric bimonoidal functor F is *strong* if $F^2_\oplus$, $F^0_\oplus$, $F^2_\otimes$, and $F^0_\otimes$ are natural isomorphisms.

Lemma 5.4.5. *There is a strong symmetric bimonoidal equivalence* $\mathsf{C} \longrightarrow \mathsf{A}$ *with underlying functor* ι.

Proof. Define the following morphisms.

$$\left(\mathbb{0}^{\mathsf{A}} = \varnothing \xrightarrow{\iota^0_\oplus} \iota\mathbb{0} = \{(\mathbb{0})\}\right) = 1_{\mathbb{0}} \in \mathsf{A}(\mathbb{0}^{\mathsf{A}}; \iota\mathbb{0}) = \mathsf{C}(\mathbb{0}; \mathbb{0})$$
$$\left(\mathbb{1}^{\mathsf{A}} = \{\varnothing\} \xrightarrow{\iota^0_\otimes} \iota\mathbb{1} = \{(\mathbb{1})\}\right) = 1_{\mathbb{1}} \in \mathsf{A}(\mathbb{1}^{\mathsf{A}}; \iota\mathbb{1}) = \mathsf{C}(\mathbb{1}; \mathbb{1})$$

These are isomorphisms, but not identity morphisms. Define the morphisms

$$\iota X \oplus^{\mathsf{A}} \iota Y = \{(X), (Y)\} \xrightarrow{\iota^2_\oplus} \iota(X \oplus Y) \in \mathsf{A}\big(\iota X \oplus^{\mathsf{A}} \iota Y; \iota(X \oplus Y)\big) = \mathsf{C}(X \oplus Y; X \oplus Y)$$
$$\iota X \otimes^{\mathsf{A}} \iota Y = \{(X, Y)\} \xrightarrow{\iota^2_\otimes} \iota(X \otimes Y) \in \mathsf{A}\big(\iota X \otimes^{\mathsf{A}} \iota Y; \iota(X \otimes Y)\big) = \mathsf{C}(X \otimes Y; X \otimes Y)$$

for objects $X, Y \in \mathsf{C}$ as, respectively, $1_{X\oplus Y}$ and $1_{X\otimes Y}$. These are natural isomorphisms, but not identity morphisms.

Consider the triple $(\iota, \iota^2_\oplus, \iota^0_\oplus)$.

- The associativity axiom (1.2.14) holds because each composite is
$$\begin{aligned}1_{X\oplus(Y\oplus Z)} &\in \mathsf{A}\big((\iota X \oplus^{\mathsf{A}} \iota Y) \oplus^{\mathsf{A}} \iota Z; \iota(X\oplus(Y\oplus Z))\big)\\ &= \mathsf{C}\big(X\oplus(Y\oplus Z); X\oplus(Y\oplus Z)\big).\end{aligned}$$
- The left unity axiom (1.2.15) holds because each composite is
$$1_X \in \mathsf{A}\big(\mathbb{0}^{\mathsf{A}} \oplus^{\mathsf{A}} \iota X; \iota X\big) = \mathsf{C}(X;X).$$
- The right unity axiom (1.2.15) holds for the analogous reason.
- The axiom (1.2.26) holds because each composite is
$$\xi^\oplus_{X,Y} = \mathsf{A}\big(\iota X \oplus^{\mathsf{A}} \iota Y; \iota(Y\oplus X)\big) = \mathsf{C}(X\oplus Y; Y\oplus X).$$

Therefore, $(\iota, \iota^2_\oplus, \iota^0_\oplus)$ is a strong symmetric monoidal functor. The same reasoning, applied to the multiplicative structures in C and A, shows that $(\iota, \iota^2_\otimes, \iota^0_\otimes)$ is a strong symmetric monoidal functor.

The multiplicative zero axiom (5.1.2) holds for ι because each composite is
$$1_{\mathbb{0}} \in \mathsf{A}\big(\iota A \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}}; \iota\mathbb{0}\big) = \mathsf{C}(\mathbb{0};\mathbb{0}).$$
The distributivity axiom (5.1.3) holds for ι because each composite is
$$\begin{aligned}1_{AC\oplus BC} &\in \mathsf{A}\big((\iota A \oplus^{\mathsf{A}} \iota B) \otimes^{\mathsf{A}} \iota C; \iota(AC\oplus BC)\big)\\ &= \mathsf{C}(AC\oplus BC; AC\oplus BC).\end{aligned}$$
Therefore, the tuple
$$(\iota, \iota^2_\oplus, \iota^0_\oplus, \iota^2_\otimes, \iota^0_\otimes)$$
is a strong symmetric bimonoidal functor. Finally, ι is an equivalence of categories by Lemma 5.4.3. □

Combining Proposition 5.3.9 and Lemmas 5.4.3 through 5.4.5, we obtain the following strictification result.

Theorem 5.4.6 (Right Bipermutative Strictification)**.** *Suppose C is a tight symmetric bimonoidal category. Then there is an adjoint equivalence*

$$\begin{tikzcd} \mathsf{A} \arrow[r, shift left, "\pi"] & \mathsf{C} \arrow[l, shift left, "\iota"] \end{tikzcd}$$

with

- *A the right bipermutative category in Proposition 5.3.9;*
- *counit $\varepsilon : \pi\iota \longrightarrow 1_{\mathsf{C}}$ the identity natural transformation;*
- *π the unitary symmetric bimonoidal equivalence in Lemma 5.4.4; and*
- *ι the strong symmetric bimonoidal equivalence in Lemma 5.4.5.*

A minor variation of the constructions in this chapter yields the following strictification result to left bipermutative categories as in Definition 2.5.11.

Theorem 5.4.7 (Left Bipermutative Strictification)**.** *Suppose C is a tight symmetric bimonoidal category. Then there is an adjoint equivalence*

$$\begin{tikzcd} \mathsf{A}_l \arrow[r, shift left, "\pi"] & \mathsf{C} \arrow[l, shift left, "\iota"] \end{tikzcd}$$

with

- A_l *a left bipermutative category;*
- *counit* $\varepsilon : \pi\iota \longrightarrow 1_{\mathsf{C}}$ *the identity natural transformation;*
- π *a unitary symmetric bimonoidal equivalence; and*
- ι *a strong symmetric bimonoidal equivalence.*

Proof. The proof is essentially the same as that of Theorem 5.4.6, with the following modifications.

- In A_l, the objects, the additive zero, the multiplicative unit, and the sum are the same as those in A in Definition 5.2.3.
- The product in A_l is redefined in such a way that the left distributive law holds strictly, that is,

$$\underline{a} \otimes^{\mathsf{A}_l} \underline{b} = \left\{(a^1, b^1), \ldots, (a^r, b^1), \ldots, (a^1, b^s), \ldots, (a^r, b^s)\right\} \tag{5.4.8}$$

  instead of (5.2.7).
- Lemma 5.3.4 is replaced by the left distributive law for morphisms.
- Axiom (2.1.5) is used to define δ^r in terms of δ^l, which is the identity in A_l.
- Lemma 5.3.7 is replaced by the assertion that (2.1.6) holds in A_l.
- Proposition 5.3.9 is replaced by the assertion that A_l is a left bipermutative category.

Other structures are defined as in A, and all the proofs require minimal or no changes. □

5.5. Strictification of Tight Bimonoidal Categories

In this section, we prove nonsymmetric analogues of Theorems 5.4.6 and 5.4.7. To state these results, recall from Definition 2.1.2 that a bimonoidal category C has the same definition as a symmetric bimonoidal category, except that

- the multiplicative structure is a monoidal category instead of a symmetric one, and
- the axioms (2.1.5) and (2.1.19) are omitted.

A tight bimonoidal category—that is, one with δ^l and δ^r invertible—is called *right rigid* if

- both the additive and the multiplicative structures are strict monoidal, and
- $\lambda^{\bullet}$, $\rho^{\bullet}$, and δ^r are identity natural transformations.

Similarly, a tight bimonoidal category is *left rigid* if it satisfies these conditions but with $\delta^l = 1$ instead of $\delta^r = 1$. The first main result of this section is Theorem 5.5.11. It states that each tight bimonoidal category is adjoint equivalent, via suitable bimonoidal functors, to a right rigid bimonoidal category. Theorem 5.5.12 is the left rigid analogue.

Here is an outline of this section.

- In Definition 5.5.1, we define bimonoidal functors. These are the nonsymmetric analogues of symmetric bimonoidal functors in Definition 5.1.1. The Strictification Theorems 5.5.11 and 5.5.12 are phrased in terms of these functors.

- In Definition 5.5.5, for a tight bimonoidal category C, we define the object A as in Sections 5.2 and 5.3, except for its left distributivity morphism. Proposition 5.5.10 shows that A is a right rigid bimonoidal category.
- Theorem 5.5.11 shows that C and A are adjoint equivalent via bimonoidal functors, with one unitary and the other one strong.
- With a slight modification of the definition of A and the proofs, Theorem 5.5.12 states that C is adjoint equivalent to a left rigid bimonoidal category via bimonoidal functors, with one unitary and the other one strong.

Bimonoidal Functors.

Definition 5.5.1. Suppose C and D are bimonoidal categories. A *bimonoidal functor* from C to D is a tuple

$$\left(F, F^2_{\oplus}, F^0_{\oplus}, F^2_{\otimes}, F^0_{\otimes}\right) : \mathsf{C} \longrightarrow \mathsf{D}$$

consisting of the following data.

- $$F_{\oplus} = (F, F^2_{\oplus}, F^0_{\oplus}) : (\mathsf{C}, \oplus) \longrightarrow (\mathsf{D}, \oplus)$$
  is a symmetric monoidal functor from the additive structure of C to the additive structure of D.
- $$F_{\otimes} = (F, F^2_{\otimes}, F^0_{\otimes}) : (\mathsf{C}, \otimes) \longrightarrow (\mathsf{D}, \otimes)$$
  is a monoidal functor from the multiplicative structure of C to the multiplicative structure of D.

These data are required to make the diagrams (5.1.2), (5.1.3), (5.1.5), and (5.1.6) commutative. This finishes the definition of a bimonoidal functor.

Moreover, we define the following.

- *Robust*, *strong*, *unitary*, and *strict* bimonoidal functors are defined as in Definition 5.1.1.
- A *bimonoidal equivalence* is a bimonoidal functor whose underlying functor is an equivalence of categories.
- Composites of bimonoidal functors are defined as in Definition 5.1.7. $\diamond$

Explanation 5.5.2. There are two differences between a bimonoidal functor and a *symmetric* bimonoidal functor as in Definition 5.1.1. In a symmetric bimonoidal functor F,

- $F_{\otimes}$ is a symmetric monoidal functor, and
- F is only assumed to satisfy the multiplicative zero axiom (5.1.2) and the distributivity axiom (5.1.3).

In Proposition 5.1.4, using the multiplicative symmetry $\xi^{\otimes}$, we showed that (5.1.2) and (5.1.3) are equivalent to, respectively, (5.1.5) and (5.1.6). However, in a bimonoidal category, the multiplicative structure is only assumed to be a monoidal category. So there is no nonsymmetric analogue of Proposition 5.1.4. This is why, in Definition 5.5.1 of a bimonoidal functor, all four axioms are included. $\diamond$

Lemma 5.5.3. *Suppose*

$$\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$$

are bimonoidal functors. Then the composite $GF : \mathsf{C} \longrightarrow \mathsf{E}$ *is also a bimonoidal functor. Moreover, if both* F *and* G *are robust (respectively, strong, unitary, or strict), then so is* GF*.*

Proof. The proof of Lemma 5.1.9 shows that the composite GF satisfies the two axioms (5.1.2) and (5.1.3). Two similar diagrams prove the other two axioms (5.1.5) and (5.1.6). The key point is that the multiplicative symmetry $\xi^{\otimes}$ is not involved in these diagrams. □

As in Proposition 5.1.10, Lemma 5.5.3 implies the existence of a category of small bimonoidal categories.

Proposition 5.5.4. *There is a category* Bi *defined by the following data.*

- *The objects are small bimonoidal categories as in Definition 2.1.2.*
- *The morphisms are bimonoidal functors as in Definition 5.5.1.*
- *Identity morphisms are identity functors with identity monoidal structures.*
- *Composition is as in Definition 5.5.1.*

Moreover, Bi *has the following wide subcategories:*

- Bi_{r} *with robust bimonoidal functors as morphisms.*
- $\mathsf{Bi}_{\mathsf{sg}}$ *with strong bimonoidal functors as morphisms.*
- Bi_{u} *with unitary bimonoidal functors as morphisms.*
- $\mathsf{Bi}_{\mathsf{st}}$ *with strict bimonoidal functors as morphisms.*

Strictified Bimonoidal Categories. Next we define the object that will serve as the strictification of a tight bimonoidal category C. In the following definition, A does not have a multiplicative symmetry. Also note that the paths that define Mac Lane coherence isomorphisms $\cong^{\oplus}_{\mathsf{ML}}$ and Laplaza coherence isomorphisms $\cong_{\mathsf{Lap}}$ in Definition 5.2.24 are nonsymmetric paths in the sense of Definition 3.10.2. Therefore, they make sense in a bimonoidal category.

Definition 5.5.5. Suppose C is a tight bimonoidal category as in Definition 2.1.2. Define the data

$$\big(\mathsf{A}, (\oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \xi^{\oplus\mathsf{A}}), (\otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}), (\lambda^{\bullet\mathsf{A}} = 1, \rho^{\bullet\mathsf{A}} = 1), (\delta^{r\mathsf{A}} = 1, \delta^{l\mathsf{A}})\big)$$

as follows.

- The category A and its structures $\oplus^{\mathsf{A}}$, $\mathbb{0}^{\mathsf{A}}$, $\xi^{\oplus\mathsf{A}}$, $\otimes^{\mathsf{A}}$, $\mathbb{1}^{\mathsf{A}}$, $\lambda^{\bullet\mathsf{A}} = 1$, $\rho^{\bullet\mathsf{A}} = 1$, and $\delta^{r\mathsf{A}} = 1$ are defined as in Definitions 5.2.3, 5.2.16, 5.2.21, 5.2.24, 5.2.26, 5.2.34, 5.3.5, and 5.4.1.
- Using the function π in (5.2.17), $\delta^{l\mathsf{A}}$ is defined as the following composite in C for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$.

$$\begin{tikzcd}
\pi\big(\underline{a} \otimes^{\mathsf{A}} (\underline{b} \oplus^{\mathsf{A}} \underline{c})\big) \arrow[r, "\delta^{l\mathsf{A}}_{\underline{a};\underline{b};\underline{c}}"] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] & \pi\big((\underline{a} \otimes^{\mathsf{A}} \underline{b}) \oplus^{\mathsf{A}} (\underline{a} \otimes^{\mathsf{A}} \underline{c})\big) \\
\pi\underline{a} \otimes (\pi\underline{b} \oplus \pi\underline{c}) \arrow[r, "\delta^{l}_{\pi\underline{a};\pi\underline{b};\pi\underline{c}}"] & (\pi\underline{a} \otimes \pi\underline{b}) \oplus (\pi\underline{a} \otimes \pi\underline{c}) \arrow[u, "\cong_{\mathsf{Lap}}"']
\end{tikzcd} \tag{5.5.6}$$

This finishes the definition of A. ◇

Explanation 5.5.7.

- As in the symmetric case, morphisms in A are defined as morphisms in C, as in the equality (5.2.22), using the function π in (5.2.17). It interprets each object $\underline{a} \in$ A as the right normalized sum (5.2.19) of the right normalized products (5.2.18) of the alphabets in each monomial in $\underline{a}$.
- In the symmetric case, A has a multiplicative symmetry $\xi^{\otimes \mathsf{A}}$ as in (5.2.36), which is defined using the multiplicative symmetry in the given tight symmetric bimonoidal category. Since C is now a tight bimonoidal category, which does not have a multiplicative symmetry, A also does not have a multiplicative symmetry.
- For the same reason, instead of using the diagram (5.3.6), in the current nonsymmetric setting, we define the left distributivity morphism $\delta^{l\mathsf{A}}$ directly as in (5.5.6). ◇

We will show shortly that A in Definition 5.5.5 is right rigid in the sense of the next definition.

Definition 5.5.8. A *right rigid bimonoidal category* is a tight bimonoidal category C as in Definition 2.1.2 that satisfies the following three conditions.

- The additive structure $(\mathsf{C}, \oplus)$ is a permutative category.
- The multiplicative structure $(\mathsf{C}, \otimes)$ is a strict monoidal category.
- $\lambda^\bullet$, $\rho^\bullet$, and δ^r are the identity natural transformations.

A *left rigid bimonoidal category* is defined in the same way as a right rigid bimonoidal category, but with $\delta^l = 1$ instead of $\delta^r = 1$. ◇

Example 5.5.9.

- By Proposition 2.5.7, each right bipermutative category is a right rigid bimonoidal category after forgetting the multiplicative symmetry $\xi^\otimes$.
- By Proposition 2.5.16, each left bipermutative category is a left rigid bimonoidal category after forgetting the multiplicative symmetry $\xi^\otimes$. ◇

Next is the nonsymmetric analogue of Proposition 5.3.9.

Proposition 5.5.10. *For each tight bimonoidal category* C*, the data*

$$\big(\mathsf{A}, (\oplus^\mathsf{A}, \mathbb{0}^\mathsf{A}, \xi^{\oplus\mathsf{A}}), (\otimes^\mathsf{A}, \mathbb{1}^\mathsf{A}), (\lambda^{\bullet\mathsf{A}} = 1, \rho^{\bullet\mathsf{A}} = 1), (\delta^{r\mathsf{A}} = 1, \delta^{l\mathsf{A}})\big)$$

in Definition 5.5.5 form a right rigid bimonoidal category.

Proof. The left distributivity morphism $\delta^{l\mathsf{A}}$ defined in (5.5.6) is a natural transformation by

- the naturality of δ^l in C,
- the definitions of $\oplus^\mathsf{A}$ and $\otimes^\mathsf{A}$ on morphisms in (5.2.29),
- Definition 5.4.1 of π as the identity assignment on morphisms, and
- the Coherence Theorem 3.10.7 for bimonoidal categories.

Since C is tight, both δ^l and δ^r are natural isomorphisms in C. This implies that each component of $\delta^{l\mathsf{A}}$ is an isomorphism. Therefore, $\delta^{l\mathsf{A}}$ is a natural isomorphism.

The rest of the proof is obtained from that of Proposition 5.3.9 as follows.

- We use the Coherence Theorem 3.10.7 for bimonoidal categories instead of the Coherence Theorem 3.9.1 for symmetric bimonoidal categories.
- We reuse the proof of Lemma 5.3.1 to infer that
  - the additive structure of A is a permutative category, and
  - the multiplicative structure of A is a strict monoidal category.

- We reuse the proofs of Lemmas 5.3.3 and 5.3.4 to infer that
  - the left multiplicative zero $\lambda^{\bullet\mathsf{A}} = 1$,
  - the right multiplicative zero $\rho^{\bullet\mathsf{A}} = 1$, and
  - the right distributivity morphism $\delta^{r\mathsf{A}} = 1$

  are well-defined natural transformations.

It remains to check that A satisfies the 22 Laplaza axioms in Definition 2.1.2, excluding (2.1.5) and (2.1.19), for a bimonoidal category.

- Each of the following 12 Laplaza axioms holds in A because each edge involved is an identity morphism: (2.1.8), (2.1.11), (2.1.14), (2.1.16)–(2.1.18), (2.1.20)–(2.1.22), (2.1.24), (2.1.26), and (2.1.28).
- Consider the axioms (2.1.15), (2.1.23), (2.1.25), and (2.1.27).
  - (2.1.15) is equivalent to the equality $\delta^{l\mathsf{A}}_{\mathbb{0}^{\mathsf{A}};-;-} = 1$.
  - (2.1.23) is equivalent to the equality $\delta^{l\mathsf{A}}_{-;\mathbb{0}^{\mathsf{A}};-} = 1$.
  - (2.1.25) is equivalent to the equality $\delta^{l\mathsf{A}}_{-;-;\mathbb{0}^{\mathsf{A}}} = 1$.
  - (2.1.27) is equivalent to the equality $\delta^{l\mathsf{A}}_{\mathbb{1}^{\mathsf{A}};-;-} = 1$.

  Each of these four equalities follows from the definition (5.5.6) of $\delta^{l\mathsf{A}}$ and an application of the Coherence Theorem 3.10.7.
- (2.1.7) is proved by reusing the proof of Lemma 5.3.7 and using Theorem 3.10.7 instead of Theorem 3.9.1.

There are five remaining axioms to check in A: (2.1.6), (2.1.9), (2.1.10), (2.1.12), and (2.1.13). Each of these axioms is a consequence of the same axiom in C, along with coherence, naturality, and functoriality properties. To prove these axioms in detail, consider arbitrary objects $\underline{a}, \underline{b}, \underline{c}, \underline{d} \in \mathsf{A}$. To save space in the following diagrams, we abbreviate $\oplus^{\mathsf{A}}$ to $\oplus$, and both $\otimes$ and $\otimes^{\mathsf{A}}$ to concatenation. Moreover, for an object $\underline{e} \in \mathsf{A}$, we write $\pi\underline{e}$ as $\underline{e}'$. For example,

$$\begin{aligned}
\pi\big(\underline{a} \otimes^{\mathsf{A}} (\underline{b} \oplus^{\mathsf{A}} \underline{c})\big) &= [\underline{a}(\underline{b} \oplus \underline{c})]',\\
\pi\underline{a} \otimes (\pi\underline{b} \oplus \pi\underline{c}) &= \underline{a}'(\underline{b}' \oplus \underline{c}'), \quad \text{and}\\
\pi\big((\underline{a} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{a} \otimes^{\mathsf{A}} \underline{b})\big) &= (\underline{ac} \oplus \underline{ab})'.
\end{aligned}$$

With these conventions, the axiom (2.1.6) in A states the commutativity of the outer diagram in C below.

$$\begin{tikzcd}
{[\underline{a}(\underline{b}\oplus\underline{c})]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"'] & \underline{a}'(\underline{b}'\oplus\underline{c}') \arrow[r, "\delta^l"] & \underline{a}'\underline{b}'\oplus\underline{a}'\underline{c}' \arrow[r, "\cong_{\mathsf{Lap}}"] \arrow[ddd, "\xi^{\oplus}"'] & (\underline{ab}\oplus\underline{ac})' \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"] \\
\underline{a}'(\underline{b}'\oplus\underline{c}') \arrow[ur, "1"'] \arrow[d, "1\xi^{\oplus}"'] & {} \arrow[d, phantom, "(2.1.6)"] & {} \arrow[dr, phantom, "3.10.7"] & (\underline{ab})'\oplus(\underline{ac})' \arrow[d, "\xi^{\oplus}"] \\
\underline{a}'(\underline{c}'\oplus\underline{b}') \arrow[dr, "1"] \arrow[d, "\cong_{\mathsf{Lap}}"'] & {} & & (\underline{ac})'\oplus(\underline{ab})' \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"] \\
{[\underline{a}(\underline{c}\oplus\underline{b})]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] & \underline{a}'(\underline{c}'\oplus\underline{b}') \arrow[r, "\delta^l"] & \underline{a}'\underline{c}'\oplus\underline{a}'\underline{b}' \arrow[r, "\cong_{\mathsf{Lap}}"] & (\underline{ac}\oplus\underline{ab})'
\end{tikzcd}$$

The middle subdiagram is commutative by the axiom (2.1.6) in C. The right subdiagram is commutative by the Coherence Theorem 3.10.7. The two unlabeled triangles are commutative by definition. Similar notations are used in the remaining diagrams in this proof.

Using the strict associativity of $\oplus^{\mathsf{A}}$, the axiom (2.1.9) in A states the commutativity of the outer diagram in C below.

$$\begin{tikzcd}
{[\underline{a}(\underline{b}\oplus\underline{c}\oplus\underline{d})]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"'] & \underline{a}'[(\underline{b}\oplus\underline{c})'\oplus\underline{d}'] \arrow[r, "\delta^l"] \arrow[dr, phantom, "\mathsf{nat}"] & \underline{a}'(\underline{b}\oplus\underline{c})'\oplus\underline{a}'\underline{d}' \arrow[r, "\cong_{\mathsf{Lap}}"] & {[\underline{a}(\underline{b}\oplus\underline{c})\oplus\underline{ad}]'} \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"] \\
\underline{a}'[\underline{b}'\oplus(\underline{c}\oplus\underline{d})'] \arrow[dd, "\delta^l"'] & \underline{a}'[(\underline{b}'\oplus\underline{c}')\oplus\underline{d}'] \arrow[u, "\cong_{\mathsf{Lap}}"'] \arrow[r, "\delta^l"] \arrow[d, "1\alpha^{\oplus}"'] \arrow[dr, phantom, "(2.1.9)"] & \underline{a}'(\underline{b}'\oplus\underline{c}')\oplus\underline{a}'\underline{d}' \arrow[u, "\cong_{\mathsf{Lap}}"] \arrow[r, "1\oplus\cong_{\mathsf{Lap}}"] \arrow[d, "\delta^l\oplus 1"] \arrow[dr, phantom, "\mathsf{fun}"] & \underline{a}'(\underline{b}'\oplus\underline{c}')\oplus(\underline{ad})' \arrow[dd, "\delta^l\oplus 1"] \\
 & \underline{a}'[\underline{b}'\oplus(\underline{c}'\oplus\underline{d}')] \arrow[ul, "\cong_{\mathsf{Lap}}"'] \arrow[d, "\delta^l"'] & (\underline{a}'\underline{b}'\oplus\underline{a}'\underline{c}')\oplus\underline{a}'\underline{d}' \arrow[d, "\alpha^{\oplus}"] \arrow[dr, "1\oplus\cong_{\mathsf{Lap}}"] & \\
\underline{a}'\underline{b}'\oplus\underline{a}'(\underline{c}\oplus\underline{d})' \arrow[d, "\cong_{\mathsf{Lap}}"'] & \underline{a}'\underline{b}'\oplus\underline{a}'(\underline{c}'\oplus\underline{d}') \arrow[r, "1\oplus\delta^l"] \arrow[d, "\cong_{\mathsf{Lap}}\oplus 1"'] \arrow[dr, phantom, "\mathsf{fun}"] & \underline{a}'\underline{b}'\oplus(\underline{a}'\underline{c}'\oplus\underline{a}'\underline{d}') \arrow[d, "\cong_{\mathsf{Lap}}\oplus 1"] & (\underline{a}'\underline{b}'\oplus\underline{a}'\underline{c}')\oplus(\underline{ad})' \arrow[d, "\cong_{\mathsf{Lap}}"] \\
{[\underline{ab}\oplus\underline{a}(\underline{c}\oplus\underline{d})]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] & (\underline{ab})'\oplus\underline{a}'(\underline{c}'\oplus\underline{d}') \arrow[r, "1\oplus\delta^l"] & (\underline{ab})'\oplus(\underline{a}'\underline{c}'\oplus\underline{a}'\underline{d}') \arrow[r, "\cong_{\mathsf{Lap}}"] & (\underline{ab}\oplus\underline{ac}\oplus\underline{ad})'
\end{tikzcd}$$

In the middle column, the top two subdiagrams are commutative by the naturality of δ^l and the axiom (2.1.9) in C. The two subdiagrams labeled by fun are commutative by the functoriality of $\oplus$ in C. Each of the other four subdiagrams is commutative by Theorem 3.10.7.

Using the strict associativity of $\otimes^{\mathsf{A}}$, the axiom (2.1.10) in A states the commutativity of the outer diagram in C below.

$$\begin{tikzcd}
{[\underline{ab}(\underline{c}\oplus\underline{d})]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"'] & (\underline{ab})'(\underline{c}'\oplus\underline{d}') \arrow[r, "\delta^l"] & (\underline{ab})'\underline{c}'\oplus(\underline{ab})'\underline{d}' \arrow[r, "\cong_{\mathsf{Lap}}"] & (\underline{abc}\oplus\underline{abd})' \\
\underline{a}'[\underline{b}'(\underline{c}'\oplus\underline{d}')] \arrow[d, "1\delta^l"'] & (\underline{a}'\underline{b}')(\underline{c}'\oplus\underline{d}') \arrow[l, "\alpha^{\otimes}"'] \arrow[u, "\cong_{\mathsf{Lap}}"] \arrow[r, "\delta^l"] \arrow[d, phantom, "(2.1.10)"] & (\underline{a}'\underline{b}')\underline{c}'\oplus(\underline{a}'\underline{b}')\underline{d}' \arrow[d, "\alpha^{\otimes}\oplus\alpha^{\otimes}"] & \\
\underline{a}'(\underline{b}'\underline{c}'\oplus\underline{b}'\underline{d}') \arrow[rr, "\delta^l"] \arrow[d, "\cong_{\mathsf{Lap}}"'] \arrow[dr, "\cong_{\mathsf{Lap}}"] & \arrow[d, phantom, "\mathsf{nat}"] & \underline{a}'(\underline{b}'\underline{c}')\oplus\underline{a}'(\underline{b}'\underline{d}') \arrow[dr, "\cong_{\mathsf{Lap}}"] & \\
{[\underline{a}(\underline{bc}\oplus\underline{bd})]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] & \underline{a}'[(\underline{bc})'\oplus(\underline{bd})'] \arrow[rr, "\delta^l"] & & \underline{a}'(\underline{bc})'\oplus\underline{a}'(\underline{bd})' \arrow[uuu, "\cong_{\mathsf{Lap}}"']
\end{tikzcd}$$

The middle rectangle is commutative by (2.1.10) in C. The bottom parallelogram is commutative by the naturality of δ^l. Each of the other three subdiagrams is commutative by Theorem 3.10.7.

Using the strict associativity of $\otimes^{\mathsf{A}}$ and that $\delta^{r\mathsf{A}} = 1$, the axiom (2.1.12) in A states the commutativity of the outer diagram in C below.

$$\begin{tikzcd}
{[\underline{a}(\underline{b} \oplus \underline{c})\underline{d}]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] \arrow[ddd, "1"'] & {[\underline{a}'(\underline{b}' \oplus \underline{c}')]\underline{d}'} \arrow[r, "\delta^l 1"] \arrow[d, "\alpha^{\otimes}"'] \arrow[dr, phantom, "(2.1.12)"] & (\underline{a}'\underline{b}' \oplus \underline{a}'\underline{c}')\underline{d}' \arrow[r, "\cong_{\mathsf{Lap}}"] \arrow[d, "\delta^r"] & {[(\underline{ab} \oplus \underline{ac})\underline{d}]'} \arrow[ddd, "1"] \\
& {\underline{a}'[(\underline{b}' \oplus \underline{c}')\underline{d}']} \arrow[d, "1\delta^r"'] & (\underline{a}'\underline{b}')\underline{d}' \oplus (\underline{a}'\underline{c}')\underline{d}' \arrow[d, "\alpha^{\otimes} \oplus \alpha^{\otimes}"] & \\
& \underline{a}'(\underline{b}'\underline{d}' \oplus \underline{c}'\underline{d}') \arrow[r, "\delta^l"] \arrow[d, "\cong_{\mathsf{Lap}}"'] \arrow[dr, phantom, "\mathsf{nat}"] & \underline{a}'(\underline{b}'\underline{d}') \oplus \underline{a}'(\underline{c}'\underline{d}') \arrow[d, "\cong_{\mathsf{Lap}}"] & \\
{[\underline{a}(\underline{bd} \oplus \underline{cd})]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] & {\underline{a}'[(\underline{bd})' \oplus (\underline{cd})']} \arrow[r, "\delta^l"] & \underline{a}'(\underline{bd})' \oplus \underline{a}'(\underline{cd})' \arrow[r, "\cong_{\mathsf{Lap}}"] & (\underline{abd} \oplus \underline{acd})'
\end{tikzcd}$$

In the middle column, the top and the bottom rectangles are commutative by, respectively, (2.1.12) and the naturality of δ^l in C. Each of the other two rectangles is commutative by Theorem 3.10.7.

When applied to $\underline{a}$, $\underline{b}$, and $\underline{c} \oplus \underline{d}$, $\delta^{r\mathsf{A}} = 1$ is the identity morphism

$$(\underline{a} \oplus \underline{b})(\underline{c} \oplus \underline{d}) \xrightarrow{\delta^{r\mathsf{A}}} \underline{a}(\underline{c} \oplus \underline{d}) \oplus \underline{b}(\underline{c} \oplus \underline{d}).$$

With this identity morphism and $\alpha^{\oplus\mathsf{A}} = 1$ taken into account, the axiom (2.1.13) in A states the commutativity of the outer diagram in C below.

$$\begin{tikzcd}
{[(\underline{a} \oplus \underline{b})(\underline{c} \oplus \underline{d})]'} \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"'] & \underline{a}'(\underline{c}' \oplus \underline{d}') \oplus \underline{b}'(\underline{c}' \oplus \underline{d}') \arrow[r, "\delta^l \oplus \delta^l"] \arrow[dr, phantom, "(2.1.13)"] & (\underline{a}'\underline{c}' \oplus \underline{a}'\underline{d}') \oplus (\underline{b}'\underline{c}' \oplus \underline{b}'\underline{d}') \arrow[r, "\cong_{\mathsf{Lap}}"] \arrow[d, "(1 \oplus \alpha^{-\oplus})\alpha^{\oplus}"] & (\underline{ac} \oplus \underline{ad} \oplus \underline{bc} \oplus \underline{bd})' \arrow[d, "\cong_{\mathsf{Lap}}"] \\
(\underline{a} \oplus \underline{b})'(\underline{c}' \oplus \underline{d}') \arrow[d, "\delta^l"'] & (\underline{a}' \oplus \underline{b}')(\underline{c}' \oplus \underline{d}') \arrow[l, "\cong_{\mathsf{Lap}}"'] \arrow[u, "\delta^r"'] \arrow[d, "(\delta^r \oplus \delta^r)\delta^l"'] & {\underline{a}'\underline{c}' \oplus [(\underline{a}'\underline{d}' \oplus \underline{b}'\underline{c}') \oplus \underline{b}'\underline{d}']} \arrow[d, "1 \oplus (\xi^{\oplus} \oplus 1)"] & {(\underline{ac})' \oplus [((\underline{ad})' \oplus (\underline{bc})') \oplus (\underline{bd})']} \arrow[d, "1 \oplus (\xi^{\oplus} \oplus 1)"] \\
(\underline{a} \oplus \underline{b})'\underline{c}' \oplus (\underline{a} \oplus \underline{b})'\underline{d}' \arrow[d, "\cong_{\mathsf{Lap}}"'] & (\underline{a}'\underline{c}' \oplus \underline{b}'\underline{c}') \oplus (\underline{a}'\underline{d}' \oplus \underline{b}'\underline{d}') \arrow[r, "(1 \oplus \alpha^{-\oplus})\alpha^{\oplus}"] & {\underline{a}'\underline{c}' \oplus [(\underline{b}'\underline{c}' \oplus \underline{a}'\underline{d}') \oplus \underline{b}'\underline{d}']} \arrow[dr, "\cong_{\mathsf{Lap}}"] & {(\underline{ac})' \oplus [((\underline{bc})' \oplus (\underline{ad})') \oplus (\underline{bd})']} \arrow[d, "\cong_{\mathsf{Lap}}"] \\
{[(\underline{a} \oplus \underline{b})\underline{c} \oplus (\underline{a} \oplus \underline{b})\underline{d}]'} \arrow[rrr, "1"] & & & (\underline{ac} \oplus \underline{bc} \oplus \underline{ad} \oplus \underline{bd})'
\end{tikzcd}$$

The middle subdiagram is commutative by (2.1.13) in C. Each of the other three subdiagrams is commutative by Theorem 3.10.7. We have proved that A satisfies the 22 Laplaza axioms for a bimonoidal category. $\square$

Strictification Theorems. We are now ready for the main strictification results of this section. The next theorem is the nonsymmetric analogue of Theorem 5.4.6.

Theorem 5.5.11 (Right Rigid Strictification)**.** *Suppose* C *is a tight bimonoidal category. Then there is an adjoint equivalence*

$$\begin{tikzcd} \mathsf{A} \arrow[r, shift left, "\pi"] & \mathsf{C} \arrow[l, shift left, "\iota"] \end{tikzcd}$$

with

- A *the right rigid bimonoidal category in Proposition 5.5.10;*
- *counit* $\varepsilon : \pi\iota \longrightarrow 1_{\mathsf{C}}$ *the identity natural transformation;*
- π *the unitary bimonoidal equivalence as in Definition 5.4.1 and Lemma 5.4.4; and*

- *ι the strong bimonoidal equivalence as in Definition 5.4.2 and Lemma 5.4.5.*

Proof. The proof of Lemma 5.4.3 shows that the pair (π, ι) is an adjoint equivalence, and its counit is the identity.

Reusing the proof of Lemma 5.4.4, to show that π is a unitary bimonoidal equivalence, it remains to prove the axioms (5.1.5) and (5.1.6) for π.

- The axiom (5.1.5) holds for π because
  - $\pi^0_\oplus$ and $\lambda^{\bullet\mathsf{A}}$ are identity natural transformations, and
  - there is an equality

$$\pi^2_\otimes = \lambda^\bullet_{\pi\underline{a}} : (\pi\mathbb{0}^{\mathsf{A}}) \otimes (\pi\underline{a}) = \mathbb{0} \otimes (\pi\underline{a}) \longrightarrow \mathbb{0} = \pi(\mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a}).$$

- The axiom (5.1.6) holds by Theorem 3.10.7 and the definition (5.5.6) of $\delta^{l\mathsf{A}}$.

Reusing the proof of Lemma 5.4.5, to show that ι is a strong bimonoidal equivalence, it remains to prove the axioms (5.1.5) and (5.1.6) for ι.

- The axiom (5.1.5) holds for ι because each of the two composites is $1_{\mathbb{0}}$.
- The axiom (5.1.6) holds by Theorem 3.10.7 and the fact that each of the four vertical morphisms is an identity morphism in C.

This finishes the proof. □

As in the symmetric case, the previous theorem has a left rigid analogue. The next theorem is the nonsymmetric analogue of Theorem 5.4.7.

Theorem 5.5.12 (Left Rigid Strictification)**.** *Suppose C is a tight bimonoidal category. Then there is an adjoint equivalence*

$$\begin{tikzcd} \mathsf{A}_l \arrow[r, shift left, "\pi"] & \mathsf{C} \arrow[l, shift left, "\iota"] \end{tikzcd}$$

with

- *A_l a left rigid bimonoidal category as in Definition 5.5.8;*
- *counit $\varepsilon : \pi\iota \longrightarrow 1_{\mathsf{C}}$ the identity natural transformation;*
- *π a unitary bimonoidal equivalence; and*
- *ι a strong bimonoidal equivalence.*

Proof. As in the symmetric case in Theorem 5.4.7, this proof is obtained from that of Theorem 5.5.11 by redefining the product in A using (5.4.8), so the left distributive law holds strictly.

Moreover, similar to (5.5.6), the right distributivity morphism in A_l is defined as the composite

$$\begin{tikzcd} \pi\big((\underline{a} \oplus^{\mathsf{A}} \underline{b}) \otimes^{\mathsf{A}} \underline{c}\big) \arrow[r, "\delta^{r\mathsf{A}_l}_{\underline{a};\underline{b};\underline{c}}"] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] & \pi\big((\underline{a} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{b} \otimes^{\mathsf{A}} \underline{c})\big) \\ (\pi\underline{a} \oplus \pi\underline{b}) \otimes \pi\underline{c} \arrow[r, "\delta^r_{\pi\underline{a};\pi\underline{b};\pi\underline{c}}"] & (\pi\underline{a} \otimes \pi\underline{c}) \oplus (\pi\underline{b} \otimes \pi\underline{c}) \arrow[u, "\cong_{\mathsf{Lap}}"'] \end{tikzcd}$$

in C for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}_l$. Other structures are defined as in A in Definition 5.5.5, and all the proofs require minimal or no changes. □

5.6. Notes

5.6.1 (Symmetric Bimonoidal Functors)**.** Our Definition 5.1.1 of a symmetric bimonoidal functor is more general than May's in [**May77**, Def. 6.3.4], which corresponds to our unitary symmetric bimonoidal functors. A *homomorphism between symmetric rig categories* in [**Elg21**, Section 2] corresponds to our strong symmetric bimonoidal functor. $\diamond$

5.6.2 (Strictification Theorem)**.** The proof of the Strictification Theorem 5.4.6 is outlined in [**May77**, Prop. 6.3.5]. We emphasize the necessity of the tightness assumption, that is, the invertibility of the distributivity morphisms. As we saw in (5.2.29) and (5.2.36), the products of morphisms and the symmetry isomorphism in the associated right bipermutative category A use the inverse of a Laplaza coherence isomorphism. By definition, $\cong_{\mathsf{Lap}}^{-1}$ is only defined if the distributivity morphisms δ^l and δ^r are natural isomorphisms.

Since May only gave an outline of the strictification theorem, it may not be apparent that May's proof actually uses the Coherence Theorem 3.9.1 many times. In [**May77**, Prop. 6.3.5], May implicitly used Theorem 3.9.1 when he defined

- the bottom displayed morphism on page 156 there, and
- the second displayed morphism on page 157 there.

These correspond to, respectively, our $f \otimes^{\mathsf{A}} g$ in (5.2.29) and $\xi^{\otimes \mathsf{A}}_{\underline{a};\underline{b}}$ in (5.2.36). Furthermore, showing that A has the desired property of a strictification of a tight symmetric bimonoidal category also uses Theorem 3.9.1 multiple times. See Explanation 5.2.31 and Lemmas 5.3.1, 5.3.4, 5.3.8, and 5.4.4. $\diamond$

5.6.3 (Elmendorf-Mandell Bipermutative Categories)**.** Elmendorf and Mandell in [**EM06**, Def. 3.6] defined their version of bipermutative categories in which the distributivity morphisms

- are not invertible in general, and
- point in the opposite direction as those in (2.1.4).

See Definition II.9.3.2 for the precise definition. Implicitly citing May's result in [**May77**, Prop. 6.3.5], it is stated in [**EM06**, page 178] that their bipermutative categories can be strictified to equivalent right bipermutative categories. As stated in Note 5.6.2, strictification to equivalent right bipermutative categories only applies to *tight* symmetric bimonoidal categories because some constructions and proofs in the strictification require the invertibility of the distributivity morphisms. Therefore, the correct statement is that bipermutative categories in the sense of [**EM06**, Def. 3.6] with invertible distributivity morphisms, which are examples of tight symmetric bimonoidal categories, can be strictified to equivalent right bipermutative categories. This is a special case of our Theorem 5.4.6. See Corollary II.9.3.13. $\diamond$

5.6.4 (Multi-Object Version of Strictification)**.** Guillou proved a multi-object version of Theorem 5.5.11 involving categories weakly enriched in symmetric monoidal categories or in permutative categories in [**Gui10**, Theorem 1.2]. The setting is that of bicategories enriched in the monoidal bicategory of symmetric monoidal categories. Guillou gave two proofs of that theorem, one using (i) the strictification of bicategories to biequivalent 2-categories, and the other using (ii) the Yoneda embedding for bicategories. As we explained in detail in [**JY21**, Chapter 8], both

(i) and (ii) depend on the Bicategorical Pasting Theorem 3.6.6 in that book, which is itself a coherence theorem that requires a careful proof. $\diamond$

Part 2

Bicategorical Aspects of Symmetric Bimonoidal Categories

CHAPTER 6

Definitions from Bicategory Theory

Part 2 concerns 2-/bicategorical aspects of symmetric bimonoidal categories.

(1) In Chapter 7, we will define a suitable 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ of flat small symmetric bimonoidal categories. The main goal of that chapter is to show that the symmetric bimonoidal category Σ in Proposition 2.4.8 is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$. In other words, Σ is a bicategorical analogue of an initial object in the 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$. This theorem confirms a conjecture due to John Baez.

(2) In Chapter 8, we will show that for a tight symmetric bimonoidal category C, the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory. It has nonnegative integers as objects, and $n \times m$ matrices with entries objects/morphisms in C as 1-/2-cells $m \longrightarrow n$. Horizontal composition is defined by matrix multiplication with a suitable bracketing for addition. The monoidal composition of the monoidal bicategory is given by matrix tensor product. The braiding of the symmetric monoidal bicategory is induced by permutation matrices in $\mathsf{Mat}^{\mathsf{C}}$. In general, the structure morphisms in $\mathsf{Mat}^{\mathsf{C}}$ are not identities. As an example, the category $\mathsf{2Vect}_{\mathsf{c}}$ of coordinatized 2-vector spaces in Examples 8.4.13 and 8.15.5 is a symmetric monoidal bicategory.

(3) Concepts about 2-/bicategories are also used in
   - Chapters II.7 and II.8, for the braided versions of Baez's Conjecture and the matrix construction $\mathsf{Mat}^{\mathsf{C}}$, and
   - Parts III.1 and III.2, for enriched monoidal categories, multicategories, and the Elmendorf-Mandell K-theory multifunctor.

Without assuming any prior knowledge of 2-dimensional categories, the purpose of this chapter is to recall the relevant concepts that we will use in later chapters. For open questions related to 2-categories and bicategories, see Questions III.A.1.1, III.A.1.2, III.A.1.4, III.A.1.5, III.A.2.4, III.A.4.1, III.A.5.4, and III.A.5.5.

Organization. In Section 6.1, we define bicategories and 2-categories, which are due to Bénabou [**Bén67, Bén65**], and provide some examples. In a bicategory, in addition to objects and morphisms between objects, there are also morphisms between morphisms called 2-cells. Just as a category may be regarded as a monoid with multiple objects, a bicategory may be regarded as a monoidal category with multiple objects. A 2-category is a particularly nice bicategory in which the associator and the unitors are identities. The main 2-category in Chapter 7 is the 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ in Definition 7.1.8. The main bicategory in Chapter 8 is the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$, whose horizontal composition is given by the matrix product.

In Section 6.2, we define bicategorical analogues of functors and natural transformations. Due to the presence of 2-cells, there are several versions of functors—lax, pseudo, (strictly) unitary, and strict—between bicategories, depending on the extent to which horizontal composites of 1-cells and identity 1-cells are preserved. A strict functor between 2-categories is called a 2-functor. In Chapter 7, $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ is an example of a 2-functor. Similarly, transformations between lax functors can be strong or lax, depending on whether the component 2-cells are invertible or not. We also briefly discuss pasting diagrams in bicategories.

In Section 6.3, we define modifications, adjunctions, and adjoint equivalences in bicategories. Modifications are morphisms between lax transformations. Here is an analogy to keep track of the various concepts. If a bicategory is regarded as a topological space, then lax functors, lax transformations, and modifications are analogues of, respectively, continuous maps, homotopies between continuous maps, and homotopies between homotopies between continuous maps. Moreover, the usual concepts of adjunctions and adjoint equivalences also make sense inside a bicategory.

In Section 6.4, we define a monoidal bicategory, which is a one-object tricategory. Since we do not discuss tricategories in this book, we will define a monoidal bicategory with all of its structures explicitly spelled out. It involves the concepts of pseudofunctors, modifications, and adjoint equivalences in a bicategory. A monoidal bicategory has a base bicategory B and a pseudofunctor $\boxtimes : \mathsf{B} \times \mathsf{B} \longrightarrow \mathsf{B}$ called the monoidal composition, along with other related structures. The monoidal composition $\boxtimes$ in a monoidal bicategory is an extra level of composition on top of the horizontal composition in the base bicategory. As an analogy, consider complex matrices. There is a base level matrix multiplication AB, which is only defined when the number of columns in A is equal to the number of rows in B. On top of that, the matrix tensor product $A \otimes B$, which is also known as the Kronecker product, is always defined regardless of the sizes of A and B.

In Section 6.5, we define braided, sylleptic, and symmetric monoidal bicategories. Braided and symmetric monoidal bicategories are bicategorical analogues of braided and symmetric monoidal categories. In the bicategorical setting, there is the intermediate concept of a sylleptic monoidal bicategory. The symmetry axiom $\xi_{Y,X}\xi_{X,Y} = 1$ is replaced by an invertible modification ν, called the syllepsis, that connects the braiding-square to the identity, and that satisfies its own coherence axioms. A symmetric monoidal bicategory is a sylleptic monoidal bicategory whose syllepsis satisfies an additional coherence axiom. The main theorem of Chapter 8 is that, for a tight symmetric bimonoidal category C, the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory. Its monoidal composition is given by the matrix tensor product for matrices in C. The braiding is given by permutation matrices interpreted using $\mathbb{0}$ and $\mathbb{1}$ in C.

In Section 6.6, we define the Gray tensor product for 2-categories and Gray monoids. The Gray tensor product is an interpolation between what we call the box product and the Cartesian product for 2-categories. The 1-category $\mathsf{2Cat}$ of small 2-categories and 2-functors is a symmetric monoidal closed category when equipped with the Gray tensor product. Its monoids are called Gray monoids.

In Section 6.7, we first define permutative Gray monoids and explain them with complete detail. Permutative Gray monoids are to Gray monoids as permutative categories are to strict monoidal categories. The last part of this section contains a brief discussion of strictification related to symmetric monoidal bicategories, permutative Gray monoids, and permutative 2-categories.

References. Most of the topics in this chapter are discussed in much greater detail in the book [**JY21**], where we refer the reader for further discussion, proofs, and examples. Here is a road map between this chapter and the book [**JY21**].

Topics	This chapter	Sections in [**JY21**]
Bicategories	6.1	2.1–2.2
2-categories	6.1	2.3
Pasting diagrams	6.2	Chapter 3
Lax functors	6.2	4.1
Lax transformations	6.2	4.2
Modifications	6.3	4.4
Adjunctions	6.3	6.1–6.2
Monoidal bicategories	6.4	11.1, 11.2, and 12.1
Mates	6.5	6.1
Braided, sylleptic, and symmetric monoidal bicategories	6.5	12.1
The Gray tensor product and Gray monoids	6.6	12.2

The definitions of permutative Gray monoids and permutative 2-categories are from [**GJO17b**]. The articles [**Lac10, Lei∞**] are useful guides for 2-categories and bicategories.

6.1. Bicategories and 2-Categories

In this section, we recall the definitions and some basic examples of bicategories and 2-categories.

Convention 6.1.1. $\mathbf{1}$ denotes the category with one object $*$ and only its identity morphism. For a category C, we identify the categories $\mathsf{C} \times \mathbf{1}$ and $\mathbf{1} \times \mathsf{C}$ with C, and regard the canonical isomorphisms between them as 1_{C}. ◇

A bicategory as defined next is a many-object version of a monoidal category in Definition 1.2.1.

Definition 6.1.2. A *bicategory* is a tuple

$$\big(\mathsf{B}, 1, c, a, \ell, r\big)$$

consisting of the following data.

Objects: B is equipped with a class $\mathsf{Ob}(\mathsf{B})$, whose elements are called *objects* or *0-cells* in B. If $X \in \mathsf{Ob}(\mathsf{B})$, we also write $X \in \mathsf{B}$.

The Hom Categories: For each pair of objects $X, Y \in \mathsf{B}$, B is equipped with a category $\mathsf{B}(X,Y)$, which is called a *hom category*.

- Its objects are called *1-cells* in B. Such a 1-cell f is denoted by either

  $f : X \longrightarrow Y$ or $X \xrightarrow{f} Y$.

- Its morphisms are called *2-cells* in B. For 1-cells $f, f' \in \mathsf{B}(X,Y)$, we display each 2-cell $\alpha : f \longrightarrow f'$ in diagrams as

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

  with a double arrow for the 2-cell.
- Composition and identity morphisms in the category $\mathsf{B}(X,Y)$ are called *vertical composition* and *identity 2-cells*, respectively.
- An isomorphism in $\mathsf{B}(X,Y)$ is called an *invertible 2-cell*, and its inverse is called a *vertical inverse*.
- For a 1-cell f, its identity 2-cell is denoted by 1_f.

Identity 1-Cells: For each object $X \in \mathsf{B}$,

$$\mathbf{1} \xrightarrow{1_X} \mathsf{B}(X,X)$$

is a functor. We identify the functor 1_X with the 1-cell $1_X(*) \in \mathsf{B}(X,X)$, which is called the *identity 1-cell of* X.

The Horizontal Composition: For each triple of objects $X, Y, Z \in \mathsf{B}$,

$$\mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \xrightarrow{c_{XYZ}} \mathsf{B}(X,Z)$$

is a functor, which is called the *horizontal composition*. For 1-cells $f \in \mathsf{B}(X,Y)$ and $g \in \mathsf{B}(Y,Z)$, and 2-cells $\alpha \in \mathsf{B}(X,Y)$ and $\beta \in \mathsf{B}(Y,Z)$, we use the notations

$$c_{XYZ}(g,f) = g \circ f \quad \text{or} \quad gf, \quad \text{and}$$
$$c_{XYZ}(\beta,\alpha) = \beta * \alpha.$$

The Associator: For objects $W, X, Y, Z \in \mathsf{B}$,

$$c_{WXZ}\big(c_{XYZ} \times \mathrm{Id}_{\mathsf{B}(W,X)}\big) \xrightarrow{a_{WXYZ}} c_{WYZ}\big(\mathrm{Id}_{\mathsf{B}(Y,Z)} \times c_{WXY}\big)$$

is a natural isomorphism, which is called the *associator*, between functors

$$\mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \times \mathsf{B}(W,X) \longrightarrow \mathsf{B}(W,Z).$$

The Unitors: For each pair of objects $X, Y \in \mathsf{B}$,

$$c_{XYY}\big(1_Y \times \mathrm{Id}_{\mathsf{B}(X,Y)}\big) \xrightarrow{\ell_{XY}} \mathrm{Id}_{\mathsf{B}(X,Y)} \xleftarrow{r_{XY}} c_{XXY}\big(\mathrm{Id}_{\mathsf{B}(X,Y)} \times 1_X\big)$$

are natural isomorphisms, which are called the *left unitor* and the *right unitor*, respectively.

The subscripts in c will often be omitted. The subscripts in a, ℓ, and r will often be used to denote their components. The above data are required to satisfy the following two axioms for 1-cells $f \in \mathsf{B}(V,W)$, $g \in \mathsf{B}(W,X)$, $h \in \mathsf{B}(X,Y)$, and $k \in \mathsf{B}(Y,Z)$.

The Unity Axiom: The middle unity diagram

$$
\begin{tikzcd}
(g1_W)f \arrow[rr, "a_{g,1_W,f}"] \arrow[dr, "r_g * 1_f"'] & & g(1_W f) \arrow[dl, "1_g * \ell_f"] \\
& gf &
\end{tikzcd}
\tag{6.1.3}
$$

in $\mathsf{B}(V,X)$ is commutative.

The Pentagon Axiom: The diagram

$$
\begin{tikzcd}
& (kh)(gf) \arrow[dr, "a_{k,h,gf}"] & \\
((kh)g)f \arrow[ur, "a_{kh,g,f}"] \arrow[d, "a_{k,h,g} * 1_f"'] & & k(h(gf)) \\
(k(hg))f \arrow[rr, "a_{k,hg,f}"] & & k((hg)f) \arrow[u, "1_k * a_{h,g,f}"']
\end{tikzcd}
\tag{6.1.4}
$$

in $\mathsf{B}(V,Z)$ is commutative.

This finishes the definition of a bicategory. We sometimes abbreviate a bicategory as above to B. Moreover, a bicategory B is

- *locally small* if each hom category is a small category, and
- *small* if it is locally small and if $\mathsf{Ob}(\mathsf{B})$ is a set. ◇

Explanation 6.1.5. In a bicategory B, we assume that the hom categories $\mathsf{B}(X,Y)$ for objects $X,Y \in \mathsf{B}$ are disjoint. If not, we tacitly replace them with their disjoint union. ◇

Definition 6.1.6. A bicategory B' is called a *subbicategory* of a bicategory B if the following statements hold.

- $\mathsf{Ob}(\mathsf{B}')$ is a subclass of $\mathsf{Ob}(\mathsf{B})$.
- For objects $X,Y \in \mathsf{B}'$, $\mathsf{B}'(X,Y)$ is a subcategory of $\mathsf{B}(X,Y)$.
- The identity 1-cell of X in B' is equal to the identity 1-cell of X in B.
- For objects X,Y,Z in B', the horizontal composition c'_{XYZ} in B' makes the diagram

$$
\begin{tikzcd}
\mathsf{B}'(Y,Z) \times \mathsf{B}'(X,Y) \arrow[r, "c'_{XYZ}"] \arrow[d] & \mathsf{B}'(X,Z) \arrow[d] \\
\mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \arrow[r, "c_{XYZ}"] & \mathsf{B}(X,Z)
\end{tikzcd}
$$

  commutative, in which the unnamed arrows are subcategory inclusions.
- Each component of the associator in B' is equal to the corresponding component of the associator in B, and similarly for the left unitors and the right unitors.

This finishes the definition of a subbicategory. Moreover, B' is called a *full* subbicategory of B if for each pair of objects $X,Y \in \mathsf{B}'$, $\mathsf{B}'(X,Y)$ is a full subcategory of $\mathsf{B}(X,Y)$. ◇

Explanation 6.1.7. A subbicategory B' of a bicategory B is full if for each pair of objects $X,Y \in \mathsf{B}'$, and for each pair of 1-cells $f,g \in \mathsf{B}'(X,Y)$, the subcategory inclusion yields an equality

$$\mathsf{B}'(X,Y)(f,g) = \mathsf{B}(X,Y)(f,g)$$

of sets of 2-cells. ◇

Definition 6.1.8. A *2-category* is a bicategory $(\mathsf{B}, 1, c, a, \ell, r)$ in which the associator a, the left unitor ℓ, and the right unitor r are identity natural transformations. ◇

Definition 6.1.9. Suppose B and B' are 2-categories. Then B' is called a *(full) sub-2-category* of B if it is a (full) subbicategory of B in the sense of Definition 6.1.6. ◇

A 2-category can also be characterized explicitly as follows.

Proposition 6.1.10. *A 2-category* B *contains precisely the following data:*

- *A class* $\mathsf{Ob}(\mathsf{B})$ *of objects.*
- *For objects* $X, Y \in \mathsf{B}$, *a class* $\mathsf{B}(X,Y)$ *of 1-cells from* X *to* Y.
- *An identity 1-cell* $1_X \in \mathsf{B}(X,X)$ *for each object* X.
- *For 1-cells* $f, f' \in \mathsf{B}(X,Y)$, *a set* $\mathsf{B}(X,Y)(f,f')$ *of 2-cells from* f *to* f'.
- *An identity 2-cell* $1_f \in \mathsf{B}(X,Y)(f,f)$ *for each 1-cell* $f \in \mathsf{B}(X,Y)$ *and each pair of objects* X, Y.
- *For objects* X *and* Y, *and 1-cells* $f, f', f'' \in \mathsf{B}(X,Y)$, *an assignment*

$$\begin{array}{rcl} \mathsf{B}(X,Y)(f',f'') \times \mathsf{B}(X,Y)(f,f') & \xrightarrow{\;v\;} & \mathsf{B}(X,Y)(f,f'') \\ (\alpha', \alpha) & \longmapsto & \alpha'\alpha, \end{array}$$

  which is called the vertical composition.
- *For objects* X, Y, *and* Z, *an assignment*

$$\begin{array}{rcl} \mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) & \xrightarrow{\;c_1\;} & \mathsf{B}(X,Z) \\ (g,f) & \longmapsto & gf, \end{array}$$

  which is called the horizontal composition of 1-cells.
- *For objects* X, Y, *and* Z, *and 1-cells* $f, f' \in \mathsf{B}(X,Y)$ *and* $g, g' \in \mathsf{B}(Y,Z)$, *an assignment*

$$\begin{array}{rcl} \mathsf{B}(Y,Z)(g,g') \times \mathsf{B}(X,Y)(f,f') & \xrightarrow{\;c_2\;} & \mathsf{B}(X,Z)(gf, g'f') \\ (\beta, \alpha) & \longmapsto & \beta * \alpha, \end{array}$$

  which is called the horizontal composition of 2-cells.

These data are required to satisfy the axioms (i)–(vi) below.

(i) *The vertical composition is associative and unital with respect to the identity 2-cells.*

(ii) *The horizontal composition preserves identity 2-cells and vertical composition.*

(iii) *The horizontal composition of 1-cells is associative, in the sense that for 1-cells* $f \in \mathsf{B}(W,X)$, $g \in \mathsf{B}(X,Y)$, *and* $h \in \mathsf{B}(Y,Z)$, *there is an equality*

$$(hg)f = h(gf) \in \mathsf{B}(W,Z).$$

(iv) *The horizontal composition of 2-cells is associative, in the sense that for 2-cells* $\alpha \in \mathsf{B}(W,X)(f,f')$, $\beta \in \mathsf{B}(X,Y)(g,g')$, *and* $\gamma \in \mathsf{B}(Y,Z)(h,h')$, *there is an equality*

$$(\gamma * \beta) * \alpha = \gamma * (\beta * \alpha)$$

in $\mathsf{B}(W,Z)\big((hg)f, h'(g'f')\big)$.

(v) *The horizontal composition of 1-cells is unital with respect to the identity 1-cells, in the sense that there are equalities*

$$1_Y f = f = f 1_X$$

for each $f \in \mathsf{B}(X,Y)$.

(vi) *The horizontal composition of 2-cells is unital with respect to the identity 2-cells of the identity 1-cells, in the sense that there are equalities*

$$1_{1_Y} * \alpha = \alpha = \alpha * 1_{1_X}$$

for each $\alpha \in \mathsf{B}(X,Y)(f,f')$.

This finishes the list of axioms of a 2-category.

For a 2-category B, its *underlying 1-category* is the category with

- objects $\mathsf{Ob}(\mathsf{B})$,
- morphisms the 1-cells in B,
- identity morphisms the identity 1-cells in B, and
- categorical composition the horizontal composition of 1-cells in B.

By a Cat-*category* we mean a category enriched in the symmetric monoidal category $(\mathsf{Cat}, \times, *)$ of small categories with the Cartesian product as the monoidal product. Discussion of enriched categories can be found in **[Kel05]**, **[JY21**, 1.3], and Section III.1.2.

Proposition 6.1.11. *A locally small 2-category is precisely a* Cat*-category.*

Example 6.1.12 (Categories)**.** Each category C yields a 2-category when each morphism set $\mathsf{C}(X,Y)$ is regarded as a category with only identity morphisms. In this 2-category, each 2-cell is an identity 2-cell. In particular, the terminal category $\mathbf{1}$ in Convention 6.1.1 yields a 2-category with one object $*$, one 1-cell 1_*, and one 2-cell 1_{1_*}. ◇

Example 6.1.13 (Small Categories)**.** There is a 2-category Cat defined by the following data.

- It has small categories as objects, functors as 1-cells, and natural transformations as 2-cells.
- Composition of functors are horizontal composition of 1-cells.
- Vertical and horizontal composition of natural transformations are those of 2-cells.
- Identity functors are identity 1-cells.
- Identity natural transformations are identity 2-cells.

From now on, the notation Cat denotes either this 2-category or its underlying 1-category of small categories and functors. ◇

Example 6.1.14 (Monoidal Categories)**.** There is a 2-category MCat with

- small monoidal categories as objects,
- (identity) monoidal functors as (identity) 1-cells,
- monoidal natural transformations as 2-cells, and
- composition of monoidal functors as horizontal composition of 1-cells.

The rest of the structure is defined as in Cat. ◇

Example 6.1.15 (Symmetric Monoidal Categories)**.** SMCat is the 2-category with

- small symmetric monoidal categories as objects,
- (identity) symmetric monoidal functors as (identity) 1-cells,

- monoidal natural transformations as 2-cells, and
- composition of symmetric monoidal functors as horizontal composition of 1-cells.

The rest of the structure is defined as in Cat. $\diamond$

Example 6.1.16 (One-Object Bicategories)**.** Each monoidal category

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$$

yields a one-object bicategory $\Sigma\mathsf{C}$. It has unique hom category $\Sigma\mathsf{C}(*,*) = \mathsf{C}$, identity 1-cell $1_* = \mathbb{1}$, horizontal composition $\otimes$, associator α, left unitor λ, and right unitor ρ. Conversely, for each one-object bicategory, the unique hom category has the structure of a monoidal category as in the previous sentence. $\diamond$

Example 6.1.17 (Bimodules)**.** There is a bicategory Bimod whose objects are unital and associative rings. For two rings R and S, the hom category $\mathsf{Bimod}(R,S)$ has (R,S)-bimodules as objects and (R,S)-bimodule homomorphisms as morphisms.

- For an (R,S)-bimodule M and an (S,T)-bimodule N, the horizontal composition of 1-cells is
$$c_{RST}(N,M) = M \otimes_S N$$
as an (R,T)-bimodule.
- The horizontal composition of 2-cells is given by tensoring of bimodule homomorphisms over a ring.
- Components of the associator, the left unitor, and the right unitor are the usual isomorphisms:
$$\begin{aligned} M \otimes_S (N \otimes_T P) &\cong (M \otimes_S N) \otimes_T P \\ M \otimes_S S &\cong M \\ R \otimes_R M &\cong M. \end{aligned}$$

Note that the associator seemingly moves brackets from right to left. Moreover, the left unitor has S on the right, and the right unitor has R on the left. This switch of directions is due to the definition of the horizontal composition c_{RST}. $\diamond$

6.2. Lax Functors, Lax Transformations, and Pastings

In this section, we define lax functors between bicategories and lax transformations between lax functors, and briefly discuss pasting diagrams.

Lax Functors. The following definition of a lax functor is modeled after the concept of a monoidal functor in Definition 1.2.11.

Definition 6.2.1. Suppose $(\mathsf{B}, 1, c, a, \ell, r)$ and $(\mathsf{B}', 1', c', a', \ell', r')$ are bicategories as in Definition 6.1.2. A *lax functor*

$$(F, F^2, F^0) : \mathsf{B} \longrightarrow \mathsf{B}'$$

from B to B' is a triple consisting of the following data.

Objects: $F : \mathsf{Ob}(\mathsf{B}) \longrightarrow \mathsf{Ob}(\mathsf{B}')$ is a function on objects.

The Local Functors: For each pair of objects X, Y in B, it is equipped with a *local functor*

$$F : \mathsf{B}(X,Y) \longrightarrow \mathsf{B}'(FX, FY).$$

The Laxity Constraints: For all objects X, Y, Z in B, it is equipped with natural transformations

$$\begin{tikzcd}
\mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \arrow[r, "c"] \arrow[d, "F \times F"'] \arrow[dr, phantom, "F^2 \nearrow"] & \mathsf{B}(X,Z) \arrow[d, "F"] \\
\mathsf{B}'(FY,FZ) \times \mathsf{B}'(FX,FY) \arrow[r, "c'"'] & \mathsf{B}'(FX,FZ)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[r, "1_X"] \arrow[dr, bend right, "1'_{FX}"'] \arrow[dr, phantom, "F^0 \nearrow"] & \mathsf{B}(X,X) \arrow[d, "F"] \\
& \mathsf{B}'(FX,FX)
\end{tikzcd}$$

with component 2-cells

$$Fg \circ Ff \xrightarrow{F^2_{g,f}} F(gf)$$
$$1'_{FX} \xrightarrow{F^0_X} F1_X.$$

We call F^2 the *lax functoriality constraint* and F^0 the *lax unity constraint*.

The above data are required to make the following three diagrams commutative for all 1-cells $f \in \mathsf{B}(W,X)$, $g \in \mathsf{B}(X,Y)$, and $h \in \mathsf{B}(Y,Z)$.

Lax Associativity:

(6.2.2)
$$\begin{tikzcd}
(Fh \circ Fg) \circ Ff \arrow[r, "a'"] \arrow[d, "F^2_{h,g} * 1_{Ff}"'] & Fh \circ (Fg \circ Ff) \arrow[d, "1_{Fh} * F^2_{g,f}"] \\
F(hg) \circ Ff \arrow[d, "F^2_{hg,f}"'] & Fh \circ F(gf) \arrow[d, "F^2_{h,gf}"] \\
F((hg)f) \arrow[r, "Fa"] & F(h(gf))
\end{tikzcd}$$

in $B'(FW,FZ)$.

Lax Left and Right Unity:

(6.2.3)
$$\begin{tikzcd}
1'_{FX} \circ Ff \arrow[r, "\ell'"] \arrow[d, "F^0_X * 1_{Ff}"'] & Ff \\
F1_X \circ Ff \arrow[r, "F^2_{1_X,f}"] & F(1_X \circ f) \arrow[u, "F\ell"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Ff \circ 1'_{FW} \arrow[r, "r'"] \arrow[d, "1_{Ff} * F^0_W"'] & Ff \\
Ff \circ F1_W \arrow[r, "F^2_{f,1_W}"] & F(f \circ 1_W) \arrow[u, "Fr"']
\end{tikzcd}$$

in $\mathsf{B}'(FW,FX)$.

This finishes the definition of a lax functor.

Moreover, we define the following.

- A lax functor is *unitary* (respectively, *strictly unitary*) if each lax unity constraint F^0_X is an invertible 2-cell (respectively, identity 2-cell).
- A *pseudofunctor* is a lax functor in which F^2 and F^0 are natural isomorphisms.
- A *strict functor* is a lax functor in which F^2 and F^0 are identity natural transformations.
- A strict functor between two 2-categories is called a *2-functor*. ◇

Explanation 6.2.4. The lax unity constraint F^0 in a lax functor is completely determined by the component 2-cells F^0_X for objects $X \in \mathsf{B}$. Its naturality condition is redundant because the terminal category **1** only has the identity morphism of $*$. See [**JY21**, Explanation 4.1.5(3)]. $\diamond$

Proposition 6.2.5. *For 2-categories* A *and* B, *a 2-functor* $F : \mathsf{A} \longrightarrow \mathsf{B}$ *consists of precisely the following data.*

- *A function* $F : \mathsf{Ob}(\mathsf{A}) \longrightarrow \mathsf{Ob}(\mathsf{B})$ *on objects.*
- *A functor* $F : \mathsf{A}(X,Y) \longrightarrow \mathsf{B}(FX,FY)$ *for each pair of objects* X, Y *in* A.

These data are subject to the following two conditions.

(1) *F is a functor between the underlying 1-categories of* A *and* B.
(2) *F preserves horizontal compositions of 2-cells.*

Next we define composites of lax functors.

Definition 6.2.6. Suppose

$$\mathsf{B} \xrightarrow{(F,F^2,F^0)} \mathsf{C} \xrightarrow{(G,G^2,G^0)} \mathsf{D}$$

are lax functors between bicategories. The *composite*

$$\mathsf{B} \xrightarrow{(GF,(GF)^2,(GF)^0)} \mathsf{D}$$

is defined as follows.

Objects: $GF : \mathsf{Ob}(\mathsf{B}) \longrightarrow \mathsf{Ob}(\mathsf{D})$ is the composite of the functions F and G on objects.

The Local Functors: For objects X, Y in B, it is equipped with the composite functor

$$\begin{tikzcd} \mathsf{B}(X,Y) \arrow[r, "F"] \arrow[rr, bend left, "GF"] & \mathsf{C}(FX,FY) \arrow[r, "G"] & \mathsf{D}(GFX,GFY). \end{tikzcd}$$

The Lax Unity Constraint: For each object X in B, it is equipped with the vertically composed 2-cell

$$\begin{tikzcd} 1_{GFX} \arrow[r, "G^0_{FX}"] \arrow[rr, bend left, "(GF)^0_X"] & G1_{FX} \arrow[r, "G(F^0_X)"] & GF1_X \end{tikzcd} \tag{6.2.7}$$

in $\mathsf{D}(GFX,GFX)$.

The Lax Functoriality Constraint: For 1-cells $(g,f) \in \mathsf{B}(Y,Z) \times \mathsf{B}(X,Y)$, it has the vertically composed 2-cell

$$\begin{tikzcd} GFg \circ GFf \arrow[r, "G^2_{Fg,Ff}"] \arrow[rr, bend left, "(GF)^2_{g,f}"] & G(Fg \circ Ff) \arrow[r, "G(F^2_{g,f})"] & GF(gf) \end{tikzcd} \tag{6.2.8}$$

in $\mathsf{D}(GFX,GFZ)$.

The finishes the definition of the composite. $\diamond$

Lemma 6.2.9. *Suppose* $F : \mathsf{B} \longrightarrow \mathsf{C}$ *and* $G : \mathsf{C} \longrightarrow \mathsf{D}$ *are lax functors between bicategories.*

(1) *The composite* $(GF,(GF)^2,(GF)^0)$ *is a lax functor from* B *to* D.
(2) *If both F and G are pseudofunctors (respectively, strict, unitary, or strictly unitary), then so is the composite GF.*

Example 6.2.10 (Identity Strict Functors)**.** Each bicategory B has an identity strict functor $1_{\mathsf{B}} : \mathsf{B} \longrightarrow \mathsf{B}$.

- It is the identity function on the objects in B.
- It is the identity functor on $\mathsf{B}(X, Y)$ for objects X, Y in B.
- For composable 1-cells (g, f), the component $(1_{\mathsf{B}})^2_{g,f}$ is the identity 2-cell $1_{gf} = 1_g * 1_f$.
- The component $(1_{\mathsf{B}})^0_X$ is the identity 2-cell 1_{1_X}.

For 1_{B}, the lax associativity diagram (6.2.2) follows from the naturality of the associator a, and both lax unity diagrams (6.2.3) are commutative by definition. $\diamond$

Theorem 6.2.11. *There is a category* Bicat *with*

- *small bicategories as objects,*
- *lax functors between them as morphisms,*
- *composites of lax functors as in Definition 6.2.6, and*
- *identity strict functors in Example 6.2.10 as identity morphisms.*

Furthermore, Bicat *contains the wide subcategories*

(i) $\mathsf{Bicat}^{\mathsf{u}}$ *with unitary lax functors as morphisms,*
(ii) $\mathsf{Bicat}^{\mathsf{su}}$ *with strictly unitary lax functors as morphisms,*
(iii) $\mathsf{Bicat}^{\mathsf{ps}}$ *with pseudofunctors as morphisms,*
(iv) $\mathsf{Bicat}^{\mathsf{sup}}$ *with strictly unitary pseudofunctors as morphisms, and*
(v) $\mathsf{Bicat}^{\mathsf{st}}$ *with strict functors as morphisms.*

Pastings. Some of the upcoming definitions use the concept of a *pasting diagram* in a bicategory. This is an efficient and visual way to represent iterated vertical composition of 2-cells, each being the horizontal composition of one 2-cell and a finite number of identity 2-cells. This topic is discussed at length in **[JY21**, Ch. 3], where we refer the reader for detailed discussion and many examples. In particular, the Bicategorical Pasting Theorem 3.6.6 there states that each pasting diagram in a bicategory has a unique composite, once a bracketing is chosen for the (co)domain composite 1-cell.

Convention 6.2.12. Pasting diagrams use the left normalized bracketing (5.2.13) for the iterated horizontal composition of (co)domain 1-cells, unless a different bracketing is specified. $\diamond$

Example 6.2.13. Consider the following data in a bicategory B.

- Objects A_1, A_2, A_3, and A_4.
- 1-cells $f : A_1 \longrightarrow A_3$, $g_1 : A_1 \longrightarrow A_2$, $g_2 : A_2 \longrightarrow A_3$, $g_3 : A_2 \longrightarrow A_3$, $g_4 : A_3 \longrightarrow A_4$, and $h : A_2 \longrightarrow A_4$.
- 2-cells $\theta_1 : f \longrightarrow g_2 g_1$, $\theta : g_2 \longrightarrow g_3$, and $\theta_2 : g_4 g_3 \longrightarrow h$.

The pasting diagram

$$\begin{tikzcd}
A_1 \arrow[r, "g_1"] \arrow[rr, bend left=70, "f"] \arrow[rr, phantom, bend left=35, "\Downarrow\theta_1"] & A_2 \arrow[r, bend left, "g_2"] \arrow[r, bend right, "g_3"'] \arrow[r, phantom, "\Downarrow\theta"] \arrow[rr, bend right=70, "h"'] \arrow[rr, phantom, bend right=35, "\Downarrow\theta_2"] & A_3 \arrow[r, "g_4"] & A_4
\end{tikzcd}$$

represents the following vertical composite 2-cell.

$$\left(\theta_2 * 1_{g_1}\right)\left(a^{-1}_{g_4,g_3,g_1}\right)\left(1_{g_4} * (\theta * 1_{g_1})\right)\left(1_{g_4} * \theta_1\right) : g_4 f \longrightarrow h g_1$$

Note that a component of the inverse of the associator is inserted because the codomain of $1_{g_4} * (\theta * 1_{g_1})$ is the 1-cell $g_4(g_3 g_1)$, while the domain of $\theta_2 * 1_{g_1}$ is the 1-cell $(g_4 g_3) g_1$. $\diamond$

Lax Transformations. The next definition is the bicategorical analogue of a natural transformation in Definition 1.1.7.

Definition 6.2.14. Suppose (F, F^2, F^0) and (G, G^2, G^0) are lax functors $\mathsf{B} \longrightarrow \mathsf{B}'$. A *lax transformation* $\alpha : F \longrightarrow G$ consists of the following data.

Components: It is equipped with a component 1-cell $\alpha_X \in \mathsf{B}'(FX, GX)$ for each object X in B.

The Lax Naturality Constraints: For each pair of objects X, Y in B, it is equipped with a natural transformation

$$\alpha : \alpha_X^* G \longrightarrow (\alpha_Y)_* F : \mathsf{B}(X, Y) \longrightarrow \mathsf{B}'(FX, GY)$$

with a component 2-cell

$$\alpha_f : (Gf)\alpha_X \longrightarrow \alpha_Y(Ff)$$

as in the following diagram for each 1-cell $f \in \mathsf{B}(X, Y)$.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\
GX \arrow[r, "Gf"'] \arrow[ur, phantom, "\overset{\alpha_f}{\Nearrow}"] & GY
\end{tikzcd}$$

The above data are required to satisfy the following two pasting diagram equalities for all objects X, Y, Z, and 1-cells $f \in \mathsf{B}(X, Y)$ and $g \in \mathsf{B}(Y, Z)$.

Lax Unity:

(6.2.15)
$$
\begin{tikzcd}
FX \arrow[rr, bend left, "F1_X"] \arrow[d, "\alpha_X"'] \arrow[drr, phantom, "\overset{\alpha_{1_X}}{\nearrow}"] & & FX \arrow[d, "\alpha_X"] \\
GX \arrow[rr, bend left, "G1_X"] \arrow[rr, bend right, "1_{GX}"'] \arrow[rr, phantom, "\Uparrow G^0"] & & GX
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FX \arrow[rr, bend left, "F1_X"] \arrow[rr, bend right, "1_{FX}"'] \arrow[rr, phantom, "\Uparrow F^0"] \arrow[d, "\alpha_X"'] \arrow[drr, bend right, "\alpha_X"'] & & FX \arrow[d, "\alpha_X"] \arrow[dl, phantom, "\nearrow r^{-1}"] \\
GX \arrow[rr, bend right, "1_{GX}"'] \arrow[r, phantom, "\nearrow \ell"] & {} & GX
\end{tikzcd}
$$

Lax Naturality:

(6.2.16)
$$
\begin{tikzcd}
FX \arrow[rr, "F(gf)"] \arrow[dd, "\alpha_X"'] \arrow[ddrr, phantom, "\overset{\alpha_{gf}}{\nearrow}"] & & FZ \arrow[dd, "\alpha_Z"] \\
& & \\
GX \arrow[rr, "G(gf)"] \arrow[dr, "Gf"'] & \Uparrow G^2 & GZ \\
& GY \arrow[ur, "Gg"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FX \arrow[rr, "F(gf)"] \arrow[dr, "Ff"'] \arrow[dd, "\alpha_X"'] & \Uparrow F^2 & FZ \arrow[dd, "\alpha_Z"] \\
\nearrow \alpha_f & FY \arrow[ur, "Fg"'] \arrow[dd, "\alpha_Y"] & \nearrow \alpha_g \\
GX \arrow[dr, "Gf"'] & & GZ \\
& GY \arrow[ur, "Gg"'] &
\end{tikzcd}
$$

This finishes the definition of a lax transformation.

Moreover, we define the following.

- A *strong transformation* is a lax transformation in which each component α_f is an invertible 2-cell.
- A *strict transformation* is a lax transformation in which each component α_f is an identity 2-cell.
- A *2-natural transformation* is a strict transformation between 2-functors between 2-categories. ◇

Next is the bicategorical analogue of the identity natural transformation of a functor.

Lemma 6.2.17. *Suppose* $(F, F^2, F^0) : \mathsf{B} \longrightarrow \mathsf{B}'$ *is a lax functor between bicategories. Then there is a strong transformation*

$$1_F : F \longrightarrow F$$

defined by the following data.

- *For each object X in B, the component 1-cell $(1_F)_X$ is the identity 1-cell $1_{FX} \in \mathsf{B}(FX, FX)$.*
- *For each 1-cell $f \in \mathsf{B}(X, Y)$, the component 2-cell is the vertical composite*

(6.2.18)
$$
\begin{tikzcd}
(Ff)1_{FX} \arrow[r, "r_{Ff}"] \arrow[rr, bend left, "(1_F)_f"] & Ff \arrow[r, "\ell^{-1}_{Ff}"] & 1_{FY}(Ff)
\end{tikzcd}
$$

in $\mathsf{B}'(FX, FY)$.

Definition 6.2.19. For a lax functor F, the strong transformation $1_F : F \longrightarrow F$ in Lemma 6.2.17 is called the *identity transformation* of F. ◇

Note that if F is a 2-functor, then the identity transformation 1_F is a 2-natural transformation. For a general lax functor F, 1_F is a strong transformation that is not usually strict. Next we define composition of lax transformations.

Definition 6.2.20. Suppose $\alpha : F \longrightarrow G$ and $\beta : G \longrightarrow H$ are lax transformations for lax functors $F, G, H : \mathsf{B} \longrightarrow \mathsf{B}'$. The *horizontal composite* $\beta\alpha : F \longrightarrow H$ is defined with the following data.

Component 1-Cells: For each object X in B, it is equipped with the horizontal composite 1-cell

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[rr, bend left, "(\beta\alpha)_X"] & GX \arrow[r, "\beta_X"] & HX \end{tikzcd}$$

in $\mathsf{B}'(FX, HX)$.

Component 2-Cells: For each 1-cell $f \in \mathsf{B}(X,Y)$, $(\beta\alpha)_f$ is the 2-cell

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"] \arrow[dd, bend right=60, "\beta_X\alpha_X"'] & FY \arrow[d, "\alpha_Y"'] \arrow[dd, bend left=60, "\beta_Y\alpha_Y"] \\ GX \arrow[r, "Gf"] \arrow[d, "\beta_X"] \arrow[ur, phantom, "\Nearrow \alpha_f"] & GY \arrow[d, "\beta_Y"'] \\ HX \arrow[r, "Hf"'] \arrow[ur, phantom, "\Nearrow \beta_f"] & HY \end{tikzcd} \tag{6.2.21}$$

whose vertical boundaries are bracketed as indicated. ◇

Lemma 6.2.22. *In Definition 6.2.20, $\beta\alpha : F \longrightarrow H$ is a lax transformation, which is strong if both α and β are strong.*

The 2-Category of 2-Categories.

Proposition 6.2.23. *For 2-functors $F, G : \mathsf{A} \longrightarrow \mathsf{B}$ between 2-categories, a 2-natural transformation $\alpha : F \longrightarrow G$ consists of exactly a component 1-cell $\alpha_X \in \mathsf{B}(FX, GX)$ for each object X in A such that the following two conditions are satisfied.*

1-Cell Naturality: *For each 1-cell $f \in \mathsf{A}(X,Y)$, the two composite 1-cells*

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"] & GY \end{tikzcd} \tag{6.2.24}$$

in $\mathsf{B}(FX, GY)$ are equal.

2-Cell Naturality: *For each 2-cell $\theta : f \longrightarrow g$ in $\mathsf{A}(X,Y)$, the diagram*

$$\begin{tikzcd} (Gf)\alpha_X \arrow[r, "1"] \arrow[d, "G\theta * 1_{\alpha_X}"'] & \alpha_Y(Ff) \arrow[d, "1_{\alpha_Y} * F\theta"] \\ (Gg)\alpha_X \arrow[r, "1"] & \alpha_Y(Fg) \end{tikzcd} \tag{6.2.25}$$

in $\mathsf{B}(FX, GY)$ is commutative.

Definition 6.2.26. Denote by $\mathsf{2Cat}$ the 2-category of small 2-categories, 2-functors, and 2-natural transformations.

- Identity 1-cells are the identity strict functors in Example 6.2.10.
- Identity 2-cells are the identity transformations in Lemma 6.2.17.
- Horizontal composition of 2-functors is as in Definition 6.2.6.
- Horizontal and vertical compositions of 2-natural transformations are those of Cat-categories.

A 2-natural transformation $\alpha : F \longrightarrow G$ is a *2-natural isomorphism* if there exists a 2-natural transformation $\beta : G \longrightarrow F$ such that the equalities

$$\beta\alpha = 1_F \quad \text{and} \quad \alpha\beta = 1_G$$

hold. The notation $\mathsf{2Cat}$ is also used for the 1-category of small 2-categories and 2-functors. ◇

Explanation 6.2.27.

(1) For 2-natural transformations $\alpha : F \longrightarrow G$ and $\beta : G \longrightarrow H$ for 2-functors $F, G, H : \mathsf{A} \longrightarrow \mathsf{B}$ between 2-categories, the *vertical composite*

$$\beta\alpha : F \longrightarrow H$$

is defined by the component 1-cells

$$(\beta\alpha)_X = \beta_X\alpha_X \in \mathsf{B}(FX, HX)$$

for objects $X \in \mathsf{A}$.

For a 2-natural transformation $\alpha' : F' \longrightarrow G'$ for 2-functors $F', G' : \mathsf{B} \longrightarrow \mathsf{C}$ between 2-categories, the *horizontal composite*

$$\alpha' * \alpha : F'F \longrightarrow G'G$$

is defined by the component 1-cells

$$\left(\alpha' * \alpha\right)_X = \alpha'_{GX}\left(F'\alpha_X\right) \in \mathsf{B}(F'FX, G'GX)$$

for objects $X \in \mathsf{A}$.

(2) If 2-natural transformations are regarded as lax transformations, then the vertical composition of 2-natural transformations in Definition 6.2.26 is the horizontal composition of lax transformations in Definition 6.2.20.

(3) A 2-natural transformation α is a 2-natural isomorphism if and only if each component 1-cell $\alpha_X : FX \longrightarrow GX$ is invertible. In other words, there exists a 1-cell $\beta_X : GX \longrightarrow FX$ such that the equalities

$$\beta_X\alpha_X = 1_{FX} \quad \text{and} \quad \alpha_X\beta_X = 1_{GX}$$

hold. ◇

6.3. Modifications and Adjunctions

In this section, we first define morphisms between lax transformations called modifications. Then we define adjunctions and adjoint equivalences in a bicategory. These concepts will be used in the definition of a monoidal bicategory in Section 6.4.

Definition 6.3.1. Suppose $\alpha,\beta : F \longrightarrow G$ are lax transformations for lax functors $F,G : \mathsf{B} \longrightarrow \mathsf{B}'$. A *modification* $\Gamma : \alpha \longrightarrow \beta$ consists of a component 2-cell

$$\Gamma_X : \alpha_X \longrightarrow \beta_X$$

in $\mathsf{B}'(FX, GX)$ for each object X in B, that satisfies the following *modification axiom* for each 1-cell $f \in \mathsf{B}(X,Y)$.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, bend right=60, "\alpha_X"'] \arrow[dr, phantom, "\alpha_f \Nearrow"] & FY \arrow[d, bend right=40, "\alpha_Y"'] \arrow[d, bend left=40, "\beta_Y"] \arrow[d, phantom, "\overset{\Gamma_Y}{\Rightarrow}"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd} \quad = \quad \begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, bend right=40, "\alpha_X"'] \arrow[d, bend left=40, "\beta_X"] \arrow[d, phantom, "\overset{\Gamma_X}{\Rightarrow}"] \arrow[dr, phantom, "\beta_f \Nearrow"] & FY \arrow[d, bend left=60, "\beta_Y"] \\ GX \arrow[r, "Gf"'] & GY \end{tikzcd} \tag{6.3.2}$$

A modification is *invertible* if each component Γ_X is an invertible 2-cell. ◇

Definition 6.3.3. Suppose

- $F,G,H : \mathsf{B} \longrightarrow \mathsf{B}'$ are lax functors, and
- $\alpha,\beta,\gamma : F \longrightarrow G$ are lax transformations.

Identity Modifications: The *identity modification* of α, denoted by $1_\alpha : \alpha \longrightarrow \alpha$, consists of the identity 2-cell

$$(1_\alpha)_X = 1_{\alpha_X} : \alpha_X \longrightarrow \alpha_X$$

in $\mathsf{B}'(FX, GX)$ for each object X in B.

Vertical Composition: Suppose $\Gamma : \alpha \longrightarrow \beta$ and $\Omega : \beta \longrightarrow \gamma$ are modifications. The *vertical composite*

$$\Omega\Gamma : \alpha \longrightarrow \gamma$$

consists of the vertical composite 2-cell

$$\begin{tikzcd} \alpha_X \arrow[r, "\Gamma_X"] \arrow[rr, bend left, "(\Omega\Gamma)_X"] & \beta_X \arrow[r, "\Omega_X"] & \gamma_X \end{tikzcd} \tag{6.3.4}$$

in $\mathsf{B}'(FX, GX)$ for each object X in B.

Horizontal Composition: With $\Gamma : \alpha \longrightarrow \beta$ as above, suppose $\Gamma' : \alpha' \longrightarrow \beta'$ is a modification for lax transformations $\alpha',\beta' : G \longrightarrow H$. The *horizontal composite*

$$\Gamma' * \Gamma : \alpha'\alpha \longrightarrow \beta'\beta$$

consists of the horizontal composite 2-cell

$$(\Gamma' * \Gamma)_X = \Gamma'_X * \Gamma_X : (\alpha'\alpha)_X \longrightarrow (\beta'\beta)_X \tag{6.3.5}$$

in $\mathsf{B}'(FX, HX)$ for each object X in B. Here $\alpha'\alpha, \beta'\beta : F \longrightarrow H$ are the horizontal composite lax transformations in Definition 6.2.20. ◇

For a small category C and a category D, there is a category with functors $\mathsf{C} \longrightarrow \mathsf{D}$ as objects and natural transformations as morphisms. Next is the bicategorical analogue of that fact.

Definition 6.3.6. Suppose B and B' are bicategories with $\mathsf{Ob}(\mathsf{B})$ a set. Define

$$\mathsf{Bicat}(\mathsf{B},\mathsf{B}')$$

with the following data.

Objects: The objects in $\mathsf{Bicat}(\mathsf{B},\mathsf{B}')$ are lax functors $\mathsf{B} \longrightarrow \mathsf{B}'$.

The Hom Categories: For lax functors $F, G : \mathsf{B} \longrightarrow \mathsf{B}'$, $\mathsf{Bicat}(\mathsf{B},\mathsf{B}')(F,G)$ is the category with
- lax transformations $F \longrightarrow G$ as objects,
- modifications $\Gamma : \alpha \longrightarrow \beta$ between such lax transformations as morphisms,
- vertical composition of modifications as composition, and
- identity modifications as identity morphisms.

In other words, in $\mathsf{Bicat}(\mathsf{B},\mathsf{B}')$ 1-cells are lax transformations between lax functors from B to B', and 2-cells are modifications between them.

Identity 1-Cells: For each lax functor $F : \mathsf{B} \longrightarrow \mathsf{B}'$, its identity 1-cell is the identity transformation $1_F : F \longrightarrow F$ in Definition 6.2.19.

Horizontal Composition: For lax functors $F, G, H : \mathsf{B} \longrightarrow \mathsf{B}'$, the horizontal composition

$$\mathsf{Bicat}(\mathsf{B},\mathsf{B}')(G,H) \times \mathsf{Bicat}(\mathsf{B},\mathsf{B}')(F,G) \xrightarrow{c} \mathsf{Bicat}(\mathsf{B},\mathsf{B}')(F,H)$$

is given by
- the horizontal composition of lax transformations in Definition 6.2.20 for 1-cells and
- the horizontal composition of modifications in Definition 6.3.3 for 2-cells.

The Associator: For
- lax functors $F, G, H, I : \mathsf{B} \longrightarrow \mathsf{B}'$ and
- lax transformations $\alpha : F \longrightarrow G$, $\beta : G \longrightarrow H$, and $\gamma : H \longrightarrow I$,

the component

$$a_{\gamma,\beta,\alpha} : (\gamma\beta)\alpha \longrightarrow \gamma(\beta\alpha)$$

of the associator a is the modification with, for each object $X \in \mathsf{B}$, the component 2-cell

$$a_{\gamma_X,\beta_X,\alpha_X} : (\gamma_X\beta_X)\alpha_X \longrightarrow \gamma_X(\beta_X\alpha_X) \quad \text{in} \quad \mathsf{B}'(FX, IX),$$

which is a component of the associator in B'.

The Unitors: For each lax transformation α as above, the component

$$\ell_\alpha : 1_G\alpha \longrightarrow \alpha$$

of the left unitor ℓ is the modification with, for each object $X \in \mathsf{B}$, the component 2-cell

$$\ell_{\alpha_X} : 1_{GX}\alpha_X \longrightarrow \alpha_X \quad \text{in} \quad \mathsf{B}'(FX, GX),$$

which is a component of the left unitor in B'. The right unitor r is defined analogously using the right unitor in B'.

This finishes the definition of $\mathsf{Bicat}(\mathsf{B},\mathsf{B}')$. $\diamond$

Theorem 6.3.7. *Suppose* B *and* B' *are bicategories such that* B *has a set of objects. Then* $\mathsf{Bicat}(\mathsf{B},\mathsf{B}')$ *with the structure in Definition 6.3.6 is a bicategory, which is furthermore a 2-category if* B' *is a 2-category.*

Moreover, the bicategory $\mathsf{Bicat}(\mathsf{B},\mathsf{B}')$ *contains a subbicategory* $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B},\mathsf{B}')$ *with*
- *pseudofunctors* $\mathsf{B} \longrightarrow \mathsf{B}'$ *as objects,*
- *strong transformations between such pseudofunctors as 1-cells, and*
- *modifications between such strong transformations as 2-cells.*

This is a sub-2-category of $\mathsf{Bicat}(\mathsf{B},\mathsf{B}')$ *if* B' *is a 2-category.*

Motivation 6.3.8. Next we define adjunction and adjoint equivalence in a bicategory. In a category C, an isomorphism is a morphism that is strictly invertible. A similar definition exists in a bicategory B. We call a 1-cell $f : X \longrightarrow Y$ in B an *isomorphism* if there exists a 1-cell $g : Y \longrightarrow X$ such that the equalities

$$gf = 1_X \quad \text{and} \quad fg = 1_Y$$

of 1-cells hold. However, this notion of sameness is far too strong to hold in practice. A more reasonable concept of sameness should replace the above equalities with invertible 2-cells that are compatible in some sense. The next definition of an adjoint equivalence in a bicategory is modeled after the concept of an adjoint equivalence in Definition 1.1.12. ◇

Definition 6.3.9. An *internal adjunction*, which is also called an *adjunction*, $f \dashv g$ in a bicategory B is a quadruple $(f, g, \eta, \varepsilon)$ consisting of

- 1-cells $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ and
- 2-cells $\eta : 1_X \longrightarrow gf$ and $\varepsilon : fg \longrightarrow 1_Y$.

These data are subject to the following two axioms, in the form of commutative triangles. These are known as the *triangle identities*.

$$\begin{tikzcd}
f1_X \arrow[r, "1_f * \eta"] \arrow[ddrr, "r_f"'] & f(gf) \arrow[r, "a^{-1}_{f,g,f}"] & (fg)f \arrow[d, "\varepsilon * 1_f"] \\
& & 1_Y f \arrow[d, "\ell_f"] \\
& & f
\end{tikzcd}
\qquad
\begin{tikzcd}
1_X g \arrow[r, "\eta * 1_g"] \arrow[ddrr, "\ell_g"'] & (gf)g \arrow[r, "a_{g,f,g}"] & g(fg) \arrow[d, "1_g * \varepsilon"] \\
& & g1_Y \arrow[d, "r_g"] \\
& & g
\end{tikzcd} \tag{6.3.10}$$

In (6.3.10) the left and the right axioms are called, respectively, the *left triangle identity* and the *right triangle identity*. The 1-cell f is called the *left adjoint*, and g is called the *right adjoint*. The 2-cell η, respectively ε, is called the *unit*, respectively *counit*.

An adjunction $(f, g, \eta, \varepsilon)$ is called an *internal equivalence* or *adjoint equivalence* if η and ε are isomorphisms. We say that f and g are members of an adjoint equivalence in this case, and we let $f^\bullet$ denote an adjoint. ◇

6.4. Monoidal Bicategories

In this section, we define monoidal bicategories, which are one-object tricategories. However, this compact definition depends on the concept of a tricategory, which we do not discuss in this book. The following definition spells out explicitly the structure of a monoidal bicategory.

Definition 6.4.1. A *monoidal bicategory* is a tuple

$$\big(\mathsf{B}, \boxtimes, 1_\boxtimes, a, \ell, r, \pi, \mu, \lambda, \rho\big)$$

consisting of the following data.

The Base Bicategory: It has a bicategory B, which is called the *base bicategory*. To avoid confusion, we may refer to the associator, the left unitor, and the right unitor in this bicategory B as, respectively, the *base associator*, the *base left unitor*, and the *base right unitor*. The n-fold product bicategory $\mathsf{B} \times \cdots \times \mathsf{B}$ is written as B^n below, and $\mathbf{1}$ is the 2-category with only one object $*$, the identity 1-cell 1_*, and the identity 2-cell 1_{1_*}.

The Monoidal Composition: It has a pseudofunctor

$$\mathsf{B} \times \mathsf{B} \xrightarrow{(\boxtimes, \boxtimes^2, \boxtimes^0)} \mathsf{B},$$

which is called the *monoidal composition*.

The Monoidal Identity: It has a pseudofunctor

$$\mathbf{1} \xrightarrow{(1_\boxtimes, 1_\boxtimes^2, 1_\boxtimes^0)} \mathsf{B},$$

which is called the *monoidal identity*. The object $1_\boxtimes(*) \in \mathsf{B}$ is also denoted by $1_\boxtimes$, which is called the *identity object*.

The Monoidal Associator: It has an adjoint equivalence $(a, a^\bullet, \eta^a, \varepsilon^a)$ with left adjoint

$$\begin{tikzcd} \mathsf{B}^3 \arrow[r, "\boxtimes \times 1"] \arrow[d, "1 \times \boxtimes"'] \arrow[dr, phantom, "\overset{a}{\swarrow}"] & \mathsf{B}^2 \arrow[d, "\boxtimes"] \\ \mathsf{B}^2 \arrow[r, "\boxtimes"'] & \mathsf{B} \end{tikzcd}$$

in the bicategory $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B}^3, \mathsf{B})$, which is called the *monoidal associator*. Its left and right adjoints have the following component 1-cells for objects $A, B, C \in \mathsf{B}$.

$$(C \boxtimes B) \boxtimes A \underset{a^\bullet_{C,B,A}}{\overset{a_{C,B,A}}{\rightleftarrows}} C \boxtimes (B \boxtimes A) \in \mathsf{B}$$

The Monoidal Unitors: It has adjoint equivalences $(\ell, \ell^\bullet, \eta^\ell, \varepsilon^\ell)$ and $(r, r^\bullet, \eta^r, \varepsilon^r)$ with respective left adjoints

$$\begin{tikzcd} & \mathsf{B}^2 \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\Downarrow\ell"] & \\ \mathsf{B} \arrow[ur, "1_\boxtimes \times 1"] \arrow[rr, "1"'] & {} & \mathsf{B} \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{B}^2 \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\Downarrow r"] & \\ \mathsf{B} \arrow[ur, "1 \times 1_\boxtimes"] \arrow[rr, "1"'] & {} & \mathsf{B} \end{tikzcd}$$

in the bicategory $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B}, \mathsf{B})$, which are called the *left monoidal unitor* and the *right monoidal unitor*, respectively. Their left and right adjoints have component 1-cells

$$1_\boxtimes \boxtimes A \underset{\ell^\bullet_A}{\overset{\ell_A}{\rightleftarrows}} A \underset{r^\bullet_A}{\overset{r_A}{\leftrightarrows}} A \boxtimes 1_\boxtimes \in \mathsf{B}.$$

The Pentagonator: It has an invertible modification π, which is called the *pentagonator*, with the following component 2-cells for objects $A, B, C, D \in \mathsf{B}$.

$$\begin{tikzcd}
& (D\boxtimes(C\boxtimes B))\boxtimes A \arrow[rr, "a_{D,C\boxtimes B,A}"] & & D\boxtimes((C\boxtimes B)\boxtimes A) \arrow[dr, "1_D\boxtimes a_{C,B,A}"] & \\
((D\boxtimes C)\boxtimes B)\boxtimes A \arrow[ur, "a_{D,C,B}\boxtimes 1_A"] \arrow[drr, "a_{D\boxtimes C,B,A}"'] & & \Downarrow\pi_{D,C,B,A} & & D\boxtimes(C\boxtimes(B\boxtimes A)) \\
& & (D\boxtimes C)\boxtimes(B\boxtimes A) \arrow[urr, "a_{D,C,B\boxtimes A}"'] & &
\end{tikzcd}$$

The 2-Unitors: It has invertible modifications μ, λ, and ρ, which are called, respectively, the *middle 2-unitor*, the *left 2-unitor*, and the *right 2-unitor*, with the following component 2-cells in B.

$$\begin{tikzcd}
& (B\boxtimes 1_\boxtimes)\boxtimes A \arrow[r, "a_{B,1_\boxtimes,A}"] \arrow[d, phantom, "\Downarrow\mu_{B,A}"] & B\boxtimes(1_\boxtimes\boxtimes A) \arrow[dr, "1_B\boxtimes\ell_A"] & \\
B\boxtimes A \arrow[ur, "r^\bullet_B\boxtimes 1_A"] \arrow[rrr, "1_{B\boxtimes A}"'] & & & B\boxtimes A
\end{tikzcd}$$

$$\begin{tikzcd}
(1_\boxtimes\boxtimes B)\boxtimes A \arrow[rr, "\ell_B\boxtimes 1_A"] \arrow[dr, bend right, "a_{1_\boxtimes,B,A}"'] & \Downarrow\lambda_{B,A} & B\boxtimes A \\
& 1_\boxtimes\boxtimes(B\boxtimes A) \arrow[ur, bend right, "\ell_{B\boxtimes A}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B\boxtimes A \arrow[rr, "1_B\boxtimes r^\bullet_A"] \arrow[dr, bend right, "r^\bullet_{B\boxtimes A}"'] & \Downarrow\rho_{B,A} & B\boxtimes(A\boxtimes 1_\boxtimes) \\
& (B\boxtimes A)\boxtimes 1_\boxtimes \arrow[ur, bend right, "a_{B,A,1_\boxtimes}"'] &
\end{tikzcd}$$

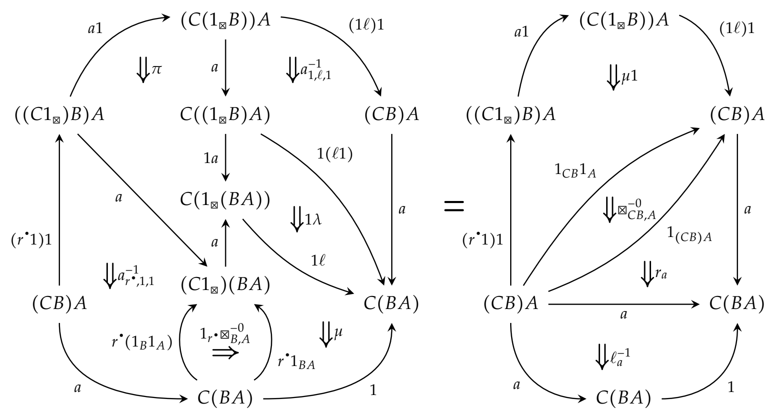
The above data are required to satisfy the following three pasting diagram equalities for objects $A, B, C, D, E \in \mathsf{B}$, with $\boxtimes$ abbreviated to concatenation, and iterates of $\boxtimes$ denoted by parentheses.

The Non-Abelian 4-Cocycle Condition:

$$\begin{gathered}
\begin{tikzcd}[column sep=small]
& (E((DC)B))A \arrow[r, "(1a)1"] \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\pi"] & (E(D(CB)))A \arrow[r, "a"] \arrow[d, phantom, "\Downarrow a^{-1}_{1,a,1}"] & E((D(CB))A) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow 1\pi"] & \\
((E(DC))B)A \arrow[ur, "a1"] \arrow[dr, "a"] \arrow[dd, phantom, "\Downarrow a^{-1}_{a,1,1}"] & & E(((DC)B)A) \arrow[ur, "1(a1)"] \arrow[dr, "1a"] & & E(D((CB)A)) \arrow[dd, "1(1a)"] \\
& (E(DC))(BA) \arrow[rr, "a"] & & E((DC)(BA)) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow\pi"] & \\
(((ED)C)B)A \arrow[uu, "(a1)1"] \arrow[dr, "a"'] & & & & E(D(C(BA))) \\
& ((ED)C)(BA) \arrow[uu, bend left, "a(1_B1_A)"] \arrow[uu, bend right, "a1_{BA}"'] \arrow[uu, phantom, "\overset{1_a(\boxtimes^{-0}_{B,A})}{\Rightarrow}"] \arrow[rr, "a"] & & (ED)(C(BA)) \arrow[ur, "a"'] &
\end{tikzcd} \\
\| \\
\begin{tikzcd}[column sep=small]
& (E((DC)B))A \arrow[r, "(1a)1"] & (E(D(CB)))A \arrow[r, "a"] & E((D(CB))A) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow\pi"] & \\
((E(DC))B)A \arrow[ur, "a1"] \arrow[r, phantom, "\overset{\pi 1}{\Rightarrow}"] & ((ED)(CB))A \arrow[ur, "a1"] \arrow[drr, "a"] \arrow[dd, phantom, "\Downarrow\pi"] & & & E(D((CB)A)) \arrow[dd, "1(1a)"] \\
& & & (ED)((CB)A) \arrow[ur, "a"] \arrow[r, phantom, "\Downarrow a_{1,1,a}"] \arrow[dd, bend right, "1_{ED}a"'] \arrow[dd, bend left, "(1_E1_D)a"] \arrow[dd, phantom, "\overset{(\boxtimes^{-0}_{E,D})1_a}{\Leftarrow}"] & \\
(((ED)C)B)A \arrow[uu, "(a1)1"] \arrow[uur, "a1"] \arrow[dr, "a"'] & & & & E(D(C(BA))) \\
& ((ED)C)(BA) \arrow[rr, "a"] & & (ED)(C(BA)) \arrow[ur, "a"'] &
\end{tikzcd}
\end{gathered} \tag{6.4.2}$$

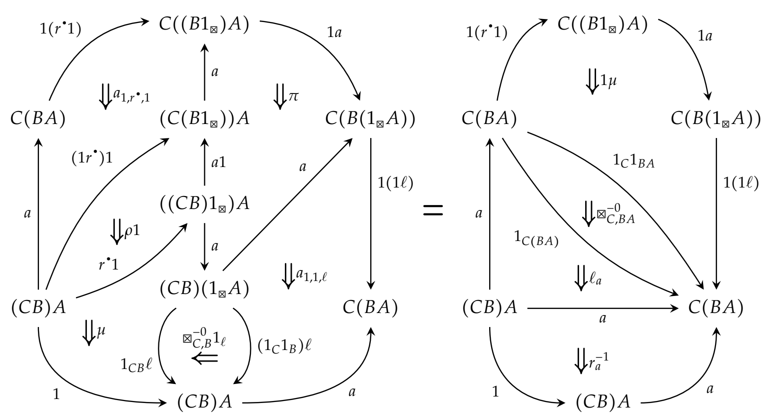
Left Normalization:

(6.4.3)



Right Normalization:

(6.4.4)



This finishes the definition of a monoidal bicategory. ◇

Remark 6.4.5. The common notation $1_\boxtimes : \mathbf{1} \longrightarrow \mathsf{B}$ for the identity, which is a pseudofunctor, and for the identity object $1_\boxtimes \in \mathsf{B}$ is not to be confused with the identity strong transformation of the pseudofunctor $\boxtimes : \mathsf{B} \times \mathsf{B} \longrightarrow \mathsf{B}$. ◇

Explanation 6.4.6 (Data)**.** The monoidal associator and the monoidal unitors in Definition 6.4.1 refer to $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B}^n, \mathsf{B})$ in Theorem 6.3.7, which assumes that B has a set of objects. This reference to the bicategory $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B}^n, \mathsf{B})$ is simply a matter of convenience to enable us to use the concept of an adjoint equivalence in Definition 6.3.9. For the definition of a monoidal bicategory, this smallness assumption is actually unnecessary.

Indeed, recall

- the composition of pseudofunctors in Definition 6.2.6;
- the identity strong transformation in Lemma 6.2.17;
- the horizontal composition of strong transformations in Definition 6.2.20; and
- the two types of compositions of modifications in Definition 6.3.3.

Using these concepts, in the definition of the monoidal associator $(a, a^\bullet, \eta^a, \varepsilon^a)$, we can equivalently require the following conditions.

(i) a and $a^\bullet$ are strong transformations as follows.

$$\boxtimes(\boxtimes\times 1) \underset{a^\bullet}{\overset{a}{\rightleftarrows}} \boxtimes(1\times\boxtimes)$$

(ii) η^a and ε^a are invertible modifications as follows.

$$1_{\boxtimes(\boxtimes\times 1)} \xrightarrow{\eta^a} a^\bullet a$$
$$aa^\bullet \xrightarrow{\varepsilon^a} 1_{\boxtimes(1\times\boxtimes)}$$

(iii) The triangle identities (6.3.10) hold. In those two diagrams, the modifications denoted by a, ℓ, and r are defined componentwise in B using, respectively, its associator, left unitor, and right unitor.

Similar remarks apply to the monoidal unitors $(\ell, \ell^\bullet, \eta^\ell, \varepsilon^\ell)$ and $(r, r^\bullet, \eta^r, \varepsilon^r)$. In summary, with the interpretation above, in the definition of a monoidal bicategory, it is *not* necessary to assume that B has a set of objects. ◇

Explanation 6.4.7 (Axioms)**.** Consider the axioms in Definition 6.4.1.

- The non-abelian 4-cocycle condition (6.4.2) will be abbreviated to NB4 from now on.
- $a_{a,1,1}$, $a_{1,a,1}$, and $a_{1,1,a}$ in NB4 are component 2-cells of the monoidal associator a, which is a strong transformation.
- $\boxtimes^{-0}_{B,A}$ is the inverse of the (B,A) component of $\boxtimes^0$.
- In the left normalization axiom (6.4.3) and the right normalization axiom (6.4.4), ℓ_a and r_a are components of, respectively, the left unitor and the right unitor in the base bicategory B.
- The 2-cell 1π in the top side of NB4 is *not* $1 \boxtimes \pi$. To interpret it correctly, first note that π has a component 2-cell

$$\big[(1_D \boxtimes a_{C,B,A})a_{D,C\boxtimes B,A}\big](a_{D,C,B}\boxtimes 1_A) \xrightarrow{\pi_{D,C,B,A}} a_{D,C,B\boxtimes A}a_{D\boxtimes C,B,A}$$

in $\mathsf{B}\big(((DC)B)A, D(C(BA))\big)$. Then 1π is defined as the following vertical composite 2-cell in $\mathsf{B}\big(E(((DC)B)A), E(D(C(BA)))\big)$, with $\boxtimes^{-2}$ the inverse of $\boxtimes^2$.

(6.4.8)
$$\begin{tikzcd}
\big[(1_E\boxtimes(1_D\boxtimes a_{C,B,A}))(1_E\boxtimes a_{D,C\boxtimes B,A})\big]\big[1_E\boxtimes(a_{D,C,B}\boxtimes 1_A)\big] \arrow[d, "\boxtimes^2 * 1"] \arrow[dddd, bend right, "1\pi"'] \\
\big[(1_E 1_E)\boxtimes((1_D\boxtimes a_{C,B,A})a_{D,C\boxtimes B,A})\big]\big[1_E\boxtimes(a_{D,C,B}\boxtimes 1_A)\big] \arrow[d, "\boxtimes^2"] \\
\big[(1_E 1_E)1_E\big]\boxtimes\big[((1_D\boxtimes a_{C,B,A})a_{D,C\boxtimes B,A})(a_{D,C,B}\boxtimes 1_A)\big] \arrow[d, "(\ell_{1_E} * 1)\boxtimes\pi_{D,C,B,A}"] \\
(1_E 1_E)\boxtimes(a_{D,C,B\boxtimes A}a_{D\boxtimes C,B,A}) \arrow[d, "\boxtimes^{-2}"] \\
(1_E\boxtimes a_{D,C,B\boxtimes A})(1_E\boxtimes a_{D\boxtimes C,B,A})
\end{tikzcd}$$

- The 2-cells
  - $\pi 1$ in the bottom side of NB4 (6.4.2),
  - 1λ and $\mu 1$ in the left normalization axiom (6.4.3), and
  - $\rho 1$ and 1μ in the right normalization axiom (6.4.4)

  are interpreted in a similar way to (6.4.8). $\diamond$

6.5. Symmetric Monoidal Bicategories

In this section, we define braided, sylleptic, and symmetric analogues of monoidal bicategories.

- Braided monoidal bicategories are in Definition 6.5.3.
- Sylleptic monoidal bicategories are in Definition 6.5.7.
- Symmetric monoidal bicategories are in Definition 6.5.9.

Motivation 6.5.1. A braided monoidal category in Definition II.1.3.15 has an underlying monoidal category and a natural isomorphism $\xi : X \otimes Y \longrightarrow Y \otimes X$, which is called the braiding, that satisfies two hexagon axioms. In a similar manner, a braided monoidal bicategory has

- an underlying monoidal bicategory;
- an adjoint equivalence β, which is called the braiding, that relates the monoidal composition $\boxtimes$ and its opposite; and
- two invertible modifications, which are called the left and the right hexagonators, that fill the diagrams in the hexagon axioms with invertible 2-cells.

Moreover, there are four coherence axioms for the braiding and the hexagonators.

In the 1-categorical setting, a symmetric monoidal category is a braided monoidal category whose braiding satisfies the symmetry axiom, $\xi_{Y,X}\xi_{X,Y} = 1$. In the monoidal bicategorical setting, the symmetry axiom is replaced by an invertible modification ν, which is called the syllepsis, that satisfies two coherence axioms. This intermediate structure is called a sylleptic monoidal bicategory. A symmetric monoidal bicategory is a sylleptic monoidal bicategory whose syllepsis satisfies an additional axiom. $\diamond$

The definition of a braided monoidal bicategory requires the theory of *mates* discussed in detail in **[JY21**, Section 6.1]. Briefly, a mate of a 2-cell θ is obtained by replacing some of the boundary 1-cells in θ by their adjoints as in Definition 6.3.9, by pasting with the (co)units, their inverses, and the left and the right unitors. The following preliminary observation contains examples of mates of the pentagonator π that are used in the definition of a braided monoidal bicategory. The symbols $\boxtimes$ for the monoidal composition among the objects are omitted to save space.

Lemma 6.5.2. *In a monoidal bicategory* B*, the pentagonator* π *induces invertible* 2*-cells* π_n *for* $1 \le n \le 10$*, with component* 2*-cells as displayed below.*

$$\begin{tikzcd}
(D(CB))A \arrow[d, "a^{\bullet} \boxtimes 1"'] & & D((CB)A) \arrow[ll, "a^{\bullet}"'] \\
((DC)B)A \arrow[rr, phantom, "\Uparrow \pi_1"] & & D(C(BA)) \arrow[u, "1 \boxtimes a^{\bullet}"'] \arrow[dl, "a^{\bullet}"] \\
& (DC)(BA) \arrow[ul, "a^{\bullet}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(D(CB))A \arrow[d, "a^{\bullet} \boxtimes 1"'] & & D((CB)A) \arrow[ll, "a^{\bullet}"'] \\
((DC)B)A \arrow[rr, phantom, "\pi_2 \Swarrow"] & & D(C(BA)) \arrow[u, "1 \boxtimes a^{\bullet}"'] \\
& (DC)(BA) \arrow[ul, "a^{\bullet}"] \arrow[ur, "a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(D(CB))A \arrow[rr, "a"] & & D((CB)A) \\
((DC)B)A \arrow[u, "a \boxtimes 1"] \arrow[rr, phantom, "\pi_3 \Leftarrow"] & & D(C(BA)) \arrow[u, "1 \boxtimes a^{\bullet}"'] \\
& (DC)(BA) \arrow[ul, "a^{\bullet}"] \arrow[ur, "a"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(D(CB))A \arrow[rr, "a"] \arrow[d, "a^{\bullet} \boxtimes 1"'] & & D((CB)A) \arrow[d, "1 \boxtimes a"] \\
((DC)B)A \arrow[rr, phantom, "\pi_4 \Leftarrow"] & & D(C(BA)) \arrow[dl, "a^{\bullet}"] \\
& (DC)(BA) \arrow[ul, "a^{\bullet}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
(D(CB))A & & D((CB)A) \arrow[ll, "a^{\bullet}"'] \\
((DC)B)A \arrow[u, "a \boxtimes 1"] \arrow[rr, phantom, "\pi_5 \Rightarrow"] & & D(C(BA)) \arrow[u, "1 \boxtimes a^{\bullet}"'] \\
& (DC)(BA) \arrow[ul, "a^{\bullet}"] \arrow[ur, "a"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(D(CB))A \arrow[d, "a^{\bullet} \boxtimes 1"'] & & D((CB)A) \arrow[ll, "a^{\bullet}"'] \arrow[d, "1 \boxtimes a"] \\
((DC)B)A \arrow[rr, phantom, "\Rightarrow \pi_6"] \arrow[dr, "a"] & & D(C(BA)) \arrow[dl, "a^{\bullet}"] \\
& (DC)(BA) &
\end{tikzcd}$$

$$\begin{tikzcd}
(D(CB))A \arrow[rr, "a"] \arrow[d, "a^{\bullet} \boxtimes 1"'] & & D((CB)A) \arrow[d, "1 \boxtimes a"] \\
((DC)B)A \arrow[rr, phantom, "\pi_7 \Rightarrow"] \arrow[dr, "a"'] & & D(C(BA)) \arrow[dl, "a^{\bullet}"] \\
& (DC)(BA) &
\end{tikzcd}
\qquad
\begin{tikzcd}
(D(CB))A \arrow[rr, "a"] \arrow[d, "a^{\bullet} \boxtimes 1"'] & & D((CB)A) \\
((DC)B)A \arrow[rr, phantom, "\Uparrow \pi_8"] \arrow[dr, "a"'] & & D(C(BA)) \arrow[u, "1 \boxtimes a^{\bullet}"'] \\
& (DC)(BA) \arrow[ur, "a"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
(D(CB))A \arrow[d, "a^{\bullet} \boxtimes 1"'] & & D((CB)A) \arrow[ll, "a^{\bullet}"'] \arrow[d, "1 \boxtimes a"] \\
((DC)B)A \arrow[rr, phantom, "\pi_9 \Leftarrow"] & & D(C(BA)) \arrow[dl, "a^{\bullet}"] \\
& (DC)(BA) \arrow[ul, "a^{\bullet}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(D(CB))A \arrow[rr, "a"] \arrow[d, "a^{\bullet} \boxtimes 1"'] & & D((CB)A) \arrow[d, "1 \boxtimes a"] \\
((DC)B)A \arrow[rr, phantom, "\pi_{10} \Rightarrow"] \arrow[dr, "a"'] & & D(C(BA)) \\
& (DC)(BA) \arrow[ur, "a"'] &
\end{tikzcd}$$

Definition 6.5.3. A *braided monoidal bicategory* is a quadruple

$$\big(\mathsf{B}, \beta, R_{-|--}, R_{--|-}\big)$$

consisting of the following data.

(1) B is a monoidal bicategory $\big(\mathsf{B}, \boxtimes, 1_{\boxtimes}, a, \ell, r, \pi, \mu, \lambda, \rho\big)$ as in Definition 6.4.1.
(2) $(\beta, \beta^{\bullet}, \eta^{\beta}, \varepsilon^{\beta})$ is an adjoint equivalence as in

$$\begin{tikzcd}
\mathsf{B}^2 \arrow[rr, "\boxtimes"] \arrow[dr, bend right, "\tau"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\beta"] & & \mathsf{B} \\
& \mathsf{B}^2 \arrow[ur, bend right, "\boxtimes"'] &
\end{tikzcd}$$

in $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B}^2,\mathsf{B})$, which is called the *braiding*, in which τ switches the two arguments. For objects $A, B \in \mathsf{B}$, the component 1-cells of β and $\beta^\bullet$ are

$$\begin{tikzcd} A \boxtimes B \arrow[r, shift left, "\beta_{A,B}"] & B \boxtimes A. \arrow[l, shift left, "\beta^\bullet_{A,B}"] \end{tikzcd}$$

(3) $R_{-|--}$ is an invertible modification, which is called the *left hexagonator*. Its components are invertible 2-cells

$$\begin{tikzcd}[column sep=small] & (B \boxtimes A) \boxtimes C \arrow[rr, "a_{B,A,C}"] & & B \boxtimes (A \boxtimes C) \arrow[dr, "1_B \boxtimes \beta_{A,C}"] & \\ (A \boxtimes B) \boxtimes C \arrow[ur, "\beta_{A,B} \boxtimes 1_C"] \arrow[dr, "a_{A,B,C}"'] & & \Downarrow R_{A|B,C} & & B \boxtimes (C \boxtimes A) \\ & A \boxtimes (B \boxtimes C) \arrow[rr, "\beta_{A,B\boxtimes C}"] & & (B \boxtimes C) \boxtimes A \arrow[ur, "a_{B,C,A}"'] & \end{tikzcd}$$

in $\mathsf{B}\big((A \boxtimes B) \boxtimes C, B \boxtimes (C \boxtimes A)\big)$ with the left normalized bracketings in the (co)domain, for objects $A, B, C \in \mathsf{B}$.

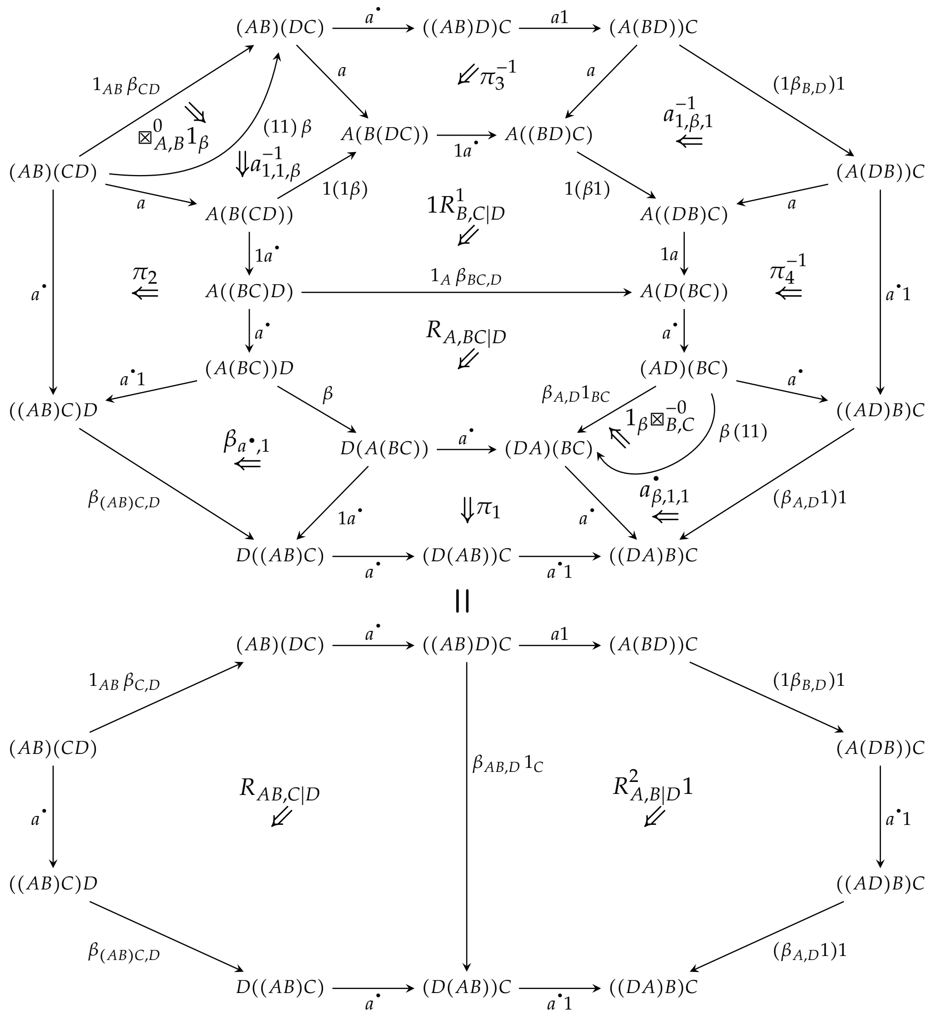
(4) $R_{--|-}$ is an invertible modification, which is called the *right hexagonator*. Its components are invertible 2-cells

$$\begin{tikzcd}[column sep=small] & A \boxtimes (C \boxtimes B) \arrow[rr, "a^\bullet_{A,C,B}"] & & (A \boxtimes C) \boxtimes B \arrow[dr, "\beta_{A,C} \boxtimes 1_B"] & \\ A \boxtimes (B \boxtimes C) \arrow[ur, "1_A \boxtimes \beta_{B,C}"] \arrow[dr, "a^\bullet_{A,B,C}"'] & & \Downarrow R_{A,B|C} & & (C \boxtimes A) \boxtimes B \\ & (A \boxtimes B) \boxtimes C \arrow[rr, "\beta_{A\boxtimes B,C}"] & & C \boxtimes (A \boxtimes B) \arrow[ur, "a^\bullet_{C,A,B}"'] & \end{tikzcd}$$

in $\mathsf{B}\big(A \boxtimes (B \boxtimes C), (C \boxtimes A) \boxtimes B\big)$ with the left normalized bracketings in the (co)domain.

The above data are required to satisfy the following four pasting diagram equalities for objects $A, B, C, D \in \mathsf{B}$, with $\boxtimes$ abbreviated to concatenation, and iterates of $\boxtimes$ denoted by parentheses. The 2-cells π_n in Lemma 6.5.2 are used in the axioms below.

The (3,1)-Crossing Axiom:



In the above pasting diagrams, $R^1_{B,C|D}$ and $R^2_{A,B|D}$ are mates of the indicated components of the right hexagonator. The 2-cells $1R^1_{B,C|D}$ and $R^2_{A,B|D}1$ are interpreted in a similar way to (6.4.8).

The (1,3)-Crossing Axiom:

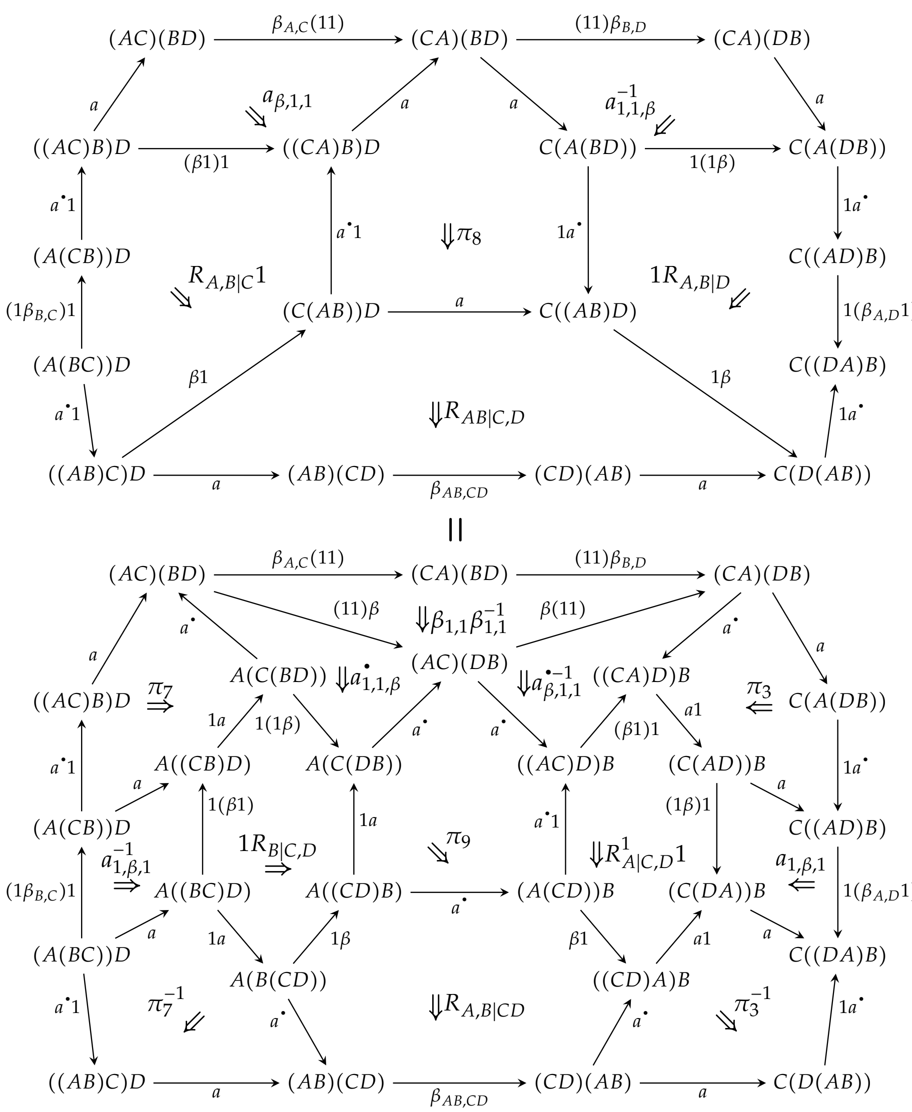
$$
\begin{tikzcd}
& (BA)(CD) \arrow[r, "a"] & B(A(CD)) \arrow[r, "1a^\bullet"] \arrow[ddd, "1_B\,\beta_{A,CD}"] & B((AC)D) \arrow[dr, "1(\beta_{A,C}1)"] & \\
(AB)(CD) \arrow[ur, "\beta_{A,B}\,1_{CD}"] \arrow[d, "a"'] & \overset{R_{A|B,CD}}{\Swarrow} & & \overset{1R^1_{A|C,D}}{\Swarrow} & B((CA)D) \arrow[d, "1a"] \\
A(B(CD)) \arrow[dr, "\beta_{A,B(CD)}"'] & & & & B(C(AD)) \arrow[dl, "1(1\beta_{A,D})"] \\
& (B(CD))A \arrow[r, "a"'] & B((CD)A) \arrow[r, "1a"'] & B(C(DA)) &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}
& (BA)(CD) \arrow[rr, "a"] \arrow[dr, "a^\bullet"] & & B(A(CD)) \arrow[rr, "1a^\bullet"] & & B((AC)D) \arrow[dl, "a^\bullet"'] \arrow[ddr, "1(\beta_{A,C}1)"] & \\
& & ((BA)C)D \arrow[rr, "a1"'] & \Swarrow\pi_5^{-1} & (B(AC))D \arrow[ddr, "(1\beta)1"'] & \overset{a^{\bullet-1}_{1,\beta,1}}{\Leftarrow} & \\
(AB)(CD) \arrow[uur, "\beta_{A,B}\,1_{CD}"] \arrow[uur, bend right, "\beta(11)"'] \arrow[dr, "a^\bullet"'] \arrow[dddd, "a"'] & 1_\beta\boxtimes^0_{C,D}\Searrow & \Downarrow a^{\bullet-1}_{\beta,1,1} & & & & B((CA)D) \arrow[dl, "a^\bullet"] \arrow[dddd, "1a"] \\
& ((AB)C)D \arrow[uur, "(\beta1)1"'] \arrow[d, "a1"] & & \overset{R^2_{A|B,C}1}{\Swarrow} & & (B(CA))D \arrow[d, "a^\bullet1"'] & \\
\overset{\pi_3^{-1}}{\Leftarrow} & (A(BC))D \arrow[rrrr, "\beta_{A,BC}\,1_D"] \arrow[d, "a"] & & & & ((BC)A)D \arrow[d, "a"'] & \overset{\pi_6^{-1}}{\Leftarrow} \\
& A((BC)D) \arrow[ddr, "\beta"] & & \overset{R_{A|BC,D}}{\Swarrow} & & (BC)(AD) \arrow[ddl, "1_{BC}\beta_{A,D}"'] \arrow[ddl, bend left, "(11)\beta"] & \\
A(B(CD)) \arrow[ur, "1a^\bullet"] \arrow[ddr, "\beta_{A,B(CD)}"'] & & & & \Nwarrow\boxtimes^{-0}_{B,C}1_\beta & & B(C(AD)) \arrow[ul, "a^\bullet"'] \arrow[ddl, "1(1\beta_{A,D})"] \\
& \Downarrow\beta^{-1}_{1,a^\bullet} & ((BC)D)A \arrow[rr, "a"] & \Downarrow\pi_{10} & (BC)(DA) \arrow[dr, "a"'] & \overset{\left(a^{\bullet-1}_{1,1,\beta}\right)'}{\Nwarrow} & \\
& (B(CD))A \arrow[ur, "a^\bullet1"'] \arrow[rr, "a"'] & & B((CD)A) \arrow[rr, "1a"'] & & B(C(DA)) &
\end{tikzcd}
$$

In the above pasting diagrams, $R^1_{A|C,D}$ and $R^2_{A|B,C}$ are mates of the indicated components of the left hexagonator. The 2-cells $1R^1_{A|C,D}$ and $R^2_{A|B,C}1$ are interpreted in a similar way to (6.4.8). The 2-cell $\left(a^{\bullet-1}_{1,1,\beta}\right)'$ is a mate of $a^{\bullet-1}_{1,1,\beta}$.

The (2,2)-Crossing Axiom:



In the above pasting diagrams, $R^1_{A|C,D}$ is a mate of the indicated component of the left hexagonator. The 2-cells $R_{A,B|C}1$, $1R_{A,B|D}$, $1R_{B|C,D}$, and $R^1_{A|C,D}1$ are interpreted in a similar way to (6.4.8).

The Yang-Baxter Axiom:

$$
\begin{tikzcd}[column sep=small]
& (AB)C \arrow[r, "a"] \arrow[dl, "\beta_{A,B}1"'] & A(BC) \arrow[dr, "1\beta_{B,C}"] \arrow[dddl, "\beta"'] & \\
(BA)C \arrow[d, "a"'] \arrow[r, phantom, "R_{A|B,C} \atop \Rightarrow"] & & \arrow[r, phantom, "\beta_{1,\beta} \atop \Rightarrow"] & A(CB) \arrow[d, "a^{\bullet}"] \arrow[dddddll, "\beta"'] \\
B(AC) \arrow[dd, "1\beta_{A,C}"'] & & & (AC)B \arrow[dd, "\beta_{A,C}1"] \\
& (BC)A \arrow[dl, "a"] \arrow[ddl, "1"] \arrow[ddd, "\beta 1"] & & \\
B(CA) \arrow[d, "a^{\bullet}"'] \arrow[r, phantom, "\Rightarrow \atop \eta^{-a}"] & \arrow[r, phantom, "\Rightarrow \atop r"] & \arrow[r, phantom, "(R^1_{A|C,B})^{-1} \atop \Rightarrow"] & (CA)B \arrow[d, "a"] \\
(BC)A \arrow[dr, "\beta_{B,C}1"'] & & & C(AB) \arrow[dl, "1\beta_{A,B}"] \\
& (CB)A \arrow[r, "a"'] & C(BA) &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& (AB)C \arrow[r, "a"] \arrow[dl, "\beta_{A,B}1"'] \arrow[ddd, "\beta 1"] \arrow[dddddrr, "\beta"] & A(BC) \arrow[dr, "1\beta_{B,C}"] & \\
(BA)C \arrow[d, "a"'] \arrow[ddr, "1"] \arrow[r, phantom, "\ell \atop \Rightarrow"] & & \arrow[r, phantom, "(R^3_{A,B|C})^{-1} \atop \Rightarrow"] & A(CB) \arrow[d, "a^{\bullet}"] \\
B(AC) \arrow[dr, "a^{\bullet}"'] \arrow[dd, "1\beta_{A,C}"'] \arrow[r, phantom, "\eta^{-a} \atop \Rightarrow"] & & & (AC)B \arrow[dd, "\beta_{A,C}1"] \\
& (BA)C \arrow[dddr, "\beta"] & \arrow[r, phantom, "\beta^{-1}_{\beta,1} \atop \Rightarrow"] & \\
B(CA) \arrow[d, "a^{\bullet}"'] \arrow[r, phantom, "R^1_{B,A|C} \atop \Rightarrow"] & & & (CA)B \arrow[d, "a"] \\
(BC)A \arrow[dr, "\beta_{B,C}1"'] & & & C(AB) \arrow[dl, "1\beta_{A,B}"] \\
& (CB)A \arrow[r, "a"'] & C(BA) &
\end{tikzcd}
$$

- ℓ and r are, respectively, the left and the right unitors in the base bicategory B.
- $\eta^a : 1 \longrightarrow a^{\bullet}a$ is the unit of the adjoint equivalence $(a, a^{\bullet}, \eta^a, \varepsilon^a)$, whose inverse is denoted by η^{-a}.
- $R^1_{A|C,B}$ is a mate of the indicated component of the left hexagonator.
- $R^1_{B,A|C}$ and $R^3_{A,B|C}$ are mates of the indicated components of the right hexagonator.

This finishes the definition of a braided monoidal bicategory. $\diamond$

Explanation 6.5.4 (Mates)**.** In the (3,1)-crossing axiom, $R^1_{B,C|D}$ is the composite of the following pasting diagram in B, with all the $\boxtimes$ symbols omitted to save space.

$$
\begin{tikzcd}
& B(DC) \arrow[r, "a^{\bullet}"] & (BD)C \arrow[dr, "\beta_{B,D}1_C"] & & \\
B(CD) \arrow[ur, "1_B\beta_{C,D}"] \arrow[dr, "a^{\bullet}"'] & & \arrow[l, phantom, "\Downarrow R_{B,C|D}"] & (DB)C \arrow[dr, "a"] & \\
& (BC)D \arrow[r, "\beta_{BC,D}"] \arrow[rrr, bend right, "\beta_{BC,D}"'] & D(BC) \arrow[ur, "a^{\bullet}"] \arrow[rr, "1"'] \arrow[r, phantom, "\Downarrow \varepsilon^a"] \arrow[d, phantom, "\Downarrow \ell"] & & D(BC) \\
& & {} & &
\end{tikzcd}
$$

Other mates of components of the left or the right hexagonators are defined in a similar way. $\diamond$

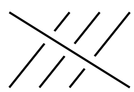
Explanation 6.5.5 (Visualization)**.** Consider Definition 6.5.3.

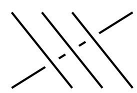
The Braiding: The braiding β may be visualized as the generating braid $\times$ in the braid group B_2. However, in this case, the braiding does not admit a strict inverse. Instead, it is the left adjoint of an adjoint equivalence with right adjoint $\beta^{\bullet}$.

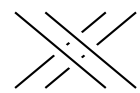
The Hexagonators: The left hexagonator $R_{-|--}$ may be visualized as the braid with the last two strings crossing over the first string. The domain of $R_{-|--}$ corresponds to first crossing the second string over the first string, followed by crossing the last string over the first string. The codomain corresponds to crossing the last two strings over the first string in one step.

The right hexagonator $R_{--|-}$ admits a similar interpretation using the braid . The domain corresponds to crossing one string over two strings to its left, one string at a time. The codomain corresponds to crossing one string over two strings in one step.

The Crossing Axioms: The (3,1)-crossing axiom may be visualized using the left-most picture below.

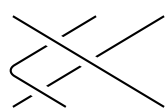

(3,1)-crossing

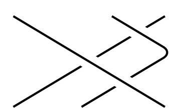

(1,3)-crossing


(2,2)-crossing

The common domain of the two pasting diagrams in the (3,1)-crossing axiom corresponds to crossing one string over three strings, one string at a time. The common codomain corresponds to crossing one string over three strings in one step. The two pasting diagrams correspond to two ways to transform from the common domain to the common codomain using the structures in a braided monoidal bicategory. The (1,3)-crossing axiom and the (2,2)-crossing axiom admit similar interpretations, using the middle and the right pictures above.

The Yang-Baxter Axiom: The Yang-Baxter axiom may be visualized using the following pictures.


domain


codomain

The common domain of the two pasting diagrams in the Yang-Baxter axiom corresponds to the left picture above, while the common codomain corresponds to the right picture. The two pasting diagrams correspond to two ways to transform from the domain to the codomain using the structures in a braided monoidal bicategory. $\diamond$

Explanation 6.5.6 (Braiding and Size)**.** Similar to Explanation 6.4.6, the assumption that the tuple $(\beta, \beta^{\bullet}, \eta^{\beta}, \varepsilon^{\beta})$ be an adjoint equivalence is equivalent to the following three statements.

(i) β and $\beta^{\bullet}$ are strong transformations as follows.

$$\begin{tikzcd} \boxtimes \arrow[r, shift left, "\beta"] & \boxtimes\tau \arrow[l, shift left, "\beta^{\bullet}"] \end{tikzcd}$$

(ii) η^β and ε^β are invertible modifications as follows.

$$1_\boxtimes \xrightarrow{\eta^\beta} \beta^\bullet\beta$$
$$\beta\beta^\bullet \xrightarrow{\varepsilon^\beta} 1_{\boxtimes\tau}$$

(iii) The triangle identities (6.3.10) hold. In those two diagrams, the modifications denoted by a, ℓ, and r are defined componentwise in B using, respectively, its associator, left unitor, and right unitor.

In particular, in the definition of a braided monoidal bicategory, it is *not* necessary to assume that B has a set of objects. ⋄

Definition 6.5.7. A *sylleptic monoidal bicategory* is a quintuple

$$\big(\mathsf{B}, \beta, R_{-|--}, R_{--|-}, \nu\big)$$

consisting of the following data.

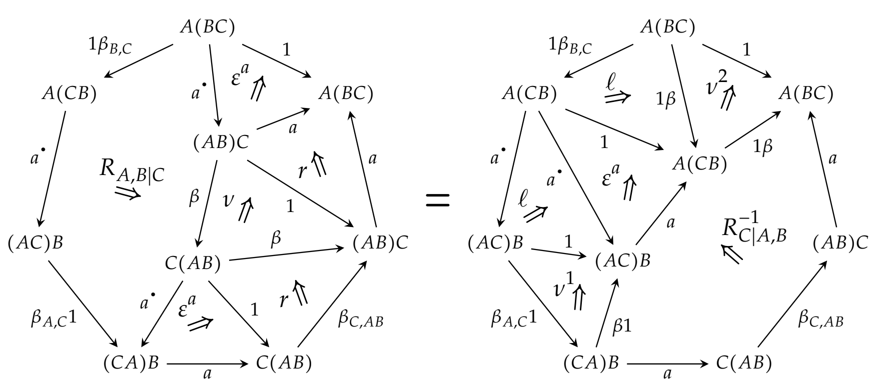
(1) $(\mathsf{B}, \beta, R_{-|--}, R_{--|-})$ is a braided monoidal bicategory.
(2) ν is an invertible modification, which is called the *syllepsis*. Its components are invertible 2-cells

$$\begin{tikzcd} A \boxtimes B \arrow[r, bend left, "\beta_{A,B}"] \arrow[rr, bend right, "1_{A\boxtimes B}"'] & B \boxtimes A \arrow[d, phantom, "\Downarrow\nu_{A,B}"] \arrow[r, bend left, "\beta_{B,A}"] & A \boxtimes B \end{tikzcd}$$

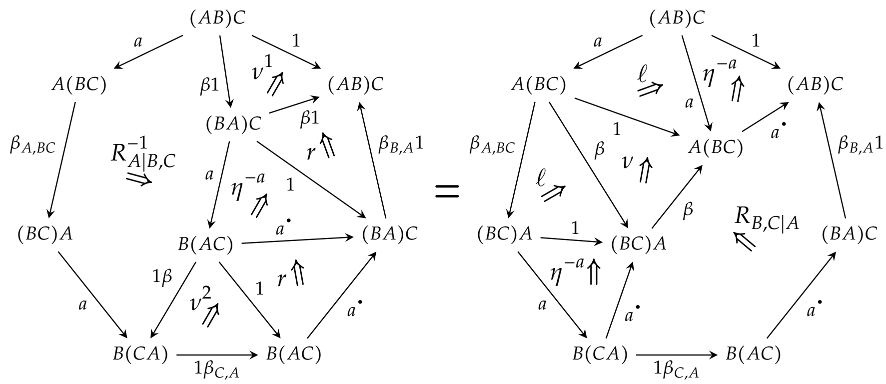
in $\mathsf{B}(A \boxtimes B, A \boxtimes B)$.

The following two pasting diagram equalities are required to hold for objects $A, B, C \in \mathsf{B}$, with the same conventions as in the axioms in Definition 6.5.3.

The (2,1)-Syllepsis Axiom:



The (1,2)-Syllepsis Axiom:



The 2-cells ν^1 and ν^2 are induced by the syllepsis ν, and are interpreted in a similar way to (6.4.8) using $\boxtimes^2$, $\boxtimes^{-0}$, and ℓ. This finishes the definition of a sylleptic monoidal bicategory. $\diamond$

Explanation 6.5.8 (Visualization)**.** Consider Definition 6.5.7.

The Braiding: Due to the existence of the syllepsis, the braiding β in a sylleptic monoidal bicategory may be visualized as the virtual crossing $\times$.

The Syllepsis: It is the isomorphism $\cong$ | | that straightens its domain to the identity.

The Axioms: In the (2,1)-syllepsis axiom, the common domain is the left picture below

(2,1)-syllepsis (1,2)-syllepsis

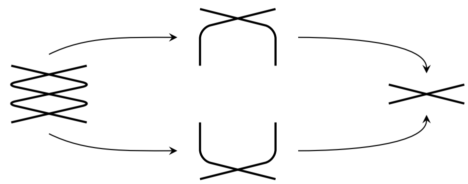
with common codomain the identity |||. The two pasting diagrams correspond to two ways to transform from the domain to the codomain using the structures in a sylleptic monoidal bicategory. The (1,2)-syllepsis axiom admits a similar interpretation with common domain the right picture above. $\diamond$

Definition 6.5.9. A *symmetric monoidal bicategory* is a sylleptic monoidal bicategory as in Definition 6.5.7 that satisfies the *triple braid axiom*

(6.5.10)
$$
\begin{tikzcd}
AB \arrow[rrr, "\beta"] \arrow[dr, "\beta"'] \arrow[drr, "1" near start] & & & BA \\
& BA \arrow[r, "\beta"'] \arrow[u, phantom, "\Uparrow\nu"] & AB \arrow[ur, "\beta"'] \arrow[ur, phantom, "\Uparrow r", shift left=4] &
\end{tikzcd}
=
\begin{tikzcd}
AB \arrow[rrr, "\beta"] \arrow[dr, "\beta"'] & \Uparrow\ell & & BA \\
& BA \arrow[urr, "1"] \arrow[r, "\beta"'] & AB \arrow[ur, "\beta"'] \arrow[u, phantom, "\Uparrow\nu"] &
\end{tikzcd}
$$

for objects A and B in B. $\diamond$

Explanation 6.5.11 (Visualization)**.** The triple braid axiom (6.5.10) may be visualized as the commutativity of the following diagram.



In other words, given three consecutive virtual crossings, straightening the first two virtual crossings is the same as straightening the last two virtual crossings. $\diamond$

6.6. The Gray Tensor Product

In this section, we define the Gray tensor product for 2-categories and the corresponding Gray monoids. The Gray tensor product is a weakening of the Cartesian product for 2-categories. The 1-category $2\mathsf{Cat}$ of small 2-categories and 2-functors becomes a symmetric monoidal closed category with the Gray tensor product, and with the internal hom given by strict functors, strong transformations, and modifications of 2-categories. From now on, whenever a 2-category is regarded as a Cat-category via Proposition 6.1.11, we automatically assume that it is locally small without explicitly stating it. Similarly, whenever a 2-category is regarded as an object in the 1-category $2\mathsf{Cat}$, it is automatically assumed to be small. Whenever necessary, we use Grothendieck's *Axiom of Universes*—that every set belongs to some universe—to move to a bigger universe. More discussion of universes may be found in [**JY21**, Section 1.1].

We begin by defining a simpler product that will be used in the definition of the Gray tensor product.

The Box Product.

Definition 6.6.1. Suppose C and D are 2-categories. The *box product* $\mathsf{C} \square \mathsf{D}$ is the Cat-enriched pushout induced by the inclusions $\mathsf{Ob}\,\mathsf{C} \longrightarrow \mathsf{C}$ and $\mathsf{Ob}\,\mathsf{D} \longrightarrow \mathsf{D}$ in the following diagram.

$$\begin{tikzcd}
\mathsf{Ob}\,\mathsf{C} \times \mathsf{Ob}\,\mathsf{D} \arrow[r] \arrow[d] & \mathsf{C} \times (\mathsf{Ob}\,\mathsf{D}) \arrow[d] \\
(\mathsf{Ob}\,\mathsf{C}) \times \mathsf{D} \arrow[r] & \mathsf{C} \square \mathsf{D}
\end{tikzcd} \tag{6.6.2}$$

This finishes the definition of the box product. $\diamond$

Explanation 6.6.3.

(1) The 1-cells of $\mathsf{C} \times \mathsf{Ob}\,\mathsf{D}$ are given by $(f, 1_Y)$ for a 1-cell $f \in \mathsf{C}(X, X')$ and an object $Y \in \mathsf{D}$, and the 2-cells of $\mathsf{C} \times \mathsf{Ob}\,\mathsf{D}$ are given by $(\alpha, 1_{1_Y})$ for a 2-cell $\alpha \in \mathsf{C}(X, X')(f_1, f_2)$ and an object $Y \in \mathsf{D}$. We denote their images in $\mathsf{C} \square \mathsf{D}$ as $f \square Y$ and $\alpha \square Y$, respectively, and do likewise for 1-cells and 2-cells in $(\mathsf{Ob}\,\mathsf{C}) \times \mathsf{D}$.

(2) Unpacking Definition 6.6.1, we can describe $\mathsf{C} \square \mathsf{D}$ as follows.
Objects: The objects are pairs (X, Y), written $X \square Y$, with $X \in \mathsf{C}$ and $Y \in \mathsf{D}$.
1-Cells: The 1-cells are generated under composition by pairs consisting of a 1-cell and an object, which are called *basic 1-cells* and written as
- $f \square Y : X \square Y \longrightarrow X' \square Y$, for $f \in \mathsf{C}(X, X')$ and $Y \in \mathsf{D}$; and
- $X \square g : X \square Y \longrightarrow X \square Y'$, for $g \in \mathsf{D}(Y, Y')$ and $X \in \mathsf{C}$.

Because the arrows in (6.6.2) are 2-functors, these 1-cells are subject to the following conditions.

(i) For $X \in \mathsf{C}$ and $Y \in \mathsf{D}$, we have

$$1_X \square Y = 1_{X\square Y} = X \square 1_Y.$$

(ii) For $f \in \mathsf{C}(X, X')$, $f' \in \mathsf{C}(X', X'')$, and $Y \in \mathsf{D}$, we have

$$(f' \square Y)(f \square Y) = (f'f) \square Y.$$

(iii) For $g \in \mathsf{D}(Y, Y')$, $g' \in \mathsf{D}(Y', Y'')$, and $X \in \mathsf{C}$, we have

$$(X \square g')(X \square g) = X \square (g'g).$$

2-Cells: The 2-cells are generated under horizontal and vertical composition by pairs consisting of a 2-cell and an object, which are called *basic 2-cells* and written as

- $\alpha \square Y : f_1 \square Y \longrightarrow f_2 \square Y$ for $\alpha \in \mathsf{C}(X, X')(f_1, f_2)$ and $Y \in \mathsf{D}$; and
- $X \square \beta : X \square g_1 \longrightarrow X \square g_2$ for $\beta \in \mathsf{D}(Y, Y')(g_1, g_2)$ and $X \in \mathsf{C}$.

Because the arrows in (6.6.2) are 2-functors, these 2-cells are subject to the following conditions.

(i) For $f \in \mathsf{C}(X, X')$ and $g \in \mathsf{D}(Y, Y')$, we have

$$1_f \square Y = 1_{f\square Y} \quad \text{and} \quad X \square 1_g = 1_{X\square g}.$$

(ii) For $\alpha \in \mathsf{C}(X, X')(f_1, f_2)$, $\alpha' \in \mathsf{C}(X', X'')(f_1', f_2')$, and $Y \in \mathsf{D}$, we have

$$(\alpha' \square Y) * (\alpha \square Y) = (\alpha' * \alpha) \square Y.$$

(iii) For $\beta \in \mathsf{D}(Y, Y')(g_1, g_2)$, $\beta' \in \mathsf{D}(Y', Y'')(g_1', g_2')$, and $X \in \mathsf{C}$, we have

$$(X \square \beta') * (X \square \beta) = X \square (\beta' * \beta).$$

(iv) For $\alpha \in \mathsf{C}(X, X')(f_1, f_2)$ and $\alpha' \in \mathsf{C}(X, X')(f_2, f_3)$, we have

$$(\alpha' \square Y)(\alpha \square Y) = (\alpha'\alpha) \square Y.$$

(v) For $\beta \in \mathsf{D}(Y, Y')(g_1, g_2)$ and $\beta' \in \mathsf{D}(Y, Y')(g_2, g_3)$, we have

$$(X \square \beta')(X \square \beta) = X \square (\beta'\beta).$$

This concludes the unpacking of Definition 6.6.1.

(3) By the universal property of the pushout, there is a 2-functor

$$j : \mathsf{C} \square \mathsf{D} \longrightarrow \mathsf{C} \times \mathsf{D}$$

that is bijective on objects. It sends a 1-cell $f \square Y$ to $f \times 1_Y$, a 2-cell $\alpha \square Y$ to $\alpha \times 1_{1_Y}$, and similarly for $X \square g$ or $X \square \beta$. The composites

$$(f \square Y')(X \square g) \quad \text{and} \quad (X' \square g)(f \square Y)$$

are distinct in $\mathsf{C} \square \mathsf{D}$, but both are mapped by j to $f \times g$. This observation is a basis for Motivation 6.6.4 below. ◇

Defining the Gray Tensor Product.

Motivation 6.6.4. We now turn to the definition of the Gray tensor product. This will have the same 0-cells and 1-cells as the box product, but additional 2-cells. Recall that the two composites in the square below are unrelated in $\mathsf{C} \square \mathsf{D}$, and their images in $\mathsf{C} \times \mathsf{D}$ are equal.

$$\begin{tikzcd}
X \square Y \arrow[r, "f \square Y"] \arrow[d, "X \square g"'] & X' \square Y \arrow[d, "X' \square g"] \\
X \square Y' \arrow[r, "f \square Y'"] & X' \square Y'
\end{tikzcd}$$

In the Gray tensor product, the corresponding square is filled by a generally non-trivial isomorphism $\Sigma_{f,g}$. In this way, the Gray tensor product is an intermediary between $\mathsf{C} \square \mathsf{D}$ and $\mathsf{C} \times \mathsf{D}$. ◇

Definition 6.6.5. For two 2-categories C and D, the *Gray tensor product* $\mathsf{C} \otimes \mathsf{D}$ is a 2-category defined as follows. The objects and 1-cells of $\mathsf{C} \otimes \mathsf{D}$ are the same as those of $\mathsf{C} \square \mathsf{D}$, now denoted with $\otimes$ instead of $\square$.

The 2-cells are defined in two stages as follows. The *proto-2-cells* are generated under horizontal composition by

- the basic 2-cells of $\mathsf{C} \square \mathsf{D}$, which are now denoted $\alpha \otimes Y$ and $X \otimes \beta$; and
- the *transition 2-cells*

  $$\Sigma_{f,g} : (f \otimes Y')(X \otimes g) \longrightarrow (X' \otimes g)(f \otimes Y) \quad \text{and}$$

  $$\Sigma_{f,g}^{-1} : (X' \otimes g)(f \otimes Y) \longrightarrow (f \otimes Y')(X \otimes g)$$

  for each pair of nonidentity 1-cells $f \in \mathsf{C}(X, X')$ and $g \in \mathsf{D}(Y, Y')$. If either f or g is an identity 1-cell, then $\Sigma_{f,g}$ is the respective identity 2-cell.

This horizontal composition is required to be associative and unital, satisfying the relations induced by $\square$, that is,

- $(\alpha' \otimes Y) * (\alpha \otimes Y) = (\alpha' * \alpha) \otimes Y$ and
- $(X \otimes \beta') * (X \otimes \beta) = X \otimes (\beta' * \beta)$

for horizontally composable 2-cells α and α' in C, respectively β and β' in D.

The 2-cells of $\mathsf{C} \otimes \mathsf{D}$ are equivalence classes of vertical composites of proto-2-cells, where the equivalence relation is the smallest one that includes the following.

(1) The vertical composites $\Sigma_{f,g}\Sigma_{f,g}^{-1}$ and $\Sigma_{f,g}^{-1}\Sigma_{f,g}$ are equivalent to the respective identities.
(2) The basic 2-cells from $\mathsf{C} \square \mathsf{D}$ satisfy the vertical composition relations induced by $\square$, namely,
  - $(\alpha_2 \otimes Y)(\alpha_1 \otimes Y) \sim (\alpha_2\alpha_1) \otimes Y$ and
  - $(X \otimes \beta_2)(X \otimes \beta_1) \sim X \otimes (\beta_2\beta_1)$.
(3) For $f \in \mathsf{C}(X, X')$, $f' \in \mathsf{C}(X', X'')$, and $g \in \mathsf{D}(Y, Y')$, we have

  $$\big(\Sigma_{f',g} * (1_f \otimes Y)\big)\big((1_{f'} \otimes Y') * \Sigma_{f,g}\big) \sim \Sigma_{f'f,g}.$$

(4) For $g \in \mathsf{D}(Y, Y')$, $g' \in \mathsf{D}(Y', Y'')$, and $f \in \mathsf{C}(X, X')$, we have

  $$\big((X' \otimes 1_{g'}) * \Sigma_{f,g}\big)\big(\Sigma_{f,g'} * (X \otimes 1_g)\big) \sim \Sigma_{f,g'g}.$$

(5) For f, f', g, and g' as above, we have
$$\big((X''\otimes 1_{g'}) * (1_{f'}\otimes Y') * \Sigma_{f,g}\big)\big(\Sigma_{f',g'} * (1_f \otimes Y') * (X\otimes 1_g)\big) \sim$$
$$\big(\Sigma_{f',g'} * (X'\otimes 1_g) * (1_f\otimes Y)\big)\big((1_{f'}\otimes Y'') * (X'\otimes 1_{g'}) * \Sigma_{f,g}\big).$$
(6) For $\alpha \in \mathsf{C}(X,X')(f_1,f_2)$ and $\beta \in \mathsf{D}(Y,Y')(g_1,g_2)$, we have
$$\big((X'\otimes\beta) * (\alpha\otimes Y)\big)\Sigma_{f_1,g_1} \sim \Sigma_{f_2,g_2}\big((\alpha\otimes Y') * (X\otimes\beta)\big).$$
(7) The equivalence relation is closed under vertical composition.
(8) For any horizontally composable proto-2-cells λ and λ', we have
$$(1*\lambda)(\lambda'*1) \sim (\lambda'*\lambda) \sim (\lambda'*1)(1*\lambda).$$

Each 2-cell Λ is represented by a vertical composite of proto-2-cells, $\lambda_1\cdots\lambda_n$, and thus the vertical composition of 2-cells is defined by concatenation. The horizontal composition of 2-cells is defined by

$$(\lambda_1'\lambda_2') * (\lambda_1\lambda_2) = (\lambda_1' * \lambda_1)(\lambda_2' * \lambda_2) \tag{6.6.6}$$

for appropriately composable proto-2-cells λ_1, λ_2, λ_1', and λ_2'. This extends to define horizontal composition of general vertical composites
$$(\lambda_1'\cdots\lambda_n') * (\lambda_1\cdots\lambda_m)$$
with proto-2-cells

- λ_i in $(\mathsf{C}\otimes\mathsf{D})(X\otimes Y, X'\otimes Y')$ and
- λ_i' in $(\mathsf{C}\otimes\mathsf{D})(X'\otimes Y', X''\otimes Y'')$

by inserting appropriate identity 2-cells so that $m = n$, and then by induction on (6.6.6).

Condition (8) implies that this definition is independent of how identities are inserted and satisfies the middle four exchange property, that is, that the horizontal composition preserves vertical composition of 2-cells. Preservation of units follows from the corresponding properties of $\square$, together with conditions (3) and (4) with f' and g' being identities. This finishes the definition of the Gray tensor product $\mathsf{C}\otimes\mathsf{D}$. ◇

Explanation 6.6.7 (The Transition 2-Cells)**.** Since $\Sigma_{f,g}$ is an identity 2-cell whenever f or g is an identity 1-cell, conditions (3), (4), and (5) are equivalent to the requirement that all possible composites formed from the following pasting diagram are equal to $\Sigma_{f'f,g'g}$ for all f,f',g, and g'.

$$\begin{tikzcd}
X\otimes Y \arrow[r, "f\otimes Y"] \arrow[d, "X\otimes g"'] \arrow[dr, phantom, "\Nearrow\Sigma_{f,g}"] & X'\otimes Y \arrow[r, "f'\otimes Y"] \arrow[d, "X'\otimes g"] \arrow[dr, phantom, "\Nearrow\Sigma_{f',g}"] & X''\otimes Y \arrow[d, "X''\otimes g"] \\
X\otimes Y' \arrow[r, "f\otimes Y'"] \arrow[d, "X\otimes g'"'] \arrow[dr, phantom, "\Nearrow\Sigma_{f,g'}"] & X'\otimes Y' \arrow[r, "f'\otimes Y'"] \arrow[d, "X'\otimes g'"] \arrow[dr, phantom, "\Nearrow\Sigma_{f',g'}"] & X''\otimes Y' \arrow[d, "X''\otimes g'"] \\
X\otimes Y'' \arrow[r, "f\otimes Y''"] & X'\otimes Y'' \arrow[r, "f'\otimes Y''"] & X''\otimes Y''
\end{tikzcd} \tag{6.6.8}$$

In particular, condition (5) means that the two ways of forming a vertical composite from $\Sigma_{f',g'} * \Sigma_{f,g}$ are equal. Furthermore, condition (6) means that the composites of the following pasting diagrams are equal in $\mathsf{C} \otimes \mathsf{D}$.

(6.6.9)

$$
\begin{tikzcd}[column sep=large, row sep=huge]
X \otimes Y \arrow[r, bend left=40, "f_2 \otimes Y"] \arrow[r, bend right=30, "f_1 \otimes Y"'] \arrow[r, phantom, "\Uparrow \alpha \otimes Y"] \arrow[d, bend right=40, "X \otimes g_1"'] \arrow[dr, phantom, "\Sigma_{f_1,g_1}"] & X' \otimes Y \arrow[d, bend right=40, "X' \otimes g_1"'] \arrow[d, bend left=40, "X' \otimes g_2"] \arrow[d, phantom, "\overset{X' \otimes \beta}{\Rightarrow}"] \\
X \otimes Y' \arrow[r, bend right=30, "f_1 \otimes Y'"'] & X' \otimes Y'
\end{tikzcd}
=
\begin{tikzcd}[column sep=large, row sep=huge]
X \otimes Y \arrow[r, bend left=30, "f_2 \otimes Y"] \arrow[d, bend right=40, "X \otimes g_1"'] \arrow[d, bend left=40, "X \otimes g_2"] \arrow[d, phantom, "\overset{\Rightarrow}{X \otimes \beta}"] \arrow[dr, phantom, "\Sigma_{f_2,g_2}"] & X' \otimes Y \arrow[d, bend left=40, "X' \otimes g_2"] \\
X \otimes Y' \arrow[r, bend left=30, "f_2 \otimes Y'"] \arrow[r, bend right=30, "f_1 \otimes Y'"'] \arrow[r, phantom, "\Uparrow \alpha \otimes Y'"] & X' \otimes Y'
\end{tikzcd}
$$

◇

Proposition 6.6.10. *For 2-categories* C *and* D*, the Gray tensor product* $\mathsf{C} \otimes \mathsf{D}$ *is a 2-category.*

Notation 6.6.11. For bicategories C and D, we let $\mathsf{Hom}(\mathsf{C}, \mathsf{D})$ denote the full subbicategory of $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{C}, \mathsf{D})$ consisting of strict functors, strong transformations, and modifications. Recall from Theorem 6.3.7 that $\mathsf{Hom}(\mathsf{C}, \mathsf{D})$ is a 2-category whenever D is a 2-category. ◇

Recall from Definition 1.2.28 the concept of a symmetric monoidal closed category.

Theorem 6.6.12. *There is a symmetric monoidal closed category*

$$\mathsf{Gray} = \big(\mathsf{2Cat}, \otimes, \mathbf{1}, a, \lambda, \rho, \xi, \mathsf{Hom} \big)$$

with the following data.

- *The underlying category is* $\mathsf{2Cat}$*, the 1-category of small 2-categories and 2-functors.*
- *The monoidal product is the Gray tensor product* $\otimes$*.*
- *The unit object is the terminal 2-category* $\mathbf{1}$*.*
- *The associativity isomorphism* a*, the left unit isomorphism* λ*, and the right unit isomorphism* ρ *are induced by those of the Cartesian product.*
- *For 2-categories* C *and* D*, the* (C, D)*-component*

  $$\mathsf{C} \otimes \mathsf{D} \xrightarrow{\xi_{\mathsf{C},\mathsf{D}}} \mathsf{D} \otimes \mathsf{C}$$

  of the symmetry isomorphism ξ *is defined on generating cells as follows for objects* $X \in \mathsf{C}$ *and* $Y \in \mathsf{D}$*, 1-cells* $f \in \mathsf{C}$ *and* $g \in \mathsf{D}$*, and 2-cells* $\alpha \in \mathsf{C}$ *and* $\beta \in \mathsf{D}$*.*

  $$\begin{aligned} \xi(X \otimes Y) &= Y \otimes X & & \\ \xi(f \otimes Y) &= Y \otimes f & \qquad \xi(X \otimes g) &= g \otimes X \\ \xi(\alpha \otimes Y) &= Y \otimes \alpha & \qquad \xi(X \otimes \beta) &= \beta \otimes X \\ \xi\Sigma_{f,g} &= \Sigma_{g,f}^{-1} & & \end{aligned}$$

- *The internal hom is* Hom *in Notation 6.6.11.*

Gray Monoids. Recall from Definition 1.2.8 that a monoid in a monoidal category consists of an object together with multiplication and unit morphisms satisfying axioms for associativity and unity.

Definition 6.6.13. A *Gray monoid* is a monoid $(\mathsf{C}, \odot, I)$ in Gray. $\diamond$

Explanation 6.6.14 (Monoids in Gray)**.** Rewriting Definition 1.2.8 in this context, a Gray monoid is a triple $(\mathsf{C}, \odot, I)$ consisting of a 2-category C and 2-functors

$$\begin{aligned} \odot : \mathsf{C} \otimes \mathsf{C} &\longrightarrow \mathsf{C} \\ I : \mathbf{1} &\longrightarrow \mathsf{C} \end{aligned}$$

such that the following diagrams of 2-categories and 2-functors commute.

$$\begin{tikzcd}
(\mathsf{C}\otimes\mathsf{C})\otimes\mathsf{C} \arrow[r, "a"] \arrow[dd, "\odot\otimes 1"'] & \mathsf{C}\otimes(\mathsf{C}\otimes\mathsf{C}) \arrow[d, "1\otimes\odot"] \\
& \mathsf{C}\otimes\mathsf{C} \arrow[d, "\odot"] \\
\mathsf{C}\otimes\mathsf{C} \arrow[r, "\odot"] & \mathsf{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1}\otimes\mathsf{C} \arrow[r, "I\otimes 1"] \arrow[dr, "\lambda"'] & \mathsf{C}\otimes\mathsf{C} \arrow[d, "\odot"] & \mathsf{C}\otimes\mathbf{1} \arrow[l, "1\otimes I"'] \arrow[dl, "\rho"] \\
& \mathsf{C} &
\end{tikzcd} \tag{6.6.15}$$

$\diamond$

Explanation 6.6.16 (Data and Axioms for Gray Monoids)**.** We have an even more explicit list of data and axioms by unpacking the definition of the Gray tensor product $\otimes$. A Gray monoid

$$(\mathsf{C}, \odot, I)$$

consists of a 2-category C together with the following data.

The Unit: A distinguished object I, which is called the *Gray unit*.

Objects: For each pair of objects W and X, an object $W \odot X$.

1-Cells: For each object W and 1-cell $f : X \longrightarrow X'$, 1-cells

$$\begin{aligned} W \odot f : W \odot X &\longrightarrow W \odot X' \quad \text{and} \\ f \odot W : X \odot W &\longrightarrow X' \odot W. \end{aligned}$$

2-Cells: For each object W and 2-cell $\alpha : f_1 \longrightarrow f_2$ in $\mathsf{C}(X, X')$, *basic 2-cells*

$$\begin{aligned} W \odot \alpha : W \odot f_1 &\longrightarrow W \odot f_2 \quad \text{and} \\ \alpha \odot W : f_1 \odot W &\longrightarrow f_2 \odot W. \end{aligned}$$

For each 1-cell $f : X \longrightarrow X'$ and 1-cell $g : Y \longrightarrow Y'$, a 2-cell isomorphism

$$\Sigma_{f,g} : (f \odot Y')(X \odot g) \xrightarrow{\cong} (X' \odot g)(f \odot Y), \tag{6.6.17}$$

which is called a *transition 2-cell*.

These data are subject to the following axioms.

(1) For each object W, the assignments on cells

$$\begin{aligned} W \odot - : \mathsf{C} &\longrightarrow \mathsf{C} \quad \text{and} \\ - \odot W : \mathsf{C} &\longrightarrow \mathsf{C} \end{aligned}$$

are 2-functors. By Proposition 6.2.5, 2-functoriality of $W \odot -$ is equivalent to the following five equalities for each object X along with appropriately composable 1-cells f and g, and 2-cells α, α', and γ.

$$\begin{aligned} W \odot 1_X &= 1_{W \odot X} \\ W \odot 1_f &= 1_{W \odot f} \\ W \odot (gf) &= (W \odot g)(W \odot f) \\ W \odot (\alpha\gamma) &= (W \odot \alpha)(W \odot \gamma) \\ W \odot (\alpha * \alpha') &= (W \odot \alpha) * (W \odot \alpha') \end{aligned}$$

2-functoriality of $- \odot W$ is equivalent to the following five equalities, provided that the compositions are defined.

$$\begin{aligned} 1_X \odot W &= 1_{X \odot W} \\ 1_f \odot W &= 1_{f \odot W} \\ (gf) \odot W &= (g \odot W)(f \odot W) \\ (\alpha\gamma) \odot W &= (\alpha \odot W)(\gamma \odot W) \\ (\alpha * \alpha') \odot W &= (\alpha \odot W) * (\alpha' \odot W) \end{aligned}$$

(2) The Gray unit I is strict. That is, for each object X, 1-cell f, and 2-cell α, we have the following equalities.

$$\begin{aligned} I \odot X = X &= X \odot I \\ I \odot f = f &= f \odot I \\ I \odot \alpha = \alpha &= \alpha \odot I \end{aligned}$$

(3) The product $\odot$ is strictly associative. That is, for objects Z, W, and X, we have

$$(Z \odot W) \odot X = Z \odot (W \odot X).$$

For each 1-cell f and 2-cell α, we have the following equalities.

$$\begin{aligned} (Z \odot W) \odot f &= Z \odot (W \odot f) & \qquad (Z \odot W) \odot \alpha &= Z \odot (W \odot \alpha) \\ (Z \odot f) \odot W &= Z \odot (f \odot W) & \qquad (Z \odot \alpha) \odot W &= Z \odot (\alpha \odot W) \\ (f \odot Z) \odot W &= f \odot (Z \odot W) & \qquad (\alpha \odot Z) \odot W &= \alpha \odot (Z \odot W) \end{aligned}$$

(4) For 1-cells $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$, and $h : Z \longrightarrow Z'$, we have the following equalities.

$$\begin{aligned} \Sigma_{f,g} \odot Z &= \Sigma_{f,g \odot Z} \\ \Sigma_{f \odot Y,h} &= \Sigma_{f,Y \odot h} \\ \Sigma_{X \odot g,h} &= X \odot \Sigma_{g,h} \end{aligned}$$

(5) For $f \in \mathsf{C}(X, X')$, $f' \in \mathsf{C}(X', X'')$, $g \in \mathsf{C}(Y, Y')$, and $g' \in \mathsf{C}(Y', Y'')$, we have the following two equalities of pasting diagrams.

$$\begin{tikzcd} X \odot Y \arrow[r, "f \odot Y"] \arrow[d, "X \odot g"'] & X' \odot Y \arrow[r, "f' \odot Y"] \arrow[d, "X' \odot g"] & X'' \odot Y \arrow[d, "X'' \odot g"] \\ X \odot Y' \arrow[r, "f \odot Y'"'] \arrow[ur, phantom, "\Nearrow \Sigma_{f,g}"] & X' \odot Y' \arrow[r, "f' \odot Y'"'] \arrow[ur, phantom, "\Nearrow \Sigma_{f',g}"] & X'' \odot Y' \end{tikzcd} = \begin{tikzcd} X \odot Y \arrow[r, "(f'f) \odot Y"] \arrow[d, "X \odot g"'] & X'' \odot Y \arrow[d, "X'' \odot g"] \\ X \odot Y' \arrow[r, "(f'f) \odot Y'"'] \arrow[ur, phantom, "\Nearrow \Sigma_{f'f,g}"] & X'' \odot Y' \end{tikzcd}$$

$$
\begin{tikzcd}
X \odot Y \arrow[r, "f \odot Y"] \arrow[d, "X \odot g"'] \arrow[dr, phantom, "\Nearrow \Sigma_{f,g}"] & X' \odot Y \arrow[d, "X' \odot g"] \\
X \odot Y' \arrow[r, "f \odot Y'"] \arrow[d, "X \odot g'"'] \arrow[dr, phantom, "\Nearrow \Sigma_{f,g'}"] & X' \odot Y' \arrow[d, "X' \odot g'"] \\
X \odot Y'' \arrow[r, "f \odot Y''"] & X' \odot Y''
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \odot Y \arrow[r, "f \odot Y"] \arrow[d, "X \odot (g'g)"'] \arrow[dr, phantom, "\Nearrow \Sigma_{f,g'g}"] & X' \odot Y \arrow[d, "X' \odot (g'g)"] \\
X \odot Y'' \arrow[r, "f \odot Y''"] & X' \odot Y''
\end{tikzcd}
$$

(6) For $\alpha \in \mathsf{C}(X,X')(f_1,f_2)$ and $\beta \in \mathsf{D}(Y,Y')(g_1,g_2)$, we have the following equality of pasting diagrams.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
X \odot Y \arrow[r, bend left, "f_2 \odot Y"] \arrow[r, bend right, "f_1 \odot Y"'] \arrow[r, phantom, "\Uparrow \alpha \odot Y"] \arrow[d, bend right, "X \odot g_1"'] \arrow[dr, phantom, "\Nearrow \Sigma_{f_1,g_1}"] & X' \odot Y \arrow[d, bend right, "X' \odot g_1"'] \arrow[d, bend left, "X' \odot g_2"] \arrow[d, phantom, "X' \odot \beta \atop \Rightarrow"] \\
X \odot Y' \arrow[r, bend right, "f_1 \odot Y'"'] & X' \odot Y'
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=huge]
X \odot Y \arrow[r, bend left, "f_2 \odot Y"] \arrow[d, bend right, "X \odot g_1"'] \arrow[d, bend left, "X \odot g_2"] \arrow[d, phantom, "\Rightarrow \atop X \odot \beta"] \arrow[dr, phantom, "\Sigma_{f_2,g_2} \Nearrow"] & X' \odot Y \arrow[d, bend left, "X' \odot g_2"] \\
X \odot Y' \arrow[r, bend left, "f_2 \odot Y'"] \arrow[r, bend right, "f_1 \odot Y'"'] \arrow[r, phantom, "\Uparrow \alpha \odot Y'"] & X' \odot Y'
\end{tikzcd}
$$

This finishes the list of axioms of a Gray monoid. In particular, condition (5) together with the invertibility of $\Sigma_{f,g}$ implies that $\Sigma_{1,g}$ and $\Sigma_{f,1}$ are identity 2-cells. $\diamond$

6.7. Permutative Gray Monoids and 2-Categories

In this section, we recall from **[GJO17b**, Def. 3.28 and 3.45] the concepts of

- permutative Gray monoids in Definition 6.7.1 and
- permutative 2-categories in Definition 6.7.16.

This section ends with a brief discussion of strictification related to these structures.

Permutative Gray Monoids. Recall from Definition 1.2.18 that a permutative category is a symmetric monoidal category whose underlying monoidal category is strict. Permutative Gray monoids are their analogues in the context of 2-categories with the Gray tensor product $\otimes$.

Definition 6.7.1. A *permutative Gray monoid* is a quadruple

$$(\mathsf{C}, \odot, I, \beta)$$

consisting of the following data.

- $(\mathsf{C}, \odot, I)$ is a Gray monoid as in Definition 6.6.13.

- $\beta : \odot \longrightarrow \odot\,\xi$ is a 2-natural isomorphism, which is called the *Gray symmetry*, as in

$$\begin{tikzcd} \mathsf{C} \otimes \mathsf{C} \arrow[rr, "\xi"] \arrow[dr, "\odot"'] & \arrow[d, phantom, "\Rightarrow^{\beta}"] & \mathsf{C} \otimes \mathsf{C} \arrow[dl, "\odot"] \\ & \mathsf{C} & \end{tikzcd}$$

with ξ the symmetry isomorphism in Gray in Theorem 6.6.12.

These data are subject to the following three pasting diagram axioms in 2Cat, in which each unlabeled region is strictly commutative.

The Symmetry Axiom: The following pasting diagram is equal to $1_{\odot}$.

(6.7.2)
$$\begin{tikzcd} \mathsf{C} \otimes \mathsf{C} \arrow[rr, "\xi"] \arrow[rrrr, bend left, "1"] \arrow[drr, bend right, "\odot"'] & \arrow[d, phantom, "\Rightarrow^{\beta}"] & \mathsf{C} \otimes \mathsf{C} \arrow[rr, "\xi"] \arrow[d, "\odot"] & \arrow[d, phantom, "\Rightarrow^{\beta}"] & \mathsf{C} \otimes \mathsf{C} \arrow[dll, bend left, "\odot"] \\ & \, & \mathsf{C} & \, & \end{tikzcd}$$

The Unit Axiom: The following pasting diagram is equal to 1_{λ}, with λ and ρ the left and the right unit isomorphisms in Gray.

(6.7.3)
$$\begin{tikzcd} \mathbf{1} \otimes \mathsf{C} \arrow[r, "I \otimes 1"] \arrow[rrrr, bend left, "\xi"] \arrow[drr, bend right, "\lambda"'] & \mathsf{C} \otimes \mathsf{C} \arrow[rr, "\xi"] \arrow[dr, "\odot"'] & \arrow[d, phantom, "\Rightarrow^{\beta}"] & \mathsf{C} \otimes \mathsf{C} \arrow[dl, "\odot"] & \mathsf{C} \otimes \mathbf{1} \arrow[l, "1 \otimes I"'] \arrow[dll, bend left, "\rho"] \\ & & \mathsf{C} & & \end{tikzcd}$$

The Hexagon Axiom:

(6.7.4)
$$
\begin{tikzcd}
& (C \circledast C) \circledast C \arrow[rrrr, "a"] & & & & C \circledast (C \circledast C) \arrow[dr, "1 \circledast \xi"] & \\
(C \circledast C) \circledast C \arrow[ur, "\xi \circledast 1"] \arrow[dd, "\odot \circledast 1"'] \arrow[drrr, "a"] & & & & & & C \circledast (C \circledast C) \arrow[dd, "a^{-1}"] \\
& & & C \circledast (C \circledast C) \arrow[d, "1 \circledast \odot"'] \arrow[drrr, "\xi"] & & & \\
C \circledast C \arrow[dr, "\odot"'] & & & C \circledast C \arrow[dll, "\odot"'] \arrow[drr, "\xi"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & & & (C \circledast C) \circledast C \arrow[dl, "\odot \circledast 1"] \\
& C & & {} & & C \circledast C \arrow[llll, "\odot"] &
\end{tikzcd}
$$
$$\Vert$$
$$
\begin{tikzcd}
& (C \circledast C) \circledast C \arrow[rrrr, "a"] \arrow[ddl, bend left=60, "\odot \circledast 1"] & & & & C \circledast (C \circledast C) \arrow[dr, "1 \circledast \xi"] \arrow[ddll, bend right, "1 \circledast \odot"'] & \\
(C \circledast C) \circledast C \arrow[ur, "\xi \circledast 1"] \arrow[dd, "\odot \circledast 1"'] \arrow[r, phantom, "\overset{\beta \circledast 1}{\Rightarrow}"] & {} & & & & \overset{1 \circledast \beta}{\Rightarrow} & C \circledast (C \circledast C) \arrow[dd, "a^{-1}"] \arrow[dlll, "1 \circledast \odot"] \\
& & & C \circledast C \arrow[ddll, "\odot"] & & & \\
C \circledast C \arrow[dr, "\odot"'] & & & & & & (C \circledast C) \circledast C \arrow[dl, "\odot \circledast 1"] \\
& C & & & & C \circledast C \arrow[llll, "\odot"] &
\end{tikzcd}
$$

This finishes the definition of a permutative Gray monoid. ◇

Explanation 6.7.5 (Data)**.** The Gray symmetry $\beta : \odot \longrightarrow \odot\xi$ in a permutative Gray monoid is a 2-natural isomorphism. By Proposition 6.2.23 and Explanation 6.6.16, it consists of a component 1-cell

(6.7.6)
$$X \odot Y \xrightarrow{\beta_{X,Y}} Y \odot X \quad \in C$$

for each pair of objects $X, Y \in C$ such that the following invertibility and naturality conditions are satisfied.

Invertibility: Each $\beta_{X,Y}$ is invertible. In other words, there exists a 1-cell

$$Y \odot X \xrightarrow{\beta^{-1}_{X,Y}} X \odot Y \quad \in C$$

such that

(6.7.7)
$$\beta^{-1}_{X,Y}\beta_{X,Y} = 1_{X\odot Y} \quad \text{and} \quad \beta_{X,Y}\beta^{-1}_{X,Y} = 1_{Y\odot X}.$$

In fact, this inverse is given by

$$\beta^{-1}_{X,Y} = \beta_{Y,X}$$

by the symmetry axiom as interpreted in (6.7.12) below.

1-Cell Naturality: For each object W and 1-cell $f : X \longrightarrow X'$ in C, the following diagram in C is commutative.

$$\begin{tikzcd}
X \odot W \arrow[r, "\beta_{X,W}"] \arrow[d, "f \odot W"'] & W \odot X \arrow[r, "\beta_{W,X}"] \arrow[d, "W \odot f"] & X \odot W \arrow[d, "f \odot W"] \\
X' \odot W \arrow[r, "\beta_{X',W}"] & W \odot X' \arrow[r, "\beta_{W,X'}"] & X' \odot W
\end{tikzcd} \tag{6.7.8}$$

In fact, the commutativity of the right square follows from that of the left square and the symmetry axiom (6.7.12). Therefore, only the left square is necessary.

2-Cell Naturality: There are two types of 2-cell naturality conditions, one for basic 2-cells and one for transition 2-cells.

(1) For each object W and 2-cell $\alpha : f_1 \longrightarrow f_2 \in \mathsf{C}(X, X')$, the following equality holds in $\mathsf{C}(X \odot W, W \odot X')$.

$$(W \odot \alpha) * 1_{\beta_{X,W}} = 1_{\beta_{X',W}} * (\alpha \odot W) \tag{6.7.9}$$

This is equivalent to the following pasting diagram equality, with the unlabeled subregions commutative by (6.7.8).

$$\begin{tikzcd}
& X \odot W \arrow[dl, bend right, "\beta_{X,W}"'] \arrow[dr, bend left, "f_2 \odot W"] & \\
W \odot X \arrow[dr, bend right, "W \odot f_1"'] \arrow[dr, bend left, "W \odot f_2"] \arrow[dr, phantom, "\overset{W \odot \alpha}{\Rightarrow}"] & & X' \odot W \arrow[dl, bend left, "\beta_{X',W}"] \\
& W \odot X' &
\end{tikzcd}
=
\begin{tikzcd}
& X \odot W \arrow[dl, bend right, "\beta_{X,W}"'] \arrow[dr, bend left, "f_2 \odot W"] \arrow[dr, bend right, "f_1 \odot W"'] \arrow[dr, phantom, "\overset{\alpha \odot W}{\Rightarrow}"] & \\
W \odot X \arrow[dr, bend right, "W \odot f_1"'] & & X' \odot W \arrow[dl, bend left, "\beta_{X',W}"] \\
& W \odot X' &
\end{tikzcd}$$

(2) For each pair of 1-cells $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$, the following equality holds in $\mathsf{C}(X \odot Y, Y' \odot X')$.

$$\Sigma^{-1}_{g,f} * 1_{\beta_{X,Y}} = 1_{\beta_{X',Y'}} * \Sigma_{f,g}. \tag{6.7.10}$$

This is equivalent to the following pasting diagram equality, with the unlabeled subregions commutative by (6.7.8).

$$\begin{tikzcd}
& X \odot Y \arrow[dl, "X \odot g"'] \arrow[dr, "f \odot Y"] \arrow[dd, "\beta_{X,Y}"] & \\
X \odot Y' \arrow[dd, "\beta_{X,Y'}"'] & & X' \odot Y \arrow[dd, "\beta_{X',Y}"] \\
& Y \odot X \arrow[dl, "g \odot X"'] \arrow[dr, "Y \odot f"] & \\
Y' \odot X \arrow[dr, "Y' \odot f"'] \arrow[rr, phantom, "\overset{\Sigma^{-1}_{g,f}}{\Rightarrow}"] & & Y \odot X' \arrow[dl, "g \odot X'"] \\
& Y' \odot X' &
\end{tikzcd}
=
\begin{tikzcd}
& X \odot Y \arrow[dl, "X \odot g"'] \arrow[dr, "f \odot Y"] & \\
X \odot Y' \arrow[dd, "\beta_{X,Y'}"'] \arrow[dr, "f \odot Y'"'] \arrow[rr, phantom, "\overset{\Sigma_{f,g}}{\Rightarrow}"] & & X' \odot Y \arrow[dd, "\beta_{X',Y}"] \arrow[dl, "X' \odot g"] \\
& X' \odot Y' \arrow[dd, "\beta_{X',Y'}"] & \\
Y' \odot X \arrow[dr, "Y' \odot f"'] & & Y \odot X' \arrow[dl, "g \odot X'"] \\
& Y' \odot X' &
\end{tikzcd}$$

This completes the conditions for $\beta : \odot \longrightarrow \odot\xi$ to be a 2-natural isomorphism. $\diamond$

Explanation 6.7.11 (Axioms)**.** Consider the axioms of a permutative Gray monoid in Definition 6.7.1.

The Symmetry Axiom: In (6.7.2), the equality $\xi\xi = 1$ follows from the symmetry axiom (1.2.20) in Gray. In terms of component 1-cells, the symmetry axiom means the commutativity of the diagram

(6.7.12)
$$\begin{tikzcd} X \odot Y \arrow[rr, "1_{X\odot Y}"] \arrow[dr, "\beta_{X,Y}"'] & & X \odot Y \\ & Y \odot X \arrow[ur, "\beta_{Y,X}"'] & \end{tikzcd}$$

for objects $X, Y \in \mathsf{C}$. This is equivalent to the invertibility condition (6.7.7).

The Unit Axiom: Consider (6.7.3).

- The top subregion is commutative by the naturality of ξ.
- The left and the right subregions are commutative by the unity axiom of the Gray monoid $(\mathsf{C}, \odot, I)$, that is, the right diagram in (6.6.15).
- The equality $\rho\xi = \lambda$ follows from the symmetry axiom (1.2.20) and the unit axiom (1.2.21) in Gray.

Since λ and ρ in Gray are induced by those of the Cartesian product, in terms of component 1-cells the unit axiom means the commutativity of the diagram

(6.7.13)
$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, equal] & X \arrow[d, equal] \\ I \odot X \arrow[r, "\beta_{I,X}"] & X \odot I \end{tikzcd}$$

for each object $X \in \mathsf{C}$, with I the Gray unit of C. It can be shown, using an argument similar to the proof of Proposition II.1.3.21, which is the corresponding fact for braided monoidal categories, that the unit axiom follows from the symmetry axiom and the hexagon axiom.

The Hexagon Axiom: Consider (6.7.4).

- In the top pasting diagram, the top hexagon is commutative by the symmetry axiom (1.2.20) and the hexagon axiom (1.2.22) in Gray. The quadrilateral is commutative by the naturality of ξ.
- The other three unlabeled subregions in (6.7.4) are commutative by the associativity axiom of the Gray monoid $(\mathsf{C}, \odot, I)$, that is, the left diagram in (6.6.15).

In terms of component 1-cells, the hexagon axiom means that the diagram

(6.7.14)
$$\begin{tikzcd} & (Y \odot X) \odot Z \arrow[r, "1"] & Y \odot (X \odot Z) \arrow[dr, "Y \odot \beta_{X,Z}"] & \\ (X \odot Y) \odot Z \arrow[ur, "\beta_{X,Y} \odot Z"] \arrow[dr, "1"'] & & & Y \odot (Z \odot X) \\ & X \odot (Y \odot Z) \arrow[r, "\beta_{X, Y\odot Z}"] & (Y \odot Z) \odot X \arrow[ur, "1"'] & \end{tikzcd}$$

is commutative for objects $X, Y, Z \in \mathsf{C}$.

This finishes the explanation of the axioms of a permutative Gray monoid. ◇

Permutative 2-Categories. The 1-category $\mathsf{2Cat}$ has another symmetric monoidal structure other than Gray in Theorem 6.6.12.

Definition 6.7.15. Denote by $(\mathsf{2Cat},\times)$ the symmetric monoidal category with the following data.

- The underlying category is $\mathsf{2Cat}$.
- The monoidal product is the Cartesian product $\times$.
- The monoidal unit is the terminal 2-category $\mathbf{1}$.
- The associativity isomorphism a is the identity.
- The left and the right unit isomorphisms λ and ρ are defined by dropping the $\mathbf{1}$ argument.
- The symmetry isomorphism ξ is given by swapping the two arguments.

This finishes the definition of $(\mathsf{2Cat},\times)$. ◇

Definition 6.7.16. A *permutative 2-category* is a quadruple

$$(\mathsf{C},\boxdot,I,\beta)$$

consisting of the following data.

- $(\mathsf{C},\boxdot,I)$ is a monoid in $(\mathsf{2Cat},\times)$ with
  - C a small 2-category,
  - multiplication 2-functor $-\boxdot- : \mathsf{C}\times\mathsf{C}\longrightarrow\mathsf{C}$, and
  - unit 2-functor $I:\mathbf{1}\longrightarrow\mathsf{C}$.
- $\beta:\boxdot\longrightarrow\boxdot\,\xi$ is a 2-natural isomorphism as follows.

$$\begin{tikzcd}
\mathsf{C}\times\mathsf{C} \arrow[rr, "\xi"] \arrow[dr, "\boxdot"'] & \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] & \mathsf{C}\times\mathsf{C} \arrow[dl, "\boxdot"] \\
& \mathsf{C} &
\end{tikzcd}$$

These data are subject to the same three axioms as for permutative Gray monoids in Definition 6.7.1 with $(\otimes,\odot)$ replaced by $(\times,\boxdot)$. This finishes the definition of a permutative 2-category. ◇

Next is the analogue of Explanations 6.6.14 and 6.6.16 with the Cartesian product instead of the Gray tensor product.

Explanation 6.7.17 (Monoid)**.** A monoid

$$(\mathsf{C},\boxdot,I)$$

in $(\mathsf{2Cat},\times)$ consists of a small 2-category C together with the following data.

The Unit: A distinguished object I.

Objects: For each pair of objects W and X, an object $W\boxdot X$.

1-Cells: For each pair of 1-cells $f:X\longrightarrow X'$ and $g:Y\longrightarrow Y'$, a 1-cell

$$f\boxdot g : X\boxdot Y\longrightarrow X'\boxdot Y'.$$

2-Cells: For each pair of 2-cells $\alpha:f_1\longrightarrow f_2$ in $\mathsf{C}(X,X')$ and $\gamma:g_1\longrightarrow g_2$ in $\mathsf{C}(Y,Y')$, a 2-cell

$$\alpha\boxdot\gamma : f_1\boxdot g_1\longrightarrow f_2\boxdot g_2.$$

These data are subject to the following 2-functoriality, unity, and associativity conditions.

The 2-Functoriality of the Multiplication:

$$- \boxdot - : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$$

is a 2-functor. This is equivalent to the following list of equalities whenever they are defined for objects W and X, 1-cells f, f', g, and g', and 2-cells α, α', $\tilde{\alpha}$, γ, γ', and $\tilde{\gamma}$.

$$\begin{gathered} 1_W \boxdot 1_X = 1_{W \boxdot X} \\ 1_f \boxdot 1_g = 1_{f \boxdot g} \\ (f'f) \boxdot (g'g) = (f' \boxdot g')(f \boxdot g) \\ (\alpha'\alpha) \boxdot (\gamma'\gamma) = (\alpha' \boxdot \gamma')(\alpha \boxdot \gamma) \\ (\tilde{\alpha} * \alpha) \boxdot (\tilde{\gamma} * \gamma) = (\tilde{\alpha} \boxdot \tilde{\gamma}) * (\alpha \boxdot \gamma) \end{gathered} \tag{6.7.18}$$

Unity: For each object X, 1-cell f, and 2-cell α, the following equalities hold.

$$\begin{gathered} I \boxdot X = X = X \boxdot I \\ 1_I \boxdot f = f = f \boxdot 1_I \\ 1_{1_I} \boxdot \alpha = \alpha = \alpha \boxdot 1_{1_I} \end{gathered} \tag{6.7.19}$$

Associativity: For objects X, Y, and Z, 1-cells f, g, and h, and 2-cells α, γ, and θ, the following equalities hold.

$$\begin{gathered} (X \boxdot Y) \boxdot Z = X \boxdot (Y \boxdot Z) \\ (f \boxdot g) \boxdot h = f \boxdot (g \boxdot h) \\ (\alpha \boxdot \gamma) \boxdot \theta = \alpha \boxdot (\gamma \boxdot \theta) \end{gathered} \tag{6.7.20}$$

This finishes the explicit description of a monoid in $(\mathsf{2Cat}, \times)$. ◇

Next is the analogue of Explanations 6.7.5 and 6.7.11 with the Cartesian product instead of the Gray tensor product.

Explanation 6.7.21 (Permutative 2-Category)**.** For a monoid $(\mathsf{C}, \boxdot, I)$ in $(\mathsf{2Cat}, \times)$ as in Explanation 6.7.17, consider the 2-natural isomorphism $\beta : \boxdot \longrightarrow \boxdot\xi$ of a permutative 2-category in Definition 6.7.16. It consists of a component 1-cell

$$X \boxdot Y \xrightarrow{\beta_{X,Y}} Y \boxdot X \quad \in \mathsf{C}$$

for each pair of objects $X, Y \in \mathsf{C}$ such that the following invertibility and naturality conditions are satisfied.

Invertibility: The following diagram is commutative for objects $X, Y \in \mathsf{C}$.

$$\begin{tikzcd} X \boxdot Y \arrow[rr, "1_{X \boxdot Y}"] \arrow[dr, "\beta_{X,Y}"'] & & X \boxdot Y \\ & Y \boxdot X \arrow[ur, "\beta_{Y,X}"'] & \end{tikzcd} \tag{6.7.22}$$

1-Cell Naturality: For 1-cells $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ in C, the following diagram in $\mathsf{C}(X \boxdot Y, Y' \boxdot X')$ is commutative.

$$\begin{tikzcd} X \boxdot Y \arrow[r, "\beta_{X,Y}"] \arrow[d, "f \boxdot g"'] & Y \boxdot X \arrow[d, "g \boxdot f"] \\ X' \boxdot Y' \arrow[r, "\beta_{X',Y'}"] & Y' \boxdot X' \end{tikzcd} \tag{6.7.23}$$

2-Cell Naturality: For 2-cells
- $\alpha : f_1 \longrightarrow f_2$ in $\mathsf{C}(X, X')$ and
- $\gamma : g_1 \longrightarrow g_2$ in $\mathsf{C}(Y, Y')$,

the following equality holds in $\mathsf{C}(X \boxdot Y, Y' \boxdot X')$.

$$(\gamma \boxdot \alpha) * 1_{\beta_{X,Y}} = 1_{\beta_{X',Y'}} * (\alpha \boxdot \gamma) \tag{6.7.24}$$

This is equivalent to the following pasting diagram equality, with the unlabeled subregions commutative by (6.7.23).

$$\begin{tikzcd}[column sep=small] & X \boxdot Y \arrow[dl, bend right, "\beta_{X,Y}"'] \arrow[dr, bend left, "f_2 \boxdot g_2"] & \\ Y \boxdot X \arrow[dr, bend right, "g_1 \boxdot f_1"'] \arrow[dr, bend left, "g_2 \boxdot f_2"] \arrow[dr, phantom, "\overset{\gamma \boxdot \alpha}{\Rightarrow}"] & & X' \boxdot Y' \arrow[dl, bend left, "\beta_{X',Y'}"] \\ & Y' \boxdot X' & \end{tikzcd} = \begin{tikzcd}[column sep=small] & X \boxdot Y \arrow[dl, bend right, "\beta_{X,Y}"'] \arrow[dr, bend left, "f_2 \boxdot g_2"] \arrow[dr, bend right, "f_1 \boxdot g_1"'] \arrow[dr, phantom, "\overset{\alpha \boxdot \gamma}{\Rightarrow}"] & \\ Y \boxdot X \arrow[dr, bend right, "g_1 \boxdot f_1"'] & & X' \boxdot Y' \arrow[dl, bend left, "\beta_{X',Y'}"] \\ & Y' \boxdot X' & \end{tikzcd}$$

This completes the conditions for $\beta : \boxdot \longrightarrow \boxdot\xi$ to be a 2-natural isomorphism.

The three axioms of a permutative 2-category are equivalent to, respectively, the commutative diagrams (6.7.12), (6.7.13), and (6.7.14) with $\odot$ replaced by $\boxdot$. Note that the symmetry axiom is the same as the invertibility condition (6.7.22). $\diamond$

Strictification. We close this chapter with a brief discussion of strictification related to symmetric monoidal bicategories, permutative Gray monoids, and permutative 2-categories. Since we will not use these results in this book, we will not provide detailed definitions and statements here. The reader is referred to the cited references for detail.

A 2-functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ between permutative Gray monoids is a *permutative Gray functor* if it strictly preserves the Gray unit, the multiplication $\odot$, and the Gray symmetry β. There is a 1-category PGray, which is denoted by *PermGrayMon* in **[GJO17b]**, of permutative Gray monoids and permutative Gray functors.

A *quasi-strict symmetric monoidal 2-category* [**SP∞**, Def. 2.28] is a symmetric monoidal bicategory B as in Definition 6.5.9 that satisfies the following conditions.

- The underlying monoidal bicategory of B is a Gray monoid as in Definition 6.6.13.
- The braided monoidal bicategory structure of B as in Definition 6.5.3 is a *semistrict braided monoidal 2-category* in the sense of [**Cra98**, Def. 2.2].
- The left hexagonator $R_{-|--}$, the right hexagonator $R_{--|-}$, and the syllepsis ν are all identities.
- The component 2-cells of the braiding β in Definition 6.5.3 of the forms $\beta_{f,1}$ and $\beta_{1,g}$ are identities.
- Transition 2-cells (6.6.17) of the forms $\Sigma_{f,\beta}$ and $\Sigma_{\beta,g}$ are identities for 1-cells f and g, and for component 1-cells of the braiding β.

A *quasi-strict symmetric monoidal 2-functor* $F : \mathsf{C} \longrightarrow \mathsf{D}$ between quasi-strict symmetric monoidal 2-categories is a 2-functor of the underlying 2-categories that strictly preserves all the structures, with naturality constraints either the identity (when this makes sense) or the unique coherence isomorphisms in D. There is a 1-category $\mathsf{2Cat}^{\mathsf{qst}}$, which is denoted by $qsSM2Cat$ in [**GJO17b**], of quasi-strict symmetric monoidal 2-categories and quasi-strict symmetric monoidal 2-functors.

By [**GJO17b**, Theorem 3.43], there is an isomorphism

$$\mathsf{PGray} \cong \mathsf{2Cat}^{\mathsf{qst}}$$

between the 1-categories of permutative Gray monoids and the 1-category of quasi-strict symmetric monoidal 2-categories. Moreover, by [**SP∞**, Theorem 2.97] every symmetric monoidal bicategory is connected to a quasi-strict symmetric monoidal 2-category via a zigzag of symmetric monoidal strict functors. Therefore, every symmetric monoidal bicategory is symmetric monoidal biequivalent to a permutative Gray monoid.

On the other hand, *not* every permutative Gray monoid is symmetric monoidal biequivalent to a permutative 2-category; see [**SP∞**, Ex. 2.30]. However, every permutative Gray monoid is *weakly equivalent* to a permutative 2-category in a homotopical sense; see [**GJO17b**, Theorem 1.1]. Therefore, permutative 2-categories model all weak homotopy types, but not all categorical equivalence types, of permutative Gray monoids.

CHAPTER 7

Baez's Conjecture

The purpose of this chapter is to prove a conjecture due to John Baez on the existence of an initial object, in a suitably weakened sense, in a suitable 2-category of symmetric bimonoidal categories. We will prove two versions of Baez's Conjecture. Denote by $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ the 2-category in Definition 7.1.8 with

- flat small symmetric bimonoidal categories as objects,
- robust symmetric bimonoidal functors as 1-cells, and
- bimonoidal natural transformations as 2-cells.

Denote by $\varnothing$ the empty 2-category with no objects, no 1-cells, and no 2-cells. The first version of Baez's Conjecture is Theorem 7.8.1. It states that the left bipermutative category Σ in Proposition 2.4.8 is a lax bicolimit of the unique 2-functor

$$\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}.$$

The second version is Theorem 7.8.3. It states that the right bipermutative category Σ' in Proposition 2.4.23 is also a lax bicolimit of the unique 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$. For an open question related to Baez's Conjecture, see Question III.A.2.6.

Motivation. To motivate Baez's Conjecture, recall that a *rig* is a ring without additive inverses. Among rigs, $\mathbb{N}$, consisting of the natural numbers with its usual addition and multiplication, is an initial object. Indeed, given a rig R, a rig map $\mathbb{N} \longrightarrow R$ must send 0 and 1 in $\mathbb{N}$ to those in R. The image of each $n \in \mathbb{N}$ for $n \geq 2$ must be the sum

$$1 + \cdots + 1 \in R$$

of n copies of $1 \in R$, since the rig map preserves addition.

Since symmetric bimonoidal categories are categorical analogues of multiplicatively commutative rigs, it is natural to ask whether there is an analogue of $\mathbb{N}$ that is initial among symmetric bimonoidal categories. Using our terminology, the following conjecture due to John Baez predicts the existence of a symmetric bimonoidal analogue of $\mathbb{N}$.

> **Baez's Conjecture** [**Bae18**]. The groupoid of finite sets and bijections is an initial object, in a suitably weakened sense, in a 2-category of symmetric bimonoidal categories.

In this chapter, we will prove two precise versions of this conjecture.

Restrictions. Let us discuss some conditions that we need to impose on the statement of Baez's Conjecture.

Smallness: The 2-category Cat has *small* categories as objects. The smallness condition ensures that for each pair of functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ with C and D

small categories, there is a *set* of natural transformations $F \longrightarrow G$. Similarly, to discuss a 2-category of symmetric bimonoidal categories, as we will see in Proposition 7.1.7, we need to restrict to small symmetric bimonoidal categories.

Finite Ordinals: The groupoid of finite sets and bijections is not small, since every infinite set has infinitely many finite subsets. Therefore, for Baez's Conjecture we should consider a small version of this groupoid, that is, a groupoid of representatives of finite sets and bijections. There are two such candidates:

- the left bipermutative category Σ in Proposition 2.4.8 and
- the right bipermutative category Σ' in Proposition 2.4.23.

Both of them have objects $n \geq 0$, and the morphisms are given by the symmetric groups. The main difference between them is that the object $m \otimes n = mn$ is interpreted as an $n \times m$ matrix in Σ and as an $m \times n$ matrix in Σ'. See Explanations 2.4.7 and 2.4.22. We will show that each of Σ and Σ' satisfies Baez's Conjecture.

Flatness: The Coherence Theorems 3.9.1 and 4.4.3, which have a monomorphism assumption, are used in the proof of Baez's Conjecture. Therefore, we will restrict to small symmetric bimonoidal categories that are *flat* in the sense of Definition 3.9.9. The flatness assumption means that if we start with a component of either distributivity morphism δ^l or δ^r, and take iterated sums and products with a finite number of identity morphisms, then the result is still a monomorphism. For example, tight symmetric bimonoidal categories—that is, those with δ^l and δ^r natural isomorphisms—are flat.

Robustness: In the proof of Baez's Conjecture, we need the symmetric bimonoidal functors in question to have some invertible structure morphisms. To be specific, we want our symmetric bimonoidal functors to be *robust* in the sense of Definition 5.1.1. This means that the following structure morphisms are isomorphisms:

- the additive monoidal constraint
$$G^2_\oplus : GA \oplus GB \longrightarrow G(A \oplus B),$$
- the additive zero constraint
$$G^0_\oplus : \mathbb{0} \longrightarrow G(0),$$
and
- the multiplicative unit constraint
$$G^0_\otimes : \mathbb{1} \longrightarrow G(1).$$

A strong symmetric bimonoidal functor is automatically robust. However, Baez's Conjecture does *not* need the invertibility of the multiplicative monoidal constraint
$$G^2_\otimes : GA \otimes GB \longrightarrow G(A \otimes B).$$

Lax Bicolimit: There is the issue of which kind of weak colimits we should use. For a general 2-functor $F : \mathsf{A} \longrightarrow \mathsf{B}$ between 2-categories with $\mathsf{Ob}(\mathsf{A})$ a set, there are five different kinds of colimits for F [**JY21**, Chapter 5]:

- *lax bicolimits* involving lax transformations between cones and equivalences of categories;

- *lax colimits* involving lax transformations between cones and isomorphisms of categories;
- *pseudo bicolimits* involving strong transformations between cones and equivalences of categories;
- *pseudo colimits* involving strong transformations between cones and isomorphisms of categories; and
- *2-colimits* in terms of 2-natural transformations between cones and isomorphisms of categories.

Since the structure morphisms $G^2_\oplus$, $G^0_\oplus$, and $G^0_\otimes$ are isomorphisms instead of identity morphisms, we should use bicolimits instead of colimits. Moreover, the domain being the empty 2-category $\varnothing$ implies that lax bicolimits and pseudo bicolimits are the same in the current setting. Therefore, Baez's Conjecture involves lax bicolimits.

Precise Formulation. With the discussion above in mind, the precise version of Baez's Conjecture, Theorem 7.8.1, states that the left bipermutative category Σ is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}^{\mathsf{fsy}}_{\mathsf{r}}$. More concretely, this means that for each flat small symmetric bimonoidal category C, the unique functor

$$\mathsf{Bi}^{\mathsf{fsy}}_{\mathsf{r}}(\Sigma, \mathsf{C}) \xrightarrow{T} \mathbf{1} \tag{7.0.1}$$

to the terminal category $\mathbf{1}$ is an equivalence of categories, that is, fully faithful and essentially surjective. The second version of Baez's Conjecture, Theorem 7.8.3, states the same thing for the right bipermutative category Σ'.

While the statement of Baez's Conjecture requires the smallness assumption for the 2-category $\mathsf{Bi}^{\mathsf{fsy}}_{\mathsf{r}}$, its proof only uses smallness at the end to make sure that C is an object in $\mathsf{Bi}^{\mathsf{fsy}}_{\mathsf{r}}$. See Remark 7.8.2.

The validity of Baez's Conjecture may seem obvious at first glance because of the definitions (7.2.3) and (7.2.4). We stress that this is not the case, since it relies on nontrivial coherence properties of symmetric bimonoidal categories and symmetric bimonoidal functors. In fact, its proof involves both Coherence Theorems 3.9.1 and 4.4.3 for symmetric bimonoidal categories, although we do not need to use the Strictification Theorem 5.4.6. See Note 7.9.1. The proof in this chapter also requires a version of Epstein's Coherence Theorem 1.3.12 for symmetric bimonoidal functors in Theorem 7.5.8.

Outline. In Section 7.1, the category $\mathsf{Bi}^{\mathsf{sy}}$ of small symmetric bimonoidal categories and symmetric bimonoidal functors in Proposition 5.1.10 is extended to a 2-category. The 2-cells in this 2-category are bimonoidal natural transformations, which are monoidal natural transformations with respect to both the additive structures and the multiplicative structures. The wide subcategories $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{r}}$, $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{sg}}$, $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{u}}$, and $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{st}}$ with, respectively, robust, strong, unitary, and strict symmetric bimonoidal functors extend to sub-2-categories of the 2-category $\mathsf{Bi}^{\mathsf{sy}}$. Furthermore, the sub-2-category $\mathsf{Bi}^{\mathsf{fsy}}_{\mathsf{r}}$ of $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{r}}$ has *flat* small symmetric bimonoidal categories as objects.

Now we provide an outline of the proof of Baez's Conjecture, which is divided into two parts:

(1) the essential surjectivity of the functor T in (7.0.1) and
(2) the fully faithfulness of the functor T.

The first part begins in Section 7.2. For each symmetric bimonoidal category C, in this section, we construct a strong symmetric monoidal functor

$$F_\oplus = (F, F^2_\oplus, F^0_\oplus) : \Sigma \longrightarrow \mathsf{C}$$

between the additive structures. The flatness of C is not needed in this section because the Coherence Theorems 3.9.1 and 4.4.3 are not used yet. The functor F sends

- $0 \in \Sigma$ to $\mathbb{0} \in \mathsf{C}$,
- $1 \in \Sigma$ to $\mathbb{1} \in \mathsf{C}$, and
- $n \in \Sigma$ for $n \geq 2$ to a sum of n copies of $\mathbb{1} \in \mathsf{C}$ with some preselected convention for additive bracketing.

Mac Lane's Coherence Theorems 1.3.3 and 1.3.8 are used (i) to define the assignment of F on morphisms and the additive monoidal constraint $F^2_\oplus$, and (ii) to prove that F is a symmetric monoidal functor.

In Section 7.3, the strong symmetric monoidal functor $F_\oplus$ is extended to a symmetric monoidal functor

$$F_\otimes = (F, F^2_\otimes, F^0_\otimes) : \Sigma \longrightarrow \mathsf{C}$$

between the multiplicative structures, where C is now assumed to be flat. The definition of the multiplicative monoidal constraint $F^2_\otimes$ involves both $\oplus$ and $\otimes$. The Coherence Theorem 4.4.3 is used in the proofs of (i) the naturality of $F^2_\otimes$ and (ii) the associativity axiom (1.2.14) of $F_\otimes$ in Lemmas 7.3.15, 7.3.24, and 7.3.25. The Coherence Theorem 3.9.1 is used in the proof of Lemma 7.3.28 to show that F satisfies the axiom (1.2.26) of a symmetric monoidal functor.

In Section 7.4, we show that

$$(F, F^2_\otimes, F^0_\otimes, F^2_\oplus, F^0_\oplus) : \Sigma \longrightarrow \mathsf{C}$$

is a robust symmetric bimonoidal functor. The proof of the distributivity axiom (5.1.3), in the equivalent form (5.1.6), for F requires the Coherence Theorem 3.9.1. See Lemma 7.4.3, which is used at the end of the proof of Proposition 7.4.4. The canonical robust symmetric bimonoidal functor F implies that the functor T in (7.0.1) is essentially surjective.

The second part of the proof of Baez's Conjecture begins in Section 7.5. To show that the functor T in (7.0.1) is fully faithful, we will need to use a coherence property of symmetric bimonoidal functors. In this section, we formulate and prove this coherence property; see Theorem 7.5.8. When we actually apply this result in the proof of Baez's Conjecture, the domain of the symmetric bimonoidal functor is Σ. Theorem 7.5.8 is stated and proved in the much more general case of symmetric bimonoidal functors $G : \mathsf{C} \longrightarrow \mathsf{D}$ with C and D flat. The Coherence Theorem 4.4.3 is used to make sure that the diagram (7.5.2) is well defined; see Explanation 7.5.5. Moreover, Epstein's Coherence Theorem 1.3.12 is used near the end of the proof of Theorem 7.5.8.

In Section 7.6, we show that for any symmetric bimonoidal category C and any two robust symmetric bimonoidal functors $G, H : \Sigma \longrightarrow \mathsf{C}$, there is at most one bimonoidal natural transformation $G \longrightarrow H$, which must be a bimonoidal natural isomorphism. See Lemma 7.6.3. The main ingredient of this observation is Lemma 7.6.2, where we give an explicit description of the components of such a bimonoidal natural transformation. It is at this point that we need the *robust* condition on symmetric bimonoidal functors. The statement of Lemma 7.6.2 states precisely where we need the invertibility of the structure morphisms $G^2_\oplus$, $G^0_\oplus$, and $G^0_\otimes$.

In Section 7.7, we observe that the robust symmetric bimonoidal functor $F : \Sigma \longrightarrow \mathsf{C}$ with C flat is initial in a suitable sense. More precisely, in Lemma 7.7.9 we show that for each robust symmetric bimonoidal functor $G : \Sigma \longrightarrow \mathsf{C}$, there exists a unique bimonoidal natural transformation $\theta^G : F \longrightarrow G$, which is furthermore an isomorphism. Epstein's Coherence Theorem 1.3.12 is used in Lemmas 7.7.3 and 7.7.4 to prove (i) the naturality of θ^G and (ii) that θ^G is a monoidal natural transformation for the additive structures. Theorem 7.5.8 is used in Lemma 7.7.6 to prove that θ^G is a monoidal natural transformation for the multiplicative structures.

In Section 7.8, we collect the results from earlier sections to finish the proof of Baez's Conjecture, which is Theorem 7.8.1. Another version using Σ' is Theorem 7.8.3. Elgueta **[Elg21]** has a more restricted version of Baez's Conjecture for bimonoidal categories without multiplicative symmetries. Its relationship to this chapter is discussed in Section 7.9.

Reading Guide. As in Chapters 3, 4, and 5, we offer the following possible alternative to reading this chapter strictly linearly.

(1) Read Definitions 7.2.2 and 7.3.12 of $F : \Sigma \longrightarrow \mathsf{C}$.
(2) Read Definition 7.7.1 of $\theta^G : F \longrightarrow G$.
(3) Read Section 7.8, which contains both versions of Baez's Conjecture.
(4) Go back and read the parts skipped earlier.

7.1. The 2-Category of Symmetric Bimonoidal Categories

In this section, we define the 2-category of small symmetric bimonoidal categories and some of its sub-2-categories. The 2-category for Baez's Conjecture is in Definition 7.1.8. Proposition 5.1.10 contains the 1-categories of small symmetric bimonoidal categories and different subclasses of symmetric bimonoidal functors. Therefore, our task is to define the 2-cells, which are suitable natural transformations. For monoidal functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, recall from Definition 1.2.16 that a monoidal natural transformation $\theta : F \longrightarrow G$ is a natural transformation that is also compatible with the structure morphisms (F^2, F^0) of F and (G^2, G^0) of G.

Convention 7.1.1. For the rest of this chapter, unless otherwise specified, C and D are arbitrary symmetric bimonoidal categories as in Definition 2.1.2. Sometimes they are required to be small or flat (as in Definition 3.9.9) as specified. To clarify that some structure belongs to a particular category, we decorate it with the name of the category. For example, $\mathbb{0}^{\mathsf{C}}$ and $\mathbb{1}^{\mathsf{C}}$ are, respectively, the additive zero and the multiplicative unit in C. ◇

Recall from Definition 5.1.1 that a symmetric bimonoidal functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor equipped with

- an additive symmetric monoidal functor structure
$$F_\oplus = (F, F^2_\oplus, F^0_\oplus) : (\mathsf{C}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus) \longrightarrow (\mathsf{D}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus)$$
and
- a multiplicative symmetric monoidal functor structure
$$F_\otimes = (F, F^2_\otimes, F^0_\otimes) : (\mathsf{C}, \otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \xi^\otimes) \longrightarrow (\mathsf{D}, \otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \xi^\otimes)$$

that are compatible with the multiplicative zeros (5.1.2) and the distributivity morphisms (5.1.3).

Definition 7.1.2. Suppose $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ are two symmetric bimonoidal functors.

(1) A *bimonoidal natural transformation* $\theta : F \longrightarrow G$ is a natural transformation of the underlying functors such that both
- $\theta : F_\oplus \longrightarrow G_\oplus$ and
- $\theta : F_\otimes \longrightarrow G_\otimes$

are monoidal natural transformations.

(2) A *bimonoidal natural isomorphism* is an invertible bimonoidal natural transformation. $\diamond$

Explanation 7.1.3. In Definition 7.1.2, the assumption that

$$F_\oplus = (F, F^2_\oplus, F^0_\oplus) \xrightarrow{\theta} (G, G^2_\oplus, G^0_\oplus) = G_\oplus$$

is a monoidal natural transformation means that the diagrams

$$\begin{tikzcd} FX \oplus FY \arrow[r, "\theta_X \oplus \theta_Y"] \arrow[d, "F^2_\oplus"'] & GX \oplus GY \arrow[d, "G^2_\oplus"] \\ F(X \oplus Y) \arrow[r, "\theta_{X \oplus Y}"] & G(X \oplus Y) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{0}^{\mathsf{D}} \arrow[r, "F^0_\oplus"] \arrow[d, equal] & F\mathbb{0}^{\mathsf{C}} \arrow[d, "\theta_{\mathbb{0}^{\mathsf{C}}}"] \\ \mathbb{0}^{\mathsf{D}} \arrow[r, "G^0_\oplus"] & G\mathbb{0}^{\mathsf{C}} \end{tikzcd} \tag{7.1.4}$$

are commutative in D for all objects $X, Y \in \mathsf{C}$. Similarly, the assumption that

$$F_\otimes = (F, F^2_\otimes, F^0_\otimes) \xrightarrow{\theta} (G, G^2_\otimes, G^0_\otimes) = G_\otimes$$

is a monoidal natural transformation means that the diagrams

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\theta_X \otimes \theta_Y"] \arrow[d, "F^2_\otimes"'] & GX \otimes GY \arrow[d, "G^2_\otimes"] \\ F(X \otimes Y) \arrow[r, "\theta_{X \otimes Y}"] & G(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}^{\mathsf{D}} \arrow[r, "F^0_\otimes"] \arrow[d, equal] & F\mathbb{1}^{\mathsf{C}} \arrow[d, "\theta_{\mathbb{1}^{\mathsf{C}}}"] \\ \mathbb{1}^{\mathsf{D}} \arrow[r, "G^0_\otimes"] & G\mathbb{1}^{\mathsf{C}} \end{tikzcd} \tag{7.1.5}$$

are commutative in D. Moreover, a bimonoidal natural transformation $\theta : F \longrightarrow G$ is a bimonoidal natural isomorphism if and only if each component of θ is an isomorphism in D. $\diamond$

Recall from Proposition 5.1.10 that $\mathsf{Bi}^{\mathsf{sy}}$ is the 1-category of *small* symmetric bimonoidal categories and symmetric bimonoidal functors. Also recall its wide subcategories $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{r}}$, $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{sg}}$, $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{u}}$, and $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{st}}$ with, respectively, robust, strong, unitary, and strict symmetric bimonoidal functors. Now we extend these 1-categories to 2-categories as follows.

Definition 7.1.6. Define the 2-categorical data for $\mathsf{Bi}^{\mathsf{sy}}$ as follows.

Objects: Objects are small symmetric bimonoidal categories.

1-Cells: The 1-cells in $\mathsf{Bi}^{\mathsf{sy}}(\mathsf{C}, \mathsf{D})$ are the symmetric bimonoidal functors $\mathsf{C} \longrightarrow \mathsf{D}$.

Identity 1-Cells: $1_{\mathsf{C}} \in \mathsf{Bi}^{\mathsf{sy}}(\mathsf{C}, \mathsf{C})$ is the identity symmetric bimonoidal functor.

1-Cell Composition: Horizontal composition of 1-cells is as in Definition 5.1.7.

2-Cells: The 2-cells in $\mathsf{Bi}^{\mathsf{sy}}(\mathsf{C}, \mathsf{D})(F, G)$ are the bimonoidal natural transformations $F \longrightarrow G$ in Definition 7.1.2.

Identity 2-Cells: For each 1-cell $F : \mathsf{C} \longrightarrow \mathsf{D}$, the identity 2-cell

$$1_F \in \mathsf{Bi}^{\mathsf{sy}}(\mathsf{C},\mathsf{D})(F,F)$$

is the identity natural transformation of F.

2-Cell Compositions: Vertical and horizontal compositions of 2-cells are those of natural transformations in Definition 1.1.8.

This finishes the definition of the 2-categorical data for $\mathsf{Bi}^{\mathsf{sy}}$.

Moreover, similar definitions define the 2-categorical data for

- $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{r}}$ with robust symmetric bimonoidal functors as 1-cells,
- $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{sg}}$ with strong symmetric bimonoidal functors as 1-cells,
- $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{u}}$ with unitary symmetric bimonoidal functors as 1-cells, and
- $\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{st}}$ with strict symmetric bimonoidal functors as 1-cells. ◇

Proposition 7.1.7. *With the data in Definition 7.1.6,* $\mathsf{Bi}^{\mathsf{sy}}$ *is a 2-category that contains the following full sub-2-categories:*

$$\mathsf{Bi}^{\mathsf{sy}}_{\mathsf{st}} \subset \mathsf{Bi}^{\mathsf{sy}}_{\mathsf{u}} \subset \mathsf{Bi}^{\mathsf{sy}}_{\mathsf{sg}} \subset \mathsf{Bi}^{\mathsf{sy}}_{\mathsf{r}} \subset \mathsf{Bi}^{\mathsf{sy}}.$$

Proof. The smallness assumption on symmetric bimonoidal categories ensures that for each pair of 1-cells $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, there is a *set* of 2-cells in $\mathsf{Bi}^{\mathsf{sy}}(\mathsf{C},\mathsf{D})(F,G)$. By Proposition 5.1.10, the objects, 1-cells, identity 1-cells, and their horizontal composition in $\mathsf{Bi}^{\mathsf{sy}}$ constitute a 1-category. Vertical and horizontal compositions of 2-cells are well defined because monoidal natural transformations are closed under these compositions. Since symmetric monoidal functors and monoidal natural transformations satisfy axioms (i)–(vi) in Proposition 6.1.10, so do the 1-cells and 2-cells in $\mathsf{Bi}^{\mathsf{sy}}$. Therefore, $\mathsf{Bi}^{\mathsf{sy}}$ is a 2-category.

The existence of the four sub-2-categories follows from the fact that, in each case, the 1-cells are closed under horizontal composition by Lemma 5.1.9. □

For Baez's Conjecture, in addition to restricting the 1-cells to the robust ones, we also need to restrict the objects as follows. Recall from Definition 3.9.9 that a symmetric bimonoidal category is *flat* if each iterated sum and product of a component of either δ^l or δ^r (2.1.4) with a finite number of identity morphisms is a monomorphism. For example,

- flatness ensures that the morphisms in (3.9.5) are monomorphisms, and
- tight symmetric bimonoidal categories—that is, those with δ^l and δ^r natural isomorphisms—are flat.

The Coherence Theorems 3.9.1 and 4.4.3 apply to flat symmetric bimonoidal categories because the value of each δ-prime edge is of the type in the definition of flatness. Also recall from Definition 5.1.1 that a *robust* symmetric bimonoidal functor F has $F^2_\oplus$, $F^0_\oplus$, and $F^0_\otimes$ isomorphisms.

Definition 7.1.8. Denote by $\mathsf{Bi}^{\mathsf{fsy}}_{\mathsf{r}}$ the full sub-2-category of $\mathsf{Bi}^{\mathsf{sy}}$ with

- flat small symmetric bimonoidal categories as objects and
- robust symmetric bimonoidal functors as 1-cells. ◇

The rest of this section contains examples. By Proposition 2.3.2, each distributive symmetric monoidal category, whose monoidal product is denoted by $\otimes$, yields a tight symmetric bimonoidal category with additive structure given by coproducts, that is, $(\oplus, \mathbb{0}) = (\amalg, \varnothing)$. Moreover, by Proposition 5.1.11, each symmetric monoidal functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ between distributive symmetric monoidal categories canonically extends to a symmetric bimonoidal functor with the additive structure $(F^2_\oplus, F^0_\oplus)$ in (5.1.12). The next observation does not require the relevant categories to be small.

Proposition 7.1.9. *Suppose $\theta : F \longrightarrow G$ is a monoidal natural transformation between symmetric monoidal functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ with C and D distributive symmetric monoidal categories. Then θ is a bimonoidal natural transformation when F and G are regarded as symmetric bimonoidal functors as in Proposition 5.1.11.*

Proof. By assumption, $\theta : F_\otimes \longrightarrow G_\otimes$ is a monoidal natural transformation.

For $\theta : F_\oplus \longrightarrow G_\oplus$, we check the commutativity of the diagrams in (7.1.4).

(1) The right diagram in (7.1.4) is commutative by the universal property of the initial object in D.
(2) The left diagram in (7.1.4) is commutative by
  - the universal property of coproducts in D,
  - the definitions of $F^2_\oplus$ and $G^2_\oplus$ in (5.1.12), and
  - the naturality of θ.

Therefore, $\theta : F_\oplus \longrightarrow G_\oplus$ is a monoidal natural transformation. $\square$

Example 7.1.10 (Distributive Categories)**.** Suppose $F, G : \mathsf{C} \longrightarrow D$ are functors between distributive categories as in Example 2.3.5, such that the natural morphisms

$$F* \xrightarrow{\ t\ } *$$
$$F(A \times B) \xrightarrow{\ p\ } FA \times FB$$

for objects $A, B \in \mathsf{C}$, which are dual to those in (5.1.12), are isomorphisms, and similarly for G. As we mentioned in Example 5.1.14,

$$F_\otimes = \big(F, p^{-1}, t^{-1}\big),\ G_\otimes = \big(G, p^{-1}, t^{-1}\big) : (\mathsf{C}, \times, *) \longrightarrow (\mathsf{D}, \times, *)$$

are symmetric monoidal functors. If $\theta : F \longrightarrow G$ is a natural transformation, then it is also a monoidal natural transformation $\theta : F_\otimes \longrightarrow G_\otimes$ for the following reasons.

(1) The right diagram in (7.1.5) is commutative by
  - the universal property of the terminal object $*$ in D and
  - the invertibility of t for both F and G.
(2) The left diagram in (7.1.5) is commutative by
  - the universal property of products in D,
  - the invertibility of p for both F and G, and
  - the naturality of θ.

By Proposition 7.1.9, θ is a bimonoidal natural transformation when F and G are regarded as symmetric bimonoidal functors as in Proposition 5.1.11. $\diamond$

7.2. The Additive Structure

This section contains the first step of the proof of Baez's Conjecture. For each symmetric bimonoidal category C, we show that there exists a canonical strong symmetric monoidal functor $F_\oplus : \Sigma \longrightarrow \mathsf{C}$ between the additive structures of Σ and C. Here Σ is the tight symmetric bimonoidal category in Definition 2.4.1 and Proposition 2.4.8. It is furthermore a left bipermutative category in the sense of Definition 2.5.11. The flatness assumption on C is not needed until we define the multiplicative structure $F_\otimes$ in Section 7.3. The smallness of C is not needed until Section 7.8.

Definition 7.2.1. For each integer $n \geq 0$, define the following object in C.

$$\overline{n} = \begin{cases} \mathbb{0} & \text{if } n = 0, \\ \mathbb{1} & \text{if } n = 1, \text{ and} \\ (\mathbb{1} \oplus \cdots \oplus \mathbb{1})_{\mathsf{lt}} & \text{if } n > 1. \end{cases}$$

In the last case, $\overline{n}$ is the left normalized sum (5.2.13) of n copies of the multiplicative unit $\mathbb{1}$ in C. ◇

In the next definition and the rest of this chapter, a *Mac Lane coherence isomorphism* means a component of a permuted canonical map as in Definition 1.3.6, applied to the additive structure

$$(\mathsf{C}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus),$$

which is often abbreviated to $(\mathsf{C}, \oplus, \mathbb{0})$ or $(\mathsf{C}, \oplus)$. This is an adaptation of Definition 5.2.24 to the current context.

Definition 7.2.2. Using the additive structures in Σ and C, define the data

$$F_\oplus = (F, F_\oplus^2, F_\oplus^0) : \Sigma \longrightarrow \mathsf{C}$$

of a symmetric monoidal functor as follows.

The Functor: The functor $F : \Sigma \longrightarrow \mathsf{C}$ is defined as follows.

Objects: For each $n \geq 0$, define

$$F(n) = \overline{n} \in \mathsf{C} \tag{7.2.3}$$

with $\overline{n}$ as in Definition 7.2.1.

Morphisms: For each morphism $\sigma \in \Sigma(n,n) = \Sigma_n$, define the morphism

$$F(n) = \overline{n} \xrightarrow{F(\sigma)} \overline{n} = F(n) \in \mathsf{C} \tag{7.2.4}$$

as the unique Mac Lane coherence isomorphism in the symmetric monoidal category $(\mathsf{C}, \oplus, \mathbb{0})$ that permutes the n copies of $\mathbb{1}$ in $\overline{n}$ as $\sigma \in \Sigma_n$ permutes n letters.

The Additive Zero Constraint: The morphism

$$\mathbb{0} \xrightarrow{F_\oplus^0} F(0) = \mathbb{0} \in \mathsf{C} \tag{7.2.5}$$

is the identity morphism $1_\mathbb{0}$.

The Additive Monoidal Constraint: For $m, n \geq 0$, define the morphism

$$\overline{m} \oplus \overline{n} = F(m) \oplus F(n) \xrightarrow{F_\oplus^2} F(m+n) = \overline{m+n} \in \mathsf{C} \tag{7.2.6}$$

as the unique Mac Lane coherence isomorphism in $(\mathsf{C}, \oplus, \mathbb{0})$ that does *not* involve $\xi^{\pm\oplus}$.

This finishes the definition of $F_\oplus$. ◇

Explanation 7.2.7. Consider the definition (7.2.6) of $F_\oplus^2$.

(1) If $m = n = 0$, then
$$\mathbb{0} \oplus \mathbb{0} \xrightarrow{F^2_\oplus} \mathbb{0}$$
is the left additive zero $\lambda^\oplus_\mathbb{0}$, which is also equal to the right additive zero $\rho^\oplus_\mathbb{0}$ by (1.2.6).

(2) If $m = 0$ and $n > 0$, then
$$\mathbb{0} \oplus \overline{n} \xrightarrow{F^2_\oplus} \overline{n}$$
is the left additive zero $\lambda^\oplus_{\overline{n}}$.

(3) If $m > 0$ and $n = 0$, then
$$\overline{m} \oplus \mathbb{0} \xrightarrow{F^2_\oplus} \overline{m}$$
is the right additive zero $\rho^\oplus_{\overline{m}}$.

(4) If $m, n > 0$, then
$$\Big(\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} = \overline{m} \oplus \overline{n} \xrightarrow{F^2_\oplus} \overline{m+n} = \Big(\bigoplus_{i=1}^{m+n} \mathbb{1}\Big)_{\mathsf{lt}} \tag{7.2.8}$$

- is the identity morphism $1_{\overline{m+1}}$ if $n = 1$, and
- involves $\alpha^{-\oplus}_{\overline{m},\overline{k},\mathbb{1}}$ if $n > 1$, with $\alpha^{-\oplus}$ the inverse of $\alpha^\oplus$ and $1 \leq k \leq n-1$.

For example, in the last case, if $n = 3$, then $F^2_\oplus$ is the following composite in C.
$$\begin{tikzcd}
\overline{m} \oplus \big((\mathbb{1} \oplus \mathbb{1}) \oplus \mathbb{1}\big) \arrow[d, "\alpha^{-\oplus}_{\overline{m},\mathbb{1}\oplus\mathbb{1},\mathbb{1}}"'] & \\
\big(\overline{m} \oplus (\mathbb{1} \oplus \mathbb{1})\big) \oplus \mathbb{1} \arrow[r, "\alpha^{-\oplus}_{\overline{m},\mathbb{1},\mathbb{1}} \oplus 1_\mathbb{1}"] & \big((\overline{m} \oplus \mathbb{1}) \oplus \mathbb{1}\big) \oplus \mathbb{1} = \overline{m+3}
\end{tikzcd}$$

We stress that $F^2_\oplus$ does not involve the additive symmetry $\xi^\oplus$ and its inverse. ◇

Lemma 7.2.9. *For each symmetric bimonoidal category* C*,*
$$F_\oplus : \Sigma \longrightarrow \mathsf{C}$$
in Definition 7.2.2 is a strong symmetric monoidal functor.

Proof. The functoriality of $F : \Sigma \longrightarrow \mathsf{C}$ follows from

- the definitions of identity morphisms and composition in Σ as those of permutations and
- the uniqueness in Theorem 1.3.8 for the symmetric monoidal category $(\mathsf{C}, \oplus, \mathbb{0})$.

By definition, $F^0_\oplus$ is the identity morphism $1_\mathbb{0}$, and $F^2_\oplus$ is componentwise an isomorphism.

The uniqueness in Theorem 1.3.8 for $(\mathsf{C}, \oplus, \mathbb{0})$ implies the following statements.

- $F^2_\oplus$ is a natural isomorphism.
- The associativity axiom (1.2.14) holds for $(F, F^2_\oplus)$.
- The compatibility axiom (1.2.26) holds for $(F, F^2_\oplus)$ with the additive symmetries $\xi^\oplus$ in Σ and C.

The unity axiom (1.2.15) for $F_\oplus$ follows from

- the definition $F^0_\oplus = 1_{\mathbb{0}}$,
- that $\lambda^\oplus$ and $\rho^\oplus$ in Σ are identities, and
- the first three cases in Explanation 7.2.7.

Therefore, $F_\oplus$ is a strong symmetric monoidal functor between the additive structures. □

7.3. The Multiplicative Structure

The purpose of this section is to extend the strong symmetric monoidal functor $F_\oplus : \Sigma \longrightarrow \mathsf{C}$ in Lemma 7.2.9 to a symmetric monoidal functor $F_\otimes : \Sigma \longrightarrow \mathsf{C}$ between the multiplicative structures of Σ and C.

Convention 7.3.1. For the rest of this chapter, unless otherwise specified, C denotes an arbitrary *flat* symmetric bimonoidal category as in Definition 3.9.9. ⋄

Flatness implies that the monomorphism condition in the Coherence Theorems 3.9.1 and 4.4.3 are satisfied. Here is a brief outline of this section.

- The multiplicative structure $F_\otimes$ is in Definition 7.3.12, after the preliminary Definition 7.3.3 and Lemmas 7.3.6 and 7.3.7.
- We verify that $F^2_\otimes$ is well defined in Lemma 7.3.15.
- We prove a factorization of $F^2_\otimes$ in Lemma 7.3.21.
- The associativity axiom (1.2.14) and the unity axioms (1.2.15) for a monoidal functor are verified in Lemmas 7.3.25 and 7.3.27, respectively.
- We show that $F_\otimes$ is a symmetric monoidal functor in Lemma 7.3.28.

Recall from (5.2.13) that the subscript lt means left normalized bracketing.

Motivation 7.3.2. The definition of the multiplicative monoidal constraint $F^2_\otimes$ is more complicated than that of $F^2_\oplus$ because it involves a mixture of $\oplus$ and $\otimes$. Here we motivate its definition.

By definition, $F^2_\otimes$ involves a structure morphism

$$\begin{tikzcd}
F(m) \otimes F(n) = \overline{m} \otimes \overline{n} \arrow[d, equal] & \overline{mn} = F(mn) \arrow[d, equal] \\
\Big(\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "F^2_\otimes"] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}}
\end{tikzcd}$$

in C for $m, n > 0$, in addition to the cases where either m or n is 0. A natural way to define this morphism is to first distribute, using δ^l and δ^r, the product in the domain all the way down to a sum of mn copies of $\mathbb{1} \otimes \mathbb{1}$. Applying the multiplicative unit

$$\mathbb{1} \otimes \mathbb{1} \xrightarrow{\lambda^\otimes_{\mathbb{1}}} \mathbb{1}$$

in each summand results in the desired codomain $\overline{mn}$.

In the first step of distributing the domain, we need to specify

- an order of the mn copies of $\mathbb{1} \otimes \mathbb{1}$ and
- an additive bracketing of their sum.

For each such choice, there exist multiple ways to distributive the domain to the sum of mn copies of $\mathbb{1} \otimes \mathbb{1}$. In the second step, in addition to applying copies of $\lambda^\otimes_{\mathbb{1}}$, we also need to make sure that the additive bracketings match. One can certainly choose a specific path that achieves these two steps.

However, in order to show that F is a symmetric bimonoidal functor and to prove Baez's Conjecture using F, we need to verify the commutativity of a number of diagrams. This involves applying the Coherence Theorems 3.9.1 and 4.4.3. Therefore, we define $F^2_{\otimes}$ in terms of a coherence property, rather than as a specific path, that will make it easier to apply those theorems. As we will see in Definitions 7.3.3 and 7.3.12, $F^2_{\otimes}$ uses the *distortion* of a path in Definition 4.3.1, which is a crucial ingredient of Theorem 4.4.3. This in turn uses

- the graph in Definition 3.1.9,
- the distortion category $\mathcal{D}$ in Section 4.2, and
- the left normalized bracketing (5.2.13) applied to elements in the free $\{\oplus,\otimes\}$-algebra as in Remark 5.2.14.

The reader may want to review those definitions before proceeding. $\diamond$

We will return to the definition of $F^2_{\otimes}$ after some preliminary definitions and lemmas.

Definition 7.3.3. Suppose $m, n \geq 1$.

(1) Define the set

$$B = \{0^B, 1^B, b_1, \ldots, b_n\}$$

with $n+2$ elements.

(2) With each 1^B_i denoting a copy of 1^B, define

$$\Big(\bigoplus_{i=1}^{m} 1^B_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} b_j\Big)_{\mathsf{lt}} \xrightarrow{\ P\ } \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} b_j\Big)_{\mathsf{lt}} \tag{7.3.4}$$

as any path in $\mathsf{Gr}(B)$ whose distortion is the identity.

(3) Define the function $\varphi : B \longrightarrow \mathsf{Ob}(\mathsf{C})$ by

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^B \text{ and} \\ \mathbb{1} & \text{if } x \in \{1^B, b_1, \ldots, b_n\}. \end{cases} \tag{7.3.5}$$

The value of the path P in C is defined as in Definition 3.1.14 via the associated graph morphism $\varphi : \mathsf{Gr}(B) \longrightarrow \mathsf{C}$. $\diamond$

First we check that the path P in (7.3.4) is well defined. Recall from Definition 4.2.1 that each object in the distortion category $\mathcal{D}$ is a sequence of nonnegative integers. The next lemma uses the graph morphism

$$\mathsf{Gr}(B) \xrightarrow{\ \vartheta\ } \mathcal{D}$$

in Definition 4.3.1, applied to the set B in Definition 7.3.3 (1).

Lemma 7.3.6. *The domain and the codomain of P in (7.3.4) are both sent by the graph morphism ϑ to the object*

$$(1, \ldots, 1) \in \mathcal{D}$$

with length mn and each entry equal to 1.

Proof. The definitions (3.1.16), (4.2.8), and (4.3.2) imply the following equalities of objects in $\mathcal{D}$.

$$\vartheta\Big(\bigoplus_{i=1}^{m} 1_i^B\Big)_{\mathsf{lt}} = (\overbrace{0,\ldots,0}^{m})$$

$$\vartheta\Big(\bigoplus_{j=1}^{n} b_j\Big)_{\mathsf{lt}} = (\overbrace{1,\ldots,1}^{n})$$

These equalities and (4.2.15) imply that the domain of P is sent by ϑ to the object

$$\begin{aligned}(\overbrace{0,\ldots,0}^{m}) \otimes (\overbrace{1,\ldots,1}^{n}) &= (\overbrace{0+1,\ldots,0+1}^{mn})\\ &= (\overbrace{1,\ldots,1}^{mn})\\ &= \vartheta\Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} b_j\Big)_{\mathsf{lt}}.\end{aligned}$$

This proves the assertion. $\square$

We also need to make sure that such a path P actually exists.

Lemma 7.3.7. *There exists a path P as in (7.3.4) whose distortion is the identity.*

Proof. Consider the path

(7.3.8) $$P = (P_4, P_3, P_2, P_1)$$

in $\mathsf{Gr}(B)$ defined below.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m} 1_i^B\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} b_j\Big)_{\mathsf{lt}} \arrow[rr, "P"] \arrow[d, "P_1"'] & & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} b_j\Big)_{\mathsf{lt}} \\
\Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{i=1}^{m} 1_i^B\Big)_{\mathsf{lt}} \otimes b_j\Big)_{\mathsf{lt}} \arrow[dr, "P_2"'] & & \Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{i=1}^{m} b_j\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[u, "P_4"'] \\
& \Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{i=1}^{m} (1_i^B \otimes b_j)\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[ur, "P_3"'] &
\end{tikzcd}$$

(1) The path P_1 is the identity if $n = 1$. If $n > 1$, then P_1 has $n-1$ prime edges, each containing an instance of the left distributivity

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m} 1_i^B\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{l=1}^{n-j+1} b_l\Big)_{\mathsf{lt}} \arrow[d, "\delta^l_{(\oplus_{i=1}^m 1_i^B)_{\mathsf{lt}},\, (\oplus_{l=1}^{n-j} b_l)_{\mathsf{lt}},\, b_{n-j+1}}"] \\
\Big[\Big(\bigoplus_{i=1}^{m} 1_i^B\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{l=1}^{n-j} b_l\Big)_{\mathsf{lt}}\Big] \oplus \Big[\Big(\bigoplus_{i=1}^{m} 1_i^B\Big)_{\mathsf{lt}} \otimes b_{n-j+1}\Big]
\end{tikzcd}$$

for $1 \le j \le n-1$.

(2) The path P_2 is the identity if $m = 1$. If $m > 1$, then P_2 has $(m-1)n$ prime edges, each containing an instance of the right distributivity

$$\begin{tikzcd} \Big(\bigoplus_{k=1}^{m-i+1} 1_k^B\Big)_{\mathsf{lt}} \otimes b_j \arrow[d, "\delta^r_{(\oplus_{k=1}^{m-i} 1_k^B)_{\mathsf{lt}},\, 1^B_{m-i+1},\, b_j}"] \\ \Big[\Big(\bigoplus_{k=1}^{m-i} 1_k^B\Big)_{\mathsf{lt}} \otimes b_j\Big] \oplus \big(1^B_{m-i+1} \otimes b_j\big) \end{tikzcd} \tag{7.3.9}$$

for $1 \leq j \leq n$ and $1 \leq i \leq m-1$.

(3) The path P_3 has mn prime edges, each containing an instance of the left multiplicative unit

$$1_i^B \otimes b_j \xrightarrow{\lambda^{\otimes}_{b_j}} b_j$$

for $1 \leq j \leq n$ and $1 \leq i \leq m$.

(4) The path P_4 is the identity if either $m = 1$ or $n = 1$. If $m, n > 1$, then P_4 has $(m-1)(n-1)$ prime edges, each containing an instance of $\alpha^{-\oplus}$.

Since this path P has the desired domain and codomain, it remains to show that its distortion is the identity. We will show that each of P_1, P_2, P_3, and P_4 has identity distortion.

The distortion category $\mathcal{D}$ is a left bipermutative category by Theorem 4.2.29. Its only nonidentity structure isomorphisms are δ^r in (4.2.24), $\xi^{\oplus}$ in (4.2.10), and $\xi^{\otimes}$ in (4.2.17). Therefore, the paths P_1, P_3, and P_4 have identity distortions.

To see that the path P_2 has identity distortion, recall that P_2 is made up of prime edges containing δ^r in (7.3.9). By Definition 3.1.14 and Lemma 4.2.27, it suffices to show that the first entry of each distortion $\vartheta\delta^r$ is the identity permutation. By Definition 3.1.14, (4.2.8), and (4.3.2), the distortion of δ^r in (7.3.9) is the following morphism in $\mathcal{D}$.

$$\begin{aligned} \vartheta\delta^r_{(\oplus_{k=1}^{m-i} 1_k^B)_{\mathsf{lt}},\, 1^B_{m-i+1},\, b_j} &= \delta^r_{\vartheta(\oplus_{k=1}^{m-i} 1_k^B)_{\mathsf{lt}},\, \vartheta 1^B_{m-i+1},\, \vartheta b_j} \\ &= \delta^r_{(0,\ldots,0),(0),(1)} \end{aligned}$$

Here $(0,\ldots,0)$ has length $m-i$. By Lemma 4.2.27, the first entry of this δ^r in $\mathcal{D}$ is the permutation

$$\big(\xi^{\otimes}_{1,m-i} \oplus \xi^{\otimes}_{1,1}\big)\xi^{\otimes}_{m-i+1,1} \in \Sigma_{m-i+1}. \tag{7.3.10}$$

This is the identity permutation because both $\xi^{\otimes}_{?,1}$ and $\xi^{\otimes}_{1,?}$ are identity permutations by definition (2.4.5). □

Example 7.3.11. For the case $m = n = 2$, the path P in (7.3.8) has the following 8 prime edges.

$$\begin{tikzcd}
(1^B \oplus 1^B) \otimes (b_1 \oplus b_2) \arrow[rr, "P"] \arrow[d, "\delta^l_{1^B \oplus 1^B, b_1, b_2}"', "= P_1"] & & {[(b_1 \oplus b_1) \oplus b_2] \oplus b_2} \\
{[(1^B \oplus 1^B) \otimes b_1] \oplus [(1^B \oplus 1^B) \otimes b_2]} \arrow[d, "\delta^r_{1^B, 1^B, b_1} \oplus 1"'] & & (b_1 \oplus b_1) \oplus (b_2 \oplus b_2) \arrow[u, "P_4 = "', "\alpha^{-\oplus}_{b_1 \oplus b_1, b_2, b_2}"] \\
{[(1^B \otimes b_1) \oplus (1^B \otimes b_1)] \oplus [(1^B \oplus 1^B) \otimes b_2]} \arrow[d, "1 \oplus \delta^r_{1^B, 1^B, b_2}"'] & & {(b_1 \oplus b_1) \oplus [b_2 \oplus (1^B \otimes b_2)]} \arrow[u, "1 \oplus (1 \oplus \lambda^{\otimes}_{b_2})"'] \\
{[(1^B \otimes b_1) \oplus (1^B \otimes b_1)] \oplus [(1^B \otimes b_2) \oplus (1^B \otimes b_2)]} \arrow[dr, "(\lambda^{\otimes}_{b_1} \oplus 1) \oplus 1"'] & & {(b_1 \oplus b_1) \oplus [(1^B \otimes b_2) \oplus (1^B \otimes b_2)]} \arrow[u, "1 \oplus (\lambda^{\otimes}_{b_2} \oplus 1)"'] \\
& {[b_1 \oplus (1^B \otimes b_1)] \oplus [(1^B \otimes b_2) \oplus (1^B \otimes b_2)]} \arrow[ur, "(1 \oplus \lambda^{\otimes}_{b_1}) \oplus 1"'] &
\end{tikzcd}$$

The path P_2 has two prime edges, and the path P_3 has four prime edges. ◇

Now we define the multiplicative structure of F.

Definition 7.3.12. For a flat symmetric bimonoidal category C, extend the functor F in Definition 7.2.2 to the data

$$F_{\otimes} = (F, F^2_{\otimes}, F^0_{\otimes}) : (\Sigma, \otimes, 1) \longrightarrow (\mathsf{C}, \otimes, \mathbb{1})$$

of a symmetric monoidal functor as follows.

The Multiplicative Unit Constraint: The morphism

$$\mathbb{1} \xrightarrow{F^0_{\otimes}} F(1) = \mathbb{1} \in \mathsf{C} \tag{7.3.13}$$

is the identity morphism $1_{\mathbb{1}}$.

The Multiplicative Monoidal Constraint: For $m, n \geq 0$, define the morphism

$$\overline{m} \otimes \overline{n} = F(m) \otimes F(n) \xrightarrow{F^2_{\otimes}} F(mn) = \overline{mn} \in \mathsf{C} \tag{7.3.14}$$

as follows.

(1) If $m = 0$, then $F^2_{\otimes}$ is the left multiplicative zero

$$\mathbb{0} \otimes \overline{n} \xrightarrow{\lambda^{\bullet}_{\overline{n}}} \mathbb{0}.$$

(2) If $n = 0$, then $F^2_{\otimes}$ is the right multiplicative zero

$$\overline{m} \otimes \mathbb{0} \xrightarrow{\rho^{\bullet}_{\overline{m}}} \mathbb{0}.$$

If $m = n = 0$, then $\lambda^{\bullet}_{\mathbb{0}} = \rho^{\bullet}_{\mathbb{0}}$ by the axiom (2.1.14) in C.

(3) If $m, n > 0$, then

$$\Big(\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} \xrightarrow{F^2_{\otimes}} \Big(\bigoplus_{j=1}^{n} \bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}}$$

is the value in C of any path P as in (7.3.4).

This finishes the definition of $F_\otimes$. $\diamond$

Lemma 7.3.15. *For a flat symmetric bimonoidal category* C, $F_\otimes^2$ *in (7.3.14) is well defined and is a natural transformation.*

Proof. To see that $F_\otimes^2$ is well defined in the case $m,n>0$, recall from Lemma 7.3.7 that such a path P in $\mathsf{Gr}(B)$ with identity distortion exists. In fact, we constructed an explicit path (7.3.8) with identity distortion. Suppose P' is any other path in $\mathsf{Gr}(B)$ with the same (co)domain as P and with identity distortion. Since C is assumed to be flat, the Coherence Theorem 4.4.3 implies that P and P' have the same value in C. Moreover, the value of P in C has the stated (co)domain of $F_\otimes^2$ by the definition (7.3.5) of the function $\varphi : B \longrightarrow \mathsf{Ob}(\mathsf{C})$. Therefore, $F_\otimes^2$ is well defined.

To see that $F_\otimes^2$ is a natural transformation, first note that the cases with either $m=0$ or $n=0$ follow from the naturality of $\lambda^\bullet$ and $\rho^\bullet$.

For $m,n>0$, by definition (7.2.4) to check the naturality of $F_\otimes^2$, we need to show the commutativity of the following diagram in C for permutations $\sigma\in\Sigma_m$ and $\tau\in\Sigma_n$.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^m \mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "F_\otimes^2"] \arrow[d, "\cong_{\mathsf{ML}}^\sigma \otimes \cong_{\mathsf{ML}}^\tau"'] & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m \mathbb{1}\Big)_{\mathsf{lt}} \arrow[d, "\cong_{\mathsf{ML}}^{\sigma\otimes\tau}"] \\
\Big(\bigoplus_{i=1}^m \mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "F_\otimes^2"] & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m \mathbb{1}\Big)_{\mathsf{lt}}
\end{tikzcd} \tag{7.3.16}$$

Consider (7.3.16).

- $\cong_{\mathsf{ML}}^\sigma : \overline{m} \longrightarrow \overline{m}$ is the unique Mac Lane coherence isomorphism in the symmetric monoidal category $(\mathsf{C},\oplus,\mathbb{0})$ that additively permutes the m copies of $\mathbb{1}$ as $\sigma\in\Sigma_m$ permutes m letters.
- $\cong_{\mathsf{ML}}^\tau : \overline{n} \longrightarrow \overline{n}$ is interpreted in the same way using $\tau\in\Sigma_n$.
- $\cong_{\mathsf{ML}}^{\sigma\otimes\tau} : \overline{mn} \longrightarrow \overline{mn}$ is interpreted in the same way using the permutation $\sigma\otimes\tau\in\Sigma_{mn}$ in (2.4.4). From Explanation 2.4.7, if $\sigma\otimes\tau$ is regarded as acting on the entries of an $n\times m$ matrix, then τ permutes the n rows, and σ permutes the m columns.

To show the commutativity of the diagram (7.3.16), we first realize its four morphisms as the values in C of the four corresponding sides in the following diagram in $\mathsf{Gr}(B)$, with each b_j^i a copy of $b_j\in B$.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^m 1_i^B\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n b_j\Big)_{\mathsf{lt}} \arrow[rr, two heads, "P"] \arrow[d, "P_{\sigma\otimes 1}"'] & & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m b_j^i\Big)_{\mathsf{lt}} \arrow[dd, "P_{\sigma\otimes\tau}"] \\
\Big(\bigoplus_{i=1}^m 1_{\sigma^{-1}(i)}^B\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n b_j\Big)_{\mathsf{lt}} \arrow[d, "P_{1\otimes\tau}"'] & & \\
\Big(\bigoplus_{i=1}^m 1_{\sigma^{-1}(i)}^B\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n b_{\tau^{-1}(j)}\Big)_{\mathsf{lt}} \arrow[rr, two heads, "P'"] & & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m b_{\tau^{-1}(j)}^{\sigma^{-1}(i)}\Big)_{\mathsf{lt}}
\end{tikzcd} \tag{7.3.17}$$

Consider (7.3.17).

- Both paths P and P' have identity distortion. They exist by Lemma 7.3.7.
- $P_{\sigma\otimes 1}$ is a path in which each prime edge has an instance of $\alpha^{\pm\oplus}$ or $\xi^{\pm\oplus}$ applied to the 1_i^B's. They are permuted as $\sigma \in \Sigma_m$ permutes m letters. Such a path exists because each symmetric group is generated by adjacent transpositions that swap two consecutive letters. The same remark applies in the next two items.
- $P_{1\otimes\tau}$ is a path in which each prime edge has an instance of $\alpha^{\pm\oplus}$ or $\xi^{\pm\oplus}$ applied to the b_j's. They are permuted as $\tau \in \Sigma_n$ permutes n letters.
- $P_{\sigma\otimes\tau}$ is a path in which each prime edge has an instance of $\alpha^{\pm\oplus}$ or $\xi^{\pm\oplus}$, such that the b_j^i's are permuted by $\sigma\otimes\tau \in \Sigma_{mn}$ as indicated by the subscript and superscript.

Now we check that the four morphisms in the diagram (7.3.16) are the values in C of the four corresponding paths in (7.3.17).

- The values in C of the paths P and P', which have identity distortion, are both $F_\otimes^2$ by the latter's definition.
- The value in C of the left vertical path $(P_{1\otimes\tau}, P_{\sigma\otimes 1})$ is $\cong_{\mathsf{ML}}^{\sigma} \otimes \cong_{\mathsf{ML}}^{\tau}$ by
  - the definitions of the paths $P_{\sigma\otimes 1}$ and $P_{1\otimes\tau}$,
  - the functoriality of $\otimes$ in C, and
  - the uniqueness part in the Symmetric Coherence Theorem 1.3.8.
- Similarly, the value in C of the right vertical path $P_{\sigma\otimes\tau}$ is $\cong_{\mathsf{ML}}^{\sigma\otimes\tau}$ by Theorem 1.3.8.

Therefore, the values in C of the four sides in the diagram (7.3.17) are the four corresponding morphisms in the diagram (7.3.16).

The Coherence Theorem 4.4.3 is applicable to C because it is assumed to be flat. Together with the discussion in the previous paragraph, to show that the diagram (7.3.16) is commutative, it suffices to show that the diagram (7.3.17) is commutative in the distortion category $\mathcal{D}$. Furthermore, since P and P' have identity distortion, it suffices to show that the two vertical paths in (7.3.17) have the same distortion.

By Lemma 7.3.6, all five vertices in (7.3.17) are sent by the graph morphism $\vartheta : \mathsf{Gr}(B) \longrightarrow \mathcal{D}$ in Definitions 4.3.1 and 7.3.3 to the same object $(1,\ldots,1) \in \mathcal{D}$. Each entry of this object is 1, and the symmetric group Σ_1 is the trivial group. Therefore, it remains to show that the distortions

$$\vartheta(P_{1\otimes\tau}, P_{\sigma\otimes 1}) \quad \text{and} \quad \vartheta P_{\sigma\otimes\tau}$$

have the same first entry. By (4.2.16),

- the first entry of $\vartheta P_{\sigma\otimes 1}$ is $\sigma \otimes \mathrm{id}_n$, and
- the first entry of $\vartheta P_{1\otimes\tau}$ is $\mathrm{id}_m \otimes \tau$.

By (4.2.4), the first entry of $\vartheta(P_{1\otimes\tau}, P_{\sigma\otimes 1})$ is the permutation

$$\begin{aligned}(\mathrm{id}_m \otimes \tau)(\sigma \otimes \mathrm{id}_n) &= (\mathrm{id}_m\sigma) \otimes (\tau\mathrm{id}_n)\\ &= \sigma\otimes\tau \in \Sigma_{mn}.\end{aligned}$$

Since this is equal to the first entry of $\vartheta P_{\sigma\otimes\tau}$, the two vertical paths in (7.3.17) have the same distortion. □

To prove the rest of the properties of $F_\otimes^2$, we will factor $F_\otimes^2$ using the following definition, which should be compared to Definition 7.3.3.

Definition 7.3.18. Suppose C is a flat symmetric bimonoidal category, and $m,n > 0$.

- Define the set
$$B' = \{0^{B'}, 1^{B'}, a_1, \ldots, a_m, b_1, \ldots, b_n\}$$
with $m+n+2$ elements.
- Define
$$\Big(\bigoplus_{i=1}^m a_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n b_j\Big)_{\mathsf{lt}} \xrightarrow{\;Q\;} \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m (a_i \otimes b_j)\Big)_{\mathsf{lt}} \tag{7.3.19}$$
as any path in $\mathsf{Gr}(B')$ with identity distortion.
- Define the function $\varphi : B' \longrightarrow \mathsf{Ob}(\mathsf{C})$ by
$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^{B'} \text{ and} \\ \mathbb{1} & \text{if } x \in B' \smallsetminus \{0^{B'}\}. \end{cases} \tag{7.3.20}$$
- Define $F'_\otimes$ as the value in C of any path Q in (7.3.19), defined via the associated graph morphism $\varphi : \mathsf{Gr}(B') \longrightarrow \mathsf{C}$ in Definition 3.1.14.
- Define $F''_\otimes$ as the left normalized sum of mn copies of the left multiplicative unit $\lambda^\otimes_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1}$. ◇

Lemma 7.3.21. *For a flat symmetric bimonoidal category* C *and* $m,n > 0$*, the morphism*
$$F^2_\otimes : \overline{m} \otimes \overline{n} \longrightarrow \overline{mn}$$
in (7.3.14) factors as follows, where $F'_\otimes$ *and* $F''_\otimes$ *are as in Definition 7.3.18.*

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^m \mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n \mathbb{1}\Big)_{\mathsf{lt}} \arrow[rr, "F^2_\otimes"] \arrow[dr, "F'_\otimes"'] & & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m \mathbb{1}\Big)_{\mathsf{lt}} \\
& \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m (\mathbb{1} \otimes \mathbb{1})\Big)_{\mathsf{lt}} \arrow[ur, "F''_\otimes"'] &
\end{tikzcd}$$

Proof. The existence of a path Q with identity distortion as in (7.3.19) is proved by the construction (7.3.8) with only (P_4, P_2, P_1) and with a_i replacing 1^B_i for $1 \leq i \leq m$. The Coherence Theorem 4.4.3 implies that $F'_\otimes$ is well defined. In other words, if Q' is any other path with the same (co)domain as Q and with identity distortion, then Q and Q' have the same value in C. In particular, we may choose the path Q such that each of its prime edges contains an instance of δ^l, δ^r, or $\alpha^{-\oplus}$.

The morphism $F^2_\otimes$ factors as $F''_\otimes F'_\otimes$ by (i) the construction of the path (7.3.8), whose value in C is $F^2_\otimes$, and (ii) the naturality of $\alpha^\oplus$ in C. □

The next definition and lemma will be used to show that $F^2_\otimes$ satisfies the associativity axiom (1.2.14).

Definition 7.3.22. Suppose $m,n,p > 0$.

- Define the set
$$B'' = \{0^{B''}, 1^{B''}, a_1, \ldots, a_m, b_1, \ldots, b_n, c_1, \ldots, c_p\}$$
with $m+n+p+2$ elements.

- Define the following diagram in $\mathsf{Gr}(B'')$.

$$\begin{tikzcd}
\Big[\big(\bigoplus_{i=1}^{m} a_i\big)_{\mathsf{lt}} \otimes \big(\bigoplus_{j=1}^{n} b_j\big)_{\mathsf{lt}}\Big] \otimes \big(\bigoplus_{k=1}^{p} c_k\big)_{\mathsf{lt}} \arrow[r, "\alpha^{\otimes}"] \arrow[d, "Q_1 \otimes 1"'] & \big(\bigoplus_{i=1}^{m} a_i\big)_{\mathsf{lt}} \otimes \Big[\big(\bigoplus_{j=1}^{n} b_j\big)_{\mathsf{lt}} \otimes \big(\bigoplus_{k=1}^{p} c_k\big)_{\mathsf{lt}}\Big] \arrow[d, "1 \otimes Q_3"] \\
\big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} (a_i \otimes b_j)\big)_{\mathsf{lt}} \otimes \big(\bigoplus_{k=1}^{p} c_k\big)_{\mathsf{lt}} \arrow[d, "Q_2"'] & \big(\bigoplus_{i=1}^{m} a_i\big)_{\mathsf{lt}} \otimes \big(\bigoplus_{k=1}^{p}\bigoplus_{j=1}^{n} (b_j \otimes c_k)\big)_{\mathsf{lt}} \arrow[d, "Q_4"] \\
\big(\bigoplus_{k=1}^{p}\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} (a_i \otimes b_j) \otimes c_k\big)_{\mathsf{lt}} \arrow[r, "R"] & \big(\bigoplus_{k=1}^{p}\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} a_i \otimes (b_j \otimes c_k)\big)_{\mathsf{lt}}
\end{tikzcd} \tag{7.3.23}$$

  - Up to a change of symbols, each of the paths Q_1, Q_2, Q_3, and Q_4 is the path Q in (7.3.19).
  - $Q_1 \otimes 1$ and $1 \otimes Q_3$ are defined in Notation 3.3.10.
  - The path R has mnp prime edges, each containing one instance of the multiplicative associativity

$$(a_i \otimes b_j) \otimes c_k \xrightarrow{\alpha^{\otimes}_{a_i,b_j,c_k}} a_i \otimes (b_j \otimes c_k)$$

  for $1 \le i \le m$, $1 \le j \le n$, and $1 \le k \le p$. ◇

The following lemma uses Definition 3.1.14.

Lemma 7.3.24. *The diagram (7.3.23) is commutative in each flat symmetric bimonoidal category* C *with respect to each function* $\varphi : B'' \longrightarrow \mathsf{Ob}(\mathsf{C})$ *that satisfies*

$$\varphi(0^{B''}) = \mathbb{0} \quad \textit{and} \quad \varphi(1^{B''}) = \mathbb{1}.$$

Proof. First we make two remarks.

- By Definition 7.3.18, Q has identity distortion. Therefore, each of the paths $Q_1 \otimes 1$, Q_2, $1 \otimes Q_3$, and Q_4 has identity distortion.
- $\alpha^{\otimes}$ in the distortion category is the identity by Definition 4.2.14. Therefore, the path R has identity distortion.

It follows that both the left-bottom path and the top-right path in (7.3.23) have identity distortion. Since C is assumed to be flat, the Coherence Theorem 4.4.3 implies that these two paths have the same value in C with respect to any choice of the function φ satisfying the stated conditions.

Alternatively, we may also use the Coherence Theorem 3.9.1 by noting that the lower right vertex in (7.3.23) is regular in the sense of Definition 3.1.25. Therefore, the upper left vertex is also regular by Lemma 3.1.29. Theorem 3.9.1 now implies that the two paths have the same value in C. □

Lemma 7.3.25. *For a flat symmetric bimonoidal category* C, $F_{\otimes}$ *in Definition 7.3.12 satisfies the associativity axiom (1.2.14).*

Proof. The axiom (1.2.14) for $F_\otimes$ means the commutativity of the diagram

$$\begin{tikzcd}
(\overline{m} \otimes \overline{n}) \otimes \overline{p} \arrow[r, "\alpha^\otimes"] \arrow[d, "F^2_\otimes \otimes 1"'] & \overline{m} \otimes (\overline{n} \otimes \overline{p}) \arrow[d, "1 \otimes F^2_\otimes"] \\
\overline{mn} \otimes \overline{p} \arrow[d, "F^2_\otimes"'] & \overline{m} \otimes \overline{np} \arrow[d, "F^2_\otimes"] \\
\overline{mnp} \arrow[r, "1"] & \overline{mnp}
\end{tikzcd} \tag{7.3.26}$$

in C for $m, n, p \geq 0$. There are four cases.

(1) If $m = 0$, then (7.3.26) is commutative by the axiom (2.1.22) and the naturality of $\lambda^\bullet$ in C.
(2) If $n = 0$, then (7.3.26) is commutative by the axiom (2.1.21) in C.
(3) If $p = 0$, then (7.3.26) is commutative by the axiom (2.1.20) and the naturality of $\rho^\bullet$ in C.

For the case $m, n, p > 0$, first we make some remarks.

(i) The diagram

$$\begin{tikzcd}
(\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1} \arrow[r, "\alpha^\otimes_{\mathbb{1},\mathbb{1},\mathbb{1}}"] \arrow[d, "\lambda^\otimes_{\mathbb{1}} \otimes 1_{\mathbb{1}}"'] & \mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[d, "1_{\mathbb{1}} \otimes \lambda^\otimes_{\mathbb{1}}"] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "1_{\mathbb{1} \otimes \mathbb{1}}"] \arrow[d, "\lambda^\otimes_{\mathbb{1}}"'] & \mathbb{1} \otimes \mathbb{1} \arrow[d, "\lambda^\otimes_{\mathbb{1}}"] \\
\mathbb{1} \arrow[r, "1_{\mathbb{1}}"] & \mathbb{1}
\end{tikzcd}$$

in C is commutative. Indeed, the top square is commutative by the unity axiom (1.2.2) and the equality $\lambda^\otimes_{\mathbb{1}} = \rho^\otimes_{\mathbb{1}}$ in (1.2.6). The bottom square is commutative by definition.

(ii) $F^2_\otimes$ factors as $F''_\otimes F'_\otimes$ by Lemma 7.3.21, where $F'_\otimes$ (respectively, $F''_\otimes$) only involves δ^l, δ^r, and $\alpha^{-\oplus}$ (respectively, $\lambda^\otimes_{\mathbb{1}}$).

(iii) $\alpha^\oplus$, δ^l, and δ^r are natural transformations in C.

These remarks imply that, to prove the commutativity of the diagram (7.3.26) when $m, n, p > 0$, it suffices to show that the diagram (7.3.23) is commutative in C with respect to the function $\varphi : B'' \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^{B''} \text{ and} \\ \mathbb{1} & \text{if } x \in B'' \setminus \{0^{B''}\}. \end{cases}$$

Therefore, Lemma 7.3.24 finishes the proof. □

Lemma 7.3.27. *In the context of Definition 7.3.12, $F_\otimes$ satisfies the unity axioms (1.2.15).*

Proof. Recall that $F^0_\otimes$ is the identity morphism $1_{\mathbb{1}}$, and $\lambda^\otimes$ in Σ is the identity by Definition 2.4.1. Therefore, the left unity axiom in (1.2.15) for $F_\otimes$ means that the two morphisms

$$\begin{tikzcd}
\mathbb{1} \otimes \overline{n} \arrow[r, shift left, "\lambda^\otimes_{\overline{n}}"] \arrow[r, shift right, "F^2_\otimes"'] & \overline{n}
\end{tikzcd}$$

in C are equal for each $n \geq 0$. There are two cases.

(1) If $n = 0$, then this $F^2_\otimes$ is $\rho^\bullet_{\mathbb{1}}$, which is equal to $\lambda^\otimes_{\mathbb{0}}$ by the axiom (2.1.18) in C.
(2) If $n > 0$, then the equality $\lambda^\otimes_{\overline{n}} = F^2_\otimes$ follows from
   - the definition of $F^2_\otimes$ (7.3.14) in the case $m, n > 0$ and
   - the fact that $\lambda^\otimes$ in the distortion category is the identity by Definition 4.2.14.

This proves the left unity axiom in (1.2.15).

The right unity axiom in (1.2.15) is proved in the same way using

- the axiom (2.1.17) instead of (2.1.18) and
- the fact that $\rho^\otimes$ in the distortion category is the identity.

Therefore, $F_\otimes$ satisfies the unity axioms (1.2.15). □

Lemma 7.3.28. *For a flat symmetric bimonoidal category* C*,* $F_\otimes$ *in Definition 7.3.12 is a symmetric monoidal functor.*

Proof. By Lemmas 7.3.15, 7.3.25, and 7.3.27, $F_\otimes$ is a monoidal functor. It remains to check the axiom (1.2.26) of a symmetric monoidal functor. This axiom means that the diagram

$$\begin{tikzcd} \overline{m} \otimes \overline{n} \arrow[r, "\xi^\otimes_{\overline{m},\overline{n}}"] \arrow[d, "F^2_\otimes"'] & \overline{n} \otimes \overline{m} \arrow[d, "F^2_\otimes"] \\ \overline{mn} \arrow[r, "F\xi^\otimes_{m,n}"] & \overline{nm} \end{tikzcd} \tag{7.3.29}$$

in C is commutative for $m, n \geq 0$. In the bottom horizontal arrow, $\xi^\otimes_{m,n} \in \Sigma_{mn}$ is the permutation in (2.4.5). It may be interpreted as taking the transpose of an $n \times m$ matrix as in Explanation 2.4.16. There are three cases.

(1) If $m = 0$, then (7.3.29) becomes the following diagram.

$$\begin{tikzcd} \mathbb{0} \otimes \overline{n} \arrow[r, "\xi^\otimes_{\mathbb{0},\overline{n}}"] \arrow[d, "\lambda^\bullet_{\overline{n}}"'] & \overline{n} \otimes \mathbb{0} \arrow[d, "\rho^\bullet_{\overline{n}}"] \\ \mathbb{0} \arrow[r, "1"] & \mathbb{0} \end{tikzcd}$$

This diagram is commutative by
   - the symmetry axiom $\xi^\otimes \xi^\otimes = 1$ in (1.2.20) and
   - the axiom $\rho^\bullet = \lambda^\bullet \xi^\otimes_{-,\mathbb{0}}$ in (2.1.19).

(2) Similarly, if $n = 0$, then (7.3.29) is commutative by the axiom (2.1.19).

For the case $m, n > 0$, by Lemma 7.3.21, the diagram (7.3.29) is the outer diagram below.

$$
\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "\xi^{\otimes}_{\overline{m},\overline{n}}"] \arrow[d, "F'_{\otimes}"'] & \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}} \arrow[d, "F'_{\otimes}"] \\
\Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} (\mathbb{1}\otimes\mathbb{1})\Big)_{\mathsf{lt}} \arrow[r, "\xi^{\otimes}_{m,n}"] \arrow[d, "(\oplus_j\oplus_i\lambda^{\otimes}_{\mathbb{1}})_{\mathsf{lt}}"'] & \Big(\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} (\mathbb{1}\otimes\mathbb{1})\Big)_{\mathsf{lt}} \arrow[d, "(\oplus_i\oplus_j\lambda^{\otimes}_{\mathbb{1}})_{\mathsf{lt}}"] \\
\Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "\xi^{\otimes}_{m,n}"] & \Big(\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}}
\end{tikzcd}
\tag{7.3.30}
$$

Consider (7.3.30).

- The upper left $F'_{\otimes}$ is the value in C of the path Q in (7.3.19) with identity distortion.
- The upper right $F'_{\otimes}$ is the same, but with the roles of the a_i's and of the b_j's switched.
- Each of the middle and the bottom horizontal arrows $\xi^{\otimes}_{m,n}$ is the unique Mac Lane coherence isomorphism in $(\mathsf{C}, \oplus, \mathbb{0})$ that additively permutes the mn summands as the transpose of an $n \times m$ matrix.

The bottom square in (7.3.30) is commutative by the naturality of Mac Lane coherence isomorphisms. It remains to show that the top square in (7.3.30) is commutative.

By Proposition II.1.3.26, $\xi^{\otimes}_{\mathbb{1},\mathbb{1}} = 1_{\mathbb{1}\otimes\mathbb{1}}$. It follows that the top square in (7.3.30) is the value in C of the following diagram in $\mathsf{Gr}(B')$ with B' as in Definition 7.3.18.

$$
\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m} a_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} b_j\Big)_{\mathsf{lt}} \arrow[r, "\xi^{\otimes}"] \arrow[d, "Q"'] & \Big(\bigoplus_{j=1}^{n} b_j\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{i=1}^{m} a_i\Big)_{\mathsf{lt}} \arrow[d, "Q'"] \\
\Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} (a_i\otimes b_j)\Big)_{\mathsf{lt}} \arrow[r, "R"] & \Big(\bigoplus_{i=1}^{m}\bigoplus_{j=1}^{n} (b_j\otimes a_i)\Big)_{\mathsf{lt}}
\end{tikzcd}
\tag{7.3.31}
$$

Consider (7.3.31).

- Q is the path in (7.3.19).
- Q' is the same but with the roles of the a_i's and of the b_j's switched.
- Each prime edge in the path R contains an instance of one of the following:
  - $\alpha^{\pm\oplus}$ or $\xi^{\oplus}$, which additively permute the mn summands as the transpose of an $n \times m$ matrix, or
  - $\xi^{\otimes}_{a_i,b_j} : a_i \otimes b_j \longrightarrow b_j \otimes a_i$ for $1 \le i \le m$ and $1 \le j \le n$.
- The lower right vertex is regular in the sense of Definition 3.1.25. By Lemma 3.1.29, the upper left vertex is also regular.

Since C is assumed to be flat, the Coherence Theorem 3.9.1 implies that the diagram (7.3.31) is commutative in C. So the top square in (7.3.30) is commutative. □

7.4. Weakly Initial Symmetric Bimonoidal Category

The purpose of this section is to prove half of Baez's Conjecture. For each flat symmetric bimonoidal category C as in Definition 3.9.9, we prove that the data

$$(F, F^2_\oplus, F^0_\oplus, F^2_\otimes, F^0_\otimes) : \Sigma \longrightarrow \mathsf{C}$$

in Definitions 7.2.2 and 7.3.12 constitute a robust symmetric bimonoidal functor as in Definition 5.1.1. The next definition and lemma will be used to prove the distributivity axiom (5.1.3) for F. Recall from (5.2.13) that the subscript lt means left normalized bracketing.

Definition 7.4.1. Suppose $m, n, p > 0$.

- Define the set

$$B' = \left\{0^{B'}, 1^{B'}, a_1, \ldots, a_m, b_1, \ldots, b_{n+p}\right\}$$

with $m+n+p+2$ elements. This is the set B' in Definition 7.3.18, but with $n+p$ in place of n.

- Define the following diagram in $\mathsf{Gr}(B')$.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^m a_i\Big)_{\mathsf{lt}} \otimes \Big[\Big(\bigoplus_{j=1}^n b_j\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{k=1}^p b_{n+k}\Big)_{\mathsf{lt}}\Big] \arrow[r, "\delta^l"] \arrow[d, "1 \otimes R_1"'] & \Big[\Big(\bigoplus_{i=1}^m a_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n b_j\Big)_{\mathsf{lt}}\Big] \oplus \Big[\Big(\bigoplus_{i=1}^m a_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{k=1}^p b_{n+k}\Big)_{\mathsf{lt}}\Big] \arrow[d, "Q_2 \oplus 1"] \\
\Big(\bigoplus_{i=1}^m a_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n+p} b_j\Big)_{\mathsf{lt}} \arrow[d, "Q_1"'] & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m (a_i \otimes b_j)\Big)_{\mathsf{lt}} \oplus \Big[\Big(\bigoplus_{i=1}^m a_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{k=1}^p b_{n+k}\Big)_{\mathsf{lt}}\Big] \arrow[d, "1 \oplus Q_3"] \\
\Big(\bigoplus_{j=1}^{n+p} \bigoplus_{i=1}^m (a_i \otimes b_j)\Big)_{\mathsf{lt}} & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m (a_i \otimes b_j)\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{k=1}^p \bigoplus_{i=1}^m (a_i \otimes b_{n+k})\Big)_{\mathsf{lt}} \arrow[l, "R_2"']
\end{tikzcd} \tag{7.4.2}$$

  - Up to a change of symbols, each of the paths Q_1, Q_2, and Q_3 is the path Q in (7.3.19).
  - Each prime edge in each of the paths R_1 and R_2 contains an instance of $\alpha^{-\oplus}$.
  - The paths $Q_2 \oplus 1$, $1 \oplus Q_3$, and $1 \otimes R_1$ are as in Notation 3.3.10. $\diamond$

Lemma 7.4.3. *The diagram (7.4.2) is commutative in each flat symmetric bimonoidal category C with respect to each function $\varphi : B' \longrightarrow \mathsf{Ob}(\mathsf{C})$ that satisfies*

$$\varphi(0^{B'}) = \mathbb{0} \quad \textit{and} \quad \varphi(1^{B'}) = \mathbb{1}.$$

Proof. The existence of the paths Q_1, Q_2, and Q_3 is proved by the construction (7.3.8) with only (P_4, P_2, P_1) and with a_i replacing 1_i^B for $1 \le i \le m$. Each of the paths R_1 and R_2 exists because, in each case, the domain and the codomain agree up to additive bracketings.

The lower left vertex in (7.4.2) is regular in the sense of Definition 3.1.25. Therefore, the upper left vertex is also regular by Lemma 3.1.29. Since C is assumed to be flat, the Coherence Theorem 3.9.1 implies that the two paths have the same value in C. $\square$

Next is the main result of this section that establishes the first half of Baez's Conjecture.

Proposition 7.4.4. *For each flat symmetric bimonoidal category* C*, the data*

$$(F, F^2_\oplus, F^0_\oplus, F^2_\otimes, F^0_\otimes) : \Sigma \longrightarrow \mathsf{C}$$

in Definitions 7.2.2 and 7.3.12 constitute a robust symmetric bimonoidal functor.

Proof. First recall the following facts.

- Lemma 7.2.9 shows that $F_\oplus = (F, F^2_\oplus, F^0_\oplus)$ is a strong symmetric monoidal functor.
- Lemma 7.3.28 shows that $F_\otimes = (F, F^2_\otimes, F^0_\otimes)$ is a symmetric monoidal functor.
- $F^0_\otimes$ is the identity morphism $1_{\mathbb{1}}$ by definition.

Therefore, it remains to check the two axioms in Definition 5.1.1.

The multiplicative zero axiom (5.1.2) holds because

- $F^0_\oplus : \mathbb{0} \longrightarrow \mathbb{0}$ is the identity morphism $1_{\mathbb{0}}$, and
- the right multiplicative zero $\rho^\bullet$ in Σ is the identity.

So each of the two composites in (5.1.2) is the same component of $\rho^\bullet$.

By Proposition 5.1.4, the distributivity axiom (5.1.3) is equivalent to the commutativity of the diagram (5.1.6). Since δ^l in Σ is the identity by Definition 2.4.1, the diagram (5.1.6) is the following diagram in C for $m, n, p \geq 0$.

(7.4.5)
$$\begin{tikzcd}
\overline{m} \otimes (\overline{n} \oplus \overline{p}) \arrow[rr, "\delta^l_{\overline{m},\overline{n},\overline{p}}"] \arrow[d, "1 \otimes F^2_\oplus"'] & & (\overline{m} \otimes \overline{n}) \oplus (\overline{m} \otimes \overline{p}) \arrow[d, "F^2_\otimes \oplus F^2_\otimes"] \\
\overline{m} \otimes \overline{n+p} \arrow[d, "F^2_\otimes"'] & & \overline{mn} \oplus \overline{mp} \arrow[d, "F^2_\oplus"] \\
\overline{m(n+p)} \arrow[rr, "F\delta^l", "="'] & & \overline{mn+mp}
\end{tikzcd}$$

To prove the commutativity of the diagram (7.4.5), consider the following four cases.

(1) If $m = 0$, then (7.4.5) is commutative by
  - the axiom (2.1.15) in C and
  - the naturality of $\lambda^\bullet$.

(2) If $n = 0$, then (7.4.5) factors as follows.

$$\begin{tikzcd}
\overline{m} \otimes (\mathbb{0} \oplus \overline{p}) \arrow[rr, "\delta^l_{\overline{m},\mathbb{0},\overline{p}}"] \arrow[d, "1 \otimes \lambda^\oplus_{\overline{p}}"'] & & (\overline{m} \otimes \mathbb{0}) \oplus (\overline{m} \otimes \overline{p}) \arrow[dl, "\rho^\bullet_{\overline{m}} \oplus 1"'] \arrow[d, "\rho^\bullet_{\overline{m}} \oplus F^2_\otimes"] \\
\overline{m} \otimes \overline{p} \arrow[d, "F^2_\otimes"'] & \mathbb{0} \oplus (\overline{m} \otimes \overline{p}) \arrow[l, "\lambda^\oplus_{\overline{m} \otimes \overline{p}}"'] \arrow[r, "1 \oplus F^2_\otimes"] & \mathbb{0} \oplus \overline{mp} \arrow[d, "\lambda^\oplus_{\overline{mp}}"] \\
\overline{mp} \arrow[rr, "="] & & \overline{mp}
\end{tikzcd}$$

- The upper left trapezoid is commutative by the axiom (2.1.23) in C.
- The upper right triangle is commutative by the functoriality of $\oplus$.
- The bottom rectangle is commutative by the naturality of $\lambda^\oplus$.

(3) If $p = 0$, then (7.4.5) factors as follows.

$$\begin{tikzcd}
\overline{m} \otimes (\overline{n} \oplus \mathbb{0}) \arrow[rr, "\delta^l_{\overline{m},\overline{n},\mathbb{0}}"] \arrow[d, "1\otimes\rho^{\oplus}_{\overline{n}}"'] & & (\overline{m} \otimes \overline{n}) \oplus (\overline{m} \otimes \mathbb{0}) \arrow[dl, "1\oplus\rho^{\bullet}_{\overline{m}}"'] \arrow[d, "F^2_{\otimes}\oplus\rho^{\bullet}_{\overline{m}}"] \\
\overline{m} \otimes \overline{n} \arrow[d, "F^2_{\otimes}"'] & (\overline{m} \otimes \overline{n}) \oplus \mathbb{0} \arrow[l, "\rho^{\oplus}_{\overline{m}\otimes\overline{n}}"'] \arrow[r, "F^2_{\otimes}\oplus 1"] & \overline{mn} \oplus \mathbb{0} \arrow[d, "\rho^{\oplus}_{\overline{mn}}"] \\
\overline{mn} \arrow[rr, "="] & & \overline{mn}
\end{tikzcd}$$

- The upper left trapezoid is commutative by the axiom (2.1.25) in C.
- The upper right triangle is commutative by the functoriality of $\oplus$.
- The bottom rectangle is commutative by the naturality of $\rho^{\oplus}$.

For the case $m, n, p > 0$, we use
- the description of $F^2_{\oplus}$ in Explanation 7.2.7 and
- the factorization of $F^2_{\otimes}$ in Lemma 7.3.21.

They imply that the diagram (7.4.5) is the outer diagram below.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m}\mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big[\Big(\bigoplus_{j=1}^{n}\mathbb{1}\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{k=1}^{p}\mathbb{1}\Big)_{\mathsf{lt}}\Big] \arrow[r, "\delta^l"] \arrow[d, "1\otimes F^2_{\oplus}"'] & \Big[\Big(\bigoplus_{i=1}^{m}\mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n}\mathbb{1}\Big)_{\mathsf{lt}}\Big] \oplus \Big[\Big(\bigoplus_{i=1}^{m}\mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{k=1}^{p}\mathbb{1}\Big)_{\mathsf{lt}}\Big] \arrow[d, "F'_{\otimes}\oplus 1"] \\
\Big(\bigoplus_{i=1}^{m}\mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n+p}\mathbb{1}\Big)_{\mathsf{lt}} \arrow[d, "F'_{\otimes}"'] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m}(\mathbb{1}\otimes\mathbb{1})\Big)_{\mathsf{lt}} \oplus \Big[\Big(\bigoplus_{i=1}^{m}\mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{k=1}^{p}\mathbb{1}\Big)_{\mathsf{lt}}\Big] \arrow[d, "1\oplus F'_{\otimes}"] \\
\Big(\bigoplus_{j=1}^{n+p}\bigoplus_{i=1}^{m}(\mathbb{1}\otimes\mathbb{1})\Big)_{\mathsf{lt}} \arrow[d, "(\oplus_j\oplus_i\lambda^{\otimes}_{\mathbb{1}})_{\mathsf{lt}}"'] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m}(\mathbb{1}\otimes\mathbb{1})\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{k=1}^{p}\bigoplus_{i=1}^{m}(\mathbb{1}\otimes\mathbb{1})\Big)_{\mathsf{lt}} \arrow[l, "\cong_{\mathsf{ML}}"'] \arrow[d, "(\oplus_j\oplus_i\lambda^{\otimes}_{\mathbb{1}})_{\mathsf{lt}}\oplus(\oplus_k\oplus_i\lambda^{\otimes}_{\mathbb{1}})_{\mathsf{lt}}"] \\
\Big(\bigoplus_{j=1}^{n+p}\bigoplus_{i=1}^{m}\mathbb{1}\Big)_{\mathsf{lt}} & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m}\mathbb{1}\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{k=1}^{p}\bigoplus_{i=1}^{m}\mathbb{1}\Big)_{\mathsf{lt}} \arrow[l, "F^2_{\oplus}"']
\end{tikzcd} \tag{7.4.6}$$

In this diagram, each of the two $F^2_{\oplus}$ and $\cong_{\mathsf{ML}}$ is the unique Mac Lane coherence isomorphism involving only $\alpha^{-\oplus}$ that moves the additive brackets. The naturality of Mac Lane coherence isomorphisms implies that the bottom rectangle in (7.4.6) is commutative.

It remains to show that the top square in (7.4.6) is commutative. In the context of Definition 7.4.1, consider the function $\varphi : B' \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as in (7.3.20) and the associated graph morphism $\varphi : \mathsf{Gr}(B') \longrightarrow \mathsf{C}$. Then the top square in (7.4.6) is the value in C of the diagram (7.4.2) in $\mathsf{Gr}(B')$. By Lemma 7.4.3, the top square in (7.4.6) is commutative. $\square$

7.5. Coherence of Symmetric Bimonoidal Functors

The purpose of this section is to prove a coherence property of symmetric bimonoidal functors. In Section 7.7, we will use this coherence property to prove that the canonical robust symmetric bimonoidal functor $F : \Sigma \longrightarrow \mathsf{C}$ in Proposition 7.4.4 is initial in a suitable sense. See Lemma 7.7.9 for the precise statement.

The desired coherence property is about the diagram (7.5.2) defined below. It is used in the proof of Lemma 7.7.6. Recall from (5.2.13) that the subscript lt means left normalized bracketing.

Definition 7.5.1. Suppose given the following data.

- C and D are flat symmetric bimonoidal categories, and

$$(G, G^2_\oplus, G^0_\oplus, G^2_\otimes, G^0_\otimes) : \mathsf{C} \longrightarrow \mathsf{D}$$

  is a symmetric bimonoidal functor.
- $A_i, B_j \in \mathsf{C}$ are objects for $1 \le i \le m$ and $1 \le j \le n$ with $m, n \ge 1$.

Define the following diagram in D.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^m GA_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n GB_j\Big)_{\mathsf{lt}} \arrow[r, "p_{m,n}"] \arrow[d, "G_\oplus \otimes G_\oplus"'] & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m (GA_i \otimes GB_j)\Big)_{\mathsf{lt}} \arrow[d, "(\oplus_j \oplus_i G^2_\otimes)_{\mathsf{lt}}"] \\
G\Big(\bigoplus_{i=1}^m A_i\Big)_{\mathsf{lt}} \otimes G\Big(\bigoplus_{j=1}^n B_j\Big)_{\mathsf{lt}} \arrow[d, "G^2_\otimes"'] & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m G(A_i \otimes B_j)\Big)_{\mathsf{lt}} \arrow[d, "G_\oplus"] \\
G\Big[\Big(\bigoplus_{i=1}^m A_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n B_j\Big)_{\mathsf{lt}}\Big] \arrow[r, "Gq_{m,n}"] & G\Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m (A_i \otimes B_j)\Big)_{\mathsf{lt}}
\end{tikzcd} \tag{7.5.2}$$

Consider (7.5.2).

(1) Each of the three morphisms $G_\oplus$ is a composite of morphisms, each being the sum of identity morphisms and at most one component of the additive monoidal constraint $G^2_\oplus$.

(2) The morphism

$$\Big(\bigoplus_{i=1}^m GA_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n GB_j\Big)_{\mathsf{lt}} \xrightarrow{p_{m,n}} \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m (GA_i \otimes GB_j)\Big)_{\mathsf{lt}} \tag{7.5.3}$$

is the value in D of the path Q in (7.3.19), taken with respect to the function $\varphi^p : B' \longrightarrow \mathsf{Ob}(\mathsf{D})$ defined by

$$\varphi^p(x) = \begin{cases} \mathbb{0}^{\mathsf{D}} & \text{if } x = 0^{B'}, \\ \mathbb{1}^{\mathsf{D}} & \text{if } x = 1^{B'}, \\ GA_i & \text{if } x = a_i \text{ for } 1 \le i \le m, \text{ and} \\ GB_j & \text{if } x = b_j \text{ for } 1 \le j \le n. \end{cases}$$

(3) The morphism

$$\Big(\bigoplus_{i=1}^m A_i\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n B_j\Big)_{\mathsf{lt}} \xrightarrow{q_{m,n}} \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m (A_i \otimes B_j)\Big)_{\mathsf{lt}} \tag{7.5.4}$$

is the value in C of the path Q in (7.3.19), taken with respect to the function $\varphi^q : B' \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by

$$\varphi^q(x) = \begin{cases} \mathbb{0}^{\mathsf{C}} & \text{if } x = 0^{B'}, \\ \mathbb{1}^{\mathsf{C}} & \text{if } x = 1^{B'}, \\ A_i & \text{if } x = a_i \text{ for } 1 \le i \le m, \text{ and} \\ B_j & \text{if } x = b_j \text{ for } 1 \le j \le n. \end{cases}$$

The finishes the definition of the diagram (7.5.2). ◇

Explanation 7.5.5. Consider Definition 7.5.1.

(1) Consider the morphism

$$\Big(\bigoplus_{i=1}^{m} GA_i\Big)_{\mathsf{lt}} \xrightarrow{G_\oplus} G\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}} \in \mathsf{D}$$

in the diagram (7.5.2).

- If $m = 1$, then $G_\oplus$ is the identity morphism 1_{GA_1}.
- Inductively, if $m \ge 2$, then $G_\oplus$ is the composite below.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m} GA_i\Big)_{\mathsf{lt}} \arrow[d, equal] & G\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}} \\
\Big(\bigoplus_{i=1}^{m-1} GA_i\Big)_{\mathsf{lt}} \oplus GA_m \arrow[r, "G_\oplus \oplus 1"] & G\Big(\bigoplus_{i=1}^{m-1} A_i\Big)_{\mathsf{lt}} \oplus GA_m \arrow[u, "G^2_\oplus"']
\end{tikzcd} \tag{7.5.6}$$

For example, if $m = 2$, then $G_\oplus$ is the additive monoidal constraint

$$GA_1 \oplus GA_2 \xrightarrow{G^2_\oplus} G(A_1 \oplus A_2).$$

The flatness of C and D are *not* needed to define the morphism $G_\oplus$. Moreover, if $G^2_\oplus$ is an isomorphism, then so is $G_\oplus$.

(2) We use the symbol $G_\oplus$ for both the additive structure $(G, G^2_\oplus, G^0_\oplus)$ of G and the morphism in the diagram (7.5.2). This should not cause any confusion, since the morphism $G_\oplus$ is entirely made up of the additive monoidal constraint $G^2_\oplus$ and identity morphisms.

(3) In the definitions of $p_{m,n}$ and $q_{m,n}$, the existence of the path Q is explained in the first paragraph of the proof of Lemma 7.3.21. The Coherence Theorem 4.4.3, which is applicable by the flatness assumption of C and D, ensures that the value of Q in each of C and D is unique regardless of how Q is chosen. ◇

The objective of this section is to show that the diagram (7.5.2) is commutative. First we consider the following preliminary case.

Lemma 7.5.7. *If $n = 1$, then the diagram (7.5.2) is commutative.*

Proof. We proceed by induction on $m \ge 1$. If $m = 1$, then all three $G_\oplus$'s, q, and g are identity morphisms. So both composites in the diagram (7.5.2) are the morphism

$$GA_1 \otimes GB_1 \xrightarrow{G^2_\otimes} G(A_1 \otimes B_1) \in \mathsf{D}.$$

Inductively, suppose $m \geq 2$. To save space below, we write B_1 as B, $G?$ as $?^G$, and $\otimes$ as concatenation, with $\otimes$ taking precedence over $\oplus$. For example, the upper left corner in (7.5.2) is written as

$$\Big(\bigoplus_{i=1}^{m} A_i^G\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n} B_j^G\Big)_{\mathsf{lt}}.$$

With these abbreviations, the diagram (7.5.2) with $n = 1$ is the outer diagram below.

$$\begin{tikzcd}
\Big(\bigoplus\limits_{i=1}^{m} A_i^G\Big)_{\mathsf{lt}} B^G \arrow[r, "\delta^r"] \arrow[d, "(G_\oplus \oplus 1)1"'] & \Big(\bigoplus\limits_{i=1}^{m-1} A_i^G\Big)_{\mathsf{lt}} B^G \oplus A_m^G B^G \arrow[r, "p_{m-1,1} \oplus 1"] \arrow[d, "G_\oplus 1 \oplus 1"] & \Big(\bigoplus\limits_{i=1}^{m-1} A_i^G B^G\Big)_{\mathsf{lt}} \oplus A_m^G B^G \arrow[d, "(\oplus_i G_\otimes^2)_{\mathsf{lt}} \oplus G_\otimes^2"] \\
\Big[\Big(\bigoplus\limits_{i=1}^{m-1} A_i\Big)_{\mathsf{lt}}^G \oplus A_m^G\Big] B^G \arrow[r, "\delta^r"] \arrow[d, "G_\oplus^2 1"'] & \Big(\bigoplus\limits_{i=1}^{m-1} A_i\Big)_{\mathsf{lt}}^G B^G \oplus A_m^G B^G \arrow[d, "G_\otimes^2 \oplus 1"] & \Big(\bigoplus\limits_{i=1}^{m-1} (A_i B)^G\Big)_{\mathsf{lt}} \oplus (A_m B)^G \arrow[d, "G_\oplus \oplus 1"] \\
\Big(\bigoplus\limits_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G B^G \arrow[dd, "G_\otimes^2"'] & \Big[\Big(\bigoplus\limits_{i=1}^{m-1} A_i\Big)_{\mathsf{lt}} B\Big]^G \oplus A_m^G B^G \arrow[r, "q_{m-1,1}^G \oplus G_\otimes^2"] \arrow[d, "1 \oplus G_\otimes^2"] & \Big(\bigoplus\limits_{i=1}^{m-1} A_i B\Big)_{\mathsf{lt}}^G \oplus (A_m B)^G \arrow[dd, "G_\oplus^2"] \\
& \Big[\Big(\bigoplus\limits_{i=1}^{m-1} A_i\Big)_{\mathsf{lt}} B\Big]^G \oplus (A_m B)^G \arrow[ur, "q_{m-1,1}^G \oplus 1"'] \arrow[d, "G_\oplus^2"] & \\
\Big[\Big(\bigoplus\limits_{i=1}^{m} A_i\Big)_{\mathsf{lt}} B\Big]^G \arrow[r, "G\delta^r"] & \Big[\Big(\bigoplus\limits_{i=1}^{m-1} A_i\Big)_{\mathsf{lt}} B \oplus A_m B\Big]^G \arrow[r, "{[q_{m-1,1} \oplus 1]^G}"] & \Big(\bigoplus\limits_{i=1}^{m} A_i B\Big)_{\mathsf{lt}}^G
\end{tikzcd}$$

Consider the previous diagram.

- The upper left rectangle is commutative by the naturality of δ^r.
- The lower left rectangle is commutative by the distributivity axiom (5.1.3) for G and the functoriality of $\oplus$.
- The upper right square is commutative by the induction hypothesis and the functoriality of $\oplus$.
- The middle right triangle is commutative by the functoriality of $\oplus$.
- The lower right trapezoid is commutative by the naturality of $G_\oplus^2$.

This finishes the induction step. □

Theorem 7.5.8. *The diagram (7.5.2) is commutative for $m, n \geq 1$.*

Proof. We proceed by induction on $n \geq 1$. The case $n = 1$ is proved in Lemma 7.5.7.

Inductively, suppose $n \geq 2$. Using the notation in the proof of Lemma 7.5.7, we factor the diagram (7.5.2) into six subdiagrams as follows.

$$
\begin{tikzcd}[column sep=small]
\Big(\bigoplus_{i=1}^{m} A_i^G\Big)_{\mathsf{lt}}\Big[\Big(\bigoplus_{j=1}^{n-1} B_j^G\Big)_{\mathsf{lt}} \oplus B_n^G\Big] \arrow[r, "\delta^l"] \arrow[d, "G_\oplus(G_\oplus \oplus 1)"'] \arrow[dr, phantom, "\boxed{\mathrm{I}}"] & \bullet \arrow[r, "p_{m,n-1} \oplus p_{m,1}"] \arrow[d] & \bullet \arrow[r, "R_1"] \arrow[d] \arrow[dr, phantom, "\boxed{\mathrm{V}}"] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} A_i^G B_j^G\Big)_{\mathsf{lt}} \arrow[d, "(\oplus_j \oplus_i G_\otimes^2)_{\mathsf{lt}}"] \\
\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G\Big[\Big(\bigoplus_{j=1}^{n-1} B_j\Big)_{\mathsf{lt}}^G \oplus B_n^G\Big] \arrow[r] \arrow[d, "1G_\oplus^2"'] \arrow[dr, phantom, "\boxed{\mathrm{II}}"] & \bullet \arrow[d] \arrow[r, phantom, "\boxed{\mathrm{III}}"] & \bullet \arrow[r] \arrow[d] \arrow[ddr, phantom, "\boxed{\mathrm{VI}}"] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} (A_i B_j)^G\Big)_{\mathsf{lt}} \arrow[dd, "G_\oplus"] \\
\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G\Big(\bigoplus_{j=1}^{n} B_j\Big)_{\mathsf{lt}}^G \arrow[d, "G_\otimes^2"'] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\mathrm{IV}}"] & \bullet \arrow[d] & \\
\Big[\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n} B_j\Big)_{\mathsf{lt}}\Big]^G \arrow[r, "G\delta^l"] & \bullet \arrow[r, "{[q_{m,n-1} \oplus q_{m,1}]^G}"] & \bullet \arrow[r, "R_2^G"] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} A_i B_j\Big)_{\mathsf{lt}}^G
\end{tikzcd}
$$

The subdiagrams $\boxed{\mathrm{I}}$ and $\boxed{\mathrm{II}}$ are as follows.

$$
\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m} A_i^G\Big)_{\mathsf{lt}}\Big[\Big(\bigoplus_{j=1}^{n-1} B_j^G\Big)_{\mathsf{lt}} \oplus B_n^G\Big] \arrow[r, "\delta^l"] \arrow[d, "G_\oplus(G_\oplus \oplus 1)"] \arrow[dr, phantom, "\boxed{\mathrm{I}}"] & \Big(\bigoplus_{i=1}^{m} A_i^G\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n-1} B_j^G\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} A_i^G\Big)_{\mathsf{lt}} B_n^G \arrow[d, "G_\oplus G_\oplus \oplus G_\oplus 1"'] \\
\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G\Big[\Big(\bigoplus_{j=1}^{n-1} B_j\Big)_{\mathsf{lt}}^G \oplus B_n^G\Big] \arrow[r, "\delta^l"] \arrow[d, "1G_\oplus^2"] & \Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G\Big(\bigoplus_{j=1}^{n-1} B_j\Big)_{\mathsf{lt}}^G \oplus \Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G B_n^G \arrow[d, "G_\otimes^2 \oplus G_\otimes^2"'] \\
\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G\Big(\bigoplus_{j=1}^{n} B_j\Big)_{\mathsf{lt}}^G \arrow[d, "G_\otimes^2"] \arrow[r, phantom, "\boxed{\mathrm{II}}"] & \Big[\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n-1} B_j\Big)_{\mathsf{lt}}\Big]^G \oplus \Big[\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}} B_n\Big]^G \arrow[d, "G_\oplus^2"'] \\
\Big[\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n} B_j\Big)_{\mathsf{lt}}\Big]^G \arrow[r, "G\delta^l"] & \Big[\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n-1} B_j\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}} B_n\Big]^G
\end{tikzcd}
$$

The subdiagram $\boxed{\mathrm{I}}$ is commutative by the naturality of δ^l. The subdiagram $\boxed{\mathrm{II}}$ is commutative by (5.1.6), which is equivalent to the distributivity axiom (5.1.3) for G.

The subdiagrams $\boxed{\mathrm{III}}$ and $\boxed{\mathrm{IV}}$ are as follows.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m} A_i^G\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n-1} B_j^G\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} A_i^G\Big)_{\mathsf{lt}} B_n^G \arrow[r, "p_{m,n-1}\oplus p_{m,1}"] \arrow[d, "G_\oplus G_\oplus \oplus G_\oplus 1"'] & \Big(\bigoplus_{j=1}^{n-1}\bigoplus_{i=1}^{m} A_i^G B_j^G\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} A_i^G B_n^G\Big)_{\mathsf{lt}} \arrow[d, "(\oplus_j \oplus_i G_\otimes^2)_{\mathsf{lt}} \oplus (\oplus_i G_\otimes^2)_{\mathsf{lt}}"'] \\
\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G\Big(\bigoplus_{j=1}^{n-1} B_j\Big)_{\mathsf{lt}}^G \oplus \Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}^G B_n^G \arrow[r, phantom, "\boxed{\mathrm{III}}"] \arrow[d, "G_\otimes^2 \oplus G_\otimes^2"'] & \Big(\bigoplus_{j=1}^{n-1}\bigoplus_{i=1}^{m} (A_i B_j)^G\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} (A_i B_n)^G\Big)_{\mathsf{lt}} \arrow[d, "G_\oplus \oplus G_\oplus"'] \\
\Big[\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n-1} B_j\Big)_{\mathsf{lt}}\Big]^G \oplus \Big[\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}} B_n\Big]^G \arrow[r, "q_{m,n-1}^G \oplus q_{m,1}^G"] \arrow[d, "G_\oplus^2"'] \arrow[dr, phantom, "\boxed{\mathrm{IV}}"] & \Big(\bigoplus_{j=1}^{n-1}\bigoplus_{i=1}^{m} A_i B_j\Big)_{\mathsf{lt}}^G \oplus \Big(\bigoplus_{i=1}^{m} A_i B_n\Big)_{\mathsf{lt}}^G \arrow[d, "G_\oplus^2"'] \\
\Big[\Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}}\Big(\bigoplus_{j=1}^{n-1} B_j\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} A_i\Big)_{\mathsf{lt}} B_n\Big]^G \arrow[r, "{[q_{m,n-1}\oplus q_{m,1}]^G}"] & \Big[\Big(\bigoplus_{j=1}^{n-1}\bigoplus_{i=1}^{m} A_i B_j\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} A_i B_n\Big)_{\mathsf{lt}}\Big]^G
\end{tikzcd}$$

The subdiagram $\boxed{\mathrm{III}}$ is commutative by

- the induction hypothesis for the summands involving $p_{m,n-1}$ and $q^G_{m,n-1}$;
- the case $n = 1$ in Lemma 7.5.7 for the summands involving $p_{m,1}$ and $q^G_{m,1}$; and
- the functoriality of $\oplus$.

The subdiagram $\boxed{\mathrm{IV}}$ is commutative by the naturality of $G^2_\oplus$.

The subdiagrams $\boxed{\mathrm{V}}$ and $\boxed{\mathrm{VI}}$ are as follows.

$$\begin{tikzcd}
\Big(\bigoplus_{j=1}^{n-1}\bigoplus_{i=1}^{m} A_i^G B_j^G\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} A_i^G B_n^G\Big)_{\mathsf{lt}} \arrow[r, "R_1"] \arrow[d, "(\oplus_j \oplus_i G_\otimes^2)_{\mathsf{lt}} \oplus (\oplus_i G_\otimes^2)_{\mathsf{lt}}"'] \arrow[dr, phantom, "\boxed{\mathrm{V}}"] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} A_i^G B_j^G\Big)_{\mathsf{lt}} \arrow[d, "(\oplus_j \oplus_i G_\otimes^2)_{\mathsf{lt}}"'] \\
\Big(\bigoplus_{j=1}^{n-1}\bigoplus_{i=1}^{m} (A_i B_j)^G\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} (A_i B_n)^G\Big)_{\mathsf{lt}} \arrow[r, "R"] \arrow[d, "G_\oplus \oplus G_\oplus"'] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} (A_i B_j)^G\Big)_{\mathsf{lt}} \arrow[dd, "G_\oplus"'] \\
\Big(\bigoplus_{j=1}^{n-1}\bigoplus_{i=1}^{m} A_i B_j\Big)_{\mathsf{lt}}^G \oplus \Big(\bigoplus_{i=1}^{m} A_i B_n\Big)_{\mathsf{lt}}^G \arrow[d, "G_\oplus^2"'] \arrow[r, phantom, "\boxed{\mathrm{VI}}"] & \\
\Big[\Big(\bigoplus_{j=1}^{n-1}\bigoplus_{i=1}^{m} A_i B_j\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{i=1}^{m} A_i B_n\Big)_{\mathsf{lt}}\Big]^G \arrow[r, "R_2^G"] & \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} A_i B_j\Big)_{\mathsf{lt}}^G
\end{tikzcd}$$

Each of R_1, R, and R_2 is a composite of morphisms, each being the sum of identity morphisms and at most one component of $\alpha^{-\oplus}$. Each of them exists because in each case the domain and the codomain agree up to additive bracketings. The subdiagram $\boxed{\mathrm{V}}$ is commutative by the naturality of $\alpha^{\oplus}$. The subdiagram $\boxed{\mathrm{VI}}$ is commutative by Epstein's Coherence Theorem 1.3.12 for the additive structure $(G, G^2_\oplus)$. This finishes the induction step. □

7.6. Uniqueness of 2-Cells

In this section, we show that for each symmetric bimonoidal category C and robust symmetric bimonoidal functors $G, H : \Sigma \longrightarrow \mathsf{C}$, there is at most one bimonoidal natural transformation $G \longrightarrow H$. Moreover, such a bimonoidal natural transformation must be an isomorphism. The main part is Lemma 7.6.2, where we describe explicitly the components of such a bimonoidal natural transformation.

Notation 7.6.1. Recall from Definition 5.1.1 that a robust symmetric bimonoidal functor G has invertible structure morphisms $G^2_\oplus$, $G^0_\oplus$, and $G^0_\otimes$.

- We write the inverses of $G^2_\oplus$, $G^0_\oplus$, and $G^0_\otimes$ as, respectively, $G^{-2}_\oplus$, $G^{-0}_\oplus$, and $G^{-0}_\otimes$.
- The inverse of the morphism $G_\oplus$ in Definition 7.5.1 and Explanation 7.5.5 is denoted by $G^{-1}_\oplus$. ⋄

Recall from Definition 2.4.18 that the objects in Σ are $n \geq 0$.

Lemma 7.6.2. *Suppose given the following data.*

- C *is a symmetric bimonoidal category as in Definition 2.1.2.*
- $G, H : \Sigma \longrightarrow \mathsf{C}$ *are symmetric bimonoidal functors with G robust.*
- $\pi : G \longrightarrow H$ *is a bimonoidal natural transformation as in Definition 7.1.2.*

Then the following statements hold.

(1) π_0 *is the following composite in* C.

$$\begin{tikzcd} G(0) \arrow[r, "G^{-0}_\oplus"] \arrow[rr, bend left, "\pi_0"] & \mathbb{0} \arrow[r, "H^0_\oplus"] & H(0) \end{tikzcd}$$

(2) π_1 *is the following composite in* C.

$$\begin{tikzcd} G(1) \arrow[r, "G^{-0}_\otimes"] \arrow[rr, bend left, "\pi_1"] & \mathbb{1} \arrow[r, "H^0_\otimes"] & H(1) \end{tikzcd}$$

(3) *For each $n \geq 2$, π_n is the following composite in* C.

$$\begin{tikzcd} G(n) \arrow[r, "\pi_n"] \arrow[d, "G^{-1}_\oplus"'] & H(n) \\ \Big(\bigoplus_{i=1}^n G(1)\Big)_{\mathsf{lt}} \arrow[r, "(\oplus_i \pi_1)_{\mathsf{lt}}"] & \Big(\bigoplus_{i=1}^n H(1)\Big)_{\mathsf{lt}} \arrow[u, "H_\oplus"'] \end{tikzcd}$$

Proof. The first two assertions for π_0 and π_1 follow from

- the right diagrams in (7.1.4) and (7.1.5), respectively, for the monoidal natural transformations $\pi : G_\oplus \longrightarrow H_\oplus$ and $\pi : G_\otimes \longrightarrow H_\otimes$; and
- the invertibility of $G^0_\oplus$ and $G^0_\otimes$.

The last assertion about π_n is proved by induction on $n \geq 2$. The case $n = 2$ follows from

- the left diagram in (7.1.4),

- the invertibility of $G_{\oplus}^{2}$, and
- the fact that $2 = 1 \oplus 1$ in Σ.

Inductively, suppose $n > 2$. In Σ, $n = (n-1) \oplus 1$. Consider the following diagram in C.

$$\begin{tikzcd}
G(n) \arrow[rr, "\pi_n"] \arrow[d, "G_{\oplus}^{-2}"] \arrow[ddd, bend right=60, "G_{\oplus}^{-1}"'] && H(n) \\
G(n-1) \oplus G(1) \arrow[rr, "\pi_{n-1} \oplus \pi_1"] \arrow[d, "G_{\oplus}^{-1} \oplus 1"] && H(n-1) \oplus H(1) \arrow[u, "H_{\oplus}^{2}"'] \\
\Big(\bigoplus_{i=1}^{n-1} G(1)\Big)_{\mathsf{lt}} \oplus G(1) \arrow[rr, "\big(\bigoplus_{i=1}^{n-1} \pi_1\big)_{\mathsf{lt}} \oplus \pi_1"] \arrow[d, "1"] && \Big(\bigoplus_{i=1}^{n-1} H(1)\Big)_{\mathsf{lt}} \oplus H(1) \arrow[u, "H_{\oplus} \oplus 1"'] \\
\Big(\bigoplus_{i=1}^{n} G(1)\Big)_{\mathsf{lt}} \arrow[rr, "\big(\bigoplus_{i=1}^{n} \pi_1\big)_{\mathsf{lt}}"] && \Big(\bigoplus_{i=1}^{n} H(1)\Big)_{\mathsf{lt}} \arrow[u, "1"'] \arrow[uuu, bend right=60, "H_{\oplus}"']
\end{tikzcd}$$

Consider the previous diagram.

- The top rectangle is commutative by the left diagram in (7.1.4) and the invertibility of $G_{\oplus}^{2}$.
- The middle rectangle is commutative by the induction hypothesis for the summands involving π_{n-1} and the functoriality of $\oplus$.
- The bottom rectangle is commutative by the definition (5.2.13) of the left normalized bracketing.
- By (7.5.6), the left and the right vertical composites are, respectively, $G_{\oplus}^{-1}$ and $H_{\oplus}$.

This finishes the induction step. $\square$

Lemma 7.6.3. *In the context of Lemma 7.6.2, the following statements hold.*

(1) π is the only bimonoidal natural transformation $G \longrightarrow H$.
(2) If H is also robust, then π is a bimonoidal natural isomorphism.

Proof. The first assertion follows from Lemma 7.6.2 because each component of π is uniquely expressed in terms of the structure morphisms of G and H. The second assertion holds because, if H is also robust, then each component of π is a composite of isomorphisms. $\square$

7.7. Initial 1-Cell

The purpose of this section is to establish the second half of Baez's Conjecture. In Lemma 7.7.9, we observe that the robust symmetric bimonoidal functor $F : \Sigma \longrightarrow \mathsf{C}$ in Definitions 7.2.2 and 7.3.12 and Proposition 7.4.4 is initial in a suitable sense. To prove the initial property of F, first we define the components of the expected unique bimonoidal natural transformation.

Definition 7.7.1. Suppose

- C is a flat symmetric bimonoidal category and
- $G : \Sigma \longrightarrow \mathsf{C}$ is a symmetric bimonoidal functor.

Define $\theta^G : F \longrightarrow G$ with the component morphisms

$$F(n) = \overline{n} \xrightarrow{\theta^G_n} G(n) \in \mathsf{C}$$

for $n \geq 0$ as follows.

- θ^G_0 is the additive zero constraint

$$F(0) = \mathbb{0} \xrightarrow{G^0_\oplus} G(0).$$

- θ^G_1 is the multiplicative unit constraint

$$F(1) = \mathbb{1} \xrightarrow{G^0_\otimes} G(1).$$

- For each $n \geq 2$, θ^G_n is the following composite in C, with $G_\oplus$ the morphism in (7.5.6).

$$\begin{tikzcd}
F(n) = \overline{n} \arrow[r, "\theta^G_n"] \arrow[d, equal] & G(n) \\
\Big(\bigoplus_{j=1}^n \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "(\oplus_j G^0_\otimes)_{\mathsf{lt}}"] & \Big(\bigoplus_{j=1}^n G(1)\Big)_{\mathsf{lt}} \arrow[u, "G_\oplus"']
\end{tikzcd}$$

This finishes the definition of θ^G. ◇

Explanation 7.7.2. The components of θ^G in Definition 7.7.1 are the results of applying the three diagrams in Lemma 7.6.2 to F and G because of the following statements.

(1) $F^0_\oplus$ is the identity morphism $1_{\mathbb{0}}$ by (7.2.5).
(2) $F^0_\otimes$ is the identity morphism $1_{\mathbb{1}}$ by (7.3.13).
(3) For each $n \geq 2$, the morphism

$$\Big(\bigoplus_{j=1}^n F(1)\Big)_{\mathsf{lt}} = \overline{n} \xrightarrow{F_\oplus} \overline{n} = F(n)$$

is the identity morphism $1_{\overline{n}}$ by (7.2.8) and Explanation 7.5.5. ◇

We now show that θ^G is a bimonoidal natural transformation. To clarify the argument, we separate the proof into several lemmas. Recall from Definition 2.4.1 that morphisms in Σ are permutations.

Lemma 7.7.3. *$\theta^G : F \longrightarrow G$ in Definition 7.7.1 is a natural transformation.*

Proof. The naturality of θ^G with respect to morphisms in $\Sigma(n,n) = \Sigma_n$ for $n = 0, 1$ holds because Σ_0 and Σ_1 are the trivial group.

Suppose $\sigma \in \Sigma_n$ is a permutation for some $n \geq 2$. The naturality of θ^G with respect to σ is the outer diagram below.

$$\begin{tikzcd}
F(n) = \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "F(\sigma)"] \arrow[d, "(\oplus_j G^0_\otimes)_{\mathsf{lt}}"'] & \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} = F(n) \arrow[d, "(\oplus_j G^0_\otimes)_{\mathsf{lt}}"] \\
\Big(\bigoplus_{j=1}^{n} G(1)\Big)_{\mathsf{lt}} \arrow[r, "\sigma"] \arrow[d, "G_\oplus"'] & \Big(\bigoplus_{j=1}^{n} G(1)\Big)_{\mathsf{lt}} \arrow[d, "G_\oplus"] \\
G(n) \arrow[r, "G(\sigma)"] & G(n)
\end{tikzcd}$$

Consider the previous diagram.

- In the top square, each of $F(\sigma)$ and σ is the unique Mac Lane coherence isomorphism in $(\mathsf{C}, \oplus)$ that additively permutes the n copies of $\mathbb{1}$, or of $G(1)$, as $\sigma \in \Sigma_n$ permutes n letters. The top square is commutative by the naturality of $\alpha^\oplus$ and $\xi^\oplus$ in C.
- In the bottom square, since

$$n = \Big(\bigoplus_{j=1}^{n} 1\Big)_{\mathsf{lt}} \in \Sigma,$$

by (7.5.6), each of the two composites is the same component of a G-coherent map in the sense of Definition 1.3.11. Therefore, the bottom square is commutative by Epstein's Coherence Theorem 1.3.12 for the symmetric monoidal functor $(G, G^2_\oplus) : \Sigma \longrightarrow \mathsf{C}$.

Therefore, θ^G is a natural transformation. □

Lemma 7.7.4. *In Definition 7.7.1,*

$$\theta^G : (F, F^2_\oplus, F^0_\oplus) \longrightarrow (G, G^2_\oplus, G^0_\oplus)$$

is a monoidal natural transformation.

Proof. By Lemma 7.7.3, it remains to check the two axioms in Definition 1.2.16 for the additive structures. The right diagram in (1.2.17) is commutative by the definition of $\theta^G_0 = G^0_\oplus$, since $F^0_\oplus : \mathbb{0} \longrightarrow F(0)$ is the identity morphism $1_\mathbb{0}$.

The left diagram in (1.2.17) is the following diagram in C for $m, n \geq 0$.

$$\begin{tikzcd}
F(m) \oplus F(n) = \overline{m} \oplus \overline{n} \arrow[r, "\theta^G_m \oplus \theta^G_n"] \arrow[d, "F^2_\oplus"'] & G(m) \oplus G(n) \arrow[d, "G^2_\oplus"] \\
F(m+n) = \overline{m+n} \arrow[r, "\theta^G_{m+n}"] & G(m+n)
\end{tikzcd} \tag{7.7.5}$$

To prove its commutativity, we consider all the possible cases.

If $m = 0$, then (7.7.5) is the outer diagram below.

$$\begin{tikzcd}
\mathbb{0} \oplus \overline{n} \arrow[rr, "G^0_\oplus \oplus \theta^G_n"] \arrow[dr, "1\oplus\theta^G_n"] \arrow[dd, "\lambda^\oplus_{\overline{n}}"'] & & G(0) \oplus G(n) \arrow[dd, "G^2_\oplus"] \\
& \mathbb{0} \oplus G(n) \arrow[ur, "G^0_\oplus \oplus 1"] \arrow[dr, "\lambda^\oplus_{G(n)}"] & \\
\overline{n} \arrow[rr, "\theta^G_n"] & & G(n)
\end{tikzcd}$$

Starting from the top and going counterclockwise, the three triangles in the diagram above are commutative by, respectively,

- the functoriality of $\oplus$,
- the naturality of $\lambda^\oplus$ in C, and
- the left unity axiom (1.2.15) for $(G, G^2_\oplus, G^0_\oplus)$ and the fact that $\lambda^\oplus$ is the identity in Σ.

If $n = 0$, then (7.7.5) decomposes analogously. The resulting diagram is commutative by the functoriality of $\oplus$, the naturality of $\rho^\oplus$ in C, the right unity axiom (1.2.15) for $(G, G^2_\oplus, G^0_\oplus)$, and the fact that $\rho^\oplus$ is the identity in Σ.

If $m, n \geq 1$, then (7.7.5) is the outer diagram below.

$$\begin{tikzcd}
\Big(\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "(\oplus_i G^0_\otimes)_{\mathsf{lt}} \oplus (\oplus_j G^0_\otimes)_{\mathsf{lt}}"] \arrow[d, "F^2_\oplus"'] & \Big(\bigoplus_{i=1}^{m} G(1)\Big)_{\mathsf{lt}} \oplus \Big(\bigoplus_{j=1}^{n} G(1)\Big)_{\mathsf{lt}} \arrow[r, "G_\oplus \oplus G_\oplus"] \arrow[d, "\cong_{\mathsf{ML}}"] & G(m) \oplus G(n) \arrow[d, "G^2_\oplus"] \\
\Big(\bigoplus_{k=1}^{m+n} \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "(\oplus_k G^0_\otimes)_{\mathsf{lt}}"] & \Big(\bigoplus_{k=1}^{m+n} G(1)\Big)_{\mathsf{lt}} \arrow[r, "G_\oplus"] & G(m+n)
\end{tikzcd}$$

Consider the previous diagram.

- Both $F^2_\oplus$ and $\cong_{\mathsf{ML}}$ are the unique Mac Lane coherence isomorphism in $(\mathsf{C}, \oplus)$ that involve only $\alpha^{-\oplus}$ and move the additive brackets in the same way. The left square is commutative by the naturality of $\alpha^\oplus$.
- In the right square, if $m = 1$, then the top left $G_\oplus$ is the identity morphism. If $n = 1$, then the top right $G_\oplus$ is the identity morphism. This square is commutative by Epstein's Coherence Theorem 1.3.12 for the symmetric monoidal functor $(G, G^2_\oplus) : \Sigma \longrightarrow \mathsf{C}$.

This finishes the proof of the commutativity of (7.7.5). $\square$

Lemma 7.7.6. *In Definition 7.7.1,*

$$\theta^G : (F, F^2_\otimes, F^0_\otimes) \longrightarrow (G, G^2_\otimes, G^0_\otimes)$$

is a monoidal natural transformation.

Proof. By Lemma 7.7.3, it remains to check the two axioms in Definition 1.2.16 for the multiplicative structures. The right diagram in (1.2.17) is commutative by the definition of $\theta^G_1 = G^0_\otimes$, since $F^0_\otimes : \mathbb{1} \longrightarrow F(1)$ is the identity morphism $1_{\mathbb{1}}$.

The left diagram in (1.2.17) is the following diagram in C for $m, n \geq 0$.

(7.7.7)
$$\begin{tikzcd}
F(m) \otimes F(n) = \overline{m} \otimes \overline{n} \arrow[r, "\theta^G_m \otimes \theta^G_n"] \arrow[d, "F^2_\otimes"'] & G(m) \otimes G(n) \arrow[d, "G^2_\otimes"] \\
F(mn) = \overline{mn} \arrow[r, "\theta^G_{mn}"] & G(mn)
\end{tikzcd}$$

To prove its commutativity, we consider all the possible cases.

If $m = 0$, then (7.7.7) is the outer diagram below.

$$\begin{tikzcd}
\mathbb{0} \otimes \overline{n} \arrow[rr, "G^0_\oplus \otimes \theta^G_n"] \arrow[dr, "1 \otimes \theta^G_n"] \arrow[dd, "\lambda^\bullet_{\overline{n}}"'] & & G(0) \otimes G(n) \arrow[dd, "G^2_\otimes"] \\
& \mathbb{0} \otimes G(n) \arrow[ur, "G^0_\oplus \otimes 1"] \arrow[dl, "\lambda^\bullet_{G(n)}"] & \\
\mathbb{0} \arrow[rr, "G^0_\oplus"] & & G(0)
\end{tikzcd}$$

Starting from the top and going counterclockwise, the three triangles in the diagram above are commutative by, respectively,

- the functoriality of $\otimes$,
- the naturality of $\lambda^\bullet$ in C, and
- (5.1.5), which is equivalent to the multiplicative zero axiom (5.1.2) for G, and the fact that $\lambda^\bullet$ is the identity in Σ.

If $n = 0$, then (7.7.7) decomposes analogously. The resulting diagram is commutative by the functoriality of $\otimes$, the naturality of $\rho^\bullet$ in C, the axiom (5.1.2) for G, and the fact that $\rho^\bullet$ is the identity in Σ.

If $m = n = 1$, then (7.7.7) is the outer diagram below.

(7.7.8)
$$\begin{tikzcd}
\mathbb{1} \otimes \mathbb{1} \arrow[rr, "G^0_\otimes \otimes G^0_\otimes"] \arrow[dr, "1 \otimes G^0_\otimes"] \arrow[dd, "\lambda^\otimes_{\mathbb{1}}"'] & & G(1) \otimes G(1) \arrow[dd, "G^2_\otimes"] \\
& \mathbb{1} \otimes G(1) \arrow[ur, "G^0_\otimes \otimes 1"] \arrow[dr, "\lambda^\otimes_{G(1)}"] & \\
\mathbb{1} \arrow[rr, "G^0_\otimes"] & & G(1)
\end{tikzcd}$$

Starting from the top and going counterclockwise, the three triangles in the diagram above are commutative by, respectively,

- the functoriality of $\otimes$,
- the naturality of $\lambda^\otimes$ in C, and
- the left unity axiom (1.2.15) for $(G, G^2_\otimes, G^0_\otimes)$ and the fact that $\lambda^\otimes$ is the identity in Σ.

If $m, n \geq 1$, then (7.7.7) is the outer diagram below.

$$\begin{tikzcd}
\overline{m} \otimes \overline{n} \arrow[r, "(\oplus_i G^0_\otimes)_{\mathsf{lt}} \otimes (\oplus_j G^0_\otimes)_{\mathsf{lt}}"] \arrow[d, "F'_\otimes"] \arrow[rr, bend left, "\theta^G_m \otimes \theta^G_n"] \arrow[dd, bend right=60, "F^2_\otimes"'] & \Big(\bigoplus_{i=1}^m G(1)\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^n G(1)\Big)_{\mathsf{lt}} \arrow[r, "G_\oplus \otimes G_\oplus"] \arrow[d, "p_{m,n}"] & G(m) \otimes G(n) \arrow[dd, "G^2_\otimes"] \\
\Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m \mathbb{1} \otimes \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "(\oplus_j \oplus_i G^0_\otimes \otimes G^0_\otimes)_{\mathsf{lt}}"] \arrow[d, "(\oplus_j \oplus_i \lambda^\otimes_{\mathbb{1}})_{\mathsf{lt}}"] & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m G(1) \otimes G(1)\Big)_{\mathsf{lt}} \arrow[d, "(\oplus_j \oplus_i G^2_\otimes)_{\mathsf{lt}}"] & \\
\Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m \mathbb{1}\Big)_{\mathsf{lt}} \arrow[r, "(\oplus_j \oplus_i G^0_\otimes)_{\mathsf{lt}}"] & \Big(\bigoplus_{j=1}^n \bigoplus_{i=1}^m G(1)\Big)_{\mathsf{lt}} \arrow[r, "G_\oplus"] & G(mn)
\end{tikzcd}$$

Consider the previous diagram.

- The top rectangle is commutative by the functoriality of $\otimes$ and the definition of θ^G_n for $n \geq 1$. In the top right horizontal morphism, the left $G_\oplus$ is the identity morphism if $m = 1$, and the right $G_\oplus$ is the identity morphism if $n = 1$.
- The left rectangle is the factorization of $F^2_\otimes$ in Lemma 7.3.21.
- The morphism $p_{m,n}$ is the one in (7.5.3) with

$$A_i = 1 = B_j \quad \text{for} \quad 1 \leq i \leq m \quad \text{and} \quad 1 \leq j \leq n.$$

  Both $F'_\otimes$ and $p_{m,n}$ are defined using the path Q in (7.3.19), which may be chosen as consisting of δ^l, δ^r, and $\alpha^{-\oplus}$. The upper left square is commutative by the naturality of δ^l, δ^r, and $\alpha^\oplus$ in C.
- The lower left square is commutative by the case $m = n = 1$ in (7.7.8), applied to each $1 \leq i \leq m$ and $1 \leq j \leq n$.
- The right rectangle is the diagram (7.5.2) for $G : \Sigma \longrightarrow \mathsf{C}$ with each A_i and each B_j equal to 1. This rectangle is commutative by Theorem 7.5.8. Note that in this case, the morphism $q_{m,n}$ in (7.5.4), which is defined by the path Q in (7.3.19), is the identity morphism. Indeed, the only nonidentity structure morphisms in Σ are δ^r, $\xi^\oplus$, and $\xi^\otimes$, the last two of which are not in Q. Each δ^r in $q_{m,n}$ is the identity permutation, as we showed in (7.3.10).

This finishes the proof of the commutativity of (7.7.7). $\square$

Lemma 7.7.9. *Suppose $G : \Sigma \longrightarrow \mathsf{C}$ is a robust symmetric bimonoidal functor with C a flat symmetric bimonoidal category. Then there exists a unique bimonoidal natural transformation*

$$\theta : F \longrightarrow G.$$

Moreover, θ is as in Definition 7.7.1, and is a bimonoidal natural isomorphism.

Proof. In Lemmas 7.7.3, 7.7.4, and 7.7.6, we showed that $\theta^G : F \longrightarrow G$ in Definition 7.7.1 is a bimonoidal natural transformation. The uniqueness and the invertibility of θ^G are established in Lemma 7.6.3. $\square$

7.8. Bi-Initial Symmetric Bimonoidal Category

In this section, we assemble the results so far in this chapter to prove Baez's Conjecture. In fact, we prove two versions, one with Σ and the other with Σ'.

Recall the following.

- Σ is the small tight symmetric bimonoidal category in Proposition 2.4.8.
- $\varnothing$ is the empty 2-category, with no objects, no 1-cells, and no 2-cells.
- $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ is the 2-category in Definition 7.1.8 with
  - flat small symmetric bimonoidal categories in Definition 3.9.9 as objects,
  - robust symmetric bimonoidal functors in Definition 5.1.1 as 1-cells, and
  - bimonoidal natural transformations in Definition 7.1.2 as 2-cells.

In the following assertion, the precise definition of a *lax bicolimit* can be found in [**JY21**, Section 5.2]. Since we only need it for a special case, we will unpack the following statement in the proof in terms of 1-categorical concepts.

Theorem 7.8.1 (Baez's Conjecture)**.** *Σ is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$.*

Proof. Since $\varnothing$ is the empty 2-category, the assertion means that for each flat small symmetric bimonoidal category C, the unique functor

$$\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}(\Sigma, \mathsf{C}) \xrightarrow{\quad T \quad} \mathbf{1}$$

to the terminal category is an equivalence of categories, that is, fully faithful and essentially surjective.

Since $\mathbf{1}$ is the terminal category, the essential surjectivity of T means the existence of a robust symmetric bimonoidal functor $\Sigma \longrightarrow \mathsf{C}$. Even without the smallness assumption on C, this is true by Proposition 7.4.4, where we constructed a canonical robust symmetric bimonoidal functor $F : \Sigma \longrightarrow \mathsf{C}$.

The fully faithfulness of T means that for each pair of robust symmetric bimonoidal functors $G, H : \Sigma \longrightarrow \mathsf{C}$, there exists a unique bimonoidal natural transformation $G \longrightarrow H$. Even without the smallness assumption on C, such a bimonoidal natural transformation is given by the vertical composite

$$G \xrightarrow{\quad (\theta^G)^{-1} \quad} F \xrightarrow{\quad \theta^H \quad} H$$

with θ^G and θ^H from Lemma 7.7.9. Its uniqueness follows from Lemma 7.6.3. The proof of Baez's Conjecture is now complete. □

Remark 7.8.2. We actually proved something stronger than the statement of Theorem 7.8.1. In the proof of Baez's Conjecture, the smallness of C is only used at the very end to make sure that it is an object in the 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$. The construction of F starting in Section 7.2 and the proofs of its properties through Lemma 7.7.9 do not require any smallness condition. ◇

Recall the small tight symmetric bimonoidal category Σ' in Definition 2.4.18 and Proposition 2.4.23. It is, furthermore, a right bipermutative category in the sense of Definition 2.5.2. In Proposition 5.1.16, we observed that Σ and Σ' are canonically isomorphic by exhibiting a pair of inverse isomorphisms. Therefore, Theorem 7.8.1 also holds for Σ'.

Theorem 7.8.3 (Baez's Conjecture, Version 2)**.** *Σ' is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$. In other words, for each flat small symmetric bimonoidal category C, the unique functor*

$$\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}(\Sigma', \mathsf{C}) \longrightarrow \mathbf{1}$$

is an equivalence of categories.

7.9. Notes

7.9.1 (Nonsymmetric Version)**.** In [**Elg21**, Theorem 4.3.3], Elgueta proved a version of Baez's Conjecture that is different from our Theorems 7.8.1 and 7.8.3 in several aspects. Here we discuss these differences.

Objects. Elgueta's version is about a 2-category **RigCat** whose objects are *rig categories*. A rig category is different from a symmetric bimonoidal category in two ways.

- A rig category does not have a multiplicative symmetry $\xi^{\otimes}$, so the multiplicative structure is a monoidal category, instead of a symmetric monoidal category. The axioms (2.1.5) (relating δ^l and δ^r) and (2.1.19) (symmetry of the multiplicative zeros) involving $\xi^{\otimes}$ are omitted.
- In a rig category, the distributivity morphisms δ^l and δ^r are natural isomorphisms, instead of monomorphisms as in (2.1.4).

Elgueta did not explicitly state that his rig categories should be small. However, as discussed in the introduction of this chapter, to form the 2-category **RigCat**, its objects should be small rig categories.

In our versions of Baez's Conjecture, the objects in the 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ are flat small symmetric bimonoidal categories. In particular, δ^l and δ^r in (2.1.4) are natural monomorphisms. Instead of asking for their invertibility, we require the much weaker condition of flatness in Definition 3.9.9.

1-Cells. For 1-cells, Elgueta uses what he calls *homomorphisms*. These are functors whose additive structure is a strong symmetric monoidal functor, and whose multiplicative structure is a strong monoidal functor.

In contrast, our 1-cells in $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$ are robust symmetric bimonoidal functors in Definition 5.1.1.

- The additive structure is a strong symmetric monoidal functor, just like Elgueta's homomorphism.
- The multiplicative structure is a *symmetric* monoidal functor whose multiplicative unit constraint

  $$G^0_{\otimes} : \mathbb{1} \longrightarrow G(1)$$

  is an isomorphism. However, the multiplicative monoidal constraint $G^2_{\otimes}$ is not required to be invertible. In particular, in Lemma 7.3.28, we need to check the axiom (1.2.26) of a symmetric monoidal functor for $F_{\otimes}$, which is the diagram (7.3.29).

2-Cells. Elgueta's 2-cells are basically the same as our bimonoidal natural transformations in Definition 7.1.2, which he calls *rig transformations*. The only difference is that rig transformations go between his homomorphisms.

Proof Strategy. Elgueta's proof [**Elg21**, start of Section 4] involves replacing his rig category with an equivalent semistrict rig category, using a strictification theorem due to Guillou [**Gui10**]. See Note 5.6.4. This is closely related to the restriction on the objects of **RigCat**. As we pointed out multiple times in Chapter 5, in particular in Note 5.6.2, the Strictification Theorem 5.4.6 only applies to *tight* symmetric bimonoidal categories, that is, those with δ^l and δ^r natural isomorphisms. The same is true for the multiplicatively nonsymmetric version in [**Gui10**, Def. 2.1]. Therefore, using strictification means that one must insist on the invertibility of the distributivity morphisms.

In contrast, our Theorems 7.8.1 and 7.8.3 do *not* use the Strictification Theorem 5.4.6. Instead, we work directly with a flat small symmetric bimonoidal category in the entire proof. Here are some advantages of this approach:

(1) It allows us to be very specific about where and how the Coherence Theorems 3.9.1 and 4.4.3 and Epstein's Coherence Theorem 1.3.12 are used.
(2) We can make precise all the relevant structure morphisms with respect to the actual symmetric bimonoidal category in question, instead of an equivalent one. See Definitions 7.2.2, 7.3.12, and 7.7.1.
(3) As we pointed out above, we only need our small symmetric bimonoidal categories to be flat, which is a much weaker assumption than the invertibility of the distributivity morphisms. $\diamond$

7.9.2 (Nonsymmetric Version via Sheet Diagrams)**.** The nonsymmetric version of Baez's Conjecture [**Elg21**] discussed in Note 7.9.1 also follows from the sheet diagram results in [**CDH∞**, Section 8]. Both papers [**CDH∞, Elg21**] use the strictification theorem in [**Gui10**]. For strictification of tight bimonoidal categories, our Theorems 5.5.11 and 5.5.12 may also be used. $\diamond$

CHAPTER 8

Symmetric Monoidal Bicategorification

The main theorem of this chapter, Theorem 8.15.4, says that a matrix construction

$$\mathsf{C} \longmapsto \mathsf{Mat}^{\mathsf{C}}$$

sends each tight symmetric bimonoidal category C as in Definition 2.1.2 to a symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$ as in Definition 6.5.9. This process is called *symmetric monoidal bicategorification* because it starts with a 1-categorical structure, namely a tight symmetric bimonoidal category C, and produces a symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$. There is also a braided analogue in Theorem II.8.4.7.

The main tools for the constructions and proofs in this chapter are the Coherence Theorems 3.9.1 and 3.10.7 and the definitions in Section 3.1. In fact, the many constructions and proofs in this chapter provide a good illustration of the utility of the graph theoretic framework introduced in Section 3.1. For open questions related to the matrix construction, see Appendix III.A. In the rest of this introduction, we discuss aspects of Theorem 8.15.4 without going into too much detail and provide an outline and a reading guide for this chapter.

Motivation. To motivate the Bicategorification Theorem 8.15.4, consider a rig $(R,+,\times,0_R,1_R)$, that is, a ring without additive inverses, such as the set of natural numbers $\mathbb{N}$ with its usual addition and multiplication. Matrices with entries in R have two natural types of products.

- For an $n \times m$ matrix $A = (A_{ji})$ and a $p \times n$ matrix $B = (B_{kj})$, there is the usual matrix product BA, which is a $p \times m$ matrix, whose (k,i)-entry is the sum
$$(BA)_{ki} = \sum_{j=1}^{n} B_{kj}A_{ji}.$$
- For an $n \times m$ matrix A and a $q \times p$ matrix $B = (B_{lk})$, the matrix tensor product, which is also called the Kronecker product,
$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1m}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nm}B \end{bmatrix}$$
is the $nq \times mp$ block matrix obtained from A by replacing each entry A_{ji} with the scalar product $A_{ji}B = \big(A_{ji}B_{lk}\big)_{l,k}$.

These products satisfy some well-known properties, including the following associativity properties, where the first one assumes that the matrix products are defined.

$$\begin{gathered} (AB)C = A(BC) \\ (A \otimes B) \otimes C = A \otimes (B \otimes C) \end{gathered} \tag{8.0.1}$$

Moreover, if R is a multiplicatively commutative rig, such as $\mathbb{N}$, then the *mixed-product property*

$$(B \otimes B')(A \otimes A') = BA \otimes B'A' \tag{8.0.2}$$

holds whenever the matrix products BA and $B'A'$ are defined. These properties imply that for a commutative rig R, there is a permutative category

$$\big(\mathsf{Mat}^R, \otimes, 1, \xi\big)$$

as in Definition 1.2.18 with the following data.

- The objects in Mat^R are nonnegative integers.
- The morphisms $m \longrightarrow n$ are $n \times m$ matrices with entries in R.
- The categorical composition sends (B, A) to the matrix product BA.
- The identity morphism on n is the $n \times n$ identity matrix 1_n with the unit 1_R along the diagonal and 0_R in every other entry.
- The monoidal unit is the integer 1.
- The monoidal product $\otimes : (\mathsf{Mat}^R)^2 \longrightarrow \mathsf{Mat}^R$ is defined by
  - $m \otimes n = mn$ on objects and
  - the matrix tensor product on morphisms.
- The symmetry isomorphism $\xi_{m,n} : mn \longrightarrow nm$ is the permutation matrix obtained from the identity matrix 1_{mn} by permuting its columns using the permutation in (2.4.5).

For example, the mixed-product property (8.0.2) says that $\otimes$ preserves categorical composition. The three symmetric monoidal category axioms for Mat^R are checked in the proof of Proposition 2.4.8. In particular, the hexagon axiom (1.2.22) for Mat^R corresponds to the permutation in (2.4.10).

Symmetric bimonoidal categories are categorical analogues of commutative rigs. For a symmetric bimonoidal category C as in Definition 2.1.2, matrices whose entries are all objects, or all morphisms, in C make sense. Using the sum $\oplus$ and the product $\otimes$ in C, and fixing a convention for iterated sums, the matrix product makes sense for matrices with objects or morphisms in C. Similarly, using $\otimes$ in C, the matrix tensor product makes sense for matrices of objects or morphisms in C. To avoid confusion with the product $\otimes$ in C, for matrices A and B of objects or morphisms in C, their matrix tensor product is denoted by $A \boxtimes B$. The existence of the permutative category Mat^R for a commutative rig R leads to a natural question.

> For a symmetric bimonoidal category C, is there a symmetric monoidal structure on the collection $\mathsf{Mat}^{\mathsf{C}}$ of matrices whose entries are objects or morphisms in C?

A commutative rig R is a *set* with an addition and a commutative multiplication. The matrix construction Mat^R produces a permutative category, that is, a *category* with a multiplication that is symmetric in a categorical sense. Heuristically, we should expect an analogous level-shifting if we start with a symmetric bimonoidal category C. In other words, we expect $\mathsf{Mat}^{\mathsf{C}}$ to be a *bicategory* with a multiplication that is symmetric in a bicategorical sense. Since the two monoidal structures in C are not strict in general, the associativity properties (8.0.1) and the mixed-product property (8.0.2) should hold up to natural isomorphisms. The goal of this chapter is to answer the above question positively: if C is a tight symmetric bimonoidal category, then $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory as in Definition 6.5.9. In general, all the structure morphisms in the symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$ are not identities.

Formulation. For each tight symmetric bimonoidal category C, the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ is defined by the following data.

- The objects are nonnegative integers.
- A 1-cell $m \longrightarrow n$ is an $n \times m$ matrix $A = (A_{ji})$ with each entry A_{ji} an object in C.
- For 1-cells $A = (A_{ji}), B = (B_{ji}) : m \longrightarrow n$, a 2-cell $f = (f_{ji}) : A \longrightarrow B$ is an $n \times m$ matrix with each (j,i)-entry a morphism $f_{ji} : A_{ji} \longrightarrow B_{ji}$ in C.
- The horizontal composition

$$\mathsf{Mat}^{\mathsf{C}}_{n,p} \times \mathsf{Mat}^{\mathsf{C}}_{m,n} \longrightarrow \mathsf{Mat}^{\mathsf{C}}_{m,p}$$

  is given by the usual matrix product (that is, multiplication of a $p \times n$ matrix with an $n \times m$ matrix to yield a $p \times m$ matrix), defined via $\oplus$ and $\otimes$ in C, applied to matrices of objects or morphisms in C, with the left normalized bracketing for iterated sums.
- The vertical composition of 2-cells is given by entrywise composition of morphisms in C.
- The structure morphisms in C induce canonical associator a, left unitor ℓ, and right unitor r in $\mathsf{Mat}^{\mathsf{C}}$, making it into a bicategory, called the *matrix bicategory*. See Theorem 8.4.12, which holds more generally for tight bimonoidal categories because the multiplicative symmetry $\xi^{\otimes}$ is not needed up to this point.
- For the monoidal bicategory structure, the monoidal composition

$$\mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}} \xrightarrow{\boxtimes} \mathsf{Mat}^{\mathsf{C}}$$

  is given by
  - $m \boxtimes n = mn$ on objects and
  - the matrix tensor product on 1-cells and 2-cells, defined via $\otimes$ in C.
- The structure morphisms in C induce the monoidal associator $a^{\boxtimes}$, the left monoidal unitor $\ell^{\boxtimes}$, the right monoidal unitor $r^{\boxtimes}$, the pentagonator π, the middle 2-unitor μ, the left 2-unitor $\lambda^{\boxtimes}$, and the right 2-unitor $\rho^{\boxtimes}$. With these structures, $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory; see Theorem 8.12.9.
- This monoidal bicategory is equipped with a braiding β induced by permutation matrices, which are defined via the additive zero $\mathbb{0}$ and the multiplicative unit $\mathbb{1}$ in C. The structure morphisms in C induce canonical left and right hexagonators, $R_{-|--}$ and $R_{--|-}$, and a syllepsis ν.

Theorem 8.15.4 states that, with these structures, $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory.

Connections. The constructions and results in this chapter have interesting connections with algebraic K-theory and linear algebra.

For each object A in a bicategory B, the hom category $\mathsf{B}(A,A)$ inherits the structure of a monoidal category. Its monoidal product is a restriction of the horizontal composition in B. For the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$, this means that each $\mathsf{Mat}^{\mathsf{C}}_{n,n}$ is a monoidal category; see Corollary 8.4.14. Its objects are $n \times n$ matrices with entries in C, and its morphisms are $n \times n$ matrices of morphisms in C between corresponding objects. The monoidal product is given by the horizontal composition in $\mathsf{Mat}^{\mathsf{C}}$, that is, the matrix product. The existence of the monoidal category $\mathsf{Mat}^{\mathsf{C}}_{n,n}$ is stated in [**BDR04**, Prop. 3.3] without proof. It is an important ingredient in the algebraic K-theory of C as defined in [**BDR04**, Def. 3.12].

In the braided monoidal bicategory structure of $\mathsf{Mat}^{\mathsf{C}}$, the braiding β is a strong transformation. Its component 1-cells $\beta_{m,n}$ (8.13.23) are permutation matrices in $\mathsf{Mat}^{\mathsf{C}}$. In the usual setting of linear algebra, a column permutation of a matrix A is equal to the matrix product AP for a suitable permutation matrix P. In the setting of $\mathsf{Mat}^{\mathsf{C}}$, this equality is replaced by a natural isomorphism (8.13.13), and similarly for row permutations as in (8.13.16). The component 2-cells $\beta_{A,B}$ of the braiding involves connecting $(B \boxtimes A)\beta_{m,n}$ and $\beta_{m',n'}(A \boxtimes B)$. This in turn involves connecting a column permutation of $B \boxtimes A$ and a row permutation of $A \boxtimes B$. These column and row permutations are generalizations to $\mathsf{Mat}^{\mathsf{C}}$ of examples of the Khatri-Rao product and the face product, which are both related to the Tracy-Singh product in linear algebra. See Explanation 8.13.21.

Non-Strictness of Structures. The symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$ is a genuine symmetric monoidal bicategory in the following sense.

(i) The left unitor ℓ, the right unitor r, and especially the associator a are not identities in general. This means that in general $\mathsf{Mat}^{\mathsf{C}}$ is a genuine bicategory and not a 2-category. In fact, the associator a, with entries in (8.3.5), involves $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, δ^{r}, and δ^{-l}. It is rare for a symmetric bimonoidal category, even a right or a left bipermutative category as in Definitions 2.5.2 and 2.5.11, to have all of these structure morphisms equal to the identity.

(ii) The monoidal composition $\boxtimes$ is a pseudofunctor, but not a strict functor in general. Its lax functoriality constraint $\boxtimes^2$, with entries in (8.6.20), involves $\alpha^{\oplus}$, $\alpha^{\pm\otimes}$, $\xi^{\otimes}$, δ^{-r}, and δ^{-l} in most cases. Once again, it is rare to have all of these structure morphisms equal to the identity at the same time.

(iii) The symmetric monoidal bicategorical data $a^{\boxtimes}$, $\ell^{\boxtimes}$, $r^{\boxtimes}$, π, μ, $\lambda^{\boxtimes}$, $\rho^{\boxtimes}$, β, $R_{-|--}$, $R_{--|-}$, and ν are also far from strict in general.

For example, suppose C is the tight symmetric bimonoidal category $\mathsf{Vect}^{\mathbb{C}}$ in Example 2.1.32, with finite dimensional complex vector spaces as objects and $\mathbb{C}$-linear maps as morphisms. Then $\mathsf{Mat}^{\mathsf{C}}$ is a genuine symmetric monoidal bicategory as described above. The 1-cells in $\mathsf{Mat}^{\mathsf{C}}$ are called coordinatized 2-vector spaces. See Examples 8.4.13 and 8.15.5.

Symmetry and Tightness. Let us discuss why the multiplicative symmetry $\xi^{\otimes}$ and the tightness assumption in C are needed to construct the symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$.

(i) The lax functoriality constraint

$$(B \boxtimes B')(A \boxtimes A') \xrightarrow{\boxtimes^2_{(B,B'),(A,A')}} BA \boxtimes B'A' \in \mathsf{Mat}^{\mathsf{C}}_{mm',pp'}$$

of the monoidal composition $\boxtimes$ is the analogue of the mixed-product property (8.0.2) in $\mathsf{Mat}^{\mathsf{C}}$. The positions of the middle two matrices B'

and A are switched. In a typical entry, this switch uses the multiplicative symmetry $\xi^{\otimes}$ in (8.6.17).

(ii) The component 2-cells of the braiding β (8.13.24) involves an entrywise application of $\xi^{\otimes}$.

(iii) Recall that a symmetric bimonoidal category is *tight* if the distributivity morphisms δ^r and δ^l are natural isomorphisms, instead of natural monomorphisms. The Coherence Theorems 3.9.1 and 4.4.3 do not require tightness, but only a condition about monomorphisms. For instance, flatness as in Definition 3.9.9 is enough for the two coherence theorems. However, as we mentioned above, in $\mathsf{Mat}^{\mathsf{C}}$,
- the associator a involves δ^{-l}, and
- the lax functoriality constraint $\boxtimes^2$ of the monoidal composition involves both δ^{-r} and δ^{-l}.

Therefore, the distributivity morphisms in C must be invertible for these constructions.

Constructions and Proofs. To define the many parts of the symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$, in each case, we write down the matrices that we want to connect. Then we use the most straightforward combination of structure morphisms in C to connect those matrices. For example, for the left unitor this step is performed in (8.2.3) and (8.2.4). The difficult part is in showing that the structures so defined have the required properties for $\mathsf{Mat}^{\mathsf{C}}$ to be a symmetric monoidal bicategory. We accomplish this by consistently reducing each such proof to an instance of the Coherence Theorem 3.9.1 or the simpler case in Proposition 3.5.33. These coherence results are formulated in terms of the graph theoretic framework in Section 3.1, which we use throughout this chapter.

There is an additional subtlety in a number of proofs in this chapter that arises from the following two facts.

- The associator a in $\mathsf{Mat}^{\mathsf{C}}$, with entries in (8.3.5), involves both δ^r and δ^{-l}.
- The lax functoriality constraint $\boxtimes^2$, with entries in (8.6.20), of the monoidal composition $\boxtimes$ in $\mathsf{Mat}^{\mathsf{C}}$ involves δ^{-r} and δ^{-l} in most cases.

The Coherence Theorem 3.9.1 and the simpler case in Proposition 3.5.33 are stated in terms of paths in $\mathsf{Gr}(X)$ in Definition 3.1.9. By Definition 3.1.6, the elementary edges δ^l and δ^r do *not* have formal inverses, so paths cannot have formal inverses of δ^l and δ^r. As a result, in some of the proofs that involve the associator a or the lax functoriality constraint $\boxtimes^2$, we will need to subdivide the diagram in question with additional morphisms in such a way that the coherence results can be applied in each subdiagram.

Relatively simple examples of this subdivision process are the following subdivided diagrams:

- (8.4.5) for the unity axiom in $\mathsf{Mat}^{\mathsf{C}}$ and
- the diagrams in cases (6)–(8) in the proof of Lemma 8.4.8 for the pentagon axiom.

More involved and typical examples are the following subdivided diagrams:

- (8.4.10) for the general case of the pentagon axiom in $\mathsf{Mat}^{\mathsf{C}}$;
- (8.7.26) for the lax associativity axiom for the pseudofunctor $\boxtimes$;
- (8.8.28) for the lax naturality axiom for the monoidal associator $a^{\boxtimes}$;

- (8.8.44) for the modification axiom for the unit $\eta^a : 1_{\boxtimes(\boxtimes\times 1)} \longrightarrow a^{\boxtimes\bullet} a^{\boxtimes}$; and
- (8.10.8) for the modification axiom for the pentagonator π.

Notation Guide. In Sections 8.1 through 8.5,

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}), (\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^l, \delta^r)\big)$$

denotes a bimonoidal category as in Definition 2.1.2. In the remaining sections, C is a symmetric bimonoidal category with multiplicative symmetry $\xi^{\otimes}$. If we want C to be *flat* as in Definition 3.9.9, or *tight*—that is, to have invertible distributivity morphisms δ^l and δ^r—we will state so explicitly, sometimes at the beginning of a section. In the absence of clarifying parentheses, $\otimes$ takes precedence over $\oplus$, and $\otimes$ is sometimes omitted to save space.

For the bicategory $\mathsf{Mat}^{\mathsf{C}}$, we sometimes add the adjective *base* before the associator and the unitors. To avoid confusion with the two monoidal products $\{\oplus, \otimes\}$ in C, the monoidal composition in $\mathsf{Mat}^{\mathsf{C}}$ is denoted by $\boxtimes$. Similarly, to avoid confusion with the base associator and the base unitors, the monoidal associator $a^{\boxtimes}$ and the monoidal unitors $\ell^{\boxtimes}$ and $r^{\boxtimes}$ are decorated with the superscript $\boxtimes$. Moreover, the left and the right 2-unitors $\lambda^{\boxtimes}$ an $\rho^{\boxtimes}$ are also decorated with $\boxtimes$ to avoid confusion with the various λ and ρ in C.

Outline. In Section 8.1, we define the hom categories $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ and the matrix product, which will form the horizontal composition. We also discuss coherence properties of the matrix product when one of the matrices is a 0 matrix. The left unitor ℓ and the right unitor r in $\mathsf{Mat}^{\mathsf{C}}$ are defined in Section 8.2. The associator a is defined in Section 8.3. The unity axiom (6.1.3) and the pentagon axiom (6.1.4) are verified in Section 8.4. By the end of this section, we will have proved that $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory.

In the next few sections, we construct a monoidal bicategory structure on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$. The monoidal identity is constructed in Section 8.5. The matrix tensor product and the monoidal composition $\boxtimes$ are defined in Section 8.6. The lax associativity axiom (6.2.2) and the lax unity axiom (6.2.3) for the pseudofunctor $\boxtimes$ are verified in Section 8.7. The monoidal associator $a^{\boxtimes}$ is constructed in Section 8.8. The left monoidal unitor $\ell^{\boxtimes}$ and the right monoidal unitor $r^{\boxtimes}$ are constructed in Section 8.9. The pentagonator is defined in Section 8.10. The middle 2-unitor μ, the left 2-unitor $\lambda^{\boxtimes}$, and the right 2-unitor $\rho^{\boxtimes}$ are defined in Section 8.11. The fact that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory is stated and verified in Section 8.12.

Permutation matrices, row and column permutations, and the braiding β are defined in Section 8.13. In Section 8.14, we define the left and the right hexagonators, and verify that $\mathsf{Mat}^{\mathsf{C}}$ is a braided monoidal bicategory. In Section 8.15, we define the syllepsis ν, and conclude the proof of the Bicategorification Theorem 8.15.4.

Conventions. The following conventions are in effect throughout this chapter.

- A *Mac Lane coherence isomorphism*, which is denoted by $\cong^{\oplus}_{\mathsf{ML}}$, means the value in C of a path that only involves identities, $\alpha^{\pm\oplus}$, $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, and $\xi^{\pm\oplus}$.
- A *Laplaza coherence isomorphism*, which is denoted by $\cong_{\mathsf{Lap}}$, means the value in C of a path that does *not* involve $\xi^{\pm\otimes}$.

These are adaptations of the notions in Definition 5.2.24 to the current context.

Reading Guide. Instead of reading this chapter linearly, the reader may first read the key definitions and statements of results listed below to obtain a general idea of the structure of $\mathsf{Mat}^{\mathsf{C}}$.

(1) The base bicategory.
- Definition 8.1.1: The hom categories $\mathsf{Mat}^{\mathsf{C}}_{m,n}$.
- Definition 8.1.3: The matrix product and the identity matrices.
- Lemmas 8.2.1 and 8.2.7: The base unitors ℓ and r.
- Lemma 8.3.1: The base associator a.
- Definition 8.4.11: The matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$.

(2) The monoidal bicategory.
- Definition 8.5.1: The monoidal identity $1_{\boxtimes} : \mathbf{1} \longrightarrow \mathsf{Mat}^{\mathsf{C}}$.
- Definition 8.6.1: The matrix tensor product $A \boxtimes B$.
- Lemma 8.6.8: The lax unity constraint $\boxtimes^0$.
- Definition 8.6.19: The monoidal composition $\boxtimes$.
- Definitions 8.8.33 and 8.8.37: The monoidal associator $(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$.
- Definitions 8.9.1, 8.9.5, and 8.9.8: The left monoidal unitor.
- Definitions 8.9.14, 8.9.17, and 8.9.20: The right monoidal unitor.
- Definition 8.10.1: The pentagonator π.
- Definitions 8.11.1, 8.11.6, and 8.11.11: The 2-unitors μ, $\lambda^{\boxtimes}$, and $\rho^{\boxtimes}$.

(3) The symmetric monoidal bicategory.
- Definition 8.13.3: Row/column permutations and permutation matrices.
- Lemmas 8.13.12 and 8.13.15: Taking matrix products with permutation matrices and row/column permutations.
- Definitions 8.13.22, 8.13.30, 8.13.35, and 8.13.38: The braiding.
- Lemma 8.14.4 and Definition 8.14.9: The left hexagonator $R_{-|--}$.
- Lemma 8.14.16 and Definition 8.14.21: The right hexagonator $R_{--|-}$.
- Definition 8.15.1: The syllepsis ν.
- Theorem 8.15.4: The Bicategorification Theorem.
- Example 8.15.5: Applying the Bicategorification Theorem.

Before reading the proofs in the rest of this chapter, the reader may want to review Section 3.1. As in Chapters 3, 4, 5, and 7, the reader may consider the many detailed constructions and proofs in this chapter as exercises with full solutions.

8.1. Matrix Construction

To construct the bicategory $\mathsf{Mat}^{\mathsf{C}}$ of matrices for a tight bimonoidal category C, in this section, we first define its hom categories and the associated matrix product. The latter will eventually be used to define the horizontal composition in the bicategory $\mathsf{Mat}^{\mathsf{C}}$. In the second half of this section, we discuss coherence properties of multiplying by a 0 matrix. These coherence properties will be used in Lemma 8.3.1 when we discuss associativity properties of the matrix product. Coherence properties of the matrix product with respect to the identity matrices and associativity will be discussed in Sections 8.2 and 8.3.

The Matrix Product.

Definition 8.1.1. Suppose C is a category. For integers $m, n \geq 0$, define a category $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ as follows.

Objects: An object in $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ is an $n \times m$ matrix

$$A = (A_{ji})_{1 \le j \le n,\, 1 \le i \le m}$$

with each A_{ji} an object in C. We call A an *$n \times m$ matrix in* C. When n and m are understood, we will also write A as (A_{ji}).

Morphisms: A morphism

$$f : A = (A_{ji}) \longrightarrow (A'_{ji}) = A'$$

is an $n \times m$ matrix

$$f = (f_{ji})_{1 \le j \le n,\, 1 \le i \le m}$$

with each $f_{ji} : A_{ji} \longrightarrow A'_{ji}$ a morphism in C.

Identity Morphisms: For an object $A = (A_{ji})$, its identity morphism is the $n \times m$ matrix

$$1_A = \big(1_{A_{ji}}\big)$$

of identity morphisms in C.

Composition: If $f : A \longrightarrow A'$ is a morphism as above and if $f' : A' \longrightarrow A''$ is another morphism, then their composite $f'f : A \longrightarrow A''$ is defined entrywise in C as

$$(f'f)_{ji} = f'_{ji} f_{ji} : A_{ji} \longrightarrow A''_{ji}.$$

This finishes the definition of the category $\mathsf{Mat}^{\mathsf{C}}_{m,n}$.

Moreover, we define the following.

- An object in $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ is also called a *matrix*.
- Consider an $n \times m$ matrix $A = (A_{ji})$.
  - $A_{ji} \in \mathsf{C}$ is called the (j,i)-*entry*.
  - For each $1 \le j \le n$, the *jth row* is the $1 \times m$ matrix $(A_{ji})_{1 \le i \le m}$.
  - For each $1 \le i \le m$, the *ith column* is the $n \times 1$ matrix $(A_{ji})_{1 \le j \le n}$.
- If C has a distinguished object $\mathbb{0}$, then the *0 matrix* $\mathbb{0}_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ is the matrix with each entry $\mathbb{0} \in \mathsf{C}$ if $m, n > 0$. If either m or n is 0, then $\mathbb{0}_{m,n}$ denotes the empty matrix.
- An $n \times n$ matrix is also called a *square matrix*.
- Consider an $n \times n$ square matrix $A = (A_{ji})$.
  - A_{jj} is said to be *on the diagonal*, and is called the *jth diagonal entry* for $1 \le j \le n$.
  - A_{ji} with $j \ne i$ is said to be *off the diagonal*, and is called an *off-diagonal entry*.
  - A_{ji} with $j > i$ is said to be *below the diagonal*, and is called a *below-diagonal entry*.
  - A_{ji} with $j < i$ is said to be *above the diagonal*, and is called an *above-diagonal entry*.

Similar terminology applies to morphisms in $\mathsf{Mat}^{\mathsf{C}}_{m,n}$. ◇

Explanation 8.1.2. Consider Definition 8.1.1.

- $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ is indeed a category because everything is defined entrywise in C.
- If either m or n is 0, then $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ is the terminal category with one object—namely, the unique empty matrix—and its identity morphism.
- If $m = n = 1$, then $\mathsf{Mat}^{\mathsf{C}}_{1,1}$ is canonically identified with C. ◇

We now extend Definition 8.1.1 to include the matrix product and identity matrices, which will be used to define, respectively, the horizontal composition and the identity 1-cells in the bicategory $\mathsf{Mat}^{\mathsf{C}}$. Recall from (5.2.13) that the subscript lt means left normalized bracketing.

Definition 8.1.3. Suppose C is a bimonoidal category, and $m, n, p \geq 0$.

- For
  - an $n \times m$ matrix $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ and
  - a $p \times n$ matrix $B = (B_{kj}) \in \mathsf{Mat}^{\mathsf{C}}_{n,p}$,

  define their *matrix product* $BA \in \mathsf{Mat}^{\mathsf{C}}_{m,p}$ whose (k,i)-entry, for $1 \leq i \leq m$ and $1 \leq k \leq p$, is the following object in C:

$$(BA)_{ki} = \begin{cases} \Big(\bigoplus_{j=1}^{n} (B_{kj} \otimes A_{ji}) \Big)_{\mathsf{lt}} & \text{if } n \geq 1 \text{ and} \\ \mathbb{0} & \text{if } n = 0. \end{cases} \tag{8.1.4}$$

  If either m or p is 0, then BA is the empty matrix.
- For morphisms
  - $f = (f_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}(A, A')$ and
  - $g = (g_{kj}) \in \mathsf{Mat}^{\mathsf{C}}_{n,p}(B, B')$,

  define their *matrix product*

$$g \star f \in \mathsf{Mat}^{\mathsf{C}}_{m,p}(BA, B'A')$$

  as the $p \times m$ matrix whose (k,i)-entry, for $1 \leq i \leq m$ and $1 \leq k \leq p$, is the following morphism in C:

$$(g \star f)_{ki} = \begin{cases} \Big(\bigoplus_{j=1}^{n} (g_{kj} \otimes f_{ji}) \Big)_{\mathsf{lt}} : (BA)_{ki} \longrightarrow (B'A')_{ki} & \text{if } n \geq 1 \text{ and} \\ 1_{\mathbb{0}} : \mathbb{0} \longrightarrow \mathbb{0} & \text{if } n = 0. \end{cases} \tag{8.1.5}$$

  If either m or p is 0, then $g \star f$ is the identity morphism of the empty matrix.
- The $n \times n$ *identity matrix* is the square matrix $\mathbb{1}^n \in \mathsf{Mat}^{\mathsf{C}}_{n,n}$ with entries

$$\mathbb{1}^n_{ji} = \begin{cases} \mathbb{1} & \text{if } i = j \text{ and} \\ \mathbb{0} & \text{if } i \neq j. \end{cases} \tag{8.1.6}$$

  If $n = 0$, then $\mathbb{1}^0$ is the unique empty matrix. ◇

Explanation 8.1.7. In Definition 8.1.3, suppose $m, p > 0$ and $n = 0$.

- Then $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ and $B \in \mathsf{Mat}^{\mathsf{C}}_{n,p}$ are both empty matrices. However, in the matrix product (8.1.4), $BA \in \mathsf{Mat}^{\mathsf{C}}_{m,p}$ is the 0 matrix $\mathbb{0}_{m,p}$ in which each entry is the additive zero $\mathbb{0} \in \mathsf{C}$.
- $f \in \mathsf{Mat}^{\mathsf{C}}_{m,n}(A, A')$ and $g \in \mathsf{Mat}^{\mathsf{C}}_{n,p}(B, B')$ are both identity morphisms of the empty matrix. In the matrix product (8.1.5), $g \star f$ is the identity morphism of the 0 matrix $\mathbb{0}_{m,p}$. ◇

Lemma 8.1.8. *Suppose* C *is a bimonoidal category, and* $m, n, p \geq 0$. *Then the matrix product*

$$\mathsf{Mat}^{\mathsf{C}}_{n,p} \times \mathsf{Mat}^{\mathsf{C}}_{m,n} \longrightarrow \mathsf{Mat}^{\mathsf{C}}_{m,p}$$

in Definition 8.1.3 is a functor.

Proof. If one of m, n, or p is 0, then functoriality is true by definition.

If $m, n, p > 0$, then the matrix product preserves identity morphisms, that is,

$$1_B \star 1_A = 1_{BA}.$$

This is true because each of 1_A, 1_B, and 1_{BA} is entrywise an identity morphism, and $\oplus$ and $\otimes$ preserves identity morphisms.

For morphisms $f' = (f'_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}(A', A'')$ and $g' = (g'_{kj}) \in \mathsf{Mat}^{\mathsf{C}}_{n,p}(B', B'')$, the functoriality of $\oplus$ and $\otimes$ implies the following equalities for $1 \leq i \leq m$ and $1 \leq k \leq p$.

$$\begin{aligned}\big[(g' \star f')(g \star f)\big]_{ki} &= \Big(\bigoplus_{j=1}^{n} (g'_{kj} \otimes f'_{ji})\Big)_{\mathsf{lt}} \Big(\bigoplus_{j=1}^{n} (g_{kj} \otimes f_{ji})\Big)_{\mathsf{lt}}\\ &= \Big(\bigoplus_{j=1}^{n} \big(g'_{kj} \otimes f'_{ji}\big)\big(g_{kj} \otimes f_{ji}\big)\Big)_{\mathsf{lt}}\\ &= \Big(\bigoplus_{j=1}^{n} \big(g'_{kj} g_{kj}\big) \otimes \big(f'_{ji} f_{ji}\big)\Big)_{\mathsf{lt}}\\ &= \big[(g'g) \star (f'f)\big]_{ki}\end{aligned} \tag{8.1.9}$$

Therefore, the matrix product preserves composition. □

Multiplying with Zero Matrices. To show that $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory and a symmetric monoidal bicategory later, we will need to understand the coherence properties of the matrix product (8.1.4) with respect to the 0 matrices, the identity matrices, and iterated products. As a warm up exercise, we first consider the matrix product with a 0 matrix. The other coherence properties will be discussed in subsequent sections. Recall that $\mathbb{0}_{m,n}$ denotes the $n \times m$ matrix with each entry the additive zero $\mathbb{0} \in \mathsf{C}$.

Lemma 8.1.10. *Suppose* C *is a bimonoidal category, and* $m, n, p \geq 0$. *Then there is a natural isomorphism*

$$\mathbb{0}_{n,p} A \xrightarrow[\cong]{\zeta^{\ell}_A} \mathbb{0}_{m,p} \quad \in \mathsf{Mat}^{\mathsf{C}}_{m,p} \tag{8.1.11}$$

for $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.

Proof. There are three cases.

- If either m or p is 0, then both $\mathbb{0}_{n,p}A$ and $\mathbb{0}_{m,p}$ are the empty matrix.
- If $m, p > 0$ and $n = 0$, then $\mathbb{0}_{n,p}$ and A are the empty matrix, but $\mathbb{0}_{n,p}A = \mathbb{0}_{m,p}$.

In each of these two cases, ζ^{ℓ}_A is the identity morphism.

For $m, n, p > 0$, suppose $A = (A_{ji})$. For $1 \leq k \leq p$ and $1 \leq i \leq m$, $\mathbb{0}_{n,p}A$ has (k, i)-entry

$$(\mathbb{0}_{n,p}A)_{ki} = \Big(\bigoplus_{j=1}^{n} (\mathbb{0} \otimes A_{ji})\Big)_{\mathsf{lt}}. \tag{8.1.12}$$

This object is isomorphic to the additive zero $\mathbb{0}$ via the following composite in C.

$$\begin{tikzcd}
(\mathbb{0}_{n,p}A)_{ki} \arrow[d, "\left(\lambda^{\bullet}_{A_{1i}}\oplus\cdots\oplus\lambda^{\bullet}_{A_{ni}}\right)_{\mathsf{lt}}"'] \arrow[dr, bend left, "(\zeta^{\ell}_A)_{ki}"] & \\
\left(\bigoplus_{j=1}^{n}\mathbb{0}\right)_{\mathsf{lt}} \arrow[r, "Z"] & \mathbb{0}
\end{tikzcd} \tag{8.1.13}$$

- If $n = 1$, then $Z = 1_{\mathbb{0}}$.
- If $n > 1$, then Z is a composite of $n-1$ morphisms, each being the sum of identity morphisms and one $\lambda^{\oplus}_{\mathbb{0}} : \mathbb{0}\oplus\mathbb{0} \longrightarrow \mathbb{0}$.

The naturality of ζ^{ℓ}_A with respect to A follows from the naturality of $\lambda^{\bullet}$ and the functoriality of $\oplus$ in C. $\square$

Next we interpret the natural isomorphism ζ^{ℓ} in Lemma 8.1.10 in terms of paths and coherence.

Example 8.1.14 (Left Zero via Paths)**.** In (8.1.13), $(\zeta^{\ell}_A)_{ki}$ is the unique value in C of any 0^X-reduction

$$\left(\bigoplus_{j=1}^{n}(0^X\otimes x_j)\right)_{\mathsf{lt}} \xrightarrow{Z^{\ell}} 0^X \tag{8.1.15}$$

in the sense of (3.1.18) and Definition 3.3.4. Here we take the set

$$X^{\ell} = \left\{0^X, 1^X, x_1, \ldots, x_n\right\} \tag{8.1.16}$$

and the function $\varphi^{\ell} : X^{\ell} \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by

$$\varphi^{\ell}(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \text{ and} \\ A_{ji} & \text{if } x = x_j \text{ for } 1 \le j \le n. \end{cases} \tag{8.1.17}$$

The existence of a 0^X-reduction Z^{ℓ} in (8.1.15) with the given (co)domain follows from Lemmas 3.3.6 and 3.3.11. The uniqueness of its value in C follows from Lemma 3.3.12. $\diamond$

Next is the case with the 0 matrix on the right side.

Lemma 8.1.18. *Suppose C is a bimonoidal category, and $m, n, q \geq 0$. Then there is a natural isomorphism*

$$A\mathbb{0}_{q,m} \xrightarrow[\cong]{\zeta^r_A} \mathbb{0}_{q,n} \quad \in \mathsf{Mat}^{\mathsf{C}}_{q,n} \tag{8.1.19}$$

for $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.

Proof. There are three cases.

- If either q or n is 0, then both $A\mathbb{0}_{q,m}$ and $\mathbb{0}_{q,n}$ are the empty matrix.
- If $q, n > 0$ and $m = 0$, then both A and $\mathbb{0}_{q,m}$ are the empty matrix, but $A\mathbb{0}_{q,m} = \mathbb{0}_{q,n}$.

In each of these two cases, ζ^r_A is the identity morphism.

For $m,n,q > 0$, suppose $A = (A_{ji})$. For $1 \leq j \leq n$ and $1 \leq l \leq q$, $A\mathbb{0}_{q,m}$ has (j,l)-entry

$$(A\mathbb{0}_{q,m})_{jl} = \Big(\bigoplus_{i=1}^{m} (A_{ji} \otimes \mathbb{0})\Big)_{\mathsf{lt}}. \tag{8.1.20}$$

This object is isomorphic to the additive zero $\mathbb{0}$ via the following composite in C.

$$\begin{tikzcd}
(A\mathbb{0}_{q,m})_{jl} \arrow[d, "(\rho^{\bullet}_{A_{j1}} \oplus \cdots \oplus \rho^{\bullet}_{A_{jm}})_{\mathsf{lt}}"'] \arrow[dr, bend left, "(\zeta^r_A)_{jl}"] & \\
\Big(\bigoplus_{i=1}^{m} \mathbb{0}\Big)_{\mathsf{lt}} \arrow[r, "Z"] & \mathbb{0}
\end{tikzcd} \tag{8.1.21}$$

- If $m = 1$, then $Z = 1_{\mathbb{0}}$.
- If $m > 1$, then Z is a composite of $m-1$ morphisms, each being the sum of identity morphisms and one $\lambda^{\oplus}_{\mathbb{0}} : \mathbb{0} \oplus \mathbb{0} \longrightarrow \mathbb{0}$.

The naturality of ζ^r_A with respect to A follows from the naturality of $\rho^{\bullet}$ and the functoriality of $\oplus$ in C. □

The following example interprets the natural isomorphism ζ^r in Lemma 8.1.18 in terms of paths and coherence.

Example 8.1.22 (Right Zero via Paths)**.** The isomorphism $(\zeta^r_A)_{jl}$ in (8.1.21) is the unique value in C of any 0^X-reduction

$$\Big(\bigoplus_{i=1}^{m} (x_i \otimes 0^X)\Big)_{\mathsf{lt}} \xrightarrow{\ Z^r\ } 0^X \tag{8.1.23}$$

in the sense of (3.1.18) and Definition 3.3.4. Here we take the set

$$X^r = \big\{0^X, 1^X, x_1, \ldots, x_m\big\} \tag{8.1.24}$$

and the function $\varphi^r : X^r \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by

$$\varphi^r(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \text{ and} \\ A_{ji} & \text{if } x = x_i \text{ for } 1 \leq i \leq m. \end{cases} \tag{8.1.25}$$

The existence and the uniqueness of its value in C of a 0^X-reduction Z^r in (8.1.23) with the given (co)domain follow from Lemmas 3.3.6, 3.3.11, and 3.3.12. ⋄

8.2. The Base Unitors

In this section, we discuss coherence properties of the matrix product (8.1.4) with respect to the identity matrices (8.1.6). These coherence properties will be used to define the left unitor and the right unitor in $\mathsf{Mat}^{\mathsf{C}}$. Recall from Definition 3.9.9 the notion of a *flat* bimonoidal category.

The Base Left Unitor.

Lemma 8.2.1. *Suppose* C *is a flat bimonoidal category, and* $m,n \geq 0$. *Then there is a natural isomorphism*

$$\mathbb{1}^n A \xrightarrow[\cong]{\ell_A} A \tag{8.2.2}$$

for $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.

Proof. If either m or n is 0, then both $\mathbb{1}^n A$ and A are the empty matrix, and ℓ_A is the identity morphism of the empty matrix.

For $m,n > 0$, $1 \leq k \leq n$, and $1 \leq i \leq m$, $\mathbb{1}^n A$ has (k,i)-entry

$$\begin{aligned}
&(\mathbb{1}^n A)_{ki}\\
&= \Big(\bigoplus_{j=1}^{n} (\mathbb{1}^n_{kj} \otimes A_{ji}) \Big)_{\mathsf{lt}}\\
&= \Big(\underbrace{(\mathbb{0} \otimes A_{1i}) \oplus \cdots \oplus (\mathbb{0} \otimes A_{k-1,i})}_{\text{empty if } k=1} \oplus (\mathbb{1} \otimes A_{ki}) \oplus \underbrace{(\mathbb{0} \otimes A_{k+1,i}) \oplus \cdots \oplus (\mathbb{0} \otimes A_{ni})}_{\text{empty if } k=n} \Big)_{\mathsf{lt}}.
\end{aligned} \tag{8.2.3}$$

This object is isomorphic to A_{ki} via the following composite in C.

$$\begin{tikzcd}
(\mathbb{1}^n A)_{ki} \arrow[d, "(\lambda^{\bullet} \oplus \cdots \oplus \lambda^{\bullet} \oplus \lambda^{\otimes} \oplus \lambda^{\bullet} \oplus \cdots \oplus \lambda^{\bullet})_{\mathsf{lt}}"'] \arrow[dr, bend left, "(\ell_A)_{ki}"] & \\
\Big(\underbrace{\mathbb{0} \oplus \cdots \oplus \mathbb{0} \oplus}_{k-1 \text{ copies of } \mathbb{0}} A_{ki} \underbrace{\oplus \mathbb{0} \oplus \cdots \oplus \mathbb{0}}_{n-k \text{ copies of } \mathbb{0}} \Big)_{\mathsf{lt}} \arrow[r, "Z"] & A_{ki}
\end{tikzcd} \tag{8.2.4}$$

- If $n = 1$, then Z is the identity morphism.
- If $n > 1$, then Z is a composite of $n-1$ morphisms, each being the sum of identity morphisms and one of $\lambda^{\oplus}_{\mathbb{0}}$, $\lambda^{\oplus}_{A_{ki}}$, or $\rho^{\oplus}_{A_{ki}}$.

The naturality of ℓ_A with respect to A follows from the naturality of $\lambda^{\bullet}$, $\lambda^{\otimes}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$, and the functoriality of $\oplus$ in C. □

Next we interpret the natural isomorphism ℓ in Lemma 8.2.1 in terms of paths and coherence.

Example 8.2.5 (The Left Unitor via Paths)**.** In (8.2.4), $(\ell_A)_{ki}$ is the unique value in C of any path

$$\begin{tikzcd}
\Big(\underbrace{(0^X \otimes x_1) \oplus \cdots \oplus (0^X \otimes x_{k-1})}_{\text{empty if } k=1} \oplus (1^X \otimes x_k) \oplus \underbrace{(0^X \otimes x_{k+1}) \oplus \cdots \oplus (0^X \otimes x_n)}_{\text{empty if } k=n} \Big)_{\mathsf{lt}} \arrow[d, "P_\ell"] \\
x_k
\end{tikzcd} \tag{8.2.6}$$

in the sense of Definition 3.1.9 and (3.1.18). Here we take the set

$$X^{\ell} = \big\{ 0^X, 1^X, x_1, \ldots, x_n \big\}$$

in (8.1.16) and the function $\varphi^{\ell} : X^{\ell} \longrightarrow \mathsf{Ob}(\mathsf{C})$ in (8.1.17). In such a path P_ℓ in $\mathsf{Gr}(X^{\ell})$,

- the domain is obtained from the last line in (8.2.3) by replacing $(\mathbb{0}, \mathbb{1}, A_{ji})$ with $(0^X, 1^X, x_j)$ for $1 \leq j \leq n$, and
- the codomain is x_k.

For example, if $k = 2$ and $n = 3$, then one such P_ℓ is the following path.

$$\begin{tikzcd}
\big[(0^X \otimes x_1) \oplus (1^X \otimes x_2)\big] \oplus (0^X \otimes x_3) \arrow[rr, two heads, "P_\ell"] \arrow[d, "{[\lambda^\bullet_{x_1} \oplus 1] \oplus 1}"'] & & x_2 \\
\big[0^X \oplus (1^X \otimes x_2)\big] \oplus (0^X \otimes x_3) \arrow[d, "1 \oplus \lambda^\bullet_{x_3}"'] & & x_2 \oplus 0^X \arrow[u, "\rho^\oplus_{x_2}"'] \\
\big[0^X \oplus (1^X \otimes x_2)\big] \oplus 0^X \arrow[rr, "{[1 \oplus \lambda^\otimes_{x_2}] \oplus 1}"] & & \big[0^X \oplus x_2\big] \oplus 0^X \arrow[u, "\lambda^\oplus_{x_2} \oplus 1"']
\end{tikzcd}$$

There are other paths in $\mathsf{Gr}(X^\ell)$ with the same (co)domain as P_ℓ. For instance, after the first two prime edges, we can also use the prime edges involving $\lambda^\oplus_{1^X \otimes x_2}$ and $\rho^\oplus_{1^X \otimes x_2}$ before $\lambda^\otimes_{x_2}$. As we will explain below, all such paths yield the same morphism in C, so it does not matter which one we choose.

In general, one such path P_ℓ whose value in C gives the morphisms in (8.2.4) is as follows.

- Each of the first $n-1$ prime edges in P_ℓ contains an instance of the left multiplicative zero

$$0^X \otimes x_j \xrightarrow{\lambda^\bullet_{x_j}} 0^X \quad \text{for} \quad 1 \leq j \neq k \leq n.$$

- The nth prime edge in P_ℓ contains an instance of the left multiplicative unit

$$1^X \otimes x_k \xrightarrow{\lambda^\otimes_{x_k}} x_k.$$

- Each of the last $n-1$ prime edges in P_ℓ contains an instance of the left additive zero $\lambda^\oplus$ or the right additive zero $\rho^\oplus$. Together they remove the $n-1$ copies of 0^X.

This construction ensures the existence of at least one path P_ℓ with the given (co)domain.

There are infinitely many different paths in $\mathsf{Gr}(X^\ell)$ with the same (co)domain as P_ℓ. The canonical isomorphism $(\ell_A)_{ki}$ in (8.2.4) is the *unique* value in C of any path in $\mathsf{Gr}(X^\ell)$ with the given (co)domain. Indeed, since the codomain x_k is nonsymmetric regular in the sense of Definition 3.10.2, so is the domain by Lemma 3.10.3. The uniqueness of the value in C of any such path follows from the Bimonoidal Coherence Theorem 3.10.7, which is applicable by the flatness assumption on C. ◇

The Base Right Unitor. Next is the case with the identity matrix on the right. It will be used to define the right unitor in $\mathsf{Mat}^{\mathsf{C}}$.

Lemma 8.2.7. *Suppose* C *is a flat bimonoidal category, and* $m, n \geq 0$. *Then there is a natural isomorphism*

$$A\mathbb{1}^m \xrightarrow[\cong]{r_A} A \tag{8.2.8}$$

for $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.

Proof. If either m or n is 0, then both $A\mathbb{1}^m$ and A are the empty matrix, and r_A is the identity morphism of the empty matrix.

For $m,n > 0$, $1 \le j \le n$, and $1 \le h \le m$, $A\mathbb{1}^m$ has (j,h)-entry

$$\begin{aligned}
&(A\mathbb{1}^m)_{jh}\\
&= \Big(\bigoplus_{i=1}^{m} \big(A_{ji} \otimes \mathbb{1}^m_{ih}\big)\Big)_{\mathsf{lt}}\\
&= \Big(\underbrace{(A_{j1} \otimes \mathbb{0}) \oplus \cdots \oplus (A_{j,h-1} \otimes \mathbb{0})}_{\text{empty if } h=1} \oplus (A_{jh} \otimes \mathbb{1}) \oplus \underbrace{(A_{j,h+1} \otimes \mathbb{0}) \oplus \cdots \oplus (A_{jm} \otimes \mathbb{0})}_{\text{empty if } h=m}\Big)_{\mathsf{lt}}.
\end{aligned} \tag{8.2.9}$$

This object is isomorphic to A_{jh} via the following composite in C.

$$\begin{tikzcd}
(A\mathbb{1}^m)_{jh} \arrow[d, "{(\rho^{\bullet}\oplus\cdots\oplus\rho^{\bullet}\oplus\rho^{\otimes}\oplus\rho^{\bullet}\oplus\cdots\oplus\rho^{\bullet})_{\mathsf{lt}}}"'] \arrow[dr, bend left, "(r_A)_{jh}"] & \\
\Big(\underbrace{\mathbb{0}\oplus\cdots\oplus\mathbb{0}\oplus}_{h-1 \text{ copies of } \mathbb{0}} A_{jh} \underbrace{\oplus\mathbb{0}\oplus\cdots\oplus\mathbb{0}}_{m-h \text{ copies of } \mathbb{0}}\Big)_{\mathsf{lt}} \arrow[r, "Z"] & A_{jh}
\end{tikzcd} \tag{8.2.10}$$

- If $m = 1$, then Z is the identity morphism.
- If $m > 1$, then Z is a composite of $m-1$ morphisms, each being the sum of identity morphisms and one of $\lambda^{\oplus}_{\mathbb{0}}$, $\lambda^{\oplus}_{A_{jh}}$, or $\rho^{\oplus}_{A_{jh}}$.

The naturality of r_A with respect to A follows from the naturality of $\rho^{\bullet}$, $\rho^{\otimes}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$, and the functoriality of $\oplus$ in C. □

The following example interprets the natural isomorphism r in Lemma 8.2.7 in terms of paths and coherence.

Example 8.2.11 (The Right Unitor via Paths)**.** The isomorphism $(r_A)_{jh}$ in (8.2.10) is the unique value in C of any path P_r as follows.

$$\begin{tikzcd}
\Big(\underbrace{(x_1 \otimes 0^X) \oplus \cdots \oplus (x_{h-1} \otimes 0^X)}_{\text{empty if } h=1} \oplus (x_h \otimes 1^X) \oplus \underbrace{(x_{h+1} \otimes 0^X) \oplus \cdots \oplus (x_m \otimes 0^X)}_{\text{empty if } h=m}\Big)_{\mathsf{lt}} \arrow[d, "P_r"] \\
x_h
\end{tikzcd} \tag{8.2.12}$$

Here we take the set

$$X^r = \{0^X, 1^X, x_1, \ldots, x_m\}$$

in (8.1.24) and the function $\varphi^r : X^r \longrightarrow \mathsf{Ob}(\mathsf{C})$ in (8.1.25). In such a path P_r in $\mathsf{Gr}(X^r)$,

- the domain is obtained from the last line in (8.2.9) by replacing $(\mathbb{0}, \mathbb{1}, A_{ji})$ with $(0^X, 1^X, x_i)$ for $1 \le i \le m$, and
- the codomain is x_h.

One such path P_r whose value in C gives the morphisms in (8.2.10) is as follows.

- Each of the first $m-1$ prime edges in P_r contains an instance of the right multiplicative zero

$$x_i \otimes 0^X \xrightarrow{\rho^{\bullet}_{x_i}} 0^X \quad \text{for} \quad 1 \le i \ne h \le m.$$

- The mth prime edge in P_r contains an instance of the right multiplicative unit

$$x_h \otimes 1^X \xrightarrow{\rho^{\otimes}_{x_h}} x_h.$$

- Each of the last $m-1$ prime edges in P_r contains an instance of the left additive zero $\lambda^{\oplus}$ or the right additive zero $\rho^{\oplus}$. Together they remove the $m-1$ copies of 0^X.

This construction ensures the existence of at least one path P_r with the given (co)domain.

There are infinitely many different paths in $\mathsf{Gr}(X^r)$ with the same (co)domain as P_r. The canonical isomorphism $(r_A)_{jh}$ in (8.2.10) is the *unique* value in C of any path in $\mathsf{Gr}(X^r)$ with the given (co)domain. Indeed, since the codomain x_h is nonsymmetric regular in the sense of Definition 3.10.2, so is the domain by Lemma 3.10.3. The uniqueness of the value in C of any such path follows from the Bimonoidal Coherence Theorem 3.10.7, which is applicable by the flatness assumption on C. ◇

8.3. The Base Associator

In this section, we discuss coherence properties of the matrix product (8.1.4) with respect to associativity. The next lemma will be used to define the associator in $\mathsf{Mat}^{\mathsf{C}}$. Recall from Definition 2.1.2 that a *tight* bimonoidal category is one in which the distributivity morphisms δ^l and δ^r are natural isomorphisms, not just monomorphisms. Also recall from the introduction of this chapter our conventions for *Mac Lane coherence isomorphisms* and *Laplaza coherence isomorphisms*.

Lemma 8.3.1. *Suppose C is a tight bimonoidal category. For $m,n,p,q \ge 0$, there is a natural isomorphism*

$$(CB)A \xrightarrow[\cong]{a_{C,B,A}} C(BA) \;\in \mathsf{Mat}^{\mathsf{C}}_{m,q} \tag{8.3.2}$$

for $(A,B,C) \in \mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{n,p} \times \mathsf{Mat}^{\mathsf{C}}_{p,q}$.

Proof. There are five cases.

(1) If either m or q is 0, then both $(CB)A$ and $C(BA)$ are the empty matrix, and $a_{C,B,A}$ is the identity morphism of the empty matrix.
(2) If $m,q > 0$ and $n = p = 0$, then both $(CB)A$ and $C(BA)$ are the 0 matrix $\mathbb{0}_{m,q}$, and $a_{C,B,A}$ is the identity morphism.
(3) If $m,p,q > 0$ and $n = 0$, then $a_{C,B,A}$ is

$$(CB)A = \mathbb{0}_{m,q} \xrightarrow[\cong]{(\zeta^r_C)^{-1}} C\mathbb{0}_{m,p} = C(BA)$$

with ζ^r the natural isomorphism in (8.1.19).

(4) If $m,n,q > 0$ and $p = 0$, then $a_{C,B,A}$ is

$$(CB)A = \mathbb{0}_{n,q}A \xrightarrow[\cong]{\zeta^\ell_A} \mathbb{0}_{m,q} = C(BA)$$

with ζ^ℓ the natural isomorphism in (8.1.11).

In each of the above cases, a is natural.

For the remaining case with $m,n,p,q > 0$, consider the following data.

- $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ is an $n \times m$ matrix.
- $B = (B_{kj}) \in \mathsf{Mat}^{\mathsf{C}}_{n,p}$ is a $p \times n$ matrix.
- $C = (C_{lk}) \in \mathsf{Mat}^{\mathsf{C}}_{p,q}$ is a $q \times p$ matrix.

We first compute the entries of the matrix products $(CB)A$ and $C(BA) \in \mathsf{Mat}^{\mathsf{C}}_{m,q}$. For $1 \le i \le m$ and $1 \le l \le q$, the (l,i)-entry of $(CB)A$ is the following object in C.

$$\begin{aligned}[(CB)A]_{li} &= \Big(\bigoplus_{j=1}^{n} (CB)_{lj} \otimes A_{ji}\Big)_{\mathsf{lt}} \\ &= \Big(\bigoplus_{j=1}^{n} \Big(\bigoplus_{k=1}^{p} C_{lk} \otimes B_{kj}\Big)_{\mathsf{lt}} \otimes A_{ji}\Big)_{\mathsf{lt}}\end{aligned} \tag{8.3.3}$$

Similarly, the (l,i)-entry of $C(BA)$ is the following object in C.

$$\begin{aligned}[C(BA)]_{li} &= \Big(\bigoplus_{k=1}^{p} C_{lk} \otimes (BA)_{ki}\Big)_{\mathsf{lt}} \\ &= \Big(\bigoplus_{k=1}^{p} C_{lk} \otimes \Big(\bigoplus_{j=1}^{n} B_{kj} \otimes A_{ji}\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}}\end{aligned} \tag{8.3.4}$$

The (l,i)-entry of $a_{C,B,A}$ is the following composite of isomorphisms in C.

$$\begin{tikzcd}
{[(CB)A]_{li}} \arrow[rr, "(a_{C,B,A})_{li}"] \arrow[d, equal] & & {[C(BA)]_{li}} \arrow[d, equal] \\
\Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} C_{lk} \otimes B_{kj}\Big)_{\mathsf{lt}} \otimes A_{ji}\Big)_{\mathsf{lt}} \arrow[d, "a^1"] & & \Big(\bigoplus_{k=1}^{p} C_{lk} \otimes \Big(\bigoplus_{j=1}^{n} B_{kj} \otimes A_{ji}\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \\
\Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} (C_{lk} \otimes B_{kj}) \otimes A_{ji}\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[dr, "a^2"'] & & \Big(\bigoplus_{k=1}^{p}\Big(\bigoplus_{j=1}^{n} C_{lk} \otimes (B_{kj} \otimes A_{ji})\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[u, "a^4"'] \\
& \Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} C_{lk} \otimes (B_{kj} \otimes A_{ji})\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[ur, "a^3"'] &
\end{tikzcd} \tag{8.3.5}$$

Consider the diagram (8.3.5).

(1) a^1 is the identity morphism if $p = 1$. If $p > 1$, then a^1 is a composite of $n(p-1)$ morphisms, each being the sum of identity morphisms and one

component of the right distributivity morphism

$$\begin{tikzcd}[row sep=large] \Big[\Big(\bigoplus_{k=1}^{p'-1} C_{lk} \otimes B_{kj}\Big)_{\mathsf{lt}} \oplus \big(C_{lp'} \otimes B_{p'j}\big)\Big] \otimes A_{ji} \arrow[d, "\delta^r_{\left(\oplus_{k=1}^{p'-1} C_{lk}\otimes B_{kj}\right)_{\mathsf{lt}},\, (C_{lp'}\otimes B_{p'j}),\, A_{ji}}"] \\ \Big[\Big(\bigoplus_{k=1}^{p'-1} C_{lk} \otimes B_{kj}\Big)_{\mathsf{lt}} \otimes A_{ji}\Big] \oplus \Big[\big(C_{lp'} \otimes B_{p'j}\big) \otimes A_{ji}\Big] \end{tikzcd} \tag{8.3.6}$$

for $1 \leq j \leq n$ and $2 \leq p' \leq p$. This is an isomorphism because C is tight.

(2) a^2 is a composite of np morphisms, each being the sum of identity morphisms and one component of the multiplicative associativity

$$\big(C_{lk} \otimes B_{kj}\big) \otimes A_{ji} \xrightarrow{\alpha^{\otimes}_{C_{lk},B_{kj},A_{ji}}} C_{lk} \otimes \big(B_{kj} \otimes A_{ji}\big) \tag{8.3.7}$$

for $1 \leq j \leq n$ and $1 \leq k \leq p$. Equivalently, by the functoriality of $\oplus$, a^2 is the sum

$$a^2 = \Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} \alpha^{\otimes}_{C_{lk},B_{kj},A_{ji}}\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}}.$$

(3) a^3 is a Mac Lane coherence isomorphism $\cong^{\oplus}_{\mathsf{ML}}$ in $(\mathsf{C}, \oplus)$ that involves only $\alpha^{\pm\oplus}$ and $\xi^{\oplus}$. It regards the np objects $C_{lk} \otimes \big(B_{kj} \otimes A_{ji}\big)$ as formal variables. Its existence and uniqueness follow from Theorem 1.3.8.

(4) a^4 is the identity morphism if $n = 1$. If $n > 1$, then a^4 is a composite of $p(n-1)$ morphisms, each being the sum of identity morphisms and one instance of

$$\begin{tikzcd}[row sep=large] \Big[C_{lk} \otimes \Big(\bigoplus_{j=1}^{n'-1} B_{kj} \otimes A_{ji}\Big)_{\mathsf{lt}}\Big] \oplus \Big[C_{lk} \otimes \big(B_{kn'} \otimes A_{n'i}\big)\Big] \arrow[d, "(\delta^l)^{-1}_{C_{lk},\, \left(\oplus_{j=1}^{n'-1} B_{kj}\otimes A_{ji}\right)_{\mathsf{lt}},\, (B_{kn'}\otimes A_{n'i})}"] \\ C_{lk} \otimes \Big(\bigoplus_{j=1}^{n'} B_{kj} \otimes A_{ji}\Big)_{\mathsf{lt}} \end{tikzcd} \tag{8.3.8}$$

for $1 \leq k \leq p$ and $2 \leq n' \leq n$. The tightness assumption on C is used here to make sure that the left distributivity morphism δ^l is invertible.

The naturality of $a_{C,B,A}$ follows from the naturality of $\alpha^{\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, δ^l, and δ^r, and the functoriality of $\oplus$ in C. □

Next we interpret the natural isomorphism a in Lemma 8.3.1 in terms of paths and coherence.

Example 8.3.9 (The Associator via Paths)**.** To realize $(a_{C,B,A})_{li}$ in (8.3.5) as the values in C of paths, for $1 \leq i \leq m$ and $1 \leq l \leq q$, we consider the set

$$X^a = \big\{0^X, 1^X, a_{ji}, b_{kj}, c_{lk}\big\}_{1 \leq j \leq n,\, 1 \leq k \leq p} \tag{8.3.10}$$

with $n + np + p + 2$ elements and the function $\varphi^a : X^a \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as

$$\varphi^a(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \\ A_{ji} & \text{if } x = a_{ji} \text{ for } 1 \leq j \leq n, \\ B_{kj} & \text{if } x = b_{kj} \text{ for } 1 \leq j \leq n \text{ and } 1 \leq k \leq p, \text{ and} \\ C_{lk} & \text{if } x = c_{lk} \text{ for } 1 \leq k \leq p. \end{cases} \tag{8.3.11}$$

Define the following four paths in $\mathsf{Gr}(X^a)$, in which the direction of P_a^4 is not an error.

$$\begin{tikzcd}[column sep=tiny] \Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} c_{lk} \otimes b_{kj}\Big)_{\mathsf{lt}} \otimes a_{ji}\Big)_{\mathsf{lt}} \arrow[d, "P_a^1"'] & & \Big(\bigoplus_{k=1}^{p} c_{lk} \otimes \Big(\bigoplus_{j=1}^{n} b_{kj} \otimes a_{ji}\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[d, "P_a^4"] \\ \Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} (c_{lk} \otimes b_{kj}) \otimes a_{ji}\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[dr, "P_a^2"'] & & \Big(\bigoplus_{k=1}^{p}\Big(\bigoplus_{j=1}^{n} c_{lk} \otimes (b_{kj} \otimes a_{ji})\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \\ & \Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} c_{lk} \otimes (b_{kj} \otimes a_{ji})\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[ur, "P_a^3"'] & \end{tikzcd} \tag{8.3.12}$$

Consider (8.3.12).

(1) For each $1 \leq h \leq 3$, the path P_a^h is obtained from the corresponding isomorphism a^h in (8.3.5) by replacing the objects (A_{ji}, B_{kj}, C_{lk}) in C with the elements (a_{ji}, b_{kj}, c_{lk}) in X^a.
   - Each prime edge in P_a^1 is an identity or involves an instance of δ^r.
   - Each prime edge in P_a^2 is an identity or involves an instance of $\alpha^{\otimes}$.
   - Each prime edge in P_a^3 is an identity or involves an instance of $\alpha^{\pm\oplus}$ or $\xi^{\oplus}$.

   The value of the path P_a^h in C is the isomorphism a^h in (8.3.5). Since $\xi^{\pm\otimes}$ are not involved, the path (P_a^3, P_a^2, P_a^1) in $\mathsf{Gr}(X^a)$ realizes the composite $a^3a^2a^1$ in C as a Laplaza coherence isomorphism $\cong_{\mathsf{Lap}}$.

(2) The path P_a^4 is obtained from the isomorphism a^4 in (8.3.5) by
   - replacing the objects (A_{ji}, B_{kj}, C_{lk}) in C with the elements (a_{ji}, b_{kj}, c_{lk}) in X^a and
   - reversing the direction of each $(\delta^l)^{-1}$ in (8.3.8) and replacing it with δ^l in $\mathsf{Gr}(X^a)$.

   The reason that P_a^4 is defined in this way is that the elementary edges δ^l and δ^r do *not* have formal inverses; see Definitions 3.1.6 and 3.1.10. The value of P_a^4 in C is $(a^4)^{-1}$. Since $\xi^{\pm\otimes}$ are not involved, the value of P_a^4 in C is a Laplaza coherence isomorphism. Therefore, a^4 is an inverse Laplaza coherence isomorphism $\cong_{\mathsf{Lap}}^{-1}$.

In summary, using the paths P_a^h in $\mathsf{Gr}(X^a)$ for $1 \leq h \leq 4$ in (8.3.12), the isomorphism $(a_{C,B,A})_{li}$ in (8.3.5) is the following composite of a Laplaza coherence isomorphism followed by an inverse Laplaza coherence isomorphism.

$$\begin{tikzcd}[column sep=small]
{[(CB)A]_{li}} \arrow[rr, "(a_{C,B,A})_{li}"] \arrow[d, equal] & & {[C(BA)]_{li}} \arrow[d, equal] \\
\Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} C_{lk}\otimes B_{kj}\Big)_{\mathsf{lt}}\otimes A_{ji}\Big)_{\mathsf{lt}} \arrow[dr, "a^3a^2a^1"', "\cong_{\mathsf{Lap}}"] & & \Big(\bigoplus_{k=1}^{p} C_{lk}\otimes\Big(\bigoplus_{j=1}^{n} B_{kj}\otimes A_{ji}\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \\
& \Big(\bigoplus_{k=1}^{p}\Big(\bigoplus_{j=1}^{n} C_{lk}\otimes\big(B_{kj}\otimes A_{ji}\big)\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[ur, "\cong^{-1}_{\mathsf{Lap}}", "a^4"'] &
\end{tikzcd} \tag{8.3.13}$$

The decomposition (8.3.13) has the following uniqueness property. Suppose given a solid arrow diagram in C of the following form.

$$\begin{tikzcd}
& \mathsf{dom}(a^4) \arrow[dr, bend left, "\cong^{-1}_{\mathsf{Lap}}", "a^4"'] \arrow[dd, dotted, "\cong_{\mathsf{Lap}}\text{ or }\cong^{-1}_{\mathsf{Lap}}"'] & \\
{[(CB)A]_{li}} \arrow[ur, bend left, "\cong_{\mathsf{Lap}}", "a^3a^2a^1"'] \arrow[dr, bend right, "\cong_{\mathsf{Lap}}"'] & & {[C(BA)]_{li}} \\
& \bullet \arrow[ur, bend right, "\cong^{-1}_{\mathsf{Lap}}"'] &
\end{tikzcd} \tag{8.3.14}$$

- The top composite is the factorization of $(a_{C,B,A})_{li}$ in (8.3.13).
- The bottom composite is some Laplaza coherence isomorphism followed by some inverse Laplaza coherence isomorphism, defined using the same set X^a in (8.3.10) and the same function φ^a in (8.3.11).

Suppose, furthermore, that there exists a dotted arrow as indicated that is either a Laplaza coherence isomorphism or an inverse Laplaza coherence isomorphism, defined with the same set X^a and function φ^a. Then the bottom composite in (8.3.14) is also equal to $(a_{C,B,A})_{li}$.

Indeed, first note that the codomain of P^1_a is nonsymmetric regular in the sense of Definition 3.10.2. Lemma 3.10.3 then implies that each vertex in (8.3.12) is nonsymmetric regular. By the tightness assumption on C, the Bimonoidal Coherence Theorem 3.9.1 implies that each of the two subdiagrams in (8.3.14) is commutative in C. This uniqueness property and similar technique will be used in later sections to prove (monoidal) bicategory axioms related to a. ◇

Example 8.3.15 (Bipermutative Categories)**.** Suppose C is a right or left bipermutative category as in Definitions 2.5.2 and 2.5.11. Then $\mathbb{1}$ is a strict multiplicative unit, and $\mathbb{0}$ is a strict additive zero and a strict multiplicative zero. It follows from (8.1.13), (8.1.21), (8.2.4), and (8.2.10) that there are equalities as follows for $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.

$$\begin{aligned}
\mathbb{0}_{n,p}A &= \mathbb{0}_{m,p} \in \mathsf{Mat}^{\mathsf{C}}_{m,p}\\
A\mathbb{0}_{q,m} &= \mathbb{0}_{q,n} \in \mathsf{Mat}^{\mathsf{C}}_{q,n}\\
\mathbb{1}^n A = A &= A\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,n}
\end{aligned}$$

Moreover, in the context of Lemma 8.3.1, the following statements hold.

- If one of m, n, p, q is 0 (the first four cases), then $a_{C,B,A}$ is the identity morphism.

- If $m, n, p, q > 0$, then $a_{C,B,A}$ in (8.3.5) is *not* the identity in general because it involves both δ^r in (8.3.6) and δ^l in (8.3.8).

This means that even for a right or a left bipermutative category C, the bicategory $\mathsf{Mat}^{\mathsf{C}}$ in Definition 8.4.11 below is *not* in general a 2-category in the sense of Definition 6.1.8. ◇

8.4. The Matrix Bicategory

In this section, we assemble the constructions in the previous sections to construct the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ for a tight bimonoidal category C; see Theorem 8.4.12. As an immediate consequence, we obtain a monoidal category $\mathsf{Mat}^{\mathsf{C}}_{n,n}$ with objects $n \times n$ matrices in C and with the monoidal product given by the matrix product; see Corollary 8.4.14. This result on $\mathsf{Mat}^{\mathsf{C}}_{n,n}$ is stated in [**BDR04**, Prop. 3.3], with the additional assumption that C is symmetric.

Motivation 8.4.1. Recall from Definition 6.1.2 the notion of a bicategory, which includes the unity axiom (6.1.3) and the pentagon axiom (6.1.4). Each of these axioms asserts that some diagram in some category $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ is commutative. Since morphisms in each $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ are defined entrywise in C, checking those two axioms amounts to checking some commutative diagrams in C. In each case, we will realize the relevant diagram in C using paths in $\mathsf{Gr}(X)$ in the sense of Definition 3.1.14 for a suitable set X of formal variables. The Bimonoidal Coherence Theorem 3.10.7 then implies the commutativity of the diagram in C. ◇

The Unity Axiom. The unity axiom in $\mathsf{Mat}^{\mathsf{C}}$ is proved in the next lemma. Recall the matrix product and the identity matrices in Definition 8.1.3 and Explanation 8.1.7 and the natural isomorphisms ℓ, r, and a in, respectively, Lemmas 8.2.1, 8.2.7, and 8.3.1.

Lemma 8.4.2. *For each tight bimonoidal category* C*, the diagram*

$$\begin{tikzcd}[column sep=small] (B\mathbb{1}^n)A \arrow[rr, "a_{B,\mathbb{1}^n,A}"] \arrow[dr, "r_B \star 1_A"'] & & B(\mathbb{1}^n A) \arrow[dl, "1_B \star \ell_A"] \\ & BA & \end{tikzcd} \tag{8.4.3}$$

in $\mathsf{Mat}^{\mathsf{C}}_{m,p}$ *is commutative for matrices* $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ *and* $B \in \mathsf{Mat}^{\mathsf{C}}_{n,p}$.

Proof. If one of m, n, or p is 0, then all three morphisms in (8.4.3) are identity morphisms.

Suppose $m, n, p > 0$. It suffices to show that the (k,i)-entries of the two composites in (8.4.3) are equal for $1 \le i \le m$ and $1 \le k \le p$. Using (8.1.4), (8.1.5), (8.3.3), and (8.3.4), the (k,i)-entry of the diagram (8.4.3) is the following diagram in C.

(8.4.4)
$$\begin{tikzcd}
{[(B\mathbb{1}^n)A]_{ki}} \arrow[rr, "(a_{B,\mathbb{1}^n,A})_{ki}"] \arrow[d, equal] & & {[B(\mathbb{1}^n A)]_{ki}} \arrow[d, equal] \\
\Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{l=1}^{n} B_{kl}\otimes \mathbb{1}^n_{lj}\Big)_{\mathsf{lt}}\otimes A_{ji}\Big)_{\mathsf{lt}} \arrow[d, "\Big(\bigoplus_{j=1}^{n}(r_B)_{kj}\otimes 1_{A_{ji}}\Big)_{\mathsf{lt}}"] & & \Big(\bigoplus_{l=1}^{n} B_{kl}\otimes\Big(\bigoplus_{j=1}^{n}\mathbb{1}^n_{lj}\otimes A_{ji}\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[d, "\Big(\bigoplus_{l=1}^{n}1_{B_{kl}}\otimes(\ell_A)_{li}\Big)_{\mathsf{lt}}"] \\
\Big(\bigoplus_{j=1}^{n} B_{kj}\otimes A_{ji}\Big)_{\mathsf{lt}} \arrow[r, "="] & (BA)_{ki} & \Big(\bigoplus_{l=1}^{n} B_{kl}\otimes A_{li}\Big)_{\mathsf{lt}} \arrow[l, "="']
\end{tikzcd}$$

To show that this diagram is commutative, we first realize its edges as the values in C of paths, in the sense of (3.1.18), defined as follows.

Consider the set

$$X = \big\{0^X, 1^X, a_1, \ldots, a_n, b_1, \ldots, b_n\big\}$$

and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \\ A_{ji} & \text{if } x = a_j \text{ for } 1 \le j \le n, \text{ and} \\ B_{kj} & \text{if } x = b_j \text{ for } 1 \le j \le n. \end{cases}$$

There is a diagram

(8.4.5)
$$\begin{tikzcd}
& \Big(\bigoplus_{l=1}^{n}\Big(\bigoplus_{j=1}^{n} b_l\otimes(\delta^X_{lj}\otimes a_j)\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[dd, two heads, "M"] & \\
\Big(\bigoplus_{j=1}^{n}\Big(\bigoplus_{l=1}^{n} b_l\otimes\delta^X_{lj}\Big)_{\mathsf{lt}}\otimes a_j\Big)_{\mathsf{lt}} \arrow[ur, two heads, bend left, "(P^3_a, P^2_a, P^1_a)"] \arrow[dr, two heads, bend right, "R"'] & & \Big(\bigoplus_{l=1}^{n} b_l\otimes\Big(\bigoplus_{j=1}^{n}\delta^X_{lj}\otimes a_j\Big)_{\mathsf{lt}}\Big)_{\mathsf{lt}} \arrow[ul, two heads, bend right, "P^4_a"'] \arrow[dl, two heads, bend left, "L"] \\
& \Big(\bigoplus_{j=1}^{n} b_j\otimes a_j\Big)_{\mathsf{lt}} &
\end{tikzcd}$$

in $\mathsf{Gr}(X)$ defined as follows.

(1) The element $\delta^X_{lj} \in X$ is defined as

(8.4.6)
$$\delta^X_{lj} = \begin{cases} 0^X & \text{if } l \neq j \text{ and} \\ 1^X & \text{if } l = j. \end{cases}$$

(2) The component $(a_{B,\mathbb{1}^n,A})_{ki}$ in (8.4.4) decomposes as $a^4a^3a^2a^1$ as in (8.3.5). The paths P^h_a for $1 \le h \le 4$ in (8.4.5) are the ones in (8.3.12) with the symbols $(k,p,l,c_{lk},b_{kj},a_{ji})$ replaced by $(l,n,k,b_l,\delta^X_{lj},a_j)$. As we explained in Example 8.3.9,
- the path (P^3_a, P^2_a, P^1_a) has value $a^3a^2a^1$ in C, and
- the path P^4_a has value $(a^4)^{-1}$ in C.

(3) For $1 \leq j \leq n$, the component $(r_B)_{kj}$ in (8.4.4) is the isomorphism in (8.2.10) with the symbols (A,m,j,h) replaced by (B,n,k,j). By Example 8.2.11, each $(r_B)_{kj}$ is the value in C of a path P_r^j as in (8.2.12), defined with the subset $\{0^X,1^X,b_1,\ldots,b_n\}$. The path R in (8.4.5) is defined using
  - the paths P_r^j for $1 \leq j \leq n$ and
  - the constructions in Notation 3.3.10

  such that the value of R in C is the left edge in (8.4.4).

(4) For $1 \leq l \leq n$, the component $(\ell_A)_{li}$ in (8.4.4) is the isomorphism in (8.2.4) with the symbol k replaced by l. By Example 8.2.5, each $(\ell_A)_{li}$ is the value in C of a path P_ℓ^l as in (8.2.6), defined with the subset $\{0^X,1^X,a_1,\ldots,a_n\}$. The path L in (8.4.5) is defined using the paths
  - P_ℓ^l for $1 \leq l \leq n$ and
  - the constructions in Notation 3.3.10

  such that the value of L in C is the right edge in (8.4.4).

(5) Since δ_{lj}^X in (8.4.6) is either 0^X or 1^X, we define M in (8.4.5) as the path with the following prime edges.
  - For $1 \leq l \neq j \leq n$, M has prime edges containing
  $$b_l \otimes (0^X \otimes a_j) \xrightarrow{1_{b_l} \otimes \lambda^{\bullet}_{a_j}} b_l \otimes 0^X \xrightarrow{\rho^{\bullet}_{b_l}} 0^X.$$
  - For $1 \leq l = j \leq n$, M has prime edges containing
  $$b_j \otimes (1^X \otimes a_j) \xrightarrow{1_{b_j} \otimes \lambda^{\otimes}_{a_j}} b_j \otimes a_j.$$
  - After the above prime edges, M has prime edges containing $\lambda^{\oplus}$ or $\rho^{\oplus}$ that remove the $n(n-1)$ copies of 0^X additively.

The bottom middle vertex in (8.4.5) is nonsymmetric regular in the sense of Definition 3.10.2. By Lemma 3.10.3, the domain of R and the domain of L are also nonsymmetric regular. Using the tightness assumption on C, Theorem 3.10.7 implies that the diagram (8.4.5) is commutative in C. This, in turn, implies that the diagram (8.4.4) is commutative. □

The Pentagon Axiom. Next we prove the pentagon axiom (6.1.4) in $\mathsf{Mat}^{\mathsf{C}}$. To clarify the cases, we separate the proof into two lemmas. For each tight bimonoidal category C, consider the diagram

(8.4.7)
$$\begin{tikzcd}
& (DC)(BA) \arrow[dr, "a_{D,C,BA}"] & \\
((DC)B)A \arrow[ur, "a_{DC,B,A}"] \arrow[d, "a_{D,C,B} \star 1_A"'] & & D(C(BA)) \\
(D(CB))A \arrow[r, "a_{D,CB,A}"] & D((CB)A) \arrow[ur, "1_D \star a_{C,B,A}"'] &
\end{tikzcd}$$

in $\mathsf{Mat}^{\mathsf{C}}_{m,t}$ for
$$(A,B,C,D) \in \mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{n,p} \times \mathsf{Mat}^{\mathsf{C}}_{p,q} \times \mathsf{Mat}^{\mathsf{C}}_{q,t}.$$

The following proof involves the natural isomorphisms ζ^ℓ in (8.1.11) and ζ^r in (8.1.19).

Lemma 8.4.8. *If at least one of m, n, p, q, and t is 0, then the pentagon (8.4.7) is commutative.*

Proof. We consider all the possible cases.

(1) If either m or t is 0, then each edge in (8.4.7) is the identity morphism of the empty matrix.

(2) If $m, t > 0$ and $n = p = q = 0$, then each edge in (8.4.7) is the identity morphism of the 0 matrix $\mathbb{0}_{m,t}$.

(3) If $m, n, t > 0$ and $p = q = 0$, then the pentagon (8.4.7) is the following diagram.

$$\begin{tikzcd}
& \mathbb{0}_{m,t} \arrow[dr, "1"] & \\
\mathbb{0}_{n,t}A \arrow[ur, "\zeta^\ell_A"] \arrow[d, "1_{\mathbb{0}_{n,t}} \star 1_A"'] & & \mathbb{0}_{m,t} \\
\mathbb{0}_{n,t}A \arrow[r, "\zeta^\ell_A"] & \mathbb{0}_{m,t} \arrow[ur, "1"'] &
\end{tikzcd}$$

Since

$$1_{\mathbb{0}_{n,t}} \star 1_A = 1_{\mathbb{0}_{n,t}A}$$

by Lemma 8.1.8, the above pentagon is commutative.

(4) If $m, q, t > 0$ and $n = p = 0$, then the pentagon (8.4.7) is the following diagram.

$$\begin{tikzcd}
& \mathbb{0}_{m,t} \arrow[dr, "(\zeta^r_D)^{-1}"] & \\
\mathbb{0}_{m,t} \arrow[ur, "1"] \arrow[d, "1"'] & & D\mathbb{0}_{m,q} \\
\mathbb{0}_{m,t} \arrow[r, "(\zeta^r_D)^{-1}"] & D\mathbb{0}_{m,q} \arrow[ur, "1_D \star 1_{\mathbb{0}_{m,q}}"'] &
\end{tikzcd}$$

As in the previous case, this pentagon is commutative by Lemma 8.1.8.

(5) If $m, p, t > 0$ and $n = q = 0$, then the pentagon (8.4.7) is the following diagram.

$$\begin{tikzcd}
& \mathbb{0}_{m,t}\mathbb{0}_{m,t} \arrow[dr, "\zeta^\ell_{\mathbb{0}_{m,t}}"] & \\
\mathbb{0}_{m,t} \arrow[ur, "(\zeta^r_{\mathbb{0}_{m,t}})^{-1}"] \arrow[d, "1"'] & & \mathbb{0}_{m,t} \\
\mathbb{0}_{m,t} \arrow[r, "1"] & \mathbb{0}_{m,t} \arrow[ur, "1"'] &
\end{tikzcd}$$

Since

$$\lambda^\bullet_{\mathbb{0}} = \rho^\bullet_{\mathbb{0}} : \mathbb{0} \otimes \mathbb{0} \longrightarrow \mathbb{0}$$

by the axiom (2.1.14) in C, each entry of $\zeta^\ell_{\mathbb{0}_{m,t}}$ in (8.1.13) is equal to the corresponding entry of $\zeta^r_{\mathbb{0}_{m,t}}$ in (8.1.21). Therefore, the above pentagon is commutative.

(6) If $m, p, q, t > 0$ and $n = 0$, then the pentagon (8.4.7) is the left diagram below.

$$\begin{tikzcd}[column sep=small]
& (DC)\mathbb{0}_{m,p} \arrow[dr, "a_{D,C,\mathbb{0}_{m,p}}"] & \\
\mathbb{0}_{m,t} \arrow[ur, "(\zeta^r_{DC})^{-1}"] \arrow[d, "1"'] & & D(C\mathbb{0}_{m,p}) \\
\mathbb{0}_{m,t} \arrow[rr, "(\zeta^r_D)^{-1}"] & & D\mathbb{0}_{m,q} \arrow[u, "1_D \star (\zeta^r_C)^{-1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& v_1 \arrow[dl, ->>, "(Z^r_{DC})_{ki}"'] \arrow[r, ->>, "(P^3_a, P^2_a, P^1_a)"] & v_2 \arrow[dll, dashed, ->>] & v_3 \arrow[l, ->>, "P^4_a"'] \arrow[d, ->>, "\left(\bigoplus_{l=1}^{q} 1_{D_{kl}} \otimes (Z^r_C)_{li}\right)_{\mathsf{lt}}"] \\
0^X \arrow[d, "1"'] & & & \\
0^X & & & v_4 \arrow[lll, ->>, "(Z^r_D)_{ki}"]
\end{tikzcd}$$

It suffices to prove the commutativity of the left pentagon in the (k,i)-entry for $1 \le i \le m$ and $1 \le k \le t$. As in Example 8.1.22, the (k,i)-entry of ζ^r_{DC} is the unique value in C of a 0^X-reduction $(Z^r_{DC})_{ki}$. Similarly, $(Z^r_D)_{ki}$ and $(Z^r_C)_{li}$ are 0^X-reductions whose values in C are, respectively, $(\zeta^r_D)_{ki}$ and $(\zeta^r_C)_{li}$.

As in Example 8.3.9, the entry $(a_{D,C,\mathbb{0}_{m,p}})_{ki}$ is the value in C of a zigzag of paths $\left(P^4_a, (P^3_a, P^2_a, P^1_a)\right)$. With the appropriate inverses taken into account, the (k,i)-entry of the left pentagon is the value in C of the right pentagon for some elements v_1, v_2, v_3, and v_4. By Lemma 3.10.3, each of the four vertices $v_?$ has the same support as 0^X. By Lemmas 3.3.6 and 3.3.11, v_2 has a 0^X-reduction denoted by the dashed arrow. By Theorem 3.10.7, the following two statements hold.

- The two paths from v_1 to the lower left 0^X have the same value in C.
- The two paths from v_3 to the lower left 0^X have the same value in C.

Therefore, the (k,i)-entry of the left pentagon is commutative.

(7) If $m, n, q, t > 0$ and $p = 0$, then the pentagon (8.4.7) is the diagram below.

$$\begin{tikzcd}
& \mathbb{0}_{m,t} \arrow[dr, "(\zeta^r_D)^{-1}"] & \\
\mathbb{0}_{n,t}A \arrow[ur, "\zeta^\ell_A"] \arrow[d, "(\zeta^r_D)^{-1} \star 1_A"'] & & D\mathbb{0}_{m,q} \\
(D\mathbb{0}_{n,q})A \arrow[rr, "a_{D,\mathbb{0}_{n,q},A}"] & & D(\mathbb{0}_{n,q}A) \arrow[u, "1_D \star \zeta^\ell_A"']
\end{tikzcd}$$

As in case (6), it suffices to show the commutativity of the pentagon above in the (k,i)-entry, which is the value in C of the pentagon below with the appropriate inverses taken into account.

$$\begin{tikzcd}
& 0^X & \\
v_1 \arrow[ur, ->>, "(Z^\ell_A)_{ki}"] & & v_5 \arrow[ul, ->>, "(Z^r_D)_{ki}"'] \\
v_2 \arrow[u, ->>, "\left(\bigoplus_{j=1}^{n} (Z^r_D)_{kj} \otimes 1_{A_{ji}}\right)_{\mathsf{lt}}"] \arrow[r, ->>, "(P^3_a, P^2_a, P^1_a)"] & v_3 \arrow[uu, dashed, ->>] & v_4 \arrow[l, ->>, "P^4_a"'] \arrow[u, ->>, "\left(\bigoplus_{l=1}^{q} 1_{D_{kl}} \otimes (Z^\ell_A)_{li}\right)_{\mathsf{lt}}"']
\end{tikzcd}$$

Here $(Z^\ell_A)_{ki}$ is any 0^X-reduction as in (8.1.15) whose value in C is the (k,i)-entry of ζ^ℓ_A, and similarly for $(Z^\ell_A)_{li}$. Because of the top 0^X, each of the other five vertices has the same support as 0^X, and v_3 has a 0^X-reduction denoted by the dashed arrow. For each $h \in \{2,4\}$, the two paths from v_h

to the top 0^X have the same value in C by Theorem 3.10.7. Therefore, the (k,i)-entry of the pentagon is commutative.

(8) If $m,n,p,t > 0$ and $q = 0$, then the pentagon (8.4.7) is the diagram below.

$$\begin{tikzcd}
& \mathbb{0}_{p,t}(BA) \arrow[dr, "\zeta^{\ell}_{BA}"] & \\
(\mathbb{0}_{p,t}B)A \arrow[ur, "a_{\mathbb{0}_{p,t},B,A}"] \arrow[d, "\zeta^{\ell}_{B} \star 1_A"'] & & \mathbb{0}_{m,t} \\
\mathbb{0}_{n,t}A \arrow[rr, "\zeta^{\ell}_{A}"] & & \mathbb{0}_{m,t} \arrow[u, "1"']
\end{tikzcd}$$

As in cases (6) and (7), it suffices to show the commutativity of the pentagon above in the (k,i)-entry, which is the value in C of the pentagon below with the appropriate inverses taken into account.

$$\begin{tikzcd}
& v_2 \arrow[dr, dashed] & v_1 \arrow[l, "P^4_a"'] \arrow[d, "(Z^{\ell}_{BA})_{ki}"] \\
v_3 \arrow[ur, "(P^3_a, P^2_a, P^1_a)"] \arrow[d, "\Big(\bigoplus_{j=1}^{n} (Z^{\ell}_{B})_{kj} \otimes 1_{A_{ji}}\Big)_{\mathsf{lt}}"'] & & 0^X \\
v_4 \arrow[rr, "(Z^{\ell}_{A})_{ki}"] & & 0^X \arrow[u, "1"']
\end{tikzcd}$$

Because of the right 0^X, each of the four vertices $v_?$ has the same support as 0^X, and v_2 has a 0^X-reduction denoted by the dashed arrow. For each $h \in \{1,3\}$, the two paths from v_h to the right 0^X have the same value in C by Theorem 3.10.7. Therefore, the (k,i)-entry of the pentagon is commutative.

All the cases have been proved. □

Lemma 8.4.9. *If $m,n,p,q,t > 0$, then the pentagon (8.4.7) is commutative.*

Proof. It suffices to show that the (t',m')-entry of the pentagon (8.4.7) is commutative for $1 \le m' \le m$ and $1 \le t' \le t$. First we realize the (t',m')-entry of the pentagon (8.4.7) as the value in C of a diagram in $\mathsf{Gr}(X)$ with the following data.

- The set of formal variables is

$$X = \left\{0^X, 1^X, a_{jm'}, b_{kj}, c_{lk}, d_{t'l}\right\}$$

in which $1 \le j \le n$, $1 \le k \le p$, and $1 \le l \le q$.
- The function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is defined by

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \\ A_{jm'} & \text{if } x = a_{jm'} \text{ for } 1 \le j \le n, \\ B_{kj} & \text{if } x = b_{kj} \text{ for } 1 \le k \le p \text{ and } 1 \le j \le n, \\ C_{lk} & \text{if } x = c_{lk} \text{ for } 1 \le l \le q \text{ and } 1 \le k \le p, \text{ and} \\ D_{t'l} & \text{if } x = d_{t'l} \text{ for } 1 \le l \le q. \end{cases}$$

By Example 8.3.9, the (t', m')-entry of the pentagon (8.4.7) is the value in C of the outer pentagon in $\mathsf{Gr}(X)$ below with the appropriate inverses taken into account.

$$
\begin{tikzcd}
& & w_1 \arrow[dl, two heads] \arrow[dr, two heads] & & \\
& v_1 \arrow[dr, dashed, two heads] & & v_5 \arrow[dl, dashed, two heads] & \\
w_2 \arrow[ur, two heads] \arrow[d, two heads] & & w & & w_5 \arrow[ul, two heads] \arrow[d, two heads] \\
v_2 \arrow[urr, dashed, two heads] & & & & v_4 \arrow[ull, dashed, two heads] \\
w_3 \arrow[u, two heads] \arrow[rr, two heads] & & v_3 \arrow[uu, dashed, two heads] & & w_4 \arrow[u, two heads] \arrow[ll, two heads]
\end{tikzcd}
\tag{8.4.10}
$$

The corner vertices in (8.4.10) and their images under φ are as follows.

$$
\begin{aligned}
w_1 &= \Big[\bigoplus_{k=1}^{p}\Big(\bigoplus_{l=1}^{q} d_{t'l}\otimes c_{lk}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} b_{kj}\otimes a_{jm'}\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} & \varphi w_1 &= \big[(DC)(BA)\big]_{t'm'}\\
w_2 &= \Big[\bigoplus_{j=1}^{n}\Big\{\bigoplus_{k=1}^{p}\Big(\bigoplus_{l=1}^{q} d_{t'l}\otimes c_{lk}\Big)_{\mathsf{lt}} \otimes b_{kj}\Big\}_{\mathsf{lt}} \otimes a_{jm'}\Big]_{\mathsf{lt}} & \varphi w_2 &= \big[((DC)B)A\big]_{t'm'}\\
w_3 &= \Big[\bigoplus_{j=1}^{n}\Big\{\bigoplus_{l=1}^{q} d_{t'l}\otimes\Big(\bigoplus_{k=1}^{p} c_{lk}\otimes b_{kj}\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}} \otimes a_{jm'}\Big]_{\mathsf{lt}} & \varphi w_3 &= \big[(D(CB))A\big]_{t'm'}\\
w_4 &= \Big[\bigoplus_{l=1}^{q} d_{t'l}\otimes\Big\{\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} c_{lk}\otimes b_{kj}\Big)_{\mathsf{lt}} \otimes a_{jm'}\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}} & \varphi w_4 &= \big[D((CB)A)\big]_{t'm'}\\
w_5 &= \Big[\bigoplus_{l=1}^{q} d_{t'l}\otimes\Big\{\bigoplus_{k=1}^{p} c_{lk}\otimes\Big(\bigoplus_{j=1}^{n} b_{kj}\otimes a_{jm'}\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}} & \varphi w_5 &= \big[D(C(BA))\big]_{t'm'}
\end{aligned}
$$

The intermediate vertices in (8.4.10) are defined as follows.

$$
\begin{aligned}
v_1 &= \Big[\bigoplus_{k=1}^{p}\Big\{\bigoplus_{j=1}^{n}\Big(\bigoplus_{l=1}^{q} d_{t'l}\otimes c_{lk}\Big)_{\mathsf{lt}} \otimes \big(b_{kj}\otimes a_{jm'}\big)\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}\\
v_2 &= \Big[\bigoplus_{j=1}^{n}\Big\{\bigoplus_{l=1}^{q}\Big(\bigoplus_{k=1}^{p} d_{t'l}\otimes\big(c_{lk}\otimes b_{kj}\big)\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}} \otimes a_{jm'}\Big]_{\mathsf{lt}}\\
v_3 &= \Big[\bigoplus_{l=1}^{q}\Big\{\bigoplus_{j=1}^{n} d_{t'l}\otimes\Big[\Big(\bigoplus_{k=1}^{p} c_{lk}\otimes b_{kj}\Big)_{\mathsf{lt}} \otimes a_{jm'}\Big]\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}\\
v_4 &= \Big[\bigoplus_{l=1}^{q} d_{t'l}\otimes\Big\{\bigoplus_{k=1}^{p}\Big(\bigoplus_{j=1}^{n} c_{lk}\otimes\big(b_{kj}\otimes a_{jm'}\big)\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}\\
v_5 &= \Big[\bigoplus_{l=1}^{q}\Big\{\bigoplus_{k=1}^{p} d_{t'l}\otimes\Big[c_{lk}\otimes\Big(\bigoplus_{j=1}^{n} b_{kj}\otimes a_{jm'}\Big)_{\mathsf{lt}}\Big]\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}
\end{aligned}
$$

The middle vertex in (8.4.10) is defined as follows.

$$
w = \Big[\bigoplus_{l=1}^{q}\Big\{\bigoplus_{k=1}^{p}\Big(\bigoplus_{j=1}^{n} d_{t'l}\otimes\big[c_{lk}\otimes\big(b_{kj}\otimes a_{jm'}\big)\big]\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}
$$

For example, the (t', m')-entry

$$
\varphi w_2 = \big[((DC)B)A\big]_{t'm'} \xrightarrow{(a_{DC,B,A})_{t'm'}} \big[(DC)(BA)\big]_{t'm'} = \varphi w_1
$$

is the value in C of the zigzag of paths

$$w_2 \xrightarrow{(P_a^3,\, P_a^2,\, P_a^1)} v_1 \twoheadleftarrow^{P_a^4} w_1$$

as in (8.3.12), up to a change of symbols.

To show the commutativity of the outer pentagon in (8.4.10) in C, we join the intermediate vertices $v_?$ at the center as follows. The middle vertex w is not the domain of any distributivity morphism δ^l and δ^r. This fact implies that for each $1 \le s \le 5$, there is a path $v_s \twoheadrightarrow w$ that is denoted by a dashed arrow in (8.4.10) and is defined as follows.

(1) Consider the path $v_1 \twoheadrightarrow w$.

(i) For $1 \le k \le p$ and $1 \le j \le n$, it contains the following instances of δ^r,

$$\begin{gathered}
\Big[\Big(\bigoplus_{l=1}^{q'-1} d_{t'l} \otimes c_{lk}\Big)_{\mathsf{lt}} \oplus \big(d_{t'q'} \otimes c_{q'k}\big)\Big] \otimes \big(b_{kj} \otimes a_{jm'}\big) \\
\Big\downarrow \delta^r_{\left(\oplus_{l=1}^{q'-1} d_{t'l} \otimes c_{lk}\right)_{\mathsf{lt}},\, (d_{t'q'} \otimes c_{q'k}),\, (b_{kj} \otimes a_{jm'})} \\
\Big[\Big(\bigoplus_{l=1}^{q'-1} d_{t'l} \otimes c_{lk}\Big)_{\mathsf{lt}} \otimes \big(b_{kj} \otimes a_{jm'}\big)\Big] \oplus \Big[\big(d_{t'q'} \otimes c_{q'k}\big) \otimes \big(b_{kj} \otimes a_{jm'}\big)\Big]
\end{gathered}$$

for $2 \le q' \le q$ if $q > 1$, or the identity if $q = 1$.

(ii) It contains the following npq instances of $\alpha^{\otimes}$.

$$\big(d_{t'l} \otimes c_{lk}\big) \otimes \big(b_{kj} \otimes a_{jm'}\big) \xrightarrow{\alpha^{\otimes}_{d_{t'l},\, c_{lk},\, (b_{kj} \otimes a_{jm'})}} d_{t'l} \otimes \big[c_{lk} \otimes \big(b_{kj} \otimes a_{jm'}\big)\big]$$

(iii) It contains $\alpha^{\pm\oplus}$ and $\xi^{\oplus}$ that move brackets and permute additively.

(2) Consider the path $v_2 \twoheadrightarrow w$.

(i) For $1 \le j \le n$, it contains the following instances of δ^r,

$$\begin{gathered}
\Big[\Big\{\bigoplus_{l=1}^{q'-1} \Big(\bigoplus_{k=1}^{p} d_{t'l} \otimes \big(c_{lk} \otimes b_{kj}\big)\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}} \oplus \Big(\bigoplus_{k=1}^{p} d_{t'q'} \otimes \big(c_{q'k} \otimes b_{kj}\big)\Big)_{\mathsf{lt}}\Big] \otimes a_{jm'} \\
\Big\downarrow \delta^r_{\{\cdots\}_{\mathsf{lt}},\, (\cdots)_{\mathsf{lt}},\, a_{jm'}} \\
\Big[\Big\{\bigoplus_{l=1}^{q'-1} \Big(\bigoplus_{k=1}^{p} d_{t'l} \otimes \big(c_{lk} \otimes b_{kj}\big)\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}} \otimes a_{jm'}\Big] \oplus \Big[\Big(\bigoplus_{k=1}^{p} d_{t'q'} \otimes \big(c_{q'k} \otimes b_{kj}\big)\Big)_{\mathsf{lt}} \otimes a_{jm'}\Big]
\end{gathered}$$

for $2 \le q' \le q$ if $q > 1$, or the identity if $q = 1$.

(ii) For $1 \le j \le n$ and $1 \le l \le q$, it contains the following instances of δ^r,

$$\begin{gathered}
\Big[\Big(\bigoplus_{k=1}^{p'-1} d_{t'l} \otimes \big(c_{lk} \otimes b_{kj}\big)\Big)_{\mathsf{lt}} \oplus \Big(d_{t'l} \otimes \big(c_{lp'} \otimes b_{p'j}\big)\Big)\Big] \otimes a_{jm'} \\
\Big\downarrow \delta^r_{(\cdots)_{\mathsf{lt}},\, (\cdots),\, a_{jm'}} \\
\Big[\Big(\bigoplus_{k=1}^{p'-1} d_{t'l} \otimes \big(c_{lk} \otimes b_{kj}\big)\Big)_{\mathsf{lt}} \otimes a_{jm'}\Big] \oplus \Big[\Big(d_{t'l} \otimes \big(c_{lp'} \otimes b_{p'j}\big)\Big) \otimes a_{jm'}\Big]
\end{gathered}$$

for $2 \leq p' \leq p$ if $p > 1$, or the identity if $p = 1$.

(iii) It contains the following $2npq$ instances of $\alpha^{\otimes}$.

$$\Big(d_{t'l} \otimes \big(c_{lk} \otimes b_{kj}\big)\Big) \otimes a_{jm'} \xrightarrow{\alpha^{\otimes}} d_{t'l} \otimes \Big(\big(c_{lk} \otimes b_{kj}\big) \otimes a_{jm'}\Big)$$

$$\big(c_{lk} \otimes b_{kj}\big) \otimes a_{jm'} \xrightarrow{\alpha^{\otimes}} c_{lk} \otimes \big(b_{kj} \otimes a_{jm'}\big)$$

(iv) It contains $\alpha^{\pm\oplus}$ and $\xi^{\oplus}$ that move brackets and permute additively.

(3) Consider the path $v_3 \twoheadrightarrow w$.

(i) For $1 \leq l \leq q$ and $1 \leq j \leq n$, it contains the following instances of δ^r,

$$\begin{array}{c} \Big[\Big(\bigoplus_{k=1}^{p'-1} c_{lk} \otimes b_{kj}\Big)_{\mathsf{lt}} \oplus \big(c_{lp'} \otimes b_{p'j}\big)\Big] \otimes a_{jm'} \\ \Big\downarrow {\scriptstyle \delta^r_{(\cdots)_{\mathsf{lt}},\, (c_{lp'} \otimes b_{p'j}),\, a_{jm'}}} \\ \Big[\Big(\bigoplus_{k=1}^{p'-1} c_{lk} \otimes b_{kj}\Big)_{\mathsf{lt}} \otimes a_{jm'}\Big] \oplus \Big[\big(c_{lp'} \otimes b_{p'j}\big) \otimes a_{jm'}\Big] \end{array}$$

for $2 \leq p' \leq p$ if $p > 1$, or the identity if $p = 1$.

(ii) For $1 \leq l \leq q$ and $1 \leq j \leq n$, it contains the following instances of δ^l,

$$\begin{array}{c} d_{t'l} \otimes \Big[\Big(\bigoplus_{k=1}^{p'-1} \big(c_{lk} \otimes b_{kj}\big) \otimes a_{jm'}\Big)_{\mathsf{lt}} \oplus \Big(\big(c_{lp'} \otimes b_{p'j}\big) \otimes a_{jm'}\Big)\Big] \\ \Big\downarrow {\scriptstyle \delta^l_{d_{t'l},\, (\cdots)_{\mathsf{lt}},\, ((c_{lp'} \otimes b_{p'j}) \otimes a_{jm'})}} \\ \Big[d_{t'l} \otimes \Big(\bigoplus_{k=1}^{p'-1} \big(c_{lk} \otimes b_{kj}\big) \otimes a_{jm'}\Big)_{\mathsf{lt}}\Big] \oplus \Big[d_{t'l} \otimes \Big(\big(c_{lp'} \otimes b_{p'j}\big) \otimes a_{jm'}\Big)\Big] \end{array}$$

for $2 \leq p' \leq p$ if $p > 1$, or the identity if $p = 1$.

(iii) It contains the following npq instances of $\alpha^{\otimes}$.

$$\big(c_{lk} \otimes b_{kj}\big) \otimes a_{jm'} \xrightarrow{\alpha^{\otimes}} c_{lk} \otimes \big(b_{kj} \otimes a_{jm'}\big)$$

(iv) It contains $\alpha^{\pm\oplus}$ and $\xi^{\oplus}$ that move brackets and permute additively.

(4) Consider the path $v_4 \twoheadrightarrow w$.

(i) For $1 \leq l \leq q$, it contains the following instances of δ^l,

$$\begin{array}{c} d_{t'l} \otimes \Big[\Big\{\bigoplus_{k=1}^{p'-1} \Big(\bigoplus_{j=1}^{n} c_{lk} \otimes \big(b_{kj} \otimes a_{jm'}\big)\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}} \oplus \Big(\bigoplus_{j=1}^{n} c_{lp'} \otimes \big(b_{p'j} \otimes a_{jm'}\big)\Big)_{\mathsf{lt}}\Big] \\ \Big\downarrow {\scriptstyle \delta^l_{d_{t'l},\, \{\cdots\}_{\mathsf{lt}},\, (\cdots)_{\mathsf{lt}}}} \\ \Big[d_{t'l} \otimes \Big\{\bigoplus_{k=1}^{p'-1} \Big(\bigoplus_{j=1}^{n} c_{lk} \otimes \big(b_{kj} \otimes a_{jm'}\big)\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}}\Big] \oplus \Big[d_{t'l} \otimes \Big(\bigoplus_{j=1}^{n} c_{lp'} \otimes \big(b_{p'j} \otimes a_{jm'}\big)\Big)_{\mathsf{lt}}\Big] \end{array}$$

for $2 \leq p' \leq p$ if $p > 1$, or the identity if $p = 1$.

(ii) For $1 \leq l \leq q$ and $1 \leq k \leq p$, it contains the following instances of δ^l,

$$\begin{tikzcd}
d_{t'l} \otimes \Big[\Big(\bigoplus_{j=1}^{n'-1} c_{lk} \otimes (b_{kj} \otimes a_{jm'}) \Big)_{\mathsf{lt}} \oplus \big(c_{lk} \otimes (b_{kn'} \otimes a_{n'm'}) \big) \Big] \arrow[d, "\delta^l_{d_{t'l}, (\cdots)_{\mathsf{lt}}, (\cdots)}"] \\
\Big[d_{t'l} \otimes \Big(\bigoplus_{j=1}^{n'-1} c_{lk} \otimes (b_{kj} \otimes a_{jm'}) \Big)_{\mathsf{lt}} \Big] \oplus \Big[d_{t'l} \otimes \big(c_{lk} \otimes (b_{kn'} \otimes a_{n'm'}) \big) \Big]
\end{tikzcd}$$

for $2 \leq n' \leq n$ if $n > 1$, or the identity if $n = 1$.

(5) Consider the path $v_5 \longrightarrow\!\!\!\!\rightarrow w$.

(i) For $1 \leq l \leq q$ and $1 \leq k \leq p$, it contains the following instances of δ^l,

$$\begin{tikzcd}
c_{lk} \otimes \Big[\Big(\bigoplus_{j=1}^{n'-1} b_{kj} \otimes a_{jm'} \Big)_{\mathsf{lt}} \oplus \big(b_{kn'} \otimes a_{n'm'} \big) \Big] \arrow[d, "\delta^l_{c_{lk}, (\cdots)_{\mathsf{lt}}, (b_{kn'} \otimes a_{n'm'})}"] \\
\Big[c_{lk} \otimes \Big(\bigoplus_{j=1}^{n'-1} b_{kj} \otimes a_{jm'} \Big)_{\mathsf{lt}} \Big] \oplus \Big[c_{lk} \otimes \big(b_{kn'} \otimes a_{n'm'} \big) \Big]
\end{tikzcd}$$

for $2 \leq n' \leq n$ if $n > 1$, or the identity if $n = 1$.

(ii) For $1 \leq l \leq q$ and $1 \leq k \leq p$, it contains the following instances of δ^l,

$$\begin{tikzcd}
d_{t'l} \otimes \Big[\Big(\bigoplus_{j=1}^{n'-1} c_{lk} \otimes (b_{kj} \otimes a_{jm'}) \Big)_{\mathsf{lt}} \oplus \big(c_{lk} \otimes (b_{kn'} \otimes a_{n'm'}) \big) \Big] \arrow[d, "\delta^l_{d_{t'l}, (\cdots)_{\mathsf{lt}}, (c_{lk} \otimes (b_{kn'} \otimes a_{n'm'}))}"] \\
\Big[d_{t'l} \otimes \Big(\bigoplus_{j=1}^{n'-1} c_{lk} \otimes (b_{kj} \otimes a_{jm'}) \Big)_{\mathsf{lt}} \Big] \oplus \Big[d_{t'l} \otimes \big(c_{lk} \otimes (b_{kn'} \otimes a_{n'm'}) \big) \Big]
\end{tikzcd}$$

for $2 \leq n' \leq n$ if $n > 1$, or the identity if $n = 1$.

The vertex w is nonsymmetric regular in the sense of Definition 3.10.2. By Lemma 3.10.3, each vertex w_s for $1 \leq s \leq 5$ is also nonsymmetric regular. By the tightness assumption on C, Theorem 3.10.7 implies that each of the five subdiagrams in (8.4.10) is commutative in C. Therefore, the (t', m')-entry of the pentagon (8.4.7) is commutative. □

The Matrix Bicategory. Recall the concept of a bicategory in Definition 6.1.2.

Definition 8.4.11. Suppose C is a tight bimonoidal category. Define the *matrix bicategory*

$$\big(\mathsf{Mat}^{\mathsf{C}}, \mathbb{1}, c, a, \ell, r \big)$$

as consisting of the following data.

Objects: The objects in $\mathsf{Mat}^{\mathsf{C}}$ are nonnegative integers $n \geq 0$.

The Hom Categories: For $m, n \geq 0$, the hom category $\mathsf{Mat}^{\mathsf{C}}(m,n)$ is the category $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ in Definition 8.1.1.

The Identity 1-Cells: For each object $n \geq 0$, its identity 1-cell 1_n is the $n \times n$ identity matrix $\mathbb{1}^n \in \mathsf{Mat}^{\mathsf{C}}_{n,n}$ defined entrywise in (8.1.6).

The Horizontal Composition: For objects $m, n, p \geq 0$, the horizontal composition

$$\mathsf{Mat}^{\mathsf{C}}_{n,p} \times \mathsf{Mat}^{\mathsf{C}}_{m,n} \xrightarrow{\ c\ } \mathsf{Mat}^{\mathsf{C}}_{m,p}$$

is given by the matrix product in Definition 8.1.3.

The Associator: For objects $m, n, p, q \geq 0$, the associator is the natural isomorphism a in Lemma 8.3.1.

The Unitors: For objects $m, n \geq 0$, the left unitor ℓ and the right unitor r are the natural isomorphisms in, respectively, Lemmas 8.2.1 and 8.2.7.

This finishes the definition of $\mathsf{Mat}^{\mathsf{C}}$. $\diamond$

Theorem 8.4.12. *In Definition 8.4.11, $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory.*

Proof. The horizontal composition, which is given by the matrix product, is a functor by Lemma 8.1.8. Lemmas 8.2.1, 8.2.7, and 8.3.1 establish the naturality of ℓ, r, and a, respectively. The unity axiom (6.1.3) is proved in Lemma 8.4.2. The pentagon axiom (6.1.4) is proved in Lemmas 8.4.8 and 8.4.9. $\square$

Example 8.4.13 (Coordinatized 2-Vector Spaces)**.** Suppose C is the tight symmetric bimonoidal category $\mathsf{Vect}^{\mathbb{C}}$ of finite dimensional complex vector spaces in Example 2.1.32. Then the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ is the bicategory of *coordinatized 2-vector spaces* in [**JY21**, Ex. 2.1.28], denoted by $2\mathsf{Vect}_{\mathsf{c}}$ there. This is one incarnation of 2-vector spaces [**KV94**, Def. 5.2]. We will revisit 2-vector spaces in Example 8.15.5. $\diamond$

If A is an object in a bicategory B, then the hom category $\mathsf{B}(A, A)$ becomes a monoidal category with

- monoidal unit the identity 1-cell 1_A,
- monoidal product the horizontal composition

$$\mathsf{B}(A,A) \times \mathsf{B}(A,A) \longrightarrow \mathsf{B}(A,A),$$

 and
- associativity isomorphism and left/right unit isomorphisms the restrictions of, respectively, the associator and the left/right unitors of B.

The unity axiom (1.2.2) and the pentagon axiom (1.2.3) for the monoidal category $\mathsf{B}(A, A)$ are special cases of the corresponding axioms (6.1.3) and (6.1.4) for B. When applied to the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ in Theorem 8.4.12, we obtain the following result, which is stated in [**BDR04**, Prop. 3.3] without proof, under the additional assumption that C is symmetric.

Corollary 8.4.14. *For each tight bimonoidal category C and each $n \geq 1$, the category $\mathsf{Mat}^{\mathsf{C}}_{n,n}$ in Definition 8.1.1 is a monoidal category with*

- *monoidal unit the identity matrix $\mathbb{1}^n$ in (8.1.6),*
- *monoidal product the matrix product in (8.1.4) and (8.1.5),*
- *left unit isomorphism in (8.2.2),*
- *right unit isomorphism in (8.2.8), and*
- *associativity isomorphism in (8.3.2).*

In the rest of this chapter, we will extend the bicategory $\mathsf{Mat}^{\mathsf{C}}$ in Theorem 8.4.12 to a symmetric monoidal bicategory.

8.5. The Monoidal Identity

For each tight bimonoidal category C as in Definition 2.1.2, we observed in Theorem 8.4.12 that $\mathsf{Mat}^{\mathsf{C}}$ in Definition 8.4.11 is a bicategory in the sense of Definition 6.1.2. The main goal for the rest of this chapter is to show that, if C is also symmetric, then $\mathsf{Mat}^{\mathsf{C}}$ has the structure of a symmetric monoidal bicategory as in Definition 6.5.9. For the next several sections, we construct a monoidal bicategory structure as in Definition 6.4.1 on $\mathsf{Mat}^{\mathsf{C}}$. In this section, we construct its monoidal identity, which is a pseudofunctor in the sense of Definition 6.2.1.

Definition 8.5.1. For each tight bimonoidal category C, define the data of a lax functor

$$\mathbf{1} \xrightarrow{(1_\boxtimes, 1^2_\boxtimes, 1^0_\boxtimes)} \mathsf{Mat}^{\mathsf{C}}$$

as follows.

Object: The identity object $1_\boxtimes(*)$ is the integer $1 \in \mathsf{Mat}^{\mathsf{C}}$.

1-Cell: The identity 1-cell $1_* \in \mathbf{1}(*,*)$ is sent by $1_\boxtimes$ to the 1×1 identity matrix

$$\mathbb{1}^1 = (\mathbb{1}) \in \mathsf{Mat}^{\mathsf{C}}_{1,1}$$

whose only entry is the multiplicative unit $\mathbb{1} \in \mathsf{C}$.

2-Cell: The identity 2-cell $1_{1_*} \in \mathbf{1}(1_*, 1_*)$ is sent by $1_\boxtimes$ to the 1×1 matrix

$$(1_\mathbb{1}) \in \mathsf{Mat}^{\mathsf{C}}_{1,1}(\mathbb{1}^1, \mathbb{1}^1)$$

whose only entry is the identity morphism $1_\mathbb{1} : \mathbb{1} \longrightarrow \mathbb{1}$ in C.

The Lax Unity Constraint: Define

$$1_{1_\boxtimes(*)} = 1_1 = \mathbb{1}^1 = (\mathbb{1}) \xrightarrow{1^0_\boxtimes} (\mathbb{1}) = \mathbb{1}^1 = 1_\boxtimes(1_*)$$

as the identity 2-cell $(1_\mathbb{1})$.

The Lax Functoriality Constraint: Using the matrix product (8.1.4), define

$$1_\boxtimes(1_*)1_\boxtimes(1_*) = (\mathbb{1})(\mathbb{1}) = \big(\mathbb{1} \otimes \mathbb{1}\big) \xrightarrow{1^2_\boxtimes} (\mathbb{1}) = 1_\boxtimes(1_*) = 1_\boxtimes(1_* 1_*)$$

as the 1×1 matrix whose only entry is the left multiplicative unit

$$\lambda^\otimes_\mathbb{1} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1} \in \mathsf{C}.$$

This finishes the definition of the tuple $(1_\boxtimes, 1^2_\boxtimes, 1^0_\boxtimes)$. ◇

Lemma 8.5.2. *The tuple $(1_\boxtimes, 1^2_\boxtimes, 1^0_\boxtimes)$ in Definition 8.5.1 is a strictly unitary pseudofunctor.*

Proof. The assignment

$$\mathbf{1}(*,*) \xrightarrow{1_\boxtimes} \mathsf{Mat}^{\mathsf{C}}_{1,1}$$

sends the identity 2-cell 1_{1_*} to the identity 2-cell $(1_\mathbb{1})$, which is also equal to the vertical composition

$$(1_\mathbb{1})(1_\mathbb{1}) = (1_\mathbb{1} 1_\mathbb{1}).$$

Therefore, $1_\boxtimes$ is a functor.

The lax functoriality constraint $1^2_{\boxtimes}$ is natural because the only 2-cell in $\mathbf{1}$ is the identity 2-cell of 1_*. To show that $1_{\boxtimes}$ is a lax functor, it remains to check the two axioms (6.2.2) and (6.2.3).

Since

$$1_{\boxtimes}(1_*) = \mathbb{1}^1 = (\mathbb{1}),$$

the lax associativity axiom (6.2.2) in the current setting is the left diagram below in $\mathsf{Mat}^{\mathsf{C}}_{1,1}$.

$$\begin{tikzcd}
(\mathbb{1}^1\mathbb{1}^1)\mathbb{1}^1 \arrow[r, "a"] \arrow[d, "1^2_{\boxtimes}*1"'] & \mathbb{1}^1(\mathbb{1}^1\mathbb{1}^1) \arrow[d, "1*1^2_{\boxtimes}"] \\
\mathbb{1}^1\mathbb{1}^1 \arrow[d, "1^2_{\boxtimes}"'] & \mathbb{1}^1\mathbb{1}^1 \arrow[d, "1^2_{\boxtimes}"] \\
\mathbb{1}^1 \arrow[r, "1_{\boxtimes}(a)"] & \mathbb{1}^1
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathbb{1}\otimes\mathbb{1})\otimes\mathbb{1} \arrow[r, "\alpha^{\otimes}_{\mathbb{1},\mathbb{1},\mathbb{1}}"] \arrow[d, "\lambda^{\otimes}_{\mathbb{1}}\otimes 1_{\mathbb{1}}"'] & \mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{1}) \arrow[d, "1_{\mathbb{1}}\otimes\lambda^{\otimes}_{\mathbb{1}}"] \\
\mathbb{1}\otimes\mathbb{1} \arrow[r, "1"] \arrow[d, "\lambda^{\otimes}_{\mathbb{1}}"'] & \mathbb{1}\otimes\mathbb{1} \arrow[d, "\lambda^{\otimes}_{\mathbb{1}}"] \\
\mathbb{1} \arrow[r, "1"] & \mathbb{1}
\end{tikzcd} \tag{8.5.3}$$

The associator in the matrix bicategory is defined in Lemma 8.3.1. Using its explicit construction in (8.3.5), the only entry of a in (8.5.3) is the multiplicative associativity $\alpha^{\otimes}_{\mathbb{1},\mathbb{1},\mathbb{1}}$, which corresponds to (8.3.7). Moreover, the associator in $\mathbf{1}$ is the identity. Therefore, in (8.5.3) the commutativity of the left diagram is equivalent to the commutativity of the outer diagram in C on the right.

In the right diagram in (8.5.3), the following statements hold.

- The bottom square is commutative by definition.
- The top square is commutative by
  - the unity axiom (1.2.2) in $(\mathsf{C}, \otimes, \mathbb{1})$ and
  - the equality $\lambda^{\otimes}_{\mathbb{1}} = \rho^{\otimes}_{\mathbb{1}}$ in (1.2.6), applied to the top left vertical morphism.

This proves the lax associativity axiom (6.2.2).

In the current setting, the lax unity axiom (6.2.3) consists of the following two diagrams in $\mathsf{Mat}^{\mathsf{C}}_{1,1}$.

$$\begin{tikzcd}
\mathbb{1}^1\mathbb{1}^1 \arrow[r, "\ell"] \arrow[d, "1^0_{\boxtimes}*1"'] & \mathbb{1}^1 \\
\mathbb{1}^1\mathbb{1}^1 \arrow[r, "1^2_{\boxtimes}"] & \mathbb{1}^1 \arrow[u, "1_{\boxtimes}(\ell)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1}^1\mathbb{1}^1 \arrow[r, "r"] \arrow[d, "1^0_{\boxtimes}*1"'] & \mathbb{1}^1 \\
\mathbb{1}^1\mathbb{1}^1 \arrow[r, "1^2_{\boxtimes}"] & \mathbb{1}^1 \arrow[u, "1_{\boxtimes}(r)"']
\end{tikzcd} \tag{8.5.4}$$

Both ℓ and r in $\mathbf{1}$ are the identity. The unitors ℓ and r in $\mathsf{Mat}^{\mathsf{C}}$ are defined in Lemmas 8.2.1 and 8.2.7, respectively. Using their explicit constructions in (8.2.4) and (8.2.10), the two diagrams in (8.5.4) are commutative if and only if the following two diagrams in C are commutative.

$$\begin{tikzcd}
\mathbb{1}\otimes\mathbb{1} \arrow[r, "\lambda^{\otimes}_{\mathbb{1}}"] \arrow[d, "1"'] & \mathbb{1} \\
\mathbb{1}\otimes\mathbb{1} \arrow[r, "\lambda^{\otimes}_{\mathbb{1}}"] & \mathbb{1} \arrow[u, "1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1}\otimes\mathbb{1} \arrow[r, "\rho^{\otimes}_{\mathbb{1}}"] \arrow[d, "1"'] & \mathbb{1} \\
\mathbb{1}\otimes\mathbb{1} \arrow[r, "\lambda^{\otimes}_{\mathbb{1}}"] & \mathbb{1} \arrow[u, "1"']
\end{tikzcd}$$

The left square is commutative by definition. The right square is commutative by the equality $\lambda^{\otimes}_{\mathbb{1}} = \rho^{\otimes}_{\mathbb{1}}$ in (1.2.6). This proves the lax unity axiom (6.2.3), so $1_{\boxtimes}$ is a lax functor.

Since $1^0_{\boxtimes}$ is the identity 2-cell $(1_{\mathbb{1}})$ and since $1^2_{\boxtimes} = (\lambda^{\otimes}_{\mathbb{1}})$ is an isomorphism, $1_{\boxtimes}$ is a strictly unitary pseudofunctor. □

8.6. The Monoidal Composition

For the rest of this chapter, C denotes a tight *symmetric* bimonoidal category as in Definition 2.1.2. We are in the process of constructing a monoidal bicategory structure on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$. In Section 8.5, we constructed its monoidal identity. In this section, we construct the monoidal composition

$$(\boxtimes, \boxtimes^2, \boxtimes^0) : \mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}} \longrightarrow \mathsf{Mat}^{\mathsf{C}}.$$

Here is an outline of this section.

- We define the matrix tensor product for matrices in C in Definition 8.6.1 and check that it is a functor in Lemma 8.6.7.
- Using the matrix tensor product, the lax unity constraint $\boxtimes^0$ is constructed in Lemma 8.6.8.
- The lax functoriality constraint $\boxtimes^2$ is constructed in several steps in Lemmas 8.6.12, 8.6.13, 8.6.16, and 8.6.21.
- The data $(\boxtimes, \boxtimes^2, \boxtimes^0)$ are assembled in Definition 8.6.19.

To see that the triple $(\boxtimes, \boxtimes^2, \boxtimes^0)$ is a pseudofunctor in the sense of Definition 6.2.1, we will check the lax associativity axiom (6.2.2) and the lax unity axiom (6.2.3) in Section 8.7. The multiplicative symmetry $\xi^{\otimes}$ is first used in Lemma 8.6.16.

The Matrix Tensor Product. The construction uses the following generalizations of the scalar product and the tensor product of complex matrices.

Definition 8.6.1. Suppose $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ and $B = (B_{lk}) \in \mathsf{Mat}^{\mathsf{C}}_{p,q}$ for some $m, n, p, q \geq 0$, and C is a tight symmetric bimonoidal category.

- For each object $C \in \mathsf{C}$, define the *scalar product*

$$C \boxtimes A = \big(C \otimes A_{ji}\big) \in \mathsf{Mat}^{\mathsf{C}}_{m,n} \tag{8.6.2}$$

  as the $n \times m$ matrix obtained from A by replacing each entry A_{ji} by the product $C \otimes A_{ji}$.
- Define the *matrix tensor product*

$$A \boxtimes B = \big(A_{ji} \boxtimes B\big)_{1 \leq j \leq n,\, 1 \leq i \leq m} \in \mathsf{Mat}^{\mathsf{C}}_{mp,nq} \tag{8.6.3}$$

  as the $nq \times mp$ matrix obtained from A by replacing each entry A_{ji} by the scalar product $A_{ji} \boxtimes B$. In other words, it has entries

$$(A \boxtimes B)_{(j-1)q+l,\,(i-1)p+k} = A_{ji} \otimes B_{lk} \in \mathsf{C} \tag{8.6.4}$$

  for $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq k \leq p$, and $1 \leq l \leq q$.

The same notation and terminology apply if C, A, and B are morphisms in, respectively, C, $\mathsf{Mat}^{\mathsf{C}}_{m,n}$, and $\mathsf{Mat}^{\mathsf{C}}_{p,q}$. ◇

Explanation 8.6.5. Suppose $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ is displayed using the following matrix notation.

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix}$$

Then the scalar product in (8.6.2) and the matrix tensor product in (8.6.3) are displayed as follows.

$$C \boxtimes A = \begin{bmatrix} C \otimes A_{11} & \cdots & C \otimes A_{1m} \\ \vdots & \ddots & \vdots \\ C \otimes A_{n1} & \cdots & C \otimes A_{nm} \end{bmatrix} \qquad A \boxtimes B = \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1m} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{n1} \boxtimes B & \cdots & A_{nm} \boxtimes B \end{bmatrix}$$

If either m or n is 0, then $C \boxtimes A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ is the empty matrix. If one of m, n, p, or q is 0, then $A \boxtimes B \in \mathsf{Mat}^{\mathsf{C}}_{mp,nq}$ is the empty matrix. $\diamond$

Example 8.6.6. Suppose R is a commutative ring, regarded as a tight symmetric bimonoidal category with its additive and multiplicative structures, and with only identity morphisms. Then the scalar product (8.6.2) and the matrix tensor product (8.6.3) in Mat^R are the usual ones for matrices with entries in R. For example, if $R = \mathbb{C}$, then the matrix tensor product in $\mathsf{Mat}^{\mathbb{C}}$ is the usual tensor product of complex matrices. $\diamond$

Lemma 8.6.7. *For $m, n, p, q \geq 0$, the matrix tensor product*

$$\mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{p,q} \xrightarrow{\ \boxtimes\ } \mathsf{Mat}^{\mathsf{C}}_{mp,nq}$$

in (8.6.3) is a functor.

Proof. Suppose $f : A \longrightarrow A' \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ and $g : B \longrightarrow B' \in \mathsf{Mat}^{\mathsf{C}}_{p,q}$ are morphisms. Then for $1 \leq j \leq n$, $1 \leq i \leq m$, $1 \leq l \leq q$, and $1 \leq k \leq p$,

$$(f \boxtimes g)_{(j-1)q+l,\,(i-1)p+k} = f_{ji} \otimes g_{lk} : A_{ji} \otimes B_{lk} \longrightarrow A'_{ji} \otimes B'_{lk} \in \mathsf{C}.$$

Therefore,

$$f \boxtimes g : A \boxtimes B \longrightarrow A' \boxtimes B' \in \mathsf{Mat}^{\mathsf{C}}_{mp,nq}$$

is a well-defined morphism. That $\boxtimes$ preserves identity morphisms and composition follows from (i) Definition 8.1.1 that these notions are defined entrywise in C and (ii) the functoriality of $\otimes$. $\square$

The Lax Unity Constraint. Recall from (8.1.6) the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ with m copies of the multiplicative unit $\mathbb{1}$ along the diagonal and the additive zero $\mathbb{0}$ in every other entry. The next lemma will be used to define the lax unity constraint of the monoidal composition in $\mathsf{Mat}^{\mathsf{C}}$.

Lemma 8.6.8. *For $m, p \geq 0$, there is a canonical isomorphism*

$$\mathbb{1}^{mp} \xrightarrow[\cong]{\boxtimes^0_{(m,p)}} \mathbb{1}^m \boxtimes \mathbb{1}^p \in \mathsf{Mat}^{\mathsf{C}}_{mp,mp} \tag{8.6.9}$$

with each entry $\lambda^{-\otimes}_{\mathbb{1}}$, $\rho^{-\bullet}_{\mathbb{1}}$, $\lambda^{-\bullet}_{\mathbb{1}}$, or $\lambda^{-\bullet}_{\mathbb{0}}$ if $m, p > 0$.

Proof. If either m or p is 0, then $\mathbb{1}^{mp}$ and $\mathbb{1}^m \boxtimes \mathbb{1}^p$ are both the empty matrix. In this case, $\boxtimes^0_{(m,p)}$ is the identity morphism.

For $m, p > 0$, $\mathbb{1}^m \boxtimes \mathbb{1}^p$ is computed as follows, with each $\mathbb{1}_i$ a copy of $\mathbb{1}$.

$$\begin{aligned}\mathbb{1}^m \boxtimes \mathbb{1}^p &= \begin{bmatrix} \mathbb{1}_1 \boxtimes \begin{bmatrix} \mathbb{1}_1 & \cdots & \mathbb{0} \\ \vdots & \ddots & \vdots \\ \mathbb{0} & \cdots & \mathbb{1}_p \end{bmatrix} & \cdots & \mathbb{0} \boxtimes \begin{bmatrix} \mathbb{1}_1 & \cdots & \mathbb{0} \\ \vdots & \ddots & \vdots \\ \mathbb{0} & \cdots & \mathbb{1}_p \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \mathbb{0} \boxtimes \begin{bmatrix} \mathbb{1}_1 & \cdots & \mathbb{0} \\ \vdots & \ddots & \vdots \\ \mathbb{0} & \cdots & \mathbb{1}_p \end{bmatrix} & \cdots & \mathbb{1}_m \boxtimes \begin{bmatrix} \mathbb{1}_1 & \cdots & \mathbb{0} \\ \vdots & \ddots & \vdots \\ \mathbb{0} & \cdots & \mathbb{1}_p \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{1}_1 \otimes \mathbb{1}_1 &&&&& \\ & \ddots &&&& \\ && \mathbb{1}_1 \otimes \mathbb{1}_p &&& \\ &&& \ddots && \\ &&&& \mathbb{1}_m \otimes \mathbb{1}_1 & \\ &&&&& \ddots \\ &&&&&& \mathbb{1}_m \otimes \mathbb{1}_p \end{bmatrix}\end{aligned} \tag{8.6.10}$$

In other words, $\mathbb{1}^m \boxtimes \mathbb{1}^p$ is an $mp \times mp$ matrix with

- each diagonal entry $\mathbb{1} \otimes \mathbb{1}$; and
- each off-diagonal entry
  - $\mathbb{1} \otimes \mathbb{0}$ (= each off-diagonal entry in each of the m diagonal blocks),
  - $\mathbb{0} \otimes \mathbb{1}$ (= each diagonal entry in each off-diagonal block), or
  - $\mathbb{0} \otimes \mathbb{0}$ (= each off-diagonal entry in each off-diagonal block).

The desired isomorphism $\boxtimes^0_{(m,p)}$ in (8.6.9) has

- each diagonal entry $\lambda^{-\otimes}_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1} \otimes \mathbb{1}$ and
- each off-diagonal entry

$$\mathbb{0} \xrightarrow{\rho^{-\bullet}_{\mathbb{1}}} \mathbb{1} \otimes \mathbb{0}, \qquad \mathbb{0} \xrightarrow{\lambda^{-\bullet}_{\mathbb{1}}} \mathbb{0} \otimes \mathbb{1}, \qquad \text{or} \qquad \mathbb{0} \xrightarrow{\lambda^{-\bullet}_{\mathbb{0}}} \mathbb{0} \otimes \mathbb{0}.$$

Here $\lambda^{-\otimes}$, $\lambda^{-\bullet}$, and $\rho^{-\bullet}$ are the inverses of, respectively, the left multiplicative unit $\lambda^{\otimes}$, the left multiplicative zero $\lambda^{\bullet}$, and the right multiplicative zero $\rho^{\bullet}$ in C. □

The Lax Functoriality Constraint. To define the lax functoriality constraint $\boxtimes^2$ of the monoidal composition in $\mathsf{Mat}^{\mathsf{C}}$, we first compute explicitly the entries of its (co)domain. Suppose A, B, A', and B' are arbitrary 1-cells in $\mathsf{Mat}^{\mathsf{C}}$ as follows.

$$\begin{gathered} m \xrightarrow{A} n \xrightarrow{B} p \\ m' \xrightarrow{A'} n' \xrightarrow{B'} p' \end{gathered} \tag{8.6.11}$$

In other words, they are matrices in C as follows.

- $(B, A) = \big((B_{kj}), (A_{ji})\big) \in \mathsf{Mat}^{\mathsf{C}}_{n,p} \times \mathsf{Mat}^{\mathsf{C}}_{m,n}$
- $(B', A') = \big((B'_{k'j'}), (A'_{j'i'})\big) \in \mathsf{Mat}^{\mathsf{C}}_{n',p'} \times \mathsf{Mat}^{\mathsf{C}}_{m',n'}$

Using the matrix product (8.1.4) and the matrix tensor product (8.6.3), there are the following matrices in C.

- $\big(B \boxtimes B', A \boxtimes A'\big) \in \mathsf{Mat}^{\mathsf{C}}_{nn',pp'} \times \mathsf{Mat}^{\mathsf{C}}_{mm',nn'}$
- $(B \boxtimes B')(A \boxtimes A') \in \mathsf{Mat}^{\mathsf{C}}_{mm',pp'}$
- $(BA, B'A') \in \mathsf{Mat}^{\mathsf{C}}_{m,p} \times \mathsf{Mat}^{\mathsf{C}}_{m',p'}$
- $BA \boxtimes B'A' \in \mathsf{Mat}^{\mathsf{C}}_{mm',pp'}$

The next lemma describes the entries in $(B \boxtimes B')(A \boxtimes A')$, which is the domain of $\boxtimes^2_{(B,B'),(A,A')}$. Recall the left normalized bracketing in (5.2.13).

Lemma 8.6.12. *In the setting of (8.6.11), for $1 \leq i \leq m$, $1 \leq i' \leq m'$, $1 \leq k \leq p$, and $1 \leq k' \leq p'$, the following equality of objects in C holds.*

$$\begin{aligned}&\big[(B \boxtimes B')(A \boxtimes A')\big]_{(k-1)p'+k',\,(i-1)m'+i'}\\ &= \begin{cases} \mathbb{0} & \text{if } n \text{ or } n' \text{ is } 0 \text{, and}\\ \left[\displaystyle\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\Big[\big(B_{kj}\otimes B'_{k'j'}\big)\otimes\big(A_{ji}\otimes A'_{j'i'}\big)\Big]\right]_{\mathsf{lt}} & \text{if } n, n' > 0.\end{cases}\end{aligned}$$

Proof. If one of m, m', p, or p' is 0, then $(B \boxtimes B')(A \boxtimes A')$ is the empty matrix, in which case there is nothing to prove. So we assume that $m, m', p, p' > 0$.

If either n or n' is 0, then $nn' = 0$, and $(B \boxtimes B')(A \boxtimes A')$ is the 0 matrix $\mathbb{0}_{mm',pp'}$ by definition (8.1.4). So we furthermore assume that $n, n' > 0$. By the definition (8.1.4) of the matrix product, the entry of $(B \boxtimes B')(A \boxtimes A')$ in the statement of the lemma is the unique entry of the matrix product of

- the $[(k-1)p' + k']$th row in $B \boxtimes B'$ and
- the $[(i-1)m' + i']$th column in $A \boxtimes A'$.

In other words, it is the unique entry of the following matrix product, in which the $\otimes$ symbol in each entry is abbreviated to concatenation.

$$\Big[\overbrace{B_{k1}B'_{k'1}\cdots B_{k1}B'_{k'n'}}^{n' \text{ entries}}\ \cdots\ \overbrace{B_{kn}B'_{k'1}\cdots B_{kn}B'_{k'n'}}^{n' \text{ entries}}\Big]\begin{bmatrix} A_{1i}A'_{1i'}\\ \vdots\\ A_{1i}A'_{n'i'}\\ \vdots\\ A_{ni}A'_{1i'}\\ \vdots\\ A_{ni}A'_{n'i'}\end{bmatrix}$$

The unique entry of this matrix product is the left normalized sum stated in the lemma. $\square$

The next lemma describes the entries in $BA \boxtimes B'A'$, which is the codomain of $\boxtimes^2_{(B,B'),(A,A')}$.

Lemma 8.6.13. *In the setting of (8.6.11), for $1 \leq i \leq m$, $1 \leq i' \leq m'$, $1 \leq k \leq p$, and $1 \leq k' \leq p'$, the following equality of objects in C holds.*

$$\begin{aligned}&\big(BA \boxtimes B'A'\big)_{(k-1)p'+k',\,(i-1)m'+i'}\\ &= \begin{cases} \mathbb{0}\otimes\mathbb{0} & \text{if } n = n' = 0,\\ \mathbb{0}\otimes\left[\displaystyle\bigoplus_{j'=1}^{n'}\big(B'_{k'j'}\otimes A'_{j'i'}\big)\right]_{\mathsf{lt}} & \text{if } n = 0 \text{ and } n' > 0,\\ \left[\displaystyle\bigoplus_{j=1}^{n}\big(B_{kj}\otimes A_{ji}\big)\right]_{\mathsf{lt}}\otimes\mathbb{0} & \text{if } n > 0 \text{ and } n' = 0 \text{, and}\\ \left[\displaystyle\bigoplus_{j=1}^{n}\big(B_{kj}\otimes A_{ji}\big)\right]_{\mathsf{lt}}\otimes\left[\displaystyle\bigoplus_{j'=1}^{n'}\big(B'_{k'j'}\otimes A'_{j'i'}\big)\right]_{\mathsf{lt}} & \text{if } n, n' > 0.\end{cases}\end{aligned}$$

Proof. If one of m, m', p, or p' is 0, then $BA \boxtimes B'A'$ is the empty matrix, in which case there is nothing to prove. So we assume that $m, m', p, p' > 0$.

By the definition (8.6.4) of the matrix tensor product, the entry of $BA \boxtimes B'A'$ in the lemma is the product

$$(BA)_{ki} \otimes (B'A')_{k'i'}.$$

By definition (8.1.4) this product is as stated in the statement of the lemma. Note that if $n = 0$, then BA is the 0 matrix $\mathbb{0}_{m,p}$. If $n' = 0$, then $B'A'$ is the 0 matrix $\mathbb{0}_{m',p'}$. □

To define the lax functoriality constraint $\boxtimes^2_{(B,B'),(A,A')}$, we connect the objects in Lemmas 8.6.12 and 8.6.13 using paths in the sense of Definition 3.1.14 with the following setting.

Definition 8.6.14. In the setting of (8.6.11), suppose $m, n, p, m', n', p' > 0$.

- Define the set of formal variables

$$X^{\boxtimes} = \left\{0^X, 1^X, a_{ji}, b_{kj}, a'_{j'i'}, b'_{k'j'}\right\}$$

  with $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$, $1 \le i' \le m'$, $1 \le j' \le n'$, and $1 \le k' \le p'$.
- Define the function $\varphi^{\boxtimes} : X^{\boxtimes} \longrightarrow \mathsf{Ob}(\mathsf{C})$ as follows.

$$\varphi^{\boxtimes}(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{ji} & \text{if } x = a_{ji}. \\ B_{kj} & \text{if } x = b_{kj}. \\ A'_{j'i'} & \text{if } x = a'_{j'i'}. \\ B'_{k'j'} & \text{if } x = b'_{k'j'}. \end{cases}$$

Paths in $\mathsf{Gr}(X^{\boxtimes})$ take values in C via the function $\varphi^{\boxtimes}$. ⋄

Motivation 8.6.15. It is possible to define an explicit morphism in C from the object in Lemma 8.6.12 to the object in Lemma 8.6.13. If $m, n, p, m', n', p' > 0$, then this involves

- permuting $B'_{k'j'}$ and A_{ji} in each object

$$\left(B_{kj} \otimes B'_{k'j'}\right) \otimes \left(A_{ji} \otimes A'_{j'i'}\right)$$

  via $\alpha^{\pm\otimes}$ and $\xi^{\otimes}$ and
- factoring via the inverses of δ^l and δ^r.

However, to prove that $\boxtimes^2$ has the desired properties of the lax functoriality constraint, we will need to use the Coherence Theorem 3.9.1. So we need to realize such a morphism in C using paths in $\mathsf{Gr}(X^{\boxtimes})$.

Recall from Definition 3.1.6 that the elementary edges δ^l and δ^r do not have formal inverses in $\mathsf{Gr}(X^{\boxtimes})$. Therefore, in general, it is not possible to have a single path in $\mathsf{Gr}(X^{\boxtimes})$ whose value in C has the object in Lemma 8.6.12 as its domain and the object in Lemma 8.6.13 as its codomain. Instead, they are connected by a zigzag of paths in $\mathsf{Gr}(X^{\boxtimes})$ as follows. ⋄

Lemma 8.6.16. *In the setting of Definition 8.6.14, the following statements hold.*

(1) *There exist paths*

$$\begin{tikzcd}
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\big[(b_{kj}\otimes b'_{k'j'})\otimes(a_{ji}\otimes a'_{j'i'})\big]\Big]_{\mathsf{lt}} \arrow[dr, "P"] & \\
& \Big[\bigoplus_{j=1}^{n}\Big[\bigoplus_{j'=1}^{n'}(b_{kj}\otimes a_{ji})\otimes(b'_{k'j'}\otimes a'_{j'i'})\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \\
\Big[\bigoplus_{j=1}^{n}(b_{kj}\otimes a_{ji})\Big]_{\mathsf{lt}}\otimes\Big[\bigoplus_{j'=1}^{n'}(b'_{k'j'}\otimes a'_{j'i'})\Big]_{\mathsf{lt}} \arrow[ur, "Q"'] &
\end{tikzcd}$$

in $\mathsf{Gr}(X^{\boxtimes})$.

(2) *All the paths in* $\mathsf{Gr}(X^{\boxtimes})$ *with the same (co)domain as* P*, respectively* Q*, have the same value in* C.

Proof. For the existence of the path P, first we use prime edges involving $\alpha^{\pm\otimes}$ and $\xi^{\otimes}$ to permute each monomial

$$(b_{kj}\otimes b'_{k'j'})\otimes(a_{ji}\otimes a'_{j'i'}) \quad\text{to}\quad (b_{kj}\otimes a_{ji})\otimes(b'_{k'j'}\otimes a'_{j'i'}). \tag{8.6.17}$$

After these prime edges, we use $\alpha^{\oplus}$ to move the additive brackets to match with the additive bracketing of the codomain of P.

The path $Q=(Q_2,Q_1)$ is the following concatenated path in $\mathsf{Gr}(X^{\boxtimes})$.

$$\begin{tikzcd}
& \Big[\bigoplus_{j=1}^{n}\Big[\bigoplus_{j'=1}^{n'}(b_{kj}\otimes a_{ji})\otimes(b'_{k'j'}\otimes a'_{j'i'})\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \\
& \Big[\bigoplus_{j=1}^{n}(b_{kj}\otimes a_{ji})\otimes\Big[\bigoplus_{j'=1}^{n'}(b'_{k'j'}\otimes a'_{j'i'})\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[u, "Q_2"'] \\
\Big[\bigoplus_{j=1}^{n}(b_{kj}\otimes a_{ji})\Big]_{\mathsf{lt}}\otimes\Big[\bigoplus_{j'=1}^{n'}(b'_{k'j'}\otimes a'_{j'i'})\Big]_{\mathsf{lt}} \arrow[ur, "Q_1"'] &
\end{tikzcd}$$

- Q_1 is the identity if $n=1$. If $n>1$, then each prime edge in Q_1 contains an instance of δ^r.
- Q_2 is the identity if $n'=1$. If $n'>1$, then each prime edge in Q_2 contains an instance of δ^l.

This proves the existence of the paths P and Q.

For the second assertion, observe that the domain of P is regular in the sense of Definition 3.1.25. By Lemma 3.1.29, the domain of Q is also regular. Since C is assumed to be a tight symmetric bimonoidal category, the Coherence Theorem 3.9.1 implies the uniqueness of the value of the path P, respectively Q, in C provided that it has the stated (co)domain. □

Remark 8.6.18. One of the reasons that we need C to be a tight *symmetric* bimonoidal category is the permutation in (8.6.17). It uses the multiplicative symmetric monoidal structure $(\mathsf{C},\otimes,\alpha^{\otimes},\xi^{\otimes})$. ◇

We are now ready to define the monoidal composition in $\mathsf{Mat}^{\mathsf{C}}$ in the sense of Definition 6.4.1.

Definition 8.6.19. For a tight symmetric bimonoidal category C, define the data of a lax functor

$$\mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}} \xrightarrow{(\boxtimes,\boxtimes^2,\boxtimes^0)} \mathsf{Mat}^{\mathsf{C}}$$

as follows.

Objects: For each pair of objects $(m,p) \in \mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}}$, define the object

$$\boxtimes(m,p) = m \boxtimes p = mp \in \mathsf{Mat}^{\mathsf{C}}.$$

The Local Functors: For $m,n,p,q \geq 0$, the local functor

$$\big(\mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}}\big)\big((m,p),(n,q)\big) = \mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{p,q} \xrightarrow{\boxtimes} \mathsf{Mat}^{\mathsf{C}}_{mp,nq}$$

is defined as the matrix tensor product (8.6.3) for matrices in C and for their morphisms.

The Lax Unity Constraint: For each pair of objects $(m,p) \in \mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}}$, define the component 2-cell

$$1_{m\boxtimes p} = \mathbb{1}^{mp} \xrightarrow{\boxtimes^0_{(m,p)}} \mathbb{1}^m \boxtimes \mathbb{1}^p = \boxtimes(1_{(m,p)}) \in \mathsf{Mat}^{\mathsf{C}}_{mp,mp}$$

as the canonical isomorphism in (8.6.9).

The Lax Functoriality Constraint: In the setting of (8.6.11), the component 2-cell

$$(B \boxtimes B')(A \boxtimes A') \xrightarrow{\boxtimes^2_{(B,B'),(A,A')}} BA \boxtimes B'A' \in \mathsf{Mat}^{\mathsf{C}}_{mm',pp'}$$

is the identity morphism of the empty matrix if m, m', p, or p' is 0. If $m,m',p,p' > 0$, then its $\big((k-1)p'+k', (i-1)m'+i'\big)$-entry is

$$\begin{cases} \lambda^{-\bullet}_{\mathbb{0}} & \text{if } n = n' = 0,\\ \lambda^{-\bullet}_{(B'A')_{k'i'}} & \text{if } n = 0 \text{ and } n' > 0,\\ \rho^{-\bullet}_{(BA)_{ki}} & \text{if } n > 0 \text{ and } n' = 0, \text{ and}\\ (\varphi^{\boxtimes}Q)^{-1}(\varphi^{\boxtimes}P) & \text{if } n,n' > 0. \end{cases} \tag{8.6.20}$$

In (8.6.20), the following statements hold.
- $1 \leq i \leq m$, $1 \leq i' \leq m'$, $1 \leq k \leq p$, and $1 \leq k' \leq p'$.
- $\lambda^{-\bullet}$ and $\rho^{-\bullet}$ are the inverses of, respectively, $\lambda^{\bullet}$ and $\rho^{\bullet}$ in C.
- P and Q are the paths in Lemma 8.6.16, with $\varphi^{\boxtimes}P$ and $\varphi^{\boxtimes}Q$ their values in C as in Definitions 3.1.14 and 8.6.14.

This finishes the definition of the triple $(\boxtimes,\boxtimes^2,\boxtimes^0)$. ◇

First we check the naturality of $\boxtimes^2$.

Lemma 8.6.21. *In Definition 8.6.19, $\boxtimes^2$ is a natural isomorphism.*

Proof. Each entry of $\boxtimes^2_{(B,B'),(A,A')}$ is a well-defined isomorphism in C by Lemmas 8.6.12, 8.6.13, and 8.6.16. If m,m',p, or p' is 0, then $\mathsf{Mat}^{\mathsf{C}}_{mm',pp'}$ is the terminal category. The naturality of $\boxtimes^2$ holds in this case. So we assume that $m,m',p,p' > 0$.

The 2-cells in $\mathsf{Mat}^{\mathsf{C}}$ are entrywise morphisms in C.

- If either n or n' is 0, then the naturality of $\boxtimes^2$ follows from the naturality of $\lambda^{\bullet}$ and $\rho^{\bullet}$ in C.

- If $n, n' > 0$, then by the proof of Lemma 8.6.16, each entry of $\boxtimes^2$ is a composite of isomorphisms, each being the iterated sum and product of identity morphisms and one component of $\alpha^{\oplus}$, $\alpha^{\pm\otimes}$, $\xi^{\otimes}$, δ^{-l}, or δ^{-r}. The naturality of $\boxtimes^2$ follows from the naturality of these structure morphisms in C.

Moreover, $\boxtimes^2$ is a natural isomorphism because each of its components is an isomorphism or a composite of isomorphisms. □

8.7. The Pseudofunctoriality of the Monoidal Composition

For a tight symmetric bimonoidal category C, we are in the process of constructing a monoidal bicategory structure, in the sense of Definition 6.4.1, on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$. In Section 8.6, we defined the data $(\boxtimes, \boxtimes^2, \boxtimes^0)$ that are supposed to constitute the monoidal composition in $\mathsf{Mat}^{\mathsf{C}}$. In this section, we check that the triple $(\boxtimes, \boxtimes^2, \boxtimes^0)$ is indeed a pseudofunctor in the sense of Definition 6.2.1. We check

- the lax associativity axiom (6.2.2) in Lemmas 8.7.3, 8.7.4, 8.7.13, and 8.7.23, and
- the lax unity axiom (6.2.3) in Lemma 8.7.27.

The Lax Associativity Axiom. To check the lax associativity axiom (6.2.2) for the triple $(\boxtimes, \boxtimes^2, \boxtimes^0)$, consider arbitrary composable 1-cells in $\mathsf{Mat}^{\mathsf{C}}$ as follows.

$$\begin{array}{c} m \xrightarrow{A} n \xrightarrow{B} p \xrightarrow{C} q \\ m' \xrightarrow{A'} n' \xrightarrow{B'} p' \xrightarrow{C'} q' \end{array} \tag{8.7.1}$$

Here $A = (A_{ji})$, $A' = (A'_{j'i'})$, $B = (B_{kj})$, and $B' = (B'_{k'j'})$ are as in (8.6.11), and

$$(C, C') = \big((C_{lk}), (C'_{l'k'})\big) \in \mathsf{Mat}^{\mathsf{C}}_{p,q} \times \mathsf{Mat}^{\mathsf{C}}_{p',q'}.$$

The lax associativity axiom (6.2.2) for these 1-cells is the commutativity of the following diagram in $\mathsf{Mat}^{\mathsf{C}}_{mm',qq'}$, with a the associator in Lemma 8.3.1.

$$\begin{tikzcd}
\big[(C \boxtimes C')(B \boxtimes B')\big](A \boxtimes A') \arrow[r, "a"] \arrow[d, "\boxtimes^2 \star 1"'] & (C \boxtimes C')\big[(B \boxtimes B')(A \boxtimes A')\big] \arrow[d, "1 \star \boxtimes^2"] \\
(CB \boxtimes C'B')(A \boxtimes A') \arrow[d, "\boxtimes^2"'] & (C \boxtimes C')(BA \boxtimes B'A') \arrow[d, "\boxtimes^2"] \\
(CB)A \boxtimes (C'B')A' \arrow[r, "a \boxtimes a"] & C(BA) \boxtimes C'(B'A')
\end{tikzcd} \tag{8.7.2}$$

Since morphisms in $\mathsf{Mat}^{\mathsf{C}}_{mm',qq'}$ are entrywise morphisms in C, it suffices to prove the commutativity of (8.7.2) in a typical entry. We use (8.6.20) to interpret the entries of $\boxtimes^2$. In some diagrams, some subscripts are omitted to save space. When we restrict to a typical entry, the indices are

$$1 \le i \le m, \quad 1 \le l \le q, \quad 1 \le i' \le m', \quad \text{and} \quad 1 \le l' \le q'.$$

To clarify the many cases, we split the proof of the commutativity of (8.7.2) into several lemmas.

Lemma 8.7.3. *In the setting of (8.7.1), suppose that either*

(1) *at least one of* m, m', q, *and* q' *is* 0, *or*
(2) $m, m', q, q' > 0$, *and at most one of* n, n', p, *and* p' *is* > 0.

Then the diagram (8.7.2) *is commutative.*

Proof. If at least one of m, m', q, and q' is 0, then $\mathsf{Mat}^{\mathsf{C}}_{mm',qq'}$ is the terminal category. The diagram (8.7.2) is commutative in this case.

For the other case, suppose that $m, m', q, q' > 0$ and that at most one of n, n', p, and p' is > 0. There are 5 subcases.

(1) If $n = n' = p = p' = 0$, then (8.7.2) is the following diagram.

$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[rr, "1"] \arrow[d, "1"'] & & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{0}_{m,q} \boxtimes \mathbb{0}_{m',q'} \arrow[rr, "1 \boxtimes 1"] & & \mathbb{0}_{m,q} \boxtimes \mathbb{0}_{m',q'}
\end{tikzcd}$$

This is commutative because $1 \boxtimes 1 = 1$ by Lemma 8.6.7.

(2) If $n > 0$ and $n' = p = p' = 0$, then (8.7.2) is the left diagram below, with ζ^{ℓ} the natural isomorphism in (8.1.11).

$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[rr, "1"] \arrow[d, "1"'] & & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{0}_{n,q} A \boxtimes \mathbb{0}_{m',q'} \arrow[rr, "\zeta^{\ell}_A \boxtimes 1"] & & \mathbb{0}_{m,q} \boxtimes \mathbb{0}_{m',q'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{0} \arrow[rr, "1"] \arrow[d, "1"'] & & \mathbb{0} \arrow[d, "1"] \\
\mathbb{0} \arrow[d, "\rho^{-\bullet}_{(\mathbb{0}_{n,q} A)_{li}}"'] & & \mathbb{0} \arrow[d, "\lambda^{-\bullet}_{\mathbb{0}} = \rho^{-\bullet}_{\mathbb{0}}"] \\
(\mathbb{0}_{n,q} A)_{li} \otimes \mathbb{0} \arrow[rr, "(\zeta^{\ell}_A)_{li} \otimes 1"] & & \mathbb{0} \otimes \mathbb{0}
\end{tikzcd}$$

A typical entry of the left diagram above is the right diagram in C. The right diagram is commutative by

- the equality $\lambda^{\bullet}_{\mathbb{0}} = \rho^{\bullet}_{\mathbb{0}}$, which is the axiom (2.1.14) in C, and
- the naturality of $\rho^{\bullet}$.

(3) If $n' > 0$ and $n = p = p' = 0$, then (8.7.2) is the left diagram below.

$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[rr, "1"] \arrow[d, "1"'] & & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{0}_{m,q} \boxtimes \mathbb{0}_{n',q'} A' \arrow[rr, "1 \boxtimes \zeta^{\ell}_{A'}"] & & \mathbb{0}_{m,q} \boxtimes \mathbb{0}_{m',q'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{0} \arrow[rr, "1"] \arrow[d, "1"'] & & \mathbb{0} \arrow[d, "1"] \\
\mathbb{0} \arrow[d, "\lambda^{-\bullet}_{(\mathbb{0}_{n',q'} A')_{l'i'}}"'] & & \mathbb{0} \arrow[d, "\lambda^{-\bullet}_{\mathbb{0}}"] \\
\mathbb{0} \otimes (\mathbb{0}_{n',q'} A')_{l'i'} \arrow[rr, "1 \otimes (\zeta^{\ell}_{A'})_{l'i'}"] & & \mathbb{0} \otimes \mathbb{0}
\end{tikzcd}$$

A typical entry of the left diagram above is the right diagram in C, which is commutative by the naturality of $\lambda^{\bullet}$.

(4) If $p > 0$ and $n = n' = p' = 0$, then (8.7.2) is the left diagram below, with ζ^r the natural isomorphism in (8.1.19).

$$\begin{tikzcd}
\mathbb{O}_{mm',qq'} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{O}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{O}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & \mathbb{O}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{O}_{m,q} \boxtimes \mathbb{O}_{m',q'} \arrow[r, "(\zeta^r_C)^{-1} \boxtimes 1"] & C\mathbb{O}_{m,p} \boxtimes \mathbb{O}_{m',q'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{O} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{O} \arrow[d, "1"] \\
\mathbb{O} \arrow[d, "\lambda^{-\bullet}_{\mathbb{O}} = \rho^{-\bullet}_{\mathbb{O}}"'] & \mathbb{O} \arrow[d, "\rho^{-\bullet}_{(C\mathbb{O}_{m,p})_{li}}"] \\
\mathbb{O} \otimes \mathbb{O} \arrow[r, "(\zeta^r_C)^{-1}_{li} \otimes 1"] & (C\mathbb{O}_{m,p})_{li} \otimes \mathbb{O}
\end{tikzcd}$$

As in case (2), a typical entry of the left diagram above is the right diagram in C. It is commutative by the equality $\lambda^\bullet_{\mathbb{O}} = \rho^\bullet_{\mathbb{O}}$ in (2.1.14) and the naturality of $\rho^\bullet$.

(5) If $p' > 0$ and $n = n' = p = 0$, then (8.7.2) is the left diagram below.

$$\begin{tikzcd}
\mathbb{O}_{mm',qq'} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{O}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{O}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & \mathbb{O}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{O}_{m,q} \boxtimes \mathbb{O}_{m',q'} \arrow[r, "1 \boxtimes (\zeta^r_{C'})^{-1}"] & \mathbb{O}_{m,q} \boxtimes C'\mathbb{O}_{m',p'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{O} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{O} \arrow[d, "1"] \\
\mathbb{O} \arrow[d, "\lambda^{-\bullet}_{\mathbb{O}}"'] & \mathbb{O} \arrow[d, "\lambda^{-\bullet}_{(C'\mathbb{O}_{m',p'})_{l'i'}}"] \\
\mathbb{O} \otimes \mathbb{O} \arrow[r, "1 \otimes (\zeta^r_{C'})^{-1}_{l'i'}"] & \mathbb{O} \otimes (C'\mathbb{O}_{m',p'})_{l'i'}
\end{tikzcd}$$

A typical entry of the left diagram above is the right diagram in C, which is commutative by the naturality of $\lambda^\bullet$.

This finishes the proofs of all the subcases. □

Lemma 8.7.4. *In the setting of (8.7.1), suppose that*

- *$m, m', q, q' > 0$ and*
- *precisely two of n, n', p, and p' are 0.*

Then the diagram (8.7.2) is commutative.

Proof. There are six cases.

(1) If $n, n' > 0$ and $p = p' = 0$, then (8.7.2) is the diagram below.

$$\begin{tikzcd}
\mathbb{O}_{nn',qq'}(A \boxtimes A') \arrow[r, "\zeta^\ell_{A \boxtimes A'}"] \arrow[d, "\boxtimes^2 \star 1"'] & \mathbb{O}_{mm',qq'} \arrow[d, "1"] \\
(\mathbb{O}_{n,q} \boxtimes \mathbb{O}_{n',q'})(A \boxtimes A') \arrow[d, "\boxtimes^2"'] & \mathbb{O}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{O}_{n,q} A \boxtimes \mathbb{O}_{n',q'} A' \arrow[r, "\zeta^\ell_A \boxtimes \zeta^\ell_{A'}"] & \mathbb{O}_{m,q} \boxtimes \mathbb{O}_{m',q'}
\end{tikzcd} \tag{8.7.5}$$

Since $\boxtimes^2 1 = \boxtimes^2$, the $\big((l-1)q'+l', (i-1)m'+i'\big)$-entry of (8.7.5) is the diagram in C below.

$$\begin{tikzcd}
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'} \mathbb{0}\otimes\big(A_{ji}\otimes A'_{j'i'}\big)\Big]_{\mathsf{lt}} \arrow[r, "(\lambda^{\bullet}{,}\lambda^{\oplus})"] \arrow[d, "{\big[\oplus_j\oplus_{j'}(\lambda_{\mathbb{0}}^{-\bullet}\otimes 1)\big]_{\mathsf{lt}}}"'] & \mathbb{0} \arrow[ddd, "\lambda_{\mathbb{0}}^{-\bullet}"] \\
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'} (\mathbb{0}\otimes\mathbb{0})\otimes\big(A_{ji}\otimes A'_{j'i'}\big)\Big]_{\mathsf{lt}} \arrow[d, "(\alpha^{\oplus}{,}\alpha^{\pm\otimes}{,}\xi^{\otimes})"'] & \\
\Big[\bigoplus_{j=1}^{n}\Big(\bigoplus_{j'=1}^{n'} (\mathbb{0}\otimes A_{ji})\otimes\big(\mathbb{0}\otimes A'_{j'i'}\big)\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "(\delta^{-r}{,}\delta^{-l})"'] & \\
\Big[\bigoplus_{j=1}^{n}\big(\mathbb{0}\otimes A_{ji}\big)\Big]_{\mathsf{lt}}\otimes\Big[\bigoplus_{j'=1}^{n'}\big(\mathbb{0}\otimes A'_{j'i'}\big)\Big]_{\mathsf{lt}} \arrow[r, "(\lambda^{\bullet}{,}\lambda^{\oplus})"] & \mathbb{0}\otimes\mathbb{0}
\end{tikzcd} \tag{8.7.6}$$

Consider the diagram (8.7.6).

- By (8.1.13), each entry of each component of ζ^{ℓ} is a composite of morphisms, each being a sum of identity morphisms and one component of $\lambda^{\bullet}$ or $\lambda^{\oplus}$. The notation $(\lambda^{\bullet},\lambda^{\oplus})$ denotes a composite of morphisms, each being an iterated sum and product of identity morphisms and at most one component of $\lambda^{\bullet}$ or $\lambda^{\oplus}$.
- The morphisms $(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})$ and $(\delta^{-r},\delta^{-l})$ are interpreted similarly, with δ^{-r} and δ^{-l} the inverses of δ^{r} and δ^{l}, respectively. Each of them is a composite of morphisms, each being an iterated sum and product of identity morphisms and at most one component of one of the indicated structure morphisms in C.
- In interpreting the lower left $\boxtimes^2$ in (8.7.5), we use the paths P and Q in the proof of Lemma 8.6.16.

To prove the commutativity of (8.7.6), we realize each morphism or its inverse as the value in C of a path in $\mathsf{Gr}(X)$ using the set of formal variables

$$X = \big\{0^X, 1^X, a_{ji}, a'_{j'i'}\big\}_{1\le j\le n,\, 1\le j'\le n'}$$

and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \\ A_{ji} & \text{if } x = a_{ji}, \text{ and} \\ A'_{j'i'} & \text{if } x = a'_{j'i'}. \end{cases}$$

By Notation 3.3.10, Example 8.1.14, and Lemma 8.6.16, there is a diagram in $\mathsf{Gr}(X)$ as follows.

$$\begin{tikzcd}
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'} 0^X \otimes \big(a_{ji}\otimes a'_{j'i'}\big)\Big]_{\mathsf{lt}} \arrow[d, "L_3"'] & & 0^X \arrow[ll, twoheadrightarrow, "L_2"'] \\
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'} (0^X\otimes 0^X)\otimes \big(a_{ji}\otimes a'_{j'i'}\big)\Big]_{\mathsf{lt}} \arrow[d, "P"'] & & \\
\Big[\bigoplus_{j=1}^{n}\Big(\bigoplus_{j'=1}^{n'} (0^X\otimes a_{ji})\otimes (0^X\otimes a'_{j'i'})\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} & & \\
\Big[\bigoplus_{j=1}^{n} (0^X\otimes a_{ji})\Big]_{\mathsf{lt}} \otimes \Big[\bigoplus_{j'=1}^{n'} (0^X\otimes a'_{j'i'})\Big]_{\mathsf{lt}} \arrow[u, twoheadrightarrow, "Q"] \arrow[rr, twoheadrightarrow, "L_1"] & & 0^X\otimes 0^X \arrow[uuu, "\lambda^{\bullet}_{0^X}"']
\end{tikzcd} \tag{8.7.7}$$

Consider the diagram (8.7.7).

- The paths P and Q are those in Lemma 8.6.16, with each b_{kj} and each $b'_{k'j'}$ replaced by 0^X.
  - Each prime edge in P is an identity or has an instance of $\alpha^{\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{\otimes}$. Its value in C is the morphism $(\alpha^{\oplus}, \alpha^{\pm\otimes}, \xi^{\otimes})$ in (8.7.6).
  - Each prime edge in Q is an identity or has an instance of δ^{l} or δ^{r}. Its value in C is the *inverse* of the morphism $(\delta^{-r}, \delta^{-l})$ in (8.7.6).
- Each prime edge in L_1 is an identity or has an instance of $\lambda^{\bullet}$ or $\lambda^{\oplus}$. Its value in C is the bottom horizontal morphism in (8.7.6).
- Each prime edge in L_2 is an identity or has an instance of $\lambda^{-\bullet}$ or $\lambda^{-\oplus}$. Its value in C is the *inverse* of the top horizontal morphism in (8.7.6).
- Each prime edge in L_3 is an identity or has an instance of $\lambda^{-\bullet}$. Its value in C is the top left vertical morphism in (8.7.6).

Since the upper right vertex in (8.7.7) is 0^X, by Lemma 3.1.29, the lower left vertex has the same support as 0^X. By Proposition 3.5.33, the two paths

$$Q \quad \text{and} \quad (P, L_3, L_2, \lambda^{\bullet}_{0^X}, L_1)$$

have the same value in C. Since all the structure morphisms involved are invertible (because C is assumed to be tight), this implies that (8.7.6) is commutative.

(2) If $p, p' > 0$ and $n = n' = 0$, then (8.7.2) is the diagram below, with ζ^{-r} the inverse of ζ^r in (8.1.19).

$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[r, "\zeta^{-r}_{C\boxtimes C'}"] \arrow[d, "1"'] & (C\boxtimes C')\mathbb{0}_{mm',pp'} \arrow[d, "1\star\boxtimes^2"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & (C\boxtimes C')(\mathbb{0}_{m,p}\boxtimes \mathbb{0}_{m',p'}) \arrow[d, "\boxtimes^2"] \\
\mathbb{0}_{m,q}\boxtimes \mathbb{0}_{m',q'} \arrow[r, "\zeta^{-r}_C\boxtimes\zeta^{-r}_{C'}"] & C\mathbb{0}_{m,p}\boxtimes C'\mathbb{0}_{m',p'}
\end{tikzcd} \tag{8.7.8}$$

Composing the upper left 1 away and using the notation in (8.7.6), the $\big((l-1)q'+l', (i-1)m'+i'\big)$-entry of (8.7.8) is the diagram in C below.

$$\begin{tikzcd}
\mathbb{0} \arrow[r, "{(\lambda^{-\oplus},\rho^{-\bullet})}"] \arrow[dddd, "\lambda^{-\bullet}_{\mathbb{0}}"'] & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'} (C_{lk}\otimes C'_{l'k'})\otimes \mathbb{0}\Big]_{\mathsf{lt}} \arrow[d, "{\big[\oplus_k\oplus_{k'}(1\otimes\lambda^{-\bullet}_{\mathbb{0}})\big]_{\mathsf{lt}}}"] \\
& \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'} (C_{lk}\otimes C'_{l'k'})\otimes (\mathbb{0}\otimes\mathbb{0})\Big]_{\mathsf{lt}} \arrow[d, "{(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})}"] \\
& \Big[\bigoplus_{k=1}^{p}\Big(\bigoplus_{k'=1}^{p'} (C_{lk}\otimes \mathbb{0})\otimes (C'_{l'k'}\otimes\mathbb{0})\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "{(\delta^{-r},\delta^{-l})}"] \\
& \\
\mathbb{0}\otimes\mathbb{0} \arrow[r, "{(\lambda^{-\oplus},\rho^{-\bullet})}"] & \Big[\bigoplus_{k=1}^{p}(C_{lk}\otimes\mathbb{0})\Big]_{\mathsf{lt}}\otimes\Big[\bigoplus_{k'=1}^{p'}(C'_{l'k'}\otimes\mathbb{0})\Big]_{\mathsf{lt}}
\end{tikzcd} \tag{8.7.9}$$

To prove the commutativity of (8.7.9), we realize each morphism or its inverse as the value in C of a path in $\mathsf{Gr}(X)$ using the set of formal variables

$$X = \{0^X, 1^X, c_{lk}, c'_{l'k'}\}_{1\le k\le p,\, 1\le k'\le p'}$$

and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \\ C_{lk} & \text{if } x = c_{lk}, \text{ and} \\ C'_{l'k'} & \text{if } x = c'_{l'k'}. \end{cases}$$

Similar to (8.7.7), by Notation 3.3.10, Example 8.1.22, and Lemma 8.6.16, there is a diagram in $\mathsf{Gr}(X)$ as follows in which each path has value in C

the corresponding edge in (8.7.9) or its inverse.

$$
\begin{tikzcd}
0^X \arrow[rr, "L_2"] & & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'} (c_{lk} \otimes c'_{l'k'}) \otimes 0^X\Big]_{\mathsf{lt}} \arrow[d, "L_3"] \\
& & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'} (c_{lk} \otimes c'_{l'k'}) \otimes (0^X \otimes 0^X)\Big]_{\mathsf{lt}} \arrow[d, "P"] \\
& & \Big[\bigoplus_{k=1}^{p}\Big(\bigoplus_{k'=1}^{p'} (c_{lk} \otimes 0^X) \otimes (c'_{l'k'} \otimes 0^X)\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \\
0^X \otimes 0^X \arrow[uuu, "\lambda^{\bullet}_{0^X}"] & & \Big[\bigoplus_{k=1}^{p} (c_{lk} \otimes 0^X)\Big]_{\mathsf{lt}} \otimes \Big[\bigoplus_{k'=1}^{p'} (c'_{l'k'} \otimes 0^X)\Big]_{\mathsf{lt}} \arrow[ll, "L_1"'] \arrow[u, "Q"']
\end{tikzcd}
\tag{8.7.10}
$$

Consider the diagram (8.7.10).

- The paths P and Q are those in Lemma 8.6.16 up to a change of symbols.
  - The value of P in C is the morphism $(\alpha^{\oplus}, \alpha^{\pm\otimes}, \xi^{\otimes})$ in (8.7.9).
  - The value of Q in C is the *inverse* of the morphism $(\delta^{-r}, \delta^{-l})$ in (8.7.9).
- Each prime edge in L_1 is an identity or has an instance of $\lambda^{\oplus}$ or $\rho^{\bullet}$. Its value in C is the *inverse* of the bottom horizontal morphism in (8.7.9).
- Each prime edge in L_2 is an identity or has an instance of $\lambda^{-\oplus}$ or $\rho^{-\bullet}$. Its value in C is the top horizontal morphism in (8.7.9).
- Each prime edge in L_3 is an identity or has an instance of $\lambda^{-\bullet}$. Its value in C is the top right vertical morphism in (8.7.9).

Since the upper left vertex in (8.7.10) is 0^X, by Lemma 3.1.29, the lower right vertex has the same support as 0^X. By Proposition 3.5.33, the two paths

$$Q \quad \text{and} \quad (P, L_3, L_2, \lambda^{\bullet}_{0^X}, L_1)$$

have the same value in C. This implies that (8.7.9) is commutative.

(3) If $n, p' > 0$ and $n' = p = 0$, then (8.7.2) is the diagram below.

$$
\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{0}_{n,q} A \boxtimes \mathbb{0}_{m',q'} \arrow[r, "\zeta^{\ell}_A \boxtimes \zeta^{-r}_{C'}"] & \mathbb{0}_{m,q} \boxtimes C' \mathbb{0}_{m',p'}
\end{tikzcd}
\tag{8.7.11}
$$

First composing away the three 1's, the $\big((l-1)q'+l',(i-1)m'+i'\big)$-entry of (8.7.11) becomes the outer diagram in C below.

$$\begin{tikzcd}[column sep=small]
& \mathbb{0} \arrow[dl, bend right, "\rho^{-\bullet}"'] \arrow[d, "\rho_{\mathbb{0}}^{-\bullet} = \lambda_{\mathbb{0}}^{-\bullet}"] \arrow[dr, bend left, "\lambda^{-\bullet}"] & \\
& \mathbb{0}\otimes\mathbb{0} \arrow[dr, "1\otimes(\zeta_{C'}^{-r})_{l'i'}"] & \\
\Big[\bigoplus_{j=1}^{n}(\mathbb{0}\otimes A_{ji})\Big]_{\mathsf{lt}}\otimes\mathbb{0} \arrow[ur, "(\zeta_A^{\ell})_{li}\otimes 1"] \arrow[rr, "(\zeta_A^{\ell})_{li}\otimes(\zeta_{C'}^{-r})_{l'i'}"] & & \mathbb{0}\otimes\Big[\bigoplus_{k'=1}^{p'}(C'_{l'k'}\otimes\mathbb{0})\Big]_{\mathsf{lt}}
\end{tikzcd}$$

- The equality $\lambda_{\mathbb{0}}^{-\bullet}=\rho_{\mathbb{0}}^{-\bullet}$ follows from the axiom (2.1.14).
- The bottom subdiagram is commutative by the functoriality of $\otimes$.
- The left and the right subdiagrams are commutative by the naturality of $\rho^{\bullet}$ and $\lambda^{\bullet}$, respectively.

(4) If $n',p>0$ and $n=p'=0$, then (8.7.2) is the diagram below.

$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{0}_{m,q}\boxtimes\mathbb{0}_{n',q'}A' \arrow[r, "\zeta_C^{-r}\boxtimes\zeta_{A'}^{\ell}"] & C\mathbb{0}_{m,p}\boxtimes\mathbb{0}_{m',q'}
\end{tikzcd} \tag{8.7.12}$$

First composing away the three 1's, the $\big((l-1)q'+l',(i-1)m'+i'\big)$-entry of (8.7.12) becomes the outer diagram in C below.

$$\begin{tikzcd}[column sep=small]
& \mathbb{0} \arrow[dl, bend right, "\lambda^{-\bullet}"'] \arrow[d, "\lambda_{\mathbb{0}}^{-\bullet} = \rho_{\mathbb{0}}^{-\bullet}"] \arrow[dr, bend left, "\rho^{-\bullet}"] & \\
& \mathbb{0}\otimes\mathbb{0} \arrow[dr, "(\zeta_C^{-r})_{li}\otimes 1"] & \\
\mathbb{0}\otimes\Big[\bigoplus_{j'=1}^{n'}(\mathbb{0}\otimes A'_{j'i'})\Big]_{\mathsf{lt}} \arrow[ur, "1\otimes(\zeta_{A'}^{\ell})_{l'i'}"] \arrow[rr, "(\zeta_C^{-r})_{li}\otimes(\zeta_{A'}^{\ell})_{l'i'}"] & & \Big[\bigoplus_{k=1}^{p}(C_{lk}\otimes\mathbb{0})\Big]_{\mathsf{lt}}\otimes\mathbb{0}
\end{tikzcd}$$

As in case (3), this diagram is commutative by the functoriality of $\otimes$ and the naturality of $\lambda^{\bullet}$ and $\rho^{\bullet}$.

(5) If $n,p>0$ and $n'=p'=0$, then (8.7.2) is the left diagram below.

$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
(CB)A\boxtimes\mathbb{0}_{m',q'} \arrow[r, "a\boxtimes 1"] & C(BA)\boxtimes\mathbb{0}_{m',q'}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& \mathbb{0} \arrow[dl, "\rho^{-\bullet}"'] \arrow[dr, "\rho^{-\bullet}"] & \\
\big[(CB)A\big]_{li}\otimes\mathbb{0} \arrow[rr, "a\otimes 1"] & & \big[C(BA)\big]_{li}\otimes\mathbb{0}
\end{tikzcd}$$

First composing the three 1's away, the $\big((l-1)q'+l',(i-1)m'+i'\big)$-entry of the left diagram above is the right diagram in C. It is commutative by the naturality of $\rho^\bullet$.

(6) If $n',p'>0$ and $n=p=0$, then (8.7.2) is the left diagram below.

$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
\mathbb{0}_{m,q} \boxtimes (C'B')A' \arrow[r, "1\boxtimes a"] & \mathbb{0}_{m,q} \boxtimes C'(B'A')
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{0} \arrow[dl, "\lambda^{-\bullet}"'] \arrow[dr, "\lambda^{-\bullet}"] & \\
\mathbb{0} \otimes \big[(C'B')A'\big]_{l'i'} \arrow[rr, "1\otimes a"] & & \mathbb{0} \otimes \big[C'(B'A')\big]_{l'i'}
\end{tikzcd}$$

First composing the three 1's away, the $\big((l-1)q'+l',(i-1)m'+i'\big)$-entry of the left diagram above is the right diagram in C. It is commutative by the naturality of $\lambda^\bullet$.

This finishes the proofs of all six cases. $\square$

Lemma 8.7.13. *In the setting of (8.7.1), suppose that*

- *$m,m',q,q'>0$ and*
- *precisely one of n,n',p, and p' is 0.*

Then the diagram (8.7.2) is commutative.

Proof. There are four cases.

(1) If $n=0$ and $n',p,p'>0$, then (8.7.2) is the following diagram.

$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[r, "\zeta^{-r}_{C\boxtimes C'}"] \arrow[d, "1"'] & (C\boxtimes C')\mathbb{0}_{mm',pp'} \arrow[d, "1\star\boxtimes^2"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & (C\boxtimes C')(\mathbb{0}_{m,p}\boxtimes B'A') \arrow[d, "\boxtimes^2"] \\
\mathbb{0}_{m,q}\boxtimes (C'B')A' \arrow[r, "\zeta^{-r}_C \boxtimes a_{C',B',A'}"] & C\mathbb{0}_{m,p}\boxtimes C'(B'A')
\end{tikzcd} \tag{8.7.14}$$

Composing the top left 1 away, the $\big((l-1)q'+l',(i-1)m'+i'\big)$-entry of (8.7.14) is the diagram in C below, with the notation in (8.7.6) and with $\otimes$

abbreviated to concatenation.

(8.7.15)
$$\begin{tikzcd}
\mathbb{0} \arrow[r, "(\lambda^{-\oplus},\rho^{-\bullet})"] \arrow[d, "\lambda^{-\bullet}"] & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'} (C_{lk}C'_{l'k'})\mathbb{0}\Big]_{\mathsf{lt}} \arrow[d, "(\oplus_k \oplus_{k'} 1\lambda^{-\bullet})_{\mathsf{lt}}"] \\
\mathbb{0}\Big[\bigoplus_{j'=1}^{n'}\Big(\bigoplus_{k'=1}^{p'} C'_{l'k'}B'_{k'j'}\Big)_{\mathsf{lt}} A'_{j'i'}\Big]_{\mathsf{lt}} \arrow[d, "(\lambda^{-\oplus},\rho^{-\bullet})(a^3a^2a^1)"] & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'} (C_{lk}C'_{l'k'})\Big\{\mathbb{0}\Big(\bigoplus_{j'=1}^{n'} B'_{k'j'}A'_{j'i'}\Big)_{\mathsf{lt}}\Big\}\Big]_{\mathsf{lt}} \arrow[d, "(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})"] \\
\Big[\bigoplus_{k=1}^{p} C_{lk}\mathbb{0}\Big]_{\mathsf{lt}}\Big[\bigoplus_{k'=1}^{p'}\Big(\bigoplus_{j'=1}^{n'} C'_{l'k'}(B'_{k'j'}A'_{j'i'})\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[dr, "1a^4"] & \Big[\bigoplus_{k=1}^{p}\Big[\bigoplus_{k'=1}^{p'} (C_{lk}\mathbb{0})\Big\{C'_{l'k'}\Big(\bigoplus_{j'=1}^{n'} B'_{k'j'}A'_{j'i'}\Big)_{\mathsf{lt}}\Big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "(\delta^{-r},\delta^{-l})"] \\
 & \Big[\bigoplus_{k=1}^{p} C_{lk}\mathbb{0}\Big]_{\mathsf{lt}}\Big[\bigoplus_{k'=1}^{p'} C'_{l'k'}\Big(\bigoplus_{j'=1}^{n'} B'_{k'j'}A'_{j'i'}\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}}
\end{tikzcd}$$

In the diagram (8.7.15), the factorization

$$(a_{C',B',A'})_{l'i'} = a^4a^3a^2a^1$$

is from (8.3.5) up to a change of symbols.

- a^1 is built from identity morphisms and δ^r.
- a^2 is built from identity morphisms and $\alpha^{\otimes}$.
- a^3 is built from identity morphisms, $\alpha^{\pm\oplus}$, and $\xi^{\oplus}$.
- a^4 is built from identity morphisms and δ^{-l}.

To prove that (8.7.15) is commutative, we realize each morphism or its inverse as the value in C of a path in $\mathsf{Gr}(X)$ using the set of formal variables

$$X = \left\{0^X, 1^X, c_{lk}, a'_{j'i'}, b'_{k'j'}, c'_{l'k'}\right\}_{1\le k\le p,\, 1\le j'\le n',\, 1\le k'\le p'}$$

and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ C_{lk} & \text{if } x = c_{lk}. \\ A'_{j'i'} & \text{if } x = a'_{j'i'}. \\ B'_{k'j'} & \text{if } x = b'_{k'j'}. \\ C'_{l'k'} & \text{if } x = c'_{l'k'}. \end{cases}$$

By Notation 3.3.10 and Example 8.3.9, there is a diagram in $\mathsf{Gr}(X)$ as follows, in which the value in C of each path is the corresponding edge in

(8.7.15) or its inverse. We continue to abbreviate $\otimes$ to concatenation.

(8.7.16)
$$\begin{tikzcd}
0^X \arrow[d, "\lambda^{-\bullet}"] & \left[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}(c_{lk}c'_{l'k'})0^X\right]_{\mathsf{lt}} \arrow[l, two heads, "L_2"'] \\
0^X\left[\bigoplus_{j'=1}^{n'}\Big(\bigoplus_{k'=1}^{p'}c'_{l'k'}b'_{k'j'}\Big)_{\mathsf{lt}}a'_{j'i'}\right]_{\mathsf{lt}} \arrow[d, two heads, "L_3"] & \left[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}(c_{lk}c'_{l'k'})\Big\{0^X\Big(\bigoplus_{j'=1}^{n'}b'_{k'j'}a'_{j'i'}\Big)_{\mathsf{lt}}\Big\}\right]_{\mathsf{lt}} \arrow[u, two heads, "L_1"'] \\
\left[\bigoplus_{k=1}^{p}c_{lk}0^X\right]_{\mathsf{lt}}\left[\bigoplus_{k'=1}^{p'}\Big(\bigoplus_{j'=1}^{n'}c'_{l'k'}\big(b'_{k'j'}a'_{j'i'}\big)\Big)_{\mathsf{lt}}\right]_{\mathsf{lt}} & \left[\bigoplus_{k=1}^{p}\left[\bigoplus_{k'=1}^{p'}(c_{lk}0^X)\Big\{c'_{l'k'}\Big(\bigoplus_{j'=1}^{n'}b'_{k'j'}a'_{j'i'}\Big)_{\mathsf{lt}}\Big\}\right]_{\mathsf{lt}}\right]_{\mathsf{lt}} \arrow[u, two heads, "P^{-1}"'] \\
& \left[\bigoplus_{k=1}^{p}c_{lk}0^X\right]_{\mathsf{lt}}\left[\bigoplus_{k'=1}^{p'}c'_{l'k'}\Big(\bigoplus_{j'=1}^{n'}b'_{k'j'}a'_{j'i'}\Big)_{\mathsf{lt}}\right]_{\mathsf{lt}} \arrow[u, two heads, "Q"'] \arrow[ul, two heads, "L"]
\end{tikzcd}$$

Consider the diagram (8.7.16).

- Each prime edge in L is an identity or contains δ^l.
- P and Q are the paths in Lemma 8.6.16 up to a change of symbols, and P^{-1} is the formal inverse of P in the sense of Definition 3.1.10.
  - Each prime edge in Q is an identity or contains δ^r or δ^l.
  - Each prime edge in P^{-1} is an identity or contains $\alpha^{-\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{-\otimes}$.
- Each prime edge in L_1 is an identity or contains $\lambda^{\bullet}$.
- Each prime edge in L_2 is an identity or contains $\lambda^{\oplus}$ or $\rho^{\bullet}$.
- Each prime edge in L_3 is an identity or contains $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\lambda^{-\oplus}$, $\alpha^{\otimes}$, $\rho^{-\bullet}$, or δ^r.

Since the upper left vertex in (8.7.16) is 0^X, by Lemma 3.1.29, the lower right vertex has the same support as 0^X. By Proposition 3.5.33, the two paths

$$L \quad \text{and} \quad (L_3, \lambda^{-\bullet}, L_2, L_1, P^{-1}, Q)$$

have the same value in C. This implies that (8.7.15) is commutative.

(2) If $n' = 0$ and $n, p, p' > 0$, then (8.7.2) is the following diagram.

(8.7.17)
$$\begin{tikzcd}
\mathbb{0}_{mm',qq'} \arrow[r, "\zeta^{-r}_{C\boxtimes C'}"] \arrow[d, "1"'] & (C\boxtimes C')\mathbb{0}_{mm',pp'} \arrow[d, "1\star\boxtimes^2"] \\
\mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"'] & (C\boxtimes C')(BA\boxtimes\mathbb{0}_{m',p'}) \arrow[d, "\boxtimes^2"] \\
(CB)A\boxtimes\mathbb{0}_{m',q'} \arrow[r, "a_{C,B,A}\boxtimes\zeta^{-r}_{C'}"] & C(BA)\boxtimes C'\mathbb{0}_{m',p'}
\end{tikzcd}$$

With an argument almost identical to that of case (1), each entry of (8.7.17) is a diagram in C similar to (8.7.15). Its commutativity is proved by realizing each edge or its inverse as the value in C of a path in $\mathsf{Gr}(X)$, similar to (8.7.16), for some set X of formal variables and function φ :

$X \longrightarrow \mathsf{Ob}(\mathsf{C})$. Since one of its vertices is 0^X, this diagram is commutative in C by Lemma 3.1.29 and Proposition 3.5.33.

(3) If $p = 0$ and $n, n', p' > 0$, then (8.7.2) is the following diagram.

$$\begin{tikzcd}
\mathbb{0}_{nn',qq'}(A \boxtimes A') \arrow[r, "\zeta^{\ell}_{A\boxtimes A'}"] \arrow[d, "\boxtimes^2 \star 1"'] & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
(\mathbb{0}_{n,q} \boxtimes C'B')(A \boxtimes A') \arrow[d, "\boxtimes^2"'] & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
(\mathbb{0}_{n,q}A) \boxtimes (C'B')A' \arrow[r, "\zeta^{\ell}_A \boxtimes a_{C',B',A'}"] & \mathbb{0}_{m,q} \boxtimes C'(B'A')
\end{tikzcd} \tag{8.7.18}$$

Composing the top right 1 away, the $\big((l-1)q'+l', (i-1)m'+i'\big)$-entry of (8.7.18) is the diagram in C below, with the notation in (8.7.6) and with $\otimes$ abbreviated to concatenation.

$$\begin{tikzcd}
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'} \mathbb{0}(A_{ji}A'_{j'i'})\Big]_{\mathsf{lt}} \arrow[r, "(\lambda^{\oplus},\lambda^{\bullet})"] \arrow[d, "(\oplus_j \oplus_{j'}\ \lambda^{-\bullet}1)_{\mathsf{lt}}"] & \mathbb{0} \arrow[d, "\lambda^{-\bullet}"'] \\
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'} \Big[\mathbb{0}\Big(\bigoplus_{k'=1}^{p'} C'_{l'k'}B'_{k'j'}\Big)_{\mathsf{lt}}\Big](A_{ji}A'_{j'i'})\Big]_{\mathsf{lt}} \arrow[d, "(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})"] & \mathbb{0}\Big[\bigoplus_{k'=1}^{p'} C'_{l'k'}\Big(\bigoplus_{j'=1}^{n'} B'_{k'j'}A'_{j'i'}\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \\
\Big[\bigoplus_{j=1}^{n}\Big[\bigoplus_{j'=1}^{n'} (\mathbb{0}A_{ji})\Big\{\Big(\bigoplus_{k'=1}^{p'} C'_{l'k'}B'_{k'j'}\Big)_{\mathsf{lt}} A'_{j'i'}\Big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "(\delta^{-r},\delta^{-l})"] & \mathbb{0}\Big[\bigoplus_{k'=1}^{p'}\Big(\bigoplus_{j'=1}^{n'} C'_{l'k'}(B'_{k'j'}A'_{j'i'})\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[u, "1a^4"'] \\
\Big[\bigoplus_{j=1}^{n} \mathbb{0}A_{ji}\Big]_{\mathsf{lt}}\Big[\bigoplus_{j'=1}^{n'}\Big(\bigoplus_{k'=1}^{p'} C'_{l'k'}B'_{k'j'}\Big)_{\mathsf{lt}} A'_{j'i'}\Big]_{\mathsf{lt}} \arrow[ur, "(\lambda^{\oplus},\lambda^{\bullet})(a^3a^2a^1)"'] &
\end{tikzcd} \tag{8.7.19}$$

As in (8.7.15), the factorization

$$(a_{C',B',A'})_{l'i'} = a^4a^3a^2a^1$$

in (8.7.19) is from (8.3.5) up to a change of symbols.

To prove that (8.7.19) is commutative, we realize each morphism or its inverse as the value in C of a path in $\mathsf{Gr}(X)$ using the set of formal variables

$$X = \left\{0^X, 1^X, a_{ji}, a'_{j'i'}, b'_{k'j'}, c'_{l'k'}\right\}_{1\le j\le n,\, 1\le j'\le n',\, 1\le k'\le p'}$$

and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{ji} & \text{if } x = a_{ji}. \\ A'_{j'i'} & \text{if } x = a'_{j'i'}. \\ B'_{k'j'} & \text{if } x = b'_{k'j'}. \\ C'_{l'k'} & \text{if } x = c'_{l'k'}. \end{cases}$$

By Notation 3.3.10 and Example 8.3.9, there is a diagram in $\mathsf{Gr}(X)$ as follows, in which the value in C of each path is the corresponding edge in (8.7.19) or its inverse. We continue to abbreviate $\otimes$ to concatenation.

(8.7.20)
$$\begin{tikzcd}
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'} 0^X(a_{ji}a'_{j'i'})\Big]_{\mathsf{lt}} \arrow[r, twoheadrightarrow, "L_2"] & 0^X \arrow[d, "\lambda^{-\bullet}"'] \\
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'} \Big[0^X\Big(\bigoplus_{k'=1}^{p'} c'_{l'k'}b'_{k'j'}\Big)_{\mathsf{lt}}\Big](a_{ji}a'_{j'i'})\Big]_{\mathsf{lt}} \arrow[u, twoheadrightarrow, "L_1"'] & 0^X\Big[\bigoplus_{k'=1}^{p'} c'_{l'k'}\Big(\bigoplus_{j'=1}^{n'} b'_{k'j'}a'_{j'i'}\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, twoheadrightarrow, "L_3"'] \\
\Big[\bigoplus_{j=1}^{n}\Big[\bigoplus_{j'=1}^{n'} (0^X a_{ji})\Big\{\Big(\bigoplus_{k'=1}^{p'} c'_{l'k'}b'_{k'j'}\Big)_{\mathsf{lt}} a'_{j'i'}\Big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[u, twoheadrightarrow, "P^{-1}"'] & 0^X\Big[\bigoplus_{k'=1}^{p'}\Big(\bigoplus_{j'=1}^{n'} c'_{l'k'}(b'_{k'j'}a'_{j'i'})\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \\
\Big[\bigoplus_{j=1}^{n} 0^X a_{ji}\Big]_{\mathsf{lt}}\Big[\bigoplus_{j'=1}^{n'}\Big(\bigoplus_{k'=1}^{p'} c'_{l'k'}b'_{k'j'}\Big)_{\mathsf{lt}} a'_{j'i'}\Big]_{\mathsf{lt}} \arrow[u, twoheadrightarrow, "Q"'] \arrow[ur, twoheadrightarrow, "L"'] &
\end{tikzcd}$$

Consider the diagram (8.7.20).

- Each prime edge in L is an identity or contains $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\lambda^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\bullet}$, or δ^{r}.
- The paths Q, P^{-1}, and L_1 are interpreted as in (8.7.16).
- Each prime edge in L_2 is an identity or contains $\lambda^{\oplus}$ or $\lambda^{\bullet}$.
- Each prime edge in L_3 is an identity or contains δ^{l}.

Since the upper right vertex in (8.7.20) is 0^X, by Lemma 3.1.29, the lower left vertex has the same support as 0^X. By Proposition 3.5.33, the two paths

$$L \quad \text{and} \quad (L_3, \lambda^{-\bullet}, L_2, L_1, P^{-1}, Q)$$

have the same value in C. This implies that (8.7.19) is commutative.

(4) If $p' = 0$ and $n, n', p > 0$, then (8.7.2) is the following diagram.

(8.7.21)
$$\begin{tikzcd}
\mathbb{0}_{nn',qq'}(A \boxtimes A') \arrow[r, "\zeta^{\ell}_{A\boxtimes A'}"] \arrow[d, "\boxtimes^2 \star 1"'] & \mathbb{0}_{mm',qq'} \arrow[d, "1"] \\
(CB \boxtimes \mathbb{0}_{n',q'})(A \boxtimes A') \arrow[d, "\boxtimes^2"'] & \mathbb{0}_{mm',qq'} \arrow[d, "\boxtimes^2"] \\
(CB)A \boxtimes (\mathbb{0}_{n',q'}A') \arrow[r, "a_{C,B,A} \boxtimes \zeta^{\ell}_{A'}"] & C(BA) \boxtimes \mathbb{0}_{m',q'}
\end{tikzcd}$$

With an argument almost identical to that of case (3), each entry of (8.7.21) is a diagram in C similar to (8.7.19). Its commutativity is proved by realizing each edge or its inverse as the value in C of a path in $\mathsf{Gr}(X)$, similar to (8.7.20), for some set X of formal variables and function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$. Since one of its vertices is 0^X, this diagram is commutative in C by Lemma 3.1.29 and Proposition 3.5.33.

This finishes the proofs of all four cases. $\square$

The remaining case of the lax associativity axiom (6.2.2) for the triple $(\boxtimes, \boxtimes^2, \boxtimes)$ is the most involved. We first compute the entries of the six objects in (8.7.2).

Lemma 8.7.22. *In the setting of (8.7.1), suppose that $m, n, p, q, m', n', p', q' > 0$. Then the $\big((l-1)q'+l', (i-1)m'+i'\big)$-entries of the six objects in (8.7.2) are as follows.*

(1)

$$\begin{aligned}&\Big[\big[(C \boxtimes C')(B \boxtimes B')\big](A \boxtimes A')\Big]_{(l-1)q'+l',(i-1)m'+i'}\\ &= \left[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\left[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}\big(C_{lk}\otimes C'_{l'k'}\big)\otimes\big(B_{kj}\otimes B'_{k'j'}\big)\right]_{\mathsf{lt}}\otimes\big(A_{ji}\otimes A'_{j'i'}\big)\right]_{\mathsf{lt}}\end{aligned}$$

(2)

$$\begin{aligned}&\big[(CB \boxtimes C'B')(A \boxtimes A')\big]_{(l-1)q'+l',(i-1)m'+i'}\\ &= \left[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\left\{\Big(\bigoplus_{k=1}^{p} C_{lk}\otimes B_{kj}\Big)_{\mathsf{lt}}\otimes\Big(\bigoplus_{k'=1}^{p'} C'_{l'k'}\otimes B'_{k'j'}\Big)_{\mathsf{lt}}\right\}\otimes\big(A_{ji}\otimes A'_{j'i'}\big)\right]_{\mathsf{lt}}\end{aligned}$$

(3)

$$\begin{aligned}&\big[(CB)A \boxtimes (C'B')A'\big]_{(l-1)q'+l',(i-1)m'+i'}\\ &= \left[\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p} C_{lk}\otimes B_{kj}\Big)_{\mathsf{lt}}\otimes A_{ji}\right]_{\mathsf{lt}}\otimes\left[\bigoplus_{j'=1}^{n'}\Big(\bigoplus_{k'=1}^{p'} C'_{l'k'}\otimes B'_{k'j'}\Big)_{\mathsf{lt}}\otimes A'_{j'i'}\right]_{\mathsf{lt}}\end{aligned}$$

(4)

$$\begin{aligned}&\Big[(C \boxtimes C')\big[(B \boxtimes B')(A \boxtimes A')\big]\Big]_{(l-1)q'+l',(i-1)m'+i'}\\ &= \left[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}\big(C_{lk}\otimes C'_{l'k'}\big)\otimes\left[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\big(B_{kj}\otimes B'_{k'j'}\big)\otimes\big(A_{ji}\otimes A'_{j'i'}\big)\right]_{\mathsf{lt}}\right]_{\mathsf{lt}}\end{aligned}$$

(5)

$$\begin{aligned}&\big[(C \boxtimes C')(BA \boxtimes B'A')\big]_{(l-1)q'+l',(i-1)m'+i'}\\ &= \left[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}\big(C_{lk}\otimes C'_{l'k'}\big)\otimes\left\{\Big(\bigoplus_{j=1}^{n} B_{kj}\otimes A_{ji}\Big)_{\mathsf{lt}}\otimes\Big(\bigoplus_{j'=1}^{n'} B'_{k'j'}\otimes A'_{j'i'}\Big)_{\mathsf{lt}}\right\}\right]_{\mathsf{lt}}\end{aligned}$$

(6)

$$\begin{aligned}&\big[C(BA) \boxtimes C'(B'A')\big]_{(l-1)q'+l',(i-1)m'+i'}\\ &= \left[\bigoplus_{k=1}^{p} C_{lk}\otimes\Big(\bigoplus_{j=1}^{n} B_{kj}\otimes A_{ji}\Big)_{\mathsf{lt}}\right]_{\mathsf{lt}}\otimes\left[\bigoplus_{k'=1}^{p'} C'_{l'k'}\otimes\Big(\bigoplus_{j'=1}^{n'} B'_{k'j'}\otimes A'_{j'i'}\Big)_{\mathsf{lt}}\right]_{\mathsf{lt}}\end{aligned}$$

Proof. Each of these equalities is proved by applying the definitions of the matrix product (8.1.4) and of the matrix tensor product (8.6.4). For example, the first equality is proved as follows.

$$\begin{aligned}
&\Big[\big[(C\boxtimes C')(B\boxtimes B')\big](A\boxtimes A')\Big]_{(l-1)q'+l',(i-1)m'+i'}\\
&=\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\big[(C\boxtimes C')(B\boxtimes B')\big]_{(l-1)q'+l',(j-1)n'+j'}\otimes(A\boxtimes A')_{(j-1)n'+j',(i-1)m'+i'}\Big]_{\mathsf{lt}}\\
&=\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}(C\boxtimes C')_{(l-1)q'+l',(k-1)p'+k'}\otimes(B\boxtimes B')_{(k-1)p'+k',(j-1)n'+j'}\Big]_{\mathsf{lt}}\\
&\qquad\qquad\otimes(A\boxtimes A')_{(j-1)n'+j',(i-1)m'+i'}\Big]_{\mathsf{lt}}
\end{aligned}$$

Both of these equalities follow from the definition (8.1.4) of the matrix product. The desired expression is obtained by applying (8.6.4) to each of the three entries of the matrix tensor products. The other five equalities in the lemma are proved in the same way. □

Next we prove the remaining case of the lax associativity axiom for $(\boxtimes, \boxtimes^2, \boxtimes^0)$.

Lemma 8.7.23. *In the setting of (8.7.1), if $m, n, p, q, m', n', p', q' > 0$, then the diagram (8.7.2) is commutative.*

Proof. The $\big((l-1)q' + l', (i-1)m' + i'\big)$-entry of (8.7.2) is the following diagram in C, with (1)–(6) denoting the six objects in Lemma 8.7.22.

$$\tag{8.7.24}
\begin{tikzcd}
(1) \arrow[r, "a_{C\boxtimes C',B\boxtimes B',A\boxtimes A'}"] \arrow[d, "\boxtimes^2 \star 1"'] & (4) \arrow[d, "1 \star \boxtimes^2"] \\
(2) \arrow[d, "\boxtimes^2"'] & (5) \arrow[d, "\boxtimes^2"] \\
(3) \arrow[r, "a_{C,B,A} \boxtimes a_{C',B',A'}"] & (6)
\end{tikzcd}$$

With

- $\otimes$ abbreviated to concatenation,
- the notation in (8.7.6),
- the factorization of the associator in (8.3.5), and
- the factorization of $\boxtimes^2$ represented by the zigzag of paths in Lemma 8.6.16,

the diagram (8.7.24) factors as follows.

(8.7.25)

$$\begin{tikzcd}
(1) \arrow[r, "(\alpha^{\pm\oplus},\xi^{\oplus},\alpha^{\otimes},\delta^{r})"] \arrow[d, "(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})"'] & \Big[\bigoplus\limits_{k=1}^{p}\bigoplus\limits_{k'=1}^{p'}\Big[\bigoplus\limits_{j=1}^{n}\bigoplus\limits_{j'=1}^{n'}(C_{lk}C'_{l'k'})\big((B_{kj}B'_{k'j'})(A_{ji}A'_{j'i'})\big)\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "(\delta^{-l})"] \\
\Big[\bigoplus\limits_{j=1}^{n}\bigoplus\limits_{j'=1}^{n'}\Big[\bigoplus\limits_{k=1}^{p}\Big\{\bigoplus\limits_{k'=1}^{p'}(C_{lk}B_{kj})(C'_{l'k'}B'_{k'j'})\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}(A_{ji}A'_{j'i'})\Big]_{\mathsf{lt}} \arrow[d, "(\delta^{-r},\delta^{-l})"'] & (4) \arrow[d, "(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})"] \\
(2) \arrow[d, "(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})"'] & \Big[\bigoplus\limits_{k=1}^{p}\bigoplus\limits_{k'=1}^{p'}(C_{lk}C'_{l'k'})\Big[\bigoplus\limits_{j=1}^{n}\Big\{\bigoplus\limits_{j'=1}^{n'}(B_{kj}A_{ji})(B'_{k'j'}A'_{j'i'})\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "(\delta^{-r},\delta^{-l})"] \\
\Big[\bigoplus\limits_{j=1}^{n}\Big[\bigoplus\limits_{j'=1}^{n'}\Big\{\Big(\bigoplus\limits_{k=1}^{p}C_{lk}B_{kj}\Big)A_{ji}\Big\}\Big\{\Big(\bigoplus\limits_{k'=1}^{p'}C'_{l'k'}B'_{k'j'}\Big)_{\mathsf{lt}}A'_{j'i'}\Big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "(\delta^{-r},\delta^{-l})"'] & (5) \arrow[d, "(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})"] \\
(3) \arrow[d, "(\alpha^{\pm\oplus},\xi^{\oplus},\alpha^{\otimes},\delta^{r})"'] & \Big[\bigoplus\limits_{k=1}^{p}\Big[\bigoplus\limits_{k'=1}^{p'}\Big\{C_{lk}\Big(\bigoplus\limits_{j=1}^{n}B_{kj}A_{ji}\Big)_{\mathsf{lt}}\Big\}\Big\{C'_{l'k'}\Big(\bigoplus\limits_{j'=1}^{n'}B'_{k'j'}A'_{j'i'}\Big)_{\mathsf{lt}}\Big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "(\delta^{-r},\delta^{-l})"] \\
\Big[\bigoplus\limits_{k=1}^{p}\Big(\bigoplus\limits_{j=1}^{n}C_{lk}(B_{kj}A_{ji})\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}}\Big[\bigoplus\limits_{k'=1}^{p'}\Big(\bigoplus\limits_{j'=1}^{n'}C'_{l'k'}(B'_{k'j'}A'_{j'i'})\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[r, "(\delta^{-l})"] & (6)
\end{tikzcd}$$

For example, (δ^{-l}) denotes a composite of morphisms, each being an iterated sum and product of identity morphisms and one component of δ^{-l}.

To prove that (8.7.25) is commutative, we realize each morphism or its inverse as the value in C of a path in $\mathsf{Gr}(X)$ using the set of formal variables

$$X = \Big\{0^X, 1^X, a_{ji}, b_{kj}, c_{lk}, a'_{j'i'}, b'_{k'j'}, c'_{l'k'}\Big\}_{1\le j\le n,\,1\le k\le p,\,1\le j'\le n',\,1\le k'\le p'}$$

and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{ji} & \text{if } x = a_{ji}. \\ B_{kj} & \text{if } x = b_{kj}. \\ C_{lk} & \text{if } x = c_{lk}. \\ A'_{j'i'} & \text{if } x = a'_{j'i'}. \\ B'_{k'j'} & \text{if } x = b'_{k'j'}. \\ C'_{l'k'} & \text{if } x = c'_{l'k'}. \end{cases}$$

There is a diagram in $\mathsf{Gr}(X)$ as follows, in which the value in C of each path in the boundary is the corresponding edge in (8.7.25) or its inverse. We continue to abbreviate $\otimes$ to concatenation. Moreover, to save space, we omit the subscripts in the formal variables, so a, b, c, a', b', and c' mean, respectively, a_{ji}, b_{kj}, c_{lk}, $a'_{j'i'}$, $b'_{k'j'}$, and $c'_{l'k'}$.

(8.7.26)
$$\begin{tikzcd}
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}(cc')(bb')\Big]_{\mathsf{lt}}(aa')\Big]_{\mathsf{lt}} \arrow[r, two heads, "L_5"] & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}(cc')\big((bb')(aa')\big)\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \\
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\Big[\bigoplus_{k=1}^{p}\Big\{\bigoplus_{k'=1}^{p'}(cb)(c'b')\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}(aa')\Big]_{\mathsf{lt}} \arrow[u, two heads, "L_4"] & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}(cc')\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}(bb')(aa')\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[u, two heads, "L_{11}"'] \\
\Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\Big\{\Big(\bigoplus_{k=1}^{p}cb\Big)_{\mathsf{lt}}\Big(\bigoplus_{k'=1}^{p'}c'b'\Big)_{\mathsf{lt}}\Big\}(aa')\Big]_{\mathsf{lt}} \arrow[u, two heads, "L_3"] & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}(cc')\Big[\bigoplus_{j=1}^{n}\Big\{\bigoplus_{j'=1}^{n'}(ba)(b'a')\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[u, two heads, "L_{10}"'] \\
\Big[\bigoplus_{j=1}^{n}\Big[\bigoplus_{j'=1}^{n'}\Big\{\Big(\bigoplus_{k=1}^{p}cb\Big)a\Big\}\Big\{\Big(\bigoplus_{k'=1}^{p'}c'b'\Big)_{\mathsf{lt}}a'\Big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[u, two heads, "L_2"] & \Big[\bigoplus_{k=1}^{p}\bigoplus_{k'=1}^{p'}(cc')\Big\{\Big(\bigoplus_{j=1}^{n}ba\Big)_{\mathsf{lt}}\Big(\bigoplus_{j'=1}^{n'}b'a'\Big)_{\mathsf{lt}}\Big\}\Big]_{\mathsf{lt}} \arrow[u, two heads, "L_9"'] \\
\Big[\bigoplus_{j=1}^{n}\Big(\bigoplus_{k=1}^{p}cb\Big)_{\mathsf{lt}}a\Big]_{\mathsf{lt}}\Big[\bigoplus_{j'=1}^{n'}\Big(\bigoplus_{k'=1}^{p'}c'b'\Big)_{\mathsf{lt}}a'\Big]_{\mathsf{lt}} \arrow[u, two heads, "L_1"] \arrow[d, two heads, "L_6"'] & \Big[\bigoplus_{k=1}^{p}\Big[\bigoplus_{k'=1}^{p'}\Big\{c\Big(\bigoplus_{j=1}^{n}ba\Big)_{\mathsf{lt}}\Big\}\Big\{c'\Big(\bigoplus_{j'=1}^{n'}b'a'\Big)_{\mathsf{lt}}\Big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[u, two heads, "L_8"'] \\
\Big[\bigoplus_{k=1}^{p}\Big(\bigoplus_{j=1}^{n}c(ba)\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}}\Big[\bigoplus_{k'=1}^{p'}\Big(\bigoplus_{j'=1}^{n'}c'(b'a')\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[uuuuur, "R"] & \Big[\bigoplus_{k=1}^{p}c\Big(\bigoplus_{j=1}^{n}ba\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}}\Big[\bigoplus_{k'=1}^{p'}c'\Big(\bigoplus_{j'=1}^{n'}b'a'\Big)_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[l, two heads, "L_{12}"'] \arrow[u, two heads, "L_7"']
\end{tikzcd}$$

Consider the diagram (8.7.26).

- Each prime edge in L_1, L_3, L_7, and L_9 is an identity or contains δ^r or δ^l.
- Each prime edge in L_2, L_4, L_8, and L_{10} is an identity or contains $\alpha^{-\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{-\otimes}$.
- Each prime edge in L_5 and L_6 is an identity or contains $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, or δ^r.
- Each prime edge in L_{11} and L_{12} is an identity or contains δ^l.
- Each prime edge in the path R is an identity or contains
  - δ^r or δ^l to bring the four sums to the front;
  - $\alpha^{\pm\otimes}$ or $\xi^{\otimes}$ to permute the monomial
  $$\big[c_{lk}(b_{kj}a_{ji})\big]\big[c'_{l'k'}(b'_{k'j'}a'_{j'i'})\big] \quad \text{to} \quad (c_{lk}c'_{l'k'})\big((b_{kj}b'_{k'j'})(a_{ji}a'_{j'i'})\big);$$
  or
  - $\alpha^{\pm\oplus}$ or $\xi^{\oplus}$ to move additive brackets and to permute additively to match with the additive bracketing of the codomain of R.

In (8.7.26), the upper right vertex is regular in the sense of Definition 3.1.25, so each vertex is regular by Lemma 3.1.29. Since C is assumed to be tight, the Coherence Theorem 3.9.1 implies that each of the two subdiagrams is commutative in C. This implies that (8.7.25) is commutative. $\square$

The Lax Unity Axiom. To finish the proof that $(\boxtimes, \boxtimes^2, \boxtimes^0)$ is a pseudofunctor, next we check the lax unity axiom. Recall the identity matrix in (8.1.6).

Lemma 8.7.27. *The data $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in Definition 8.6.19 satisfy the lax unity axiom (6.2.3).*

Proof. We check the lax left unity axiom; the proof for the lax right unity axiom is almost identical.

The lax left unity axiom (6.2.3) states the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{1}^{nq}(A \boxtimes B) \arrow[rr, "\ell_{A\boxtimes B}"] \arrow[d, "\boxtimes^0_{(n,q)} \star 1"'] & & A \boxtimes B \\
(\mathbb{1}^n \boxtimes \mathbb{1}^q)(A \boxtimes B) \arrow[rr, "\boxtimes^2_{(\mathbb{1}^n,\mathbb{1}^q),(A,B)}"] & & (\mathbb{1}^n A) \boxtimes (\mathbb{1}^q B) \arrow[u, "\ell_A \boxtimes \ell_B"']
\end{tikzcd} \tag{8.7.28}$$

in $\mathsf{Mat}^{\mathsf{C}}_{mp,nq}$ for $(A,B) \in \mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{p,q}$, with ℓ the natural isomorphism in (8.2.2). If one of m, n, p, or q is 0, then $\mathsf{Mat}^{\mathsf{C}}_{mp,nq}$ is the terminal category, and (8.7.28) is commutative. For the rest of this proof, we assume that $m,n,p,q > 0$.

For $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$, and $1 \le l \le q$, it suffices to show that the (J,I)-entry of (8.7.28) is commutative in C with

$$J = (j-1)q + l \quad \text{and} \quad I = (i-1)p + k.$$

By definitions (8.1.4), (8.1.5), (8.2.4), (8.6.4), (8.6.9), and (8.6.20), the (J,I)-entry of (8.7.28) is the following diagram in C. We abbreviate $\otimes$ to concatenation and use the notation in (8.7.6). Moreover, $J' = (j'-1)q + l'$.

$$\begin{tikzcd}
\Big[\bigoplus_{j'=1}^{n}\bigoplus_{l'=1}^{q} \mathbb{1}^{nq}_{(J,J')}(A_{j'i}B_{l'k})\Big]_{\mathsf{lt}} \arrow[rr, "(\lambda^{\oplus},\rho^{\oplus},\lambda^{\otimes},\lambda^{\bullet})"] \arrow[d, "(\lambda^{-\otimes},\lambda^{-\bullet},\rho^{-\bullet})"'] & & A_{ji}B_{lk} \\
\Big[\bigoplus_{j'=1}^{n}\bigoplus_{l'=1}^{q} (\mathbb{1}^n_{jj'}\mathbb{1}^q_{ll'})(A_{j'i}B_{l'k})\Big]_{\mathsf{lt}} \arrow[d, "(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})"'] & & \Big[\bigoplus_{j'=1}^{n} \mathbb{1}^n_{jj'}A_{j'i}\Big]_{\mathsf{lt}}\Big[\bigoplus_{l'=1}^{q} \mathbb{1}^q_{ll'}B_{l'k}\Big]_{\mathsf{lt}} \arrow[u, "(\lambda^{\oplus},\rho^{\oplus},\lambda^{\otimes},\lambda^{\bullet})"'] \\
\Big[\bigoplus_{j'=1}^{n}\Big\{\bigoplus_{l'=1}^{q} (\mathbb{1}^n_{jj'}A_{j'i})(\mathbb{1}^q_{ll'}B_{l'k})\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[rr, "(\delta^{-r},\delta^{-l})"'] & & {} \arrow[u]
\end{tikzcd} \tag{8.7.29}$$

For example, $(\lambda^{\oplus},\rho^{\oplus},\lambda^{\otimes},\lambda^{\bullet})$ denotes a composite of morphisms, each being an iterated sum and product of identity morphisms and one component of $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, or $\lambda^{\bullet}$.

To prove that (8.7.29) is commutative, we realize each morphism or its inverse as the value in C of a path in $\mathsf{Gr}(X)$ using the set of formal variables

$$X = \Big\{0^X, 1^X, a_{j'i}, b_{l'k}\Big\}_{1\le j'\le n,\, 1\le l'\le q}$$

and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{j'i} & \text{if } x = a_{j'i}. \\ B_{l'k} & \text{if } x = b_{l'k}. \end{cases}$$

Below we continue to abbreviate $\otimes$ to concatenation. The element $\delta^X_{rs} \in \{0^X, 1^X\}$ is defined in (8.4.6). There is a diagram in $\mathsf{Gr}(X)$ as follows, in which the value in C

of each path is the corresponding edge in (8.7.29) or its inverse.

$$
\begin{tikzcd}
\Big[\bigoplus_{j'=1}^{n}\bigoplus_{l'=1}^{q}\delta^{X}_{(J,J')}\big(a_{j'i}b_{l'k}\big)\Big]_{\mathsf{lt}} \arrow[d, ->>, "L_3"'] & & a_{ji}b_{lk} \arrow[ll, ->>, "L_2"'] \\
\Big[\bigoplus_{j'=1}^{n}\bigoplus_{l'=1}^{q}\big(\delta^{X}_{jj'}\delta^{X}_{ll'}\big)\big(a_{j'i}b_{l'k}\big)\Big]_{\mathsf{lt}} \arrow[dr, ->>, "L_4"'] & & \Big[\bigoplus_{j'=1}^{n}\delta^{X}_{jj'}a_{j'i}\Big]_{\mathsf{lt}}\Big[\bigoplus_{l'=1}^{q}\delta^{X}_{ll'}b_{l'k}\Big]_{\mathsf{lt}} \arrow[u, ->>, "L_1"'] \arrow[dl, ->>, "L"] \\
& \Big[\bigoplus_{j'=1}^{n}\Big\{\bigoplus_{l'=1}^{q}\big(\delta^{X}_{jj'}a_{j'i}\big)\big(\delta^{X}_{ll'}b_{l'k}\big)\Big\}_{\mathsf{lt}}\Big]_{\mathsf{lt}} &
\end{tikzcd}
\tag{8.7.30}
$$

Consider the diagram (8.7.30).

- Each prime edge in L is an identity or contains δ^r or δ^l.
- Each prime edge in L_1 is an identity or contains $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, or $\lambda^{\bullet}$.
- Each prime edge in L_2 is an identity or contains $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\lambda^{-\otimes}$, or $\lambda^{-\bullet}$.
- Each prime edge in L_3 is an identity or contains $\lambda^{-\otimes}$, $\lambda^{-\bullet}$, or $\rho^{-\bullet}$.
- Each prime edge in L_4 is an identity or contains $\alpha^{\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{\otimes}$.

The vertex $a_{ji}b_{lk}$ in (8.7.30) is regular in the sense of Definition 3.1.25. By Lemma 3.1.29, the middle right vertex is also regular. Since C is assumed to be tight, the Coherence Theorem 3.9.1 implies that the two paths

$$L \quad \text{and} \quad (L_4, L_3, L_2, L_1)$$

have the same value in C. This implies that (8.7.29) is commutative. □

Lemma 8.7.31. *The data $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in Definition 8.6.19 is a pseudofunctor.*

Proof. We already proved all the necessary statements.

- Lemma 8.6.7 shows that $\boxtimes$ defines local functors.
- Each component of $\boxtimes^0$ is an isomorphism by Lemma 8.6.8.
- $\boxtimes^2$ is a natural isomorphism by Lemma 8.6.21.
- Lemmas 8.7.3, 8.7.4, 8.7.13, and 8.7.23 prove the lax associativity axiom (6.2.2).
- Lemma 8.7.27 proves the lax unity axiom (6.2.3).

Therefore, $(\boxtimes, \boxtimes^2, \boxtimes^0)$ is a pseudofunctor. □

8.8. The Monoidal Associator

Throughout this section, C denotes an arbitrary tight symmetric bimonoidal category as in Definition 2.1.2. We are in the process of constructing a monoidal bicategory structure on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$. In this section, we define the monoidal associator, in the sense of Definition 6.4.1, for the monoidal composition $\boxtimes$ on $\mathsf{Mat}^{\mathsf{C}}$. In Lemma 8.7.31, we proved that the triple

$$(\boxtimes, \boxtimes^2, \boxtimes^0) : \mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}} \longrightarrow \mathsf{Mat}^{\mathsf{C}}$$

in Definition 8.6.19 is a pseudofunctor. This induces two pseudofunctors

$$\mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}} \underset{\boxtimes(1\times\boxtimes)}{\overset{\boxtimes(\boxtimes\times 1)}{\rightrightarrows}} \mathsf{Mat}^{\mathsf{C}}.$$

As discussed in Explanation 6.4.6, the monoidal associator $(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$ consists of the following data.

(i) $a^{\boxtimes}$ and $a^{\boxtimes\bullet}$ are strong transformations as in Definition 6.2.14 as follows.

$$\boxtimes(\boxtimes \times 1) \underset{a^{\boxtimes\bullet}}{\overset{a^{\boxtimes}}{\rightleftarrows}} \boxtimes(1 \times \boxtimes)$$

These strong transformations are decorated with $\boxtimes$ to avoid confusion with the base associator (8.3.2) in $\mathsf{Mat}^{\mathsf{C}}$.

(ii) η^a and ε^a are invertible modifications as in Definition 6.3.1 as follows.

$$1_{\boxtimes(\boxtimes\times 1)} \xrightarrow{\eta^a} a^{\boxtimes\bullet}a^{\boxtimes}$$
$$a^{\boxtimes}a^{\boxtimes\bullet} \xrightarrow{\varepsilon^a} 1_{\boxtimes(1\times\boxtimes)}$$

Moreover, these data are required to satisfy the triangle identities (6.3.10).

Here is an outline of this section.

- The left adjoint $a^{\boxtimes}$ is defined in Definition 8.8.1. Lemmas 8.8.5, 8.8.11, 8.8.17, and 8.8.26 show that $a^{\boxtimes}$ is a strong transformation.
- The right adjoint $a^{\boxtimes\bullet}$ is defined in Definition 8.8.33.
- The modifications η^a and ε^a are defined in Definition 8.8.37 and verified in Lemma 8.8.39.
- The triangle identities (6.3.10) are proved in Lemma 8.8.45.

The Left Adjoint of the Monoidal Associator.

Definition 8.8.1. With respect to the pseudofunctor $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in Definition 8.6.19 and Lemma 8.7.31, define the data of a lax transformation

$$\boxtimes(\boxtimes \times 1) \xrightarrow{a^{\boxtimes}} \boxtimes(1 \times \boxtimes)$$

as follows.

Component 1-Cells: For each triple of objects $(m, n, p) \in \left(\mathsf{Mat}^{\mathsf{C}}\right)^3$, define

$$mnp = \big((m \boxtimes n) \boxtimes p\big) \xrightarrow{a^{\boxtimes}_{m,n,p}} \big(m \boxtimes (n \boxtimes p)\big) = mnp \tag{8.8.2}$$

as the identity matrix $\mathbb{1}^{mnp} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ in (8.1.6).

Component 2-Cells: For each triple of 1-cells

$$\big(A = (A_{i'i}), B = (B_{j'j}), C = (C_{k'k})\big) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'} \times \mathsf{Mat}^{\mathsf{C}}_{p,p'},$$

define the component 2-cell

$$a^{\boxtimes}_{A,B,C} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,m'n'p'}\Big([A \boxtimes (B \boxtimes C)]a^{\boxtimes}_{m,n,p}; a^{\boxtimes}_{m',n',p'}[(A \boxtimes B) \boxtimes C]\Big)$$

as the following vertical composite.

(8.8.3)
$$\begin{tikzcd}[column sep=large]
& mnp \arrow[dr, bend left, "A \boxtimes (B \boxtimes C)"] \arrow[d, phantom, "\Downarrow r_{A \boxtimes (B \boxtimes C)}"] & \\
mnp \arrow[ur, bend left, "\mathbb{1}^{mnp}"] \arrow[rr, bend left=10, "A \boxtimes (B \boxtimes C)"] \arrow[rr, bend right=10, "(A \boxtimes B) \boxtimes C"'] \arrow[rr, phantom, "\Downarrow \alpha^{-\otimes}"] \arrow[dr, bend right, "(A \boxtimes B) \boxtimes C"'] & \arrow[d, phantom, "\Downarrow \ell^{-1}_{(A \boxtimes B) \boxtimes C}"] & m'n'p' \\
& m'n'p' \arrow[ur, bend right, "\mathbb{1}^{m'n'p'}"'] &
\end{tikzcd}$$

- ℓ is the base left unitor in $\mathsf{Mat}^{\mathsf{C}}$ in (8.2.2).
- r is the base right unitor in $\mathsf{Mat}^{\mathsf{C}}$ in (8.2.8).
- $\alpha^{-\otimes}$ is the 2-cell with $((i'-1)n'p' + (j'-1)p' + k', (i-1)np + (j-1)p + k)$-entry the structure morphism in C,

$$A_{i'i} \otimes \left(B_{j'j} \otimes C_{k'k}\right) \xrightarrow{\alpha^{-\otimes}_{A_{i'i}, B_{j'j}, C_{k'k}}} \left(A_{i'i} \otimes B_{j'j}\right) \otimes C_{k'k}$$

for $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$, $1 \le i' \le m'$, $1 \le j' \le n'$, and $1 \le k' \le p'$.

This finishes the definition of $a^{\boxtimes}$. $\diamond$

Explanation 8.8.4. By (8.2.4) and (8.2.10), each entry of the 2-cell $a^{\boxtimes}_{A,B,C}$ in (8.8.3) is a composite of isomorphisms, each being an iterated sum of identity morphisms and one component of $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\alpha^{-\otimes}$, $\lambda^{-\otimes}$, $\rho^{\otimes}$, $\lambda^{-\bullet}$, or $\rho^{\bullet}$ in C. $\diamond$

Lemma 8.8.5. *For each pair of objects (m,n,p) and (m',n',p') in $\left(\mathsf{Mat}^{\mathsf{C}}\right)^3$, $a^{\boxtimes}$ in (8.8.3) is a natural isomorphism.*

Proof. The naturality of $a^{\boxtimes}$ means that, for each triple of 2-cells

$$(f,g,h) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'}(A,A') \times \mathsf{Mat}^{\mathsf{C}}_{n,n'}(B,B') \times \mathsf{Mat}^{\mathsf{C}}_{p,p'}(C,C'),$$

the diagram

(8.8.6)
$$\begin{tikzcd}
{[A \boxtimes (B \boxtimes C)]\, \mathbb{1}^{mnp}} \arrow[r, "a^{\boxtimes}_{A,B,C}"] \arrow[d, "{[f \boxtimes (g \boxtimes h)] \star 1}"'] & \mathbb{1}^{m'n'p'}\, [(A \boxtimes B) \boxtimes C] \arrow[d, "{1 \star [(f \boxtimes g) \boxtimes h]}"] \\
{[A' \boxtimes (B' \boxtimes C')]\, \mathbb{1}^{mnp}} \arrow[r, "a^{\boxtimes}_{A',B',C'}"] & \mathbb{1}^{m'n'p'}\, [(A' \boxtimes B') \boxtimes C']
\end{tikzcd}$$

in $\mathsf{Mat}^{\mathsf{C}}_{mnp,m'n'p'}$ is commutative. Since 2-cells in $\mathsf{Mat}^{\mathsf{C}}$ are entrywise morphisms in C, it suffices to prove the commutativity of (8.8.6) in each entry. Restricted to each entry, the commutativity of (8.8.6) follows from the naturality of the structure morphisms $\lambda^{\oplus}$, $\rho^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\lambda^{\bullet}$, and $\rho^{\bullet}$ in C. Moreover, since these are natural isomorphisms in C, $a^{\boxtimes}$ is a natural isomorphism. $\square$

The Lax Unity of the Left Adjoint. Before we show that $a^{\boxtimes}$ satisfies the lax unity axiom (6.2.15), we first describe the entries of the relevant 1-cells. The next few lemmas follow from the definitions of the matrix product (8.1.4) and of the matrix tensor product (8.6.4), applied to the identity matrices (8.1.6). The next lemma describes the entries in the domain 1-cell of the lax unity axiom.

Lemma 8.8.7. *For $m,n,p > 0$, each entry of the matrix product*

$$\mathbb{1}^{mnp}\mathbb{1}^{mnp} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$$

is a left normalized sum of mnp objects as follows.

(1) *For $1 \le s \le mnp$, the sth diagonal entry is*

$$\Bigg[\overbrace{\Big[\bigoplus_{i=1}^{s-1} \mathbb{0}\otimes\mathbb{0}\Big]}^{\text{empty if } s=1} \oplus (\mathbb{1}\otimes\mathbb{1}) \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{mnp} \mathbb{0}\otimes\mathbb{0}\Big]}^{\text{empty if } s=mnp}\Bigg]_{\mathsf{lt}}.$$

(2) *The (s,t)-entry with $s < t$ is*

$$\Bigg[\overbrace{\Big[\bigoplus_{i=1}^{s-1} \mathbb{0}\otimes\mathbb{0}\Big]}^{\text{empty if } s=1} \oplus (\mathbb{1}\otimes\mathbb{0}) \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{t-1} \mathbb{0}\otimes\mathbb{0}\Big]}^{\text{empty if } s=t-1} \oplus (\mathbb{0}\otimes\mathbb{1}) \oplus \overbrace{\Big[\bigoplus_{i=t+1}^{mnp} \mathbb{0}\otimes\mathbb{0}\Big]}^{\text{empty if } t=mnp}\Bigg]_{\mathsf{lt}}.$$

(3) *The (s,t)-entry with $s > t$ is*

$$\Bigg[\overbrace{\Big[\bigoplus_{i=1}^{t-1} \mathbb{0}\otimes\mathbb{0}\Big]}^{\text{empty if } t=1} \oplus (\mathbb{0}\otimes\mathbb{1}) \oplus \overbrace{\Big[\bigoplus_{i=t+1}^{s-1} \mathbb{0}\otimes\mathbb{0}\Big]}^{\text{empty if } t=s-1} \oplus (\mathbb{1}\otimes\mathbb{0}) \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{mnp} \mathbb{0}\otimes\mathbb{0}\Big]}^{\text{empty if } s=mnp}\Bigg]_{\mathsf{lt}}.$$

The lax unity axiom also involves the following matrices.

Lemma 8.8.8. *Suppose $m,n,p > 0$.*

(1) *For the matrix*

$$\mathbb{1}^m \boxtimes (\mathbb{1}^n \boxtimes \mathbb{1}^p) \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp},$$

the following two statements hold.
- *Each diagonal entry is $\mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{1})$.*
- *Each off-diagonal entry is $\mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{0})$, $\mathbb{1}\otimes(\mathbb{0}\otimes\mathbb{1})$, $\mathbb{1}\otimes(\mathbb{0}\otimes\mathbb{0})$, $\mathbb{0}\otimes(\mathbb{1}\otimes\mathbb{1})$, $\mathbb{0}\otimes(\mathbb{1}\otimes\mathbb{0})$, $\mathbb{0}\otimes(\mathbb{0}\otimes\mathbb{1})$, or $\mathbb{0}\otimes(\mathbb{0}\otimes\mathbb{0})$.*

(2) *For the matrix*

$$(\mathbb{1}^m \boxtimes \mathbb{1}^n) \boxtimes \mathbb{1}^p \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp},$$

the following two statements hold.
- *Each diagonal entry is $(\mathbb{1}\otimes\mathbb{1})\otimes\mathbb{1}$.*
- *Each off-diagonal entry is $(\mathbb{1}\otimes\mathbb{1})\otimes\mathbb{0}$, $(\mathbb{1}\otimes\mathbb{0})\otimes\mathbb{1}$, $(\mathbb{1}\otimes\mathbb{0})\otimes\mathbb{0}$, $(\mathbb{0}\otimes\mathbb{1})\otimes\mathbb{1}$, $(\mathbb{0}\otimes\mathbb{1})\otimes\mathbb{0}$, $(\mathbb{0}\otimes\mathbb{0})\otimes\mathbb{1}$, or $(\mathbb{0}\otimes\mathbb{0})\otimes\mathbb{0}$.*

The next lemma describes an intermediate 1-cell in the left-hand side of the lax unity axiom.

Lemma 8.8.9. *For $m,n,p > 0$, each entry of the matrix product*

$$\big[\mathbb{1}^m \boxtimes (\mathbb{1}^n \boxtimes \mathbb{1}^p)\big]\mathbb{1}^{mnp} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$$

is a left normalized sum of mnp objects as follows, with $z_{j,i}$ denoting the (j,i)-entry in $\mathbb{1}^m \boxtimes (\mathbb{1}^n \boxtimes \mathbb{1}^p)$.

(1) *For $1 \le s \le mnp$, the sth diagonal entry is*

$$\Bigg[\overbrace{\Big[\bigoplus_{i=1}^{s-1} z_{s,i}\otimes\mathbb{0}\Big]}^{\text{empty if } s=1} \oplus \Big[\big(\mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{1})\big)\otimes\mathbb{1}\Big] \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{mnp} z_{s,i}\otimes\mathbb{0}\Big]}^{\text{empty if } s=mnp}\Bigg]_{\mathsf{lt}}.$$

(2) *The* (s,t)*-entry with* $s < t$ *is*

$$\Big[\overbrace{\Big[\bigoplus_{i=1}^{s-1} z_{s,i} \otimes \mathbb{0}\Big]}^{\text{empty if } s=1} \oplus \Big[\big(\mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1})\big) \otimes \mathbb{0}\Big] \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{t-1} z_{s,i} \otimes \mathbb{0}\Big]}^{\text{empty if } s=t-1} \oplus (z_{s,t} \otimes \mathbb{1}) \oplus \overbrace{\Big[\bigoplus_{i=t+1}^{mnp} z_{s,i} \otimes \mathbb{0}\Big]}^{\text{empty if } t=mnp}\Big]_{\mathsf{lt}}.$$

(3) *The* (s,t)*-entry with* $s > t$ *is*

$$\Big[\overbrace{\Big[\bigoplus_{i=1}^{t-1} z_{s,i} \otimes \mathbb{0}\Big]}^{\text{empty if } t=1} \oplus (z_{s,t} \otimes \mathbb{1}) \oplus \overbrace{\Big[\bigoplus_{i=t+1}^{s-1} z_{s,i} \otimes \mathbb{0}\Big]}^{\text{empty if } t=s-1} \oplus \Big[\big(\mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1})\big) \otimes \mathbb{0}\Big] \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{mnp} z_{s,i} \otimes \mathbb{0}\Big]}^{\text{empty if } s=mnp}\Big]_{\mathsf{lt}}.$$

The next lemma describes the entries in the codomain 1-cell of the lax unity axiom.

Lemma 8.8.10. *For* $m,n,p > 0$*, each entry of the matrix product*

$$\mathbb{1}^{mnp}\big[(\mathbb{1}^m \boxtimes \mathbb{1}^n) \boxtimes \mathbb{1}^p\big] \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$$

is a left normalized sum of mnp *objects as follows, with* $y_{i,j}$ *denoting the* (i,j)*-entry in* $(\mathbb{1}^m \boxtimes \mathbb{1}^n) \boxtimes \mathbb{1}^p$*.*

(1) *For* $1 \leq s \leq mnp$*, the sth diagonal entry is*

$$\Big[\overbrace{\Big[\bigoplus_{i=1}^{s-1} \mathbb{0} \otimes y_{i,s}\Big]}^{\text{empty if } s=1} \oplus \Big[\mathbb{1} \otimes \big((\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1}\big)\Big] \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{mnp} \mathbb{0} \otimes y_{i,s}\Big]}^{\text{empty if } s=mnp}\Big]_{\mathsf{lt}}.$$

(2) *The* (s,t)*-entry with* $s < t$ *is*

$$\Big[\overbrace{\Big[\bigoplus_{i=1}^{s-1} \mathbb{0} \otimes y_{i,t}\Big]}^{\text{empty if } s=1} \oplus (\mathbb{1} \otimes y_{s,t}) \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{t-1} \mathbb{0} \otimes y_{i,t}\Big]}^{\text{empty if } s=t-1} \oplus \Big[\mathbb{0} \otimes \big((\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1}\big)\Big] \oplus \overbrace{\Big[\bigoplus_{i=t+1}^{mnp} \mathbb{0} \otimes y_{i,t}\Big]}^{\text{empty if } t=mnp}\Big]_{\mathsf{lt}}.$$

(3) *The* (s,t)*-entry with* $s > t$ *is*

$$\Big[\overbrace{\Big[\bigoplus_{i=1}^{t-1} \mathbb{0} \otimes y_{i,t}\Big]}^{\text{empty if } t=1} \oplus \Big[\mathbb{0} \otimes \big((\mathbb{1} \otimes \mathbb{1}) \otimes \mathbb{1}\big)\Big] \oplus \overbrace{\Big[\bigoplus_{i=t+1}^{s-1} \mathbb{0} \otimes y_{i,t}\Big]}^{\text{empty if } t=s-1} \oplus (\mathbb{1} \otimes y_{s,t}) \oplus \overbrace{\Big[\bigoplus_{i=s+1}^{mnp} \mathbb{0} \otimes y_{i,t}\Big]}^{\text{empty if } s=mnp}\Big]_{\mathsf{lt}}.$$

To see that $a^{\boxtimes}$ is a strong transformation, next we check the axioms in Definition 6.2.14, starting with the lax unity axiom.

Lemma 8.8.11. $a^{\boxtimes}$ *in Definition 8.8.1 satisfies the lax unity axiom (6.2.15).*

Proof. By (6.2.7) and (8.8.2), the lax unity axiom for $a^{\boxtimes}$ states the following equality of pasting diagrams in $\mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ for $m,n,p \geq 0$.

(8.8.12)
$$
\begin{tikzcd}[column sep=huge, row sep=huge]
mnp \arrow[r, bend left=60, "(\mathbb{1}^m \boxtimes \mathbb{1}^n) \boxtimes \mathbb{1}^p"] \arrow[d, "\mathbb{1}^{mnp}"'] \arrow[dr, phantom, "\Uparrow a^{\boxtimes}_{\mathbb{1}^m,\mathbb{1}^n,\mathbb{1}^p}"] & mnp \arrow[d, "\mathbb{1}^{mnp}"] \\
mnp \arrow[r, bend left=60, "\mathbb{1}^m \boxtimes (\mathbb{1}^n \boxtimes \mathbb{1}^p)"] \arrow[r, "\mathbb{1}^m \boxtimes \mathbb{1}^{np}" description] \arrow[r, bend right=60, "\mathbb{1}^{mnp}"'] \arrow[r, phantom, shift left=4ex, "\Uparrow 1 \boxtimes \boxtimes^0_{(n,p)}"] \arrow[r, phantom, shift right=4ex, "\Uparrow \boxtimes^0_{(m,np)}"] & mnp
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=huge]
mnp \arrow[r, bend left=60, "(\mathbb{1}^m \boxtimes \mathbb{1}^n) \boxtimes \mathbb{1}^p"] \arrow[r, "\mathbb{1}^{mn} \boxtimes \mathbb{1}^p" description] \arrow[r, bend right=50, "\mathbb{1}^{mnp}"'] \arrow[r, phantom, shift left=4ex, "\Uparrow \boxtimes^0_{(m,n)} \boxtimes 1"] \arrow[r, phantom, shift right=4ex, "\Uparrow \boxtimes^0_{(mn,p)}"] \arrow[d, "\mathbb{1}^{mnp}"'] \arrow[dr, bend right=30, "\mathbb{1}^{mnp}"'] \arrow[dr, phantom, "\Uparrow r^{-1}_{\mathbb{1}^{mnp}}"] & mnp \arrow[d, "\mathbb{1}^{mnp}"] \\
mnp \arrow[r, bend right=60, "\mathbb{1}^{mnp}"'] \arrow[r, phantom, shift right=3ex, "\Uparrow \ell_{\mathbb{1}^{mnp}}"] & mnp
\end{tikzcd}
$$

If m, n, or p is 0, then $\mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ is the terminal category, and the equality in (8.8.12) holds. For the rest of this proof, we assume that $m,n,p > 0$.

Since 2-cells in $\mathsf{Mat}^{\mathsf{C}}$ are entrywise morphisms in C, it suffices to prove the equality in (8.8.12) in a typical entry. The entries of the 2-cells in (8.8.12) are interpreted using the definitions of

- the lax unity constraint $\boxtimes^0$ in Lemma 8.6.8,
- the component 2-cell $a^{\boxtimes}_{A,B,C}$ in (8.8.3),
- the base left unitor ℓ in (8.2.4), and
- the base right unitor r in (8.2.10).

The entries of the 1-cells in (8.8.12) are described in Lemmas 8.8.7 through 8.8.10. We will realize each 2-cell entry in (8.8.12) using paths in $\mathsf{Gr}(X)$ with

- the set $X = \{0^X, 1^X\}$ and
- the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by $\varphi(0^X) = \mathbb{0}$ and $\varphi(1^X) = \mathbb{1}$.

The rest of this proof is split into two cases, one for the diagonal entries and one for the off-diagonal entries.

(1) For each $1 \leq s \leq mnp$, there exists a diagram in $\mathsf{Gr}(X)$ as in (8.8.13) below, whose left path (L_2, L_1), respectively right path (R_3, R_2, R_1), has value in C the sth diagonal entry of the left, respectively right, pasting diagram in (8.8.12). We abbreviate $\otimes$ to concatenation.

(8.8.13)
$$
\begin{tikzcd}[column sep=small]
\Big[\big[\bigoplus_{i=1}^{s-1} 0^X0^X\big] \oplus (1^X1^X) \oplus \big[\bigoplus_{i=s+1}^{mnp} 0^X0^X\big]\Big]_{\mathsf{lt}} \arrow[r, "R_1"] \arrow[d, ->>, "L_1"'] & 1^X \arrow[d, ->>, "R_2"] \\
\Big[\big[\bigoplus_{i=1}^{s-1} z0^X\big] \oplus \big[(1^X(1^X1^X))1^X\big] \oplus \big[\bigoplus_{i=s+1}^{mnp} z0^X\big]\Big]_{\mathsf{lt}} \arrow[d, ->>, "L_2"'] & \Big[\big[\bigoplus_{i=1}^{s-1} 0^X0^X\big] \oplus (1^X1^X) \oplus \big[\bigoplus_{i=s+1}^{mnp} 0^X0^X\big]\Big]_{\mathsf{lt}} \arrow[dl, ->>, "R_3"] \\
\Big[\big[\bigoplus_{i=1}^{s-1} 0^Xy\big] \oplus \big[1^X((1^X1^X)1^X)\big] \oplus \big[\bigoplus_{i=s+1}^{mnp} 0^Xy\big]\Big]_{\mathsf{lt}} &
\end{tikzcd}
$$

Consider the diagram (8.8.13).

- In the upper left vertex, as in Lemma 8.8.7, the sequence of symbols

$$\Big[\bigoplus_{i=1}^{s-1} 0^X 0^X\Big] \oplus$$

is empty if $s = 1$, and the sum after $(1^X 1^X)$ is empty if $s = mnp$. Similar interpretations apply to the sums in the other vertices.
- Each instance of z is an element $\delta_1^X(\delta_2^X \delta_3^X) \in X^{\mathsf{fr}}$ with each $\delta_j^X \in X = \{0^X, 1^X\}$, at least one of which is 0^X.
- Each instance of y is an element $(\delta_1^X \delta_2^X)\delta_3^X \in X^{\mathsf{fr}}$ with each $\delta_j^X \in X = \{0^X, 1^X\}$, at least one of which is 0^X.
- The value of L_1 in C is the morphism

$$\varphi L_1 = \Big(\big[(1 \boxtimes \boxtimes^0_{(n,p)}) \boxtimes^0_{(m,np)}\big] \star 1_{\mathbb{1}^{mnp}}\Big)_{s,s}.$$

Each prime edge in L_1 is an identity or contains $\lambda^{-\otimes}$, $\lambda^{-\bullet}$, or $\rho^{-\bullet}$.
- The value of L_2 in C is the morphism

$$\varphi L_2 = \big(a^{\boxtimes}_{\mathbb{1}^m, \mathbb{1}^n, \mathbb{1}^p}\big)_{s,s}.$$

Each prime edge in L_2 is an identity or contains $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\alpha^{-\otimes}$, $\lambda^{-\otimes}$, $\rho^{\otimes}$, $\lambda^{-\bullet}$, or $\rho^{\bullet}$.
- The value of R_1 in C is the morphism

$$\varphi R_1 = (\ell_{\mathbb{1}^{mnp}})_{s,s}.$$

Each prime edge in R_1 is an identity or contains $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, or $\lambda^{\bullet}$.
- The value of R_2 in C is the morphism

$$\varphi R_2 = \big(r^{-1}_{\mathbb{1}^{mnp}}\big)_{s,s}.$$

Each prime edge in R_2 is an identity or contains $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\rho^{-\otimes}$, or $\rho^{-\bullet}$.
- The value of R_3 in C is the morphism

$$\varphi R_3 = \Big(1_{\mathbb{1}^{mnp}} \star \big[(\boxtimes^0_{(m,n)} \boxtimes 1) \boxtimes^0_{(mn,p)}\big]\Big)_{s,s}.$$

Each prime edge in R_3 is an identity or contains $\lambda^{-\otimes}$, $\lambda^{-\bullet}$, or $\rho^{-\bullet}$.

Since the vertex 1^X in (8.8.13) is regular as in Definition 3.1.25, the common domain of L_1 and R_1 is also regular by Lemma 3.1.29. Since C is assumed to be tight, the Coherence Theorem 3.9.1 implies that the two paths (L_2, L_1) and (R_3, R_2, R_1) have the same value in C. This implies that the sth diagonal entries of the two pasting diagrams in (8.8.12) are equal.

(2) For $1 \leq s < t \leq mnp$, there exists a diagram in $\mathsf{Gr}(X)$ as follows.

(8.8.14)
$$\begin{tikzcd}
\Big[\big[\bigoplus_{i=1}^{s-1} 0^X 0^X\big] \oplus (1^X 0^X) \oplus \big[\bigoplus_{i=s+1}^{t-1} 0^X 0^X\big] \oplus (0^X 1^X) \oplus \big[\bigoplus_{i=t+1}^{mnp} 0^X 0^X\big]\Big]_{\mathsf{lt}} \arrow[d, twoheadrightarrow, "L_1"'] \arrow[r, twoheadrightarrow, "R_1"] & 0^X \arrow[dd, twoheadrightarrow, "R_2"] \\
\Big[\big[\bigoplus_{i=1}^{s-1} z0^X\big] \oplus \big[(1^X(1^X1^X))0^X\big] \oplus \big[\bigoplus_{i=s+1}^{t-1} z0^X\big] \oplus (z1^X) \oplus \big[\bigoplus_{i=t+1}^{mnp} z0^X\big]\Big]_{\mathsf{lt}} \arrow[dd, twoheadrightarrow, "L_2"'] & \\
 & \Big[\big[\bigoplus_{i=1}^{s-1} 0^X 0^X\big] \oplus (1^X 0^X) \oplus \big[\bigoplus_{i=s+1}^{t-1} 0^X 0^X\big] \oplus (0^X 1^X) \oplus \big[\bigoplus_{i=t+1}^{mnp} 0^X 0^X\big]\Big]_{\mathsf{lt}} \arrow[dl, twoheadrightarrow, "R_3"] \\
\Big[\big[\bigoplus_{i=1}^{s-1} 0^X y\big] \oplus (1^X y) \oplus \big[\bigoplus_{i=s+1}^{t-1} 0^X y\big] \oplus \big[0^X((1^X1^X)1^X)\big] \oplus \big[\bigoplus_{i=t+1}^{mnp} 0^X y\big]\Big]_{\mathsf{lt}} &
\end{tikzcd}$$

Consider the diagram (8.8.14).

- As in Lemma 8.8.7, the sums
$$\bigoplus_{i=1}^{s-1}, \quad \bigoplus_{i=s+1}^{t-1}, \quad \text{and} \quad \bigoplus_{i=t+1}^{mnp}$$
are empty if, respectively, $s = 1$, $s = t - 1$, and $t = mnp$.
- The elements $y, z \in X^{\mathsf{fr}}$ are interpreted as in case (1), and similarly for the paths L_1, L_2, R_1, R_2, and R_3 with the (s,t)-entry instead of the (s,s)-entry.

Since one vertex in (8.8.14) is 0^X, the common domain of L_1 and R_1 has the same support as 0^X by Lemma 3.1.29. Proposition 3.5.33 implies that the two paths (L_2, L_1) and (R_3, R_2, R_1) have the same value in C. This implies that the (s,t)-entries of the two pasting diagrams in (8.8.12) are equal.

The case with $s > t$ is proved in the same way, using the $s > t$ cases in Lemmas 8.8.7 through 8.8.10 to interpret the 1-cells.

This finishes the proof that $a^{\boxtimes}$ satisfies the lax unity axiom (6.2.15). $\square$

The Lax Naturality of the Left Adjoint. The lax naturality axiom (6.2.16) for $a^{\boxtimes}$ states that, for 1-cells

(8.8.15)
$$\begin{tikzcd}
m \arrow[d, "A"'] & m' \arrow[d, "A'"'] & m'' \arrow[d, "A''"'] \\
n \arrow[d, "B"'] & n' \arrow[d, "B'"'] & n'' \arrow[d, "B''"'] \\
p & p' & p''
\end{tikzcd}$$

in $\mathsf{Mat}^{\mathsf{C}}$, the following equality of pasting diagrams in $\mathsf{Mat}^{\mathsf{C}}_{mm'm'',pp'p''}$ holds.

(8.8.16)
$$
\begin{tikzcd}[row sep=large]
\overline{m} \arrow[rr, bend left=60, "(BA\boxtimes B'A')\boxtimes B''A''"] \arrow[rr, phantom, bend left=30, "\Uparrow a^{\boxtimes}_{BA,B'A',B''A''}"] \arrow[d, "\mathbb{1}^{\overline{m}}"'] & & \overline{p} \arrow[d, "\mathbb{1}^{\overline{p}}"] \\
\overline{m} \arrow[rr, bend left=60, "BA\boxtimes(B'A'\boxtimes B''A'')"] \arrow[rr, phantom, bend left=30, "\Uparrow 1\boxtimes\boxtimes^2"] \arrow[rr, "BA\boxtimes[(B'\boxtimes B'')(A'\boxtimes A'')]"] \arrow[rr, phantom, bend right=30, "\Uparrow\boxtimes^2"] \arrow[dr, bend right, "A\boxtimes(A'\boxtimes A'')"'] & & \overline{p} \\
& \overline{n} \arrow[ur, bend right, "B\boxtimes(B'\boxtimes B'')"'] &
\end{tikzcd}
=
\begin{tikzcd}[row sep=large]
\overline{m} \arrow[rr, bend left=60, "(BA\boxtimes B'A')\boxtimes B''A''"] \arrow[rr, phantom, bend left=30, "\Uparrow\boxtimes^2\boxtimes 1"] \arrow[rr, "{[(B\boxtimes B')(A\boxtimes A')]\boxtimes B''A''}"] \arrow[rr, phantom, bend right=20, "\Uparrow\boxtimes^2"] \arrow[dr, bend right, "(A\boxtimes A')\boxtimes A''"'] \arrow[dd, "\mathbb{1}^{\overline{m}}"'] & & \overline{p} \arrow[dd, "\mathbb{1}^{\overline{p}}"] \\
& \overline{n} \arrow[ur, bend right, "(B\boxtimes B')\boxtimes B''"'] \arrow[dd, "\mathbb{1}^{\overline{n}}"] \arrow[dl, phantom, "\overset{a^{\boxtimes}_{A,A',A''}}{\Rightarrow}"] \arrow[dr, phantom, "\overset{a^{\boxtimes}_{B,B',B''}}{\Rightarrow}"] & \\
\overline{m} \arrow[dr, bend right, "A\boxtimes(A'\boxtimes A'')"'] & & \overline{p} \\
& \overline{n} \arrow[ur, bend right, "B\boxtimes(B'\boxtimes B'')"'] &
\end{tikzcd}
$$

Consider (8.8.16).

(1) $\overline{m} = mm'm''$, $\overline{n} = nn'n''$, and $\overline{p} = pp'p''$.

(2) (6.2.8) is used to obtain the following decompositions of lax functoriality constraints for the composites, with $\boxtimes^2$ as in Definition 8.6.19.

$$\big(\boxtimes(\boxtimes\times 1)\big)^2 = \big(\boxtimes^2\boxtimes 1\big)\boxtimes^2$$
$$\big(\boxtimes(1\times\boxtimes)\big)^2 = \big(1\boxtimes\boxtimes^2\big)\boxtimes^2$$

(3) In the left pasting diagram, by (8.8.3),

$$a^{\boxtimes}_{BA,B'A',B''A''} = \ell^{-1}\alpha^{-\otimes}r$$

with
- ℓ the base left unitor in (8.2.2),
- $\alpha^{-\otimes}$ entrywise a component of $\alpha^{-\otimes}$ in C, and
- r the base right unitor in (8.2.8).

(4) In the right pasting diagram, there are two instances of the base associator a in (8.3.2) and one instance of a^{-1} that are not explicitly displayed. They are necessary because horizontal composition of 1-cells in $\mathsf{Mat}^{\mathsf{C}}$ is not strictly associative, but only up to the natural isomorphism a.

To clarify the argument of the various cases, we split the proof of the lax naturality axiom for $a^{\boxtimes}$ into two lemmas.

Lemma 8.8.17. *In the setting of (8.8.15), suppose that either*

(1) at least one of m, m', m'', p, p', or p'' is 0, or

(2) $m, m', m'', p, p', p'' > 0$, and at least one of n, n', or n'' is 0.

Then the equality in (8.8.16) holds.

Proof. If at least one of m, m', m'', p, p', or p'' is 0, then $\mathsf{Mat}^{\mathsf{C}}_{\overline{m},\overline{p}}$ is the terminal category, and the equality in (8.8.16) holds.

For the second case, suppose that $m, m', m'', p, p', p'' > 0$, and that at least one of n, n', or n'' is 0. Since $\overline{n} = 0$, the two pasting diagrams in (8.8.16) become the following ones, where $\varnothing$ denotes the empty matrix.

(8.8.18)
$$
\begin{tikzcd}[column sep=large]
\overline{m} \arrow[rr, bend left=50, "(BA\boxtimes B'A')\boxtimes B''A''"] \arrow[rr, phantom, bend left=25, "\Uparrow a^{\boxtimes}_{BA,B'A',B''A''}"] \arrow[d, "\mathbb{1}^{\overline{m}}"'] & & \overline{p} \arrow[d, "\mathbb{1}^{\overline{p}}"] \\
\overline{m} \arrow[rr, bend left=50, "BA\boxtimes(B'A'\boxtimes B''A'')"] \arrow[rr, phantom, bend left=25, "\Uparrow 1\boxtimes\boxtimes^2"] \arrow[rr, "BA\boxtimes[(B'\boxtimes B'')(A'\boxtimes A'')]"] \arrow[rr, phantom, bend right=20, "\Uparrow\boxtimes^2"] \arrow[r, bend right, "\varnothing"'] & 0 \arrow[r, bend right, "\varnothing"'] & \overline{p}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=large]
\overline{m} \arrow[rr, bend left=50, "(BA\boxtimes B'A')\boxtimes B''A''"] \arrow[rr, phantom, bend left=25, "\Uparrow\boxtimes^2\boxtimes 1"] \arrow[rr, "{[(B\boxtimes B')(A\boxtimes A')]\boxtimes B''A''}"] \arrow[dr, bend right, "\varnothing"'] \arrow[dd, "\mathbb{1}^{\overline{m}}"'] & & \overline{p} \arrow[dd, "\mathbb{1}^{\overline{p}}"] \\
& 0 \arrow[ur, bend right, "\varnothing"'] \arrow[u, phantom, "\Uparrow\boxtimes^2"] \arrow[d, "\varnothing"] \arrow[dl, phantom, "\overset{a^{\boxtimes}=1}{\Rightarrow}"] \arrow[dr, phantom, "\overset{a^{\boxtimes}=1}{\Rightarrow}"] & \\
\overline{m} \arrow[r, bend right, "\varnothing"'] & 0 \arrow[r, bend right, "\varnothing"'] & \overline{p}
\end{tikzcd}
$$

Consider (8.8.18).

- By Convention 6.2.12 and (8.1.4), each domain 1-cell is the matrix product

$$(\varnothing\varnothing)\mathbb{1}^{\overline{m}} = \mathbb{0}_{\overline{m},\overline{p}}\mathbb{1}^{\overline{m}} \in \mathsf{Mat}^{\mathsf{C}}_{\overline{m},\overline{p}}.$$

- If $n = 0$, then BA and $(B \boxtimes B')(A \boxtimes A')$ are 0 matrices.
- If $n' = 0$, then $B'A'$, $(B' \boxtimes B'')(A' \boxtimes A'')$, and $(B \boxtimes B')(A \boxtimes A')$ are 0 matrices.
- If $n'' = 0$, then $B''A''$ and $(B' \boxtimes B'')(A' \boxtimes A'')$ are 0 matrices.
- Each of the two 2-cells

$$\varnothing\varnothing = \mathbb{0}_{\overline{m},\overline{p}} \xrightarrow{\boxtimes^2} BA \boxtimes \big[(B' \boxtimes B'')(A' \boxtimes A'')\big]$$
$$\varnothing\varnothing = \mathbb{0}_{\overline{m},\overline{p}} \xrightarrow{\boxtimes^2} \big[(B \boxtimes B')(A \boxtimes A')\big] \boxtimes B''A''$$

  is entrywise a component of $\lambda^{-\bullet}$ or $\rho^{-\bullet}$.
- Each of the two 2-cells

(8.8.19)
$$
\begin{gathered}
(B' \boxtimes B'')(A' \boxtimes A'') \xrightarrow{\boxtimes^2} B'A' \boxtimes B''A'' \\
(B \boxtimes B')(A \boxtimes A') \xrightarrow{\boxtimes^2} BA \boxtimes B'A'
\end{gathered}
$$

  is as in (8.6.20).

The left and the right pasting diagrams in (8.8.18) are, respectively, the left and the right composites below in $\mathsf{Mat}^{\mathsf{C}}_{\overline{m},\overline{p}}$, with ζ^{ℓ} and ζ^{r} the natural isomorphisms in

(8.1.11) and (8.1.19).

(8.8.20)
$$\begin{tikzcd}[column sep=small]
& \varnothing(\varnothing\mathbb{1}^{\overline{m}}) = \mathbb{0}_{\overline{m},\overline{p}} \arrow[dr, bend left, "1 \star a^{\boxtimes} = 1"] & \\
(\varnothing\varnothing)\mathbb{1}^{\overline{m}} = \mathbb{0}_{\overline{m},\overline{p}}\mathbb{1}^{\overline{m}} \arrow[ur, bend left, "a = \zeta^{\ell}_{\mathbb{1}^{\overline{m}}}"] \arrow[d, "\boxtimes^2 \star 1"'] & & \varnothing(\varnothing\varnothing) \arrow[d, "a^{-1} = 1"] \\
\big[BA \boxtimes [(B' \boxtimes B'')(A' \boxtimes A'')]\big]\mathbb{1}^{\overline{m}} \arrow[d, "{[1 \boxtimes \boxtimes^2] \star 1}"'] & & (\varnothing\varnothing)\varnothing \arrow[d, "a^{\boxtimes} \star 1 = 1"] \\
\big[BA \boxtimes (B'A' \boxtimes B''A'')\big]\mathbb{1}^{\overline{m}} \arrow[d, "r"'] & & (\mathbb{1}^{\overline{p}}\varnothing)\varnothing = \mathbb{0}_{\overline{m},\overline{p}} \arrow[d, "a = \zeta^{-r}_{\mathbb{1}^{\overline{p}}}"] \\
BA \boxtimes (B'A' \boxtimes B''A'') \arrow[d, "\alpha^{-\otimes}"'] & & \mathbb{1}^{\overline{p}}(\varnothing\varnothing) = \mathbb{1}^{\overline{p}}\mathbb{0}_{\overline{m},\overline{p}} \arrow[d, "1 \star \boxtimes^2"] \\
(BA \boxtimes B'A') \boxtimes B''A'' \arrow[dr, bend right, "\ell^{-1}"'] & & \mathbb{1}^{\overline{p}}\big[[(B \boxtimes B')(A \boxtimes A')] \boxtimes B''A''\big] \arrow[dl, bend left, "{1 \star [\boxtimes^2 \boxtimes 1]}"] \\
& \mathbb{1}^{\overline{p}}\big[(BA \boxtimes B'A') \boxtimes B''A''\big] &
\end{tikzcd}$$

Using the description above of the morphisms involved, similar to the proofs in Section 8.7, the commutativity of (8.8.20) in each entry is proved by realizing each edge or its inverse as the value in C of a path in $\mathsf{Gr}(X)$ for some set X of formal variables and some function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$. The detailed argument is given below.

(1) Suppose $nn' > 0$ and $n'' = 0$. First composing the three 1's away, the diagram (8.8.20) becomes the following diagram in $\mathsf{Mat}^{\mathsf{C}}_{\overline{m},\overline{p}}$.

(8.8.21)
$$\begin{tikzcd}
\mathbb{0}_{\overline{m},\overline{p}}\mathbb{1}^{\overline{m}} \arrow[r, "\zeta^{\ell}_{\mathbb{1}^{\overline{m}}}"] \arrow[d, "\boxtimes^2 \star 1"'] & \mathbb{0}_{\overline{m},\overline{p}} \arrow[d, "\zeta^{-r}_{\mathbb{1}^{\overline{p}}}"] \\
\big[BA \boxtimes \mathbb{0}_{m'm'',p'p''}\big]\mathbb{1}^{\overline{m}} \arrow[d, "{[1 \boxtimes \boxtimes^2] \star 1}"'] & \mathbb{1}^{\overline{p}}\mathbb{0}_{\overline{m},\overline{p}} \arrow[d, "1 \star \boxtimes^2"] \\
\big[BA \boxtimes (B'A' \boxtimes \mathbb{0}_{m'',p''})\big]\mathbb{1}^{\overline{m}} \arrow[d, "r"'] & \mathbb{1}^{\overline{p}}\big[[(B \boxtimes B')(A \boxtimes A')] \boxtimes \mathbb{0}_{m'',p''}\big] \arrow[dd, "{1 \star [\boxtimes^2 \boxtimes 1]}"] \\
BA \boxtimes (B'A' \boxtimes \mathbb{0}_{m'',p''}) \arrow[d, "\alpha^{-\otimes}"'] & \\
(BA \boxtimes B'A') \boxtimes \mathbb{0}_{m'',p''} \arrow[r, "\ell^{-1}"] & \mathbb{1}^{\overline{p}}\big[(BA \boxtimes B'A') \boxtimes \mathbb{0}_{m'',p''}\big]
\end{tikzcd}$$

For indices

(8.8.22)
$$\begin{array}{lll} 1 \leq i, e \leq m & 1 \leq i', e' \leq m' & 1 \leq i'', e'' \leq m'' \\ 1 \leq k, c \leq p & 1 \leq k', c' \leq p' & 1 \leq k'', c'' \leq p'' \end{array}$$

we use the following abbreviations.

$$\begin{gathered}
K = (k-1)p'p'' + (k'-1)p'' + k'' \\
I = (i-1)m'm'' + (i'-1)m'' + i'' \\
E = (e-1)m'm'' + (e'-1)m'' + e'' \\
C = (c-1)p'p'' + (c'-1)p'' + c'' \\
\bigoplus_{\overline{e}=1}^{\overline{m}} = \bigoplus_{e=1}^{m}\bigoplus_{e'=1}^{m'}\bigoplus_{e''=1}^{m''} \qquad \bigoplus_{\overline{c}=1}^{\overline{p}} = \bigoplus_{c=1}^{p}\bigoplus_{c'=1}^{p'}\bigoplus_{c''=1}^{p''} \\
\bigoplus_{j,j'=1,1}^{n,n'} = \bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}
\end{gathered} \tag{8.8.23}$$

Using the notation in (8.7.6) for morphisms, the (K,I)-entry of (8.8.21) is the following diagram in C, with $\otimes$ abbreviated to concatenation.

$$\begin{tikzcd}
\Big[\Big(\bigoplus_{e=1}^{I-1}\mathbb{0}\mathbb{0}\Big)\oplus(\mathbb{0}\mathbb{1})\oplus\Big(\bigoplus_{e=I+1}^{\overline{m}}\mathbb{0}\mathbb{0}\Big)\Big]_{\mathsf{lt}} \arrow[r, "{(\lambda^{\oplus},\lambda^{\bullet})}"] \arrow[dd, "{(\rho^{-\bullet})}"'] & \mathbb{0} \arrow[d, "{(\lambda^{-\oplus},\rho^{-\bullet})}"] \\
& \Big[\Big(\bigoplus_{c=1}^{K-1}\mathbb{0}\mathbb{0}\Big)\oplus(\mathbb{1}\mathbb{0})\oplus\Big(\bigoplus_{c=K+1}^{\overline{p}}\mathbb{0}\mathbb{0}\Big)\Big]_{\mathsf{lt}} \arrow[d, "{(\rho^{-\bullet})}"] \\
\Big[\bigoplus_{\overline{e}=1}^{\overline{m}}\Big[\Big(\bigoplus_{j=1}^{n}B_{kj}A_{je}\Big)_{\mathsf{lt}}\mathbb{0}\Big]\mathbb{1}^{\overline{m}}_{E,I}\Big]_{\mathsf{lt}} \arrow[d, "{(\rho^{-\bullet})}"'] & \Big[\bigoplus_{\overline{c}=1}^{\overline{p}}\mathbb{1}^{\overline{p}}_{K,C}\Big[\Big\{\bigoplus_{j,j'=1,1}^{n,n'}(B_{cj}B'_{c'j'})(A_{ji}A'_{j'i'})\Big\}_{\mathsf{lt}}\mathbb{0}\Big]\Big]_{\mathsf{lt}} \arrow[d, "{(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})}"] \\
\Big[\bigoplus_{\overline{e}=1}^{\overline{m}}\Big[\Big(\bigoplus_{j=1}^{n}B_{kj}A_{je}\Big)_{\mathsf{lt}}\Big\{\Big(\bigoplus_{j'=1}^{n'}B'_{k'j'}A'_{j'e'}\Big)_{\mathsf{lt}}\mathbb{0}\Big\}\Big]\mathbb{1}^{\overline{m}}_{E,I}\Big]_{\mathsf{lt}} \arrow[d, "{(\lambda^{\oplus},\rho^{\oplus},\rho^{\otimes},\rho^{\bullet})}"'] & \Big[\bigoplus_{\overline{c}=1}^{\overline{p}}\mathbb{1}^{\overline{p}}_{K,C}\Big[\Big\{\bigoplus_{j=1}^{n}\Big(\bigoplus_{j'=1}^{n'}(B_{cj}A_{ji})(B'_{c'j'}A'_{j'i'})\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}}\mathbb{0}\Big]\Big]_{\mathsf{lt}} \arrow[d, "{(\delta^{-r},\delta^{-l})}"] \\
\Big(\bigoplus_{j=1}^{n}B_{kj}A_{ji}\Big)_{\mathsf{lt}}\Big[\Big(\bigoplus_{j'=1}^{n'}B'_{k'j'}A'_{j'i'}\Big)_{\mathsf{lt}}\mathbb{0}\Big] \arrow[d, "\alpha^{-\otimes}"'] & \Big[\bigoplus_{\overline{c}=1}^{\overline{p}}\mathbb{1}^{\overline{p}}_{K,C}\Big[\Big\{\Big(\bigoplus_{j=1}^{n}B_{cj}A_{ji}\Big)_{\mathsf{lt}}\Big(\bigoplus_{j'=1}^{n'}B'_{c'j'}A'_{j'i'}\Big)_{\mathsf{lt}}\Big\}\mathbb{0}\Big]\Big]_{\mathsf{lt}} \\
\Big[\Big(\bigoplus_{j=1}^{n}B_{kj}A_{ji}\Big)_{\mathsf{lt}}\Big(\bigoplus_{j'=1}^{n'}B'_{k'j'}A'_{j'i'}\Big)_{\mathsf{lt}}\Big]\mathbb{0} \arrow[ur, "{(\lambda^{-\oplus},\rho^{-\oplus},\lambda^{-\otimes},\lambda^{-\bullet})}"'] &
\end{tikzcd} \tag{8.8.24}$$

In the top left object, the symbols

$$\Big(\bigoplus_{e=1}^{I-1}\mathbb{0}\mathbb{0}\Big)\oplus \quad \text{and} \quad \oplus\Big(\bigoplus_{e=I+1}^{\overline{m}}\mathbb{0}\mathbb{0}\Big)$$

are empty if, respectively, $I = 1$ and $I = \overline{m}$. Similar interpretations apply to the sums in the second-to-top object in the right column.

To show that (8.8.24) is commutative, we use the set of formal variables

$$X = \Big\{0^X, 1^X, a_{je}, b_{cj}, a'_{j'e'}, b'_{c'j'}\Big\}_{1\le e\le m,\, 1\le j\le n,\, 1\le c\le p,\, 1\le e'\le m',\, 1\le j'\le n',\, 1\le c'\le p'}$$

and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{je} & \text{if } x = a_{je}. \\ B_{cj} & \text{if } x = b_{cj}. \\ A'_{j'e'} & \text{if } x = a'_{j'e'}. \\ B'_{c'j'} & \text{if } x = b'_{c'j'}. \end{cases}$$

There is a diagram in $\mathsf{Gr}(X)$ as follows. Each path has value in C the corresponding edge in (8.8.24) or its inverse, and $\delta^X_{\bullet\bullet}$ is as in (8.4.6).

$$\begin{tikzcd}
\Big[\Big(\bigoplus\limits_{e=1}^{I-1} 0^X0^X\Big) \oplus (0^X1^X) \oplus \Big(\bigoplus\limits_{e=I+1}^{\overline{m}} 0^X0^X\Big)\Big]_{\mathsf{lt}} \arrow[r, two heads, "{(\lambda^{\oplus},\lambda^{\bullet})}"] & 0^X \arrow[d, two heads, "{(\lambda^{-\oplus},\rho^{-\bullet})}"] \\
\Big[\bigoplus\limits_{\overline{e}=1}^{\overline{m}} \Big[\Big(\bigoplus\limits_{j=1}^{n} b_{kj}a_{je}\Big)_{\mathsf{lt}} 0^X\Big]\delta^X_{E,I}\Big]_{\mathsf{lt}} \arrow[u, two heads, "{(\rho^{\bullet})}"] & \Big[\Big(\bigoplus\limits_{c=1}^{K-1} 0^X0^X\Big) \oplus (1^X0^X) \oplus \Big(\bigoplus\limits_{c=K+1}^{\overline{p}} 0^X0^X\Big)\Big]_{\mathsf{lt}} \arrow[d, two heads, "{(\rho^{-\bullet})}"] \\
\Big[\bigoplus\limits_{\overline{e}=1}^{\overline{m}} \Big[\Big(\bigoplus\limits_{j=1}^{n} b_{kj}a_{je}\Big)_{\mathsf{lt}} \Big\{\Big(\bigoplus\limits_{j'=1}^{n'} b'_{k'j'}a'_{j'e'}\Big)_{\mathsf{lt}} 0^X\Big\}\Big]\delta^X_{E,I}\Big]_{\mathsf{lt}} \arrow[u, two heads, "{(\rho^{\bullet})}"] & \Big[\bigoplus\limits_{\overline{c}=1}^{\overline{p}} \delta^X_{K,C}\Big[\Big\{\bigoplus\limits_{j,j'=1,1}^{n,n'} (b_{cj}b'_{c'j'})(a_{ji}a'_{j'i'})\Big\}_{\mathsf{lt}} 0^X\Big]\Big]_{\mathsf{lt}} \arrow[d, two heads, "{(\alpha^{\oplus},\alpha^{\pm\otimes},\xi^{\otimes})}"] \\
\Big(\bigoplus\limits_{j=1}^{n} b_{kj}a_{ji}\Big)_{\mathsf{lt}} \Big[\Big(\bigoplus\limits_{j'=1}^{n'} b'_{k'j'}a'_{j'i'}\Big)_{\mathsf{lt}} 0^X\Big] \arrow[u, two heads, "{(\lambda^{-\oplus},\rho^{-\oplus},\rho^{-\otimes},\rho^{-\bullet})}"] & \Big[\bigoplus\limits_{\overline{c}=1}^{\overline{p}} \delta^X_{K,C}\Big[\Big\{\bigoplus\limits_{j=1}^{n}\Big(\bigoplus\limits_{j'=1}^{n'} (b_{cj}a_{ji})(b'_{c'j'}a'_{j'i'})\Big)_{\mathsf{lt}}\Big\}_{\mathsf{lt}} 0^X\Big]\Big]_{\mathsf{lt}} \\
\Big[\Big(\bigoplus\limits_{j=1}^{n} b_{kj}a_{ji}\Big)_{\mathsf{lt}} \Big(\bigoplus\limits_{j'=1}^{n'} b'_{k'j'}a'_{j'i'}\Big)_{\mathsf{lt}}\Big] 0^X \arrow[u, "\alpha^{\otimes}"] & \Big[\bigoplus\limits_{\overline{c}=1}^{\overline{p}} \delta^X_{K,C}\Big[\Big\{\Big(\bigoplus\limits_{j=1}^{n} b_{cj}a_{ji}\Big)_{\mathsf{lt}} \Big(\bigoplus\limits_{j'=1}^{n'} b'_{c'j'}a'_{j'i'}\Big)_{\mathsf{lt}}\Big\} 0^X\Big]\Big]_{\mathsf{lt}} \arrow[u, two heads, "{(\delta^r,\delta^l)}"] \arrow[l, two heads, "{(\lambda^{\oplus},\rho^{\oplus},\lambda^{\otimes},\lambda^{\bullet})}"]
\end{tikzcd} \tag{8.8.25}$$

In the lower right path (δ^r, δ^l), each prime edge is an identity or contains δ^r or δ^l. Similar notation is used in the other paths.

Since the upper right vertex in (8.8.25) is 0^X, the lower right vertex has the same support as 0^X by Lemma 3.1.29. Proposition 3.5.33 implies that the long concatenated path and (δ^r, δ^l) in (8.8.25) have the same value in C. This implies that (8.8.24) is commutative.

(2) Suppose $n = 0$ and $n'n'' > 0$. This case is similar to case (1). Each entry of the diagram (8.8.20) is the value in C of some diagram in $\mathsf{Gr}(X)$, similar to (8.8.25), for some set X of formal variables and some function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$. There is one path of the form (δ^r, δ^l) that arises from the 2-cell

$$(B' \boxtimes B'')(A' \boxtimes A'') \xrightarrow{\boxtimes^2} B'A' \boxtimes B''A''$$

in (8.8.20) as part of the last case of (8.6.20).

(3) The other cases have either (i) $n' = 0$, or (ii) $n' > 0$ and $n = n'' = 0$. Each of these cases is similar to case (1). In each case, each entry of (8.8.20) is the value in C of some diagram in $\mathsf{Gr}(X)$, similar to (8.8.25), for some set X of formal variables and some function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$. However, the elementary edges δ^r and δ^l do *not* appear in this diagram because each entry of each of the two 2-cells in (8.8.19) is a component of $\lambda^{-\bullet}$ or $\rho^{-\bullet}$.

This finishes the proof of the lemma. $\square$

Lemma 8.8.26. *In the setting of (8.8.15), suppose that $m, m', m'', n, n', n'', p, p', p'' > 0$. Then the equality in (8.8.16) holds.*

Proof. The two pasting diagrams in (8.8.16) are the two composites below.

(8.8.27)
$$\begin{tikzcd}[column sep=tiny]
& \big(B \boxtimes (B' \boxtimes B'')\big)\big[\big(A \boxtimes (A' \boxtimes A'')\big)\mathbb{1}^{\overline{m}}\big] \arrow[dr, "1 \star a^{\boxtimes}"] & \\
\big[\big(B \boxtimes (B' \boxtimes B'')\big)\big(A \boxtimes (A' \boxtimes A'')\big)\big]\mathbb{1}^{\overline{m}} \arrow[ur, "a"] \arrow[d, "\boxtimes^2 \star 1"'] & & \big(B \boxtimes (B' \boxtimes B'')\big)\big[\mathbb{1}^{\overline{n}}\big((A \boxtimes A') \boxtimes A''\big)\big] \arrow[d, "a^{-1}"] \\
\big[BA \boxtimes (B' \boxtimes B'')(A' \boxtimes A'')\big]\mathbb{1}^{\overline{m}} \arrow[d, "{[1 \boxtimes \boxtimes^2] \star 1}"'] & & \big[\big(B \boxtimes (B' \boxtimes B'')\big)\mathbb{1}^{\overline{n}}\big]\big((A \boxtimes A') \boxtimes A''\big) \arrow[d, "a^{\boxtimes} \star 1"] \\
\big[BA \boxtimes (B'A' \boxtimes B''A'')\big]\mathbb{1}^{\overline{m}} \arrow[d, "a^{\boxtimes}"'] & & \big[\mathbb{1}^{\overline{p}}\big((B \boxtimes B') \boxtimes B''\big)\big]\big((A \boxtimes A') \boxtimes A''\big) \arrow[d, "a"] \\
\mathbb{1}^{\overline{p}}\big[(BA \boxtimes B'A') \boxtimes B''A''\big] & & \mathbb{1}^{\overline{p}}\big[\big((B \boxtimes B') \boxtimes B''\big)\big((A \boxtimes A') \boxtimes A''\big)\big] \arrow[dl, "1 \star \boxtimes^2"] \\
& \mathbb{1}^{\overline{p}}\big[(B \boxtimes B')(A \boxtimes A') \boxtimes B''A''\big] \arrow[ul, "{1 \star [\boxtimes^2 \boxtimes 1]}"] &
\end{tikzcd}$$

Consider the diagram (8.8.27).

- Each instance of a is entrywise $a^4a^3a^2a^1$ as in (8.3.5). It involves δ^r (8.3.6) in a^1 and δ^{-l} (8.3.8) in a^4, but neither δ^{-r} nor δ^l.
- Each instance of $a^{\boxtimes} = \ell^{-1}\alpha^{-\otimes}r$ is as in (8.8.3). It has neither $\delta^{\pm r}$ nor $\delta^{\pm l}$.
- Each instance of $\boxtimes^2$ is entrywise as in the last case in (8.6.20). As explained in Lemma 8.6.16, it involves δ^{-r} and δ^{-l}, but neither δ^r nor δ^l. In particular, each entry of $(\boxtimes^2)^{-1}$ involves δ^r and δ^l, but neither δ^{-r} nor δ^{-l}.

As a brief outline, similar to Section 8.7, we prove the commutativity of each entry of the diagram (8.8.27) by

- expressing its morphisms using paths in $\mathsf{Gr}(X)$ for some set X of formal variables and some function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ and
- applying the Coherence Theorem 3.9.1.

Due to the presence of $\delta^{\pm r}$ and $\delta^{\pm l}$ in some of the morphisms as discussed in the previous paragraph, we need to consider the *inverses* of a^4 and $\boxtimes^2$. This direction reversal creates a zigzag of morphisms. To show that the resulting diagram is commutative, we need to subdivide it in such a way that the additional morphisms are also paths in $\mathsf{Gr}(X)$ with the same set X and function φ. The detailed argument is given below.

Using the notation in (8.8.23), consider the (K, I)-entry of the diagram (8.8.27). Denote by

- R_1 the codomain of r in $a^{\boxtimes} = \ell^{-1}\alpha^{-\otimes}r$ along the left side;
- R_2 the codomain of $1 \star r$ in $1 \star a^{\boxtimes}$ in the upper right;
- R_3 the codomain of $r \star 1$ in $a^{\boxtimes} \star 1$ in the middle right;
- D_1 the domain of a^4 in $a = a^4a^3a^2a^1$ in the upper left;
- D_2 the codomain of $(a^4)^{-1}$ in a^{-1} in the upper right; and
- D_3 the domain of a^4 in a in the lower right.

Using these notations and (8.8.23), the (K, I)-entry of the diagram (8.8.27) is equivalent to the outer diagram in C below.

(8.8.28)
$$\begin{tikzcd}
& R_2 \arrow[dl, "g_2"'] \arrow[dr, "f_2"] \arrow[ddrr, "h_2"] & & \\
D_1 & & D_2 & \\
R_1 \arrow[u, "f_1"] \arrow[uur, "h_1"'] \arrow[dr, "g_1"'] & & R_3 \arrow[u, "g_3"'] \arrow[dl, "f_3"] & \\
& D_3 & &
\end{tikzcd}$$

The objects and morphisms in the diagram (8.8.28) are described explicitly below.

- In addition to (8.8.23), we use the following indices and abbreviations.

(8.8.29)
$$\begin{gathered}
1 \le j, l \le n \qquad 1 \le j', l' \le n' \qquad 1 \le j'', l'' \le n'' \\
J = (j-1)n'n'' + (j'-1)n'' + j'' \\
L = (l-1)n'n'' + (l'-1)n'' + l'' \\
\bigoplus_{\bar{j}=1}^{\bar{n}} = \bigoplus_{j=1}^{n} \bigoplus_{j'=1}^{n'} \bigoplus_{j''=1}^{n''} \qquad \bigoplus_{\bar{l}=1}^{\bar{n}} = \bigoplus_{l=1}^{n} \bigoplus_{l'=1}^{n'} \bigoplus_{l''=1}^{n''}
\end{gathered}$$

- Abbreviating $\otimes$ to concatenation, the objects R_1, R_2, and R_3 are as follows.

$$\begin{aligned}
R_1 &= \big[BA \boxtimes (B'A' \boxtimes B''A'')\big]_{K,I} \\
&= (BA)_{ki}\big[(B'A')_{k'i'}(B''A'')_{k''i''}\big] \\
&= \Bigg[\bigoplus_{j=1}^{n} B_{kj}A_{ji}\Bigg]_{\mathsf{lt}}\Bigg[\Bigg\{\bigoplus_{j'=1}^{n'} B'_{k'j'}A'_{j'i'}\Bigg\}_{\mathsf{lt}}\Bigg\{\bigoplus_{j''=1}^{n''} B''_{k''j''}A''_{j''i''}\Bigg\}_{\mathsf{lt}}\Bigg] \\
R_2 &= \Big[\big(B \boxtimes (B' \boxtimes B'')\big)\big(A \boxtimes (A' \boxtimes A'')\big)\Big]_{K,I} \\
&= \Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}} \big(B \boxtimes (B' \boxtimes B'')\big)_{K,J}\big(A \boxtimes (A' \boxtimes A'')\big)_{J,I}\Bigg]_{\mathsf{lt}} \\
&= \Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}} \Big(B_{kj}(B'_{k'j'}B''_{k''j''})\Big)\Big(A_{ji}(A'_{j'i'}A''_{j''i''})\Big)\Bigg]_{\mathsf{lt}} \\
R_3 &= \Big[\big(B \boxtimes (B' \boxtimes B'')\big)\big((A \boxtimes A') \boxtimes A''\big)\Big]_{K,I} \\
&= \Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}} \big(B \boxtimes (B' \boxtimes B'')\big)_{K,J}\big((A \boxtimes A') \boxtimes A''\big)_{J,I}\Bigg]_{\mathsf{lt}} \\
&= \Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}} \Big(B_{kj}(B'_{k'j'}B''_{k''j''})\Big)\Big((A_{ji}A'_{j'i'})A''_{j''i''}\Big)\Bigg]_{\mathsf{lt}}
\end{aligned}$$

- The objects D_1, D_2, and D_3 are as follows.

$$\begin{aligned}
D_1 &= \Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}}\Bigg[\bigoplus_{\bar{e}=1}^{\overline{m}} \Big(B \boxtimes (B' \boxtimes B'')\Big)_{K,J}\Big\{\Big(A \boxtimes (A' \boxtimes A'')\Big)_{J,E}\mathbb{1}^{\overline{m}}_{E,I}\Big\}\Bigg]_{\mathsf{lt}}\Bigg]_{\mathsf{lt}} \\
&= \Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}}\Bigg[\bigoplus_{\bar{e}=1}^{\overline{m}} \Big(B_{kj}(B'_{k'j'}B''_{k''j''})\Big)\Big\{\Big(A_{je}(A'_{j'e'}A''_{j''e''})\Big)\mathbb{1}^{\overline{m}}_{E,I}\Big\}\Bigg]_{\mathsf{lt}}\Bigg]_{\mathsf{lt}} \\
D_2 &= \Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}}\Bigg[\bigoplus_{\bar{l}=1}^{\bar{n}} \Big(B \boxtimes (B' \boxtimes B'')\Big)_{K,J}\Big\{\mathbb{1}^{\bar{n}}_{J,L}\Big((A \boxtimes A') \boxtimes A''\Big)_{L,I}\Big\}\Bigg]_{\mathsf{lt}}\Bigg]_{\mathsf{lt}} \\
&= \Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}}\Bigg[\bigoplus_{\bar{l}=1}^{\bar{n}} \Big(B_{kj}(B'_{k'j'}B''_{k''j''})\Big)\Big\{\mathbb{1}^{\bar{n}}_{J,L}\Big((A_{li}A'_{l'i'})A''_{l''i''}\Big)\Big\}\Bigg]_{\mathsf{lt}}\Bigg]_{\mathsf{lt}} \\
D_3 &= \Bigg[\bigoplus_{\bar{c}=1}^{\bar{p}}\Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}} \mathbb{1}^{\bar{p}}_{K,C}\Big\{\Big((B \boxtimes B') \boxtimes B''\Big)_{C,J}\Big((A \boxtimes A') \boxtimes A''\Big)_{J,I}\Big\}\Bigg]_{\mathsf{lt}}\Bigg]_{\mathsf{lt}} \\
&= \Bigg[\bigoplus_{\bar{c}=1}^{\bar{p}}\Bigg[\bigoplus_{\bar{j}=1}^{\bar{n}} \mathbb{1}^{\bar{p}}_{K,C}\Big\{\Big((B_{cj}B'_{c'j'})B''_{c''j''}\Big)\Big((A_{ji}A'_{j'i'})A''_{j''i''}\Big)\Big\}\Bigg]_{\mathsf{lt}}\Bigg]_{\mathsf{lt}}
\end{aligned}$$

- With $\boxtimes^{-2} = (\boxtimes^2)^{-1}$ and $a^{-4} = (a^4)^{-1}$, the morphisms in the boundary are as follows.

$$\begin{aligned}
f_1 &= \big(a^3a^2a^1\big)\Big[(\boxtimes^{-2} \star 1)\big([1 \boxtimes \boxtimes^{-2}] \star 1\big)r^{-1}\Big]_{K,I}\\
f_2 &= a^{-4}\Big[(1 \star \ell^{-1})(1 \star \alpha^{-\otimes})\Big]_{K,I}\\
f_3 &= \big(a^3a^2a^1\big)\Big[(\ell^{-1} \star 1)(\alpha^{-\otimes} \star 1)\Big]_{K,I}\\
g_1 &= a^{-4}\Big[(1 \star \boxtimes^{-2})\big(1 \star [\boxtimes^{-2} \boxtimes 1]\big)\ell^{-1}\alpha^{-\otimes}\Big]_{K,I}\\
g_2 &= a^{-4}\big[1 \star r^{-1}\big]_{K,I}\\
g_3 &= \big(a^3a^2a^1\big)\big[r^{-1} \star 1\big]_{K,I}
\end{aligned} \tag{8.8.30}$$

The composite

$$g_1^{-1}f_3g_3^{-1}f_2g_2^{-1}f_1 : R_1 \longrightarrow R_1$$

in (8.8.28) is equal to going one round clockwise in the (K, I)-entry in (8.8.27), starting and ending at R_1.

- Using the notation in (8.7.6), the morphism

$$\begin{array}{c}
R_1 = \Big[\bigoplus_{j=1}^{n} B_{kj}A_{ji}\Big]_{\mathsf{lt}}\Big[\Big\{\bigoplus_{j'=1}^{n'} B'_{k'j'}A'_{j'i'}\Big\}_{\mathsf{lt}}\Big\{\bigoplus_{j''=1}^{n''} B''_{k''j''}A''_{j''i''}\Big\}_{\mathsf{lt}}\Big]\\
\Big\downarrow h_1\\
R_2 = \Big[\bigoplus_{\bar{j}=1}^{\bar{n}} \big(B_{kj}(B'_{k'j'}B''_{k''j''})\big)\big(A_{ji}(A'_{j'i'}A''_{j''i''})\big)\Big]_{\mathsf{lt}}
\end{array}$$

has the form $(\alpha^{-\oplus}, \alpha^{\pm\otimes}, \xi^{\otimes}, \delta^r, \delta^l)$. In h_1, the following statements hold.
  - The morphisms involving δ^r and δ^l bring the three sums to the front.
  - The morphisms involving $\alpha^{\pm\otimes}$ and $\xi^{\otimes}$ permute

$$(B_{kj}A_{ji})(B'_{k'j'}A'_{j'i'})(B''_{k''j''}A''_{j''i''}) \quad \text{to} \quad \big(B_{kj}(B'_{k'j'}B''_{k''j''})\big)\big(A_{ji}(A'_{j'i'}A''_{j''i''})\big)$$

    in each summand.
  - The morphisms involving $\alpha^{-\oplus}$ move the additive brackets to match with those in R_2.
- The morphism

$$\begin{array}{c}
R_2 = \Big[\bigoplus_{\bar{j}=1}^{\bar{n}} \big(B_{kj}(B'_{k'j'}B''_{k''j''})\big)\big(A_{ji}(A'_{j'i'}A''_{j''i''})\big)\Big]_{\mathsf{lt}}\\
\Big\downarrow h_2\\
R_3 = \Big[\bigoplus_{\bar{j}=1}^{\bar{n}} \big(B_{kj}(B'_{k'j'}B''_{k''j''})\big)\big((A_{ji}A'_{j'i'})A''_{j''i''}\big)\Big]_{\mathsf{lt}}
\end{array}$$

has the form $(\alpha^{-\otimes})$ with one copy of

$$A_{ji}(A'_{j'i'}A''_{j''i''}) \xrightarrow{\alpha^{-\otimes}} (A_{ji}A'_{j'i'})A''_{j''i''}$$

in each summand.

To show that the diagram (8.8.28) is commutative, we use the set

$$X = \left\{0^X, 1^X, a_{je}, a'_{j'e'}, a''_{j''e''}, b_{cj}, b'_{c'j'}, b''_{c''j''}\right\}$$

of formal variables, with the indices as in (8.8.22) and (8.8.29), and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{je} & \text{if } x = a_{je}. \\ A'_{j'e'} & \text{if } x = a'_{j'e'}. \\ A''_{j''e''} & \text{if } x = a''_{j''e''}. \\ B_{cj} & \text{if } x = b_{cj}. \\ B'_{c'j'} & \text{if } x = b'_{c'j'}. \\ B''_{c''j''} & \text{if } x = b''_{c''j''}. \end{cases}$$

By the description above of the objects and morphisms in (8.8.28), there exists a diagram in $\mathsf{Gr}(X)$ below in which the value in C of each path is the corresponding edge in (8.8.28).

(8.8.31)
$$\begin{tikzcd} & & v_2 \arrow[dll, "G_2"'] \arrow[drr, "F_2"] \arrow[ddrr, "H_2"] & & \\ u_1 & & & & u_2 \\ v_1 \arrow[u, "F_1"] \arrow[uurr, "H_1"] \arrow[drr, "G_1"'] & & & & v_3 \arrow[u, "G_3"'] \arrow[dll, "F_3"] \\ & & u_3 & & \end{tikzcd}$$

In the diagram (8.8.31), for $1 \le t \le 3$, the following statements hold.

- The images of v_t and u_t under φ are, respectively, R_t and D_t in (8.8.28). For example, the vertex
$$v_2 = \left[\bigoplus_{\bar{j}=1}^{\bar{n}} \Big(b_{kj}(b'_{k'j'}b''_{k''j''})\Big)\Big(a_{ji}(a'_{j'i'}a''_{j''i''})\Big)\right]_{\mathsf{lt}}.$$
- The images of F_t and G_t under φ are, respectively, f_t and g_t.
- The images of H_1 and H_2 under φ are, respectively, h_1 and h_2.

Moreover, by (8.2.4), (8.2.10), (8.3.5), Lemma 8.6.16, and (8.8.30), the following statements hold.

- Each prime edge in F_1 is an identity or contains $\alpha^{\pm\oplus}$, $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\xi^{\oplus}$, $\alpha^{\pm\otimes}$, $\rho^{-\otimes}$, $\xi^{-\otimes}$, $\rho^{-\bullet}$, δ^l, or δ^r.
- Each prime edge in F_2 is an identity or contains $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\alpha^{-\otimes}$, $\lambda^{-\otimes}$, $\lambda^{-\bullet}$, or δ^l.
- Each prime edge in F_3 is an identity or contains $\alpha^{\pm\oplus}$, $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\xi^{\oplus}$, $\alpha^{\pm\otimes}$, $\lambda^{-\otimes}$, $\lambda^{-\bullet}$, or δ^r.
- Each prime edge in G_1 is an identity or contains $\alpha^{-\oplus}$, $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\alpha^{\pm\otimes}$, $\lambda^{-\otimes}$, $\xi^{-\otimes}$, $\lambda^{-\bullet}$, δ^l, or δ^r.
- Each prime edge in G_2 is an identity or contains $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\rho^{-\otimes}$, $\rho^{-\bullet}$, or δ^l.

- Each prime edge in G_3 is an identity or contains $\alpha^{\pm\oplus}$, $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, $\rho^{-\otimes}$, $\rho^{-\bullet}$, or δ^r.
- Each prime edge in H_1 is an identity or contains $\alpha^{-\oplus}$, $\alpha^{\pm\otimes}$, $\xi^{\otimes}$, δ^l, or δ^r.
- Each prime edge in H_2 is an identity or contains $\alpha^{-\otimes}$.

In (8.8.31), the vertex v_2 is regular as in Definition 3.1.25, so the vertex v_1 is also regular by Lemma 3.1.29. Since C is assumed to be tight, the Coherence Theorem 3.9.1 implies that in each of the following three cases, the two paths have the same value in C.

- F_1 and $(G_2, H_1) : v_1 \twoheadrightarrow u_1$.
- G_1 and $(F_3, H_2, H_1) : v_1 \twoheadrightarrow u_3$.
- F_2 and $(G_3, H_2) : v_2 \twoheadrightarrow u_2$.

This implies that the diagram (8.8.28) is commutative. This, in turn, implies that the diagram (8.8.27) is commutative. □

Lemma 8.8.32. *$a^{\boxtimes}$ in Definition 8.8.1 is a strong transformation.*

Proof. We have proved all the necessary assertions.

- Lemma 8.8.5 shows that $a^{\boxtimes}$ in (8.8.3) is a natural isomorphism.
- Lemma 8.8.11 shows that $a^{\boxtimes}$ satisfies the lax unity axiom (6.2.15).
- Lemmas 8.8.17 and 8.8.26 show that $a^{\boxtimes}$ satisfies the lax naturality axiom (6.2.16).

Therefore, $a^{\boxtimes}$ is a strong transformation. □

The Right Adjoint of the Monoidal Associator.

Definition 8.8.33. With respect to the pseudofunctor $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in Definition 8.6.19 and Lemma 8.7.31, define the data of a lax transformation

$$\boxtimes(1 \times \boxtimes) \xrightarrow{a^{\boxtimes\bullet}} \boxtimes(\boxtimes \times 1)$$

as follows.

Component 1-Cells: For each triple of objects $(m,n,p) \in \left(\mathsf{Mat}^{\mathsf{C}}\right)^3$, define

$$\big(m \boxtimes (n \boxtimes p)\big) = mnp \xrightarrow{a^{\boxtimes\bullet}_{m,n,p}} mnp = \big((m \boxtimes n) \boxtimes p\big) \tag{8.8.34}$$

as the identity matrix $\mathbb{1}^{mnp} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ in (8.1.6).

Component 2-Cells: For each triple of 1-cells

$$\big(A = (A_{i'i}), B = (B_{j'j}), C = (C_{k'k})\big) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'} \times \mathsf{Mat}^{\mathsf{C}}_{p,p'},$$

define the component 2-cell

$$a^{\boxtimes\bullet}_{A,B,C} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,m'n'p'}\Big([(A \boxtimes B) \boxtimes C]a^{\boxtimes\bullet}_{m,n,p}; a^{\boxtimes\bullet}_{m',n',p'}[A \boxtimes (B \boxtimes C)]\Big)$$

as the following vertical composite.

$$\begin{tikzcd}[column sep=large] & mnp \arrow[dr, bend left, "(A\boxtimes B)\boxtimes C"] \arrow[d, phantom, "\Downarrow r_{(A\boxtimes B)\boxtimes C}"] & \\ mnp \arrow[ur, bend left, "\mathbb{1}^{mnp}"] \arrow[rr, bend left=10, "(A\boxtimes B)\boxtimes C"] \arrow[rr, bend right=10, "A\boxtimes(B\boxtimes C)"'] \arrow[dr, bend right, "A\boxtimes(B\boxtimes C)"'] & \Downarrow\alpha^{\otimes} \arrow[d, phantom, "\Downarrow\ell^{-1}_{A\boxtimes(B\boxtimes C)}"] & m'n'p' \\ & m'n'p' \arrow[ur, bend right, "\mathbb{1}^{m'n'p'}"'] & \end{tikzcd} \tag{8.8.35}$$

- ℓ is the base left unitor in $\mathsf{Mat}^{\mathsf{C}}$ in (8.2.2).
- r is the base right unitor in $\mathsf{Mat}^{\mathsf{C}}$ in (8.2.8).
- $\alpha^{\otimes}$ is the 2-cell with $((i'-1)n'p'+(j'-1)p'+k',(i-1)np+(j-1)p+k)$-entry the structure morphism in C,
$$\left(A_{i'i}\otimes B_{j'j}\right)\otimes C_{k'k} \xrightarrow{\alpha^{\otimes}_{A_{i'i},B_{j'j},C_{k'k}}} A_{i'i}\otimes\left(B_{j'j}\otimes C_{k'k}\right)$$
for $1\le i\le m$, $1\le j\le n$, $1\le k\le p$, $1\le i'\le m'$, $1\le j'\le n'$, and $1\le k'\le p'$.

This finishes the definition of $a^{\boxtimes\bullet}$. ◇

Lemma 8.8.36. *$a^{\boxtimes\bullet}$ in Definition 8.8.33 is a strong transformation.*

Proof. The proofs of Lemmas 8.8.5, 8.8.11, 8.8.17, and 8.8.26 for $a^{\boxtimes}$ apply to $a^{\boxtimes\bullet}$ with only cosmetic changes. □

The Unit and the Counit of the Monoidal Associator. Recall

- the identity strong transformation in Lemma 6.2.17,
- the horizontal composition of lax transformations in Definition 6.2.20,
- modification in Definition 6.3.1, and
- the base left unitor ℓ in (8.2.2).

Next we define the unit and the counit for $(a^{\boxtimes},a^{\boxtimes\bullet})$.

Definition 8.8.37. For the strong transformations

$$\boxtimes(\boxtimes\times 1) \underset{a^{\boxtimes\bullet}}{\overset{a^{\boxtimes}}{\rightleftarrows}} \boxtimes(1\times\boxtimes)$$

in Definitions 8.8.1 and 8.8.33, define the data

$$1_{\boxtimes(\boxtimes\times 1)} \xrightarrow{\eta^a} a^{\boxtimes\bullet}a^{\boxtimes} \qquad a^{\boxtimes}a^{\boxtimes\bullet} \xrightarrow{\varepsilon^a} 1_{\boxtimes(1\times\boxtimes)}$$

as consisting of the component 2-cells

$$\begin{array}{ccc} \left(1_{\boxtimes(\boxtimes\times 1)}\right)_{(m,n,p)} = \mathbb{1}^{mnp} & \qquad & (a^{\boxtimes}a^{\boxtimes\bullet})_{(m,n,p)} = \mathbb{1}^{mnp}\mathbb{1}^{mnp} \\ \eta^a_{(m,n,p)} = \Big\downarrow \ell^{-1}_{\mathbb{1}^{mnp}} & & \varepsilon^a_{(m,n,p)} = \Big\downarrow \ell_{\mathbb{1}^{mnp}} \\ (a^{\boxtimes\bullet}a^{\boxtimes})_{(m,n,p)} = \mathbb{1}^{mnp}\mathbb{1}^{mnp} & & \left(1_{\boxtimes(1\times\boxtimes)}\right)_{(m,n,p)} = \mathbb{1}^{mnp} \end{array} \tag{8.8.38}$$

in $\mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ for each triple of objects $(m,n,p)\in\left(\mathsf{Mat}^{\mathsf{C}}\right)^3$. ◇

Lemma 8.8.39. *η^a and ε^a in Definition 8.8.37 are invertible modifications.*

Proof. Each component 2-cell of each of η^a and ε^a is invertible by Lemma 8.2.1. We will show that η^a is a modification; the proof for ε^a is obtained by a slight adjustment.

By (6.2.18) applied to $1_{\boxtimes(\boxtimes\times 1)}$ and (6.2.21) applied to $a^{\boxtimes\bullet}a^{\boxtimes}$, the modification axiom (6.3.2) for $\eta^a : 1_{\boxtimes(\boxtimes\times 1)} \longrightarrow a^{\boxtimes\bullet}a^{\boxtimes}$ states that, for each triple of 1-cells

$$(A, B, C) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'} \times \mathsf{Mat}^{\mathsf{C}}_{p,p'},$$

the following equality of pasting diagrams in $\mathsf{Mat}^{\mathsf{C}}_{q,q'}$ holds, with $q = mnp$ and $q' = m'n'p'$.

(8.8.40)
$$\begin{tikzcd}
q \arrow[rr, "(A\boxtimes B)\boxtimes C"] \arrow[dd, bend right, "\mathbb{1}^q"'] \arrow[ddrr, "(A\boxtimes B)\boxtimes C"] \arrow[dd, phantom, shift left=5ex, "\overset{r}{\Rightarrow}"] & & q' \arrow[dr, "\mathbb{1}^{q'}"] \arrow[dd, bend right, "\mathbb{1}^{q'}"'] \arrow[dl, phantom, "\overset{\ell^{-1}}{\Rightarrow}"] & \\
& & \overset{\eta^a}{\Rightarrow} & q' \arrow[dl, "\mathbb{1}^{q'}"] \\
q \arrow[rr, "(A\boxtimes B)\boxtimes C"'] & & q' &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
q \arrow[rr, "(A\boxtimes B)\boxtimes C"] \arrow[dr, "\mathbb{1}^q"] \arrow[dd, bend right, "\mathbb{1}^q"'] \arrow[dr, phantom, shift right=4ex, "\overset{\eta^a}{\Rightarrow}"] & & q' \arrow[dr, "\mathbb{1}^{q'}"] & \\
& q \arrow[rr, "A\boxtimes(B\boxtimes C)"] \arrow[dl, "\mathbb{1}^q"] \arrow[ur, phantom, "\overset{a^{\boxtimes}_{A,B,C}}{\Rightarrow}"] \arrow[dr, phantom, "\overset{a^{\boxtimes\bullet}_{A,B,C}}{\Rightarrow}"] & & q' \arrow[dl, "\mathbb{1}^{q'}"] \\
q \arrow[rr, "(A\boxtimes B)\boxtimes C"'] & & q' &
\end{tikzcd}$$

On the right-hand side of (8.8.40), by Convention 6.2.12, there are

- one instance of the base associator a in (8.3.2) and
- two instances of a^{-1}

that are not explicitly displayed. If $qq' = 0$, then $\mathsf{Mat}^{\mathsf{C}}_{q,q'}$ is the terminal category, and the equality in (8.8.40) holds.

Suppose $qq' > 0$. The proof for this case proceeds along the lines of the proof of Lemma 8.8.26. Due to the presence of both the base associator a and its inverse, a typical entry of (8.8.40) involves $\delta^{\pm r}$ and $\delta^{\pm l}$. Therefore, when the entries of the 2-cells in (8.8.40) are realized as paths in $\mathsf{Gr}(X)$, the diagram will need to be further subdivided before the Coherence Theorem 3.9.1 can be applied. The detailed argument is given below.

The two pasting diagrams in (8.8.40) are the two composites in $\mathsf{Mat}^{\mathsf{C}}_{q,q'}$ below.

(8.8.41)
$$\begin{tikzcd}
& \big((A\boxtimes B)\boxtimes C\big)\big(\mathbb{1}^q\mathbb{1}^q\big) \arrow[dr, "a^{-1}"] & \\
\big((A\boxtimes B)\boxtimes C\big)\mathbb{1}^q \arrow[ur, "1\star\eta^a \,=\, 1\star\ell^{-1}"] \arrow[d, "r"'] & & \Big[\big((A\boxtimes B)\boxtimes C\big)\mathbb{1}^q\Big]\mathbb{1}^q \arrow[d, "a^{\boxtimes\bullet}\star 1"] \\
(A\boxtimes B)\boxtimes C \arrow[d, "\ell^{-1}"'] & & \Big[\mathbb{1}^{q'}\big(A\boxtimes(B\boxtimes C)\big)\Big]\mathbb{1}^q \arrow[d, "a"] \\
\mathbb{1}^{q'}\big((A\boxtimes B)\boxtimes C\big) \arrow[d, "\eta^a\star 1 \,=\, \ell^{-1}\star 1"'] & & \mathbb{1}^{q'}\Big[\big(A\boxtimes(B\boxtimes C)\big)\mathbb{1}^q\Big] \arrow[d, "1\star a^{\boxtimes}"] \\
\big(\mathbb{1}^{q'}\mathbb{1}^{q'}\big)\big((A\boxtimes B)\boxtimes C\big) & & \mathbb{1}^{q'}\Big[\mathbb{1}^{q'}\big((A\boxtimes B)\boxtimes C\big)\Big] \arrow[ll, "a^{-1}"']
\end{tikzcd}$$

To show that each entry of (8.8.41) is a commutative diagram in C, consider the following indices and abbreviations.

$$
\begin{array}{ll}
1 \le i, s_1, t_1 \le m & 1 \le i', u_1, v_1 \le m' \\
1 \le j, s_2, t_2 \le n & 1 \le j', u_2, v_2 \le n' \\
1 \le k, s_3, t_3 \le p & 1 \le k', u_3, v_3 \le p' \\
Q = (i-1)np + (j-1)p + k & Q' = (i'-1)n'p' + (j'-1)p' + k' \\
S = (s_1-1)np + (s_2-1)p + s_3 & U = (u_1-1)n'p' + (u_2-1)p' + u_3 \\
T = (t_1-1)np + (t_2-1)p + t_3 & V = (v_1-1)n'p' + (v_2-1)p' + v_3 \\
\bigoplus_{s=1}^{q} = \bigoplus_{s_1=1}^{m} \bigoplus_{s_2=1}^{n} \bigoplus_{s_3=1}^{p} & \bigoplus_{u=1}^{q'} = \bigoplus_{u_1=1}^{m'} \bigoplus_{u_2=1}^{n'} \bigoplus_{u_3=1}^{p'} \\
\bigoplus_{t=1}^{q} = \bigoplus_{t_1=1}^{m} \bigoplus_{t_2=1}^{n} \bigoplus_{t_3=1}^{p} & \bigoplus_{v=1}^{q'} = \bigoplus_{v_1=1}^{m'} \bigoplus_{v_2=1}^{n'} \bigoplus_{v_3=1}^{p'}
\end{array}
\tag{8.8.42}
$$

We also use

- the entrywise decomposition $a = a^4 a^3 a^2 a^1$ in (8.3.5),
- the definition of $a^{\boxtimes} = \ell^{-1}\alpha^{-\otimes} r$ in (8.8.3),
- the definition of $a^{\boxtimes^\bullet} = \ell^{-1}\alpha^{\otimes} r$ in (8.8.35), and
- concatenation for $\otimes$ and the shorthand $a^{-4} = (a^4)^{-1}$.

The (Q', Q)-entry of (8.8.41) is equivalent to the following diagram in C.

$$
\begin{tikzcd}
Y_1 = \Big[\bigoplus_{t=1}^{q} \big((A_{i't_1}B_{j't_2})C_{k't_3}\big)\Big[\bigoplus_{s=1}^{q} \mathbb{1}^q_{T,S}\mathbb{1}^q_{S,Q}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[r, bend left, "a^{-4}"] & \Big[\bigoplus_{t=1}^{q}\Big[\bigoplus_{s=1}^{q} \big((A_{i't_1}B_{j't_2})C_{k't_3}\big)\big(\mathbb{1}^q_{T,S}\mathbb{1}^q_{S,Q}\big)\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} = Z_1 \\
Y_2 = \Big[\bigoplus_{t=1}^{q} \big((A_{i't_1}B_{j't_2})C_{k't_3}\big)\mathbb{1}^q_{T,Q}\Big]_{\mathsf{lt}} \arrow[u, "1 \star \ell^{-1}"] \arrow[d, "r"'] & \Big[\bigoplus_{s=1}^{q}\Big[\bigoplus_{t=1}^{q} \big((A_{i't_1}B_{j't_2})C_{k't_3}\big)\mathbb{1}^q_{T,S}\Big]_{\mathsf{lt}} \mathbb{1}^q_{S,Q}\Big]_{\mathsf{lt}} = Z_2 \arrow[u, "a^3a^2a^1"'] \arrow[d, "(\ell^{-1}\alpha^{\otimes} r) \star 1"] \\
Y_3 = (A_{i'i}B_{j'j})C_{k'k} \arrow[d, "\ell^{-1}"'] & \Big[\bigoplus_{s=1}^{q}\Big[\bigoplus_{v=1}^{q'} \mathbb{1}^{q'}_{Q',V}\big(A_{v_1s_1}(B_{v_2s_2}C_{v_3s_3})\big)\Big]_{\mathsf{lt}} \mathbb{1}^q_{S,Q}\Big]_{\mathsf{lt}} = Z_3 \arrow[d, "a^3a^2a^1"] \\
Y_4 = \Big[\bigoplus_{u=1}^{q'} \mathbb{1}^{q'}_{Q',U}\big((A_{u_1i}B_{u_2j})C_{u_3k}\big)\Big]_{\mathsf{lt}} \arrow[d, "\ell^{-1} \star 1"'] & \Big[\bigoplus_{v=1}^{q'}\Big[\bigoplus_{s=1}^{q} \mathbb{1}^{q'}_{Q',V}\big\{\big(A_{v_1s_1}(B_{v_2s_2}C_{v_3s_3})\big)\mathbb{1}^q_{S,Q}\big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} = Z_4 \\
Y_5 = \Big[\bigoplus_{u=1}^{q'}\Big[\bigoplus_{v=1}^{q'} \mathbb{1}^{q'}_{Q',V}\mathbb{1}^{q'}_{V,U}\Big]_{\mathsf{lt}}\big((A_{u_1i}B_{u_2j})C_{u_3k}\big)\Big]_{\mathsf{lt}} \arrow[d, "a^3a^2a^1"'] & \Big[\bigoplus_{v=1}^{q'} \mathbb{1}^{q'}_{Q',V}\Big[\bigoplus_{s=1}^{q} \big(A_{v_1s_1}(B_{v_2s_2}C_{v_3s_3})\big)\mathbb{1}^q_{S,Q}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} = Z_5 \arrow[u, "a^{-4}"'] \arrow[d, "1 \star (\ell^{-1}\alpha^{-\otimes} r)"] \\
Y_6 = \Big[\bigoplus_{v=1}^{q'}\Big[\bigoplus_{u=1}^{q'} \mathbb{1}^{q'}_{Q',V}\big\{\mathbb{1}^{q'}_{V,U}\big((A_{u_1i}B_{u_2j})C_{u_3k}\big)\big\}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} & \Big[\bigoplus_{v=1}^{q'} \mathbb{1}^{q'}_{Q',V}\Big[\bigoplus_{u=1}^{q'} \mathbb{1}^{q'}_{V,U}\big((A_{u_1i}B_{u_2j})C_{u_3k}\big)\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} = Z_6 \arrow[l, bend left, "a^{-4}"]
\end{tikzcd}
\tag{8.8.43}
$$

To show that the diagram (8.8.43) is commutative when suitably subdivided, we use the set of formal variables

$$X = \left\{0^X, 1^X, a_{v_1 s_1}, b_{v_2 s_2}, c_{v_3 s_3}\right\},$$

with the indices as in (8.8.42), and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{v_1 s_1} & \text{if } x = a_{v_1 s_1}. \\ B_{v_2 s_2} & \text{if } x = b_{v_2 s_2}. \\ C_{v_3 s_3} & \text{if } x = c_{v_3 s_3}. \end{cases}$$

For $1 \leq l \leq 6$,

- $y_l \in X^{\mathsf{fr}}$ is obtained from Y_l in (8.8.43) by replacing the symbols
$$\left(A, B, C, \mathbb{1}^q, \mathbb{1}^{q'}\right) \quad \text{with} \quad \left(a, b, c, \delta^X, \delta^X\right),$$
where δ^X is as in (8.4.6), and
- $z_l \in X^{\mathsf{fr}}$ is obtained in a similar way from Z_l in (8.8.43).

For example,

$$y_1 = \left[\bigoplus_{t=1}^{q} \left((a_{i't_1} b_{j't_2}) c_{k't_3}\right) \left[\bigoplus_{s=1}^{q} \delta^X_{T,S} \delta^X_{S,Q}\right]_{\mathsf{lt}}\right]_{\mathsf{lt}}.$$

Using (8.2.4), (8.2.10), (8.3.5), and the notation in (8.7.6), there is a diagram in $\mathsf{Gr}(X)$ below in which the value in C of each boundary path is the corresponding edge in (8.8.43).

(8.8.44)
$$\begin{tikzcd}
y_1 \arrow[rr, twoheadrightarrow, "(\delta^l)"] & & z_1 \arrow[ddll, twoheadrightarrow, "(\lambda^{\oplus}, \rho^{\oplus}, \rho^{\otimes}, \lambda^{\bullet}, \rho^{\bullet})" description] \\
y_2 \arrow[u, twoheadrightarrow, "(\lambda^{-\oplus}, \rho^{-\oplus}, \lambda^{-\otimes}, \lambda^{-\bullet})"] \arrow[d, twoheadrightarrow, "(\lambda^{\oplus}, \rho^{\oplus}, \rho^{\otimes}, \rho^{\bullet})"'] & & z_2 \arrow[u, twoheadrightarrow, "(\alpha^{\pm\oplus}, \xi^{\oplus}, \alpha^{\otimes}, \delta^r)"'] \arrow[d, twoheadrightarrow, "(\lambda^{\pm\oplus}, \rho^{\pm\oplus}, \alpha^{\otimes}, \lambda^{-\otimes}, \rho^{\otimes}, \lambda^{-\bullet}, \rho^{\bullet})"] \\
y_3 \arrow[d, twoheadrightarrow, "(\lambda^{-\oplus}, \rho^{-\oplus}, \lambda^{-\otimes}, \lambda^{-\bullet})"'] & & z_3 \arrow[d, twoheadrightarrow, "(\alpha^{\pm\oplus}, \xi^{\oplus}, \alpha^{\otimes}, \delta^r)"] \\
y_4 \arrow[d, twoheadrightarrow, "(\lambda^{-\oplus}, \rho^{-\oplus}, \lambda^{-\otimes}, \lambda^{-\bullet})"'] & a_{i'i}\left(b_{j'j} c_{k'k}\right) \arrow[ul, "\alpha^{-\otimes}"] & z_4 \arrow[l, twoheadrightarrow, "(\lambda^{\oplus}, \rho^{\oplus}, \lambda^{\otimes}, \rho^{\otimes}, \lambda^{\bullet}, \rho^{\bullet})"] \\
y_5 \arrow[d, twoheadrightarrow, "(\alpha^{\pm\oplus}, \xi^{\oplus}, \alpha^{\otimes}, \delta^r)"'] & & z_5 \arrow[u, twoheadrightarrow, "(\delta^l)"'] \arrow[d, twoheadrightarrow, "(\lambda^{\pm\oplus}, \rho^{\pm\oplus}, \alpha^{-\otimes}, \lambda^{-\otimes}, \rho^{\otimes}, \lambda^{-\bullet}, \rho^{\bullet})"] \\
y_6 & & z_6 \arrow[ll, twoheadrightarrow, "(\delta^l)"']
\end{tikzcd}$$

Consider the diagram (8.8.44).

- As an example of the notation, in the lower left path $y_5 \longrightarrow\!\!\!\!\!\rightarrow y_6$, each prime edge is an identity or contains $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, or δ^r.
- In the path $z_1 \longrightarrow\!\!\!\!\!\rightarrow y_3$, the following statements hold.
  (i) $\rho^{\otimes}$, $\lambda^{\bullet}$, and $\rho^{\bullet}$ are used to reduce each summand
$$\left((a_{i't_1} b_{j't_2}) c_{k't_3}\right)\left(\delta^X_{T,S} \delta^X_{S,Q}\right)$$

in z_1 to

$$\begin{cases} 0^X & \text{if } T \neq S \text{ or } S \neq Q, \text{ and} \\ y_3 = \left(a_{i'i} b_{j'j}\right) c_{k'k} & \text{if } T = S = Q. \end{cases}$$

(ii) $\lambda^{\oplus}$ and $\rho^{\oplus}$ are then used to remove all the 0^X's.

- The path

$$z_4 \twoheadrightarrow a_{i'i} \left(b_{j'j} c_{k'k}\right)$$

is defined similarly.

Since the vertex y_3 is regular as in Definition 3.1.25, each vertex in (8.8.44) is regular by Lemma 3.1.29. Since C is assumed to be tight, the Coherence Theorem 3.9.1 implies that the image of (8.8.44) in C is a commutative diagram. This implies that (8.8.43) is commutative. This in turn implies that (8.8.41) is commutative. $\square$

The Triangle Identities of the Monoidal Associator.

Lemma 8.8.45. *The quadruple $(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$ satisfies the triangle identities.*

Proof. We will show the left triangle identity in (6.3.10). The proof for the right triangle identity is obtained by a slight adjustment.

The left triangle identity states the commutativity of the left diagram below.

(8.8.46)
$$\begin{tikzcd}[column sep=small]
a^{\boxtimes} 1_{\boxtimes(\boxtimes\times 1)} \arrow[rr, "r_{a^{\boxtimes}}"] \arrow[d, "1_{a^{\boxtimes}} * \eta^a"'] & & a^{\boxtimes} \\
a^{\boxtimes}(a^{\boxtimes\bullet} a^{\boxtimes}) \arrow[dr, "a^{-1}_{a^{\boxtimes}, a^{\boxtimes\bullet}, a^{\boxtimes}}"'] & & 1_{\boxtimes(1\times\boxtimes)} a^{\boxtimes} \arrow[u, "\ell_{a^{\boxtimes}}"'] \\
& (a^{\boxtimes} a^{\boxtimes\bullet}) a^{\boxtimes} \arrow[ur, "\varepsilon^a * 1_{a^{\boxtimes}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\mathbb{1}^q \mathbb{1}^q \arrow[rr, "r_{\mathbb{1}^q}"] \arrow[d, "1 \star \ell^{-1}_{\mathbb{1}^q}"'] & & \mathbb{1}^q \\
\mathbb{1}^q(\mathbb{1}^q \mathbb{1}^q) \arrow[dr, "a^{-1}_{\mathbb{1}^q, \mathbb{1}^q, \mathbb{1}^q}"'] & & \mathbb{1}^q \mathbb{1}^q \arrow[u, "\ell_{\mathbb{1}^q}"'] \\
& (\mathbb{1}^q \mathbb{1}^q) \mathbb{1}^q \arrow[ur, "\ell_{\mathbb{1}^q} \star 1"'] &
\end{tikzcd}$$

Consider the left diagram in (8.8.46).

- $1_{a^{\boxtimes}} * \eta^a$ and $\varepsilon^a * 1_{a^{\boxtimes}}$ are horizontal composites of modifications in (6.3.5).
- The component 2-cells of $\ell_{a^{\boxtimes}}$, $r_{a^{\boxtimes}}$, and $a^{-1}_{a^{\boxtimes}, a^{\boxtimes\bullet}, a^{\boxtimes}}$ are computed in the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ using, respectively, the base left unitor in (8.2.2), the base right unitor in (8.2.8), and the inverse of the base associator in (8.3.2).
- The composite of any two consecutive edges is the vertical composite of modifications in (6.3.4).

To show that the left diagram in (8.8.46) is commutative, it suffices to show that the two composites have the same component 2-cells.

In other words, it suffices to show that, for each triple of objects $(m, n, p) \in \left(\mathsf{Mat}^{\mathsf{C}}\right)^3$, the right diagram in (8.8.46) is commutative in $\mathsf{Mat}^{\mathsf{C}}_{q,q}$ with $q = mnp$. This is equivalent to showing that, for $1 \leq s, t \leq q$, the (s,t)-entry of the right diagram in (8.8.46) is commutative in C. Using the entrywise factorization $a = a^4 a^3 a^2 a^1$ in

(8.3.5) and with $\otimes$ abbreviated to concatenation, the (s,t)-entry of the right diagram in (8.8.46) is equivalent to the following diagram in C.

$$\begin{tikzcd}
\Big[\bigoplus_{i=1}^{q} \mathbb{1}^q_{si}\mathbb{1}^q_{it}\Big]_{\mathsf{lt}} \arrow[d, "(1\star\ell^{-1}_{\mathbb{1}^q})_{st}"'] & & \mathbb{1}^q_{st} \arrow[ll, "(r^{-1}_{\mathbb{1}^q})_{st}"'] \arrow[d, "(\ell^{-1}_{\mathbb{1}^q})_{st}"] \\
\Big[\bigoplus_{i=1}^{q} \mathbb{1}^q_{si}\Big[\bigoplus_{j=1}^{q} \mathbb{1}^q_{ij}\mathbb{1}^q_{jt}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, "a^{-4}"'] & & \Big[\bigoplus_{j=1}^{q} \mathbb{1}^q_{sj}\mathbb{1}^q_{jt}\Big]_{\mathsf{lt}} \arrow[d, "(\ell^{-1}_{\mathbb{1}^q}\star 1)_{st}"] \\
\Big[\bigoplus_{i=1}^{q}\Big[\bigoplus_{j=1}^{q} \mathbb{1}^q_{si}\big(\mathbb{1}^q_{ij}\mathbb{1}^q_{jt}\big)\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} & & \Big[\bigoplus_{j=1}^{q}\Big[\bigoplus_{i=1}^{q} \mathbb{1}^q_{si}\mathbb{1}^q_{ij}\Big]_{\mathsf{lt}}\mathbb{1}^q_{jt}\Big]_{\mathsf{lt}} \arrow[ll, "a^3a^2a^1"']
\end{tikzcd} \tag{8.8.47}$$

To show that (8.8.47) is commutative, we use

- the set $X = \{0^X, 1^X\}$ and
- the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ defined by $\varphi(0^X) = \mathbb{0}$ and $\varphi(1^X) = \mathbb{1}$.

By (8.2.4), (8.2.10), (8.3.5), and the notation in (8.4.6) and (8.7.6), there exists a diagram in $\mathsf{Gr}(X)$ below in which the value in C of each path is the corresponding edge in (8.8.47).

$$\begin{tikzcd}
\Big[\bigoplus_{i=1}^{q} \delta^X_{si}\delta^X_{it}\Big]_{\mathsf{lt}} \arrow[d, two heads, "(\lambda^{-\oplus},\rho^{-\oplus},\lambda^{-\otimes},\lambda^{-\bullet})"'] & & \delta^X_{st} \arrow[ll, two heads, "(\lambda^{-\oplus},\rho^{-\oplus},\rho^{-\otimes},\rho^{-\bullet})"'] \arrow[d, two heads, "(\lambda^{-\oplus},\rho^{-\oplus},\lambda^{-\otimes},\lambda^{-\bullet})"] \\
\Big[\bigoplus_{i=1}^{q} \delta^X_{si}\Big[\bigoplus_{j=1}^{q} \delta^X_{ij}\delta^X_{jt}\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \arrow[d, two heads, "(\delta^l)"'] & & \Big[\bigoplus_{j=1}^{q} \delta^X_{sj}\delta^X_{jt}\Big]_{\mathsf{lt}} \arrow[d, two heads, "(\lambda^{-\oplus},\rho^{-\oplus},\lambda^{-\otimes},\lambda^{-\bullet})"] \\
\Big[\bigoplus_{i=1}^{q}\Big[\bigoplus_{j=1}^{q} \delta^X_{si}\big(\delta^X_{ij}\delta^X_{jt}\big)\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} & & \Big[\bigoplus_{j=1}^{q}\Big[\bigoplus_{i=1}^{q} \delta^X_{si}\delta^X_{ij}\Big]_{\mathsf{lt}}\delta^X_{jt}\Big]_{\mathsf{lt}} \arrow[ll, two heads, "(\alpha^{\pm\oplus},\xi^{\oplus},\alpha^{\otimes},\delta^r)"']
\end{tikzcd} \tag{8.8.48}$$

For example, in the bottom horizontal path, each prime edge is an identity or contains $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, or δ^r.

- If $s \neq t$, then $\delta^X_{st} = 0^X$. Proposition 3.5.33 implies that the image of (8.8.48) is a commutative diagram in C.
- If $s = t$, then $\delta^X_{ss} = 1^X$, which is regular in the sense of Definition 3.1.25. Since C is assumed to be tight, the Coherence Theorem 3.9.1 implies that the image of (8.8.48) is a commutative diagram in C.

Therefore, the right, hence also the left, diagram in (8.8.46) is commutative. $\square$

Lemma 8.8.49. *The quadruple $(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$ with*

- *$a^{\boxtimes}$ in Definition 8.8.1,*
- *$a^{\boxtimes\bullet}$ in Definition 8.8.33, and*
- *η^a and ε^a in Definition 8.8.37*

is an adjoint equivalence.

Proof. We already proved the following statements.

- $a^{\boxtimes}$ is a strong transformation by Lemma 8.8.32.

- $a^{\boxtimes\bullet}$ is a strong transformation by Lemma 8.8.36.
- η^a and ε^a are invertible modifications by Lemma 8.8.39.
- The triangle identities are satisfied by Lemma 8.8.45.

This is sufficient by Explanation 6.4.6. □

8.9. The Monoidal Unitors

We continue to assume that C is an arbitrary tight symmetric bimonoidal category as in Definition 2.1.2. We are in the process of constructing a monoidal bicategory structure, in the sense of Definition 6.4.1, on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$. In this section, we define the monoidal unitors $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$ and $(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^{r}, \varepsilon^{r})$.

Recall

- the identity strict functor $1_{\mathsf{B}} : \mathsf{B} \longrightarrow \mathsf{B}$ for each bicategory B in Example 6.2.10;
- the strictly unitary pseudofunctor
$$(1_{\boxtimes}, 1^2_{\boxtimes}, 1^0_{\boxtimes}) : \mathbf{1} \longrightarrow \mathsf{Mat}^{\mathsf{C}}$$
in Definition 8.5.1 and Lemma 8.5.2; and
- the pseudofunctor
$$(\boxtimes, \boxtimes^2, \boxtimes^0) : \mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}} \longrightarrow \mathsf{Mat}^{\mathsf{C}}$$
in Definition 8.6.19 and Lemma 8.7.31.

With the composite of pseudofunctors in Definition 6.2.6, we consider the following two composites.

$$\begin{tikzcd} \mathsf{Mat}^{\mathsf{C}} \arrow[r, bend left, "1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}}"] \arrow[r, bend right, "1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes}"'] & (\mathsf{Mat}^{\mathsf{C}})^2 \arrow[r, "\boxtimes"] & \mathsf{Mat}^{\mathsf{C}} \end{tikzcd}$$

As discussed in Explanation 6.4.6, the left monoidal unitor $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$ consists of the following data.

(i) $\ell^{\boxtimes}$ and $\ell^{\boxtimes\bullet}$ are strong transformations as in Definition 6.2.14 as follows.
$$\begin{tikzcd} \boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}}) \arrow[r, shift left, "\ell^{\boxtimes}"] & 1_{\mathsf{Mat}^{\mathsf{C}}} \arrow[l, shift left, "\ell^{\boxtimes\bullet}"] \end{tikzcd}$$
These strong transformations are decorated with $\boxtimes$ to avoid confusion with the base left unitor (8.2.2) in $\mathsf{Mat}^{\mathsf{C}}$.

(ii) η^{ℓ} and ε^{ℓ} are invertible modifications as in Definition 6.3.1 as follows.
$$1_{\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})} \xrightarrow{\eta^{\ell}} \ell^{\boxtimes\bullet}\ell^{\boxtimes}$$
$$\ell^{\boxtimes}\ell^{\boxtimes\bullet} \xrightarrow{\varepsilon^{\ell}} 1_{1_{\mathsf{Mat}^{\mathsf{C}}}}$$

Moreover, these data are required to satisfy the triangle identities (6.3.10). The left monoidal unitor is defined in Definitions 8.9.1, 8.9.5, and 8.9.8.

The right monoidal unitor $(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^{r}, \varepsilon^{r})$ consists of the following data.

(i) $r^{\boxtimes}$ and $r^{\boxtimes\bullet}$ are strong transformations as follows.

$$\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes}) \underset{r^{\boxtimes\bullet}}{\overset{r^{\boxtimes}}{\rightleftarrows}} 1_{\mathsf{Mat}^{\mathsf{C}}}$$

These strong transformations are decorated with $\boxtimes$ to avoid confusion with the base right unitor (8.2.8) in $\mathsf{Mat}^{\mathsf{C}}$.

(ii) η^r and ε^r are invertible modifications as follows.

$$1_{\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})} \xrightarrow{\eta^r} r^{\boxtimes\bullet} r^{\boxtimes}$$
$$r^{\boxtimes} r^{\boxtimes\bullet} \xrightarrow{\varepsilon^r} 1_{1_{\mathsf{Mat}^{\mathsf{C}}}}$$

Moreover, these data are required to satisfy the triangle identities (6.3.10). The right monoidal unitor is defined in Definitions 8.9.14, 8.9.17, and 8.9.20.

The Left Monoidal Unitor. First we define the left adjoint of the left monoidal unitor.

Definition 8.9.1. Define the data of a lax transformation

$$\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}}) \xrightarrow{\ell^{\boxtimes}} 1_{\mathsf{Mat}^{\mathsf{C}}}$$

as follows.

Component 1-Cells: For each object $m \in \mathsf{Mat}^{\mathsf{C}}$, define

$$m = 1 \boxtimes m \xrightarrow{\ell^{\boxtimes}_m} \big(1_{\mathsf{Mat}^{\mathsf{C}}}\big)\, m = m \tag{8.9.2}$$

as the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ in (8.1.6).

Component 2-Cells: For each 1-cell $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, define

$$\ell^{\boxtimes}_A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}\Big(\big(1_{\mathsf{Mat}^{\mathsf{C}}} A\big)\, \ell^{\boxtimes}_m ; \ell^{\boxtimes}_n \big(\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})\big)(A)\Big)$$

as the following vertical composite 2-cell.

$$\begin{tikzcd}
 & m \arrow[dr, bend left, "A"] \arrow[d, phantom, "\Downarrow r_A"] & \\
m \arrow[ur, bend left, "\mathbb{1}^m"] \arrow[rr, bend left=10, "A"] \arrow[rr, bend right=10, "\mathbb{1}\boxtimes A"'] \arrow[rr, phantom, "\Downarrow \lambda^{-\otimes}"] \arrow[dr, bend right, "\mathbb{1}\boxtimes A"'] & \arrow[d, phantom, "\Downarrow \ell^{-1}_{\mathbb{1}\boxtimes A}"] & n \\
 & n \arrow[ur, bend right, "\mathbb{1}^n"'] &
\end{tikzcd} \tag{8.9.3}$$

- $\mathbb{1} \boxtimes A$ is the scalar product in (8.6.2). This makes sense because the identity matrix $\mathbb{1}^1 \in \mathsf{Mat}^{\mathsf{C}}_{1,1}$ has the unique entry $\mathbb{1} \in \mathsf{C}$, and
$$\big(\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})\big)(A) = \mathbb{1}^1 \boxtimes A = \mathbb{1} \boxtimes A = \big(\mathbb{1} \otimes A_{ji}\big)$$
by the definition (8.6.3) of the matrix tensor product.
- ℓ is the base left unitor in $\mathsf{Mat}^{\mathsf{C}}$ in (8.2.2).
- r is the base right unitor in $\mathsf{Mat}^{\mathsf{C}}$ in (8.2.8).

- $\lambda^{-\otimes}$ is the 2-cell with (j,i)-entry the structure morphism in C,

$$A_{ji} \xrightarrow{\lambda^{-\otimes}_{A_{ji}}} \mathbb{1} \otimes A_{ji}$$

for $1 \le i \le m$ and $1 \le j \le n$.

This finishes the definition of $\ell^{\boxtimes}$. ⋄

Explanation 8.9.4. The matrix $\mathbb{1} \boxtimes A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ is obtained from A by replacing the (j,i)-entry A_{ji} with the product $\mathbb{1} \otimes A_{ji}$ for each $1 \le i \le m$ and $1 \le j \le n$. ⋄

Next we define the right adjoint of the left monoidal unitor.

Definition 8.9.5. Define the data of a lax transformation

$$1_{\mathsf{Mat}^{\mathsf{C}}} \xrightarrow{\ell^{\boxtimes\bullet}} \boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})$$

as follows.

Component 1-Cells: For each object $m \in \mathsf{Mat}^{\mathsf{C}}$, define

$$m = \big(1_{\mathsf{Mat}^{\mathsf{C}}}\big)\, m \xrightarrow{\ell^{\boxtimes\bullet}_m} 1 \boxtimes m = m \tag{8.9.6}$$

as the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ in (8.1.6).

Component 2-Cells: For each 1-cell $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, define

$$\ell^{\boxtimes\bullet}_A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}\Big(\big(\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})\big)(A)\,\ell^{\boxtimes\bullet}_m ; \ell^{\boxtimes\bullet}_n\,\big(1_{\mathsf{Mat}^{\mathsf{C}}} A\big)\Big)$$

as the following vertical composite 2-cell.

$$\begin{tikzcd}[column sep=large]
 & m \arrow[dr, bend left, "\mathbb{1} \boxtimes A"] \arrow[d, phantom, "\Downarrow r_{\mathbb{1}\boxtimes A}"] & \\
m \arrow[ur, bend left, "\mathbb{1}^m"] \arrow[rr, bend left=10, "\mathbb{1} \boxtimes A"] \arrow[rr, bend right=10, "A"'] \arrow[dr, bend right, "A"'] & \Downarrow\lambda^{\otimes} \arrow[d, phantom, "\Downarrow \ell_A^{-1}"] & n \\
 & n \arrow[ur, bend right, "\mathbb{1}^n"'] &
\end{tikzcd} \tag{8.9.7}$$

In this vertical composite, $\lambda^{\otimes}$ is the 2-cell with (j,i)-entry the structure morphism in C,

$$\mathbb{1} \otimes A_{ji} \xrightarrow{\lambda^{\otimes}_{A_{ji}}} A_{ji}$$

for $1 \le i \le m$ and $1 \le j \le n$.

This finishes the definition of $\ell^{\boxtimes\bullet}$. ⋄

Next we define the unit and the counit for $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet})$.

Definition 8.9.8. Define the data

$$1_{\boxtimes(1_{\boxtimes}\times 1_{\mathsf{Mat}^{\mathsf{C}}})} \xrightarrow{\eta^{\ell}} \ell^{\boxtimes\bullet}\ell^{\boxtimes} \qquad \ell^{\boxtimes}\ell^{\boxtimes\bullet} \xrightarrow{\varepsilon^{\ell}} 1_{1_{\mathsf{Mat}^{\mathsf{C}}}}$$

as consisting of the component 2-cells

$$\begin{tikzcd} \big(1_{\boxtimes(1_\boxtimes \times 1_{\mathsf{Mat}^\mathsf{C}})}\big)_m = \mathbb{1}^m \arrow[d, "\ell^{-1}_{\mathbb{1}^m}", "\eta^\ell_m ="'] \\ (\ell^{\boxtimes\bullet}\ell^\boxtimes)_m = \mathbb{1}^m\mathbb{1}^m \end{tikzcd} \qquad \begin{tikzcd} (\ell^\boxtimes\ell^{\boxtimes\bullet})_m = \mathbb{1}^m\mathbb{1}^m \arrow[d, "\ell_{\mathbb{1}^m}", "\varepsilon^\ell_m ="'] \\ \big(1_{1_{\mathsf{Mat}^\mathsf{C}}}\big)_m = \mathbb{1}^m \end{tikzcd}$$

in $\mathsf{Mat}^\mathsf{C}_{m,m}$ for each object $m \in \mathsf{Mat}^\mathsf{C}$. $\diamond$

Lemma 8.9.9. *The quadruple $(\ell^\boxtimes, \ell^{\boxtimes\bullet}, \eta^\ell, \varepsilon^\ell)$ in Definitions 8.9.1, 8.9.5, and 8.9.8 is an adjoint equivalence.*

Proof. The statement of the lemma means the following three assertions.

(1) $\ell^\boxtimes$ and $\ell^{\boxtimes\bullet}$ are strong transformations.
(2) η^ℓ and ε^ℓ are invertible modifications.
(3) The triangle identities (6.3.10) are satisfied.

The naturality of $\ell^\boxtimes_A$ and $\ell^{\boxtimes\bullet}_A$ with respect to A follows from the naturality of $\lambda^\oplus$, $\rho^\oplus$, $\lambda^\otimes$, $\rho^\otimes$, $\lambda^\bullet$, and $\rho^\bullet$ in C.

Taking advantage of the similarity between $(\ell^\boxtimes, \ell^{\boxtimes\bullet}, \eta^\ell, \varepsilon^\ell)$ and $(a^\boxtimes, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$ in Definitions 8.8.1, 8.8.33, and 8.8.37, the above assertions are proved by simpler versions of the proofs of Lemmas 8.8.5, 8.8.11, 8.8.17, 8.8.26, 8.8.39, and 8.8.45. Since we already showed all the detailed argument in Section 8.8, below we only state the equalities that one has to check. In marginal cases with $\mathsf{Mat}^\mathsf{C}_{m,0}$ or $\mathsf{Mat}^\mathsf{C}_{0,m}$, which are both the terminal category, the equality holds automatically. In every other case, one has to show that some diagram of 2-cells in Mat^C is commutative. In each entry, the diagram is realized as paths in $\mathsf{Gr}(X)$ for some set X of formal variables and some function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$. The desired commutativity in C then follows from either

- Proposition 3.5.33, if one vertex has the same support as 0^X, or
- the Coherence Theorem 3.9.1 in every other case, where the vertices are regular.

Analogous to (8.8.12), the lax unity axiom (6.2.15) for $\ell^\boxtimes$ states the following equality of pasting diagrams in $\mathsf{Mat}^\mathsf{C}_{m,m}$ for $m \geq 0$.

$$\begin{tikzcd}[row sep=large] m \arrow[rr, bend left=50, "\mathbb{1}\boxtimes\mathbb{1}^m"] \arrow[dd, "\mathbb{1}^m"'] & \Uparrow \ell^\boxtimes_{\mathbb{1}^m} & m \arrow[dd, "\mathbb{1}^m"] \\ & & \\ m \arrow[rr, bend left=50, "\mathbb{1}^m"] \arrow[rr, bend right=50, "\mathbb{1}^m"'] & \Uparrow 1_{\mathbb{1}^m} & m \end{tikzcd} \quad = \quad \begin{tikzcd}[row sep=large] m \arrow[rr, bend left=60, "\mathbb{1}\boxtimes\mathbb{1}^m"] \arrow[rr, "\mathbb{1}\boxtimes\mathbb{1}^m" description] \arrow[rr, phantom, bend left=35, "\Uparrow 1_\mathbb{1}\boxtimes 1_{\mathbb{1}^m}"] \arrow[rr, bend right=50, "\mathbb{1}^m"'] \arrow[rr, phantom, bend right=25, "\Uparrow \boxtimes^0_{(1,m)}"] \arrow[ddrr, bend right=30, "\mathbb{1}^m"'] \arrow[dd, "\mathbb{1}^m"'] & & m \arrow[dd, "\mathbb{1}^m"] \\ & \Uparrow r^{-1}_{\mathbb{1}^m} & \\ m \arrow[rr, bend right=50, "\mathbb{1}^m"'] & \Uparrow \ell_{\mathbb{1}^m} & m \end{tikzcd} \tag{8.9.10}$$

Consider (8.9.10).

- $\ell^\boxtimes_{\mathbb{1}^m} = \ell^{-1}_{\mathbb{1}\boxtimes\mathbb{1}^m}\lambda^{-\otimes}r_{\mathbb{1}^m}$.

- Each diagonal entry of $\mathbb{1} \boxtimes \mathbb{1}^m$ is $\mathbb{1} \otimes \mathbb{1}$, and each off-diagonal entry is $\mathbb{1} \otimes \mathbb{0}$.
- $\boxtimes^0_{(1,m)}$ is
  - $\lambda_{\mathbb{1}}^{-\otimes} : \mathbb{1} \longrightarrow \mathbb{1} \otimes \mathbb{1}$ in each diagonal entry and
  - $\rho_{\mathbb{1}}^{-\bullet} : \mathbb{0} \longrightarrow \mathbb{1} \otimes \mathbb{0}$ in each off-diagonal entry.
- $1_{\mathbb{1}} \boxtimes 1_{\mathbb{1}^m}$ is entrywise an identity morphism in C.

The lax unity axiom for $\ell^{\boxtimes \bullet}$ is analogous to (8.9.10).

Analogous to (8.8.16), the lax naturality axiom (6.2.16) for $\ell^{\boxtimes}$ states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{m,p}$ for 1-cells $(A, B) \in \mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{n,p}$.

$$
\begin{tikzcd}
m \arrow[rr, bend left=60, "\mathbb{1} \boxtimes BA"] \arrow[d, "\mathbb{1}^m"'] \arrow[rr, phantom, "\Uparrow \ell^{\boxtimes}_{BA}"] & & p \arrow[d, "\mathbb{1}^p"] \\
m \arrow[rr, bend left=60, "BA"] \arrow[dr, bend right, "A"'] \arrow[rr, phantom, "\Uparrow 1_{BA}"] & & p \\
& n \arrow[ur, bend right, "B"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
m \arrow[rr, bend left=60, "\mathbb{1} \boxtimes BA"] \arrow[rr, "\mathbb{1}\mathbb{1} \boxtimes BA"] \arrow[dr, bend right, "\mathbb{1} \boxtimes A"'] \arrow[dd, "\mathbb{1}^m"'] & \Uparrow \lambda^{\otimes}_{\mathbb{1}} \boxtimes 1_{BA} \arrow[d, phantom, "\Uparrow \boxtimes^2"] & p \arrow[dd, "\mathbb{1}^p"] \\
& n \arrow[ur, bend right, "\mathbb{1} \boxtimes B"'] \arrow[dd, "\mathbb{1}^n"] & \\
m \arrow[dr, bend right, "A"'] \arrow[r, phantom, "\overset{\ell^{\boxtimes}_A}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\ell^{\boxtimes}_B}{\Rightarrow}"] & p \\
& n \arrow[ur, bend right, "B"'] &
\end{tikzcd}
\tag{8.9.11}
$$

Consider (8.9.11).

- $\ell^{\boxtimes}_{BA} = \ell^{-1}_{\mathbb{1} \boxtimes BA} \lambda^{-\otimes} r_{BA}$, $\ell^{\boxtimes}_{A} = \ell^{-1}_{\mathbb{1} \boxtimes A} \lambda^{-\otimes} r_{A}$, and $\ell^{\boxtimes}_{B} = \ell^{-1}_{\mathbb{1} \boxtimes B} \lambda^{-\otimes} r_{B}$.
- $\boxtimes^2$ is either an identity morphism or as in (8.6.20).
- For $1 \le i \le m$ and $1 \le k \le p$, the (k, i)-entry of $\mathbb{1}\mathbb{1} \boxtimes BA$ is

$$\big(\mathbb{1}\mathbb{1} \boxtimes BA\big)_{ki} = (\mathbb{1} \otimes \mathbb{1}) \otimes (BA)_{ki} \in \mathsf{C}.$$

The lax naturality axiom for $\ell^{\boxtimes \bullet}$ is analogous to (8.9.11).

Analogous to (8.8.40), the modification axiom (6.3.2) for η^{ℓ} states that for each 1-cell $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, the following pasting diagram equality holds.

$$
\begin{tikzcd}
m \arrow[r, "\mathbb{1} \boxtimes A"] \arrow[dd, bend right=40, "\mathbb{1}^m"'] \arrow[ddr, "\mathbb{1} \boxtimes A"] \arrow[dd, phantom, "\overset{r_{\mathbb{1} \boxtimes A}}{\Rightarrow}"] & n \arrow[dd, bend right=40, "\mathbb{1}^n"'] \arrow[dr, "\mathbb{1}^n"] \arrow[dd, phantom, "\overset{\ell^{-1}_{\mathbb{1} \boxtimes A}}{\Rightarrow}"'] \arrow[dr, phantom, "\overset{\ell^{-1}_{\mathbb{1}^n}}{\Rightarrow}"'] & \\
& & n \arrow[dl, "\mathbb{1}^n"] \\
m \arrow[r, "\mathbb{1} \boxtimes A"'] & n &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
m \arrow[r, "\mathbb{1} \boxtimes A"] \arrow[d, "\mathbb{1}^m"] \arrow[dd, bend right=50, "\mathbb{1}^m"'] \arrow[dr, phantom, "\overset{\ell^{\boxtimes}_A}{\Rightarrow}"] & n \arrow[d, "\mathbb{1}^n"] \\
m \arrow[r, "A"] \arrow[d, "\mathbb{1}^m"] \arrow[l, phantom, "\overset{\ell^{-1}_{\mathbb{1}^m}}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\ell^{\boxtimes \bullet}_A}{\Rightarrow}"] & n \arrow[d, "\mathbb{1}^n"] \\
m \arrow[r, "\mathbb{1} \boxtimes A"'] & n
\end{tikzcd}
\tag{8.9.12}
$$

On the right-hand side, $\ell^{\boxtimes \bullet}_A = \ell^{-1}_A \lambda^{\otimes} r_{\mathbb{1} \boxtimes A}$. The modification axiom for ε^{ℓ} is analogous to (8.9.12).

Analogous to (8.8.46), the left triangle identity in (6.3.10) for $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$ states the commutativity of the left diagram below.

$$\begin{tikzcd}[column sep=small]
\ell^{\boxtimes} 1_{\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})} \arrow[rr, "r_{\ell^{\boxtimes}}"] \arrow[d, "1_{\ell^{\boxtimes}} * \eta^{\ell}"'] & & \ell^{\boxtimes} \\
\ell^{\boxtimes}(\ell^{\boxtimes\bullet}\ell^{\boxtimes}) \arrow[dr, "a^{-1}_{\ell^{\boxtimes},\ell^{\boxtimes\bullet},\ell^{\boxtimes}}"'] & & 1_{1_{\mathsf{Mat}^{\mathsf{C}}}} \ell^{\boxtimes} \arrow[u, "\ell_{\ell^{\boxtimes}}"'] \\
& (\ell^{\boxtimes}\ell^{\boxtimes\bullet})\ell^{\boxtimes} \arrow[ur, "\varepsilon^{\ell} * 1_{\ell^{\boxtimes}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\mathbb{1}^m \mathbb{1}^m \arrow[rr, "r_{\mathbb{1}^m}"] \arrow[d, "1 \star \ell^{-1}_{\mathbb{1}^m}"'] & & \mathbb{1}^m \\
\mathbb{1}^m(\mathbb{1}^m\mathbb{1}^m) \arrow[dr, "a^{-1}_{\mathbb{1}^m,\mathbb{1}^m,\mathbb{1}^m}"'] & & \mathbb{1}^m\mathbb{1}^m \arrow[u, "\ell_{\mathbb{1}^m}"'] \\
& (\mathbb{1}^m\mathbb{1}^m)\mathbb{1}^m \arrow[ur, "\ell_{\mathbb{1}^m} \star 1"'] &
\end{tikzcd}
\tag{8.9.13}$$

At each object $m \in \mathsf{Mat}^{\mathsf{C}}$, the left diagram in (8.9.13) yields the right diagram in $\mathsf{Mat}^{\mathsf{C}}_{m,m}$. The right triangle identity in (6.3.10) is analogous to (8.9.13). □

The Right Monoidal Unitor. Next we define the right monoidal unitor in $\mathsf{Mat}^{\mathsf{C}}$, which is similar to the left monoidal unitor.

Definition 8.9.14. Define the data of a lax transformation

$$\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes}) \xrightarrow{r^{\boxtimes}} 1_{\mathsf{Mat}^{\mathsf{C}}}$$

as follows.

Component 1-Cells: For each object $m \in \mathsf{Mat}^{\mathsf{C}}$, define

$$m = m \boxtimes 1 \xrightarrow{r^{\boxtimes}_m} \big(1_{\mathsf{Mat}^{\mathsf{C}}}\big)\, m = m \tag{8.9.15}$$

as the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ in (8.1.6).

Component 2-Cells: For each 1-cell $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, define

$$r^{\boxtimes}_A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}\Big(\big(1_{\mathsf{Mat}^{\mathsf{C}}} A\big)\, r^{\boxtimes}_m ; r^{\boxtimes}_n \big(\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})\big)\,(A) \Big)$$

as the following vertical composite 2-cell.

$$\begin{tikzcd}
& m \arrow[dr, bend left, "A"] \arrow[d, phantom, "\Downarrow r_A"] & \\
m \arrow[ur, bend left, "\mathbb{1}^m"] \arrow[rr, bend left=10, "A"] \arrow[rr, bend right=10, "A \boxtimes \mathbb{1}"'] \arrow[dr, bend right, "A \boxtimes \mathbb{1}"'] & \Downarrow \rho^{-\otimes} \arrow[d, phantom, "\Downarrow \ell^{-1}_{A \boxtimes \mathbb{1}}"] & n \\
& n \arrow[ur, bend right, "\mathbb{1}^n"'] &
\end{tikzcd}
\tag{8.9.16}$$

- $A \boxtimes \mathbb{1}$ is the matrix tensor product in (8.6.3) with
$$A \boxtimes \mathbb{1} = \big(A_{ji} \otimes \mathbb{1}\big) = \big(\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})\big)\,(A).$$
Here and in what follows, the identity matrix $\mathbb{1}^1 \in \mathsf{Mat}^{\mathsf{C}}_{1,1}$ is abbreviated to $\mathbb{1}$.
- $\rho^{-\otimes}$ is the 2-cell with (j,i)-entry the structure morphism in C,
$$A_{ji} \xrightarrow{\rho^{-\otimes}_{A_{ji}}} A_{ji} \otimes \mathbb{1}$$
for $1 \leq i \leq m$ and $1 \leq j \leq n$.

This finishes the definition of $r^{\boxtimes}$. $\diamond$

Definition 8.9.17. Define the data of a lax transformation

$$1_{\mathsf{Mat}^{\mathsf{C}}} \xrightarrow{r^{\boxtimes\bullet}} \boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})$$

as follows.

Component 1-Cells: For each object $m \in \mathsf{Mat}^{\mathsf{C}}$, define

$$m = \big(1_{\mathsf{Mat}^{\mathsf{C}}}\big)\, m \xrightarrow{r^{\boxtimes\bullet}_m} m \boxtimes 1 = m \tag{8.9.18}$$

as the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ in (8.1.6).

Component 2-Cells: For each 1-cell $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, define

$$r^{\boxtimes\bullet}_A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}\Big(\big(\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})\big)(A) r^{\boxtimes\bullet}_m ; r^{\boxtimes\bullet}_n \big(1_{\mathsf{Mat}^{\mathsf{C}}} A\big) \Big)$$

as the following vertical composite 2-cell.

$$\begin{tikzcd}
 & m \arrow[dr, bend left, "A \boxtimes \mathbb{1}"] \arrow[d, phantom, "\Downarrow r_{A\boxtimes\mathbb{1}}"] & \\
m \arrow[ur, bend left, "\mathbb{1}^m"] \arrow[rr, bend left=10, "A \boxtimes \mathbb{1}"] \arrow[rr, bend right=10, "A"'] \arrow[dr, bend right, "A"'] & \Downarrow \rho^{\otimes} \arrow[d, phantom, "\Downarrow \ell_A^{-1}"] & n \\
 & n \arrow[ur, bend right, "\mathbb{1}^n"'] &
\end{tikzcd} \tag{8.9.19}$$

In this vertical composite, $\rho^{\otimes}$ is the 2-cell with (j,i)-entry the structure morphism in C,

$$A_{ji} \otimes \mathbb{1} \xrightarrow{\rho^{\otimes}_{A_{ji}}} A_{ji}$$

for $1 \le i \le m$ and $1 \le j \le n$.

This finishes the definition of $r^{\boxtimes\bullet}$. $\diamond$

Definition 8.9.20. Define the data

$$1_{\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})} \xrightarrow{\eta^r} r^{\boxtimes\bullet} r^{\boxtimes} \qquad r^{\boxtimes} r^{\boxtimes\bullet} \xrightarrow{\varepsilon^r} 1_{1_{\mathsf{Mat}^{\mathsf{C}}}}$$

as consisting of the component 2-cells

$$\begin{tikzcd}
\big(1_{\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})}\big)_m = \mathbb{1}^m \arrow[d, "\eta^r_m = \ell^{-1}_{\mathbb{1}^m}"] & (r^{\boxtimes} r^{\boxtimes\bullet})_m = \mathbb{1}^m \mathbb{1}^m \arrow[d, "\varepsilon^r_m = \ell_{\mathbb{1}^m}"] \\
(r^{\boxtimes\bullet} r^{\boxtimes})_m = \mathbb{1}^m \mathbb{1}^m & \big(1_{1_{\mathsf{Mat}^{\mathsf{C}}}}\big)_m = \mathbb{1}^m
\end{tikzcd}$$

in $\mathsf{Mat}^{\mathsf{C}}_{m,m}$ for each object $m \in \mathsf{Mat}^{\mathsf{C}}$. $\diamond$

Lemma 8.9.21. *The quadruple $(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^r, \varepsilon^r)$ in Definitions 8.9.14, 8.9.17, and 8.9.20 is an adjoint equivalence.*

Proof. The statement of the lemma means the following three assertions.

(1) $r^{\boxtimes}$ and $r^{\boxtimes\bullet}$ are strong transformations.
(2) η^r and ε^r are invertible modifications.

(3) The triangle identities (6.3.10) are satisfied.

The naturality of $r^{\boxtimes}_A$ and $r^{\boxtimes\bullet}_A$ with respect to A follows from the naturality of $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\lambda^{\bullet}$, and $\rho^{\bullet}$ in C.

The rest of the proof is similar to those in Section 8.8, in particular Lemmas 8.8.5, 8.8.11, 8.8.17, 8.8.26, 8.8.39, and 8.8.45. As in Lemma 8.9.9, below we only state the equalities that one has to check.

Analogous to (8.8.12) and (8.9.10), the lax unity axiom (6.2.15) for $r^{\boxtimes}$ states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{m,m}$ for $m \geq 0$.

(8.9.22)
$$\begin{tikzcd}[row sep=huge, column sep=huge]
m \arrow[r, bend left=60, "\mathbb{1}^m \boxtimes \mathbb{1}"] \arrow[d, "\mathbb{1}^m"'] \arrow[dr, phantom, "\Uparrow r^{\boxtimes}_{\mathbb{1}^m}"] & m \arrow[d, "\mathbb{1}^m"] \\
m \arrow[r, bend left, "\mathbb{1}^m"] \arrow[r, bend right=60, "\mathbb{1}^m"'] \arrow[r, phantom, "\Uparrow 1_{\mathbb{1}^m}"] & m
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[row sep=huge, column sep=huge]
m \arrow[r, bend left=60, "\mathbb{1}^m \boxtimes \mathbb{1}"] \arrow[r, "\mathbb{1}^m \boxtimes \mathbb{1}" description] \arrow[r, phantom, bend left=35, "\Uparrow 1_{\mathbb{1}^m} \boxtimes 1_{\mathbb{1}}"] \arrow[r, bend right=50, "\mathbb{1}^m"'] \arrow[r, phantom, bend right=25, "\Uparrow \boxtimes^0_{(m,1)}"] \arrow[d, "\mathbb{1}^m"'] \arrow[dr, bend right, "\mathbb{1}^m" description] \arrow[dr, phantom, "\Uparrow r^{-1}_{\mathbb{1}^m}"] & m \arrow[d, "\mathbb{1}^m"] \\
m \arrow[r, bend right=60, "\mathbb{1}^m"'] \arrow[r, phantom, "\Uparrow \ell_{\mathbb{1}^m}"] & m
\end{tikzcd}$$

Consider (8.9.22).

- Each diagonal entry of $\mathbb{1}^m \boxtimes \mathbb{1}$ is $\mathbb{1} \otimes \mathbb{1}$, and each off-diagonal entry is $\mathbb{0} \otimes \mathbb{1}$.
- $\boxtimes^0_{(m,1)}$ is
  - $\lambda^{-\otimes}_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1} \otimes \mathbb{1}$ in each diagonal entry and
  - $\lambda^{-\bullet}_{\mathbb{1}} : \mathbb{0} \longrightarrow \mathbb{0} \otimes \mathbb{1}$ in each off-diagonal entry.
- $1_{\mathbb{1}^m} \boxtimes 1_{\mathbb{1}}$ is entrywise an identity morphism in C.

The lax unity axiom for $r^{\boxtimes\bullet}$ is analogous to (8.9.22).

Analogous to (8.8.16) and (8.9.11), the lax naturality axiom (6.2.16) for $r^{\boxtimes}$ states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{m,p}$ for 1-cells $(A, B) \in \mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{n,p}$.

(8.9.23)
$$\begin{tikzcd}[row sep=huge, column sep=large]
m \arrow[rr, bend left=50, "BA \boxtimes \mathbb{1}"] \arrow[d, "\mathbb{1}^m"'] \arrow[drr, phantom, "\Uparrow r^{\boxtimes}_{BA}"] & & p \arrow[d, "\mathbb{1}^p"] \\
m \arrow[rr, bend left, "BA"] \arrow[rr, phantom, "\Uparrow 1_{BA}"] \arrow[dr, bend right, "A"'] & & p \\
& n \arrow[ur, bend right, "B"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[row sep=huge, column sep=large]
m \arrow[rr, bend left=50, "BA \boxtimes \mathbb{1}"] \arrow[rr, "BA \boxtimes \mathbb{1}\mathbb{1}" description] \arrow[rr, phantom, bend left=30, "\Uparrow 1_{BA} \boxtimes \lambda^{\otimes}_{\mathbb{1}}"] \arrow[dr, bend right, "A \boxtimes \mathbb{1}" description] \arrow[dd, "\mathbb{1}^m"'] & \Uparrow \boxtimes^2 & p \arrow[dd, "\mathbb{1}^p"] \\
& n \arrow[ur, bend right, "B \boxtimes \mathbb{1}" description] \arrow[d, "\mathbb{1}^n"] \arrow[dl, phantom, "\overset{r^{\boxtimes}_A}{\Rightarrow}"] \arrow[dr, phantom, "\overset{r^{\boxtimes}_B}{\Rightarrow}"] & \\
m \arrow[r, bend right, "A"'] & n \arrow[r, bend right, "B"'] & p
\end{tikzcd}$$

In (8.9.23), for $1 \leq i \leq m$ and $1 \leq k \leq p$, the (k,i)-entry of $BA \boxtimes \mathbb{1}\mathbb{1}$ is

$$\big(BA \boxtimes \mathbb{1}\mathbb{1}\big)_{ki} = (BA)_{ki} \otimes (\mathbb{1} \otimes \mathbb{1}) \in \mathsf{C}.$$

The lax naturality axiom for $r^{\boxtimes\bullet}$ is analogous to (8.9.23).

Analogous to (8.8.40) and (8.9.12), the modification axiom (6.3.2) for η^r states that for each 1-cell $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, the following pasting diagram equality holds.

(8.9.24)
$$
\begin{tikzcd}
m \arrow[rr, "A\boxtimes\mathbb{1}"] \arrow[dd, bend right=40, "\mathbb{1}^m"'] \arrow[ddrr, "A\boxtimes\mathbb{1}" near start] & & n \arrow[d, bend left, "\mathbb{1}^n"] \arrow[dd, bend right=40, "\mathbb{1}^n"'] \\
\overset{r_{A\boxtimes\mathbb{1}}}{\Rightarrow} & \overset{\ell^{-1}_{A\boxtimes\mathbb{1}}}{\Rightarrow} & n \arrow[d, bend left, "\mathbb{1}^n"] \arrow[l, phantom, "\overset{\ell^{-1}_{\mathbb{1}^n}}{\Rightarrow}"] \\
m \arrow[rr, "A\boxtimes\mathbb{1}"'] & & n
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
m \arrow[rr, "A\boxtimes\mathbb{1}"] \arrow[d, bend left, "\mathbb{1}^m"] \arrow[dd, bend right=40, "\mathbb{1}^m"'] & \overset{r^{\boxtimes}_A}{\Rightarrow} & n \arrow[d, bend left, "\mathbb{1}^n"] \\
m \arrow[rr, "A"] \arrow[d, bend left, "\mathbb{1}^m"] \arrow[l, phantom, "\overset{\ell^{-1}_{\mathbb{1}^m}}{\Rightarrow}"] & \overset{r^{\boxtimes\bullet}_A}{\Rightarrow} & n \arrow[d, bend left, "\mathbb{1}^n"] \\
m \arrow[rr, "A\boxtimes\mathbb{1}"'] & & n
\end{tikzcd}
$$

The modification axiom for ε^r is analogous to (8.9.24).

Analogous to (8.8.46) and (8.9.13), the left triangle identity in (6.3.10) for the data $(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^r, \varepsilon^r)$ states the commutativity of the left diagram below.

(8.9.25)
$$
\begin{tikzcd}[column sep=small]
r^{\boxtimes}1_{\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}}\times 1_{\boxtimes})} \arrow[rr, "r_{r^{\boxtimes}}"] \arrow[d, "1_{r^{\boxtimes}} * \eta^r"'] & & r^{\boxtimes} \\
r^{\boxtimes}(r^{\boxtimes\bullet}r^{\boxtimes}) \arrow[dr, "a^{-1}_{r^{\boxtimes},r^{\boxtimes\bullet},r^{\boxtimes}}"'] & & 1_{1_{\mathsf{Mat}^{\mathsf{C}}}}r^{\boxtimes} \arrow[u, "\ell_{r^{\boxtimes}}"'] \\
& (r^{\boxtimes}r^{\boxtimes\bullet})r^{\boxtimes} \arrow[ur, "\varepsilon^r * 1_{r^{\boxtimes}}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\mathbb{1}^m\mathbb{1}^m \arrow[rr, "r_{\mathbb{1}^m}"] \arrow[d, "1 \star \ell^{-1}_{\mathbb{1}^m}"'] & & \mathbb{1}^m \\
\mathbb{1}^m(\mathbb{1}^m\mathbb{1}^m) \arrow[dr, "a^{-1}_{\mathbb{1}^m,\mathbb{1}^m,\mathbb{1}^m}"'] & & \mathbb{1}^m\mathbb{1}^m \arrow[u, "\ell_{\mathbb{1}^m}"'] \\
& (\mathbb{1}^m\mathbb{1}^m)\mathbb{1}^m \arrow[ur, "\ell_{\mathbb{1}^m} \star 1"'] &
\end{tikzcd}
$$

At each object $m \in \mathsf{Mat}^{\mathsf{C}}$, the left diagram in (8.9.25) yields the right diagram in $\mathsf{Mat}^{\mathsf{C}}_{m,m}$. The right triangle identity in (6.3.10) is analogous to (8.9.25). □

8.10. The Pentagonator

For a tight symmetric bimonoidal category C, we continue the construction of a monoidal bicategory structure, in the sense of Definition 6.4.1, on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ in Theorem 8.4.12. In this section, we construct the pentagonator in $\mathsf{Mat}^{\mathsf{C}}$.

Definition 8.10.1. Define π as consisting of the 2-cells

$$\pi_{m,n,p,q} \in \mathsf{Mat}^{\mathsf{C}}_{s,s} \quad \text{with} \quad m,n,p,q \geq 0 \quad \text{and} \quad s = mnpq$$

given by the composite of the following pasting diagram in $\mathsf{Mat}^{\mathsf{C}}_{s,s}$.

(8.10.2)
$$
\begin{tikzcd}[column sep=small]
& (m\boxtimes(n\boxtimes p))\boxtimes q \arrow[rr, "a^{\boxtimes}_{m,np,q}"] \arrow[drr, "\mathbb{1}^s"' near end] \arrow[d, bend left, "\mathbb{1}^s"] & \Downarrow \ell_{\mathbb{1}^s} & m\boxtimes((n\boxtimes p)\boxtimes q) \arrow[d, bend right, "\mathbb{1}^s"'] \arrow[d, bend left, "\mathbb{1}^m\boxtimes a^{\boxtimes}_{n,p,q}"] \arrow[d, phantom, "\overset{\boxtimes^{-0}_{m,npq}}{\Leftarrow}"] & \\
& ((m\boxtimes n)\boxtimes p)\boxtimes q \arrow[u, bend left, "a^{\boxtimes}_{m,n,p}\boxtimes\mathbb{1}^q"] \arrow[u, phantom, "\overset{\boxtimes^{-0}_{mnp,q}}{\Rightarrow}"] \arrow[dr, "a^{\boxtimes}_{mn,p,q}"'] & \Downarrow 1_{\mathbb{1}^s\mathbb{1}^s} & m\boxtimes(n\boxtimes(p\boxtimes q)) & \\
& & (m\boxtimes n)\boxtimes(p\boxtimes q) \arrow[ur, "a^{\boxtimes}_{m,n,pq}"'] & &
\end{tikzcd}
$$

This finishes the definition of π. ◇

Explanation 8.10.3. Consider the pasting diagram in (8.10.2).

(1) By Definition 8.6.19, each vertex is the object $s \in \mathsf{Mat}^{\mathsf{C}}$.
(2) Each instance of $a^{\boxtimes}$ is a component 1-cell of the monoidal associator in (8.8.2). In particular,

$$a^{\boxtimes}_{m,n,p} = \mathbb{1}^{mnp}, \qquad a^{\boxtimes}_{n,p,q} = \mathbb{1}^{npq}, \quad \text{and}$$
$$a^{\boxtimes}_{mn,p,q} = a^{\boxtimes}_{m,np,q} = a^{\boxtimes}_{m,n,pq} = \mathbb{1}^{s}.$$

As in (8.6.10), each diagonal entry of the 1-cell $\mathbb{1}^{mnp} \boxtimes \mathbb{1}^{q}$ is $\mathbb{1} \otimes \mathbb{1}$, and each off-diagonal entry is $\mathbb{1} \otimes \mathbb{0}$, $\mathbb{0} \otimes \mathbb{1}$, or $\mathbb{0} \otimes \mathbb{0}$. A similar description applies to the 1-cell $\mathbb{1}^{m} \boxtimes \mathbb{1}^{npq}$.
(3) By Convention 6.2.12, the domain 1-cell is the horizontal composite

$$\Big(\big(\mathbb{1}^{m} \boxtimes \mathbb{1}^{npq}\big)\mathbb{1}^{s}\Big)\big(\mathbb{1}^{mnp} \boxtimes \mathbb{1}^{q}\big).$$

The codomain 1-cell is $\mathbb{1}^{s}\mathbb{1}^{s}$.
(4) Each instance of $\boxtimes^{0}$ is a component 2-cell of the lax unity constraint of the monoidal composition in (8.6.9), with $\boxtimes^{-0} = (\boxtimes^{0})^{-1}$. If $s > 0$, then each entry of each $\boxtimes^{-0}$ is
  - $\lambda^{\otimes}_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1}$ for a diagonal entry and
  - $\rho^{\bullet}_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{0} \longrightarrow \mathbb{0}$, $\lambda^{\bullet}_{\mathbb{1}} : \mathbb{0} \otimes \mathbb{1} \longrightarrow \mathbb{0}$, or $\lambda^{\bullet}_{\mathbb{0}} : \mathbb{0} \otimes \mathbb{0} \longrightarrow \mathbb{0}$ for an off-diagonal entry.
(5) $\ell_{\mathbb{1}^{s}} : \mathbb{1}^{s}\mathbb{1}^{s} \longrightarrow \mathbb{1}^{s}$ is the base left unitor in (8.2.2).
(6) $1_{\mathbb{1}^{s}\mathbb{1}^{s}}$ is the identity 2-cell of $\mathbb{1}^{s}\mathbb{1}^{s}$.
(7) Each entry of each component 2-cell of π is a composite of morphisms in C, each being an iterated sum and product of identity morphisms and one component of $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, $\lambda^{\bullet}$, or $\rho^{\bullet}$.

Moreover, each $\pi_{m,n,p,q} \in \mathsf{Mat}^{\mathsf{C}}_{s,s}$ is a well-defined 2-cell by the Bicategorical Pasting Theorem in [**JY21**, 3.6.6]. It states that each pasting diagram in a bicategory has a unique composite, once a bracketing is chosen for the (co)domain composite 1-cell. ◇

Lemma 8.10.4. *In Definition 8.10.1, π is an invertible modification.*

Proof. Each component 2-cell of π is a composite of invertible 2-cells by (8.2.2) and (8.6.9). To verify the modification axiom (6.3.2) for π, consider arbitrary 1-cells

$$\begin{tikzcd} m \arrow[d, "B^1"'] & n \arrow[d, "B^2"'] & p \arrow[d, "B^3"'] & q \arrow[d, "B^4"'] \\ m' & n' & p' & q' \end{tikzcd}$$

in $\mathsf{Mat}^{\mathsf{C}}$ with $s = mnpq$ and $s' = m'n'p'q'$. If either $s = 0$ or $s' = 0$, then $\mathsf{Mat}^{\mathsf{C}}_{s,s'}$ is the terminal category, and the modification axiom holds automatically.

Suppose $s, s' > 0$. We use the abbreviations

$$\begin{aligned} B^{12} &= B^1 \boxtimes B^2 & B^{2(34)} &= B^2 \boxtimes (B^3 \boxtimes B^4) \\ B^{[(12)3]4} &= \big[(B^1 \boxtimes B^2) \boxtimes B^3\big] \boxtimes B^4 & B^{(12)(34)} &= (B^1 \boxtimes B^2) \boxtimes (B^3 \boxtimes B^4) \\ a^{\boxtimes}_{2,3,4} &= a^{\boxtimes}_{B^2,B^3,B^4} & a^{\boxtimes}_{1,23,4} &= a^{\boxtimes}_{B^1,B^2\boxtimes B^3,B^4} \\ \boxtimes^{-2} &= (\boxtimes^2)^{-1} \end{aligned}$$

and so forth. Using (6.2.18), (8.6.20), and (8.8.3), the modification axiom (6.3.2) for the 1-cells (B^1, B^2, B^3, B^4) states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{s,s'}$.

$$\begin{tikzcd}[column sep=small]
 & & s \arrow[dll, bend right=10, "\mathbb{1}^{mnp} \boxtimes \mathbb{1}^{q}"'] \arrow[drr, bend left=10, "B^{[(12)3]4}"] \arrow[d, bend right=50, "B^{1(23)}\mathbb{1}^{mnp} \boxtimes B^4 \mathbb{1}^q"'] \arrow[d, bend left=50, "\mathbb{1}^{m'n'p'}B^{(12)3} \boxtimes \mathbb{1}^{q'}B^4"] \arrow[d, phantom, "\overset{a^{\boxtimes}_{1,2,3} \boxtimes \ell^{-1}r}{\Rightarrow}"] & & \\
s \arrow[rr, bend right=10, "B^{[1(23)]4}"'] \arrow[d, "\mathbb{1}^s"'] \arrow[r, phantom, "\overset{\boxtimes^2}{\Rightarrow}"] & \arrow[d, phantom, "\overset{a^{\boxtimes}_{1,23,4}}{\Rightarrow}"] & s' \arrow[d, "\mathbb{1}^{s'}"] & \arrow[l, phantom, "\overset{\boxtimes^{-2}}{\Rightarrow}"] & s' \arrow[ll, bend left=10, "\mathbb{1}^{m'n'p'} \boxtimes \mathbb{1}^{q'}"] \arrow[dd, "\mathbb{1}^{s'}"] \arrow[dl, phantom, "\overset{\pi_{m',n',p',q'}}{\Rightarrow}"] \\
s \arrow[rr, "B^{1[(23)4]}"] \arrow[d, "\mathbb{1}^m \boxtimes \mathbb{1}^{npq}"'] \arrow[ddrr, bend left=20, "\mathbb{1}^{m'}B^1 \boxtimes \mathbb{1}^{n'p'q'}B^{(23)4}"] \arrow[ddrr, bend right=10, "B^1\mathbb{1}^m \boxtimes B^{2(34)}\mathbb{1}^{npq}"'] \arrow[dr, phantom, "\overset{\boxtimes^2}{\Rightarrow}"'] & \arrow[d, phantom, "\overset{\ell^{-1}r \boxtimes a^{\boxtimes}_{2,3,4}}{\Rightarrow}"] & s' \arrow[dd, "\mathbb{1}^{m'} \boxtimes \mathbb{1}^{n'p'q'}"] \arrow[dl, phantom, "\overset{\boxtimes^{-2}}{\Rightarrow}"'] & & \\
s \arrow[drr, bend right=10, "B^{1[2(34)]}"'] & & & & s' \arrow[dll, bend left=10, "\mathbb{1}^{s'}"] \\
 & & s' & &
\end{tikzcd}$$

$$\parallel$$

$$\begin{tikzcd}[column sep=small]
 & & s \arrow[dll, bend right=10, "\mathbb{1}^{mnp} \boxtimes \mathbb{1}^{q}"'] \arrow[drr, bend left=10, "B^{[(12)3]4}"] \arrow[dd, "\mathbb{1}^s"] & & \\
s \arrow[d, "\mathbb{1}^s"'] & \overset{\pi_{m,n,p,q}}{\Rightarrow} & & \overset{a^{\boxtimes}_{12,3,4}}{\Rightarrow} & s' \arrow[dd, "\mathbb{1}^{s'}"] \\
s \arrow[d, "\mathbb{1}^m \boxtimes \mathbb{1}^{npq}"'] & & s \arrow[dll, "\mathbb{1}^s"'] \arrow[drr, "B^{(12)(34)}"] \arrow[d, phantom, "\overset{a^{\boxtimes}_{1,2,34}}{\Rightarrow}"] & & \\
s \arrow[drr, bend right=10, "B^{1[2(34)]}"'] & & & & s' \arrow[dll, bend left=10, "\mathbb{1}^{s'}"] \\
 & & s' & &
\end{tikzcd} \tag{8.10.5}$$

Consider (8.10.5).

- The common domain 1-cell is

$$B^{1[2(34)]}\Big[\big((\mathbb{1}^m \boxtimes \mathbb{1}^{npq})\mathbb{1}^s\big)\big(\mathbb{1}^{mnp} \boxtimes \mathbb{1}^q\big)\Big]. \tag{8.10.6}$$

  The common codomain 1-cell is

$$\big(\mathbb{1}^{s'}\mathbb{1}^{s'}\big)B^{[(12)3]4}. \tag{8.10.7}$$

- In the bottom pasting diagram, the following statements hold.
  - There is one instance of a^{-1} after $1_{B^{1[2(34)]}} * \pi_{m,n,p,q}$, with a the base associator in (8.3.2).
  - There is one instance of a after $a^{\boxtimes}_{1,2,34} * 1_{\mathbb{1}^s}$.
  - There is one instance of a^{-1} after $1_{\mathbb{1}^{s'}} * a^{\boxtimes}_{12,3,4}$.

  These a and a^{-1} are not explicitly displayed in the pasting diagram.
- In the top pasting diagram, the following statements hold.
  - There are multiple instances of the base associator and its inverse that are not explicitly displayed.
  - ℓ is the base left unitor in (8.2.2), and r is the base right unitor in (8.2.8).

It suffices to prove the equality (8.10.5) in each entry. As in the proofs in Section 8.8, in each entry, each of the two composite 2-cells in (8.10.5) is a composite of morphisms in C. The resulting diagram is the image in C of some diagram in $\mathsf{Gr}(X)$. The main subtlety comes from

- $\boxtimes^2$, which contains δ^{-r} and δ^{-l} by Lemma 8.6.16, and
- the base associator $a = a^4a^3a^2a^1$ in (8.3.5), which contains δ^r (8.3.6) in a^1 and δ^{-l} (8.3.8) in a^4.

Since the elementary edges δ^l and δ^r do not have formal inverses as in Definition 3.1.6, only the *inverses* of $\boxtimes^2$ and a^4 are represented as paths in $\mathsf{Gr}(X)$, creating a zigzag. Before the Coherence Theorem 3.9.1 can be applied, the diagram must first be subdivided such that each subdiagram consists of two parallel paths. This subdivision process is analogous to those in the proofs of Lemmas 8.4.9 and 8.8.26. The detailed argument is given below.

Consider the following indices and abbreviations.

$$\begin{gathered}
1 \le i \le m \qquad 1 \le j \le n \qquad 1 \le k \le p \qquad 1 \le l \le q \\
1 \le i' \le m' \qquad 1 \le j' \le n' \qquad 1 \le k' \le p' \qquad 1 \le l' \le q' \\
I = (i-1)npq + (j-1)pq + (k-1)q + l \\
I' = (i'-1)n'p'q' + (j'-1)p'q' + (k'-1)q' + l'
\end{gathered}$$

In the (I', I)-entry, the two pasting diagrams in (8.10.5) yield the outer diagram in C below, which we will explain in detail.

$$\begin{tikzcd}
Z_1 \arrow[r] \arrow[d, "2"'] & R_4 \arrow[dddr, "f_4"] & S_3 \arrow[l] \arrow[r] & R_5 \arrow[dddl, "f_5"'] & S_4 \arrow[l, "2"'] \arrow[d, "3"] \\
R_1 \arrow[ddrr, "f_1"] & & & & R_6 \arrow[ddll, "f_6"'] \\
S_1 \arrow[u] \arrow[d, "2"'] & & & & S_5 \arrow[u] \arrow[d, "2"] \\
R_2 \arrow[rr, "f_2"] & & Y & & R_7 \arrow[ll, "f_7"'] \\
S_2 \arrow[u] \arrow[d, "2"'] & & & & S_6 \arrow[u] \arrow[dd] \\
R_3 \arrow[uurr, "f_3"] & & & & \\
Z_2 \arrow[u] & S_8 \arrow[l] \arrow[r] & R_9 \arrow[uuu, "f_9"] & S_7 \arrow[l, "3"'] \arrow[r, "2"] & R_8 \arrow[uuull, "f_8"']
\end{tikzcd} \tag{8.10.8}$$

Consider the diagram (8.10.8).

- Z_1 is the (I', I)-entry of the common domain (8.10.6).
- Z_2 is the (I', I)-entry of the common codomain (8.10.7).
- Each arrow is an isomorphism built from the structure morphisms in C. The number 2 or 3 decorating an arrow means that it is a composite of that many morphisms.
- With the appropriate inverses taken into account, the left zigzag from Z_1 to Z_2 is the (I', I)-entry of the bottom pasting diagram in (8.10.5).
- The top-right-bottom zigzag, from Z_1 to S_4 to R_8 to Z_2, is the (I', I)-entry of the top pasting diagram in (8.10.5).

- Each R_i for $1 \leq i \leq 9$ is the codomain of an instance of $a^{-4} = (a^4)^{-1}$ (8.3.5), or of a morphism induced by a^{-4}.
- The object Y is the (I', I)-entry of $B^{[(12)3]4}$, that is,

$$Y = \big[(B^1_{i'i} \otimes B^2_{j'j}) \otimes B^3_{k'k}\big] \otimes B^4_{l'l}. \tag{8.10.9}$$

Next we describe explicitly the objects and the morphisms in (8.10.8). Then we will explain how to realize the diagram using paths in $\mathsf{Gr}(X)$ such that, by Theorem 3.9.1, each of the 9 subdiagrams is commutative in C. We will use the following abbreviations.

$$\begin{aligned} \mathbb{1}^{m,npq} &= \mathbb{1}^m \boxtimes \mathbb{1}^{npq} & \qquad \mathbb{1}^{m',n'p'q'} &= \mathbb{1}^{m'} \boxtimes \mathbb{1}^{n'p'q'} \\ \mathbb{1}^{mnp,q} &= \mathbb{1}^{mnp} \boxtimes \mathbb{1}^{q} & \qquad \mathbb{1}^{m'n'p',q'} &= \mathbb{1}^{m'n'p'} \boxtimes \mathbb{1}^{q'} \end{aligned} \tag{8.10.10}$$

Since we have specified the (I', I)-entry in (8.10.5), to save space, we will omit the subscript (I', I) in most objects and also the subscripts in π and $a^{\boxtimes}$. In displaying the objects R_i, we abbreviate $\otimes$ to concatenation. In the following diagrams, $\star$ is the matrix product of morphisms in (8.1.5), and $a = a^4a^3a^2a^1$ refers to an entry of the base associator in (8.3.5). As mentioned above, we will use $\boxtimes^{-2} = (\boxtimes^2)^{-1}$ and $a^{-4} = (a^4)^{-1}$.

The left zigzag in (8.10.8) from Z_1 to Z_2 is the following zigzag in C.

$$\begin{tikzcd}
R_1 = \Big[\bigoplus\limits_{t=1}^{s} \Big[\bigoplus\limits_{u=1}^{s} B^{1[2(34)]}_{I't}(\mathbb{1}^s_{tu}\mathbb{1}^s_{uI})\Big]_{\mathrm{lt}}\Big]_{\mathrm{lt}} & B^{1[2(34)]}(\mathbb{1}^s\mathbb{1}^s) \arrow[l, "a^{-4}"'] \\
S_1 = \big(B^{1[2(34)]}\mathbb{1}^s\big)\mathbb{1}^s \arrow[u, "a^3a^2a^1"] \arrow[d, "a^{\boxtimes}\star 1"'] & B^{1[2(34)]}\big[(\mathbb{1}^{m,npq}\mathbb{1}^s)\mathbb{1}^{mnp,q}\big] = Z_1 \arrow[u, "1\star\pi"'] \\
\big(\mathbb{1}^{s'}B^{(12)(34)}\big)\mathbb{1}^s \arrow[d, "a^3a^2a^1"'] & (\mathbb{1}^{s'}\mathbb{1}^{s'})B^{[(12)3]4} = Z_2 \arrow[d, "a^3a^2a^1"] \\
R_2 = \Big[\bigoplus\limits_{t'=1}^{s'} \Big[\bigoplus\limits_{u=1}^{s} \mathbb{1}^{s'}_{I't'}(B^{(12)(34)}_{t'u}\mathbb{1}^s_{uI})\Big]_{\mathrm{lt}}\Big]_{\mathrm{lt}} & \Big[\bigoplus\limits_{t'=1}^{s'} \Big[\bigoplus\limits_{u'=1}^{s'} \mathbb{1}^{s'}_{I't'}(\mathbb{1}^{s'}_{t'u'}B^{[(12)3]4}_{u'I})\Big]_{\mathrm{lt}}\Big]_{\mathrm{lt}} = R_3 \\
S_2 = \mathbb{1}^{s'}\big(B^{(12)(34)}\mathbb{1}^s\big) \arrow[u, "a^{-4}"] \arrow[r, "1\star a^{\boxtimes}"] & \mathbb{1}^{s'}\big(\mathbb{1}^{s'}B^{[(12)3]4}\big) \arrow[u, "a^{-4}"']
\end{tikzcd} \tag{8.10.11}$$

As discussed in the bullet point following (8.10.7), the zigzag (8.10.11) is equivalent to the (I', I)-entry of the bottom pasting diagram in (8.10.5).

The top zigzag in (8.10.8) from Z_1 to S_4 is the following zigzag in C.

$$\begin{tikzcd}
& R_5 = \Big[\bigoplus_{v=1}^{s}\Big\{\bigoplus_{t=1}^{s}\Big[\bigoplus_{u=1}^{s} B^{1[2(34)]}_{I't}\big(\mathbb{1}^{m,npq}_{tu}\mathbb{1}^{s}_{uv}\big)\Big]_{\mathrm{lt}}\Big\}_{\mathrm{lt}}\mathbb{1}^{mnp,q}_{vI}\Big]_{\mathrm{lt}} & \\
S_3 = \big[B^{1[2(34)]}\big(\mathbb{1}^{m,npq}\mathbb{1}^{s}\big)\big]\mathbb{1}^{mnp,q} \arrow[ur, "{[\oplus_v a^{-4}1]_{\mathrm{lt}}}"] \arrow[d, "a^3a^2a^1"'] & & \big[\big(B^{1[2(34)]}\mathbb{1}^{m,npq}\big)\mathbb{1}^{s}\big]\mathbb{1}^{mnp,q} \arrow[ul, "{[\oplus_v (a^3a^2a^1)1]_{\mathrm{lt}}}"'] \\
R_4 = \Big[\bigoplus_{t=1}^{s}\Big[\bigoplus_{u=1}^{s} B^{1[2(34)]}_{I't}\big((\mathbb{1}^{m,npq}\mathbb{1}^{s})_{tu}\mathbb{1}^{mnp,q}_{uI}\big)\Big]_{\mathrm{lt}}\Big]_{\mathrm{lt}} & & \\
Z_1 = B^{1[2(34)]}\big[\big(\mathbb{1}^{m,npq}\mathbb{1}^{s}\big)\mathbb{1}^{mnp,q}\big] \arrow[u, "a^{-4}"] & & \big[\big(B^{1}\mathbb{1}^{m} \boxtimes B^{2(34)}\mathbb{1}^{npq}\big)\mathbb{1}^{s}\big]\mathbb{1}^{mnp,q} = S_4 \arrow[uu, "{[\boxtimes^{-2} \star 1] \star 1}"']
\end{tikzcd} \tag{8.10.12}$$

In (8.10.12), $\boxtimes^{-2}$ comes from the lower left $\boxtimes^{2}$ in the top pasting diagram in (8.10.5). The instances of a^{-1} before $\boxtimes^{-2}$ are needed because of the bracketing of the domain 1-cell (8.10.6).

The right zigzag in (8.10.8) from S_4 to R_8 is the following zigzag in C.

$$\begin{tikzcd}
\big[\big(\mathbb{1}^{m'}B^{1} \boxtimes \mathbb{1}^{n'p'q'}B^{(23)4}\big)\mathbb{1}^{s}\big]\mathbb{1}^{mnp,q} \arrow[r, "{[\boxtimes^{-2} \star 1] \star 1}"] & \big[\big(\mathbb{1}^{m',n'p'q'}B^{1[(23)4]}\big)\mathbb{1}^{s}\big]\mathbb{1}^{mnp,q} \arrow[dd, "{[\oplus_v (a^3a^2a^1)1]_{\mathrm{lt}}}"] \\
S_4 = \big[\big(B^{1}\mathbb{1}^{m} \boxtimes B^{2(34)}\mathbb{1}^{npq}\big)\mathbb{1}^{s}\big]\mathbb{1}^{mnp,q} \arrow[u, "{[(\ell^{-1}r \boxtimes a^{\boxtimes}) \star 1] \star 1}"] & \\
& \Big[\bigoplus_{v=1}^{s}\Big\{\bigoplus_{t'=1}^{s'}\Big[\bigoplus_{t=1}^{s}\mathbb{1}^{m',n'p'q'}_{I't'}\big(B^{1[(23)4]}_{t't}\mathbb{1}^{s}_{tv}\big)\Big]_{\mathrm{lt}}\Big\}_{\mathrm{lt}}\mathbb{1}^{mnp,q}_{vI}\Big]_{\mathrm{lt}} = R_6 \\
& \big[\mathbb{1}^{m',n'p'q'}\big(B^{1[(23)4]}\mathbb{1}^{s}\big)\big]\mathbb{1}^{mnp,q} = S_5 \arrow[u, "{[\oplus_v a^{-4}1]_{\mathrm{lt}}}"'] \arrow[dd, "{[1 \star a^{\boxtimes}] \star 1}"] \\
R_8 = \Big[\bigoplus_{t'=1}^{s'}\Big[\bigoplus_{t=1}^{s}\big(\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'}\big)_{I't'}\big(B^{[1(23)]4}_{t't}\mathbb{1}^{mnp,q}_{tI}\big)\Big]_{\mathrm{lt}}\Big]_{\mathrm{lt}} & \\
& \big[\mathbb{1}^{m',n'p'q'}\big(\mathbb{1}^{s'}B^{[1(23)]4}\big)\big]\mathbb{1}^{mnp,q} \arrow[dd, "{[\oplus_v a^{-4}1]_{\mathrm{lt}}}"] \\
S_6 = \big[\big(\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'}\big)B^{[1(23)]4}\big]\mathbb{1}^{mnp,q} \arrow[uu, "a^3a^2a^1"] \arrow[dr, "{[\oplus_v (a^3a^2a^1)1]_{\mathrm{lt}}}"'] & \\
& \Big[\bigoplus_{v=1}^{s}\Big\{\bigoplus_{t'=1}^{s'}\Big[\bigoplus_{u'=1}^{s'}\mathbb{1}^{m',n'p'q'}_{I't'}\big(\mathbb{1}^{s'}_{t'u'}B^{[1(23)]4}_{u'v}\big)\Big]_{\mathrm{lt}}\Big\}_{\mathrm{lt}}\mathbb{1}^{mnp,q}_{vI}\Big]_{\mathrm{lt}} = R_7
\end{tikzcd} \tag{8.10.13}$$

In (8.10.13), $\ell^{-1}r \boxtimes a^{\boxtimes}$, $\boxtimes^{-2}$, and $a^{\boxtimes}$ are from the left side of the top pasting diagram in (8.10.5).

The bottom zigzag in (8.10.8) from R_8 to Z_2 is the following zigzag in C.

$$\begin{tikzcd}
Z_2 = (\mathbb{1}^{s'}\mathbb{1}^{s'})B^{[(12)3]4} & \Big[\bigoplus_{t'=1}^{s'}\Big[\bigoplus_{t=1}^{s}(\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'})_{I't'}(B^{[1(23)]4}_{t't}\mathbb{1}^{mnp,q}_{tI})\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} = R_8 \\
S_8 = \Big[(\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'})\mathbb{1}^{m'n'p',q'}\Big]B^{[(12)3]4} \arrow[u, "\pi \star 1"] \arrow[dd, "a^3a^2a^1"'] & (\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'})(B^{[1(23)]4}\mathbb{1}^{mnp,q}) \arrow[u, "a^{-4}"'] \\
& (\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'})(B^{1(23)}\mathbb{1}^{mnp} \boxtimes B^4\mathbb{1}^q) = S_7 \arrow[u, "1 \star \boxtimes^{-2}"'] \arrow[dd, "1 \star (a^{\boxtimes} \boxtimes \ell^{-1}r)"] \\
R_9 = \Big[\bigoplus_{t'=1}^{s'}\Big[\bigoplus_{u'=1}^{s'}(\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'})_{I't'}(\mathbb{1}^{m'n'p',q'}_{t'u'}B^{[(12)3]4}_{u'I})\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} & \\
(\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'})(\mathbb{1}^{m'n'p',q'}B^{[(12)3]4}) \arrow[u, "a^{-4}"] & (\mathbb{1}^{m',n'p'q'}\mathbb{1}^{s'})(\mathbb{1}^{m'n'p'}B^{(12)3} \boxtimes \mathbb{1}^{q'}B^4) \arrow[l, bend right, "1 \star \boxtimes^{-2}"']
\end{tikzcd} \tag{8.10.14}$$

In (8.10.14), from R_8 to Z_2, the 2-cells $\boxtimes^{-2}$, $a^{\boxtimes} \boxtimes \ell^{-1}r$, $\boxtimes^{-2}$, and π are from the top and the right parts of the top pasting diagram in (8.10.5).

The morphisms $f_i : R_i \longrightarrow Y$ for $1 \leq i \leq 9$ in (8.10.8) are defined as follows. To define the morphism

$$R_2 = \Big[\bigoplus_{t'=1}^{s'}\Big[\bigoplus_{u=1}^{s}\mathbb{1}^{s'}_{I't'}(B^{(12)(34)}_{t'u}\mathbb{1}^{s}_{uI})\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \xrightarrow{f_2} \big[(B^1_{i'i}B^2_{j'j})B^3_{k'k}\big]B^4_{l'l} = Y,$$

first observe that the summands in R_2 are as follows.

$$\mathbb{1}^{s'}_{I't'}\big(B^{(12)(34)}_{t'u}\mathbb{1}^{s}_{uI}\big) = \begin{cases} \mathbb{0}(B^{(12)(34)}_{t'u}\mathbb{0}) & \text{if } t' \neq I' \text{ and } u \neq I. \\ \mathbb{1}(B^{(12)(34)}_{t'u}\mathbb{0}) & \text{if } t' = I' \text{ and } u \neq I. \\ \mathbb{0}(B^{(12)(34)}_{t'u}\mathbb{1}) & \text{if } t' \neq I' \text{ and } u = I. \\ \mathbb{1}\Big(\big[(B^1_{i'i}B^2_{j'j})(B^3_{k'k}B^4_{l'l})\big]\mathbb{1}\Big) & \text{if } t' = I' \text{ and } u = I. \end{cases} \tag{8.10.15}$$

With the notation in (8.7.6) for morphisms, f_2 is defined as the following composite.

$$\begin{tikzcd}
R_2 \arrow[r, "f_2"] \arrow[dr, "(\lambda^{\oplus},\rho^{\oplus},\lambda^{\otimes},\rho^{\otimes},\lambda^{\bullet},\rho^{\bullet})"'] & \big[(B^1_{i'i}B^2_{j'j})B^3_{k'k}\big]B^4_{l'l} = Y \\
& (B^1_{i'i}B^2_{j'j})(B^3_{k'k}B^4_{l'l}) \arrow[u, "\alpha^{-\otimes}"']
\end{tikzcd}$$

In the morphism f_2, the following statements hold.

- $(\lambda^{\oplus},\rho^{\oplus},\lambda^{\otimes},\rho^{\otimes},\lambda^{\bullet},\rho^{\bullet})$ is a composite of morphisms, each being an iterated sum and product of identity morphisms and one component of one of the indicated structure morphisms in C.
- $\lambda^{\bullet}$ and $\rho^{\bullet}$ are applied in the first three cases of (8.10.15), when at least one of $\mathbb{1}^{s'}_{I't'}$ and $\mathbb{1}^{s}_{uI}$ is $\mathbb{0}$, to reduce that summand to $\mathbb{0}$.
- $\lambda^{\otimes}$ and $\rho^{\otimes}$ are applied in the last case of (8.10.15), when both $\mathbb{1}^{s'}_{I't'}$ and $\mathbb{1}^{s}_{uI}$ are $\mathbb{1}$, to reduce that summand to $(B^1_{i'i}B^2_{j'j})(B^3_{k'k}B^4_{l'l})$.
- $\lambda^{\oplus}$ and $\rho^{\oplus}$ are applied to additively remove all the $\mathbb{0}$'s.
- $\alpha^{-\otimes}$ moves the multiplicative brackets to match with those in Y.

The other 8 morphisms $f_i : R_i \longrightarrow Y$ are defined in almost the same way as f_2. Each such f_i is a composite consisting of

- a morphism of the form $(\lambda^{\oplus}, \rho^{\oplus}, \lambda^{\otimes}, \rho^{\otimes}, \lambda^{\bullet}, \rho^{\bullet})$ that reduces R_i to $B^{1234}_{I'I}$ and
- a morphism of the form
$$(\alpha^{-\otimes}) : B^{1234}_{I'I} \longrightarrow Y$$
that moves the multiplicative brackets to match with those in Y.

Here the superscript in $B^{1234}_{I'I}$ has the bracketing as it appears in R_i. For example, $B^{1234}_{I'I}$ is $B^{[(12)3]4}_{I'I} = Y$ in R_3 and R_9. This finishes the description of the diagram (8.10.8).

To realize the diagram (8.10.8) using paths, we use the set of formal variables

$$X = \left\{0^X, 1^X, b^1_{c'c}, b^2_{d'd}, b^3_{e'e}, b^4_{g'g}\right\} \tag{8.10.16}$$

with the indices as follows.

$$\begin{array}{llll} 1 \le c \le m & 1 \le d \le n & 1 \le e \le p & 1 \le g \le q \\ 1 \le c' \le m' & 1 \le d' \le n' & 1 \le e' \le p' & 1 \le g' \le q' \end{array}$$

Also define the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ B^1_{c'c} & \text{if } x = b^1_{c'c}. \\ B^2_{d'd} & \text{if } x = b^2_{d'd}. \\ B^3_{e'e} & \text{if } x = b^3_{e'e}. \\ B^4_{g'g} & \text{if } x = b^4_{g'g}. \end{cases}$$

To define the desired paths in $\mathsf{Gr}(X)$, in each of (8.10.11), (8.10.12), (8.10.13), and (8.10.14), we perform the following two steps.

(1) In each object, replace
   - $\left\{B^1_{c'c}, B^2_{d'd}, B^3_{e'e}, B^4_{g'g}\right\}$ with the variables $\left\{b^1_{c'c}, b^2_{d'd}, b^3_{e'e}, b^4_{g'g}\right\}$ and
   - $\{\mathbb{0}, \mathbb{1}\}$ in each identity matrix with the variables $\{0^X, 1^X\}$.

(2) Replace each morphism by a corresponding path in $\mathsf{Gr}(X)$ using
   - (8.3.12) for (a^1, a^2, a^3, a^{-4}),
   - Examples 8.2.5 and 8.2.11 for ℓ, r, and $a^{\boxtimes} = \ell^{-1}\alpha^{-\otimes} r$ (8.8.3),
   - Lemma 8.6.16 for $\boxtimes^{-2}$, and
   - Example 8.2.5 and (8.6.9) for π in (8.10.2).

Moreover, perform the same procedure on the object Y to obtain the element

$$y = \left[\left(b^1_{i'i} \otimes b^2_{j'j}\right) \otimes b^3_{k'k}\right] \otimes b^4_{l'l} \in X^{\mathsf{fr}}$$

and on the morphisms f_i for $1 \le i \le 9$ to obtain 9 corresponding paths in $\mathsf{Gr}(X)$.

After the procedure in the previous paragraph, the resulting diagram D in $\mathsf{Gr}(X)$ has precisely the same shape as (8.10.8), with each edge replaced by a path in $\mathsf{Gr}(X)$. Moreover, the image of D in C is the diagram (8.10.8). The vertex y is regular in the sense of Definition 3.1.25. By Lemma 3.1.29, each vertex in D is regular. Since C is assumed to be tight, the Coherence Theorem 3.9.1 implies that the image of D in C, that is, the diagram (8.10.8), is commutative. This finishes the proof of the equality (8.10.5). $\square$

8.11. The 2-Unitors

For a tight symmetric bimonoidal category C as in Definition 2.1.2, we are in the process of constructing a monoidal bicategory structure as in Definition 6.4.1 on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$. In Section 8.10, we constructed the pentagonator π in $\mathsf{Mat}^{\mathsf{C}}$. In this section, we construct the 2-unitors in $\mathsf{Mat}^{\mathsf{C}}$.

- The middle 2-unitor μ is in Definition 8.11.1.
- The left 2-unitor $\lambda^{\boxtimes}$ is in Definition 8.11.6.
- The right 2-unitor $\rho^{\boxtimes}$ is in Definition 8.11.11.

The left and the right 2-unitors are decorated with $\boxtimes$ to avoid confusion with the structure morphisms in the tight symmetric bimonoidal category C.

The Middle 2-Unitor.

Definition 8.11.1. Define μ as consisting of the 2-cells

$$\mu_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn} \quad \text{with} \quad m,n \geq 0$$

given by the composite of the following pasting diagram in $\mathsf{Mat}^{\mathsf{C}}_{mn,mn}$.

(8.11.2)
$$\begin{tikzcd}
(m \boxtimes 1) \boxtimes n \arrow[rr, "a^{\boxtimes}_{m,1,n}"] \arrow[drr, bend left, "\mathbb{1}^m \boxtimes \mathbb{1}^n"] \arrow[drr, "\mathbb{1}^{mn}"'] \arrow[r, phantom, "\Downarrow r_{\mathbb{1}^m \boxtimes \mathbb{1}^n}"] & & m \boxtimes (1 \boxtimes n) \arrow[d, bend left, "\mathbb{1}^m \boxtimes \ell^{\boxtimes}_n"] \\
m \boxtimes n \arrow[u, bend left, "r^{\boxtimes\bullet}_m \boxtimes \mathbb{1}^n"] \arrow[u, bend right, "\mathbb{1}^{mn}"'] \arrow[u, phantom, "\overset{\boxtimes^{-0}_{m,n}}{\Rightarrow}"] \arrow[rr, "\mathbb{1}^{m \boxtimes n}"'] \arrow[r, phantom, "\Downarrow \ell_{\mathbb{1}^{mn}}"] & \arrow[u, phantom, "\Downarrow \boxtimes^{-0}_{m,n}"] & m \boxtimes n
\end{tikzcd}$$

This finishes the definition of μ. ◇

Explanation 8.11.3. Consider the pasting diagram in (8.11.2).

(1) By Definition 8.6.19, each vertex is the object $mn \in \mathsf{Mat}^{\mathsf{C}}$.
(2) By Convention 6.2.12, (8.8.2), (8.9.2), and (8.9.18), the domain 1-cell is the horizontal composite

$$\Big((\mathbb{1}^m \boxtimes \ell^{\boxtimes}_n) a^{\boxtimes}_{m,1,n}\Big)\big(r^{\boxtimes\bullet}_m \boxtimes \mathbb{1}^n\big) = \Big((\mathbb{1}^m \boxtimes \mathbb{1}^n)\mathbb{1}^{mn}\Big)\big(\mathbb{1}^m \boxtimes \mathbb{1}^n\big).$$

The codomain 1-cell is $\mathbb{1}^{m \boxtimes n} = \mathbb{1}^{mn}$.
(3) $\boxtimes^0$ is as in (8.6.9), and $\boxtimes^{-0} = (\boxtimes^0)^{-1}$.
(4) $r_{\mathbb{1}^m \boxtimes \mathbb{1}^n} : (\mathbb{1}^m \boxtimes \mathbb{1}^n)\mathbb{1}^{mn} \longrightarrow \mathbb{1}^m \boxtimes \mathbb{1}^n$ is the base right unitor in (8.2.8).
(5) $\ell_{\mathbb{1}^{mn}} : \mathbb{1}^{mn}\mathbb{1}^{mn} \longrightarrow \mathbb{1}^{mn}$ is the base left unitor in (8.2.2).
(6) Each entry of $\mu_{m,n}$ is a composite of morphisms in C, each being an iterated sum and product of identity morphisms and one component of $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\lambda^{\bullet}$, or $\rho^{\bullet}$. ◇

Lemma 8.11.4. *In Definition 8.11.1, μ is an invertible modification.*

Proof. Each component 2-cell $\mu_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn}$ is well defined by the Bicategorical Pasting Theorem in [**JY21**, 3.6.6]. It is a composite of invertible 2-cells by (8.2.2), (8.2.8), and (8.6.9).

To verify the modification axiom (6.3.2) for μ, consider arbitrary 1-cells

$$(A, B) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'}.$$

If one of m, m', n, and n' is 0, then $\mathsf{Mat}^{\mathsf{C}}_{mn,m'n'}$ is the terminal category, and the modification axiom holds automatically.

Suppose $m, m', n, n' > 0$. We use the abbreviation

$$\mathbb{1}^{m,n} = \mathbb{1}^m \boxtimes \mathbb{1}^n$$

in (8.10.10), $\boxtimes^2$ (8.6.20), $a^{\boxtimes}$ (8.8.3), $\ell^{\boxtimes}$ (8.9.3), and $r^{\boxtimes\bullet}$ (8.9.19). The modification axiom (6.3.2) for μ for the 1-cells (A, B) states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mn,m'n'}$.

(8.11.5)

$$\begin{tikzcd}
 & & mn \arrow[dll, bend right, "\mathbb{1}^{m,n}"'] \arrow[drr, bend left, "A\boxtimes B"] \arrow[d, bend right=50, "(A\boxtimes\mathbb{1})\mathbb{1}^m\boxtimes B\mathbb{1}^n"'] \arrow[d, bend left=50, "\mathbb{1}^{m'}A\boxtimes\mathbb{1}^{n'}B"] \arrow[d, phantom, "\overset{r^{\boxtimes\bullet}_A\boxtimes\ell^{-1}_B r_B}{\Rightarrow}"] & & \\
mn \arrow[rr, bend right=10, "(A\boxtimes\mathbb{1})\boxtimes B"'] \arrow[d, "\mathbb{1}^{mn}"'] \arrow[r, phantom, "\overset{\boxtimes^2}{\Rightarrow}"] & {} & m'n' \arrow[d, "\mathbb{1}^{m'n'}"] & {} \arrow[l, phantom, "\overset{\boxtimes^{-2}}{\Rightarrow}"] & m'n' \arrow[ll, bend left=10, "\mathbb{1}^{m',n'}"] \arrow[ddd, "\mathbb{1}^{m'n'}"] \\
mn \arrow[rr, "A\boxtimes(\mathbb{1}\boxtimes B)"] \arrow[d, "\mathbb{1}^{m,n}"'] \arrow[ddrr, bend left=20, "\mathbb{1}^{m'}A\boxtimes\mathbb{1}^{n'}(\mathbb{1}\boxtimes B)"] \arrow[ddrr, bend right=20, "A\mathbb{1}^m\boxtimes B\mathbb{1}^n"'] \arrow[ur, phantom, "\overset{a^{\boxtimes}_{A,\mathbb{1},B}}{\Rightarrow}"] & {} \arrow[dd, phantom, "\overset{\ell^{-1}_A r_A\boxtimes\ell^{\boxtimes}_B}{\Rightarrow}"] & m'n' \arrow[dd, "\mathbb{1}^{m',n'}"] & {} \arrow[d, phantom, "\overset{\mu_{m',n'}}{\Rightarrow}"] & \\
mn \arrow[drr, bend right, "A\boxtimes B"'] \arrow[r, phantom, "\overset{\boxtimes^2}{\Rightarrow}"] & {} & {} \arrow[l, phantom, "\overset{\boxtimes^{-2}}{\Rightarrow}"] & {} & \\
 & & m'n' & & {} \arrow[ll, bend left=10]
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & & mn \arrow[dll, bend right, "\mathbb{1}^{m,n}"'] \arrow[drr, bend left, "A\boxtimes B"] \arrow[dddll, "\mathbb{1}^{mn}"'] \arrow[dddd, "A\boxtimes B"] & & \\
mn \arrow[d, "\mathbb{1}^{mn}"'] \arrow[r, phantom, "\overset{\mu_{m,n}}{\Rightarrow}"] & {} & & & m'n' \arrow[ddd, "\mathbb{1}^{m'n'}"] \\
mn \arrow[d, "\mathbb{1}^{m,n}"'] & & & {} \arrow[r, phantom, "\overset{\ell^{-1}_{A\boxtimes B}}{\Rightarrow}"] & {} \\
mn \arrow[drr, bend right, "A\boxtimes B"'] \arrow[r, phantom, "\overset{r_{A\boxtimes B}}{\Rightarrow}"] & {} & & & {} \arrow[dll, bend left] \\
 & & m'n' & &
\end{tikzcd}$$

Consider (8.11.5).

- The common domain 1-cell is

$$(A \boxtimes B)\Big[\big((\mathbb{1}^m \boxtimes \mathbb{1}^n)\mathbb{1}^{mn}\big)\big(\mathbb{1}^m \boxtimes \mathbb{1}^n\big)\Big].$$

  The common codomain 1-cell is

$$\mathbb{1}^{m'n'}(A \boxtimes B).$$

- In the top pasting diagram, there are multiple instances of the base associator (8.3.2) and its inverse that are not explicitly displayed.

The proof of the equality (8.11.5) is adapted from that of (8.10.5), which is the modification axiom for the pentagonator π. Since we already showed all the detail in the proof of Lemma 8.10.4, we safely skip the detail here and only outline the remaining steps.

(1) First restrict to a typical (I', I)-entry with the following indices and abbreviations.

$$\begin{array}{llll} 1 \le i \le m & 1 \le j \le n & 1 \le i' \le m' & 1 \le j' \le n' \\ I = (i-1)n + j & & I' = (i'-1)n' + j' & \end{array}$$

In the (I', I)-entry, the two pasting diagrams in (8.11.5) yield a zigzag diagram in C analogous to the diagram (8.10.8), with the object Y in (8.10.9) replaced by the object

$$(A \boxtimes B)_{I'I} = A_{i'i} \otimes B_{j'j} \in \mathsf{C}.$$

The reason that the outer diagram consists of zigzags is that instances of $\boxtimes^2$ and a^4 in (8.3.5) are expressed using their respective inverses, that is., $\boxtimes^{-2}$ and a^{-4}.

(2) The set X of formal variables in (8.10.16) is replaced by

$$X = \left\{0^X, 1^X, a_{c'c}, b_{d'd}\right\}.$$

The function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is defined as follows.

$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{c'c} & \text{if } x = a_{c'c}. \\ B_{d'd} & \text{if } x = b_{d'd}. \end{cases}$$

(3) The procedure in the proof of Lemma 8.10.4 after (8.10.16) yields a diagram D in $\mathsf{Gr}(X)$. Each subdiagram in D consists of two parallel paths with common codomain

$$y = a_{i'i} \otimes b_{j'j} \in X^{\mathsf{fr}}.$$

The image of D in C is the diagram in step (1).

(4) Since C is assumed to be tight, the Coherence Theorem 3.9.1 is applicable in each subdiagram in D because the central vertex y is regular, which implies that each vertex in D is regular.

The commutativity of D in C implies the equality (8.11.5) in the (I', I)-entry. □

The Left 2-Unitor.

Definition 8.11.6. Define $\lambda^{\boxtimes}$ as consisting of the 2-cells

$$\lambda^{\boxtimes}_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn} \quad \text{with} \quad m, n \ge 0$$

given by the composite of the following pasting diagram in $\mathsf{Mat}^{\mathsf{C}}_{mn,mn}$.

(8.11.7)
$$\begin{tikzcd}
(1 \boxtimes m) \boxtimes n \arrow[rr, bend left, "\ell^{\boxtimes}_m \boxtimes \mathbb{1}^n"] \arrow[rr, bend right, "\mathbb{1}^{mn}"'] \arrow[rr, phantom, "\Downarrow \boxtimes^{-0}_{m,n}"] \arrow[dr, bend right, "a^{\boxtimes}_{1,m,n}"'] & & m \boxtimes n \\
& 1 \boxtimes (m \boxtimes n) \arrow[ur, bend right, "\ell^{\boxtimes}_{mn}"'] \arrow[u, phantom, "\Downarrow \ell^{-1}_{\mathbb{1}^{mn}}"] &
\end{tikzcd}$$

This finishes the definition of $\lambda^{\boxtimes}$. ◇

Explanation 8.11.8. Consider the pasting diagram in (8.11.7).

(1) By Definition 8.6.19, each vertex is the object $mn \in \mathsf{Mat}^{\mathsf{C}}$.
(2) By (8.8.2) and (8.9.2), the domain 1-cell is

$$\ell^{\boxtimes}_m \boxtimes \mathbb{1}^n = \mathbb{1}^m \boxtimes \mathbb{1}^n.$$

The codomain 1-cell is the horizontal composite

$$\ell^{\boxtimes}_{mn} a^{\boxtimes}_{1,m,n} = \mathbb{1}^{mn}\mathbb{1}^{mn}.$$

(3) $\boxtimes^0$ is as in (8.6.9), and $\boxtimes^{-0} = (\boxtimes^0)^{-1}$.
(4) $\ell_{\mathbb{1}^{mn}} : \mathbb{1}^{mn}\mathbb{1}^{mn} \longrightarrow \mathbb{1}^{mn}$ is the base left unitor in (8.2.2).
(5) Each entry of $\lambda^{\boxtimes}_{m,n}$ is a composite of morphisms in C, each being an iterated sum and product of identity morphisms and one component of $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\lambda^{\pm\otimes}$, $\lambda^{\pm\bullet}$, or $\rho^{\bullet}$. $\diamond$

Lemma 8.11.9. *In Definition 8.11.6, $\lambda^{\boxtimes}$ is an invertible modification.*

Proof. Each component 2-cell $\lambda^{\boxtimes}_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn}$ is well defined by the Bicategorical Pasting Theorem in **[JY21**, 3.6.6]. It is a composite of invertible 2-cells by (8.2.2) and (8.6.9).

To verify the modification axiom (6.3.2) for $\lambda^{\boxtimes}$, consider arbitrary 1-cells

$$(A,B) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'}.$$

If one of m, m', n, and n' is 0, then $\mathsf{Mat}^{\mathsf{C}}_{mn,m'n'}$ is the terminal category, and the modification axiom holds automatically.

Suppose $m, m', n, n' > 0$. With the notation in the proof of Lemma 8.11.4, the modification axiom (6.3.2) for $\lambda^{\boxtimes}$ for the 1-cells (A,B) states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mn,m'n'}$.

$$\begin{tikzcd}
mn \arrow[rr, "(\mathbb{1}\boxtimes A)\boxtimes B"] \arrow[dd, bend right=50, "\mathbb{1}^{m,n}"'] \arrow[ddrr, "\mathbb{1}^{m'}(\mathbb{1}\boxtimes A)\boxtimes \mathbb{1}^{n'}B" near start] \arrow[ddrr, bend right=30, "A\mathbb{1}^m \boxtimes B\mathbb{1}^n"'] \arrow[dd, phantom, "\overset{\boxtimes^2}{\Rightarrow}"] & & m'n' \arrow[dr, "\mathbb{1}^{m'n'}"] \arrow[dd, bend right=30, "\mathbb{1}^{m',n'}"'] \arrow[dl, phantom, "\overset{\boxtimes^{-2}}{\Rightarrow}"] & \\
 & \overset{\ell^{\boxtimes}_A \boxtimes \ell^{-1}r}{\Rightarrow} & \overset{\lambda^{\boxtimes}_{m',n'}}{\Rightarrow} & m'n' \arrow[dl, "\mathbb{1}^{m'n'}"] \\
mn \arrow[rr, "A\boxtimes B"'] & & m'n' &
\end{tikzcd}
=
\begin{tikzcd}
mn \arrow[rrr, "(\mathbb{1}\boxtimes A)\boxtimes B"] \arrow[dd, bend right=50, "\mathbb{1}^{m,n}"'] \arrow[dr, "\mathbb{1}^{mn}"] & & & m'n' \arrow[d, "\mathbb{1}^{m'n'}"] \\
\overset{\lambda^{\boxtimes}_{m,n}}{\Rightarrow} & mn \arrow[rr, "\mathbb{1}\boxtimes(A\boxtimes B)"] \arrow[dl, "\mathbb{1}^{mn}"] \arrow[ur, phantom, "\overset{a^{\boxtimes}_{\mathbb{1},A,B}}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\ell^{\boxtimes}_{A\boxtimes B}}{\Rightarrow}"] & & m'n' \arrow[d, "\mathbb{1}^{m'n'}"] \\
mn \arrow[rrr, "A\boxtimes B"'] & & & m'n'
\end{tikzcd}
\tag{8.11.10}$$

Consider (8.11.10).

- The common domain 1-cell is

$$(A\boxtimes B)(\mathbb{1}^m \boxtimes \mathbb{1}^n).$$

- The common codomain 1-cell is

$$\big(\mathbb{1}^{m'n'}\mathbb{1}^{m'n'}\big)\big[(\mathbb{1}\boxtimes A)\boxtimes B\big].$$

- In the right pasting diagram, there are instances of the base associator (8.3.2) and its inverse that are not explicitly displayed.

The proof of the equality (8.11.10) is adapted from that of (8.10.5), which is the modification axiom for the pentagonator π, in the proof of Lemma 8.10.4. The remaining steps are outlined in the proof of Lemma 8.11.4. $\square$

The Right 2-Unitor.

Definition 8.11.11. Define $\rho^{\boxtimes}$ as consisting of the 2-cells

$$\rho^{\boxtimes}_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn} \quad \text{with} \quad m,n \geq 0$$

given by the composite of the following pasting diagram in $\mathsf{Mat}^{\mathsf{C}}_{mn,mn}$.

$$\begin{tikzcd}
m \boxtimes n \arrow[rr, bend left, "\mathbb{1}^m \boxtimes r^{\boxtimes\bullet}_n"] \arrow[rr, bend right, "\mathbb{1}^{mn}"'] \arrow[rr, phantom, "\Downarrow \boxtimes^{-0}_{m,n}"] \arrow[dr, bend right, "r^{\boxtimes\bullet}_{mn}"'] & \arrow[d, phantom, "\Downarrow \ell^{-1}_{\mathbb{1}^{mn}}"] & m \boxtimes (n \boxtimes 1) \\
& (m \boxtimes n) \boxtimes 1 \arrow[ur, bend right, "a^{\boxtimes}_{m,n,1}"'] &
\end{tikzcd} \tag{8.11.12}$$

This finishes the definition of $\rho^{\boxtimes}$. ◇

Explanation 8.11.13. Consider the pasting diagram in (8.11.12).

(1) By Definition 8.6.19, each vertex is the object $mn \in \mathsf{Mat}^{\mathsf{C}}$.

(2) By (8.8.2) and (8.9.18), the domain 1-cell is

$$\mathbb{1}^m \boxtimes r^{\boxtimes\bullet}_n = \mathbb{1}^m \boxtimes \mathbb{1}^n.$$

The codomain 1-cell is the horizontal composite

$$a^{\boxtimes}_{m,n,1} r^{\boxtimes\bullet}_{mn} = \mathbb{1}^{mn} \mathbb{1}^{mn}.$$

(3) $\rho^{\boxtimes}_{m,n} = \lambda^{\boxtimes}_{m,n}$ in (8.11.7). However, the modification axioms for $\rho^{\boxtimes}$ and $\lambda^{\boxtimes}$ are different because their (co)domain strong transformations are different. ◇

Lemma 8.11.14. *In Definition 8.11.11, $\rho^{\boxtimes}$ is an invertible modification.*

Proof. This proof is almost identical to that of Lemma 8.11.9, so we only write down the modification axiom (6.3.2) for $\rho^{\boxtimes}$ for the 1-cells (A,B) below.

$$\begin{tikzcd}
mn \arrow[rr, "A \boxtimes B"] \arrow[dd, bend right=50, "\mathbb{1}^{m,n}"'] \arrow[ddrr, bend right, "A\mathbb{1}^m \boxtimes (B \boxtimes \mathbb{1})\mathbb{1}^n"'] \arrow[ddrr, "\mathbb{1}^{m'}A \boxtimes \mathbb{1}^{n'}B"] \arrow[dd, phantom, "\overset{\boxtimes^2}{\Rightarrow}"] & \arrow[d, phantom, "\overset{\boxtimes^{-2}}{\Rightarrow}"] & m'n' \arrow[dd, bend right, "\mathbb{1}^{m',n'}"'] \arrow[dr, bend left, "\mathbb{1}^{m'n'}"] \arrow[d, phantom, "\overset{\ell^{-1}r \boxtimes r^{\boxtimes\bullet}_B}{\Rightarrow}"] & \\
& & \arrow[r, phantom, "\overset{\rho^{\boxtimes}_{m',n'}}{\Rightarrow}"] & m'n' \arrow[dl, bend left, "\mathbb{1}^{m'n'}"] \\
mn \arrow[rr, "A \boxtimes (B \boxtimes 1)"'] & & m'n' &
\end{tikzcd}
=
\begin{tikzcd}
mn \arrow[rr, "A \boxtimes B"] \arrow[dd, bend right=50, "\mathbb{1}^{m,n}"'] \arrow[d, bend left, "\mathbb{1}^{mn}"] \arrow[dd, phantom, "\overset{\rho^{\boxtimes}_{m,n}}{\Rightarrow}"] & \arrow[d, phantom, "\overset{r^{\boxtimes\bullet}_{A \boxtimes B}}{\Rightarrow}"] & m'n' \arrow[d, bend left, "\mathbb{1}^{m'n'}"] \\
mn \arrow[rr, "(A \boxtimes B) \boxtimes \mathbb{1}"] \arrow[d, bend left, "\mathbb{1}^{mn}"] & \arrow[d, phantom, "\overset{a^{\boxtimes}_{A,B,\mathbb{1}}}{\Rightarrow}"] & m'n' \arrow[d, bend left, "\mathbb{1}^{m'n'}"] \\
mn \arrow[rr, "A \boxtimes (B \boxtimes 1)"'] & & m'n'
\end{tikzcd} \tag{8.11.15}$$

The proof of the equality (8.11.15) is adapted from that of (8.10.5) in the proof of Lemma 8.10.4. The remaining steps are outlined in the proof of Lemma 8.11.4. □

8.12. The Matrix Monoidal Bicategory

For a tight symmetric bimonoidal category C, in this section, we show that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory. So far in this chapter, we have defined the following data.

- The matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ is in Definition 8.4.11 and Theorem 8.4.12.
- The monoidal identity $(1_{\boxtimes}, 1^2_{\boxtimes}, 1^0_{\boxtimes})$ is in Definition 8.5.1 and Lemma 8.5.2.

- The monoidal composition $(\boxtimes, \boxtimes^2, \boxtimes^0)$ is in Definitions 8.6.1 and 8.6.19 and Lemma 8.7.31.
- The monoidal associator $(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$ is in Section 8.8.
- The left monoidal unitor $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$ and the right monoidal unitor $(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^r, \varepsilon^r)$ are in Section 8.9.
- The pentagonator π is in Section 8.10.
- The middle 2-unitor μ, the left 2-unitor $\lambda^{\boxtimes}$, and the right 2-unitor $\rho^{\boxtimes}$ are in Section 8.11.

To show that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory, it remains to check the three axioms in Definition 6.4.1.

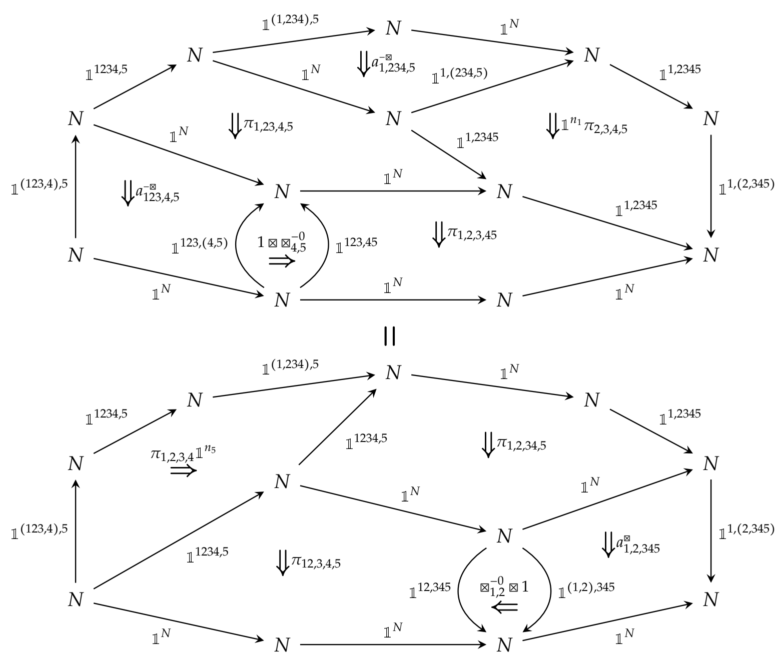
Lemma 8.12.1. $\mathsf{Mat}^{\mathsf{C}}$ *satisfies the non-abelian 4-cocycle condition (6.4.2).*

Proof. Consider arbitrary objects $n_1, n_2, n_3, n_4, n_5 \in \mathsf{Mat}^{\mathsf{C}}$ with $N = n_1n_2n_3n_4n_5$. We use the abbreviations

$$\begin{gathered}
\mathbb{1}^{(123,4),5} = \left(\mathbb{1}^{n_1n_2n_3} \boxtimes \mathbb{1}^{n_4}\right) \boxtimes \mathbb{1}^{n_5} \qquad \mathbb{1}^{1,2345} = \mathbb{1}^{n_1} \boxtimes \mathbb{1}^{n_2n_3n_4n_5} \\
\mathbb{1}^{1,(234,5)} = \mathbb{1}^{n_1} \boxtimes \left(\mathbb{1}^{n_2n_3n_4} \boxtimes \mathbb{1}^{n_5}\right) \qquad \mathbb{1}^{123,45} = \mathbb{1}^{n_1n_2n_3} \boxtimes \mathbb{1}^{n_4n_5} \\
a^{\boxtimes}_{1,2,345} = a^{\boxtimes}_{\mathbb{1}^{n_1},\mathbb{1}^{n_2},\mathbb{1}^{n_3n_4n_5}} \qquad a^{-\boxtimes}_{1,234,5} = \left(a^{\boxtimes}_{\mathbb{1}^{n_1},\mathbb{1}^{n_2n_3n_4},\mathbb{1}^{n_5}}\right)^{-1} \\
\pi_{1,23,4,5} = \pi_{n_1,n_2n_3,n_4,n_5} \qquad \boxtimes^{-0}_{1,2} = \left(\boxtimes^0_{n_1,n_2}\right)^{-1}
\end{gathered} \tag{8.12.2}$$

and so forth. The non-abelian 4-cocycle condition (6.4.2) for $\mathsf{Mat}^{\mathsf{C}}$ for the objects $(n_1, \ldots, n_5)$ states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{N,N}$.

(8.12.3)



Consider (8.12.3).

- The common (co)domain 1-cell has the left normalized bracketing (5.2.13) by Convention 6.2.12.

- As discussed in Explanation 6.4.7, both $\mathbb{1}^{n_1}\pi_{2,3,4,5}$ and $\pi_{1,2,3,4}\mathbb{1}^{n_5}$ involve $\boxtimes^{\pm 2}$ in (8.6.20) and the base left unitor ℓ in (8.2.2).
- There are many instances of the base associator in (8.3.2) and its inverse that are not explicitly displayed in each pasting diagram.

If $N = 0$, then $\mathsf{Mat}^{\mathsf{C}}_{N,N}$ is the terminal category, and the equality (8.12.3) holds automatically.

The proof of the equality (8.12.3) for the case $N > 0$ is adapted from those of Lemmas 8.8.11 and 8.10.4. Since we already showed all the detail in those lemmas, here we only outline the remaining steps.

(1) First restrict to a typical (j,i)-entry with $1 \leq i, j \leq N$. In the (j,i)-entry, the two pasting diagrams in (8.12.3) yield a zigzag diagram in C analogous to the diagram (8.10.8). The object Y in (8.10.9) is now replaced by the object

$$\begin{cases} \mathbb{1} \in \mathsf{C} & \text{if } i = j \text{ and} \\ \mathbb{0} \in \mathsf{C} & \text{if } i \neq j. \end{cases}$$

The diagram involves zigzags because instances of $\boxtimes^2$ and a^4 in (8.3.5) are expressed using their respective inverses, that is, $\boxtimes^{-2}$ and a^{-4}.

(2) The set X in (8.10.16) and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ are now defined as follows.
   - $X = \{0^X, 1^X\}$.
   - $\varphi(0^X) = \mathbb{0}$ and $\varphi(1^X) = \mathbb{1}$.

(3) The procedure in the proof of Lemma 8.10.4 after (8.10.16) yields a diagram D in $\mathsf{Gr}(X)$. Each subdiagram in D consists of two parallel paths with common codomain

$$\begin{cases} 1^X & \text{if } i = j \text{ and} \\ 0^X & \text{if } i \neq j. \end{cases}$$

The image of D in C is the diagram in step (1).

(4) If $i = j$, then the regularity of 1^X and Lemma 3.1.29 imply that each vertex in D is regular. Since C is assumed to be tight, the Coherence Theorem 3.9.1 implies that each subdiagram in D is commutative in C.

(5) If $i \neq j$, then each vertex in D has the same support as 0^X. Proposition 3.5.33 implies that each subdiagram in D is commutative in C.

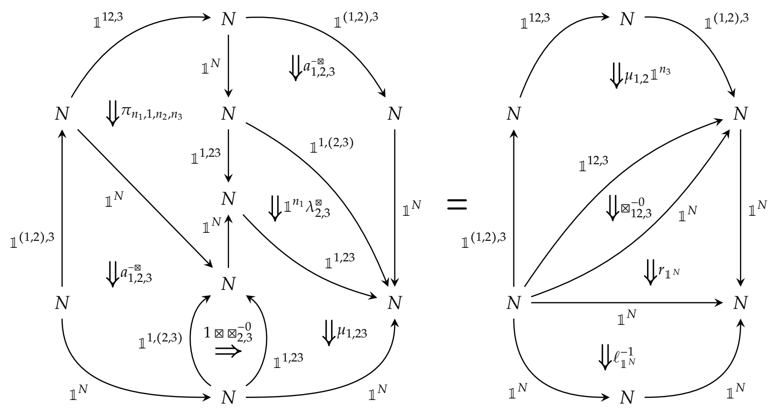
The commutativity of D in C implies the equality (8.12.3) in the (j,i)-entry. $\square$

Lemma 8.12.4. $\mathsf{Mat}^{\mathsf{C}}$ *satisfies the left normalization axiom (6.4.3).*

Proof. Consider arbitrary objects $n_1, n_2, n_3 \in \mathsf{Mat}^{\mathsf{C}}$ with $N = n_1 n_2 n_3$. We use the abbreviations in (8.12.2) and the following.

$$\lambda^{\boxtimes}_{2,3} = \lambda^{\boxtimes}_{n_2,n_3} \qquad \mu_{1,2} = \mu_{n_1,n_2} \qquad \mu_{1,23} = \mu_{n_1,n_2 n_3} \tag{8.12.5}$$

The left normalization axiom (6.4.3) for $\mathsf{Mat}^{\mathsf{C}}$ for the objects (n_1, n_2, n_3) states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{N,N}$.

(8.12.6)


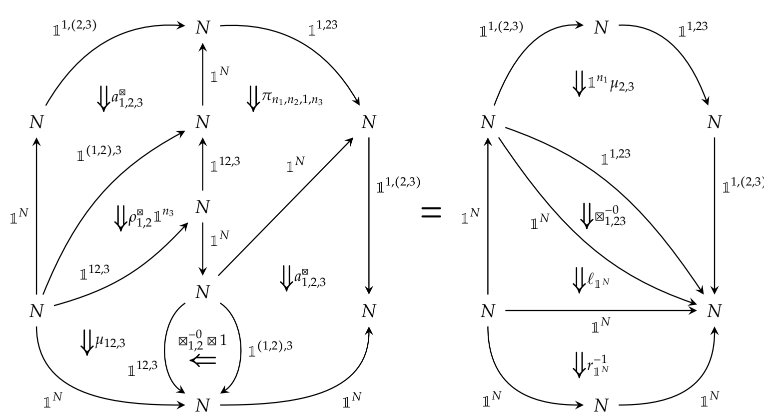
Consider (8.12.6).

- The common domain 1-cell has the left normalized bracketing (5.2.13) by Convention 6.2.12.
- As discussed in Explanation 6.4.7, both $\mathbb{1}^{n_1}\lambda^{\boxtimes}_{2,3}$ and $\mu_{1,2}\mathbb{1}^{n_3}$ involve $\boxtimes^{\pm 2}$ in (8.6.20) and the base left unitor ℓ in (8.2.2).
- There are instances of the base associator in (8.3.2) and its inverse that are not explicitly displayed.

If $N = 0$, then $\mathsf{Mat}^{\mathsf{C}}_{N,N}$ is the terminal category, and the equality (8.12.6) holds automatically. The proof of the equality (8.12.6) for the case $N > 0$ is adapted from those of Lemmas 8.8.11 and 8.10.4, using the steps (1)–(5) outlined in the proof of Lemma 8.12.1. $\square$

Lemma 8.12.7. $\mathsf{Mat}^{\mathsf{C}}$ *satisfies the right normalization axiom (6.4.4).*

Proof. Consider arbitrary objects $n_1, n_2, n_3 \in \mathsf{Mat}^{\mathsf{C}}$ with $N = n_1 n_2 n_3$. We use the abbreviations in (8.12.2), (8.12.5), and $\rho^{\boxtimes}_{1,2} = \rho^{\boxtimes}_{n_1,n_2}$. The right normalization axiom (6.4.4) for $\mathsf{Mat}^{\mathsf{C}}$ for the objects (n_1, n_2, n_3) states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{N,N}$.

(8.12.8)


Consider (8.12.8).

- The common domain 1-cell has the left normalized bracketing (5.2.13) by Convention 6.2.12.
- As discussed in Explanation 6.4.7, both $\rho^{\boxtimes}_{1,2}\mathbb{1}^{n_3}$ and $\mathbb{1}^{n_1}\mu_{2,3}$ involve $\boxtimes^{\pm 2}$ in (8.6.20) and the base left unitor ℓ in (8.2.2).
- There are instances of the base associator in (8.3.2) and its inverse that are not explicitly displayed.

If $N = 0$, then $\mathsf{Mat}^{\mathsf{C}}_{N,N}$ is the terminal category, and the equality (8.12.8) holds automatically. The proof of the equality (8.12.8) for the case $N > 0$ is adapted from those of Lemmas 8.8.11 and 8.10.4, using the steps (1)–(5) outlined in the proof of Lemma 8.12.1. □

We now assemble the results so far in this chapter to obtain the following.

Theorem 8.12.9. *For each tight symmetric bimonoidal category* C*, the matrix bicategory* $\mathsf{Mat}^{\mathsf{C}}$ *is a monoidal bicategory.*

Proof. As noted in the beginning of this section, all the monoidal bicategorical data on $\mathsf{Mat}^{\mathsf{C}}$ have already been constructed in the previous sections. The three axioms in Definition 6.4.1 are proved in Lemmas 8.12.1, 8.12.4, and 8.12.7. □

8.13. The Braiding

For each tight symmetric bimonoidal category C, in Theorem 8.12.9, we observed that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory. The goal for the rest of this chapter is to show that this structure on $\mathsf{Mat}^{\mathsf{C}}$ extends to a *symmetric* monoidal bicategory as in Definition 6.5.9. Recall the following concepts.

- A symmetric monoidal bicategory is a sylleptic monoidal bicategory as in Definition 6.5.7 that satisfies the triple braid axiom (6.5.10).
- A sylleptic monoidal bicategory is a braided monoidal bicategory as in Definition 6.5.3 together with an invertible modification ν, which is called the syllepsis, that satisfies the two axioms in Definition 6.5.7.

Therefore, our next task is to construct a braided monoidal bicategory structure on $\mathsf{Mat}^{\mathsf{C}}$. In this section, we construct the braiding $(\beta, \beta^{\bullet}, \eta^{\beta}, \varepsilon^{\beta})$ in the monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$.

From Explanation 6.5.6, the braiding consists of the following data.

(i) β and $\beta^{\bullet}$ are strong transformations as in

$$\begin{tikzcd} \boxtimes \arrow[r, shift left, "\beta"] & \boxtimes\tau \arrow[l, shift left, "\beta^{\bullet}"] \end{tikzcd}$$

in which $\tau : \left(\mathsf{Mat}^{\mathsf{C}}\right)^2 \longrightarrow \left(\mathsf{Mat}^{\mathsf{C}}\right)^2$ switches the two arguments.

(ii) η^{β} and ε^{β} are invertible modifications as follows.

$$1_{\boxtimes} \xrightarrow{\eta^{\beta}} \beta^{\bullet}\beta$$
$$\beta\beta^{\bullet} \xrightarrow{\varepsilon^{\beta}} 1_{\boxtimes\tau}$$

Moreover, these data satisfy the triangle identities (6.3.10).

Motivation 8.13.1. Let us motivate the construction of the braiding β. For complex matrices A and B, the two matrix tensor products $A \otimes B$ and $B \otimes A$ are equal up to a permutation of the entries. A permutation matrix is obtained from an identity matrix by permuting its columns. It has precisely one instance of 1 in each row and in each column, and every other entry is 0. There are permutation matrices P^1 and P^2 such that

$$A \otimes B = P^2(B \otimes A)P^1. \tag{8.13.2}$$

Here P^1 permutes the columns of $B \otimes A$, and P^2 permutes the rows.

The identity matrix $\mathbb{1}^n \in \mathsf{Mat}^{\mathsf{C}}_{n,n}$ makes sense in the matrix bicategory, and so do permutation matrices. For 1-cells A and B in $\mathsf{Mat}^{\mathsf{C}}$, the matrix tensor products $A \boxtimes B$ and $B \boxtimes A$, with entries in (8.6.4), differ by

- a permutation of the entries as in the previous paragraph and
- an entrywise multiplicative symmetry

$$A_{ji} \otimes B_{lk} \xrightarrow{\xi^{\otimes}} B_{lk} \otimes A_{ji}.$$

However, even with the $\xi^{\otimes}$ taken into account, the equality (8.13.2) does not hold strictly in $\mathsf{Mat}^{\mathsf{C}}$ in general. The reason is that $\mathbb{0}$ and $\mathbb{1}$ in C are not strict zero and strict unit, but only up to natural isomorphisms as in Definition 2.1.2. The analogue of the equality (8.13.2) is an invertible 2-cell $\beta_{A,B}$ that is a component 2-cell of the braiding β. Component 1-cells of β are analogues of the permutation matrices P^1 and P^2 in (8.13.2). $\diamond$

The rest of this section is organized as follows.

- Before we define the braiding, first we define row permutation, column permutation, and permutation matrices in $\mathsf{Mat}^{\mathsf{C}}$ in Definition 8.13.3, and discuss some of their properties.
- The left adjoint β, the right adjoint $\beta^{\bullet}$, the unit η^{β}, and the counit ε^{β} are in Definitions 8.13.22, 8.13.30, 8.13.35, and 8.13.38.

Permutation Matrices. We now begin by defining row permutation, column permutation, and permutation matrices in the setting of $\mathsf{Mat}^{\mathsf{C}}$. Recall that Σ_m denotes the symmetric group on m letters.

Definition 8.13.3. For $m, n \geq 0$, suppose $\sigma \in \Sigma_m$, $\theta \in \Sigma_n$, and $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.

- The *row permutation of A by θ*, denoted by ${}_{\theta}A$, is obtained from A by permuting its n rows by θ. Its entries are determined by

$$({}_{\theta}A)_{\theta(j),i} = A_{ji} \tag{8.13.4}$$

  for $1 \leq i \leq m$ and $1 \leq j \leq n$.
- The *column permutation of A by σ*, denoted by A^{σ}, is obtained from A by permuting its m columns by σ. Its entries are determined by

$$(A^{\sigma})_{j,\sigma(i)} = A_{ji}. \tag{8.13.5}$$

- Define the *permutation matrix of σ* as

$$\mathbb{1}^{\sigma} = (\mathbb{1}^m)^{\sigma}. \tag{8.13.6}$$

It is the column permutation of the $m \times m$ identity matrix $\mathbb{1}^m$ in (8.1.6) by σ. Its entries are given by

$$\mathbb{1}^{\sigma}_{ij} = \begin{cases} \mathbb{1} & \text{if } j = \sigma(i) \text{ and} \\ \mathbb{0} & \text{if } j \neq \sigma(i). \end{cases}$$

The notation in (8.13.4) and (8.13.5) also apply if A is a 2-cell in $\mathsf{Mat}^{\mathsf{C}}_{m,n}$. ◇

To define the (co)unit of the braiding, we will use the following preliminary observation. The transpose of a matrix A is denoted by A^T.

Lemma 8.13.7. *In the setting of Definition 8.13.3, the following statements hold.*

(1) *For $\sigma' \in \Sigma_m$ and $\theta' \in \Sigma_n$, the following equalities hold.*

$$\begin{aligned} {}_{\theta'}({}_{\theta}A) &= {}_{\theta'\theta}A \\ (A^{\sigma})^{\sigma'} &= A^{\sigma'\sigma} \end{aligned} \tag{8.13.8}$$

(2) $(\mathbb{1}^{\sigma})^T = \mathbb{1}^{\sigma^{-1}}$.

(3) *For $1 \le i \le m$, the ith diagonal entry in $\mathbb{1}^{\sigma^{-1}}\mathbb{1}^{\sigma}$ is*

$$\left(\mathbb{1}^{\sigma^{-1}}\mathbb{1}^{\sigma}\right)_{ii} = \left\{\underbrace{\left[\bigoplus_{k=1}^{\sigma^{-1}(i)-1} \mathbb{0}\mathbb{0}\right]}_{\text{empty if } \sigma^{-1}(i)=1} \oplus (\mathbb{1}\mathbb{1}) \oplus \underbrace{\left[\bigoplus_{k=\sigma^{-1}(i)+1}^{m} \mathbb{0}\mathbb{0}\right]}_{\text{empty if } \sigma^{-1}(i)=m}\right\}_{\mathsf{lt}}. \tag{8.13.9}$$

(4) *Each off-diagonal entry in $\mathbb{1}^{\sigma^{-1}}\mathbb{1}^{\sigma}$ is a left normalized sum of m objects as in either*

$$\begin{gathered} \left[\mathbb{0}\mathbb{0} \oplus \cdots \oplus \mathbb{1}\mathbb{0} \oplus \cdots \oplus \mathbb{0}\mathbb{1} \oplus \cdots \oplus \mathbb{0}\mathbb{0}\right]_{\mathsf{lt}} \quad \text{or} \\ \left[\mathbb{0}\mathbb{0} \oplus \cdots \oplus \mathbb{0}\mathbb{1} \oplus \cdots \oplus \mathbb{1}\mathbb{0} \oplus \cdots \oplus \mathbb{0}\mathbb{0}\right]_{\mathsf{lt}}. \end{gathered} \tag{8.13.10}$$

In each case, there is one instance of each of $\mathbb{1}\mathbb{0}$ and $\mathbb{0}\mathbb{1}$, and every other summand is $\mathbb{0}\mathbb{0}$.

Proof. The equalities in (8.13.8) follow from the definitions (8.13.4) and (8.13.5).

For assertion (2), the transpose $(\mathbb{1}^{\sigma})^T$ has $\mathbb{1}$ precisely in the $(\sigma(i), i)$-entries for $1 \le i \le m$, that is, the $(i, \sigma^{-1}(i))$-entries. These entries are where $\mathbb{1}^{\sigma^{-1}}$ has $\mathbb{1}$.

The last two assertions follow from assertion (2) and (8.1.4). □

Component 2-cells of the braiding β involve the natural isomorphisms in Lemmas 8.13.12 and 8.13.15 below.

Motivation 8.13.11. Suppose for the moment that $\mathbb{0}$ and $\mathbb{1}$ in C are strict zero and strict unit. This is true, for example, if C is a right bipermutative category or a left bipermutative category as in Definitions 2.5.2 and 2.5.11. Then the matrix product $A\mathbb{1}^{\sigma}$ is equal to A^{σ}, the column permutation of A by σ. In the general case, when $\mathbb{0}$ and $\mathbb{1}$ are not strict zero and strict unit, $A\mathbb{1}^{\sigma}$ and A^{σ} are connected by a natural isomorphism as in the next lemma. Similar remarks apply to $\mathbb{1}^{\theta}A$ and ${}_{\theta^{-1}}A$. ◇

Lemma 8.13.12. *For $\sigma \in \Sigma_m$ and $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, there is a natural isomorphism*

$$A\mathbb{1}^{\sigma} \xrightarrow[\cong]{r^{\sigma}_A} A^{\sigma} \tag{8.13.13}$$

in $\mathsf{Mat}^{\mathsf{C}}_{m,n}$.

Proof. If either m or n is 0, then $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ is the terminal category, and r^{σ}_A is the identity morphism of the empty matrix.

For $m,n > 0$, $1 \le i \le m$, and $1 \le j \le n$, the (j,i)-entry of $A\mathbb{1}^{\sigma}$ is as follows.

$$\begin{aligned}(A\mathbb{1}^{\sigma})_{ji} &= \Big[\bigoplus_{k=1}^{m} A_{jk}\mathbb{1}^{\sigma}_{ki}\Big]_{\mathsf{lt}}\\ &= \Big\{\underbrace{\Big[\bigoplus_{k=1}^{\sigma^{-1}(i)-1} A_{jk}\mathbb{0}\Big]}_{\text{empty if } \sigma^{-1}(i)=1} \oplus \big(A_{j,\sigma^{-1}(i)}\mathbb{1}\big) \oplus \underbrace{\Big[\bigoplus_{k=\sigma^{-1}(i)+1}^{m} A_{jk}\mathbb{0}\Big]}_{\text{empty if } \sigma^{-1}(i)=m}\Big\}_{\mathsf{lt}}\end{aligned}$$

This object is isomorphic to A^{σ}_{ji} via the following composite in C.

$$\begin{tikzcd}[column sep=large]
(A\mathbb{1}^{\sigma})_{ji} \arrow[rr, "(r^{\sigma}_A)_{ji}"] \arrow[d, "(\rho^{\bullet}\oplus\cdots\oplus\rho^{\bullet}\oplus\rho^{\otimes}\oplus\rho^{\bullet}\oplus\cdots\oplus\rho^{\bullet})_{\mathsf{lt}}"'] & & A^{\sigma}_{ji} \arrow[d, equal] \\
\Big(\underbrace{\mathbb{0}\oplus\cdots\oplus\mathbb{0}\oplus}_{\sigma^{-1}(i)-1 \text{ copies of } \mathbb{0}} A_{j,\sigma^{-1}(i)} \underbrace{\oplus\mathbb{0}\oplus\cdots\oplus\mathbb{0}}_{m-\sigma^{-1}(i) \text{ copies of } \mathbb{0}}\Big)_{\mathsf{lt}} \arrow[rr, "Z"] & & A_{j,\sigma^{-1}(i)}
\end{tikzcd} \tag{8.13.14}$$

If $m = 1$, then Z is the identity morphism. If $m > 1$, then Z is a composite of $m-1$ morphisms, each being the sum of identity morphisms and one component of $\lambda^{\oplus}$ or $\rho^{\oplus}$.

The naturality of r^{σ}_A with respect to A follows from the naturality of $\rho^{\bullet}$, $\rho^{\otimes}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$, and the functoriality of $\oplus$ in C. $\square$

In the next lemma, note that ${}_{\theta^{-1}}A$ is the row permutation of A by θ^{-1}.

Lemma 8.13.15. *For $\theta \in \Sigma_n$ and $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, there is a natural isomorphism*

$$\mathbb{1}^{\theta}A \xrightarrow[\cong]{\ell^{\theta}_A} {}_{\theta^{-1}}A \tag{8.13.16}$$

in $\mathsf{Mat}^{\mathsf{C}}_{m,n}$.

Proof. If either m or n is 0, then $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ is the terminal category, and ℓ^{θ}_A is the identity morphism of the empty matrix.

For $m,n > 0$, $1 \le i \le m$, and $1 \le j \le n$, the (j,i)-entry of $\mathbb{1}^{\theta}A$ is as follows.

$$\begin{aligned}\big(\mathbb{1}^{\theta}A\big)_{ji} &= \Big[\bigoplus_{l=1}^{n} \mathbb{1}^{\theta}_{jl}A_{li}\Big]_{\mathsf{lt}}\\ &= \Big\{\underbrace{\Big[\bigoplus_{l=1}^{\theta(j)-1} \mathbb{0}A_{li}\Big]}_{\text{empty if } \theta(j)=1} \oplus \big(\mathbb{1}A_{\theta(j),i}\big) \oplus \underbrace{\Big[\bigoplus_{l=\theta(j)+1}^{n} \mathbb{0}A_{li}\Big]}_{\text{empty if } \theta(j)=n}\Big\}_{\mathsf{lt}}\end{aligned}$$

This object is isomorphic to $\big({}_{\theta^{-1}}A\big)_{ji}$ via the following composite in C.

$$
\begin{tikzcd}
(\mathbb{1}^{\theta}A)_{ji} \arrow[rr, "(\ell^{\theta}_{A})_{ji}"] \arrow[d, "\left(\lambda^{\bullet}\oplus\cdots\oplus\lambda^{\bullet}\oplus\lambda^{\otimes}\oplus\lambda^{\bullet}\oplus\cdots\oplus\lambda^{\bullet}\right)_{\mathsf{lt}}"'] && \big({}_{\theta^{-1}}A\big)_{ji} \arrow[d, equal] \\
\Big(\underbrace{\mathbb{0}\oplus\cdots\oplus\mathbb{0}\oplus}_{\theta(j)-1 \text{ copies of } \mathbb{0}} A_{\theta(j),i} \underbrace{\oplus\mathbb{0}\oplus\cdots\oplus\mathbb{0}}_{n-\theta(j) \text{ copies of } \mathbb{0}}\Big)_{\mathsf{lt}} \arrow[rr, "Z"] && A_{\theta(j),i}
\end{tikzcd}
\tag{8.13.17}
$$

If $n = 1$, then Z is the identity morphism. If $n > 1$, then Z is a composite of $n-1$ morphisms, each being the sum of identity morphisms and one component of $\lambda^{\oplus}$ or $\rho^{\oplus}$.

The naturality of ℓ^{θ}_{A} with respect to A follows from the naturality of $\lambda^{\bullet}$, $\lambda^{\otimes}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$, and the functoriality of $\oplus$ in C. □

Remark 8.13.18. In Lemma 8.13.12, if σ is the identity permutation id_m, then $\mathbb{1}^{\mathrm{id}_m} = \mathbb{1}^m$, and $r^{\mathrm{id}_m} = r$, which is the base right unitor in (8.2.8). In Lemma 8.13.15, if θ is the identity permutation id_n, then $\mathbb{1}^{\mathrm{id}_n} = \mathbb{1}^n$, and $\ell^{\mathrm{id}_n} = \ell$, which is the base left unitor in (8.2.2). ⋄

Denote by $\tau_{m,n} \in \Sigma_{mn}$ the permutation in (2.4.5). It rearranges n intervals of m objects each to m intervals of n objects each. Note that

$$
(\tau_{m,n})^{-1} = \tau_{n,m}. \tag{8.13.19}
$$

The next lemma makes precise the two bullet points in Motivation 8.13.1. Namely, the matrix tensor products $A \boxtimes B$ and $B \boxtimes A$ differ by a column permutation, a row permutation, and a multiplicative symmetry in each entry.

Lemma 8.13.20. *Suppose $A \in \mathsf{Mat}^{\mathsf{C}}_{m,m'}$ and $B \in \mathsf{Mat}^{\mathsf{C}}_{n,n'}$ are either both 1-cells or both 2-cells. Then there are equalities*

$$
\begin{aligned}
(B \boxtimes A)^{\tau_{m,n}}_{i'+(j'-1)m',\,j+(i-1)n} &= B_{j'j} \otimes A_{i'i} \\
\big({}_{\tau_{n',m'}}(A \boxtimes B)\big)_{i'+(j'-1)m',\,j+(i-1)n} &= A_{i'i} \otimes B_{j'j}
\end{aligned}
$$

for $1 \le i \le m$, $1 \le j \le n$, $1 \le i' \le m'$, and $1 \le j' \le n'$.

Proof. The first equality follows from (2.4.5), (8.13.5), and (8.6.4) as follows.

$$
\begin{aligned}
(B \boxtimes A)^{\tau_{m,n}}_{i'+(j'-1)m',\,j+(i-1)n} &= (B \boxtimes A)^{\tau_{m,n}}_{i'+(j'-1)m',\,\tau_{m,n}(i+(j-1)m)} \\
&= (B \boxtimes A)_{i'+(j'-1)m',\,i+(j-1)m} \\
&= B_{j'j} \otimes A_{i'i}
\end{aligned}
$$

The second equality follows similarly from (2.4.5), (8.13.4), and (8.6.4) as follows.

$$
\begin{aligned}
\big({}_{\tau_{n',m'}}(A \boxtimes B)\big)_{i'+(j'-1)m',\,j+(i-1)n} &= \big({}_{\tau_{n',m'}}(A \boxtimes B)\big)_{\tau_{n',m'}(j'+(i'-1)n'),\,j+(i-1)n} \\
&= (A \boxtimes B)_{j'+(i'-1)n',\,j+(i-1)n} \\
&= A_{i'i} \otimes B_{j'j}
\end{aligned}
$$

This finishes the proof of the lemma. □

Explanation 8.13.21. There is a more conceptual way to represent the two matrices in Lemma 8.13.20. Suppose $A_{\bullet i} \in \mathsf{Mat}^{\mathsf{C}}_{m',1}$ is the ith column of A for $1 \le i \le m$. Then

$$(B \boxtimes A)^{\tau_{m,n}} = \left[\begin{array}{c|c|c} B \boxtimes A_{\bullet 1} & \cdots & B \boxtimes A_{\bullet m} \end{array}\right].$$

From left to right, the ith block is $B \boxtimes A_{\bullet i}$. In linear algebra, this column-wise matrix tensor product is an example of the Khatri-Rao product [**KR68**], which is related to the Tracy-Singh product [**Liu99, TS72**].

Similarly, suppose $B_{j'\bullet} \in \mathsf{Mat}^{\mathsf{C}}_{1,n}$ is the j'th row of B for $1 \le j' \le n'$. Then

$${}_{\tau_{n',m'}}(A \boxtimes B) = \left[\begin{array}{c} A \boxtimes B_{1\bullet} \\ \hline \vdots \\ \hline A \boxtimes B_{n'\bullet} \end{array}\right].$$

From top to bottom, the j'th block is $A \boxtimes B_{j'\bullet}$. In linear algebra, this row-wise matrix tensor product is an example of the face product in [**Sly99**]. $\diamond$

The Left Adjoint of the Braiding. Recall the monoidal composition $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in Definition 8.6.19 and Lemma 8.7.31.

Definition 8.13.22. Define the data of a lax transformation

$$\boxtimes \xrightarrow{\ \beta\ } \boxtimes\tau$$

as follows, in which $\tau : \left(\mathsf{Mat}^{\mathsf{C}}\right)^2 \longrightarrow \left(\mathsf{Mat}^{\mathsf{C}}\right)^2$ switches the two arguments.

Component 1-Cells: For each pair of objects $(m, n) \in \left(\mathsf{Mat}^{\mathsf{C}}\right)^2$, define

$$mn = m \boxtimes n \xrightarrow{\ \beta_{m,n}\ } n \boxtimes m = nm$$

as the permutation matrix

$$\beta_{m,n} = \mathbb{1}^{\tau_{m,n}} \in \mathsf{Mat}^{\mathsf{C}}_{mn,nm} \tag{8.13.23}$$

in (8.13.6), with $\tau_{m,n} \in \Sigma_{mn}$ the permutation in (2.4.5).

Component 2-Cells: For 1-cells $A \in \mathsf{Mat}^{\mathsf{C}}_{m,m'}$ and $B \in \mathsf{Mat}^{\mathsf{C}}_{n,n'}$, define the component 2-cell

$$\beta_{A,B} \in \mathsf{Mat}^{\mathsf{C}}_{mn,n'm'}\big((B \boxtimes A)\beta_{m,n};\, \beta_{m',n'}(A \boxtimes B)\big)$$

as the following vertical composite.

$$\begin{tikzcd}[column sep=huge]
& nm \arrow[dr, bend left, "B \boxtimes A"] \arrow[d, phantom, "\Downarrow r^{\tau_{m,n}}_{B\boxtimes A}"] & \\
mn \arrow[ur, bend left, "\beta_{m,n}"] \arrow[rr, bend left=10, "(B \boxtimes A)^{\tau_{m,n}}"] \arrow[rr, bend right=10, "{}_{\tau_{n',m'}}(A \boxtimes B)"'] \arrow[rr, phantom, "\Downarrow \xi^{\otimes}"] \arrow[dr, bend right, "A \boxtimes B"'] & {} \arrow[d, phantom, "\Downarrow (\ell^{\tau_{m',n'}}_{A\boxtimes B})^{-1}"] & n'm' \\
& m'n' \arrow[ur, bend right, "\beta_{m',n'}"'] &
\end{tikzcd} \tag{8.13.24}$$

- $r^{\tau_{m,n}}_{B\boxtimes A}$ is the natural isomorphism in (8.13.13).
- $\ell^{\tau_{m',n'}}_{A\boxtimes B}$ is the natural isomorphism in (8.13.16), along with (8.13.19).

- $\xi^{\otimes}$ is the 2-cell with $\big(i' + (j'-1)m', j + (i-1)n\big)$-entry the structure morphism in C,

$$B_{j'j} \otimes A_{i'i} \xrightarrow{\xi^{\otimes}_{B_{j'j},A_{i'i}}} A_{i'i} \otimes B_{j'j}$$

  for $1 \le i \le m$, $1 \le j \le n$, $1 \le i' \le m'$, and $1 \le j' \le n'$.

This finishes the definition of β. ◇

Explanation 8.13.25. Consider Definition 8.13.22.

(1) The entries of $\beta_{m,n} = \mathbb{1}^{\tau_{m,n}}$ are determined as follows.

$$(\beta_{m,n})_{pq} = \begin{cases} \mathbb{1} & \text{if } (p,q) = \big(i+(j-1)m, j+(i-1)n\big) \\ & \text{ for some } 1 \le i \le m \text{ and } 1 \le j \le n \text{, and} \\ \mathbb{0} & \text{otherwise.} \end{cases} \tag{8.13.26}$$

(2) By (8.13.14) and (8.13.17), each entry of $\beta_{A,B}$ in (8.13.24) is a composite of isomorphisms, each being an iterated sum of identity morphisms and at most one component of $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\lambda^{-\otimes}$, $\rho^{\otimes}$, $\xi^{\otimes}$, $\lambda^{-\bullet}$, or $\rho^{\bullet}$ in C.
(3) The 2-cell $\xi^{\otimes}$ in (8.13.24) is well defined by Lemma 8.13.20.
(4) There is a strong similarity between β and $a^{\boxtimes}$ in Definition 8.8.1. Since the proofs in Section 8.8 for $a^{\boxtimes}$ have all the detail, we will safely skip most of the detail below. ◇

Lemma 8.13.27. *In Definition 8.13.22, $\beta : \boxtimes \longrightarrow \boxtimes\tau$ is a strong transformation.*

Proof. Similar to Lemma 8.8.5, β in (8.13.24) is a natural isomorphism because the structure isomorphisms $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\xi^{\otimes}$, $\lambda^{\bullet}$, and $\rho^{\bullet}$ in C are natural.

The lax unity axiom (6.2.15) for β states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mn,nm}$ for $m,n \ge 0$.

$$\begin{tikzcd} mn \arrow[r, bend left, "\mathbb{1}^m \boxtimes \mathbb{1}^n"] \arrow[d, "\beta_{m,n}"'] \arrow[dr, phantom, "\Uparrow\beta_{\mathbb{1}^m,\mathbb{1}^n}"] & mn \arrow[d, "\beta_{m,n}"] \\ nm \arrow[r, bend left, "\mathbb{1}^n \boxtimes \mathbb{1}^m"] \arrow[r, bend right, "\mathbb{1}^{nm}"'] \arrow[r, phantom, "\Uparrow\boxtimes^0_{(n,m)}"] & nm \end{tikzcd} \quad = \quad \begin{tikzcd} mn \arrow[r, bend left, "\mathbb{1}^m \boxtimes \mathbb{1}^n"] \arrow[r, bend right, "\mathbb{1}^{mn}"'] \arrow[r, phantom, "\Uparrow\boxtimes^0_{(m,n)}"] \arrow[d, "\beta_{m,n}"'] \arrow[dr, "\beta_{m,n}" description] & mn \arrow[d, "\beta_{m,n}"] \arrow[dl, phantom, "\Uparrow r^{-1}_{\beta_{m,n}}"] \\ nm \arrow[r, bend right, "\mathbb{1}^{nm}"'] \arrow[r, phantom, "\Uparrow\ell_{\beta_{m,n}}"] & nm \end{tikzcd} \tag{8.13.28}$$

The proof of this equality is adapted from that of (8.8.12), which is the lax unity axiom for $a^{\boxtimes}$. Instead of diagonal entries, here we consider entries of the form

$$\big(i+(j-1)m, j+(i-1)n\big) \quad \text{for} \quad 1 \le i \le m \quad \text{and} \quad 1 \le j \le n.$$

These entries are where $\mathbb{1}$ appears in $\beta_{m,n}$ (8.13.26). Instead of off-diagonal entries, here we consider entries where $\mathbb{0}$ appears in $\beta_{m,n}$.

The lax naturality axiom (6.2.16) for β states that for 1-cells

$$m \xrightarrow{A} n \xrightarrow{B} p$$
$$m' \xrightarrow{A'} n' \xrightarrow{B'} p'$$

in $\mathsf{Mat}^{\mathsf{C}}$, the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mm',p'p}$ holds.

$$
\begin{tikzcd}[column sep=large]
mm' \arrow[rr, bend left, "BA \boxtimes B'A'"] \arrow[d, "\beta_{m,m'}"'] \arrow[drr, phantom, "\Uparrow \beta_{BA,B'A'}" near start] & & pp' \arrow[d, "\beta_{p,p'}"] \\
m'm \arrow[rr, bend left=40, "B'A' \boxtimes BA"] \arrow[r, bend right, "A' \boxtimes A"'] \arrow[rr, phantom, "\Uparrow \boxtimes^2"] & n'n \arrow[r, bend right, "B' \boxtimes B"'] & p'p
\end{tikzcd}
=
\begin{tikzcd}[column sep=large]
mm' \arrow[rr, bend left=40, "BA \boxtimes B'A'"] \arrow[rr, phantom, "\Uparrow \boxtimes^2", shift left=3] \arrow[r, bend right, "A \boxtimes A'"'] \arrow[d, "\beta_{m,m'}"'] \arrow[dr, phantom, "\overset{\beta_{A,A'}}{\Rightarrow}"] & nn' \arrow[r, bend right, "B \boxtimes B'"'] \arrow[d, "\beta_{n,n'}"'] \arrow[dr, phantom, "\overset{\beta_{B,B'}}{\Rightarrow}"] & pp' \arrow[d, "\beta_{p,p'}"] \\
m'm \arrow[r, bend right, "A' \boxtimes A"'] & n'n \arrow[r, bend right, "B' \boxtimes B"'] & p'p
\end{tikzcd}
\tag{8.13.29}
$$

The proof of this equality is adapted from that of (8.8.16), which is the lax naturality axiom for $a^{\boxtimes}$, in Lemmas 8.8.17 and 8.8.26. Instead of the subdivided diagram (8.8.28), here we have a zigzag diagram analogous to (8.10.8). The object Y in (8.10.9) is replaced by the following object for $1 \le i \le m$, $1 \le i' \le m'$, $1 \le k \le p$, and $1 \le k' \le p'$.

$$
\begin{aligned}
&\Big[\big((B' \boxtimes B)(A' \boxtimes A)\big)^{\tau_{m,m'}}\Big]_{(k+(k'-1)p,\, i'+(i-1)m')}\\
&= \Big[\big((B' \boxtimes B)(A' \boxtimes A)\big)^{\tau_{m,m'}}\Big]_{(k+(k'-1)p,\, \tau_{m,m'}(i+(i'-1)m))}\\
&= \big[(B' \boxtimes B)(A' \boxtimes A)\big]_{(k+(k'-1)p,\, i+(i'-1)m)}\\
&= \Big[\bigoplus_{j'=1}^{n'} \bigoplus_{j=1}^{n} (B'_{k'j'} B_{kj})(A'_{j'i'} A_{ji})\Big]_{\mathsf{lt}}
\end{aligned}
$$

This object is a typical entry of the column permutation of $(B' \boxtimes B)(A' \boxtimes A)$ by $\tau_{m,m'}$. The three equalities above follow from, respectively, (2.4.5), (8.13.5), and (8.1.4) and (8.6.4). $\square$

The Right Adjoint of the Braiding.

Definition 8.13.30. Define the data of a lax transformation

$$\boxtimes\tau \xrightarrow{\;\beta^{\bullet}\;} \boxtimes$$

as follows, in which $\tau : \big(\mathsf{Mat}^{\mathsf{C}}\big)^2 \longrightarrow \big(\mathsf{Mat}^{\mathsf{C}}\big)^2$ switches the two arguments.

Component 1-Cells: For each pair of objects $(m,n) \in \big(\mathsf{Mat}^{\mathsf{C}}\big)^2$, define

$$nm = n \boxtimes m \xrightarrow{\;\beta^{\bullet}_{m,n}\;} m \boxtimes n = mn \tag{8.13.31}$$

as the permutation matrix

$$\beta^{\bullet}_{m,n} = \beta_{n,m} = \mathbb{1}^{\tau_{n,m}} \in \mathsf{Mat}^{\mathsf{C}}_{nm,mn}$$

in (8.13.6). Here $\tau_{n,m} = (\tau_{m,n})^{-1}$ with $\tau_{m,n} \in \Sigma_{mn}$ the permutation in (2.4.5).

Component 2-Cells: For 1-cells $A \in \mathsf{Mat}^{\mathsf{C}}_{m,m'}$ and $B \in \mathsf{Mat}^{\mathsf{C}}_{n,n'}$, define the component 2-cell

$$\beta^{\bullet}_{A,B} = \beta_{B,A} \in \mathsf{Mat}^{\mathsf{C}}_{nm,m'n'}\big((A \boxtimes B)\beta^{\bullet}_{m,n}; \beta^{\bullet}_{m',n'}(B \boxtimes A)\big).$$

This finishes the definition of $\beta^{\bullet}$. $\diamond$

Explanation 8.13.32. Using (8.13.24) the component 2-cell $\beta^{\bullet}_{A,B} = \beta_{B,A}$ is the following vertical composite.

$$
\begin{tikzcd}
& mn \arrow[dr, bend left=20, "A \boxtimes B"] \arrow[d, phantom, near end, "\Downarrow r^{\tau_{n,m}}_{A\boxtimes B}"] & \\
nm \arrow[ur, bend left=20, "\beta^{\bullet}_{m,n} = \beta_{n,m}"] \arrow[rr, bend left=8, "(A \boxtimes B)^{\tau_{n,m}}"] \arrow[rr, bend right=8, "\tau_{m',n'}(B \boxtimes A)"'] \arrow[rr, phantom, "\Downarrow \xi^{\otimes}"] \arrow[dr, bend right=20, "B \boxtimes A"'] & {} & m'n' \\
& n'm' \arrow[ur, bend right=20, "\beta^{\bullet}_{m',n'} = \beta_{n',m'}"'] \arrow[u, phantom, near end, "\Downarrow (\ell^{\tau_{n',m'}}_{B\boxtimes A})^{-1}"] &
\end{tikzcd} \tag{8.13.33}
$$

- $r^{\tau_{n,m}}_{A\boxtimes B}$ is the natural isomorphism in (8.13.13).
- $\ell^{\tau_{n',m'}}_{B\boxtimes A}$ is the natural isomorphism in (8.13.16), along with (8.13.19).
- $\xi^{\otimes}$ is the 2-cell with $\big(j' + (i'-1)n', i + (j-1)m\big)$-entry the structure morphism in C,

  $$A_{i'i} \otimes B_{j'j} \xrightarrow{\xi^{\otimes}_{A_{i'i}, B_{j'j}}} B_{j'j} \otimes A_{i'i}$$

  for $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq i' \leq m'$, and $1 \leq j' \leq n'$. This is well defined by Lemma 8.13.20.

By (8.13.14) and (8.13.17), each entry of $\beta^{\bullet}_{A,B}$ is a composite of isomorphisms, each being an iterated sum of identity morphisms and at most one component of $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\lambda^{-\otimes}$, $\rho^{\otimes}$, $\xi^{\otimes}$, $\lambda^{-\bullet}$, or $\rho^{\bullet}$ in C. ◇

Lemma 8.13.34. *In Definition 8.13.30, $\beta^{\bullet} : \boxtimes\tau \longrightarrow \boxtimes$ is a strong transformation.*

Proof. The proof of Lemma 8.13.27 applies here by switching symbols. In the lax unity axiom (8.13.28), m and n are switched. In the lax naturality axiom (8.13.29), the symbols (m, n, p, A, B) are switched with (m', n', p', A', B'). □

The Unit and the Counit of the Braiding. Recall that $\tau_{m,n} \in \Sigma_{mn}$ is the permutation in (2.4.5). The next two definitions refer to the strong transformations

$$\boxtimes \underset{\beta^{\bullet}}{\overset{\beta}{\rightleftarrows}} \boxtimes\tau$$

in Definitions 8.13.22 and 8.13.30.

Definition 8.13.35. Define the data

$$1_{\boxtimes} \xrightarrow{\eta^{\beta}} \beta^{\bullet}\beta$$

as consisting of the component 2-cells

$$
\begin{array}{c}
(1_{\boxtimes})_{(m,n)} = \mathbb{1}^{mn} \\
\Big\downarrow \eta^{\beta}_{(m,n)} = \Big(r^{\tau_{m,n}}_{\mathbb{1}^{\tau_{n,m}}}\Big)^{-1} \\
(\beta^{\bullet}\beta)_{(m,n)} = \beta^{\bullet}_{m,n}\beta_{m,n} = \mathbb{1}^{\tau_{n,m}}\mathbb{1}^{\tau_{m,n}}
\end{array} \tag{8.13.36}
$$

in $\mathsf{Mat}^{\mathsf{C}}_{mn,mn}$ for $m, n \geq 0$, with r as in (8.13.13). ◇

Explanation 8.13.37. Consider (8.13.36).

- $\eta^{\beta}_{(m,n)}$ is well defined because $r^{\tau_{m,n}}_{\mathbb{1}^{\tau_{n,m}}}$ is a natural isomorphism

$$\mathbb{1}^{\tau_{n,m}}\mathbb{1}^{\tau_{m,n}} \xrightarrow{r^{\tau_{m,n}}_{\mathbb{1}^{\tau_{n,m}}}} (\mathbb{1}^{\tau_{n,m}})^{\tau_{m,n}} = (\mathbb{1}^{mn})^{\tau_{m,n}\tau_{n,m}} = \mathbb{1}^{mn}.$$

  - In the domain, the entries of $\mathbb{1}^{\tau_{n,m}}\mathbb{1}^{\tau_{m,n}}$ are described in (8.13.9) and (8.13.10).
  - In the codomain, the equalities follow from (8.13.8) and (8.13.19).
- The kth diagonal entry of $\eta^{\beta}_{(m,n)}$ has the following form.

$$\mathbb{1} \xrightarrow{\quad(\lambda^{-\oplus},\rho^{-\oplus},\rho^{-\otimes},\rho^{-\bullet})\quad} \big[(\beta^{\bullet}\beta)_{(m,n)}\big]_{kk}$$

  It is a composite of morphisms, each being an iterated sum of identity morphisms and at most one component of $\lambda^{-\oplus}$, $\rho^{-\oplus}$, $\rho^{-\otimes}$, or $\rho^{-\bullet}$.
- Similarly, each off-diagonal entry of $\eta^{\beta}_{(m,n)}$ has the form

$$\mathbb{0} \xrightarrow{\quad(\lambda^{-\oplus},\rho^{-\otimes},\rho^{-\bullet})\quad} \big[(\beta^{\bullet}\beta)_{(m,n)}\big]_{kl}$$

  for $1 \le k \ne l \le mn$. ◇

Definition 8.13.38. Define the data

$$\beta\beta^{\bullet} \xrightarrow{\varepsilon^{\beta}} 1_{\boxtimes\tau}$$

as consisting of the component 2-cells

$$\begin{array}{c} (\beta\beta^{\bullet})_{(m,n)} = \beta_{m,n}\beta^{\bullet}_{m,n} = \mathbb{1}^{\tau_{m,n}}\mathbb{1}^{\tau_{n,m}} \\ \Big\downarrow \varepsilon^{\beta}_{(m,n)} = r^{\tau_{n,m}}_{\mathbb{1}^{\tau_{m,n}}} \\ (1_{\boxtimes\tau})_{(m,n)} = \mathbb{1}^{nm} = \mathbb{1}^{mn} \end{array} \tag{8.13.39}$$

in $\mathsf{Mat}^{\mathsf{C}}_{mn,mn}$ for $m,n \ge 0$, with r as in (8.13.13). ◇

Explanation 8.13.40. Consider (8.13.39).

- $\varepsilon^{\beta}_{(m,n)}$ is well defined by (8.13.8) and (8.13.19).
- The kth diagonal entry of $\varepsilon^{\beta}_{(m,n)}$ has the following form.

$$\big[(\beta\beta^{\bullet})_{(m,n)}\big]_{kk} \xrightarrow{\quad(\lambda^{\oplus},\rho^{\oplus},\rho^{\otimes},\rho^{\bullet})\quad} \mathbb{1}$$

- Each off-diagonal entry of $\varepsilon^{\beta}_{(m,n)}$ has the form

$$\big[(\beta\beta^{\bullet})_{(m,n)}\big]_{kl} \xrightarrow{\quad(\lambda^{\oplus},\rho^{\otimes},\rho^{\bullet})\quad} \mathbb{0}$$

  for $1 \le k \ne l \le mn$. ◇

Lemma 8.13.41. *η^{β} and ε^{β} in, respectively, Definitions 8.13.35 and 8.13.38 are invertible modifications.*

Proof. Each component 2-cell of each of $\eta^\beta : 1_\boxtimes \longrightarrow \beta^\bullet\beta$ and $\varepsilon^\beta : \beta\beta^\bullet \longrightarrow 1_{\boxtimes\tau}$ is invertible by (8.13.13).

The modification axiom (6.3.2) for η^β states that for each pair of 1-cells

$$(A, B) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'}$$

the following pasting diagram equality holds in $\mathsf{Mat}^{\mathsf{C}}_{mn,m'n'}$.

$$\begin{tikzcd}
mn \arrow[rr, "A \boxtimes B"] \arrow[dd, bend right=40, "\mathbb{1}^{mn}"'] \arrow[ddrr, "A \boxtimes B"'] \arrow[dr, phantom, "\overset{r}{\Rightarrow}"', near end] & & m'n' \arrow[dr, "\beta_{m',n'}"] \arrow[dd, bend right=40, "\mathbb{1}^{m'n'}"'] \arrow[d, phantom, "\overset{\ell^{-1}}{\Rightarrow}"', near start] & \\
 & & \arrow[r, phantom, "\overset{\eta^\beta_{(m',n')}}{\Rightarrow}"] & n'm' \arrow[dl, "\beta_{n',m'}"] \\
mn \arrow[rr, "A \boxtimes B"'] & & m'n' &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
mn \arrow[rrr, "A \boxtimes B"] \arrow[dd, bend right=40, "\mathbb{1}^{mn}"'] \arrow[dr, "\beta_{m,n}"] & \arrow[d, phantom, "\overset{\eta^\beta_{(m,n)}}{\Rightarrow}"'] & \arrow[d, phantom, "\overset{\beta_{A,B}}{\Rightarrow}"] & m'n' \arrow[dr, "\beta_{m',n'}"] & \\
 & nm \arrow[rrr, "B \boxtimes A"] \arrow[dl, "\beta_{n,m}"] & \arrow[d, phantom, "\overset{\beta^\bullet_{A,B} = \beta_{B,A}}{\Rightarrow}"] & & n'm' \arrow[dl, "\beta_{n',m'}"] \\
mn \arrow[rrr, "A \boxtimes B"'] & & & m'n' &
\end{tikzcd} \tag{8.13.42}$$

The proof of this equality is adapted from that of (8.8.40), the modification axiom for η^a. The proof of the modification axiom for ε^β is obtained similarly. $\square$

Lemma 8.13.43. *The quadruple $(\beta, \beta^\bullet, \eta^\beta, \varepsilon^\beta)$ satisfies the triangle identities.*

Proof. The left triangle identity (6.3.10) states the commutativity of the left diagram below.

$$\begin{tikzcd}
\beta 1_\boxtimes \arrow[rr, "r_\beta"] \arrow[d, "1_\beta * \eta^\beta"'] & & \beta \\
\beta(\beta^\bullet\beta) \arrow[dr, "a^{-1}_{\beta,\beta^\bullet,\beta}"'] & & 1_{\boxtimes\tau}\beta \arrow[u, "\ell_\beta"'] \\
 & (\beta\beta^\bullet)\beta \arrow[ur, "\varepsilon^\beta * 1_\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta_{m,n}\mathbb{1}^{mn} \arrow[rr, "r_{\beta_{m,n}}"] \arrow[d, "1 \star \eta^\beta_{(m,n)}"'] & & \beta_{m,n} \\
\beta_{m,n}(\beta_{n,m}\beta_{m,n}) \arrow[dr, "a^{-1}_{\beta_{m,n},\beta_{n,m},\beta_{m,n}}"'] & & \mathbb{1}^{nm}\beta_{m,n} \arrow[u, "\ell_{\beta_{m,n}}"'] \\
 & (\beta_{m,n}\beta_{n,m})\beta_{m,n} \arrow[ur, "\varepsilon^\beta_{(m,n)} \star 1"'] &
\end{tikzcd}$$

To show that the left diagram is commutative, it suffices to show that the two composites have the same component 2-cells. So it suffices to show that the right diagram is commutative for $m, n \geq 0$. The proof of the commutativity of the right diagram is adapted from that of (8.8.46), which is the left triangle identity for $a^\boxtimes$. The proof for the right triangle identity is obtained similarly. $\square$

Lemma 8.13.44. *The quadruple $(\beta, \beta^\bullet, \eta^\beta, \varepsilon^\beta)$ with*

- *β in Definition 8.13.22,*
- *$\beta^\bullet$ in Definition 8.13.30,*
- *η^β in Definition 8.13.35, and*
- *ε^β in Definition 8.13.38*

is an adjoint equivalence.

Proof. This follows from Lemmas 8.13.27, 8.13.34, 8.13.41, and 8.13.43. $\square$

8.14. The Matrix Braided Monoidal Bicategory

We continue to assume that C is a tight symmetric bimonoidal category. In Section 8.13, we constructed the braiding $(\beta, \beta^{\bullet}, \eta^{\beta}, \varepsilon^{\beta})$. In this section, we construct the left hexagonator and the right hexagonator and observe that $\mathsf{Mat}^{\mathsf{C}}$ is a braided monoidal bicategory as in Definition 6.5.3.

- The left hexagonator is in Definition 8.14.9.
- The right hexagonator is in Definition 8.14.21.
- The verification of the braided monoidal bicategory axioms is in Theorem 8.14.26.

The Left Hexagonator. To define the left hexagonator, we will make use of an auxiliary 2-cell $h_{m|n,p}$ in Lemma 8.14.4 below.

Motivation 8.14.1. The left hexagonator $R_{-|--}$ in a braided monoidal bicategory is the 2-cell analogue of the left hexagon diagram in (II.1.3.17). With the braiding denoted by β, the left hexagonator compares $\beta_{A,B} \boxtimes 1_C$ and $1_B \boxtimes \beta_{A,C}$ with $\beta_{A,B\boxtimes C}$. In the context of the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$, the left hexagonator involves a comparison 2-cell

$$\big(\mathbb{1}^n \boxtimes \beta_{m,p}\big)\big(\beta_{m,n} \boxtimes \mathbb{1}^p\big) \xrightarrow{h_{m|n,p}} \beta_{m,np} \quad \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$$

for objects $m,n,p \in \mathsf{Mat}^{\mathsf{C}}$, that is, nonnegative integers. To define this comparison 2-cell, in Lemma 8.14.2, we first describe explicitly the three 1-cells involved. The definition of $h_{m|n,p}$ is given in the proof of Lemma 8.14.4 in terms of the structure morphisms in C. $\diamond$

Lemma 8.14.2. *For $m,n,p \geq 0$, $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq p$, define the following integers.*

$$I = i + \big[k-1+(j-1)p\big]m \qquad J = k + \big[j-1+(i-1)n\big]p$$
$$J' = k + \big[i-1+(j-1)m\big]p$$

Then the following equalities hold for $1 \leq u,v \leq mnp$.

$$\big(\beta_{m,np}\big)_{uv} = \begin{cases} \mathbb{1} & \text{if } (u,v) = (I,J) \text{ for some } (i,j,k) \text{, and} \\ \mathbb{0} & \text{otherwise.} \end{cases}$$
$$\big(\mathbb{1}^n \boxtimes \beta_{m,p}\big)_{uv} = \begin{cases} \mathbb{1} \otimes \mathbb{1} & \text{if } (u,v) = (I,J') \text{ for some } (i,j,k) \text{, and} \\ \mathbb{0} \otimes \mathbb{0},\ \mathbb{0} \otimes \mathbb{1}, \text{ or } \mathbb{1} \otimes \mathbb{0} & \text{otherwise.} \end{cases} \tag{8.14.3}$$
$$\big(\beta_{m,n} \boxtimes \mathbb{1}^p\big)_{uv} = \begin{cases} \mathbb{1} \otimes \mathbb{1} & \text{if } (u,v) = (J',J) \text{ for some } (i,j,k) \text{, and} \\ \mathbb{0} \otimes \mathbb{0},\ \mathbb{0} \otimes \mathbb{1}, \text{ or } \mathbb{1} \otimes \mathbb{0} & \text{otherwise.} \end{cases}$$

Proof. If m, n, or p is 0, then $\mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ is the terminal category, and the three matrices in (8.14.3) are all empty matrices. In this case, there is nothing to prove, so we assume that $m,n,p > 0$.

With $1 \leq j \leq n$ and $1 \leq k \leq p$, the sum $k+(j-1)p$ runs through $\{1,\ldots,np\}$. Therefore, the first equality in (8.14.3) is the result of applying (8.13.26) to $\beta_{m,np}$.

For the second equality, by (8.1.6) and (8.6.3), there is an equality as follows.

$$\mathbb{1}^n \boxtimes \beta_{m,p} = \begin{bmatrix} \mathbb{1} \boxtimes \beta_{m,p} & \cdots & \mathbb{0} \boxtimes \beta_{m,p} \\ \vdots & \ddots & \vdots \\ \mathbb{0} \boxtimes \beta_{m,p} & \cdots & \mathbb{1} \boxtimes \beta_{m,p} \end{bmatrix}$$

Among its n^2 blocks,

- each of the n diagonal blocks is $\mathbb{1} \boxtimes \beta_{m,p} \in \mathsf{Mat}^{\mathsf{C}}_{mp,mp}$, and
- each off-diagonal block is $\mathbb{0} \boxtimes \beta_{m,p} \in \mathsf{Mat}^{\mathsf{C}}_{mp,mp}$.

By (8.13.26), the (I, J')-entry in $\mathbb{1}^n \boxtimes \beta_{m,p}$ is the following object in the jth diagonal block $\mathbb{1} \boxtimes \beta_{m,p}$.

$$\begin{aligned}\left(\mathbb{1}^n \boxtimes \beta_{m,p}\right)_{I,J'} &= \left(\mathbb{1} \boxtimes \beta_{m,p}\right)_{(i+(k-1)m,\, k+(i-1)p)}\\ &= \mathbb{1} \otimes \mathbb{1}\end{aligned}$$

Every other entry in the jth diagonal block $\mathbb{1} \boxtimes \beta_{m,p}$ is $\mathbb{1} \otimes \mathbb{0}$. Each entry in each off-diagonal block $\mathbb{0} \boxtimes \beta_{m,p}$ is either $\mathbb{0} \otimes \mathbb{0}$ or $\mathbb{0} \otimes \mathbb{1}$. This proves the second equality in (8.14.3).

For the last equality in (8.14.3), by (8.6.3), $\beta_{m,n} \boxtimes \mathbb{1}^p$ has $(mn)^2$ blocks. By (8.13.26),

- its $\big(i + (j-1)m, j + (i-1)n\big)$-th block is $\mathbb{1} \boxtimes \mathbb{1}^p \in \mathsf{Mat}^{\mathsf{C}}_{p,p}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and
- every other block is $\mathbb{0} \boxtimes \mathbb{1}^p \in \mathsf{Mat}^{\mathsf{C}}_{p,p}$.

In the $\big(i + (j-1)m, j + (i-1)n\big)$-th block of $\beta_{m,n} \boxtimes \mathbb{1}^p$, namely, $\mathbb{1} \boxtimes \mathbb{1}^p$,

- for $1 \leq k \leq p$, the (k,k)-entry is
$$(\beta_{m,n} \boxtimes \mathbb{1}^p)_{J',J} = \mathbb{1} \otimes \mathbb{1},$$
and
- every other entry is $\mathbb{1} \otimes \mathbb{0}$.

Each entry in every other block of $\beta_{m,n} \boxtimes \mathbb{1}^p$, namely, $\mathbb{0} \boxtimes \mathbb{1}^p$, is either $\mathbb{0} \otimes \mathbb{1}$ or $\mathbb{0} \otimes \mathbb{0}$. This proves the last equality in (8.14.3). □

Using Lemma 8.14.2, we now define the comparison 2-cell $h_{m|n,p}$.

Lemma 8.14.4. *For $m, n, p \geq 0$, there is an isomorphism*

$$\left(\mathbb{1}^n \boxtimes \beta_{m,p}\right)\left(\beta_{m,n} \boxtimes \mathbb{1}^p\right) \xrightarrow{h_{m|n,p}} \beta_{m,np} \quad \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp} \tag{8.14.5}$$

that is defined entrywise by the structure morphisms in C.

Proof. If m, n, or p is 0, then $\mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ is the terminal category, and $h_{m|n,p}$ is the identity morphism of the empty matrix. Next we assume that $m, n, p > 0$.

Using Lemma 8.14.2 and its notation, there are equalities as follows.

$$\begin{aligned}&\Big[\left(\mathbb{1}^n \boxtimes \beta_{m,p}\right)\left(\beta_{m,n} \boxtimes \mathbb{1}^p\right)\Big]_{I,J}\\ &= \left[\bigoplus_{t=1}^{mnp} \left(\mathbb{1}^n \boxtimes \beta_{m,p}\right)_{I,t} \otimes \left(\beta_{m,n} \boxtimes \mathbb{1}^p\right)_{t,J}\right]_{\mathsf{lt}}\\ &= \Big[\underbrace{z_1 \oplus \cdots \oplus z_{J'-1}}_{\text{empty if } J' = 1} \oplus (\mathbb{1}\mathbb{1})(\mathbb{1}\mathbb{1}) \underbrace{\oplus z_{J'+1} \oplus \cdots \oplus z_{mnp}}_{\text{empty if } J' = mnp}\Big]_{\mathsf{lt}}\end{aligned}$$

Here each z_t for $1 \leq t \leq mnp$ with $t \neq J'$ has the form $(\delta_1\delta_2)(\delta_3\delta_4)$ such that

- each $\delta_i \in \{\mathbb{0}, \mathbb{1}\}$ and
- $(\delta_1, \delta_2) \neq (\mathbb{1}, \mathbb{1}) \neq (\delta_3, \delta_4)$.

The (I,J)-entry of $h_{m|n,p}$ is the following composite isomorphism in C.

$$\begin{tikzcd}
\Big[\big(\mathbb{1}^n \boxtimes \beta_{m,p}\big)\big(\beta_{m,n} \boxtimes \mathbb{1}^p\big)\Big]_{I,J} \arrow[r, "(h_{m|n,p})_{I,J}"] \arrow[d, "(\lambda^{\otimes}{,}\lambda^{\bullet}{,}\rho^{\bullet})"'] & \big(\beta_{m,np}\big)_{I,J} \arrow[d, equal] \\
\Big(\underbrace{\mathbb{0} \oplus \cdots \oplus \mathbb{0} \oplus}_{J'-1 \text{ copies of } \mathbb{0}} \mathbb{1} \underbrace{\oplus \mathbb{0} \oplus \cdots \oplus \mathbb{0}}_{mnp - J' \text{ copies of } \mathbb{0}}\Big)_{\mathsf{lt}} \arrow[r, "(\lambda^{\oplus}{,}\rho^{\oplus})"] & \mathbb{1}
\end{tikzcd} \tag{8.14.6}$$

Here we used the notation in (8.7.6) for morphisms. For example, $(\lambda^{\otimes}, \lambda^{\bullet}, \rho^{\bullet})$ is a composite of morphisms, each being an iterated sum and product of identity morphisms and at most one component of $\lambda^{\otimes}$, $\lambda^{\bullet}$, or $\rho^{\bullet}$.

If $(u,v) \neq (I,J)$ for any (i,j,k), then Lemma 8.14.2 implies that

$$\Big[\big(\mathbb{1}^n \boxtimes \beta_{m,p}\big)\big(\beta_{m,n} \boxtimes \mathbb{1}^p\big)\Big]_{uv} = \big[y_1 \oplus \cdots \oplus y_{mnp}\big]_{\mathsf{lt}}.$$

Here each y_t for $1 \le t \le mnp$ has the form $(\delta_1\delta_2)(\delta_3\delta_4)$ with

- each $\delta_i \in \{\mathbb{0}, \mathbb{1}\}$ and
- $(\delta_1, \delta_2, \delta_3, \delta_4) \neq (\mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})$.

The (u,v)-entry of $h_{m|n,p}$ is the following composite isomorphism in C.

$$\begin{tikzcd}
\Big[\big(\mathbb{1}^n \boxtimes \beta_{m,p}\big)\big(\beta_{m,n} \boxtimes \mathbb{1}^p\big)\Big]_{uv} \arrow[r, "(h_{m|n,p})_{uv}"] \arrow[d, "(\lambda^{\bullet}{,}\rho^{\bullet})"'] & \big(\beta_{m,np}\big)_{uv} \arrow[d, equal] \\
\Big[\bigoplus_{t=1}^{mnp} \mathbb{0}\Big]_{\mathsf{lt}} \arrow[r, "(\lambda^{\oplus})"] & \mathbb{0}
\end{tikzcd} \tag{8.14.7}$$

This finishes the construction of the isomorphism $h_{m|n,p}$. $\square$

Explanation 8.14.8. While some choices are made in the definition of $h_{m|n,p}$—for example, in the order in which $\lambda^{\oplus}$ and $\rho^{\oplus}$ are applied in (8.14.6)—its value as a 2-cell in $\mathsf{Mat}^{\mathsf{C}}$ is unique. This follows from essentially the same argument as in Examples 8.1.14 and 8.2.5 using the set $X = \{0^X, 1^X\}$ of formal variables and the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ with $\varphi(0^X) = \mathbb{0}$ and $\varphi(1^X) = \mathbb{1}$.

- In the (I,J)-entry for some (i,j,k), this uniqueness follows from the regularity of 1^X, Lemma 3.1.29, and the Coherence Theorem 3.9.1.
- In every other entry, this uniqueness follows from Proposition 3.5.33. $\diamond$

Using the comparison 2-cell $h_{m|n,p}$, we now define the left hexagonator in the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$.

Definition 8.14.9. Define $R_{-|--}$ as consisting of the 2-cells $R_{m|n,p} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,npm}$ for $m,n,p \geq 0$ given by the following pasting diagram.

$$\begin{tikzcd}
(nm)p \arrow[r, "a^{\boxtimes}_{n,m,p} = \mathbb{1}^{nmp}"] \arrow[dr, "\mathbb{1}^n \boxtimes \beta_{m,p}" description] & n(mp) \arrow[d, "\mathbb{1}^n \boxtimes \beta_{m,p}"] \arrow[dl, phantom, "\Downarrow r", near start] \\
(mn)p \arrow[u, "\beta_{m,n} \boxtimes \mathbb{1}^p"] \arrow[r, "\beta_{m,np}"] \arrow[d, "\mathbb{1}^{mnp} = a^{\boxtimes}_{m,n,p}"'] \arrow[dr, "\beta_{m,np}" description] \arrow[ur, phantom, "\Downarrow h_{m|n,p}", near start] & n(pm) \arrow[dl, phantom, "\Downarrow \ell^{-1}", near start] \\
m(np) \arrow[r, "\beta_{m,np}"] \arrow[ur, phantom, "\Downarrow r^{-1}", near start] & (np)m \arrow[u, "a^{\boxtimes}_{n,p,m} = \mathbb{1}^{npm}"']
\end{tikzcd} \tag{8.14.10}$$

This finishes the definition of $R_{-|--}$. ◇

Explanation 8.14.11. Consider (8.14.10).

- By Convention 6.2.12, the (co)domain 1-cell has the left normalized bracketing.
- Each component 1-cell of β is a permutation matrix by (8.13.23).
- Each component 1-cell of $a^{\boxtimes}$ is an identity matrix by (8.8.2).
- The top 2-cell

$$r = r_{\mathbb{1}^n \boxtimes \beta_{m,p}} : \big(\mathbb{1}^n \boxtimes \beta_{m,p}\big)\mathbb{1}^{nmp} \longrightarrow \mathbb{1}^n \boxtimes \beta_{m,p}$$

  is the base right unitor in (8.2.8).
- $h_{m|n,p}$ is the comparison 2-cell in (8.14.5).
- The 2-cell

$$\ell^{-1} = \ell^{-1}_{\beta_{m,np}} : \beta_{m,np} \longrightarrow \mathbb{1}^{npm}\beta_{m,np}$$

  is the inverse of the base left unitor in (8.2.2).
- The bottom 2-cell $r^{-1} = r^{-1}_{\beta_{m,np}}$ is the inverse of the base right unitor.
- By (8.2.4), (8.2.10), (8.14.6), and (8.14.7), each entry of $R_{m|n,p}$ is a composite of morphisms, each being an iterated sum and product of identity morphisms and at most one component of $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\lambda^{\pm\otimes}$, $\rho^{\pm\otimes}$, $\lambda^{\pm\bullet}$, or $\rho^{\pm\bullet}$. ◇

Lemma 8.14.12. *In Definition 8.14.9, $R_{-|--}$ is an invertible modification.*

Proof. Each component 2-cell of $R_{-|--}$ is entrywise a composite of isomorphisms in C, so it is invertible.

Using the notation in the proof of Lemma 8.10.4, the modification axiom (6.3.2) for $R_{-|--}$ for the 1-cells

$$(B^1, B^2, B^3) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'} \times \mathsf{Mat}^{\mathsf{C}}_{p,p'}$$

states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mnp,n'p'm'}$.

$$
\begin{gathered}
\begin{tikzcd}
& (mn)p \arrow[dl, bend right=10, "\beta_{m,n}\boxtimes \mathbb{1}^{p}"'] \arrow[dr, bend left=10, "B^{(12)3}"] \arrow[d, bend right=60, "B^{21}\beta_{m,n}\boxtimes B^{3}\mathbb{1}^{p}"', ""{name=L}] \arrow[d, bend left=60, "\beta_{m',n'}B^{12}\boxtimes \mathbb{1}^{p'}B^{3}", ""'{name=R}] \arrow[d, phantom, "\overset{\beta_{B^1,B^2}\boxtimes \ell^{-1}r}{\Rightarrow}"] & \\
(nm)p \arrow[r, phantom, "\overset{\boxtimes^{2}}{\Rightarrow}"] \arrow[dr, bend right=10, "B^{(21)3}"'] \arrow[dd, "\mathbb{1}^{nmp}"'] & (n'm')p' \arrow[d, "\mathbb{1}^{n'm'p'}"] \arrow[dl, phantom, "\overset{a^{\boxtimes}_{2,1,3}}{\Rightarrow}"] & (m'n')p' \arrow[l, phantom, "\overset{\boxtimes^{-2}}{\Rightarrow}"'] \arrow[l, bend left=10, "\beta_{m',n'}\boxtimes \mathbb{1}^{p'}"] \arrow[dd, "\mathbb{1}^{m'n'p'}"] \\
& n'(m'p') \arrow[dd, "\mathbb{1}^{n'}\boxtimes \beta_{m',p'}"] \arrow[r, phantom, "\overset{R_{m'|n',p'}}{\Rightarrow}"] & \\
n(mp) \arrow[ur, "B^{2(13)}"] \arrow[d, "\mathbb{1}^{n}\boxtimes \beta_{m,p}"'] \arrow[ddr, bend left=20, "\mathbb{1}^{n'}B^{2}\boxtimes \beta_{m',p'}B^{13}"] \arrow[ddr, bend right=10, "B^{2}\mathbb{1}^{n}\boxtimes B^{31}\beta_{m,p}"'] \arrow[dr, phantom, "\overset{\ell^{-1}r\boxtimes \beta_{B^1,B^3}}{\Rightarrow}"] \arrow[d, phantom, shift left=8, "\overset{\boxtimes^{2}}{\Rightarrow}"] & \arrow[d, phantom, shift right=4, "\overset{\boxtimes^{-2}}{\Rightarrow}"] & m'(n'p') \arrow[d, "\beta_{m',n'p'}"] \\
n(pm) \arrow[dr, bend right=10, "B^{2(31)}"'] & & (n'p')m' \arrow[dl, bend left=10, "\mathbb{1}^{n'p'm'}"] \\
& n'(p'm') &
\end{tikzcd}
\\
\parallel
\\
\begin{tikzcd}
& (mn)p \arrow[dl, bend right=10, "\beta_{m,n}\boxtimes \mathbb{1}^{p}"'] \arrow[dr, bend left=10, "B^{(12)3}"] \arrow[dd, "\mathbb{1}^{mnp}"] & \\
(nm)p \arrow[dd, "\mathbb{1}^{nmp}"'] & & (m'n')p' \arrow[dd, "\mathbb{1}^{m'n'p'}"] \arrow[dl, phantom, "\overset{a^{\boxtimes}_{1,2,3}}{\Rightarrow}"] \\
& m(np) \arrow[dr, bend left=10, "B^{1(23)}"] \arrow[dd, "\beta_{m,np}"] & \\
n(mp) \arrow[dd, "\mathbb{1}^{n}\boxtimes \beta_{m,p}"'] \arrow[r, phantom, "\overset{R_{m|n,p}}{\Rightarrow}"] & & m'(n'p') \arrow[dd, "\beta_{m',n'p'}"] \arrow[dl, phantom, "\overset{\beta_{B^1,B^2\boxtimes B^3}}{\Rightarrow}"] \\
& (np)m \arrow[dl, bend right=10, "\mathbb{1}^{npm}"'] \arrow[dr, bend left=10, "B^{(23)1}"] \arrow[d, phantom, "\overset{a^{\boxtimes}_{2,3,1}}{\Rightarrow}"] & \\
n(pm) \arrow[r, bend right=10, "B^{2(31)}"'] & n'(p'm') & (n'p')m' \arrow[l, bend left=10, "\mathbb{1}^{n'p'm'}"]
\end{tikzcd}
\end{gathered}
\tag{8.14.13}
$$

For example,

$$
\begin{aligned}
B^{2(31)} &= B^{2}\boxtimes \left(B^{3}\boxtimes B^{1}\right)\\
a^{\boxtimes}_{2,1,3} &= a^{\boxtimes}_{B^2,B^1,B^3}.
\end{aligned}
$$

The proof of the equality (8.14.13) is adapted from that of (8.10.5), which is the modification axiom for the pentagonator π. When restricted to a typical entry, the two pasting diagrams in (8.14.13) yield a zigzag diagram analogous to (8.10.8). The object Y in (8.10.9) is now replaced by the corresponding entry of $B^{(12)3}$, which has the form

$$
Y = \left(B^{1}_{i'i}\otimes B^{2}_{j'j}\right)\otimes B^{3}_{k'k}.
$$

Since the proof of the equality (8.10.5) has all the detail, we safely skip the detail here. $\square$

The Right Hexagonator. Similar to the left hexagonator $R_{-|--}$, to define the right hexagonator, we will make use of a comparison 2-cell that involves the 1-cells in the following lemma.

Lemma 8.14.14. *For $m,n,p \geq 0$, $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq p$, define the following integers.*

$$\begin{aligned} K &= j + \big[i-1+(k-1)m\big]n & \qquad J &= k + \big[j-1+(i-1)n\big]p \\ K' &= j + \big[k-1+(i-1)p\big]n \end{aligned}$$

Then the following equalities hold for $1 \leq u,v \leq mnp$.

$$\begin{aligned} \big(\beta_{mn,p}\big)_{uv} &= \begin{cases} \mathbb{1} & \text{if } (u,v) = (K,J) \text{ for some } (i,j,k)\text{, and} \\ \mathbb{0} & \text{otherwise.} \end{cases} \\ \big(\mathbb{1}^m \boxtimes \beta_{n,p}\big)_{uv} &= \begin{cases} \mathbb{1} \otimes \mathbb{1} & \text{if } (u,v) = (K',J) \text{ for some } (i,j,k)\text{, and} \\ \mathbb{0} \otimes \mathbb{0},\ \mathbb{0} \otimes \mathbb{1}, \text{ or } \mathbb{1} \otimes \mathbb{0} & \text{otherwise.} \end{cases} \\ \big(\beta_{m,p} \boxtimes \mathbb{1}^n\big)_{uv} &= \begin{cases} \mathbb{1} \otimes \mathbb{1} & \text{if } (u,v) = (K,K') \text{ for some } (i,j,k)\text{, and} \\ \mathbb{0} \otimes \mathbb{0},\ \mathbb{0} \otimes \mathbb{1}, \text{ or } \mathbb{1} \otimes \mathbb{0} & \text{otherwise.} \end{cases} \end{aligned} \tag{8.14.15}$$

Proof. As in the proof of Lemma 8.14.2, we may assume that $m,n,p > 0$, since there is nothing to prove otherwise. With $1 \leq i \leq m$ and $1 \leq j \leq n$, the sum $j+(i-1)n$ runs through $\{1,\ldots,mn\}$. Therefore, the first equality in (8.14.15) is the result of applying (8.13.26) to $\beta_{mn,p}$.

The second equality in (8.14.15) is obtained from the second equality in (8.14.3) by switching (i,m) with (j,n). The third equality in (8.14.15) is obtained from the third equality in (8.14.3) by switching (j,n) with (k,p). □

Using Lemma 8.14.14, we now define the comparison 2-cell that will be used in the definition of the right hexagonator.

Lemma 8.14.16. *For $m,n,p \geq 0$, there is an isomorphism*

$$\big(\beta_{m,p} \boxtimes \mathbb{1}^n\big)\big(\mathbb{1}^m \boxtimes \beta_{n,p}\big) \xrightarrow{h_{m,n|p}} \beta_{mn,p} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp} \tag{8.14.17}$$

that is defined entrywise by the structure morphisms in C.

Proof. As in the proof of Lemma 8.14.4, we may assume that $m,n,p > 0$, since otherwise $h_{m,n|p}$ is the identity morphism of the empty matrix.

Using Lemma 8.14.14 and its notation, analogous to (8.14.6), the (K,J)-entry of $h_{m,n|p}$ is the following composite isomorphism in C.

$$\begin{tikzcd} \Big[\big(\beta_{m,p} \boxtimes \mathbb{1}^n\big)\big(\mathbb{1}^m \boxtimes \beta_{n,p}\big)\Big]_{K,J} \arrow[r, "(h_{m,n|p})_{K,J}"] \arrow[d, "(\lambda^{\otimes}{,}\lambda^{\bullet}{,}\rho^{\bullet})"'] & \big(\beta_{mn,p}\big)_{K,J} \arrow[d, equal] \\ \Big(\underbrace{\mathbb{0} \oplus \cdots \oplus \mathbb{0} \oplus}_{K'-1 \text{ copies of } \mathbb{0}} \mathbb{1} \underbrace{\oplus \mathbb{0} \oplus \cdots \oplus \mathbb{0}}_{mnp-K' \text{ copies of } \mathbb{0}}\Big)_{\mathsf{lt}} \arrow[r, "(\lambda^{\oplus}{,}\rho^{\oplus})"] & \mathbb{1} \end{tikzcd} \tag{8.14.18}$$

Suppose $(u,v) \neq (K,J)$ for any (i,j,k). Analogous to (8.14.7), the (u,v)-entry of $h_{m,n|p}$ is the following composite isomorphism in C.

$$\begin{tikzcd}
\Big[\big(\beta_{m,p} \boxtimes \mathbb{1}^n\big)\big(\mathbb{1}^m \boxtimes \beta_{n,p}\big)\Big]_{uv} \arrow[r, "(h_{m,n|p})_{uv}"] \arrow[d, "(\lambda^{\bullet}, \rho^{\bullet})"'] & \big(\beta_{mn,p}\big)_{uv} \arrow[d, equal] \\
\Big[\bigoplus_{t=1}^{mnp} \mathbb{0}\Big]_{\mathsf{lt}} \arrow[r, "(\lambda^{\oplus})"] & \mathbb{0}
\end{tikzcd} \tag{8.14.19}$$

This finishes the construction of the isomorphism $h_{m,n|p}$. $\square$

Explanation 8.14.20. As in Explanation 8.14.8, $h_{m,n|p}$ is a well-defined 2-cell in $\mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ by Proposition 3.5.33 and Theorem 3.9.1. $\diamond$

We now define the right hexagonator in the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$.

Definition 8.14.21. Define $R_{--|-}$ as consisting of the 2-cells $R_{m,n|p} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,pmn}$ for $m,n,p \geq 0$ given by the following pasting diagram.

$$\begin{tikzcd}
m(pn) \arrow[rr, "a^{\boxtimes\bullet}_{m,p,n} = \mathbb{1}^{mpn}"] \arrow[drr, "\beta_{m,p} \boxtimes \mathbb{1}^n"] & & (mp)n \arrow[d, "\beta_{m,p} \boxtimes \mathbb{1}^n"] \arrow[dl, phantom, "\Downarrow r"] \\
m(np) \arrow[u, "\mathbb{1}^m \boxtimes \beta_{n,p}"] \arrow[rr, "\beta_{mn,p}"] \arrow[drr, "\beta_{mn,p}"] \arrow[d, "\mathbb{1}^{mnp} = a^{\boxtimes\bullet}_{m,n,p}"'] \arrow[ur, phantom, "\Downarrow h_{m,n|p}"] & & (pm)n \arrow[dl, phantom, "\Downarrow \ell^{-1}"] \\
(mn)p \arrow[rr, "\beta_{mn,p}"] \arrow[ur, phantom, "\Downarrow r^{-1}"] & & p(mn) \arrow[u, "a^{\boxtimes\bullet}_{p,m,n} = \mathbb{1}^{pmn}"']
\end{tikzcd} \tag{8.14.22}$$

This finishes the definition of $R_{--|-}$. $\diamond$

Explanation 8.14.23. Consider (8.14.22).

- By Convention 6.2.12, the (co)domain 1-cell has the left normalized bracketing.
- $h_{m,n|p}$ is the comparison 2-cell in (8.14.17).
- $\ell = \ell_{\beta_{mn,p}}$ is the base left unitor (8.2.2).
- The top $r = r_{\beta_{m,p} \boxtimes \mathbb{1}^n}$ and the bottom $r = r_{\beta_{mn,p}}$ are the base right unitor (8.2.8).
- Each component 1-cell of β is a permutation matrix by (8.13.23).
- Each component 1-cell of $a^{\boxtimes\bullet}$ is an identity matrix by (8.8.34).
- By (8.2.4), (8.2.10), (8.14.18), and (8.14.19), each entry of $R_{m,n|p}$ is a composite of morphisms, each being an iterated sum and product of identity morphisms and at most one component of $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, $\lambda^{\pm\otimes}$, $\rho^{\pm\otimes}$, $\lambda^{\pm\bullet}$, or $\rho^{\pm\bullet}$. $\diamond$

Lemma 8.14.24. *In Definition 8.14.21, $R_{--|-}$ is an invertible modification.*

Proof. Each component 2-cell of $R_{--|-}$ is entrywise a composite of isomorphisms in C, so it is invertible.

Using the notation in the proof of Lemma 8.10.4, the modification axiom (6.3.2) for $R_{--|-}$ for the 1-cells

$$(B^1, B^2, B^3) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'} \times \mathsf{Mat}^{\mathsf{C}}_{p,p'}$$

states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mnp,p'm'n'}$.

$$\begin{gathered}
\begin{tikzcd}
 & m(np) \arrow[dl, bend right, "\mathbb{1}^m \boxtimes \beta_{n,p}"'] \arrow[dr, bend left, "B^{1(23)}"] \arrow[d, bend right=50, "B^1\mathbb{1}^m \boxtimes B^{32}\beta_{n,p}"'] \arrow[d, bend left=50, "\mathbb{1}^{m'}B^1 \boxtimes \beta_{n',p'}B^{23}"] \arrow[d, phantom, "\overset{\ell^{-1}r \boxtimes \beta_{B^2,B^3}}{\Rightarrow}"] & \\
m(pn) \arrow[r, phantom, "\overset{\boxtimes^2}{\Rightarrow}"] \arrow[dr, bend right, "B^{1(32)}"'] \arrow[dd, "\mathbb{1}^{mpn}"'] & & m'(n'p') \arrow[l, phantom, "\overset{\boxtimes^{-2}}{\Rightarrow}"] \arrow[dl, bend left, "\mathbb{1}^{m'} \boxtimes \beta_{n',p'}"] \arrow[dd, "\mathbb{1}^{m'n'p'}"] \\
 {} \arrow[r, phantom, "\overset{a^{\boxtimes\bullet}_{1,3,2}}{\Rightarrow}"] & m'(p'n') \arrow[d, "\mathbb{1}^{m'p'n'}"] & \\
(mp)n \arrow[r, "B^{(13)2}"] \arrow[d, "\beta_{m,p} \boxtimes \mathbb{1}^n"'] \arrow[ddr, bend right, "B^{31}\beta_{m,p} \boxtimes B^2\mathbb{1}^n"'] \arrow[ddr, bend left, "\beta_{m',p'}B^{13} \boxtimes \mathbb{1}^{n'}B^2"] & (m'p')n' \arrow[dd, "\beta_{m',p'} \boxtimes \mathbb{1}^{n'}"] \arrow[r, phantom, "\overset{R_{m',n'|p'}}{\Rightarrow}"] & (m'n')p' \arrow[d, "\beta_{m'n',p'}"] \\
(pm)n \arrow[dr, bend right, "B^{(31)2}"'] \arrow[r, phantom, "\overset{\boxtimes^2}{\Rightarrow}\ \overset{\beta_{B^1,B^3} \boxtimes \ell^{-1}r}{\Rightarrow}\ \overset{\boxtimes^{-2}}{\Rightarrow}"] & & p'(m'n') \arrow[dl, bend left, "\mathbb{1}^{p'm'n'}"] \\
 & (p'm')n' &
\end{tikzcd} \\
\| \\
\begin{tikzcd}
 & m(np) \arrow[dl, bend right, "\mathbb{1}^m \boxtimes \beta_{n,p}"'] \arrow[dr, bend left, "B^{1(23)}"] \arrow[d, "\mathbb{1}^{mnp}"] & \\
m(pn) \arrow[dd, "\mathbb{1}^{mpn}"'] & {} \arrow[r, phantom, "\overset{a^{\boxtimes\bullet}_{1,2,3}}{\Rightarrow}"] & m'(n'p') \arrow[dd, "\mathbb{1}^{m'n'p'}"] \\
 {} \arrow[r, phantom, "\overset{R_{m,n|p}}{\Rightarrow}"] & (mn)p \arrow[dr, bend left, "B^{(12)3}"] \arrow[dd, "\beta_{mn,p}"] & \\
(mp)n \arrow[dd, "\beta_{m,p} \boxtimes \mathbb{1}^n"'] & {} \arrow[r, phantom, "\overset{\beta_{B^1 \boxtimes B^2, B^3}}{\Rightarrow}"] & (m'n')p' \arrow[dd, "\beta_{m'n',p'}"] \\
 & p(mn) \arrow[dl, bend right, "\mathbb{1}^{pmn}"'] \arrow[dr, bend left, "B^{3(12)}"] \arrow[d, phantom, "\overset{a^{\boxtimes\bullet}_{3,1,2}}{\Rightarrow}"] & \\
(pm)n \arrow[dr, bend right, "B^{(31)2}"'] & {} & p'(m'n') \arrow[dl, bend left, "\mathbb{1}^{p'm'n'}"] \\
 & (p'm')n' &
\end{tikzcd}
\end{gathered} \tag{8.14.25}$$

Analogous to (8.14.13), the proof of the equality (8.14.25) is adapted from the detailed proof of (8.10.5), which is the modification axiom for the pentagonator π. □

The Axioms.

Theorem 8.14.26. *For each tight symmetric bimonoidal category* C*, the matrix bicategory* $\mathsf{Mat}^{\mathsf{C}}$ *equipped with*

- *the braiding* $(\beta, \beta^{\bullet}, \eta^{\beta}, \varepsilon^{\beta})$ *in Section 8.13,*
- *the left hexagonator* $R_{-|--}$ *in Definition 8.14.9, and*
- *the right hexagonator* $R_{--|-}$ *in Definition 8.14.21*

is a braided monoidal bicategory.

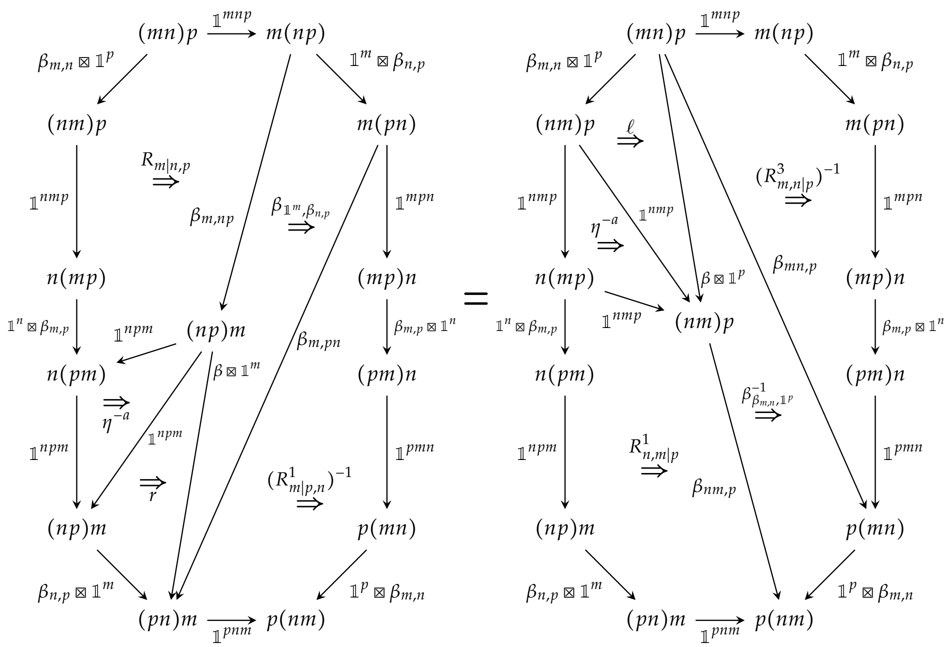
Proof. We already have the following.

- $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory by Theorem 8.12.9.
- The braiding is an adjoint equivalence by Lemma 8.13.44.

- The left hexagonator and the right hexagonator are invertible modifications by, respectively, Lemmas 8.14.12 and 8.14.24.

It remains to check the four axioms in Definition 6.5.3.

For objects $m, n, p \in \mathsf{Mat}^{\mathsf{C}}$, the Yang-Baxter axiom states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mnp,pnm}$.



The proof of this equality is adapted from that of (8.12.3), which is the non-abelian 4-cocycle condition, in Lemma 8.12.1. The key points are as follows.

- Component 1-cells of $a^{\boxtimes}$ (8.8.2) and $a^{\boxtimes\bullet}$ (8.8.34) are identity matrices.
- Each component 1-cell of the braiding β (8.13.23) is a permutation matrix. It has precisely one $\mathbb{1}$ in each row and each column and $\mathbb{0}$ in every other entry.
- Each 2-cell in the Yang-Baxter axiom is entrywise defined by the structure morphisms in C.

These facts allow us to reuse steps (1)–(5) in the proof of Lemma 8.12.1. In the last two steps, instead of considering the two cases $i = j$ and $i \neq j$, we simply observe that by Lemma 3.1.29, all the vertices in the diagram D in $\mathsf{Gr}(X)$ have the same support, which is that of either 0^X or 1^X. Therefore, either Proposition 3.5.33 (for 0^X) or the Coherence Theorem 3.9.1 (for 1^X) implies the commutativity of D in C. This, in turn, proves the Yang-Baxter axiom.

The (3,1)-crossing, the (1,3)-crossing, and the (2,2)-crossing axioms are proved in the same way by following the steps in the proof of Lemma 8.12.1 as discussed in the previous paragraph. $\square$

8.15. The Matrix Symmetric Monoidal Bicategory

For a tight symmetric bimonoidal category C as in Definition 2.1.2, we saw in Theorem 8.14.26 that the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ is a braided monoidal bicategory.

In this section, we finish the proof that $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory as in Definition 6.5.9. Since a symmetric monoidal bicategory is a sylleptic monoidal bicategory as in Definition 6.5.7 with an extra axiom, our next task is to define the syllepsis that relates the braid-square to the identity.

Definition 8.15.1. Define the data of a modification ν with component 2-cells

$$\begin{tikzcd} & n \boxtimes m \arrow[dr, bend left, "\beta_{n,m}"] & \\ m \boxtimes n \arrow[ur, bend left, "\beta_{m,n}"] \arrow[rr, bend right, "1_{m\boxtimes n}"'] & \Downarrow \nu_{m,n} & m \boxtimes n \end{tikzcd}$$

defined as

$$\beta_{n,m}\beta_{m,n} = \mathbb{1}^{\tau_{n,m}}\mathbb{1}^{\tau_{m,n}} \xrightarrow{\nu_{m,n} = r^{\tau_{m,n}}_{\mathbb{1}^{\tau_{n,m}}}} \mathbb{1}^{mn} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn}$$

for $m,n \geq 0$ and r as in (8.13.13). ⋄

Explanation 8.15.2. In the setting of Definition 8.15.1, there are equalities

$$\nu_{m,n} = r^{\tau_{m,n}}_{\mathbb{1}^{\tau_{n,m}}} = \left(\eta^{\beta}_{(m,n)}\right)^{-1} = \varepsilon^{\beta}_{(n,m)}$$

with η^{β} the unit (8.13.36) and ε^{β} the counit (8.13.39) of the braiding. By (8.13.14) and the notation for morphisms in (8.7.6),

- each diagonal entry of $\nu_{m,n}$ has the form $\left(\lambda^{\oplus},\rho^{\oplus},\rho^{\otimes},\rho^{\bullet}\right)$, and
- each off-diagonal entry of $\nu_{m,n}$ has the form $\left(\lambda^{\oplus},\rho^{\otimes},\rho^{\bullet}\right)$.

The entries of the domain 1-cell $\mathbb{1}^{\tau_{n,m}}\mathbb{1}^{\tau_{m,n}}$ are described in (8.13.9) and (8.13.10). ⋄

Lemma 8.15.3. *ν in Definition 8.15.1 is an invertible modification.*

Proof. Each component 2-cell of ν is invertible by (8.13.13). The modification axiom (6.3.2) for ν is obtained from (8.13.42), which is the modification axiom for η^{β}, by pasting

- $\left(\eta^{\beta}_{(m,n)}\right)^{-1}$ on the left and
- $\left(\eta^{\beta}_{(m',n')}\right)^{-1}$ on the right

in each pasting diagram. □

We are now ready for the main result of this chapter.

Theorem 8.15.4 (Bicategorification)**.** *For each tight symmetric bimonoidal category* C*, the quintuple*

$$\left(\mathsf{Mat}^{\mathsf{C}}, \beta, R_{-|--}, R_{--|-}, \nu\right)$$

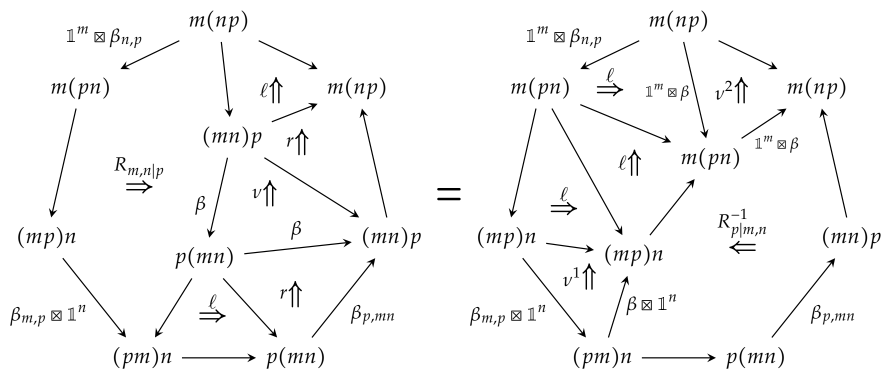
is a symmetric monoidal bicategory.

Proof. By Theorem 8.14.26, the monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$ in Theorem 8.12.9 becomes a braided monoidal bicategory when equipped with

- the braiding $(\beta,\beta^{\bullet},\eta^{\beta},\varepsilon^{\beta})$ in Section 8.13 and
- the hexagonators $R_{-|--}$ and $R_{--|-}$ in Definitions 8.14.9 and 8.14.21.

The syllepsis ν is an invertible modification by Lemma 8.15.3. It remains to check the two syllepsis axioms in Definition 6.5.7 and the triple braid axiom (6.5.10).

The (2,1)-syllepsis axiom is the next pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ for $m, n, p \geq 0$. Unlabeled edges are identity 1-cells.



In these pasting diagrams, we used the fact that each component 2-cell of the counit ε^a (8.8.38) is an instance of the base left unitor ℓ (8.2.2), applied to an identity 1-cell. The proof of the above equality is adapted from that of (8.12.3), which is the non-abelian 4-cocycle condition, in Lemma 8.12.1 by following steps (1)–(5) there. As we pointed out in the proof of Theorem 8.14.26, the key points that allow us to reuse those steps are as follows.

- Component 1-cells of $a^{\boxtimes}$ (8.8.2) and $a^{\boxtimes\bullet}$ (8.8.34) are identity matrices.
- Each component 1-cell of the braiding β (8.13.23) is a permutation matrix, with precisely one $\mathbb{1}$ in each row and each column and $\mathbb{0}$ in every other entry.
- Each 2-cell in the (2,1)-syllepsis axiom is entrywise defined by the structure morphisms in C.

The (1,2)-syllepsis axiom is proved in the same way.

The triple braid axiom (6.5.10) states the following pasting diagram equality in $\mathsf{Mat}^{\mathsf{C}}_{mn,nm}$ for $m, n \geq 0$.

$$
\begin{tikzcd}
mn \arrow[rr, "\beta_{m,n}"] \arrow[d, "\beta_{m,n}"'] \arrow[drr, "\mathbb{1}^{mn}"] \arrow[dr, phantom, "\Uparrow\nu_{m,n}"] & \arrow[d, phantom, "\Uparrow r"] & nm \\
nm \arrow[rr, "\beta_{n,m}"'] & \, & mn \arrow[u, "\beta_{m,n}"']
\end{tikzcd}
=
\begin{tikzcd}
mn \arrow[rr, "\beta_{m,n}"] \arrow[d, "\beta_{m,n}"'] \arrow[dr, phantom, "\Uparrow\ell"] & & nm \\
nm \arrow[rr, "\beta_{n,m}"'] \arrow[urr, "\mathbb{1}^{nm}"] & \arrow[u, phantom, "\Uparrow\nu_{n,m}"] & mn \arrow[u, "\beta_{m,n}"']
\end{tikzcd}
$$

The proof of this equality is also adapted from that of (8.12.3) in Lemma 8.12.1 by following steps (1)–(5) there. In the last two steps, instead of considering the two cases $i = j$ and $i \neq j$, we simply observe that by Lemma 3.1.29, all the vertices in the diagram D in $\mathsf{Gr}(X)$ have the same support, which is that of either 0^X or 1^X. Therefore, either Proposition 3.5.33 (for 0^X) or the Coherence Theorem 3.9.1 (for 1^X) implies the commutativity of D in C. This in turn proves the triple braid axiom. □

Example 8.15.5. Theorem 8.15.4 applies to the tight symmetric bimonoidal categories below.

- Distributive symmetric monoidal categories by Proposition 2.3.2.
- Symmetric monoidal closed categories with finite coproducts by Example 2.3.3.
- The category of modules over a commutative ring by Example 2.3.4.
- Distributive categories by Example 2.3.5.
- Σ and Σ' by Propositions 2.4.8 and 2.4.23.
- Right bipermutative categories by Proposition 2.5.7.
- Left bipermutative categories by Proposition 2.5.16.
- The category $\mathsf{Vect}^{\mathbb{C}}$ of finite dimensional complex vector spaces in Example 8.4.13.

In particular, when applied to the last example $\mathsf{C} = \mathsf{Vect}^{\mathbb{C}}$, Theorem 8.15.4 says that 2-vector spaces, in the form $\mathsf{Mat}^{\mathsf{Vect}^{\mathbb{C}}}$, constitute a genuine symmetric monoidal bicategory in which no structure morphisms are identities. $\diamond$

Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory

Volume II: Braided Bimonoidal Categories with Applications

Donald Yau

The author dedicates this book to Jacqueline.

Part 1

Braided Bimonoidal Categories

CHAPTER 1

Preliminaries on Braided Structures

Part 1 is about braided bimonoidal categories and some of their applications. To reduce the prerequisite to an absolute minimum, we do not assume any prior knowledge of the braid groups and braided monoidal categories. To facilitate the discussion later, in this chapter we review some properties of the braid groups and braided monoidal categories, starting from the basic definitions.

In Section 1.1, we first define the braid groups B_n algebraically. Then we discuss their geometric interpretation and some examples. Next we discuss braided analogues of block sums and block permutations, which are called sum braids and block braids, respectively. Each of these two constructions is first defined algebraically and is followed by a geometric interpretation and examples.

In Section 1.2, we discuss elementary block braids induced by the generating braid in B_2. They are the braided analogues of interval-swapping permutations. Elementary block braids are used in Definition 1.6.2 for the underlying braid of a braided canonical map, which, in turn, is used in the Braided Coherence Theorem 1.6.3. Elementary block braids will also play a crucial role in the braiding in the braided distortion category in Section 5.2.

In Section 1.3, we first recall the definition of a braided monoidal category. Then we prove several consequences of the braided monoidal category axioms, including some unity properties and two categorical versions of the third Reidemeister move. The unity properties are used in Sections 1.4 and 1.5 in the discussion of the Drinfeld center and the symmetric center. Both the unity properties and the categorical third Reidemeister move will be used in the proof of Theorem 2.2.1, which recovers all the Laplaza axioms in a braided bimonoidal category.

Monoidal, braided monoidal, and symmetric monoidal categories are related by two center constructions. In Section 1.4, we prove in detail that the Drinfeld center of a monoidal category is a braided monoidal category. There are several published accounts of this fact when the monoidal category is strict; see Note 1.7.2. However, we will need the general nonstrict version of the Drinfeld center. Since a proof for the general nonstrict case does not seem to have appeared before, we provide the detailed proof in this section.

Section 1.5 contains the observation that the symmetric center of a braided monoidal category is a symmetric monoidal category. Both the Drinfeld center and the symmetric center will be extended to the bimonoidal and the ring categorical setting in Chapter 4 and Section 9.6.

In Section 1.6, we review the Joyal-Street coherence theorem for braided monoidal categories. It will be used in the proof of the Braided Bimonoidal Coherence Theorem 5.4.4. Section 1.7 lists some references to the literature on the braid groups, the Drinfeld center, and coherence theorems for braided monoidal categories.

1.1. The Braid Groups

The purpose of this section is to recall the braid groups in its algebraic and geometric forms. Then we define sum braids and block braids, which are the braided analogues of block sums and block permutations. These constructions are needed to define the braided distortion category in Section 5.2.

The Braid Groups.

Definition 1.1.1. For $n \geq 0$, the *nth braid group* B_n is defined as follows.

- Both B_0 and B_1 are the trivial group with one element.
- For $n \geq 2$, the braid group B_n is the group generated by the generators $s_1, \ldots, s_{n-1}$, and is subject to the following *braid relations*:

$$\begin{aligned} s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2 \text{ and } 1 \leq i,j \leq n-1. \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 1 \leq i \leq n-2. \end{aligned} \tag{1.1.2}$$

Moreover, we define the following.

- An element in B_n is called a *braid*.
- $s_i \in B_n$ is called the ith *generating braid*.
- To emphasize that s_i belongs to B_n, we write it as $s_i^{(n)}$.
- The identity element in B_n is called the *identity braid* and is denoted by id or id_n. ◇

Example 1.1.3. The braid group B_2 has one generator s_1 and no relations, so it is the infinite cyclic group. ◇

Explanation 1.1.4. The braid group B_n admits the following geometric interpretation. A *topological interval* is a topological space homeomorphic to the closed unit interval $\mathcal{I} = [0,1] \subseteq \mathbb{R}$. A *geometric braid on n strings* is a subset

$$b = \coprod_{i=1}^{n} I_i \subseteq \mathbb{R}^2 \times \mathcal{I}$$

such that the following two statements hold.

(i) Each I_i, called the *ith string of b*, is a topological interval via the homeomorphism

$$I_i \subseteq \mathbb{R}^2 \times \mathcal{I} \longrightarrow \mathcal{I}$$

with the second map the projection.

(ii) There are equalities

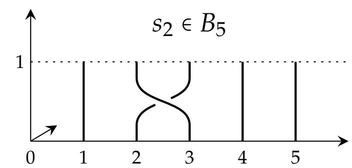
$$I_i \cap \big(\mathbb{R}^2 \times \{0\}\big) = \big\{(i,0,0)\big\} \quad \text{for } 1 \leq i \leq n \text{ and}$$

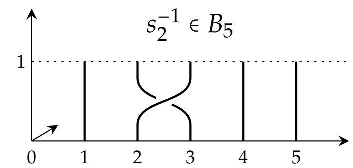
$$\Big(\coprod_{i=1}^{n} I_i\Big) \cap \big(\mathbb{R}^2 \times \{1\}\big) = \big\{(1,0,1),(2,0,1),\ldots,(n,0,1)\big\}.$$

A *braid on n strings* is an isotopy class of geometric braids on n strings. We will use the same notation for a geometric braid and its isotopy class.

Similar to the fundamental group, using the $\mathcal{I}$ coordinate, the set of braids on n strings is a group under vertical composition. This group is naturally identified with the braid group B_n, in which the generator $s_i \in B_n$ is identified with the braid on n strings with the $(i+1)$st string crossing over the ith string when viewed from bottom to top. We identify an element in the braid group B_n with the corresponding braid on n strings. ◇

Example 1.1.5. The generator $s_2 \in B_5$ and its inverse are the following braids.



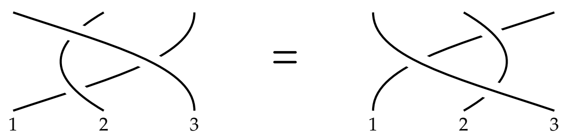


In each of these two pictures, the three axes, the horizontal dotted line, and the numbers $\{0, \ldots, 5\}$ are not part of the braid. They are there to help the reader visualize the braid. From now on, we will omit the axes and the dotted lines. $\diamond$

Example 1.1.6. If $n = 3$ and $i = 1$, then the second braid relation

$$s_1 s_2 s_1 = s_2 s_1 s_2 \in B_3$$

in (1.1.2) is the equality of the following two braids.



This equality is known as the third Reidemeister move. $\diamond$

The Sum Braids.

Motivation 1.1.7. Σ_n denotes the permutation group on n letters. For permutations $(\sigma, \tau) \in \Sigma_m \times \Sigma_n$, their *block sum* is the permutation $\sigma \oplus \tau \in \Sigma_{m+n}$ defined as

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } 1 \leq i \leq m \text{ and} \\ \tau(i-m) + m & \text{if } m+1 \leq i \leq m+n. \end{cases} \tag{1.1.8}$$

In other words, σ permutes the first m objects, and τ permutes the last n objects with the indices shifted appropriately. Next we define the braided analogue of the block sum. $\diamond$

Definition 1.1.9. For $m, n \geq 1$, the *sum braid* is the group homomorphism

$$B_m \times B_n \xrightarrow{\oplus} B_{m+n} \tag{1.1.10}$$

defined on the generators by

$$s_i^{(m)} \oplus s_j^{(n)} = s_i^{(m+n)} s_{m+j}^{(m+n)} \in B_{m+n}$$

for $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$, extended multiplicatively to all of $B_m \times B_n$. If either m or n is 0, then we define the sum braid to be the identity map. $\diamond$

The symmetric group Σ_n admits a similar generator and relation description as the braid group B_n with s_i replaced by the adjacent transposition $(i, i+1)$ and with the extra relation

$$(i, i+1)(i, i+1) = \mathrm{id}$$

for each i.

Definition 1.1.11. The group homomorphism

$$\pi : B_n \longrightarrow \Sigma_n \tag{1.1.12}$$

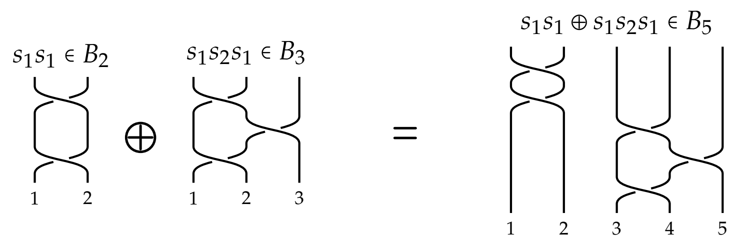
is defined by sending the generating braid $s_i \in B_n$ to the transposition $(i, i+1) \in \Sigma_n$ for $1 \le i \le n-1$, extended multiplicatively to all of B_n. For a braid $b \in B_n$, its image $\pi(b) \in \Sigma_n$ is denoted by $\overline{b}$ and is called the *underlying permutation* of b. $\diamond$

Explanation 1.1.13. The sum braid (1.1.10) extends the block sum (1.1.8) of permutations in the sense that there is a commutative diagram

$$\begin{tikzcd} B_m \times B_n \arrow[r, "\oplus"] \arrow[d, "{(\pi,\pi)}"'] & B_{m+n} \arrow[d, "\pi"] \\ \Sigma_m \times \Sigma_n \arrow[r, "\oplus"] & \Sigma_{m+n} \end{tikzcd} \tag{1.1.14}$$

of group homomorphisms, with each π the group homomorphism in (1.1.12). For braids $(\sigma, \tau) \in B_m \times B_n$, the sum braid $\sigma \oplus \tau \in B_{m+n}$ is geometrically the braid obtained by placing σ and τ side-by-side with σ on the left. $\diamond$

Example 1.1.15. The sum braid of $s_1 s_1 \in B_2$ and $s_1 s_2 s_1 \in B_3$ is geometrically the braid on the right-hand side below.



Algebraically, it is the braid

$$s_1^{(2)} s_1^{(2)} \oplus s_1^{(3)} s_2^{(3)} s_1^{(3)} = s_1^{(5)} s_1^{(5)} s_3^{(5)} s_4^{(5)} s_3^{(5)} \in B_5.$$

The superscripts are included to clarify which braid group each s_i belongs to. $\diamond$

Example 1.1.16. Since the sum braid is a group homomorphism, there are equalities

$$\begin{aligned} \sigma \oplus \tau &= (\sigma \oplus \mathrm{id}_n)(\mathrm{id}_m \oplus \tau) \\ &= (\mathrm{id}_m \oplus \tau)(\sigma \oplus \mathrm{id}_n) \end{aligned} \tag{1.1.17}$$

in B_{m+n} for braids $(\sigma, \tau) \in B_m \times B_n$. $\diamond$

The Block Braids.

Motivation 1.1.18. Suppose $n \ge 1$, $k_1, \ldots, k_n \ge 0$, and $k = k_1 + \cdots + k_n$. We write $\underline{k}$ for the sequence $(k_1, \ldots, k_n)$. The *block permutation*

$$\Sigma_n \xrightarrow{(-)\langle \underline{k} \rangle} \Sigma_k$$

is defined as follows for $\sigma \in \Sigma_n$, $1 \le j \le n$, and $1 \le i \le k_j$.

$$\big(\sigma\langle \underline{k} \rangle\big)\big(\overbrace{k_1 + \cdots + k_{j-1}}^{0 \text{ if } j=1} + i\big) = \overbrace{k_{\sigma^{-1}(1)} + \cdots + k_{\sigma^{-1}(\sigma(j)-1)}}^{0 \text{ if } \sigma(j)=1} + i \tag{1.1.19}$$

To interpret this formula, consider n consecutive intervals with the jth interval having k_j objects for each $1 \le j \le n$. The block permutation $\sigma\langle \underline{k} \rangle \in \Sigma_k$ permutes the n intervals as $\sigma \in \Sigma_n$ permutes n objects. For each $1 \le j \le n$, within the jth interval the order of the k_j objects remain unchanged. After the block permutation, the new n intervals have lengths

$$\big(k_{\sigma^{-1}(1)}, \ldots, k_{\sigma^{-1}(n)}\big).$$

The formula (1.1.19) states that $\sigma\langle \underline{k}\rangle$ sends the ith object in the original jth interval to the ith object in the new $\sigma(j)$th interval. Here are two examples.

- If $k_j = 1$ for $1 \le j \le n$, then
$$\sigma\langle 1, \ldots, 1\rangle = \sigma.$$
- If $\sigma = \mathrm{id}_n \in \Sigma_n$, then
$$\mathrm{id}_n\langle \underline{k}\rangle = \mathrm{id}_k \in \Sigma_k.$$

Next we define the braided analogue of block permutation by expanding each string into a band of several parallel strings. We first define block braid algebraically and then return to the geometric interpretation using parallel strings. ◇

Definition 1.1.20. Suppose $n \ge 1$, $k_1, \ldots, k_n \ge 0$, and $k = k_1 + \cdots + k_n$. The *block braid*

$$\begin{array}{ccc} B_n & \xrightarrow{(-)\langle \underline{k}\rangle} & B_k \\ b & \longmapsto & b\langle k_1, \ldots, k_n\rangle = b\langle \underline{k}\rangle \end{array} \tag{1.1.21}$$

is the constant function at the identity braid $\mathrm{id} \in B_{k_1}$ if $n = 1$. For $n \ge 2$, the block braid is the function defined by the following three steps.

(i) For $1 \le i \le n-1$, the generating braid $s_i \in B_n$ is sent to the product

$$\begin{aligned} &s_i\langle \underline{k}\rangle = \sigma_1 \cdots \sigma_{k_i} \in B_k \quad \text{with} \\ &\sigma_j = \prod_{m=1}^{k_{i+1}} s^{(k)}_{k_1+\cdots+k_{i-1}+j+k_{i+1}-m} \in B_k \\ &\quad = \big(s^{(k)}_{k_1+\cdots+k_{i-1}+j+k_{i+1}-1}\big)\big(s^{(k)}_{k_1+\cdots+k_{i-1}+j+k_{i+1}-2}\big)\cdots\big(s^{(k)}_{k_1+\cdots+k_{i-1}+j}\big) \end{aligned} \tag{1.1.22}$$

for $1 \le j \le k_i$. The empty product, which happens if k_i or k_{i+1} is 0, is defined as the identity.

(ii) For $1 \le i \le n-1$, the block braid for s_i^{-1} is defined as

$$s_i^{-1}\langle \underline{k}\rangle = \Big[s_i\langle \underbrace{k_1, \ldots, k_{i-1}}_{\varnothing \text{ if } i=1}, k_{i+1}, k_i, \underbrace{k_{i+2}, \ldots, k_n}_{\varnothing \text{ if } i=n-1}\rangle\Big]^{-1} \in B_k \tag{1.1.23}$$

(iii) Inductively, for $\sigma, \tau \in B_n$, suppose the block braids $\sigma\langle\cdots\rangle, \tau\langle\cdots\rangle \in B_k$ have already been defined. The block braid for the product $\sigma\tau$ is defined as

$$(\sigma\tau)\langle \underline{k}\rangle = \sigma\langle \overline{\tau}\underline{k}\rangle \cdot \tau\langle \underline{k}\rangle \in B_k \tag{1.1.24}$$

with
- $\overline{\tau} = \pi(\tau) \in \Sigma_n$ the underlying permutation of τ in (1.1.12) and
- $\overline{\tau}\underline{k} = \big(k_{\overline{\tau}^{-1}(1)}, \ldots, k_{\overline{\tau}^{-1}(n)}\big)$.

This finishes the definition of the block braid. ◇

Explanation 1.1.25. Let us provide geometric interpretation for Definition 1.1.20.

(1) The block braid (1.1.21) extends the block permutation (1.1.19) in the sense that there is a commutative diagram

$$\begin{tikzcd} B_n \arrow[r, "\langle \underline{k} \rangle"] \arrow[d, "\pi"'] & B_{k_1+\cdots+k_n} \arrow[d, "\pi"] \\ \Sigma_n \arrow[r, "\langle \underline{k} \rangle"] & \Sigma_{k_1+\cdots+k_n} \end{tikzcd} \tag{1.1.26}$$

of functions with π the underlying permutation in (1.1.12). The formula (1.1.24) implies that the block braid is *not* a group homomorphism and similarly for the block permutation. Therefore, the diagram (1.1.26) is only commutative in the category of sets and not in the category of groups.

(2) Geometrically, for a braid $b \in B_n$, the block braid $b\langle \underline{k} \rangle$ is obtained from b by replacing its jth string by k_j parallel strings for $1 \leq j \leq n$.
- In the block braid $s_i\langle \underline{k} \rangle$, each of the k_{i+1} strings in the $(i+1)$st block crosses over each of the k_i strings in the ith block.
- The braid σ_j in (1.1.22) encodes the k_{i+1} strings in the $(i+1)$st block crossing over the jth string in the ith block. More precisely, the lth generator
$$s^{(k)}_{k_1+\cdots+k_{i-1}+j+l-1} \in B_k,$$
counting from the right in σ_j, represents the lth string in the $(i+1)$st block crossing over the jth string in the ith block.
- In the formula (1.1.23), the braid
$$s_i\langle k_1, \ldots, k_{i-1}, k_{i+1}, k_i, k_{i+2}, \ldots, k_n \rangle$$
is similar to $s_i\langle \underline{k} \rangle$, except that the ith string and the $(i+1)$st string in s_i are replaced by, respectively, k_{i+1} and k_i parallel strings. Therefore, its inverse $s_i^{-1}\langle \underline{k} \rangle$ is obtained from s_i^{-1} by replacing the jth string by k_j parallel strings for $1 \leq j \leq n$.
- The formula (1.1.24) means that $(\sigma\tau)\langle \underline{k} \rangle$ is obtained from the braid $\sigma\tau \in B_n$ by replacing its jth string by k_j parallel strings for $1 \leq j \leq n$.

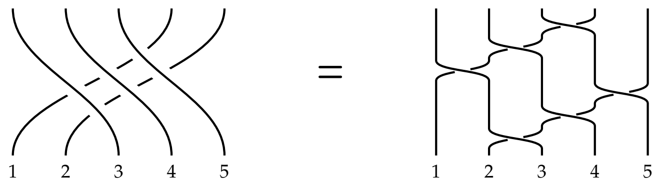
An inspection of the relevant pictures shows that the block braid is well defined, that is, respects the braid relations (1.1.2) in B_n. Moreover, the algebraic definition agrees with the geometric interpretation. ◇

Example 1.1.27. If $k_j = 1$ for $1 \leq j \leq n$ in (1.1.21), then
$$b\langle 1, \ldots, 1 \rangle = b \in B_n.$$
Indeed, by (1.1.22) and (1.1.23), this equality is true for each generating braid $s_i \in B_n$ and its inverse s_i^{-1}. The formula (1.1.24) then implies the equality for a general braid. Moreover, for the identity braid $\mathrm{id}_n \in B_n$, there is an equality
$$\mathrm{id}_n\langle \underline{k} \rangle = \mathrm{id}_k \in B_k$$
by (1.1.23) and (1.1.24). ◇

Example 1.1.28. For the generator $s_1 \in B_2$, $k_1 = 2$, and $k_2 = 3$, the block braid
$$s_1\langle 2, 3 \rangle = \overbrace{s_3^{(5)} s_2^{(5)} s_1^{(5)}}^{\sigma_1} \overbrace{s_4^{(5)} s_3^{(5)} s_2^{(5)}}^{\sigma_2} \in B_5$$

is illustrated below.

$$s_1\langle 2,3\rangle \in B_5 \qquad\qquad s_3^{(5)}s_2^{(5)}s_1^{(5)}s_4^{(5)}s_3^{(5)}s_2^{(5)}$$



- On the left-hand side, the first string in $s_1 \in B_2$ is replaced by two parallel strings, and its second string is replaced by three parallel strings. Each of the latter three strings crosses over each of the former two strings.
- On the right-hand side, from bottom to top, σ_2 contains the first three crossings, and σ_1 contains the last three crossings. ⋄

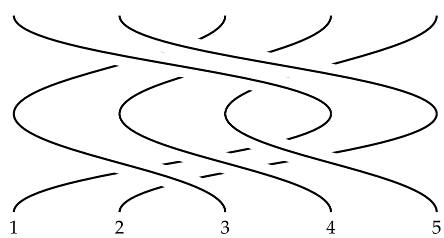
Example 1.1.29. To illustrate the formula (1.1.24) for $(\sigma\tau)\langle \underline{k}\rangle$, consider

$$\sigma = \tau = s_1 \in B_2, \quad k_1 = 2, \quad \text{and} \quad k_2 = 3.$$

The block braid

$$\begin{aligned}(s_1s_1)\langle 2,3\rangle &= s_1\langle 3,2\rangle \cdot s_1\langle 2,3\rangle \in B_5\\ &= \big(s_2^{(5)}s_1^{(5)}s_3^{(5)}s_2^{(5)}s_4^{(5)}s_3^{(5)}\big)\big(s_3^{(5)}s_2^{(5)}s_1^{(5)}s_4^{(5)}s_3^{(5)}s_2^{(5)}\big)\end{aligned}$$

is illustrated below.



In this picture, the bottom half is $s_1\langle 2,3\rangle$ as in Example 1.1.28. The top half is $s_1\langle 3,2\rangle$. The entire block braid is obtained from $s_1s_1 \in B_2$ by replacing, from the bottom, its first string by two parallel strings and its second string by three parallel strings. ⋄

1.2. Elementary Block Braids

Block braids of the type in Example 1.1.28 are part of the braiding in the braided distortion category in Section 5.2. They are also used in Definition 1.6.2 for the underlying braid of a braided canonical map and the Braided Coherence Theorem 1.6.3. In this section, we discuss some of their properties.

Motivation 1.2.1. For the transposition $\tau \in \Sigma_2$, the block permutation $\tau\langle m,n\rangle \in \Sigma_{m+n}$ swaps an interval of length m with an interval of length n and leaves the order within each interval unchanged. It is given by the formula

$$\big(\tau\langle m,n\rangle\big)(j) = \begin{cases} j+n & \text{if } 1 \le j \le m \text{ and}\\ j-m & \text{if } m+1 \le j \le m+n.\end{cases} \tag{1.2.2}$$

The generating braid $s_1 \in B_2$ has underlying braid the transposition $\tau \in \Sigma_2$. The braided analogue of the interval-swapping permutation $\tau\langle m, n\rangle \in \Sigma_{m+n}$ is defined as a block braid of s_1. ⋄

Definition 1.2.3. For the generator $s_1 \in B_2$ and $m, n \geq 0$, define the *elementary block braid*

$$b^{\oplus}_{m,n} = s_1\langle m, n\rangle \in B_{m+n} \tag{1.2.4}$$

with the right-hand side as in (1.1.22). ⋄

Explanation 1.2.5. Geometrically, $b^{\oplus}_{m,n}$ is obtained from the generating braid $s_1 \in B_2$ by replacing, from the bottom, its first string by m parallel strings and its second string by n parallel strings. Each of the latter n strings crosses over each of the former m strings. For example, $b^{\oplus}_{2,3} = s_1\langle 2, 3\rangle$ is the block braid in Example 1.1.28.

Algebraically (1.1.22), $b^{\oplus}_{m,n}$ is the product

$$\begin{aligned} b^{\oplus}_{m,n} &= b_1 \cdots b_m \in B_{m+n} \quad \text{with} \\ b_i &= \prod_{j=1}^{n} s^{(m+n)}_{i+n-j} = \big(s^{(m+n)}_{i+n-1}\big)\big(s^{(m+n)}_{i+n-2}\big)\cdots\big(s^{(m+n)}_{i}\big) \end{aligned} \tag{1.2.6}$$

for $1 \leq i \leq m$. Counting from the right, for $1 \leq l \leq n$, the lth generator $s^{(m+n)}_{i+l-1}$ in b_i represents the lth string in the second block crossing over the ith string in the first block. ⋄

Next we provide another formula for the elementary block braid $b^{\oplus}_{m,n}$ that will be useful in proving some of its properties.

Lemma 1.2.7. *For $m, n \geq 0$, there is an equality*

$$\begin{aligned} b^{\oplus}_{m,n} &= b'_n \cdots b'_1 \in B_{m+n} \quad \textit{with} \\ b'_j &= \big(s^{(m+n)}_{j}\big)\big(s^{(m+n)}_{j+1}\big)\cdots\big(s^{(m+n)}_{j+m-1}\big) \end{aligned} \tag{1.2.8}$$

for $1 \leq j \leq n$.

Proof. The formula (1.2.8) is another interpretation of $b^{\oplus}_{m,n}$ as obtained from the generating braid $s_1 \in B_2$ by replacing, from the bottom, its first string by m parallel strings and its second string by n parallel strings. For $1 \leq j \leq n$, the braid $b'_j \in B_{m+n}$ encodes the jth string in the second block crossing over the m strings in the first block. □

The next two lemmas describe how the factors of $b^{\oplus}_{m,n}$ in (1.2.6) and (1.2.8) commute with generating braids.

Lemma 1.2.9. *For $1 \leq i \leq m-1$ and $1 \leq j \leq n$, the braid b'_j in (1.2.8) satisfies the following equality.*

$$b'_j s^{(m+n)}_{j+i-1} = s^{(m+n)}_{j+i} b'_j \in B_{m+n} \tag{1.2.10}$$

Proof. Since we are working in the braid group B_{m+n} only, we omit the superscript $(m+n)$. The equality (1.2.10) follows from the following computation that uses the braid relations (1.1.2) repeatedly.

$$\begin{aligned}
b'_j s_{j+i-1} &= \overbrace{\big(s_j \cdots s_{j+i-2}\big)}^{\varnothing \text{ if } i=1} \big(s_{j+i-1}s_{j+i}\big) \overbrace{\big(s_{j+i+1} \cdots s_{j+m-1}\big)}^{\varnothing \text{ if } i=m-1} s_{j+i-1}\\
&= \big(s_j \cdots s_{j+i-2}\big)\big[\big(s_{j+i-1}s_{j+i}\big)s_{j+i-1}\big]\big(s_{j+i+1} \cdots s_{j+m-1}\big)\\
&= \big(s_j \cdots s_{j+i-2}\big)\big[s_{j+i}\big(s_{j+i-1}s_{j+i}\big)\big]\big(s_{j+i+1} \cdots s_{j+m-1}\big)\\
&= s_{j+i}\big(s_j \cdots s_{j+i-2}\big)\big(s_{j+i-1}s_{j+i}\big)\big(s_{j+i+1} \cdots s_{j+m-1}\big)\\
&= s_{j+i}b'_j
\end{aligned}$$

The second equality uses the first braid relation $m-i-1$ times. The third equality uses the second braid relation in the form

$$s_{j+i-1}s_{j+i}s_{j+i-1} = s_{j+i}s_{j+i-1}s_{j+i}.$$

The fourth equality uses the first braid relation $i-1$ times. □

An almost identical calculation as in the proof of Lemma 1.2.9 proves the following.

Lemma 1.2.11. *For $1 \leq i \leq m$ and $1 \leq j \leq n-1$, the braid b_i in (1.2.6) satisfies the following equality.*

$$b_i s_{i+j}^{(m+n)} = s_{i+j-1}^{(m+n)} b_i \in B_{m+n} \tag{1.2.12}$$

The following lemma will be used to show the naturality of the braiding in the braided distortion category in Section 5.2.

Lemma 1.2.13. *The following equality holds in B_{m+n} for $m,n \geq 0$, $\sigma \in B_m$, and $\tau \in B_n$.*

$$b^{\oplus}_{m,n}(\sigma \oplus \tau) = (\tau \oplus \sigma)b^{\oplus}_{m,n} \tag{1.2.14}$$

Proof. It suffices to prove (1.2.14) in two cases: $\tau = \mathrm{id}_n$ or $\sigma = \mathrm{id}_m$. In fact, if these two special cases are true, then the general case follows from the following computation using (1.1.17).

$$\begin{aligned}
b^{\oplus}_{m,n}(\sigma \oplus \tau) &= b^{\oplus}_{m,n}(\sigma \oplus \mathrm{id}_n)(\mathrm{id}_m \oplus \tau)\\
&= (\mathrm{id}_n \oplus \sigma)b^{\oplus}_{m,n}(\mathrm{id}_m \oplus \tau)\\
&= (\mathrm{id}_n \oplus \sigma)(\tau \oplus \mathrm{id}_m)b^{\oplus}_{m,n}\\
&= (\tau \oplus \sigma)b^{\oplus}_{m,n}
\end{aligned}$$

Since the sum braid is a group homomorphism, to prove (1.2.14) for $\tau = \mathrm{id}_n$, it suffices to consider a generating braid $\sigma = s_i^{(m)} \in B_m$ for $1 \leq i \leq m-1$. This case follows from the following computation using (1.2.8) and (1.2.10) repeatedly.

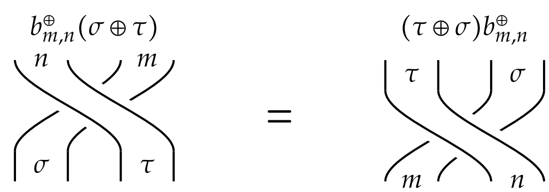
$$\begin{aligned}
b^{\oplus}_{m,n}(s_i^{(m)} \oplus \mathrm{id}_n) &= b'_n \cdots b'_2 b'_1 s_i^{(m+n)}\\
&= b'_n \cdots b'_2 s_{i+1}^{(m+n)} b'_1\\
&= b'_n \cdots b'_3 s_{i+2}^{(m+n)} b'_2 b'_1\\
&\;\;\vdots\\
&= s_{i+n}^{(m+n)} b'_n \cdots b'_1\\
&= (\mathrm{id}_n \oplus s_i^{(m)})b^{\oplus}_{m,n}.
\end{aligned}$$

Similarly, to prove the desired equality (1.2.14) for $\sigma = \mathrm{id}_m$, it suffices to consider a generating braid $\tau = s_j^{(n)} \in B_n$ for $1 \le j \le n-1$. This case follows from the following computation using (1.2.6) and (1.2.12) repeatedly.

$$\begin{aligned}
b^{\oplus}_{m,n}(\mathrm{id}_m \oplus s_j^{(n)}) &= b_1 \cdots b_{m-1} b_m s^{(m+n)}_{m+j}\\
&= b_1 \cdots b_{m-1} s^{(m+n)}_{m+j-1} b_m\\
&= b_1 \cdots b_{m-2} s^{(m+n)}_{m+j-2} b_{m-1} b_m\\
&\;\;\vdots\\
&= s_j^{(m+n)} b_1 \cdots b_m\\
&= (s_j^{(n)} \oplus \mathrm{id}_m) b^{\oplus}_{m,n}
\end{aligned}$$

This finishes the proof of the lemma. □

Explanation 1.2.15. Recall from Explanation 1.1.13 that $\sigma \oplus \tau$ is the braid with σ on the left and τ on the right. The equality (1.2.14) may be visualized as follows.



On the left-hand side, the bottom half consists of the braids σ and τ side-by-side, and the top half is $b^{\oplus}_{m,n}$. On the right-hand side, the bottom half is $b^{\oplus}_{m,n}$, and the top half is τ and σ side-by-side. ⋄

The following lemma will be used to prove the hexagon axioms in the braided distortion category in Section 5.2.

Lemma 1.2.16. *For $l, m, n \ge 0$, the following equalities hold in B_{l+m+n}.*

$$b^{\oplus}_{l+m,n} = (b^{\oplus}_{l,n} \oplus \mathrm{id}_m)(\mathrm{id}_l \oplus b^{\oplus}_{m,n}) \tag{1.2.17}$$

$$b^{\oplus}_{l,m+n} = (\mathrm{id}_m \oplus b^{\oplus}_{l,n})(b^{\oplus}_{l,m} \oplus \mathrm{id}_n) \tag{1.2.18}$$

Proof. The first equality (1.2.17) follows from the following computation using (1.2.6) and the multiplicativity of the sum braid.

$$\begin{aligned}
&(b^{\oplus}_{l,n} \oplus \mathrm{id}_m)(\mathrm{id}_l \oplus b^{\oplus}_{m,n})\\
&= \Big[\Big(\prod_{h=1}^{l}\prod_{j=1}^{n} s^{(l+n)}_{h+n-j}\Big) \oplus \mathrm{id}_m\Big]\Big[\mathrm{id}_l \oplus \Big(\prod_{i=1}^{m}\prod_{j=1}^{n} s^{(m+n)}_{i+n-j}\Big)\Big]\\
&= \Big[\prod_{h=1}^{l}\prod_{j=1}^{n} s^{(l+m+n)}_{h+n-j}\Big]\Big[\prod_{i=1}^{m}\prod_{j=1}^{n} s^{(l+m+n)}_{l+i+n-j}\Big]\\
&= \prod_{h=1}^{l+m}\prod_{j=1}^{n} s^{(l+m+n)}_{h+n-j} = b^{\oplus}_{l+m,n}
\end{aligned}$$

The equality (1.2.18) is proved using the alternative factorization in (1.2.8),

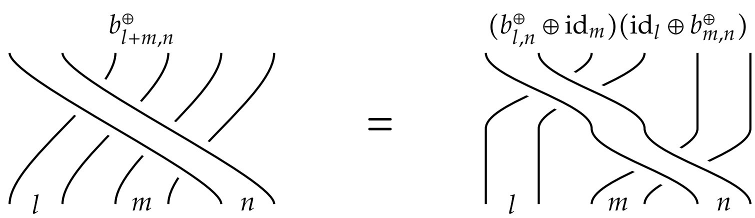
$$b^{\oplus}_{m,n} = \prod_{j=1}^{n} \prod_{i=1}^{m} s^{(m+n)}_{n-j+i} \in B_{m+n}.$$

Then (1.2.18) follows from the following computation.

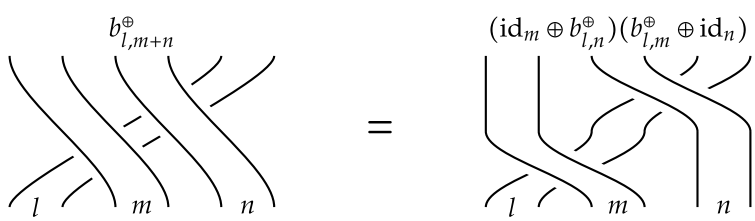
$$\begin{aligned}
&(\mathrm{id}_m \oplus b^{\oplus}_{l,n})(b^{\oplus}_{l,m} \oplus \mathrm{id}_n)\\
&= \left[\mathrm{id}_m \oplus \Big(\prod_{j=1}^{n} \prod_{h=1}^{l} s^{(l+n)}_{n-j+h}\Big)\right]\left[\Big(\prod_{i=1}^{m} \prod_{h=1}^{l} s^{(l+m)}_{m-i+h}\Big) \oplus \mathrm{id}_n\right]\\
&= \left[\prod_{j=1}^{n} \prod_{h=1}^{l} s^{(l+m+n)}_{m+n-j+h}\right]\left[\prod_{i=1}^{m} \prod_{h=1}^{l} s^{(l+m+n)}_{m-i+h}\right]\\
&= \prod_{j=1}^{m+n} \prod_{h=1}^{l} s^{(l+m+n)}_{m+n-j+h} = b^{\oplus}_{l,m+n}
\end{aligned}$$

This finishes the proof of the lemma. □

Explanation 1.2.19. The equality (1.2.17) may be visualized as follows.



Similarly, the equality (1.2.18) may be visualized as follows.



In these pictures, each label l means that that band contains l parallel strings, and similarly for m and n. ◇

1.3. Braided Monoidal Categories

In this section, we recall (braided) monoidal categories, (braided) monoidal functors, and monoidal natural transformations. Then we discuss basic properties of braided monoidal categories that we will use later.

Monoidal Categories.

Definition 1.3.1. A *monoidal category* is a tuple

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$$

consisting of

- a category C;

- a functor
$$\otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C},$$
which is called the *monoidal product*;
- an object $\mathbb{1} \in \mathsf{C}$, which is called the *monoidal unit*;
- a natural isomorphism
$$(X \otimes Y) \otimes Z \xrightarrow[\cong]{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z)$$
for all objects $X, Y, Z \in \mathsf{C}$, which is called the *associativity isomorphism*; and
- natural isomorphisms
$$\mathbb{1} \otimes X \xrightarrow[\cong]{\lambda_X} X \quad \text{and} \quad X \otimes \mathbb{1} \xrightarrow[\cong]{\rho_X} X$$
for all objects $X \in \mathsf{C}$, which are called the *left unit isomorphism* and the *right unit isomorphism*, respectively.

These data are subject to the following two axioms.

The Unity Axiom: The diagram

(1.3.2)
$$\begin{tikzcd} (X \otimes \mathbb{1}) \otimes Y \arrow[r, "\alpha_{X,\mathbb{1},Y}"] \arrow[d, "\rho_X \otimes 1_Y"'] & X \otimes (\mathbb{1} \otimes Y) \arrow[d, "1_X \otimes \lambda_Y"] \\ X \otimes Y \arrow[r, equal] & X \otimes Y \end{tikzcd}$$

is commutative for all objects $X, Y \in \mathsf{C}$.

The Pentagon Axiom: The pentagon

(1.3.3)
$$\begin{tikzcd}[column sep=small] & (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y\otimes Z}"] & \\ \big((W \otimes X) \otimes Y\big) \otimes Z \arrow[ur, "\alpha_{W\otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes 1_Z"'] & & W \otimes \big(X \otimes (Y \otimes Z)\big) \\ \big(W \otimes (X \otimes Y)\big) \otimes Z \arrow[rr, "\alpha_{W,X\otimes Y,Z}"] & & W \otimes \big((X \otimes Y) \otimes Z\big) \arrow[u, "1_W \otimes \alpha_{X,Y,Z}"'] \end{tikzcd}$$

is commutative for all objects $W, X, Y, Z \in \mathsf{C}$.

This finishes the definition of a monoidal category. A monoidal category is *strict* if α, λ, and ρ are identity natural transformations. ◇

In a monoidal category, the equality

(1.3.4)
$$\lambda_{\mathbb{1}} = \rho_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1}$$

and the commutative diagrams

(1.3.5)
$$\begin{tikzcd} (\mathbb{1} \otimes X) \otimes Y \arrow[r, "\alpha_{\mathbb{1},X,Y}"] \arrow[d, "\lambda_X \otimes 1_Y"'] & \mathbb{1} \otimes (X \otimes Y) \arrow[d, "\lambda_{X\otimes Y}"] \\ X \otimes Y \arrow[r, equal] & X \otimes Y \end{tikzcd} \qquad \begin{tikzcd} (X \otimes Y) \otimes \mathbb{1} \arrow[r, "\alpha_{X,Y,\mathbb{1}}"] \arrow[d, "\rho_{X\otimes Y}"'] & X \otimes (Y \otimes \mathbb{1}) \arrow[d, "1_X \otimes \rho_Y"] \\ X \otimes Y \arrow[r, equal] & X \otimes Y \end{tikzcd}$$

are formal consequences of the monoidal category axioms. These two diagrams are called the *left unity diagram* and the *right unity diagram*, respectively.

Definition 1.3.6. A *monoid* in a monoidal category C is a triple (X, μ, η) with
- X an object in C;

- $\mu : X \otimes X \longrightarrow X$ a morphism, which is called the *multiplication*; and
- $\eta : \mathbb{1} \longrightarrow X$ a morphism, which is called the *unit*.

These data are required to make the following associativity and unity diagrams commutative.

$$\begin{tikzcd}
(X \otimes X) \otimes X \arrow[r, "\alpha"] \arrow[dd, "\mu \otimes 1_X"'] & X \otimes (X \otimes X) \arrow[d, "1_X \otimes \mu"] \\
& X \otimes X \arrow[d, "\mu"] \\
X \otimes X \arrow[r, "\mu"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes X \arrow[r, "\lambda_X"] \arrow[d, "\eta \otimes 1_X"'] & X \arrow[d, equal] \\
X \otimes X \arrow[r, "\mu"] & X \\
X \otimes \mathbb{1} \arrow[r, "\rho_X"] \arrow[u, "1_X \otimes \eta"] & X \arrow[u, equal]
\end{tikzcd}$$

A *morphism* of monoids

$$f : (X, \mu^X, \eta^X) \longrightarrow (Y, \mu^Y, \eta^Y)$$

is a morphism $f : X \longrightarrow Y$ in C that preserves the multiplications and the units in the sense that the diagrams

$$\begin{tikzcd}
X \otimes X \arrow[r, "f \otimes f"] \arrow[d, "\mu^X"'] & Y \otimes Y \arrow[d, "\mu^Y"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta^X"] \arrow[d, equal] & X \arrow[d, "f"] \\
\mathbb{1} \arrow[r, "\eta^Y"] & Y
\end{tikzcd}$$

are commutative. $\diamond$

Definition 1.3.7. For monoidal categories C and D, a *monoidal functor*

$$(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of

- a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$;
- a natural transformation, which is called the *monoidal constraint*,

(1.3.8) $$FX \otimes FY \xrightarrow{F^2} F(X \otimes Y) \in \mathsf{D}$$

  for objects $X, Y \in \mathsf{C}$; and
- a morphism, which is called the *unit constraint*,

(1.3.9) $$\mathbb{1}^{\mathsf{D}} \xrightarrow{F^0} F\mathbb{1}^{\mathsf{C}} \in \mathsf{D}.$$

These data are required to satisfy the following associativity and unity axioms.

Associativity: The diagram

(1.3.10) $$\begin{tikzcd}
\big(FX \otimes FY\big) \otimes FZ \arrow[r, "\alpha^{\mathsf{D}}"] \arrow[d, "F^2 \otimes 1_{FZ}"'] & FX \otimes \big(FY \otimes FZ\big) \arrow[d, "1_{FX} \otimes F^2"] \\
F(X \otimes Y) \otimes FZ \arrow[d, "F^2"'] & FX \otimes F(Y \otimes Z) \arrow[d, "F^2"] \\
F\big((X \otimes Y) \otimes Z\big) \arrow[r, "F\alpha^{\mathsf{C}}"] & F\big(X \otimes (Y \otimes Z)\big)
\end{tikzcd}$$

is commutative for all objects $X, Y, Z \in \mathsf{C}$.

Unity: The diagrams

$$\begin{tikzcd} \mathbb{1}^{\mathsf{D}} \otimes FX \arrow[r, "\lambda^{\mathsf{D}}_{FX}"] \arrow[d, "F^0 \otimes 1_{FX}"'] & FX \\ F\mathbb{1}^{\mathsf{C}} \otimes FX \arrow[r, "F^2"] & F(\mathbb{1}^{\mathsf{C}} \otimes X) \arrow[u, "F\lambda^{\mathsf{C}}_X"'] \end{tikzcd} \qquad \begin{tikzcd} FX \otimes \mathbb{1}^{\mathsf{D}} \arrow[r, "\rho^{\mathsf{D}}_{FX}"] \arrow[d, "1_{FX} \otimes F^0"'] & FX \\ FX \otimes F\mathbb{1}^{\mathsf{C}} \arrow[r, "F^2"] & F(X \otimes \mathbb{1}^{\mathsf{C}}) \arrow[u, "F\rho^{\mathsf{C}}_X"'] \end{tikzcd} \tag{1.3.11}$$

are commutative for all objects $X \in \mathsf{C}$. They are called the *left unity diagram* and the *right unity diagram*, respectively.

This finishes the definition of a monoidal functor. A monoidal functor (F, F^2, F^0) is often abbreviated to F.

Moreover, a monoidal functor (F, F^2, F^0) is said to be

- *unital* if F^0 is an isomorphism;
- *strictly unital* if F^0 is the identity morphism;
- *strong* if F^0 and the components of F^2 are isomorphisms; and
- *strict* if F^0 and the components of F^2 are identity morphisms. $\diamond$

Definition 1.3.12. Suppose

$$\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$$

are monoidal functors. Their *composite*

$$(GF, (GF)^2, (GF)^0) : \mathsf{C} \longrightarrow \mathsf{E}$$

is the monoidal functor with underlying functor GF and the structure morphisms

$$\begin{tikzcd} \mathbb{1} \arrow[r, "G^0"] \arrow[rr, bend left, "(GF)^0"] & G\mathbb{1} \arrow[r, "G(F^0)"] & GF\mathbb{1} \end{tikzcd}$$

$$\begin{tikzcd} GFA \otimes GFB \arrow[r, "G^2"] \arrow[rr, bend left, "(GF)^2"] & G(FA \otimes FB) \arrow[r, "G(F^2)"] & GF(A \otimes B) \end{tikzcd}$$

for objects $A, B \in \mathsf{C}$. $\diamond$

Definition 1.3.13. For monoidal functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, a *monoidal natural transformation* $\theta : F \longrightarrow G$ is a natural transformation between the underlying functors such that the diagrams

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\theta_X \otimes \theta_Y"] \arrow[d, "F^2"'] & GX \otimes GY \arrow[d, "G^2"] \\ F(X \otimes Y) \arrow[r, "\theta_{X \otimes Y}"] & G(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}^{\mathsf{D}} \arrow[r, "F^0"] \arrow[d, equal] & F\mathbb{1}^{\mathsf{C}} \arrow[d, "\theta_{\mathbb{1}^{\mathsf{C}}}"] \\ \mathbb{1}^{\mathsf{D}} \arrow[r, "G^0"] & G\mathbb{1}^{\mathsf{C}} \end{tikzcd} \tag{1.3.14}$$

are commutative for all objects $X, Y \in \mathsf{C}$. $\diamond$

Braided Monoidal Categories.

Definition 1.3.15. A *braided monoidal category* is a pair (C, ξ) consisting of the following data.

- $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category.

- ξ is a natural isomorphism

$$X \otimes Y \xrightarrow[\cong]{\xi_{X,Y}} Y \otimes X \tag{1.3.16}$$

  for objects $X, Y \in \mathsf{C}$, which is called the *braiding*.

These data are required to satisfy the *Hexagon Axioms*, stating the commutativity of the following two diagrams, called the *left hexagon diagram* and the *right hexagon diagram*, respectively, for objects $X, Y, Z \in \mathsf{C}$.

(1.3.17)
$$\begin{tikzcd}
& (Y \otimes X) \otimes Z \arrow[r, "\alpha"] & Y \otimes (X \otimes Z) \arrow[dr, "1_Y \otimes \xi_{X,Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\xi_{X,Y} \otimes 1_Z"] \arrow[dr, "\alpha"'] & & & Y \otimes (Z \otimes X) \\
& X \otimes (Y \otimes Z) \arrow[r, "\xi_{X,Y Z}"] & (Y \otimes Z) \otimes X \arrow[ur, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}"] & (X \otimes Z) \otimes Y \arrow[dr, "\xi_{X,Z} \otimes 1_Y"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "1_X \otimes \xi_{Y,Z}"] \arrow[dr, "\alpha^{-1}"'] & & & (Z \otimes X) \otimes Y \\
& (X \otimes Y) \otimes Z \arrow[r, "\xi_{XY,Z}"] & Z \otimes (X \otimes Y) \arrow[ur, "\alpha^{-1}"'] &
\end{tikzcd}$$

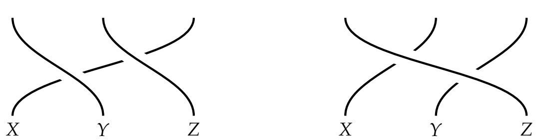
This finishes the definition of a braided monoidal category. A braided monoidal category is *strict* if the underlying monoidal category is strict. $\diamond$

Definition 1.3.18. For braided monoidal categories C and D, a *braided monoidal functor* $(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$ is a monoidal functor between the underlying monoidal categories such that the diagram

(1.3.19)
$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\xi_{FX,FY}", "\cong"'] \arrow[d, "F^2"'] & FY \otimes FX \arrow[d, "F^2"] \\
F(X \otimes Y) \arrow[r, "F\xi_{X,Y}", "\cong"'] & F(Y \otimes X)
\end{tikzcd}$$

is commutative for all objects $X, Y \in \mathsf{C}$. A braided monoidal functor is said to be *strong* (respectively, *strict*, *unital*, or *strictly unital*) if the underlying monoidal functor is so. $\diamond$

Explanation 1.3.20. The two hexagon diagrams in (1.3.17) may be visualized as the braids, read bottom-to-top,



in the braid group B_3, with the braiding ξ interpreted as the generating braid $s_1 \in B_2$. On the left, the two strings labeled by Y and Z cross over the string labeled by X. The two composites along the boundary of the left hexagon diagram (1.3.17) correspond to passing Y and Z over X either one at a time, or both at once. On the right, the string labeled by Z crosses over the two strings labeled by Y and X. The two composites along the boundary of the right hexagon diagram (1.3.17) likewise correspond to the two ways of passing Z over X and Y. $\diamond$

The rest of this section contains several useful consequences of the hexagon axioms (1.3.17). The following unit properties are from [**JS93**, Prop. 2.1], but here we provide more detail.

Proposition 1.3.21. *In each braided monoidal category (C, ξ), the following two unit diagrams are commutative for all objects $X \in \mathsf{C}$.*

(1.3.22)
$$\begin{tikzcd}
X \otimes \mathbb{1} \arrow[r, "\xi_{X,\mathbb{1}}"] \arrow[d, "\rho_X"'] & \mathbb{1} \otimes X \arrow[d, "\lambda_X"] \\
X \arrow[r, equal] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes X \arrow[r, "\xi_{\mathbb{1},X}"] \arrow[d, "\lambda_X"'] & X \otimes \mathbb{1} \arrow[d, "\rho_X"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

Proof. To show that the left unit diagram in (1.3.22) is commutative, by the naturality and the invertibility of λ, it suffices to show that the following diagram is commutative.

(1.3.23)
$$\begin{tikzcd}
\mathbb{1} \otimes (X \otimes \mathbb{1}) \arrow[r, "1_{\mathbb{1}} \otimes \xi_{X,\mathbb{1}}"] \arrow[d, "1_{\mathbb{1}} \otimes \rho_X"'] & \mathbb{1} \otimes (\mathbb{1} \otimes X) \arrow[d, "1_{\mathbb{1}} \otimes \lambda_X"] \\
\mathbb{1} \otimes X \arrow[r, equal] & \mathbb{1} \otimes X
\end{tikzcd}$$

To show that (1.3.23) is commutative, we consider the left hexagon diagram in (1.3.17) with $Y = Z = \mathbb{1}$, which is the outer diagram below. To save space, we omit the $\otimes$ symbol.

(1.3.24)
$$\begin{tikzcd}
& (\mathbb{1}X)\mathbb{1} \arrow[rrr, "\alpha_{\mathbb{1},X,\mathbb{1}}"] \arrow[drr, "\rho_{\mathbb{1}X}"] & & & \mathbb{1}(X\mathbb{1}) \arrow[dl, "1_{\mathbb{1}}\rho_X"'] \arrow[dr, "1_{\mathbb{1}}\xi_{X,\mathbb{1}}"] \arrow[d, phantom, "(1.3.23)"] & \\
(X\mathbb{1})\mathbb{1} \arrow[ur, "\xi_{X,\mathbb{1}}1_{\mathbb{1}}"] \arrow[rr, "\rho_{X\mathbb{1}}"] \arrow[dr, "\alpha_{X,\mathbb{1},\mathbb{1}}"'] & & X\mathbb{1} \arrow[r, "\xi_{X,\mathbb{1}}"] & \mathbb{1}X & & \mathbb{1}(\mathbb{1}X) \arrow[ll, "1_{\mathbb{1}}\lambda_X"'] \\
& X(\mathbb{1}\mathbb{1}) \arrow[ur, "1_X\rho_{\mathbb{1}}"'] \arrow[rrr, "\xi_{X,\mathbb{1}\mathbb{1}}"] & & & (\mathbb{1}\mathbb{1})X \arrow[ul, "\rho_{\mathbb{1}}1_X"] \arrow[ur, "\alpha_{\mathbb{1},\mathbb{1},X}"'] &
\end{tikzcd}$$

Consider the top half of the diagram (1.3.24).

- The left triangle is commutative by the naturality of ρ.
- The middle triangle is commutative by the right unity property in (1.3.5).
- The right triangle is the diagram (1.3.23) that we want to show is commutative.

Consider the bottom half of the diagram (1.3.24).

- The left triangle is commutative by the right unity property in (1.3.5).
- The middle trapezoid is commutative by the naturality of ξ.
- The right triangle is commutative by the unity axiom (1.3.2).

Moreover, the following statements hold.

- The outer diagram in (1.3.24), being the $Y = Z = \mathbb{1}$ special case of the left hexagon diagram in (1.3.17), is commutative.
- Each arrow in (1.3.24) is an isomorphism.

It follows that the diagram (1.3.23) is commutative, proving the left unit diagram in (1.3.22).

The right unit diagram in (1.3.22) is proved similarly by considering the right hexagon diagram in (1.3.17) with $X = Y = \mathbb{1}$. $\square$

Proposition 1.3.25. *For each object X in a braided monoidal category (C,ξ), the morphisms*

$$\begin{tikzcd} X \otimes \mathbb{1} \arrow[r, shift left, "\xi_{X,\mathbb{1}}"] & \mathbb{1} \otimes X \arrow[l, shift left, "\xi_{\mathbb{1},X}"] \end{tikzcd}$$

are inverses of each other.

Proof. This follows by horizontally concatenating the two squares in (1.3.22) in either order and using the invertibility of λ_X and ρ_X. □

Proposition 1.3.26. *In each braided monoidal category (C,ξ), the equality*

$$\xi_{\mathbb{1},\mathbb{1}} = 1_{\mathbb{1}\otimes\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1} \otimes \mathbb{1}$$

holds.

Proof. The desired equality follows from

- the equality $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ in (1.3.4),
- the unit property $\rho = \lambda\xi_{-,\mathbb{1}}$ in (1.3.22), and
- the invertibility of ρ.

This finishes the proof. □

The following observation is from **[JS93**, Prop. 2.7], where it was obtained as a consequence of the Braided Coherence Theorem 1.6.3. Here we provide a direct proof.

Proposition 1.3.27. *In each braided monoidal category (C,ξ), the following diagram is commutative for all objects $A, B, C \in \mathsf{C}$.*

(1.3.28)
$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[rr, "\alpha_{A,B,C}"] \arrow[dl, "\xi_{A,B}\otimes 1_C"'] & & A \otimes (B \otimes C) \arrow[dr, "1_A \otimes \xi_{B,C}"] & \\
(B \otimes A) \otimes C \arrow[dr, "\xi_{B\otimes A,C}"'] & & & & A \otimes (C \otimes B) \arrow[dl, "\xi_{A,C\otimes B}"] \\
& C \otimes (B \otimes A) \arrow[rr, "\alpha^{-1}_{C,B,A}"] & & (C \otimes B) \otimes A &
\end{tikzcd}$$

Proof. The diagram (1.3.28) is the outer diagram below, with the symbol $\otimes$ omitted.

$$\begin{tikzcd}
& (AB)C \arrow[rrrrrr, "\alpha_{A,B,C}"] \arrow[dl, "\xi_{A,B}1_C"'] & & & & & & A(BC) \arrow[dl, "\xi_{A,BC}"'] \arrow[dr, "1_A\xi_{B,C}"] & \\
(BA)C \arrow[rr, "\alpha_{B,A,C}"] \arrow[dr, "\xi_{BA,C}"'] & & B(AC) \arrow[rr, "1_B\xi_{A,C}"] & & B(CA) \arrow[rr, "\alpha^{-1}_{B,C,A}"] & & (BC)A \arrow[dr, "\xi_{B,C}1_A"'] & & A(CB) \arrow[dl, "\xi_{A,CB}"] \\
& C(BA) \arrow[rrrrrr, "\alpha^{-1}_{C,B,A}"] & & & & & & (CB)A &
\end{tikzcd}$$

- The top and the bottom left parallelograms are commutative by, respectively, the left and the right hexagon diagrams in (1.3.17).
- The right diamond is commutative by the naturality of ξ.

This finishes the proof. □

Explanation 1.3.29. The commutative diagram (1.3.28) may be visualized as the third Reidemeister move in Example 1.1.6. The left braid in Example 1.1.6 corresponds to the left-bottom composite in (1.3.28), and the right braid corresponds to the top-right composite. ⋄

The following variation of Proposition 1.3.27 is another manifestation of the third Reidemeister move in a braided monoidal category.

Proposition 1.3.30. *In each braided monoidal category* (C, ξ)*, the following diagram is commutative for all objects* $A, B, C \in \mathsf{C}$*.*

(1.3.31)
$$\begin{tikzcd}
& (A \otimes B) \otimes C \arrow[r, "\xi_{A\otimes B,C}"] \arrow[dl, "\alpha_{A,B,C}"'] & C \otimes (A \otimes B) \arrow[dr, "1_C \otimes \xi_{A,B}"] & \\
A \otimes (B \otimes C) \arrow[dr, "\xi_{A,B\otimes C}"'] & & & C \otimes (B \otimes A) \\
& (B \otimes C) \otimes A \arrow[r, "\xi_{B,C}\otimes 1_A"] & (C \otimes B) \otimes A \arrow[ur, "\alpha_{C,B,A}"'] &
\end{tikzcd}$$

Proof. The diagram (1.3.31) is the outer diagram below, with the symbol $\otimes$ omitted.

$$\begin{tikzcd}
(AB)C \arrow[rrr, "\xi_{AB,C}"] \arrow[dr, "\xi_{A,B}1_C"'] \arrow[d, "\alpha_{A,B,C}"'] & & & C(AB) \arrow[d, "1_C\xi_{A,B}"] \\
A(BC) \arrow[dr, "1_A\xi_{B,C}"] \arrow[d, "\xi_{A,BC}"'] & (BA)C \arrow[rr, "\xi_{BA,C}"] & (1.3.28) & C(BA) \\
(BC)A \arrow[rrr, "\xi_{B,C}1_A"] & A(CB) \arrow[rr, "\xi_{A,CB}"] & & (CB)A \arrow[u, "\alpha_{C,B,A}"']
\end{tikzcd}$$

In this diagram, the middle subdiagram is commutative by (1.3.28). The upper right and the lower left triangles are commutative by the naturality of ξ. □

Symmetric Monoidal Categories.

Definition 1.3.32. A *symmetric monoidal category* is a monoidal category

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$$

equipped with a natural isomorphism

$$X \otimes Y \xrightarrow[\cong]{\xi_{X,Y}} Y \otimes X$$

for objects $X, Y \in \mathsf{C}$, which is called the *braiding* or the *symmetry isomorphism*, that satisfies the following axioms.

The Symmetry Axiom: The diagram

(1.3.33)
$$\begin{tikzcd}
X \otimes Y \arrow[r, "\xi_{X,Y}"] \arrow[dr, equal] & Y \otimes X \arrow[d, "\xi_{Y,X}"] \\
& X \otimes Y
\end{tikzcd}$$

is commutative for all objects $X, Y \in \mathsf{C}$.

The Unit Axiom: The diagram

$$\begin{tikzcd} X \otimes \mathbb{1} \arrow[r, "\xi_{X,\mathbb{1}}"] \arrow[d, "\rho_X"'] & \mathbb{1} \otimes X \arrow[d, "\lambda_X"] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{1.3.34}$$

is commutative for all objects $X \in \mathsf{C}$.

The Hexagon Axiom: The diagram

$$\begin{tikzcd} & X \otimes (Z \otimes Y) \arrow[r, "1_X \otimes \xi_{Z,Y}"] & X \otimes (Y \otimes Z) \arrow[dr, "\alpha^{-1}"] & \\ (X \otimes Z) \otimes Y \arrow[ur, "\alpha"] \arrow[dr, "\xi_{X \otimes Z, Y}"'] & & & (X \otimes Y) \otimes Z \\ & Y \otimes (X \otimes Z) \arrow[r, "\alpha^{-1}"] & (Y \otimes X) \otimes Z \arrow[ur, "\xi_{Y,X} \otimes 1_Z"'] & \end{tikzcd} \tag{1.3.35}$$

is commutative for all objects $X, Y, Z \in \mathsf{C}$.

This finishes the definition of a symmetric monoidal category. A *permutative category* is a symmetric monoidal category whose underlying monoidal category is strict.

A *symmetric monoidal functor* between symmetric monoidal categories is a monoidal functor that satisfies (1.3.19). A symmetric monoidal functor is said to be *strong* (respectively, *strict*, *unital*, or *strictly unital*) if the underlying monoidal functor is so. ◇

Proposition 1.3.36. *A symmetric monoidal category is precisely a braided monoidal category whose braiding satisfies the symmetry axiom (1.3.33).*

Proof. Suppose C is a symmetric monoidal category. The symmetry axiom (1.3.33) implies the following statements.

- The hexagon axiom (1.3.35) in a symmetric monoidal category implies the right hexagon diagram in (1.3.17).
- Taking the inverse of each edge in the right hexagon diagram yields the left hexagon diagram in (1.3.17).

Therefore, C is also a braided monoidal category.

Conversely, suppose C is a braided monoidal category whose braiding satisfies the symmetry axiom.

- The unit axiom (1.3.34) holds by the left unit property in (1.3.22).
- By the symmetry axiom, the right hexagon diagram in (1.3.17) is equivalent to the hexagon axiom (1.3.35).

Therefore, C is a symmetric monoidal category. □

Example 1.3.37. Since each symmetric monoidal category is also a braided monoidal category, Propositions 1.3.21, 1.3.25 through 1.3.27, and 1.3.30 also hold for symmetric monoidal categories. In particular, the unit axiom (1.3.34) is redundant by the left unit property in (1.3.22). ◇

1.4. The Drinfeld Center

The Drinfeld center is an important construction that takes a monoidal category to a braided monoidal category. In this section, we provide a detailed proof of this fact for a general nonstrict monoidal category; see Theorem 1.4.27 and Note 1.7.2. In Chapter 4 and Section 9.6, we will extend the Drinfeld center to the bimonoidal and the ring categorical setting.

Defining the Drinfeld Center.

Motivation 1.4.1. Suppose M is a monoid in the usual set. The *center* of M, denoted $\mathsf{Z}(M)$, consists of elements $a \in M$ such that

$$ab = ba \quad \text{for each} \quad b \in M. \tag{1.4.2}$$

So an element in the center $\mathsf{Z}(M)$ commutes with each element in M. The center $\mathsf{Z}(M)$ inherits from M the structure of a commutative monoid. The Drinfeld center, which we will define shortly, is a categorification of the center construction that takes a monoidal category to a braided monoidal category. In the categorification process, the equality (1.4.2) is replaced by a natural isomorphism β^A that satisfies a coherence axiom (1.4.4), which is modeled after the left hexagon axiom (1.3.17). For a monoid M and a monoidal category C, the following table summarizes this discussion.

input	center	elements/objects
monoid M	$\mathsf{Z}(M)$ is a commutative monoid	$a \in M$ such that $ab = ba$
monoidal category C	$\overline{\mathsf{C}}$ is a braided monoidal category	$\big(A, \beta^A : A \otimes (-) \xrightarrow{\cong} (-) \otimes A\big)$

See Questions III.A.3.2 and III.A.3.3 for open questions related to the Drinfeld center. ◇

Definition 1.4.3. Suppose $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category. The *Drinfeld center* of C consists of the data of a braided monoidal category

$$\big(\overline{\mathsf{C}}, \overline{\otimes}, \overline{\mathbb{1}}, \overline{\alpha}, \overline{\lambda}, \overline{\rho}, \overline{\xi}\big)$$

defined as follows.

Objects: An object in $\overline{\mathsf{C}}$ is a pair $(A; \beta^A)$ consisting of

- an object $A \in \mathsf{C}$ and
- a natural isomorphism

$$A \otimes B \xrightarrow[\cong]{\beta^A_B} B \otimes A \quad \text{for} \quad B \in \mathsf{C}$$

such that the following hexagon is commutative for objects $B, C \in \mathsf{C}$.

$$\begin{tikzcd}[column sep=small]
& A \otimes (B \otimes C) \arrow[rr, "\beta^A_{B \otimes C}"] \arrow[dl, "\alpha^{-1}_{A,B,C}"'] && (B \otimes C) \otimes A & \\
(A \otimes B) \otimes C \arrow[dr, "\beta^A_B \otimes 1_C"'] &&&& B \otimes (C \otimes A) \arrow[ul, "\alpha^{-1}_{B,C,A}"'] \\
& (B \otimes A) \otimes C \arrow[rr, "\alpha_{B,A,C}"] && B \otimes (A \otimes C) \arrow[ur, "1_B \otimes \beta^A_C"'] &
\end{tikzcd} \tag{1.4.4}$$

We call A the *underlying object* and β^A the *A-braiding*.

Morphisms: A morphism

$$f : (A;\beta^A) \longrightarrow (B;\beta^B)$$

in $\overline{\mathsf{C}}$ is a morphism $f : A \longrightarrow B$ in C such that the following diagram is commutative for each object $C \in \mathsf{C}$.

$$\begin{tikzcd} A \otimes C \arrow[r, "f\otimes 1_C"] \arrow[d, "\beta^A_C"'] & B \otimes C \arrow[d, "\beta^B_C"] \\ C \otimes A \arrow[r, "1_C \otimes f"] & C \otimes B \end{tikzcd} \tag{1.4.5}$$

Identity Morphisms: The identity morphism of an object $(A;\beta^A) \in \overline{\mathsf{C}}$ is the identity morphism $1_A : A \longrightarrow A$ in C.

Composition: The composition of morphisms in $\overline{\mathsf{C}}$ is the composition of morphisms in C.

The Monoidal Product: For the rest of this definition, $(A;\beta^A)$, $(B;\beta^B)$, and $(C;\beta^C)$ are arbitrary objects in $\overline{\mathsf{C}}$. The functor

$$\overline{\mathsf{C}} \times \overline{\mathsf{C}} \xrightarrow{\overline{\otimes}} \overline{\mathsf{C}} \tag{1.4.6}$$

is defined as follows.

Objects: Define the object

$$(A;\beta^A) \,\overline{\otimes}\, (B;\beta^B) = (A \otimes B; \beta^{A\otimes B}) \tag{1.4.7}$$

with $\beta^{A\otimes B}$ defined by the following hexagon for objects $C \in \mathsf{C}$.

$$\begin{tikzcd} & (A\otimes B)\otimes C \arrow[r, "\beta^{A\otimes B}_C"] \arrow[dl, "\alpha_{A,B,C}"'] & C \otimes (A\otimes B) & \\ A\otimes(B\otimes C) \arrow[dr, "1_A\otimes\beta^B_C"'] & & & (C\otimes A)\otimes B \arrow[ul, "\alpha_{C,A,B}"'] \\ & A\otimes(C\otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"] & (A\otimes C)\otimes B \arrow[ur, "\beta^A_C\otimes 1_B"'] & \end{tikzcd} \tag{1.4.8}$$

Morphisms: Define the morphism

$$(A;\beta^A) \,\overline{\otimes}\, (B;\beta^B) \xrightarrow{f \overline{\otimes} g} (A';\beta^{A'}) \,\overline{\otimes}\, (B';\beta^{B'}) \tag{1.4.9}$$

as

$$A \otimes B \xrightarrow{f\otimes g} A' \otimes B' \in \mathsf{C}$$

for the following morphisms in $\overline{\mathsf{C}}$.

$$(A;\beta^A) \xrightarrow{f} (A';\beta^{A'})$$
$$(B;\beta^B) \xrightarrow{g} (B';\beta^{B'})$$

The Monoidal Unit: Define the object

$$\overline{\mathbb{1}} = (\mathbb{1}; \beta^{\mathbb{1}}) \tag{1.4.10}$$

with $\beta^{\mathbb{1}}$ defined as the following composite for objects $A \in \mathsf{C}$.

$$\begin{tikzcd} \mathbb{1} \otimes A \arrow[rr, "\beta^{\mathbb{1}}_A"] \arrow[dr, "\lambda_A"'] & & A \otimes \mathbb{1} \\ & A \arrow[ur, "\rho_A^{-1}"'] & \end{tikzcd} \tag{1.4.11}$$

The Associativity Isomorphisms: Define the morphism

$$\begin{tikzcd} \big[(A;\beta^A)\,\overline{\otimes}\,(B;\beta^B)\big]\,\overline{\otimes}\,(C;\beta^C) \arrow[d, "\overline{\alpha}_{(A;\beta^A),(B;\beta^B),(C;\beta^C)}"] \\ (A;\beta^A)\,\overline{\otimes}\,\big[(B;\beta^B)\,\overline{\otimes}\,(C;\beta^C)\big] \end{tikzcd} \tag{1.4.12}$$

as

$$(A \otimes B) \otimes C \xrightarrow{\alpha_{A,B,C}} A \otimes (B \otimes C) \in \mathsf{C}.$$

The Unit Isomorphisms: Define the left and the right unit isomorphisms

$$\overline{\mathbb{1}}\,\overline{\otimes}\,(A;\beta^A) \xrightarrow{\overline{\lambda}_{(A;\beta^A)}} (A;\beta^A) \xleftarrow{\overline{\rho}_{(A;\beta^A)}} (A;\beta^A)\,\overline{\otimes}\,\overline{\mathbb{1}} \tag{1.4.13}$$

as, respectively,

$$\mathbb{1} \otimes A \xrightarrow{\lambda_A} A \xleftarrow{\rho_A} A \otimes \mathbb{1} \in \mathsf{C}.$$

The Braiding: Define the morphism

$$(A;\beta^A)\,\overline{\otimes}\,(B;\beta^B) \xrightarrow{\overline{\xi}_{(A;\beta^A),(B;\beta^B)}} (B;\beta^B)\,\overline{\otimes}\,(A;\beta^A) \tag{1.4.14}$$

as

$$\beta^A_B : A \otimes B \longrightarrow B \otimes A \in \mathsf{C}.$$

This finishes the definition of the Drinfeld center of C. ◇

Explanation 1.4.15. Consider Definition 1.4.3.

- For an object $(A;\beta^A) \in \overline{\mathsf{C}}$, the naturality of β^A means that for each morphism $g : B \longrightarrow C$ in C, the following diagram in C is commutative.

$$\begin{tikzcd} A \otimes B \arrow[r, "\beta^A_B"] \arrow[d, "1_A \otimes g"'] & B \otimes A \arrow[d, "g \otimes 1_A"] \\ A \otimes C \arrow[r, "\beta^A_C"] & C \otimes A \end{tikzcd} \tag{1.4.16}$$

- The diagrams (1.4.4) and (1.4.8) are modeled after, respectively, the left hexagon diagram and the right hexagon diagram in (1.3.17).
- The diagram (1.4.11) defining $\beta^{\mathbb{1}}$ is modeled after the right diagram in (1.3.22).

Moreover, the diagram (1.4.18) below, characterizing $\beta^A_{\mathbb{1}}$, is modeled after the left diagram in (1.3.22). ◇

The Braided Monoidal Category Axioms. We now check in several steps that the Drinfeld center is, in fact, a braided monoidal category, starting with the following preliminary observations. For the rest of this section, in diagrams we often omit the symbol $\otimes$ and some subscripts to save space.

Lemma 1.4.17. *In the context of Definition 1.4.3, the following statements hold.*

(1) $\overline{\mathsf{C}}$ *is a category.*
(2) *For each object* $(A;\beta^A)\in\overline{\mathsf{C}}$*, the following diagram is commutative in* C*.*

$$\begin{tikzcd} A\otimes\mathbb{1} \arrow[rr, "\beta^A_{\mathbb{1}}"] \arrow[dr, "\rho_A"'] & & \mathbb{1}\otimes A \arrow[dl, "\lambda_A"] \\ & A & \end{tikzcd} \tag{1.4.18}$$

Proof. The first assertion follows from the following statements.

- $1_A : (A;\beta^A)\longrightarrow(A;\beta^A)$ satisfies (1.4.5) because $\otimes$ preserves identity morphisms.
- If
$$f:(A;\beta^A)\longrightarrow(B;\beta^B)\quad\text{and}\quad g:(B;\beta^B)\longrightarrow(C;\beta^C)$$
are morphisms in $\overline{\mathsf{C}}$, then gf satisfies (1.4.5) because $\otimes$ preserves composition and identity morphisms.
- The unity and the associativity of composition in $\overline{\mathsf{C}}$ follow from the corresponding properties in C.

Therefore, $\overline{\mathsf{C}}$ is a category.

For the second assertion, first observe that, by the naturality and the invertibility of ρ, the diagram (1.4.18) is commutativity if and only if the diagram

$$\begin{tikzcd} (A\otimes\mathbb{1})\otimes\mathbb{1} \arrow[rr, "\beta^A_{\mathbb{1}}\otimes 1_{\mathbb{1}}"] \arrow[dr, "\rho_A\otimes 1_{\mathbb{1}}"'] & & (\mathbb{1}\otimes A)\otimes\mathbb{1} \arrow[dl, "\lambda_A\otimes 1_{\mathbb{1}}"] \\ & A\otimes\mathbb{1} & \end{tikzcd} \tag{1.4.19}$$

is commutative. To show that (1.4.19) is commutative, consider the following diagram.

$$\begin{tikzcd} & A(\mathbb{1}\mathbb{1}) \arrow[rrrr, "\beta^A_{\mathbb{1}\mathbb{1}}"] \arrow[dl, "\alpha^{-1}_{A,\mathbb{1},\mathbb{1}}"'] \arrow[dr, "1_A\lambda_{\mathbb{1}}"] \arrow[d, phantom, "(1.3.2)"] & & \arrow[d, phantom, "\text{nat}"] & & (\mathbb{1}\mathbb{1})A \arrow[dl, "\lambda_{\mathbb{1}}1_A"'] \arrow[d, phantom, "(1.3.5)"] & \\ (A\mathbb{1})\mathbb{1} \arrow[rr, "\rho_A 1_{\mathbb{1}}"] \arrow[dr, "\beta^A_{\mathbb{1}}1_{\mathbb{1}}"'] & \arrow[d, phantom, "(1.4.19)"] & A\mathbb{1} \arrow[rr, "\beta^A_{\mathbb{1}}"] & & \mathbb{1}A & \arrow[d, phantom, "\text{nat}"] & \mathbb{1}(\mathbb{1}A) \arrow[ll, "\lambda_{\mathbb{1}A}"'] \arrow[ul, "\alpha^{-1}_{\mathbb{1},\mathbb{1},A}"'] \\ & (\mathbb{1}A)\mathbb{1} \arrow[ur, "\lambda_A 1_{\mathbb{1}}"'] \arrow[rrrr, "\alpha_{\mathbb{1},A,\mathbb{1}}"'] & & \arrow[u, phantom, "(1.3.5)"] & & \mathbb{1}(A\mathbb{1}) \arrow[ulll, "\lambda_{A\mathbb{1}}"'] \arrow[ur, "1_{\mathbb{1}}\beta^A_{\mathbb{1}}"'] & \end{tikzcd}$$

- The outer diagram is the commutative diagram (1.4.4) with $B = C = \mathbb{1}$.
- In the top half, from left to right, the subdiagrams are commutative by the unity axiom (1.3.2), the naturality of β^A (1.4.16) for the morphism $\lambda_{\mathbb{1}}$, and the left unity property in (1.3.5).

- In the bottom half, the left triangle is (1.4.19). The other two triangles are commutative by the left unity property in (1.3.5) and the naturality of λ.
- Each edge is an isomorphism.

Therefore, the diagram (1.4.19) is commutative. □

In the next three lemmas, we check that $\overline{\otimes}$ in (1.4.6) is a well-defined functor.

Lemma 1.4.20. *$(A \otimes B; \beta^{A\otimes B})$ in (1.4.7) is an object in $\overline{\mathsf{C}}$.*

Proof. The $(A\otimes B)$-braiding $\beta^{A\otimes B}$ in (1.4.8) is a natural isomorphism because α, β^A, and β^B are natural isomorphisms. It remains to check the axiom (1.4.4) for $\beta^{A\otimes B}$.

For objects $C, D \in \mathsf{C}$, the diagram (1.4.4) for $\beta^{A\otimes B}$ is the outer diagram below.

$$
\begin{tikzcd}[column sep=small]
(AB)(CD) \arrow[r, "\alpha"] \arrow[dd, "\alpha^{-1}"'] \arrow[ddrrrrrr, bend left=40, "\beta^{AB}_{CD}"] & A[B(CD)] \arrow[rr, "1\beta^{B}_{CD}"] \arrow[dd, "1\alpha^{-1}"'] \arrow[ddrr, phantom, "(1.4.4)"] & & A[(CD)B] \arrow[r, "\alpha^{-1}"] \arrow[dd, "1\alpha"] & {[A(CD)]B} \arrow[rr, "\beta^{A}_{CD}1"] \arrow[dd, "\alpha^{-1}1"] \arrow[ddrr, phantom, "(1.4.4)"] & & {[(CD)A]B} \arrow[dd, "\alpha"] \\
& & & & & & \\
{[(AB)C]D} \arrow[dd, "\alpha 1"'] \arrow[dddddddd, bend right=50, "\beta^{AB}_{C}1"'] & A[(BC)D] \arrow[dd, "1(\beta^{B}_{C}1)"] & & A[C(DB)] & {[(AC)D]B} \arrow[dd, "(\beta^{A}_{C}1)1"] \arrow[dddl, "\alpha"'] & & (CD)(AB) \\
& & & & & {[C(DA)]B} \arrow[uuur, "\alpha^{-1}1"] \arrow[dddr, "\alpha"] & \\
{[A(BC)]D} \arrow[uur, "\alpha"] \arrow[dd, "(1\beta^{B}_{C})1"'] & A[(CB)D] \arrow[r, "1\alpha"] & A[C(BD)] \arrow[uur, "1(1\beta^{B}_{D})"] & & {[(CA)D]B} \arrow[dd, "\alpha 1"] \arrow[dddl, "\alpha"'] & & C[D(AB)] \arrow[uu, "\alpha^{-1}"'] \\
& & & (AC)(DB) \arrow[uuu, "\alpha"'] \arrow[dd, "\beta^{A}_{C}1"] & & & \\
{[A(CB)]D} \arrow[uur, "\alpha"] \arrow[dd, "\alpha^{-1}1"'] & (AC)(BD) \arrow[uur, "\alpha"] \arrow[urr, "1\beta^{B}_{D}"'] \arrow[dd, "\beta^{A}_{C}1"'] \arrow[ddrr, phantom, "(\dagger)"] & & & {[C(AD)]B} \arrow[uuur, bend left, "(1\beta^{A}_{D})1"] \arrow[ddrr, "\alpha"] & & C[(DA)B] \arrow[uu, "1\alpha"'] \\
& & & (CA)(DB) \arrow[dddrrr, "\alpha"] & & & \\
{[(AC)B]D} \arrow[uur, "\alpha"] \arrow[dd, "(\beta^{A}_{C}1)1"'] & (CA)(BD) \arrow[urr, "1\beta^{B}_{D}"] \arrow[ddrrr, "\alpha"] & & & & & C[(AD)B] \arrow[uu, "1(\beta^{A}_{D}1)"'] \\
& & & & & & \\
{[(CA)B]D} \arrow[uur, "\alpha"] \arrow[r, "\alpha 1"'] & {[C(AB)]D} \arrow[rr, "\alpha"'] & & C[(AB)D] \arrow[r, "1\alpha"'] \arrow[uuuuuurrr, bend right=40, "1\beta^{AB}_{D}"'] & C[A(BD)] \arrow[rr, "1(1\beta^{B}_{D})"'] & & C[A(DB)] \arrow[uu, "1\alpha^{-1}"']
\end{tikzcd}
$$

- The three subdiagrams along the boundary are commutative by the definitions (1.4.8) of $\beta^{A\otimes B}_{C\otimes D}$, $\beta^{A\otimes B}_{C}$, and $\beta^{A\otimes B}_{D}$.
- The subdiagram $(\dagger)$ is commutative by the functoriality of $\otimes$.
- Two subdiagrams are commutative by the axiom (1.4.4) for β^A and β^B as indicated.
- Every other subdiagram is commutative by either the pentagon axiom (1.3.3) or the naturality of α.

Therefore, $\beta^{A\otimes B}$ satisfies the axiom (1.4.4). □

Lemma 1.4.21. *$f \,\overline{\otimes}\, g$ in (1.4.9) is a morphism in $\overline{\mathsf{C}}$.*

Proof. We must show that $f \otimes g$ satisfies the axiom (1.4.5), which is the outer diagram below.

$$
\begin{tikzcd}
(AB)C \arrow[r, "(fg)1"] \arrow[d, "\alpha"] \arrow[dddddd, bend right=60, "\beta_C^{AB}"'] \arrow[dr, phantom, "\mathrm{nat}"] & (A'B')C \arrow[d, "\alpha"'] \arrow[dddddd, bend left=60, "\beta_C^{A'B'}"] \\
A(BC) \arrow[r, "f(g1)"] \arrow[d, "1\beta_C^{B}"] \arrow[dr, phantom, "(1.4.5)"] \arrow[d, phantom, xshift=-8ex, "(1.4.8)"] & A'(B'C) \arrow[d, "1\beta_C^{B'}"'] \arrow[d, phantom, xshift=8ex, "(1.4.8)"] \\
A(CB) \arrow[r, "f(1g)"] \arrow[d, "\alpha^{-1}"] \arrow[dr, phantom, "\mathrm{nat}"] & A'(CB') \arrow[d, "\alpha^{-1}"'] \\
(AC)B \arrow[r, "(f1)g"] \arrow[d, "\beta_C^{A}1"] \arrow[dr, phantom, "(1.4.5)"] & (A'C)B' \arrow[d, "\beta_C^{A'}1"'] \\
(CA)B \arrow[r, "(1f)g"] \arrow[d, "\alpha"] \arrow[dr, phantom, "\mathrm{nat}"] & (CA')B' \arrow[d, "\alpha"'] \\
C(AB) \arrow[r, "1(fg)"] & C(A'B')
\end{tikzcd}
$$

- The left and the right rectangles are commutative by (1.4.8) for, respectively, $\beta_C^{A\otimes B}$ and $\beta_C^{A'\otimes B'}$.
- In the middle column, the following statements hold.
  - The second rectangle is commutative by (1.4.5) for g and the functoriality of $\otimes$.
  - The fourth rectangle is commutative by (1.4.5) for f and the functoriality of $\otimes$.
  - The other three rectangles are commutative by the naturality of α.

Therefore, $f \otimes g$ satisfies the axiom (1.4.5). □

Lemma 1.4.22. *In (1.4.6),*

$$- \,\overline{\otimes}\, - : \overline{\mathsf{C}} \times \overline{\mathsf{C}} \longrightarrow \overline{\mathsf{C}}$$

is a functor.

Proof. This follows from Lemmas 1.4.20 and 1.4.21 and the fact that $\otimes$ preserves identity morphisms and composition. □

In the next four lemmas, we check that the monoidal unit, the associativity isomorphisms, the left/right unit isomorphisms, and the braiding in $\overline{\mathsf{C}}$ are well defined.

Lemma 1.4.23. $\overline{\mathbb{1}} = (\mathbb{1}; \beta^{\mathbb{1}})$ *in (1.4.10) is an object in* $\overline{\mathsf{C}}$.

Proof. $\beta^{\mathbb{1}} = \rho^{-1}\lambda$ in (1.4.11) is a natural isomorphism because λ and ρ are natural isomorphisms. The following commutative diagram proves the axiom (1.4.4) for

$\beta^{\mathbb{1}}$. The unlabeled regions are commutative by the definition (1.4.11) of $\beta^{\mathbb{1}}$.

$$\begin{tikzcd}
\mathbb{1}(BC) \arrow[rr, "\lambda"] \arrow[d, "\alpha^{-1}"'] \arrow[rrrr, bend left, "\beta^{\mathbb{1}}_{BC}"] & & BC \arrow[rr, "\rho^{-1}"] \arrow[ddl, "\rho^{-1}1"'] \arrow[ddr, "1\lambda^{-1}"] \arrow[drr, "1\rho^{-1}"] \arrow[dd, phantom, "(1.3.2)"] & & (BC)\mathbb{1} \\
(\mathbb{1}B)C \arrow[urr, "\lambda 1"'] \arrow[dr, "\beta^{\mathbb{1}}_B 1"'] \arrow[ur, phantom, "(1.3.5)"] & & & & B(C\mathbb{1}) \arrow[u, "\alpha^{-1}"'] \arrow[ul, phantom, "(1.3.5)"] \\
& (B\mathbb{1})C \arrow[rr, "\alpha"] & & B(\mathbb{1}C) \arrow[ur, "1\beta^{\mathbb{1}}_C"'] &
\end{tikzcd}$$

Therefore, $\overline{\mathbb{1}} = (\mathbb{1}; \beta^{\mathbb{1}})$ is an object in $\overline{\mathsf{C}}$. $\square$

Lemma 1.4.24. *In (1.4.12),*

$$\overline{\alpha} : (- \mathbin{\overline{\otimes}} -) \mathbin{\overline{\otimes}} - \longrightarrow - \mathbin{\overline{\otimes}} (- \mathbin{\overline{\otimes}} -)$$

is a natural isomorphism.

Proof. Since α in C is a natural isomorphism, it suffices to check that each component of $\overline{\alpha}$ is a morphism in $\overline{\mathsf{C}}$. For each object $D \in \mathsf{C}$, the axiom (1.4.5) for the component (1.4.12) of $\overline{\alpha}$ is the outer diagram below.

$$\begin{tikzcd}[column sep=small]
{[(AB)C]D} \arrow[r, "\alpha 1"] \arrow[d, "\alpha"] \arrow[ddddddrrr, bend right, "\beta_D^{(AB)C}"'] & {[A(BC)]D} \arrow[r, "\alpha"] \arrow[ddddddrrr, bend left, "\beta_D^{A(BC)}"] & {A[(BC)D]} \arrow[r, "1\alpha"] & {A[B(CD)]} \arrow[r, "1(1\beta_D^C)"] & {A[B(DC)]} \arrow[d, "1\alpha^{-1}"] \\
(AB)(CD) \arrow[urrr, "\alpha"] \arrow[d, "1\beta_D^C"] & & & & {A[(BD)C]} \arrow[d, "1(\beta_D^B 1)"] \\
(AB)(DC) \arrow[uurrrr, "\alpha"] \arrow[d, "\alpha^{-1}"] & & & & {A[(DB)C]} \arrow[d, "1\alpha"] \\
{[(AB)D]C} \arrow[d, "\alpha 1"] & & & & {A[D(BC)]} \arrow[d, "\alpha^{-1}"] \\
{[A(BD)]C} \arrow[uuurrrr, "\alpha"] \arrow[d, "(1\beta_D^B)1"] & & & & (AD)(BC) \arrow[d, "\beta_D^A 1"] \\
{[A(DB)]C} \arrow[uuurrrr, "\alpha"] \arrow[d, "\alpha^{-1}1"] & & & & (DA)(BC) \arrow[d, "\alpha"] \\
{[(AD)B]C} \arrow[r, "(\beta_D^A 1)1"'] \arrow[uurrrr, "\alpha"] & {[(DA)B]C} \arrow[r, "\alpha 1"'] \arrow[urrr, "\alpha"] & {[D(AB)]C} \arrow[r, "\alpha"'] & {D[(AB)C]} \arrow[r, "1\alpha"'] & {D[A(BC)]}
\end{tikzcd}$$

- The left-bottom subdiagram along the boundary is commutative by the definitions (1.4.8) of $\beta_D^{(A\otimes B)\otimes C}$ and $\beta_D^{A\otimes B}$.
- The top-right subdiagram along the boundary is commutative by the definitions (1.4.8) of $\beta_D^{A\otimes(B\otimes C)}$ and $\beta_D^{B\otimes C}$.
- The other seven subdiagrams, from the upper left to the lower right, are commutative by the pentagon axiom (1.3.3) and the naturality of α in an alternating manner.

Therefore, each component of $\overline{\alpha}$ satisfies the axiom (1.4.5). $\square$

Lemma 1.4.25. *In (1.4.13),*

$$\overline{\lambda} : \overline{\mathbb{1}}\,\overline{\otimes}\, - \longrightarrow - \quad \textit{and} \quad \overline{\rho} : -\,\overline{\otimes}\,\overline{\mathbb{1}} \longrightarrow -$$

are natural isomorphisms.

Proof. Since λ and ρ in C are natural isomorphisms, it suffices to check that each component of each of $\overline{\lambda}$ and $\overline{\rho}$ is a morphism in $\overline{\mathsf{C}}$. For the component $\overline{\lambda}_{(A;\beta^A)}$ in (1.4.13) and an object $B \in \mathsf{C}$, the axiom (1.4.5) is the outer diagram below.

$$\begin{tikzcd}
(\mathbb{1}A)B \arrow[rr, "\lambda 1"] \arrow[d, "\alpha"'] \arrow[drrr, bend right=60, "\beta_B^{\mathbb{1}A}"'] \arrow[dr, phantom, "(1.3.5)"] & & AB \arrow[r, "\beta_B^A"] & BA \\
\mathbb{1}(AB) \arrow[urr, "\lambda"] \arrow[d, "1\beta_B^A"'] \arrow[r, phantom, "\mathrm{nat}"] & {} \arrow[r, phantom, "(1.3.5)"] & {} \arrow[r, phantom, "(1.3.2)"] & B(\mathbb{1}A) \arrow[u, "1\lambda"'] \\
\mathbb{1}(BA) \arrow[r, "\alpha^{-1}"'] \arrow[uurrr, "\lambda"] & (\mathbb{1}B)A \arrow[r, "\lambda 1"'] & BA \arrow[uur, "1"] \arrow[r, "\rho^{-1}1"'] & (B\mathbb{1})A \arrow[u, "\alpha"']
\end{tikzcd}$$

- The subdiagram along the boundary is commutative by the definitions (1.4.8) of $\beta_B^{\mathbb{1}\otimes A}$ and (1.4.11) of $\beta_B^{\mathbb{1}}$.
- The other four subdiagrams are commutative by the left unity property in (1.3.5), the naturality of λ, and the unity axiom (1.3.2).

Therefore, each component of $\overline{\lambda}$ satisfies the axiom (1.4.5).

For the component $\overline{\rho}_{(A;\beta^A)}$ in (1.4.13) and an object $B \in \mathsf{C}$, the axiom (1.4.5) is the outer diagram below.

$$\begin{tikzcd}
(A\mathbb{1})B \arrow[rr, "\rho 1"] \arrow[d, "\alpha"'] \arrow[drrr, bend right=60, "\beta_B^{A\mathbb{1}}"'] \arrow[dr, phantom, "(1.3.2)"] & & AB \arrow[r, "\beta_B^A"] \arrow[dd, "\rho^{-1}"] \arrow[r, phantom, shift right=3, "\mathrm{nat}"] & BA \\
A(\mathbb{1}B) \arrow[d, "1\lambda"'] & {} \arrow[d, phantom, "(1.3.5)"] & {} \arrow[r, phantom, "(1.3.5)"] & B(A\mathbb{1}) \arrow[u, "1\rho"'] \\
AB \arrow[r, "1\rho^{-1}"'] \arrow[uurr, "1"] & A(B\mathbb{1}) \arrow[r, "\alpha^{-1}"'] & (AB)\mathbb{1} \arrow[r, "\beta_B^A 1"'] & (BA)\mathbb{1} \arrow[u, "\alpha"'] \arrow[uu, bend left=60, "\rho"]
\end{tikzcd}$$

- The subdiagram along the boundary is commutative by the definitions (1.4.8) of $\beta_B^{A\otimes\mathbb{1}}$ and (1.4.11) of $\beta_B^{\mathbb{1}}$.
- The other four subdiagrams are commutative by the unity axiom (1.3.2), the right unity property in (1.3.5), and the naturality of ρ.

Therefore, each component of $\overline{\rho}$ satisfies the axiom (1.4.5). $\square$

Lemma 1.4.26. $\overline{\xi}$ *in (1.4.14) is a natural isomorphism.*

Proof. To check that the component

$$\overline{\xi}_{(A;\beta^A),(B;\beta^B)} = \beta_B^A : (A;\beta^A)\,\overline{\otimes}\,(B;\beta^B) \longrightarrow (B;\beta^B)\,\overline{\otimes}\,(A;\beta^A)$$

in (1.4.14) is a morphism in $\overline{\mathsf{C}}$, we check the axiom (1.4.5) for each object $C \in \mathsf{C}$. This is the outer diagram below.

$$\begin{tikzcd}
(AB)C \arrow[r, "\beta^A_B 1"] \arrow[d, "\alpha"] \arrow[ddddd, bend right=60, "\beta_C^{AB}"'] & (BA)C \arrow[d, "\alpha"'] \arrow[ddddd, bend left=60, "\beta_C^{BA}"] \\
A(BC) \arrow[d, "1\beta_C^B"] \arrow[ddr, "\beta^A_{BC}"] \arrow[r, phantom, "(1.4.4)"] & B(AC) \arrow[d, "1\beta^A_C"'] \\
A(CB) \arrow[d, "\alpha^{-1}"] \arrow[ddr, "\beta^A_{CB}"'] \arrow[r, phantom, "\text{nat}"] & B(CA) \arrow[d, "\alpha^{-1}"'] \\
(AC)B \arrow[d, "\beta^A_C 1"] & (BC)A \arrow[d, "\beta^B_C 1"'] \\
(CA)B \arrow[d, "\alpha"] \arrow[r, phantom, "(1.4.4)"] & (CB)A \arrow[d, "\alpha"'] \\
C(AB) \arrow[r, "1\beta^A_B"] & C(BA)
\end{tikzcd}$$

- The left and the right rectangles are commutative by (1.4.8) for, respectively, $\beta_C^{A\otimes B}$ and $\beta_C^{B\otimes A}$.
- In the middle column, the following statements hold.
  - The top and the bottom trapezoids are commutative by (1.4.4) for, respectively, $\beta^A_{B\otimes C}$ and $\beta^A_{C\otimes B}$.
  - The middle parallelogram is commutative by the naturality (1.4.16) of β^A for the morphism β^B_C.

Since each component of β^A is an isomorphism, we have shown that each component of $\overline{\xi}$ is an isomorphism in $\overline{\mathsf{C}}$.

To check the naturality of $\overline{\xi}$, consider morphisms $f, g \in \overline{\mathsf{C}}$ as in (1.4.9). Then the naturality of $\overline{\xi}$ is the outer diagram below.

$$\begin{tikzcd}
AB \arrow[r, "\beta^A_B"] \arrow[d, "1g"] \arrow[dd, bend right=60, "fg"'] \arrow[dr, phantom, "\text{nat}"] & BA \arrow[d, "g1"'] \arrow[dd, bend left=60, "gf"] \\
AB' \arrow[r, "\beta^A_{B'}"] \arrow[d, "f1"] \arrow[dr, phantom, "(1.4.5)"] & B'A \arrow[d, "1f"'] \\
A'B' \arrow[r, "\beta^{A'}_{B'}"] & B'A'
\end{tikzcd}$$

- The left and right rectangles are commutative by the functoriality of $\otimes$.
- The top middle rectangle is commutative by the naturality of β^A.
- The bottom middle rectangle is commutative by the axiom (1.4.5) for f applied to the object $B' \in \mathsf{C}$.

Therefore, $\overline{\xi}$ is a natural isomorphism. $\square$

We are now ready for the main result of this section.

Theorem 1.4.27. *For each monoidal category* $(\mathsf{C},\otimes,\mathbb{1},\alpha,\lambda,\rho)$*, the Drinfeld center*

$$\big(\overline{\mathsf{C}},\,\overline{\otimes}\,,\overline{\mathbb{1}},\overline{\alpha},\overline{\lambda},\overline{\rho},\overline{\xi}\big)$$

in Definition 1.4.3 is a braided monoidal category.

Proof. Using Lemmas 1.4.17 and 1.4.22 through 1.4.25, the unity axiom (I.1.2.2) and the pentagon axiom (I.1.2.3) for the data

$$\big(\overline{\mathsf{C}},\,\overline{\otimes}\,,\overline{\mathbb{1}},\overline{\alpha},\overline{\lambda},\overline{\rho}\big)$$

follow from the corresponding axioms for the monoidal category C. The braiding $\overline{\xi}$ in (1.4.14) is a natural isomorphism by Lemma 1.4.26. The left and the right hexagon diagrams (1.3.17) for $(\overline{\mathsf{C}},\overline{\xi})$ are commutative by, respectively, the axiom (1.4.4) for $\beta^{A}_{B\otimes C}$ and the definition (1.4.8) of $\beta^{A\otimes B}_{C}$. □

1.5. The Symmetric Center

Theorem 1.4.27 states that the Drinfeld center of a monoidal category is a braided monoidal category. This section contains the following analogous construction that sends a braided monoidal category to a symmetric monoidal category. In Theorems 4.5.3 and 9.6.4, we will extend the symmetric center to the bimonoidal and the ring categorical setting.

Definition 1.5.1. Suppose $(\mathsf{C},\otimes,\mathbb{1},\alpha,\lambda,\rho,\xi)$ is a braided monoidal category. The *symmetric center of* C is the full subcategory $\mathsf{C}^{\mathsf{sym}}$ consisting of objects $A \in \mathsf{C}$ such that the diagram

(1.5.2)
$$\begin{tikzcd}
A\otimes B \arrow[rr, "1"] \arrow[dr, "\xi_{A,B}"'] & & A\otimes B \\
& B\otimes A \arrow[ur, "\xi_{B,A}"'] &
\end{tikzcd}$$

is commutative for each object $B \in \mathsf{C}$. ◇

Proposition 1.5.3. *For each braided monoidal category* $(\mathsf{C},\otimes,\mathbb{1},\alpha,\lambda,\rho,\xi)$*, the symmetric center* $\mathsf{C}^{\mathsf{sym}}$ *inherits from* C *the structure of a symmetric monoidal category.*

Proof. The object $\mathbb{1} \in \mathsf{C}$ satisfies

$$\xi_{B,\mathbb{1}}\xi_{\mathbb{1},B} = 1_{\mathbb{1}\otimes B}$$

by Proposition 1.3.25, so $\mathbb{1} \in \mathsf{C}^{\mathsf{sym}}$.

For objects $A, B \in \mathsf{C}^{\mathsf{sym}}$, the monoidal product $A\otimes B$ also belongs to $\mathsf{C}^{\mathsf{sym}}$ by the following commutative diagram, with $C \in \mathsf{C}$ an arbitrary object.

$$\begin{tikzcd}
(AB)C \arrow[rr, "\xi_{AB,C}"] \arrow[d, "\alpha"] \arrow[drr, phantom, "(1.3.17)"] & & C(AB) \arrow[rr, "\xi_{C,AB}"] \arrow[d, "\alpha^{-1}"'] \arrow[drr, phantom, "(1.3.17)"] & & (AB)C \\
A(BC) \arrow[d, "1\xi_{B,C}"] \arrow[rrrr, bend right=70, "1"'] & & (CA)B \arrow[dr, "\xi_{C,A}1"] & & A(BC) \arrow[u, "\alpha^{-1}"] \\
A(CB) \arrow[r, "\alpha^{-1}"] & (AC)B \arrow[ur, "\xi_{A,C}1"] \arrow[rr, equal, "(1.5.2)", "(1.5.2)"'] & & (AC)B \arrow[r, "\alpha"] & A(CB) \arrow[u, "1\xi_{C,B}"']
\end{tikzcd}$$

- The two subdiagrams involving $\xi_{A\otimes B,C}$ and $\xi_{C,A\otimes B}$ are commutative by, respectively, the right hexagon diagram and the left hexagon diagram in (1.3.17).
- The other two subdiagrams are commutative by the axiom (1.5.2) for A and B.

Therefore, $A \otimes B$ also satisfies the axiom (1.5.2).

Restricting $(\otimes, \alpha, \lambda, \rho, \xi)$ to $\mathsf{C}^{\mathsf{sym}}$, the data

$$(\mathsf{C}^{\mathsf{sym}}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi)$$

satisfy all the axioms of a braided monoidal category by the corresponding properties in C. Moreover, the axiom (1.5.2) implies that the braiding ξ in $\mathsf{C}^{\mathsf{sym}}$ satisfies the symmetry axiom (1.3.33). Proposition 1.3.36 now implies that $\mathsf{C}^{\mathsf{sym}}$ is a symmetric monoidal category. □

1.6. Coherence of Braided Monoidal Categories

In this section, we recall a coherence theorem of braided monoidal categories due to Joyal and Street. First we recall a few definitions that are used in the coherence theorems for (symmetric) monoidal categories.

Words and Canonical Maps. A *word* of length 0 is the symbol e. A word of length 1 is the symbol $-$. Inductively, if u and v are words of lengths m and n, respectively, then $u \,\square\, v$ is a word of length $m + n$. The length of a word w is denoted by $|w|$.

For a monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$, each word w of length n determines a functor

$$w : \mathsf{C}^n \longrightarrow \mathsf{C}$$

by interpreting

- the length 0 word e as the constant functor at the monoidal unit $\mathbb{1}$;
- the length 1 word $-$ as the identity functor 1_{C}; and
- $\square$ as the monoidal product $\otimes$ in C.

Canonical maps are natural isomorphisms between words of the same length that are defined inductively by the following four conditions.

- The identity morphism of $\mathbb{1}$ is a canonical map.
- The identity natural transformation of 1_{C} is a canonical map.
- α, λ, ρ, and their inverses are canonical maps.
- Canonical maps are closed under $\otimes$ and vertical composites.

For a word w of length n and a permutation $\sigma \in \Sigma_n$, the *permuted word*

$$w\sigma : \mathsf{C}^n \longrightarrow \mathsf{C}$$

is the composite functor $w \circ \sigma$, where $\sigma : \mathsf{C}^n \longrightarrow \mathsf{C}^n$ is given by

$$\sigma(x_1, \ldots, x_n) = \big(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\big)$$

with the x_i's all objects, or all morphisms, in C.

The Braided Case.

Definition 1.6.1. In a braided monoidal category (C,ξ), a *braided canonical map* is a natural isomorphism between permuted words of the same length that has the same definition as a canonical map by also allowing the braiding ξ and its inverse. For a symmetric monoidal category, a braided canonical map is also called a *permuted canonical map*. $\diamond$

Definition 1.6.2. In a braided monoidal category $\big(\mathsf{C},\otimes,\mathbb{1},\alpha,\lambda,\rho,\xi\big)$, each braided canonical map ϕ between permuted words of the same length n has an *underlying braid* $\mathsf{br}(\phi) \in B_n$ defined as follows.

- $\mathsf{br}(1_{\mathbb{1}}) = \mathrm{id}_0 \in B_0$.
- The identity natural transformation of 1_{C} has underlying braid $\mathrm{id}_1 \in B_1$.
- The vertical composite $\phi'\phi$ of two braided canonical maps has underlying braid the product $\mathsf{br}(\phi')\mathsf{br}(\phi)$.
- For braided canonical maps ϕ_1 and ϕ_2, the underlying braid of $\phi_1 \otimes \phi_2$ is
$$\mathsf{br}(\phi_1 \otimes \phi_2) = \mathsf{br}(\phi_1) \oplus \mathsf{br}(\phi_2)$$
with $\oplus$ the sum braid (1.1.10).
- For permuted words u, v, and w, the associativity isomorphism
$$\alpha_{u,v,w} : (u \otimes v) \otimes w \longrightarrow u \otimes (v \otimes w)$$
has underlying braid
$$\mathsf{br}(\alpha_{u,v,w}) = \mathrm{id} \in B_{|u|+|v|+|w|}.$$
- The unit isomorphisms
$$\lambda : \mathbb{1} \otimes u \longrightarrow u \quad \text{and} \quad \rho : u \otimes \mathbb{1} \longrightarrow u$$
have underlying braids
$$\mathsf{br}(\lambda_u) = \mathsf{br}(\rho_u) = \mathrm{id} \in B_{|u|}.$$
- The braiding
$$\xi_{u,v} : u \otimes v \longrightarrow v \otimes u$$
has underlying braid
$$\mathsf{br}(\xi_{u,v}) = b^{\oplus}_{|u|,|v|} \in B_{|u|+|v|}$$
with the right-hand side the elementary block braid (1.2.4).

This finishes the definition of the underlying braid. $\diamond$

Theorem 1.6.3 (Braided Coherence)**.** *In a braided monoidal category* C*, two braided canonical maps with the same (co)domain are equal if their underlying braids are equal.*

Example 1.6.4. Recall the elementary block braid
$$b^{\oplus}_{m,n} = s_1^{(2)}\langle m,n\rangle \in B_{m+n}$$
with $s_1^{(2)} \in B_2$ the generating braid. The braided canonical map
$$\begin{tikzcd}[column sep=small]
& (Y \otimes X) \otimes Z \arrow[rr, "\alpha"] && Y \otimes (X \otimes Z) \arrow[dr, "1_Y \otimes \xi_{X,Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\xi_{X,Y} \otimes 1_Z"] &&&& Y \otimes (Z \otimes X)
\end{tikzcd}$$

has underlying braid

$$\begin{aligned}\big(\mathrm{id}_1 \oplus s_1^{(2)}\big)\big(s_1^{(2)} \oplus \mathrm{id}_1\big) &= s_2^{(3)} s_1^{(3)}\\ &= b_{1,2}^{\oplus} \in B_3.\end{aligned}$$

This is the left braid in Explanation 1.3.20. The braid $b_{1,2}^{\oplus}$ is also the underlying braid of the following braided canonical map.

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[d, "\alpha"'] & & Y \otimes (Z \otimes X)\\
X \otimes (Y \otimes Z) \arrow[rr, "\xi_{X,YZ}"] & & (Y \otimes Z) \otimes X \arrow[u, "\alpha"']
\end{tikzcd}$$

The equality between this braided canonical map and the previous one is the left hexagon axiom (1.3.17). $\diamond$

A consequence of Theorem 1.6.3 is the following strictification result.

Theorem 1.6.5 (Braided Strictification)**.** *For each braided monoidal category* C*, there exist a braided strict monoidal category* C_{st} *and an adjoint equivalence*

$$\mathsf{C}_{\mathsf{st}} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{C}$$

with (i) both L *and* R *strong braided monoidal functors and (ii)* $LR = 1_{\mathsf{C}}$.

1.7. Notes

1.7.1 (The Braid Groups)**.** For more detailed discussion of the braid groups, the reader is referred to [**Art47, KT08**]. $\diamond$

1.7.2 (The Drinfeld Center)**.** The Drinfeld center plays an important role in quantum group theory. For a finite dimensional Hopf algebra A with invertible antipode, the Drinfeld center of the monoidal category of A-modules is equivalent to the category of modules over the Drinfeld double of A. See [**Kas95**, XIII.5.1] for a proof. According to [**Kas95**, XIII.7] and [**Maj91**, Ex. 3.4], the Drinfeld center is due to Drinfeld in unpublished work.

The Drinfeld center is called the *center* in [**JS91c**, Prop. 4(a)]. In [**JS91c**], the commutative diagram (1.4.18) characterizing $\beta^A_{\mathbb{1}}$ is included as an axiom of β^A. Our proof of (1.4.18) shows that it is a formal consequence of the axiom (1.4.4) and other monoidal category axioms.

Under different terminology, the Drinfeld center appeared in [**Maj91**, Th. 3.3] at about the same time as [**JS91c**]. In [**Kas95, JS91c, Maj91**], the proof that the Drinfeld center is a braided monoidal category is only given when the monoidal category is strict. A detailed proof in the general nonstrict case as in Section 1.4 does not seem to have appeared before. $\diamond$

1.7.3 (Braided Coherence)**.** Theorem 1.6.3 is due to Joyal and Street [**JS93**, Cor. 2.6]. A detailed proof at a higher level of generality is in [**Yau**∞, 20.3.7 and 21.3.4]. For the Braided Strictification Theorem 1.6.5, see [**Yau**∞, 20.1.1 and 21.3.1]. Further discussion of coherence can be found in [**Kel74**] and [**Yau**∞, Part 4]. $\diamond$

CHAPTER 2

Braided Bimonoidal Categories

In this chapter, we introduce braided bimonoidal categories and prove some of their basic properties. Braided bimonoidal categories are the braided analogues of symmetric bimonoidal categories with the multiplicative structure replaced by a braided monoidal category. The first half of this chapter contains the definition of a braided bimonoidal category and its relationship with a symmetric bimonoidal category. The second half of this chapter contains the observation that an abelian category with a compatible (symmetric/braided) monoidal structure is a tight (symmetric/braided) bimonoidal category. Examples, applications, and further properties of braided bimonoidal categories are discussed in subsequent chapters.

Organization. In Section 2.1, we first recall the notion of a symmetric bimonoidal category; see Definition 2.1.1. Then we define a braided bimonoidal category. As in Definition 2.1.1, in a braided bimonoidal category, the distributivity morphisms δ^l and δ^r are only assumed to be natural monomorphisms. If δ^l and δ^r are natural isomorphisms, then the braided bimonoidal category is said to be *tight*. Our tight braided bimonoidal categories are equivalent to the BD (= braided distributive) categories in the sense of Blass and Gurevich [**BG20a**], with some presentational differences, which will be discussed in Explanation 2.1.37.

By definition, a braided bimonoidal category satisfies 12 of Laplaza's axioms in Definition 2.1.1, one from each of the 12 groups of axioms there, along with a variant of each of the axioms (2.1.4) and (2.1.18). In Definition 2.1.1, these last two Laplaza axioms are the only ones involving $\xi^{\otimes}$. In the braided case, due to the absence of the symmetry axiom (1.3.33), we also need the variants of (2.1.4) and (2.1.18) with each $\xi^{\otimes}$ pointing in the opposite direction.

In Section 2.2, we observe that each braided bimonoidal category satisfies all 24 Laplaza axioms in Definition 2.1.1; see Theorem 2.2.1. For a tight braided bimonoidal category, this theorem is due to Blass and Gurevich [**BG20a**]. This theorem is proved by adapting the lemmas in Section I.2.2 in the symmetric case with

- the axiom (2.1.4) replaced by its variant (2.1.32) and
- the axiom (2.1.18) replaced by its variant (2.1.33) in three instances.

A consequence of Theorem 2.2.1 is that a symmetric bimonoidal category is precisely a braided bimonoidal category whose braiding $\xi^{\otimes}$ satisfies the symmetry axiom (I.1.2.20); see Corollary 2.2.3. This is the bimonoidal analogue of Proposition 1.3.36 relating symmetric and braided monoidal categories.

To prepare for the discussion in Sections 2.4 and 2.5, in Section 2.3 we recall some basic concepts related to Ab-categories, abelian categories, and additive functors. Most of the properties about Ab-categories and additive functors that we will use later are in Theorem 2.3.7.

The main result of Section 2.4 is Theorem 2.4.22. It states that an abelian category with a compatible braided monoidal structure $\otimes$ is a tight braided bimonoidal category. Compatibility means that the functors $A \otimes -$ and $- \otimes A$ are additive functors for each object A. The additive symmetric monoidal structure comes from the direct sum of the abelian category. This result is due to Blass and Gurevich [**BG20a**]. Applications of Theorem 2.4.22 in quantum group theory and topological quantum computation will be discussed in Chapter 3.

In Section 2.5, we observe that the symmetric and the nonbraided analogues of Theorem 2.4.22 are also true; see Corollary 2.5.1 and Theorem 2.5.2. In other words, an abelian category with a compatible (symmetric) monoidal structure is a tight (symmetric) bimonoidal category.

The following table summarizes the main properties of braided bimonoidal categories, along with their references.

axioms	12 + 2 (2.1.29)
Laplaza axioms	all 24 (2.2.1)
key examples	2.4.22, 3.2.19, 3.4.13, 3.6.14
centers	4.4.3, 4.5.3
coherence	5.4.4
strictification	6.3.6, 6.3.7
Braided Baez Conjecture	7.3.4, 7.3.6
monoidal bicategorification	8.4.7

Reading Guide.

(1) Read Definition 2.1.1 of a symmetric bimonoidal category as a refresher.
(2) Read Definition 2.1.29 of a braided bimonoidal category.
(3) Read the statement of Theorem 2.2.1, which says that each braided bimonoidal category satisfies all the axioms of a symmetric bimonoidal category.
(4) Read Convention 2.4.1 and the statement of Theorem 2.4.22, which says that an abelian category with a compatible braided monoidal structure is a tight braided bimonoidal category.
(5) Read Corollary 2.5.1 and the statement of Theorem 2.5.2, which is the nonbraided version of Theorem 2.4.22.
(6) Go back and read the rest of this chapter.

2.1. Definitions

In this section, we first recall the definition of a symmetric bimonoidal category. Then we define braided bimonoidal categories as the braided analogues of symmetric bimonoidal categories. Our braided bimonoidal categories are more general than the BD categories in the sense of Blass and Gurevich [**BG20a**] because our distributivity morphisms δ^l and δ^r are only assumed to be natural monomorphisms, instead of isomorphisms. The differences between these two definitions are discussed in Explanation 2.1.37.

Symmetric Bimonoidal Categories. For comparison and reference, we first recall the notion of a symmetric bimonoidal category from Definition I.2.1.2.

Definition 2.1.1. A *symmetric bimonoidal category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}), (\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^{l}, \delta^{r})\big)$$

consisting of the following data.

- $(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$ is a symmetric monoidal category, which is called the *additive structure*.
- $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$ is a symmetric monoidal category, which is called the *multiplicative structure*.
- $\lambda^{\bullet}$ and $\rho^{\bullet}$ are natural isomorphisms

$$\mathbb{0} \otimes A \xrightarrow[\cong]{\lambda^{\bullet}_A} \mathbb{0} \xleftarrow[\cong]{\rho^{\bullet}_A} A \otimes \mathbb{0} \quad \text{for} \quad A \in \mathsf{C}, \tag{2.1.2}$$

which are called the *left multiplicative zero* and the *right multiplicative zero*, respectively.
- δ^{l} and δ^{r} are natural monomorphisms

$$\begin{gathered} A \otimes (B \oplus C) \xrightarrow{\delta^{l}_{A,B,C}} (A \otimes B) \oplus (A \otimes C) \\ (A \oplus B) \otimes C \xrightarrow{\delta^{r}_{A,B,C}} (A \otimes C) \oplus (B \otimes C) \end{gathered} \tag{2.1.3}$$

for objects $A, B, C \in \mathsf{C}$, which are called the *left distributivity morphism* and the *right distributivity morphism*, respectively.

To simplify the presentation, we abbreviate $\otimes$ using concatenation. In the absence of parentheses, $\otimes$ always takes precedence over $\oplus$. For example, the left distributivity morphism is abbreviated to $A(B \oplus C) \longrightarrow AB \oplus AC$.

The above data are required to make the following 24 diagrams in C commutative for all objects $A, B, C, D \in \mathsf{C}$. They are collectively known as *Laplaza's Axioms*.

Distributivity and Multiplicative Symmetry:

$$\begin{tikzcd} (A \oplus B)C \arrow[r, "\delta^{r}_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A\oplus B,C}"'] & AC \oplus BC \arrow[d, "\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}"] \\ C(A \oplus B) \arrow[r, "\delta^{l}_{C,A,B}"] & CA \oplus CB \end{tikzcd} \tag{2.1.4}$$

Distributivity and Additive Symmetry:

$$\begin{tikzcd} A(B \oplus C) \arrow[r, "\delta^{l}_{A,B,C}"] \arrow[d, "1_A \xi^{\oplus}_{B,C}"'] & AB \oplus AC \arrow[d, "\xi^{\oplus}_{AB,AC}"] \\ A(C \oplus B) \arrow[r, "\delta^{l}_{A,C,B}"] & AC \oplus AB \end{tikzcd} \tag{2.1.5}$$

$$\begin{tikzcd} (A \oplus B)C \arrow[r, "\delta^{r}_{A,B,C}"] \arrow[d, "\xi^{\oplus}_{A,B} 1_C"'] & AC \oplus BC \arrow[d, "\xi^{\oplus}_{AC,BC}"] \\ (B \oplus A)C \arrow[r, "\delta^{r}_{B,A,C}"] & BC \oplus AC \end{tikzcd} \tag{2.1.6}$$

Distributivity and Additive Associativity:

(2.1.7)
$$\begin{tikzcd}
{[(A\oplus B)\oplus C]D} \arrow[r, "\delta^r_{A\oplus B,C,D}"] \arrow[d, "\alpha^\oplus_{A,B,C}1_D"'] & (A\oplus B)D\oplus CD \arrow[r, "\delta^r_{A,B,D}\oplus 1_{CD}"] & (AD\oplus BD)\oplus CD \arrow[d, "\alpha^\oplus_{AD,BD,CD}"] \\
{[A\oplus(B\oplus C)]D} \arrow[r, "\delta^r_{A,B\oplus C,D}"] & AD\oplus(B\oplus C)D \arrow[r, "1_{AD}\oplus\delta^r_{B,C,D}"] & AD\oplus(BD\oplus CD)
\end{tikzcd}$$

(2.1.8)
$$\begin{tikzcd}
A[(B\oplus C)\oplus D] \arrow[r, "\delta^l_{A,B\oplus C,D}"] \arrow[d, "1_A\alpha^\oplus_{B,C,D}"'] & A(B\oplus C)\oplus AD \arrow[r, "\delta^l_{A,B,C}\oplus 1_{AD}"] & (AB\oplus AC)\oplus AD \arrow[d, "\alpha^\oplus_{AB,AC,AD}"] \\
A[B\oplus(C\oplus D)] \arrow[r, "\delta^l_{A,B,C\oplus D}"] & AB\oplus A(C\oplus D) \arrow[r, "1_{AB}\oplus\delta^l_{A,C,D}"] & AB\oplus(AC\oplus AD)
\end{tikzcd}$$

Distributivity and Multiplicative Associativity:

(2.1.9)
$$\begin{tikzcd}
(AB)(C\oplus D) \arrow[rr, "\delta^l_{AB,C,D}"] \arrow[d, "\alpha^\otimes_{A,B,C\oplus D}"'] & & (AB)C\oplus(AB)D \arrow[d, "\alpha^\otimes_{A,B,C}\oplus\alpha^\otimes_{A,B,D}"] \\
A[B(C\oplus D)] \arrow[r, "1_A\delta^l_{B,C,D}"] & A(BC\oplus BD) \arrow[r, "\delta^l_{A,BC,BD}"] & A(BC)\oplus A(BD)
\end{tikzcd}$$

(2.1.10)
$$\begin{tikzcd}
{[(A\oplus B)C]D} \arrow[r, "\delta^r_{A,B,C}1_D"] \arrow[d, "\alpha^\otimes_{A\oplus B,C,D}"'] & (AC\oplus BC)D \arrow[r, "\delta^r_{AC,BC,D}"] & (AC)D\oplus(BC)D \arrow[d, "\alpha^\otimes_{A,C,D}\oplus\alpha^\otimes_{B,C,D}"] \\
(A\oplus B)(CD) \arrow[rr, "\delta^r_{A,B,CD}"] & & A(CD)\oplus B(CD)
\end{tikzcd}$$

(2.1.11)
$$\begin{tikzcd}
{[A(B\oplus C)]D} \arrow[r, "\delta^l_{A,B,C}1_D"] \arrow[d, "\alpha^\otimes_{A,B\oplus C,D}"'] & (AB\oplus AC)D \arrow[r, "\delta^r_{AB,AC,D}"] & (AB)D\oplus(AC)D \arrow[d, "\alpha^\otimes_{A,B,D}\oplus\alpha^\otimes_{A,C,D}"] \\
A[(B\oplus C)D] \arrow[r, "1_A\delta^r_{B,C,D}"] & A(BD\oplus CD) \arrow[r, "\delta^l_{A,BD,CD}"] & A(BD)\oplus A(CD)
\end{tikzcd}$$

2-By-2 Distributivity:

(2.1.12)
$$\begin{tikzcd}
(A\oplus B)(C\oplus D) \arrow[r, "\delta^r_{A,B,C\oplus D}"] \arrow[d, "\delta^l_{A\oplus B,C,D}"'] & A(C\oplus D)\oplus B(C\oplus D) \arrow[d, "\delta^l_{A,C,D}\oplus\delta^l_{B,C,D}"] \\
(A\oplus B)C\oplus(A\oplus B)D \arrow[d, "\delta^r_{A,B,C}\oplus\delta^r_{A,B,D}"'] & (AC\oplus AD)\oplus(BC\oplus BD) \arrow[d, "\alpha^\oplus_{AC,AD,BC\oplus BD}"] \\
(AC\oplus BC)\oplus(AD\oplus BD) \arrow[d, "\alpha^\oplus_{AC,BC,AD\oplus BD}"'] & AC\oplus[AD\oplus(BC\oplus BD)] \arrow[d, "1_{AC}\oplus(\alpha^\oplus)^{-1}"] \\
AC\oplus[BC\oplus(AD\oplus BD)] \arrow[d, "1_{AC}\oplus(\alpha^\oplus)^{-1}"'] & AC\oplus[(AD\oplus BC)\oplus BD] \arrow[d, "1_{AC}\oplus(\xi^\oplus_{AD,BC}\oplus 1_{BD})"] \\
AC\oplus[(BC\oplus AD)\oplus BD] \arrow[r, equal] & AC\oplus[(BC\oplus AD)\oplus BD]
\end{tikzcd}$$

Multiplicative Zero of 0:

(2.1.13)
$$\begin{tikzcd}
\mathbb{0}\otimes\mathbb{0} \arrow[r, shift left, "\lambda^\bullet_{\mathbb{0}}"] \arrow[r, shift right, "\rho^\bullet_{\mathbb{0}}"'] & \mathbb{0}
\end{tikzcd}$$

Multiplicative Zero of a Sum:

(2.1.14)
$$\begin{tikzcd}
\mathbb{0}(A \oplus B) \arrow[r, "\delta^l_{\mathbb{0},A,B}"] \arrow[d, "\lambda^\bullet_{A\oplus B}"'] & \mathbb{0}A \oplus \mathbb{0}B \arrow[d, "\lambda^\bullet_A \oplus \lambda^\bullet_B"] \\
\mathbb{0} & \mathbb{0} \oplus \mathbb{0} \arrow[l, "\lambda^\oplus_{\mathbb{0}}"']
\end{tikzcd}$$

(2.1.15)
$$\begin{tikzcd}
(A \oplus B)\mathbb{0} \arrow[r, "\delta^r_{A,B,\mathbb{0}}"] \arrow[d, "\rho^\bullet_{A\oplus B}"'] & A\mathbb{0} \oplus B\mathbb{0} \arrow[d, "\rho^\bullet_A \oplus \rho^\bullet_B"] \\
\mathbb{0} & \mathbb{0} \oplus \mathbb{0} \arrow[l, "\lambda^\oplus_{\mathbb{0}}"']
\end{tikzcd}$$

Multiplicative Zero and Multiplicative Unit:

(2.1.16)
$$\begin{tikzcd}
\mathbb{0} \otimes \mathbb{1} \arrow[r, shift left, "\lambda^\bullet_{\mathbb{1}}"] \arrow[r, shift right, "\rho^\otimes_{\mathbb{0}}"'] & \mathbb{0}
\end{tikzcd}$$

(2.1.17)
$$\begin{tikzcd}
\mathbb{1} \otimes \mathbb{0} \arrow[r, shift left, "\rho^\bullet_{\mathbb{1}}"] \arrow[r, shift right, "\lambda^\otimes_{\mathbb{0}}"'] & \mathbb{0}
\end{tikzcd}$$

Symmetry of Multiplicative Zero:

(2.1.18)
$$\begin{tikzcd}
A \otimes \mathbb{0} \arrow[rr, "\xi^\otimes_{A,\mathbb{0}}"] \arrow[dr, "\rho^\bullet_A"'] & & \mathbb{0} \otimes A \arrow[dl, "\lambda^\bullet_A"] \\
& \mathbb{0} &
\end{tikzcd}$$

Multiplicative Zero and Multiplicative Associativity:

(2.1.19)
$$\begin{tikzcd}
(AB)\mathbb{0} \arrow[r, "\alpha^\otimes_{A,B,\mathbb{0}}"] \arrow[d, "\rho^\bullet_{AB}"'] & A(B\mathbb{0}) \arrow[d, "1_A\rho^\bullet_B"] \\
\mathbb{0} & A\mathbb{0} \arrow[l, "\rho^\bullet_A"']
\end{tikzcd}$$

(2.1.20)
$$\begin{tikzcd}
(A\mathbb{0})B \arrow[rr, "\alpha^\otimes_{A,\mathbb{0},B}"] \arrow[d, "\rho^\bullet_A 1_B"'] & & A(\mathbb{0}B) \arrow[d, "1_A\lambda^\bullet_B"] \\
\mathbb{0}B \arrow[dr, "\lambda^\bullet_B"'] & & A\mathbb{0} \arrow[dl, "\rho^\bullet_A"] \\
& \mathbb{0} &
\end{tikzcd}$$

(2.1.21)
$$\begin{tikzcd}
(\mathbb{0}A)B \arrow[r, "\lambda^\bullet_A 1_B"] \arrow[d, "\alpha^\otimes_{\mathbb{0},A,B}"'] & \mathbb{0}B \arrow[d, "\lambda^\bullet_B"] \\
\mathbb{0}(AB) \arrow[r, "\lambda^\bullet_{AB}"] & \mathbb{0}
\end{tikzcd}$$

Additive and Multiplicative Zero:

$$\begin{tikzcd} A(\mathbb{0} \oplus B) \arrow[r, "\delta^l_{A,\mathbb{0},B}"] \arrow[d, "1_A\lambda^{\oplus}_B"'] & A\mathbb{0} \oplus AB \arrow[d, "\rho^{\bullet}_A \oplus 1_{AB}"] \\ AB & \mathbb{0} \oplus AB \arrow[l, "\lambda^{\oplus}_{AB}"'] \end{tikzcd} \tag{2.1.22}$$

$$\begin{tikzcd} (\mathbb{0} \oplus B)A \arrow[r, "\delta^r_{\mathbb{0},B,A}"] \arrow[d, "\lambda^{\oplus}_B 1_A"'] & \mathbb{0}A \oplus BA \arrow[d, "\lambda^{\bullet}_A \oplus 1_{BA}"] \\ BA & \mathbb{0} \oplus BA \arrow[l, "\lambda^{\oplus}_{BA}"'] \end{tikzcd} \tag{2.1.23}$$

$$\begin{tikzcd} A(B \oplus \mathbb{0}) \arrow[r, "\delta^l_{A,B,\mathbb{0}}"] \arrow[d, "1_A\rho^{\oplus}_B"'] & AB \oplus A\mathbb{0} \arrow[d, "1_{AB} \oplus \rho^{\bullet}_A"] \\ AB & AB \oplus \mathbb{0} \arrow[l, "\rho^{\oplus}_{AB}"'] \end{tikzcd} \tag{2.1.24}$$

$$\begin{tikzcd} (B \oplus \mathbb{0})A \arrow[r, "\delta^r_{B,\mathbb{0},A}"] \arrow[d, "\rho^{\oplus}_B 1_A"'] & BA \oplus \mathbb{0}A \arrow[d, "1_{BA} \oplus \lambda^{\bullet}_A"] \\ BA & BA \oplus \mathbb{0} \arrow[l, "\rho^{\oplus}_{BA}"'] \end{tikzcd} \tag{2.1.25}$$

Distributivity and Multiplicative Unit:

$$\begin{tikzcd} \mathbb{1}(A \oplus B) \arrow[rr, "\delta^l_{\mathbb{1},A,B}"] \arrow[dr, "\lambda^{\otimes}_{A\oplus B}"'] & & \mathbb{1}A \oplus \mathbb{1}B \arrow[dl, "\lambda^{\otimes}_A \oplus \lambda^{\otimes}_B"] \\ & A \oplus B & \end{tikzcd} \tag{2.1.26}$$

$$\begin{tikzcd} (A \oplus B)\mathbb{1} \arrow[rr, "\delta^r_{A,B,\mathbb{1}}"] \arrow[dr, "\rho^{\otimes}_{A\oplus B}"'] & & A\mathbb{1} \oplus B\mathbb{1} \arrow[dl, "\rho^{\otimes}_A \oplus \rho^{\otimes}_B"] \\ & A \oplus B & \end{tikzcd} \tag{2.1.27}$$

This finishes the definition of a symmetric bimonoidal category.

A *bimonoidal category* has the same definition as a symmetric bimonoidal category except for the following two conditions.

- The multiplicative symmetry $\xi^{\otimes}$ is omitted, and $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes})$ is a monoidal category.
- The axioms (2.1.4) and (2.1.18) are omitted.

Moreover, we define the following.

- A (symmetric) bimonoidal category is *small* if its class of objects is a set.
- A (symmetric) bimonoidal category is *tight* if both δ^l and δ^r are natural isomorphisms.
- The objects $\mathbb{0}$ and $\mathbb{1}$ are called the *additive zero* and the *multiplicative unit*, respectively.

- $\oplus$ and $\otimes$ are called the *sum* and the *product*, respectively.
- $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, and $\xi^{\oplus}$ are called the *additive associativity isomorphism*, the *left additive zero*, the *right additive zero*, and the *additive symmetry isomorphism*, respectively.
- $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, and $\xi^{\otimes}$ are called the *multiplicative associativity isomorphism*, the *left multiplicative unit*, the *right multiplicative unit*, and the *multiplicative symmetry isomorphism*, respectively. $\diamond$

The 24 Laplaza axioms (2.1.4)–(2.1.27) are precisely the axioms (I.2.1.5)–(I.2.1.28) in the same order.

Braided Bimonoidal Categories.

Motivation 2.1.28. To motivate the definition of a braided bimonoidal category, recall from Definition 2.1.1 that a symmetric bimonoidal category C has

- an additive symmetric monoidal structure $\oplus$,
- a multiplicative symmetric monoidal structure $\otimes$,
- left and right multiplicative zeros $(\lambda^{\bullet}, \rho^{\bullet})$, and
- left and right distributivity morphisms (δ^l, δ^r).

The braided analogue replaces the multiplicative symmetric monoidal category by a braided monoidal category.

We showed in Section I.2.2 that among the 24 Laplaza axioms of a symmetric bimonoidal category, 12 of them are formal consequences of the others. In each of the 12 groups of axioms in Definition 2.1.1, only the first axiom, or an equivalent axiom within that group, is necessary. In the braided analogue, one Laplaza axiom from each of those 12 groups is assumed to hold. In Section 2.2, we will show that, analogous to Section I.2.2, in a braided bimonoidal category, the other 12 Laplaza axioms can be recovered from the assumed axioms.

Among the 24 Laplaza axioms in Definition 2.1.1, only (2.1.4) and (2.1.18) involve the multiplicative symmetry $\xi^{\otimes}$. The symmetry axiom (1.3.33) in $(\mathsf{C}, \otimes)$ states that

$$\xi^{\otimes}_{Y,X}\xi^{\otimes}_{X,Y} = 1_{X\otimes Y}.$$

This implies that each of the two axioms (2.1.4) and (2.1.18) is equivalent to the modified version with each $\xi^{\otimes}_{?,?'}$ replaced by $\xi^{\otimes}_{?',?}$ pointing in the opposite direction. In the braided analogue, $\xi^{\otimes}$ is the braiding of a braided monoidal category, so the braiding-square $\xi^{\otimes}_{Y,X}\xi^{\otimes}_{X,Y}$ is not equal to $1_{X\otimes Y}$ in general. Therefore, in the braided case, (2.1.4) and (2.1.18) are not in general equivalent to their respective modified versions. In the definition of a braided bimonoidal category, these modified versions are two additional axioms (2.1.32) and (2.1.33). We will discuss these axioms in more detail in Explanations 2.1.35 and 2.1.37. $\diamond$

Definition 2.1.29. A *braided bimonoidal category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}), (\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^l, \delta^r)\big)$$

consisting of the following data.

- $(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$ is a symmetric monoidal category, which is called the *additive structure*.
- $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$ is a braided monoidal category, which is called the *multiplicative structure*, with $\xi^{\otimes}$ called the *braiding*.

- $\lambda^{\bullet}$ and $\rho^{\bullet}$ are natural isomorphisms

$$\mathbb{0} \otimes A \xrightarrow[\cong]{\lambda^{\bullet}_A} \mathbb{0} \xleftarrow[\cong]{\rho^{\bullet}_A} A \otimes \mathbb{0} \quad \text{for} \quad A \in \mathsf{C}, \tag{2.1.30}$$

  which are called the *left multiplicative zero* and the *right multiplicative zero*, respectively.
- δ^l and δ^r are natural monomorphisms

$$\begin{aligned} A \otimes (B \oplus C) &\xrightarrow{\delta^l_{A,B,C}} (A \otimes B) \oplus (A \otimes C) \\ (A \oplus B) \otimes C &\xrightarrow{\delta^r_{A,B,C}} (A \otimes C) \oplus (B \otimes C) \end{aligned} \tag{2.1.31}$$

  for objects $A, B, C \in \mathsf{C}$, which are called the *left distributivity morphism* and the *right distributivity morphism*, respectively.

The above data are required to satisfy the following 14 axioms for all objects $A, B, C, D \in \mathsf{C}$.

12 Laplaza Axioms: The following diagrams are commutative: (2.1.4), (2.1.5), (2.1.8), (2.1.9), (2.1.12), (2.1.13), (2.1.15), (2.1.17), (2.1.18), (2.1.19), (2.1.24), and (2.1.26).

Distributivity and the Braiding: The following diagram is commutative in C.

$$\begin{tikzcd} C \otimes (A \oplus B) \arrow[r, "\delta^l_{C,A,B}"] \arrow[d, "\xi^{\otimes}_{C,A\oplus B}"'] & (C \otimes A) \oplus (C \otimes B) \arrow[d, "\xi^{\otimes}_{C,A} \oplus \xi^{\otimes}_{C,B}"] \\ (A \oplus B) \otimes C \arrow[r, "\delta^r_{A,B,C}"] & (A \otimes C) \oplus (B \otimes C) \end{tikzcd} \tag{2.1.32}$$

Multiplicative Zeros and the Braiding: The following diagram is commutative in C.

$$\begin{tikzcd} A \otimes \mathbb{0} \arrow[dr, "\rho^{\bullet}_A"'] & & \mathbb{0} \otimes A \arrow[ll, "\xi^{\otimes}_{\mathbb{0},A}"'] \arrow[dl, "\lambda^{\bullet}_A"] \\ & \mathbb{0} & \end{tikzcd} \tag{2.1.33}$$

This finishes the definition of a braided bimonoidal category. Moreover, a braided bimonoidal category is called

- *tight* if δ^l and δ^r are natural isomorphisms and
- *small* if it has a set of objects. ◇

Convention 2.1.34. We often omit the $\otimes$ symbol to save space. In the absence of clarifying parentheses, $\otimes$ takes precedence over $\oplus$. For example, $AB \oplus CD$ means $(A \otimes B) \oplus (C \otimes D)$. ◇

Explanation 2.1.35 (Axioms)**.** Consider Definition 2.1.29.

- One axiom is assumed in each of the 12 groups of Laplaza's axioms in Definition 2.1.1. We will see in Theorem 2.2.1 that all 24 Laplaza axioms hold in a braided bimonoidal category.
- The diagram (2.1.32) is a modified version of (2.1.4). In a braided monoidal category, the braiding-square

$$\xi^{\otimes}_{Y,X}\xi^{\otimes}_{X,Y} : X \otimes Y \longrightarrow X \otimes Y$$

is not the same as $1_{X\otimes Y}$ in general. Therefore, in a braided bimonoidal category, (2.1.32) and (2.1.4) are not equivalent to each other in general.

- The diagram (2.1.33) is a modified version of (2.1.18). In a braided bimonoidal category, (2.1.33) and (2.1.18) are not equivalent to each other in general because

$$\xi^{\otimes}_{\mathbb{0},A}\xi^{\otimes}_{A,\mathbb{0}} : A \otimes \mathbb{0} \longrightarrow A \otimes \mathbb{0}$$

is not equal to $1_{A\otimes\mathbb{0}}$ in general. ◇

Remark 2.1.36. The braided bimonoidal categories in Definition 2.1.29 are more general than Richter's in [**Ric10**, Def. 5.1], which correspond to right permbraided categories in Definition 5.1.11. ◇

Blass-Gurevich BD Categories.

Explanation 2.1.37. Definition 2.1.29 of a braided bimonoidal category is more general than the one introduced by Blass and Gurevich in [**BG20a**, Def. 7]. Here we discuss the differences between the two definitions.

Terminology: Our *tight* braided bimonoidal categories are equivalent to the *BD categories* in [**BG20a**], with BD standing for *braided distributive*. In [**BG20a**], they assume from the beginning that the distributivity morphisms δ^l and δ^r in BD categories are natural isomorphisms. In our braided bimonoidal categories, δ^l and δ^r are only natural monomorphisms.

Notation: The notational differences are listed in the following table, with the last column explained below.

structure	2.1.29	[**BG20a**]	primitive data
additive symmetry	$\xi^{\oplus}$	$\gamma^{\oplus}$	yes
braiding	$\xi^{\otimes}$	$\gamma^{\otimes}$	yes
left distributivity	δ^l	δ	yes
right distributivity	δ^r	$\delta^{\sharp}$	no
right multiplicative zero	$\rho^{\bullet}$	ε	yes
left multiplicative zero	$\lambda^{\bullet}$	λ^{*}	no

Distributivity: In [**BG20a**], δ^r (= their $\delta^{\sharp}$) is *not* taken as part of the data, but is defined in terms of δ^l and $\xi^{\otimes}$ using the diagram (2.1.4). Moreover, instead of the braided bimonoidal category axioms (2.1.4) and (2.1.32), in [**BG20a**, Fig. 9], they assume the commutativity of a diagram that is equivalent to the outer diagram below.

$$\begin{tikzcd}
C(A\oplus B) \arrow[rr, "\delta^l_{C,A,B}"] \arrow[drr, phantom, "(2.1.32)"] & & CA\oplus CB \arrow[d, "\xi^{\otimes}_{C,A}\oplus\xi^{\otimes}_{C,B}"] \\
(A\oplus B)C \arrow[u, "\xi^{-\otimes}_{C,A\oplus B}"] \arrow[rr, "\delta^r_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A\oplus B,C}"'] \arrow[drr, phantom, "(2.1.4)"] & & AC\oplus BC \\
C(A\oplus B) \arrow[rr, "\delta^l_{C,A,B}"] & & CA\oplus CB \arrow[u, "\xi^{-\otimes}_{A,C}\oplus\xi^{-\otimes}_{B,C}"']
\end{tikzcd} \tag{2.1.38}$$

- Their definition of δ^r in terms of δ^l and $\xi^{\otimes}$ is equivalent to the commutativity of the lower half of (2.1.38), which is equivalent to the axiom (2.1.4) by the invertibility of $\xi^{\otimes}$.
- The upper half of (2.1.38) is equivalent to (2.1.32).

Among (2.1.4), (2.1.32), and the outer diagram in (2.1.38), any two of them imply the third. The two halves of (2.1.38) provide two equivalent ways to represent δ^r in terms of δ^l and $\xi^{\otimes}$.

Multiplicative Zeros: In [**BG20a**], $\lambda^{\bullet}$ (= their λ^*) is *not* taken as part of the data, but is defined in terms of $\rho^{\bullet}$ and $\xi^{\otimes}$ using the diagram (2.1.33). Moreover, instead of the braided bimonoidal category axioms (2.1.18) and (2.1.33), in [**BG20a**, Fig. 9], they assume the commutativity of the diagram below.

(2.1.39)
$$\begin{tikzcd}
A \otimes \mathbb{0} \arrow[rr, "1_{A\otimes\mathbb{0}}"] \arrow[dr, "\xi^{\otimes}_{A,\mathbb{0}}"'] & & A \otimes \mathbb{0} \\
& \mathbb{0} \otimes A \arrow[ur, "\xi^{\otimes}_{\mathbb{0},A}"'] &
\end{tikzcd}$$

For comparison, the axiom (2.1.18) is equivalent to the left commutative diagram below by the invertibility of $\xi^{\otimes}$. The axiom (2.1.33) is the right commutative diagram below.

$$\begin{tikzcd}
A \otimes \mathbb{0} \arrow[dr, bend right, "\rho^{\bullet}_A"'] & & \mathbb{0} \otimes A \arrow[ll, "\xi^{-\otimes}_{A,\mathbb{0}}"'] \arrow[dl, bend left, "\lambda^{\bullet}_A"] \arrow[d, phantom, "(2.1.18)"] \\
& \mathbb{0} &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes \mathbb{0} \arrow[dr, bend right, "\rho^{\bullet}_A"'] & & \mathbb{0} \otimes A \arrow[ll, "\xi^{\otimes}_{\mathbb{0},A}"'] \arrow[dl, bend left, "\lambda^{\bullet}_A"] \arrow[d, phantom, "(2.1.33)"] \\
& \mathbb{0} &
\end{tikzcd}$$

By the invertibility of $\rho^{\bullet}$, among (2.1.18), (2.1.33), and (2.1.39), any two of them imply the third. The previous two commutative diagrams provide two equivalent ways to represent $\lambda^{\bullet}$ in terms of $\rho^{\bullet}$ and $\xi^{\otimes}$.

We define braided bimonoidal categories as in Definition 2.1.29, instead of the form in [**BG20a**], for three reasons.

(1) We want the definition of a braided bimonoidal category to be as similar to that of a symmetric bimonoidal category in Definition 2.1.1 as possible. In particular, we want to preserve the symmetry of the data pairs $(\lambda^{\bullet}, \rho^{\bullet})$ in (2.1.30) and (δ^l, δ^r) in (2.1.31).
(2) As explained above, instead of the axioms (2.1.32) and (2.1.33), we could also have assumed the outer diagrams in (2.1.38) and (2.1.39). We chose to adopt (2.1.32) and (2.1.33), instead of (2.1.38) and (2.1.39), because of their symmetry with (2.1.4) and (2.1.18), respectively.
(3) We want to clarify precisely where the invertibility of the distributivity morphisms δ^l and δ^r is needed, and where it is not necessary. Therefore, following Definition 2.1.1, we only assume that δ^l and δ^r are natural monomorphisms in a general braided bimonoidal category. ⋄

2.2. Recovering Laplaza's Axioms

Recall from Definition 2.1.29 that a braided bimonoidal category satisfies the following 14 axioms:

- 12 Laplaza axioms: (2.1.4), (2.1.5), (2.1.8), (2.1.9), (2.1.12), (2.1.13), (2.1.15), (2.1.17), (2.1.18), (2.1.19), (2.1.24), and (2.1.26); and
- the axioms (2.1.32) and (2.1.33), which are variants of, respectively, (2.1.4) and (2.1.18).

In this section, we observe that all 24 Laplaza axioms, (2.1.4)–(2.1.27), in Definition 2.1.1 are satisfied by each braided bimonoidal category. This is the braided analogue of Theorem I.2.2.13. A consequence of this result is Corollary 2.2.3. It states that a symmetric bimonoidal category is precisely a braided bimonoidal category whose braiding $\xi^{\otimes}$ satisfies the symmetry axiom (1.3.33). This is the bimonoidal analogue of Proposition 1.3.36.

Theorem 2.2.1. *Each braided bimonoidal category satisfies all 24 Laplaza axioms in Definition 2.1.1.*

Proof. Since 12 of the 24 Laplaza axioms are already assumed to hold in a braided bimonoidal category, we must prove that the other 12 Laplaza axioms also hold. They are listed in the leftmost column in the following table. To obtain these axioms, we reuse the proofs in Section I.2.2 with $(\mathsf{C}, \xi^{\otimes})$ a braided monoidal category, along with suitable braided bimonoidal category axioms and properties, as indicated in the next three columns in the following table. Recall that the 24 Laplaza axioms (2.1.4)–(2.1.27) are precisely (I.2.1.5)–(I.2.1.28) in the same order.

To obtain	Use the axioms	And	In the proof of
(2.1.6)	(2.1.32), (2.1.5)		Lemma I.2.2.4
(2.1.7)	(2.1.32), (2.1.8)		Lemma I.2.2.5
(2.1.10)	(2.1.32), (2.1.9)	(1.3.31)	Lemma I.2.2.6
(2.1.11)	(2.1.32), (2.1.9)	(1.3.17)	Lemma I.2.2.7
(2.1.14)	(2.1.32), (2.1.33), (2.1.15)		Lemma I.2.2.8
(2.1.16)	(2.1.17), (2.1.18)	(1.3.22)	Lemma I.2.2.9
(2.1.20), (2.1.21)	(2.1.33), (2.1.19)	(1.3.17)	Lemma I.2.2.10
(2.1.22), (2.1.23), (2.1.25)	(2.1.32), (2.1.5), (2.1.18), (2.1.24)		Lemma I.2.2.11
(2.1.27)	(2.1.32), (2.1.26)	(1.3.22)	Lemma I.2.2.12

Note the following two facts in the second column.

- Each instance of (2.1.4), which is the same as (I.2.1.5), is replaced by its variant (2.1.32).
- (2.1.18), which is the same as (I.2.1.19), is replaced by its variant (2.1.33) for (2.1.14), (2.1.20), and (2.1.21).

The following remarks are about the third column in the previous table.

- To obtain (2.1.10), (1.3.31) is used in place of the Symmetric Coherence Theorem I.1.3.8 in the proof of Lemma I.2.2.6.
- To obtain (2.1.11), the left hexagon axiom (1.3.17) is used in place of Theorem I.1.3.8 in the proof of Lemma I.2.2.7.
- To obtain (2.1.20), the right hexagon axiom (1.3.17) is used in place of Theorem I.1.3.8 in the first diagram in the proof of Lemma I.2.2.10.
- To obtain (2.1.21), the left hexagon axiom (1.3.17) is used in place of Theorem I.1.3.8 in the second diagram in the proof of Lemma I.2.2.10.
- The right diagram in (1.3.22), stating $\lambda^{\otimes} = \rho^{\otimes}\xi^{\otimes}_{\mathbb{1},-}$, for the braided monoidal category $(\mathsf{C}, \xi^{\otimes})$ is used to obtain

– (2.1.16) in the outer diagram in the proof of Lemma I.2.2.9 and
– (2.1.27) in the left and the right triangles in Lemma I.2.2.12.

This finishes the proof. □

Remark 2.2.2. The *tight* braided bimonoidal category case of Theorem 2.2.1 is due to Blass and Gurevich [**BG20a**, Th. 10]. In addition to not requiring the invertibility of the distributivity morphisms, our proof above is shorter and simpler than the one in [**BG20a**]. ◇

Recall from Definition 2.1.1 that a symmetric bimonoidal category is called *tight* if δ^l and δ^r are natural isomorphisms.

Corollary 2.2.3. *A (tight) symmetric bimonoidal category is precisely a (tight) braided bimonoidal category whose braiding satisfies the symmetry axiom (1.3.33).*

Proof. In both Definitions 2.1.1 and 2.1.29, tightness refers to the invertibility of δ^l and δ^r. Suppose C is a symmetric bimonoidal category.

- By Proposition 1.3.36, the multiplicative structure of C, which is a symmetric monoidal category, is also a braided monoidal category.
- The 12 Laplaza axioms in Definition 2.1.29 are among those of a symmetric bimonoidal category in Definition 2.1.1.
- The symmetry axiom (1.3.33) in $(\mathsf{C}, \otimes)$ implies that (2.1.32) and (2.1.33) are equivalent to (2.1.4) and (2.1.18), respectively.

Therefore, C is a braided bimonoidal category.

Conversely, suppose C is a braided bimonoidal category whose braiding satisfies the symmetry axiom. Then the multiplicative structure of C is a symmetric monoidal category by Proposition 1.3.36. Theorem 2.2.1 implies that C is a symmetric bimonoidal category. □

2.3. Abelian Categories

In this section, we discuss Ab-categories, (pre)additive categories, abelian categories, and additive functors. These concepts will be used in Section 2.4 when we discuss braided bimonoidal categories arising from abelian categories with a compatible braided monoidal structure. In the following definitions, the prototypical example to keep in mind is the category Ab of abelian groups. See Section 2.6 for a list of references on abelian categories.

Ab-Categories and Additive Functors.

Definition 2.3.1. An Ab-*category* is a category C in which each set $\mathsf{C}(A, B)$ for objects $A, B \in \mathsf{C}$ is equipped with the structure of an abelian group, written additively, such that categorical composition distributes over addition. This means that for any morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "g'"'] & C, \end{tikzcd}$$

the equalities

$$\begin{aligned} (g + g')f &= gf + g'f \\ g(f + f') &= gf + gf' \end{aligned}$$

hold in $\mathsf{C}(A,C)$. The additive unit in the abelian group $\mathsf{C}(A,B)$ is written as $\mathbb{0} : A \longrightarrow B$ and is called the *zero morphism*. ◇

Lemma 2.3.2. *Suppose $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are morphisms in an* Ab*-category* C*. Then the following equalities hold in* $\mathsf{C}(A,C)$*.*

$$\mathbb{0}f = \mathbb{0} = g\mathbb{0}$$
$$g(-f) = -(gf) = (-g)f$$

Proof. The equalities $\mathbb{0}f = \mathbb{0} = g\mathbb{0}$ follow from the following computation.

$$\mathbb{0}f = (\mathbb{0}+\mathbb{0})f = \mathbb{0}f + \mathbb{0}f$$
$$g\mathbb{0} = g(\mathbb{0}+\mathbb{0}) = g\mathbb{0} + g\mathbb{0}$$

The previous equalities and the computation

$$\mathbb{0} = g\mathbb{0} = g(f+(-f)) = gf + g(-f)$$
$$\mathbb{0} = \mathbb{0}f = (g+(-g))f = gf + (-g)f$$

imply $g(-f) = -(gf) = (-g)f$. □

Definition 2.3.3. An *additive functor* $F : \mathsf{C} \longrightarrow \mathsf{D}$ between Ab-categories is a functor such that the function

$$\mathsf{C}(A,B) \xrightarrow{F} \mathsf{D}(FA,FB)$$

is a group homomorphism for each pair of objects $A,B \in \mathsf{C}$. ◇

The concept in the next definition is modeled after the direct sum of two modules over a ring.

Definition 2.3.4. For objects $A,B \in \mathsf{C}$ with C an Ab-category, a *direct sum* of A and B is a quintuple

$$(C, p_1, p_2, i_1, i_2)$$

as in the diagram

$$\begin{tikzcd} A \arrow[r, shift right, "i_1"'] & C \arrow[l, shift right, "p_1"'] \arrow[r, shift left, "p_2"] & B \arrow[l, shift left, "i_2"] \end{tikzcd}$$

in C that satisfies the following equalities.

$$i_1p_1 + i_2p_2 = 1_C$$

$$p_ki_j = \begin{cases} 1_A & \text{if } k = j = 1, \\ 1_B & \text{if } k = j = 2, \text{ and} \\ \mathbb{0} & \text{if } k \neq j. \end{cases} \tag{2.3.5}$$

In this case, we define the following.

- C is called a *direct sum object* of A and B, which is denoted by $A \oplus B$.
- p_1 and p_2 are called the *projections*.
- i_1 and i_2 are called the *inclusions*.

If we want to emphasize the objects A and B, we will write p_1 as $p_1^{A,B}$ and similarly for $p_2^{A,B}$, $i_1^{A,B}$, and $i_2^{A,B}$. ◇

The following definition extends the concept of a direct sum to morphisms.

Definition 2.3.6. Suppose C is an Ab-category, and $A, A', B, B' \in \mathsf{C}$ are objects such that the direct sums $A \oplus A'$ and $B \oplus B'$ exist. For morphisms $f : A \longrightarrow B$ and $f' : A' \longrightarrow B'$, define the *direct sum morphism*

$$A \oplus A' \xrightarrow{f \oplus f'} B \oplus B'$$

by requiring that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
A \oplus A' \arrow[u, "p_1"] \arrow[r, "f \oplus f'"] \arrow[d, "p_2"'] & B \oplus B' \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd}$$

be commutative. $\diamond$

In other words, the direct sum morphism $f \oplus f'$ is defined by the projections. The following result collects some of the basic properties of Ab-categories, direct sums, and additive functors.

Theorem 2.3.7. *Suppose C is an Ab-category, and $A, B, C \in \mathsf{C}$ are objects. Then the following statements hold.*

(1) *The following three statements are equivalent:*
  (i) *A and B have a product (C, p_1, p_2) as in the diagram*

$$A \xleftarrow{p_1} C \xrightarrow{p_2} B.$$

  (ii) *A and B have a coproduct (C, i_1, i_2) as in the diagram*

$$A \xrightarrow{i_1} C \xleftarrow{i_2} B.$$

  (iii) *A and B have a direct sum (C, p_1, p_2, i_1, i_2) as in Definition 2.3.4.*

(2) *In the context of Definition 2.3.6, the diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_1"'] & B \arrow[d, "i_1"] \\
A \oplus A' \arrow[r, "f \oplus f'"] & B \oplus B' \\
A' \arrow[u, "i_2"] \arrow[r, "f'"] & B' \arrow[u, "i_2"']
\end{tikzcd}$$

is commutative.

(3) *Suppose the direct sums $A \oplus A$ and $B \oplus B$ exist. Then for morphisms $f, g : A \longrightarrow B$, the diagram*

(2.3.8)
$$\begin{tikzcd}
A \arrow[r, "f+g"] \arrow[d, "\Delta_A"'] & B \\
A \times A = A \oplus A \arrow[r, "f \oplus g"] & B \oplus B = B \amalg B \arrow[u, "\nabla_B"']
\end{tikzcd}$$

is commutative, with

- *$\Delta_A : A \longrightarrow A \times A$ the diagonal defined by*

$$p_1\Delta_A = p_2\Delta_A = 1_A$$

and

- *$\nabla_B : B \amalg B \longrightarrow B$ the codiagonal defined by*

$$\nabla_B i_1 = \nabla_B i_2 = 1_B.$$

(4) *For a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ between Ab-categories with C having all direct sums, F is additive if and only if it preserves direct sums.*

Proof. Proof of (1). First we observe that a direct sum yields a (co)product.

A direct sum is a (co)product. Suppose (C, p_1, p_2, i_1, i_2) is a direct sum. We check that (C, p_1, p_2) is a product of A and B. Given morphisms $f_1 : D \longrightarrow A$ and $f_2 : D \longrightarrow B$, we must show that there is a unique morphism $f : D \longrightarrow C$ such that the solid-arrow diagram

(2.3.9)
$$\begin{tikzcd} & D \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ A \arrow[r, dashed, "i_1"', shift right] & C \arrow[l, "p_1"', shift right] \arrow[r, "p_2", shift left] & B \arrow[l, dashed, "i_2", shift left] \end{tikzcd}$$

is commutative. We will show that the morphism

$$f = i_1 f_1 + i_2 f_2 : D \longrightarrow C$$

has these properties. By Lemma 2.3.2 and the axioms (2.3.5) of a direct sum, we have

$$\begin{aligned} p_1 f &= p_1(i_1 f_1 + i_2 f_2) \\ &= p_1 i_1 f_1 + p_1 i_2 f_2 \\ &= 1_A f_1 + \mathbb{0} f_2 = f_1. \end{aligned}$$

A similar computation shows that $p_2 f = f_2$. This shows that the solid-arrow diagram (2.3.9) is commutative. For uniqueness, suppose $g : D \longrightarrow C$ also makes the solid-arrow diagram (2.3.9) commutative. By (2.3.5), we have

$$\begin{aligned} g &= 1_C g \\ &= (i_1 p_1 + i_2 p_2) g \\ &= i_1 p_1 g + i_2 p_2 g \\ &= i_1 f_1 + i_2 f_2 = f. \end{aligned}$$

This shows the uniqueness of f, so (C, p_1, p_2) is a product of A and B. A dual argument proves that (C, i_1, i_2) is a coproduct of A and B.

A (co)product is a direct sum. To prove the converse, suppose $(A \times B, p_1, p_2)$ is a product of A and B. By the universal property of a product, the morphisms $1_A : A \longrightarrow A$ and $\mathbb{0} : A \longrightarrow B$ yield a unique morphism $i_1 : A \longrightarrow A \times B$ such that the top half of the diagram

$$\begin{tikzcd}
& A \arrow[dl, "1_A"'] \arrow[d, "i_1"] \arrow[dr, "\mathbb{0}"] & \\
A & A \times B \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\
& B \arrow[ul, "\mathbb{0}"] \arrow[u, "i_2"'] \arrow[ur, "1_B"'] &
\end{tikzcd}$$

is commutative. Similarly, the morphisms $\mathbb{0} : B \longrightarrow A$ and $1_B : B \longrightarrow B$ yield a unique morphism $i_2 : B \longrightarrow A \times B$ such that the bottom half of the previous diagram is commutative. This commutative diagram means that the direct sum axioms (2.3.5), with the possible exception of the first equality there, are satisfied.

By the universal property of a product, to prove the first equality in (2.3.5), it suffices to show that

$$p_j(i_1p_1 + i_2p_2) = p_j \quad \text{for} \quad j = 1, 2.$$

The $j = 1$ case is proved by the following computation, with the last equality by Lemma 2.3.2.

$$\begin{aligned}
p_1(i_1p_1 + i_2p_2) &= p_1i_1p_1 + p_1i_2p_2 \\
&= 1_Ap_1 + \mathbb{0}p_2 = p_1
\end{aligned}$$

A similar computation proves the $j = 2$ case. Therefore, $(A \times B, p_1, p_2, i_1, i_2)$ is a direct sum of A and B. A dual argument shows that a coproduct of A and B is also a direct sum. This proves assertion (1).

Proof of (2). In the context of Definition 2.3.6, there are equalities as follows, with the last equality by Lemma 2.3.2.

$$\begin{aligned}
(f \oplus f')i_1 &= 1_{B \oplus B'}(f \oplus f')i_1 \\
&= (i_1p_1 + i_2p_2)(f \oplus f')i_1 \\
&= i_1p_1(f \oplus f')i_1 + i_2p_2(f \oplus f')i_1 \\
&= i_1fp_1i_1 + i_2f'p_2i_1 \\
&= i_1f1_A + i_2f'\mathbb{0} \\
&= i_1f
\end{aligned}$$

A similar computation shows that $(f \oplus f')i_2 = i_2f'$.

Proof of (3). Using parts (1) and (2), the commutativity of the diagram (2.3.8) is proved by the following computation.

$$\begin{aligned}
\nabla_B(f \oplus g)\Delta_A &= \nabla_B(f \oplus g)1_{A \oplus A}\Delta_A \\
&= \nabla_B(f \oplus g)(i_1p_1 + i_2p_2)\Delta_A \\
&= \nabla_B(f \oplus g)i_1p_1\Delta_A + \nabla_B(f \oplus g)i_2p_2\Delta_A \\
&= \nabla_Bi_1fp_1\Delta_A + \nabla_Bi_2gp_2\Delta_A \\
&= 1_Bf1_A + 1_Bg1_A \\
&= f + g
\end{aligned}$$

Proof of (4). Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor between Ab-categories with C having all direct sums. First suppose F is an additive functor. Then F preserves direct sums because the direct sum axioms (2.3.5) consist of zero morphisms, identity morphisms, composites, and sums of morphisms, all of which are preserved by an additive functor.

Conversely, suppose that F preserves direct sums. To show that F is an additive functor, we must show that, for objects $A, B \in \mathsf{C}$, the function

$$\mathsf{C}(A,B) \xrightarrow{F} \mathsf{D}(FA, FB)$$

preserves

- the zero morphism and
- sums of morphisms.

Suppose $(A \oplus B, p_1, p_2, i_1, i_2)$ is a direct sum of A and B, which exists by the assumption on C. By the assumption on F, the tuple

$$\big(F(A \oplus B), Fp_1, Fp_2, Fi_1, Fi_2\big)$$

is a direct sum of FA and FB. By the last direct sum axiom in (2.3.5) and the functoriality of F, we have

$$\mathbb{0} = (Fp_2)(Fi_1) = F(p_2 i_1) = F\mathbb{0}.$$

This shows that F preserves the zero morphism.

For the preservation of sums of morphisms, suppose $f, g : A \longrightarrow B$ are morphisms. Using part (3) twice, the assumption that F preserves direct sums, and the functoriality of F, the following equalities prove that F preserves sums of morphisms.

$$\begin{aligned} F(f+g) &= F\big(\nabla_B (f \oplus g) \Delta_A\big) \\ &= (F\nabla_B) F(f \oplus g)(F\Delta_A) \\ &= \nabla_{FB}(Ff \oplus Fg)\Delta_{FA} \\ &= Ff + Fg \end{aligned}$$

This finishes the proof. □

Explanation 2.3.10. Consider Theorem 2.3.7.

- Statement (1) says that, in an Ab-category, a product, a coproduct, and a direct sum are equivalent.
- Statement (2) says that the direct sum morphism $f \oplus f'$, which is defined in terms of the projections, can also be characterized in terms of the inclusions.
- Statement (3) says that, for morphisms $f, g : A \longrightarrow B$, the sum $f + g$ factors through the direct sum morphism $f \oplus g$.
- Statement (4) says that, for a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ between Ab-categories with C having all direct sums, F preserves the abelian group structures in the morphism sets if and only if it preserves direct sums. ⋄

Additive Categories. Recall that a *zero object* $\mathbb{0}$ in a category C is an object that is both initial and terminal. In other words, for each object $A \in \mathsf{C}$, there are unique morphisms $A \longrightarrow \mathbb{0} \longrightarrow A$.

Definition 2.3.11.

- A *preadditive category* is an Ab-category with a zero object.
- An *additive category* is a preadditive category in which any two objects have a direct sum. $\diamond$

Lemma 2.3.12. *For objects A and B in a preadditive category C with a zero object $\mathbb{0}$, the composite*

$$A \longrightarrow \mathbb{0} \longrightarrow B$$

is the zero morphism in $\mathsf{C}(A,B)$.

Proof. By the universal properties of a zero object, we have

$$\mathsf{C}(A,\mathbb{0}) = * \quad \text{and} \quad \mathsf{C}(\mathbb{0},B) = *.$$

So the unique morphisms

$$A \xrightarrow{t_A} \mathbb{0} \quad \text{and} \quad \mathbb{0} \xrightarrow{i_B} B$$

are also the zero morphisms in, respectively, $\mathsf{C}(A,\mathbb{0})$ and $\mathsf{C}(\mathbb{0},B)$. Since composition with a zero morphism yields a zero morphism by Lemma 2.3.2, the composite $i_B t_A$ is the zero morphism. $\square$

In a category C with a zero object $\mathbb{0}$, the composite $A \longrightarrow \mathbb{0} \longrightarrow B$ is denoted by $\mathbb{0} : A \longrightarrow B$ for objects A and B. The following concepts are categorical analogues of (co)kernels of linear maps between modules.

Definition 2.3.13. Suppose C is a category with a zero object $\mathbb{0}$, and $f : A \longrightarrow B$ is a morphism in C. Define the following.

- A *kernel* of f is an equalizer of the morphisms $f, \mathbb{0} : A \longrightarrow B$.
- A *cokernel* of f is a coequalizer of the morphisms $f, \mathbb{0} : A \longrightarrow B$. $\diamond$

Explanation 2.3.14. A kernel, if it exists, is unique up to a unique isomorphism. More explicitly, a kernel of $f : A \longrightarrow B$ is a morphism $c : C \longrightarrow A$ such that the following two conditions hold.

(i) $fc = \mathbb{0} : C \longrightarrow B$.
(ii) For each morphism $c' : C' \longrightarrow A$ with $fc' = \mathbb{0} : C' \longrightarrow B$, there exists a unique morphism $c'' : C' \longrightarrow C$ such that the following diagram is commutative.

$$\begin{tikzcd}
C' \arrow[rr, "c''"] \arrow[dr, "c'"] \arrow[ddr, bend right, "\mathbb{0}"'] & & C \arrow[dl, "c"'] \arrow[ddl, bend left, "\mathbb{0}"] \\
& A \arrow[d, "f"] & \\
& B &
\end{tikzcd}$$

A kernel is a monomorphism.

Similarly, a cokernel, if it exists, is unique up to a unique isomorphism. More explicitly, a cokernel of $f : A \longrightarrow B$ is a morphism $d : B \longrightarrow D$ such that the following two conditions hold.

(i) $df = \mathbb{0} : A \longrightarrow D$.

(ii) For each morphism $d' : B \longrightarrow D'$ with $d'f = \mathbb{0} : A \longrightarrow D'$, there exists a unique morphism $d'' : D \longrightarrow D'$ such that the following diagram is commutative.

$$\begin{tikzcd} & A \arrow[d, "f"] \arrow[ddl, bend right, "\mathbb{0}"'] \arrow[ddr, bend left, "\mathbb{0}"] & \\ & B \arrow[dl, "d"'] \arrow[dr, "d'"] & \\ D \arrow[rr, "d''"] & & D' \end{tikzcd}$$

A cokernel is an epimorphism. ◇

Abelian Categories. Recall from Definition 2.3.11 that an additive category is an Ab-category with a zero object in which any two objects have a direct sum.

Definition 2.3.15. An *abelian category* is an additive category in which the following three conditions are satisfied.

- Each morphism has a kernel and a cokernel.
- Each monomorphism is a kernel of some morphism.
- Each epimorphism is a cokernel of some morphism.

This finishes the definition of an abelian category. ◇

Example 2.3.16. In the abelian category Ab of abelian groups, kernels and cokernels are taken in the usual sense of basic algebra. Monomorphisms and epimorphisms are precisely, respectively, injective and surjective group homomorphisms. Suppose $f : A \longrightarrow B$ is a morphism in Ab.

- If f is a monomorphism, then it is the kernel of the quotient map
$$B \longrightarrow \mathrm{cok}(f)$$
to the cokernel.
- If f is an epimorphism, then it is the cokernel of the inclusion
$$\ker(f) \longrightarrow A$$
from the kernel. ◇

Lemma 2.3.17. *Each abelian category has all finite limits and finite colimits, with products and coproducts given by direct sums.*

Proof. By Theorem 2.3.7 (1), an abelian category C has all finite (co)products given by iterated direct sums. For morphisms $f, g : A \longrightarrow B$ in C,

- an equalizer is given by a kernel of $f - g$, and
- a coequalizer is given by a cokernel of $f - g$.

With finite (co)products and (co)equalizers, C has all finite (co)limits. □

Example 2.3.18. Suppose A and B are abelian categories.

- The opposite category A^{op} is an abelian category.
- For each small category D, the functor category A^{D} is an abelian category, with $\mathbb{0}$, direct sums, and (co)kernels defined pointwise in A.
- The category Ch^{A} of chain complexes in A is an abelian category, with $\mathbb{0}$, direct sums, and (co)kernels taken degreewise.
- The Cartesian product $\mathsf{A} \times \mathsf{B}$ is an abelian category.

- For a ring R, the category of left R-modules is an abelian category, with $\mathbb{0}$, direct sums, and (co)kernels taken in the usual sense. Similarly, the category of right R-modules is an abelian category.
- The category of finitely generated left modules over a left Noetherian ring is an abelian category. $\diamond$

2.4. Abelian Categories with a Braiding

The main result of this section, Theorem 2.4.22, states that an abelian category with a compatible braided monoidal structure has the structure of a tight braided bimonoidal category. The additive symmetric monoidal structure is given by the direct sum of the abelian category. This theorem is due to Blass and Gurevich [**BG20a**, Th. 12]. It provides a natural class of examples of tight braided bimonoidal categories. Applications of Theorem 2.4.22 are provided in Chapter 3. Recall from Definitions 1.3.15, 2.3.3, and 2.3.15 the notions of a braided monoidal category, an additive functor, and an abelian category.

Convention 2.4.1. Throughout this section, assume that the tuple

$$(\mathsf{C}, \mathbb{0}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$$

consists of

- an abelian category C with a zero object $\mathbb{0}$ and
- a braided monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$.

Moreover, for each object $A \in \mathsf{C}$, it is assumed that the functors

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "A\otimes -"] \arrow[r, shift right, "-\otimes A"'] & \mathsf{C} \end{tikzcd}$$

are additive functors. $\diamond$

This section is organized as follows.

- Using the direct sum of the abelian category, the additive symmetric monoidal structure $\oplus$ on C is defined in Definition 2.4.2. Its detail is in Explanation 2.4.3. The essence of the additive structure is that it is entirely defined in terms of the projections of the direct sum.
- Using both the direct sum $\oplus$ and the braided monoidal structure $\otimes$, the distributivity morphisms δ^l and δ^r are defined in (2.4.9). Lemma 2.4.10 shows that they are natural isomorphisms. It is here that we use the assumption that $- \otimes A$ and $A \otimes -$ are additive functors.
- The multiplicative zeros $\lambda^{\bullet}$ and $\rho^{\bullet}$ are in Definition 2.4.11, and are verified to be natural isomorphisms in Lemma 2.4.12. To show that they have invertible components, we have to use the assumption that $- \otimes A$ and $A \otimes -$ preserve zero morphisms.
- The second half of this section contains the proof of the 14 braided bimonoidal category axioms in C.

The Additive Structure. In an abelian category, any two objects have a direct sum as in Definition 2.3.4, which is both a product and a coproduct.

Definition 2.4.2. With C as in Convention 2.4.1, denote by

$$(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$$

the symmetric monoidal category with the monoidal product given by the direct sum $\oplus$ on objects. This is called the *additive structure* of C. ⋄

Explanation 2.4.3. A more detailed description of the additive structure of C is given as follows, with $A, B, C \in \mathsf{C}$ arbitrary objects.

The Sum: The object $A \oplus B$ is their direct sum in (2.3.5).

For morphisms $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ in C, the morphism $f \oplus g$ is defined by the following commutative diagram.

(2.4.4)
$$\begin{tikzcd}
 & A \arrow[r, "f"] & A' \\
A \oplus B \arrow[ur, bend left, "p_1"] \arrow[r, "f\oplus g"] \arrow[dr, bend right, "p_2"'] & A' \oplus B' \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\
 & B \arrow[r, "g"'] & B'
\end{tikzcd}$$

The Additive Associativity: $\alpha^{\oplus}_{A,B,C}$ is defined by the following commutative diagram.

(2.4.5)
$$\begin{tikzcd}
A \oplus B \arrow[rr, "p_1"] & & A \\
(A \oplus B) \oplus C \arrow[u, "p_1"] \arrow[rr, "\alpha^{\oplus}_{A,B,C}"] \arrow[dr, "p_2"] \arrow[dd, "p_1"'] & & A \oplus (B \oplus C) \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
 & C & B \oplus C \arrow[l, "p_2"'] \arrow[d, "p_1"] \\
A \oplus B \arrow[rr, "p_2"] & & B
\end{tikzcd}$$

The Additive Zeros: The left and the right unit isomorphisms

(2.4.6)
$$\mathbb{0} \oplus A \xrightarrow{\lambda^{\oplus}_A = p_2} A \xleftarrow{\rho^{\oplus}_A = p_1} A \oplus \mathbb{0}$$

are, respectively, the projections to the second factor and the first factor.

The Additive Symmetry: $\xi^{\oplus}_{A,B}$ is defined by the following commutative diagram.

(2.4.7)
$$\begin{tikzcd}
 & & B \\
A \oplus B \arrow[urr, bend left, "p_2"] \arrow[r, "\xi^{\oplus}_{A,B}"] \arrow[drr, bend right, "p_1"'] & B \oplus A \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\
 & & A
\end{tikzcd}$$

This finishes the description of the additive structure of C. ⋄

Distributivity. Next we define the distributivity morphisms in C.

Definition 2.4.8. With C as in Convention 2.4.1 and Definition 2.4.2, define the distributivity morphisms $\delta^l_{A,B,C}$ and $\delta^r_{A,B,C}$ by the commutative diagrams

$$\begin{tikzcd}
& & A \otimes B \\
A \otimes (B \oplus C) \arrow[urr, bend left, "1_A \otimes p_1^{B,C}"] \arrow[drr, bend right, "1_A \otimes p_2^{B,C}"'] \arrow[r, "\delta^l_{A,B,C}"] & (A \otimes B) \oplus (A \otimes C) \arrow[ur, "p_1^{A\otimes B, A\otimes C}"'] \arrow[dr, "p_2^{A\otimes B, A\otimes C}"] & \\
& & A \otimes C
\end{tikzcd}
\qquad
\begin{tikzcd}
& & A \otimes C \\
(A \oplus B) \otimes C \arrow[urr, bend left, "p_1^{A,B} \otimes 1_C"] \arrow[drr, bend right, "p_2^{A,B} \otimes 1_C"'] \arrow[r, "\delta^r_{A,B,C}"] & (A \otimes C) \oplus (B \otimes C) \arrow[ur, "p_1^{A\otimes C, B\otimes C}"'] \arrow[dr, "p_2^{A\otimes C, B\otimes C}"] & \\
& & B \otimes C
\end{tikzcd} \tag{2.4.9}$$

for objects $A, B, C \in \mathsf{C}$. ◇

Lemma 2.4.10. *In Definition 2.4.8, δ^l and δ^r are natural isomorphisms.*

Proof. We prove the assertion for δ^l; the proof for δ^r is essentially the same.

To see that $\delta^l_{A,B,C}$ is an isomorphism, recall that $A \otimes -$ is an additive functor. So it preserves the equations in (2.3.5) that characterize the direct sum $B \oplus C$. By Theorem 2.3.7(1), this implies that the diagram

$$A \otimes B \xleftarrow{1_A \otimes p_1^{B,C}} A \otimes (B \oplus C) \xrightarrow{1_A \otimes p_2^{B,C}} A \otimes C$$

is a product of $A \otimes B$ and $A \otimes C$. Therefore, $A \otimes (B \oplus C)$ is uniquely isomorphic to the direct sum $(A \otimes B) \oplus (A \otimes C)$, which is also the product. This unique isomorphism is $\delta^l_{A,B,C}$ by definition.

To show that δ^l is a natural transformation, consider morphisms

$$f : A \longrightarrow A', \quad g : B \longrightarrow B', \quad \text{and} \quad h : C \longrightarrow C'.$$

We must show that the rectangle (†) in the following diagram is commutative. The symbol $\otimes$ is omitted among objects to save space.

$$\begin{tikzcd}
& & & AB \arrow[dddd, bend left=60, "f \otimes g"] \\
& A(B \oplus C) \arrow[urr, bend left, "1_A \otimes p_1^{B,C}"] \arrow[r, "\delta^l_{A,B,C}"] \arrow[ur, phantom, "(2.4.9)"] \arrow[d, "f \otimes (g \oplus h)"] \arrow[ddl, bend right, "1_A \otimes p_1^{B,C}"'] & AB \oplus AC \arrow[ur, "p_1^{AB,AC}"] \arrow[d, "(f \otimes g) \oplus (f \otimes h)"] \arrow[r, phantom, "(2.4.4)"] \arrow[dl, phantom, "(\dagger)"] & \\
& A'(B' \oplus C') \arrow[r, "\delta^l_{A',B',C'}"] \arrow[dr, "1_{A'} \otimes p_1^{B',C'}"'] \arrow[dl, phantom, "(2.4.4)"] & A'B' \oplus A'C' \arrow[d, "p_1^{A'B',A'C'}"] \arrow[dl, phantom, "(2.4.9)"] & \\
AB \arrow[rr, bend right, "f \otimes g"'] & & A'B' &
\end{tikzcd}$$

This diagram shows that the two composites in (†) composed with the projection $p_1^{A'B',A'C'}$ are equal to each other. An analogous diagram shows that the two composites in (†) composed with the other projection $p_2^{A'B',A'C'}$ are equal to each other. Since the direct sum $A'B' \oplus A'C'$ is also the product of $A'B'$ and $A'C'$, it follows that (†) is commutative. This proves the naturality of δ^l. □

The Multiplicative Zeros. Next we define the multiplicative zeros in C.

Definition 2.4.11. With C as in Convention 2.4.1 and an object $A \in \mathsf{C}$, define the left and the right multiplicative zeros

$$\mathbb{0} \otimes A \xrightarrow{\lambda^{\bullet}_{A}} \mathbb{0} \xleftarrow{\rho^{\bullet}_{A}} A \otimes \mathbb{0}$$

as the unique morphisms to the zero object $\mathbb{0} \in \mathsf{C}$. ◇

Lemma 2.4.12. *In Definition 2.4.11, $\lambda^{\bullet}$ and $\rho^{\bullet}$ are natural isomorphisms.*

Proof. The inverse

$$\lambda^{-\bullet}_{A} : \mathbb{0} \longrightarrow \mathbb{0} \otimes A$$

is defined as the unique morphism from the initial object $\mathbb{0}$.

- The composite
$$\lambda^{\bullet}_{A}\lambda^{-\bullet}_{A} : \mathbb{0} \longrightarrow \mathbb{0}$$
is the identity morphism because $\mathbb{0}$ is a zero object.
- The zero morphism in $\mathsf{C}(\mathbb{0}, \mathbb{0})$, which has only one element, is also the identity morphism $1_{\mathbb{0}}$. Since $- \otimes A$ is an additive functor, the identity morphism $1_{\mathbb{0} \otimes A}$ is equal to the zero morphism in $\mathsf{C}(\mathbb{0} \otimes A, \mathbb{0} \otimes A)$. This zero morphism is equal to the composite $\lambda^{-\bullet}_{A}\lambda^{\bullet}_{A}$ by Lemma 2.3.12. So $\lambda^{-\bullet}_{A}\lambda^{\bullet}_{A}$ is the identity morphism.

Therefore, $\lambda^{\bullet}_{A}$ is indeed an isomorphism with inverse $\lambda^{-\bullet}_{A}$. The naturality of $\lambda^{\bullet}$ follows from the fact that $\mathbb{0}$ is a terminal object.

The proof for $\rho^{\bullet}$ is almost the same as that for $\lambda^{\bullet}$. □

The Braided Bimonoidal Category Axioms. We now assume the setting of Convention 2.4.1 and Definitions 2.4.2, 2.4.8, and 2.4.11. Together they contain the data part of a braided bimonoidal category C. To show that C is a tight braided bimonoidal category, we check the 14 axioms in Definition 2.1.29, starting with the trivial ones.

Lemma 2.4.13. *The axioms (2.1.13), (2.1.15), (2.1.17), (2.1.18), (2.1.19), and (2.1.33) hold in* C.

Proof. In each of these diagrams, the common codomain of the two composites is the terminal object $\mathbb{0}$. □

In the next several lemmas, we omit the symbol $\otimes$ among objects and morphisms.

Lemma 2.4.14. C *satisfies the axiom (2.1.24).*

Proof. The diagram (2.1.24) is the outer diagram below.

$$\begin{tikzcd}
A(B \oplus \mathbb{0}) \arrow[rr, "\delta^{l}_{A,B,\mathbb{0}}"] \arrow[d, "1_A\rho^{\oplus}_{B} = 1_A p_1^{B,\mathbb{0}}"'] \arrow[dr, phantom, "(2.4.9)"] & & AB \oplus A\mathbb{0} \arrow[d, "1_{AB} \oplus \rho^{\bullet}_{A}"] \arrow[dll, "p_1^{AB,A\mathbb{0}}"'] \\
AB & \arrow[ur, phantom, "(2.4.4)"] & AB \oplus \mathbb{0} \arrow[ll, "\rho^{\oplus}_{AB} = p_1^{AB,\mathbb{0}}"]
\end{tikzcd}$$

By (2.4.4), (2.4.6), and (2.4.9), the above diagram shows that (2.1.24) is commutative. □

Motivation 2.4.15. In each of the remaining proofs of the braided bimonoidal category axioms in C, the idea is the same as in the proof of the naturality of δ^l in Lemma 2.4.10. In other words, in each of the desired axioms, we consider the two composites composed with a projection of the common codomain to each of its direct sum factors, and show that they are equal to each other. The fact that the direct sum is also the product, as in Theorem 2.3.7 (1), then implies the desired axiom. $\diamond$

Lemma 2.4.16. C *satisfies the axiom (2.1.26).*

Proof. The diagram (2.1.26) is the middle triangle in the diagram below.

$$\begin{tikzcd}
& \mathbb{1}A \arrow[ddd, bend left=80, "\lambda^{\otimes}_A"] \arrow[d, phantom, "(2.4.9)"] & \\
\mathbb{1}(A\oplus B) \arrow[ur, "1_{\mathbb{1}}p_1^{A,B}"] \arrow[rr, "\delta^l_{\mathbb{1},A,B}"] \arrow[dr, "\lambda^{\otimes}_{A\oplus B}"'] & {} & \mathbb{1}A\oplus\mathbb{1}B \arrow[ul, "p_1^{\mathbb{1}A,\mathbb{1}B}"'] \arrow[dl, "\lambda^{\otimes}_A\oplus\lambda^{\otimes}_B"] \\
& A\oplus B \arrow[u, phantom, "(2.1.26)"] \arrow[d, "p_1^{A,B}"'] & {} \\
& A \arrow[ur, phantom, "(2.4.4)"] &
\end{tikzcd}$$

The outer diagram is commutative by the naturality of $\lambda^{\otimes}$. This diagram shows that the two composites in (2.1.26) composed with the projection $p_1^{A,B}$ are equal to each other. An analogous diagram shows that the two composites in (2.1.26) composed with the other projection $p_2^{A,B}$ are equal to each other. Since the direct sum $A \oplus B$ is also the product $A \times B$, it follows that (2.1.26) is commutative. $\square$

Lemma 2.4.17. C *satisfies the axioms (2.1.4) and (2.1.32).*

Proof. We will prove (2.1.4); the proof of its variant (2.1.32) is almost the same.

The diagram (2.1.4) is the rectangle in the following diagram.

$$\begin{tikzcd}
& & {} & AC \arrow[ddd, bend left=40, "\xi^{\otimes}_{A,C}"] \\
& (A\oplus B)C \arrow[urr, bend left, "p_1^{A,B}1_C"] \arrow[ur, phantom, "(2.4.9)"] \arrow[r, "\delta^r_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A\oplus B,C}"] \arrow[ddl, bend right=50, "p_1^{A,B}1_C"'] \arrow[dr, phantom, "(2.1.4)"] & AC\oplus BC \arrow[ur, "p_1^{AC,BC}"] \arrow[d, "\xi^{\otimes}_{A,C}\oplus\xi^{\otimes}_{B,C}"] \arrow[r, phantom, "(2.4.4)"] & {} \\
& C(A\oplus B) \arrow[r, "\delta^l_{C,A,B}"] \arrow[dr, "1_C p_1^{A,B}"'] \arrow[dr, phantom, "(2.4.9)", near start, shift left=3ex] \arrow[dl, phantom, "\mathrm{nat}"] & CA\oplus CB \arrow[d, "p_1^{CA,CB}"] & \\
AC \arrow[rr, bend right=15, "\xi^{\otimes}_{A,C}"'] & & CA &
\end{tikzcd}$$

This diagram shows that the two composites in (2.1.4) composed with the projection $p_1^{CA,CB}$ are equal to each other. An analogous diagram shows that the two composites in (2.1.4) composed with the other projection $p_2^{CA,CB}$ are equal to each other. Since the direct sum $CA \oplus CB$ is also the product $CA \times CB$, it follows that (2.1.4) is commutative. $\square$

Lemma 2.4.18. C *satisfies the axiom (2.1.5).*

Proof. The diagram (2.1.5) is the top rectangle in the following diagram.

$$
\begin{tikzcd}
A(B\oplus C) \arrow[r, "\delta^l_{A,B,C}"] \arrow[d, "1_A\xi^\oplus_{B,C}"] \arrow[dd, bend right=60, "1_A p_2^{B,C}"'] \arrow[dr, phantom, "(2.1.5)"] & AB\oplus AC \arrow[d, "\xi^\oplus_{AB,AC}"'] \arrow[dd, bend left=60, "p_2^{AB,AC}"] \\
A(C\oplus B) \arrow[r, "\delta^l_{A,C,B}"] \arrow[d, "1_A p_1^{C,B}"] \arrow[dr, phantom, "(2.4.9)"] \arrow[d, phantom, "(2.4.7)"', shift right=40] & AC\oplus AB \arrow[d, "p_1^{AC,AB}"'] \arrow[d, phantom, "(2.4.7)", shift left=40] \\
AC \arrow[r, equal] & AC
\end{tikzcd}
$$

The outer diagram is commutative by (2.4.9). The above diagram shows that the two composites in (2.1.5) composed with the projection $p_1^{AC,AB}$ are equal to each other. An analogous diagram shows that the two composites in (2.1.5) composed with the other projection $p_2^{AC,AB}$ are equal to each other. Since $AC \oplus AB$ is also the product $AC \times AB$, it follows that (2.1.5) is commutative. □

Lemma 2.4.19. C *satisfies the axiom (2.1.8).*

Proof. The diagram (2.1.8) is the middle rectangle in the following diagram.

$$
\begin{tikzcd}
A(B\oplus C) \arrow[r, equal] \arrow[ddd, bend right=60, "1_A p_1^{B,C}"'] & A(B\oplus C) \arrow[r, "\delta^l_{A,B,C}"] & AB\oplus AC \arrow[ddd, bend left=60, "p_1^{AB,AC}"] \\
A[(B\oplus C)\oplus D] \arrow[u, "1_A p_1^{B\oplus C,D}"] \arrow[ur, phantom, "(2.4.9)"] \arrow[r, "\delta^l_{A,B\oplus C,D}"] \arrow[d, "1_A\alpha^\oplus_{B,C,D}"] \arrow[d, phantom, "(2.4.5)"', shift right=40] \arrow[drr, phantom, "(2.1.8)"] & A(B\oplus C)\oplus AD \arrow[u, "p_1^{A(B\oplus C),AD}"] \arrow[ur, phantom, "(2.4.4)"] \arrow[r, "\delta^l_{A,B,C}\oplus 1_{AD}"] & (AB\oplus AC)\oplus AD \arrow[u, "p_1^{AB\oplus AC,AD}"'] \arrow[d, "\alpha^\oplus_{AB,AC,AD}"'] \arrow[d, phantom, "(2.4.5)", shift left=40] \\
A[B\oplus(C\oplus D)] \arrow[r, "\delta^l_{A,B,C\oplus D}"] \arrow[d, "1_A p_1^{B,C\oplus D}"] \arrow[dr, phantom, "(2.4.9)"] & AB\oplus A(C\oplus D) \arrow[r, "1_{AB}\oplus\delta^l_{A,C,D}"] \arrow[d, "p_1^{AB,A(C\oplus D)}"] \arrow[dr, phantom, "(2.4.4)"] & AB\oplus(AC\oplus AD) \arrow[d, "p_1^{AB,AC\oplus AD}"'] \\
AB \arrow[r, equal] & AB \arrow[r, equal] & AB
\end{tikzcd}
$$

The outer diagram is commutative by (2.4.9). The above diagram shows that the two composites in (2.1.8) composed with the projection $p_1^{AB,AC\oplus AD}$ are equal to each other. Two analogous diagrams show that the two composites in (2.1.8) composed with either iterated projections below are equal to each other.

$$
AB\oplus(AC\oplus AD) \xrightarrow{p_2^{AB,AC\oplus AD}} AC\oplus AD \xrightarrow{p_1^{AC,AD}} AC
$$

$$
AB\oplus(AC\oplus AD) \xrightarrow{p_2^{AB,AC\oplus AD}} AC\oplus AD \xrightarrow{p_2^{AC,AD}} AD
$$

Since $AB \oplus (AC \oplus AD)$ is also the product $AB \times (AC \times AD)$, it follows that (2.1.8) is commutative. □

Lemma 2.4.20. C *satisfies the axiom (2.1.9).*

Proof. The diagram (2.1.9) is the middle left rectangle in the following diagram.

$$
\begin{tikzcd}
(AB)(C\oplus D) \arrow[rr, "\delta^l_{AB,C,D}"] \arrow[d, "\alpha^{\otimes}_{A,B,C\oplus D}"] \arrow[rrr, bend left, "1_{AB}p_1^{C,D}"] \arrow[rr, phantom, bend left=20, "(2.4.9)"] \arrow[drr, phantom, "(2.1.9)"] & & (AB)C\oplus(AB)D \arrow[r, "p_1^{(AB)C,(AB)D}"] \arrow[d, "\alpha^{\otimes}_{A,B,C}\oplus\alpha^{\otimes}_{A,B,D}"] \arrow[dr, phantom, "(2.4.4)"] & (AB)C \arrow[ddl, "\alpha^{\otimes}_{A,B,C}"] \\
A[B(C\oplus D)] \arrow[r, "1_A\delta^l_{B,C,D}"] \arrow[d, "1_A(1_Bp_1^{C,D})"] \arrow[dr, phantom, "(2.4.9)"] & A(BC\oplus BD) \arrow[r, "\delta^l_{A,BC,BD}"] \arrow[d, "1_Ap_1^{BC,BD}"] \arrow[dr, phantom, "(2.4.9)"] & A(BC)\oplus A(BD) \arrow[d, "p_1^{A(BC),A(BD)}"] & \\
A(BC) \arrow[r, equal] & A(BC) \arrow[r, equal] & A(BC) &
\end{tikzcd}
$$

The outer diagram is commutative by the naturality of $\alpha^{\otimes}$. The above diagram shows that the two composites in (2.1.9) composed with the projection $p_1^{A(BC),A(BD)}$ are equal to each other. An analogous diagram shows that the two composites in (2.1.9) composed with the other projection $p_2^{A(BC),A(BD)}$ are equal to each other. Since $A(BC)\oplus A(BD)$ is also the product $A(BC)\times A(BD)$, it follows that (2.1.9) is commutative. $\square$

Lemma 2.4.21. C *satisfies the axiom (2.1.12).*

Proof. The diagram (2.1.12) is the outer diagram below.

$$
\begin{tikzcd}
(A\oplus B)(C\oplus D) \arrow[rrrr, "\delta^r"] \arrow[rrr, "p_21"] \arrow[ddrr, "p_2p_1"] \arrow[dddr, "1p_1"] \arrow[dd, "\delta^l"] & & & & A(C\oplus D)\oplus B(C\oplus D) \arrow[dl, "p_2"'] \arrow[dd, "\delta^l\oplus\delta^l"] \\
& & \text{(†)} & B(C\oplus D) \arrow[dl, "1p_1"'] \arrow[dd, "\delta^l"] & \\
(A\oplus B)C\oplus(A\oplus B)D \arrow[dr, "p_1"] \arrow[dd, "\delta^r\oplus\delta^r"] & \text{(†)} & BC \arrow[dd, equal] & & (AC\oplus AD)\oplus(BC\oplus BD) \arrow[dl, "p_2"'] \arrow[dd, "\alpha^{\oplus}"] \\
& (A\oplus B)C \arrow[ur, "p_21"] \arrow[dd, "\delta^r"] & & BC\oplus BD \arrow[dl, "p_1"'] & \\
(AC\oplus BC)\oplus(AD\oplus BD) \arrow[dr, "p_1"] \arrow[dd, "\alpha^{\oplus}"] & & BC \arrow[dd, equal] & & AC\oplus[AD\oplus(BC\oplus BD)] \arrow[dl, "p_2"'] \arrow[dd, "1\oplus\alpha^{-\oplus}"] \\
& AC\oplus BC \arrow[ur, "p_2"] & & AD\oplus(BC\oplus BD) \arrow[uu, "p_2"] \arrow[dd, "\alpha^{-\oplus}"] & \\
AC\oplus[BC\oplus(AD\oplus BD)] \arrow[dr, "p_2"] \arrow[ddrr, "1\oplus\alpha^{-\oplus}"'] & & BC & & AC\oplus[(AD\oplus BC)\oplus BD] \arrow[dl, "p_2"'] \arrow[ddll, "1\oplus(\xi^{\oplus}\oplus 1)"] \\
& BC\oplus(AD\oplus BD) \arrow[ur, "p_1"] & & (AD\oplus BC)\oplus BD \arrow[ul, "p_2p_1"'] & \\
& & AC\oplus[(BC\oplus AD)\oplus BD] \arrow[uu, "p_1p_1p_2"] & &
\end{tikzcd}
$$

The two triangles labeled by (†) are commutative by the functoriality of $\otimes$. Every other subdiagram is commutative by (2.4.4), (2.4.5), (2.4.7), and (2.4.9). The above diagram shows that the two composites in (2.1.12) composed with the following iterated projection are equal to each other.

$$AC \oplus [(BC \oplus AD) \oplus BD] \xrightarrow{p_2} (BC \oplus AD) \oplus BD \xrightarrow{p_1} BC \oplus AD \xrightarrow{p_1} BC$$

Three analogous diagrams show that the two composites in (2.1.12) composed with each of the following three iterated projections are equal to each other.

$$\begin{gathered}AC \oplus [(BC \oplus AD) \oplus BD] \xrightarrow{p_1} AC\\ AC \oplus [(BC \oplus AD) \oplus BD] \xrightarrow{p_2} (BC \oplus AD) \oplus BD \xrightarrow{p_2} BD\\ AC \oplus [(BC \oplus AD) \oplus BD] \xrightarrow{p_2} (BC \oplus AD) \oplus BD \xrightarrow{p_1} BC \oplus AD \xrightarrow{p_2} AD\end{gathered}$$

Since $AC \oplus [(BC \oplus AD) \oplus BD]$ is also the product $AC \times [(BC \times AD) \times BD]$, it follows that (2.1.12) is commutative. □

We are now ready for the main result of this section.

Theorem 2.4.22. *Under Convention 2.4.1, with*

- *the additive structure in Definition 2.4.2,*
- *the distributivity morphisms δ^l and δ^r in Definition 2.4.8, and*
- *the multiplicative zeros $\lambda^\bullet$ and $\rho^\bullet$ in Definition 2.4.11,*

C *is a tight braided bimonoidal category.*

Proof. This is a consequence of the following facts.

- Since the direct sum $\oplus$ is also the product, the additive structure is a symmetric monoidal structure on C.
- Lemmas 2.4.10 and 2.4.12 show that δ^l, δ^r, $\lambda^\bullet$, and $\rho^\bullet$ are natural isomorphisms.
- The 14 axioms of a braided bimonoidal category in Definition 2.1.29 are verified in Lemmas 2.4.13, 2.4.14, and 2.4.16 through 2.4.21.

Therefore, C is a tight braided bimonoidal category. □

2.5. Abelian Categories with a Monoidal Structure

Theorem 2.4.22 states that an abelian category with a compatible braided monoidal structure is a tight braided bimonoidal category. This section contains the symmetric and the nonbraided analogues of Theorem 2.4.22. Recall from Definition 2.1.1 the notion of a tight (symmetric) bimonoidal category.

Corollary 2.5.1. *Under the assumptions of Theorem 2.4.22, if the braided monoidal structure on C is symmetric monoidal, then C is a tight symmetric bimonoidal category.*

Proof. This is a consequence of Proposition 1.3.36, Corollary 2.2.3, and Theorem 2.4.22. □

Next is the nonbraided version of Theorem 2.4.22.

Theorem 2.5.2. *Suppose the tuple*

$$(\mathsf{C}, \mathbb{0}, \otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes)$$

consists of

- *an abelian category C with a zero object $\mathbb{0}$ and*
- *a monoidal structure $(\otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes)$*

such that, for each object $A \in \mathsf{C}$, the functors

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "A\otimes -"] \arrow[r, shift right, "-\otimes A"'] & \mathsf{C} \end{tikzcd}$$

are additive functors. Then with

- *the additive structure in Definition 2.4.2,*
- *the distributivity morphisms δ^l and δ^r in Definition 2.4.8, and*
- *the multiplicative zeros $\lambda^\bullet$ and $\rho^\bullet$ in Definition 2.4.11,*

C is a tight bimonoidal category.

Proof. The additive structure is a symmetric monoidal category because the direct sum is a product of the abelian category. Lemmas 2.4.10 and 2.4.12 show that δ^l, δ^r, $\lambda^\bullet$, and $\rho^\bullet$ are natural isomorphisms. We must check the 22 Laplaza axioms in Definition 2.1.1 *not* including (2.1.4) and (2.1.18), which involve $\xi^\otimes$.

- Lemma 2.4.18 proves the axiom (2.1.5). The axiom (2.1.6) is proved by almost the same argument.
- Lemma 2.4.19 proves the axiom (2.1.8). The axiom (2.1.7) is proved by almost the same argument.
- Lemma 2.4.20 proves the axiom (2.1.9). The axioms (2.1.10)–(2.1.11) are proved by almost the same argument.
- Lemma 2.4.21 proves the axiom (2.1.12).
- The axioms (2.1.13)–(2.1.17) and (2.1.19)–(2.1.21) hold because, in each case, the common codomain of the two composites is the terminal object $\mathbb{0}$.
- Lemma 2.4.14 proves the axiom (2.1.24). The axioms (2.1.22), (2.1.23), and (2.1.25) are proved by almost the same argument.
- Lemma 2.4.16 proves the axiom (2.1.26). The axiom (2.1.27) is proved by almost the same argument.

Therefore, C is a tight bimonoidal category. □

In summary, an abelian category with a compatible (symmetric) monoidal structure $\otimes$—in the sense that $A \otimes -$ and $- \otimes A$ are additive functors for each object A—has the structure of a tight (symmetric) bimonoidal category.

2.6. Notes

2.6.1 (Abelian Categories)**.** The definitions and results in Section 2.3 are from [**ML98**, VIII.2–3], where a direct sum is called a *biproduct*. For more discussion of abelian categories, the reader is referred to

- the papers [**Buc55, Gro57**] and
- the books [**BK00**, 2.2–2.3], [**Fre03**, Ch. 2], [**ML98**, VIII], [**Mit65**, I.20], and [**Sch72**, Ch. 12].

In [**Buc55**], an abelian category is called an exact category, which differs from modern usage of that term.

Some authors assume a different set of axioms for an abelian category. For example, in [**Fre03**, Ch. 2] and [**Sch72**, Ch. 12], a category C is *abelian* if the following conditions hold:

(i) It has a zero object.
(ii) Each pair of objects has a product and a coproduct.

(iii) Each morphism has a kernel and a cokernel.
(iv) Each monomorphism is a kernel of some morphism.
(v) Each epimorphism is a cokernel of some morphism.

This definition does not assume that C is an Ab-category. However, the results in [**Fre03**, 2.3–2.4] and [**Sch72**, 12.5] show that this definition and Definition 2.3.15 of an abelian category are equivalent. Furthermore, axiom (ii) can be replaced by either one of the following two weaker axioms:

- Each pair of objects has a product.
- Each pair of objects has a coproduct. $\diamond$

2.6.2 (Mitchell's Embedding Theorem)**.** A central result about abelian categories is the embedding theorem due to Mitchell [**Mit64**], which is also known as the Freyd-Mitchell Embedding Theorem. It states that each small abelian category admits a full and exact embedding into the category of modules over some ring. For a textbook account of this theorem, see [**Mit65**, VI.7] and [**Fre03**, 7.34]. $\diamond$

CHAPTER 3

Applications to Quantum Groups and Topological Quantum Computation

This chapter contains the following applications of braided bimonoidal categories to quantum group theory and topological quantum computation.

Quantum Groups. A central fact in quantum group theory states that, for each braided bialgebra A, which is also known as a quasitriangular bialgebra, the category $\mathsf{Mod}(A)$ of left A-modules is a braided monoidal category with the tensor product. In Theorem 3.2.19, we observe that, together with the direct sum of left A-modules, $\mathsf{Mod}(A)$ becomes a tight braided bimonoidal category. This assertion can be verified directly from the definition, but we will obtain it as a corollary of Theorem 2.4.22. The nonbraided and the symmetric analogues—for modules over, respectively, bialgebras and symmetric bialgebras—are also true.

Topological Quantum Computation. We observe that Fibonacci anyons and Ising anyons each form a tight braided bimonoidal category; see Theorems 3.4.13 and 3.6.14. These anyons are two of the most popular models in topological quantum computation. Providing a simple mathematical framework for anyon models in topological quantum computation is the main motivation for introducing tight braided bimonoidal categories in [**BG20a**]. Fibonacci anyons are generated under the direct sum $\oplus$ by the vacuum $\mathbb{1}$ and the non-abelian anyon τ, with the fusion rule

$$\tau \otimes \tau = \mathbb{1} \oplus \tau.$$

Ising anyons are generated under the direct sum by the vacuum $\mathbb{1}$, the non-abelian anyon σ, and the fermion ψ. The fusion rules are

$$\begin{aligned} &\sigma \otimes \sigma = \mathbb{1} \oplus \psi, \\ &\sigma \otimes \psi = \sigma = \psi \otimes \sigma, \quad \text{and} \\ &\psi \otimes \psi = \mathbb{1}. \end{aligned}$$

In each case, $\mathbb{1}$ is a strict two-sided unit for $\otimes$. These tight braided bimonoidal categories of Fibonacci anyons and Ising anyons are obtained as corollaries of Theorem 2.4.22.

These applications in quantum group theory and topological quantum computation provide specific examples of tight braided bimonoidal categories that are important outside of pure category theory. Connection of braided bimonoidal categories with homotopy theory and algebraic K-theory will be discussed in Part III.2.

Organization. In preparation for Theorem 3.2.19, in Section 3.1 we recall the definitions of a bialgebra, a braided bialgebra, and a symmetric bialgebra. In the literature, braided and symmetric bialgebras are also called quasitriangular and triangular bialgebras, respectively. Then we discuss some well-known examples from quantum group theory, including:

- cocommutative bialgebras, group bialgebras, the universal enveloping bialgebras, and Sweedler's 4-dimensional non-(co)commutative bialgebra, which are symmetric bialgebras; and
- the anyonic quantum groups, which are braided bialgebras.

In Section 3.2, we first prove the well-known fact in quantum group theory that, for a bialgebra A, the category $\mathsf{Mod}(A)$ of left A-modules is a monoidal category. It is, furthermore, symmetric or braided if A is a symmetric or braided bialgebra, respectively. See Propositions 3.2.6, 3.2.12, and 3.2.13. Then we equip $\mathsf{Mod}(A)$ with an additive structure $\oplus$, multiplicative zeros, and distributivity morphisms. The main observation is Theorem 3.2.19. It states that, for a bialgebra A, the category $\mathsf{Mod}(A)$ is a tight bimonoidal category, which is, furthermore, symmetric or braided if A is a symmetric or braided bialgebra, respectively.

In preparation for Theorem 3.4.13, in Section 3.3 we first define the abelian category $\mathcal{F}^{\mathsf{any}}$ of Fibonacci anyons. Then we equip it with a monoidal structure $\otimes$, using the fusion rule $\tau \otimes \tau = \mathbb{1} \oplus \tau$ and the requirement that $\mathbb{1}$ is a strict two-sided unit. Some sample calculation involving $\otimes$ is provided, both as examples and as preparation for the proofs of the (braided) monoidal category axioms. This section ends with Lemma 3.3.27, which proves in detail that $(\mathcal{F}^{\mathsf{any}}, \otimes)$ is a monoidal category.

Section 3.4 begins with the definition of the braiding β in $\mathcal{F}^{\mathsf{any}}$. Lemma 3.4.5 shows that $(\mathcal{F}^{\mathsf{any}}, \otimes, \beta)$ is a braided monoidal category. As a consequence of Theorem 2.4.22, we obtain Theorem 3.4.13, which states that $\mathcal{F}^{\mathsf{any}}$ is a tight braided bimonoidal category.

In Section 3.5, we define the abelian category $\mathcal{I}^{\mathsf{any}}$ of Ising anyons and equip it with a monoidal structure. After some sample calculation, we carefully check the pentagon axiom (1.3.3) in Lemma 3.5.27. In Section 3.6, we equip the monoidal category $\mathcal{I}^{\mathsf{any}}$ with a braiding and prove the hexagon axioms (1.3.17) in Lemma 3.6.7. Theorem 3.6.14 states that $\mathcal{I}^{\mathsf{any}}$ is a tight braided bimonoidal category.

Section 3.7 lists some references for the literature on quantum groups and topological quantum computation. Convention 2.1.34 is still in effect, so $\otimes$ is sometimes abbreviated to concatenation.

Reading Guide.

Quantum groups:

(1) Read Definitions 3.1.6 and 3.1.19 of a braided bialgebra.
(2) Read Definition 3.2.2 and the statement of Proposition 3.2.6 for the multiplicative structure on $\mathsf{Mod}(A)$.
(3) Read Definition 3.2.8 and the statement of Proposition 3.2.12 for the braided monoidal structure on $\mathsf{Mod}(A)_{\otimes}$.
(4) Read Definition 3.2.14 and the statement of Theorem 3.2.19 for the tight braided bimonoidal structure on $\mathsf{Mod}(A)$.

Fibonacci anyons:

(1) Read Definitions 3.3.1, 3.3.3, 3.3.7, and 3.3.23 and the statement of Lemma 3.3.27 for the monoidal category $\mathcal{F}^{\mathsf{any}}$.
(2) Read Definition 3.4.1 and the statements of Lemma 3.4.5 and Theorem 3.4.13 for the tight braided bimonoidal structure on $\mathcal{F}^{\mathsf{any}}$.

Ising anyons:

(1) For the monoidal structure, read Definitions 3.5.1, 3.5.5, and 3.5.21 and the statement of Lemma 3.5.27.
(2) Read Definition 3.6.1 and the statements of Lemma 3.6.7 and Theorem 3.6.14 for the tight braided bimonoidal structure on $\mathcal{I}^{\mathsf{any}}$.

Then go back and read the rest of this chapter.

3.1. Braided Bialgebras

In preparation for Section 3.2, where we apply Theorems 2.4.22 and 2.5.2 and Corollary 2.5.1 to the category of modules over a (symmetric/braided) bialgebra, in this section we review the definitions of (symmetric/braided) bialgebras and some examples. In the literature, braided and symmetric bialgebras are also known as, respectively, quasitriangular and triangular bialgebras. For their connection with quantum group theory, the reader is referred to the references in Note 3.7.1.

Convention 3.1.1. Throughout this section, $\Bbbk$ is a fixed ground field.

- $\mathsf{Vect}^{\Bbbk}$ is the category of $\Bbbk$-vector spaces and $\Bbbk$-linear maps. If there is no danger of confusion, we drop $\Bbbk$ and call them vector spaces and linear maps.
- For vector spaces A and B, their direct sum and tensor product as vector spaces are written as, respectively $A \oplus B$ and $A \otimes B$.
- Following common practice, for another vector space C, using the canonical isomorphism

$$\begin{aligned}(A \otimes B) \otimes C &\xrightarrow{\cong} A \otimes (B \otimes C)\\ (a \otimes b) \otimes c &\longmapsto a \otimes (b \otimes c),\end{aligned}$$

  we omit parentheses and write $A \otimes B \otimes C$ for either one of these two canonically isomorphic vector spaces. A similar convention applies to direct sums and morphisms.
- $A^{\otimes n} = A \otimes \cdots \otimes A$ with n copies of A, and $A^{\otimes 0} = \Bbbk$.
- $A^{\oplus n} = A \oplus \cdots \oplus A$ with n copies of A, and $A^{\oplus 0} = \mathbb{0}$ (the $\mathbb{0}$ $\Bbbk$-vector space).
- $\mathsf{Vect}^{\Bbbk}_{\otimes} = (\mathsf{Vect}^{\Bbbk}, \otimes, \Bbbk, \xi^{\otimes})$ is the symmetric monoidal category of vector spaces with the tensor product.
- $\mathsf{Vect}^{\Bbbk}_{\oplus} = (\mathsf{Vect}^{\Bbbk}, \oplus, \mathbb{0}, \xi^{\oplus})$ is the symmetric monoidal category of vector spaces with the direct sum. ◇

Algebras and Coalgebras.

Definition 3.1.2.

(1) A $\Bbbk$-*algebra*, or simply an *algebra*, is a monoid in $\mathsf{Vect}^{\Bbbk}_{\otimes}$.
(2) A $\Bbbk$-*coalgebra*, or simply a *coalgebra*, is a comonoid in $\mathsf{Vect}^{\Bbbk}_{\otimes}$.
(3) A $\Bbbk$-(co)algebra is *(co)commutative* if it is so as a (co)monoid. ◇

Explanation 3.1.3.

(1) Interpreting Definition I.1.2.8 in $\mathsf{Vect}^{\Bbbk}_{\otimes}$, a $\Bbbk$-algebra is a triple (A, μ, η) consisting of
   - a $\Bbbk$-vector space A;
   - a linear map $\mu : A \otimes A \longrightarrow A$, which is called the *multiplication*; and
   - a linear map $\eta : \Bbbk \longrightarrow A$, which is called the *unit*.

   These data are required to make the following associativity and unity diagrams commutative.

$$\begin{tikzcd} A^{\otimes 3} \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & A^{\otimes 2} \arrow[d, "\mu"] \\ A^{\otimes 2} \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \otimes A \arrow[r, "\eta\otimes 1"] \arrow[d, "\lambda"'] & A^{\otimes 2} \arrow[d, "\mu"] & A \otimes \Bbbk \arrow[l, "1\otimes\eta"'] \arrow[d, "\rho"] \\ A \arrow[r, equal] & A \arrow[r, equal] & A \end{tikzcd}$$

(2) Interpreting Definition I.1.2.9 in $\mathsf{Vect}^{\Bbbk}_{\otimes}$, a $\Bbbk$-coalgebra is a triple (C, Δ, ε) consisting of
   - a $\Bbbk$-vector space C;
   - a linear map $\Delta : C \longrightarrow C \otimes C$, which is called the *comultiplication*; and
   - a linear map $\varepsilon : C \longrightarrow \Bbbk$, which is called the *counit*.

   These data are required to make the following coassociativity and counity diagrams commutative.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C^{\otimes 2} \arrow[d, "\Delta\otimes 1"] \\ C^{\otimes 2} \arrow[r, "1\otimes\Delta"] & C^{\otimes 3} \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, equal] & C \arrow[r, equal] \arrow[d, "\Delta"] & C \\ \Bbbk \otimes C \arrow[u, "\lambda"] & C^{\otimes 2} \arrow[l, "\varepsilon\otimes 1"'] \arrow[r, "1\otimes\varepsilon"] & C \otimes \Bbbk \arrow[u, "\rho"'] \end{tikzcd} \tag{3.1.4}$$

   Moreover, interpreting (I.1.2.24), a coalgebra is *cocommutative* if the diagram

$$\begin{tikzcd} C \arrow[r, equal] \arrow[d, "\Delta"'] & C \arrow[d, "\Delta"] \\ C^{\otimes 2} \arrow[r, "\xi^{\otimes}"] & C^{\otimes 2} \end{tikzcd} \tag{3.1.5}$$

   is commutative. ◇

Bialgebras.

Definition 3.1.6. A $\Bbbk$-*bialgebra*, or simply a *bialgebra*, is a tuple

$$(A, \mu, \eta, \Delta, \varepsilon)$$

consisting of
- an algebra (A, μ, η) and
- a coalgebra (A, Δ, ε)

such that Δ and ε are morphisms of algebras. ◇

Explanation 3.1.7. Consider Definition 3.1.6.

(1) $\Bbbk$ is regarded as an algebra with
   - the canonical isomorphism $\Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk$, sending $1 \otimes 1$ to 1, as the multiplication, and
   - the identity morphism $1 : \Bbbk \longrightarrow \Bbbk$ as the unit.

(2) The tensor product $A^{\otimes 2}$ is regarded as an algebra with

- multiplication the composite

$$A^{\otimes 2} \otimes A^{\otimes 2} \xrightarrow{1 \otimes \xi^{\otimes} \otimes 1} A^{\otimes 2} \otimes A^{\otimes 2} \xrightarrow{\mu \otimes \mu} A^{\otimes 2} \tag{3.1.8}$$

  and
- unit the composite

$$\Bbbk \xrightarrow{\cong} \Bbbk^{\otimes 2} \xrightarrow{\eta^{\otimes 2}} A^{\otimes 2}. \tag{3.1.9}$$

(3) The comultiplication $\Delta : A \longrightarrow A^{\otimes 2}$ is a morphism of algebras if and only if the following two diagrams are commutative.

$$\begin{tikzcd} A^{\otimes 2} \arrow[r, "\Delta^{\otimes 2}"] \arrow[dd, "\mu"'] & A^{\otimes 4} \arrow[d, "1\otimes\xi^{\otimes}\otimes 1"] \\ & A^{\otimes 4} \arrow[d, "\mu\otimes\mu"] \\ A \arrow[r, "\Delta"] & A^{\otimes 2} \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "\eta"] \arrow[d, "\cong"'] & A \arrow[d, "\Delta"] \\ \Bbbk^{\otimes 2} \arrow[r, "\eta^{\otimes 2}"] & A^{\otimes 2} \end{tikzcd} \tag{3.1.10}$$

(4) The counit $\varepsilon : A \longrightarrow \Bbbk$ is a morphism of algebras if and only if the following two diagrams are commutative.

$$\begin{tikzcd} A^{\otimes 2} \arrow[r, "\varepsilon^{\otimes 2}"] \arrow[d, "\mu"'] & \Bbbk^{\otimes 2} \arrow[d, "\cong"] \\ A \arrow[r, "\varepsilon"] & \Bbbk \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \arrow[r, "\eta"] \arrow[dr, equal] & A \arrow[d, "\varepsilon"] \\ & \Bbbk \end{tikzcd} \tag{3.1.11}$$

Therefore, a bialgebra $(A, \mu, \eta, \Delta, \varepsilon)$ is simultaneously an algebra (A, μ, η) and a coalgebra (A, Δ, ε) such that the four diagrams in (3.1.10) and (3.1.11) are commutative. ◇

Notation 3.1.12.

- For elements x and y in an algebra (A, μ, η), we write

$$\mu(x, y) = xy \quad \text{and} \quad \eta(1_{\Bbbk}) = 1_A = 1,$$

  with $1_{\Bbbk}$ the multiplicative unit in $\Bbbk$.
- For an element $S = \sum_i s_i' \otimes s_i'' \in A^{\otimes 2}$ for an algebra A, we define the element

$$S^{\mathsf{op}} = \xi^{\otimes} S = \sum_i s_i'' \otimes s_i' \in A^{\otimes 2} \tag{3.1.13}$$

  and the following elements in $A^{\otimes 3}$.

$$\begin{aligned} S_{12} &= \sum_i s_i' \otimes s_i'' \otimes 1 \\ S_{13} &= \sum_i s_i' \otimes 1 \otimes s_i'' \\ S_{23} &= \sum_i 1 \otimes s_i' \otimes s_i'' \end{aligned} \tag{3.1.14}$$

- For an element x in a coalgebra (C, Δ, ε), we use Sweedler's notation for comultiplication:

$$\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}. \tag{3.1.15}$$

◇

Example 3.1.16.

- The four commutative diagrams in (3.1.10) and (3.1.11) defining a bialgebra mean the following four equalities for elements $x, y \in A$.

$$\begin{aligned} \sum_{(xy)} (xy)^{(1)} \otimes (xy)^{(2)} &= \sum_{(x),(y)} \left(x^{(1)}y^{(1)}\right) \otimes \left(x^{(2)}y^{(2)}\right) \in A^{\otimes 2} \\ \Delta(1_A) &= 1_A \otimes 1_A \in A^{\otimes 2} \\ \varepsilon(xy) &= \varepsilon(x)\varepsilon(y) \in \Bbbk \\ \varepsilon(1_A) &= 1_{\Bbbk} \in \Bbbk \end{aligned} \tag{3.1.17}$$

- The counity (3.1.4) in a coalgebra C means the following equalities for $x \in C$.

$$\sum_{(x)} \varepsilon(x^{(1)})x^{(2)} = x = \sum_{(x)} x^{(1)}\varepsilon(x^{(2)}) \tag{3.1.18}$$

◇

Braided and Symmetric Bialgebras.

Definition 3.1.19. Suppose $(A, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra.

- The *opposite comultiplication* is defined as

$$\Delta^{\mathsf{op}} = \xi^{\otimes}\Delta : A \longrightarrow A^{\otimes 2}.$$

- An *R-matrix* is an invertible element R in the algebra $A^{\otimes 2}$ such that the equality

$$\Delta^{\mathsf{op}}(a)R = R\Delta(a) \tag{3.1.20}$$

  holds for each element $a \in A$.
- A is *quasi-cocommutative* if it is equipped with an R-matrix.
- A *braided bialgebra* is a quasi-cocommutative bialgebra

$$(A, \mu, \eta, \Delta, \varepsilon, R)$$

  whose R-matrix R satisfies the following two equalities in $A^{\otimes 3}$ using the notation in (3.1.14).

$$\begin{aligned} (\Delta \otimes 1)(R) &= R_{13}R_{23} \\ (1 \otimes \Delta)(R) &= R_{13}R_{12} \end{aligned} \tag{3.1.21}$$

- A *symmetric bialgebra* is a braided bialgebra whose R-matrix satisfies the equality

$$R^{\mathsf{op}} = R^{-1} \in A^{\otimes 2} \tag{3.1.22}$$

  using the notation in (3.1.13). ◇

Explanation 3.1.23. For a quasi-cocommutative bialgebra A with R-matrix R, we write

$$R = \sum_i s_i \otimes t_i \in A^{\otimes 2}.$$

- The R-matrix axiom (3.1.20) is the equality

$$\sum_{(a),i} a^{(2)}s_i \otimes a^{(1)}t_i = \sum_{(a),i} s_i a^{(1)} \otimes t_i a^{(2)} \tag{3.1.24}$$

  in $A^{\otimes 2}$ for each element $a \in A$. The R-matrix measures how far A is from being cocommutative. In fact, A is cocommutative (that is, $\Delta^{\mathrm{op}} = \Delta$) if and only if $1 \otimes 1$ is an R-matrix.
- The braided bialgebra axioms (3.1.21) mean the following equalities in $A^{\otimes 3}$.

$$\begin{aligned}\sum_{i,(s_i)} s_i^{(1)} \otimes s_i^{(2)} \otimes t_i &= \sum_{i,j} s_i \otimes s_j \otimes t_i t_j\\ \sum_{i,(t_i)} s_i \otimes t_i^{(1)} \otimes t_i^{(2)} &= \sum_{i,j} s_i s_j \otimes t_j \otimes t_i\end{aligned} \tag{3.1.25}$$

  The braided bialgebra axioms (3.1.21) are algebraic analogues of, respectively, the right and the left hexagon diagrams in (1.3.17).
- The symmetric bialgebra axiom (3.1.22) means the equality

$$\sum_{i,j} t_j s_i \otimes s_j t_i = 1 \otimes 1 \in A^{\otimes 2}. \tag{3.1.26}$$

  This is an algebraic analogue of the symmetry axiom (1.3.33). $\diamond$

Examples. The rest of this section contains examples of braided and symmetric bialgebras.

Example 3.1.27 (Cocommutative Bialgebras)**.** Suppose $(A, \mu, \eta, \Delta, \varepsilon)$ is a cocommutative bialgebra, that is, a bialgebra satisfying $\Delta^{\mathrm{op}} = \Delta$. Then A equipped with the element

$$R = 1 \otimes 1 \in A^{\otimes 2}$$

is a symmetric bialgebra. Indeed, the R-matrix axiom (3.1.20) holds by cocommutativity. The braided bialgebra axioms (3.1.21) and the symmetric bialgebra axiom (3.1.22) hold because $\Delta(1) = 1 \otimes 1$ by (3.1.17) and $\mu(1,1) = 1$. $\diamond$

Example 3.1.28 (Group Bialgebras)**.** For a finite group G, the group bialgebra $\Bbbk G$ is the cocommutative bialgebra defined as follows.

- Its vector space basis is the set G. A typical element in $\Bbbk G$ is a linear combination $\sum_{g \in G} a_g g$ for some scalars $a_g \in \Bbbk$.
- Its algebra structure is induced by the group multiplication and unit in G, with multiplication extended linearly to all of $\Bbbk G$. Associativity and unity hold because they do in $\Bbbk$ and G. This is a noncommutative algebra in general, unless G is an abelian group.
- Its coalgebra structure is determined by

$$\begin{aligned}\Delta(g) &= g \otimes g \quad \text{and}\\ \varepsilon(g) &= 1 \quad \text{for} \quad g \in G,\end{aligned}$$

  extended linearly to all of $\Bbbk G$. Coassociativity, counity (3.1.4), and cocommutativity (3.1.5) hold because they hold for the basis elements.
- ε is an algebra morphism (3.1.11) by definition.

- Δ is an algebra morphism because, with $e \in G$ denoting the unit,

$$\begin{aligned}\Delta(e) &= e \otimes e \quad \text{and}\\ \Delta(gh) &= gh \otimes gh\\ &= (g \otimes g)(h \otimes h)\\ &= \Delta(g)\Delta(h)\end{aligned}$$

for $g, h \in G$.

Therefore, by Example 3.1.27, $(\Bbbk G, e \otimes e)$ is a symmetric bialgebra. $\diamond$

Example 3.1.29 (Universal Enveloping Bialgebras)**.** Suppose $(\mathfrak{g}, [-,-])$ is a Lie algebra over $\Bbbk$. Its *universal enveloping bialgebra* $U\mathfrak{g}$ is the cocommutative bialgebra defined as follows.

- As an algebra, it is the quotient

$$\frac{\bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}}{\big([x,y] - xy + yx : x, y \in \mathfrak{g}\big)}$$

of the tensor algebra $\bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$ by the relation

$$[x,y] = xy - yx \quad \text{for} \quad x, y \in \mathfrak{g},$$

with $\mathfrak{g}^{\otimes 0} = \Bbbk$. This is a noncommutative algebra in general.

- Its coalgebra structure is determined by

$$\begin{aligned}\Delta(x) &= x \otimes 1 + 1 \otimes x \quad \text{and}\\ \varepsilon(x) &= 0 \quad \text{for} \quad x \in \mathfrak{g},\end{aligned}$$

extended as algebra morphisms to all of $U\mathfrak{g}$ using (3.1.10) and (3.1.11). It follows from the definition that ε is well defined.

Extending Δ as an algebra morphism means that we define

$$\begin{aligned}\Delta(1) &= 1 \otimes 1 \in (U\mathfrak{g})^{\otimes 2} \quad \text{and}\\ \Delta(xy) &= \Delta(x)\Delta(y)\\ &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y)\end{aligned}$$

for $x, y \in \mathfrak{g}$. This definition respects the defining relation in $U\mathfrak{g}$ by the following computation.

$$\begin{aligned}&\Delta(xy - yx) = \Delta(xy) - \Delta(yx)\\ &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) - (y \otimes 1 + 1 \otimes y)(x \otimes 1 + 1 \otimes x)\\ &= xy \otimes 1 + y \otimes x + x \otimes y + 1 \otimes xy - yx \otimes 1 - x \otimes y - y \otimes x - 1 \otimes yx\\ &= (xy - yx) \otimes 1 + 1 \otimes (xy - yx)\\ &= [x,y] \otimes 1 + 1 \otimes [x,y]\\ &= \Delta([x,y])\end{aligned}$$

Counity (3.1.4) holds because

$$\varepsilon(1) = 1 \quad \text{and} \quad \varepsilon(x) = 0 \quad \text{for} \quad x \in \mathfrak{g}.$$

Coassociativity holds because it holds for the algebra generators $x \in \mathfrak{g}$:

$$\begin{aligned}(\Delta \otimes 1)\Delta(x) &= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x\\ &= (1 \otimes \Delta)\Delta(x).\end{aligned}$$

Cocommutativity (3.1.5) holds because it holds for the algebra generators $x \in \mathfrak{g}$.

Therefore, by Example 3.1.27, $\big(U\mathfrak{g}, 1 \otimes 1\big)$ is a symmetric bialgebra. ◇

Next is a noncommutative and non-cocommutative symmetric bialgebra.

Example 3.1.30 (Sweedler's 4-Dimensional Bialgebra)**.** Assuming 2 is invertible in $\Bbbk$, Sweedler's 4-dimensional noncommutative, non-cocommutative bialgebra $(H_4, \mu, \eta, \Delta, \varepsilon)$ is defined as follows.

- H_4 is a 4-dimensional vector space with a basis $\{1, x, y, xy\}$.
- 1 is the multiplicative unit. The rest of the multiplicative structure in H_4 is defined by the following multiplication table.

	x	y	xy
x	1	xy	y
y	$-xy$	0	0
xy	$-y$	0	0

  In other words, as an algebra H_4 is generated by $\{x, y\}$, subject to the three relations

$$x^2 = 1, \quad y^2 = 0, \quad \text{and} \quad yx = -xy. \tag{3.1.31}$$

  This is a noncommutative algebra because $yx = -xy$.
- Its coalgebra structure is determined as follows, extended multiplicatively to all of H_4 using (3.1.10) and (3.1.11).

$$\begin{aligned}\Delta(x) &= x \otimes x & \varepsilon(x) &= 1\\ \Delta(y) &= 1 \otimes y + y \otimes x & \varepsilon(y) &= 0\end{aligned}$$

  A short calculation shows that Δ and ε respect the three defining relations (3.1.31) in H_4, and are coassociative and counital (3.1.4). This coalgebra is non-cocommutative because $\Delta^{\mathsf{op}}(y) \neq \Delta(y)$.

For each scalar $c \in \Bbbk$, consider the element

$$\begin{aligned}R_c &= R_{c,1} + R_{c,2} \in H_4^{\otimes 2} \quad \text{with}\\ R_{c,1} &= \frac{1}{2}\big(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x\big) \quad \text{and}\\ R_{c,2} &= \frac{c}{2}\big(y \otimes y + y \otimes xy + xy \otimes xy - xy \otimes y\big).\end{aligned} \tag{3.1.32}$$

Then (H_4, R_c) is a symmetric bialgebra. In other words, R_c satisfies the axioms (3.1.20), (3.1.21), and (3.1.22), with $R_c^{-1} = R_c^{\mathsf{op}}$.

For the R-matrix axiom (3.1.20), it suffices to check it for the algebra generators x and y. For the generator x, it follows from the following computation.

$$\begin{aligned}&\Delta^{\mathsf{op}}(x)R_c = (x \otimes x)(R_{c,1} + R_{c,2})\\ &= \frac{1}{2}\big(x \otimes x + x \otimes 1 + 1 \otimes x - 1 \otimes 1\big) + \frac{c}{2}\big(xy \otimes xy + xy \otimes y + y \otimes y - y \otimes xy\big)\\ &= (R_{c,1} + R_{c,2})(x \otimes x) = R_c\Delta(x)\end{aligned}$$

Similarly, the equalities

$$\Delta^{\mathsf{op}}(y)R_{c,2} = 0 = R_{c,2}\Delta(y)$$

imply the following equalities, which prove (3.1.20) for the generator y.

$$\begin{aligned}\Delta^{\mathsf{op}}(y)R_c &= (y\otimes 1 + x\otimes y)R_{c,1}\\ &= \frac{1}{2}\big(y\otimes 1 + 1\otimes y + x\otimes y + y\otimes x + xy\otimes x - x\otimes xy + 1\otimes xy - xy\otimes 1\big)\\ &= R_{c,1}(1\otimes y + y\otimes x) = R_c\Delta(y)\end{aligned}$$

The two braided bialgebra axioms (3.1.21) are proved similarly using the equalities

$$\begin{aligned}\Delta(xy) &= \Delta(x)\Delta(y)\\ &= (x\otimes x)(1\otimes y + y\otimes x)\\ &= x\otimes xy + xy\otimes 1.\end{aligned}$$

In each axiom in (3.1.21), on the right-hand side (that is, either $R_{13}R_{23}$ or $R_{13}R_{12}$ for $R = R_c$), the following statements hold.

- The term with coefficient $c^2/4$ is 0 because $y^2 = 0$.
- For terms with coefficients $1/4$ or $c/4$, half of them cancel out due to opposite signs. The remaining terms add to the left-hand side, which is either one of the following two elements in $H_4^{\otimes 3}$.

$$\begin{aligned}(\Delta\otimes 1)R_c &= \frac{1}{2}\big(1\otimes 1\otimes 1 + 1\otimes 1\otimes x + x\otimes x\otimes 1 - x\otimes x\otimes x\big)\\ &\quad + \frac{c}{2}\big(1\otimes y\otimes y + y\otimes x\otimes y + 1\otimes y\otimes xy + y\otimes x\otimes xy + \Delta(xy)\otimes xy - \Delta(xy)\otimes y\big)\\ (1\otimes\Delta)R_c &= \frac{1}{2}\big(1\otimes 1\otimes 1 + 1\otimes x\otimes x + x\otimes 1\otimes 1 - x\otimes x\otimes x\big)\\ &\quad + \frac{c}{2}\big(y\otimes 1\otimes y + y\otimes y\otimes x + y\otimes\Delta(xy) + xy\otimes\Delta(xy) - xy\otimes 1\otimes y - xy\otimes y\otimes x\big)\end{aligned}$$

The proof for the symmetric bialgebra axiom (3.1.22) is similar and is outlined as follows.

$$\begin{aligned}R_c^{\mathsf{op}}R_c &= \big(R_{c,1}^{\mathsf{op}} + R_{c,2}^{\mathsf{op}}\big)\big(R_{c,1} + R_{c,2}\big)\\ &= \underbrace{R_{c,1}^{\mathsf{op}}R_{c,1}}_{1\otimes 1} + \underbrace{R_{c,2}^{\mathsf{op}}R_{c,1} + R_{c,1}^{\mathsf{op}}R_{c,2}}_{0} + \underbrace{R_{c,2}^{\mathsf{op}}R_{c,2}}_{0}\end{aligned}$$

In summary, (H_4, R_c) is a symmetric bialgebra for each scalar $c \in \Bbbk$. ◇

Example 3.1.33 (Anyonic Quantum Groups)**.** For the cyclic group $\mathbb{Z}_n$ of order $n \geq 2$ generated by x, the group bialgebra $\mathbb{C}\mathbb{Z}_n$ over the field $\mathbb{C}$ in Example 3.1.28 is both commutative (because $\mathbb{Z}_n$ is abelian) and cocommutative (because it is a group bialgebra). In addition to the symmetric bialgebra structure with R-matrix $1\otimes 1$ in Example 3.1.28, it is also a braided bialgebra with the nonstandard R-matrix

$$R = \frac{1}{n}\sum_{p,q=0}^{n-1} e^{-2\pi ipq/n}x^p \otimes x^q \in (\mathbb{C}\mathbb{Z}_n)^{\otimes 2}. \tag{3.1.34}$$

Its inverse is

$$R^{-1} = \frac{1}{n}\sum_{r,s=0}^{n-1} e^{-2\pi irs/n}x^{-r} \otimes x^s.$$

Indeed, the R-matrix axiom (3.1.20) holds by (co)commutativity.

To check the invertibility of R and the braided bialgebra axioms (3.1.21), we use the equality

$$\frac{1}{n}\sum_{q=0}^{n-1} e^{-2\pi i q(p-r)/n} = \delta_{p,r} = \begin{cases} 1 & \text{if } p = r \text{ and} \\ 0 & \text{if } p \neq r. \end{cases} \tag{3.1.35}$$

Then

$$RR^{-1} = 1 \otimes 1$$

by the following computation, with the last equality by (3.1.35).

$$\begin{aligned} RR^{-1} &= \frac{1}{n^2}\Big(\sum_{p,q=0}^{n-1} e^{-2\pi i pq/n} x^p \otimes x^q\Big)\Big(\sum_{r,s=0}^{n-1} e^{-2\pi i rs/n} x^{-r} \otimes x^s\Big) \\ &= \frac{1}{n^2}\sum_{p,q,r,s} e^{-2\pi i(pq+rs)/n} x^{p-r} \otimes x^{q+s} \\ &= \frac{1}{n^2}\sum_{p,q,r,s} e^{-2\pi i q(p-r)/n} \cdot e^{-2\pi i r(q+s)/n} x^{p-r} \otimes x^{q+s} \\ &= 1 \otimes 1. \end{aligned}$$

The first braided bialgebra axiom (3.1.21) follows from almost the same computation as follows, with the third equality by (3.1.35).

$$\begin{aligned} R_{13}R_{23} &= \Big(\frac{1}{n}\sum_{p,q=0}^{n-1} e^{-2\pi i pq/n} x^p \otimes 1 \otimes x^q\Big)\Big(\frac{1}{n}\sum_{r,s=0}^{n-1} e^{-2\pi i rs/n} 1 \otimes x^r \otimes x^s\Big) \\ &= \frac{1}{n^2}\sum_{p,q,r,s} e^{-2\pi i q(p-r)/n} \cdot e^{-2\pi i r(q+s)/n} x^p \otimes x^r \otimes x^{q+s} \\ &= \frac{1}{n}\sum_{p,t} e^{-2\pi i pt/n} x^p \otimes x^p \otimes x^t \\ &= (\Delta \otimes 1)R \end{aligned}$$

A similar computation proves the other braided bialgebra axiom (3.1.21). Therefore, $(\mathbb{C}\mathbb{Z}_n, R)$ is a braided bialgebra.

Note that $R^{\mathsf{op}} = R$. If $n = 2$, then $R^{\mathsf{op}} = R^{-1}$, and $(\mathbb{C}\mathbb{Z}_2, R)$ is a symmetric bialgebra. If $n > 2$, then $(\mathbb{C}\mathbb{Z}_n, R)$ is not symmetric. $\diamond$

3.2. Modules over Braided Bialgebras

In this section, we apply the main results in Sections 2.4 and 2.5 to the category $\mathsf{Mod}(A)$ of left modules over a (symmetric/braided) bialgebra A. The conclusion is that $\mathsf{Mod}(A)$ is a tight (symmetric/braided) bimonoidal category.

- For a bialgebra A, in Definition 3.2.2 we equip the category of left A-modules with a multiplicative structure $\mathsf{Mod}(A)_\otimes$. Proposition 3.2.6 shows that $\mathsf{Mod}(A)_\otimes$ is a monoidal category.
- For a braided bialgebra (A, R), in Definition 3.2.8 we equip the monoidal category $\mathsf{Mod}(A)_\otimes$ with a braiding. Proposition 3.2.12 shows that $\mathsf{Mod}(A)_\otimes$ is a braided monoidal category. If (A, R) is furthermore a symmetric bialgebra, then Proposition 3.2.13 shows that $\mathsf{Mod}(A)_\otimes$ is a symmetric monoidal category.

- In Definition 3.2.14, we define the additive structure, the multiplicative zeros, and the distributivity morphisms in $\mathsf{Mod}(A)$. Theorem 3.2.19 shows that for a bialgebra A, $\mathsf{Mod}(A)$ is a tight bimonoidal category. It is furthermore a tight symmetric/braided bimonoidal category if (A,R) is a symmetric/braided bialgebra.
- Example 3.2.20 applies Theorem 3.2.19 to the symmetric/braided bialgebras in Examples 3.1.27 through 3.1.30 and 3.1.33.

Convention 3.1.1 is still in effect, with $\Bbbk$ the fixed ground field. Also recall Sweedler's notation (3.1.15) for comultiplication.

Modules over Bialgebras.

Motivation 3.2.1. For an algebra A, the tensor product $A^{\otimes 2}$ is an algebra with the multiplication in (3.1.8) and the unit in (3.1.9). For left A-modules M and N, the tensor product $M \otimes N$ in $\mathsf{Vect}^{\Bbbk}_{\otimes}$ becomes a left $A^{\otimes 2}$-module with the action

$$(a \otimes b)(x \otimes y) = ax \otimes by$$

for $a, b \in A$, $x \in M$, and $y \in N$. If A is, furthermore, a bialgebra with comultiplication $\Delta : A \longrightarrow A^{\otimes 2}$, then one can use $\Delta(a)$ for $a \in A$ to define an A-action on $M \otimes N$. This makes the category of left A-modules into a monoidal category, as we now explain in detail. ◇

Definition 3.2.2. Suppose $(A, \mu, \eta, \Delta, \varepsilon)$ is a bialgebra as in Definition 3.1.6, and $\mathsf{Mod}(A)$ is the category of left A-modules and A-linear maps. Define the data of a monoidal category

$$\mathsf{Mod}(A)_{\otimes} = \big(\mathsf{Mod}(A), \otimes, \Bbbk, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}\big), \tag{3.2.3}$$

which is called the *multiplicative structure* on $\mathsf{Mod}(A)$, as follows.

The Monoidal Product: The functor

$$- \otimes - : \mathsf{Mod}(A) \times \mathsf{Mod}(A) \longrightarrow \mathsf{Mod}(A)$$

is defined as follows.

Objects: For two left A-modules M and N, $M \otimes N$ is their tensor product in $\mathsf{Vect}^{\Bbbk}_{\otimes}$, with the following left A-action for $a \in A$, $x \in M$, and $y \in N$.

$$\begin{array}{ccc} A \otimes M \otimes N & \xrightarrow{\mu_{M \otimes N}} & M \otimes N \\ a \otimes x \otimes y & \longmapsto & \Delta(a)(x \otimes y) = \sum_{(a)} a^{(1)} x \otimes a^{(2)} y \end{array} \tag{3.2.4}$$

Morphisms: For two left A-module morphisms f and g, $f \otimes g$ is defined as their tensor product in $\mathsf{Vect}^{\Bbbk}_{\otimes}$.

The Monoidal Unit: The left A-module structure on $\Bbbk$ is defined as follows for $a \in A$ and $c \in \Bbbk$.

$$\begin{array}{ccccc} A \otimes \Bbbk & \xrightarrow{\varepsilon \otimes 1} & \Bbbk \otimes \Bbbk & \xrightarrow{\cong} & \Bbbk \\ a \otimes c & \longmapsto & & \longrightarrow & \varepsilon(a)c \end{array} \tag{3.2.5}$$

Associativity and Unity: The associativity isomorphism $\alpha^{\otimes}$, the left unit isomorphism $\lambda^{\otimes}$, and the right unit isomorphism $\rho^{\otimes}$ are the ones in $\mathsf{Vect}^{\Bbbk}_{\otimes}$.

The finishes the definition of $\mathsf{Mod}(A)_{\otimes}$. ◇

Proposition 3.2.6. *For each bialgebra A, $\mathsf{Mod}(A)_{\otimes}$ is a monoidal category.*

Proof. To see that $\otimes$ in $\mathsf{Mod}(A)_\otimes$ is a well-defined functor, first observe that $M \otimes N$ with the left A-action in (3.2.4) is a left A-module because the comultiplication $\Delta : A \longrightarrow A^{\otimes 2}$ is an algebra morphism (3.1.10), corresponding to the first two conditions in (3.1.17). For two left A-module morphisms f and g, $f \otimes g$ is a morphism of left A-modules by the A-linearity of f and g. Moreover, $\otimes$ in $\mathsf{Mod}(A)_\otimes$ preserves identity morphisms and composition by the functoriality of $\otimes$ in $\mathsf{Vect}^{\Bbbk}_\otimes$.

The left A-module structure on $\Bbbk$ in (3.2.5) is well defined because the counit $\varepsilon : A \longrightarrow \Bbbk$ is an algebra morphism (3.1.11), corresponding to the last two conditions in (3.1.17).

The A-linearity of each component of $\alpha^\otimes$ in $\mathsf{Mod}(A)_\otimes$ follows from the coassociativity (3.1.4) of the comultiplication Δ. The A-linearity of each component of each of $\lambda^\otimes$ and $\rho^\otimes$ in $\mathsf{Mod}(A)_\otimes$ follows from the counity (3.1.4) of the coalgebra (A, Δ, ε), corresponding to the condition (3.1.18). The naturality and the invertibility of $\alpha^\otimes$, $\lambda^\otimes$, and $\rho^\otimes$ in $\mathsf{Mod}(A)_\otimes$ follow from their corresponding properties in $\mathsf{Vect}^{\Bbbk}_\otimes$.

Finally, the unity axiom (1.3.2) and the pentagon axiom (1.3.3) in $\mathsf{Mod}(A)_\otimes$ follow from the corresponding axioms in $\mathsf{Vect}^{\Bbbk}_\otimes$. $\square$

Modules over Braided Bialgebras.

Motivation 3.2.7. In a braided bialgebra (A, R) as in Definition 3.1.19, the R-matrix R controls the non-cocommutativity of the coalgebra structure of A. As we will explain shortly, the R-matrix R defines a braiding in the monoidal category $\mathsf{Mod}(A)_\otimes$, which makes it into a braided monoidal category. This explains the terminology of a braided bialgebra. In Proposition 3.2.12, we will show that the axioms for R correspond to properties in $\mathsf{Mod}(A)_\otimes$ as follows.

- The R-matrix axiom (3.1.20) corresponds to the A-linearity of the braiding.
- The invertibility of R corresponds to the invertibility of the brading.
- The braided bialgebra axioms (3.1.21) correspond to the hexagon axioms (1.3.17).

If, in addition, (A, R) is a symmetric bialgebra, then the symmetric bialgebra axiom (3.1.22) implies that the braiding satisfies the symmetry axiom (1.3.33). In this case, $\mathsf{Mod}(A)_\otimes$ is a symmetric monoidal category. $\diamond$

Definition 3.2.8. Suppose $(A, \mu, \eta, \Delta, \varepsilon, R)$ is a braided bialgebra with R-matrix R. Define the *braiding* in $\mathsf{Mod}(A)_\otimes$ by

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\xi^\otimes_{M,N}} & N \otimes M \\ x \otimes y & \longmapsto & \xi^\otimes\big(R(x \otimes y)\big) \end{array} \tag{3.2.9}$$

for left A-modules M and N, $x \in M$, and $y \in N$, with $\xi^\otimes$ the symmetry isomorphism in $\mathsf{Vect}^{\Bbbk}_\otimes$. $\diamond$

Explanation 3.2.10. The braiding (3.2.9) is equal to

$$\xi^\otimes_{M,N}(x \otimes y) = \sum_i t_i y \otimes s_i x = R^{\mathsf{op}} \xi^\otimes (x \otimes y) \tag{3.2.11}$$

with

$$R = \sum_i s_i \otimes t_i \in A^{\otimes 2} \quad \text{and} \quad R^{\mathsf{op}} = \sum_i t_i \otimes s_i$$

as in (3.1.13). ◇

Proposition 3.2.12. *For a braided bialgebra $(A, \mu, \eta, \Delta, \varepsilon, R)$, when equipped with the braiding in (3.2.9), the pair*

$$\big(\mathsf{Mod}(A)_{\otimes}, \xi^{\otimes}\big)$$

is a braided monoidal category.

Proof. By Proposition 3.2.6, $\mathsf{Mod}(A)_{\otimes}$ is a monoidal category.

For left A-modules M and N, the A-linearity of $\xi^{\otimes}_{M,N}$ in (3.2.9) follows from the following computation for $a \in A$, $x \in M$, and $y \in N$, using the R-matrix axiom (3.1.20) and the left A-action on $M \otimes N$ in (3.2.4).

$$\begin{aligned}
\xi^{\otimes}_{M,N}\big(a(x \otimes y)\big) &= \xi^{\otimes}\big(R\Delta(a)(x \otimes y)\big)\\
&= \xi^{\otimes}\big(\Delta^{\mathsf{op}}(a)R(x \otimes y)\big)\\
&= \Delta(a)\big(\xi^{\otimes}R(x \otimes y)\big)\\
&= a\big(\xi^{\otimes}_{M,N}(x \otimes y)\big)
\end{aligned}$$

The invertibility of $\xi^{\otimes}_{M,N}$ follows from the invertibility of $\xi^{\otimes}$ in $\mathsf{Vect}^{\Bbbk}_{\otimes}$ and of $R \in A^{\otimes 2}$. The inverse of $\xi^{\otimes}_{M,N}$ is $R^{-1}\xi^{\otimes}(-)$. Moreover, the braiding in (3.2.9) is natural with respect to left A-module morphisms by the A-linearity of the morphisms involved. Therefore, the braiding is a natural isomorphism.

For another left A-module P and an element $z \in P$, the right hexagon axiom (1.3.17) starting from $M \otimes N \otimes P$ follows from the following computation, with the third equality by the first braided bialgebra axiom in (3.1.25).

$$\begin{aligned}
&(\xi^{\otimes}_{M,P} \otimes 1)(1 \otimes \xi^{\otimes}_{N,P})(x \otimes y \otimes z)\\
&\qquad = (\xi^{\otimes}_{M,P} \otimes 1)\Big(\sum_j x \otimes t_j z \otimes s_j y\Big)\\
&\qquad = \sum_{i,j} t_i t_j z \otimes s_i x \otimes s_j y\\
&\qquad = \sum_{i,(s_i)} t_i z \otimes s_i^{(1)} x \otimes s_i^{(2)} y\\
&\qquad = \sum_i t_i z \otimes s_i(x \otimes y)\\
&\qquad = \xi^{\otimes}_{M \otimes N,P}(x \otimes y \otimes z)
\end{aligned}$$

Similarly, the left hexagon axiom (1.3.17) starting from $M \otimes N \otimes P$ follows from the following computation, with the third equality by the second braided bialgebra axiom in (3.1.25).

$$\begin{aligned}
&(1\otimes\xi^{\otimes}_{M,P})(\xi^{\otimes}_{M,N}\otimes 1)(x\otimes y\otimes z)\\
&\quad= (1\otimes\xi^{\otimes}_{M,P})\Big(\sum_j t_j y\otimes s_j x\otimes z\Big)\\
&\quad= \sum_{i,j} t_j y\otimes t_i z\otimes s_i s_j x\\
&\quad= \sum_{i,(t_i)} t_i^{(1)} y\otimes t_i^{(2)} z\otimes s_i x\\
&\quad= \sum_i t_i(y\otimes z)\otimes s_i x\\
&\quad= \xi^{\otimes}_{M,N\otimes P}(x\otimes y\otimes z)
\end{aligned}$$

Therefore, $(\mathsf{Mod}(A)_\otimes,\xi^\otimes)$ is a braided monoidal category. $\square$

Recall from Definition 3.1.19 that a symmetric bialgebra is a braided bialgebra (A,R) whose R-matrix R satisfies $R^{\mathsf{op}}=R^{-1}$.

Proposition 3.2.13. *For a symmetric bialgebra $(A,\mu,\eta,\Delta,\varepsilon,R)$, when equipped with the braiding in (3.2.9), the pair*

$$\big(\mathsf{Mod}(A)_\otimes,\xi^\otimes\big)$$

is a symmetric monoidal category.

Proof. By Propositions 1.3.36 and 3.2.12, it suffices to check that the braiding in (3.2.9) satisfies the symmetry axiom (1.3.33). For left A-modules M and N, the symmetry axiom is proved by the following computation, using the formulation of the braiding in (3.2.11).

$$\begin{aligned}
\xi^{\otimes}_{N,M}\xi^{\otimes}_{M,N} &= R^{\mathsf{op}}\xi^{\otimes}R^{\mathsf{op}}\xi^{\otimes}\\
&= R^{\mathsf{op}}R\xi^{\otimes}\xi^{\otimes}
\end{aligned}$$

This is equal to $1_{M\otimes N}$ by (i) the symmetry axiom (1.3.33) of $\xi^\otimes$ in $\mathsf{Vect}^{\Bbbk}_{\otimes}$ and (ii) the symmetric bialgebra axiom $R^{\mathsf{op}}=R^{-1}$ in (3.1.22). $\square$

Bimonoidal Structure.

Definition 3.2.14.

(1) For an algebra A, the *additive structure* on $\mathsf{Mod}(A)$ is the symmetric monoidal category

$$\mathsf{Mod}(A)_\oplus=\big(\mathsf{Mod}(A),\oplus,\mathbb{0},\alpha^\oplus,\lambda^\oplus,\rho^\oplus,\xi^\oplus\big) \tag{3.2.15}$$

with
- $\oplus$ the direct sum of left A-modules and A-linear maps;
- $\mathbb{0}$ the zero left A-module; and
- $(\alpha^\oplus,\lambda^\oplus,\rho^\oplus,\xi^\oplus)$ defined as in $\mathsf{Vect}^{\Bbbk}_{\oplus}$ via the underlying vector spaces.

(2) Suppose A is a bialgebra.
  (i) Define the left and the right multiplicative zeros

$$\mathbb{0}\otimes M \xrightarrow{\lambda^{\bullet}_M} \mathbb{0} \xleftarrow{\rho^{\bullet}_M} M\otimes\mathbb{0} \tag{3.2.16}$$

as the unique morphisms to the zero A-module for $M\in\mathsf{Mod}(A)$, with $\otimes$ the monoidal product in $\mathsf{Mod}(A)_\otimes$ in Definition 3.2.2.

(ii) Define the left distributivity morphism as

$$\begin{tikzcd}[row sep=tiny] M \otimes (N \oplus P) \arrow[r, "\delta^l_{M,N,P}"] & (M \otimes N) \oplus (M \otimes P) \\ x \otimes (y \oplus z) \arrow[r, mapsto] & (x \otimes y) \oplus (x \otimes z) \end{tikzcd} \tag{3.2.17}$$

and the right distributivity morphism as

$$\begin{tikzcd}[row sep=tiny] (M \oplus N) \otimes P \arrow[r, "\delta^r_{M,N,P}"] & (M \otimes P) \oplus (N \otimes P) \\ (x \oplus y) \otimes z \arrow[r, mapsto] & (x \otimes z) \oplus (y \otimes z) \end{tikzcd} \tag{3.2.18}$$

for $M, N, P \in \mathsf{Mod}(A)$ and $(x, y, z) \in M \times N \times P$. $\diamond$

We are now ready for the main observation of this section. Recall from Definitions 2.1.1 and 2.1.29 the notion of a tight (symmetric/braided) bimonoidal category.

Theorem 3.2.19. *Suppose A is a bialgebra.*

(1) Then the category $\mathsf{Mod}(A)$ equipped with
- *the additive structure $\mathsf{Mod}(A)_\oplus$ in (3.2.15),*
- *the multiplicative structure $\mathsf{Mod}(A)_\otimes$ in (3.2.3),*
- *the multiplicative zeros $\lambda^\bullet$ and $\rho^\bullet$ in (3.2.16), and*
- *the distributivity morphisms δ^l in (3.2.17) and δ^r in (3.2.18)*

is a tight bimonoidal category.

(2) If (A, R) is a braided bialgebra, then $\mathsf{Mod}(A)$ with the braiding (3.2.9) is a tight braided bimonoidal category.

(3) If (A, R) is a symmetric bialgebra, then $\mathsf{Mod}(A)$ with the braiding (3.2.9) is a tight symmetric bimonoidal category.

Proof. We observed in Proposition 3.2.6 that $\mathsf{Mod}(A)_\otimes$ is a monoidal category. Moreover, the category $\mathsf{Mod}(A)$ is an abelian category with the usual notions of direct sums $\oplus$, zero module $\mathbb{0}$, and (co)kernels.

- Its additive structure in the sense of Definition 2.4.2 is equal to $\mathsf{Mod}(A)_\oplus$ in (3.2.15).
- Its distributivity morphisms in the sense of Definition 2.4.8 are equal to those in (3.2.17) and (3.2.18).
- Its multiplicative zeros in the sense of Definition 2.4.11 are equal to those in (3.2.16).
- For each $M \in \mathsf{Mod}(A)$, the functors $M \otimes -$ and $- \otimes M$ on $\mathsf{Mod}(A)$ are additive functors, that is, preserve zero morphisms and addition of morphisms.

Therefore, assertion (1) follows from Theorem 2.5.2.

For assertion (2), since (A, R) is a braided bialgebra, $\mathsf{Mod}(A)_\otimes$ is a braided monoidal category by Proposition 3.2.12. Therefore, $\mathsf{Mod}(A)$ is a tight braided bimonoidal category by Theorem 2.4.22.

For assertion (3), since (A, R) is a symmetric bialgebra, $\mathsf{Mod}(A)_\otimes$ is a symmetric monoidal category by Proposition 3.2.13. Therefore, $\mathsf{Mod}(A)$ is a tight symmetric bimonoidal category by Corollary 2.5.1. $\square$

Example 3.2.20. Below are examples of tight symmetric/braided bimonoidal categories that arise as the categories of modules over some symmetric/braided bialgebras.

(1) A cocommutative bialgebra A with the R-matrix $1 \otimes 1$ is a symmetric bialgebra by Example 3.1.27. Therefore, $\mathsf{Mod}(A)$ is a tight symmetric bimonoidal category. For example, this applies to
   - the group bialgebra $\Bbbk G$ of a finite group G in Example 3.1.28 and
   - the universal enveloping bialgebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ in Example 3.1.29.

   For left A-modules M and N, the braiding
$$M \otimes N \xrightarrow[\cong]{\xi^{\otimes}_{M,N}} N \otimes M$$
   sends $x \otimes y$ to $y \otimes x$ for $(x,y) \in M \times N$.

(2) Assuming 2 is invertible in $\Bbbk$, for each scalar $c \in \Bbbk$, (H_4, R_c) is a symmetric bialgebra with
   - H_4 Sweedler's 4-dimensional bialgebra in Example 3.1.30 and
   - R_c the R-matrix in (3.1.32).

   Therefore, $\mathsf{Mod}(H_4)$ is a tight symmetric bimonoidal category. Its braiding sends $x \otimes y$ to $\xi^{\otimes}\big(R_c(x \otimes y)\big)$.

(3) The group bialgebra $\mathbb{C}\mathbb{Z}_n$ in Example 3.1.33 with the nonstandard R-matrix R in (3.1.34) is a braided bialgebra that is symmetric only when $n = 2$. Therefore, $\mathsf{Mod}(\mathbb{C}\mathbb{Z}_n)$ is a tight braided bimonoidal category if $n > 2$, and is a tight symmetric bimonoidal category if $n = 2$. The braiding sends $x \otimes y$ to $\xi^{\otimes}\big(R(x \otimes y)\big)$. $\diamond$

3.3. Fibonacci Anyons: The Monoidal Structure

This section and Section 3.4 contain an application of Theorem 2.4.22 to Fibonacci anyons. They form a model for universal quantum computation. The main result is Theorem 3.4.13. It states that there is a tight braided bimonoidal category $\mathcal{F}^{\mathsf{any}}$ in which the vacuum $\mathbb{1}$ and the non-abelian anyon τ generate all the objects under the direct sum. The fusion rule $\tau \otimes \tau = \mathbb{1} \oplus \tau$ uses both the additive structure $\oplus$ and the multiplicative structure $\otimes$.

In this section, we first define the abelian category $\mathcal{F}^{\mathsf{any}}$. Then we equip $\mathcal{F}^{\mathsf{any}}$ with a monoidal structure $\otimes$, along with some sample calculation. The pentagon axiom (1.3.3) in $\mathcal{F}^{\mathsf{any}}$ is proved in detail in Lemma 3.3.27. The braiding in $\mathcal{F}^{\mathsf{any}}$ is discussed in Section 3.4. The reader is referred to the references in Note 3.7.3 for further discussion of topological quantum computation.

The Abelian Category of Fibonacci Anyons. In the category of Fibonacci anyons, there are two simple objects. Each of the two simple objects is based on a copy of the skeleton of $\mathsf{Vect}^{\mathbb{C}}$ defined next. Recall from Definition 2.3.15 the notion of an abelian category.

Definition 3.3.1. Suppose $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}$ is the abelian category defined by the following data.

Objects: The objects in $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}$ are nonnegative integers $n \geq 0$.
Morphisms: $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}(m,n)$ is the abelian group of $\mathbb{C}$-linear maps $\mathbb{C}^m \longrightarrow \mathbb{C}^n$.
Identities: The identity morphism 1_n is the identity map of $\mathbb{C}^n$.

Composition: Composition is that of $\mathbb{C}$-linear maps.
The Zero Object: It is the object 0.
The Direct Sums: $m \oplus n = m + n$ for objects $m, n \geq 0$.

This finishes the definition of $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}$. ◇

Explanation 3.3.2. Consider the abelian category $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}$ in Definition 3.3.1.

- $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}$ is a skeleton of $\mathsf{Vect}^{\mathbb{C}}$, the category of finite dimensional complex vector spaces and linear maps. In fact, the assignments sending m to $\mathbb{C}^m$ and a linear map to itself is an equivalence of categories $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}} \longrightarrow \mathsf{Vect}^{\mathbb{C}}$. This justifies the notation for $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}$.
- In the context of Definition 2.3.4, the inclusions

$$m \xrightarrow{i_1} m \oplus n \xleftarrow{i_2} n$$

  are, respectively,
  - the inclusion $i_1 : \mathbb{C}^m \longrightarrow \mathbb{C}^{m+n}$ in the first m coordinates and
  - the inclusion $i_2 : \mathbb{C}^n \longrightarrow \mathbb{C}^{m+n}$ in the last n coordinates.

  The projections are defined similarly by the first m coordinates and the last n coordinates.
- Kernels and cokernels are the usual ones for $\mathbb{C}$-linear maps. Monomorphisms are injective linear maps. Epimorphisms are surjective linear maps. ◇

Definition 3.3.3. The abelian category of *Fibonacci anyons* is the Cartesian product

$$\mathcal{F}^{\mathsf{any}} = \mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}} \times \mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}.$$

Moreover, define the following objects in $\mathcal{F}^{\mathsf{any}}$:

- the *additive zero* $\mathbb{0} = (0;0)$,
- the *vacuum* $\mathbb{1} = (1;0)$, and
- the *non-abelian anyon* $\tau = (0;1)$.

The first and the second components of $\mathcal{F}^{\mathsf{any}}$ are called the $\mathbb{1}$*-component* and the τ*-component*, respectively. ◇

Explanation 3.3.4. The direct sum in $\mathcal{F}^{\mathsf{any}}$ is taken componentwise.

- For $m, n, p \geq 0$, we write

$$(m;n)^{\oplus p} = (mp;np)$$

  for the direct sum of p copies of $(m;n)$, with $(m;n)^{\oplus 0} = \mathbb{0}$.
- Each object $(m;n) \in \mathcal{F}^{\mathsf{any}}$ can be expressed as a sum

$$(m;n) = \mathbb{1}^{\oplus m} \oplus \tau^{\oplus n} = \tau^{\oplus n} \oplus \mathbb{1}^{\oplus m}. \tag{3.3.5}$$

  For example, $\mathbb{1} \oplus \tau = (1;1)$.
- A morphism in $\mathcal{F}^{\mathsf{any}}$ is a pair

$$(m;n) \xrightarrow{(f;g)} (p;q) \tag{3.3.6}$$

  of linear maps $f : \mathbb{C}^m \longrightarrow \mathbb{C}^p$ and $g : \mathbb{C}^n \longrightarrow \mathbb{C}^q$.
- The only morphism between $\mathbb{1}$ and τ in either direction is the zero morphism. ◇

The Monoidal Product.

Definition 3.3.7. Define the functor

$$\mathcal{F}^{\mathsf{any}} \times \mathcal{F}^{\mathsf{any}} \xrightarrow{\otimes} \mathcal{F}^{\mathsf{any}} \tag{3.3.8}$$

by the following three rules.

(i) For each object $(m;n) \in \mathcal{F}^{\mathsf{any}}$,

$$\mathbb{0} \otimes (m;n) = \mathbb{0} = (m;n) \otimes \mathbb{0}. \tag{3.3.9}$$

(ii) On the objects $\{\mathbb{1}, \tau\}$, it is defined by the following four equalities.

$$\begin{cases} \mathbb{1} \otimes \mathbb{1} = \mathbb{1} \\ \mathbb{1} \otimes \tau = \tau = \tau \otimes \mathbb{1} \\ \tau \otimes \tau = \mathbb{1} \oplus \tau \qquad \text{(the fusion rule)} \end{cases} \tag{3.3.10}$$

(iii) $\otimes$ distributes over $\oplus$ in $\mathcal{F}^{\mathsf{any}}$ strictly on both sides.

This finishes the definition of $\otimes$ in $\mathcal{F}^{\mathsf{any}}$. ◇

Explanation 3.3.11. Consider $\otimes$ in Definition 3.3.7.

- The fusion rule, $\tau \otimes \tau = \mathbb{1} \oplus \tau$, is physically interpreted as follows. When two copies of the anyon τ are fused together, the result is either the vacuum $\mathbb{1}$ or τ with a certain probability.
- The first three equalities in (3.3.10), the strict distributivity of $\otimes$ over $\oplus$ on both sides, and (3.3.5) imply that the vacuum $\mathbb{1} \in \mathcal{F}^{\mathsf{any}}$ is a strict two-sided unit for $\otimes$.
- By (3.3.5) and the strict distributivity of $\otimes$ over $\oplus$, the rules (3.3.9) and (3.3.10) uniquely determine $\otimes$. The general formula for $\otimes$ is given in (3.3.13) below. The iterated tensor products of copies of τ are computed in Lemma 3.3.17. ◇

Lemma 3.3.12. *For $m, m', n, n' \geq 0$, the equality*

$$(m;n) \otimes (m';n') = \mathbb{1}^{\oplus(mm'+nn')} \oplus \tau^{\oplus(mn'+nm'+nn')} \tag{3.3.13}$$

holds in $\mathcal{F}^{\mathsf{any}}$.

Proof. This follows from the following computation using (3.3.10) and the strict distributivity of $\otimes$ over $\oplus$.

$$\begin{aligned}
&(m;n) \otimes (m';n') \\
&= (\mathbb{1}^{\oplus m} \oplus \tau^{\oplus n}) \otimes (\mathbb{1}^{\oplus m'} \oplus \tau^{\oplus n'}) \\
&= (\mathbb{1}^{\oplus m} \otimes \mathbb{1}^{\oplus m'}) \oplus (\mathbb{1}^{\oplus m} \otimes \tau^{\oplus n'}) \oplus (\tau^{\oplus n} \otimes \mathbb{1}^{\oplus m'}) \oplus (\tau^{\oplus n} \otimes \tau^{\oplus n'}) \\
&= (\mathbb{1} \otimes \mathbb{1})^{\oplus mm'} \oplus (\mathbb{1} \otimes \tau)^{\oplus mn'} \oplus (\tau \otimes \mathbb{1})^{\oplus nm'} \oplus (\tau \otimes \tau)^{\oplus nn'} \\
&= \mathbb{1}^{\oplus mm'} \oplus \tau^{\oplus mn'} \oplus \tau^{\oplus nm'} \oplus (\mathbb{1} \oplus \tau)^{\oplus nn'} \\
&= \mathbb{1}^{\oplus(mm'+nn')} \oplus \tau^{\oplus(mn'+nm'+nn')}.
\end{aligned}$$

This proves (3.3.13). □

Recall from Definition 2.3.3 the concept of an additive functor.

Lemma 3.3.14. *The functors*

$$\mathcal{F}^{\mathsf{any}} \underset{-\otimes(m;n)}{\overset{(m;n)\otimes -}{\rightrightarrows}} \mathcal{F}^{\mathsf{any}}$$

are additive functors for each object $(m;n) \in \mathcal{F}^{\mathsf{any}}$.

Proof. This follows from (3.3.6) and (3.3.13). □

Iterated Monoidal Products. To define the associativity isomorphism and to check the pentagon axiom (1.3.3) and the hexagon axioms (1.3.17) later, we will need to use iterated monoidal products of copies of τ. These products have a simple description in terms of the Fibonacci sequence, which we recall next.

Motivation 3.3.15. The *Fibonacci sequence* $\{F_n\}_{n\geq 0}$ is defined recursively by:

$$F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \quad \text{for} \quad n \geq 1. \tag{3.3.16}$$

The first ten terms are $0, 1, 1, 2, 3, 5, 8, 13$, 21, and 34. The next observation says that iterated monoidal products of τ can be expressed in terms of the Fibonacci sequence, justifying the names for the category $\mathcal{F}^{\mathsf{any}}$ and the anyon τ. ◇

Lemma 3.3.17. *The Fibonacci anyon* $\tau \in \mathcal{F}^{\mathsf{any}}$ *satisfies*

$$\tau^{\otimes n} = \mathbb{1}^{\oplus F_{n-1}} \oplus \tau^{\oplus F_n} \quad \textit{for} \quad n \geq 1,$$

with

- $\{F_n\}$ *the Fibonacci sequence (3.3.16) and*
- $\tau^{\otimes n}$ *having either the right normalized bracketing (I.5.2.12) or the left normalized bracketing (I.5.2.13).*

Proof. For $n = 1$ and $n = 2$, the stated formula says, respectively,

$$\tau = \mathbb{1}^{\oplus 0} \oplus \tau \quad \text{and} \quad \tau^{\otimes 2} = \mathbb{1} \oplus \tau.$$

These formulas hold by definition. Inductively, for $n \geq 2$ and the right normalized bracketing, we compute as follows.

$$\begin{aligned}
\tau^{\otimes(n+1)} &= \tau \otimes \tau^{\otimes n} \\
&= \tau \otimes \left(\mathbb{1}^{\oplus F_{n-1}} \oplus \tau^{\oplus F_n}\right) \\
&= \left(\tau \otimes \mathbb{1}^{\oplus F_{n-1}}\right) \oplus \left(\tau \otimes \tau^{\oplus F_n}\right) \\
&= (\tau \otimes \mathbb{1})^{\oplus F_{n-1}} \oplus (\tau \otimes \tau)^{\oplus F_n} \\
&= \tau^{\oplus F_{n-1}} \oplus (\mathbb{1} \oplus \tau)^{\oplus F_n} \\
&= \mathbb{1}^{\oplus F_n} \oplus \tau^{\oplus (F_{n-1} + F_n)}
\end{aligned}$$

Since $F_{n+1} = F_n + F_{n-1}$, this finishes the induction step. The computation for the left normalized bracketing is almost identical. □

Example 3.3.18. The cases $n = 2, 3, 4$, and 5 of Lemma 3.3.17 are as follows.

$$\begin{aligned}
\tau^{\otimes 2} &= \mathbb{1} \oplus \tau & \qquad \tau^{\otimes 4} &= \mathbb{1}^{\oplus 2} \oplus \tau^{\oplus 3} \\
\tau^{\otimes 3} &= \mathbb{1} \oplus \tau^{\oplus 2} & \qquad \tau^{\otimes 5} &= \mathbb{1}^{\oplus 3} \oplus \tau^{\oplus 5}
\end{aligned}$$

◇

Example 3.3.19. There are the following two ways to compute $\tau^{\otimes 3}$, depending on the bracketing. Each of τ_1 and τ_2 below denotes a copy of τ.

$$\begin{aligned}(\tau\otimes\tau)\otimes\tau &= (\mathbb{1}\oplus\tau)\otimes\tau\\ &= (\mathbb{1}\otimes\tau)\oplus(\tau\otimes\tau)\\ &= \tau_1\oplus(\mathbb{1}\oplus\tau_2)\\ &= \mathbb{1}\oplus(\tau_1\oplus\tau_2)\\ \tau\otimes(\tau\otimes\tau) &= \tau\otimes(\mathbb{1}\oplus\tau)\\ &= (\tau\otimes\mathbb{1})\oplus(\tau\otimes\tau)\\ &= \tau_1\oplus(\mathbb{1}\oplus\tau_2)\\ &= \mathbb{1}\oplus(\tau_1\oplus\tau_2)\end{aligned}\tag{3.3.20}$$

These expressions for $\tau^{\otimes 3}$ will be used in (3.3.25) below to define the associativity isomorphism. ◇

Example 3.3.21. There are five ways to compute $\tau^{\otimes 4}$, depending on the bracketing. The following computation gives one such expression for $\tau^{\otimes 4}$. Each $\mathbb{1}_i$ denotes a copy of $\mathbb{1}$, and each τ_j denotes a copy of τ.

$$\begin{aligned}(\tau\otimes\tau)\otimes(\tau\otimes\tau) &= (\mathbb{1}\oplus\tau)\otimes(\mathbb{1}\oplus\tau)\\ &= (\mathbb{1}\otimes\mathbb{1})\oplus(\mathbb{1}\otimes\tau)\oplus(\tau\otimes\mathbb{1})\oplus(\tau\otimes\tau)\\ &= \mathbb{1}_1\oplus\tau_1\oplus\tau_2\oplus(\mathbb{1}_2\oplus\tau_3)\\ &= (\mathbb{1}_1\oplus\mathbb{1}_2)\oplus(\tau_1\oplus\tau_2\oplus\tau_3).\end{aligned}\tag{3.3.22}$$

For each of the four-fold tensor products

$$\big((\tau\otimes\tau)\otimes\tau\big)\otimes\tau \quad\text{and}\quad \tau\otimes\big((\tau\otimes\tau)\otimes\tau\big),$$

there is an analogous computation involving the following steps.

- Start with the expression for $(\tau\otimes\tau)\otimes\tau$ in (3.3.20).
- Apply $-\otimes\tau$ in the first case and $\tau\otimes -$ in the second case, respectively.
- Use the rules in (3.3.10) and the distributivity of $\otimes$ over $\oplus$.

For example,

$$\begin{aligned}\big((\tau\otimes\tau)\otimes\tau\big)\otimes\tau &= (\mathbb{1}\oplus\tau\oplus\tau)\otimes\tau\\ &= (\mathbb{1}\otimes\tau)\oplus(\tau\otimes\tau)\oplus(\tau\otimes\tau)\\ &= \tau_1\oplus(\mathbb{1}_1\oplus\tau_2)\oplus(\mathbb{1}_2\oplus\tau_3)\\ &= (\mathbb{1}_1\oplus\mathbb{1}_2)\oplus(\tau_1\oplus\tau_2\oplus\tau_3).\end{aligned}$$

In this computation, as in (3.3.20) and (3.3.22), we do not additively permute two copies of τ or two copies of $\mathbb{1}$.

For each of the four-fold tensor products

$$\big(\tau\otimes(\tau\otimes\tau)\big)\otimes\tau \quad\text{and}\quad \tau\otimes\big(\tau\otimes(\tau\otimes\tau)\big),$$

there is an analogous computation, starting with the expression for $\tau\otimes(\tau\otimes\tau)$ in (3.3.20). These five expressions for $\tau^{\otimes 4}$ will be used in Lemma 3.3.27 below to verify the pentagon axiom in $\mathcal{F}^{\mathsf{any}}$. ◇

The Associativity Isomorphism. The tensor product $\otimes$ in (3.3.8) has the vacuum $\mathbb{1}$ as a strict two-sided unit. Next we define the associativity isomorphism in the sense of Definition I.1.2.1.

Definition 3.3.23. For the functor $\otimes$ in (3.3.8), define a natural isomorphism

$$(- \otimes -) \otimes - \xrightarrow[\cong]{\alpha} - \otimes (- \otimes -) \tag{3.3.24}$$

by the following four rules.

(i) Using the expressions in (3.3.20) and $1 \in \mathbb{C}$ as the basis vector for each copy of $\mathbb{C}$, the component

$$\mathbb{C}\mathbb{1} \oplus \mathbb{C}\tau_1 \oplus \mathbb{C}\tau_2 \xrightarrow{\alpha_{\tau,\tau,\tau}} \mathbb{C}\mathbb{1} \oplus \mathbb{C}\tau_1 \oplus \mathbb{C}\tau_2$$

is defined by the matrix

$$\alpha_{\tau,\tau,\tau} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & q & q^{1/2} \\ 0 & q^{1/2} & -q \end{bmatrix} \quad \text{with} \quad q = \frac{\sqrt{5}-1}{2}. \tag{3.3.25}$$

(ii) Each component of α with at least one subscript $\mathbb{1}$ is the identity morphism.

(iii) Each component of α with at least one subscript $\mathbb{0}$ is the identity morphism of $\mathbb{0}$.

(iv) All other components of α are determined by the above rules and the naturality requirement of α.

This finishes the definition of α. $\diamond$

Explanation 3.3.26. Consider α in Definition 3.3.23.

- With the computation in (3.3.20), the component $\alpha_{\tau,\tau,\tau}$ is a morphism
$$\alpha_{\tau,\tau,\tau} : (1;2) \longrightarrow (1;2) \in \mathcal{F}^{\mathsf{any}}.$$
In other words, $\alpha_{\tau,\tau,\tau}$ consists of a pair of linear maps as follows.
  - The entry 1 in the matrix (3.3.25) means that the $\mathbb{1}$-component of $\alpha_{\tau,\tau,\tau}$ is the identity map $\mathbb{C} \longrightarrow \mathbb{C}$.
  - Using the standard basis of $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$, the τ-component of
  $$\alpha_{\tau,\tau,\tau} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
  is given by the lower-right 2×2 submatrix in (3.3.25).
- The number q in (3.3.25) is the positive root of the quadratic equation
$$x^2 + x - 1 = 0.$$
It is also the reciprocal of the golden ratio $(\sqrt{5}+1)/2$.
- The matrix $\alpha_{\tau,\tau,\tau}$ is orthogonal. In other words, its transpose, which is equal to itself, is its inverse. It follows that $\alpha_{\tau,\tau,\tau}$ is an isomorphism. $\diamond$

Lemma 3.3.27. *The quadruple*

$$\left(\mathcal{F}^{\mathsf{any}}, \otimes, \mathbb{1}, \alpha\right)$$

consisting of

- $\mathcal{F}^{\mathsf{any}}$ *and* $\mathbb{1} = (1;0)$ *in Definition 3.3.3,*
- $- \otimes - : \mathcal{F}^{\mathsf{any}} \times \mathcal{F}^{\mathsf{any}} \longrightarrow \mathcal{F}^{\mathsf{any}}$ *in (3.3.8), and*
- α *in (3.3.24)*

is a monoidal category with identities for the left and the right unit isomorphisms.

Proof. The naturality of α is part of its definition. Each of its components is an isomorphism because it is a direct sum of identity morphisms and copies of $\alpha_{\tau,\tau,\tau}$.

The unity axiom (1.3.2) holds because both the left and the right unit isomorphisms are identities. It remains to check the pentagon axiom (1.3.3). By Definition 3.3.23 (ii)–(iv) and the strict distributivity of $\otimes$ over $\oplus$, it suffices to check the pentagon axiom when all four objects involved are copies of τ. In the following computation, we use the expressions in Example 3.3.21 for the five ordered bases of the vector space

$$(\mathbb{C}\mathbb{1}_1 \oplus \mathbb{C}\mathbb{1}_2) \oplus (\mathbb{C}\tau_1 \oplus \mathbb{C}\tau_2 \oplus \mathbb{C}\tau_3)$$

corresponding to the five objects in the pentagon axiom involving only copies of τ.

With the order of the basis vectors in each basis taken into account, the upper path in this pentagon axiom (1.3.3) is given by the following matrix. Each unspecified entry is 0.

$$\begin{aligned}
&(\alpha_{\tau,\tau,\tau\otimes\tau})(\alpha_{\tau\otimes\tau,\tau,\tau}) = (\alpha_{\tau,\tau,\mathbb{1}\oplus\tau})(\alpha_{\mathbb{1}\oplus\tau,\tau,\tau})\\
&= \big(\alpha_{\tau,\tau,\mathbb{1}} \oplus \alpha_{\tau,\tau,\tau}\big)\big(\alpha_{\mathbb{1},\tau,\tau} \oplus \alpha_{\tau,\tau,\tau}\big)\\
&= \big(1_{\tau\otimes\tau} \oplus \alpha_{\tau,\tau,\tau}\big)\big(1_{\tau\otimes\tau} \oplus \alpha_{\tau,\tau,\tau}\big)\\
&= \begin{bmatrix} 1 &&&& \\ & 1 &&& \\ && q && q^{1/2} \\ &&& 1 & \\ && q^{1/2} && -q \end{bmatrix}
\begin{bmatrix} 1 &&&& \\ & 1 &&& \\ &&& 1 & \\ && q && q^{1/2} \\ && q^{1/2} && -q \end{bmatrix}
= \begin{bmatrix} 1 &&&& \\ & 1 &&& \\ && q & q & -q^{3/2} \\ && q && q^{1/2} \\ && -q^{3/2} & q^{1/2} & q^2 \end{bmatrix}
\end{aligned} \tag{3.3.28}$$

The lower path in the pentagon axiom is given by the following matrix.

$$\begin{aligned}
&(1_\tau \otimes \alpha_{\tau,\tau,\tau})(\alpha_{\tau,\tau\otimes\tau,\tau})(\alpha_{\tau,\tau,\tau} \otimes 1_\tau)\\
&= (1_\tau \otimes \alpha_{\tau,\tau,\tau})(\alpha_{\tau,\mathbb{1}\oplus\tau,\tau})(\alpha_{\tau,\tau,\tau} \otimes 1_\tau)\\
&= \big(1_\tau \otimes \alpha_{\tau,\tau,\tau}\big)\big(\alpha_{\tau,\mathbb{1},\tau} \oplus \alpha_{\tau,\tau,\tau}\big)\big(\alpha_{\tau,\tau,\tau} \otimes 1_\tau\big)\\
&= \big(1_\tau \otimes \alpha_{\tau,\tau,\tau}\big)\big(1_{\tau\otimes\tau} \oplus \alpha_{\tau,\tau,\tau}\big)\big(\alpha_{\tau,\tau,\tau} \otimes 1_\tau\big)\\
&= \begin{bmatrix} q & q^{1/2} &&& \\ q^{1/2} & -q &&& \\ && 1 && \\ &&& q & q^{1/2} \\ &&& q^{1/2} & -q \end{bmatrix}
\begin{bmatrix} 1 &&&& \\ & 1 &&& \\ && q && q^{1/2} \\ &&& 1 & \\ && q^{1/2} && -q \end{bmatrix}
\begin{bmatrix} q & q^{1/2} &&& \\ q^{1/2} & -q &&& \\ && 1 && \\ &&& q & q^{1/2} \\ &&& q^{1/2} & -q \end{bmatrix}\\
&= \begin{bmatrix} q^2+q &&&& \\ & q+q^2 &&& \\ && q & q & -q^{3/2} \\ && q && q^{3/2}+q^{5/2} \\ && -q^{3/2} & q^{3/2}+q^{5/2} & q-q^3 \end{bmatrix}
\end{aligned} \tag{3.3.29}$$

The last matrix in (3.3.28) is equal to the last matrix in (3.3.29) because

$$q^2 + q = 1.$$

This finishes the proof of the pentagon axiom. □

3.4. Fibonacci Anyons: The Braided Bimonoidal Structure

In this section, we first equip the monoidal category $\mathcal{F}^{\mathsf{any}}$ of Fibonacci anyons in Lemma 3.3.27 with a braiding. Then we observe that the abelian category $\mathcal{F}^{\mathsf{any}}$ with this braiding is a tight braided bimonoidal category. Therefore, $\mathcal{F}^{\mathsf{any}}$ provides a specific connection between braided bimonoidal categories and topological quantum computation.

Definition 3.4.1. For the functor $\otimes$ in (3.3.8), define a natural isomorphism

$$(m;n) \otimes (m';n') \xrightarrow[\cong]{\beta_{(m;n),(m';n')}} (m';n') \otimes (m;n) \tag{3.4.2}$$

for objects $(m;n), (m';n') \in \mathcal{F}^{\mathsf{any}}$ by the following four rules.

(i) Using $1 \in \mathbb{C}$ as the basis vector for each copy of $\mathbb{C}$, the component

$$\mathbb{1} \oplus \tau = \tau \otimes \tau \xrightarrow{\beta_{\tau,\tau}} \tau \otimes \tau = \mathbb{1} \oplus \tau$$

is defined by the matrix

$$\beta_{\tau,\tau} = \begin{bmatrix} z^2 & 0 \\ 0 & z \end{bmatrix} \quad \text{with} \quad z = e^{3\pi i/5}. \tag{3.4.3}$$

(ii) Each component of β with at least one subscript $\mathbb{1}$ is the identity morphism.

(iii) Each component of β with at least one subscript $\mathbb{0}$ is the identity morphism of $\mathbb{0}$.

(iv) All other components of β are determined by the above rules and the naturality requirement of β.

This finishes the definition of β. $\diamond$

Explanation 3.4.4. Consider β in Definition 3.4.1.

- z in (3.4.3) is a primitive tenth root of unity. It follows that $\beta_{\tau,\tau}$ has order 10.
- The $\mathbb{1}$-component of $\beta_{\tau,\tau}$ is the linear map $\mathbb{C} \longrightarrow \mathbb{C}$ that multiplies by z^2. The τ-component of $\beta_{\tau,\tau}$ is the linear map $\mathbb{C} \longrightarrow \mathbb{C}$ that multiplies by z. It follows that $\beta_{\tau,\tau}$ is an isomorphism. $\diamond$

Lemma 3.4.5. *The quintuple*

$$\big(\mathcal{F}^{\mathsf{any}}, \otimes, \mathbb{1}, \alpha, \beta\big)$$

consisting of

- *the monoidal category* $(\mathcal{F}^{\mathsf{any}}, \otimes, \mathbb{1}, \alpha)$ *in Lemma 3.3.27 and*
- β *in (3.4.2)*

is a braided monoidal category.

Proof. The naturality of β is part of its definition. Each of its components is an isomorphism because it is a direct sum of identity morphisms and copies of $\beta_{\tau,\tau}$. It remains to check the two hexagon axioms (1.3.17). As in the proof of Lemma 3.3.27, it suffices to check each hexagon axiom when all three objects involved are copies of τ.

The left hexagon in (1.3.17) involving only τ is the following diagram.

$$\begin{tikzcd}[column sep=small]
& (\tau \otimes \tau) \otimes \tau \arrow[rr, "\alpha_{\tau,\tau,\tau}"] & & \tau \otimes (\tau \otimes \tau) \arrow[dr, "1_\tau \otimes \beta_{\tau,\tau}"] & \\
(\tau \otimes \tau) \otimes \tau \arrow[ur, "\beta_{\tau,\tau} \otimes 1_\tau"] \arrow[dr, "\alpha_{\tau,\tau,\tau}"'] & & & & \tau \otimes (\tau \otimes \tau) \\
& \tau \otimes (\tau \otimes \tau) \arrow[rr, "\beta_{\tau,\tau\otimes\tau}"] & & (\tau \otimes \tau) \otimes \tau \arrow[ur, "\alpha_{\tau,\tau,\tau}"'] &
\end{tikzcd} \tag{3.4.6}$$

Using the two ordered bases of

$$\mathbb{C}\mathbb{1} \oplus (\mathbb{C}\tau_1 \oplus \mathbb{C}\tau_2)$$

in (3.3.20), with $1 \in \mathbb{C}$ as the basis vector for each copy of $\mathbb{C}$, the lower path in the left hexagon diagram (3.4.6) is given by the following matrix.

$$\begin{aligned}
&(\alpha_{\tau,\tau,\tau})(\beta_{\tau,\tau\otimes\tau})(\alpha_{\tau,\tau,\tau}) \\
&= (\alpha_{\tau,\tau,\tau})(\beta_{\tau,\mathbb{1}\oplus\tau})(\alpha_{\tau,\tau,\tau}) \\
&= (\alpha_{\tau,\tau,\tau})(\beta_{\tau,\mathbb{1}} \oplus \beta_{\tau,\tau})(\alpha_{\tau,\tau,\tau}) \\
&= (\alpha_{\tau,\tau,\tau})(1_\tau \oplus \beta_{\tau,\tau})(\alpha_{\tau,\tau,\tau}) \\
&= \begin{bmatrix} 1 & & \\ & q & q^{1/2} \\ & q^{1/2} & -q \end{bmatrix} \begin{bmatrix} z^2 & & \\ & 1 & \\ & & z \end{bmatrix} \begin{bmatrix} 1 & & \\ & q & q^{1/2} \\ & q^{1/2} & -q \end{bmatrix} = \begin{bmatrix} z^2 & & \\ & q^2+qz & q^{3/2}(1-z) \\ & q^{3/2}(1-z) & q+q^2z \end{bmatrix}
\end{aligned} \tag{3.4.7}$$

The upper path in the diagram (3.4.6) is given by the following matrix.

$$\begin{aligned}
&(1_\tau \otimes \beta_{\tau,\tau})(\alpha_{\tau,\tau,\tau})(\beta_{\tau,\tau}\otimes 1_\tau) \\
&= \begin{bmatrix} z & & \\ & z^2 & \\ & & z \end{bmatrix} \begin{bmatrix} 1 & & \\ & q & q^{1/2} \\ & q^{1/2} & -q \end{bmatrix} \begin{bmatrix} z & & \\ & z^2 & \\ & & z \end{bmatrix} \\
&= \begin{bmatrix} z^2 & & \\ & qz^4 & q^{1/2}z^3 \\ & q^{1/2}z^3 & -qz^2 \end{bmatrix}
\end{aligned} \tag{3.4.8}$$

The last matrix in (3.4.7) is equal to the last matrix in (3.4.8) if and only if the following three equalities hold.

$$\begin{aligned}
q^2 + qz &= qz^4 \\
q^{3/2}(1-z) &= q^{1/2}z^3 \\
q + q^2z &= -qz^2
\end{aligned} \tag{3.4.9}$$

To prove these equalities, first we note that

$$z^5 = e^{3\pi i} = -1 \quad \text{and} \quad z^8 = -z^3. \tag{3.4.10}$$

Moreover, since

$$z^4 = e^{12\pi i/5} = e^{2\pi i/5},$$

it follows that

$$z^4 - z = e^{2\pi i/5} - e^{3\pi i/5}$$

is a positive real number.

The first desired equality in (3.4.9) is equivalent to

$$q = z^4 - z. \tag{3.4.11}$$

Since q and $z^4 - z$ are positive real numbers, (3.4.11) holds if $z^4 - z$ is also a root of $x^2 + x - 1 = 0$. The following computation shows that this is indeed the case, using (3.4.10) in the second and the last steps.

$$\begin{aligned}
(z^4-z)^2 + (z^4-z) - 1 &= z^8 - 2z^5 + z^2 + z^4 - z - 1 \\
&= 1 - z + z^2 - z^3 + z^4 \\
&= \frac{1+z^5}{1+z} = 0
\end{aligned} \tag{3.4.12}$$

This proves the first equality in (3.4.9).

Using $q = z^4 - z$, the second desired equality in (3.4.9) is equivalent to

$$z^4 - z = \frac{z^3}{1-z},$$

which is equivalent to

$$1 - z + z^2 - z^3 + z^4 = 0.$$

The last equality is verified in (3.4.12) above.

The last desired equality in (3.4.9) is equivalent to the equality

$$q = -\frac{1}{z} - z,$$

which by (3.4.10) is equivalent to $q = z^4 - z$. The last equality is verified in (3.4.11) and (3.4.12) above. This finishes the proof of the left hexagon axiom (3.4.6) involving only copies of τ.

The right hexagon diagram (1.3.17) involving only τ is obtained from the left hexagon diagram (3.4.6) by replacing

- $\beta_{\tau,\tau\otimes\tau}$ by $\beta^{-1}_{\tau\otimes\tau,\tau}$ and
- each of the other two $\beta_{\tau,\tau}$ by $\beta^{-1}_{\tau,\tau}$.

The matrix for $\beta^{-1}_{\tau,\tau}$ is

$$\beta^{-1}_{\tau,\tau} = \begin{bmatrix} z^{-2} & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

Therefore, the lower and the upper paths in the right hexagon diagram involving only τ are obtained from, respectively, (3.4.7) and (3.4.8) by replacing each z by z^{-1}. Equating them leads to the same three equations in (3.4.9) with each z replaced by z^{-1}. These equalities are verified by the same steps that proved (3.4.9) above by replacing z by z^{-1}. This is valid by the equalities

$$z^{-5} = -1 \quad \text{and} \quad z^{-8} = -z^{-3},$$

which are analogues of (3.4.10), and the fact that

$$z^{-4} - z^{-1} = e^{-2\pi i/5} - e^{-3\pi i/5}$$

is a positive real number. $\square$

We are now ready for the main result of this section.

Theorem 3.4.13. *The abelian category $\mathcal{F}^{\mathsf{any}}$ in Definition 3.3.3, when equipped with the braided monoidal structure in Lemma 3.4.5, is a tight braided bimonoidal category.*

Proof. We combine the following facts.

- $\mathcal{F}^{\mathsf{any}}$ is an abelian category by definition.
- It is a braided monoidal category by Lemma 3.4.5.
- The functors $(m;n)\otimes -$ and $-\otimes(m;n)$ are additive functors for each object $(m;n) \in \mathcal{F}^{\mathsf{any}}$ by Lemma 3.3.14.

Therefore, the assertion follows from Theorem 2.4.22. $\square$

3.5. Ising Anyons: The Monoidal Structure

This section and Section 3.6 contain an application of Theorem 2.4.22 to Ising anyons. These anyons form another anyon model for topological quantum computation that has both a non-abelian anyon σ and a fermion ψ. The main result is Theorem 3.6.14. It states that there is a tight braided bimonoidal category $\mathcal{I}^{\mathsf{any}}$ in which the vacuum $\mathbb{1}$, the non-abelian anyon σ, and the fermion ψ generate all the objects under the direct sum.

In this section, we first define the abelian category $\mathcal{I}^{\mathsf{any}}$. Then we equip $\mathcal{I}^{\mathsf{any}}$ with a monoidal structure $\otimes$, along with some sample calculation. The pentagon axiom (1.3.3) in $\mathcal{I}^{\mathsf{any}}$ is proved in detail in Lemma 3.5.27. The braiding in $\mathcal{I}^{\mathsf{any}}$ is discussed in Section 3.6. The reader is referred to the references in Notes 3.7.3 and 3.7.4 for further discussion of topological quantum computation and Ising anyons.

The Abelian Category of Ising Anyons. Recall from Definition 3.3.1 the abelian category $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}$. Its objects are nonnegative integers $n \geq 0$. A morphism $m \longrightarrow n$ is a $\mathbb{C}$-linear map $\mathbb{C}^m \longrightarrow \mathbb{C}^n$, with direct sum $m \oplus n = m + n$.

Definition 3.5.1. The abelian category of *Ising anyons* is the Cartesian product

$$\mathcal{I}^{\mathsf{any}} = \mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}} \times \mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}} \times \mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}.$$

Moreover, define the following objects in $\mathcal{I}^{\mathsf{any}}$:

- the *additive zero* $\mathbb{0} = (0;0;0)$,
- the *vacuum* $\mathbb{1} = (1;0;0)$,
- the *non-abelian anyon* $\sigma = (0;1;0)$, and
- the *fermion* $\psi = (0;0;1)$.

The first, the second, and the third components of $\mathcal{I}^{\mathsf{any}}$ are called, respectively, the $\mathbb{1}$*-component*, the σ*-component*, and the ψ*-component*. ◇

Explanation 3.5.2. The direct sum in the abelian category $\mathcal{I}^{\mathsf{any}}$ is taken componentwise.

- For $k, m, n, p \geq 0$, we write

$$(m;n;p)^{\oplus k} = (mk;nk;pk)$$

  for the direct sum of k copies of $(m;n;p)$, with $(m;n;p)^{\oplus 0} = \mathbb{0}$.
- Each object $(m;n;p) \in \mathcal{I}^{\mathsf{any}}$ can be expressed as a sum

$$(m;n;p) = \mathbb{1}^{\oplus m} \oplus \sigma^{\oplus n} \oplus \psi^{\oplus p}. \tag{3.5.3}$$

  For example, $\mathbb{1} \oplus \psi = (1;0;1)$.
- A morphism in $\mathcal{I}^{\mathsf{any}}$ is a triple

$$(m;n;p) \xrightarrow{(f;g;h)} (r;s;t) \tag{3.5.4}$$

  of linear maps $f : \mathbb{C}^m \longrightarrow \mathbb{C}^r$, $g : \mathbb{C}^n \longrightarrow \mathbb{C}^s$, and $h : \mathbb{C}^p \longrightarrow \mathbb{C}^t$. ◇

The Monoidal Product.

Definition 3.5.5. Define the functor

$$\mathcal{I}^{\mathsf{any}} \times \mathcal{I}^{\mathsf{any}} \xrightarrow{\otimes} \mathcal{I}^{\mathsf{any}} \tag{3.5.6}$$

by the following three rules.

(i) For each object $(m;n;p) \in \mathcal{I}^{\text{any}}$,

$$\mathbb{0} \otimes (m;n;p) = \mathbb{0} = (m;n;p) \otimes \mathbb{0}. \tag{3.5.7}$$

(ii) On the objects $\{\mathbb{1}, \sigma, \psi\}$, it is defined by the following equalities.

$$\text{Two-sided unit:} \qquad \mathbb{1} \otimes x = x = x \otimes \mathbb{1} \quad \text{for} \quad x \in \{\mathbb{1}, \sigma, \psi\}.$$

$$\text{The fusion rules:} \qquad \begin{cases} \sigma \otimes \sigma = \mathbb{1} \oplus \psi \\ \sigma \otimes \psi = \sigma = \psi \otimes \sigma \\ \psi \otimes \psi = \mathbb{1}. \end{cases} \tag{3.5.8}$$

(iii) $\otimes$ distributes over $\oplus$ in $\mathcal{I}^{\text{any}}$ strictly on both sides.

This finishes the definition of $\otimes$ in $\mathcal{I}^{\text{any}}$. ◇

Explanation 3.5.9. Consider $\otimes$ in Definition 3.5.5.

- The first line in (3.5.8), the strict distributivity of $\otimes$ over $\oplus$ on both sides, and (3.5.3) imply that the vacuum $\mathbb{1} \in \mathcal{I}^{\text{any}}$ is a strict two-sided unit for $\otimes$.
- By (3.5.3) and the strict distributivity of $\otimes$ over $\oplus$, the rules (3.5.7) and (3.5.8) uniquely determine $\otimes$. The general formula for $\otimes$ is given in (3.5.14) below. The iterated tensor products of copies of the non-abelian anyon σ are computed in Lemma 3.5.16. ◇

Example 3.5.10. The expressions below are obtained by applying the fusion rules (3.5.8) and the strict distributivity of $\otimes$ over $\oplus$. They will be used in (3.5.22) to define the associativity isomorphism α in $\mathcal{I}^{\text{any}}$. The first two sets of expressions correspond to nonidentity components of α.

$$\begin{gathered} \begin{cases} (\sigma \otimes \psi) \otimes \sigma = \sigma \otimes \sigma = \mathbb{1} \oplus \psi \\ \sigma \otimes (\psi \otimes \sigma) = \sigma \otimes \sigma = \mathbb{1} \oplus \psi \end{cases} \\ \begin{cases} (\psi \otimes \sigma) \otimes \psi = \sigma \otimes \psi = \sigma \\ \psi \otimes (\sigma \otimes \psi) = \psi \otimes \sigma = \sigma \end{cases} \end{gathered} \tag{3.5.11}$$

The following expressions correspond to identity components of α.

$$\begin{gathered} \begin{cases} (\psi \otimes \sigma) \otimes \sigma = \sigma \otimes \sigma = \mathbb{1} \oplus \psi \\ \psi \otimes (\sigma \otimes \sigma) = \psi \otimes (\mathbb{1} \oplus \psi) = \psi \oplus \mathbb{1} \end{cases} \\ \begin{cases} (\sigma \otimes \sigma) \otimes \psi = (\mathbb{1} \oplus \psi) \otimes \psi = \psi \oplus \mathbb{1} \\ \sigma \otimes (\sigma \otimes \psi) = \sigma \otimes \sigma = \mathbb{1} \oplus \psi \end{cases} \\ \begin{cases} (\psi \otimes \psi) \otimes \sigma = \mathbb{1} \otimes \sigma = \sigma \\ \psi \otimes (\psi \otimes \sigma) = \psi \otimes \sigma = \sigma \end{cases} \\ \begin{cases} (\sigma \otimes \psi) \otimes \psi = \sigma \otimes \psi = \sigma \\ \sigma \otimes (\psi \otimes \psi) = \sigma \otimes \mathbb{1} = \sigma \end{cases} \\ \begin{cases} (\psi \otimes \psi) \otimes \psi = \mathbb{1} \otimes \psi = \psi \\ \psi \otimes (\psi \otimes \psi) = \psi \otimes \mathbb{1} = \psi \end{cases} \end{gathered} \tag{3.5.12}$$

Note that in $(\psi \otimes \sigma) \otimes \sigma$, the object $\mathbb{1}$ is listed first. On the other hand, in $\psi \otimes (\sigma \otimes \sigma)$, the object ψ is listed first. We will need to take this order into account when we write down the matrices using these symbols as ordered bases. A similar remark applies to the pair with $(\sigma \otimes \sigma) \otimes \psi$ and $\sigma \otimes (\sigma \otimes \psi)$. $\diamond$

Lemma 3.5.13. *For $m, m', n, n', p, p' \geq 0$, the following equality holds in $\mathcal{I}^{\mathsf{any}}$.*

$$\begin{aligned}&(m;n;p) \otimes (m';n';p')\\&= \mathbb{1}^{\oplus(mm'+nn'+pp')} \oplus \sigma^{\oplus(mn'+nm'+np'+pn')} \oplus \psi^{\oplus(mp'+nn'+pm')}\end{aligned} \tag{3.5.14}$$

Proof. This follows from the following computation using (3.5.8) and the strict distributivity of $\otimes$ over $\oplus$.

$$\begin{aligned}&(m;n;p) \otimes (m';n';p')\\&= (\mathbb{1}^{\oplus m} \oplus \sigma^{\oplus n} \oplus \psi^{\oplus p}) \otimes (\mathbb{1}^{\oplus m'} \oplus \sigma^{\oplus n'} \oplus \psi^{\oplus p'})\\&= (\mathbb{1}^{\oplus m} \otimes \mathbb{1}^{\oplus m'}) \oplus (\mathbb{1}^{\oplus m} \otimes \sigma^{\oplus n'}) \oplus (\mathbb{1}^{\oplus m} \otimes \psi^{\oplus p'})\\&\quad \oplus (\sigma^{\oplus n} \otimes \mathbb{1}^{\oplus m'}) \oplus (\sigma^{\oplus n} \otimes \sigma^{\oplus n'}) \oplus (\sigma^{\oplus n} \otimes \psi^{\oplus p'})\\&\quad \oplus (\psi^{\oplus p} \otimes \mathbb{1}^{\oplus m'}) \oplus (\psi^{\oplus p} \otimes \sigma^{\oplus n'}) \oplus (\psi^{\oplus p} \otimes \psi^{\oplus p'})\\&= \mathbb{1}^{\oplus mm'} \oplus \sigma^{\oplus mn'} \oplus \psi^{\oplus mp'}\\&\quad \oplus \sigma^{\oplus nm'} \oplus (\mathbb{1} \oplus \psi)^{\oplus nn'} \oplus \sigma^{\oplus np'}\\&\quad \oplus \psi^{\oplus pm'} \oplus \sigma^{\oplus pn'} \oplus \mathbb{1}^{\oplus pp'}\end{aligned}$$

This is equal to the expression in (3.5.14). $\square$

Lemma 3.5.15. *For each object $(m;n;p) \in \mathcal{I}^{\mathsf{any}}$, the following functors are additive.*

$$\mathcal{I}^{\mathsf{any}} \underset{-\otimes(m;n;p)}{\overset{(m;n;p)\otimes -}{\rightrightarrows}} \mathcal{I}^{\mathsf{any}}$$

Proof. This follows from (3.5.4) and (3.5.14). $\square$

Lemma 3.5.16. *The following equalities hold in $\mathcal{I}^{\mathsf{any}}$ for $n \geq 1$, with each $\otimes$-product either left normalized (I.5.2.13) or right normalized (I.5.2.12).*

$$\begin{aligned}\sigma^{\otimes 2n} &= \mathbb{1}^{\oplus 2^{n-1}} \oplus \psi^{\oplus 2^{n-1}}\\ \sigma^{\otimes(2n+1)} &= \sigma^{\oplus 2^n}\end{aligned} \tag{3.5.17}$$

Proof. The proof proceeds by induction. If $n = 1$, then $\sigma^{\otimes 2} = \mathbb{1} \oplus \psi$ by one of the fusion rules (3.5.8). Moreover, there are equalities as follows.

$$\begin{aligned}\sigma \otimes (\sigma \otimes \sigma) &= \sigma \otimes (\mathbb{1} \oplus \psi)\\&= (\sigma \otimes \mathbb{1}) \oplus (\sigma \otimes \psi)\\&= \sigma \oplus \sigma\\(\sigma \otimes \sigma) \otimes \sigma &= (\mathbb{1} \oplus \psi) \otimes \sigma\\&= (\mathbb{1} \otimes \sigma) \oplus (\psi \otimes \sigma)\\&= \sigma \oplus \sigma\end{aligned} \tag{3.5.18}$$

The induction step for the right normalized bracketing is proved as follows.

$$\begin{aligned}
\sigma^{\otimes 2(n+1)} &= \sigma \otimes \sigma^{\otimes(2n+1)} \\
&= \sigma \otimes \sigma^{\oplus 2^n} \\
&= (\sigma \otimes \sigma)^{\oplus 2^n} \\
&= (\mathbb{1} \oplus \psi)^{\oplus 2^n} \\
&= \mathbb{1}^{\oplus 2^n} \oplus \psi^{\oplus 2^n} \\
\sigma^{\otimes 2(n+1)+1} &= \sigma \otimes \sigma^{\otimes 2(n+1)} \\
&= \sigma \otimes (\mathbb{1}^{\oplus 2^n} \oplus \psi^{\oplus 2^n}) \\
&= (\sigma \otimes \mathbb{1})^{\oplus 2^n} \oplus (\sigma \otimes \psi)^{\oplus 2^n} \\
&= \sigma^{\oplus 2^n} \oplus \sigma^{\oplus 2^n} \\
&= \sigma^{\oplus 2^{n+1}}
\end{aligned}$$

The computation for the left normalized bracketing is almost identical. □

Example 3.5.19. Using (3.5.18), there are the following five ways to compute $\sigma^{\otimes 4}$, depending on the bracketing. Each $\mathbb{1}_i$ is a copy of $\mathbb{1}$, and each ψ_j is a copy of ψ.

$$\begin{aligned}
[(\sigma \otimes \sigma) \otimes \sigma] \otimes \sigma &= (\sigma \oplus \sigma) \otimes \sigma \\
&= (\sigma \otimes \sigma) \oplus (\sigma \otimes \sigma) \\
&= (\mathbb{1}_1 \oplus \psi_1) \oplus (\mathbb{1}_2 \oplus \psi_2) \\
(\sigma \otimes \sigma) \otimes (\sigma \otimes \sigma) &= (\mathbb{1} \oplus \psi) \otimes (\mathbb{1} \oplus \psi) \\
&= (\mathbb{1} \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes \psi) \oplus (\psi \otimes \mathbb{1}) \oplus (\psi \otimes \psi) \\
&= \mathbb{1}_1 \oplus \psi_1 \oplus \psi_2 \oplus \mathbb{1}_2 \\
&= (\mathbb{1}_1 \oplus \psi_1) \oplus (\mathbb{1}_2 \oplus \psi_2) \\
\sigma \otimes [\sigma \otimes (\sigma \otimes \sigma)] &= \sigma \otimes (\sigma \oplus \sigma) \\
&= (\sigma \otimes \sigma) \oplus (\sigma \otimes \sigma) \\
&= (\mathbb{1}_1 \oplus \psi_1) \oplus (\mathbb{1}_2 \oplus \psi_2) \\
[\sigma \otimes (\sigma \otimes \sigma)] \otimes \sigma &= (\sigma \oplus \sigma) \otimes \sigma \\
&= (\sigma \otimes \sigma) \oplus (\sigma \otimes \sigma) \\
&= (\mathbb{1}_1 \oplus \psi_1) \oplus (\mathbb{1}_2 \oplus \psi_2) \\
\sigma \otimes [(\sigma \otimes \sigma) \otimes \sigma] &= \sigma \otimes (\sigma \oplus \sigma) \\
&= (\sigma \otimes \sigma) \oplus (\sigma \otimes \sigma) \\
&= (\mathbb{1}_1 \oplus \psi_1) \oplus (\mathbb{1}_2 \oplus \psi_2)
\end{aligned} \tag{3.5.20}$$

These four-fold tensor products will be used in Lemma 3.5.27 below to verify the pentagon axiom in $\mathcal{I}^{\text{any}}$. ⋄

The Associativity Isomorphism. Unless otherwise specified below, $1 \in \mathbb{C}$ is the basis vector for each copy of $\mathbb{C}$.

Definition 3.5.21. For the functor $\otimes$ in (3.5.6), define a natural isomorphism

$$(-\otimes -)\otimes - \xrightarrow[\cong]{\alpha} -\otimes(-\otimes -) \tag{3.5.22}$$

by the following rules.

(i) Using the expressions in (3.5.18), the component

$$\sigma\oplus\sigma = (\sigma\otimes\sigma)\otimes\sigma \xrightarrow{\alpha_{\sigma,\sigma,\sigma}} \sigma\otimes(\sigma\otimes\sigma) = \sigma\oplus\sigma$$

is defined by the matrix

$$\alpha_{\sigma,\sigma,\sigma} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{3.5.23}$$

(ii) The following two components of α use the expressions in (3.5.11).

- The component

$$\mathbb{1}\oplus\psi = (\sigma\otimes\psi)\otimes\sigma \xrightarrow{\alpha_{\sigma,\psi,\sigma}} \sigma\otimes(\psi\otimes\sigma) = \mathbb{1}\oplus\psi$$

is defined by the matrix

$$\alpha_{\sigma,\psi,\sigma} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.5.24}$$

- The component

$$\sigma = (\psi\otimes\sigma)\otimes\psi \xrightarrow{\alpha_{\psi,\sigma,\psi}} \psi\otimes(\sigma\otimes\psi) = \sigma$$

is defined as multiplication by -1.

(iii) Using the expressions in (3.5.12), every other component of α with subscripts in $\{\sigma,\psi\}$ is the identity morphism.

(iv) Each component of α with at least one subscript in $\{\mathbb{0},\mathbb{1}\}$ is the identity morphism.

(v) All other components of α are determined by the above rules and the naturality requirement of α.

This finishes the definition of α. ◇

Explanation 3.5.25. Definition 3.5.21 (iii) means that each of the following five components of α is the identity morphism.

$$\begin{aligned}
\mathbb{1}\oplus\psi = (\psi\otimes\sigma)\otimes\sigma &\xrightarrow{\alpha_{\psi,\sigma,\sigma}} \psi\otimes(\sigma\otimes\sigma) = \psi\oplus\mathbb{1}\\
\psi\oplus\mathbb{1} = (\sigma\otimes\sigma)\otimes\psi &\xrightarrow{\alpha_{\sigma,\sigma,\psi}} \sigma\otimes(\sigma\otimes\psi) = \mathbb{1}\oplus\psi\\
\sigma = (\psi\otimes\psi)\otimes\sigma &\xrightarrow{\alpha_{\psi,\psi,\sigma}} \psi\otimes(\psi\otimes\sigma) = \sigma\\
\sigma = (\sigma\otimes\psi)\otimes\psi &\xrightarrow{\alpha_{\sigma,\psi,\psi}} \sigma\otimes(\psi\otimes\psi) = \sigma\\
\psi = (\psi\otimes\psi)\otimes\psi &\xrightarrow{\alpha_{\psi,\psi,\psi}} \psi\otimes(\psi\otimes\psi) = \psi
\end{aligned} \tag{3.5.26}$$

Moreover, the matrix $\alpha_{\sigma,\sigma,\sigma}$ in (3.5.23) is orthogonal. In other words, its transpose, which is equal to itself, is its inverse. ◇

Lemma 3.5.27. *The quadruple*

$$(\mathcal{I}^{\mathsf{any}},\otimes,\mathbb{1},\alpha)$$

consisting of

- *$\mathcal{I}^{\mathsf{any}}$ and $\mathbb{1} = (1;0;0)$ in Definition 3.5.1,*
- *$- \otimes - : \mathcal{I}^{\mathsf{any}} \times \mathcal{I}^{\mathsf{any}} \longrightarrow \mathcal{I}^{\mathsf{any}}$ in (3.5.6), and*
- *α in (3.5.22)*

is a monoidal category with identities for the left and the right unit isomorphisms.

Proof. The naturality of α is part of its definition. Each of its components is an isomorphism because it is a direct sum of identity morphisms and copies of $\alpha_{\sigma,\sigma,\sigma}$ in (3.5.23), $\alpha_{\sigma,\psi,\sigma}$ in (3.5.24), and $\alpha_{\psi,\sigma,\psi} = -1$.

The unity axiom (1.3.2) holds because both the left and the right unit isomorphisms are identities. It remains to check the pentagon axiom (1.3.3). By Definition 3.5.21 (iii)–(v) and the strict distributivity of $\otimes$ over $\oplus$, it suffices to check the pentagon axiom when all four objects involved are in $\{\sigma, \psi\}$ with at least one σ. There are 15 cases, which we divide into four groups depending on the number of ψ factors. To save space below, we omit the $\otimes$ symbols among objects, so, for example, $\sigma\sigma = \sigma \otimes \sigma$. We write Id for an identity matrix. In each matrix, an empty entry means 0.

Case 1

If all four objects involved are σ, then the pentagon (1.3.3) in $\mathcal{I}^{\mathsf{any}}$ is the following diagram.

$$\begin{tikzcd}
 & (\sigma\sigma)(\sigma\sigma) \arrow[dr, "\alpha_{\sigma,\sigma,\sigma\sigma}"] & \\
{[(\sigma\sigma)\sigma]\sigma} \arrow[ur, "\alpha_{\sigma\sigma,\sigma,\sigma}"] \arrow[d, "\alpha_{\sigma,\sigma,\sigma}\otimes 1_\sigma"'] & & \sigma[\sigma(\sigma\sigma)] \\
{[\sigma(\sigma\sigma)]\sigma} \arrow[rr, "\alpha_{\sigma,\sigma\sigma,\sigma}"] & & \sigma[(\sigma\sigma)\sigma] \arrow[u, "1_\sigma\otimes\alpha_{\sigma,\sigma,\sigma}"']
\end{tikzcd}$$

To prove the commutativity of this pentagon, we consistently use the ordered bases $\{\mathbb{1}_1, \psi_1, \mathbb{1}_2, \psi_2\}$ in (3.5.20). In the upper path of the pentagon, the two maps are given by the following matrices.

$$\alpha_{\sigma\sigma,\sigma,\sigma} = \alpha_{\mathbb{1}\oplus\psi,\sigma,\sigma} = \alpha_{\mathbb{1},\sigma,\sigma} \oplus \alpha_{\psi,\sigma,\sigma} = \mathrm{Id}$$
$$\alpha_{\sigma,\sigma,\sigma\sigma} = \alpha_{\sigma,\sigma,\mathbb{1}\oplus\psi} = \alpha_{\sigma,\sigma,\mathbb{1}} \oplus \alpha_{\sigma,\sigma,\psi} = \begin{bmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & 1 & & \end{bmatrix}$$

Therefore, the matrix for the upper path of the pentagon is the same as that for $\alpha_{\sigma,\sigma,\sigma\sigma}$.

Note that the matrix for $\alpha_{\sigma,\sigma,\sigma\sigma}$ is not the identity matrix for the following reasons.

- In the domain $(\sigma\sigma)(\sigma\sigma)$, the object $\mathbb{1}$ in the second $\sigma\sigma = \mathbb{1} \oplus \psi$ yields $\{\mathbb{1}_1, \psi_2\}$.
- In the codomain $\sigma[\sigma(\sigma\sigma)]$, the object $\mathbb{1}$ in the rightmost $\sigma\sigma = \mathbb{1} \oplus \psi$ yields $\{\mathbb{1}_1, \psi_1\}$.

This role-reversal involving ψ_1 and ψ_2 and the consistent usage of the ordered bases $\{\mathbb{1}_1, \psi_1, \mathbb{1}_2, \psi_2\}$ lead to the above matrix for $\alpha_{\sigma,\sigma,\sigma\sigma}$. Analogous remarks will apply in some of the matrices below.

In the lower path of the pentagon, the three maps are given by the following matrices involving $\alpha_{\sigma,\sigma,\sigma}$ in (3.5.23) and $\alpha_{\sigma,\psi,\sigma}$ in (3.5.24).

$$\alpha_{\sigma,\sigma,\sigma} \otimes 1_\sigma = 1_\sigma \otimes \alpha_{\sigma,\sigma,\sigma} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & 1 & & -1 \end{bmatrix}$$

$$\alpha_{\sigma,\sigma\sigma,\sigma} = \alpha_{\sigma,\mathbb{1}\oplus\psi,\sigma} = \alpha_{\sigma,\mathbb{1},\sigma} \oplus \alpha_{\sigma,\psi,\sigma} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

Therefore, the matrix for the lower path of the pentagon is the product

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & 1 & & -1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \\ & 1 & & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & & & \\ & & & 2 \\ & & 2 & \\ & 2 & & \end{bmatrix}.$$

This is equal to the matrix for the upper path, that is, the matrix for $\alpha_{\sigma,\sigma,\sigma\sigma}$.

Case 2

Next are the four subcases involving one ψ and three copies of σ. We use the ordered bases in (3.5.11), (3.5.12), and (3.5.18).

The pentagon

$$\begin{tikzcd}
& (\psi\sigma)(\sigma\sigma) \arrow[dr, "\alpha_{\psi,\sigma,\sigma\sigma}"] & \\
{[(\psi\sigma)\sigma]\sigma} \arrow[ur, "\alpha_{\psi\sigma,\sigma,\sigma}"] \arrow[d, "\alpha_{\psi,\sigma,\sigma}\otimes 1_\sigma"'] & & \psi[\sigma(\sigma\sigma)] \\
{[\psi(\sigma\sigma)]\sigma} \arrow[rr, "\alpha_{\psi,\sigma\sigma,\sigma}"] & & \psi[(\sigma\sigma)\sigma] \arrow[u, "1_\psi \otimes \alpha_{\sigma,\sigma,\sigma}"']
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(\alpha_{\psi,\sigma,\sigma\sigma})(\alpha_{\psi\sigma,\sigma,\sigma}) = (\alpha_{\psi,\sigma,\mathbb{1}\oplus\psi})(\alpha_{\sigma,\sigma,\sigma}) = (\alpha_{\psi,\sigma,\mathbb{1}} \oplus \alpha_{\psi,\sigma,\psi})(\alpha_{\sigma,\sigma,\sigma}) \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}(\mathrm{Id})\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \\
&= (\alpha_{\sigma,\sigma,\sigma})(\alpha_{\psi,\mathbb{1},\sigma} \oplus \alpha_{\psi,\psi,\sigma})(\alpha_{\psi,\sigma,\sigma} \otimes 1_\sigma) \\
&= (\alpha_{\sigma,\sigma,\sigma})(\alpha_{\psi,\mathbb{1}\oplus\psi,\sigma})(\alpha_{\psi,\sigma,\sigma} \otimes 1_\sigma) \\
&= (1_\psi \otimes \alpha_{\sigma,\sigma,\sigma})(\alpha_{\psi,\sigma\sigma,\sigma})(\alpha_{\psi,\sigma,\sigma} \otimes 1_\sigma)
\end{aligned}$$

The pentagon

$$\begin{tikzcd}
& (\sigma\psi)(\sigma\sigma) \arrow[dr, "\alpha_{\sigma,\psi,\sigma\sigma}"] & \\
{[(\sigma\psi)\sigma]\sigma} \arrow[ur, "\alpha_{\sigma\psi,\sigma,\sigma}"] \arrow[d, "\alpha_{\sigma,\psi,\sigma}\otimes 1_\sigma"'] & & \sigma[\psi(\sigma\sigma)] \\
{[\sigma(\psi\sigma)]\sigma} \arrow[rr, "\alpha_{\sigma,\psi\sigma,\sigma}"] & & \sigma[(\psi\sigma)\sigma] \arrow[u, "1_\sigma \otimes \alpha_{\psi,\sigma,\sigma}"']
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(\alpha_{\sigma,\psi,\sigma\sigma})(\alpha_{\sigma\psi,\sigma,\sigma}) = (\alpha_{\sigma,\psi,\mathbb{1}\oplus\psi})(\alpha_{\sigma,\sigma,\sigma}) \\
&= (\alpha_{\sigma,\psi,\mathbb{1}} \oplus \alpha_{\sigma,\psi,\psi})(\alpha_{\sigma,\sigma,\sigma}) = \mathrm{Id}(\alpha_{\sigma,\sigma,\sigma}) \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\
&= (1_\sigma \otimes \alpha_{\psi,\sigma,\sigma})(\alpha_{\sigma,\sigma,\sigma})(\alpha_{\sigma,\psi,\sigma} \otimes 1_\sigma) \\
&= (1_\sigma \otimes \alpha_{\psi,\sigma,\sigma})(\alpha_{\sigma,\psi\sigma,\sigma})(\alpha_{\sigma,\psi,\sigma} \otimes 1_\sigma)
\end{aligned}$$

The pentagon

$$\begin{tikzcd}
 & (\sigma\sigma)(\psi\sigma) \arrow[dr, "\alpha_{\sigma,\sigma,\psi\sigma}"] & \\
{[(\sigma\sigma)\psi]\sigma} \arrow[ur, "\alpha_{\sigma\sigma,\psi,\sigma}"] \arrow[d, "\alpha_{\sigma,\sigma,\psi} \otimes 1_\sigma"'] & & \sigma[\sigma(\psi\sigma)] \\
{[\sigma(\sigma\psi)]\sigma} \arrow[rr, "\alpha_{\sigma,\sigma\psi,\sigma}"] & & \sigma[(\sigma\psi)\sigma] \arrow[u, "1_\sigma \otimes \alpha_{\sigma,\psi,\sigma}"']
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(\alpha_{\sigma,\sigma,\psi\sigma})(\alpha_{\sigma\sigma,\psi,\sigma}) = (\alpha_{\sigma,\sigma,\sigma})(\alpha_{\mathbb{1}\oplus\psi,\psi,\sigma}) \\
&= (\alpha_{\sigma,\sigma,\sigma})(\alpha_{\mathbb{1},\psi,\sigma} \oplus \alpha_{\psi,\psi,\sigma}) = (\alpha_{\sigma,\sigma,\sigma})\mathrm{Id} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \\
&= (1_\sigma \otimes \alpha_{\sigma,\psi,\sigma})(\alpha_{\sigma,\sigma,\sigma})(\alpha_{\sigma,\sigma,\psi} \otimes 1_\sigma) \\
&= (1_\sigma \otimes \alpha_{\sigma,\psi,\sigma})(\alpha_{\sigma,\sigma\psi,\sigma})(\alpha_{\sigma,\sigma,\psi} \otimes 1_\sigma)
\end{aligned}$$

The pentagon

$$\begin{tikzcd}
 & (\sigma\sigma)(\sigma\psi) \arrow[dr, "\alpha_{\sigma,\sigma,\sigma\psi}"] & \\
{[(\sigma\sigma)\sigma]\psi} \arrow[ur, "\alpha_{\sigma\sigma,\sigma,\psi}"] \arrow[d, "\alpha_{\sigma,\sigma,\sigma} \otimes 1_\psi"'] & & \sigma[\sigma(\sigma\psi)] \\
{[\sigma(\sigma\sigma)]\psi} \arrow[rr, "\alpha_{\sigma,\sigma\sigma,\psi}"] & & \sigma[(\sigma\sigma)\psi] \arrow[u, "1_\sigma \otimes \alpha_{\sigma,\sigma,\psi}"']
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(\alpha_{\sigma,\sigma,\sigma\psi})(\alpha_{\sigma\sigma,\sigma,\psi}) = (\alpha_{\sigma,\sigma,\sigma})(\alpha_{\mathbb{1}\oplus\psi,\sigma,\psi}) = (\alpha_{\sigma,\sigma,\sigma})(\alpha_{\mathbb{1},\sigma,\psi} \oplus \alpha_{\psi,\sigma,\psi}) \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}(\mathrm{Id})\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= (1_\sigma \otimes \alpha_{\sigma,\sigma,\psi})(\alpha_{\sigma,\mathbb{1},\psi} \oplus \alpha_{\sigma,\psi,\psi})(\alpha_{\sigma,\sigma,\sigma} \otimes 1_\psi) \\
&= (1_\sigma \otimes \alpha_{\sigma,\sigma,\psi})(\alpha_{\sigma,\mathbb{1}\oplus\psi,\psi})(\alpha_{\sigma,\sigma,\sigma} \otimes 1_\psi) \\
&= (1_\sigma \otimes \alpha_{\sigma,\sigma,\psi})(\alpha_{\sigma,\sigma\sigma,\psi})(\alpha_{\sigma,\sigma,\sigma} \otimes 1_\psi)
\end{aligned}$$

Case 3

Next are the six subcases involving two copies of ψ and two copies of σ.

The pentagon

$$\begin{tikzcd}
& (\psi\psi)(\sigma\sigma) \arrow[dr, "\alpha_{\psi,\psi,\sigma\sigma}"] & \\
[(\psi\psi)\sigma]\sigma \arrow[ur, "\alpha_{\psi\psi,\sigma,\sigma}"] \arrow[d, "\alpha_{\psi,\psi,\sigma}\otimes 1_{\sigma}"'] & & \psi[\psi(\sigma\sigma)] \\
[\psi(\psi\sigma)]\sigma \arrow[rr, "\alpha_{\psi,\psi\sigma,\sigma}"] & & \psi[(\psi\sigma)\sigma] \arrow[u, "1_{\psi}\otimes\alpha_{\psi,\sigma,\sigma}"']
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(\alpha_{\psi,\psi,\sigma\sigma})(\alpha_{\psi\psi,\sigma,\sigma}) = (\alpha_{\psi,\psi,\mathbb{1}\oplus\psi})(\alpha_{\mathbb{1},\sigma,\sigma}) \\
&= (\alpha_{\psi,\psi,\mathbb{1}} \oplus \alpha_{\psi,\psi,\psi})(\alpha_{\mathbb{1},\sigma,\sigma}) \\
&= (\mathrm{Id})(\mathrm{Id}) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}(\mathrm{Id}) \\
&= (1_{\psi} \otimes \alpha_{\psi,\sigma,\sigma})(\alpha_{\psi,\sigma,\sigma})(\alpha_{\psi,\psi,\sigma} \otimes 1_{\sigma}) \\
&= (1_{\psi} \otimes \alpha_{\psi,\sigma,\sigma})(\alpha_{\psi,\psi\sigma,\sigma})(\alpha_{\psi,\psi,\sigma} \otimes 1_{\sigma})
\end{aligned}$$

The pentagon

$$\begin{tikzcd}
& (\psi\sigma)(\psi\sigma) \arrow[dr, "\alpha_{\psi,\sigma,\psi\sigma}"] & \\
[(\psi\sigma)\psi]\sigma \arrow[ur, "\alpha_{\psi\sigma,\psi,\sigma}"] \arrow[d, "\alpha_{\psi,\sigma,\psi}\otimes 1_{\sigma}"'] & & \psi[\sigma(\psi\sigma)] \\
[\psi(\sigma\psi)]\sigma \arrow[rr, "\alpha_{\psi,\sigma\psi,\sigma}"] & & \psi[(\sigma\psi)\sigma] \arrow[u, "1_{\psi}\otimes\alpha_{\sigma,\psi,\sigma}"']
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(\alpha_{\psi,\sigma,\psi\sigma})(\alpha_{\psi\sigma,\psi,\sigma}) = (\alpha_{\psi,\sigma,\sigma})(\alpha_{\sigma,\psi,\sigma}) \\
&= \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}(-\mathrm{Id}) \\
&= (1_{\psi} \otimes \alpha_{\sigma,\psi,\sigma})(\alpha_{\psi,\sigma,\sigma})(\alpha_{\psi,\sigma,\psi} \otimes 1_{\sigma}) \\
&= (1_{\psi} \otimes \alpha_{\sigma,\psi,\sigma})(\alpha_{\psi,\sigma\psi,\sigma})(\alpha_{\psi,\sigma,\psi} \otimes 1_{\sigma})
\end{aligned}$$

The pentagon

$$\begin{tikzcd}
& (\psi\sigma)(\sigma\psi) \arrow[dr, "\alpha_{\psi,\sigma,\sigma\psi}"] & \\
[(\psi\sigma)\sigma]\psi \arrow[ur, "\alpha_{\psi\sigma,\sigma,\psi}"] \arrow[d, "\alpha_{\psi,\sigma,\sigma}\otimes 1_{\psi}"'] & & \psi[\sigma(\sigma\psi)] \\
[\psi(\sigma\sigma)]\psi \arrow[rr, "\alpha_{\psi,\sigma\sigma,\psi}"] & & \psi[(\sigma\sigma)\psi] \arrow[u, "1_{\psi}\otimes\alpha_{\sigma,\sigma,\psi}"']
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(\alpha_{\psi,\sigma,\sigma\psi})(\alpha_{\psi\sigma,\sigma,\psi}) = (\alpha_{\psi,\sigma,\sigma})(\alpha_{\sigma,\sigma,\psi}) \\
&= \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}(\mathrm{Id})\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \\
&= (1_{\psi} \otimes \alpha_{\sigma,\sigma,\psi})(\alpha_{\psi,\mathbb{1},\psi} \oplus \alpha_{\psi,\psi,\psi})(\alpha_{\psi,\sigma,\sigma} \otimes 1_{\psi}) \\
&= (1_{\psi} \otimes \alpha_{\sigma,\sigma,\psi})(\alpha_{\psi,\mathbb{1}\oplus\psi,\psi})(\alpha_{\psi,\sigma,\sigma} \otimes 1_{\psi}) \\
&= (1_{\psi} \otimes \alpha_{\sigma,\sigma,\psi})(\alpha_{\psi,\sigma\sigma,\psi})(\alpha_{\psi,\sigma,\sigma} \otimes 1_{\psi})
\end{aligned}$$

The pentagon

$$
\begin{tikzcd}
& (\sigma\psi)(\psi\sigma) \arrow[dr, "\alpha_{\sigma,\psi,\psi\sigma}"] & \\
{[(\sigma\psi)\psi]\sigma} \arrow[ur, "\alpha_{\sigma\psi,\psi,\sigma}"] \arrow[d, "\alpha_{\sigma,\psi,\psi}\otimes 1_\sigma"'] & & \sigma[\psi(\psi\sigma)] \\
{[\sigma(\psi\psi)]\sigma} \arrow[rr, "\alpha_{\sigma,\psi\psi,\sigma}"] & & \sigma[(\psi\psi)\sigma] \arrow[u, "1_\sigma\otimes\alpha_{\psi,\psi,\sigma}"']
\end{tikzcd}
$$

is commutative by the following computation.

$$
\begin{aligned}
(\alpha_{\sigma,\psi,\psi\sigma})(\alpha_{\sigma\psi,\psi,\sigma}) &= (\alpha_{\sigma,\psi,\sigma})(\alpha_{\sigma,\psi,\sigma}) \\
&= \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = (\mathrm{Id})(\mathrm{Id})(\mathrm{Id}) \\
&= (1_\sigma \otimes \alpha_{\psi,\psi,\sigma})(\alpha_{\sigma,\mathbb{1},\sigma})(\alpha_{\sigma,\psi,\psi}\otimes 1_\sigma) \\
&= (1_\sigma \otimes \alpha_{\psi,\psi,\sigma})(\alpha_{\sigma,\psi\psi,\sigma})(\alpha_{\sigma,\psi,\psi}\otimes 1_\sigma)
\end{aligned}
$$

The pentagon

$$
\begin{tikzcd}
& (\sigma\psi)(\sigma\psi) \arrow[dr, "\alpha_{\sigma,\psi,\sigma\psi}"] & \\
{[(\sigma\psi)\sigma]\psi} \arrow[ur, "\alpha_{\sigma\psi,\sigma,\psi}"] \arrow[d, "\alpha_{\sigma,\psi,\sigma}\otimes 1_\psi"'] & & \sigma[\psi(\sigma\psi)] \\
{[\sigma(\psi\sigma)]\psi} \arrow[rr, "\alpha_{\sigma,\psi\sigma,\psi}"] & & \sigma[(\psi\sigma)\psi] \arrow[u, "1_\sigma\otimes\alpha_{\psi,\sigma,\psi}"']
\end{tikzcd}
\tag{3.5.28}
$$

is commutative by the following computation.

$$
\begin{aligned}
(\alpha_{\sigma,\psi,\sigma\psi})(\alpha_{\sigma\psi,\sigma,\psi}) &= (\alpha_{\sigma,\psi,\sigma})(\alpha_{\sigma,\sigma,\psi}) \\
&= \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = (-\mathrm{Id})\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\
&= (1_\sigma \otimes \alpha_{\psi,\sigma,\psi})(\alpha_{\sigma,\sigma,\psi})(\alpha_{\sigma,\psi,\sigma}\otimes 1_\psi) \\
&= (1_\sigma \otimes \alpha_{\psi,\sigma,\psi})(\alpha_{\sigma,\psi\sigma,\psi})(\alpha_{\sigma,\psi,\sigma}\otimes 1_\psi)
\end{aligned}
$$

The pentagon

$$
\begin{tikzcd}
& (\sigma\sigma)(\psi\psi) \arrow[dr, "\alpha_{\sigma,\sigma,\psi\psi}"] & \\
{[(\sigma\sigma)\psi]\psi} \arrow[ur, "\alpha_{\sigma\sigma,\psi,\psi}"] \arrow[d, "\alpha_{\sigma,\sigma,\psi}\otimes 1_\psi"'] & & \sigma[\sigma(\psi\psi)] \\
{[\sigma(\sigma\psi)]\psi} \arrow[rr, "\alpha_{\sigma,\sigma\psi,\psi}"] & & \sigma[(\sigma\psi)\psi] \arrow[u, "1_\sigma\otimes\alpha_{\sigma,\psi,\psi}"']
\end{tikzcd}
$$

is commutative by the following computation.

$$
\begin{aligned}
(\alpha_{\sigma,\sigma,\psi\psi})(\alpha_{\sigma\sigma,\psi,\psi}) &= (\alpha_{\sigma,\sigma,\mathbb{1}})(\alpha_{\mathbb{1}\oplus\psi,\psi,\psi}) \\
&= (\alpha_{\sigma,\sigma,\mathbb{1}})(\alpha_{\mathbb{1},\psi,\psi}\oplus\alpha_{\psi,\psi,\psi}) \\
&= (\mathrm{Id})(\mathrm{Id}) = (\mathrm{Id})\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \\
&= (1_\sigma \otimes \alpha_{\sigma,\psi,\psi})(\alpha_{\sigma,\sigma,\psi})(\alpha_{\sigma,\sigma,\psi}\otimes 1_\psi) \\
&= (1_\sigma \otimes \alpha_{\sigma,\psi,\psi})(\alpha_{\sigma,\sigma\psi,\psi})(\alpha_{\sigma,\sigma,\psi}\otimes 1_\psi)
\end{aligned}
$$

Case 4

Next are the four pentagons involving three copies of ψ and one σ.

$$\begin{tikzcd}[column sep=small]
& & (\psi\psi)(\psi\sigma) \arrow[drr, "\alpha_{\psi,\psi,\psi\sigma}", "1"'] & & \\
{[(\psi\psi)\psi]\sigma} \arrow[urr, "\alpha_{\psi\psi,\psi,\sigma}", "1"'] \arrow[dr, "\alpha_{\psi,\psi,\psi} \otimes 1_\sigma"', "1"] & & & & \psi[\psi(\psi\sigma)] \\
& {[\psi(\psi\psi)]\sigma} \arrow[rr, "1", "\alpha_{\psi,\psi\psi,\sigma}"'] & & \psi[(\psi\psi)\sigma] \arrow[ur, "1", "1_\psi \otimes \alpha_{\psi,\psi,\sigma}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & (\psi\psi)(\sigma\psi) \arrow[drr, "\alpha_{\psi,\psi,\sigma\psi}", "1"'] & & \\
{[(\psi\psi)\sigma]\psi} \arrow[urr, "\alpha_{\psi\psi,\sigma,\psi}", "1"'] \arrow[dr, "\alpha_{\psi,\psi,\sigma} \otimes 1_\psi"', "1"] & & & & \psi[\psi(\sigma\psi)] \\
& {[\psi(\psi\sigma)]\psi} \arrow[rr, "-1", "\alpha_{\psi,\psi\sigma,\psi}"'] & & \psi[(\psi\sigma)\psi] \arrow[ur, "-1", "1_\psi \otimes \alpha_{\psi,\sigma,\psi}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& & (\psi\sigma)(\psi\psi) \arrow[drr, "\alpha_{\psi,\sigma,\psi\psi}", "1"'] & & \\
{[(\psi\sigma)\psi]\psi} \arrow[urr, "\alpha_{\psi\sigma,\psi,\psi}", "1"'] \arrow[dr, "\alpha_{\psi,\sigma,\psi} \otimes 1_\psi"', "-1"] & & & & \psi[\sigma(\psi\psi)] \\
& {[\psi(\sigma\psi)]\psi} \arrow[rr, "-1", "\alpha_{\psi,\sigma\psi,\psi}"'] & & \psi[(\sigma\psi)\psi] \arrow[ur, "1", "1_\psi \otimes \alpha_{\sigma,\psi,\psi}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & (\sigma\psi)(\psi\psi) \arrow[drr, "\alpha_{\sigma,\psi,\psi\psi}", "1"'] & & \\
{[(\sigma\psi)\psi]\psi} \arrow[urr, "\alpha_{\sigma\psi,\psi,\psi}", "1"'] \arrow[dr, "\alpha_{\sigma,\psi,\psi} \otimes 1_\psi"', "1"] & & & & \sigma[\psi(\psi\psi)] \\
& {[\sigma(\psi\psi)]\psi} \arrow[rr, "1", "\alpha_{\sigma,\psi\psi,\psi}"'] & & \sigma[(\psi\psi)\psi] \arrow[ur, "1", "1_\sigma \otimes \alpha_{\psi,\psi,\psi}"'] &
\end{tikzcd}$$

In each of these four pentagons, by the fusion rules

$$\psi\psi = \mathbb{1} \quad \text{and} \quad \sigma\psi = \sigma = \psi\sigma$$

in (3.5.8), each map is either the identity 1 or its negative −1, as indicted by the label next to each arrow. In each case, both the upper path and the lower path are equal to the identity map. This finishes the proof that $\mathcal{I}^{\mathsf{any}}$ is a monoidal category. □

3.6. Ising Anyons: The Braided Bimonoidal Structure

In this section, we first equip the monoidal category $\mathcal{I}^{\mathsf{any}}$ of Ising anyons in Lemma 3.5.27 with a braiding. Then we observe that the abelian category $\mathcal{I}^{\mathsf{any}}$ with this braiding is a tight braided bimonoidal category. Unless otherwise specified below, $1 \in \mathbb{C}$ is the basis vector.

Definition 3.6.1. For the functor $\otimes$ in (3.5.6), define a natural isomorphism

$$x \otimes y \xrightarrow[\cong]{\beta_{x,y}} y \otimes x \tag{3.6.2}$$

for objects $x, y \in \mathcal{I}^{\mathsf{any}}$ by the following three rules.

(i) The components of β with subscripts in $\{\sigma, \psi\}$ are defined as follows.

- The component

$$\mathbb{1} \oplus \psi = \sigma \otimes \sigma \xrightarrow{\beta_{\sigma,\sigma}} \sigma \otimes \sigma = \mathbb{1} \oplus \psi$$

is defined as the matrix

$$\beta_{\sigma,\sigma} = \begin{bmatrix} w^{-1} & \\ & w^3 \end{bmatrix} \quad \text{with} \quad w = e^{\pi i/8}. \tag{3.6.3}$$

- Both components

$$\sigma = \sigma \otimes \psi \underset{\beta_{\psi,\sigma}}{\overset{\beta_{\sigma,\psi}}{\rightleftarrows}} \psi \otimes \sigma = \sigma \tag{3.6.4}$$

are defined as multiplication by

$$w^{-4} = e^{-\pi i/2} = -i.$$

- The component

$$\mathbb{1} = \psi \otimes \psi \xrightarrow{\beta_{\psi,\psi}} \psi \otimes \psi = \mathbb{1} \tag{3.6.5}$$

  is defined as multiplication by -1.

(ii) Each component of β with at least one subscript in $\{\mathbb{0}, \mathbb{1}\}$ is the identity morphism.

(iii) All other components of β are determined by the above rules and the naturality requirement of β.

This finishes the definition of β. $\diamond$

Explanation 3.6.6. Consider β in Definition 3.6.1.

- w in (3.6.3) is a primitive 16th root of unity, and $\beta_{\sigma,\sigma}$ has order 16.
- $\beta_{\psi,\psi} = -1$ in (3.6.5) has order 2. $\diamond$

Lemma 3.6.7. *The quintuple*

$$\left(\mathcal{I}^{\mathsf{any}}, \otimes, \mathbb{1}, \alpha, \beta\right)$$

consisting of

- *the monoidal category* $(\mathcal{I}^{\mathsf{any}}, \otimes, \mathbb{1}, \alpha)$ *in Lemma 3.5.27 and*
- β *in (3.6.2)*

is a braided monoidal category.

Proof. The naturality of β is part of its definition. Each of its components is an isomorphism because it is a direct sum of identity morphisms and copies of $\beta_{\sigma,\sigma}$ in (3.6.3), $\beta_{\sigma,\psi}$ and $\beta_{\psi,\sigma}$ in (3.6.4), and $\beta_{\psi,\psi}$ in (3.6.5).

It remains to verify the two hexagon axioms (1.3.17). By Definition 3.6.1 (ii)–(iii) and the strict distributivity of $\otimes$ over $\oplus$, it suffices to check each hexagon axiom when all three objects involved are in $\{\sigma, \psi\}$. First we consider the left hexagon axiom. We divide the 8 cases into 3 groups, depending on the number of ψ factors. As in the proof of Lemma 3.5.27, to save space, we omit the $\otimes$ symbols among objects

Case 1

First are the four left hexagon diagrams with three copies of ψ, or with two copies of ψ and one σ.

$$\begin{tikzcd}
& (\psi\psi)\psi \arrow[r, "\alpha_{\psi,\psi,\psi}", "1"'] & \psi(\psi\psi) \arrow[dr, "1_\psi \otimes \beta_{\psi,\psi}", "-1"'] & \\
(\psi\psi)\psi \arrow[ur, "\beta_{\psi,\psi} \otimes 1_\psi", "-1"'] \arrow[dr, "\alpha_{\psi,\psi,\psi}"', "1"] & & & \psi(\psi\psi) \\
& \psi(\psi\psi) \arrow[r, "1", "\beta_{\psi,\psi\psi}"'] & (\psi\psi)\psi \arrow[ur, "1", "\alpha_{\psi,\psi,\psi}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (\psi\psi)\sigma \arrow[r, "\alpha_{\psi,\psi,\sigma}", "1"'] & \psi(\psi\sigma) \arrow[dr, "1_\psi \otimes \beta_{\psi,\sigma}", "-i"'] & \\
(\psi\psi)\sigma \arrow[ur, "\beta_{\psi,\psi} \otimes 1_\sigma", "-1"'] \arrow[dr, "\alpha_{\psi,\psi,\sigma}"', "1"] & & & \psi(\sigma\psi) \\
& \psi(\psi\sigma) \arrow[r, "-i", "\beta_{\psi,\psi\sigma}"'] & (\psi\sigma)\psi \arrow[ur, "-1", "\alpha_{\psi,\sigma,\psi}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (\sigma\psi)\psi \arrow[r, "\alpha_{\sigma,\psi,\psi}", "1"'] & \sigma(\psi\psi) \arrow[dr, "1_\sigma \otimes \beta_{\psi,\psi}", "-1"'] & \\
(\psi\sigma)\psi \arrow[ur, "\beta_{\psi,\sigma} \otimes 1_\psi", "-i"'] \arrow[dr, "\alpha_{\psi,\sigma,\psi}"', "-1"] & & & \sigma(\psi\psi) \\
& \psi(\sigma\psi) \arrow[r, "-i", "\beta_{\psi,\sigma\psi}"'] & (\sigma\psi)\psi \arrow[ur, "1", "\alpha_{\sigma,\psi,\psi}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (\psi\sigma)\psi \arrow[r, "\alpha_{\psi,\sigma,\psi}", "-1"'] & \psi(\sigma\psi) \arrow[dr, "1_\psi \otimes \beta_{\sigma,\psi}", "-i"'] & \\
(\sigma\psi)\psi \arrow[ur, "\beta_{\sigma,\psi} \otimes 1_\psi", "-i"'] \arrow[dr, "\alpha_{\sigma,\psi,\psi}"', "1"] & & & \psi(\psi\sigma) \\
& \sigma(\psi\psi) \arrow[r, "1", "\beta_{\sigma,\psi\psi}"'] & (\psi\psi)\sigma \arrow[ur, "1", "\alpha_{\psi,\psi,\sigma}"'] &
\end{tikzcd}$$

In each of these four left hexagon diagrams, each map is the identity, multiplication by -1, or multiplication by $-i$, as indicated by the label next to each edge. In each case, the upper path and the lower path are equal.

Case 2

Next are the three left hexagon diagrams with one ψ and two copies of σ.

The left hexagon diagram

(3.6.8)
$$\begin{tikzcd}
& (\sigma\psi)\sigma \arrow[rr, "\alpha_{\sigma,\psi,\sigma}"] && \sigma(\psi\sigma) \arrow[dr, "1_\sigma \otimes \beta_{\psi,\sigma}"] & \\
(\psi\sigma)\sigma \arrow[ur, "\beta_{\psi,\sigma} \otimes 1_\sigma"] \arrow[dr, "\alpha_{\psi,\sigma,\sigma}"'] &&&& \sigma(\sigma\psi) \\
& \psi(\sigma\sigma) \arrow[rr, "\beta_{\psi,\sigma\sigma}"] && (\sigma\sigma)\psi \arrow[ur, "\alpha_{\sigma,\sigma,\psi}"'] &
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(1_\sigma \otimes \beta_{\psi,\sigma})(\alpha_{\sigma,\psi,\sigma})(\beta_{\psi,\sigma} \otimes 1_\sigma) = (-i)\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}(-i) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} \\
&= \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = (\alpha_{\sigma,\sigma,\psi})(\beta_{\psi,\mathbb{1}} \oplus \beta_{\psi,\psi})(\alpha_{\psi,\sigma,\sigma}) \\
&= (\alpha_{\sigma,\sigma,\psi})(\beta_{\psi,\mathbb{1}\oplus\psi})(\alpha_{\psi,\sigma,\sigma}) = (\alpha_{\sigma,\sigma,\psi})(\beta_{\psi,\sigma\sigma})(\alpha_{\psi,\sigma,\sigma})
\end{aligned}$$

The left hexagon diagram

(3.6.9)
$$\begin{tikzcd}
& (\psi\sigma)\sigma \arrow[rr, "\alpha_{\psi,\sigma,\sigma}"] && \psi(\sigma\sigma) \arrow[dr, "1_\psi \otimes \beta_{\sigma,\sigma}"] & \\
(\sigma\psi)\sigma \arrow[ur, "\beta_{\sigma,\psi} \otimes 1_\sigma"] \arrow[dr, "\alpha_{\sigma,\psi,\sigma}"'] &&&& \psi(\sigma\sigma) \\
& \sigma(\psi\sigma) \arrow[rr, "\beta_{\sigma,\psi\sigma}"] && (\psi\sigma)\sigma \arrow[ur, "\alpha_{\psi,\sigma,\sigma}"'] &
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(1_\psi \otimes \beta_{\sigma,\sigma})(\alpha_{\psi,\sigma,\sigma})(\beta_{\sigma,\psi} \otimes 1_\sigma) = \begin{bmatrix} w^{-1} & \\ & w^3 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}(w^{-4}) \\
&= \begin{bmatrix} & w^{-5} \\ w^{-1} & \end{bmatrix} = \begin{bmatrix} & -w^3 \\ w^{-1} & \end{bmatrix} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} w^{-1} & \\ & w^3 \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \\
&= (\alpha_{\psi,\sigma,\sigma})(\beta_{\sigma,\sigma})(\alpha_{\sigma,\psi,\sigma}) = (\alpha_{\psi,\sigma,\sigma})(\beta_{\sigma,\psi\sigma})(\alpha_{\sigma,\psi,\sigma})
\end{aligned}$$

We used the equalities

(3.6.10)
$$w^8 = e^{\pi i} = -1 \quad \text{and} \quad w^3 = -w^{-5}.$$

The left hexagon diagram

(3.6.11)
$$\begin{tikzcd}
& (\sigma\sigma)\psi \arrow[rr, "\alpha_{\sigma,\sigma,\psi}"] && \sigma(\sigma\psi) \arrow[dr, "1_\sigma \otimes \beta_{\sigma,\psi}"] & \\
(\sigma\sigma)\psi \arrow[ur, "\beta_{\sigma,\sigma} \otimes 1_\psi"] \arrow[dr, "\alpha_{\sigma,\sigma,\psi}"'] &&&& \sigma(\psi\sigma) \\
& \sigma(\sigma\psi) \arrow[rr, "\beta_{\sigma,\sigma\psi}"] && (\sigma\psi)\sigma \arrow[ur, "\alpha_{\sigma,\psi,\sigma}"'] &
\end{tikzcd}$$

is commutative by the following computation.

$$\begin{aligned}
&(1_\sigma \otimes \beta_{\sigma,\psi})(\alpha_{\sigma,\sigma,\psi})(\beta_{\sigma,\sigma} \otimes 1_\psi) = (w^{-4})\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\begin{bmatrix} w^{-1} & \\ & w^3 \end{bmatrix}\\
&= \begin{bmatrix} & w^{-1} \\ w^{-5} & \end{bmatrix} = \begin{bmatrix} & w^{-1} \\ -w^3 & \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\begin{bmatrix} w^{-1} & \\ & w^3 \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}\\
&= (\alpha_{\sigma,\psi,\sigma})(\beta_{\sigma,\sigma})(\alpha_{\sigma,\sigma,\psi}) = (\alpha_{\sigma,\psi,\sigma})(\beta_{\sigma,\sigma\psi})(\alpha_{\sigma,\sigma,\psi})
\end{aligned}$$

Case 3

Next is the left hexagon diagram with three copies of σ.

(3.6.12)
$$\begin{tikzcd}
& (\sigma\sigma)\sigma \arrow[rr, "\alpha_{\sigma,\sigma,\sigma}"] & & \sigma(\sigma\sigma) \arrow[dr, "1_\sigma \otimes \beta_{\sigma,\sigma}"] & \\
(\sigma\sigma)\sigma \arrow[ur, "\beta_{\sigma,\sigma} \otimes 1_\sigma"] \arrow[dr, "\alpha_{\sigma,\sigma,\sigma}"'] & & & & \sigma(\sigma\sigma) \\
& \sigma(\sigma\sigma) \arrow[rr, "\beta_{\sigma,\sigma\sigma}"] & & (\sigma\sigma)\sigma \arrow[ur, "\alpha_{\sigma,\sigma,\sigma}"'] &
\end{tikzcd}$$

Using the equalities

(3.6.13)
$$w^2 = e^{\pi i/4} = \frac{1+i}{\sqrt{2}} \quad \text{and} \quad w^{-2} = \frac{1-i}{\sqrt{2}} = -w^6,$$

the previous left hexagon diagram is commutative by the following computation.

$$\begin{aligned}
&(1_\sigma \otimes \beta_{\sigma,\sigma})(\alpha_{\sigma,\sigma,\sigma})(\beta_{\sigma,\sigma} \otimes 1_\sigma)\\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} w^{-1} & \\ & w^3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} w^{-1} & \\ & w^3 \end{bmatrix}\\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} w^{-2} & w^2 \\ w^2 & -w^6 \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}\\
&= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & \\ & -i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\\
&= (\alpha_{\sigma,\sigma,\sigma})(\beta_{\sigma,\mathbb{1}} \oplus \beta_{\sigma,\psi})(\alpha_{\sigma,\sigma,\sigma})\\
&= (\alpha_{\sigma,\sigma,\sigma})(\beta_{\sigma,\mathbb{1}\oplus\psi})(\alpha_{\sigma,\sigma,\sigma})\\
&= (\alpha_{\sigma,\sigma,\sigma})(\beta_{\sigma,\sigma\sigma})(\alpha_{\sigma,\sigma,\sigma})
\end{aligned}$$

This finishes the proof of the left hexagon axiom (1.3.17) in $(\mathcal{I}^{\mathsf{any}}, \beta)$.

Each of the eight right hexagon diagrams (1.3.17) involving only σ and ψ is obtained from the left hexagon counterpart by replacing each instance of $\beta_{?,?'}$ by $\beta^{-1}_{?',?}$. In other words, we replace

- $\beta_{\sigma,\sigma} = \begin{bmatrix} w^{-1} & \\ & w^3 \end{bmatrix}$ by $\beta^{-1}_{\sigma,\sigma} = \begin{bmatrix} w & \\ & w^{-3} \end{bmatrix}$;
- $\beta_{\sigma,\psi} = w^{-4} = -i$ by $\beta^{-1}_{\psi,\sigma} = w^4 = i$;
- $\beta_{\psi,\sigma} = w^{-4} = -i$ by $\beta^{-1}_{\sigma,\psi} = w^4 = i$; and
- $\beta_{\psi,\psi} = -1$ by $\beta^{-1}_{\psi,\psi} = -1$.

With these replacements, the computation above, using (3.6.10) and (3.6.13) in the last three cases, shows that each of the eight right hexagon diagrams involving only σ and ψ is commutative. This finishes the proof that $(\mathcal{I}^{\mathsf{any}}, \beta)$ is a braided monoidal category. $\square$

We are now ready for the main result of this section.

Theorem 3.6.14. *The abelian category $\mathcal{I}^{\mathsf{any}}$ in Definition 3.5.1, when equipped with the braided monoidal structure in Lemma 3.6.7, is a tight braided bimonoidal category.*

Proof. We combine the following facts.

- $\mathcal{I}^{\mathsf{any}}$ is an abelian category by definition.
- It is a braided monoidal category by Lemma 3.6.7.
- The functors $x \otimes -$ and $- \otimes x$ are additive functors for each object $x \in \mathcal{I}^{\mathsf{any}}$ by Lemma 3.5.15.

Therefore, the assertion follows from Theorem 2.4.22. □

3.7. Notes

3.7.1 (Braided Bialgebras)**.** The reader is referred to the books [**Abe80, Kas95, Maj95, Mon93, Swe69**] for further discussion and examples of (braided) bialgebras. Braided bialgebras and symmetric bialgebras were introduced by Drinfeld [**Dri87, Dri89**], who called them *quasitriangular bialgebras* and *triangular bialgebras*, respectively. The bialgebra H_4 in Example 3.1.30 is from [**Swe69**]. The R-matrix R_c in (3.1.32) is from [**Rad93**]. The nonstandard R-matrix in (3.1.34) is from [**Maj93**, Prop. 2.1]. ◇

3.7.2 (Modules over Braided Bialgebras)**.** Propositions 3.2.6, 3.2.12, and 3.2.13 on the (braided/symmetric) monoidal structure on $\mathsf{Mod}(A)$ are from quantum group theory; see for example [**Kas95**, Prop. III.5.1 and VIII.3.1]. Theorem 3.2.19 on the (braided/symmetric) bimonoidal structure on $\mathsf{Mod}(A)$ is new, especially as consequences of Theorems 2.4.22 and 2.5.2 and Corollary 2.5.1. ◇

3.7.3 (Topological Quantum Computation)**.** For general surveys, the reader is referred to [**FKLW03, LP17, NSS⁺08, Pac12, PP11, Wan10**]. Computation related to the Fibonacci anyons, such as that in Sections 3.3 and 3.4, can be found in [**BG16, BG20b, TTWL08**]. ◇

3.7.4 (Ising Anyons)**.** In defining the associativity isomorphism α in (3.5.22) and the braiding β in (3.6.2) for the Ising anyons, we follow the sign conventions in [**Wan10**, 1.5.1]. These signs are different from those in [**Pac12**, 4.3] as follows.

- In [**Pac12**], the ψ-component of $\alpha_{\sigma,\psi,\sigma}$ ($= (F^{\psi}_{\sigma\psi\sigma})^{\sigma}_{\sigma}$ there) is not explicitly specified. We define the ψ-component of $\alpha_{\sigma,\psi,\sigma}$ to be -1 in (3.5.24).
- In [**Pac12**], $\beta_{\sigma,\psi}$ ($= R^{\sigma}_{\sigma\psi}$ there) is i. In (3.6.4), we define
$$\beta_{\sigma,\psi} = -i = \beta_{\psi,\sigma}.$$
- In [**Pac12**], $\beta_{\psi,\psi}$ is not explicitly specified. In (3.6.5), we define
$$\beta_{\psi,\psi} = -1.$$

The specific signs that we adopted for α in (3.5.22) and β in (3.6.2) are important for the following reasons.

(1) The pentagon (3.5.28), starting at $[(\sigma\psi)\sigma]\psi$, shows that one must define the ψ-component of $\alpha_{\sigma,\psi,\sigma}$ to be -1.
(2) The hexagon (3.6.8), starting at $(\psi\sigma)\sigma$, shows that one must define $\beta_{\psi,\psi} = -1$.

(3) As opposed to our definition in (3.6.4) with $\beta_{\sigma,\psi} = -i$, if we were to define

$$\beta_{\sigma,\psi} = i = w^4$$

as in [**Pac12**, 4.3], then the following would happen.

(i) The hexagon (3.6.9), starting at $(\sigma\psi)\sigma$, would *not* be commutative. The reason is that the $(2,1)$-entry in the upper path would be w^7, and the $(2,1)$-entry in the lower path is still w^{-1}. However, $w^7 \neq w^{-1}$ by (3.6.10).

(ii) A similar discussion using the $(1,2)$-entry shows that the hexagon (3.6.11), starting at $(\sigma\sigma)\psi$, would *not* be commutative.

(iii) The hexagon (3.6.12), involving three copies of σ, would *not* be commutative. The upper path, involving $\alpha_{\sigma,\sigma,\sigma}$ and $\beta_{\sigma,\sigma}$, remains the same. However, in the lower path, the matrix for $\beta_{\sigma,\sigma\sigma} = \beta_{\sigma,\mathbb{1}} \oplus \beta_{\sigma,\psi}$ would become $\begin{bmatrix} 1 & \\ & i \end{bmatrix}$. So the lower path would not be equal to the upper path.

Therefore, the choice $\beta_{\sigma,\psi} = i$ would not yield a braided monoidal category of Ising anyons. $\diamond$

CHAPTER 4

Bimonoidal Centers

In this chapter, we extend the Drinfeld center and the symmetric center constructions in Section 1.4 to the bimonoidal setting. Section 1.4 contains the following two results about monoidal categories.

- Theorem 1.4.27 states that the Drinfeld center $\overline{\mathsf{C}}$ in Definition 1.4.3 of each monoidal category C is a braided monoidal category.
- Proposition 1.5.3 states that the symmetric center $\mathsf{C}^{\mathsf{sym}}$ in Definition 1.5.1 of each braided monoidal category is a symmetric monoidal category.

To extend these constructions and results to the bimonoidal setting, recall from Definition 2.1.1 that a bimonoidal category is defined in a similar way as a symmetric bimonoidal category, except for the following two conditions:

- There is no multiplicative symmetry $\xi^{\otimes}$, so the multiplicative structure is a monoidal category.
- The axioms (2.1.4) and (2.1.18), which involve $\xi^{\otimes}$, are omitted.

A bimonoidal category is *tight* if the distributivity morphisms δ^l and δ^r are natural isomorphisms.

The main result of this chapter is Theorem 4.4.3. It states that the bimonoidal Drinfeld center of each tight bimonoidal category is a tight braided bimonoidal category. The symmetric analogue of this result is Theorem 4.5.3. It states that the bimonoidal symmetric center of a braided bimonoidal category is a symmetric bimonoidal category. Notice that tightness is not required in Theorem 4.5.3. The following table summaries the center constructions mentioned above.

– category	center	reference
monoidal	braided monoidal	1.4.27
braided monoidal	symmetric monoidal	1.5.3
tight bimonoidal	tight braided bimonoidal	4.4.3
braided bimonoidal	symmetric bimonoidal	4.5.3

For open questions related to the center constructions, see Question III.A.3.2.

Motivation. To motivate the definition of the bimonoidal Drinfeld center, recall from Definition 1.4.3 and Theorem 1.4.27 that the Drinfeld center $\overline{\mathsf{C}}$ of a monoidal category C is a braided monoidal category. The hexagon axioms (1.3.17) are built into the definition of $\overline{\mathsf{C}}$ as follows.

(1) An object in $\overline{\mathsf{C}}$ is a pair $(A;\beta^A)$ consisting of
   - an object $A \in \mathsf{C}$ and
   - a natural isomorphism

$$\beta^A : A \otimes - \xrightarrow{\cong} - \otimes A,$$

which is called the A-braiding,

such that $\beta^A_{B\otimes C}$ satisfies the left hexagon axiom in the form (1.4.4).

(2) The monoidal product $\overline{\otimes}$ of two objects in $\overline{\mathsf{C}}$ involves the $(A \otimes B)$-braiding $\beta^{A\otimes B}$ that is defined by the right hexagon axiom in the form (1.4.8).

In other words, the left hexagon axiom is an axiom of the A-braiding, and the right hexagon axiom is the definition of the $(A \otimes B)$-braiding.

The monoidal category axioms of C imply those of $\overline{\mathsf{C}}$. Since the two hexagon axioms (1.3.17) are built into the definition of the Drinfeld center $\overline{\mathsf{C}}$, checking that it is a braided monoidal category is mostly about checking that its various parts are well defined. These categorical diagram-chasing proofs involve repeated usage of the pentagon axiom (1.3.3), the unity axiom (1.3.2), and the unity properties (I.1.2.7) in C.

In a similar manner, the bimonoidal Drinfeld center of a tight bimonoidal category C is a tight braided bimonoidal category $\overline{\mathsf{C}}^{\mathsf{bi}}$. Among the 14 axioms of a braided bimonoidal category in Definition 2.1.29, only four of them—namely, (2.1.4), (2.1.18), and their variants (2.1.32) and (2.1.33)—involve the braiding $\xi^{\otimes}$. These axioms are incorporated into the definition of the bimonoidal Drinfeld center as follows.

In each object $(A; \beta^A)$, in addition to the compatibility with a product $B \otimes C$, we also assume that the A-braiding β^A is compatible with (i) the sum $B \oplus C$ and (ii) the additive zero $\mathbb{0}$.

- The condition (i) corresponds to the axiom (2.1.32) relating δ^l and δ^r via the braiding and requires the invertibility of δ^r. This is why we need the bimonoidal category C to be tight.
- The condition (ii) corresponds to the axiom (2.1.18) relating $\rho^{\bullet}$ and $\lambda^{\bullet}$ via the braiding.

The axioms

- (2.1.4) relating δ^r and δ^l via the braiding and
- (2.1.33) relating $\lambda^{\bullet}$ and $\rho^{\bullet}$ via the braiding

are built into the definitions of, respectively, the $(A \oplus B)$-braiding $\beta^{A\oplus B}$ in (4.1.8) and the $\mathbb{0}$-braiding $\beta^{\mathbb{0}}$ in (4.1.11).

With the four axioms (2.1.4), (2.1.18), (2.1.32), and (2.1.33) built into the definition of the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathsf{bi}}$, the other 10 Laplaza axioms in Definition 2.1.29 follow from the corresponding properties in C. Therefore, showing that $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a tight braided bimonoidal category is mostly about showing that its various parts are well defined. These proofs in Sections 4.2 through 4.4 use all 22 Laplaza axioms in the tight bimonoidal category C.

Interpretation. There are several ways to interpret Theorems 4.4.3 and 4.5.3 conceptually.

(i) They provide evidence that the axioms of a braided bimonoidal category in Definition 2.1.29 are well chosen, in the sense that both the Drinfeld center and the symmetric center extend naturally to the bimonoidal setting.

(ii) They provide further illustration of Laplaza's axioms in Definition 2.1.1. In the course of defining the bimonoidal Drinfeld center and verifying that it is well defined, we will use all 24 Laplaza axioms, the variant (2.1.32) of (2.1.4), and the variant (2.1.33) of (2.1.18).

(iii) Theorems 4.4.3 and 4.5.3 provide further justification for distinguishing between
- braided bimonoidal categories, with δ^l and δ^r natural monomorphisms, and
- tight braided bimonoidal categories, with δ^l and δ^r invertible.

In the definition of the bimonoidal Drinfeld center, the invertibility of the distributivity morphisms are used in (4.1.3) and (4.1.8). This is why tightness is required in Theorem 4.4.3. On the other hand, the proof of Theorem 4.5.3 involving the bimonoidal symmetric center uses the monomorphism assumption on δ^r, but not the invertibility of δ^l and δ^r. Therefore, this result does not require tightness.

Organization. The bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathsf{bi}}$ of a tight bimonoidal category C is defined in Section 4.1. Section 4.2 proves that the additive structure of $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a symmetric monoidal category. Section 4.3 proves that the multiplicative structure of $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a braided monoidal category. Section 4.4 proves that the multiplicative zeros and the distributivity morphisms are natural isomorphisms in $\overline{\mathsf{C}}^{\mathsf{bi}}$ and the main Theorem 4.4.3. Section 4.5 proves Theorem 4.5.3, which states that the bimonoidal symmetric center of a braided bimonoidal category is a symmetric bimonoidal category. Convention 2.1.34 is still in effect, so $\otimes$ is sometimes abbreviated to concatenation.

Reading Guide.

(1) Read Definition 4.1.2 and the statement of Theorem 4.4.3 for the bimonoidal Drinfeld center.
(2) Read Definition 4.5.1 and the statement of Theorem 4.5.3 for the bimonoidal symmetric center.
(3) Go back and read the rest of this chapter.

4.1. The Bimonoidal Drinfeld Center: Definition

In this section, we define the bimonoidal Drinfeld center, followed by some commentary.

Motivation 4.1.1. Recall from Definition 1.4.3 that, for a monoidal category $(\mathsf{C}, \otimes)$, an object in the Drinfeld center $\overline{\mathsf{C}}$ of C is a pair $(A; \beta^A)$ with $A \in \mathsf{C}$ an object and

$$\beta^A : A \otimes (-) \longrightarrow (-) \otimes A$$

a natural isomorphism that satisfies a version of the left hexagon axiom. For the Drinfeld center of a bimonoidal category C, which we will define shortly, we also need to make sure that β^A is compatible with the monoidal structure $(\oplus, \mathbb{0})$ in the sense of (4.1.3) and (4.1.4) below. $\diamond$

Definition 4.1.2. Suppose C is a tight bimonoidal category as in Definition 2.1.1. The *bimonoidal Drinfeld center* of C consists of the data of a braided bimonoidal category

$$\big(\overline{\mathsf{C}}^{\mathsf{bi}}, (\overline{\oplus}, \overline{\mathbb{0}}, \alpha^{\overline{\oplus}}, \lambda^{\overline{\oplus}}, \rho^{\overline{\oplus}}, \xi^{\overline{\oplus}}), (\overline{\otimes}, \mathbb{1}, \alpha^{\overline{\otimes}}, \lambda^{\overline{\otimes}}, \rho^{\overline{\otimes}}, \xi^{\overline{\otimes}}), (\overline{\lambda}^{\bullet}, \overline{\rho}^{\bullet}), (\overline{\delta}^{l}, \overline{\delta}^{r})\big)$$

defined as follows.

Objects: An object in $\overline{\mathsf{C}}^{\mathrm{bi}}$ is a pair $(A;\beta^A)$ consisting of
- an object $A \in \mathsf{C}$ and
- a natural isomorphism

$$A \otimes B \xrightarrow[\cong]{\beta^A_B} B \otimes A \quad \text{for} \quad B \in \mathsf{C}$$

such that (1.4.4) and the following two diagrams are commutative for objects $B, C \in \mathsf{C}$.

(4.1.3)
$$\begin{tikzcd}
A \otimes (B \oplus C) \arrow[rr, "\beta^A_{B\oplus C}"] \arrow[d, "\delta^l_{A,B,C}"'] & & (B \oplus C) \otimes A \\
(A \otimes B) \oplus (A \otimes C) \arrow[rr, "\beta^A_B \oplus \beta^A_C"] & & (B \otimes A) \oplus (C \otimes A) \arrow[u, "\delta^{-r}_{B,C,A}"']
\end{tikzcd}$$

(4.1.4)
$$\begin{tikzcd}
A \otimes \mathbb{0} \arrow[rr, "\beta^A_{\mathbb{0}}"] \arrow[dr, "\rho^\bullet_A"'] & & \mathbb{0} \otimes A \\
& \mathbb{0} \arrow[ur, "\lambda^{-\bullet}_A"'] &
\end{tikzcd}$$

We call A the *underlying object* and β^A the *A-braiding*.

Morphisms: A morphism

$$f : (A;\beta^A) \longrightarrow (B;\beta^B)$$

in $\overline{\mathsf{C}}^{\mathrm{bi}}$ is a morphism $f : A \longrightarrow B$ in C such that (1.4.5) is commutative for each object $C \in \mathsf{C}$.

Identity Morphisms: The identity morphism of an object $(A;\beta^A) \in \overline{\mathsf{C}}^{\mathrm{bi}}$ is the identity morphism $1_A : A \longrightarrow A$ in C.

Composition: The composition of morphisms in $\overline{\mathsf{C}}^{\mathrm{bi}}$ is the composition of morphisms in C.

The Multiplicative Structure: The data

$$(\overline{\otimes}, \overline{\mathbb{1}}, \alpha^{\overline{\otimes}}, \lambda^{\overline{\otimes}}, \rho^{\overline{\otimes}}, \xi^{\overline{\otimes}}) \tag{4.1.5}$$

are defined as in Definition 1.4.3, with $\overline{\alpha}$ now denoted by $\alpha^{\overline{\otimes}}$ and similarly for $\overline{\lambda}$, $\overline{\rho}$, and $\overline{\xi}$.

The Additive Monoidal Product: For the rest of this definition, $(A;\beta^A)$, $(B;\beta^B)$, and $(C;\beta^C)$ are arbitrary objects in $\overline{\mathsf{C}}^{\mathrm{bi}}$. The functor

$$\overline{\mathsf{C}}^{\mathrm{bi}} \times \overline{\mathsf{C}}^{\mathrm{bi}} \xrightarrow{\overline{\oplus}} \overline{\mathsf{C}}^{\mathrm{bi}} \tag{4.1.6}$$

is defined as follows.

Objects: Define the object

$$(A;\beta^A) \,\overline{\oplus}\, (B;\beta^B) = (A \oplus B; \beta^{A\oplus B}) \tag{4.1.7}$$

with $\beta^{A\oplus B}$ defined by the following diagram for objects $C \in \mathsf{C}$.

$$\begin{tikzcd}
(A \oplus B) \otimes C \arrow[r, "\beta^{A\oplus B}_C"] \arrow[d, "\delta^r_{A,B,C}"'] & C \otimes (A \oplus B) \\
(A \otimes C) \oplus (B \otimes C) \arrow[r, "\beta^A_C \oplus \beta^B_C"] & (C \otimes A) \oplus (C \otimes B) \arrow[u, "\delta^{-l}_{C,A,B}"']
\end{tikzcd} \tag{4.1.8}$$

Morphisms: Define the morphism

$$(A;\beta^A) \mathbin{\overline{\oplus}} (B;\beta^B) \xrightarrow{f \mathbin{\overline{\oplus}} g} (A';\beta^{A'}) \mathbin{\overline{\oplus}} (B';\beta^{B'}) \tag{4.1.9}$$

as

$$A \oplus B \xrightarrow{f \oplus g} A' \oplus B'$$

in C for the following morphisms in $\overline{\mathsf{C}}^{\mathrm{bi}}$.

$$(A;\beta^A) \xrightarrow{f} (A';\beta^{A'})$$
$$(B;\beta^B) \xrightarrow{g} (B';\beta^{B'})$$

The Additive Unit: Define the object

$$\overline{\mathbb{0}} = (\mathbb{0};\beta^{\mathbb{0}}) \in \overline{\mathsf{C}}^{\mathrm{bi}} \tag{4.1.10}$$

with $\beta^{\mathbb{0}}$ defined as the following composite for objects $A \in \mathsf{C}$.

$$\begin{tikzcd}
\mathbb{0} \otimes A \arrow[rr, "\beta^{\mathbb{0}}_A"] \arrow[dr, "\lambda^{\bullet}_A"'] & & A \otimes \mathbb{0} \\
& \mathbb{0} \arrow[ur, "\rho^{-\bullet}_A"'] &
\end{tikzcd} \tag{4.1.11}$$

The Additive Associativity: Define the morphism

$$\begin{tikzcd}
\big[(A;\beta^A) \mathbin{\overline{\oplus}} (B;\beta^B)\big] \mathbin{\overline{\oplus}} (C;\beta^C) \arrow[d, "\alpha^{\overline{\oplus}}_{(A;\beta^A),(B;\beta^B),(C;\beta^C)}"] \\
(A;\beta^A) \mathbin{\overline{\oplus}} \big[(B;\beta^B) \mathbin{\overline{\oplus}} (C;\beta^C)\big]
\end{tikzcd} \tag{4.1.12}$$

in $\overline{\mathsf{C}}^{\mathrm{bi}}$ as

$$(A \oplus B) \oplus C \xrightarrow{\alpha^{\oplus}_{A,B,C}} A \oplus (B \oplus C) \in \mathsf{C}.$$

The Additive Zeros: Define the morphisms

$$\overline{\mathbb{0}} \mathbin{\overline{\oplus}} (A;\beta^A) \xrightarrow{\lambda^{\overline{\oplus}}_{(A;\beta^A)}} (A;\beta^A) \xleftarrow{\rho^{\overline{\oplus}}_{(A;\beta^A)}} (A;\beta^A) \mathbin{\overline{\oplus}} \overline{\mathbb{0}} \tag{4.1.13}$$

in $\overline{\mathsf{C}}^{\mathrm{bi}}$ as, respectively,

$$\mathbb{0} \oplus A \xrightarrow{\lambda^{\oplus}_A} A \xleftarrow{\rho^{\oplus}_A} A \oplus \mathbb{0} \in \mathsf{C}.$$

The Additive Symmetry: Define the morphism

$$(A;\beta^A)\,\overline{\oplus}\,(B;\beta^B) \xrightarrow{\xi^{\overline{\oplus}}_{(A;\beta^A),(B;\beta^B)}} (B;\beta^B)\,\overline{\oplus}\,(A;\beta^A) \tag{4.1.14}$$

in $\overline{\mathsf{C}}^{\mathrm{bi}}$ as

$$\xi^{\oplus}_{A,B} : A \oplus B \longrightarrow B \oplus A \in \mathsf{C}.$$

The Multiplicative Zeros: Define the morphisms

$$\overline{\mathbb{0}}\,\overline{\otimes}\,(A;\beta^A) \xrightarrow{\overline{\lambda}^{\bullet}_{(A;\beta^A)}} \overline{\mathbb{0}} \xleftarrow{\overline{\rho}^{\bullet}_{(A;\beta^A)}} (A;\beta^A)\,\overline{\otimes}\,\overline{\mathbb{0}} \tag{4.1.15}$$

in $\overline{\mathsf{C}}^{\mathrm{bi}}$ as, respectively,

$$\mathbb{0} \otimes A \xrightarrow{\lambda^{\bullet}_A} \mathbb{0} \xleftarrow{\rho^{\bullet}_A} A \otimes \mathbb{0} \in \mathsf{C}.$$

The Left Distributivity: Define the morphism

$$\begin{gathered}(A;\beta^A)\,\overline{\otimes}\,\big[(B;\beta^B)\,\overline{\oplus}\,(C;\beta^C)\big]\\ \big\downarrow \overline{\delta}^{l}_{(A;\beta^A),(B,\beta^B),(C,\beta^C)}\\ \big[(A;\beta^A)\,\overline{\otimes}\,(B;\beta^B)\big]\,\overline{\oplus}\,\big[(A;\beta^A)\,\overline{\otimes}\,(C;\beta^C)\big]\end{gathered} \tag{4.1.16}$$

in $\overline{\mathsf{C}}^{\mathrm{bi}}$ as

$$A \otimes (B \oplus C) \xrightarrow{\delta^{l}_{A,B,C}} (A \otimes B) \oplus (A \otimes C) \in \mathsf{C}.$$

The Right Distributivity: Define the morphism

$$\begin{gathered}\big[(A;\beta^A)\,\overline{\oplus}\,(B;\beta^B)\big]\,\overline{\otimes}\,(C;\beta^C)\\ \big\downarrow \overline{\delta}^{r}_{(A;\beta^A),(B,\beta^B),(C,\beta^C)}\\ \big[(A;\beta^A)\,\overline{\otimes}\,(C;\beta^C)\big]\,\overline{\oplus}\,\big[(B;\beta^B)\,\overline{\otimes}\,(C;\beta^C)\big]\end{gathered} \tag{4.1.17}$$

in $\overline{\mathsf{C}}^{\mathrm{bi}}$ as

$$(A \oplus B) \otimes C \xrightarrow{\delta^{r}_{A,B,C}} (A \otimes C) \oplus (B \otimes C) \in \mathsf{C}.$$

This finishes the definition of the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathrm{bi}}$. ◇

Explanation 4.1.18. Consider Definition 4.1.2.

(1) $\overline{\mathsf{C}}^{\mathrm{bi}}$ is a category as in Lemma 1.4.17.
(2) Consider the multiplicative structure (4.1.5).
- $\overline{\otimes}$ and $\overline{\mathbb{1}} = (\mathbb{1};\beta^{\mathbb{1}})$ are as in (1.4.6) and (1.4.10), respectively. In the definitions (1.4.8) of $\beta^{A\otimes B}_C$ and (1.4.11) of $\beta^{\mathbb{1}}_A$, the morphisms α, λ, and ρ mean, respectively, $\alpha^{\otimes}$, $\lambda^{\otimes}$, and $\rho^{\otimes}$.
- $\alpha^{\overline{\otimes}}$, $\lambda^{\overline{\otimes}}$, and $\rho^{\overline{\otimes}}$ are defined as, respectively, $\alpha^{\otimes}$, $\lambda^{\otimes}$, and $\rho^{\otimes}$ in C using the underlying objects.
- The braiding $\xi^{\overline{\otimes}}_{(A;\beta^A),-}$ is the A-braiding β^A as in (1.4.14).

However, Theorem 1.4.27 is *not* enough to conclude that the multiplicative structure (4.1.5) is a braided monoidal category because of the axioms (4.1.3) and (4.1.4) for an object in $\overline{\mathsf{C}}^{\mathsf{bi}}$. These axioms are not in Definition 1.4.3 of the Drinfeld center of a monoidal category. Therefore, we still need to check that $\beta^{\mathbb{1}}$ and $\beta^{A\otimes B}$, for objects $(A;\beta^A)$ and $(B;\beta^B)$ in $\overline{\mathsf{C}}^{\mathsf{bi}}$, satisfy the axioms (4.1.3) and (4.1.4).

(3) The bimonoidal category C is assumed to be tight for two reasons.
  - The axiom (4.1.3) of $\beta^A_{B\oplus C}$ uses the invertibility of δ^r.
  - The definition (4.1.8) of $\beta^{A\oplus B}_C$ uses the invertibility of δ^l.

(4) In an object $(A;\beta^A) \in \overline{\mathsf{C}}^{\mathsf{bi}}$, the three axioms (1.4.4), (4.1.3), and (4.1.4) determine the A-braiding of, respectively, a product $B\otimes C$, a sum $B\oplus C$, and the additive zero $\mathbb{0}$. The A-braiding of $\mathbb{1}$ is equal to $\lambda^{-\otimes}_A\rho^{\otimes}_A$, as shown in (1.4.18).

(5) The definition (4.1.8) of $\beta^{A\oplus B}_C$ and the axiom (4.1.3) of $\beta^A_{B\oplus C}$ are modeled after, respectively, the Laplaza axiom (2.1.4) and its variant (2.1.32).

(6) The axiom (4.1.4) of $\beta^A_{\mathbb{0}}$ and the definition (4.1.11) of $\beta^{\mathbb{0}}_A$ are modeled after, respectively, the Laplaza axiom (2.1.18) and its variant (2.1.33).

In the sense of the last two points, the only two Laplaza axioms involving $\xi^{\otimes}$, namely, (2.1.4) and (2.1.18), and their variants (2.1.32) and (2.1.33), are already incorporated into the definition of the bimonoidal Drinfeld center. The other 22 Laplaza axioms in Definition 2.1.1 will appear in the proofs in the next few sections. $\diamond$

4.2. The Additive Structure

In this section, we check that the additive structure of the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a symmetric monoidal category. To check that the additive structure

$$\left(\overline{\mathsf{C}}^{\mathsf{bi}}, \overline{\oplus}, \overline{\mathbb{0}}, \alpha^{\overline{\oplus}}, \lambda^{\overline{\oplus}}, \rho^{\overline{\oplus}}, \xi^{\overline{\oplus}}\right)$$

is a symmetric monoidal category, first we check that $(A\oplus B;\beta^{A\oplus B})$ in (4.1.7) is a well-defined object in $\overline{\mathsf{C}}^{\mathsf{bi}}$ for any two objects $(A;\beta^A),(B;\beta^B)\in\overline{\mathsf{C}}^{\mathsf{bi}}$.

Lemma 4.2.1. *$\beta^{A\oplus B}$ in (4.1.8) satisfies the axiom (1.4.4).*

Proof. For objects $C, D \in \mathsf{C}$, the diagram (1.4.4) for $\beta^{A\oplus B}$ is the outer diagram below.

$$\begin{tikzcd}[column sep=small]
(A\oplus B)(CD) \arrow[rrrr, bend left, "\beta^{A\oplus B}_{CD}"] \arrow[dr, "\delta^r"] \arrow[dd, "\alpha^{-\otimes}"'] & & (4.1.8) & & (CD)(A\oplus B) \\
& A(CD)\oplus B(CD) \arrow[rr, "\beta^A_{CD}\oplus\beta^B_{CD}"] \arrow[dd, "\alpha^{-\otimes}\oplus\alpha^{-\otimes}"] & & (CD)A\oplus(CD)B \arrow[ur, "\delta^{-l}"] \arrow[dr, phantom, "(2.1.9)"] & \\
{[(A\oplus B)C]D} \arrow[ur, phantom, "(2.1.10)"] \arrow[dd, "\delta^r 1"'] \arrow[dddddr, bend right, "\beta^{A\oplus B}_C 1"'] & & & & C[D(A\oplus B)] \arrow[uu, "\alpha^{-\otimes}"'] \\
(4.1.8) & (AC)D\oplus(BC)D \arrow[dd, "\beta^A_C 1\oplus\beta^B_C 1"] & (1.4.4) & C(DA)\oplus C(DB) \arrow[uu, "\alpha^{-\otimes}\oplus\alpha^{-\otimes}"'] \arrow[dr, "\delta^{-l}"] & (4.1.8) \\
{[AC\oplus BC]D} \arrow[ur, "\delta^r"] \arrow[dd, "(\beta^A_C\oplus\beta^B_C)1"'] \arrow[dr, phantom, "\mathsf{nat}"] & & & & C[DA\oplus DB] \arrow[uu, "1\delta^{-l}"'] \arrow[dl, phantom, "\mathsf{nat}"] \\
& (CA)D\oplus(CB)D \arrow[rr, "\alpha^{\otimes}\oplus\alpha^{\otimes}"] & & C(AD)\oplus C(BD) \arrow[uu, "1\beta^A_D\oplus 1\beta^B_D"] \arrow[dr, "\delta^{-l}"] & \\
{[CA\oplus CB]D} \arrow[ur, "\delta^r"] \arrow[dr, "\delta^{-l}1"] & & (2.1.11) & & C[AD\oplus BD] \arrow[uu, "1(\beta^A_D\oplus\beta^B_D)"'] \\
& {[C(A\oplus B)]D} \arrow[rr, "\alpha^{\otimes}"] & & C[(A\oplus B)D] \arrow[ur, "1\delta^r"] \arrow[uuuuur, bend right, "1\beta^{A\oplus B}_D"'] &
\end{tikzcd}$$

- The top, the left, and the right subdiagrams along the boundary are commutative by the definitions (4.1.8) of, respectively, $\beta^{A\oplus B}_{C\otimes D}$, $\beta^{A\oplus B}_C$, and $\beta^{A\oplus B}_D$.
- The middle hexagon is commutative by the axiom (1.4.4) for β^A and β^B.
- The two subdiagrams labeled by nat are commutative by the naturality of δ^l and δ^r.
- Three subdiagrams are commutative by the axioms (2.1.9), (2.1.10), and (2.1.11) in C.

Therefore, $\beta^{A\oplus B}$ satisfies the axiom (1.4.4). $\square$

Lemma 4.2.2. *$\beta^{A\oplus B}$ in (4.1.8) satisfies the axiom (4.1.3).*

Proof. For objects $C, D \in \mathsf{C}$, the diagram (4.1.3) for $\beta^{A\oplus B}$ is the outer diagram below.

$$
\begin{tikzcd}[column sep=small]
(A\oplus B)(C\oplus D) \arrow[rrr, "\beta^{A\oplus B}_{C\oplus D}"] \arrow[dr, "\delta^r"] \arrow[dddd, "\delta^l"'] & & & (C\oplus D)(A\oplus B) \\
 & A(C\oplus D)\oplus B(C\oplus D) \arrow[r, "\beta^A_{C\oplus D}\oplus\beta^B_{C\oplus D}"] \arrow[d, "\delta^l\oplus\delta^l"] \arrow[u, phantom, "(4.1.8)"] & (C\oplus D)A\oplus(C\oplus D)B \arrow[ur, "\delta^{-l}"] \arrow[d, "\delta^r\oplus\delta^r"'] & \\
 & (AC\oplus AD)\oplus(BC\oplus BD) \arrow[r, "(\beta^A_C\oplus\beta^A_D)\oplus(\beta^B_C\oplus\beta^B_D)"] \arrow[d, "\alpha^\oplus"] \arrow[u, phantom, "(4.1.3)"] \arrow[dl, phantom, "(2.1.12)"] & (CA\oplus DA)\oplus(CB\oplus DB) \arrow[d, "\alpha^\oplus"'] \arrow[dr, phantom, "\substack{(2.1.12)\\(1.3.33)}"] & \\
 & AC\oplus[AD\oplus(BC\oplus BD)] \arrow[r, "\beta^A_C\oplus[\beta^A_D\oplus(\beta^B_C\oplus\beta^B_D)]"] \arrow[dd, "1\oplus\alpha^{-\oplus}"] & CA\oplus[DA\oplus(CB\oplus DB)] \arrow[dd, "1\oplus\alpha^{-\oplus}"'] & \\
(A\oplus B)C\oplus(A\oplus B)D \arrow[dddd, "\delta^r\oplus\delta^r"'] \arrow[rrr, bend right=90, looseness=1.5, "\beta^{A\oplus B}_C\oplus\beta^{A\oplus B}_D"'] & & & C(A\oplus B)\oplus D(A\oplus B) \arrow[uuuu, "\delta^{-r}"'] \\
 & AC\oplus[(AD\oplus BC)\oplus BD] \arrow[r, "\beta^A_C\oplus[(\beta^A_D\oplus\beta^B_C)\oplus\beta^B_D]"] \arrow[d, "1\oplus(\xi^\oplus\oplus 1)"] & CA\oplus[(DA\oplus CB)\oplus DB] \arrow[d, "1\oplus(\xi^\oplus\oplus 1)"'] \arrow[dl, phantom, "\mathsf{nat}"] & \\
 & AC\oplus[(BC\oplus AD)\oplus BD] \arrow[r, "\beta^A_C\oplus[(\beta^B_C\oplus\beta^A_D)\oplus\beta^B_D]"] \arrow[d, "1\oplus\alpha^\oplus"] & CA\oplus[(CB\oplus DA)\oplus DB] \arrow[d, "1\oplus\alpha^\oplus"'] & \\
 & AC\oplus[BC\oplus(AD\oplus BD)] \arrow[r, "\beta^A_C\oplus[\beta^B_C\oplus(\beta^A_D\oplus\beta^B_D)]"] & CA\oplus[CB\oplus(DA\oplus DB)] \arrow[dr, "\alpha^{-\oplus}"] & \\
(AC\oplus BC)\oplus(AD\oplus BD) \arrow[ur, "\alpha^\oplus"'] \arrow[rrr, "(\beta^A_C\oplus\beta^B_C)\oplus(\beta^A_D\oplus\beta^B_D)", "(4.1.8)"'] & & & (CA\oplus CB)\oplus(DA\oplus DB) \arrow[uuuu, "\delta^{-l}\oplus\delta^{-l}"']
\end{tikzcd}
$$

- The top and the bottom subdiagrams along the boundary are commutative by the definitions (4.1.8) of $\beta^{A\oplus B}_{C\oplus D}$, $\beta^{A\oplus B}_C$, and $\beta^{A\oplus B}_D$.
- The left vertical trapezoid is commutative by the axiom (2.1.12) in C.
- The right vertical trapezoid is commutative by the axiom (2.1.12) and the symmetry axiom (1.3.33) for the additive symmetry $\xi^\oplus$.
- In the middle column, the following statements hold.
  - The top rectangle is commutative by the axiom (4.1.3) for $\beta^A_{C\oplus D}$ and $\beta^B_{C\oplus D}$.
  - The rectangle labeled by nat is commutative by the naturality of $\xi^\oplus$.
  - The other four unlabeled subdiagrams are commutative by the naturality of $\alpha^\oplus$.

Therefore, $\beta^{A\oplus B}$ satisfies the axiom (4.1.3). □

Lemma 4.2.3. *$(A\oplus B;\beta^{A\oplus B})$ in (4.1.7) is an object in $\overline{\mathsf{C}}^{\mathsf{bi}}$.*

Proof. $\beta^{A\oplus B}$ in (4.1.8) is a natural isomorphism because β^A, δ^l, and δ^r are natural isomorphisms. By Lemmas 4.2.1 and 4.2.2, respectively, $\beta^{A\oplus B}$ satisfies the axioms (1.4.4) and (4.1.3). It remains to check the axiom (4.1.4) for $\beta^{A\oplus B}$, which is the outer

diagram below.

$$\begin{tikzcd}[column sep=small]
(A \oplus B)\mathbb{0} \arrow[rrrr, "\beta_{\mathbb{0}}^{A\oplus B}"] \arrow[dr, "\delta^r"] \arrow[ddrr, bend right=40, "\rho^{\bullet}_{A\oplus B}"'] \arrow[drr, phantom, "(4.1.8)"] & & & & \mathbb{0}(A \oplus B) \\
(2.1.15) & A\mathbb{0} \oplus B\mathbb{0} \arrow[rr, "\beta_{\mathbb{0}}^{A} \oplus \beta_{\mathbb{0}}^{B}"] \arrow[dr, "\rho^{\bullet}_A \oplus \rho^{\bullet}_B"'] \arrow[rr, phantom, shift right=3, "(4.1.4)"] & & \mathbb{0}A \oplus \mathbb{0}B \arrow[ur, "\delta^{-l}"] & (2.1.14) \\
& & \mathbb{0} \oplus \mathbb{0} \arrow[ur, "\lambda^{-\bullet}_A \oplus \lambda^{-\bullet}_B"'] \arrow[d, "\lambda^{\oplus}_{\mathbb{0}}"] & & \\
& & \mathbb{0} \arrow[uuurr, bend right=40, "\lambda^{-\bullet}_{A\oplus B}"'] & &
\end{tikzcd}$$

- The top trapezoid is commutative by the definition (4.1.8) of $\beta_{\mathbb{0}}^{A\oplus B}$.
- The middle triangle is commutative by the axiom (4.1.4) for $\beta_{\mathbb{0}}^{A}$ and $\beta_{\mathbb{0}}^{B}$.
- The other two subdiagrams are commutative by the axioms (2.1.14) and (2.1.15) in C.

Therefore, $\beta^{A\oplus B}$ satisfies the axiom (4.1.4). $\square$

Lemma 4.2.4. *In (4.1.6),*

$$- \,\overline{\oplus}\, - : \overline{\mathsf{C}}^{\mathsf{bi}} \times \overline{\mathsf{C}}^{\mathsf{bi}} \longrightarrow \overline{\mathsf{C}}^{\mathsf{bi}}$$

is a functor.

Proof. Lemma 4.2.3 shows that $\overline{\oplus}$ is well defined on objects. To show that it is well defined on morphisms, we must show that $f \,\overline{\oplus}\, g$ in (4.1.9) is a morphism in $\overline{\mathsf{C}}^{\mathsf{bi}}$ for morphisms $f, g \in \overline{\mathsf{C}}^{\mathsf{bi}}$. In other words, we must check that $f \oplus g$ satisfies the axiom (1.4.5) for each object $C \in \mathsf{C}$, which is the outer diagram below.

$$\begin{tikzcd}
(A \oplus B)C \arrow[r, "(f\oplus g)1"] \arrow[d, "\delta^r"] \arrow[ddd, bend right=60, "\beta_C^{A\oplus B}"'] \arrow[dr, phantom, "\mathsf{nat}"] & (A' \oplus B')C \arrow[d, "\delta^r"'] \arrow[ddd, bend left=60, "\beta_C^{A'\oplus B'}"] \\
AC \oplus BC \arrow[r, "f1 \oplus g1"] \arrow[d, "\beta_C^{A} \oplus \beta_C^{B}"] \arrow[dr, phantom, "(1.4.5)"] & A'C \oplus B'C \arrow[d, "\beta_C^{A'} \oplus \beta_C^{B'}"'] \\
CA \oplus CB \arrow[r, "1f \oplus 1g"] \arrow[d, "\delta^{-l}"] \arrow[dr, phantom, "\mathsf{nat}"] & CA' \oplus CB' \arrow[d, "\delta^{-l}"'] \\
C(A \oplus B) \arrow[r, "1(f\oplus g)"] & C(A' \oplus B')
\end{tikzcd}$$

$(4.1.8)$ $(4.1.8)$

- The left and the right rectangles are commutative by (4.1.8).
- In the middle column, the following statements hold.
  - The top and the bottom rectangles are commutative by the naturality of, respectively, δ^r and δ^l.
  - The middle rectangle is commutative by the axiom (1.4.5) for f and g.

Therefore, $f \,\overline{\oplus}\, g$ is a well-defined morphism in $\overline{\mathsf{C}}^{\mathsf{bi}}$.

The fact that $\overline{\oplus}$ preserves identity morphisms and composition follows from the corresponding properties of $\oplus$ in C. $\square$

Lemma 4.2.5. $\overline{\mathbb{0}} = (\mathbb{0}; \beta^{\mathbb{0}})$ *in (4.1.10) is an object in* $\overline{\mathsf{C}}^{\mathsf{bi}}$.

Proof. By definition (4.1.11), $\beta^{\mathbb{0}} = \rho^{-\bullet}\lambda^{\bullet}$ is a natural isomorphism because $\lambda^{\bullet}$ and $\rho^{\bullet}$ are natural isomorphisms. It remains to check the axioms (1.4.4), (4.1.3), and (4.1.4) for $\beta^{\mathbb{0}}$. In the commutative diagrams below, $B, C \in \mathsf{C}$ are objects, and unlabeled regions are commutative by the definition $\beta^{\mathbb{0}} = \rho^{-\bullet}\lambda^{\bullet}$.

The following commutative diagram proves the axiom (1.4.4) for $\beta^{\mathbb{0}}$.

$$\begin{tikzcd}
 & \mathbb{0}(BC) \arrow[rrrrrr, "\beta^{\mathbb{0}}_{BC}"] \arrow[dl, "\alpha^{-\otimes}"'] \arrow[drrr, "\lambda^{\bullet}"] & & & & & & (BC)\mathbb{0} & \\
(\mathbb{0}B)C \arrow[drr, "\lambda^{\bullet}1"] \arrow[ddr, "\beta^{\mathbb{0}}_B 1"'] & & (2.1.21) & & \mathbb{0} \arrow[urrr, "\rho^{-\bullet}"] \arrow[drr, "\rho^{-\bullet}"] & & (2.1.19) & & B(C\mathbb{0}) \arrow[ul, "\alpha^{-\otimes}"'] \\
 & & \mathbb{0}C \arrow[urr, "\lambda^{\bullet}"] \arrow[dl, "\rho^{-\bullet}1"] & & (2.1.20) & & B\mathbb{0} \arrow[urr, "1\rho^{-\bullet}"] & & \\
 & (B\mathbb{0})C \arrow[rrrrrr, "\alpha^{\otimes}"] & & & & & & B(\mathbb{0}C) \arrow[ul, "1\lambda^{\bullet}"] \arrow[ur, "1\beta^{\mathbb{0}}_C"'] &
\end{tikzcd}$$

The following commutative diagram proves the axiom (4.1.3) for $\beta^{\mathbb{0}}$.

$$\begin{tikzcd}
\mathbb{0}(B \oplus C) \arrow[r, "\lambda^{\bullet}"] \arrow[d, "\delta^{l}"'] \arrow[rr, bend left, "\beta^{\mathbb{0}}_{B\oplus C}"] \arrow[dr, phantom, "(2.1.14)"] & \mathbb{0} \arrow[r, "\rho^{-\bullet}"] \arrow[d, "\lambda^{-\oplus}"'] \arrow[dr, phantom, "(2.1.15)"] & (B \oplus C)\mathbb{0} \\
\mathbb{0}B \oplus \mathbb{0}C \arrow[r, "\lambda^{\bullet} \oplus \lambda^{\bullet}"] \arrow[rr, bend right, "\beta^{\mathbb{0}}_B \oplus \beta^{\mathbb{0}}_C"'] & \mathbb{0} \oplus \mathbb{0} \arrow[r, "\rho^{-\bullet} \oplus \rho^{-\bullet}"] & B\mathbb{0} \oplus C\mathbb{0} \arrow[u, "\delta^{-r}"']
\end{tikzcd}$$

The axiom (4.1.4) follows from the definition (4.1.11) and the axiom (2.1.13):

$$\beta^{\mathbb{0}}_{\mathbb{0}} = \rho^{-\bullet}_{\mathbb{0}}\lambda^{\bullet}_{\mathbb{0}} = \lambda^{-\bullet}_{\mathbb{0}}\rho^{\bullet}_{\mathbb{0}}.$$

Therefore, $\overline{\mathbb{0}} = (\mathbb{0}; \beta^{\mathbb{0}})$ is a well-defined object in $\overline{\mathsf{C}}^{\mathrm{bi}}$. □

Lemma 4.2.6. *The additive structure in Definition 4.1.2*

$$\left(\overline{\mathsf{C}}^{\mathrm{bi}}, \overline{\oplus}, \overline{\mathbb{0}}, \alpha^{\overline{\oplus}}, \lambda^{\overline{\oplus}}, \rho^{\overline{\oplus}}, \xi^{\overline{\oplus}}\right)$$

is a symmetric monoidal category.

Proof. By Lemmas 4.2.4 and 4.2.5, $\overline{\oplus}$ is a functor, and $\overline{\mathbb{0}}$ is a well-defined object in $\overline{\mathsf{C}}^{\mathrm{bi}}$. Once we check that $\alpha^{\overline{\oplus}}$, $\lambda^{\overline{\oplus}}$, $\rho^{\overline{\oplus}}$, and $\xi^{\overline{\oplus}}$ are well-defined natural isomorphisms, the symmetric monoidal category axioms for the additive structure of $\overline{\mathsf{C}}^{\mathrm{bi}}$ will follow from the corresponding properties of C.

Since $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, and $\xi^{\oplus}$ in C are natural isomorphisms, to show that $\alpha^{\overline{\oplus}}$, $\lambda^{\overline{\oplus}}$, $\rho^{\overline{\oplus}}$, and $\xi^{\overline{\oplus}}$ are natural isomorphisms, it suffices to check that, in each case, each component is a well-defined morphism in $\overline{\mathsf{C}}^{\mathrm{bi}}$. In the following diagrams in C, each unlabeled region is commutative by a combination of

- the definition (4.1.8) of $\beta^{A\oplus B}$ or the definition (4.1.11) of $\beta^{\mathbb{0}}$,
- the functoriality of $\oplus$ in C, and
- the naturality of $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, or $\xi^{\oplus}$ in C.

The following four commutative diagrams prove the axiom (1.4.5) of a morphism in $\overline{\mathsf{C}}^{\mathsf{bi}}$ for $\alpha^{\overline{\oplus}}$ in (4.1.12), $\lambda^{\overline{\oplus}}$ and $\rho^{\overline{\oplus}}$ in (4.1.13), and $\xi^{\overline{\oplus}}$ in (4.1.14).

$$\begin{tikzcd}
{[(A\oplus B)\oplus C]D} \arrow[r, "\alpha^{\oplus}1"] \arrow[d, "\delta^r"] \arrow[ddddd, bend right=60, "\beta_D^{(A\oplus B)\oplus C}"'] & {[A\oplus (B\oplus C)]D} \arrow[d, "\delta^r"'] \arrow[ddddd, bend left=60, "\beta_D^{A\oplus (B\oplus C)}"] \\
(A\oplus B)D\oplus CD \arrow[d, "\delta^r\oplus 1"] \arrow[ddd, bend right=50, "\beta_D^{A\oplus B}\oplus\beta_D^C"'] \arrow[r, phantom, "(2.1.7)"] & AD\oplus (B\oplus C)D \arrow[d, "1\oplus\delta^r"'] \arrow[ddd, bend left=50, "\beta_D^A\oplus\beta_D^{B\oplus C}"] \\
(AD\oplus BD)\oplus CD \arrow[r, "\alpha^{\oplus}"] \arrow[d, "(\beta_D^A\oplus\beta_D^B)\oplus\beta_D^C"] & AD\oplus (BD\oplus CD) \arrow[d, "\beta_D^A\oplus(\beta_D^B\oplus\beta_D^C)"'] \\
(DA\oplus DB)\oplus DC \arrow[r, "\alpha^{\oplus}"] \arrow[d, "\delta^{-l}\oplus 1"] & DA\oplus (DB\oplus DC) \arrow[d, "1\oplus\delta^{-l}"'] \\
D(A\oplus B)\oplus DC \arrow[d, "\delta^{-l}"] \arrow[r, phantom, "(2.1.8)"] & DA\oplus D(B\oplus C) \arrow[d, "\delta^{-l}"'] \\
{D[(A\oplus B)\oplus C]} \arrow[r, "1\alpha^{\oplus}"] & {D[A\oplus (B\oplus C)]}
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbb{0}\oplus A)B \arrow[rr, "\lambda^{\oplus}1"] \arrow[d, "\delta^r"] \arrow[ddddd, bend right=60, "\beta_B^{\mathbb{0}\oplus A}"'] & & AB \arrow[dd, equal] \\
\mathbb{0}B\oplus AB \arrow[d, "\lambda^{\bullet}\oplus 1"] \arrow[dd, bend right=50, "\beta_B^{\mathbb{0}}\oplus 1"'] \arrow[r, phantom, "(2.1.23)"] & & \\
\mathbb{0}\oplus AB \arrow[r, equal] \arrow[d, "\rho^{-\bullet}\oplus 1"] & \mathbb{0}\oplus AB \arrow[r, "\lambda^{\oplus}"] \arrow[dd, "1\oplus\beta_B^A"] & AB \arrow[ddd, "\beta_B^A"'] \\
B\mathbb{0}\oplus AB \arrow[d, "1\oplus\beta_B^A"] & & \\
B\mathbb{0}\oplus BA \arrow[r, "\rho^{\bullet}\oplus 1"] \arrow[d, "\delta^{-l}"] \arrow[r, phantom, shift right=4ex, "(2.1.22)"] & \mathbb{0}\oplus BA \arrow[dr, "\lambda^{\oplus}"] & \\
B(\mathbb{0}\oplus A) \arrow[rr, "1\lambda^{\oplus}"] & & BA
\end{tikzcd}$$

$$\begin{tikzcd}
(A\oplus\mathbb{0})B \arrow[rr, "\rho^{\oplus}1"] \arrow[d, "\delta^r"] \arrow[ddddd, bend right=60, "\beta_B^{A\oplus\mathbb{0}}"'] & & AB \arrow[dd, equal] \\
AB\oplus\mathbb{0}B \arrow[d, "1\oplus\lambda^{\bullet}"] \arrow[dd, bend right=50, "1\oplus\beta_B^{\mathbb{0}}"'] \arrow[r, phantom, "(2.1.25)"] & & \\
AB\oplus\mathbb{0} \arrow[r, equal] \arrow[d, "1\oplus\rho^{-\bullet}"] & AB\oplus\mathbb{0} \arrow[r, "\rho^{\oplus}"] \arrow[dd, "\beta_B^A\oplus 1"] & AB \arrow[ddd, "\beta_B^A"'] \\
AB\oplus B\mathbb{0} \arrow[d, "\beta_B^A\oplus 1"] & & \\
BA\oplus B\mathbb{0} \arrow[r, "1\oplus\rho^{\bullet}"] \arrow[d, "\delta^{-l}"] \arrow[r, phantom, shift right=4ex, "(2.1.24)"] & BA\oplus\mathbb{0} \arrow[dr, "\rho^{\oplus}"] & \\
B(A\oplus\mathbb{0}) \arrow[rr, "1\rho^{\oplus}"] & & BA
\end{tikzcd}$$

$$\begin{tikzcd}
(A\oplus B)C \arrow[r, "\xi^{\oplus}1"] \arrow[d, "\delta^r"] \arrow[ddd, bend right=60, "\beta_C^{A\oplus B}"'] \arrow[dr, phantom, "(2.1.6)"] & (B\oplus A)C \arrow[d, "\delta^r"'] \arrow[ddd, bend left=60, "\beta_C^{B\oplus A}"] \\
AC\oplus BC \arrow[r, "\xi^{\oplus}"] \arrow[d, "\beta_C^A\oplus\beta_C^B"] & BC\oplus AC \arrow[d, "\beta_C^B\oplus\beta_C^A"'] \\
CA\oplus CB \arrow[r, "\xi^{\oplus}"] \arrow[d, "\delta^{-l}"] \arrow[dr, phantom, "(2.1.5)"] & CB\oplus CA \arrow[d, "\delta^{-l}"'] \\
C(A\oplus B) \arrow[r, "1\xi^{\oplus}"] & C(B\oplus A)
\end{tikzcd}$$

Therefore, each component of each of $\alpha^{\overline{\oplus}}$, $\lambda^{\overline{\oplus}}$, $\rho^{\overline{\oplus}}$, and $\xi^{\overline{\oplus}}$ is a morphism in $\overline{\mathsf{C}}^{\mathsf{bi}}$. $\square$

4.3. The Multiplicative Structure

In this section, we check that the multiplicative structure of the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a braided monoidal category. Recall that the multiplicative structure (4.1.5)

$$\big(\overline{\otimes}, \overline{\mathbb{1}}, \alpha^{\overline{\otimes}}, \lambda^{\overline{\otimes}}, \rho^{\overline{\otimes}}, \xi^{\overline{\otimes}}\big)$$

in $\overline{\mathsf{C}}^{\mathsf{bi}}$ is as in Definition 1.4.3 using the multiplicative structure $(\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes})$. We first check that

$$- \,\overline{\otimes}\, - : \overline{\mathsf{C}}^{\mathsf{bi}} \times \overline{\mathsf{C}}^{\mathsf{bi}} \longrightarrow \overline{\mathsf{C}}^{\mathsf{bi}}$$

is well defined on objects.

Lemma 4.3.1. *For objects $(A;\beta^A), (B;\beta^B) \in \overline{\mathsf{C}}^{\mathsf{bi}}$,*

$$(A \otimes B; \beta^{A\otimes B})$$

in (1.4.7) is an object in $\overline{\mathsf{C}}^{\mathsf{bi}}$.

Proof. In Lemma 1.4.20, we observed that

- $\beta^{A\otimes B}$ in (1.4.8) is a natural isomorphism, and
- $\beta^{A\otimes B}$ satisfies the axiom (1.4.4).

The following two commutative diagrams verify the other two axioms (4.1.3) and (4.1.4) for $\beta^{A\otimes B}$. The unlabeled regions are commutative by a combination of the functoriality of $\oplus$, the definition (1.4.8) of $\beta^{A\otimes B}$, and the naturality of δ^l or δ^r.

$$\begin{tikzcd}
(AB)(C\oplus D) \arrow[r, "\alpha^{\otimes}"] \arrow[d, "\delta^l"'] \arrow[rrr, bend left=20, "\beta^{AB}_{C\oplus D}"] \arrow[dr, phantom, "(2.1.9)"] & A[B(C\oplus D)] \arrow[r, "1\beta^B_{C\oplus D}"] \arrow[d, "1\delta^l"] \arrow[r, phantom, shift right=6, "(4.1.3)"] & A[(C\oplus D)B] \arrow[r, "\alpha^{-\otimes}"] \arrow[ddl, "1\delta^r"'] & {[A(C\oplus D)]B} \arrow[d, "\beta^A_{C\oplus D}1"] \arrow[dl, "\delta^l 1"'] \arrow[dl, phantom, shift right=6, "(4.1.3)"] \\
(AB)C\oplus(AB)D \arrow[d, "\alpha^{\otimes}\oplus\alpha^{\otimes}"'] \arrow[ddrrr, bend right=40, "\beta^{AB}_C\oplus\beta^{AB}_D"'] & A(BC\oplus BD) \arrow[dl, "\delta^l"'] \arrow[d, "1(\beta^B_C\oplus\beta^B_D)"] & (AC\oplus AD)B \arrow[d, "(\beta^A_C\oplus\beta^A_D)1"] \arrow[ddl, "\delta^r"'] & {[(C\oplus D)A]B} \arrow[dl, "\delta^r 1"'] \arrow[d, "\alpha^{\otimes}"] \\
A(BC)\oplus A(BD) \arrow[d, "1\beta^B_C\oplus 1\beta^B_D"'] & A(CB\oplus DB) \arrow[dl, "\delta^l"'] \arrow[d, phantom, "(2.1.11)"] & (CA\oplus DA)B \arrow[d, "\delta^r"'] \arrow[dr, phantom, "(2.1.10)"] & (C\oplus D)(AB) \\
A(CB)\oplus A(DB) \arrow[r, "\alpha^{-\otimes}\oplus\alpha^{-\otimes}"'] & (AC)B\oplus(AD)B \arrow[r, "\beta^A_C 1\oplus\beta^A_D 1"'] & (CA)B\oplus(DA)B \arrow[r, "\alpha^{\otimes}\oplus\alpha^{\otimes}"'] & C(AB)\oplus D(AB) \arrow[u, "\delta^{-r}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(AB)\mathbb{0} \arrow[r, "\alpha^{\otimes}"] \arrow[d, "\rho^{\bullet}"'] \arrow[ddd, phantom] \arrow[dddr, bend left=40, "\beta^{AB}_{\mathbb{0}}"] \arrow[dr, phantom, "(2.1.19)"] & A(B\mathbb{0}) \arrow[r, "1\beta^B_{\mathbb{0}}"] \arrow[d, "1\rho^{\bullet}"'] \arrow[r, phantom, shift right=6, "(4.1.4)"] & A(\mathbb{0}B) \arrow[d, "\alpha^{-\otimes}"] \\
\mathbb{0} \arrow[d, equal] \arrow[dr, phantom, "(2.1.20)"] & A\mathbb{0} \arrow[l, "\rho^{\bullet}"'] \arrow[ur, "1\lambda^{-\bullet}"'] & (A\mathbb{0})B \arrow[dl, "\rho^{\bullet}1"'] \arrow[dd, "\beta^A_{\mathbb{0}}1"] \arrow[dd, phantom, shift right=10, "(4.1.4)"] \\
\mathbb{0} \arrow[d, equal] \arrow[dr, phantom, "(2.1.21)"] & \mathbb{0}B \arrow[l, "\lambda^{\bullet}"'] \arrow[dr, "\lambda^{-\bullet}1"] & \\
\mathbb{0} \arrow[r, "\lambda^{-\bullet}"'] & \mathbb{0}(AB) & (\mathbb{0}A)B \arrow[l, "\alpha^{\otimes}"]
\end{tikzcd}$$

Therefore, $(A \otimes B; \beta^{A\otimes B})$ in (1.4.7) is an object in $\overline{\mathsf{C}}^{\mathsf{bi}}$. $\square$

Lemma 4.3.2. *$\overline{\mathbb{1}} = (\mathbb{1};\beta^{\mathbb{1}})$ in (1.4.10) is an object in $\overline{\mathsf{C}}^{\mathsf{bi}}$.*

Proof. In Lemma 1.4.23, we observed that

- $\beta^{\mathbb{1}} = \rho^{-\otimes}\lambda^{\otimes}$ in (1.4.11) is a natural isomorphism, and
- $\beta^{\mathbb{1}}$ satisfies the axiom (1.4.4).

The following two commutative diagrams verify the other two axioms (4.1.3) and (4.1.4) for $\beta^{\mathbb{1}}$, with the unlabeled regions commutative by the definition $\beta^{\mathbb{1}} = \rho^{-\otimes}\lambda^{\otimes}$ and the functoriality of $\oplus$.

$$\begin{tikzcd}
\mathbb{1}(A\oplus B) \arrow[r, "\lambda^{\otimes}"] \arrow[d, "\delta^l"'] \arrow[rr, bend left, "\beta^{\mathbb{1}}_{A\oplus B}"] \arrow[dr, phantom, "(2.1.26)"] & A\oplus B \arrow[r, "\rho^{-\otimes}"] \arrow[d, equal] \arrow[dr, phantom, "(2.1.27)"] & (A\oplus B)\mathbb{1} \\
\mathbb{1}A\oplus\mathbb{1}B \arrow[r, "\lambda^{\otimes}\oplus\lambda^{\otimes}"] \arrow[rr, bend right, "\beta^{\mathbb{1}}_A\oplus\beta^{\mathbb{1}}_B"'] & A\oplus B \arrow[r, "\rho^{-\otimes}\oplus\rho^{-\otimes}"] & A\mathbb{1}\oplus B\mathbb{1} \arrow[u, "\delta^{-r}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1}\otimes\mathbb{0} \arrow[r, "\lambda^{\otimes}_{\mathbb{0}}"] \arrow[rr, bend left, "\beta^{\mathbb{1}}_{\mathbb{0}}"] \arrow[dr, bend right, "\rho^{\bullet}_{\mathbb{1}}"'] \arrow[dr, phantom, "(2.1.17)"] & \mathbb{0} \arrow[r, "\rho^{-\otimes}_{\mathbb{0}}"] \arrow[d, equal] \arrow[dr, phantom, "(2.1.16)"] & \mathbb{0}\otimes\mathbb{1} \\
& \mathbb{0} \arrow[ur, bend right, "\lambda^{-\bullet}_{\mathbb{1}}"'] &
\end{tikzcd}$$

Therefore, $\overline{\mathbb{1}} = (\mathbb{1};\beta^{\mathbb{1}})$ in (1.4.10) is an object in $\overline{\mathsf{C}}^{\mathsf{bi}}$. □

Lemma 4.3.3. *The multiplicative structure*

$$\left(\overline{\mathsf{C}}^{\mathsf{bi}}, \overline{\otimes}, \overline{\mathbb{1}}, \alpha^{\overline{\otimes}}, \lambda^{\overline{\otimes}}, \rho^{\overline{\otimes}}, \xi^{\overline{\otimes}}\right)$$

in Definition 4.1.2 is a braided monoidal category.

Proof. By Lemmas 1.4.21 and 4.3.1, $\overline{\otimes}$ is well defined. It preserves identity morphisms and composition by the corresponding properties of $\otimes$ in C. Lemma 4.3.2 shows that $\overline{\mathbb{1}}$ is an object in $\overline{\mathsf{C}}^{\mathsf{bi}}$. Lemmas 1.4.24 through 1.4.26 show that $\alpha^{\overline{\otimes}}$, $\lambda^{\overline{\otimes}}$, $\rho^{\overline{\otimes}}$, and $\xi^{\overline{\otimes}}$ are natural isomorphisms.

The unity axiom (1.3.2) and the pentagon axiom (1.3.3) for the data

$$\left(\overline{\mathsf{C}}^{\mathsf{bi}}, \overline{\otimes}, \overline{\mathbb{1}}, \alpha^{\overline{\otimes}}, \lambda^{\overline{\otimes}}, \rho^{\overline{\otimes}}\right)$$

follow from the corresponding properties of the multiplicative structure of C. The left and the right hexagon diagrams (1.3.17) for $(\overline{\mathsf{C}}^{\mathsf{bi}}, \xi^{\overline{\otimes}})$ are commutative by, respectively, the axiom (1.4.4) for $\beta^{A}_{B\otimes C}$ and the definition (1.4.8) of $\beta^{A\otimes B}_{C}$. □

4.4. The Multiplicative Zeros and Distributivity

In this section, we finish the proof that the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a tight braided bimonoidal category. First we check that the multiplicative zeros and the distributivity morphisms are natural isomorphisms.

The Multiplicative Zeros.

Lemma 4.4.1. *In (4.1.15),*

$$\overline{\lambda}^{\bullet} : \overline{\mathbb{0}}\,\overline{\otimes} - \longrightarrow \overline{\mathbb{0}} \quad \textit{and} \quad \overline{\rho}^{\bullet} : -\,\overline{\otimes}\,\overline{\mathbb{0}} \longrightarrow \overline{\mathbb{0}}$$

are natural isomorphisms.

Proof. Since $\lambda^\bullet$ and $\rho^\bullet$ in C are natural isomorphisms, it suffices to check that each component of each of $\overline{\lambda}^\bullet$ and $\overline{\rho}^\bullet$ is a morphism in $\overline{\mathsf{C}}^{\mathrm{bi}}$. The following two commutative diagrams prove the axiom (1.4.5) for each component of $\overline{\lambda}^\bullet$ and $\overline{\rho}^\bullet$. The unlabeled regions are commutative by a combination of

- the definition (1.4.8) of $\beta^{A\otimes B}$ or the definition (4.1.11) of $\beta^{\mathbb{0}}$,
- the functoriality of $\otimes$ in C, and
- the naturality of $\lambda^\bullet$ or $\rho^\bullet$ in C.

$$\begin{tikzcd}
(\mathbb{0}A)B \arrow[r, "\lambda^\bullet 1"] \arrow[d, "\alpha^\otimes"] \arrow[dddddd, bend right=60, "\beta_B^{\mathbb{0}A}"'] \arrow[dr, phantom, "(2.1.21)"] & \mathbb{0}B \arrow[d, "\lambda^\bullet"] \arrow[dddddd, bend left=60, "\beta_B^{\mathbb{0}}"] \\
\mathbb{0}(AB) \arrow[r, "\lambda^\bullet"] \arrow[d, "1\beta_B^A"] & \mathbb{0} \arrow[d, equal] \\
\mathbb{0}(BA) \arrow[r, "\lambda^\bullet"] \arrow[d, "\alpha^{-\otimes}"] & \mathbb{0} \arrow[dd, equal] \\
(\mathbb{0}B)A \arrow[d, "\lambda^\bullet 1"] \arrow[dd, bend right=60, "\beta_B^{\mathbb{0}}1"'] \arrow[r, phantom, "(2.1.21)"] & \\
\mathbb{0}A \arrow[r, "\lambda^\bullet"] \arrow[d, "\rho^{-\bullet}1"] & \mathbb{0} \arrow[dd, "\rho^{-\bullet}"] \\
(B\mathbb{0})A \arrow[d, "\alpha^\otimes"] \arrow[r, phantom, "(2.1.20)"] & \\
B(\mathbb{0}A) \arrow[r, "1\lambda^\bullet"] & B\mathbb{0}
\end{tikzcd}
\qquad
\begin{tikzcd}
(A\mathbb{0})B \arrow[r, "\rho^\bullet 1"] \arrow[d, "\alpha^\otimes"] \arrow[dddddd, bend right=60, "\beta_B^{A\mathbb{0}}"'] & \mathbb{0}B \arrow[dd, "\lambda^\bullet"] \arrow[dddddd, bend left=60, "\beta_B^{\mathbb{0}}"] \\
A(\mathbb{0}B) \arrow[d, "1\lambda^\bullet"] \arrow[dd, bend right=60, "1\beta_B^{\mathbb{0}}"'] \arrow[r, phantom, "(2.1.20)"] & \\
A\mathbb{0} \arrow[r, "\rho^\bullet"] \arrow[d, "1\rho^{-\bullet}"] & \mathbb{0} \arrow[dd, equal] \\
A(B\mathbb{0}) \arrow[d, "\alpha^{-\otimes}"] \arrow[r, phantom, "(2.1.19)"] & \\
(AB)\mathbb{0} \arrow[r, "\rho^\bullet"] \arrow[d, "\beta_B^A 1"] & \mathbb{0} \arrow[d, equal] \\
(BA)\mathbb{0} \arrow[r, "\rho^\bullet"] \arrow[d, "\alpha^\otimes"] \arrow[dr, phantom, "(2.1.19)"] & \mathbb{0} \arrow[d, "\rho^{-\bullet}"] \\
B(A\mathbb{0}) \arrow[r, "1\rho^\bullet"] & B\mathbb{0}
\end{tikzcd}$$

Therefore, $\overline{\lambda}^\bullet$ and $\overline{\rho}^\bullet$ are natural isomorphisms. $\square$

Distributivity.

Lemma 4.4.2. *$\overline{\delta}^l$ in (4.1.16) and $\overline{\delta}^r$ in (4.1.17) are natural isomorphisms.*

Proof. The following two commutative diagrams prove the axiom (1.4.5) for each component of $\overline{\delta}^l$ and $\overline{\delta}^r$.

$$\begin{tikzcd}
{[A(B\oplus C)]D} \arrow[r, "\delta^l 1"] \arrow[d, "\alpha^{\otimes}"] \arrow[ddddddd, bend right=60, "\beta_D^{A(B\oplus C)}"'] \arrow[dr, phantom, "(2.1.11)"] & (AB\oplus AC)D \arrow[d, "\delta^r"'] \arrow[ddddddd, bend left=60, "\beta_D^{AB\oplus AC}"] \\
A[(B\oplus C)D] \arrow[d, "1\delta^r"] \arrow[ddd, bend right=40, "1\beta_D^{B\oplus C}"'] & (AB)D\oplus(AC)D \arrow[d, "\alpha^{\otimes}\oplus\alpha^{\otimes}"'] \arrow[ddddd, bend left=40, "\beta_D^{AB}\oplus\beta_D^{AC}"] \\
A(BD\oplus CD) \arrow[r, "\delta^l"] \arrow[d, "1(\beta_D^B\oplus\beta_D^C)"] & A(BD)\oplus A(CD) \arrow[d, "1\beta_D^B\oplus 1\beta_D^C"'] \\
A(DB\oplus DC) \arrow[r, "\delta^l"] \arrow[d, "1\delta^{-l}"] \arrow[dr, phantom, "(2.1.9)"] & A(DB)\oplus A(DC) \arrow[d, "\alpha^{-\otimes}\oplus\alpha^{-\otimes}"'] \\
A[D(B\oplus C)] \arrow[d, "\alpha^{-\otimes}"] & (AD)B\oplus(AD)C \arrow[d, "\beta_D^A 1\oplus\beta_D^A 1"'] \\
(AD)(B\oplus C) \arrow[ur, "\delta^l"] \arrow[d, "\beta_D^A 1"] & (DA)B\oplus(DA)C \arrow[d, "\alpha^{\otimes}\oplus\alpha^{\otimes}"'] \\
(DA)(B\oplus C) \arrow[ur, "\delta^l"] \arrow[d, "\alpha^{\otimes}"] \arrow[dr, phantom, "(2.1.9)"] & D(AB)\oplus D(AC) \arrow[d, "\delta^{-l}"'] \\
D[A(B\oplus C)] \arrow[r, "1\delta^l"] & D(AB\oplus AC)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[(A\oplus B)C]D} \arrow[r, "\delta^r 1"] \arrow[d, "\alpha^{\otimes}"] \arrow[ddddddd, bend right=60, "\beta_D^{(A\oplus B)C}"'] \arrow[dr, phantom, "(2.1.10)"] & (AC\oplus BC)D \arrow[d, "\delta^r"'] \arrow[ddddddd, bend left=60, "\beta_D^{AC\oplus BC}"] \\
(A\oplus B)(CD) \arrow[d, "1\beta_D^C"] \arrow[dr, "\delta^r"] & (AC)D\oplus(BC)D \arrow[d, "\alpha^{\otimes}\oplus\alpha^{\otimes}"'] \arrow[ddddd, bend left=40, "\beta_D^{AC}\oplus\beta_D^{BC}"] \\
(A\oplus B)(DC) \arrow[d, "\alpha^{-\otimes}"] \arrow[dr, "\delta^r"] & A(CD)\oplus B(CD) \arrow[d, "1\beta_D^C\oplus 1\beta_D^C"'] \\
{[(A\oplus B)D]C} \arrow[d, "\delta^r 1"] \arrow[ddd, bend right=40, "\beta_D^{A\oplus B}1"'] \arrow[r, phantom, "(2.1.10)"] & A(DC)\oplus B(DC) \arrow[d, "\alpha^{-\otimes}\oplus\alpha^{-\otimes}"'] \\
(AD\oplus BD)C \arrow[r, "\delta^r"] \arrow[d, "(\beta_D^A\oplus\beta_D^B)1"] & (AD)C\oplus(BD)C \arrow[d, "\beta_D^A 1\oplus\beta_D^B 1"'] \\
(DA\oplus DB)C \arrow[r, "\delta^r"] \arrow[d, "\delta^{-l}1"] \arrow[dr, phantom, "(2.1.11)"] & (DA)C\oplus(DB)C \arrow[d, "\alpha^{\otimes}\oplus\alpha^{\otimes}"'] \\
{[D(A\oplus B)]C} \arrow[d, "\alpha^{\otimes}"] & D(AC)\oplus D(BC) \arrow[d, "\delta^{-l}"'] \\
D[(A\oplus B)C] \arrow[r, "1\delta^r"] & D(AC\oplus BC)
\end{tikzcd}$$

In the above diagrams, the unlabeled regions are commutative by a combination of

- the definition (1.4.8) of $\beta^{A\otimes B}$ or the definition (4.1.8) of $\beta^{A\oplus B}$,
- the functoriality of $\otimes$ in C, and
- the naturality of δ^l or δ^r in C.

Since δ^l and δ^r in C are natural isomorphisms, it follows that $\overline{\delta}^l$ and $\overline{\delta}^r$ are natural isomorphisms. $\square$

The Main Result. Recall from Definitions 2.1.1 and 2.1.29 the notions of a tight bimonoidal category and a tight braided bimonoidal category. We are now ready for the main result of this chapter.

Theorem 4.4.3. *For each tight bimonoidal category C, the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathrm{bi}}$ in Definition 4.1.2 is a tight braided bimonoidal category.*

Proof. We already have the following results.

- Lemma 4.2.6 shows that the additive structure of $\overline{\mathsf{C}}^{\mathrm{bi}}$ is a symmetric monoidal category.
- Lemma 4.3.3 shows that the multiplicative structure of $\overline{\mathsf{C}}^{\mathrm{bi}}$ is a braided monoidal category.
- Lemmas 4.4.1 and 4.4.2 show that $\overline{\lambda}^{\bullet}$, $\overline{\rho}^{\bullet}$, $\overline{\delta}^l$, and $\overline{\delta}^r$ are natural isomorphisms.

The braiding $\xi^{\overline{\otimes}}$ in (1.4.14) is the only structure morphism in $\overline{\mathsf{C}}^{\mathrm{bi}}$ that is not defined by its counterpart in C. Among the 12 Laplaza axioms in Definition 2.1.29, only

(2.1.4) and (2.1.18) involve the braiding. Therefore, the other 10 Laplaza axioms there hold in $\overline{\mathsf{C}}^{\mathrm{bi}}$ as they do in C.

Consider the remaining 4 axioms.

- (2.1.4) holds in $\overline{\mathsf{C}}^{\mathrm{bi}}$ by the definition (4.1.8) of $\beta^{A\oplus B}$.
- (2.1.18) holds in $\overline{\mathsf{C}}^{\mathrm{bi}}$ by the axiom (4.1.4) for $\beta^{A}_{\mathbb{0}}$.
- (2.1.32) holds in $\overline{\mathsf{C}}^{\mathrm{bi}}$ by the axiom (4.1.3) for $\beta^{A}_{B\oplus C}$.
- (2.1.33) holds in $\overline{\mathsf{C}}^{\mathrm{bi}}$ by the definition (4.1.11) of $\beta^{\mathbb{0}}$.

Therefore, $\overline{\mathsf{C}}^{\mathrm{bi}}$ is a tight braided bimonoidal category. $\square$

Example 4.4.4 (Abelian Categories with a Monoidal Structure)**.** Suppose C is an abelian category equipped with a compatible monoidal structure as in Theorem 2.5.2. Then C is also a tight bimonoidal category. By Theorem 4.4.3, its bimonoidal Drinfeld center is a tight braided bimonoidal category. $\diamond$

4.5. The Bimonoidal Symmetric Center

Recall from Definition 1.5.1 that the symmetric center $\mathsf{C}^{\mathsf{sym}}$ of a braided monoidal category C is the full subcategory consisting of objects $A \in \mathsf{C}$ satisfying the symmetry axiom in the sense of (1.5.2). In Proposition 1.5.3, we observed that the symmetric center $\mathsf{C}^{\mathsf{sym}}$ inherits from C the structure of a symmetric monoidal category. This section contains the bimonoidal analogue of that result.

Definition 4.5.1. For a braided bimonoidal category C as in Definition 2.1.29, the *bimonoidal symmetric center of* C is the full subcategory $\mathsf{C}^{\mathsf{sym}}$ consisting of objects $A \in \mathsf{C}$ such that the symmetry axiom

$$\xi^{\otimes}_{-,A}\xi^{\otimes}_{A,-} = 1 : A\otimes - \longrightarrow A\otimes - \tag{4.5.2}$$

holds. $\diamond$

Theorem 4.5.3. *For each braided bimonoidal category C as in Definition 2.1.29, the bimonoidal symmetric center $\mathsf{C}^{\mathsf{sym}}$ inherits from C the structure of a symmetric bimonoidal category.*

Proof. First we check that the additive structure of C restricts to one on $\mathsf{C}^{\mathsf{sym}}$. For each object $A \in \mathsf{C}$, the following diagram in C is commutative.

$$\begin{tikzcd}
\mathbb{0}A \arrow[rr, "\xi^{\otimes}_{\mathbb{0},A}"] \arrow[d, "\lambda^{\bullet}_A"'] \arrow[dr, phantom, "(2.1.33)"] & & A\mathbb{0} \arrow[d, "\xi^{\otimes}_{A,\mathbb{0}}"] \arrow[dll, "\rho^{\bullet}_A"'] \\
\mathbb{0} & \arrow[ur, phantom, "(2.1.18)"] & \mathbb{0}A \arrow[ll, "\lambda^{\bullet}_A"]
\end{tikzcd}$$

The above commutative diagram and the invertibility of $\lambda^{\bullet}$ imply the symmetry axiom (4.5.2) for $\mathbb{0}$, that is,

$$\xi^{\otimes}_{A,\mathbb{0}}\xi^{\otimes}_{\mathbb{0},A} = 1 : \mathbb{0}\otimes A \longrightarrow \mathbb{0}\otimes A,$$

so $\mathbb{0} \in \mathsf{C}^{\mathsf{sym}}$.

To check that $\mathsf{C}^{\mathsf{sym}}$ is closed under $\oplus$, suppose $A, B \in \mathsf{C}^{\mathsf{sym}}$, so each of them satisfies the symmetry axiom (4.5.2). For each object $C \in \mathsf{C}$, the following diagram in C is commutative.

$$\begin{tikzcd}
(A \oplus B)C \arrow[r, "\xi^{\otimes}_{A\oplus B,C}"] \arrow[d, "\delta^r"'] \arrow[dr, phantom, "(2.1.4)"] & C(A \oplus B) \arrow[r, "\xi^{\otimes}_{C,A\oplus B}"] \arrow[d, "\delta^l"'] \arrow[dr, phantom, "(2.1.32)"] & (A \oplus B)C \arrow[d, "\delta^r"] \\
AC \oplus BC \arrow[r, "\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}"] \arrow[rr, bend right, "1"'] & CA \oplus CB \arrow[r, "\xi^{\otimes}_{C,A} \oplus \xi^{\otimes}_{C,B}", "(4.5.2)"'] & AC \oplus BC
\end{tikzcd}$$

The above commutative diagram and the fact that δ^r is a natural monomorphism imply the symmetry axiom (4.5.2) for $A \oplus B$, that is,

$$\xi^{\otimes}_{C,A\oplus B}\xi^{\otimes}_{A\oplus B,C} = 1 : (A \oplus B)C \longrightarrow (A \oplus B)C,$$

so $A \oplus B \in \mathsf{C}^{\mathsf{sym}}$. Therefore, restricting $(\oplus, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$ to $\mathsf{C}^{\mathsf{sym}}$, the additive structure

$$\left(\mathsf{C}^{\mathsf{sym}}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}\right)$$

satisfies all the symmetric monoidal category axioms as they do in C.

Since the multiplicative structure of C is a braided monoidal category, Proposition 1.5.3 shows that the multiplicative structure

$$\left(\mathsf{C}^{\mathsf{sym}}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}\right)$$

is a symmetric monoidal category. Equipped with the restrictions of δ^l, δ^r, $\lambda^{\bullet}$, and $\rho^{\bullet}$, $\mathsf{C}^{\mathsf{sym}}$ satisfies all 14 axioms of a braided bimonoidal category in Definition 2.1.29 as they do in C. Moreover, since the braiding $\xi^{\otimes}$ in $\mathsf{C}^{\mathsf{sym}}$ satisfies the symmetry axiom (4.5.2), $\mathsf{C}^{\mathsf{sym}}$ is a symmetric bimonoidal category by Corollary 2.2.3. □

Example 4.5.4 (Abelian Categories with a Braiding)**.** As in Convention 2.4.1, suppose C is an abelian category equipped with a compatible braided monoidal structure. Theorem 2.4.22 shows that C is a tight braided bimonoidal category. By Theorem 4.5.3, its bimonoidal symmetric center is a tight symmetric bimonoidal category. ◇

Example 4.5.5 (Anyonic Quantum Groups)**.** Recall from Example 3.1.33 that the group bialgebra $\mathbb{C}\mathbb{Z}_n$ of the cyclic group $\mathbb{Z}_n$ of order n is a braided bialgebra when equipped with the nonstandard R-matrix R in (3.1.34). As discussed in Example 3.2.20, the category $\mathsf{Mod}(\mathbb{C}\mathbb{Z}_n)$ of left $\mathbb{C}\mathbb{Z}_n$-modules is a tight braided bimonoidal category. By Theorem 4.5.3, its bimonoidal symmetric center is a tight symmetric bimonoidal category. ◇

Example 4.5.6 (Fibonacci Anyons)**.** By Theorem 3.4.13, the Fibonacci anyons $\mathcal{F}^{\mathsf{any}}$ form a tight braided bimonoidal category. By Theorem 4.5.3, its bimonoidal symmetric center is a tight symmetric bimonoidal category. ◇

Example 4.5.7 (Ising Anyons)**.** Theorem 3.6.14 shows that the Ising anyons $\mathcal{I}^{\mathsf{any}}$ form a tight braided bimonoidal category. By Theorem 4.5.3, its bimonoidal symmetric center is a tight symmetric bimonoidal category. ◇

CHAPTER 5

Coherence of Braided Bimonoidal Categories

In this chapter, we discuss the first coherence theorem for braided bimonoidal categories as in Definition 2.1.29. Using terminology that will be introduced in this chapter, the main Theorem 5.4.4 states that, for each braided bimonoidal category C that satisfies a monomorphism assumption, any two parallel paths with the same braided distortion have the same value in C. The monomorphism assumption is automatically satisfied if C is tight. Theorem 5.4.4 is the braided version of Theorem I.4.4.3, which is the analogous statement for a symmetric bimonoidal category. It is also a bimonoidal analogue of the Joyal-Street Coherence Theorem 1.6.3 for braided monoidal categories. For open questions related to coherence of braided bimonoidal categories, see Questions III.A.1.6 and III.A.5.6.

The Blass-Gurevich Conjecture. Recall from Explanation 2.1.37 that *BD categories* in the sense of [**BG20a**] are our tight braided bimonoidal categories, up to some presentational differences. In [**BG20a**, Conjecture 3], Blass and Gurevich conjectured that there should be a coherence theorem for BD categories. The main Theorem 5.4.4 of this chapter proves the Blass-Gurevich Conjecture in the form of commutative formal diagrams in braided bimonoidal categories that satisfy a monomorphism assumption.

The Coherence Theorem 5.4.4 will play an important role in later chapters.

- Using Theorem 5.4.4, in Theorems 6.3.6 and 6.3.7, we prove the Blass-Gurevich Conjecture in the form of strictification results for tight braided bimonoidal categories. These theorems are the bimonoidal analogues of the Strictification Theorem 1.6.5 for braided monoidal categories. They are also the braided analogues of Theorems I.5.4.6 and I.5.4.7, which are strictification results for tight symmetric bimonoidal categories.
- In Chapter 7, we use Theorem 5.4.4 to prove a braided version of Baez's Conjecture (Theorem I.7.8.1).
- In Chapter 8, we use Theorem 5.4.4 to prove a braided version of Theorem I.8.12.9, which states that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory if C is a tight braided bimonoidal category.

In the rest of this introduction, we motivate Theorem 5.4.4 and outline the rest of this chapter.

Motivation. To motivate Theorem 5.4.4, first recall that the Braided Coherence Theorem 1.6.3 states that two parallel braided canonical maps in a braided monoidal category are equal if their underlying braids are equal. In particular, the value of a braided canonical map is not determined by the domain and the codomain, unlike the Coherence Theorem I.1.3.8 for symmetric monoidal categories. Instead, the value of a braided canonical map in a braided monoidal category is determined by how it braids the tensor factors. This suggests that a coherence theorem for braided bimonoidal categories in the form of commutative formal diagrams should *not* be simply about the domain and the codomain. Therefore, we are not seeking a braided analogue of Laplaza's First Coherence Theorem I.3.9.1 for symmetric bimonoidal categories.

In a braided bimonoidal category C (Definition 2.1.29), the multiplicative structure $(\mathsf{C}, \otimes)$ is a braided monoidal category, and the additive structure $(\mathsf{C}, \oplus)$ is a symmetric monoidal category. Using the distributivity morphisms δ^l and δ^r, a formal expression involving the sum $\oplus$ and the product $\otimes$ may be expanded to a sum of monomials as follows.

$$\underline{a} = \bigoplus_{i=1}^{m} \left(a_1^i \otimes \cdots \otimes a_{r_i}^i\right) \tag{5.0.1}$$

Here the sum has some additive bracketing, and each monomial

$$a^i = a_1^i \otimes \cdots \otimes a_{r_i}^i$$

for $1 \leq i \leq m$ has some multiplicative bracketing. Similar to the Coherence Theorem 1.6.3 for braided monoidal categories, a bimonoidal analogue of the underlying braid of a braided canonical map should involve braiding the r_i tensor factors in each monomial a^i in $\underline{a}$. So it involves the product

$$B_{r_1} \times \cdots \times B_{r_m}$$

of m braid groups as in Definition 1.1.1. Moreover, we also need to know how a path permutes the set of monomials $\{a^1, \ldots, a^m\}$ in $\underline{a}$. So a bimonoidal analogue of an underlying braid should also involve the symmetric group Σ_m. This leads to the braided analogue $\mathcal{D}^{\mathsf{br}}$ of the distortion category $\mathcal{D}$ in Definition I.4.2.1. The latter is the main ingredient in formulating Laplaza's Second Coherence Theorem I.4.4.3 for symmetric bimonoidal categories.

As in the distortion category $\mathcal{D}$, the *braided distortion category* $\mathcal{D}^{\mathsf{br}}$ has as its objects finite sequences

$$\underline{r} = (r_1, \ldots, r_m)$$

of nonnegative integers. The length m of this sequence records the number of monomials in $\underline{a}$. The ith entry $r_i \geq 0$ records the number of tensor factors in the ith monomial a^i in $\underline{a}$. Unlike the distortion category $\mathcal{D}$, morphisms $\underline{r} \longrightarrow \underline{s} \in \mathcal{D}^{\mathsf{br}}$ are finite sequences

$$\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) \in \Sigma_m \times B_{r_1} \times \cdots \times B_{r_m}$$

such that $\sigma\underline{r} = \underline{s}$. In terms of the polynomial $\underline{a}$ in (5.0.1), $\underline{\sigma}$ first braids the tensor factors in each monomial a^i via the braid $\sigma_i \in B_{r_i}$, which is called the *ith braid component* in $\underline{\sigma}$. Then it permutes the m resulting monomials using the permutation $\sigma \in \Sigma_m$, which is called the *permutation component* in $\underline{\sigma}$. The rest of the structure of $\mathcal{D}^{\mathsf{br}}$ also needs to be suitably redefined relative to $\mathcal{D}$ because of the presence of the braid groups.

The graph theoretic notion of a *path* in $\mathsf{Gr}(X)$ in Section I.3.1 is the bimonoidal analogue of a canonical map in a monoidal category in Section 1.6. The bimonoidal analogue of the underlying braid of a braided canonical map is the *braided distortion* of a path in $\mathsf{Gr}(X)$. The braided distortion of a path is its image in the braided distortion category $\mathcal{D}^{\mathsf{br}}$, which will be defined precisely in Definition 5.3.15. The bimonoidal analogue of two parallel braided canonical maps having the same underlying braids consists of two parallel paths in $\mathsf{Gr}(X)$ with the same braided distortion. Under this assumption, Theorem 5.4.4 states that these two paths have the same value in the given braided bimonoidal category C, as long as it satisfies a monomorphism condition.

Proof Strategy. Since Theorem 5.4.4 is the braided version of the Second Coherence Theorem I.4.4.3 for symmetric bimonoidal categories, we will reuse parts of the proof of the latter. The proof of Theorem I.4.4.3 uses the same five reduction steps as in the proof of the First Coherence Theorem I.3.9.1 for symmetric bimonoidal categories, which involve many concepts and preliminary results in Chapter I.3. The proof of Theorem 5.4.4 also uses these five reduction steps, the third of which requires the monomorphism assumption on C. In the proof of Theorem 5.4.4, we will explain in detail how the concepts and results in Chapter I.3 are adapted to the braided context.

In the proof of Theorem I.4.4.3, after the five reduction steps, we are reduced to dealing with polynomials as in (5.0.1). To finish the proof, the Coherence Theorem I.1.3.8 for symmetric monoidal categories is used twice, once for the additive structure applied to the set of monomials and once for the multiplicative structure applied to the tensor factors in each monomial. Simiarly, for Theorem 5.4.4, after the five reduction steps, we use Theorem I.1.3.8 for the additive structure and the Braided Coherence Theorem 1.6.3 for the multiplicative structure. This step is carried out in Lemma 5.4.2.

Organization. An outline of the rest of this chapter follows.

In Section 5.1, we introduce left permutative braided categories, which are also called left permbraided categories, and the right analogue. They are the braided analogues of left and right bipermutative categories in Section I.2.5. By Definition 5.1.2 and Proposition 5.1.10, a left permbraided category is a tight braided bimonoidal category that satisfies the following conditions.

- Its additive structure is a permutative category.
- Its multiplicative structure is a braided strict monoidal category.
- The structure morphisms $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^{l}, $\xi^{\otimes}_{-,\mathbb{0}}$, and $\xi^{\otimes}_{\mathbb{0},-}$ are identities.

In a right permbraided category, the right distributivity morphism δ^{r}, instead of δ^{l}, is the identity. Left and right permbraided categories serve two purposes. First, the braided distortion category $\mathcal{D}^{\mathsf{br}}$ is a left permbraided category. Second, in Theorems 6.3.6 and 6.3.7, we prove strictification results from tight braided bimonoidal categories to right and left permbraided categories.

In Section 5.2, as the first step in formulating Theorem 5.4.4, we define the braided distortion category $\mathcal{D}^{\mathsf{br}}$. The main observation in this section is Theorem 5.2.30, which states that $\mathcal{D}^{\mathsf{br}}$ is a left permbraided category. This is the braided analogue of Theorem I.4.2.29, which states that the distortion category $\mathcal{D}$ is a left bipermutative category. In particular, $\mathcal{D}^{\mathsf{br}}$ is a tight braided bimonoidal category and satisfies all 24 Laplaza axioms in Definition 2.1.1. Moreover, its only nonidentity structure morphisms are the additive symmetry $\xi^{\oplus}$, the braiding $\xi^{\otimes}$, and the right distributivity morphism δ^{r}. Among these three structure morphisms, only the braiding $\xi^{\otimes}$ in (5.2.18) involves nonidentity braid components, which are the elementary block braids in (1.2.4).

In Section 5.3, as the next step in formulating Theorem 5.4.4, we define the braided distortion of a path in $\mathsf{Gr}(X)$. For the reader's convenience, we first recall some relevant graph theoretic definitions from Chapter I.3. For a path in $\mathsf{Gr}(X)$, its braided distortion is defined as its value in the braided distortion category $\mathcal{D}^{\mathsf{br}}$ under the graph morphism $\vartheta : \mathsf{Gr}(X) \longrightarrow \mathcal{D}^{\mathsf{br}}$ in (5.3.17).

In Section 5.4, we first prove Lemma 5.4.2, which is a preliminary case of Theorem 5.4.4, with assumptions not only on the paths but also on the domain and the codomain. The proof of Theorem 5.4.4 involves reduction steps that reduce to the setting of Lemma 5.4.2. The monomorphism assumption in Theorem 5.4.4 is automatically satisfied if C is tight, which means that δ^l and δ^r are natural isomorphisms, not just monomorphisms. After listing some examples, we finish this section with an explanation of why the First Coherence Theorem I.3.9.1 for symmetric bimonoidal categories does *not* have a braided analogue.

Section 5.5 provides a conceptual description of the braided distortion category as a Grothendieck construction over the finite ordinal category Σ. This observation is a repackaging of the definition of $\mathcal{D}^{\mathsf{br}}$, but it provides a better perspective about the relationship between $\mathcal{D}^{\mathsf{br}}$ and Σ. It is the braided version of Propositions I.4.6.5 and I.4.6.7 for, respectively, the distortion category $\mathcal{D}$ and the additive distortion category $\mathcal{D}^{\mathsf{ad}}$.

Convention 2.1.34 is still in effect, so $\otimes$ is sometimes abbreviated to concatenation.

Reading Guide.

(1) Read Definition 5.1.2 and the statements of Propositions 5.1.8 and 5.1.10 for left permbraided categories.
(2) Read Definition 5.1.11 and the statements of Propositions 5.1.17 and 5.1.19 for right permbraided categories.
(3) Read Definitions 5.2.2, 5.2.8, 5.2.14, and 5.2.25 and the statement of Theorem 5.2.30 for the braided distortion category $\mathcal{D}^{\mathsf{br}}$.
(4) Read Definition 5.3.15 for the braided distortion of a path.
(5) Read Convention 5.4.1 and Theorem 5.4.4 for the coherence of braided bimonoidal categories.
(6) Go back and read the rest of this chapter.

5.1. Permutative Braided Categories

In this section, we define left and right permutative braided categories and observe that they are tight braided bimonoidal categories. The braided distortion category in Section 5.2 is an example of a left permutative braided category.

Motivation 5.1.1. The Coherence Theorem 5.4.4 for braided bimonoidal categories is formulated in terms of the braided analogue $\mathcal{D}^{\mathsf{br}}$ of the distortion category $\mathcal{D}$ in Section I.4.2. In Theorem I.4.2.29, we observed that $\mathcal{D}$ is a left bipermutative category as in Definition I.2.5.11. In a left bipermutative category, the additive structure and the multiplicative structure are both permutative categories. Moreover, the multiplicative zeros and the left distributivity morphisms are the identities, and it is assumed to satisfy a small list of axioms. In Proposition I.2.5.16, we checked that each left bipermutative category is a tight symmetric bimonoidal category.

In the braided distortion category $\mathcal{D}^{\mathsf{br}}$, the multiplicative structure is a braided strict monoidal category. There should be an analogue of a left bipermutative category with the multiplicative structure given by a braided strict monoidal category, of which $\mathcal{D}^{\mathsf{br}}$ is an example. We make this concept precise below and observe that it is a tight braided bimonoidal category as in Definition 2.1.29. ⋄

Left Permutative Braided Categories.

Definition 5.1.2. A *left permutative braided category*, or a *left permbraided category* for short, is a tuple

$$\big(\mathsf{C},(\oplus,\mathbb{0},\xi^{\oplus}),(\otimes,\mathbb{1},\xi^{\otimes}),(\lambda^{\bullet},\rho^{\bullet}),(\delta^{l},\delta^{r})\big)$$

consisting of the following data.

- $(\mathsf{C},\oplus,\mathbb{0},\xi^{\oplus})$ is a permutative category.
- $(\mathsf{C},\otimes,\mathbb{1},\xi^{\otimes})$ is a braided strict monoidal category.
- $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^{l}, and δ^{r} are natural transformations as in (2.1.30) and (2.1.31).

The above data are required to satisfy the following four conditions.

(1) $\lambda^{\bullet}$ and $\rho^{\bullet}$ are both equal to the identity natural transformation of the constant functor $\mathsf{C} \longrightarrow \mathsf{C}$ at $\mathbb{0}$.
(2) δ^{l} is the identity natural transformation.
(3) For each object A, the morphisms

$$\begin{tikzcd} A \otimes \mathbb{0} \arrow[r, shift left, "\xi^{\otimes}_{A,\mathbb{0}}"] & \mathbb{0} \otimes A \arrow[l, shift left, "\xi^{\otimes}_{\mathbb{0},A}"] \end{tikzcd}$$

are both equal to the identity morphism of $\mathbb{0}$.
(4) The axioms (2.1.4), (2.1.5), (2.1.12), and (2.1.32) are satisfied.

This finishes the definition of a left permbraided category. ⋄

Explanation 5.1.3. In Definition 5.1.2, suppose A,B,C, and D are objects in C.

- The axiom (2.1.4) is the following commutative diagram.

(5.1.4)
$$\begin{tikzcd} (A \oplus B)C \arrow[r, "\delta^{r}_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A\oplus B,C}"'] & AC \oplus BC \arrow[d, "\xi^{\otimes}_{A,C}\oplus\xi^{\otimes}_{B,C}"] \\ C(A \oplus B) \arrow[r, "\delta^{l}_{C,A,B}", "="'] & CA \oplus CB \end{tikzcd}$$

  Since the braiding $\xi^{\otimes}$ is a natural isomorphism, it follows that δ^{r} is also a natural isomorphism.
- The axiom (2.1.32) is the following commutative diagram.

(5.1.5)
$$\begin{tikzcd} C(A \oplus B) \arrow[r, "\delta^{l}_{C,A,B}", "="'] \arrow[d, "\xi^{\otimes}_{C,A\oplus B}"'] & CA \oplus CB \arrow[d, "\xi^{\otimes}_{C,A}\oplus\xi^{\otimes}_{C,B}"] \\ (A \oplus B)C \arrow[r, "\delta^{r}_{A,B,C}"] & AC \oplus BC \end{tikzcd}$$

- The axiom (2.1.5) is the following commutative diagram.

$$\begin{tikzcd} A(B\oplus C) \arrow[r, "\delta^l_{A,B,C}", "="'] \arrow[d, "1_A\xi^{\oplus}_{B,C}"'] & AB\oplus AC \arrow[d, "\xi^{\oplus}_{AB,AC}"] \\ A(C\oplus B) \arrow[r, "\delta^l_{A,C,B}", "="'] & AC\oplus AB \end{tikzcd} \tag{5.1.6}$$

- The axiom (2.1.12) is the following commutative diagram.

$$\begin{tikzcd} (A\oplus B)(C\oplus D) \arrow[rr, "\delta^r_{A,B,C\oplus D}"] \arrow[d, "\delta^l_{A\oplus B,C,D}"', "="] & & A(C\oplus D)\oplus B(C\oplus D) \arrow[d, "="', "\delta^l_{A,C,D}\oplus\delta^l_{B,C,D}"] \\ (A\oplus B)C\oplus(A\oplus B)D \arrow[dr, "\delta^r_{A,B,C}\oplus\delta^r_{A,B,D}"'] & & AC\oplus AD\oplus BC\oplus BD \arrow[dl, "1_{AC}\oplus\xi^{\oplus}_{AD,BC}\oplus 1_{BD}"] \\ & AC\oplus BC\oplus AD\oplus BD & \end{tikzcd} \tag{5.1.7}$$

◇

Proposition 5.1.8. *Each left bipermutative category is a left permbraided category.*

Proof. This follows from Definitions I.2.5.11 and 5.1.2 and Proposition 1.3.36. In a left bipermutative category, the axiom (2.1.32) holds because it is equivalent to (2.1.4) when $\xi^{\otimes}$ satisfies the symmetry axiom (1.3.33). □

Example 5.1.9. The finite ordinal category Σ in Definition I.2.4.1 and the distortion category $\mathcal{D}$ in Theorem I.4.2.29 are left bipermutative categories, hence also left permbraided categories. In Theorem 5.2.30 below, we will show that the braided distortion category $\mathcal{D}^{\mathrm{br}}$ is a left permbraided category. ◇

Proposition 5.1.10. *Each left permbraided category is a tight braided bimonoidal category.*

Proof. Each left permbraided category C has the data of a braided bimonoidal category as in Definition 2.1.29. The left distributivity morphism δ^l is the identity by assumption, and δ^r is a natural isomorphism as discussed in Explanation 5.1.3. Consider the 14 axioms in Definition 2.1.29.

- (2.1.4), (2.1.5), (2.1.12), and (2.1.32) hold by assumption.
- (2.1.8), (2.1.9), (2.1.13), (2.1.17), (2.1.18), (2.1.19), (2.1.24), (2.1.26), and (2.1.33) hold because $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^l, $\xi^{\otimes}_{-,\mathbb{0}}$, and $\xi^{\otimes}_{\mathbb{0},-}$ are the identities.
- (2.1.15) holds because $\rho^{\bullet}$ and $\delta^r_{A,B,\mathbb{0}}$ are both equal to the identity morphism of $\mathbb{0}$. The equality $\delta^r_{A,B,\mathbb{0}} = 1_{\mathbb{0}}$ follows from (5.1.4) and $\xi^{\otimes}_{-,\mathbb{0}} = 1_{\mathbb{0}}$.

Therefore, C is a tight braided bimonoidal category. □

Right Permutative Braided Categories. Next we discuss the right version of a left permbraided category.

Definition 5.1.11. A *right permutative braided category*, or a *right permbraided category* for short, is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\lambda^{\bullet}, \rho^{\bullet}), (\delta^l, \delta^r)\big)$$

consisting of the following data.

- $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ is a permutative category.

- $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a braided strict monoidal category.
- $\lambda^{\bullet}, \rho^{\bullet}, \delta^l$, and δ^r are natural transformations as in (2.1.30) and (2.1.31).

The above data are required to satisfy the following four conditions.

(1) $\lambda^{\bullet}$ and $\rho^{\bullet}$ are both equal to the identity natural transformation of the constant functor $\mathsf{C} \longrightarrow \mathsf{C}$ at $\mathbb{0}$.
(2) δ^r is the identity natural transformation.
(3) For each object A, the morphisms

$$\begin{tikzcd} A \otimes \mathbb{0} \arrow[r, shift left, "\xi^{\otimes}_{A,\mathbb{0}}"] & \mathbb{0} \otimes A \arrow[l, shift left, "\xi^{\otimes}_{\mathbb{0},A}"] \end{tikzcd}$$

are both equal to the identity morphism of $\mathbb{0}$.
(4) The axioms (2.1.4), (2.1.6), (2.1.12), and (2.1.32) are satisfied.

This finishes the definition of a right permbraided category. ◇

Explanation 5.1.12. In Definition 5.1.11, suppose A, B, C, and D are objects in C.

- The axiom (2.1.4) is the following commutative diagram.

$$\begin{tikzcd} (A \oplus B)C \arrow[r, "\delta^r_{A,B,C}", "="'] \arrow[d, "\xi^{\otimes}_{A\oplus B,C}"'] & AC \oplus BC \arrow[d, "\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}"] \\ C(A \oplus B) \arrow[r, "\delta^l_{C,A,B}"] & CA \oplus CB \end{tikzcd} \tag{5.1.13}$$

Since the braiding $\xi^{\otimes}$ is a natural isomorphism, it follows that δ^l is also a natural isomorphism.

- The axiom (2.1.32) is the following commutative diagram.

$$\begin{tikzcd} C(A \oplus B) \arrow[r, "\delta^l_{C,A,B}"] \arrow[d, "\xi^{\otimes}_{C,A\oplus B}"'] & CA \oplus CB \arrow[d, "\xi^{\otimes}_{C,A} \oplus \xi^{\otimes}_{C,B}"] \\ (A \oplus B)C \arrow[r, "\delta^r_{A,B,C}", "="'] & AC \oplus BC \end{tikzcd} \tag{5.1.14}$$

- The axiom (2.1.6) is the following commutative diagram.

$$\begin{tikzcd} (A \oplus B)C \arrow[r, "\delta^r_{A,B,C}", "="'] \arrow[d, "\xi^{\oplus}_{A,B} 1_C"'] & AC \oplus BC \arrow[d, "\xi^{\oplus}_{AC,BC}"] \\ (B \oplus A)C \arrow[r, "\delta^r_{B,A,C}", "="'] & BC \oplus AC \end{tikzcd} \tag{5.1.15}$$

- The axiom (2.1.12) is the following commutative diagram.

$$\begin{tikzcd} (A \oplus B)(C \oplus D) \arrow[rr, "\delta^r_{A,B,C\oplus D}", "="'] \arrow[d, "\delta^l_{A\oplus B,C,D}"'] & & A(C \oplus D) \oplus B(C \oplus D) \arrow[d, "\delta^l_{A,C,D} \oplus \delta^l_{B,C,D}"] \\ (A \oplus B)C \oplus (A \oplus B)D \arrow[dr, "\delta^r_{A,B,C} \oplus \delta^r_{A,B,D}"', "="] & & AC \oplus AD \oplus BC \oplus BD \arrow[dl, "1_{AC} \oplus \xi^{\oplus}_{AD,BC} \oplus 1_{BD}"] \\ & AC \oplus BC \oplus AD \oplus BD & \end{tikzcd} \tag{5.1.16}$$

◇

Proposition 5.1.17. *Each right bipermutative category is a right permbraided category.*

Proof. This follows from Definitions I.2.5.2 and 5.1.11 and Proposition 1.3.36. In a right bipermutative category, the axiom (2.1.32) holds because it is equivalent to (2.1.4) when $\xi^{\otimes}$ satisfies the symmetry axiom (1.3.33). □

Example 5.1.18. The categories

- Σ' in Definition I.2.4.18 and
- $\mathsf{Vect}_{\mathsf{c}}^{\mathbb{C}}$ of coordinatized finite dimensional complex vector spaces in Example I.2.5.9

are right bipermutative categories, hence also right permbraided categories. ◇

Proposition 5.1.19. *Each right permbraided category is a tight braided bimonoidal category.*

Proof. Each right permbraided category C has the data of a braided bimonoidal category as in Definition 2.1.29. The right distributivity morphism δ^r is the identity by assumption, and δ^l is a natural isomorphism as discussed in Explanation 5.1.12. Consider the 14 axioms in Definition 2.1.29.

- (2.1.4), (2.1.12), and (2.1.32) hold by assumption.
- (2.1.5) follows from (2.1.6) and (2.1.32) as in the proof of Lemma I.2.2.4.
- (2.1.8) follows from (2.1.7) and (2.1.32) as in the proof of Lemma I.2.2.5. The axiom (2.1.7) holds because δ^r is the identity.
- (2.1.9) follows from (2.1.10), (1.3.31), and (2.1.32) as in the proof of Lemma I.2.2.6. The axiom (2.1.10) holds because δ^r is the identity.
- (2.1.13), (2.1.15), (2.1.17), (2.1.18), (2.1.19), and (2.1.33) hold because $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^r, $\xi^{\otimes}_{-,\mathbb{0}}$, and $\xi^{\otimes}_{\mathbb{0},-}$ are the identities.
- (2.1.24) follows from (2.1.18), (2.1.25), and (2.1.32) as in the last diagram in the proof of Lemma I.2.2.11. The axiom (2.1.25) holds because δ^r and $\lambda^{\bullet}$ are the identities.
- (2.1.26) follows from (2.1.27), (1.3.22), and (2.1.32) as in the proof of Lemma I.2.2.12. The axiom (2.1.27) holds because δ^r is the identity.

Therefore, C is a tight braided bimonoidal category. □

Remark 5.1.20. Right permbraided categories in Definition 5.1.11 are precisely Richter's braided bimonoidal categories in [**Ric10**, Def. 5.1]. Braided bimonoidal categories in Definition 2.1.29 are strictly more general than Richter's. ◇

5.2. The Braided Distortion Category

The purpose of this section is to define the braided version of the distortion category in Section I.4.2. The braided distortion category will be the main ingredient in formulating and proving the Coherence Theorem 5.4.4 for braided bimonoidal categories. This section is organized as follows.

- The underlying category $\mathcal{D}^{\mathsf{br}}$ of the braided distortion category is defined in Definition 5.2.2.
- The additive structure in $\mathcal{D}^{\mathsf{br}}$ is defined in Definition 5.2.8.
- The multiplicative structure in $\mathcal{D}^{\mathsf{br}}$ is defined in Definition 5.2.14 and is shown to be a braided strict monoidal category in Lemma 5.2.21.
- The multiplicative zeros, $\lambda^{\bullet}$ and $\rho^{\bullet}$, and the distributivity morphisms, δ^l and δ^r, are defined in Definition 5.2.25.

- The explicit formula for δ^r is in Lemma 5.2.28.
- Theorem 5.2.30 shows that $\mathcal{D}^{\mathsf{br}}$ is a left permbraided category in the sense of Definition 5.1.2.
- Corollaries 5.2.33 and 5.2.34 state that $\mathcal{D}^{\mathsf{br}}$ is a tight braided bimonoidal category and that it satisfies all 24 Laplaza axioms in Definition 2.1.1.

Motivation 5.2.1. In the distortion category $\mathcal{D}$ in Definition I.4.2.1, an object is a finite sequence $\underline{r} = (r_1, \ldots, r_m)$ with length $|\underline{r}| = m \geq 0$ and with each $r_i \geq 0$. We think of $\underline{r}$ as a sum of m monomials, with the ith monomial having r_i factors. For objects $\underline{r}$ and $\underline{s}$ with the same length m, the morphism set $\mathcal{D}(\underline{r};\underline{s})$ is the set of finite sequences of permutations

$$\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) \in \Sigma_m \times \Sigma_{r_1} \times \cdots \times \Sigma_{r_m}$$

such that

$$\sigma\underline{r} = \left(r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(m)}\right) = \underline{s}.$$

For each $1 \leq i \leq m$, $\sigma_i \in \Sigma_{r_i}$ permutes the r_i factors in the ith monomial in $\underline{r}$. The first entry $\sigma \in \Sigma_m$ permutes the m resulting monomials in $\underline{r}$ and leaves the order of the factors in each monomial unchanged.

In a braided bimonoidal category, the additive structure is still a symmetric monoidal category, but the multiplicative structure is a braided monoidal category. Therefore, in the braided analogue of the distortion category, σ should still be a permutation, but each σ_i should be an element in the braid group B_{r_i} in Definition 1.1.1. In the braided distortion category, we think of σ_i as braiding the r_i factors in the ith monomial in $\underline{r}$. In other words, its underlying permutation $\pi(\sigma_i)$ in (1.1.12) permutes the r_i factors in the ith monomial in $\underline{r}$. Composition and other structures must take into account the fact that σ_i belongs to the braid group B_{r_i} and not the symmetric group Σ_{r_i}.

As we will explain in Section 5.4, this braided version of the distortion category is precisely the concept needed to formulate a coherence theorem for braided bimonoidal categories. ⋄

The Underlying Category.

Definition 5.2.2. Define the *braided distortion category* $\mathcal{D}^{\mathsf{br}}$ as follows.

Objects: An object in $\mathcal{D}^{\mathsf{br}}$ is a finite sequence

$$\underline{r} = (r_1, \ldots, r_m)$$

with $m \geq 0$ and each $r_i \geq 0$ for $1 \leq i \leq m$. We call m the *length* of $\underline{r}$, which is denoted by $|\underline{r}|$. The unique sequence with length 0 is denoted by $\varnothing$.

Morphisms: Suppose $\underline{s} = (s_1, \ldots, s_n)$ is an object in $\mathcal{D}^{\mathsf{br}}$. With $\underline{r}$ as above, the morphism set $\mathcal{D}^{\mathsf{br}}(\underline{r};\underline{s})$ is defined as follows.

- If $m \neq n$, then $\mathcal{D}^{\mathsf{br}}(\underline{r};\underline{s})$ is empty.
- If $m = n$, then $\mathcal{D}^{\mathsf{br}}(\underline{r};\underline{s})$ is the set of finite sequences

$$\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) \in \Sigma_m \times B_{r_1} \times \cdots \times B_{r_m}$$

such that

$$\sigma\underline{r} = \left(r_{\sigma^{-1}(1)}, \ldots, r_{\sigma^{-1}(m)}\right) = \underline{s}. \tag{5.2.3}$$

Here Σ_m is the mth symmetric group, and each B_{r_i} is the r_ith braid group in Definition 1.1.1. We call σ the *permutation component* and σ_i the *ith braid component* of $\underline{\sigma}$.

Identities: The identity morphism of an object $\underline{r}$ as above is the sequence

$$1_{\underline{r}} = \big(\mathrm{id}_m; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}\big) \in \Sigma_m \times B_{r_1} \times \cdots \times B_{r_m} \tag{5.2.4}$$

with $1_\varnothing = (\mathrm{id}_0;)$.

Composition: Suppose given morphisms

$$\underline{r} \xrightarrow{\underline{\sigma}} \underline{s} \xrightarrow{\underline{\tau}} \underline{t}$$

in $\mathcal{D}^{\mathsf{br}}$ with $\underline{\sigma}$ as above and with

$$\underline{\tau} = (\tau; \tau_1, \ldots, \tau_m) \in \Sigma_m \times B_{r_{\sigma^{-1}(1)}} \times \cdots \times B_{r_{\sigma^{-1}(m)}}.$$

Their composite

$$\Big(\underline{r} \xrightarrow{\underline{\tau\sigma}} \underline{t} \Big) \in \mathcal{D}^{\mathsf{br}}(\underline{r}; \underline{t})$$

is defined as

$$\underline{\tau\sigma} = \big(\tau\sigma; \tau_{\sigma(1)}\sigma_1, \ldots, \tau_{\sigma(m)}\sigma_m\big) \in \Sigma_m \times B_{r_1} \times \cdots \times B_{r_m}. \tag{5.2.5}$$

This finishes the definition of the braided distortion category. ◇

Explanation 5.2.6. Consider Definition 5.2.2.

- We regard an object $\underline{r} = (r_1, \ldots, r_m) \in \mathcal{D}^{\mathsf{br}}$ as a sum of m monomials, with the ith monomial having r_i factors for $1 \leq i \leq m$.
- A morphism

  $$\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) : \underline{r} \longrightarrow \underline{s} \in \mathcal{D}^{\mathsf{br}}$$

  is regarded as a two-step process.
  - Starting with $\underline{r}$, first it braids the r_i factors in the ith monomial in $\underline{r}$ via the braid $\sigma_i \in B_{r_i}$ for $1 \leq i \leq m$.
  - Then it additively permutes the m resulting monomials via $\sigma \in \Sigma_m$.

  The result is $\underline{s}$.
- The composition (5.2.5) has the same formula (I.4.2.4) as in the distortion category $\mathcal{D}$, except that each $\tau_{\sigma(i)}\sigma_i$ here is a product in the braid group B_{r_i} instead of the symmetric group Σ_{r_i}. ◇

Lemma 5.2.7. *The braided distortion category $\mathcal{D}^{\mathsf{br}}$ in Definition 5.2.2 is a groupoid.*

Proof. The proof is the same as that of Lemma I.4.2.5, which states that the distortion category $\mathcal{D}$ is a groupoid. We simply note that, in each morphism $(\sigma; \sigma_1, \ldots, \sigma_m)$, each entry σ_i for $1 \leq i \leq m$ is in a braid group. □

In what follows, as in Lemma 5.2.7, we often reuse proofs in the symmetric case and simply point out the necessary modifications.

The Additive Structure.

Definition 5.2.8. Define the additive structure

$$\big(\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}\big)$$

in the braided distortion category $\mathcal{D}^{\mathsf{br}}$ as follows.

The Sum: The functor

$$\mathcal{D}^{\mathsf{br}} \times \mathcal{D}^{\mathsf{br}} \xrightarrow{\oplus} \mathcal{D}^{\mathsf{br}}$$

is defined as follows.

Objects: For objects $\underline{r} = (r_1, \ldots, r_m)$ and $\underline{r}' = (r'_1, \ldots, r'_n)$ in $\mathcal{D}^{\mathsf{br}}$, define the object

$$\underline{r} \oplus \underline{r}' = \big(r_1, \ldots, r_m, r'_1, \ldots, r'_n\big) \tag{5.2.9}$$

with length $|\underline{r}| + |\underline{r}'| = m + n$.

Morphisms: For morphisms $\underline{\sigma} = (\sigma; \sigma_1, \ldots, \sigma_m) \in \mathcal{D}^{\mathsf{br}}(\underline{r}; \underline{s})$ and

$$\underline{\sigma}' = (\sigma'; \sigma'_1, \ldots, \sigma'_n) \in \Sigma_n \times B_{r'_1} \times \cdots \times B_{r'_n}$$

in $\mathcal{D}^{\mathsf{br}}(\underline{r}'; \underline{s}')$ with $|\underline{r}'| = |\underline{s}'| = n$, define the morphism

$$\begin{aligned} \underline{\sigma} \oplus \underline{\sigma}' &= \big(\sigma \oplus \sigma'; \sigma_1, \ldots, \sigma_m, \sigma'_1, \ldots, \sigma'_n\big) \\ &\in \Sigma_{m+n} \times B_{r_1} \times \cdots \times B_{r_m} \times B_{r'_1} \times \cdots \times B_{r'_n} \end{aligned} \tag{5.2.10}$$

in $\mathcal{D}^{\mathsf{br}}\big(\underline{r} \oplus \underline{r}'; \underline{s} \oplus \underline{s}'\big)$. Here $\sigma \oplus \sigma' \in \Sigma_{m+n}$ is the block sum in (1.1.8).

The Additive Zero: The object $\mathbb{0}$ is defined as the empty sequence $\varnothing \in \mathcal{D}^{\mathsf{br}}$.

Associativity and Unity: The natural transformations $\alpha^{\oplus}$, $\lambda^{\oplus}$, and $\rho^{\oplus}$, with components, respectively,

$$(\underline{r} \oplus \underline{r}') \oplus \underline{r}'' \xrightarrow{\alpha^{\oplus}_{\underline{r},\underline{r}',\underline{r}''}} \underline{r} \oplus (\underline{r}' \oplus \underline{r}'')$$
$$\varnothing \oplus \underline{r} \xrightarrow{\lambda^{\oplus}_{\underline{r}}} \underline{r}$$
$$\underline{r} \oplus \varnothing \xrightarrow{\rho^{\oplus}_{\underline{r}}} \underline{r}$$

are defined as the identities.

The Additive Symmetry: The natural transformation $\xi^{\oplus}$ has components

$$\begin{tikzcd} \underline{r} \oplus \underline{r}' \arrow[r, "\xi^{\oplus}_{\underline{r},\underline{r}'}"] \arrow[d, equal] & \underline{r}' \oplus \underline{r} \arrow[d, equal] \\ (r_1, \ldots, r_m, r'_1, \ldots, r'_n) & (r'_1, \ldots, r'_n, r_1, \ldots, r_m) \end{tikzcd}$$

defined as

$$\begin{aligned} \xi^{\oplus}_{\underline{r},\underline{r}'} &= \big(\xi^{\oplus}_{m,n}; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}, \mathrm{id}_{r'_1}, \ldots, \mathrm{id}_{r'_n}\big) \\ &\in \Sigma_{m+n} \times B_{r_1} \times \cdots \times B_{r_m} \times B_{r'_1} \times \cdots \times B_{r'_n} \end{aligned} \tag{5.2.11}$$

with $\xi^{\oplus}_{m,n} \in \Sigma_{m+n}$ the block permutation $\tau\langle m, n\rangle$ in (1.2.2).

This finishes the definition of $\big(\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}\big)$ in $\mathcal{D}^{\mathsf{br}}$. ◇

Explanation 5.2.12. Consider Definition 5.2.8 using Explanation 5.2.6.

(1) The sum $\underline{r} \oplus \underline{r}'$ in (5.2.9) corresponds to adding $\underline{r}$ and $\underline{r}'$ such that, in each of $\underline{r}$ and $\underline{r}'$, the order of the monomials and the order of the factors in each monomial remain unchanged.

(2) Consider the sum

$$\underline{\sigma} \oplus \underline{\sigma}' : \underline{r} \oplus \underline{r}' \longrightarrow \underline{s} \oplus \underline{s}'$$

in (5.2.10).

- For each $1 \le i \le m$, $\sigma_i \in B_{r_i}$ braids the r_i factors in the ith monomial in $\underline{r}$.

- For each $1 \leq j \leq n$, $\sigma'_j \in B_{r'_j}$ braids the r'_j factors in the jth monomial in $\underline{r}'$.
- Then the block sum $\sigma \oplus \sigma' \in \Sigma_{m+n}$ permutes
  - the first m resulting monomials via $\sigma \in \Sigma_m$ and
  - the last n resulting monomials via $\sigma' \in \Sigma_n$.

(3) The additive symmetry $\xi^{\oplus}_{\underline{r},\underline{r}'}$ in (5.2.11) swaps the places of $\underline{r}$ and $\underline{r}'$ and leaves the orders of both the monomials and the factors in each monomial in each of $\underline{r}$ and $\underline{r}'$ unchanged. The formula (5.2.11) is the same as in the distortion category (I.4.2.10), except that each $\mathrm{id}_?$ here is an identity braid instead of an identity permutation. $\diamond$

Lemma 5.2.13. *In Definition 5.2.8, the tuple*

$$\left(\mathcal{D}^{\mathsf{br}}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}\right)$$

is a permutative category.

Proof. We reuse the proof of Lemma I.4.2.12, which states that the distortion category $\mathcal{D}$ with its additive structure is a permutative category. In the proofs of the naturality of $\xi^{\oplus}$ and of the symmetric monoidal category axioms, we use the fact that each component of $\xi^{\oplus}$ in (5.2.11) has a sequence of identity braids after the permutation component $\xi^{\oplus}_{m,n}$. $\square$

The Multiplicative Structure.

Definition 5.2.14. Define the multiplicative structure

$$\left(\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}\right)$$

in the braided distortion category $\mathcal{D}^{\mathsf{br}}$ as follows.

The Product: The functor

$$\mathcal{D}^{\mathsf{br}} \times \mathcal{D}^{\mathsf{br}} \xrightarrow{\ \otimes\ } \mathcal{D}^{\mathsf{br}}$$

is defined as follows.

Objects: For objects $\underline{r} = (r_1, \ldots, r_m)$ and $\underline{r}' = (r'_1, \ldots, r'_n)$ in $\mathcal{D}^{\mathsf{br}}$, define the object

$$\underline{r} \otimes \underline{r}' = \left(\left\{r_i + r'_1\right\}_{i=1}^m, \ldots, \left\{r_i + r'_n\right\}_{i=1}^m\right) \tag{5.2.15}$$

with length $|\underline{r}| \cdot |\underline{r}'| = mn$. The $\big(i + (j-1)m\big)$th entry is $r_i + r'_j$ for each $1 \leq i \leq m$ and $1 \leq j \leq n$.

Morphisms: For morphisms $\underline{\sigma} \in \mathcal{D}^{\mathsf{br}}(\underline{r};\underline{s})$ and $\underline{\sigma}' \in \mathcal{D}^{\mathsf{br}}(\underline{r}';\underline{s}')$ as in Definition 5.2.8, define the morphism

$$\underline{\sigma} \otimes \underline{\sigma}' = \left(\sigma \otimes \sigma'; \left\{\sigma_i \oplus \sigma'_1\right\}_{i=1}^m, \ldots, \left\{\sigma_i \oplus \sigma'_n\right\}_{i=1}^m\right) \in \mathcal{D}^{\mathsf{br}}\left(\underline{r} \otimes \underline{r}'; \underline{s} \otimes \underline{s}'\right) \tag{5.2.16}$$

with the following entries.

- The permutation component $\sigma \otimes \sigma' \in \Sigma_{mn}$ is defined by

$$(\sigma \otimes \sigma')\big(i + (j-1)m\big) = \sigma(i) + \big(\sigma'(j) - 1\big)m \tag{5.2.17}$$

  for $1 \leq i \leq m$ and $1 \leq j \leq n$.
- The $\big(i + (j-1)m\big)$th braid component $\sigma_i \oplus \sigma'_j \in B_{r_i + r'_j}$ is the sum braid (1.1.10).

The Multiplicative Unit: The object $\mathbb{1}$ is the sequence $(0) \in \mathcal{D}^{\mathsf{br}}$ with length 1 and the entry 0.

Associativity and Unity: The natural transformations $\alpha^{\otimes}$, $\lambda^{\otimes}$, and $\rho^{\otimes}$, with components, respectively,

$$\begin{aligned} (\underline{r} \otimes \underline{r}') \otimes \underline{r}'' &\xrightarrow{\alpha^{\otimes}_{\underline{r},\underline{r}',\underline{r}''}} \underline{r} \otimes (\underline{r}' \otimes \underline{r}'') \\ (0) \otimes \underline{r} &\xrightarrow{\lambda^{\otimes}_{\underline{r}}} \underline{r} \\ \underline{r} \otimes (0) &\xrightarrow{\rho^{\otimes}_{\underline{r}}} \underline{r} \end{aligned}$$

are defined as the identities.

The Braiding: The natural transformation $\xi^{\otimes}$ has components

$$\begin{array}{ccc} \underline{r} \otimes \underline{r}' & \xrightarrow{\qquad \xi^{\otimes}_{\underline{r},\underline{r}'} \qquad} & \underline{r}' \otimes \underline{r} \\ \parallel & & \parallel \\ \Big(\{r_i + r'_1\}_{i=1}^{m}, \ldots, \{r_i + r'_n\}_{i=1}^{m}\Big) & & \Big(\{r'_j + r_1\}_{j=1}^{n}, \ldots, \{r'_j + r_m\}_{j=1}^{n}\Big) \end{array}$$

defined as

$$\xi^{\otimes}_{\underline{r},\underline{r}'} = \Big(\xi^{\otimes}_{m,n}; \{b^{\oplus}_{r_i,r'_1}\}_{i=1}^{m}, \ldots, \{b^{\oplus}_{r_i,r'_n}\}_{i=1}^{m}\Big) \tag{5.2.18}$$

with the following entries.

- The permutation component $\xi^{\otimes}_{m,n} \in \Sigma_{mn}$ is defined by

$$\xi^{\otimes}_{m,n}\big(i + (j-1)m\big) = j + (i-1)n \tag{5.2.19}$$

  for $1 \leq i \leq m$ and $1 \leq j \leq n$.
- The $\big(i + (j-1)m\big)$th braid component

$$b^{\oplus}_{r_i,r'_j} = s_1^{(2)}\langle r_i, r'_j\rangle \in B_{r_i + r'_j}$$

  is an elementary block braid (1.2.4).

This finishes the definition of $\big(\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}\big)$ in $\mathcal{D}^{\mathsf{br}}$. ◇

Explanation 5.2.20. Consider Definition 5.2.14 using Explanation 5.2.6.

(1) Regarding $\underline{r}$ as a sum of m monomials and $\underline{r}'$ as a sum of n monomials, the object $\underline{r} \otimes \underline{r}'$ in (5.2.15) is an expanded form of their product. The resulting mn monomials are arranged into an $n \times m$ matrix.

$$\underline{r} \otimes \underline{r}' = \begin{bmatrix} r_1 + r'_1 & \cdots & r_m + r'_1 \\ \vdots & \ddots & \vdots \\ r_1 + r'_n & \cdots & r_m + r'_n \end{bmatrix}$$

For $1 \leq i \leq m$ and $1 \leq j \leq n$, the (j,i)-entry of the matrix is the product of the ith monomial in $\underline{r}$ with the jth monomial in $\underline{r}'$, which has $r_i + r'_j$ factors.

(2) Consider the morphism $\underline{\sigma} \otimes \underline{\sigma}'$ in (5.2.16).

- The $\big(i + (j-1)m\big)$th braid component $\sigma_i \oplus \sigma'_j \in B_{r_i + r'_j}$ acts on the (j,i)-entry $r_i + r'_j$ of $\underline{r} \otimes \underline{r}'$. It braids the first r_i factors via $\sigma_i \in B_{r_i}$ and the last r'_j factors via $\sigma'_j \in B_{r'_j}$.

- In the permutation component $\sigma \otimes \sigma'$ in (5.2.17),
  - the factor σ permutes the m columns in the $n \times m$ matrix, and
  - the factor σ' permutes the n rows in the $n \times m$ matrix.

(3) Consider the braiding $\xi^{\otimes}_{\underline{r},\underline{r}'}$ in (5.2.18).

- The $\big(i+(j-1)m\big)$th braid component $b^{\oplus}_{r_i,r'_j}$ acts on the (j,i)-entry $r_i + r'_j$ of $\underline{r} \otimes \underline{r}'$ by braiding the jth monomial in $\underline{r}'$ over the ith monomial in $\underline{r}$.
- The permutation component $\xi^{\otimes}_{m,n}$ in (5.2.19) takes the transpose of the resulting $n \times m$ matrix. ⋄

The proof of the next lemma refers to the left bipermutative category Σ in Proposition I.2.4.8.

Lemma 5.2.21. *In Definition 5.2.14, the tuple*

$$\big(\mathcal{D}^{\mathsf{br}}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}\big)$$

is a braided strict monoidal category.

Proof. We follow the proof of Lemma I.4.2.19, which states that the distortion category $\mathcal{D}$ with its multiplicative structure is a permutative category, with the following notes and adjustments.

- To show that $\otimes$ preserves composition in $\mathcal{D}^{\mathsf{br}}$, we note that the sum braid $\oplus$ in (1.1.10) is multiplicative.
- To show that $\alpha^{\otimes} = 1$ is well defined, we note that the sum braid $\oplus$ is strictly associative.
- Each component $\xi^{\otimes}_{\underline{r},\underline{r}'}$ in (5.2.18) is an isomorphism with inverse

$$\xi^{-\otimes}_{\underline{r},\underline{r}'} = \Big(\xi^{\otimes}_{n,m}; \big\{b^{-\oplus}_{r_1,r'_j}\big\}_{j=1}^{n}, \ldots, \big\{b^{-\oplus}_{r_m,r'_j}\big\}_{j=1}^{n}\Big). \tag{5.2.22}$$

  Here

$$\xi^{\otimes}_{n,m} = \big(\xi^{\otimes}_{m,n}\big)^{-1} \in \Sigma_{mn},$$

  which follows from the definition (5.2.19). The $\big(j+(i-1)n\big)$th braid component is defined as

$$\begin{aligned} b^{-\oplus}_{r_i,r'_j} &= \big(s_1\langle r_i, r'_j\rangle\big)^{-1} \\ &= s_1^{-1}\langle r'_j, r_i\rangle \in B_{r_i + r'_j}. \end{aligned} \tag{5.2.23}$$

  This is the inverse of $b^{\oplus}_{r_i,r'_j}$ defined in (1.1.23).
- In the proof of the naturality of the braiding $\xi^{\otimes}$ in $\mathcal{D}^{\mathsf{br}}$, the permutation component is the same as in the proof of Lemma I.4.2.19. The required equality in each braid component follows from (1.2.14).
- In the proof of the right hexagon axiom (1.3.17) in $\mathcal{D}^{\mathsf{br}}$, the permutation component is the equality

$$\xi^{\otimes}_{lm,n} = \big(\xi^{\otimes}_{l,n} \otimes \mathrm{id}_m\big)\big(\mathrm{id}_l \otimes \xi^{\otimes}_{m,n}\big) \in \Sigma_{lmn}$$

  for $l, m, n \geq 0$. This is the hexagon axiom (I.2.4.10) in the multiplicative structure in the left bipermutative category Σ. The required equality in each braid component follows from (1.2.17).

- In the proof of the left hexagon axiom (1.3.17) in $\mathcal{D}^{\mathsf{br}}$, the permutation component is the equality
$$\xi^{\otimes}_{l,mn} = \big(\mathrm{id}_m \otimes \xi^{\otimes}_{l,n}\big)\big(\xi^{\otimes}_{l,m} \otimes \mathrm{id}_n\big) \in \Sigma_{lmn}$$
for $l, m, n \geq 0$. This is the left hexagon axiom in the multiplicative structure in the left bipermutative category Σ. The left hexagon axiom holds in $(\Sigma, \otimes)$ by Proposition 1.3.36. The required equality in each braid component follows from (1.2.18).

This finishes the proof. $\square$

Explanation 5.2.24. The multiplicative structure in the braided distortion category $\mathcal{D}^{\mathsf{br}}$ in Lemma 5.2.21 is genuinely a braided monoidal category and not a symmetric monoidal category. The reason is that the braiding-square $\xi^{\otimes}_{\underline{r}',\underline{r}}\xi^{\otimes}_{\underline{r},\underline{r}'}$ is not the identity in general. In fact, its typical braid component is
$$b^{\oplus}_{r'_j,r_i} b^{\oplus}_{r_i,r'_j} = s_1\langle r'_j, r_i\rangle \cdot s_1\langle r_i, r'_j\rangle \in B_{r_i+r'_j}.$$
This is not the identity braid in general. For example, $b^{\oplus}_{3,2}b^{\oplus}_{2,3} \in B_5$ is the nonidentity braid in Example 1.1.29. $\diamond$

The Multiplicative Zeros and Distributivity.

Definition 5.2.25. Define the natural transformations
$$\big(\lambda^{\bullet}, \rho^{\bullet}, \delta^l, \delta^r\big)$$
for the braided distortion category $\mathcal{D}^{\mathsf{br}}$ as follows, for objects $\underline{r}, \underline{r}', \underline{r}'' \in \mathcal{D}^{\mathsf{br}}$.

The Multiplicative Zeros and the Left Distributivity: $\lambda^{\bullet}$, $\rho^{\bullet}$, and δ^l, with components, respectively,
$$\mathbb{0} \otimes \underline{r} \xrightarrow{\lambda^{\bullet}_{\underline{r}}} \mathbb{0}$$
$$\underline{r} \otimes \mathbb{0} \xrightarrow{\rho^{\bullet}_{\underline{r}}} \mathbb{0}$$
$$\underline{r} \otimes (\underline{r}' \oplus \underline{r}'') \xrightarrow{\delta^l_{\underline{r},\underline{r}',\underline{r}''}} (\underline{r} \otimes \underline{r}') \oplus (\underline{r} \otimes \underline{r}'')$$
are defined as the identities.

The Right Distributivity: δ^r has components the composites

(5.2.26)
$$\begin{tikzcd}
(\underline{r} \oplus \underline{r}') \otimes \underline{r}'' \arrow[r, "\delta^r_{\underline{r},\underline{r}',\underline{r}''}"] \arrow[d, "\xi^{\otimes}_{\underline{r}\oplus\underline{r}',\underline{r}''}"'] & (\underline{r} \otimes \underline{r}'') \oplus (\underline{r}' \otimes \underline{r}'') \\
\underline{r}'' \otimes (\underline{r} \oplus \underline{r}') \arrow[r, "\delta^l_{\underline{r}'',\underline{r},\underline{r}'}", "="'] & (\underline{r}'' \otimes \underline{r}) \oplus (\underline{r}'' \otimes \underline{r}') \arrow[u, "\xi^{-\otimes}_{\underline{r},\underline{r}''} \oplus \xi^{-\otimes}_{\underline{r}',\underline{r}''}"']
\end{tikzcd}$$

with $\xi^{\otimes}$ the braiding in (5.2.18).

This finishes the definition of $(\lambda^{\bullet}, \rho^{\bullet}, \delta^l, \delta^r)$ for $\mathcal{D}^{\mathsf{br}}$. $\diamond$

Lemma 5.2.27. *The natural transformations $(\lambda^{\bullet}, \rho^{\bullet}, \delta^l, \delta^r)$ in Definition 5.2.25 are well defined.*

Proof. We reuse the proof of Lemma I.4.2.25, which states that the constructions $(\lambda^{\bullet}, \rho^{\bullet}, \delta^l, \delta^r)$ in the distortion category $\mathcal{D}$ are natural transformations. $\square$

Next we provide an explicit description of the right distributivity morphism in the braided distortion category. The following observation is the braided analogue of Lemma I.4.2.27.

Lemma 5.2.28. *For objects $\underline{r}, \underline{r}', \underline{r}'' \in \mathcal{D}^{\mathrm{br}}$ with lengths $|\underline{r}| = m$, $|\underline{r}'| = n$, and $|\underline{r}''| = p$, the right distributivity morphism*

$$\begin{array}{c} (\underline{r} \oplus \underline{r}') \otimes \underline{r}'' = \Big(\{r_i + r_1''\}_{i=1}^m, \{r_j' + r_1''\}_{j=1}^n, \ldots, \{r_i + r_p''\}_{i=1}^m, \{r_j' + r_p''\}_{j=1}^m \Big) \\ \Big\downarrow \delta^r_{\underline{r},\underline{r}',\underline{r}''} \\ (\underline{r} \otimes \underline{r}'') \oplus (\underline{r}' \otimes \underline{r}'') = \Big(\{r_i + r_1''\}_{i=1}^m, \ldots, \{r_i + r_p''\}_{i=1}^m, \{r_j' + r_1''\}_{j=1}^n, \ldots, \{r_j' + r_p''\}_{j=1}^n \Big) \end{array}$$

in (5.2.26) is given by

$$\begin{gathered} \Big((\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,n}) \xi^{\otimes}_{m+n,p} ; \{\mathrm{id}_{r_i + r_1''}\}_{i=1}^m, \{\mathrm{id}_{r_j' + r_1''}\}_{j=1}^n, \ldots, \{\mathrm{id}_{r_i + r_p''}\}_{i=1}^m, \{\mathrm{id}_{r_j' + r_p''}\}_{j=1}^n \Big) \\ \in \Sigma_{(m+n)p} \times \prod_{k=1}^p \Bigg[\Big(\prod_{i=1}^m B_{r_i + r_k''} \Big) \times \Big(\prod_{j=1}^n B_{r_j' + r_k''} \Big) \Bigg]. \end{gathered}$$

Here

$$\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,n} \in \Sigma_{pm+pn}$$

is the block sum in (1.1.8).

Proof. By (5.2.10), (5.2.18), and (5.2.22), the two vertical morphisms in (5.2.26) are as follows.

$$\begin{aligned} \xi^{\otimes}_{\underline{r} \oplus \underline{r}', \underline{r}''} &= \Big(\xi^{\otimes}_{m+n,p} ; \{b^{\oplus}_{r_i, r_1''}\}_{i=1}^m, \{b^{\oplus}_{r_j', r_1''}\}_{j=1}^n, \ldots, \{b^{\oplus}_{r_i, r_p''}\}_{i=1}^m, \{b^{\oplus}_{r_j', r_p''}\}_{j=1}^n \Big) \\ \xi^{-\otimes}_{\underline{r}, \underline{r}''} \oplus \xi^{-\otimes}_{\underline{r}', \underline{r}''} &= \Big(\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,n} ; \{b^{-\oplus}_{r_1, r_k''}\}_{k=1}^p, \ldots, \{b^{-\oplus}_{r_m, r_k''}\}_{k=1}^p, \{b^{-\oplus}_{r_1', r_k''}\}_{k=1}^p, \ldots, \{b^{-\oplus}_{r_n', r_k''}\}_{k=1}^p \Big) \end{aligned}$$

The stated formula for $\delta^r_{\underline{r},\underline{r}',\underline{r}''}$ now follows from (5.2.5) and the fact that the braid $b^{-\oplus}_{a,b}$ in (5.2.23) is the inverse of the elementary block braid $b^{\oplus}_{a,b}$. □

Explanation 5.2.29. Consider Lemma 5.2.28.

(1) The right distributivity morphism δ^r has only identity braid components. Its only nonidentity part is the permutation component

$$(\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,n}) \xi^{\otimes}_{m+n,p} \in \Sigma_{(m+n)p}.$$

As in Explanation 5.2.20, suppose
- A is the $p \times m$ matrix with entries those in $\underline{r} \otimes \underline{r}''$, and
- B is the $p \times n$ matrix with entries those in $\underline{r}' \otimes \underline{r}''$.

Then the domain and the codomain of $\delta^r_{\underline{r},\underline{r}',\underline{r}''}$ are the following arrangements.

$$\begin{aligned} (\underline{r} \oplus \underline{r}') \otimes \underline{r}'' &= [A \,|\, B] \\ (\underline{r} \otimes \underline{r}'') \oplus (\underline{r}' \otimes \underline{r}'') &= \left[\frac{A}{B} \right] \end{aligned}$$

(2) The formula for δ^r is the same as in Lemma I.4.2.27 for δ^r in the distortion category $\mathcal{D}$, except that each $\mathrm{id}_?$ here is an identity braid instead of an identity permutation.

(3) Among the structure morphisms $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\xi^{\otimes}$, $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^l, and δ^r in $\mathcal{D}^{\mathrm{br}}$, only
- $\xi^{\oplus}$ in (5.2.11),

- $\xi^{\otimes}$ in (5.2.18), and
- δ^r in (5.2.26) and Lemma 5.2.28

are nonidentities. Moreover, among these three structure morphisms, only $\xi^{\otimes}$ has nonidentity braid components. $\diamond$

The Main Result. Next is the braided analogue of Theorem I.4.2.29, which states that the distortion category $\mathcal{D}$ is a left bipermutative category. Recall the notion of a left permbraided category in Definition 5.1.2.

Theorem 5.2.30. *The braided distortion category $\mathcal{D}^{\mathsf{br}}$ with*

- *the underlying category in Definition 5.2.2,*
- *the additive structure in Definition 5.2.8,*
- *the multiplicative structure in Definition 5.2.14, and*
- *the natural transformations $(\lambda^{\bullet}, \rho^{\bullet}, \delta^l, \delta^r)$ in Definition 5.2.25*

is a left permbraided category.

Proof. By Lemmas 5.2.7, 5.2.13, 5.2.21, and 5.2.27, it remains to check conditions (3) and (4) in Definition 5.1.2 for $\mathcal{D}^{\mathsf{br}}$.

For each object $\underline{r} \in \mathcal{D}^{\mathsf{br}}$, both $\xi^{\otimes}_{\underline{r},\varnothing}$ and $\xi^{\otimes}_{\varnothing,\underline{r}}$ are equal to $(\mathrm{id}_0;) = 1_{\varnothing}$ by definition (5.2.18). This proves condition (3) in $\mathcal{D}^{\mathsf{br}}$.

For condition (4), the axiom (2.1.4) holds by the definition (5.2.26) of δ^r.

For the axioms (2.1.5) and (2.1.12), we reuse the proofs of these axioms in the distortion category $\mathcal{D}$ in Theorem I.4.2.29. For (2.1.12), instead of Lemma I.4.2.27, we use the corresponding Lemma 5.2.28 here.

Finally, the axiom (2.1.32) asserts the commutativity of the diagram

$$\begin{tikzcd}
\underline{r}'' \otimes (\underline{r} \oplus \underline{r}') \arrow[r, "\delta^l_{\underline{r}'',\underline{r},\underline{r}'}", "="'] \arrow[d, "\xi^{\otimes}_{\underline{r}'',\underline{r}\oplus\underline{r}'}"'] & (\underline{r}'' \otimes \underline{r}) \oplus (\underline{r}'' \otimes \underline{r}') \arrow[d, "\xi^{\otimes}_{\underline{r}'',\underline{r}} \oplus \xi^{\otimes}_{\underline{r}'',\underline{r}'}"] \\
(\underline{r} \oplus \underline{r}') \otimes \underline{r}'' \arrow[r, "\delta^r_{\underline{r},\underline{r}',\underline{r}''}"] & (\underline{r} \otimes \underline{r}'') \oplus (\underline{r}' \otimes \underline{r}'')
\end{tikzcd} \tag{5.2.31}$$

for objects $\underline{r}, \underline{r}', \underline{r}'' \in \mathcal{D}^{\mathsf{br}}$. To prove the commutativity of (5.2.31), first note that, by definition (5.2.18), the right vertical morphism is the sum of the following two morphisms, where $|\underline{r}| = m$, $|\underline{r}'| = n$, and $|\underline{r}''| = p$.

$$\begin{aligned}
\xi^{\otimes}_{\underline{r}'',\underline{r}} &= \left(\xi^{\otimes}_{p,m}; \left\{b^{\oplus}_{r''_k,r_1}\right\}_{k=1}^{p}, \ldots, \left\{b^{\oplus}_{r''_k,r_m}\right\}_{k=1}^{p}\right) \\
\xi^{\otimes}_{\underline{r}'',\underline{r}'} &= \left(\xi^{\otimes}_{p,n}; \left\{b^{\oplus}_{r''_k,r'_1}\right\}_{k=1}^{p}, \ldots, \left\{b^{\oplus}_{r''_k,r'_n}\right\}_{k=1}^{p}\right)
\end{aligned}$$

By definition (5.2.10), their sum is the following morphism.

$$\begin{aligned}
&\xi^{\otimes}_{\underline{r}'',\underline{r}} \oplus \xi^{\otimes}_{\underline{r}'',\underline{r}'} \\
&= \left(\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,n}; \left\{b^{\oplus}_{r''_k,r_1}\right\}_{k=1}^{p}, \ldots, \left\{b^{\oplus}_{r''_k,r_m}\right\}_{k=1}^{p}, \left\{b^{\oplus}_{r''_k,r'_1}\right\}_{k=1}^{p}, \ldots, \left\{b^{\oplus}_{r''_k,r'_n}\right\}_{k=1}^{p}\right)
\end{aligned} \tag{5.2.32}$$

On the other hand, by (5.2.9) and (5.2.18), the left vertical morphism in (5.2.31) is the following morphism.

$$\xi^{\otimes}_{\underline{r}'',\underline{r}\oplus\underline{r}'} = \left(\xi^{\otimes}_{p,m+n}; \left\{b^{\oplus}_{r''_k,r_1}\right\}_{k=1}^{p}, \ldots, \left\{b^{\oplus}_{r''_k,r_m}\right\}_{k=1}^{p}, \left\{b^{\oplus}_{r''_k,r'_1}\right\}_{k=1}^{p}, \ldots, \left\{b^{\oplus}_{r''_k,r'_n}\right\}_{k=1}^{p}\right)$$

Combining this with (5.2.5) and the formula for $\delta^r_{\underline{r},\underline{r}',\underline{r}''}$ in Lemma 5.2.28, the composite $\delta^r_{\underline{r},\underline{r}',\underline{r}''}\xi^{\otimes}_{\underline{r}'',\underline{r}\oplus\underline{r}'}$ is the following morphism.

$$\Big((\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,n})\underbrace{\xi^{\otimes}_{m+n,p}\xi^{\otimes}_{p,m+n}}_{\mathrm{id}_{p(m+n)}}; \{b^{\oplus}_{r''_k,r_1}\}_{k=1}^{p}, \ldots, \{b^{\oplus}_{r''_k,r_m}\}_{k=1}^{p}, \{b^{\oplus}_{r''_k,r'_1}\}_{k=1}^{p}, \ldots, \{b^{\oplus}_{r''_k,r'_n}\}_{k=1}^{p}\Big)$$

This is equal to the morphism in (5.2.32) because

$$\xi^{\otimes}_{m+n,p} = \big(\xi^{\otimes}_{p,m+n}\big)^{-1} \in \Sigma_{p(m+n)},$$

which follows from the definition (5.2.19). This proves that (5.2.31) is commutative. □

Corollary 5.2.33. *The braided distortion category $\mathcal{D}^{\mathrm{br}}$ is a tight braided bimonoidal category.*

Proof. This follows from Proposition 5.1.10 and Theorem 5.2.30. □

Corollary 5.2.34. *The braided distortion category $\mathcal{D}^{\mathrm{br}}$ satisfies all 24 Laplaza axioms in Definition 2.1.1.*

Proof. This follows from Theorem 2.2.1 and Corollary 5.2.33. □

5.3. The Braided Distortion of a Path

In preparation for the Coherence Theorem 5.4.4, in this section, we define the braided distortion of a path in $\mathsf{Gr}(X)$. For the reader's convenience, first we recall some graph theoretic definitions from Chapter I.3, including that of the graph $\mathsf{Gr}(X)$ of a set X with two distinguished elements $\{0^X, 1^X\}$. Then we define the value of a path $P \in \mathsf{Gr}(X)$ in a braided bimonoidal category. In Definition 5.3.15, we define the braided distortion of a path $P \in \mathsf{Gr}(X)$ using a particular function $\vartheta : X \longrightarrow \mathsf{Ob}(\mathcal{D}^{\mathrm{br}})$, with $\mathcal{D}^{\mathrm{br}}$ the braided distortion category in Section 5.2. This section ends with some examples.

Elementary Graph. The following definitions are from Section I.3.1.

Definition 5.3.1. For a set S, its *free $\{\oplus,\otimes\}$-algebra* is the set S^{fr} defined inductively by the following two conditions.

- $S \subset S^{\mathsf{fr}}$.
- If $a, b \in S^{\mathsf{fr}}$, then the symbols

$$a \oplus b \quad \text{and} \quad a \otimes b$$

  also belong to S^{fr}. They are called, respectively, the *sum* and the *product* of a and b.

We sometimes abbreviate $a \otimes b$ to ab. In the absence of clarifying parentheses, $\otimes$ takes precedence over $\oplus$. ◇

Definition 5.3.2. A *graph* $G = (V, E)$ is a pair consisting of the following data.

- V is a class. An element in V is called a *vertex* in G.
- E assigns to each ordered pair (u, v) with $u, v \in V$ a set $E(u, v)$, an element of which is called an *edge* with *domain* u and *codomain* v. We also denote such an edge by

$$u \longrightarrow v, \quad e : u \longrightarrow v, \quad \text{or} \quad u \xrightarrow{e} v$$

if e is the name of the edge.

A *path* in such a graph is a nonempty finite sequence of edges $(e_n,\ldots,e_1)$ as in

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_n.$$

Such a path is denoted by

$$v_0 \twoheadrightarrow v_n$$

and is said to have *length* n, domain v_0, and codomain v_n. ◇

Example 5.3.3. Each category C has an associated graph (V,E) with V the class of objects in C, and $E(u,v)=\mathsf{C}(u,v)$ for objects $u,v\in\mathsf{C}$. A nonempty finite sequence of composable morphisms in C yields a path in the associated graph. ◇

Definition 5.3.4. Suppose X is a set with two distinguished elements 0^X and 1^X, called the *additive zero* and the *multiplicative unit*, respectively. The *elementary graph of* X, which is denoted by $\mathsf{Gr}^{\mathsf{el}}(X)$, is the graph defined as follows.

Vertices: The set of vertices in $\mathsf{Gr}^{\mathsf{el}}(X)$ is the free $\{\oplus,\otimes\}$-algebra X^{fr} of X.

Edges: Edges in $\mathsf{Gr}^{\mathsf{el}}(X)$ are of the following types for all $x,y,z\in X^{\mathsf{fr}}$.

The Additive Structure:

$$(x\oplus y)\oplus z \underset{\alpha^{-\oplus}_{x,y,z}}{\overset{\alpha^{\oplus}_{x,y,z}}{\rightleftarrows}} x\oplus(y\oplus z) \qquad x\oplus y \underset{\xi^{-\oplus}_{x,y}}{\overset{\xi^{\oplus}_{x,y}}{\rightleftarrows}} y\oplus x$$

$$0^X\oplus x \underset{\lambda^{-\oplus}_{x}}{\overset{\lambda^{\oplus}_{x}}{\rightleftarrows}} x \underset{\rho^{-\oplus}_{x}}{\overset{\rho^{\oplus}_{x}}{\leftrightarrows}} x\oplus 0^X$$

The Multiplicative Structure:

$$(x\otimes y)\otimes z \underset{\alpha^{-\otimes}_{x,y,z}}{\overset{\alpha^{\otimes}_{x,y,z}}{\rightleftarrows}} x\otimes(y\otimes z) \qquad x\otimes y \underset{\xi^{-\otimes}_{x,y}}{\overset{\xi^{\otimes}_{x,y}}{\rightleftarrows}} y\otimes x$$

$$1^X\otimes x \underset{\lambda^{-\otimes}_{x}}{\overset{\lambda^{\otimes}_{x}}{\rightleftarrows}} x \underset{\rho^{-\otimes}_{x}}{\overset{\rho^{\otimes}_{x}}{\leftrightarrows}} x\otimes 1^X$$

The Multiplicative Zeros:

$$0^X\otimes x \underset{\lambda^{-\bullet}_{x}}{\overset{\lambda^{\bullet}_{x}}{\rightleftarrows}} 0^X \underset{\rho^{-\bullet}_{x}}{\overset{\rho^{\bullet}_{x}}{\leftrightarrows}} x\otimes 0^X$$

The Distributivity Morphisms:

$$x\otimes(y\oplus z) \xrightarrow{\delta^{l}_{x,y,z}} (x\otimes y)\oplus(x\otimes z)$$

$$(x\oplus y)\otimes z \xrightarrow{\delta^{r}_{x,y,z}} (x\otimes z)\oplus(y\otimes z)$$

Identities:

$$x \xrightarrow{1_x} x$$

This finishes the definition of $\mathsf{Gr}^{\mathsf{el}}(X)$.

Moreover, we define the following.

- The set of edges in $\mathsf{Gr}^{\mathsf{el}}(X)$ is denoted by $\mathsf{E}_{\mathsf{el}}(X)$, the elements of which are called *elementary edges*.
- $\alpha^{\oplus}$ and $\alpha^{-\oplus}$ are *formal inverses* of each other, and similarly for the other 9 pairs of elementary edges in the first three groups above.
- 1_x is called the *identity* of x.
- The names in Definition 2.1.29 are reused for elementary edges. For example, $\xi^{\otimes}$ is called the braiding. $\diamond$

Graph.

Definition 5.3.5. With $(X, 0^X, 1^X)$ as in Definition 5.3.4, consider the free $\{\oplus, \otimes\}$-algebra $\mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ of the set $\mathsf{E}_{\mathsf{el}}(X)$ of elementary edges. The *domain* and the *codomain* of an element $f \in \mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ are elements in X^{fr} defined inductively as follows.

- For an elementary edge $f \in \mathsf{E}_{\mathsf{el}}(X)$, its (co)domain are those of f in the elementary graph $\mathsf{Gr}^{\mathsf{el}}(X)$.
- Suppose $f_1, f_2 \in \mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ with
  - $u_i \in X^{\mathsf{fr}}$ the domain of f_i and
  - $v_i \in X^{\mathsf{fr}}$ the codomain of f_i

  already defined for $i = 1, 2$. Then
  - $f_1 \oplus f_2$ has domain $u_1 \oplus u_2$ and codomain $v_1 \oplus v_2$, and
  - $f_1 \otimes f_2$ has domain $u_1 \otimes u_2$ and codomain $v_1 \otimes v_2$. $\diamond$

Definition 5.3.6. Continuing Definition 5.3.5, *identity prime edges* and *nonidentity prime edges* are elements in $\mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ defined inductively by the following four conditions.

- Elementary edges of the type 1_x for $x \in X^{\mathsf{fr}}$ are identity prime edges.
- Elementary edges not of the type 1_x for $x \in X^{\mathsf{fr}}$ are nonidentity prime edges.
- If $e_1, e_2 \in \mathsf{E}^{\mathsf{fr}}_{\mathsf{el}}(X)$ are identity prime edges, then so are $e_1 \oplus e_2$ and $e_2 \otimes e_2$.
- If f is a nonidentity prime edge and if e is an identity prime edge, then

$$f \oplus e, \quad e \oplus f, \quad f \otimes e, \quad \text{and} \quad e \otimes f$$

  are nonidentity prime edges.

Moreover, we define the following.

- A *prime edge* means either an identity prime edge or a nonidentity prime edge. An identity prime edge is also called an *identity*.
- The set of prime edges is denoted by $\mathsf{E}^{\mathsf{pr}}(X)$. $\diamond$

Note that each nonidentity prime edge involves a finite number of identity elementary edges and only one nonidentity elementary edge.

Definition 5.3.7. With $(X, 0^X, 1^X)$ as in Definition 5.3.4, the *graph of* X, which is denoted by $\mathsf{Gr}(X)$, is the graph with the following vertices and edges.

Vertices: The set of vertices in $\mathsf{Gr}(X)$ is the free $\{\oplus, \otimes\}$-algebra X^{fr} of X.

Edges: The set of edges in $\mathsf{Gr}(X)$ is the set $\mathsf{E}^{\mathsf{pr}}(X)$ of prime edges as in Definition 5.3.6, with (co)domain as in Definition 5.3.5. $\diamond$

The following notions are from Definitions I.3.6.2, I.3.8.1, and I.3.8.8 and Explanation I.3.8.9.

Definition 5.3.8. In the context of Definition 5.3.7, define the following notions in $\mathsf{Gr}(X)$.

- A *δ-prime edge* is a prime edge that involves either δ^l or δ^r.
- An element $a \in X^{\mathsf{fr}}$ is *δ-reduced* if it is not the domain of any δ-prime edges.
- A *1^X-prime edge* is a prime edge that involves either $\lambda^{\otimes}$ or $\rho^{\otimes}$.
- An element $a \in X^{\mathsf{fr}}$ is *1^X-reduced* if it is not the domain of any 1^X-prime edges.
- A *1^X-free path* in $\mathsf{Gr}(X)$ is a path in which each prime edge is either an identity or involves a single instance of $\alpha^{\pm\oplus}$, $\xi^{\pm\oplus}$, $\alpha^{\pm\otimes}$, or $\xi^{\pm\otimes}$. $\diamond$

The Value of a Path. The graph of X is interpreted in a braided bimonoidal category via the following concept.

Definition 5.3.9. Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs. A *graph morphism* $f : G_1 \longrightarrow G_2$ consists of functions

- $f_V : V_1 \longrightarrow V_2$ and
- $f_E : E_1(u,v) \longrightarrow E_2\big(f_V u, f_V v\big)$ for $u, v \in V_1$.

Both f_V and f_E are denoted by the same symbol f below. $\diamond$

The definitions so far in this section are purely graph theoretic. Braided bimonoidal categories are involved in the next two definitions, which are the braided versions of Definitions I.3.1.14 and I.4.3.1.

Definition 5.3.10. Suppose given data (X, C, φ) as follows.

- X is a set with two distinguished elements $\{0^X, 1^X\}$ as in Definition 5.3.4.
- C is an arbitrary braided bimonoidal category as in Definition 2.1.29, equipped with the graph structure in Example 5.3.3.
- $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is any function such that

$$\varphi(0^X) = \mathbb{0} \quad \text{and} \quad \varphi(1^X) = \mathbb{1}. \tag{5.3.11}$$

Extend φ to a graph morphism

$$\mathsf{Gr}(X) \xrightarrow{\ \varphi\ } \mathsf{C}$$

as follows.

Vertices: For $x, y \in X^{\mathsf{fr}}$ such that $\varphi x, \varphi y \in \mathsf{Ob}(\mathsf{C})$ are already defined, we define

$$\begin{aligned} \varphi(x \oplus y) &= \varphi x \oplus \varphi y \quad \text{and} \\ \varphi(x \otimes y) &= \varphi x \otimes \varphi y. \end{aligned} \tag{5.3.12}$$

Elementary Edges: φ sends each elementary edge to the structure morphism in C with the same name and with the subscripts replaced by their images under φ.

Prime Edges: If $f_1, f_2 \in \mathsf{E}^{\mathsf{pr}}(X)$ are prime edges with at most one of them nonidentity and with $\varphi(f_1)$ and $\varphi(f_2)$ already defined, then we define

$$\begin{aligned} \varphi(f_1 \oplus f_2) &= \varphi(f_1) \oplus \varphi(f_2) \quad \text{and} \\ \varphi(f_1 \otimes f_2) &= \varphi(f_1) \otimes \varphi(f_2). \end{aligned} \tag{5.3.13}$$

This finishes the definition of the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$.

Moreover, we define the following.

- For a path $P = (f_n, \ldots, f_1)$ in $\mathsf{Gr}(X)$ with domain u and codomain v, its *value* in C is the composite morphism

$$\varphi P = \varphi(f_n) \circ \cdots \circ \varphi(f_1) \in \mathsf{C}(\varphi u; \varphi v). \tag{5.3.14}$$

- A diagram with vertices and edges in $\mathsf{Gr}(X)$ is *commutative in* C if its image under the graph morphism φ is a commutative diagram in C.
- A diagram with vertices and edges in $\mathsf{Gr}(X)$ is *braided commutative* if it is commutative in each braided bimonoidal category C and for each function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ satisfying (5.3.11). ◇

In the next definition, $\mathcal{D}^{\mathsf{br}}$ is the braided distortion category in Section 5.2.

Definition 5.3.15. Suppose X is a set with two distinguished elements $\{0^X, 1^X\}$.

- Define the function

$$\vartheta : X \longrightarrow \mathsf{Ob}(\mathcal{D}^{\mathsf{br}})$$

as follows.

$$\vartheta(x) = \begin{cases} (1) & \text{if } x \in X \smallsetminus \{0^X, 1^X\}, \\ \varnothing & \text{if } x = 0^X, \text{ and} \\ (0) & \text{if } x = 1^X. \end{cases} \tag{5.3.16}$$

- Using the same symbol, define the associated graph morphism

$$\mathsf{Gr}(X) \xrightarrow{\ \vartheta\ } \mathcal{D}^{\mathsf{br}} \tag{5.3.17}$$

as in Definition 5.3.10, applied to ϑ in (5.3.16) and $\mathcal{D}^{\mathsf{br}}$.
- For a path P in $\mathsf{Gr}(X)$, its value $\vartheta P \in \mathcal{D}^{\mathsf{br}}$ in the sense of (5.3.14) is called the *braided distortion of* P.

This finishes the definition. ◇

Examples. Suppose X is a set with two distinguished elements $\{0^X, 1^X\}$. In the following examples, the symbol $\otimes$ in X^{fr} is often omitted.

Example 5.3.18. For elements $r, s, t, u, v, w, x, y, z \in X \smallsetminus \{0^X, 1^X\}$, consider the elements

$$\begin{aligned} a &= (uv1^X) \oplus (1^X1^X1^X) \oplus (w1^Xx1^Xy) \oplus (z0^X), \\ b &= r \oplus (0^X0^X) \oplus (st), \quad \text{and} \\ c &= (r \oplus s)(t \oplus u) \oplus 1^X \oplus 0^X \end{aligned}$$

in X^{fr}, with any additive bracketing and any multiplicative bracketing within each monomial in a. Using the graph morphism ϑ in (5.3.17), there are the following objects in the braided distortion category $\mathcal{D}^{\mathsf{br}}$.

$$\begin{aligned} \vartheta(a) &= (2,0,3) \\ \vartheta(b) &= (1,2) \\ \vartheta(c) &= (2,2,2,2,0) \\ \vartheta(a \oplus b) &= (2,0,3,1,2) \\ \vartheta(a \otimes b) &= (3,1,4,4,2,5) \end{aligned}$$

With more detail, $\vartheta(c) \in \mathcal{D}^{\mathrm{br}}$ is computed as follows.

$$\begin{aligned}\vartheta(c) &= \big\{[\vartheta(r) \oplus \vartheta(s)] \otimes [\vartheta(t) \oplus \vartheta(u)]\big\} \oplus \vartheta(1^X) \oplus \vartheta(0^X)\\ &= \big\{[(1) \oplus (1)] \otimes [(1) \oplus (1)]\big\} \oplus (0) \oplus \varnothing\\ &= \big\{(1,1) \otimes (1,1)\big\} \oplus (0)\\ &= (1+1, 1+1, 1+1, 1+1) \oplus (0)\\ &= (2,2,2,2,0)\end{aligned}$$

The other objects $\vartheta(a)$, $\vartheta(b)$, $\vartheta(a \oplus b)$, and $\vartheta(a \otimes b)$ are computed similarly. $\diamond$

Convention 5.3.19. For Lemma 5.3.20 below, we interpret the diagrams (2.1.4)–(2.1.27) in Definition 2.1.1, (2.1.32), and (2.1.33) in $\mathsf{Gr}(X)$ as follows.

- We interpret each object there as an element in X^{fr} with $A, B, C, D \in X^{\mathsf{fr}}$, and with $\mathbb{0}$ and $\mathbb{1}$ interpreted as 0^X and 1^X, respectively.
- We interpret each morphism there as the corresponding prime edge in $\mathsf{Gr}(X)$ as in Definition 5.3.6, with one kind of exceptions as stated next.
- If a morphism is the sum of two nonidentity structure morphisms, then we interpret it as a path of length 2 in $\mathsf{Gr}(X)$ consisting of the two corresponding nonidentity prime edges. For example, the morphism

  $$AC \oplus BC \xrightarrow{\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}} CA \oplus CB$$

  in (2.1.4) is interpreted as the path

  $$AC \oplus BC \xrightarrow{\xi^{\otimes}_{A,C} \oplus 1_{BC}} CA \oplus BC \xrightarrow{1_{CA} \oplus \xi^{\otimes}_{B,C}} CA \oplus CB$$

  of length 2 in $\mathsf{Gr}(X)$.

In this way, we interpret each of those 26 diagrams as consisting of two paths in $\mathsf{Gr}(X)$ with a common domain and a common codomain. $\diamond$

Lemma 5.3.20. *With Convention 5.3.19, in each of (2.1.4)–(2.1.27), (2.1.32), and (2.1.33), the two paths in* $\mathsf{Gr}(X)$ *have the same braided distortion.*

Proof. The assertion means that, if we apply $\vartheta : \mathsf{Gr}(X) \longrightarrow \mathcal{D}^{\mathrm{br}}$ in (5.3.17) to each of those 26 diagrams, then the result is a commutative diagram in $\mathcal{D}^{\mathrm{br}}$. Therefore, the assertion follows from Corollaries 5.2.33 and 5.2.34. $\square$

5.4. The Coherence Theorem

The purpose of this section is to prove Theorem 5.4.4, which is our main coherence result for braided bimonoidal categories in this chapter. It is the braided version of the Coherence Theorem I.4.4.3 for symmetric bimonoidal categories. It asserts that for a braided bimonoidal category C that satisfies a monomorphism assumption, which is automatically true if C is tight, any two parallel paths in $\mathsf{Gr}(X)$ with the same braided distortion also have the same value in C.

The proof of Theorem 5.4.4 is adapted from that of Theorem I.4.4.3 and reuses a large part of Chapter I.3, with appropriate adjustments to be explained in detail below. We first prove a preliminary version of Theorem 5.4.4 in Lemma 5.4.2. In the proof of Theorem 5.4.4, we reduce to the special setting of Lemma 5.4.2.

After Theorem 5.4.4 and some examples, we explain why Laplaza's First Coherence Theorem I.3.9.1 does *not* have a braided analogue.

Convention 5.4.1. Suppose (X, C, φ) consists of the following data.

- X is a set with two distinguished elements $\{0^X, 1^X\}$ as in Definition 5.3.4.
- C is a braided bimonoidal category as in Definition 2.1.29, equipped with the graph structure in Example 5.3.3.
- $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is a function such that
$$\varphi(0^X) = \mathbb{0} \quad \text{and} \quad \varphi(1^X) = \mathbb{1}.$$

Moreover, suppose $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ is the graph morphism in Definition 5.3.10. The *value* in C of each path $P \in \mathsf{Gr}(X)$ is defined in (5.3.14) using the graph morphism φ. ◇

Definitions 5.3.8, 5.3.10, and 5.3.15 are used in the following preliminary case of the coherence theorem. It is the braided analogue of Lemma I.4.4.1.

Lemma 5.4.2. *With Convention 5.4.1, suppose*

$$\begin{tikzcd} a \arrow[r, bend left, "P_1", twoheadrightarrow] \arrow[r, bend right, "P_2"', twoheadrightarrow] & b \end{tikzcd}$$

are two paths in $\mathsf{Gr}(X)$ *such that the following two statements hold.*

- *P_1 and P_2 have the same braided distortion and are 1^X-free paths.*
- *a and b contain no 0^X and are δ-reduced and 1^X-reduced.*

Then P_1 and P_2 have the same value in C.

Proof. We reuse most of the proof of Lemma I.4.4.1. Each of a and b is a polynomial, that is, a finite $\oplus$ of monomials, each being a finite $\otimes$ of elements in X. Moreover, in a and b, each monomial is either equal to 1^X, or contains no 0^X and 1^X.

As in the proof of Lemma I.4.4.1, for each $i = 1, 2$, there is a diagram in $\mathsf{Gr}(X)$

$$\begin{tikzcd} a \arrow[rr, twoheadrightarrow, "P_i"] \arrow[dr, twoheadrightarrow, "P_i'"'] & & b \\ & c_i \arrow[ur, twoheadrightarrow, "P_i''"'] & \end{tikzcd}$$

such that the following three statements hold.

- The diagram is braided commutative, in particular in C and $\mathcal{D}^{\mathsf{br}}$.
- $P_i' : a \twoheadrightarrow c_i$ consists of identities and prime edges involving $\alpha^{\pm\oplus}$ or $\xi^{\pm\oplus}$.
- $P_i'' : c_i \twoheadrightarrow b$ consists of identities and prime edges involving $\alpha^{\pm\otimes}$ or $\xi^{\pm\otimes}$.

We aim to show that P_1' and P_2' have the same value in C and similarly for P_1'' and P_2''.

Suppose that P_1 has braided distortion

$$\vartheta(P_1) = \big(\sigma; \sigma_1, \ldots, \sigma_m\big) \in \Sigma_m \times B_{r_1} \times \cdots \times B_{r_m},$$

which is a morphism $\vartheta a \longrightarrow \vartheta b$ in the braided distortion category $\mathcal{D}^{\mathsf{br}}$. The proof of Lemma I.4.4.1 shows the following equalities.

$$\begin{aligned} \vartheta(P_1') &= \vartheta(P_2') = \big(\sigma; \mathrm{id}_{r_1}, \ldots, \mathrm{id}_{r_m}\big) \\ \vartheta(P_1'') &= \vartheta(P_2'') = \big(\mathrm{id}_m; \sigma_{\sigma^{-1}(1)}, \ldots, \sigma_{\sigma^{-1}(m)}\big) \\ c_1 &= c_2 \in X^{\mathsf{fr}} \end{aligned} \tag{5.4.3}$$

We write c for the element $c_1 = c_2$.

The first line in (5.4.3) implies that P'_1 and $P'_2 : a \twoheadrightarrow c$ both additively permute the set of monomials in a via $\sigma \in \Sigma_m$. The Symmetric Coherence Theorem I.1.3.8, applied to the symmetric monoidal category

$$(\mathsf{C}, \oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus}),$$

implies that P'_1 and P'_2 have the same value in C.

The second line in (5.4.3) implies that P''_1 and $P''_2 : c \twoheadrightarrow b$, when restricted to the ith monomial in c for each $1 \leq i \leq m$, both have underlying braid

$$\sigma_{\sigma^{-1}(i)} \in B_{r_{\sigma^{-1}(i)}}$$

in the sense of Definition 1.6.2. The Braided Coherence Theorem 1.6.3, applied to the braided monoidal category

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$$

and to each monomial in c, implies that the paths P''_1 and P''_2 have the same value in C. Therefore, $P_1 = (P''_1, P'_1)$ and $P_2 = (P''_2, P'_2)$ have the same value in C. □

Next is our main coherence result for braided bimonoidal categories in this chapter. It is the braided analogue of Theorem I.4.4.3. The monomorphism assumption in the next theorem is automatically satisfied if C is tight.

Theorem 5.4.4 (Braided Bimonoidal Coherence)**.** *With Convention 5.4.1, suppose the value of each δ-prime edge is a monomorphism in C. If*

$$\begin{tikzcd} a \arrow[r, two heads, bend left, "P_1"] \arrow[r, two heads, bend right, "P_2"'] & b \end{tikzcd}$$

are two paths in $\mathsf{Gr}(X)$ with the same braided distortion, then their values in C are equal.

Proof. We reuse the strategy in the proof of Theorem I.4.4.3, which consists of two main steps. First we remark that the braided distortion category $\mathcal{D}^{\mathsf{br}}$ is a groupoid by Lemma 5.2.7 and is a tight braided bimonoidal category by Corollary 5.2.33. Therefore, if two diagrams in $\mathsf{Gr}(X)$,

$$\begin{tikzcd} a \arrow[r, two heads, "P_i"] \arrow[d, two heads, "Q_a"'] & b \arrow[d, two heads, "Q_b"] \\ a' \arrow[r, two heads, "R_i"] & b' \end{tikzcd} \qquad \text{for} \quad i = 1, 2,$$

are braided commutative in the sense of Definition 5.3.10, in particular in $\mathcal{D}^{\mathsf{br}}$, such that P_1 and P_2 have the same braided distortion (that is, $\vartheta P_1 = \vartheta P_2$ in $\mathcal{D}^{\mathsf{br}}$), then R_1 and R_2 also have the same braided distortion.

Next we observe that, with appropriate adjustments as explained below, the five reduction steps in the proof of Theorem I.3.9.1 are still valid in the current setting. The hypothesis that the value of each δ-prime edge is a monomorphism in C is used in the third reduction step involving the diagram (I.3.9.2). Those five reduction steps in Theorem I.3.9.1 involve a number of definitions and preliminary results in Chapter I.3 about the given triple (X, C, φ). To reuse them here, we note that they fall into two groups.

(i) Concepts and statements that only involve $\mathsf{Gr}(X)$ and not C, such as Definition I.3.3.4 and Lemma I.3.3.5, are reused here with no changes.
(ii) Other concepts and statements involve both $\mathsf{Gr}(X)$ and C, such as Definition I.3.4.5 and Lemma I.3.4.8.

Definitions and assertions in group (ii) are reused here with the following notes and adjustments.

Laplaza's axioms: C here is a braided bimonoidal category. By Theorem 2.2.1, C satisfies all 24 Laplaza axioms in Definition 2.1.1 of a symmetric bimonoidal category, in addition to the axioms (2.1.32) and (2.1.33). In the proofs in Chapter I.3, whenever an axiom in Definition 2.1.1, which is the same as Definition I.2.1.2, is used to infer the commutativity of a diagram in C, we can still do that here.

Commutativity: In Definition I.3.1.14, a diagram in $\mathsf{Gr}(X)$ is said to be *commutative* if it is commutative in each symmetric bimonoidal category C and for each function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ satisfying $\varphi(0^X) = \mathbb{0}$ and $\varphi(1^X) = \mathbb{1}$. This concept of commutativity is used in Definitions I.3.4.5, I.3.7.1, and I.3.8.11, and a number of related assertions. In the current setting, we replace commutativity by the corresponding concept of *braided commutativity* in Definition 5.3.10.

Symmetry: When the symmetry axiom (1.3.33) is used with either axioms (2.1.4) or (2.1.18)—which are the same as, respectively, (I.2.1.5) and (I.2.1.19)—in the following places, here we replace that part of the proof by an appropriate braided bimonoidal category axiom.

- In (I.3.5.15), the left diagram is commutative by the axiom (2.1.33).
- In the proof of Lemma I.3.6.12, in
  - the diagram (I.3.6.18) and
  - the last paragraph case I.(iii) with $(f, g) = (\xi^{-\otimes}, \delta^r)$,

  we use the axiom (2.1.32) instead of (2.1.4).

Symmetric Coherence: When the Symmetric Coherence Theorem I.1.3.8 is used in the following places, here we replace it by the Braided Coherence Theorem 1.6.3 for the braided monoidal category $(\mathsf{C}, \otimes, \xi^{\otimes})$.

- In Lemmas I.3.4.11 and I.3.4.12, we now use Theorem 1.6.3 along with the fact that the first braid group B_1 is the trivial group.
- In the proofs of Lemma I.3.8.13 case (3) and of Proposition I.3.8.14, Theorem 1.6.3 is applicable because, in each case, the two braided canonical maps in question have the same underlying braid.

With the remark in the first paragraph of this proof and the adjustments above, we can reuse the five reduction steps in the proof of Theorem I.3.9.1. After the five reduction steps, we reduce to the setting of Lemma 5.4.2, which finishes the proof. □

Next is the braided version of Definition I.3.9.9.

Definition 5.4.5. A braided bimonoidal category C is called *flat* if each iterated sum and product of a component of δ^l or δ^r with a finite number of identity morphisms is a monomorphism. ⋄

Example 5.4.6. Tight braided bimonoidal categories—that is, those with δ^l and δ^r natural isomorphisms—are flat. If C is a flat braided bimonoidal category, then the value of each δ-prime edge is a monomorphism in C. Therefore, Theorem 5.4.4

applies to each flat, in particular, tight, braided bimonoidal category. For example, Theorem 5.4.4 applies in the following cases.

- Theorem 2.4.22 shows that an abelian category equipped with a compatible braided monoidal structure is a tight braided bimonoidal category. This includes
  - the category $\mathsf{Mod}(A)$ in Theorem 3.2.19 (2) for a braided bialgebra (A, R),
  - the category $\mathcal{F}^{\mathsf{any}}$ of Fibonacci anyons in Theorem 3.4.13, and
  - the category $\mathcal{I}^{\mathsf{any}}$ of Ising anyons in Theorem 3.6.14.
- Theorem 4.4.3 shows that, for each tight bimonoidal category C, the bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a tight braided bimonoidal category.
- By Proposition 5.1.10, each left permbraided category is a tight braided bimonoidal category, and similarly for each right permbraided category by Proposition 5.1.19. $\diamond$

The Regular Version Fails. Theorem 5.4.4 is the braided version of Laplaza's Second Coherence Theorem I.4.4.3 for symmetric bimonoidal categories. They impose a condition on the two paths P_1 and P_2, namely, that they have the same braided distortion in the sense of Definition 5.3.15, or that they have the same distortion in the symmetric case. However, no conditions are imposed on their domain a and codomain b.

It is natural to ask whether or not Laplaza's First Coherence Theorem I.3.9.1 also has a braided version. In Theorem I.3.9.1, no conditions are imposed on the two paths P_1 and P_2. Instead, it is assumed that their domain $a \in X^{\mathsf{fr}}$ is *regular* in the sense of Definition I.3.1.25. The braided version of Theorem I.3.9.1, if it is true, would assert the following.

> For (X, C, φ) as in Theorem 5.4.4 with $a \in X^{\mathsf{fr}}$ regular, but without any assumption on P_1 and P_2, the values of P_1 and P_2 in C are equal.

This braided version is *not* true. For example, for distinct elements $x, y \in X$, there are two paths

$$\begin{tikzcd} a = x \otimes y \arrow[rr, "1_a"] \arrow[dr, "\xi^{\otimes}_{x,y}"'] & & x \otimes y \\ & y \otimes x \arrow[ur, "\xi^{\otimes}_{y,x}"'] & \end{tikzcd}$$

in $\mathsf{Gr}(X)$ with $a = x \otimes y$ regular. However, in a braided monoidal category C, it is not true in general that $\xi^{\otimes}_{y,x}\xi^{\otimes}_{x,y} = 1_{x \otimes y}$, which is the symmetry axiom (1.3.33).

A natural followup question is the following.

> Where does the proof of Theorem I.3.9.1 fail for a braided bimonoidal category?

The five reduction steps in the proof of Theorem I.3.9.1 are still valid in the braided case, as we explained in the proof of Theorem 5.4.4. This allows us to reduce to the setting of Lemma 5.4.2, but without the assumption $\vartheta P_1 = \vartheta P_2$. In the absence of this assumption, the last paragraph in the proof of Lemma 5.4.2 fails for the following reason. Without assuming $\vartheta P_1 = \vartheta P_2$, we can no longer infer that P_1'' and $P_2'' : c \longrightarrow b$ have the same underlying braid when restricted to each monomial in c. Therefore, we cannot apply the Braided Coherence Theorem 1.6.3 to infer that

P_1'' and P_2'' have the same value in C. This is where the proof of Theorem I.3.9.1 fails for a general braided bimonoidal category. In the actual proof of Theorem I.3.9.1, this corresponds to the second bullet point in the last paragraph.

In fact, the failure of the last part of the proof of Theorem I.3.9.1 in the braided case as discussed above is precisely why Theorem 5.4.4 has the assumption that P_1 and P_2 have the same braided distortion.

5.5. Braided Distortion as a Grothendieck Construction

In this section, we provide a conceptual explanation of the braided distortion category $\mathcal{D}^{\mathsf{br}}$ (Definition 5.2.2) as a Grothendieck construction over the finite ordinal category. This description of $\mathcal{D}^{\mathsf{br}}$ is not used in the rest of this work.

The Grothendieck Construction. Recall from Definition I.4.6.1 that, for a category C and a functor $F : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$, the *Grothendieck construction* of F is the category $\int_{\mathsf{C}} F$ with objects (A, X) with $A \in \mathsf{C}$ and $X \in FA$. A morphism

$$(f, p) : (A, X) \longrightarrow (B, Y) \in \int_{\mathsf{C}} F$$

consists of

- a morphism $f : A \longrightarrow B$ in C and
- a morphism $p : X \longrightarrow (Ff)(Y)$ in FA.

The identity morphism of an object (A, X) is the pair $(1_A, 1_X)$ of identity morphisms. For another morphism

$$(g, q) : (B, Y) \longrightarrow (C, Z),$$

the composition with (f, p) is defined as

$$(g, q) \circ (f, p) = \big(gf, (Ff)(q) \circ p\big) : (A, X) \longrightarrow (C, Z).$$

Proposition I.4.6.5 shows that there is a canonical isomorphism of categories

$$\mathcal{D} \cong \int_{\Sigma} F$$

between

- the distortion category $\mathcal{D}$ in Definition I.4.2.1 and
- the Grothendieck construction of some functor

$$F : \Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}.$$

Here Σ is the finite ordinal category in Definition I.2.4.1.

- Its objects are nonnegative integers $n \geq 0$.
- Its morphisms are permutations
$$\Sigma(n, n) = \Sigma_n$$
with $\Sigma(m, n) = \varnothing$ if $m \neq n$.
- Identity morphisms are identity permutations.
- Composition is the product in each symmetric group Σ_n.

The functor F is defined by the following assignments on objects and morphisms.

$$n \longmapsto \Sigma^{\times n}$$
$$\big(\Sigma(n, n) = \Sigma_n \ni \sigma\big) \longmapsto \big(\sigma^{-1} \in \mathsf{Cat}(\Sigma^{\times n}, \Sigma^{\times n})\big)$$

Here σ^{-1} is the functor that permutes the factors via $\sigma^{-1} \in \Sigma_n$. Moreover, Proposition I.4.6.7 is the additive version for the additive distortion category $\mathcal{D}^{\mathsf{ad}}$.

The Braided Distortion Category. The description of the braided distortion category $\mathcal{D}^{\mathsf{br}}$ as a Grothendieck construction over Σ involves the following braided version of Σ. Recall the nth braid group B_n in Definition 1.1.1.

Definition 5.5.1. The *braid category* $\mathcal{B}$ is the category defined as follows.

- Its objects are nonnegative integers $m \geq 0$.
- For $m, n \geq 0$, the morphism set is defined as

$$\mathcal{B}(m,n) = \begin{cases} B_m & \text{if } m = n \text{ and} \\ \varnothing & \text{if } m \neq n. \end{cases}$$

- Identity morphisms are identity braids.
- Composition is the product in each braid group B_m.

This finishes the definition of $\mathcal{B}$. ◇

We will use the following braided analogue of the functor F to relate the braided distortion category $\mathcal{D}^{\mathsf{br}}$, the finite ordinal category Σ, and the braid category $\mathcal{B}$. The n-fold Cartesian product of $\mathcal{B}$ is denoted by $\mathcal{B}^{\times n}$, with $\mathcal{B}^{\times 0} = *$.

Definition 5.5.2. Define a functor

$$F^{\mathsf{br}} : \Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$$

by the following assignments on objects and morphisms.

$$\begin{aligned} n &\longmapsto \mathcal{B}^{\times n} \\ \big(\Sigma(n,n) = \Sigma_n \ni \sigma\big) &\longmapsto \big(\sigma^{-1} \in \mathsf{Cat}(\mathcal{B}^{\times n}, \mathcal{B}^{\times n})\big) \end{aligned}$$

Here σ^{-1} is the functor that permutes the factors via $\sigma^{-1} \in \Sigma_n$, that is,

$$\sigma^{-1}(x_1, \ldots, x_n) = \big(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\big).$$

This finishes the definition of the functor F^{br}. ◇

The following observation provides a conceptual explanation of the braided distortion category $\mathcal{D}^{\mathsf{br}}$ as a Grothendieck construction over Σ.

Proposition 5.5.3. *There is a canonical isomorphism of categories*

$$\mathcal{D}^{\mathsf{br}} \cong \int_{\Sigma} F^{\mathsf{br}} \tag{5.5.4}$$

with $F^{\mathsf{br}} : \Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ *the functor in Definition 5.5.2.*

Proof. This follows from unpacking the definition of $\int_{\Sigma} F^{\mathsf{br}}$. In more detail, an object in $\int_{\Sigma} F^{\mathsf{br}}$ is a pair $(m, \underline{r})$ consisting of

- an object $m \geq 0$ in Σ and
- an object $\underline{r} \in F^{\mathsf{br}}(m) = \mathcal{B}^{\times m}$, that is, a sequence $(r_1, \ldots, r_m)$ with each $r_j \geq 0$.

A morphism

$$\underline{\sigma} : (m, \underline{r}) \longrightarrow (n, \underline{s}) \in \int_{\Sigma} F^{\mathsf{br}}$$

consists of

- a morphism $\sigma \in \Sigma(m,n)$, that is, a permutation $\sigma \in \Sigma_m$ with $m = n$, and

- a morphism
$$(\sigma_1,\ldots,\sigma_m) : \underline{r} \longrightarrow \sigma^{-1}\underline{s} = \big(s_{\sigma(1)},\ldots,s_{\sigma(m)}\big) \in \mathcal{B}^{\times m},$$
that is, a braid
$$\sigma_j \in \mathcal{B}(r_j,r_j) = B_{r_j} \quad \text{for each} \quad 1 \le j \le m$$
with $r_j = s_{\sigma(j)}$.

So objects and morphisms in $\int_\Sigma F^{\mathsf{br}}$ are the same as those in the braided distortion category $\mathcal{D}^{\mathsf{br}}$ (Definition 5.2.2). Similarly, the identity morphisms and composition in $\int_\Sigma F^{\mathsf{br}}$ are the same as those in $\mathcal{D}^{\mathsf{br}}$. □

The following table summarizes the relationship between the additive distortion category $\mathcal{D}^{\mathsf{ad}}$, the distortion category $\mathcal{D}$, and the braided distortion category $\mathcal{D}^{\mathsf{br}}$. See also Question III.A.5.6.

	$\mathcal{D}^{\mathsf{ad}}$ (I.4.5.2)	$\mathcal{D}$ (I.4.2.1)	$\mathcal{D}^{\mathsf{br}}$ (5.2.2)
functor $\Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$	F^{ad} (I.4.6.6)	F (I.4.6.3)	F^{br} (5.5.2)
sends n to	$\mathbb{N}^{\times n}$	$\Sigma^{\times n}$	$\mathcal{B}^{\times n}$
sends $\sigma \in \Sigma(n,n) = \Sigma_n$ to	σ^{-1}	σ^{-1}	σ^{-1}
Grothendieck construction	$\int_\Sigma F^{\mathsf{ad}}$ (I.4.6.7)	$\int_\Sigma F$ (I.4.6.5)	$\int_\Sigma F^{\mathsf{br}}$ (5.5.3)

Braided Structure on the Braid Category. Since the braided distortion category $\mathcal{D}^{\mathsf{br}}$ is a left permbraided category, which is, in particular, a tight braided bimonoidal category (Corollary 5.2.33), the braid category $\mathcal{B}$ in Definition 5.5.1 also has a braided structure.

Proposition 5.5.5. *The multiplicative structure on the braided distortion category $\mathcal{D}^{\mathsf{br}}$ restricts to a braided strict monoidal structure on the braid category $\mathcal{B}$.*

Proof. Under the canonical isomorphism $\mathcal{D}^{\mathsf{br}} \cong \int_\Sigma F^{\mathsf{br}}$ in (5.5.4), the braid category $\mathcal{B}$ is the full subcategory of $\mathcal{D}^{\mathsf{br}}$ consisting of objects of length 1, that is, nonnegative integers. Objects of length 1 are closed under the multiplicative structure on $\mathcal{D}^{\mathsf{br}}$ in Definition 5.2.14. Since $\mathcal{D}^{\mathsf{br}}$ with its multiplicative structure is a braided strict monoidal category by Lemma 5.2.21, so is the braid category $\mathcal{B}$. □

Explanation 5.5.6. Restricting Definition 5.2.14 to objects of length 1 yields the following explicit description of the braided strict monoidal structure on the braid category $\mathcal{B}$.

- The monoidal product $\otimes : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ is given on objects and morphisms by, respectively,
$$m \otimes n = m + n \quad \text{and}$$
$$\sigma \otimes \tau = \sigma \oplus \tau$$
with $\sigma \oplus \tau$ the sum braid (1.1.10).
- The monoidal unit is 0.
- The braiding ξ at a pair of nonnegative integers (m,n) is the elementary block braid
$$b^{\oplus}_{m,n} : m + n \longrightarrow n + m$$
in B_{m+n} in (1.2.4).

This braided strict monoidal structure on the braid category $\mathcal{B}$ was first described in [**JS93**]. It is proved there that $\mathcal{B}$ is the free braided strict monoidal category on one object. For a more general discussion of coherence of monoidal structure, see [**Yau**∞, 21.3]. $\diamond$

CHAPTER 6

Strictification of Tight Braided Bimonoidal Categories

Recall from Definition 2.1.29 that a braided bimonoidal category is *tight* if the distributivity morphisms δ^l and δ^r are natural isomorphisms, not just monomorphisms. Examples of tight braided bimonoidal categories include those in Theorems 2.4.22, 3.2.19, 3.4.13, 3.6.14, and 4.4.3. By Definition 5.1.11 and Proposition 5.1.19, a *right permbraided category* is a tight braided bimonoidal category with

- a permutative category as its additive structure,
- a braided strict monoidal category as its multiplicative structure, and
- identities for the structure morphisms $\lambda^\bullet$, $\rho^\bullet$, δ^r, $\zeta^\otimes_{-,\mathbb{0}}$, and $\zeta^\otimes_{\mathbb{0},-}$.

The main Theorem 6.3.6 in this chapter states that each tight braided bimonoidal category is canonically equivalent to a right permbraided category. The left variant is Theorem 6.3.7. It states that each tight braided bimonoidal category is canonically equivalent to a left permbraided category.

The Blass-Gurevich Conjecture. As stated in the introduction in Chapter 5, the Blass-Gurevich Conjecture in [**BG20a**] states that there should be a coherence theorem for their BD categories, which are our tight braided bimonoidal categories. Theorem 5.4.4 is one positive answer to the Blass-Gurevich Conjecture in the form of commutative formal diagrams in braided bimonoidal categories that satisfy a monomorphism assumption. The main results in this chapter, Theorems 6.3.6 and 6.3.7, are two further positive answers to the Blass-Gurevich Conjecture in the form of strictification of tight braided bimonoidal categories.

There are several more ways to interpret Theorems 6.3.6 and 6.3.7.

- They are the bimonoidal analogues of the Braided Strictification Theorem 1.6.5 for braided monoidal categories.
- They are the braided analogues of Theorems I.5.4.6 and I.5.4.7, which are the strictification results for tight symmetric bimonoidal categories to, respectively, right and left bipermutative categories.
- As we will explain below, the proofs of Theorems 6.3.6 and 6.3.7 crucially depend on the Coherence Theorem 5.4.4 for braided bimonoidal categories. Therefore, the main results in this chapter are illustrations of the practical usage of Theorem 5.4.4.

Motivation. To motivate the Strictification Theorems 6.3.6 and 6.3.7, first consider the known cases of monoidal categories, braided monoidal categories, and symmetric bimonoidal categories.

The Strictification Theorem I.1.3.5 for a monoidal category C (see [**ML98**, XI.3 Theorem 1]) asserts the existence of an adjoint equivalence of strong monoidal functors between C and a strict monoidal category C_{st}. The proof explicitly constructs C_{st} and the adjoint equivalence. The objects in C_{st} are finite sequences of objects in C, with the monoidal product given by concatenation of sequences. By taking iterated monoidal products in C with the right normalized bracketing (6.2.2), each object in C_{st} has a canonically associated object in C. Morphisms in C_{st} are defined as morphisms in C between the associated objects. The nontrivial part involves using Mac Lane's Coherence Theorem I.1.3.3 for monoidal categories (see [**ML98**, VII.2 Cor.])

- to define the monoidal product of two morphisms in C_{st} and to see that it is well defined;
- to prove the strict associativity and unity of the monoidal product in C_{st} for morphisms; and
- to define the strong monoidal functors between C and C_{st}.

The key observation is that the coherence theorem for monoidal categories, in the form of commutative formal diagrams, is used in the construction of the strict monoidal category C_{st} and the adjoint equivalence with C.

The Strictification Theorem 1.6.5 for braided monoidal categories relies on the Joyal-Street Braided Coherence Theorem 1.6.3 in a similar way. The underlying strict monoidal category is also C_{st}. The Braided Coherence Theorem 1.6.3 is used in the construction of the braiding and the proof of the hexagon axioms in C_{st}. For an explicit proof, see the references in Note 1.7.3.

A symmetric bimonoidal analogue of Mac Lane's Coherence Theorem I.1.3.3 is Laplaza's First Coherence Theorem I.3.9.1. The latter states that, in each symmetric bimonoidal category C that satisfies a monomorphism assumption, which is automatically true if C is tight, any two parallel paths in $\mathsf{Gr}(X)$ with a regular domain, in the sense of Definition I.3.1.25, have the same value in C. Similar to the strictification of monoidal categories, the Strictification Theorem I.5.4.6 of tight symmetric bimonoidal categories to right bipermutative categories uses Theorem I.3.9.1 many times. For a tight symmetric bimonoidal category C, its associated right bipermutative category A has as its objects formal polynomials in the objects in C. They are interpreted in C using suitable additive and multiplicative bracketings. In the proof of the Strictification Theorem I.5.4.6, the Coherence Theorem I.3.9.1 is used in Explanations I.5.2.31 and I.5.2.37 and Lemmas I.5.2.33, I.5.3.1, I.5.3.4, I.5.3.7, I.5.3.8, and I.5.4.4. Each time Theorem I.3.9.1 is used to infer the commutativity of a diagram in C.

For a tight braided bimonoidal category C, the associated right permbraided category A has the same underlying category and additive structure as in the symmetric case; see Definitions 6.2.3 and 6.2.10. In particular, its objects are formal polynomials in the objects in C. The rest of the construction of A—including the multiplicative zeros $\lambda^{\bullet}$ and $\rho^{\bullet}$, the distributivity morphisms δ^{l} and δ^{r}, and the proof that it is a right permbraided category—follows a similar outline, but it is different in one crucial aspect. As we mentioned in the previous paragraph, in the symmetric case, we use Theorem I.3.9.1 many times to infer the commutativity of formal diagrams in C. However, as discussed near the end of Section 5.4, Theorem I.3.9.1 does *not* have a literal braided analogue, with the word braided replacing the word symmetric in the statement.

A careful examination of the proof of Theorem I.5.4.6 reveals that each instance of Theorem I.3.9.1 may be replaced by Laplaza's Second Coherence Theorem I.4.4.3. We use Theorem I.3.9.1 instead of Theorem I.4.4.3 mainly for convenience. In each instance, by assigning a separate formal variable to each alphabet involved, we make sure that the common domain of the parallel paths is regular as in Definition I.3.1.25. Therefore, we can simply use Theorem I.3.9.1 each time. To use Theorem I.4.4.3 instead, we would have to check that, in each instance, the two paths in question have the same distortion in the sense of Definition I.4.3.1. This is possible, but it involves more work.

The Braided Bimonoidal Coherence Theorem 5.4.4 is the braided version of Theorem I.4.4.3. Therefore, in the proof of Theorem 6.3.6, each time Theorem I.3.9.1 is used in the proof of Theorem I.5.4.6, we now use Theorem 5.4.4. In each instance, we check that the two paths in question have the same braided distortion as in Definition 5.3.15. See the proofs of Lemmas 6.2.29, 6.2.35, 6.2.37, 6.2.38, and 6.3.4.

Organization. An outline of the rest of this chapter follows.

To prepare for Theorem 6.3.6, in Section 6.1, we discuss braided bimonoidal functors. They are the same as symmetric bimonoidal functors in Definition I.5.1.1, but they apply to braided bimonoidal categories. After providing equivalent characterizations of the axioms and defining composition, we finish this section with some examples. Proposition 6.1.12 shows that, for abelian categories with a compatible braided monoidal structure, a braided monoidal functor that is also an additive functor canonically extends to a braided bimonoidal functor.

As the first main step for Theorem 6.3.6, in Section 6.2, for each tight braided bimonoidal category C, we construct an explicit right permbraided category A. An important difference between the braided case and the symmetric case is in the definition of the left distributivity morphism δ^l in A. In the symmetric case (I.5.3.6), δ^l is defined in terms of δ^r and the multiplicative symmetry $\xi^{\otimes}$ in A using the diagram (2.1.4). Therefore, that axiom is automatically true in A. On the other hand, in the braided case (6.2.34), δ^l is defined in terms of δ^r and the braiding $\xi^{\otimes}$ in A using the diagram (2.1.32) in the definition of a braided bimonoidal category. So we have to check the axiom (2.1.4) in A, which is Lemma 6.2.38.

In Section 6.3, we finish the proof of Theorem 6.3.6 by constructing an adjoint equivalence between C and A. The two functors involved are both braided bimonoidal equivalences. The Strictification Theorem 6.3.7 to left permbraided categories has almost the same proof as that of Theorem 6.3.6, and we explain the necessary adjustments. This time δ^r is defined in terms of δ^l and $\xi^{\otimes}$ using (2.1.4), and we have to prove the axiom (2.1.32).

As before, we often omit the $\otimes$ symbol to save space. In the absence of clarifying parentheses, $\otimes$ takes precedence over $\oplus$. For example, $AB \oplus CD$ means $(A \otimes B) \oplus (C \otimes D)$.

Reading Guide.

(1) Read Definitions 6.1.1 and 6.1.8 of braided bimonoidal functors and their composites.
(2) Read Convention 6.2.1 and Definitions 6.2.3, 6.2.9, 6.2.10, 6.2.19, 6.2.22, and 6.2.31 and the statement of Proposition 6.2.39 for the associated right permbraided category.
(3) Read Definitions 6.3.1 and 6.3.2 and the statements of Theorems 6.3.6 and 6.3.7 for the strictification of tight braided bimonoidal categories.
(4) Go back and read the rest of this chapter.

6.1. Braided Bimonoidal Functors

In this section, we define functors between braided bimonoidal categories and observe that there is a category $\mathsf{Bi}^{\mathsf{br}}$ of small braided bimonoidal categories and braided bimonoidal functors. Examples of braided bimonoidal functors are discussed in the second half of this section. Recall from Definitions 1.3.7 and 1.3.18 the concept of a braided monoidal functor and from Definition 2.1.29 the concept of a braided bimonoidal category.

Definition 6.1.1. Suppose C and D are braided bimonoidal categories. A *braided bimonoidal functor* from C to D is a tuple

$$\big(F, F^2_\oplus, F^0_\oplus, F^2_\otimes, F^0_\otimes\big) : \mathsf{C} \longrightarrow \mathsf{D}$$

consisting of the following data.

The Additive Structure:

$$F_\oplus = (F, F^2_\oplus, F^0_\oplus) : (\mathsf{C}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus) \longrightarrow (\mathsf{D}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus)$$

is a symmetric monoidal functor.

The Multiplicative Structure:

$$F_\otimes = (F, F^2_\otimes, F^0_\otimes) : (\mathsf{C}, \otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \xi^\otimes) \longrightarrow (\mathsf{D}, \otimes, \mathbb{1}, \alpha^\otimes, \lambda^\otimes, \rho^\otimes, \xi^\otimes)$$

is a braided monoidal functor.

These data are required to make the following two diagrams in D commutative for all objects $A, B, C \in \mathsf{C}$.

Multiplicative Zero:

$$\begin{tikzcd}
(FA)\mathbb{0} \arrow[rr, "1_{FA}F^0_\oplus"] \arrow[d, "\rho^\bullet_{FA}"'] & & (FA)(F\mathbb{0}) \arrow[d, "F^2_\otimes"] \\
\mathbb{0} \arrow[dr, "F^0_\oplus"'] & & F(A\mathbb{0}) \arrow[dl, "F\rho^\bullet_A"] \\
& F\mathbb{0} &
\end{tikzcd} \tag{6.1.2}$$

Distributivity:

$$\begin{tikzcd}
(FA \oplus FB)(FC) \arrow[r, "\delta^r"] \arrow[d, "F^2_\oplus 1_{FC}"'] & (FA)(FC) \oplus (FB)(FC) \arrow[d, "F^2_\otimes \oplus F^2_\otimes"] \\
F(A \oplus B)(FC) \arrow[d, "F^2_\otimes"'] & F(AC) \oplus F(BC) \arrow[d, "F^2_\oplus"] \\
F\big((A \oplus B)C\big) \arrow[r, "F\delta^r"] & F(AC \oplus BC)
\end{tikzcd} \tag{6.1.3}$$

This finishes the definition of a braided bimonoidal functor.

Moreover, we define the following.

- A braided bimonoidal functor as above is sometimes abbreviated to F.
- A braided bimonoidal functor F is
  - *robust* if $F^2_\oplus$, $F^0_\oplus$, and $F^0_\otimes$ are isomorphisms;
  - *strong* (respectively, *strict*) if both $F_\oplus$ and $F_\otimes$ are so;

– *unitary* if it is strong and if $F^0_\oplus$ and $F^0_\otimes$ are identities; and
– a *braided bimonoidal equivalence* if it is also an equivalence of categories. ◇

Explanation 6.1.4. A braided bimonoidal functor F is strong (respectively, strict) if $F^2_\oplus$, $F^0_\oplus$, $F^2_\otimes$, and $F^0_\otimes$ are isomorphisms (respectively, identities). It is unitary if $F^0_\oplus$ and $F^0_\otimes$ are identities, and $F^2_\oplus$ and $F^2_\otimes$ are isomorphisms. ◇

While the axioms (6.1.2) and (6.1.3) are stated in terms of $\rho^\bullet$ and δ^r, they are equivalent to the left versions in the following sense.

Proposition 6.1.5. *Suppose*

$$\big(F, F^2_\oplus, F^0_\oplus, F^2_\otimes, F^0_\otimes\big) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of the same data as in Definition 6.1.1.

(1) The multiplicative zero axiom (6.1.2) is equivalent to the commutativity of the following diagram for all objects $A \in \mathsf{C}$.

$$\begin{tikzcd}
\mathbb{0}(FA) \arrow[rr, "F^0_\oplus 1_{FA}"] \arrow[d, "\lambda^\bullet_{FA}"'] & & (F\mathbb{0})(FA) \arrow[d, "F^2_\otimes"] \\
\mathbb{0} \arrow[dr, "F^0_\oplus"'] & & F(\mathbb{0}A) \arrow[dl, "F\lambda^\bullet_A"] \\
& F\mathbb{0} &
\end{tikzcd} \tag{6.1.6}$$

(2) The distributivity axiom (6.1.3) is equivalent to the commutativity of the following diagram for all objects $A, B, C \in \mathsf{C}$.

$$\begin{tikzcd}
(FA)(FB \oplus FC) \arrow[r, "\delta^l"] \arrow[d, "1_{FA}F^2_\oplus"'] & (FA)(FB) \oplus (FA)(FC) \arrow[d, "F^2_\otimes \oplus F^2_\otimes"] \\
(FA)F(B \oplus C) \arrow[d, "F^2_\otimes"'] & F(AB) \oplus F(AC) \arrow[d, "F^2_\oplus"] \\
F\big(A(B \oplus C)\big) \arrow[r, "F\delta^l"] & F(AB \oplus AC)
\end{tikzcd} \tag{6.1.7}$$

Proof. We reuse the proof of Proposition I.5.1.4, which is the symmetric bimonoidal functor analogue, with the following changes.

- To prove the equivalence between (6.1.2) and (6.1.6), in the first diagram in the proof of Proposition I.5.1.4, we use the braided bimonoidal category axiom (2.1.33) instead of (2.1.18), which is the same as (I.2.1.19), and the symmetry axiom (1.3.33).
- To prove the equivalence between (6.1.3) and (6.1.7), in the second diagram in the proof of Proposition I.5.1.4, we use the braided bimonoidal category axiom (2.1.32) instead of (2.1.4), which is the same as (I.2.1.5), and the symmetry axiom. □

Recall from Definition 1.3.12 the composite of two monoidal functors.

Definition 6.1.8. Suppose

$$\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$$

are braided bimonoidal functors. The *composite*

$$\big(GF, (GF)^2_\oplus, (GF)^0_\oplus, (GF)^2_\otimes, (GF)^0_\otimes\big) : \mathsf{C} \longrightarrow \mathsf{E}$$

is defined by the following composites of monoidal functors.

$$\begin{aligned}\big(GF, (GF)^2_\oplus, (GF)^0_\oplus\big) &= G_\oplus \circ F_\oplus\\ \big(GF, (GF)^2_\otimes, (GF)^0_\otimes\big) &= G_\otimes \circ F_\otimes\end{aligned}$$

The finishes the definition of the composite. ◇

The next two results are the braided analogues of Lemma I.5.1.9 and Proposition I.5.1.10, whose proofs are reused here without any changes.

Lemma 6.1.9. *In Definition 6.1.8, $GF : \mathsf{C} \longrightarrow \mathsf{E}$ is a braided bimonoidal functor. Moreover, if both F and G are robust (respectively, strong, unitary, or strict), then so is GF.*

A subcategory is called *wide* if it contains all the objects in the larger category. A braided bimonoidal category is *small* if it has a set of objects.

Proposition 6.1.10. *There is a category $\mathsf{Bi}^{\mathsf{br}}$ with the following data.*

- *The objects are small braided bimonoidal categories as in Definition 2.1.29.*
- *The morphisms are braided bimonoidal functors as in Definition 6.1.1.*
- *Identity morphisms are identity functors with identity monoidal structures.*
- *Composition is as in Definition 6.1.8.*

Moreover, $\mathsf{Bi}^{\mathsf{br}}$ has the wide subcategories

- *$\mathsf{Bi}^{\mathsf{br}}_{\mathsf{r}}$ with robust braided bimonoidal functors as morphisms;*
- *$\mathsf{Bi}^{\mathsf{br}}_{\mathsf{sg}}$ with strong braided bimonoidal functors as morphisms;*
- *$\mathsf{Bi}^{\mathsf{br}}_{\mathsf{u}}$ with unitary braided bimonoidal functors as morphisms; and*
- *$\mathsf{Bi}^{\mathsf{br}}_{\mathsf{st}}$ with strict braided bimonoidal functors as morphisms.*

Examples. The rest of this section contains examples of braided bimonoidal functors.

Example 6.1.11. A symmetric monoidal category is also a braided monoidal category by Proposition 1.3.36, and similarly in the bimonoidal setting by Corollary 2.2.3. Therefore, by Definitions I.5.1.1, 1.3.18, 1.3.32, and 6.1.1, between symmetric bimonoidal categories, a braided bimonoidal functor is the same thing as a symmetric bimonoidal functor. ◇

In Theorem 2.4.22, we observed that an abelian category with a compatible braided monoidal structure is a tight braided bimonoidal category. Examples of such braided bimonoidal categories include

- the category $\mathsf{Mod}(A)$ in Theorem 3.2.19 (2) for a braided bialgebra (A, R),
- the category $\mathcal{F}^{\mathsf{any}}$ of Fibonacci anyons in Theorem 3.4.13, and
- the category $\mathcal{I}^{\mathsf{any}}$ of Ising anyons in Theorem 3.6.14.

Recall from Definition 2.3.3 the notion of an additive functor.

Proposition 6.1.12. *Suppose the triple $(\mathsf{C}, \mathsf{D}, F_\otimes)$ consists of the following data.*

- *Each of C and D is an abelian category with a compatible braided monoidal structure in the sense of Convention 2.4.1.*

- *$F_\otimes = (F, F^2_\otimes, F^0_\otimes) : \mathsf{C} \longrightarrow \mathsf{D}$ is a braided monoidal functor such that $F : \mathsf{C} \longrightarrow \mathsf{D}$ is also an additive functor.*

Then F canonically extends to a braided bimonoidal functor

$$(F, F^2_\otimes, F^0_\otimes, F^2_\oplus, F^0_\oplus) : \mathsf{C} \longrightarrow \mathsf{D}$$

with C and D regarded as tight braided bimonoidal categories as in Theorem 2.4.22.

Proof. First we define the additive structure morphisms $F^0_\oplus$ and $F^2_\oplus$.

- $F^0_\oplus : \mathbb{0} \longrightarrow F\mathbb{0} \in \mathsf{D}$ is the unique morphism from the zero object in D.
- The additive monoidal constraint $F^2_\oplus$ is defined by the commutative diagrams

$$\begin{tikzcd} FA_1 \oplus FA_2 \arrow[rr, "F^2_\oplus"] & & F(A_1 \oplus A_2) \\ & FA_k \arrow[ul, "i_k^{FA_1,FA_2}"] \arrow[ur, "Fi_k^{A_1,A_2}"'] & \end{tikzcd} \tag{6.1.13}$$

  for objects $A_1, A_2 \in \mathsf{C}$ and $k = 1, 2$, with each i_k the kth factor inclusion in Definition 2.3.4.

The naturality and the invertibility of $F^2_\oplus$ follow from Theorem 2.3.7 (4), which states that F preserves direct sums. In particular, $F^2_\oplus$ can also be characterized by the commutative diagrams

$$\begin{tikzcd} FA_1 \oplus FA_2 \arrow[rr, "F^2_\oplus"] \arrow[dr, "p_k^{FA_1,FA_2}"'] & & F(A_1 \oplus A_2) \arrow[dl, "Fp_k^{A_1,A_2}"] \\ & FA_k & \end{tikzcd} \tag{6.1.14}$$

with each p_k the kth factor projection.

The associativity axiom (1.3.10), the unity axiom (1.3.11), and the compatibility axiom (1.3.19) with $\xi^\oplus$ for $F_\oplus = (F, F^2_\oplus, F^0_\oplus)$ follow from the direct sum axioms (2.3.5) and Explanation 2.4.3. Therefore,

$$(\mathsf{C}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus) \xrightarrow{F_\oplus} (\mathsf{D}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus)$$

is a symmetric monoidal functor.

The multiplicative zero axiom (6.1.2) is the outer diagram below for $A \in \mathsf{C}$.

$$\begin{tikzcd} (FA)\mathbb{0} \arrow[rr, "1_{FA}F^0_\oplus"] \arrow[d, "\rho^\bullet_{FA}"'] & & (FA)(F\mathbb{0}) \arrow[d, "F^2_\otimes"] \\ \mathbb{0} \arrow[dr, "F^0_\oplus"'] & & F(A\mathbb{0}) \arrow[ll] \arrow[dl, "F\rho^\bullet_A"] \\ & F\mathbb{0} & \end{tikzcd}$$

- The middle horizontal arrow is the unique morphism to the zero object in D. This implies that the top square is commutative.

- In the bottom triangle,

$$\rho^{\bullet}_{A} : A \otimes \mathbb{0} \longrightarrow \mathbb{0}$$

  is the unique morphism to the zero object by Definition 2.4.11. This is also the zero morphism in the trivial abelian group $\mathsf{C}(A \otimes \mathbb{0}, \mathbb{0})$. Since F is additive, it preserves zero morphisms. Lemma 2.3.12 implies that the zero morphism $F\rho^{\bullet}_{A}$ is equal to the composite

$$F(A\mathbb{0}) \longrightarrow \mathbb{0} \longrightarrow F\mathbb{0}.$$

  Therefore, the bottom triangle is commutative.

This proves the axiom (6.1.2).

Next we consider the distributivity axiom (6.1.3). Since F preserves direct sums, it suffices to show that the two composites in (6.1.3) composed with the projection to the object $F(AC)$ are equal and similarly for $F(BC)$. Composed with the projection to $F(AC)$, (6.1.3) becomes the outer diagram below.

$$\begin{tikzcd}
(FA \oplus FB)(FC) \arrow[r, "\delta^r"] \arrow[d, "F^2_{\oplus} 1_{FC}"'] \arrow[dr, "p_1^{FA,FB} 1_{FC}"] & (FA)(FC) \oplus (FB)(FC) \arrow[r, "F^2_{\otimes} \oplus F^2_{\otimes}"] \arrow[d, "p_1^{(FA)(FC),(FB)(FC)}"] \arrow[dr, phantom, "(2.4.4)"] & F(AC) \oplus F(BC) \arrow[dr, "F^2_{\oplus}"] \arrow[dd, "p_1^{F(AC),F(BC)}"'] & \\
F(A \oplus B)(FC) \arrow[r, "(Fp_1^{A,B}) 1_{FC}"'] \arrow[d, "F^2_{\otimes}"'] & (FA)(FC) \arrow[dr, "F^2_{\otimes}"] \arrow[d, phantom, "\text{nat}"] & & F(AC \oplus BC) \arrow[dl, "Fp_1^{AC,BC}"] \\
F((A \oplus B)C) \arrow[rr, "F(p_1^{A,B} 1_C)"] \arrow[dr, "F\delta^r"'] & {} \arrow[d, phantom, "(2.4.9)"] & F(AC) & \\
& F(AC \oplus BC) \arrow[ur, "Fp_1^{AC,BC}"'] & &
\end{tikzcd}$$

As indicated, this diagram is commutative by

- the naturality of $F^2_{\otimes}$ and
- the definitions of a direct sum of morphisms (2.4.4), δ^r (2.4.9), and $F^2_{\oplus}$ (6.1.14).

A similar diagram with p_2 instead of p_1 shows that the two composites in (6.1.3) composed with the projection to the object $F(BC)$ are equal. This proves the axiom (6.1.3). We have proved that F is a braided bimonoidal functor when equipped with the additive structure $F_{\oplus}$ and the multiplicative structure $F_{\otimes}$. $\square$

Corollary 6.1.15. *In Proposition 6.1.12, suppose the braided monoidal structures on* C *and* D *are symmetric monoidal. Then the extension of F is a symmetric bimonoidal functor.*

Proof. By Corollary 2.5.1, C and D are tight symmetric bimonoidal categories. Therefore, the assertion follows from Example 6.1.11 and Proposition 6.1.12. $\square$

6.2. Associated Right Permutative Braided Category

In this section, for each tight braided bimonoidal category C, we construct an associated right permbraided category A in the sense of Definition 5.1.11. By Proposition 5.1.19, each right permbraided category is a tight braided bimonoidal category whose structure morphisms $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^r, $\xi^{\otimes}_{-,\mathbb{0}}$, and $\xi^{\otimes}_{\mathbb{0},-}$ are identities. In Theorem 6.3.6, we will observe that A is a strictification of C in a precise sense.

This section is organized as follows.

- Definition 6.2.3 defines the underlying category of A.
- The additive structure of A is in Definition 6.2.10 and is verified to be a permutative category in Lemma 6.2.18.
- The multiplicative structure of A is in Definition 6.2.22 and is verified to be a braided strict monoidal category in Lemma 6.2.29.
- The multiplicative zeros and the distributivity morphisms in A are in Definition 6.2.31 and are verified to be well defined in Lemma 6.2.35.
- The other axioms are checked in Lemmas 6.2.36 through 6.2.38.
- Proposition 6.2.39 states that A is a right permbraided category.

Convention 6.2.1. For the rest of this chapter, unless specified otherwise, C is a tight braided bimonoidal category as in Definition 2.1.29. $\diamond$

The Underlying Category. The morphisms in the associated right permutative braided category A uses the following notation. With $\odot \in \{\oplus, \otimes\}$, the *right normalized bracketing* is defined inductively by

$$(x_1 \odot \cdots \odot x_k)_{\mathsf{rt}} = \begin{cases} x_1 & \text{if } k = 1 \text{ and} \\ x_1 \odot (x_2 \odot \cdots \odot x_k)_{\mathsf{rt}} & \text{if } k > 1. \end{cases} \tag{6.2.2}$$

The next definition is the same as in the symmetric case in Definitions I.5.2.3, I.5.2.16, and I.5.2.21.

Definition 6.2.3. For a tight braided bimonoidal category C, define

- the category A and
- the function $\pi : \mathsf{Ob}(\mathsf{A}) \longrightarrow \mathsf{Ob}(\mathsf{C})$

as follows.

Objects: An object $\underline{a} \in \mathsf{Ob}(\mathsf{A})$ is a finite sequence

$$\underline{a} = \{a^1, \ldots, a^r\} \tag{6.2.4}$$

with *additive length* $|\underline{a}| = r \geq 0$, such that for each $1 \leq i \leq r$, a^i is a finite sequence

$$a^i = (a^i_1, \ldots, a^i_{m_i}) \tag{6.2.5}$$

with *multiplicative length* $|a^i| = m_i \geq 0$ and each $a^i_j \in \mathsf{Ob}(\mathsf{C})$. We call

- a^i the *ith monomial* in $\underline{a}$ and
- a^i_j the *jth alphabet* in a^i.

Realization: Define the *realization function*

$$\mathsf{Ob}(\mathsf{A}) \xrightarrow{\ \pi\ } \mathsf{Ob}(\mathsf{C}) \tag{6.2.6}$$

as follows for $\underline{a} = \{a^1, \ldots, a^r\} \in \mathsf{Ob}(\mathsf{A})$ as in (6.2.4).

$$\begin{aligned} \pi a^i &= \begin{cases} \mathbb{1} \in \mathsf{C} & \text{if } m_i = 0 \text{ and} \\ (a^i_1 \otimes \cdots \otimes a^i_{m_i})_{\mathsf{rt}} \in \mathsf{C} & \text{if } m_i > 0. \end{cases} \\ \pi \underline{a} &= \begin{cases} \mathbb{0} \in \mathsf{C} & \text{if } r = 0 \text{ and} \\ (\pi a^1 \oplus \cdots \oplus \pi a^r)_{\mathsf{rt}} \in \mathsf{C} & \text{if } r > 0. \end{cases} \end{aligned} \tag{6.2.7}$$

For each $1 \leq i \leq r$, πa^i uses the right normalized bracketing (6.2.2) in $(\mathsf{C}, \otimes)$. The object $\pi \underline{a}$ uses the right normalized bracketing in $(\mathsf{C}, \oplus)$.

Morphisms: For objects $\underline{a}, \underline{b} \in \mathsf{Ob}(\mathsf{A})$, define the *morphism* set

$$\mathsf{A}(\underline{a}; \underline{b}) = \mathsf{C}(\pi \underline{a}; \pi \underline{b}) \tag{6.2.8}$$

with π the realization function in (6.2.6).

Identity Morphisms: Define $1_{\underline{a}} = 1_{\pi \underline{a}} \in \mathsf{A}(\underline{a}; \underline{a})$.

Composition: Define the *composition* in A

$$\mathsf{A}(\underline{b}; \underline{c}) \times \mathsf{A}(\underline{a}; \underline{b}) = \mathsf{C}(\pi \underline{b}; \pi \underline{c}) \times \mathsf{C}(\pi \underline{a}; \pi \underline{b}) \longrightarrow \mathsf{C}(\pi \underline{a}; \pi \underline{c}) = \mathsf{A}(\underline{a}; \underline{c})$$

as the composition in C.

This finishes the definition of (A, π). ◇

Since C is a category, so is A.

Mac Lane Coherence Isomorphisms. Recall from (5.3.14) the *value* in C of a path in $\mathsf{Gr}(X)$. Each time the value in C of a path is considered, it is assumed that the following data (X, φ) are given.

- X is a set of formal variables with two distinguished elements $\{0^X, 1^X\}$.
- $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is a function that satisfies

$$\varphi(0^X) = \mathbb{0} \quad \text{and} \quad \varphi(1^X) = \mathbb{1}.$$

The function φ is extended additively and multiplicatively to a graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ as in (5.3.12) and (5.3.13). The additive structure in A involves the following notion; see (6.2.14) and (6.2.16).

Definition 6.2.9. A *Mac Lane coherence isomorphism* in C, which is denoted by $\cong^{\oplus}_{\mathsf{ML}}$, is the value $\varphi P : \varphi u \longrightarrow \varphi v$ in C of a path $P : u \longrightarrow v$ in $\mathsf{Gr}(X)$ that satisfies the following three conditions.

(i) P only involves identities, $\alpha^{\pm\oplus}$, $\lambda^{\pm\oplus}$, $\rho^{\pm\oplus}$, and $\xi^{\pm\oplus}$.

(ii) In addition to the distinguished elements $\{0^X, 1^X\}$, the set X contains a specific element x_m for each monomial m in each object in A that appears in φu.

(iii) For each monomial m in (ii), the equality

$$\varphi(x_m) = \pi(m) \in \mathsf{C}$$

holds, with $\pi(m)$ as in (6.2.7).

This finishes the definition of a Mac Lane coherence isomorphism. ◇

The Additive Structure.

Definition 6.2.10. Continuing Definition 6.2.3, define the *additive structure*

$$\left(\mathbb{0}^{\mathsf{A}}, \oplus^{\mathsf{A}}, \xi^{\oplus \mathsf{A}}\right)$$

in A as follows.

The Additive Zero: Define

$$\mathbb{0}^{\mathsf{A}} = \varnothing \in \mathsf{Ob}(\mathsf{A})$$

as the unique object with additive length 0.

The Sum on Objects: For objects $\underline{a} = \{a^1, \ldots, a^r\}$ and $\underline{b} = \{b^1, \ldots, b^s\} \in \mathsf{Ob}(\mathsf{A})$, define their *sum* by concatenation, that is,

$$\underline{a} \oplus^{\mathsf{A}} \underline{b} = \left\{a^1, \ldots, a^r, b^1, \ldots, b^s\right\} \in \mathsf{Ob}(\mathsf{A}), \tag{6.2.11}$$

which has additive length $|\underline{a}| + |\underline{b}|$.

The Sum on Morphisms: For morphisms

$$\begin{aligned} \underline{a} &\xrightarrow{f} \underline{b} \in \mathsf{A}(\underline{a};\underline{b}) = \mathsf{C}(\pi\underline{a};\pi\underline{b}) \quad \text{and} \\ \underline{c} &\xrightarrow{g} \underline{d} \in \mathsf{A}(\underline{c};\underline{d}) = \mathsf{C}(\pi\underline{c};\pi\underline{d}), \end{aligned} \tag{6.2.12}$$

their *sum*

$$\underline{a} \oplus^{\mathsf{A}} \underline{c} \xrightarrow{f \oplus^{\mathsf{A}} g} \underline{b} \oplus^{\mathsf{A}} \underline{d} \in \mathsf{A}\big(\underline{a} \oplus^{\mathsf{A}} \underline{c}; \underline{b} \oplus^{\mathsf{A}} \underline{d}\big) = \mathsf{C}\big(\pi(\underline{a} \oplus^{\mathsf{A}} \underline{c}); \pi(\underline{b} \oplus^{\mathsf{A}} \underline{d})\big) \tag{6.2.13}$$

is defined as the following composite in C, with $\cong^{\oplus}_{\mathsf{ML}}$ as in Definition 6.2.9.

$$\begin{tikzcd} \pi(\underline{a} \oplus^{\mathsf{A}} \underline{c}) \arrow[r, "f \oplus^{\mathsf{A}} g"] \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"'] & \pi(\underline{b} \oplus^{\mathsf{A}} \underline{d}) \\ \pi\underline{a} \oplus \pi\underline{c} \arrow[r, "f \oplus g"] & \pi\underline{b} \oplus \pi\underline{d} \arrow[u, "\cong^{\oplus}_{\mathsf{ML}}"'] \end{tikzcd} \tag{6.2.14}$$

The Additive Symmetry: For objects $\underline{a}, \underline{b} \in \mathsf{Ob}(\mathsf{A})$, define the morphism

$$\underline{a} \oplus^{\mathsf{A}} \underline{b} \xrightarrow{\xi^{\oplus\mathsf{A}}_{\underline{a};\underline{b}}} \underline{b} \oplus^{\mathsf{A}} \underline{a} \in \mathsf{A}\big(\underline{a} \oplus^{\mathsf{A}} \underline{b}; \underline{b} \oplus^{\mathsf{A}} \underline{a}\big) = \mathsf{C}\big(\pi(\underline{a} \oplus^{\mathsf{A}} \underline{b}); \pi(\underline{b} \oplus^{\mathsf{A}} \underline{a})\big) \tag{6.2.15}$$

as the following composite in C, with $\cong^{\oplus}_{\mathsf{ML}}$ as in Definition 6.2.9.

$$\begin{tikzcd} \pi(\underline{a} \oplus^{\mathsf{A}} \underline{b}) \arrow[r, "\xi^{\oplus\mathsf{A}}_{\underline{a};\underline{b}}"] \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"'] & \pi(\underline{b} \oplus^{\mathsf{A}} \underline{a}) \\ \pi\underline{a} \oplus \pi\underline{b} \arrow[r, "\xi^{\oplus}_{\pi\underline{a};\pi\underline{b}}"] & \pi\underline{b} \oplus \pi\underline{a} \arrow[u, "\cong^{\oplus}_{\mathsf{ML}}"'] \end{tikzcd} \tag{6.2.16}$$

This finishes the definition of the additive structure in A. ◇

Explanation 6.2.17. In the definition (6.2.14) of $f \oplus^{\mathsf{A}} g$, suppose $\underline{a} = \{a^1, \ldots, a^r\}$ and $\underline{c} = \{c^1, \ldots, c^t\} \in \mathsf{A}$. The Mac Lane coherence isomorphism

$$\pi(\underline{a} \oplus^{\mathsf{A}} \underline{c}) \xrightarrow{\cong^{\oplus}_{\mathsf{ML}}} \pi\underline{a} \oplus \pi\underline{c}$$

is defined as follows. Conditions (i)–(iii) below refer to those in Definition 6.2.9.

- By (ii), the set X of formal variables is
$$X = \left\{0^X, 1^X, x_1, \ldots, x_r, y_1, \ldots, y_t\right\},$$
with one element in $X \setminus \{0^X, 1^X\}$ for each monomial in each of $\underline{a}$ and $\underline{c}$.
- By (iii), the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is defined as
$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \\ \pi a^i & \text{if } x = x_i \text{ for } 1 \le i \le r, \text{ and} \\ \pi c^k & \text{if } x = y_k \text{ for } 1 \le k \le t. \end{cases}$$

- By (i) and Mac Lane's Coherence Theorem I.1.3.3, the left vertical $\cong^{\oplus}_{\mathsf{ML}}$ in (6.2.14) is
  - $\lambda^{-\oplus}_{\pi\underline{c}}$ if $\underline{a} = \mathbb{0}^{\mathsf{A}}$ and
  - $\rho^{-\oplus}_{\pi\underline{a}}$ if $\underline{c} = \mathbb{0}^{\mathsf{A}}$.

  If $\underline{a} \neq \mathbb{0}^{\mathsf{A}} \neq \underline{c}$, then this instance of $\cong^{\oplus}_{\mathsf{ML}}$ involves only identity morphisms and $\alpha^{-\oplus}$.

Each of the other three instances of $\cong^{\oplus}_{\mathsf{ML}}$ in (6.2.14) and (6.2.16) admits a similar description. $\diamond$

Recall from Definition 1.3.32 that a *permutative category* is a symmetric monoidal category whose underlying monoidal category is strict; that is, the associativity isomorphism and the left/right unit isomorphisms are identities.

Lemma 6.2.18. *In the context of Definitions 6.2.3 and 6.2.10, the quadruple*

$$(\mathsf{A}, \oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \xi^{\oplus\mathsf{A}})$$

is a permutative category.

Proof. The proofs of Lemmas I.5.2.8, I.5.2.23, I.5.2.33, and I.5.3.1, where C is a tight symmetric bimonoidal category, are still valid in the current context. $\square$

Laplaza Coherence Isomorphisms. We continue to assume that C is a tight braided bimonoidal category. The multiplicative structure in A involves the following notion; see (6.2.25) and (6.2.27).

Definition 6.2.19. A *Laplaza coherence isomorphism* in C, which is denoted by $\cong_{\mathsf{Lap}}$, is the value $\varphi P : \varphi v \longrightarrow \varphi w$ in C of a path $P : v \longrightarrow w$ in $\mathsf{Gr}(X)$ that satisfies the following three conditions.

(i) P does *not* involve $\xi^{\pm\otimes}$.
(ii) In addition to the distinguished elements $\{0^X, 1^X\}$, the set X contains a specific element x_a for each alphabet a in each object in A that appears in φv.
(iii) For each alphabet a in (ii), the equality
$$\varphi(x_a) = a \in \mathsf{C}$$
holds.

The inverse of a Laplaza coherence isomorphism is denoted by $\cong^{-1}_{\mathsf{Lap}}$. $\diamond$

Explanation 6.2.20. By Definition 5.3.4, the elementary edges δ^l and δ^r in $\mathsf{Gr}(X)$ do *not* have formal inverses. Therefore, a Laplaza coherence isomorphism also does not involve δ^{-l} and δ^{-r}. On the other hand, the inverse of a Laplaza coherence isomorphism may involve δ^{-l} and δ^{-r}, but neither δ^l nor δ^r. This is why we need to distinguish between $\cong_{\mathsf{Lap}}$ and $\cong^{-1}_{\mathsf{Lap}}$. The tightness assumption on C is needed to make sure that the distributivity morphisms δ^l and δ^r are invertible, so $\cong^{-1}_{\mathsf{Lap}}$ is defined. $\diamond$

Lemma 6.2.21. *The following statements hold for Laplaza coherence isomorphisms.*

(1) A Mac Lane coherence isomorphism as in Definition 6.2.9 is also a Laplaza coherence isomorphism.
(2) Laplaza coherence isomorphisms are closed under $\oplus$, $\otimes$, and composition.
(3) The braided distortion of each path that yields a Laplaza coherence isomorphism has identity braid components.

Proof. Statement (1) holds by definition.

For statement (2), Laplaza coherence isomorphisms are closed under composition because two paths P and Q in $\mathsf{Gr}(X)$ can be concatenated to the path (Q, P) as long as the codomain of P is the domain of Q. Closure under $\oplus$ and $\otimes$ follows from the definition (5.3.13) of the graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$ at prime edges.

For statement (3), the braided distortion of a path is its value in the braided distortion category as in Definition 5.3.15. As noted in Explanation 5.2.29 (3), in the braided distortion category $\mathcal{D}^{\mathsf{br}}$, the braiding $\xi^{\otimes}$ in (5.2.18) is the only structure isomorphism with nonidentity braid components in the sense of Definition 5.2.2. By definition, a Laplaza coherence isomorphism is the value in C of a path that does not involve $\xi^{\pm\otimes}$. So the braided distortion of such a path has only identity braids in its braid components. $\square$

The Multiplicative Structure.

Definition 6.2.22. Continuing Definition 6.2.3, define the *multiplicative structure*

$$\left(\mathbb{1}^{\mathsf{A}}, \otimes^{\mathsf{A}}, \xi^{\otimes\mathsf{A}}\right)$$

in A as follows.

The Multiplicative Unit: Define

$$\mathbb{1}^{\mathsf{A}} = \{\varnothing\} \in \mathsf{Ob}(\mathsf{A})$$

as the object with additive length 1 whose only monomial has multiplicative length 0.

The Product on Objects: For objects $\underline{a} = \{a^1, \ldots, a^r\}$ and $\underline{b} = \{b^1, \ldots, b^s\}$ in A, define their *product* by

$$\underline{a} \otimes^{\mathsf{A}} \underline{b} = \left\{(a^1, b^1), \ldots, (a^1, b^s), \ldots, (a^r, b^1), \ldots, (a^r, b^s)\right\} \in \mathsf{Ob}(\mathsf{A}) \tag{6.2.23}$$

with additive length $|\underline{a}||\underline{b}|$. For $1 \leq i \leq r$ and $1 \leq j \leq s$, the $(j + (i-1)s)$th monomial in $\underline{a} \otimes^{\mathsf{A}} \underline{b}$ is the concatenation (a^i, b^j) with multiplicative length $|a^i| + |b^j|$.

The Product on Morphisms: For morphisms f and g as in (6.2.12), their *product*

$$\underline{a} \otimes^{\mathsf{A}} \underline{c} \xrightarrow{f \otimes^{\mathsf{A}} g} \underline{b} \otimes^{\mathsf{A}} \underline{d} \in \mathsf{A}\big(\underline{a} \otimes^{\mathsf{A}} \underline{c}; \underline{b} \otimes^{\mathsf{A}} \underline{d}\big) = \mathsf{C}\big(\pi(\underline{a} \otimes^{\mathsf{A}} \underline{c}); \pi(\underline{b} \otimes^{\mathsf{A}} \underline{d})\big) \tag{6.2.24}$$

is defined as the following composite in C.

$$\begin{tikzcd} \pi(\underline{a} \otimes^{\mathsf{A}} \underline{c}) \arrow[r, "f \otimes^{\mathsf{A}} g"] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{d}) \\ \pi\underline{a} \otimes \pi\underline{c} \arrow[r, "f \otimes g"] & \pi\underline{b} \otimes \pi\underline{d} \arrow[u, "\cong_{\mathsf{Lap}}"'] \end{tikzcd} \tag{6.2.25}$$

The Braiding: For objects $\underline{a}, \underline{b} \in \mathsf{Ob}(\mathsf{A})$, define the morphism

$$\underline{a} \otimes^{\mathsf{A}} \underline{b} \xrightarrow{\xi^{\otimes\mathsf{A}}_{\underline{a};\underline{b}}} \underline{b} \otimes^{\mathsf{A}} \underline{a} \in \mathsf{A}\big(\underline{a} \otimes^{\mathsf{A}} \underline{b}; \underline{b} \otimes^{\mathsf{A}} \underline{a}\big) = \mathsf{C}\big(\pi(\underline{a} \otimes^{\mathsf{A}} \underline{b}); \pi(\underline{b} \otimes^{\mathsf{A}} \underline{a})\big) \tag{6.2.26}$$

as the following composite in C.

$$
\begin{tikzcd}
\pi(\underline{a} \otimes^{\mathsf{A}} \underline{b}) \arrow[r, "\xi^{\otimes \mathsf{A}}_{\underline{a};\underline{b}}"] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{a}) \\
\pi\underline{a} \otimes \pi\underline{b} \arrow[r, "\xi^{\otimes}_{\pi\underline{a};\pi\underline{b}}"] & \pi\underline{b} \otimes \pi\underline{a} \arrow[u, "\cong_{\mathsf{Lap}}"']
\end{tikzcd} \tag{6.2.27}
$$

This finishes the definition of the multiplicative structure in A. ◇

Explanation 6.2.28. In the definition (6.2.25) of $f \otimes^{\mathsf{A}} g$, suppose

- $\underline{b} = \{b^1, \ldots, b^s\} \in \mathsf{A}$ with
$$b^j = \left(b^j_1, \ldots, b^j_{n_j}\right)$$
for $1 \le j \le s$ and
- $\underline{d} = \{d^1, \ldots, d^u\} \in \mathsf{A}$ with
$$d^l = \left(d^l_1, \ldots, d^l_{q_l}\right)$$
for $1 \le l \le u$.

The Laplaza coherence isomorphism

$$\pi\underline{b} \otimes \pi\underline{d} \xrightarrow{\cong_{\mathsf{Lap}}} \pi(\underline{b} \otimes^{\mathsf{A}} \underline{d})$$

is defined as follows. Conditions (i)–(iii) below refer to those in Definition 6.2.19.

- By (ii), the set X of formal variables is
$$X = \{0^X, 1^X\} \coprod \{x^j_1, \ldots, x^j_{n_j}\}_{1 \le j \le s} \coprod \{y^l_1, \ldots, y^l_{q_l}\}_{1 \le l \le u}.$$
There is one element in $X \smallsetminus \{0^X, 1^X\}$ for each alphabet in each of $\underline{b}$ and $\underline{d}$.
- By (iii), the function $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ is defined as
$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X, \\ \mathbb{1} & \text{if } x = 1^X, \\ b^j_h & \text{if } x = x^j_h \text{ for } 1 \le j \le s \text{ and } 1 \le h \le n_j, \text{ and} \\ d^l_k & \text{if } x = y^l_k \text{ for } 1 \le l \le u \text{ and } 1 \le k \le q_l. \end{cases}$$
- By (i), the right vertical $\cong_{\mathsf{Lap}}$ in (6.2.25) is
  - $\lambda^{\bullet}_{\pi\underline{d}}$ if $\underline{b} = \mathbb{0}^{\mathsf{A}}$,
  - $\lambda^{\otimes}_{\pi\underline{d}}$ if $\underline{b} = \mathbb{1}^{\mathsf{A}}$,
  - $\rho^{\bullet}_{\pi\underline{b}}$ if $\underline{d} = \mathbb{0}^{\mathsf{A}}$, and
  - $\rho^{\otimes}_{\pi\underline{b}}$ if $\underline{d} = \mathbb{1}^{\mathsf{A}}$.

  If $\underline{b}, \underline{d} \notin \{\mathbb{0}^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}\}$, then this instance of $\cong_{\mathsf{Lap}}$ involves identity morphisms, $\alpha^{\pm\oplus}$, $\xi^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, δ^l, and δ^r. Moreover, $\lambda^{\otimes}$ and $\rho^{\otimes}$ are involved if and only if, respectively, $\underline{b}$ and $\underline{d}$ contain monomials of multiplicative length 0.

Each of the other three instances of $\cong_{\mathsf{Lap}}$ in (6.2.25) and (6.2.27) admits a similar description. ◇

Recall from Definition 1.3.15 that a braided monoidal category is *strict* if the underlying monoidal category is strict; that is, the associativity isomorphism and the left/right unit isomorphisms are identities.

Lemma 6.2.29. *In the context of Definitions 6.2.3 and 6.2.22, the quadruple*

$$(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}, \xi^{\otimes\mathsf{A}})$$

is a braided strict monoidal category.

Proof. We reuse the proofs of Lemmas I.5.2.8, I.5.2.33, and I.5.3.1 with the following adjustments. First note that in (6.2.25) and (6.2.27), $\cong_{\mathsf{Lap}}$ and $\cong_{\mathsf{Lap}}^{-1}$ are well defined. Indeed, each of the four instances of $\cong_{\mathsf{Lap}}$ is the value in C of a path whose braided distortion has only identity braid components by Lemma 6.2.21(3). The permutation component is uniquely determined by the monomials of the objects involved because there is a unique permutation between any two permuted words of the same length. The Coherence Theorem 5.4.4, which is applicable because C is assumed to be tight, then implies that each instance of $\cong_{\mathsf{Lap}}$ is well defined in C.

To check that $(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}})$ is a strict monoidal category, first note that the functoriality of

$$- \otimes^{\mathsf{A}} - : \mathsf{A} \times \mathsf{A} \longrightarrow \mathsf{A}$$

follows from the functoriality of $\otimes$ in C.

It is shown in Lemma I.5.2.8 that $\otimes^{\mathsf{A}}$ is strictly associative on objects, with $\mathbb{1}^{\mathsf{A}}$ as a strict two-sided unit. The naturality of the left unit isomorphism $\lambda^{\otimes\mathsf{A}} = 1$ and the right unit isomorphism $\rho^{\otimes\mathsf{A}} = 1$ follow from the naturality of $\lambda^{\otimes}$ and $\rho^{\otimes}$ in C.

The naturality of the multiplicative associativity $\alpha^{\otimes\mathsf{A}} = 1$ in A follows from the following diagram in C for morphisms $f_i \in \mathsf{A}(\underline{a}_i; \underline{b}_i)$ for $1 \le i \le 3$.

$$\begin{tikzcd}
\pi\big((\underline{a}_1 \otimes^{\mathsf{A}} \underline{a}_2) \otimes^{\mathsf{A}} \underline{a}_3\big) \arrow[r, "="] \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"'] & \pi\big(\underline{a}_1 \otimes^{\mathsf{A}} (\underline{a}_2 \otimes^{\mathsf{A}} \underline{a}_3)\big) \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"] \\
\pi(\underline{a}_1 \otimes^{\mathsf{A}} \underline{a}_2) \otimes \pi\underline{a}_3 \arrow[d, "\cong_{\mathsf{Lap}}^{-1} \otimes 1"'] & \pi\underline{a}_1 \otimes \pi(\underline{a}_2 \otimes^{\mathsf{A}} \underline{a}_3) \arrow[d, "1 \otimes \cong_{\mathsf{Lap}}^{-1}"] \\
(\pi\underline{a}_1 \otimes \pi\underline{a}_2) \otimes \pi\underline{a}_3 \arrow[r, "\alpha^{\otimes}"] \arrow[d, "(f_1 \otimes f_2) \otimes f_3"'] & \pi\underline{a}_1 \otimes (\pi\underline{a}_2 \otimes \pi\underline{a}_3) \arrow[d, "f_1 \otimes (f_2 \otimes f_3)"] \\
(\pi\underline{b}_1 \otimes \pi\underline{b}_2) \otimes \pi\underline{b}_3 \arrow[r, "\alpha^{\otimes}"] \arrow[d, "\cong_{\mathsf{Lap}} \otimes 1"'] & \pi\underline{b}_1 \otimes (\pi\underline{b}_2 \otimes \pi\underline{b}_3) \arrow[d, "1 \otimes \cong_{\mathsf{Lap}}"] \\
\pi(\underline{b}_1 \otimes^{\mathsf{A}} \underline{b}_2) \otimes \pi\underline{b}_3 \arrow[d, "\cong_{\mathsf{Lap}}"'] & \pi\underline{b}_1 \otimes \pi(\underline{b}_2 \otimes^{\mathsf{A}} \underline{b}_3) \arrow[d, "\cong_{\mathsf{Lap}}"] \\
\pi\big((\underline{b}_1 \otimes^{\mathsf{A}} \underline{b}_2) \otimes^{\mathsf{A}} \underline{b}_3\big) \arrow[r, "="] & \pi\big(\underline{b}_1 \otimes^{\mathsf{A}} (\underline{b}_2 \otimes^{\mathsf{A}} \underline{b}_3)\big)
\end{tikzcd} \tag{6.2.30}$$

- By (6.2.25), the left vertical composite defines $(f_1 \otimes^{\mathsf{A}} f_2) \otimes^{\mathsf{A}} f_3$, and the right vertical composite defines $f_1 \otimes^{\mathsf{A}} (f_2 \otimes^{\mathsf{A}} f_3)$.
- The middle rectangle is commutative by the naturality of $\alpha^{\otimes}$ in C.
- In the bottom rectangle, each of the two composites is a Laplaza coherence isomorphism. As in the first paragraph of this proof, Lemma 6.2.21 and Theorem 5.4.4 imply that the two composites are equal.
- The top rectangle is commutative for the same reason as for the bottom rectangle, after replacing each $\cong_{\mathsf{Lap}}^{-1}$ with its inverse $\cong_{\mathsf{Lap}}$ in the opposite direction.

The unity axiom (1.3.2) and the pentagon axiom (1.3.3) hold in $(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}})$ because each edge involved is the identity morphism. Therefore, $(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}})$ is a strict monoidal category.

Similar to (6.2.30), the naturality and the invertibility of $\xi^{\otimes \mathsf{A}}$ in (6.2.26) follow from the corresponding properties of $\xi^{\otimes}$ in C and the fact that each $\cong_{\mathsf{Lap}}$ is well defined.

The left hexagon diagram (1.3.17) for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$ is the outer diagram in C below.

$$\begin{tikzcd}
(\pi\underline{b} \otimes \pi\underline{a}) \otimes \pi\underline{c} \arrow[r, "\cong_{\mathsf{Lap}} \otimes 1"] \arrow[drrr, "\alpha^{\otimes}"] & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{a}) \otimes \pi\underline{c} \arrow[rr, "\cong_{\mathsf{Lap}}"] & & \pi[(\underline{b} \otimes^{\mathsf{A}} \underline{a}) \otimes^{\mathsf{A}} \underline{c}] \arrow[d, "\alpha^{\otimes \mathsf{A}} = 1"] \\
(\pi\underline{a} \otimes \pi\underline{b}) \otimes \pi\underline{c} \arrow[u, "\xi^{\otimes} 1"] \arrow[ddr, "\alpha^{\otimes}"] & & & \pi[\underline{b} \otimes^{\mathsf{A}} (\underline{a} \otimes^{\mathsf{A}} \underline{c})] \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"] \arrow[dddd, bend left, "1_{\underline{b}} \otimes^{\mathsf{A}} \xi^{\otimes \mathsf{A}}_{\underline{a};\underline{c}}"] \\
\pi(\underline{a} \otimes^{\mathsf{A}} \underline{b}) \otimes \pi\underline{c} \arrow[u, "\cong_{\mathsf{Lap}}^{-1} \otimes 1"] & & & \pi\underline{b} \otimes \pi(\underline{a} \otimes^{\mathsf{A}} \underline{c}) \arrow[d, "1 \otimes \cong_{\mathsf{Lap}}^{-1}"] \\
\pi[(\underline{a} \otimes^{\mathsf{A}} \underline{b}) \otimes^{\mathsf{A}} \underline{c}] \arrow[u, "\cong_{\mathsf{Lap}}^{-1}"] \arrow[d, "\alpha^{\otimes \mathsf{A}} = 1"] \arrow[uuurrr, bend left, "\xi^{\otimes \mathsf{A}}_{\underline{a};\underline{b}} \otimes^{\mathsf{A}} 1_{\underline{c}}"] & \pi\underline{a} \otimes (\pi\underline{b} \otimes \pi\underline{c}) \arrow[ddl, "1 \otimes \cong_{\mathsf{Lap}}"] \arrow[dr, "\xi^{\otimes}"] & (1.3.17) & \pi\underline{b} \otimes (\pi\underline{a} \otimes \pi\underline{c}) \arrow[d, "1\xi^{\otimes}"] \\
\pi[\underline{a} \otimes^{\mathsf{A}} (\underline{b} \otimes^{\mathsf{A}} \underline{c})] \arrow[d, "\cong_{\mathsf{Lap}}^{-1}"] \arrow[drr, bend right, "\alpha^{\otimes \mathsf{A}} \xi^{\otimes}_{\underline{a};\underline{b} \otimes^{\mathsf{A}} \underline{c}}"'] & (\dagger) & (\pi\underline{b} \otimes \pi\underline{c}) \otimes \pi\underline{a} \arrow[r, "\alpha^{\otimes}"] \arrow[dl, "\cong_{\mathsf{Lap}} \otimes 1"] & \pi\underline{b} \otimes (\pi\underline{c} \otimes \pi\underline{a}) \arrow[d, "1 \otimes \cong_{\mathsf{Lap}}"] \\
\pi\underline{a} \otimes \pi(\underline{b} \otimes^{\mathsf{A}} \underline{c}) \arrow[r, "\xi^{\otimes}"] & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{c}) \otimes \pi\underline{a} \arrow[r, "\cong_{\mathsf{Lap}}"] & \pi[\underline{b} \otimes^{\mathsf{A}} (\underline{c} \otimes^{\mathsf{A}} \underline{a})] & \pi\underline{b} \otimes \pi(\underline{c} \otimes^{\mathsf{A}} \underline{a}) \arrow[l, "\cong_{\mathsf{Lap}}"']
\end{tikzcd}$$

- By (6.2.25) and (6.2.27), the three boundary regions are the definitions of the indicated morphisms.
- The subdiagram $(\dagger)$ is commutative by the naturality of $\xi^{\otimes}$ in C.
- The middle subdiagram is an instance of the left hexagon diagram in the multiplicative structure in C, which is a braided monoidal category.
- In each of the three remaining subdiagrams, for each instance of $\cong_{\mathsf{Lap}}^{-1}$, we consider its inverse $\cong_{\mathsf{Lap}}$ in the opposite direction. Each of these three subdiagrams is commutative by Lemma 6.2.21 and Theorem 5.4.4 as in (6.2.30).

The right hexagon axiom (1.3.17) in A follows similarly from that in the multiplicative structure in C, the naturality of $\xi^{\otimes}$ in C, Lemma 6.2.21, and Theorem 5.4.4. Therefore, $(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}, \xi^{\otimes \mathsf{A}})$ is a braided strict monoidal category. $\square$

The Multiplicative Zeros and Distributivity.

Definition 6.2.31. Continuing Definitions 6.2.3, 6.2.10, and 6.2.22, define the structure morphisms $\lambda^{\bullet \mathsf{A}}$, $\rho^{\bullet \mathsf{A}}$, $\delta^{l \mathsf{A}}$, and $\delta^{r \mathsf{A}}$ in A as follows.

The Multiplicative Zeros: $\lambda^{\bullet \mathsf{A}}$ and $\rho^{\bullet \mathsf{A}}$ have components

$$\mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a} \xrightarrow[=]{\lambda^{\bullet \mathsf{A}}_{\underline{a}}} \mathbb{0}^{\mathsf{A}} \xleftarrow[=]{\rho^{\bullet \mathsf{A}}_{\underline{a}}} \underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}} \tag{6.2.32}$$

the identity morphism of $\mathbb{0} \in \mathsf{C}$ for objects $\underline{a} \in \mathsf{A}$.

Right Distributivity: $\delta^{r\mathsf{A}}$ has components the identity morphisms

$$(\underline{a} \oplus^{\mathsf{A}} \underline{b}) \otimes^{\mathsf{A}} \underline{c} \xrightarrow[=]{\delta^{r\mathsf{A}}_{\underline{a};\underline{b};\underline{c}}} (\underline{a} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{b} \otimes^{\mathsf{A}} \underline{c}) \tag{6.2.33}$$

in C for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$.

Left Distributivity: $\delta^{l\mathsf{A}}$ has components the composites

$$\begin{tikzcd} \underline{a} \otimes^{\mathsf{A}} (\underline{b} \oplus^{\mathsf{A}} \underline{c}) \arrow[r, "\delta^{l\mathsf{A}}_{\underline{a};\underline{b};\underline{c}}"] \arrow[d, "\xi^{\otimes\mathsf{A}}_{\underline{a};\underline{b}\oplus^{\mathsf{A}}\underline{c}}"'] & (\underline{a} \otimes^{\mathsf{A}} \underline{b}) \oplus^{\mathsf{A}} (\underline{a} \otimes^{\mathsf{A}} \underline{c}) \\ (\underline{b} \oplus^{\mathsf{A}} \underline{c}) \otimes^{\mathsf{A}} \underline{a} \arrow[r, "\delta^{r\mathsf{A}}_{\underline{b};\underline{c};\underline{a}}", "="'] & (\underline{b} \otimes^{\mathsf{A}} \underline{a}) \oplus^{\mathsf{A}} (\underline{c} \otimes^{\mathsf{A}} \underline{a}) \arrow[u, "\xi^{-\otimes\mathsf{A}}_{\underline{a};\underline{b}} \oplus^{\mathsf{A}} \xi^{-\otimes\mathsf{A}}_{\underline{a};\underline{c}}"'] \end{tikzcd} \tag{6.2.34}$$

in C for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$, with $\xi^{-\otimes\mathsf{A}}$ the inverse of $\xi^{\otimes\mathsf{A}}$ in (6.2.26).

This finishes the definition of $\lambda^{\bullet\mathsf{A}}$, $\rho^{\bullet\mathsf{A}}$, $\delta^{l\mathsf{A}}$, and $\delta^{r\mathsf{A}}$. $\diamond$

Lemma 6.2.35. *In Definition 6.2.31, $\lambda^{\bullet\mathsf{A}}$, $\rho^{\bullet\mathsf{A}}$, $\delta^{l\mathsf{A}}$, and $\delta^{r\mathsf{A}}$ are natural isomorphisms.*

Proof. By Lemma I.5.2.8, the following equalities hold for objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$.

$$\mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a} = \mathbb{0}^{\mathsf{A}} = \underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}}$$

$$(\underline{a} \oplus^{\mathsf{A}} \underline{b}) \otimes^{\mathsf{A}} \underline{c} = (\underline{a} \otimes^{\mathsf{A}} \underline{c}) \oplus^{\mathsf{A}} (\underline{b} \otimes^{\mathsf{A}} \underline{c})$$

Since $\pi(\mathbb{0}^{\mathsf{A}}) = \mathbb{0} \in \mathsf{C}$, the identity morphisms in (6.2.32) and (6.2.33) are well defined.

To see that $\lambda^{\bullet\mathsf{A}}$ is natural, for a morphism $f \in \mathsf{A}(\underline{a};\underline{b})$, $1_{\mathbb{0}^{\mathsf{A}}} \otimes^{\mathsf{A}} f$ is the following composite in C by (6.2.25).

$$\begin{tikzcd} \pi(\mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a}) \arrow[r, "\lambda^{-\bullet}_{\pi\underline{a}}"] \arrow[d, equal] & \pi\mathbb{0}^{\mathsf{A}} \otimes \pi\underline{a} \arrow[r, "1_{\mathbb{0}} \otimes f"] \arrow[d, equal] & \pi\mathbb{0}^{\mathsf{A}} \otimes \pi\underline{b} \arrow[r, "\lambda^{\bullet}_{\pi\underline{b}}"] \arrow[d, equal] & \pi(\mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{b}) \arrow[d, equal] \\ \mathbb{0} & \mathbb{0} \otimes \pi\underline{a} & \mathbb{0} \otimes \pi\underline{b} & \mathbb{0} \end{tikzcd}$$

This composite is equal to $1_{\mathbb{0}} \in \mathsf{C}$ by the naturality of $\lambda^{\bullet}$ in C. Similarly, the naturality of $\rho^{\bullet}$ in C implies the naturality of $\rho^{\bullet\mathsf{A}}$.

The naturality of $\delta^{r\mathsf{A}}$ in (6.2.33) is proved by the following diagram in C for morphisms $f_i \in \mathsf{A}(\underline{a}_i;\underline{b}_i)$ for $1 \le i \le 3$, with $\otimes$ abbreviated to concatenation.

$$\begin{tikzcd} \pi((\underline{a}_1 \oplus^{\mathsf{A}} \underline{a}_2) \otimes^{\mathsf{A}} \underline{a}_3) \arrow[r, "="] \arrow[d, "\cong^{-1}_{\mathrm{Lap}}"'] & \pi((\underline{a}_1 \otimes^{\mathsf{A}} \underline{a}_3) \oplus^{\mathsf{A}} (\underline{a}_2 \otimes^{\mathsf{A}} \underline{a}_3)) \arrow[d, "\cong^{\oplus}_{\mathrm{ML}}"] \\ \pi(\underline{a}_1 \oplus^{\mathsf{A}} \underline{a}_2)(\pi\underline{a}_3) \arrow[d, "\cong^{\oplus}_{\mathrm{ML}} \otimes 1"'] & \pi(\underline{a}_1 \otimes^{\mathsf{A}} \underline{a}_3) \oplus \pi(\underline{a}_2 \otimes^{\mathsf{A}} \underline{a}_3) \arrow[d, "\cong^{-1}_{\mathrm{Lap}} \oplus \cong^{-1}_{\mathrm{Lap}}"] \\ (\pi\underline{a}_1 \oplus \pi\underline{a}_2)(\pi\underline{a}_3) \arrow[r, "\delta^r"] \arrow[d, "(f_1 \oplus f_2) f_3"'] & (\pi\underline{a}_1)(\pi\underline{a}_3) \oplus (\pi\underline{a}_2)(\pi\underline{a}_3) \arrow[d, "f_1 f_3 \oplus f_2 f_3"] \\ (\pi\underline{b}_1 \oplus \pi\underline{b}_2)(\pi\underline{b}_3) \arrow[r, "\delta^r"] \arrow[d, "\cong^{\oplus}_{\mathrm{ML}} \otimes 1"'] & (\pi\underline{b}_1)(\pi\underline{b}_3) \oplus (\pi\underline{b}_2)(\pi\underline{b}_3) \arrow[d, "\cong_{\mathrm{Lap}} \oplus \cong_{\mathrm{Lap}}"] \\ \pi(\underline{b}_1 \oplus^{\mathsf{A}} \underline{b}_2)(\pi\underline{b}_3) \arrow[d, "\cong_{\mathrm{Lap}}"'] & \pi(\underline{b}_1 \otimes^{\mathsf{A}} \underline{b}_3) \oplus \pi(\underline{b}_2 \otimes^{\mathsf{A}} \underline{b}_3) \arrow[d, "\cong^{\oplus}_{\mathrm{ML}}"] \\ \pi((\underline{b}_1 \oplus^{\mathsf{A}} \underline{b}_2) \otimes^{\mathsf{A}} \underline{b}_3) \arrow[r, "="] & \pi((\underline{b}_1 \otimes^{\mathsf{A}} \underline{b}_3) \oplus^{\mathsf{A}} (\underline{b}_2 \otimes^{\mathsf{A}} \underline{b}_3)) \end{tikzcd}$$

- By Lemma 6.2.21, (6.2.14), and (6.2.25), the left vertical composite is $(f_1 \oplus^{\mathsf{A}} f_2) \otimes^{\mathsf{A}} f_3$, and the right vertical composite is $(f_1 \otimes^{\mathsf{A}} f_3) \oplus^{\mathsf{A}} (f_2 \otimes^{\mathsf{A}} f_3)$.
- The middle rectangle is commutative by the naturality of δ^r in C.
- The top and the bottom rectangles are commutative by Lemma 6.2.21 and Theorem 5.4.4 as in (6.2.30).

The naturality of $\delta^{l\mathsf{A}}$ in (6.2.34) follows from that of $\xi^{\otimes\mathsf{A}}$ in Lemma 6.2.29 and $\delta^{r\mathsf{A}}$, and the functoriality of $\oplus^{\mathsf{A}}$ and $\otimes^{\mathsf{A}}$. $\square$

The Right Permutative Braided Category Axioms. With C still assumed to be a tight braided bimonoidal category, we now show in several steps that A as in Definitions 6.2.3, 6.2.10, 6.2.22, and 6.2.31 is a right permbraided category as in Definition 5.1.11.

Lemma 6.2.36. *For each object $\underline{a} \in \mathsf{A}$, the morphisms*

$$\underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}} \underset{\xi^{\otimes\mathsf{A}}_{\mathbb{0}^{\mathsf{A}};\underline{a}}}{\overset{\xi^{\otimes\mathsf{A}}_{\underline{a};\mathbb{0}^{\mathsf{A}}}}{\rightleftarrows}} \mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a}$$

are both equal to $1_{\mathbb{0}^{\mathsf{A}}}$.

Proof. By (6.2.27) with $\underline{b} = \mathbb{0}^{\mathsf{A}}$, the morphism $\xi^{\otimes\mathsf{A}}_{\underline{a};\mathbb{0}^{\mathsf{A}}}$ is the following composite in C.

$$\mathbb{0} = \pi(\underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}}) \xrightarrow{\rho^{-\bullet}_{\pi\underline{a}}} \pi\underline{a} \otimes \mathbb{0} \xrightarrow{\xi^{\otimes}_{\pi\underline{a};\mathbb{0}}} \mathbb{0} \otimes \pi\underline{a} \xrightarrow{\lambda^{\bullet}_{\pi\underline{a}}} \pi(\mathbb{0}^{\mathsf{A}} \otimes^{\mathsf{A}} \underline{a}) = \mathbb{0}$$

This is equal to $1_{\mathbb{0}} = 1_{\mathbb{0}^{\mathsf{A}}}$ by the axiom (2.1.18) in C. Similarly, the morphism $\xi^{\otimes\mathsf{A}}_{\mathbb{0}^{\mathsf{A}};\underline{a}}$ is equal to $1_{\mathbb{0}^{\mathsf{A}}}$ by the axiom (2.1.33) in C. $\square$

Lemma 6.2.37. A *satisfies the axioms (2.1.32), (2.1.6), and (2.1.12).*

Proof. The axiom (2.1.32) holds by the definition (6.2.34) of $\delta^{l\mathsf{A}}$.

The axiom (2.1.6) is proved as in Lemma I.5.3.7, which is the symmetric case. In the diagram in that proof, the top and the bottom subdiagrams are now commutative by Lemma 6.2.21 and Theorem 5.4.4 as in the proof of Lemma 6.2.29.

The axiom (2.1.12) is proved by reusing the proof of Lemma I.5.3.8, which is the symmetric case, with the following changes in the diagram in that proof.

- The left vertical morphism $\xi^{\otimes} \oplus \xi^{\otimes}$ is replaced by
$$\xi^{-\otimes}_{\pi(\underline{a}\oplus^{\mathsf{A}}\underline{b});\pi\underline{c}} \oplus \xi^{-\otimes}_{\pi(\underline{a}\oplus^{\mathsf{A}}\underline{b});\pi\underline{d}}.$$
- The right vertical morphism $(\xi^{\otimes} \oplus \xi^{\otimes}) \oplus (\xi^{\otimes} \oplus \xi^{\otimes})$ is replaced by
$$(\xi^{-\otimes}_{\pi\underline{a};\pi\underline{c}} \oplus \xi^{-\otimes}_{\pi\underline{a};\pi\underline{d}}) \oplus (\xi^{-\otimes}_{\pi\underline{b};\pi\underline{c}} \oplus \xi^{-\otimes}_{\pi\underline{b};\pi\underline{d}}).$$

These changes are necessary because $\delta^{l\mathsf{A}}$ in (6.2.34) involves $\xi^{-\otimes\mathsf{A}}_{\underline{a};\underline{b}}$ and $\xi^{-\otimes\mathsf{A}}_{\underline{a};\underline{c}}$, hence also $\xi^{-\otimes}$ in C. In a braided monoidal category, $\xi^{\otimes}_{?,?'}$ and $\xi^{-\otimes}_{?',?}$ are not equal in general.

The diagram in Lemma I.5.3.8 is divided into four subdiagrams from top to bottom. In the current context, they are commutative for the following reasons.

- The first and the third subdiagrams do not involve $\xi^{\pm\otimes}$. They are commutative by Lemma 6.2.21 and Theorem 5.4.4 as in Lemma 6.2.29.

- The second subdiagram is commutative by Theorem 5.4.4 because the two paths that yield the two composites have the same braided distortion in the sense of Definition 5.3.15. Indeed, in the braided distortions, the permutation component is uniquely determined by the monomials of the objects involved because there is a unique permutation between any two permuted words of the same length. Moreover, by (5.2.18), the braided distortion of each path has as its braid components the elementary block braids (1.2.4)

$$\Big\{\big\{b^{\oplus}_{|a^i|,|c^k|}\big\}_{i=1}^{r}, \big\{b^{\oplus}_{|b^j|,|c^k|}\big\}_{j=1}^{s}\Big\}_{k=1}^{t} \qquad \Big\{\big\{b^{\oplus}_{|a^i|,|d^l|}\big\}_{i=1}^{r}, \big\{b^{\oplus}_{|b^j|,|d^l|}\big\}_{j=1}^{s}\Big\}_{l=1}^{u}$$

  if

$$\begin{aligned} \underline{a} &= \{a^1, \ldots, a^r\}, \\ \underline{b} &= \{b^1, \ldots, b^s\}, \\ \underline{c} &= \{c^1, \ldots, c^t\}, \quad \text{and} \\ \underline{d} &= \{d^1, \ldots, d^u\} \in \mathsf{A}. \end{aligned}$$

  Therefore, Theorem 5.4.4 implies that the two composites are equal.
- The bottom subdiagram in Lemma I.5.3.8 is commutative for the same reason as for the second subdiagram. The braided distortions of the two paths that yield the two composites have the same permutation component, and their braid components are as follows by (5.2.22).

$$\Big\{\big\{b^{-\oplus}_{|a^i|,|c^k|}\big\}_{k=1}^{t}\Big\}_{i=1}^{r} \quad \Big\{\big\{b^{-\oplus}_{|b^j|,|c^k|}\big\}_{k=1}^{t}\Big\}_{j=1}^{s} \quad \Big\{\big\{b^{-\oplus}_{|a^i|,|d^l|}\big\}_{l=1}^{u}\Big\}_{i=1}^{r} \quad \Big\{\big\{b^{-\oplus}_{|b^j|,|d^l|}\big\}_{l=1}^{u}\Big\}_{j=1}^{s}$$

This proves the axiom (2.1.12). □

Lemma 6.2.38. A *satisfies the axiom (2.1.4).*

Proof. For objects $\underline{a}, \underline{b}, \underline{c} \in \mathsf{A}$, the diagram (2.1.4) is the outer diagram in C below. To save space, we abbreviate $\oplus^{\mathsf{A}}$ and $\otimes^{\mathsf{A}}$ to, respectively, $\oplus$ and $\otimes$, and write $\otimes$ in C as concatenation.

$$
\begin{tikzcd}[column sep=large]
\pi[(\underline{a}\oplus\underline{b})\otimes\underline{c}] \arrow[r, "\delta^{r\mathsf{A}}", "="'] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] \arrow[ddd, bend right=60, "\xi^{\otimes\mathsf{A}}_{\underline{a}\oplus\underline{b};\underline{c}}"'] & \pi[(\underline{a}\otimes\underline{c})\oplus(\underline{b}\otimes\underline{c})] \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, phantom, "5.4.4"] \arrow[rdddd, bend left=40, "\xi^{\otimes\mathsf{A}}_{\underline{a};\underline{c}}\oplus\xi^{\otimes\mathsf{A}}_{\underline{b};\underline{c}}"] & \pi(\underline{a}\otimes\underline{c})\oplus\pi(\underline{b}\otimes\underline{c}) \arrow[d, "\cong^{-1}_{\mathsf{Lap}}\oplus\cong^{-1}_{\mathsf{Lap}}"] \\
\pi(\underline{a}\oplus\underline{b})(\pi\underline{c}) \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, "\xi^{\otimes}"'] & (\pi\underline{a}\oplus\pi\underline{b})(\pi\underline{c}) \arrow[r, "\delta^{r}"] \arrow[d, phantom, "5.4.4"] & (\pi\underline{a})(\pi\underline{c})\oplus(\pi\underline{b})(\pi\underline{c}) \arrow[d, "\xi^{\otimes}\oplus\xi^{\otimes}"] \\
(\pi\underline{c})\pi(\underline{a}\oplus\underline{b}) \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, "\cong_{\mathsf{Lap}}"'] & (\pi\underline{c})(\pi\underline{a}\oplus\pi\underline{b}) \arrow[r, "\delta^{l}"] & (\pi\underline{c})(\pi\underline{a})\oplus(\pi\underline{c})(\pi\underline{b}) \arrow[d, "\cong_{\mathsf{Lap}}\oplus\cong_{\mathsf{Lap}}"] \\
\pi[\underline{c}\otimes(\underline{a}\oplus\underline{b})] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"'] \arrow[rrd, bend right=90, looseness=1.5, "\delta^{l\mathsf{A}}_{\underline{c};\underline{a};\underline{b}}"'] & & \pi(\underline{c}\otimes\underline{a})\oplus\pi(\underline{c}\otimes\underline{b}) \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"] \\
(\pi\underline{c})\pi(\underline{a}\oplus\underline{b}) \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, "\xi^{\otimes}"'] \arrow[dr, phantom, "\mathsf{nat}"] & (\pi\underline{c})(\pi\underline{a}\oplus\pi\underline{b}) \arrow[d, "\xi^{\otimes}"] \arrow[rdd, "\delta^{l}"] & \pi[(\underline{c}\otimes\underline{a})\oplus(\underline{c}\otimes\underline{b})] \\
\pi(\underline{a}\oplus\underline{b})(\pi\underline{c}) \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, "\cong_{\mathsf{Lap}}"'] & (\pi\underline{a}\oplus\pi\underline{b})(\pi\underline{c}) \arrow[rdd, "\delta^{r}"] \arrow[r, phantom, "(2.1.32)"] \arrow[d, phantom, "5.4.4"] & \pi(\underline{c}\otimes\underline{a})\oplus\pi(\underline{c}\otimes\underline{b}) \arrow[u, "\cong^{\oplus}_{\mathsf{ML}}"'] \\
\pi[(\underline{a}\oplus\underline{b})\otimes\underline{c}] \arrow[d, "\delta^{r\mathsf{A}}"', "="] & {} & (\pi\underline{c})(\pi\underline{a})\oplus(\pi\underline{c})(\pi\underline{b}) \arrow[u, "\cong_{\mathsf{Lap}}\oplus\cong_{\mathsf{Lap}}"'] \\
\pi[(\underline{a}\otimes\underline{c})\oplus(\underline{b}\otimes\underline{c})] \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] & \pi(\underline{a}\otimes\underline{c})\oplus\pi(\underline{b}\otimes\underline{c}) \arrow[r, "\cong^{-1}_{\mathsf{Lap}}\oplus\cong^{-1}_{\mathsf{Lap}}"] & (\pi\underline{a})(\pi\underline{c})\oplus(\pi\underline{b})(\pi\underline{c}) \arrow[u, "\xi^{-\otimes}\oplus\xi^{-\otimes}"']
\end{tikzcd}
$$

- The three boundary regions define the indicated morphisms by (6.2.14), (6.2.27), (6.2.33), and (6.2.34).
- The middle unlabeled subdiagram is commutative by definition.
- As in the proof of Lemma 6.2.29, the top rectangle and the bottom trapezoid are commutative by Theorem 5.4.4 and Lemma 6.2.21.
- In the second rectangle from the top, as in the proof of Lemma 6.2.37, the two paths that yield the two composites have the same permutation component in the braided distortions. Moreover, the braided distortion of each path has as its braid components the following elementary block braids (1.2.4).

$$\Big\{\big\{b^{\oplus}_{|a^i|,|c^k|}\big\}_{i=1}^{r}, \big\{b^{\oplus}_{|b^j|,|c^k|}\big\}_{j=1}^{s}\Big\}_{k=1}^{t}$$

  Therefore, these paths have the same value in C by Theorem 5.4.4.
- The subdiagram with label nat is commutative by the naturality of $\xi^{\otimes}$ in C.
- The remaining subdiagram is commutative by the axiom (2.1.32) in C.

This finishes the proof. $\square$

Recall from Definition 5.1.11 and Proposition 5.1.19 that a right permbraided category is a tight braided bimonoidal category in which the structure morphisms $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^{r}, $\xi^{\otimes}_{-,\mathbb{0}}$, and $\xi^{\otimes}_{\mathbb{0},-}$ are identities.

Proposition 6.2.39. *Associated to each tight braided bimonoidal category* C*, the tuple*

$$\big(\mathsf{A}, (\oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \xi^{\oplus\mathsf{A}}), (\otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}, \xi^{\otimes\mathsf{A}}), \lambda^{\bullet\mathsf{A}}, \rho^{\bullet\mathsf{A}}, \delta^{r\mathsf{A}}, \delta^{l\mathsf{A}}\big)$$

in Definitions 6.2.3, 6.2.10, 6.2.22, and 6.2.31 is a right permbraided category.

Proof. We already checked all the conditions for A to be a right permbraided category.

- Lemma 6.2.18 shows that $(\mathsf{A}, \oplus^{\mathsf{A}}, \mathbb{0}^{\mathsf{A}}, \xi^{\oplus\mathsf{A}})$ is a permutative category.
- Lemma 6.2.29 shows that $(\mathsf{A}, \otimes^{\mathsf{A}}, \mathbb{1}^{\mathsf{A}}, \xi^{\otimes\mathsf{A}})$ is a braided strict monoidal category.
- Lemma 6.2.35 shows that the natural isomorphisms $\lambda^{\bullet\mathsf{A}} = 1$, $\rho^{\bullet\mathsf{A}} = 1$, $\delta^{r\mathsf{A}} = 1$, and $\delta^{l\mathsf{A}}$ are well defined.
- The other axioms in Definition 5.1.11 are verified in Lemmas 6.2.36 through 6.2.38.

Therefore, A is a right permbraided category. □

6.3. Strictification

In this section, we finish the proof that each tight braided bimonoidal category is equivalent to a right permbraided category. There is also a variant that involves a left permbraided category. These results are the braided analogues of Theorems I.5.4.6 and I.5.4.7, which are strictification results for tight symmetric bimonoidal categories. We continue to assume that C is a tight braided bimonoidal category. Moreover, A denotes the associated right permbraided category in Proposition 6.2.39. First we define the functors that constitute an adjoint equivalence between them.

Definition 6.3.1. Define the functor

$$\pi : \mathsf{A} \longrightarrow \mathsf{C}$$

as follows.

- Its assignment on objects is the realization function in (6.2.6).
- For objects $\underline{a}, \underline{b} \in \mathsf{A}$, its assignment on morphisms

$$\mathsf{A}(\underline{a}; \underline{b}) \xrightarrow{\ \pi\ } \mathsf{C}(\pi\underline{a}; \pi\underline{b})$$

  is the identity function using (6.2.8).

The fact that π is a functor is part of Definition 6.2.3. ◇

Definition 6.3.2. Define the functor

$$\iota : \mathsf{C} \longrightarrow \mathsf{A}$$

as follows.

- Using the notations in (6.2.4) and (6.2.5), for each object $X \in \mathsf{C}$, define the object

$$\iota X = \{(X)\} \in \mathsf{A}.$$

  It has additive length 1, and its only monomial has multiplicative length 1 consisting of X.
- On morphism sets, it is the identity function

$$\mathsf{C}(X; Y) = \mathsf{C}(\pi\iota X; \pi\iota Y) = \mathsf{A}(\iota X; \iota Y).$$

The fact that ι is a functor is part of Definition 6.2.3. ◇

Lemma 6.3.3. *The functors in Definitions 6.3.1 and 6.3.2 form an adjoint equivalence*

$$\mathsf{A} \underset{\iota}{\overset{\pi}{\rightleftarrows}} \mathsf{C}$$

with counit $\varepsilon : \pi\iota \longrightarrow 1_{\mathsf{C}}$ *the identity natural transformation.*

Proof. The proof of Lemma I.5.4.3, which is the symmetric case, is still valid in the current context. ☐

Recall from Definition 6.1.1 that a *braided bimonoidal functor* $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor between braided bimonoidal categories that is equipped with

- a symmetric monoidal functor structure $(F, F^2_\oplus, F^0_\oplus)$ between the additive structures of C and D and
- a braided monoidal functor structure $(F, F^2_\otimes, F^0_\otimes)$ between the multiplicative structures of C and D.

Moreover, the data $(F, F^2_\oplus, F^0_\oplus, F^2_\otimes, F^0_\otimes)$ are required to satisfy the multiplicative zero axiom (6.1.2) and the distributivity axiom (6.1.3). It is said to be *unitary* if $F^0_\oplus$ and $F^0_\otimes$ are identities and if $F^2_\oplus$ and $F^2_\otimes$ are isomorphisms. It is a *braided bimonoidal equivalence* if F is an equivalence of categories.

Lemma 6.3.4. *There is a unitary braided bimonoidal equivalence*

$$\left(\pi, \pi^2_\oplus, \pi^0_\oplus, \pi^2_\otimes, \pi^0_\otimes\right) : \mathsf{A} \longrightarrow \mathsf{C}.$$

Proof. Using the equalities

$$\pi\mathbb{0}^{\mathsf{A}} = \mathbb{0} \quad \text{and} \quad \pi\mathbb{1}^{\mathsf{A}} = \mathbb{1},$$

we define the unit constraints

$$\mathbb{0} \xrightarrow{\pi^0_\oplus} \pi\mathbb{0}^{\mathsf{A}} \quad \text{and} \quad \mathbb{1} \xrightarrow{\pi^0_\otimes} \pi\mathbb{1}^{\mathsf{A}}$$

as the identity morphisms of, respectively, $\mathbb{0}$ and $\mathbb{1}$ in C.

For objects $\underline{a}, \underline{b} \in \mathsf{A}$, we define the monoidal constraints

$$\pi\underline{a} \oplus \pi\underline{b} \xrightarrow{\pi^2_\oplus} \pi(\underline{a} \oplus^{\mathsf{A}} \underline{b})$$

as a Mac Lane coherence isomorphism $\cong^{\oplus}_{\mathsf{ML}}$ as in Definition 6.2.9 and

$$\pi\underline{a} \otimes \pi\underline{b} \xrightarrow{\pi^2_\otimes} \pi(\underline{a} \otimes^{\mathsf{A}} \underline{b})$$

as a Laplaza coherence isomorphism $\cong_{\mathsf{Lap}}$ as in Definition 6.2.19. These are natural isomorphisms by

- the definition (6.2.14) of $\oplus^{\mathsf{A}}$ on morphisms,
- the definition (6.2.25) of $\otimes^{\mathsf{A}}$ on morphisms, and
- Definition 6.3.1 of π as the identity assignment on morphisms.

The triple $(\pi, \pi^2_\oplus, \pi^0_\oplus)$ is a symmetric monoidal functor between the additive structures because each diagram in (1.3.10), (1.3.11), and (1.3.19) is commutative by the Coherence Theorem I.1.3.8 for the symmetric monoidal category $(\mathsf{C}, \oplus)$.

Next we check that the triple $(\pi, \pi^2_\otimes, \pi^0_\otimes)$ is a braided monoidal functor.

- The unity axioms (1.3.11) hold because each vertical morphism there is an identity morphism.

- By Lemma 6.2.29, $\alpha^{\otimes \mathsf{A}} = 1$. In the associativity axiom (1.3.10), each of the two composites is a Laplaza coherence isomorphism. They are equal by Lemma 6.2.21 and Theorem 5.4.4 as in the first paragraph in the proof of Lemma 6.2.29.
- By the definition (6.2.27) of $\xi^{\otimes \mathsf{A}}$, the compatibility axiom (1.3.19) with the braidings is the following diagram in C for objects $\underline{a}, \underline{b} \in \mathsf{A}$.

$$\begin{tikzcd}
\pi\underline{a} \otimes \pi\underline{b} \arrow[rrr, "\xi^{\otimes}"] \arrow[d, "\cong_{\mathsf{Lap}}"'] \arrow[dr, equal] & & & \pi\underline{b} \otimes \pi\underline{a} \arrow[d, "\cong_{\mathsf{Lap}}"] \\
\pi(\underline{a} \otimes^{\mathsf{A}} \underline{b}) \arrow[r, "\cong_{\mathsf{Lap}}^{-1}"] & \pi\underline{a} \otimes \pi\underline{b} \arrow[r, "\xi^{\otimes}"] & \pi\underline{b} \otimes \pi\underline{a} \arrow[r, "\cong_{\mathsf{Lap}}"] & \pi(\underline{b} \otimes^{\mathsf{A}} \underline{a})
\end{tikzcd}$$

This diagram is commutative by definition.

The multiplicative zero axiom (6.1.2) for π holds because

- $\pi^0_{\oplus}$ and $\rho^{\bullet \mathsf{A}}$ are the identities, and
- there is an equality

$$\pi^2_{\otimes} = \rho^{\bullet}_{\pi\underline{a}} : (\pi\underline{a}) \otimes (\pi\mathbb{0}^{\mathsf{A}}) = (\pi\underline{a}) \otimes \mathbb{0} \longrightarrow \mathbb{0} = \pi(\underline{a} \otimes^{\mathsf{A}} \mathbb{0}^{\mathsf{A}}).$$

In the distributivity axiom (6.1.3) for π, $\delta^{r\mathsf{A}} = 1$ by definition (6.2.33). Each of the two composites is a Laplaza coherence isomorphism. They are equal by Lemma 6.2.21 and Theorem 5.4.4 as in the proof of Lemma 6.2.29.

Finally, π is an equivalence of categories by Lemma 6.3.3. □

Recall from Definition 6.1.1 that a braided bimonoidal functor F is *strong* if $F^2_{\oplus}$, $F^0_{\oplus}$, $F^2_{\otimes}$, and $F^0_{\otimes}$ are natural isomorphisms.

Lemma 6.3.5. *There is a strong braided bimonoidal equivalence*

$$\left(\iota, \iota^2_{\oplus}, \iota^0_{\oplus}, \iota^2_{\otimes}, \iota^0_{\otimes}\right) : \mathsf{C} \longrightarrow \mathsf{A}.$$

Proof. The proof of Lemma I.5.4.5, which is the symmetric case, is still valid in the current context. □

Combining Proposition 6.2.39 and Lemmas 6.3.3 through 6.3.5, we obtain the following strictification result to right permbraided categories as in Definition 5.1.11.

Theorem 6.3.6 (Right Permbraided Strictification)**.** *Suppose* C *is a tight braided bimonoidal category. Then there is an adjoint equivalence*

$$\begin{tikzcd} \mathsf{A} \arrow[r, shift left, "\pi"] & \mathsf{C} \arrow[l, shift left, "\iota"] \end{tikzcd}$$

with

- A *the right permbraided category in Proposition 6.2.39,*
- *counit* $\varepsilon : \pi\iota \longrightarrow 1_{\mathsf{C}}$ *the identity natural transformation,*
- π *the unitary braided bimonoidal equivalence in Lemma 6.3.4, and*
- ι *the strong braided bimonoidal equivalence in Lemma 6.3.5.*

A minor variation of the constructions in this chapter yields the following strictification result to left permbraided categories as in Definition 5.1.2.

Theorem 6.3.7 (Left Permbraided Strictification)**.** *Suppose* C *is a tight braided bimonoidal category. Then there is an adjoint equivalence*

$$\begin{tikzcd} \mathsf{A}_l \arrow[r, shift left, "\pi"] & \mathsf{C} \arrow[l, shift left, "\iota"] \end{tikzcd}$$

with

- A_l *a left permbraided category,*
- *counit* $\varepsilon : \pi\iota \longrightarrow 1_{\mathsf{C}}$ *the identity natural transformation,*
- π *a unitary braided bimonoidal equivalence, and*
- ι *a strong braided bimonoidal equivalence.*

Proof. The proof is essentially the same as that of Theorem 6.3.6, with the following modifications.

- The category A_l and its additive zero, multiplicative unit, sum, additive symmetry, and braiding, are the same as those in A in Definitions 6.2.3, 6.2.10, and 6.2.22.
- The product in A_l is redefined in such a way that the left distributive law holds strictly, that is, as

$$\underline{a} \otimes^{\mathsf{A}_l} \underline{b} = \big\{(a^1,b^1),\ldots,(a^r,b^1),\ldots,(a^1,b^s),\ldots,(a^r,b^s)\big\}$$

  instead of (6.2.23).
- δ^l in A_l is defined as the identity natural transformation.
- Instead of (6.2.34), here the axiom (2.1.4) is used to define $\delta^{r\mathsf{A}_l}$ in terms of $\xi^{\otimes \mathsf{A}_l}$ and $\delta^{l\mathsf{A}_l} = 1$. In other words, we define $\delta^{r\mathsf{A}_l}$ as having the following components for $\underline{a},\underline{b},\underline{c} \in \mathsf{A}_l$.

$$\begin{tikzcd} (\underline{a} \oplus^{\mathsf{A}_l} \underline{b}) \otimes^{\mathsf{A}_l} \underline{c} \arrow[r, "\delta^{r\mathsf{A}_l}_{\underline{a};\underline{b};\underline{c}}"] \arrow[d, "\xi^{\otimes \mathsf{A}_l}_{\underline{a}\oplus^{\mathsf{A}_l}\underline{b};\,\underline{c}}"'] & (\underline{a} \otimes^{\mathsf{A}_l} \underline{c}) \oplus^{\mathsf{A}_l} (\underline{b} \otimes^{\mathsf{A}_l} \underline{c}) \\ \underline{c} \otimes^{\mathsf{A}_l} (\underline{a} \oplus^{\mathsf{A}_l} \underline{b}) \arrow[r, "\delta^{l\mathsf{A}_l}_{\underline{c};\underline{a};\underline{b}}", "="'] & (\underline{c} \otimes^{\mathsf{A}_l} \underline{a}) \oplus^{\mathsf{A}_l} (\underline{c} \otimes^{\mathsf{A}_l} \underline{b}) \arrow[u, "\xi^{-\otimes \mathsf{A}_l}_{\underline{a};\underline{c}} \oplus^{\mathsf{A}_l} \xi^{-\otimes \mathsf{A}_l}_{\underline{b};\underline{c}}"'] \end{tikzcd}$$

  All other structures in A_l are defined as in A.

- The diagram in the proof of Lemma 6.2.38 is adapted to form the following diagram in C that proves the axiom (2.1.32) in A_l.

$$\begin{tikzcd}[column sep=small]
\pi[\underline{a}\otimes(\underline{b}\oplus\underline{c})] \arrow[r, "\delta^{l\mathsf{A}_l}", "="'] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"] \arrow[ddd, bend right=60, "\xi^{\otimes\mathsf{A}_l}_{\underline{a};\underline{b}\oplus\underline{c}}"'] \arrow[dr, phantom, "5.4.4"] & \pi[(\underline{a}\otimes\underline{b})\oplus(\underline{a}\otimes\underline{c})] \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[ddddr, bend left=40, "\xi^{\otimes\mathsf{A}_l}_{\underline{a};\underline{b}}\oplus\xi^{\otimes\mathsf{A}_l}_{\underline{a};\underline{c}}"] & \pi(\underline{a}\otimes\underline{b})\oplus\pi(\underline{a}\otimes\underline{c}) \arrow[d, "\cong^{-1}_{\mathsf{Lap}}\oplus\cong^{-1}_{\mathsf{Lap}}"] \\
(\pi\underline{a})\pi(\underline{b}\oplus\underline{c}) \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, "\xi^{\otimes}"] \arrow[dr, phantom, "5.4.4"] & (\pi\underline{a})(\pi\underline{b}\oplus\pi\underline{c}) \arrow[r, "\delta^{l}"] & (\pi\underline{a})(\pi\underline{b})\oplus(\pi\underline{a})(\pi\underline{c}) \arrow[d, "\xi^{\otimes}\oplus\xi^{\otimes}"] \\
\pi(\underline{b}\oplus\underline{c})(\pi\underline{a}) \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, "\cong_{\mathsf{Lap}}"] & (\pi\underline{b}\oplus\pi\underline{c})(\pi\underline{a}) \arrow[r, "\delta^{r}"] & (\pi\underline{b})(\pi\underline{a})\oplus(\pi\underline{c})(\pi\underline{a}) \arrow[d, "\cong_{\mathsf{Lap}}\oplus\cong_{\mathsf{Lap}}"] \\
\pi[(\underline{b}\oplus\underline{c})\otimes\underline{a}] \arrow[d, "\cong^{-1}_{\mathsf{Lap}}"] \arrow[dr, bend right=90, looseness=1.5, "\delta^{r\mathsf{A}_l}_{\underline{b};\underline{c};\underline{a}}"'] & & \pi(\underline{b}\otimes\underline{a})\oplus\pi(\underline{c}\otimes\underline{a}) \arrow[d, "\cong^{\oplus}_{\mathsf{ML}}"] \\
\pi(\underline{b}\oplus\underline{c})(\pi\underline{a}) \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, "\xi^{\otimes}"] \arrow[dr, phantom, "\mathsf{nat}"] & (\pi\underline{b}\oplus\pi\underline{c})(\pi\underline{a}) \arrow[d, "\xi^{\otimes}"] \arrow[ddr, "\delta^{r}"] & \pi[(\underline{b}\otimes\underline{a})\oplus(\underline{c}\otimes\underline{a})] \\
(\pi\underline{a})\pi(\underline{b}\oplus\underline{c}) \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] \arrow[d, "\cong_{\mathsf{Lap}}"] & (\pi\underline{a})(\pi\underline{b}\oplus\pi\underline{c}) \arrow[ddr, "\delta^{l}"] \arrow[r, phantom, "(2.1.4)"] & \pi(\underline{b}\otimes\underline{a})\oplus\pi(\underline{c}\otimes\underline{a}) \arrow[u, "\cong^{\oplus}_{\mathsf{ML}}"'] \\
\pi[\underline{a}\otimes(\underline{b}\oplus\underline{c})] \arrow[d, "\delta^{l\mathsf{A}_l}", "="'] \arrow[r, phantom, "5.4.4"] & & (\pi\underline{b})(\pi\underline{a})\oplus(\pi\underline{c})(\pi\underline{a}) \arrow[u, "\cong_{\mathsf{Lap}}\oplus\cong_{\mathsf{Lap}}"'] \\
\pi[(\underline{a}\otimes\underline{b})\oplus(\underline{a}\otimes\underline{c})] \arrow[r, "\cong^{\oplus}_{\mathsf{ML}}"] & \pi(\underline{a}\otimes\underline{b})\oplus\pi(\underline{a}\otimes\underline{c}) \arrow[r, "\cong^{-1}_{\mathsf{Lap}}\oplus\cong^{-1}_{\mathsf{Lap}}"] & (\pi\underline{a})(\pi\underline{b})\oplus(\pi\underline{a})(\pi\underline{c}) \arrow[u, "\xi^{-\otimes}\oplus\xi^{-\otimes}"']
\end{tikzcd}$$

In the second rectangle from the top, in the braided distortion of each of the two paths that yield the two composites, the braid components are the following elementary block braids.

$$\Big\{\big\{b^{\oplus}_{|a^i|,|b^j|}\big\}_{i=1}^{r}\Big\}_{j=1}^{s} \qquad \Big\{\big\{b^{\oplus}_{|a^i|,|c^k|}\big\}_{i=1}^{r}\Big\}_{k=1}^{t}$$

Other proofs require minimal or no changes. □

CHAPTER 7

The Braided Baez Conjecture

The purpose of this chapter is to prove the braided version of Baez's Conjecture; see Theorem 7.3.4. It states that in a suitable 2-category of small braided bimonoidal categories, the left bipermutative category Σ in Proposition I.2.4.8 is an initial object in the bicategorical sense. This theorem is the braided version of Baez's Conjecture, Theorem I.7.8.1, which is the same statement for symmetric bimonoidal categories. There is another version that involves the right bipermutative category Σ' in Proposition I.2.4.23; see Theorem 7.3.6. For an open question related to the braided version of Baez's Conjecture, see Question III.A.2.6.

Proof Strategy. Since the main Theorem 7.3.4 is the braided analogue of Baez's Conjecture, Theorem I.7.8.1, as in Chapter 6, we will adapt the proof in the symmetric case by making suitable adjustments when necessary. To understand what adjustments we must make, let us first discuss the statement of the Braided Baez Conjecture.

In Definition 7.1.5, we will define a 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$ with

- *flat* small braided bimonoidal categories in Definition 5.4.5 as objects,
- *robust* braided bimonoidal functors in Definition 6.1.1 as 1-cells, and
- bimonoidal natural transformations in Definition 7.1.2 as 2-cells.

The flatness condition on the objects ensures that the Coherence Theorem 5.4.4 for braided bimonoidal categories is applicable. The robustness condition on the 1-cells allows us to use the inverses of the structure morphisms $G^2_{\oplus}$, $G^0_{\oplus}$, and $G^0_{\otimes}$ of a braided bimonoidal functor G. This is necessary for the uniqueness part of the Braided Baez Conjecture.

Since Σ is a small left bipermutative category, it is also an object in $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$. With $\varnothing$ denoting the empty 2-category, the main Theorem 7.3.4 in this chapter states that Σ is a lax bicolimit of the unique 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$. Restating this in 1-categorical language, the assertion is that, for each flat small braided bimonoidal category C, the unique functor

$$\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}(\Sigma, \mathsf{C}) \xrightarrow{\;T\;} \mathbf{1}$$

to the terminal category $\mathbf{1}$ is an equivalence of categories. In other words, the functor T is (i) fully faithful on morphisms and (ii) essentially surjective on objects.

In the proof of Baez's Conjecture in Chapter I.7, the fully faithfulness of the functor T occupies Sections I.7.5 through I.7.7. A careful examination of those sections reveals that they still hold in the braided context essentially without any changes. The reason is that they do not involve the Coherence Theorems I.3.9.1 and I.4.4.3 for symmetric bimonoidal categories.

The proof of the essential surjectivity of the functor T in Chapter I.7 occupies Sections I.7.2 through I.7.4. The construction of the strong symmetric monoidal functor

$$F_\oplus : (\Sigma, \oplus) \longrightarrow (\mathsf{C}, \oplus)$$

between the additive structures in Section I.7.2 does not use the Coherence Theorems I.3.9.1 and I.4.4.3. So this part of the proof is also reused in this chapter with minimal changes; see Lemma 7.2.4.

On the other hand,

- the construction of the symmetric monoidal functor

$$F_\otimes : (\Sigma, \otimes) \longrightarrow (\mathsf{C}, \otimes)$$

  between the multiplicative structures in Section I.7.3 and
- the proof that $F : \Sigma \longrightarrow \mathsf{C}$ is a robust symmetric bimonoidal functor in Section I.7.4

both use the distortion of a path in Definition I.4.3.1 and Theorems I.3.9.1 and I.4.4.3. To use these constructions and proofs in the braided setting, the following changes are necessary.

(i) Instead of the distortion of a path, here we use the *braided* distortion of a path in Definition 5.3.15. In particular, this involves the braided distortion category $\mathcal{D}^{\mathsf{br}}$ in Section 5.2, which in turn involves the braid groups.

(ii) Instead of Theorems I.3.9.1 and I.4.4.3, here we use Theorem 5.4.4, which is our main coherence theorem for braided bimonoidal categories. Each time either Theorem I.3.9.1 or Theorem I.4.4.3 is used in Sections I.7.3 and I.7.4, here we have to check that the two paths in question have the same braided distortion in order to use Theorem 5.4.4.

Furthermore, as in Chapter I.7, the proof of the Braided Baez Conjecture does *not* use the Strictification Theorems 6.3.6 and 6.3.7 for tight braided bimonoidal categories. This approach has two advantages.

- Both of those strictification theorems require tightness, that is, the invertibility of the distributivity morphisms δ^l and δ^r in the braided bimonoidal categories. In the Braided Baez Conjecture, Theorem 7.3.4, the small braided bimonoidal categories are flat in the sense of Definition 5.4.5, which is a much weaker assumption than tightness.
- By not using the strictification theorems, we work directly with a flat small braided bimonoidal category C instead of an equivalent one.

Organization. The rest of this chapter contains the following sections.

In Section 7.1, we define several 2-categories of braided bimonoidal categories, braided bimonoidal functors, and bimonoidal natural transformations. The 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$ that appears in the Braided Baez Conjecture is in Definition 7.1.5.

Section 7.2 proves the first half of the Braided Baez Conjecture concerning the existence of an appropriate 1-cell in $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$. For each flat braided bimonoidal category C, Lemma 7.2.11 states that there is an explicitly constructed robust braided bimonoidal functor

$$(F, F_\oplus^2, F_\oplus^0, F_\otimes^2, F_\otimes^0) : \Sigma \longrightarrow \mathsf{C}.$$

The additive and multiplicative structures,

$$F_\oplus = (F, F_\oplus^2, F_\oplus^0) \quad \text{and} \quad F_\otimes = (F, F_\otimes^2, F_\otimes^0),$$

are in, respectively, Definitions 7.2.3 and 7.2.8.

Section 7.3 proves the second half of the Braided Baez Conjecture concerning the existence and the uniqueness of appropriate 2-cells in $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$. For any two robust braided bimonoidal functors $G, H : \Sigma \longrightarrow \mathsf{C}$, the objective is to show that there exists a unique 2-cell $G \longrightarrow H$ in $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$. This is carried out in Lemmas 7.3.1 and 7.3.3 and Theorem 7.3.4. The second version of the Braided Baez Conjecture is Theorem 7.3.6. It involves the right bipermutative category Σ' in Proposition I.2.4.23.

Reading Guide.

(1) Read Convention 7.1.1 and Definitions 7.1.2, 7.1.3, and 7.1.5 for the 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$ of flat small braided bimonoidal categories, robust braided bimonoidal functors, and bimonoidal natural transformations.
(2) Read Theorems 7.3.4 and 7.3.6 for the two versions of the Braided Baez Conjecture.
(3) Go back and read the rest of this chapter.

7.1. The 2-Category of Braided Bimonoidal Categories

In this section, we define the 2-category $\mathsf{Bi}^{\mathsf{br}}$ of small braided bimonoidal categories, braided bimonoidal functors, and bimonoidal natural transformations. The Braided Baez Conjecture, Theorem 7.3.4, involves the full sub-2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$ in Definition 7.1.5.

- The objects in $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$ are *flat* small braided bimonoidal categories as in Definition 5.4.5. The flatness assumption ensures that the Coherence Theorem 5.4.4 is applicable.
- The 1-cells in $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$ are *robust* braided bimonoidal functors as in Definition 6.1.1. The robustness condition on a braided bimonoidal functor F means that the structure morphisms $F^2_\oplus$, $F^0_\oplus$, and $F^0_\otimes$ are isomorphisms.

Proposition 7.1.7 provides a class of examples of bimonoidal natural transformations in the context of Theorem 2.4.22.

Convention 7.1.1. The following conventions are in effect throughout this chapter.

(1) Unless otherwise specified, C and D are arbitrary braided bimonoidal categories as in Definition 2.1.29. Sometimes they are required to be small or flat, as in Definition 5.4.5, as specified.
(2) A *Mac Lane coherence isomorphism* means a component of a permuted canonical map as in Definition 1.6.1, applied to the additive structure
$$(\mathsf{C}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus),$$
which is often abbreviated to $(\mathsf{C}, \oplus)$. This is an adaptation of Definition 6.2.9 to the current context. ⋄

For monoidal functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, recall from Definition 1.3.13 that a monoidal natural transformation $\theta : F \longrightarrow G$ is a natural transformation that is also compatible with the structure morphisms (F^2, F^0) of F and (G^2, G^0) of G.

Recall from Definition 6.1.1 that a braided bimonoidal functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor equipped with

- a symmetric monoidal functor structure

$$F_\oplus = (F, F^2_\oplus, F^0_\oplus) : (\mathsf{C}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus) \longrightarrow (\mathsf{D}, \oplus, \mathbb{0}, \alpha^\oplus, \lambda^\oplus, \rho^\oplus, \xi^\oplus)$$

and
- a braided monoidal functor structure

$$F_{\otimes} = (F, F^2_{\otimes}, F^0_{\otimes}) : (\mathsf{C}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}) \longrightarrow (\mathsf{D}, \otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes}).$$

These data satisfy the multiplicative zero axiom (6.1.2) and the distributivity axiom (6.1.3).

Next is the braided version of Definition I.7.1.2.

Definition 7.1.2. Suppose $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ are two braided bimonoidal functors.

(1) A *bimonoidal natural transformation* $\theta : F \longrightarrow G$ is a natural transformation of the underlying functors such that both
   - $\theta : F_{\oplus} \longrightarrow G_{\oplus}$ and
   - $\theta : F_{\otimes} \longrightarrow G_{\otimes}$

   are monoidal natural transformations.

(2) A *bimonoidal natural isomorphism* is an invertible bimonoidal natural transformation. ◇

Recall from Proposition 6.1.10 the 1-category $\mathsf{Bi}^{\mathsf{br}}$ of *small* braided bimonoidal categories and braided bimonoidal functors. Also recall its wide subcategories $\mathsf{Bi}^{\mathsf{br}}_{\mathsf{r}}$, $\mathsf{Bi}^{\mathsf{br}}_{\mathsf{sg}}$, $\mathsf{Bi}^{\mathsf{br}}_{\mathsf{u}}$, and $\mathsf{Bi}^{\mathsf{br}}_{\mathsf{st}}$ with, respectively, robust, strong, unitary, and strict braided bimonoidal functors. Now we extend these 1-categories to 2-categories as follows.

Definition 7.1.3. Define the 2-categorical data for $\mathsf{Bi}^{\mathsf{br}}$ as follows.

Objects: Objects are small braided bimonoidal categories.

1-Cells: The 1-cells in $\mathsf{Bi}^{\mathsf{br}}(\mathsf{C}, \mathsf{D})$ are the braided bimonoidal functors $\mathsf{C} \longrightarrow \mathsf{D}$.

Identity 1-Cells: $1_{\mathsf{C}} \in \mathsf{Bi}^{\mathsf{br}}(\mathsf{C}, \mathsf{C})$ is the identity braided bimonoidal functor.

1-Cell Composition: Horizontal composition of 1-cells is as in Definition 6.1.8.

2-Cells: The 2-cells in $\mathsf{Bi}^{\mathsf{br}}(\mathsf{C}, \mathsf{D})(F, G)$ are the bimonoidal natural transformations $F \longrightarrow G$ in Definition 7.1.2.

Identity 2-Cells: For each 1-cell $F : \mathsf{C} \longrightarrow \mathsf{D}$, the identity 2-cell

$$1_F \in \mathsf{Bi}^{\mathsf{br}}(\mathsf{C}, \mathsf{D})(F, F)$$

is the identity natural transformation of F.

2-Cell Compositions: Vertical and horizontal compositions of 2-cells are those of natural transformations in Definition I.1.1.8.

This finishes the definition of the 2-categorical data for $\mathsf{Bi}^{\mathsf{br}}$.

Moreover, similar definitions define the 2-categorical data for

- $\mathsf{Bi}^{\mathsf{br}}_{\mathsf{r}}$ with robust braided bimonoidal functors as 1-cells,
- $\mathsf{Bi}^{\mathsf{br}}_{\mathsf{sg}}$ with strong braided bimonoidal functors as 1-cells,
- $\mathsf{Bi}^{\mathsf{br}}_{\mathsf{u}}$ with unitary braided bimonoidal functors as 1-cells, and
- $\mathsf{Bi}^{\mathsf{br}}_{\mathsf{st}}$ with strict braided bimonoidal functors as 1-cells. ◇

Proposition 7.1.4. *With the data in Definition 7.1.3, $\mathsf{Bi}^{\mathsf{br}}$ is a 2-category that contains the following full sub-2-categories:*

$$\mathsf{Bi}^{\mathsf{br}}_{\mathsf{st}} \subset \mathsf{Bi}^{\mathsf{br}}_{\mathsf{u}} \subset \mathsf{Bi}^{\mathsf{br}}_{\mathsf{sg}} \subset \mathsf{Bi}^{\mathsf{br}}_{\mathsf{r}} \subset \mathsf{Bi}^{\mathsf{br}}.$$

Proof. The proof of Proposition I.7.1.7, which is the symmetric case for $\mathsf{Bi}^{\mathsf{sy}}$, is still valid in the current context. Instead of Lemma I.5.1.9, here we use the corresponding Lemma 6.1.9 to infer the existence of the four sub-2-categories. □

Recall from Definition 5.4.5 that a braided bimonoidal category is *flat* if each iterated sum and product of a component of either δ^l or δ^r (2.1.31) with a finite number of identity morphisms is a monomorphism. For example, tight braided bimonoidal categories—that is, those with δ^l and δ^r natural isomorphisms—are flat. The Coherence Theorem 5.4.4 applies to flat braided bimonoidal categories. Also recall from Definition 6.1.1 that a *robust* braided bimonoidal functor F has $F^2_\oplus$, $F^0_\oplus$, and $F^0_\otimes$ isomorphisms.

Definition 7.1.5. Denote by $\mathsf{Bi}^{\mathsf{fbr}}_{\mathsf{r}}$ the full sub-2-category of $\mathsf{Bi}^{\mathsf{br}}$ with

- flat small braided bimonoidal categories as objects and
- robust braided bimonoidal functors as 1-cells. ⋄

Example 7.1.6.

(1) The tight, in particular flat, braided bimonoidal category $\mathcal{F}^{\mathsf{any}}$ of Fibonacci anyons in Theorem 3.4.13 is small by Definition 3.3.3. In fact, it has a countable set of objects.

(2) The tight, in particular flat, braided bimonoidal category $\mathcal{I}^{\mathsf{any}}$ of Ising anyons in Theorem 3.6.14 is small by Definition 3.5.1. It also has a countable set of objects. ⋄

In Theorem 2.4.22, we observed that an abelian category with a compatible braided monoidal structure is a tight braided bimonoidal category. Recall from Definition 2.3.3 the notion of an additive functor. The next observation provides a class of examples of bimonoidal natural transformations between braided bimonoidal functors.

Proposition 7.1.7. *Suppose the quintuple*

$$(\mathsf{C}, \mathsf{D}, F_\otimes, G_\otimes, \theta)$$

consists of the following data.

- *Each of* C *and* D *is an abelian category with a compatible braided monoidal structure in the sense of Convention 2.4.1.*
- *Each of*
$$F_\otimes = (F, F^2_\otimes, F^0_\otimes) \quad \text{and} \quad G_\otimes = (G, G^2_\otimes, G^0_\otimes)$$
is a braided monoidal functor $\mathsf{C} \longrightarrow \mathsf{D}$ *that is also an additive functor.*
- $\theta : F_\otimes \longrightarrow G_\otimes$ *is a monoidal natural transformation.*

Then θ *is a bimonoidal natural transformation*

$$\theta : (F, F^2_\otimes, F^0_\otimes, F^2_\oplus, F^0_\oplus) \longrightarrow (G, G^2_\otimes, G^0_\otimes, G^2_\oplus, G^0_\oplus)$$

between the canonically extended braided bimonoidal functors.

Proof. By Proposition 6.1.12, F canonically extends to a braided bimonoidal functor

$$(F, F^2_\otimes, F^0_\otimes, F^2_\oplus, F^0_\oplus) : \mathsf{C} \longrightarrow \mathsf{D}$$

with C and D regarded as tight braided bimonoidal categories by Theorem 2.4.22, and similarly for G. Since $\theta : F_\otimes \longrightarrow G_\otimes$ is a monoidal natural transformation by assumption, it remains to check that

$$\theta : F_\oplus = (F, F^2_\oplus, F^0_\oplus) \longrightarrow (G, G^2_\oplus, G^0_\oplus) = G_\oplus$$

is a monoidal natural transformation.

The compatibility condition of θ with $F^0_\oplus$ and $G^0_\oplus$ asserts the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{0} \arrow[r, "F^0_\oplus"] \arrow[d, equal] & F\mathbb{0} \arrow[d, "\theta_{\mathbb{0}}"] \\
\mathbb{0} \arrow[r, "G^0_\oplus"] & G\mathbb{0}
\end{tikzcd}$$

in D. This diagram is commutative because each composite is the unique morphism from the zero object $\mathbb{0}$ in D.

The compatibility condition of θ with $F^2_\oplus$ and $G^2_\oplus$ asserts the commutativity of the following diagram in D for objects $A, B \in$ C.

$$\begin{tikzcd}
FA \oplus FB \arrow[r, "\theta_A \oplus \theta_B"] \arrow[d, "F^2_\oplus"'] & GA \oplus GB \arrow[d, "G^2_\oplus"] \\
F(A \oplus B) \arrow[r, "\theta_{A \oplus B}"] & G(A \oplus B)
\end{tikzcd} \tag{7.1.8}$$

The additive functors F and G preserve direct sums by Theorem 2.3.7 (4). Therefore, it suffices to show that the two composites in (7.1.8) composed with the projection to $GA \in$ D are equal, and similarly for GB. When composed with the projection to GA, (7.1.8) is the outer diagram in D below.

$$\begin{tikzcd}
FA \oplus FB \arrow[rrr, "\theta_A \oplus \theta_B"] \arrow[d, "F^2_\oplus"'] \arrow[drr, "p_1^{FA,FB}"] & & & GA \oplus GB \arrow[r, dash] \arrow[dd, "p_1^{GA,GB}"'] & {} \arrow[d, "G^2_\oplus"] \\
F(A \oplus B) \arrow[rr, "Fp_1^{A,B}"'] \arrow[d, "\theta_{A \oplus B}"'] \arrow[urr, phantom, "(6.1.14)"] & & FA \arrow[dr, "\theta_A"] \arrow[ur, phantom, "(2.4.4)"] & & G(A \oplus B) \arrow[d, dash, "Gp_1^{A,B}"] \arrow[ul, phantom, "(6.1.14)"] \\
G(A \oplus B) \arrow[rrr, "Gp_1^{A,B}"] \arrow[urr, phantom, "\mathsf{nat}"] & & & GA & {} \arrow[l]
\end{tikzcd}$$

As indicated, this diagram is commutative by

- the naturality of θ and
- the definitions of a direct sum of morphisms (2.4.4), $F^2_\oplus$ (6.1.14), and $G^2_\oplus$.

A similar diagram with p_2 instead of p_1 shows that the two composites in (7.1.8) composed with the projection to the object GB are equal. $\square$

7.2. Weakly Initial Braided Bimonoidal Category

In this section, we prove the first half of the Braided Baez Conjecture, Theorem 7.3.4. For each flat braided bimonoidal category C as in Definition 5.4.5, we construct a robust braided bimonoidal functor

$$F : \Sigma \longrightarrow \mathsf{C}$$

as in Definition 6.1.1, with Σ the left bipermutative category in Definition I.2.4.1 and Proposition I.2.4.8. See Lemma 7.2.11.

Recall that each left bipermutative category, such as Σ, is a tight, in particular flat, braided bimonoidal category by Propositions 5.1.8 and 5.1.10. Moreover, Σ is small because it has a countable set of objects. After recalling the definition of Σ, we define the additive structure of F in Definition 7.2.3 and the multiplicative structure of F in Definition 7.2.8.

The Finite Ordinal Category. Let us first recall the structure of Σ from Definition I.2.4.1. It is the category with

- objects $n \geq 0$ and
- morphisms

$$\Sigma(m,n) = \begin{cases} \Sigma_n & \text{if } m = n \text{ and} \\ \varnothing & \text{if } m \neq n. \end{cases}$$

Here Σ_n is the symmetric group on n letters.

In the context of Definition 2.1.1, its symmetric bimonoidal structure is as follows.

- The functor
$$- \oplus - : \Sigma \times \Sigma \longrightarrow \Sigma$$
is given by
  - $m \oplus n = m + n$ on objects and
  - the block sums (1.1.8) on morphisms.

  The additive zero is 0.
- The additive symmetry
$$m + n \xrightarrow{\xi^{\oplus}_{m,n}} n + m$$
is the interval-swapping permutation $\tau\langle m,n\rangle \in \Sigma_{m+n}$ in (1.2.2).
- The functor
$$- \otimes - : \Sigma \times \Sigma \longrightarrow \Sigma$$
is given by
  - $m \otimes n = mn$ on objects and
  - the permutations in (5.2.17) on morphisms.

  The multiplicative unit is 1. We think of $m \otimes n$ as an $n \times m$ matrix. For permutations $\sigma \in \Sigma_m$ and $\sigma' \in \Sigma_n$, the morphism
$$m \otimes n \xrightarrow{\sigma \otimes \sigma'} m \otimes n$$
permutes the n rows via σ' and the m columns via σ.
- The multiplicative symmetry
$$mn \xrightarrow{\xi^{\otimes}_{m,n}} nm$$
is the permutation $\xi^{\otimes}_{m,n} \in \Sigma_{mn}$ in (5.2.19). It corresponds to taking the transpose of an $n \times m$ matrix.
- The right distributivity morphism
$$(m+n)p \xrightarrow{\delta^r_{m,n,p}} mp + np$$
is the permutation in $\Sigma_{(m+n)p}$ given by
$$(\xi^{\otimes}_{p,m} \oplus \xi^{\otimes}_{p,n})\xi^{\otimes}_{m+n,p} : \begin{cases} i + (k-1)(m+n) \longmapsto i + (k-1)m \\ j + m + (k-1)(m+n) \longmapsto j + (k-1)n + pm \end{cases}$$
for $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq p$. It appears as the permutation component in δ^r in the braided distortion category $\mathcal{D}^{\mathrm{br}}$ in Lemma 5.2.28; see also Explanation 5.2.29 (1).

- $\alpha^{\oplus}$, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\alpha^{\otimes}$, $\lambda^{\otimes}$, $\rho^{\otimes}$, $\lambda^{\bullet}$, $\rho^{\bullet}$, and δ^{l} are the identities.

By Proposition I.2.4.8, Σ is a left bipermutative category in the sense of Definition I.2.5.11.

The Additive Structure. The definition of the desired braided bimonoidal functor $F : \Sigma \longrightarrow \mathsf{C}$ uses the following notions.

With $\odot \in \{\oplus, \otimes\}$, the *left normalized bracketing* is defined inductively by

$$(x_1 \odot \cdots \odot x_k)_{\mathsf{lt}} = \begin{cases} x_1 & \text{if } k = 1 \text{ and} \\ (x_1 \odot \cdots \odot x_{k-1})_{\mathsf{lt}} \odot x_k & \text{if } k > 1. \end{cases} \tag{7.2.1}$$

Definition 7.2.2. For each integer $n \geq 0$, define the following object in C.

$$\overline{n} = \begin{cases} \mathbb{0} & \text{if } n = 0, \\ \mathbb{1} & \text{if } n = 1, \text{ and} \\ (\mathbb{1} \oplus \cdots \oplus \mathbb{1})_{\mathsf{lt}} & \text{if } n > 1. \end{cases}$$

In the last case, $\overline{n}$ is the left normalized sum (7.2.1) of n copies of $\mathbb{1} \in \mathsf{C}$. $\diamond$

Recall from Convention 7.1.1 that, in this chapter, a *Mac Lane coherence isomorphism* is a component of a permuted canonical map in the additive structure $(\mathsf{C}, \oplus)$. The next definition is the same as Definition I.7.2.2, except that C is a braided bimonoidal category here.

Definition 7.2.3. Using the additive structures in Σ and C, define the data

$$F_{\oplus} = (F, F^2_{\oplus}, F^0_{\oplus}) : (\Sigma, \oplus) \longrightarrow (\mathsf{C}, \oplus)$$

of a symmetric monoidal functor as follows.

The Functor: The functor $F : \Sigma \longrightarrow \mathsf{C}$ is defined as follows.

Objects: For each $n \geq 0$, define

$$F(n) = \overline{n} \in \mathsf{C}$$

with $\overline{n}$ as in Definition 7.2.2.

Morphisms: For each morphism $\sigma \in \Sigma(n, n)$, define the morphism

$$F(n) = \overline{n} \xrightarrow{F(\sigma)} \overline{n} = F(n) \in \mathsf{C}$$

as the Mac Lane coherence isomorphism that additively permutes the n copies of $\mathbb{1}$ in $\overline{n}$ as $\sigma \in \Sigma_n$ permutes n letters. Its existence and uniqueness are guaranteed by the Symmetric Coherence Theorem I.1.3.8 in the symmetric monoidal category $(\mathsf{C}, \oplus)$.

The Additive Zero Constraint: The morphism

$$\mathbb{0} \xrightarrow{F^0_{\oplus}} F(0) = \mathbb{0} \in \mathsf{C}$$

is the identity morphism $1_{\mathbb{0}}$.

The Additive Monoidal Constraint: For $m, n \geq 0$, define the morphism

$$\overline{m} \oplus \overline{n} = F(m) \oplus F(n) \xrightarrow{F^2_{\oplus}} F(m+n) = \overline{m+n} \in \mathsf{C}$$

as the Mac Lane coherence isomorphism that does *not* involve $\xi^{\pm\oplus}$. Its existence and uniqueness are guaranteed by Mac Lane's Coherence Theorem I.1.3.3 in the monoidal category $(\mathsf{C}, \oplus)$.

This finishes the definition of $F_{\oplus}$. $\diamond$

Lemma 7.2.4. *For each braided bimonoidal category* C*,*

$$F_{\oplus} : (\Sigma, \oplus) \longrightarrow (\mathsf{C}, \oplus)$$

in Definition 7.2.3 is a strong symmetric monoidal functor.

Proof. The proof of Lemma I.7.2.9, which is the case with C a symmetric bimonoidal category, is still valid in the current context. $\square$

The Multiplicative Structure.

Convention 7.2.5. For the rest of this chapter, unless otherwise specified, C is a flat braided bimonoidal category as in Definition 5.4.5. $\diamond$

The multiplicative structure of $F : \Sigma \longrightarrow \mathsf{C}$ requires some preliminary notions. Next is the braided version of Definition I.7.3.3. Recall from Definition 5.3.15 the *braided distortion* of a path $P \in \mathsf{Gr}(X)$. It is the *value* of P, in the sense of (5.3.14), in the braided distortion category $\mathcal{D}^{\mathsf{br}}$ via the graph morphism $\vartheta : \mathsf{Gr}(X) \longrightarrow \mathcal{D}^{\mathsf{br}}$ in (5.3.17).

Definition 7.2.6. Suppose $m, n \geq 1$.

- Define the set
$$B = \{0^B, 1^B, b_1, \ldots, b_n\}$$
with $n+2$ elements.
- With each 1_i^B denoting a copy of 1^B, define

$$\Big(\bigoplus_{i=1}^{m} 1_i^B\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} b_j\Big)_{\mathsf{lt}} \xrightarrow{\quad P \quad} \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} b_j\Big)_{\mathsf{lt}} \tag{7.2.7}$$

as any path in $\mathsf{Gr}(B)$ whose braided distortion is the identity morphism
$$1_{(1,\ldots,1)} = \big(\mathrm{id}_{mn}; \overbrace{\mathrm{id}_1, \ldots, \mathrm{id}_1}^{mn}\big) \in \Sigma_{mn} \times B_1^{\times mn}$$
of $(1, \ldots, 1) \in \mathcal{D}^{\mathsf{br}}$.
- Define the function
$$\varphi : B \longrightarrow \mathsf{Ob}(\mathsf{C})$$
by
$$\varphi(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^B \text{ and} \\ \mathbb{1} & \text{if } x \in \{1^B, b_1, \ldots, b_n\}. \end{cases}$$
The value of the path P in C is defined as in Definition 5.3.10 via the associated graph morphism $\varphi : \mathsf{Gr}(B) \longrightarrow \mathsf{C}$. $\diamond$

Next we define the multiplicative structure of F.

Definition 7.2.8. For a flat braided bimonoidal category C, extend the functor F in Definition 7.2.3 to the data

$$F_{\otimes} = (F, F_{\otimes}^2, F_{\otimes}^0) : (\Sigma, \otimes) \longrightarrow (\mathsf{C}, \otimes)$$

of a braided monoidal functor between the multiplicative structures as follows.

The Multiplicative Unit Constraint: The morphism

$$\mathbb{1} \xrightarrow{F^0_\otimes} F(1) = \mathbb{1} \in \mathsf{C}$$

is the identity morphism $1_{\mathbb{1}}$.

The Multiplicative Monoidal Constraint: For $m, n \geq 0$, define the morphism

$$\overline{m} \otimes \overline{n} = F(m) \otimes F(n) \xrightarrow{F^2_\otimes} F(mn) = \overline{mn} \in \mathsf{C}$$

as follows.

- If $m = 0$, then $F^2_\otimes$ is the left multiplicative zero

$$\mathbb{0} \otimes \overline{n} \xrightarrow{\lambda^\bullet_{\overline{n}}} \mathbb{0}.$$

- If $n = 0$, then $F^2_\otimes$ is the right multiplicative zero

$$\overline{m} \otimes \mathbb{0} \xrightarrow{\rho^\bullet_{\overline{m}}} \mathbb{0}.$$

  If $m = n = 0$, then $\lambda^\bullet_{\mathbb{0}} = \rho^\bullet_{\mathbb{0}}$ by the axiom (2.1.13) in C.
- If $m, n > 0$, then

$$\Big(\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}} \otimes \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} \xrightarrow{F^2_\otimes} \Big(\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{m} \mathbb{1}\Big)_{\mathsf{lt}}$$

  is the value in C of any path P as in (7.2.7) with identity braided distortion.

This finishes the definition of $F_\otimes$. ◇

Lemma 7.2.9. *For each flat braided bimonoidal category* C*,*

$$F_\otimes : (\Sigma, \otimes) \longrightarrow (\mathsf{C}, \otimes)$$

in Definition 7.2.8 is a braided monoidal functor.

Proof. We reuse the proofs in Section I.7.3 for the symmetric case, with the following notes and adjustments.

- Each instance of the distortion of a path in $\mathsf{Gr}(X)$ is replaced by the *braided* distortion of that path via the graph morphism

$$\vartheta : \mathsf{Gr}(X) \longrightarrow \mathcal{D}^{\mathsf{br}}$$

  in (5.3.17). In particular, the path Q in (I.7.3.19) has as its braided distortion the identity morphism

$$1_{(2,\ldots,2)} = \big(\mathrm{id}_{mn}; \overbrace{\mathrm{id}_2, \ldots, \mathrm{id}_2}^{mn}\big) \in \Sigma_{mn} \times B_2^{\times mn} \tag{7.2.10}$$

  of $(2, \ldots, 2) \in \mathcal{D}^{\mathsf{br}}$.
- By Theorem 2.2.1, C satisfies all 24 Laplaza axioms in Definition 2.1.1. Some of those axioms are used in the proofs of the associativity axiom (1.3.10) and the unity axioms (1.3.11) of $F_\otimes$ in Lemmas I.7.3.25 and I.7.3.27. We can still use those axioms here.

Moreover, each instance of Laplaza's Second Coherence Theorem I.4.4.3 for symmetric bimonoidal categories is replaced by its braided analogue here, namely, the Coherence Theorem 5.4.4 for braided bimonoidal categories, which is applicable because C is flat. In each case, as listed below, the paths in question have the same braided distortion.

- Lemma I.7.3.15 shows that $F^2_\otimes$ is well defined and is natural.
  - In the first paragraph of its proof and (I.7.3.17), P and P' both have braided distortion the identity morphism.
  - The last paragraph of its proof shows that the two paths have the same permutation component in their braided distortions. Moreover, since B_1 is the trivial group, each of their braid components is the identity braid $\mathrm{id}_1 \in B_1$. Therefore, the two paths have the same braided distortion.
- Similarly, in the proofs of Lemmas I.7.3.21, I.7.3.24, and I.7.3.25 for the associativity axiom (1.3.10) for $F_\otimes$, each path has braided distortion the identity morphism.

Furthermore, the proof of Lemma I.7.3.28 is reused here with the following adjustments.

- In case (1), the diagram is commutative by the braided bimonoidal category axiom (2.1.33), which states that $\lambda^\bullet = \rho^\bullet \xi^\otimes_{\mathbb{0},-}$.
- In the diagram (I.7.3.31), both vertical paths Q and Q' have braided distortions the identity morphism of $(2,\ldots,2) \in \mathcal{D}^{\mathrm{br}}$. The top path $\xi^\otimes$ and the bottom path R both have braided distortions the morphism

$$\big(\xi^\otimes_{m,n}; \overbrace{s_1,\ldots,s_1}^{mn}\big) \in \Sigma_{mn} \times B_2^{\times mn}.$$

  Here
  - $\xi^\otimes_{m,n} \in \Sigma_{mn}$ is the permutation in (5.2.19), and
  - $s_1 \in B_2$ is the generating braid as in Definition 1.1.1.
- In the last paragraph, we use Theorem 5.4.4 instead of Laplaza's First Coherence Theorem I.3.9.1 by noting that the two paths in (I.7.3.31) have the same braided distortion. □

Lemma 7.2.11. *For each flat braided bimonoidal category* C*, the data*

$$(F, F^2_\oplus, F^0_\oplus, F^2_\otimes, F^0_\otimes) : \Sigma \longrightarrow \mathsf{C}$$

in Definitions 7.2.3 and 7.2.8 constitute a robust braided bimonoidal functor.

Proof. We already have the following.

- Lemma 7.2.4 shows that

$$F_\oplus = (F, F^2_\oplus, F^0_\oplus)$$

  is a strong symmetric monoidal functor.
- Lemma 7.2.9 shows that

$$F_\otimes = (F, F^2_\otimes, F^0_\otimes)$$

  is a braided monoidal functor.
- $F^0_\otimes$ is the identity morphism $1_{\mathbb{1}}$ by definition.

Therefore, it remains to check the multiplicative zero axiom (6.1.2) and the distributivity axiom (6.1.3).

To prove these two axioms, as in the proof of Lemma 7.2.9, we reuse the proofs of Lemma I.7.4.3 and Proposition I.7.4.4 in the symmetric case with the following notes and adjustments.

- In Lemma I.7.4.3, to show that the diagram (I.7.4.2) is commutative in C in the current context, observe that each of its six paths has braided distortion an identity morphism in $\mathcal{D}^{\mathsf{br}}$.
  - For the paths Q_1, $Q_2 \oplus 1$, and $1 \oplus Q_3$ in that diagram, this is true by (7.2.10).
  - The paths $1 \otimes R_1$ and R_2 involve only identities and $\alpha^{-\oplus}$.

  In the braided distortion category $\mathcal{D}^{\mathsf{br}}$, the only nonidentity structure morphisms are $\xi^{\oplus}$ (5.2.11), $\xi^{\otimes}$ (5.2.18), and δ^r (5.2.26). Therefore, in the diagram (I.7.4.2), each of the two composite paths has identity braided distortion. It is commutative in C by Theorem 5.4.4.
- The proof of Proposition I.7.4.4 uses Lemma I.7.4.3 and Laplaza's axioms in Definition 2.1.1, which also hold in the braided bimonoidal category C by Theorem 2.2.1. □

7.3. Bi-Initial Braided Bimonoidal Category

In this section, we finish the proof of the Braided Baez Conjecture in Theorem 7.3.4. In the language of bicategory theory, it states that the left bipermutative category Σ in Proposition I.2.4.8 is a bi-initial object in the 2-category $\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$ in Definition 7.1.5. There is also a second version involving the right bipermutative category Σ' in Proposition I.2.4.23; see Theorem 7.3.6. These theorems are the braided analogues of Theorems I.7.8.1 and I.7.8.3.

As we will explain in the proof, Theorem 7.3.4 has

- an existence part for 1-cells and
- a uniqueness part for 2-cells.

The existence part is already established in Lemma 7.2.11. The uniqueness part is proved in Lemmas 7.3.1 and 7.3.3 below.

Lemma 7.3.1. *Suppose given the following data.*

- C *is a braided bimonoidal category as in Definition 2.1.29.*
- $G, H : \Sigma \longrightarrow \mathsf{C}$ *are robust braided bimonoidal functors.*
- $\pi : G \longrightarrow H$ *is a bimonoidal natural transformation as in Definition 7.1.2.*

Then the following statements hold.

(1) π_0 *is the following composite in* C.

$$\begin{tikzcd} G(0) \arrow[r, "G_{\oplus}^{-0}"] \arrow[rr, bend left, "\pi_0"] & \mathbb{0} \arrow[r, "H_{\oplus}^{0}"] & H(0) \end{tikzcd}$$

(2) π_1 *is the following composite in* C.

$$\begin{tikzcd} G(1) \arrow[r, "G_{\otimes}^{-0}"] \arrow[rr, bend left, "\pi_1"] & \mathbb{1} \arrow[r, "H_{\otimes}^{0}"] & H(1) \end{tikzcd}$$

(3) *For each $n \geq 2$, π_n is the following composite in* C.

$$\begin{tikzcd} G(n) \arrow[rr, "\pi_n"] \arrow[d, "G_\oplus^{-1}"'] && H(n) \\ \Big(\bigoplus_{i=1}^{n} G(1)\Big)_{\mathsf{lt}} \arrow[rr, "(\oplus_i \pi_1)_{\mathsf{lt}}"] && \Big(\bigoplus_{i=1}^{n} H(1)\Big)_{\mathsf{lt}} \arrow[u, "H_\oplus"'] \end{tikzcd}$$

Here $G_\oplus$ means

- $G_\oplus^2$ *if $n = 2$ and*
- *inductively, $G_\oplus^2(G_\oplus \oplus 1)$ if $n > 2$.*

$H_\oplus$ is defined in the same way using $H_\oplus^2$.

(4) *π is the only bimonoidal natural transformation $G \longrightarrow H$.*

(5) *π is a bimonoidal natural isomorphism.*

Proof. The proofs of Lemmas I.7.6.2 and I.7.6.3 in the symmetric case are still valid in the current context. □

In the rest of this section, $F : \Sigma \longrightarrow \mathsf{C}$ is the robust braided bimonoidal functor in Lemma 7.2.11.

Definition 7.3.2. Suppose

- C is a flat braided bimonoidal category, and
- $G : \Sigma \longrightarrow \mathsf{C}$ is a robust braided bimonoidal functor.

Define

$$\theta^G : F \longrightarrow G$$

with the component morphisms

$$F(n) = \overline{n} \xrightarrow{\theta_n^G} G(n) \in \mathsf{C}$$

for $n \geq 0$ as follows.

- θ_0^G is the additive zero constraint

$$F(0) = \mathbb{0} \xrightarrow{G_\oplus^0} G(0).$$

- θ_1^G is the multiplicative unit constraint

$$F(1) = \mathbb{1} \xrightarrow{G_\otimes^0} G(1).$$

- For each $n \geq 2$, θ_n^G is the following composite in C.

$$\begin{tikzcd} F(n) = \overline{n} \arrow[rr, "\theta_n^G"] \arrow[d, equal] && G(n) \\ \Big(\bigoplus_{j=1}^{n} \mathbb{1}\Big)_{\mathsf{lt}} \arrow[rr, "(\oplus_j G_\otimes^0)_{\mathsf{lt}}"] && \Big(\bigoplus_{j=1}^{n} G(1)\Big)_{\mathsf{lt}} \arrow[u, "G_\oplus"'] \end{tikzcd}$$

This finishes the definition of θ^G. ◇

Lemma 7.3.3. *In the context of Definition 7.3.2, the following statements hold.*

(1) *$\theta^G : F \longrightarrow G$ is the unique bimonoidal natural transformation from F to G.*

(2) *θ^G is a bimonoidal natural isomorphism.*

Proof. We reuse the proofs of Theorem I.7.5.8 and Lemmas I.7.7.3, I.7.7.4, and I.7.7.6 in the symmetric case to show that

$$\theta^G : F_\oplus \longrightarrow G_\oplus \quad \text{and} \quad \theta^G : F_\otimes \longrightarrow G_\otimes$$

are both monoidal natural transformations. The uniqueness and the invertibility of θ^G follow from Lemma 7.3.1. □

Recall the following.

- $\varnothing$ is the empty 2-category, with no objects, no 1-cells, and no 2-cells.
- $\mathsf{Bi}_\mathsf{r}^\mathsf{fbr}$ is the 2-category in Definition 7.1.5 with
  - flat small braided bimonoidal categories in Definition 5.4.5 as objects,
  - robust braided bimonoidal functors in Definition 6.1.1 as 1-cells, and
  - bimonoidal natural transformations in Definition 7.1.2 as 2-cells.

We are now ready for the braided version of Baez's Conjecture, Theorem I.7.8.1.

Theorem 7.3.4 (Braided Baez Conjecture)**.** *Σ is a lax bicolimit of the 2-functor*

$$\varnothing \longrightarrow \mathsf{Bi}_\mathsf{r}^\mathsf{fbr}.$$

Proof. Using **[JY21**, Sections 4.1 and 5.2] to unpack the assertion in 1-categorical terms, it means that for each flat small braided bimonoidal category C, the unique functor

$$\mathsf{Bi}_\mathsf{r}^\mathsf{fbr}(\Sigma, \mathsf{C}) \xrightarrow{\;T\;} \mathbf{1}$$

to the terminal category $\mathbf{1}$ is an equivalence of categories. In other words, it is fully faithful on morphisms and essentially surjective on objects.

Since $\mathbf{1}$ is the terminal category, the essential surjectivity of T means the existence of a robust braided bimonoidal functor $\Sigma \longrightarrow \mathsf{C}$. Even without the smallness assumption on C, this is true by Lemma 7.2.11, where we constructed a canonical robust braided bimonoidal functor $F : \Sigma \longrightarrow \mathsf{C}$.

The fully faithfulness of the functor T means that, for each pair $G, H : \Sigma \longrightarrow \mathsf{C}$ of robust braided bimonoidal functors, there exists a unique bimonoidal natural transformation $G \longrightarrow H$. Even without the smallness assumption on C, such a bimonoidal natural transformation is given by the vertical composite

$$G \xrightarrow{(\theta^G)^{-1}} F \xrightarrow{\theta^H} H$$

with θ^G and θ^H from Lemma 7.3.3. Its uniqueness follows from Lemma 7.3.1. □

Remark 7.3.5. In the proof of the Braided Baez Conjecture, the smallness assumption of C is only used to make sure that it is an object in the 2-category $\mathsf{Bi}_\mathsf{r}^\mathsf{fbr}$. The construction of $F : \Sigma \longrightarrow \mathsf{C}$ and the proofs of its properties do not require any smallness condition. ◇

Recall Σ' in Definition I.2.4.18 and Proposition I.2.4.23. It is a small right bipermutative category in the sense of Definition I.2.5.2, so it is also an object in $\mathsf{Bi}_\mathsf{r}^\mathsf{fbr}$ by Propositions 5.1.17 and 5.1.19. Moreover, as symmetric bimonoidal categories, Σ' is canonically isomorphic to Σ by Propositions I.5.1.15 and I.5.1.16. Therefore, Theorem 7.3.4 also holds for Σ'.

Theorem 7.3.6 (Braided Baez Conjecture, Version 2)**.** Σ' *is a lax bicolimit of the 2-functor*

$$\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}.$$

In other words, for each flat small braided bimonoidal category C*, the unique functor*

$$\mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}(\Sigma', \mathsf{C}) \longrightarrow \mathbf{1}$$

is an equivalence of categories.

CHAPTER 8

Monoidal Bicategorification

The main Theorem 8.4.7 in this chapter states that, for each tight braided bimonoidal category C as in Definition 2.1.29, the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ in Chapter I.8, with appropriate adjustments, is a monoidal bicategory in the sense of Definition I.6.4.1. This is an extension of Theorem I.8.12.9, which states that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory for each tight symmetric bimonoidal category C. While most of the constructions in the symmetric case in Chapter I.8 remain the same in the braided case, there are several crucial differences, which we discuss next.

Coherence: In Chapter I.8, most of the commutative diagrams are proved by invoking the Coherence Theorems I.3.9.1 and I.4.4.3 and Proposition I.3.5.33 for symmetric bimonoidal categories, or Theorem I.3.10.7 for bimonoidal categories. In this chapter, we use Theorem 5.4.4, which is our main coherence result for braided bimonoidal categories. Each time this coherence result is used, we have to check that the relevant paths have the same braided distortion in the sense of Definition 5.3.15.

The Monoidal Composition: In Chapter I.8, the lax functoriality constraint $\boxtimes^2$ in (I.8.6.20) of the monoidal composition $\boxtimes$ in $\mathsf{Mat}^{\mathsf{C}}$ involves the paths P and Q in Lemma I.8.6.16. The same paths are used in the braided case in Lemma 8.2.14, except that here we must specify their braided distortions. See (8.2.15) and (8.2.16). The additional requirement on the paths P and Q is needed in the current context because the Coherence Theorem 5.4.4 has a hypothesis about the braided distortions of the paths.

The Braiding: When C is a tight symmetric bimonoidal category, Theorem I.8.15.4 states that $\mathsf{Mat}^{\mathsf{C}}$ is a *symmetric* monoidal bicategory. Its braiding β involves the braiding $\xi^{\otimes}$ in C in a natural way; see (I.8.13.24). One may optimistically guess that, when C is a tight braided bimonoidal category, the same construction would yield a *braided* monoidal bicategory structure in $\mathsf{Mat}^{\mathsf{C}}$. However, this guess is false. As we will explain at the end of Section 8.4, in the current context, β in Definition I.8.13.22 is *not* in general a strong transformation because it fails to satisfy the lax naturality axiom. Furthermore, the unit $\eta^{\beta} : 1_{\boxtimes} \longrightarrow \beta^{\bullet}\beta$ as in Definition I.8.13.35 is *not* a modification. Therefore, Theorem I.8.15.4 does not seem to have a braided analogue.

The following table summarizes the main results for the matrix construction.

tight bimonoidal category C	bicategory $\mathsf{Mat}^{\mathsf{C}}$
plain	plain (I.8.4.12)
braided	monoidal (8.4.7)
symmetric	symmetric monoidal (I.8.15.4)

For open questions related to the matrix construction, see Appendix III.A.

Organization. A description of the rest of this chapter follows.

In Section 8.1, the main result is Theorem 8.1.13. It states that, for each tight braided bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ equipped with the structure in Sections I.8.1 through I.8.4 is a bicategory.

In Section 8.2, we define the monoidal identity $(1_{\boxtimes}, 1^2_{\boxtimes}, 1^0_{\boxtimes})$ and the monoidal composition $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in the bicategory $\mathsf{Mat}^{\mathsf{C}}$ by extending the constructions in Sections I.8.5 through I.8.7.

In Section 8.3, we define the monoidal associator $(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$, the left monoidal unitor $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$, and the right monoidal unitor $(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^r, \varepsilon^r)$ in the bicategory $\mathsf{Mat}^{\mathsf{C}}$ by extending the constructions in Sections I.8.8 and I.8.9.

In Section 8.4, we define the pentagonator π, the middle 2-unitor μ, the left 2-unitor $\lambda^{\boxtimes}$, and the right 2-unitor $\rho^{\boxtimes}$ in the bicategory $\mathsf{Mat}^{\mathsf{C}}$ by extending the constructions in Sections I.8.10 and I.8.11. After proving the main Theorem 8.4.7, we explain in detail the reasons why, for a tight braided bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ is not a braided monoidal bicategory.

Reading Guide.

(1) Read Definitions 8.1.5 and 8.1.6 and the statements of Lemma 8.1.10 and Theorem 8.1.13 for the bicategory $\mathsf{Mat}^{\mathsf{C}}$.
(2) Read Definitions 8.2.5 and 8.2.7 and Lemma 8.2.10 for the monoidal identity, the matrix tensor product, and the lax unity constraint.
(3) Read Definitions 8.2.13 and 8.2.18 for the monoidal composition.
(4) Read Definitions 8.3.1, 8.3.5, and 8.3.7 for the monoidal associator.
(5) Read Definitions 8.3.9 through 8.3.11 for the left monoidal unitor.
(6) Read Definitions 8.3.13 through 8.3.15 for the right monoidal unitor.
(7) Read Definitions 8.4.1 and 8.4.3 through 8.4.5 for the pentagonator and the 2-unitors.
(8) Read the statement of Theorem 8.4.7, which says that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory.
(9) Go back and read the rest of this chapter.

8.1. Matrix Bicategories

The main observation in this section is Theorem 8.1.13. It states that the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory for each tight braided bimonoidal category C. We actually proved a more general result in Theorem I.8.4.12, where C is only assumed to be a tight bimonoidal category. In this section, we give another proof of this result in the braided context using the Coherence Theorem 5.4.4 for braided bimonoidal categories.

We first recall the definition of a bicategory from Chapter I.6. The related definitions of a lax functor, a lax transformation, a modification, an adjoint equivalence, and so forth, can also be found in Chapter I.6 and [**JY21**]. Then we define the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ for each tight braided bimonoidal category C and prove Theorem 8.1.13.

Bicategories. One way to understand the definition of a bicategory is that a monoidal category is precisely a one-object bicategory. The unity axiom and the pentagon axiom in the definition of a bicategory are conceptually identical to those in a monoidal category in (1.3.2) and (1.3.3).

Definition 8.1.1. A *bicategory* is a tuple

$$\big(\mathsf{B}, 1, c, a, \ell, r\big)$$

consisting of the following data.

- B is equipped with a class $\mathsf{Ob}(\mathsf{B})$ of *objects*.
- For each pair of objects $X, Y \in \mathsf{B}$, it is equipped with a *hom category* $\mathsf{B}(X,Y)$.
  - Its objects $f : X \longrightarrow Y$ are called *1-cells*.
  - Its morphisms $\alpha : f \longrightarrow f'$ are called *2-cells*.
  - Its composition is called the *vertical composition*.
- Each object $X \in \mathsf{B}$ is equipped with an *identity 1-cell*

  $$1_X : X \longrightarrow X.$$

- For each triple of objects $X, Y, Z \in \mathsf{B}$,

  $$\mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \xrightarrow{c_{XYZ}} \mathsf{B}(X,Z)$$

  is a functor, which is called the *horizontal composition*. For
  - 1-cells $f \in \mathsf{B}(X,Y)$ and $g \in \mathsf{B}(Y,Z)$ and
  - 2-cells $\alpha \in \mathsf{B}(X,Y)$ and $\beta \in \mathsf{B}(Y,Z)$,

  the horizontal compositions are denoted by gf and $\beta * \alpha$.
- It is equipped with a natural isomorphism

  $$(hg)f \xrightarrow{a_{h,g,f}} h(gf)$$

  for 1-cells

  $$(h,g,f) \in \mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \times \mathsf{B}(W,X).$$

  It is called the *associator*.
- It is equipped with two natural isomorphisms

  $$1_Y f \xrightarrow{\ell_f} f \xleftarrow{r_f} f1_X$$

  for 1-cells $f \in \mathsf{B}(X,Y)$, which are called the *left unitor* and the *right unitor*, respectively.

The above data are required to satisfy the following two axioms for 1-cells $f \in \mathsf{B}(V,W)$, $g \in \mathsf{B}(W,X)$, $h \in \mathsf{B}(X,Y)$, and $k \in \mathsf{B}(Y,Z)$.

The Unity Axiom: The middle unity diagram

(8.1.2)
$$\begin{tikzcd}
(g1_W)f \arrow[rr, "a_{g,1_W,f}"] \arrow[dr, "r_g * 1_f"'] & & g(1_W f) \arrow[dl, "1_g * \ell_f"] \\
& gf &
\end{tikzcd}$$

in $\mathsf{B}(V,X)$ is commutative.

The Pentagon Axiom: The diagram

$$
\begin{tikzcd}
& (kh)(gf) \arrow[dr, "a_{k,h,gf}"] & \\
((kh)g)f \arrow[ur, "a_{kh,g,f}"] \arrow[d, "a_{k,h,g} * 1_f"'] & & k(h(gf)) \\
(k(hg))f \arrow[r, "a_{k,hg,f}"] & k((hg)f) \arrow[ur, "1_k * a_{h,g,f}"'] &
\end{tikzcd}
\tag{8.1.3}
$$

in $\mathsf{B}(V, Z)$ is commutative.

This finishes the definition of a bicategory. We sometimes abbreviate a bicategory as above to B. A *2-category* is a bicategory in which a, ℓ, and r are identity natural transformations. ◇

Example 8.1.4. Suppose B and B' are bicategories such that B has a set of objects. Then there is a bicategory $\mathsf{Bicat}(\mathsf{B}, \mathsf{B}')$ with the following data.

- Its objects are lax functors $\mathsf{B} \longrightarrow \mathsf{B}'$.
- 1-cells in $\mathsf{Bicat}(\mathsf{B}, \mathsf{B}')(F, G)$ are lax transformations $F \longrightarrow G$.
- 2-cells $\alpha \longrightarrow \beta$ are modifications for 1-cells $\alpha, \beta : F \longrightarrow G$.
- Identity 1-cells are identity strong transformations of lax functors.
- Vertical composition is that of modifications.
- Horizontal composition is that of lax transformations for 1-cells and that of modifications for 2-cells.
- The associator, the left unitor, and the right unitor are invertible modifications whose component 2-cells are defined in B'.

Moreover, $\mathsf{Bicat}(\mathsf{B}, \mathsf{B}')$ is a 2-category if B' is a 2-category.

The bicategory $\mathsf{Bicat}(\mathsf{B}, \mathsf{B}')$ contains a subbicategory $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B}, \mathsf{B}')$ with

- pseudofunctors $\mathsf{B} \longrightarrow \mathsf{B}'$ as objects,
- strong transformations between such pseudofunctors as 1-cells, and
- modifications between such strong transformations as 2-cells.

This is a sub-2-category of $\mathsf{Bicat}(\mathsf{B}, \mathsf{B}')$ if B' is a 2-category. ◇

The Matrix Product. Next we define the hom categories in $\mathsf{Mat}^{\mathsf{C}}$.

Definition 8.1.5. Suppose C is a category. For integers $m, n \geq 0$, define a category $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ as follows.

Objects: An object in $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ is an $n \times m$ matrix

$$A = (A_{ji})_{1 \leq j \leq n,\, 1 \leq i \leq m}$$

with each A_{ji}, which is called the (j, i)*-entry* in A, an object in C. We call A an $n \times m$ *matrix in* C. When n and m are understood, we will also write A as (A_{ji}).

Morphisms: A morphism

$$f : A = (A_{ji}) \longrightarrow (A'_{ji}) = A'$$

is an $n \times m$ matrix

$$f = (f_{ji})_{1 \leq j \leq n,\, 1 \leq i \leq m}$$

with each $f_{ji} : A_{ji} \longrightarrow A'_{ji}$ a morphism in C.

Identity Morphisms: For an object $A = (A_{ji})$, its identity morphism is the $n \times m$ matrix
$$1_A = \big(1_{A_{ji}}\big)$$
of identity morphisms in C.

Composition: If $f : A \longrightarrow A'$ is a morphism as above and if $f' : A' \longrightarrow A''$ is another morphism, then their composite $f'f : A \longrightarrow A''$ is defined entrywise in C as
$$(f'f)_{ji} = f'_{ji} f_{ji} : A_{ji} \longrightarrow A''_{ji}.$$

This finishes the definition of the category $\mathsf{Mat}^{\mathsf{C}}_{m,n}$.

If C has a distinguished object $\mathbb{0}$, then the *0 matrix* $\mathbb{0}_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ is the matrix with each entry $\mathbb{0} \in \mathsf{C}$ if $m, n > 0$. If either m or n is 0, then $\mathbb{0}_{m,n}$ denotes the empty matrix. $\diamond$

To define the horizontal composition in $\mathsf{Mat}^{\mathsf{C}}$, we extend the usual matrix product to $\mathsf{Mat}^{\mathsf{C}}$.

Definition 8.1.6. Suppose C is a bimonoidal category, and $m, n, p \geq 0$.

- For
  - an $n \times m$ matrix $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ and
  - a $p \times n$ matrix $B = (B_{kj}) \in \mathsf{Mat}^{\mathsf{C}}_{n,p}$,

  define their *matrix product*
$$BA \in \mathsf{Mat}^{\mathsf{C}}_{m,p}$$
  whose (k,i)-entry, for $1 \leq i \leq m$ and $1 \leq k \leq p$, is the following object in C:
$$(BA)_{ki} = \begin{cases} \Big(\bigoplus_{j=1}^{n} (B_{kj} \otimes A_{ji})\Big)_{\mathsf{lt}} & \text{if } n \geq 1 \text{ and} \\ \mathbb{0} & \text{if } n = 0. \end{cases} \tag{8.1.7}$$
  The subscript lt denotes the left normalized sum in (7.2.1). If either m or p is 0, then BA is the empty matrix.
- For morphisms
  - $f = (f_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}(A, A')$ and
  - $g = (g_{kj}) \in \mathsf{Mat}^{\mathsf{C}}_{n,p}(B, B')$,

  define their *matrix product*
$$g \star f \in \mathsf{Mat}^{\mathsf{C}}_{m,p}(BA, B'A')$$
  as the $p \times m$ matrix whose (k,i)-entry, for $1 \leq i \leq m$ and $1 \leq k \leq p$, is the following morphism in C:
$$(g \star f)_{ki} = \begin{cases} \Big(\bigoplus_{j=1}^{n} (g_{kj} \otimes f_{ji})\Big)_{\mathsf{lt}} : (BA)_{ki} \longrightarrow (B'A')_{ki} & \text{if } n \geq 1 \text{ and} \\ 1_{\mathbb{0}} : \mathbb{0} \longrightarrow \mathbb{0} & \text{if } n = 0. \end{cases} \tag{8.1.8}$$
  If either m or p is 0, then $g \star f$ is the identity morphism of the empty matrix.
- The $n \times n$ *identity matrix* is the square matrix $\mathbb{1}^n \in \mathsf{Mat}^{\mathsf{C}}_{n,n}$ with entries
$$\mathbb{1}^n_{ji} = \begin{cases} \mathbb{1} & \text{if } i = j \text{ and} \\ \mathbb{0} & \text{if } i \neq j. \end{cases} \tag{8.1.9}$$

Here $\mathbb{1}$ is the multiplicative unit in C, and $\mathbb{0}$ is the additive zero in C. If $n = 0$, then $\mathbb{1}^0$ is the unique empty matrix. ◇

The Bicategory Structure. The left unitor and the right unitor in $\mathsf{Mat}^{\mathsf{C}}$ are, respectively, ℓ and r in (8.1.11) below, and the associator in $\mathsf{Mat}^{\mathsf{C}}$ is in (8.1.12) below. The next lemma combines Lemmas I.8.1.8, I.8.1.10, I.8.1.18, I.8.2.1, I.8.2.7, and I.8.3.1.

Lemma 8.1.10. *Suppose C is a tight bimonoidal category, and $m, n, p, q \geq 0$.*

(1) *The matrix product*

$$\mathsf{Mat}^{\mathsf{C}}_{n,p} \times \mathsf{Mat}^{\mathsf{C}}_{m,n} \longrightarrow \mathsf{Mat}^{\mathsf{C}}_{m,p}$$

in Definition 8.1.6 is a functor.

(2) *There are natural isomorphisms*

$$\begin{gathered} \mathbb{0}_{n,p}A \xrightarrow{\zeta^{\ell}_A} \mathbb{0}_{m,p} \\ A\mathbb{0}_{q,m} \xrightarrow{\zeta^{r}_A} \mathbb{0}_{q,n} \\ \mathbb{1}^n A \xrightarrow{\ell_A} A \xleftarrow{r_A} A\mathbb{1}^m \end{gathered} \tag{8.1.11}$$

for $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$ such that the following statements hold.

- *ζ^{ℓ}_A involves only identities, $\lambda^{\oplus}_{\mathbb{0}}$, and $\lambda^{\bullet}$.*
- *ζ^{r}_A involves only identities, $\lambda^{\oplus}_{\mathbb{0}}$, and $\rho^{\bullet}$.*
- *ℓ_A involves only identities, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, and $\lambda^{\bullet}$.*
- *r_A involves only identities, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\rho^{\otimes}$, and $\rho^{\bullet}$.*

(3) *There is a natural isomorphism*

$$(CB)A \xrightarrow{a_{C,B,A}} C(BA) \in \mathsf{Mat}^{\mathsf{C}}_{m,q} \tag{8.1.12}$$

for

$$(A, B, C) \in \mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{n,p} \times \mathsf{Mat}^{\mathsf{C}}_{p,q}$$

such that the following statements hold.

- *If either m or q is 0, or if $m, q > 0$ and $n = p = 0$, then $a_{C,B,A}$ is the identity morphism.*
- *If $m, p, q > 0$ and $n = 0$, then $a_{C,B,A} = (\zeta^{r}_C)^{-1}$.*
- *If $m, n, q > 0$ and $p = 0$, then $a_{C,B,A} = \zeta^{\ell}_A$.*
- *If $m, n, p, q > 0$, then each entry of $a_{C,B,A}$ factors as $a^4a^3a^2a^1$ such that the following statements hold.*
  - *a^1 involves only identities and δ^r.*
  - *a^2 involves only identities and $\alpha^{\otimes}$.*
  - *a^3 involves only identities, $\alpha^{\pm\oplus}$, and $\xi^{\oplus}$.*
  - *a^4 involves only identities and δ^{-l}.*

Next is the main result of this section. It states that each tight braided bimonoidal category C has an associated matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$.

Theorem 8.1.13. *Suppose C is a tight braided bimonoidal category. Then there is a bicategory*

$$\big(\mathsf{Mat}^{\mathsf{C}}, \mathbb{1}, c, a, \ell, r\big)$$

with the following data.

- *The objects in* $\mathsf{Mat}^{\mathsf{C}}$ *are nonnegative integers* $n \geq 0$.
- *For* $m, n \geq 0$*, the hom category* $\mathsf{Mat}^{\mathsf{C}}(m,n)$ *is the category* $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ *in Definition 8.1.5.*
- *For each object* $n \geq 0$*, its identity 1-cell* 1_n *is the* $n \times n$ *identity matrix* $\mathbb{1}^n \in \mathsf{Mat}^{\mathsf{C}}_{n,n}$ *defined entrywise in (8.1.9).*
- *The horizontal composition* c *is the matrix product in Lemma 8.1.10 (1).*
- *The left unitor* ℓ *and the right unitor* r *are the natural isomorphisms in (8.1.11).*
- *The associator* a *is the natural isomorphism in (8.1.12).*

Proof. This theorem holds more generally for tight bimonoidal categories by Theorem I.8.4.12. Here we give another proof using the Braided Bimonoidal Coherence Theorem 5.4.4.

To prove the unity axiom (8.1.2) and the pentagon axiom (8.1.3), we reuse the proofs of Lemmas I.8.4.2, I.8.4.8, and I.8.4.9, with the following notes and adjustments.

Coherence: These lemmas are proved using Theorem I.3.10.7 for bimonoidal categories. Here we use Theorem 5.4.4, which is our main coherence result for braided bimonoidal categories. To use this coherence result in the braided context, in each case, we check that the paths in question have the same braided distortion in the sense of Definition 5.3.15. In the rest of this proof, we describe the braided distortions of the paths that are used in the proofs of Lemmas I.8.4.2, I.8.4.8, and I.8.4.9. In each case, the (co)domain of the paths in the braided distortion category $\mathcal{D}^{\mathsf{br}}$ is computed using the definitions (5.3.16) of the function ϑ, its extension by additivity and multiplicativity (5.3.12), the sum (5.2.9), and the product (5.2.15) in $\mathcal{D}^{\mathsf{br}}$.

The Left and the Right Zeros: For the path Z^ℓ in (I.8.1.15) for an entry of ζ^ℓ_A in (8.1.11), its braided distortion is the identity morphism

$$1_\varnothing = (\mathrm{id}_0;) \in \Sigma_0$$

of $\varnothing \in \mathcal{D}^{\mathsf{br}}$. Recall from Definition 5.2.8 that the empty sequence $\varnothing$ is the additive zero $\mathbb{0}$ in the braided distortion category $\mathcal{D}^{\mathsf{br}}$.

Similarly, for the path Z^r in (I.8.1.23) for an entry of ζ^r_A in (8.1.11), its braided distortion is the identity morphism

$$1_\varnothing \in \mathcal{D}^{\mathsf{br}}(\varnothing;\varnothing).$$

The Unitors: For the path P_ℓ in (I.8.2.6) for an entry of the left unitor ℓ_A in (8.1.11), its braided distortion is the identity morphism

$$1_{(1)} = (\mathrm{id}_1;\mathrm{id}_1) \in \Sigma_1 \times B_1$$

of $(1) \in \mathcal{D}^{\mathsf{br}}$, which has length 1 and unique entry 1.

Similarly, for the path P_r in (I.8.2.12) for an entry of the right unitor r_A in (8.1.11), its braided distortion is the identity morphism

$$1_{(1)} \in \mathcal{D}^{\mathsf{br}}\big((1);(1)\big).$$

The Associator: Consider the paths in (I.8.3.12) for an entry of the associator a in (8.1.12).

- The path P_a^1 consists of identities and δ^r. Its braided distortion is the identity morphism

$$1_{(3,\ldots,3)} = (\mathrm{id}_{pn}; \overbrace{\mathrm{id}_3, \ldots, \mathrm{id}_3}^{pn}) \in \Sigma_{pn} \times B_3^{\times pn}$$

of the object

$$(3, \ldots, 3) \in \mathcal{D}^{\mathsf{br}}$$

with pn copies of 3. This follows from Lemma 5.2.28, which describes δ^r in $\mathcal{D}^{\mathsf{br}}$, and the fact that $\xi^{\otimes}_{-,1}$ and $\xi^{\otimes}_{1,-}$ are identity permutations.
- The path P_a^2 consists of identities and $\alpha^{\otimes}$. Its braided distortion is the identity morphism of $(3, \ldots, 3) \in \mathcal{D}^{\mathsf{br}}$ because $\alpha^{\otimes}$ in $\mathcal{D}^{\mathsf{br}}$ is the identity.
- The path P_a^4 consists of identities and δ^l. Its braided distortion is the identity morphism of $(3, \ldots, 3) \in \mathcal{D}^{\mathsf{br}}$ because δ^l in $\mathcal{D}^{\mathsf{br}}$ is the identity.
- The path P_a^3 consists of identities, $\alpha^{\pm\oplus}$, and $\xi^{\oplus}$. In $\mathcal{D}^{\mathsf{br}}$, $\alpha^{\oplus}$ is the identity, and $\xi^{\oplus}$ (5.2.11) has identity braid components. The braided distortion of P_a^3 is the morphism

$$\big(\xi^{\otimes}_{p,n}; \overbrace{\mathrm{id}_3, \ldots, \mathrm{id}_3}^{pn}\big) \in \Sigma_{pn} \times B_3^{\times pn}$$

in $\mathcal{D}^{\mathsf{br}}\big((3, \ldots, 3); (3, \ldots, 3)\big)$. Its permutation component $\xi^{\otimes}_{p,n}$ is defined in (5.2.19), and it corresponds to taking the transpose of an $n \times p$ matrix.

The Unity Axiom: In the proof of Lemma I.8.4.2 for the unity axiom, consider the diagram (I.8.4.5).

- Each of the paths R, L, and M, has braided distortion the identity morphism

$$1_{(2,\ldots,2)} = (\mathrm{id}_n; \overbrace{\mathrm{id}_2, \ldots, \mathrm{id}_2}^{n}) \in \Sigma_n \times B_2^{\times n}$$

of the object

$$(2, \ldots, 2) \in \mathcal{D}^{\mathsf{br}}$$

with n copies of 2.
- Due to the presence of δ^X_{lj}, each of the paths P_a^h for $1 \le h \le 4$ also has braided distortion the identity morphism of $(2, \ldots, 2) \in \mathcal{D}^{\mathsf{br}}$.

Therefore, each of the five paths in (I.8.4.5) has identity braided distortion.

The Pentagon Axiom: In the proof of Lemma I.8.4.8, in each of cases (6)–(8), in the second pentagon, each path has braided distortion $1_{\varnothing}$.

In the proof of Lemma I.8.4.9, in the subdivided pentagon (I.8.4.10), each path (both solid and dashed) has identity braid components. The reason is that they do not involve the braiding $\xi^{\otimes}$, which is the only structure morphism in $\mathcal{D}^{\mathsf{br}}$ with nonidentity braid components. See Explanation 5.2.29 (3). In each of the five quadrilaterals, the two composite paths have the same permutation components because there is a unique permuted canonical map between two permuted words of the same length. Therefore, in each of the five quadrilaterals, the two composite paths have the same braided distortion.

With the adjustments described above and the Coherence Theorem 5.4.4, the proofs of Lemmas I.8.4.2, I.8.4.8, and I.8.4.9 imply that $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory. □

8.2. The Monoidal Identity and the Monoidal Composition

We continue to assume that C is a tight braided bimonoidal category as in Definition 2.1.29. In Theorem 8.1.13, we observed that $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory. In the rest of this chapter, we equip $\mathsf{Mat}^{\mathsf{C}}$ with the structure of a monoidal bicategory. In this section, we define the monoidal identity and the monoidal composition in $\mathsf{Mat}^{\mathsf{C}}$. We first recall the definition of a monoidal bicategory from Chapter I.6.

Monoidal Bicategories. Conceptually, monoidal bicategories are to bicategories as monoidal categories are to categories. Monoidal bicategories may be defined as one-object tricategories. The following is the explicit definition of a monoidal bicategory. Denote by $\mathbf{1}$ the bicategory with one object $*$, one 1-cell 1_*, and one 2-cell 1_{1_*}.

Definition 8.2.1. A *monoidal bicategory* is a tuple

$$\big(\mathsf{B}, \boxtimes, 1_{\boxtimes}, a, \ell, r, \pi, \mu, \lambda, \rho\big)$$

consisting of the following data.

- B is a bicategory, which is called the *base bicategory*.
- It is equipped with a pseudofunctor

  $$\mathsf{B} \times \mathsf{B} \xrightarrow{(\boxtimes, \boxtimes^2, \boxtimes^0)} \mathsf{B},$$

  which is called the *monoidal composition*.
- It is equipped with a pseudofunctor

  $$\mathbf{1} \xrightarrow{(1_{\boxtimes}, 1_{\boxtimes}^2, 1_{\boxtimes}^0)} \mathsf{B},$$

  which is called the *monoidal identity*. The object $1_{\boxtimes}(*) \in \mathsf{B}$ is denoted by $1_{\boxtimes}$, which is called the *identity object*.
- It is equipped with an adjoint equivalence $(a, a^{\bullet}, \eta^a, \varepsilon^a)$ with left adjoint

  $$\begin{tikzcd} \mathsf{B}^3 \arrow[r, "\boxtimes \times 1"] \arrow[d, "1 \times \boxtimes"'] \arrow[dr, phantom, "\overset{a}{\swarrow}"] & \mathsf{B}^2 \arrow[d, "\boxtimes"] \\ \mathsf{B}^2 \arrow[r, "\boxtimes"'] & \mathsf{B} \end{tikzcd}$$

  in the bicategory $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B}^3, \mathsf{B})$, which is called the *monoidal associator*. Its left and right adjoints have the following component 1-cells for objects $A, B, C \in \mathsf{B}$.

  $$(C \boxtimes B) \boxtimes A \underset{a^{\bullet}_{C,B,A}}{\overset{a_{C,B,A}}{\rightleftarrows}} C \boxtimes (B \boxtimes A) \in \mathsf{B}$$

- It has adjoint equivalences $(\ell,\ell^\bullet,\eta^\ell,\varepsilon^\ell)$ and $(r,r^\bullet,\eta^r,\varepsilon^r)$ with respective left adjoints

$$\begin{tikzcd} & \mathsf{B}^2 \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\Downarrow\ell"] & \\ \mathsf{B} \arrow[ur, "1_\boxtimes \times 1"] \arrow[rr, "1"'] & {} & \mathsf{B} \end{tikzcd} \qquad \begin{tikzcd} & \mathsf{B}^2 \arrow[dr, "\boxtimes"] \arrow[d, phantom, "\Downarrow r"] & \\ \mathsf{B} \arrow[ur, "1 \times 1_\boxtimes"] \arrow[rr, "1"'] & {} & \mathsf{B} \end{tikzcd}$$

in the bicategory $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B},\mathsf{B})$, which are called the *left monoidal unitor* and the *right monoidal unitor*, respectively. Their left and right adjoints have component 1-cells

$$\begin{tikzcd} 1_\boxtimes \boxtimes A \arrow[r, shift left, "\ell_A"] & A \arrow[l, shift left, "\ell^\bullet_A"] \arrow[r, shift right, "r^\bullet_A"'] & A \boxtimes 1_\boxtimes \arrow[l, shift right, "r_A"'] \end{tikzcd} \in \mathsf{B}.$$

- It has an invertible modification π, which is called the *pentagonator*, with the following component 2-cells for objects $A,B,C,D \in \mathsf{B}$.

$$\begin{tikzcd}[column sep=small] (D \boxtimes (C \boxtimes B)) \boxtimes A \arrow[rr, "a_{D,C\boxtimes B,A}"] & & D \boxtimes ((C \boxtimes B) \boxtimes A) \arrow[d, "1_D \boxtimes a_{C,B,A}"] \\ ((D \boxtimes C) \boxtimes B) \boxtimes A \arrow[u, "a_{D,C,B} \boxtimes 1_A"] \arrow[dr, "a_{D\boxtimes C,B,A}"'] & \Downarrow\pi_{D,C,B,A} & D \boxtimes (C \boxtimes (B \boxtimes A)) \\ & (D \boxtimes C) \boxtimes (B \boxtimes A) \arrow[ur, "a_{D,C,B\boxtimes A}"'] & \end{tikzcd}$$

- It has invertible modifications μ, λ, and ρ, which are called, respectively, the *middle 2-unitor*, the *left 2-unitor*, and the *right 2-unitor*, with the following component 2-cells in B.

$$\begin{tikzcd} (B \boxtimes 1_\boxtimes) \boxtimes A \arrow[r, "a_{B,1_\boxtimes,A}"] & B \boxtimes (1_\boxtimes \boxtimes A) \arrow[d, "1_B \boxtimes \ell_A"] \\ B \boxtimes A \arrow[u, "r^\bullet_B \boxtimes 1_A"] \arrow[r, "1_{B\boxtimes A}"'] \arrow[ur, phantom, "\Downarrow\mu_{B,A}"] & B \boxtimes A \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] (1_\boxtimes \boxtimes B) \boxtimes A \arrow[rr, "\ell_B \boxtimes 1_A"] \arrow[dr, bend right, "a_{1_\boxtimes,B,A}"'] & \Downarrow\lambda_{B,A} & B \boxtimes A \\ & 1_\boxtimes \boxtimes (B \boxtimes A) \arrow[ur, bend right, "\ell_{B\boxtimes A}"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] B \boxtimes A \arrow[rr, "1_B \boxtimes r^\bullet_A"] \arrow[dr, bend right, "r^\bullet_{B\boxtimes A}"'] & \Downarrow\rho_{B,A} & B \boxtimes (A \boxtimes 1_\boxtimes) \\ & (B \boxtimes A) \boxtimes 1_\boxtimes \arrow[ur, bend right, "a_{B,A,1_\boxtimes}"'] & \end{tikzcd}$$

The above data are required to satisfy the following three pasting diagram equalities for objects $A,B,C,D,E \in \mathsf{B}$, with $\boxtimes$ abbreviated to concatenation and iterates of $\boxtimes$ denoted by parentheses.

The Non-Abelian 4-Cocycle Condition:

(8.2.2)

$$
\begin{tikzcd}[column sep=small]
& & (E(D(CB)))A \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow a^{-1}_{1,a,1}"] & & \\
& (E((DC)B))A \arrow[ur, "(1a)1"] \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\pi"] & {} & E((D(CB))A) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow 1\pi"] & \\
((E(DC))B)A \arrow[ur, "a1"] \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow a^{-1}_{a,1,1}"] & {} & E(((DC)B)A) \arrow[ur, "1(a1)"] \arrow[dr, "1a"] & {} & E(D((CB)A)) \arrow[dd, "1(1a)"] \\
{} & (E(DC))(BA) \arrow[rr, "a"] & & E((DC)(BA)) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow\pi"] & \\
(((ED)C)B)A \arrow[uu, "(a1)1"] \arrow[dr, "a"'] & & & {} & E(D(C(BA))) \\
& ((ED)C)(BA) \arrow[u, bend left, "a(1_B1_A)"] \arrow[u, bend right, "a1_{BA}"'] \arrow[u, phantom, "1_a(\boxtimes^{-0}_{B,A}) \Rightarrow"] \arrow[rr, "a"] & & (ED)(C(BA)) \arrow[ur, "a"'] &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}[column sep=small]
& & (E(D(CB)))A \arrow[dr, "a"] & & \\
& (E((DC)B))A \arrow[ur, "(1a)1"] \arrow[d, phantom, "\overset{\pi 1}{\Rrightarrow}"] & & E((D(CB))A) \arrow[dr, "1a"] \arrow[d, phantom, "\Downarrow\pi"] & \\
((E(DC))B)A \arrow[ur, "a1"] & ((ED)(CB))A \arrow[uur, "a1"'] \arrow[drr, "a"] \arrow[dd, phantom, "\Downarrow\pi"] & & {} & E(D((CB)A)) \arrow[dd, "1(1a)"] \\
& & & (ED)((CB)A) \arrow[ur, "a"] \arrow[dd, bend right, "1_{ED}a"'] \arrow[dd, bend left, "(1_E1_D)a"] \arrow[dd, phantom, "(\boxtimes^{-0}_{E,D})1_a \Leftarrow"] \arrow[r, phantom, "\Downarrow a_{1,1,a}"] & {} \\
(((ED)C)B)A \arrow[uu, "(a1)1"] \arrow[uur, "a1"'] \arrow[dr, "a"'] & {} & & & E(D(C(BA))) \\
& ((ED)C)(BA) \arrow[rr, "a"] & & (ED)(C(BA)) \arrow[ur, "a"'] &
\end{tikzcd}
$$

Left Normalization:

(8.2.3)

$$
\begin{tikzcd}[column sep=small]
& (C(1_\boxtimes B))A \arrow[d, "a"] \arrow[ddr, bend left, "(1\ell)1"] & \\
((C1_\boxtimes)B)A \arrow[ur, bend left, "a1"] \arrow[ddr, "a"] \arrow[r, phantom, "\Downarrow\pi"] & C((1_\boxtimes B)A) \arrow[d, "1a"] \arrow[ddr, bend left, "1(\ell 1)"] \arrow[r, phantom, "\Downarrow a^{-1}_{1,\ell,1}"] & (CB)A \arrow[dd, "a"] \\
& C(1_\boxtimes(BA)) \arrow[dr, bend right, "1\ell"'] \arrow[r, phantom, "\Downarrow 1\lambda"] & {} \\
{} & (C1_\boxtimes)(BA) \arrow[u, "a"] & {} \\
(CB)A \arrow[uuu, "(r^\bullet 1)1"] \arrow[dr, bend right, "a"'] \arrow[u, phantom, "\Downarrow a^{-1}_{r^\bullet,1,1}"] & & C(BA) \arrow[u, phantom, "\Downarrow\mu"] \\
& C(BA) \arrow[uu, bend left, "r^\bullet(1_B1_A)"] \arrow[uu, bend right, "r^\bullet 1_{BA}"'] \arrow[uu, phantom, "1_{r^\bullet}\boxtimes^{-0}_{B,A} \Rightarrow"] \arrow[ur, bend right, "1"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=small]
& (C(1_\boxtimes B))A \arrow[dr, bend left, "(1\ell)1"] \arrow[d, phantom, "\Downarrow\mu 1"] & \\
((C1_\boxtimes)B)A \arrow[ur, bend left, "a1"] & {} & (CB)A \arrow[d, "a"] \\
(CB)A \arrow[u, "(r^\bullet 1)1"] \arrow[urr, bend left, "1_{CB}1_A"] \arrow[urr, bend right, "1_{(CB)A}"'] \arrow[urr, phantom, "\Downarrow\boxtimes^{-0}_{CB,A}"] \arrow[rr, "a"] \arrow[dr, bend right, "a"'] & \Downarrow r_a & C(BA) \\
& C(BA) \arrow[ur, bend right, "1"'] \arrow[u, phantom, "\Downarrow\ell_a^{-1}"] &
\end{tikzcd}
$$

Right Normalization:

$$\begin{tikzcd}
& C((B1_{\boxtimes})A) \arrow[dr, bend left, "1a"] & \\
C(BA) \arrow[ur, bend left, "1(r^{\bullet}1)"] \arrow[r, phantom, "\Downarrow a_{1,r^{\bullet},1}"] & (C(B1_{\boxtimes}))A \arrow[u, "a"] \arrow[r, phantom, "\Downarrow \pi"] & C(B(1_{\boxtimes}A)) \arrow[dd, "1(1\ell)"] \\
& ((CB)1_{\boxtimes})A \arrow[u, "a1"] \arrow[d, "a"] & \\
(CB)A \arrow[uu, "a"] \arrow[uur, bend left, "(1r^{\bullet})1"] \arrow[ur, bend right, "r^{\bullet}1"'] \arrow[dr, bend right, "1"'] \arrow[ur, phantom, "\Downarrow \rho 1"] \arrow[dr, phantom, "\Downarrow \mu"] & (CB)(1_{\boxtimes}A) \arrow[uur, "a"] \arrow[d, bend right=50, "1_{CB}\ell"'] \arrow[d, bend left=50, "(1_C 1_B)\ell"] \arrow[d, phantom, "\overset{\boxtimes^{-0}_{C,B}1_{\ell}}{\Leftarrow}"] \arrow[r, phantom, "\Downarrow a_{1,1,\ell}"] & C(BA) \\
& (CB)A \arrow[ur, bend right, "a"'] &
\end{tikzcd} \tag{8.2.4}$$

$$=$$

$$\begin{tikzcd}
& C((B1_{\boxtimes})A) \arrow[dr, bend left, "1a"] \arrow[d, phantom, "\Downarrow 1\mu"] & \\
C(BA) \arrow[ur, bend left, "1(r^{\bullet}1)"] \arrow[drr, bend left, "1_C 1_{BA}"] \arrow[drr, bend right=15, "1_{C(BA)}"'] \arrow[r, phantom, "\Downarrow \boxtimes^{-0}_{C,BA}"] & {} & C(B(1_{\boxtimes}A)) \arrow[d, "1(1\ell)"] \\
(CB)A \arrow[u, "a"] \arrow[rr, "a"'] \arrow[dr, bend right, "1"'] \arrow[r, phantom, "\Downarrow \ell_a"] & {} \arrow[d, phantom, "\Downarrow r_a^{-1}"] & C(BA) \\
& (CB)A \arrow[ur, bend right, "a"'] &
\end{tikzcd}$$

This finishes the definition of a monoidal bicategory. $\diamond$

The Monoidal Identity. For a tight braided bimonoidal category C as in Definition 2.1.29, we now begin to define the structure of a monoidal bicategory on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ in Theorem 8.1.13, starting with the monoidal identity.

Definition 8.2.5. Define the data of a lax functor

$$\mathbf{1} \xrightarrow{(1_{\boxtimes}, 1_{\boxtimes}^2, 1_{\boxtimes}^0)} \mathsf{Mat}^{\mathsf{C}}$$

as follows.

Object: The identity object $1_{\boxtimes}(*)$ is the integer $1 \in \mathsf{Mat}^{\mathsf{C}}$.

1-Cell: The identity 1-cell $1_* \in \mathbf{1}(*, *)$ is sent by $1_{\boxtimes}$ to the 1×1 identity matrix

$$\mathbb{1}^1 = (\mathbb{1}) \in \mathsf{Mat}^{\mathsf{C}}_{1,1}$$

whose only entry is the multiplicative unit $\mathbb{1} \in \mathsf{C}$.

2-Cell: The identity 2-cell $1_{1_*} \in \mathbf{1}(1_*, 1_*)$ is sent by $1_{\boxtimes}$ to the 1×1 matrix

$$(1_{\mathbb{1}}) \in \mathsf{Mat}^{\mathsf{C}}_{1,1}(\mathbb{1}^1, \mathbb{1}^1)$$

whose only entry is the identity morphism

$$1_{\mathbb{1}} : \mathbb{1} \longrightarrow \mathbb{1} \in \mathsf{C}.$$

The Lax Unity Constraint: Define

$$1_{1_{\boxtimes}(*)} = 1_1 = \mathbb{1}^1 = (\mathbb{1}) \xrightarrow{1_{\boxtimes}^0} (\mathbb{1}) = \mathbb{1}^1 = 1_{\boxtimes}(1_*)$$

as the identity 2-cell $(1_{\mathbb{1}})$.

The Lax Functoriality Constraint: Using the matrix product (I.8.1.4), define

$$1_{\boxtimes}(1_*) 1_{\boxtimes}(1_*) = (\mathbb{1})(\mathbb{1}) = \big(\mathbb{1} \otimes \mathbb{1}\big) \xrightarrow{1_{\boxtimes}^2} (\mathbb{1}) = 1_{\boxtimes}(1_*) = 1_{\boxtimes}(1_* 1_*)$$

as the 1×1 matrix whose only entry is the left multiplicative unit

$$\lambda^{\otimes}_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1} \in \mathsf{C}.$$

This finishes the definition of the tuple $(1_{\boxtimes}, 1_{\boxtimes}^2, 1_{\boxtimes}^0)$. $\diamond$

Next is the braided version of Lemma I.8.5.2, whose proof is reused here without any changes.

Lemma 8.2.6. *The tuple*

$$(1_\boxtimes, 1^2_\boxtimes, 1^0_\boxtimes)$$

in Definition 8.2.5 is a strictly unitary pseudofunctor.

The Matrix Tensor Product. To define the monoidal composition $\boxtimes$ in $\mathsf{Mat}^{\mathsf{C}}$, we extend the matrix product to $\mathsf{Mat}^{\mathsf{C}}$. The following is Definition I.8.6.1 applied to a tight braided bimonoidal category C.

Definition 8.2.7. Suppose

$$A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n} \quad \text{and} \quad B = (B_{lk}) \in \mathsf{Mat}^{\mathsf{C}}_{p,q}$$

for some $m, n, p, q \geq 0$.

- For each object $C \in \mathsf{C}$, define the *scalar product*

  $$C \boxtimes A = \big(C \otimes A_{ji}\big) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$$

  as the $n \times m$ matrix obtained from A by replacing each entry A_{ji} by the product $C \otimes A_{ji}$.
- Define the *matrix tensor product*

$$A \boxtimes B = \big(A_{ji} \boxtimes B\big)_{1 \leq j \leq n,\, 1 \leq i \leq m} \in \mathsf{Mat}^{\mathsf{C}}_{mp,nq} \tag{8.2.8}$$

  as the $nq \times mp$ matrix obtained from A by replacing each entry A_{ji} by the scalar product $A_{ji} \boxtimes B$. In other words, it has entries

  $$(A \boxtimes B)_{(j-1)q+l,\,(i-1)p+k} = A_{ji} \otimes B_{lk} \in \mathsf{C}$$

  for $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq k \leq p$, and $1 \leq l \leq q$.

The same notation and terminology apply if C, A, and B are morphisms in, respectively, C, $\mathsf{Mat}^{\mathsf{C}}_{m,n}$, and $\mathsf{Mat}^{\mathsf{C}}_{p,q}$. ◇

Next is the braided version of Lemma I.8.6.7, whose proof is reused here without any changes.

Lemma 8.2.9. *For $m, n, p, q \geq 0$, the matrix tensor product*

$$\mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{p,q} \xrightarrow{\ \boxtimes\ } \mathsf{Mat}^{\mathsf{C}}_{mp,nq}$$

in (8.2.8) is a functor.

The Lax Unity Constraint. The next lemma will be used to define the lax unity constraint of $\boxtimes$. It is the braided version of Lemma I.8.6.8, whose proof is reused here without any changes.

Lemma 8.2.10. *For $m, p \geq 0$, there is a canonical isomorphism*

$$\mathbb{1}^{mp} \xrightarrow[\cong]{\ \boxtimes^0_{(m,p)}\ } \mathbb{1}^m \boxtimes \mathbb{1}^p \in \mathsf{Mat}^{\mathsf{C}}_{mp,mp} \tag{8.2.11}$$

with each entry $\lambda^{-\otimes}_{\mathbb{1}}$, $\rho^{-\bullet}_{\mathbb{1}}$, $\lambda^{-\bullet}_{\mathbb{1}}$, or $\lambda^{-\bullet}_{\mathbb{0}}$ if $m, p > 0$.

The Lax Functoriality Constraint. To define the lax functoriality constraint $\boxtimes^2$ of the monoidal composition in $\mathsf{Mat}^{\mathsf{C}}$, suppose $A = (A_{ji})$, $B = (B_{kj})$, $A' = (A'_{j'i'})$, and $B' = (B'_{k'j'})$ are arbitrary 1-cells in $\mathsf{Mat}^{\mathsf{C}}$ as follows.

$$\begin{gathered} m \xrightarrow{A} n \xrightarrow{B} p \\ m' \xrightarrow{A'} n' \xrightarrow{B'} p' \end{gathered} \tag{8.2.12}$$

To define the lax functoriality constraint $\boxtimes^2_{(B,B'),(A,A')}$ in most cases, we use paths in the sense of Definition 5.3.10 with the following setting.

Definition 8.2.13. In the setting of (8.2.12), suppose $m, n, p, m', n', p' > 0$.

- Define the set of formal variables
$$X^{\boxtimes} = \left\{0^X, 1^X, a_{ji}, b_{kj}, a'_{j'i'}, b'_{k'j'}\right\}$$
with $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$, $1 \le i' \le m'$, $1 \le j' \le n'$, and $1 \le k' \le p'$.
- Define the function $\varphi^{\boxtimes} : X^{\boxtimes} \longrightarrow \mathsf{Ob}(\mathsf{C})$ as follows.
$$\varphi^{\boxtimes}(x) = \begin{cases} \mathbb{0} & \text{if } x = 0^X. \\ \mathbb{1} & \text{if } x = 1^X. \\ A_{ji} & \text{if } x = a_{ji}. \\ B_{kj} & \text{if } x = b_{kj}. \\ A'_{j'i'} & \text{if } x = a'_{j'i'}. \\ B'_{k'j'} & \text{if } x = b'_{k'j'}. \end{cases}$$

Paths in $\mathsf{Gr}(X^{\boxtimes})$ take values in C in the sense of (5.3.14), via the graph morphism $\varphi^{\boxtimes} : \mathsf{Gr}(X^{\boxtimes}) \longrightarrow \mathsf{C}$. ⋄

Recall from Definition 5.3.15 the notion of the *braided distortion* of a path. It is the value of the path in the braided distortion category $\mathcal{D}^{\mathrm{br}}$ under the graph morphism ϑ in (5.3.17). The paths in the next lemma will be used to define most cases of $\boxtimes^2$. Recall that the subscript lt means the left normalized sum in (7.2.1).

Lemma 8.2.14. *In the setting of Definition 8.2.13, the following statements hold.*

(1) There exist paths

$$\begin{tikzcd}[column sep=small] \Big[\bigoplus_{j=1}^{n}\bigoplus_{j'=1}^{n'}\big[(b_{kj}\otimes b'_{k'j'})\otimes(a_{ji}\otimes a'_{j'i'})\big]\Big]_{\mathsf{lt}} \arrow[dr, "P"] & \\ & \Big[\bigoplus_{j=1}^{n}\Big[\bigoplus_{j'=1}^{n'}(b_{kj}\otimes a_{ji})\otimes(b'_{k'j'}\otimes a'_{j'i'})\Big]_{\mathsf{lt}}\Big]_{\mathsf{lt}} \\ \Big[\bigoplus_{j=1}^{n}(b_{kj}\otimes a_{ji})\Big]_{\mathsf{lt}}\otimes\Big[\bigoplus_{j'=1}^{n'}(b'_{k'j'}\otimes a'_{j'i'})\Big]_{\mathsf{lt}} \arrow[ur, "Q"'] & \end{tikzcd}$$

in $\mathsf{Gr}(X^{\boxtimes})$ such that the following statements hold.

- *The braided distortion of P is the morphism*

$$\big(\mathrm{id}_{nn'}; \overbrace{s_2, \ldots, s_2}^{nn'}\big) \in \Sigma_{nn'} \times B_4^{\times nn'} \tag{8.2.15}$$

in

$$\mathcal{D}^{\mathsf{br}}\big((4,\ldots,4);(4,\ldots,4)\big),$$

with $s_2 \in B_4$ *the second braid generator as in Definition 1.1.1. In the object* $(4,\ldots,4)$*, there are* nn' *copies of 4.*

- *The braided distortion of* Q *is the morphism*

$$\Big(\xi^{\otimes}_{n,n'};\overbrace{\mathrm{id}_4,\ldots,\mathrm{id}_4}^{nn'}\Big) \in \Sigma_{nn'} \times B_4^{\times nn'} \tag{8.2.16}$$

in $\mathcal{D}^{\mathsf{br}}\big((4,\ldots,4);(4,\ldots,4)\big)$*, with permutation component* $\xi^{\otimes}_{n,n'}$ *defined in (5.2.19).*

(2) All the paths in $\mathsf{Gr}(X^{\boxtimes})$ *with the same (co)domain and braided distortion as* P*, respectively* Q*, have the same value in* C*.*

Proof. This is the braided analogue of Lemma I.8.6.16, with the additional conditions about braided distortions. The desired paths P and Q with the stated braided distortions are the paths in the proof of Lemma I.8.6.16.

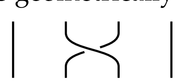
- The path P involves only identities, $\alpha^{\pm\otimes}$, and $\xi^{\otimes}$. In the braided distortion category $\mathcal{D}^{\mathsf{br}}$, $\alpha^{\otimes}$ is the identity. The nn' braid components $s_2 \in B_4$ in P arise from braiding the middle two variables from

$$\big(b_{kj} \otimes b'_{k'j'}\big) \otimes \big(a_{ji} \otimes a'_{j'i'}\big) \quad \text{to} \quad \big(b_{kj} \otimes a_{ji}\big) \otimes \big(b'_{k'j'} \otimes a'_{j'i'}\big).$$

  The braided distortion of the path P is as in (8.2.15) by
  - the definition (5.2.18) of the braiding $\xi^{\otimes}$ in $\mathcal{D}^{\mathsf{br}}$;
  - $\xi^{\otimes}_{1,1} = \mathrm{id}_1 \in \Sigma_1$;
  - $b^{\oplus}_{1,1} = s_1 \in B_2$ as in (1.2.4); and
  - $\mathrm{id}_1 \oplus s_1 \oplus \mathrm{id}_1 = s_2 \in B_4$ as in Definition 1.1.9.
- The path Q involves only identities, δ^l, and δ^r. In the braided distortion category $\mathcal{D}^{\mathsf{br}}$, δ^l is the identity, and δ^r has identity braid components by Lemma 5.2.28. Its permutation component is $\xi^{\otimes}_{n,n'}$, which corresponds to taking the transpose of an $n' \times n$ matrix, by
  - the definitions of the sum (5.2.9) and the product (5.2.15) in $\mathcal{D}^{\mathsf{br}}$ and
  - the fact that there is a unique permuted canonical map between two permuted words of the same length.

The second assertion, which is about the uniqueness of the values in C of P and Q with the prescribed braided distortions, follows from the Coherence Theorem 5.4.4. It is applicable because C is assumed to be tight. □

Explanation 8.2.17. For the path P in Lemma 8.2.14, each braid component $s_2 \in B_4$ in its braided distortion (8.2.15) is geometrically the following braid.



As in Example 1.1.5, geometric braids are read bottom-to-top. So $s_2 \in B_4$ has a right-handed crossing—that is, a crossing with the right string over the left string—involving the second and the third strings. ◇

Next is the monoidal composition in $\mathsf{Mat}^{\mathsf{C}}$ in the sense of Definition 8.2.1.

Definition 8.2.18. For a tight braided bimonoidal category C, define the data of a lax functor

$$\mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}} \xrightarrow{(\boxtimes,\boxtimes^2,\boxtimes^0)} \mathsf{Mat}^{\mathsf{C}}$$

as follows.

Objects: For each pair of objects $(m,p) \in \mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}}$, define the object

$$\boxtimes(m,p) = m \boxtimes p = mp \in \mathsf{Mat}^{\mathsf{C}}.$$

The Local Functors: For $m,n,p,q \geq 0$, the local functor

$$\big(\mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}}\big)\big((m,p),(n,q)\big) = \mathsf{Mat}^{\mathsf{C}}_{m,n} \times \mathsf{Mat}^{\mathsf{C}}_{p,q} \xrightarrow{\boxtimes} \mathsf{Mat}^{\mathsf{C}}_{mp,nq}$$

is defined as the matrix tensor product in Lemma 8.2.9.

The Lax Unity Constraint: For each pair of objects $(m,p) \in \mathsf{Mat}^{\mathsf{C}} \times \mathsf{Mat}^{\mathsf{C}}$, define the component 2-cell

$$1_{m\boxtimes p} = \mathbb{1}^{mp} \xrightarrow{\boxtimes^0_{(m,p)}} \mathbb{1}^m \boxtimes \mathbb{1}^p = \boxtimes(1_{(m,p)}) \in \mathsf{Mat}^{\mathsf{C}}_{mp,mp}$$

as the canonical isomorphism in (8.2.11).

The Lax Functoriality Constraint: In the setting of (8.2.12), the component 2-cell

$$(B \boxtimes B')(A \boxtimes A') \xrightarrow{\boxtimes^2_{(B,B'),(A,A')}} BA \boxtimes B'A' \in \mathsf{Mat}^{\mathsf{C}}_{mm',pp'}$$

is the identity morphism of the empty matrix if $m, m', p,$ or p' is 0. If $m,m',p,p' > 0$, then its $\big((k-1)p'+k',(i-1)m'+i'\big)$-entry is

$$\begin{cases} \lambda^{-\bullet}_{\mathbb{0}} & \text{if } n = n' = 0, \\ \lambda^{-\bullet}_{(B'A')_{k'i'}} & \text{if } n = 0 \text{ and } n' > 0, \\ \rho^{-\bullet}_{(BA)_{ki}} & \text{if } n > 0 \text{ and } n' = 0, \text{ and} \\ (\varphi^{\boxtimes}Q)^{-1}(\varphi^{\boxtimes}P) & \text{if } n,n' > 0. \end{cases}$$

- The indices are $1 \leq i \leq m$, $1 \leq i' \leq m'$, $1 \leq k \leq p$, and $1 \leq k' \leq p'$.
- $\lambda^{-\bullet}$ and $\rho^{-\bullet}$ are the inverses of, respectively, $\lambda^{\bullet}$ and $\rho^{\bullet}$ in C.
- P and Q are the paths in Lemma 8.2.14 with braided distortions, respectively, (8.2.15) and (8.2.16). The symbols $\varphi^{\boxtimes}P$ and $\varphi^{\boxtimes}Q$ are their values in C as in (5.3.14) via the graph morphism $\varphi^{\boxtimes}$ from Definition 8.2.13.

This finishes the definition of the tuple $(\boxtimes,\boxtimes^2,\boxtimes^0)$. ◇

Lemma 8.2.19. *The data*

$$(\boxtimes,\boxtimes^2,\boxtimes^0)$$

in Definition 8.2.18 is a pseudofunctor.

Proof. We reuse the proofs of Lemma I.8.6.21 and the lemmas in Section I.8.7 in the symmetric case, with the following notes and adjustments.

Coherence: As in the proof of Theorem 8.1.13, here we use the Coherence Theorem 5.4.4 for braided bimonoidal categories in place of Theorems I.3.9.1 and I.4.4.3 and Proposition I.3.5.33 for symmetric bimonoidal categories. It remains to check that, in each case, the paths in question have the same braided distortion in the sense of Definition 5.3.15.

Lax Associativity: In (I.8.7.7), (I.8.7.10), (I.8.7.16), and (I.8.7.20), each path has braided distortion the identity morphism $1_\varnothing \in \mathcal{D}^{\mathsf{br}}(\varnothing;\varnothing)$ by the equalities

$$\varnothing \otimes \underline{r} = \varnothing = \varnothing \oplus \varnothing$$

for all $\underline{r} \in \mathcal{D}^{\mathsf{br}}$.

Consider (I.8.7.26).

- Each of the 12 vertices is sent by the graph morphism ϑ in (5.3.17) to the object

$$(6,\ldots,6) \in \mathcal{D}^{\mathsf{br}}$$

with $pp'nn'$ copies of 6.

- The braided distortions of the parallel paths

$$(R, L_6) \quad \text{and} \quad (L_5, L_4, L_3, L_2, L_1)$$

have the same permutation component in $\Sigma_{pp'nn'}$ because there is a unique permuted canonical map between two permuted words of the same length. Each of them has the braid components

$$\big(s_4s_2s_3,\ldots,s_4s_2s_3\big) \in B_6^{\times pp'nn'} \tag{8.2.20}$$

with $pp'nn'$ copies of the braid $s_4s_2s_3 \in B_6$. This braid arises from braiding $cbac'b'a'$ to $cc'bb'aa'$. Therefore, these two parallel paths have the same braided distortion.

- Similarly, the braided distortions of the parallel paths

$$(R, L_{12}) \quad \text{and} \quad (L_{11}, L_{10}, L_9, L_8, L_7)$$

have the same permutation component. Moreover, each of them has the braid components (8.2.20).

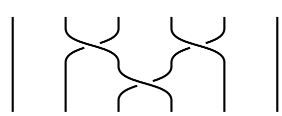
Lax Unity: In (I.8.7.30), each of the five paths has braided distortion the identity morphism

$$1_{(2)} = (\mathrm{id}_1;\mathrm{id}_2) \in \Sigma_1 \times B_2$$

in $\mathcal{D}^{\mathsf{br}}\big((2);(2)\big)$.

Therefore, in each case, the two paths have the same braided distortion, and Theorem 5.4.4 applies. $\square$

Explanation 8.2.21. Each braid $s_4s_2s_3 \in B_6$ in (8.2.20) is geometrically the following braid.



Note that there is an equality

$$s_4s_2s_3 = s_2s_4s_3 \in B_6$$

by the first braid relation (1.1.2). $\diamond$

8.3. The Monoidal Associator and the Monoidal Unitors

With C still assumed to be a tight braided bimonoidal category, in this section we define the monoidal associator and the monoidal unitors in $\mathsf{Mat}^{\mathsf{C}}$. The monoidal associator is a quadruple $(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$ consisting of the following data.

(i) $a^{\boxtimes}$ and $a^{\boxtimes\bullet}$ are strong transformations as in Definition I.6.2.14 as follows.

$$\begin{tikzcd} \boxtimes(\boxtimes \times 1) \arrow[r, shift left, "a^{\boxtimes}"] & \boxtimes(1 \times \boxtimes) \arrow[l, shift left, "a^{\boxtimes\bullet}"] \end{tikzcd}$$

(ii) η^a and ε^a are invertible modifications as in Definition I.6.3.1 as follows.

$$1_{\boxtimes(\boxtimes\times 1)} \xrightarrow{\eta^a} a^{\boxtimes\bullet}a^{\boxtimes}$$
$$a^{\boxtimes}a^{\boxtimes\bullet} \xrightarrow{\varepsilon^a} 1_{\boxtimes(1\times\boxtimes)}$$

Moreover, these data are required to satisfy the triangle identities (I.6.3.10).

The Left Adjoint of the Monoidal Associator. Next is Definition I.8.8.1 for a tight braided bimonoidal category C.

Definition 8.3.1. With respect to the pseudofunctor $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in Lemma 8.2.19, define the data of a lax transformation

$$\boxtimes(\boxtimes \times 1) \xrightarrow{a^{\boxtimes}} \boxtimes(1 \times \boxtimes)$$

as follows.

Component 1-Cells: For each triple of objects $(m,n,p) \in \left(\mathsf{Mat}^{\mathsf{C}}\right)^3$, define

$$mnp = \big((m \boxtimes n) \boxtimes p\big) \xrightarrow{a^{\boxtimes}_{m,n,p}} \big(m \boxtimes (n \boxtimes p)\big) = mnp$$

as the identity matrix $\mathbb{1}^{mnp} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ in (8.1.9).

Component 2-Cells: For each triple of 1-cells

$$\big(A = (A_{i'i}), B = (B_{j'j}), C = (C_{k'k})\big) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'} \times \mathsf{Mat}^{\mathsf{C}}_{p,p'},$$

define the component 2-cell

$$a^{\boxtimes}_{A,B,C} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,m'n'p'}\Big([A \boxtimes (B \boxtimes C)]a^{\boxtimes}_{m,n,p}; a^{\boxtimes}_{m',n',p'}[(A \boxtimes B) \boxtimes C]\Big)$$

as the following vertical composite.

$$\begin{tikzcd}[column sep=large] & mnp \arrow[dr, bend left, "A \boxtimes (B \boxtimes C)"] \arrow[d, phantom, "\Downarrow r_{A \boxtimes (B \boxtimes C)}"] & \\ mnp \arrow[ur, bend left, "\mathbb{1}^{mnp}"] \arrow[rr, bend left=10, "A \boxtimes (B \boxtimes C)"] \arrow[rr, bend right=10, "(A \boxtimes B) \boxtimes C"'] \arrow[rr, phantom, "\Downarrow \alpha^{-\otimes}"] \arrow[dr, bend right, "(A \boxtimes B) \boxtimes C"'] & \arrow[d, phantom, "\Downarrow \ell^{-1}_{(A \boxtimes B) \boxtimes C}"] & m'n'p' \\ & m'n'p' \arrow[ur, bend right, "\mathbb{1}^{m'n'p'}"'] & \end{tikzcd}$$

- ℓ and r are as in (8.1.11).

- $\alpha^{-\otimes}$ is the 2-cell with $((i'-1)n'p'+(j'-1)p'+k',(i-1)np+(j-1)p+k)$-entry the structure morphism in C,

$$A_{i'i}\otimes\big(B_{j'j}\otimes C_{k'k}\big)\xrightarrow{\alpha^{-\otimes}_{A_{i'i},B_{j'j},C_{k'k}}}\big(A_{i'i}\otimes B_{j'j}\big)\otimes C_{k'k}$$

for $1\le i\le m$, $1\le j\le n$, $1\le k\le p$, $1\le i'\le m'$, $1\le j'\le n'$, and $1\le k'\le p'$.

This finishes the definition of $a^{\boxtimes}$. $\diamond$

Lemma 8.3.2. *$a^{\boxtimes}$ in Definition 8.3.1 is a strong transformation.*

Proof. We reuse the proofs of Lemmas I.8.8.5, I.8.8.11, I.8.8.17, and I.8.8.26 in the symmetric case, with the following notes and adjustments.

Coherence: As in the proofs of Theorem 8.1.13 and Lemma 8.2.19, here we use the Coherence Theorem 5.4.4 for braided bimonoidal categories. So we have to check that, in each case, the paths in question have the same braided distortion as in Definition 5.3.15. Moreover, the equality of the permutation components follows from the uniqueness of permuted canonical maps between two permuted words of the same length.

Lax Unity: In (I.8.8.13), each path has braided distortion

$$1_{(0)}\in\mathcal{D}^{\mathsf{br}}\big((0);(0)\big).$$

In (I.8.8.14), each path has braided distortion

$$1_{\varnothing}\in\mathcal{D}^{\mathsf{br}}(\varnothing;\varnothing).$$

Lax Naturality: In (I.8.8.25), each path has braided distortion

$$1_{\varnothing}\in\mathcal{D}^{\mathsf{br}}(\varnothing;\varnothing).$$

Consider (I.8.8.31).

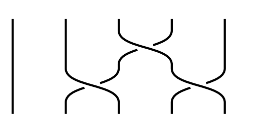
- Each of the six vertices is sent by ϑ to $(6,\ldots,6)\in\mathcal{D}^{\mathsf{br}}$ with $\overline{n}$ copies of 6.
- The braided distortions of the parallel paths F_1 and (G_2,H_1) have braid components

$$\big(s_3s_2s_4,\ldots,s_3s_2s_4\big)\in B_6^{\times\overline{n}}. \tag{8.3.3}$$

  Each braid $s_3s_2s_4\in B_6$ arises from braiding $bab'a'b''a''$ to $bb'b''aa'a''$.
- The braided distortions of the parallel paths G_1 and (F_3,H_2,H_1) also have the braid components (8.3.3).
- Each braid component in the braided distortions of the parallel paths F_2 and (G_3,H_2) is the identity braid $\mathrm{id}_6\in B_6$.

Therefore, in each case, the two paths have the same braided distortion, and Theorem 5.4.4 applies. $\square$

Explanation 8.3.4. Each braid $s_3s_2s_4\in B_6$ in (8.3.3) is geometrically the following braid.



Note that there is an equality

$$s_3 s_2 s_4 = s_3 s_4 s_2 \in B_6$$

by the first braid relation (1.1.2). ◇

The Right Adjoint of the Monoidal Associator.

Definition 8.3.5. With respect to the pseudofunctor $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in Lemma 8.2.19, define the data of a lax transformation

$$\boxtimes(1 \times \boxtimes) \xrightarrow{a^{\boxtimes\bullet}} \boxtimes(\boxtimes \times 1)$$

as follows.

Component 1-Cells: For each triple of objects $(m, n, p) \in (\mathsf{Mat}^{\mathsf{C}})^3$, define

$$\big(m \boxtimes (n \boxtimes p)\big) = mnp \xrightarrow{a^{\boxtimes\bullet}_{m,n,p}} mnp = \big((m \boxtimes n) \boxtimes p\big)$$

as the identity matrix $\mathbb{1}^{mnp} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ in (8.1.9).

Component 2-Cells: For each triple of 1-cells

$$\big(A = (A_{i'i}), B = (B_{j'j}), C = (C_{k'k})\big) \in \mathsf{Mat}^{\mathsf{C}}_{m,m'} \times \mathsf{Mat}^{\mathsf{C}}_{n,n'} \times \mathsf{Mat}^{\mathsf{C}}_{p,p'},$$

define the component 2-cell

$$a^{\boxtimes\bullet}_{A,B,C} \in \mathsf{Mat}^{\mathsf{C}}_{mnp,m'n'p'}\Big([(A \boxtimes B) \boxtimes C]a^{\boxtimes\bullet}_{m,n,p}; a^{\boxtimes\bullet}_{m',n',p'}[A \boxtimes (B \boxtimes C)]\Big)$$

as the following vertical composite.

$$\begin{tikzcd}
 & mnp \arrow[dr, bend left, "(A\boxtimes B)\boxtimes C"] \arrow[d, phantom, "\Downarrow r_{(A\boxtimes B)\boxtimes C}"] & \\
mnp \arrow[ur, bend left, "\mathbb{1}^{mnp}"] \arrow[rr, bend left=10, "(A\boxtimes B)\boxtimes C"] \arrow[rr, bend right=10, "A\boxtimes(B\boxtimes C)"'] \arrow[rr, phantom, "\Downarrow\alpha^{\otimes}"] \arrow[dr, bend right, "A\boxtimes(B\boxtimes C)"'] & \arrow[d, phantom, "\Downarrow \ell^{-1}_{A\boxtimes(B\boxtimes C)}"] & m'n'p' \\
 & m'n'p' \arrow[ur, bend right, "\mathbb{1}^{m'n'p'}"'] &
\end{tikzcd}$$

- ℓ and r are as in (8.1.11).
- $\alpha^{\otimes}$ is the 2-cell with $((i'-1)n'p' + (j'-1)p' + k', (i-1)np + (j-1)p + k)$-entry the structure morphism in C,

$$\big(A_{i'i} \otimes B_{j'j}\big) \otimes C_{k'k} \xrightarrow{\alpha^{\otimes}_{A_{i'i},B_{j'j},C_{k'k}}} A_{i'i} \otimes \big(B_{j'j} \otimes C_{k'k}\big)$$

for $1 \le i \le m$, $1 \le j \le n$, $1 \le k \le p$, $1 \le i' \le m'$, $1 \le j' \le n'$, and $1 \le k' \le p'$.

This finishes the definition of $a^{\boxtimes\bullet}$. ◇

A minor modification of the proof of Lemma 8.3.2 yields the next lemma.

Lemma 8.3.6. *$a^{\boxtimes\bullet}$ in Definition 8.3.5 is a strong transformation.*

The Unit and the Counit of the Monoidal Associator.

Definition 8.3.7. For the strong transformations

$$\boxtimes(\boxtimes \times 1) \underset{a^{\boxtimes\bullet}}{\overset{a^{\boxtimes}}{\rightleftarrows}} \boxtimes(1 \times \boxtimes)$$

in Definitions 8.3.1 and 8.3.5, define the data

$$1_{\boxtimes(\boxtimes\times 1)} \xrightarrow{\eta^a} a^{\boxtimes\bullet}a^{\boxtimes} \qquad a^{\boxtimes}a^{\boxtimes\bullet} \xrightarrow{\varepsilon^a} 1_{\boxtimes(1\times\boxtimes)}$$

as consisting of the component 2-cells

$$\begin{tikzcd}[row sep=large]
\big(1_{\boxtimes(\boxtimes\times 1)}\big)_{(m,n,p)} = \mathbb{1}^{mnp} \arrow[d, "\eta^a_{(m,n,p)} = " ', "\ell^{-1}_{\mathbb{1}^{mnp}}"] \\
(a^{\boxtimes\bullet}a^{\boxtimes})_{(m,n,p)} = \mathbb{1}^{mnp}\mathbb{1}^{mnp}
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=large]
(a^{\boxtimes}a^{\boxtimes\bullet})_{(m,n,p)} = \mathbb{1}^{mnp}\mathbb{1}^{mnp} \arrow[d, "\varepsilon^a_{(m,n,p)} = " ', "\ell_{\mathbb{1}^{mnp}}"] \\
\big(1_{\boxtimes(1\times\boxtimes)}\big)_{(m,n,p)} = \mathbb{1}^{mnp}
\end{tikzcd}$$

in $\mathsf{Mat}^{\mathsf{C}}_{mnp,mnp}$ for each triple of objects $(m,n,p) \in \big(\mathsf{Mat}^{\mathsf{C}}\big)^3$. Here ℓ is as in (8.1.11). ◇

Lemma 8.3.8. *The quadruple*

$$(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$$

in Definitions 8.3.1, 8.3.5, and 8.3.7 is an adjoint equivalence.

Proof. We reuse the proofs of Lemmas I.8.8.39 and I.8.8.45 in the symmetric case, with the following notes and adjustments.

Coherence: As in the proofs of Theorem 8.1.13 and Lemmas 8.2.19 and 8.3.2, here we use the Coherence Theorem 5.4.4 for braided bimonoidal categories. In each case, we check that the paths in question have the same braided distortion as in Definition 5.3.15. Moreover, the equality of the permutation components follows from the uniqueness of permuted canonical maps between two permuted words of the same length.

The Modification Axiom: In (I.8.8.44), each path has braided distortion the identity morphism $1_{(3)} \in \mathcal{D}^{\mathsf{br}}\big((3);(3)\big)$.

The Triangle Identities: In (I.8.8.48), there are two cases.

- If $s \neq t$, then each path has braided distortion

$$1_{\varnothing} \in \mathcal{D}^{\mathsf{br}}(\varnothing;\varnothing).$$

- If $s = t$, then each path has braided distortion

$$1_{(0)} \in \mathcal{D}^{\mathsf{br}}((0);(0)).$$

This finishes the proof. □

The Left Monoidal Unitor. Next we define the left adjoint of the left monoidal unitor.

Definition 8.3.9. Define the data of a lax transformation

$$\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}}) \xrightarrow{\ell^{\boxtimes}} 1_{\mathsf{Mat}^{\mathsf{C}}}$$

as follows.

Component 1-Cells: For each object $m \in \mathsf{Mat}^{\mathsf{C}}$, define

$$m = 1 \boxtimes m \xrightarrow{\ell^{\boxtimes}_m} \left(1_{\mathsf{Mat}^{\mathsf{C}}}\right) m = m$$

as the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ in (8.1.9).

Component 2-Cells: For each 1-cell $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, define

$$\ell^{\boxtimes}_A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}\Big(\left(1_{\mathsf{Mat}^{\mathsf{C}}} A\right) \ell^{\boxtimes}_m ; \ell^{\boxtimes}_n \left(\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})\right)(A)\Big)$$

as the following vertical composite 2-cell.

$$\begin{tikzcd}
& m \arrow[dr, bend left, "A"] \arrow[d, phantom, "\Downarrow r_A", near start] & \\
m \arrow[ur, bend left, "\mathbb{1}^m"] \arrow[rr, bend left=10, "A"] \arrow[rr, bend right=10, "\mathbb{1} \boxtimes A"'] \arrow[rr, phantom, "\Downarrow \lambda^{-\otimes}"] \arrow[dr, bend right, "\mathbb{1} \boxtimes A"'] & & n \\
& n \arrow[ur, bend right, "\mathbb{1}^n"'] \arrow[u, phantom, "\Downarrow \ell^{-1}_{\mathbb{1} \boxtimes A}", near start] &
\end{tikzcd}$$

- By the definition (8.2.8) of the matrix tensor product,

$$\left(\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})\right)(A) = \mathbb{1}^1 \boxtimes A = \mathbb{1} \boxtimes A = \left(\mathbb{1} \otimes A_{ji}\right).$$

- ℓ and r are as in (8.1.11).
- $\lambda^{-\otimes}$ is the 2-cell with (j,i)-entry the structure morphism in C,

$$A_{ji} \xrightarrow{\lambda^{-\otimes}_{A_{ji}}} \mathbb{1} \otimes A_{ji}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

This finishes the definition of $\ell^{\boxtimes}$. $\diamond$

Next we define the right adjoint of the left monoidal unitor.

Definition 8.3.10. Define the data of a lax transformation

$$1_{\mathsf{Mat}^{\mathsf{C}}} \xrightarrow{\ell^{\boxtimes\bullet}} \boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})$$

as follows.

Component 1-Cells: For each object $m \in \mathsf{Mat}^{\mathsf{C}}$, define

$$m = \left(1_{\mathsf{Mat}^{\mathsf{C}}}\right) m \xrightarrow{\ell^{\boxtimes\bullet}_m} 1 \boxtimes m = m$$

as the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ in (8.1.9).

Component 2-Cells: For each 1-cell $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, define

$$\ell^{\boxtimes\bullet}_A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}\Big(\left(\boxtimes(1_{\boxtimes} \times 1_{\mathsf{Mat}^{\mathsf{C}}})\right)(A) \ell^{\boxtimes\bullet}_m ; \ell^{\boxtimes\bullet}_n \left(1_{\mathsf{Mat}^{\mathsf{C}}} A\right)\Big)$$

as the following vertical composite 2-cell.

$$\begin{tikzcd}[column sep=large]
& m \arrow[dr, bend left, "\mathbb{1}\boxtimes A"] & \\
m \arrow[ur, bend left, "\mathbb{1}^m"] \arrow[rr, bend left=15, "\mathbb{1}\boxtimes A"] \arrow[rr, bend right=15, "A"'] \arrow[dr, bend right, "A"'] \arrow[rr, phantom, "\Downarrow\lambda^{\otimes}"] & \Downarrow r_{\mathbb{1}\boxtimes A} \quad \Downarrow\ell_A^{-1} & n \\
& n \arrow[ur, bend right, "\mathbb{1}^n"'] &
\end{tikzcd}$$

In this vertical composite, $\lambda^{\otimes}$ is the 2-cell with (j,i)-entry the structure morphism in C,

$$\mathbb{1}\otimes A_{ji} \xrightarrow{\lambda^{\otimes}_{A_{ji}}} A_{ji}$$

for $1 \le i \le m$ and $1 \le j \le n$.

This finishes the definition of $\ell^{\boxtimes\bullet}$. $\diamond$

Next we define the unit and the counit for $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet})$.

Definition 8.3.11. Define the data

$$1_{\boxtimes(1_{\boxtimes}\times 1_{\mathsf{Mat}^{\mathsf{C}}})} \xrightarrow{\eta^{\ell}} \ell^{\boxtimes\bullet}\ell^{\boxtimes} \qquad \ell^{\boxtimes}\ell^{\boxtimes\bullet} \xrightarrow{\varepsilon^{\ell}} 1_{1_{\mathsf{Mat}^{\mathsf{C}}}}$$

as consisting of the component 2-cells

$$\begin{array}{ccc}
\left(1_{\boxtimes(1_{\boxtimes}\times 1_{\mathsf{Mat}^{\mathsf{C}}})}\right)_m = \mathbb{1}^m & \qquad & (\ell^{\boxtimes}\ell^{\boxtimes\bullet})_m = \mathbb{1}^m\mathbb{1}^m \\
\eta^{\ell}_m = \Big\downarrow \ell^{-1}_{\mathbb{1}^m} & & \varepsilon^{\ell}_m = \Big\downarrow \ell_{\mathbb{1}^m} \\
(\ell^{\boxtimes\bullet}\ell^{\boxtimes})_m = \mathbb{1}^m\mathbb{1}^m & & \left(1_{1_{\mathsf{Mat}^{\mathsf{C}}}}\right)_m = \mathbb{1}^m
\end{array}$$

in $\mathsf{Mat}^{\mathsf{C}}_{m,m}$ for each object $m \in \mathsf{Mat}^{\mathsf{C}}$. Here ℓ is as in (8.1.11). $\diamond$

Lemma 8.3.12. *The quadruple*

$$(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$$

in Definitions 8.3.9 through 8.3.11 is an adjoint equivalence.

Proof. Similar to Lemma I.8.9.9, which is the symmetric case, the proof is adapted from that of Lemma 8.3.8 for the monoidal associator. $\square$

The Right Monoidal Unitor. Next we define the right monoidal unitor in $\mathsf{Mat}^{\mathsf{C}}$, which is similar to the left monoidal unitor.

Definition 8.3.13. Define the data of a lax transformation

$$\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes}) \xrightarrow{r^{\boxtimes}} 1_{\mathsf{Mat}^{\mathsf{C}}}$$

as follows.

Component 1-Cells: For each object $m \in \mathsf{Mat}^{\mathsf{C}}$, define

$$m = m \boxtimes 1 \xrightarrow{r^{\boxtimes}_m} \left(1_{\mathsf{Mat}^{\mathsf{C}}}\right) m = m$$

as the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ in (8.1.9).

Component 2-Cells: For each 1-cell $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, define

$$r^{\boxtimes}_A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}\Big(\big(1_{\mathsf{Mat}^{\mathsf{C}}} A\big)\, r^{\boxtimes}_m ; r^{\boxtimes}_n \left(\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})\right)(A)\Big)$$

as the following vertical composite 2-cell.

$$\begin{tikzcd}
& m \arrow[dr, bend left, "A"] \arrow[d, phantom, "\Downarrow r_A"] & \\
m \arrow[ur, bend left, "\mathbb{1}^m"] \arrow[rr, bend left=10, "A"] \arrow[rr, bend right=10, "A \boxtimes \mathbb{1}"'] \arrow[dr, bend right, "A \boxtimes \mathbb{1}"'] & \Downarrow \rho^{-\otimes} & n \\
& n \arrow[ur, bend right, "\mathbb{1}^n"'] \arrow[u, phantom, "\Downarrow \ell^{-1}_{A \boxtimes \mathbb{1}}"] &
\end{tikzcd}$$

- $A \boxtimes \mathbb{1}$ is the matrix tensor product in (8.2.8) with
$$A \boxtimes \mathbb{1} = \big(A_{ji} \otimes \mathbb{1}\big) = \big(\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})\big)(A).$$
- $\rho^{-\otimes}$ is the 2-cell with (j,i)-entry the structure morphism in C,
$$A_{ji} \xrightarrow{\rho^{-\otimes}_{A_{ji}}} A_{ji} \otimes \mathbb{1}$$
for $1 \le i \le m$ and $1 \le j \le n$.

This finishes the definition of $r^{\boxtimes}$. ◇

Definition 8.3.14. Define the data of a lax transformation

$$1_{\mathsf{Mat}^{\mathsf{C}}} \xrightarrow{r^{\boxtimes\bullet}} \boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})$$

as follows.

Component 1-Cells: For each object $m \in \mathsf{Mat}^{\mathsf{C}}$, define

$$m = \big(1_{\mathsf{Mat}^{\mathsf{C}}}\big)\, m \xrightarrow{r^{\boxtimes\bullet}_m} m \boxtimes 1 = m$$

as the identity matrix $\mathbb{1}^m \in \mathsf{Mat}^{\mathsf{C}}_{m,m}$ in (8.1.9).

Component 2-Cells: For each 1-cell $A = (A_{ji}) \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$, define

$$r^{\boxtimes\bullet}_A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}\Big(\big(\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})\big)(A) r^{\boxtimes\bullet}_m ; r^{\boxtimes\bullet}_n \big(1_{\mathsf{Mat}^{\mathsf{C}}} A\big)\Big)$$

as the following vertical composite 2-cell.

$$\begin{tikzcd}
& m \arrow[dr, bend left, "A \boxtimes \mathbb{1}"] \arrow[d, phantom, "\Downarrow r_{A \boxtimes \mathbb{1}}"] & \\
m \arrow[ur, bend left, "\mathbb{1}^m"] \arrow[rr, bend left=10, "A \boxtimes \mathbb{1}"] \arrow[rr, bend right=10, "A"'] \arrow[dr, bend right, "A"'] & \Downarrow \rho^{\otimes} & n \\
& n \arrow[ur, bend right, "\mathbb{1}^n"'] \arrow[u, phantom, "\Downarrow \ell^{-1}_{A}"] &
\end{tikzcd}$$

In this vertical composite, $\rho^{\otimes}$ is the 2-cell with (j,i)-entry the structure morphism in C,

$$A_{ji} \otimes \mathbb{1} \xrightarrow{\rho^{\otimes}_{A_{ji}}} A_{ji}$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$.

This finishes the definition of $r^{\boxtimes \bullet}$. $\diamond$

Definition 8.3.15. Define the data

$$1_{\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})} \xrightarrow{\eta^r} r^{\boxtimes \bullet} r^{\boxtimes} \qquad r^{\boxtimes} r^{\boxtimes \bullet} \xrightarrow{\varepsilon^r} 1_{1_{\mathsf{Mat}^{\mathsf{C}}}}$$

as consisting of the component 2-cells

$$\begin{tikzcd}
\big(1_{\boxtimes(1_{\mathsf{Mat}^{\mathsf{C}}} \times 1_{\boxtimes})}\big)_m = \mathbb{1}^m \arrow[d, "\ell^{-1}_{\mathbb{1}^m}", "\eta^r_m ="'] \\
\big(r^{\boxtimes \bullet} r^{\boxtimes}\big)_m = \mathbb{1}^m \mathbb{1}^m
\end{tikzcd}
\qquad
\begin{tikzcd}
\big(r^{\boxtimes} r^{\boxtimes \bullet}\big)_m = \mathbb{1}^m \mathbb{1}^m \arrow[d, "\ell_{\mathbb{1}^m}", "\varepsilon^r_m ="'] \\
\big(1_{1_{\mathsf{Mat}^{\mathsf{C}}}}\big)_m = \mathbb{1}^m
\end{tikzcd}$$

in $\mathsf{Mat}^{\mathsf{C}}_{m,m}$ for each object $m \in \mathsf{Mat}^{\mathsf{C}}$. Here ℓ is as in (8.1.11). $\diamond$

Lemma 8.3.16. *The quadruple*

$$(r^{\boxtimes}, r^{\boxtimes \bullet}, \eta^r, \varepsilon^r)$$

in Definitions 8.3.13 through 8.3.15 is an adjoint equivalence.

Proof. Similar to Lemma I.8.9.21, which is the symmetric case, the proof is adapted from that of Lemma 8.3.8 for the monoidal associator. $\square$

8.4. Matrix Monoidal Bicategories

We continue to assume that C is a tight braided bimonoidal category. In this section, we define the remaining structure that makes $\mathsf{Mat}^{\mathsf{C}}$ into a monoidal bicategory, namely, the pentagonator and the three 2-unitors. Then we prove the main Theorem 8.4.7, which states that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory as in Definition 8.2.1. This section ends with a discussion of why $\mathsf{Mat}^{\mathsf{C}}$ is *not* a braided monoidal bicategory in general.

The Pentagonator.

Definition 8.4.1. Define π as consisting of the 2-cells

$$\pi_{m,n,p,q} \in \mathsf{Mat}^{\mathsf{C}}_{s,s} \quad \text{with} \quad m,n,p,q \geq 0 \quad \text{and} \quad s = mnpq$$

given by the composite of the following pasting diagram in $\mathsf{Mat}^{\mathsf{C}}_{s,s}$.

$$\begin{tikzcd}[column sep=small]
& (m \boxtimes (n \boxtimes p)) \boxtimes q \arrow[rr, "a^{\boxtimes}_{m,np,q}"] \arrow[rrrd, "\mathbb{1}^s"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow \ell_{\mathbb{1}^s}"] & & m \boxtimes ((n \boxtimes p) \boxtimes q) \arrow[rd, bend right, "\mathbb{1}^s"'] \arrow[rd, bend left, "\mathbb{1}^m \boxtimes a^{\boxtimes}_{n,p,q}"] \arrow[rd, phantom, "\overset{\boxtimes^{-0}_{m,npq}}{\Leftarrow}"] & \\
((m \boxtimes n) \boxtimes p) \boxtimes q \arrow[ru, bend left, "a^{\boxtimes}_{m,n,p} \boxtimes \mathbb{1}^q"] \arrow[ru, bend right, "\mathbb{1}^s"'] \arrow[ru, phantom, "\overset{\boxtimes^{-0}_{mnp,q}}{\Rightarrow}"] \arrow[rd, "a^{\boxtimes}_{mn,p,q}"'] & & \Downarrow 1_{\mathbb{1}^s \mathbb{1}^s} & & m \boxtimes (n \boxtimes (p \boxtimes q)) \\
& & (m \boxtimes n) \boxtimes (p \boxtimes q) \arrow[rru, "a^{\boxtimes}_{m,n,pq}"'] & &
\end{tikzcd}$$

Here $\boxtimes^{-0}$ is the inverse of $\boxtimes^0$ in (8.2.11), and ℓ is as in (8.1.11). This finishes the definition of π. ◇

Lemma 8.4.2. *In Definition 8.4.1, π is an invertible modification.*

Proof. We reuse the proof of Lemma I.8.10.4 in the symmetric case, with the following notes and adjustments.

Coherence: We use the Coherence Theorem 5.4.4 for braided bimonoidal categories, as in the proofs of Theorem 8.1.13 and Lemmas 8.2.19, 8.3.2, and 8.3.8.

The Modification Axiom: In (I.8.10.8), which is realized as a diagram in $\mathsf{Gr}(X)$ using the variables in (I.8.10.16), each path has braided distortion

$$1_{(4)} \in \mathcal{D}^{\mathsf{br}}\big((4);(4)\big).$$

Therefore, Theorem 5.4.4 applies. □

The 2-Unitors. Next we define the middle 2-unitor, the left 2-unitor, and the right 2-unitor in $\mathsf{Mat}^{\mathsf{C}}$. In the next three definitions, we use $\boxtimes^0$ in (8.2.11), and ℓ and r in (8.1.11).

Definition 8.4.3. Define μ as consisting of the 2-cells

$$\mu_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn} \quad \text{with} \quad m,n \geq 0$$

given by the composite of the following pasting diagram in $\mathsf{Mat}^{\mathsf{C}}_{mn,mn}$.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(m \boxtimes 1) \boxtimes n \arrow[r, bend left=10, "a^{\boxtimes}_{m,1,n}"] \arrow[dr, bend left=15, "\mathbb{1}^m \boxtimes \mathbb{1}^n"] \arrow[dr, bend right=15, "\mathbb{1}^{mn}"'] \arrow[r, phantom, shift right=4ex, "\Downarrow r_{\mathbb{1}^m \boxtimes \mathbb{1}^n}"] \arrow[dr, phantom, "\Downarrow \boxtimes^{-0}_{m,n}"] & m \boxtimes (1 \boxtimes n) \arrow[d, bend left=40, "\mathbb{1}^m \boxtimes \ell^{\boxtimes}_n"] \\
m \boxtimes n \arrow[u, bend left=40, "r^{\boxtimes\bullet}_m \boxtimes \mathbb{1}^n"] \arrow[u, bend right=30, "\mathbb{1}^{mn}"'] \arrow[u, phantom, "\overset{\boxtimes^{-0}_{m,n}}{\Rightarrow}"] \arrow[r, bend right=10, "\mathbb{1}^{m\boxtimes n}"'] \arrow[r, phantom, shift left=4ex, "\Downarrow \ell_{\mathbb{1}^{mn}}"] & m \boxtimes n
\end{tikzcd}$$

This finishes the definition of μ. ◇

Definition 8.4.4. Define $\lambda^{\boxtimes}$ as consisting of the 2-cells

$$\lambda^{\boxtimes}_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn} \quad \text{with} \quad m,n \geq 0$$

given by the composite of the following pasting diagram in $\mathsf{Mat}^{\mathsf{C}}_{mn,mn}$.

$$\begin{tikzcd}[column sep=huge, row sep=large]
(1 \boxtimes m) \boxtimes n \arrow[rr, bend left=15, "\ell^{\boxtimes}_m \boxtimes \mathbb{1}^n"] \arrow[rr, bend right=10, "\mathbb{1}^{mn}"'] \arrow[rr, phantom, "\Downarrow \boxtimes^{-0}_{m,n}"] \arrow[dr, bend right, "a^{\boxtimes}_{1,m,n}"'] & \arrow[d, phantom, "\Downarrow \ell^{-1}_{\mathbb{1}^{mn}}"] & m \boxtimes n \\
& 1 \boxtimes (m \boxtimes n) \arrow[ur, bend right, "\ell^{\boxtimes}_{mn}"'] &
\end{tikzcd}$$

This finishes the definition of $\lambda^{\boxtimes}$. ◇

Definition 8.4.5. Define $\rho^{\boxtimes}$ as consisting of the 2-cells

$$\rho^{\boxtimes}_{m,n} \in \mathsf{Mat}^{\mathsf{C}}_{mn,mn} \quad \text{with} \quad m,n \geq 0$$

given by the composite of the following pasting diagram in $\mathsf{Mat}^{\mathsf{C}}_{mn,mn}$.

$$\begin{tikzcd}
m \boxtimes n \arrow[rr, bend left, "\mathbb{1}^m \boxtimes r_n^{\boxtimes\bullet}"] \arrow[rr, bend right, "\mathbb{1}^{mn}"'] \arrow[rr, phantom, "\Downarrow \boxtimes^{-0}_{m,n}"] \arrow[dr, bend right, "r^{\boxtimes\bullet}_{mn}"'] & & m \boxtimes (n \boxtimes 1) \\
& (m \boxtimes n) \boxtimes 1 \arrow[ur, bend right, "a^{\boxtimes}_{m,n,1}"'] \arrow[u, phantom, "\Downarrow \ell^{-1}_{\mathbb{1}^{mn}}"] &
\end{tikzcd}$$

This finishes the definition of $\rho^{\boxtimes}$. $\diamond$

Lemma 8.4.6. *μ, $\lambda^{\boxtimes}$, and $\rho^{\boxtimes}$ in Definitions 8.4.3 through 8.4.5 are invertible modifications.*

Proof. As in Lemma 8.4.2 for the pentagonator π, we reuse the proofs in the symmetric cases in Lemmas I.8.11.4, I.8.11.9, and I.8.11.14, and use the Coherence Theorem 5.4.4. In steps (1)–(3) in the proof of Lemma I.8.11.4, each path has braided distortion the identity morphism

$$1_{(2)} \in \mathcal{D}^{\mathsf{br}}\big((2);(2)\big).$$

A similar remark applies in the proofs of Lemmas I.8.11.9 and I.8.11.14. Therefore, Theorem 5.4.4 applies. $\square$

The Main Theorem.

Theorem 8.4.7. *For each tight braided bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ equipped with*

- *the bicategory structure in Theorem 8.1.13,*
- *the monoidal identity $(1_{\boxtimes}, 1^2_{\boxtimes}, 1^0_{\boxtimes})$ in Lemma 8.2.6,*
- *the monoidal composition $(\boxtimes, \boxtimes^2, \boxtimes^0)$ in Lemma 8.2.19,*
- *the monoidal associator $(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$ in Lemma 8.3.8,*
- *the left monoidal unitor $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$ in Lemma 8.3.12,*
- *the right monoidal unitor $(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^r, \varepsilon^r)$ in Lemma 8.3.16,*
- *the pentagonator π in Lemma 8.4.2, and*
- *the 2-unitors μ, $\lambda^{\boxtimes}$, and $\rho^{\boxtimes}$ in Lemma 8.4.6*

is a monoidal bicategory.

Proof. It remains to check the three axioms (8.2.2), (8.2.3), and (8.2.4) in Definition 8.2.1. We reuse the proofs of Lemmas I.8.12.1, I.8.12.4, and I.8.12.7 in the symmetric case, with the following notes and adjustments.

Coherence: We use the Coherence Theorem 5.4.4 for braided bimonoidal categories, as in the earlier proofs in this chapter.

The Non-Abelian 4-Cocycle Condition: In steps (1)–(5) in Lemma I.8.12.1, each path has braided distortion
- $1_{(0)} \in \mathcal{D}^{\mathsf{br}}\big((0);(0)\big)$ if $i = j$ and
- $1_{\varnothing} \in \mathcal{D}^{\mathsf{br}}(\varnothing;\varnothing)$ if $i \neq j$.

Left and Right Normalizations: The same remark applies in the proofs of Lemmas I.8.12.4 and I.8.12.7.

This finishes the proof. $\square$

Not a Braided Monoidal Bicategory. In Theorem I.8.15.4, we proved that, for each tight symmetric bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ equipped with

- the braiding $(\beta, \beta^{\bullet}, \eta^{\beta}, \varepsilon^{\beta})$ in Section I.8.13,
- the left hexagonator $R_{-|--}$ in Definition I.8.14.9,
- the right hexagonator $R_{--|-}$ in Definition I.8.14.21, and
- the syllepsis ν in Definition I.8.15.1

is a symmetric monoidal bicategory. Moreover, for each tight braided bimonoidal category C, Theorem 8.4.7 states that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory. A natural question is whether $\mathsf{Mat}^{\mathsf{C}}$, when equipped with the braiding and the hexagonators above, is a *braided* monoidal bicategory in the sense of Definition I.6.5.3. The answer is no in general. To explain this negative answer, let us first recall from Definition I.8.13.22 the braiding β in $\mathsf{Mat}^{\mathsf{C}}$ in the symmetric case.

Definition 8.4.8. Define the data of a lax transformation

$$\boxtimes \xrightarrow{\ \beta\ } \boxtimes\tau$$

as follows, in which

$$\tau : \left(\mathsf{Mat}^{\mathsf{C}}\right)^2 \longrightarrow \left(\mathsf{Mat}^{\mathsf{C}}\right)^2$$

switches the two arguments.

Component 1-Cells: For each pair of objects $(m,n) \in \left(\mathsf{Mat}^{\mathsf{C}}\right)^2$, define

$$mn = m \boxtimes n \xrightarrow{\ \beta_{m,n}\ } n \boxtimes m = nm$$

as the permutation matrix

$$\beta_{m,n} = \mathbb{1}^{\tau_{m,n}} \in \mathsf{Mat}^{\mathsf{C}}_{mn,nm}.$$

It is obtained from the identity matrix $\mathbb{1}^{mn}$ in (8.1.9) by permuting its columns by the permutation $\xi^{\otimes}_{m,n}$ in (5.2.19).

Component 2-Cells: For 1-cells $A \in \mathsf{Mat}^{\mathsf{C}}_{m,m'}$ and $B \in \mathsf{Mat}^{\mathsf{C}}_{n,n'}$, define the component 2-cell

$$\beta_{A,B} \in \mathsf{Mat}^{\mathsf{C}}_{mn,n'm'}\big((B \boxtimes A)\beta_{m,n};\, \beta_{m',n'}(A \boxtimes B)\big)$$

as the following vertical composite.

$$\begin{tikzcd}
& nm \arrow[dr, bend left, "B \boxtimes A"] \arrow[d, phantom, "\Downarrow r^{\tau_{m,n}}_{B\boxtimes A}"] & \\
mn \arrow[ur, bend left, "\beta_{m,n}"] \arrow[rr, bend left=10, "(B \boxtimes A)^{\tau_{m,n}}"] \arrow[rr, bend right=10, "{}_{\tau_{n',m'}}(A \boxtimes B)"'] \arrow[rr, phantom, "\Downarrow \xi^{\otimes}"] \arrow[dr, bend right, "A \boxtimes B"'] & \arrow[d, phantom, "\Downarrow (\ell^{\tau_{m',n'}}_{A\boxtimes B})^{-1}"] & n'm' \\
& m'n' \arrow[ur, bend right, "\beta_{m',n'}"'] &
\end{tikzcd}$$

- $(B \boxtimes A)^{\tau_{m,n}}$ is obtained from $B \boxtimes A$ by permuting its columns by the permutation $\xi^{\otimes}_{m,n}$.
- $r^{\tau_{m,n}}_{B\boxtimes A}$ is the natural isomorphism in (I.8.13.13). It consists of identities, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\rho^{\otimes}$, and $\rho^{\bullet}$.
- ${}_{\tau_{n',m'}}(A \boxtimes B)$ is obtained from $A \boxtimes B$ by permuting its rows by the permutation $\xi^{\otimes}_{n',m'}$.

- $\ell^{\tau_{m',n'}}_{A\boxtimes B}$ is the natural isomorphism in (I.8.13.16). It consists of identities, $\lambda^{\oplus}$, $\rho^{\oplus}$, $\lambda^{\otimes}$, and $\lambda^{\bullet}$.
- $\xi^{\otimes}$ is the 2-cell with $\big(i' + (j'-1)m', j + (i-1)n\big)$-entry the braiding

$$B_{j'j} \otimes A_{i'i} \xrightarrow{\xi^{\otimes}_{B_{j'j}, A_{i'i}}} A_{i'i} \otimes B_{j'j}$$

in C for $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq i' \leq m'$, and $1 \leq j' \leq n'$.

This finishes the definition of β. $\diamond$

When C is a tight symmetric bimonoidal category, Lemma I.8.13.27 states that $\beta : \boxtimes \longrightarrow \boxtimes\tau$ is a strong transformation. However, when C is a tight braided bimonoidal category, β is *not* in general a strong transformation because it does not satisfy the lax naturality axiom (I.6.2.16), which is the pasting diagram equality (I.8.13.29). Indeed, for this pasting diagram equality to hold in general (that is, for an arbitrary tight braided bimonoidal category C), it would have to hold for the braided distortion category $\mathcal{D}^{\mathsf{br}}$, which is a tight braided bimonoidal category by Corollary 5.2.33. In other words, after representing a typical entry of each side as the value in C of a path, the two resulting paths in $\mathsf{Gr}(X)$ should have the same braided distortion. However, in the process of checking that their braid components are equal, we would need the following two braids in B_4 to be equal.

Although these two braids in B_4 have the same underlying permutation, they are not equal as braids. Indeed, the second and the third strings from the bottom wrap around each other on the left side, but not on the right side.

Moreover, changing either

- β in such a way that its typical component 2-cell involves $\xi^{-\otimes}$ instead of $\xi^{\otimes}$, or
- $\boxtimes^2$ in such a way that each braid component in (8.2.15) involves $s_2^{-1} \in B_4$ instead of s_2,

or both, would lead to similar problems of trying to equate braids that are not equal.

Furthermore, for the unit

$$\eta^{\beta} : 1_{\boxtimes} \longrightarrow \beta^{\bullet}\beta$$

in Definition I.8.13.35, the modification axiom (I.6.3.2) is the pasting diagram equality (I.8.13.42). After representing a typical entry of each side as the value in C of a path, we would need to show that the two resulting paths have the same braided distortion. If these braided distortions were equal, then their braid components would yield the equality

$$\mathrm{id}_2 = s_1 s_1 \in B_2,$$

which is false. Therefore, for a tight braided bimonoidal category C, η^{β} is *not* a modification in general.

In summary, for a tight braided bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory, but not in general a braided monoidal bicategory with the quadruple $(\beta, \beta^{\bullet}, \eta^{\beta}, \varepsilon^{\beta})$ in Section I.8.13.

Part 2

E_n-Monoidal Categories

CHAPTER 9

Ring, Bipermutative, and Braided Ring Categories

In this chapter, we discuss another categorification of a rig, called a *ring category*, and their permutative and braided analogues, called, respectively, a *bipermutative category* and a *braided ring category*. They are different from their bimonoidal counterparts in several ways.

Strict Additive and Multiplicative Structures: Like a bimonoidal category, a ring category C also has a symmetric monoidal structure $(\oplus, \mathbb{0}, \xi^{\oplus})$ and another monoidal structure $(\otimes, \mathbb{1})$. These two monoidal structures in a ring category are assumed to be *strict* monoidal from the start.

Strict Multiplicative Zeros: The left multiplicative zero $\lambda^{\bullet}$ and the right multiplicative zero $\rho^{\bullet}$ in a ring category are assumed to be the identity natural transformations. This is called the multiplicative zero axiom (9.1.4).

Factorizations: More significantly, instead of distributivity morphisms δ^l and δ^r as in a bimonoidal category, a ring category has *factorization morphisms*

$$(A \otimes C) \oplus (B \otimes C) \xrightarrow{\partial^l_{A,B,C}} (A \oplus B) \otimes C$$
$$(A \otimes B) \oplus (A \otimes C) \xrightarrow{\partial^r_{A,B,C}} A \otimes (B \oplus C)$$

that go in the opposite direction as δ^r and δ^l. These are natural transformations, but not natural isomorphisms in general.

Due to the differences between factorization morphisms and distributivity morphisms, a ring category is in general not a bimonoidal category, or vice versa. We will observe that *tight* ring categories, that is, those with invertible factorization morphisms, form a subclass of tight bimonoidal categories. Therefore, the strictification theorems for tight bimonoidal categories also apply to tight ring categories. There are similar results for the bipermutative and the braided cases. These subclass inclusions and strictification results are summarized in the following table.

tight – categories	are special cases of tight – categories	strictify to right/left – categories
ring	bimonoidal (9.1.15)	rigid bimonoidal (9.1.20)
bipermutative	symmetric bimonoidal (9.3.7)	bipermutative (9.3.13)
braided ring	braided bimonoidal (9.5.6)	permbraided (9.5.11)

For open questions related to coherence of ring, bipermutative, and braided ring categories, see Question III.A.2.1.

Connection with Algebraic K-Theory. Ring categories, bipermutative categories, and braided ring categories will play important roles in our discussion of algebraic K-theory in Part III.2. The ring and bipermutative categories in this chapter were introduced by Elmendorf-Mandell **[EM06, EM09]** in the context of multiplicative infinite loop space theory and algebraic K-theory. As proved in **[EM06, EM09]**, the Elmendorf-Mandell K-theory multifunctor sends

- a small ring category to a strict ring symmetric spectrum and
- a small bipermutative category to an E_∞-symmetric spectrum.

In Part III.2, we will discuss in detail the Segal K-theory functor and the Elmendorf-Mandell K-theory multifunctor and extend the above statements to the E_2 case. So the K-theory multifunctor sends a small braided ring category to an E_2-symmetric spectrum.

Here is a conceptual way to understand the connection between braided ring categories and E_2-symmetric spectra. A central fact in algebraic K-theory is that the Segal K-theory, which is equivalent to the Elmendorf-Mandell K-theory, of a small permutative category is a symmetric spectrum. To have an E_2 structure on the K-theory symmetric spectrum, one should start with a small permutative category with an E_2 structure. By the work of Boardman-Vogt [**BV73**] and May [**May72**], E_2 structure corresponds to double loop spaces. Their categorical counterparts are braided monoidal categories by the work of Fiedorowicz [**Fie**∞, Theorem 2]. Therefore, a small permutative category with a compatible braided monoidal structure should yield an E_2-symmetric spectrum via the K-theory multifunctor. The precise categorical notion is a braided ring category.

Moreover, the general E_n analogues for $1 < n < \infty$ are also true. They involve E_n-monoidal categories that we will discuss in Chapter 10. We will show that the K-theory multifunctor sends a small E_n-monoidal category to an E_n-symmetric spectrum.

Organization. A summary of the rest of this chapter follows.

Section 9.1 introduces ring categories in the sense of Elmendorf-Mandell and discusses their relationship with bimonoidal categories. Theorem 9.1.15 identifies tight ring categories, which have invertible factorization morphisms ∂^l and ∂^r, with a subcalss of tight bimonoidal categories. As a result, right and left rigid bimonoidal categories (Definition I.5.5.8) are tight ring categories with, respectively, $\partial^l = 1$ and $\partial^r = 1$. Moreover, tight ring categories can be strictified to adjoint equivalent right or left rigid bimonoidal categories by the Rigid Strictification Theorems I.5.5.11 and I.5.5.12.

Section 9.2 observes that each small permutative category C yields an endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ with

- as its objects strictly unital symmetric monoidal functors $F : \mathsf{C} \longrightarrow \mathsf{C}$ and
- as its morphisms monoidal natural transformations.

The *strictly unital* condition means that the unit constraint

$$F^0 : \mathbb{0} \longrightarrow F\mathbb{0}$$

is the identity; see (9.2.4). The existence of the endomorphism ring category is analogous to the fact that each commutative monoid A has an endomorphism rig $\mathsf{End}(A)$ of monoid endomorphisms $A \longrightarrow A$. The most nontrivial part of the proof that $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ is a ring category involves checking that the right factorization morphism ∂^r is entrywise a monoidal natural transformation; see (9.2.16). The endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ is, in general, not tight because ∂^r is usually not invertible. A tight version $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ is obtained in Theorem 9.2.20 by restricting to strictly unital *strong* symmetric monoidal functors $F : \mathsf{C} \longrightarrow \mathsf{C}$.

The strong condition means that, in addition to $F^0 = 1_{\mathbb{0}}$, the monoidal constraint F^2 is a natural isomorphism.

Section 9.3 introduces bipermutative categories in the sense of Elmendorf-Mandell and discusses their relationship with symmetric bimonoidal categories. In addition to an underlying ring category, a bipermutative category C is equipped with a multiplicative symmetry

$$\xi^{\otimes}_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$$

that makes the multiplicative structure into a permutative category, along with two axioms involving $\xi^{\otimes}$. Theorem 9.3.7 identifies tight bipermutative categories with a subclass of tight symmetric bimonoidal categories. As a result, right and left bipermutative categories (Definitions I.2.5.2 and I.2.5.11) are tight bipermutative categories with, respectively, $\partial^l = 1$ and $\partial^r = 1$. Moreover, tight bipermutative categories can be strictified to adjoint equivalent right or left bipermutative categories by the Bipermutative Strictification Theorems I.5.4.6 and I.5.4.7.

Section 9.4 shows that, in a bipermutative category, about half of the ring category axioms are redundant. This is analogous to the fact, established in Section I.2.2, that half of the 24 Laplaza axioms in a symmetric bimonoidal category are redundant. In a ring category (Definition 9.1.2), most of the axioms have both a left version for ∂^l and a right version for ∂^r. In a bipermutative category, using the multiplicative symmetry $\xi^{\otimes}$ and the axiom (9.3.4) relating ∂^l and ∂^r, some of the ring category axioms for ∂^l are equivalent to the ∂^r versions. One of these redundancies was already observed by Elmendorf-Mandell; see Note 9.7.4.

Section 9.5 introduces the braided analogue of a ring category, in which the multiplicative structure is now a braided strict monoidal category. A bipermutative category is precisely a braided ring category whose braiding satisfies the symmetry axiom $\xi^{\otimes}\xi^{\otimes} = 1$. Like in a bipermutative category, Theorem 9.5.5 shows that, in a braided ring category, about half of the ring category axioms are redundant. Theorem 9.5.6 identifies tight braided ring categories with a subclass of tight braided bimonoidal categories. As a result, left and right permbraided categories (Definitions 5.1.2 and 5.1.11) are tight braided ring categories with, respectively, $\partial^r = 1$ and $\partial^l = 1$. Moreover, tight braided ring categories can be strictified to adjoint equivalent right or left permbraided categories by the Permbraided Strictification Theorems 6.3.6 and 6.3.7.

Section 9.6 extends the Drinfeld center (Theorem 1.4.27) and the symmetric center (Proposition 1.5.3) for monoidal categories to the setting of ring categories. Corollary 9.6.1 shows that the bimonoidal Drinfeld center (Theorem 4.4.3) of each tight ring category is a tight braided ring category. Theorem 9.6.4 shows that for a braided ring category whose left factorization morphism ∂^l is a natural epimorphism, the symmetric center is a bipermutative category.

Section 9.7 discusses some differences between **[EM06]** and Sections 9.1 through 9.4. Note 9.7.5 points out that our braided ring categories are more general than Richter's braided bimonoidal categories in [**Ric10**, Def. 5.1].

Reading Guide.

(1) For ring categories, read Definition 9.1.2 and the statements of Theorem 9.1.15 and Corollaries 9.1.19 and 9.1.20.

(2) For bipermutative categories, read Definition 9.3.2 and the statements of Theorem 9.3.7 and Corollaries 9.3.12 and 9.3.13.
(3) For braided ring categories, read Definition 9.5.1 and the statements of Proposition 9.5.4, Theorems 9.5.5 and 9.5.6, and Corollaries 9.5.10 and 9.5.11.
(4) Go back and read the rest of this chapter.

9.1. Ring Categories

In this section, we define ring categories in the sense of Elmendorf-Mandell [**EM06**, Def. 3.3] and discuss their relationship with bimonoidal categories in Definition 2.1.1. A ring category has a symmetric strict monoidal structure $(\oplus, \mathbb{0})$, a multiplicative strict monoidal structure $(\otimes, \mathbb{1})$, and strict multiplicative zeros. Moreover, a ring category has *factorization* morphisms that go in the opposite direction as the distributivity morphisms in a bimonoidal category. Therefore, a ring category is, in general, not a bimonoidal category, or vice versa.

The main observation in this section is Theorem 9.1.15. It identifies *tight* ring categories—those with invertible factorization morphisms—with the subclass of tight bimonoidal categories with

- strict additive and multiplicative structures and
- identities for the left and right multiplicative zeros.

As a result, both right and left rigid bimonoidal categories (Definition I.5.5.8) are tight ring categories with identity as one factorization morphism; see Corollary 9.1.19. Moreover, the Rigid Strictification Theorems I.5.5.11 and I.5.5.12 apply to tight ring categories. Therefore, each tight ring category is adjoint equivalent to one in which the left, or the right, factorization morphism is the identity; see Corollary 9.1.20.

Motivation 9.1.1. In a bimonoidal category, the distributivity morphisms δ^l and δ^r in (2.1.3) are categorifications of the distributive properties

$$x(y+z) = xy + xz$$
$$(x+y)z = xz + yz$$

in a rig. In a ring category, which we will define shortly, we read these equalities from right to left. In other words, we factor x out in the first equality and z out in the second equality. Therefore, the corresponding categorified natural transformations in (9.1.3) are called factorization morphisms. $\diamond$

Definition. Recall from Definitions 1.3.1 and 1.3.32 strict monoidal and permutative categories. In a strict monoidal category, the associativity isomorphism α, the left unit isomorphism λ, and the right unit isomorphism ρ are identity natural transformations, so we omit them from the notation. Also recall that $*$ denotes the terminal category, with only one object and its identity morphism

Definition 9.1.2. A *ring category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$

consisting of the following data.

The Additive Structure: $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ is a permutative category, with $\oplus$, $\mathbb{0}$, and $\xi^{\oplus}$ called, respectively, the *sum*, the *additive zero*, and the *additive symmetry*.

The Multiplicative Structure: $(\mathsf{C}, \otimes, \mathbb{1})$ is a strict monoidal category, with $\otimes$ and $\mathbb{1}$ called, respectively, the *product* and the *multiplicative unit*.

The Factorization Morphisms: ∂^l and ∂^r are natural transformations

$$\begin{aligned} (A \otimes C) \oplus (B \otimes C) &\xrightarrow{\partial^l_{A,B,C}} (A \oplus B) \otimes C \\ (A \otimes B) \oplus (A \otimes C) &\xrightarrow{\partial^r_{A,B,C}} A \otimes (B \oplus C) \end{aligned} \tag{9.1.3}$$

for objects $A, B, C \in \mathsf{C}$, which are called the *left factorization morphism* and the *right factorization morphism*, respectively.

To simplify the presentation, we often abbreviate $\otimes$ to concatenation, with $\otimes$ always taking precedence over $\oplus$ in the absence of clarifying parentheses. For example, the left factorization morphism is abbreviated to

$$AC \oplus BC \longrightarrow (A \oplus B)C.$$

The subscripts in $\xi^{\oplus}$, ∂^l, and ∂^r are sometimes omitted.

The above data are required to satisfy the following seven axioms for all objects A, A', A'', B, B', B'', C, and C' in C.

The Multiplicative Zero Axiom: The diagram of functors

$$\begin{tikzcd} * \times \mathsf{C} \arrow[r, "\cong"] \arrow[d, "\mathbb{0} \times 1_{\mathsf{C}}"'] & \mathsf{C} \arrow[d, "\mathbb{0}"] & \mathsf{C} \times * \arrow[l, "\cong"'] \arrow[d, "1_{\mathsf{C}} \times \mathbb{0}"] \\ \mathsf{C} \times \mathsf{C} \arrow[r, "\otimes"] & \mathsf{C} & \mathsf{C} \times \mathsf{C} \arrow[l, "\otimes"'] \end{tikzcd} \tag{9.1.4}$$

is commutative. In this diagram, the top horizontal isomorphisms drop the $*$ argument. Each $\mathbb{0}$ denotes the constant functor at $\mathbb{0} \in \mathsf{C}$ and $1_{\mathbb{0}}$.

The Zero Factorization Axiom:

$$\begin{aligned} \partial^l_{\mathbb{0},B,C} &= 1_{B \otimes C} & \partial^r_{\mathbb{0},B,C} &= 1_{\mathbb{0}} \\ \partial^l_{A,\mathbb{0},C} &= 1_{A \otimes C} & \partial^r_{A,\mathbb{0},C} &= 1_{A \otimes C} \\ \partial^l_{A,B,\mathbb{0}} &= 1_{\mathbb{0}} & \partial^r_{A,B,\mathbb{0}} &= 1_{A \otimes B} \end{aligned} \tag{9.1.5}$$

The three equalities for ∂^l are called the *left zero factorization axioms*. The three equalities for ∂^r are called the *right zero factorization axioms*.

The Unit Factorization Axiom:

$$\begin{aligned} \partial^l_{A,B,\mathbb{1}} &= 1_{A \oplus B} \\ \partial^r_{\mathbb{1},B,C} &= 1_{B \oplus C} \end{aligned} \tag{9.1.6}$$

These are called, respectively, the *left* and the *right* unit factorization axioms.

The Symmetry Factorization Axiom: The following two diagrams in C are commutative.

$$\begin{tikzcd} AC \oplus BC \arrow[r, "\partial^l"] \arrow[d, "\xi^{\oplus}"'] & (A \oplus B)C \arrow[d, "\xi^{\oplus} 1_C"] \\ BC \oplus AC \arrow[r, "\partial^l"] & (B \oplus A)C \end{tikzcd} \qquad \begin{tikzcd} AB \oplus AC \arrow[r, "\partial^r"] \arrow[d, "\xi^{\oplus}"'] & A(B \oplus C) \arrow[d, "1_A \xi^{\oplus}"] \\ AC \oplus AB \arrow[r, "\partial^r"] & A(C \oplus B) \end{tikzcd} \tag{9.1.7}$$

These are called, respectively, the *left* and the *right* symmetry factorization axioms.

The Internal Factorization Axiom: The following two diagrams in C are commutative.

(9.1.8)
$$\begin{tikzcd}
AB \oplus A'B \oplus A''B \arrow[r, "\partial^l \oplus 1"] \arrow[d, "1 \oplus \partial^l"'] & (A \oplus A')B \oplus A''B \arrow[d, "\partial^l"] \\
AB \oplus (A' \oplus A'')B \arrow[r, "\partial^l"] & (A \oplus A' \oplus A'')B
\end{tikzcd}
\qquad
\begin{tikzcd}
AB \oplus AB' \oplus AB'' \arrow[r, "\partial^r \oplus 1"] \arrow[d, "1 \oplus \partial^r"'] & A(B \oplus B') \oplus AB'' \arrow[d, "\partial^r"] \\
AB \oplus A(B' \oplus B'') \arrow[r, "\partial^r"] & A(B \oplus B' \oplus B'')
\end{tikzcd}$$

These are called, respectively, the *left* and the *right* internal factorization axioms.

The External Factorization Axiom: The three diagrams in C below are commutative.

$$\begin{tikzcd}
ABC \oplus A'BC \arrow[r, "\partial^l_{A,A',BC}"] \arrow[d, "\partial^l_{AB,A'B,C}"'] & (A \oplus A')BC \arrow[d, equal] \\
(AB \oplus A'B)C \arrow[r, "\partial^l_{A,A',B} 1_C"] & (A \oplus A')BC
\end{tikzcd} \tag{9.1.9}$$

$$\begin{tikzcd}
ABC \oplus AB'C \arrow[r, "\partial^l_{AB,AB',C}"] \arrow[d, "\partial^r_{A,BC,B'C}"'] & (AB \oplus AB')C \arrow[d, "\partial^r 1_C"] \\
A(BC \oplus B'C) \arrow[r, "1_A \partial^l_{B,B',C}"] & A(B \oplus B')C
\end{tikzcd} \tag{9.1.10}$$

$$\begin{tikzcd}
ABC \oplus ABC' \arrow[r, "\partial^r_{AB,C,C'}"] \arrow[d, "\partial^r_{A,BC,BC'}"'] & AB(C \oplus C') \arrow[d, equal] \\
A(BC \oplus BC') \arrow[r, "1_A \partial^r_{B,C,C'}"] & AB(C \oplus C')
\end{tikzcd} \tag{9.1.11}$$

These are called, respectively, the *left*, the *middle*, and the *right* external factorization axioms.

The 2-By-2 Factorization Axiom: The following diagram in C is commutative.

$$\begin{tikzcd}[column sep=small]
& A(B \oplus B') \oplus A'(B \oplus B') \arrow[dr, "\partial^l"] & \\
AB \oplus AB' \oplus A'B \oplus A'B' \arrow[ur, "\partial^r \oplus \partial^r"] \arrow[d, "1 \oplus \xi^{\oplus} \oplus 1"'] & & (A \oplus A')(B \oplus B') \\
AB \oplus A'B \oplus AB' \oplus A'B' \arrow[dr, "\partial^l \oplus \partial^l"'] & & \\
& (A \oplus A')B \oplus (A \oplus A')B' \arrow[uur, "\partial^r"'] &
\end{tikzcd} \tag{9.1.12}$$

This finishes the definition of a ring category.

Moreover, a ring category as above is said to be

- *small* if it has a set of objects and
- *tight* if ∂^l and ∂^r in (9.1.3) are natural isomorphisms. ◇

Explanation 9.1.13. Consider Definition 9.1.2 of a ring category.

(1) The factorization morphisms ∂^l and ∂^r in (9.1.3) are not required to be invertible in general. Moreover, they go in the opposite direction as the distributivity morphisms δ^r and δ^l, respectively, in (2.1.3) and (2.1.31).
   - The left factorization morphism ∂^l goes from $- \otimes C$ on the inside to the outside.
   - The right factorization morphism ∂^r goes from $A \otimes -$ on the inside to the outside.

   The direction of the factorization morphisms is similar to the monoidal constraint of a monoidal functor in (1.3.8).

(2) The multiplicative zero axiom (9.1.4) means the equalities

$$\begin{gathered} \mathbb{0} \otimes A = \mathbb{0} = A \otimes \mathbb{0} \\ 1_{\mathbb{0}} \otimes f = 1_{\mathbb{0}} = f \otimes 1_{\mathbb{0}} \end{gathered} \tag{9.1.14}$$

   for objects $A \in \mathsf{C}$ and morphisms $f \in \mathsf{C}$. $\diamond$

Relationship with Bimonoidal Categories. Recall from Definition 2.1.1 that a bimonoidal category has the same definition as a symmetric bimonoidal category, except that

- the multiplicative structure is a monoidal category, and
- the two Laplaza axioms (2.1.4) and (2.1.18) are omitted.

It is called *tight* if both distributivity morphisms δ^l and δ^r in (2.1.3) are invertible. We now observe that tight ring categories can be identified with a subclass of tight bimonoidal categories.

Theorem 9.1.15. *There is a canonical bijective correspondence between*

(1) the class of tight ring categories in Definition 9.1.2 and

(2) the class of tight bimonoidal categories in Definition 2.1.1 with
   - *a permutative category as the additive structure,*
   - *a strict monoidal category as the multiplicative structure, and*
   - $\lambda^{\bullet} = 1$ *and* $\rho^{\bullet} = 1$.

The correspondence between the factorization morphisms (∂^l, ∂^r) in (1) and the distributivity morphisms (δ^l, δ^r) in (2) is given by the equalities

$$\delta^l = (\partial^r)^{-1} \quad \text{and} \quad \delta^r = (\partial^l)^{-1}. \tag{9.1.16}$$

Proof. First suppose C is a tight ring category. Consider the data part of a bimonoidal category.

- C is already equipped with a permutative structure as its additive structure and a strict monoidal structure as its multiplicative structure.
- The left and right distributivity morphisms, δ^l and δ^r, are defined as in (9.1.16). They are natural isomorphisms because the factorization morphisms ∂^r and ∂^l are assumed to be so.
- The left and right multiplicative zeros, $\lambda^{\bullet}$ and $\rho^{\bullet}$ as in (2.1.2), are defined as the identity natural transformations. The fact that they are well-defined natural isomorphisms is equivalent to the multiplicative zero axiom (9.1.4), or equivalently (9.1.14).

With the above structure, next we observe that C satisfies the 22 Laplaza axioms for a bimonoidal category, namely, (2.1.5)–(2.1.27) excluding (2.1.18). The symmetry axiom (1.3.33) in the permutative category $(\mathsf{C}, \xi^{\oplus})$, namely,

$$\xi^{\oplus}_{A,B} = \xi^{-\oplus}_{B,A},$$

will be used below without further comment.

- The Laplaza axioms (2.1.6) and (2.1.5) are equivalent to the symmetry factorization axiom (9.1.7).
- The Laplaza axioms (2.1.7) and (2.1.8) are equivalent to the internal factorization axiom (9.1.8).
- The Laplaza axioms (2.1.9)–(2.1.11) are equivalent to the external factorization axioms (9.1.11), (9.1.9), and (9.1.10), respectively.
- (2.1.12) is equivalent to the 2-by-2 factorization axiom (9.1.12).
- (2.1.13) and (2.1.16)–(2.1.21), excluding (2.1.18), hold by $\lambda^{\bullet} = 1$ and $\rho^{\bullet} = 1$.
- Given that $\lambda^{\bullet} = 1$ and $\rho^{\bullet} = 1$, the axioms (2.1.14), (2.1.15), and (2.1.22)–(2.1.25) are equivalent to the zero factorization axiom (9.1.5).
- The last two Laplaza axioms (2.1.26) and (2.1.27) are equivalent to the unit factorization axiom (9.1.6).

Therefore, C yields a tight bimonoidal category as in (2) in the statement of the theorem.

Conversely, suppose given a tight bimonoidal category as in (2) in the statement of the theorem. The argument above shows that it yields a tight ring category with the factorization morphisms

$$\partial^l = (\delta^r)^{-1} \quad \text{and} \quad \partial^r = (\delta^l)^{-1}, \tag{9.1.17}$$

as stated in (9.1.16). □

Example 9.1.18. The additive distortion category $\mathcal{D}^{\mathrm{ad}}$ in Section I.4.5 is a small tight bimonoidal category that satisfies the conditions in Theorem 9.1.15(2). So $\mathcal{D}^{\mathrm{ad}}$ is a small tight ring category with factorization morphisms

- $\partial^l = (\delta^r)^{-1}$ with δ^r in (I.4.5.4) and
- $\partial^r = (\delta^l)^{-1} = 1$. ⋄

Strictification. Recall from Definition I.5.5.8 that a *right rigid bimonoidal category* is a tight bimonoidal category with

- a permutative category as its additive structure,
- a strict monoidal category as its multiplicative structure, and
- identities for the structure morphisms $\lambda^{\bullet}$, $\rho^{\bullet}$, and δ^r.

A *left* rigid bimonoidal category is defined in the same way but with $\delta^l = 1$ instead of $\delta^r = 1$. The next observation follows from Theorem 9.1.15 and provides examples of tight ring categories.

Corollary 9.1.19. *Each right (respectively, left) rigid bimonoidal category yields a tight ring category with factorization morphisms determined by (9.1.17).*

Corollary 9.1.20. *Each tight ring category, when regarded as a tight bimonoidal category as in Theorem 9.1.15 (2), is adjoint equivalent to a right (respectively, left) rigid bimonoidal category via strong bimonoidal functors.*

Proof. Apply the Rigid Strictification Theorems I.5.5.11 and I.5.5.12. □

In other words, each tight ring category is adjoint equivalent, via strong bimonoidal functors, to one with $\partial^l = 1$, respectively, $\partial^r = 1$.

9.2. Endomorphism Ring Categories

In this section, we observe in Theorems 9.2.14 and 9.2.20 that each small permutative category C yields

- an endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ and
- a tight endomorphism ring category $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$.

The objects in the endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ are strictly unital symmetric monoidal functors in the sense of Definitions 1.3.7 and 1.3.32. They are symmetric monoidal functors

$$(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{C}$$

such that the unit constraint

$$F^0 : \mathbb{0} \longrightarrow F\mathbb{0}$$

is the identity morphism, while F^2 is a natural transformation. The morphisms are monoidal natural transformations. Theorem 9.2.14 shows that $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ is a ring category. We emphasize that $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ is usually *not* a tight ring category because its right factorization morphism (9.2.11) is, in general, not invertible.

To obtain a tight ring category, we consider the full subcategory $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ of $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ whose objects $(F, F^2, F^0 = 1)$ satisfy the additional condition that F^2 is a natural isomorphism. Theorem 9.2.20 says that $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ is a tight ring category.

Motivation 9.2.1. As motivation for the relevant definitions, consider a commutative monoid $(A, +, 0)$, such as the set $\mathbb{N}$ of natural numbers with its addition. The set $\mathsf{End}(A)$ of monoid morphisms $A \longrightarrow A$ has a canonical rig (that is, ring without additive inverse) structure defined as follows.

- The addition is given by pointwise addition in A, that is,

  $$(f+g)(a) = f(a) + g(a)$$

  for monoid morphisms $f, g : A \longrightarrow A$ and elements $a \in A$.
- The additive zero is the constant map at $0 \in A$.
- The multiplication is given by composition of monoid morphisms, that is,

  $$(fg)(a) = f(g(a)).$$
- The multiplicative unit is the identity map

  $$1_A : A \longrightarrow A.$$

All the rig axioms for $\mathsf{End}(A)$ follow from the fact that A is a commutative monoid and that composition is strictly associative. We call $\mathsf{End}(A)$ the *endomorphism rig* of A. Note that even if A is commutative, $\mathsf{End}(A)$ is in general *not* a commutative rig because composition is rarely commutative.

A permutative category, that is, a symmetric strict monoidal category, is a categorification of a commutative monoid. By the above example of the endomorphism rig $\mathsf{End}(A)$, we expect a permutative category C to yield an endomorphism ring category, with suitable functors $\mathsf{C} \longrightarrow \mathsf{C}$ as objects, pointwise sum, and composition as the product on objects. This is, in fact, true by Theorem 9.2.14. To explain the detail, first we make precise the relevant structure. $\diamond$

Definition 9.2.2. For permutative categories $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ and $(\mathsf{D}, \oplus, \mathbb{0}, \xi^{\oplus})$ with C small, define the category

$$\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{D})$$

as follows.

Objects: Its objects are strictly unital symmetric monoidal functors $\mathsf{C} \longrightarrow \mathsf{D}$.

Morphisms: Its morphisms are monoidal natural transformations.

Identities: The identity morphism of an object $F : \mathsf{C} \longrightarrow \mathsf{D}$ is the identity natural transformation of F.

Composition: Composition is the vertical composition of natural transformations (Definition I.1.1.8).

This finishes the definition of the category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{D})$. ◇

Explanation 9.2.3. Consider Definition 9.2.2.

(1) Interpreting Definitions 1.3.7 and 1.3.32 here, a strictly unital symmetric monoidal functor

$$(F, F^2) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of
- a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ and
- a natural transformation

$$FA \oplus FB \xrightarrow{F^2_{A,B}} F(A \oplus B)$$

for objects $A, B \in \mathsf{C}$, which is called the *monoidal constraint*.

It is required that (i) the equalities

$$\begin{aligned} F(\mathbb{0}) &= \mathbb{0} \\ F^2_{\mathbb{0},B} &= 1_{FB} \\ F^2_{A,\mathbb{0}} &= 1_{FA} \end{aligned} \tag{9.2.4}$$

hold and (ii) the associativity and symmetry diagrams

$$\begin{tikzcd} FA \oplus FB \oplus FC \arrow[r, "1 \oplus F^2"] \arrow[d, "F^2 \oplus 1"'] & FA \oplus F(B \oplus C) \arrow[d, "F^2"] \\ F(A \oplus B) \oplus FC \arrow[r, "F^2"] & F(A \oplus B \oplus C) \end{tikzcd} \qquad \begin{tikzcd} FA \oplus FB \arrow[r, "\xi^{\oplus}"] \arrow[d, "F^2"'] & FB \oplus FA \arrow[d, "F^2"] \\ F(A \oplus B) \arrow[r, "F\xi^{\oplus}"] & F(B \oplus A) \end{tikzcd} \tag{9.2.5}$$

are commutative for objects $A, B, C \in \mathsf{C}$. We emphasize that the monoidal constraint F^2 is *not* required to be a natural isomorphism.

(2) Interpreting Definition 1.3.13 here, a morphism

$$\theta : (F, F^2) \longrightarrow (G, G^2) \in \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{D})$$

is a natural transformation $\theta : F \longrightarrow G$ such that

$$\theta_{\mathbb{0}} = 1_{\mathbb{0}} : \mathbb{0} \longrightarrow \mathbb{0} \tag{9.2.6}$$

and the diagram

$$\begin{tikzcd} FA \oplus FB \arrow[r, "\theta_A \oplus \theta_B"] \arrow[d, "F^2"'] & GA \oplus GB \arrow[d, "G^2"] \\ F(A \oplus B) \arrow[r, "\theta_{A \oplus B}"] & G(A \oplus B) \end{tikzcd} \tag{9.2.7}$$

is commutative for objects $A, B \in \mathsf{C}$. The smallness of C ensures that there is only a set of such natural transformations. $\diamond$

Definition 9.2.8. For a small permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$, define the *endomorphism ring category*

$$\big(\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}), (\boxplus, \mathbb{0}, \xi^{\boxplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$

by the following ring category data on the category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$.

The Additive Zero: The object

$$\mathbb{0} \in \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$$

is the constant functor $\mathsf{C} \longrightarrow \mathsf{C}$ at the additive zero $\mathbb{0} \in \mathsf{C}$ and its identity morphism $1_{\mathbb{0}}$, with monoidal constraint

$$1_{\mathbb{0}} : \mathbb{0} \oplus \mathbb{0} = \mathbb{0} \longrightarrow \mathbb{0}.$$

The Sum: The functor

$$- \boxplus - : \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}) \times \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}) \longrightarrow \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$$

is defined by

$$(F, F^2) \boxplus (G, G^2) = \big(F \boxplus G, (F \boxplus G)^2\big)$$

for strictly unital symmetric monoidal functors

$$(F, F^2), (G, G^2) : \mathsf{C} \longrightarrow \mathsf{C},$$

with

$$(F \boxplus G)(A) = FA \oplus GA$$

for objects and morphisms $A \in \mathsf{C}$. Its monoidal constraint is the composite

$$\begin{tikzcd}[column sep=small]
(F \boxplus G)(A) \oplus (F \boxplus G)(B) \arrow[rr, "(F \boxplus G)^2_{A,B}"] \arrow[d, equal] & & (F \boxplus G)(A \oplus B) \arrow[d, equal] \\
FA \oplus GA \oplus FB \oplus GB \arrow[dr, "1 \oplus \xi^{\oplus} \oplus 1"'] & & F(A \oplus B) \oplus G(A \oplus B) \\
& FA \oplus FB \oplus GA \oplus GB \arrow[ur, "F^2 \oplus G^2"'] &
\end{tikzcd} \tag{9.2.9}$$

in C for objects $A, B \in \mathsf{C}$.

For morphisms

$$\theta : F \longrightarrow F_1 \quad \text{and} \quad \phi : G \longrightarrow G_1 \in \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}),$$

that is, monoidal natural transformations, their sum

$$\theta \boxplus \phi : F \boxplus G \longrightarrow F_1 \boxplus G_1$$

has component morphisms

$$(\theta \boxplus \phi)_A = \theta_A \oplus \phi_A : FA \oplus GA \longrightarrow F_1 A \oplus G_1 A$$

for objects $A \in \mathsf{C}$.

The Additive Symmetry: The natural transformation

$$\xi^{\boxplus}_{F,G} : F \boxplus G \longrightarrow G \boxplus F$$

has component morphisms

$$(\xi^{\boxplus}_{F,G})_A = \xi^{\oplus}_{FA,GA} : FA \oplus GA \xrightarrow{\cong} GA \oplus FA$$

for objects $A \in \mathsf{C}$.

The Multiplicative Unit: The object

$$\mathbb{1} \in \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$$

is the identity symmetric monoidal functor 1_{C}.

The Product: The functor

$$- \otimes - : \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}) \times \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}) \longrightarrow \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$$

is defined as the composite

$$(F, F^2) \otimes (G, G^2) = \big(FG, (FG)^2\big)$$

of strictly unital symmetric monoidal functors as in Definition 1.3.12.

For morphisms $\theta : F \longrightarrow F_1$ and $\phi : G \longrightarrow G_1$ as above, their product is the horizontal composite natural transformation

$$\theta \otimes \phi = \theta * \phi : FG \longrightarrow F_1 G_1$$

in Definition I.1.1.8.

Left Factorization: For objects (F, F^2), (F', F'^2), and (G, G^2) in $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$, the left factorization morphism

$$(F \otimes G) \boxplus (F' \otimes G) \xrightarrow{\partial^l_{F,F',G}} (F \boxplus F') \otimes G \tag{9.2.10}$$

has as its components the identity morphisms $1_{FG(A) \oplus F'G(A)}$ for $A \in \mathsf{C}$.

Right Factorization: For another object (G', G'^2) in $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$, the right factorization morphism

$$(F \otimes G) \boxplus (F \otimes G') \xrightarrow{\partial^r_{F,G,G'}} F \otimes (G \boxplus G')$$

has as its components the morphisms

$$FG(A) \oplus FG'(A) \xrightarrow{F^2_{GA,G'A}} F(GA \oplus G'A) \tag{9.2.11}$$

for $A \in \mathsf{C}$.

This finishes the definition of the endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$. ◇

Explanation 9.2.12. In Definition 9.2.8, the monoidal constraint of the product $(F, F^2) \otimes (G, G^2)$ is the composite

$$\begin{tikzcd}[column sep=small] FG(A) \oplus FG(B) \arrow[rr, "(FG)^2_{A,B}"] \arrow[dr, "F^2_{GA,GB}"'] & & FG(A \oplus B) \\ & F(GA \oplus GB) \arrow[ur, "F(G^2_{A,B})"'] & \end{tikzcd} \tag{9.2.13}$$

in C for objects $A, B \in \mathsf{C}$.

The product morphism

$$\theta \otimes \phi : FG \longrightarrow F_1 G_1$$

has component morphisms

$$\begin{aligned}(\theta * \phi)_A &= F_1(\phi_A) \circ \theta_{GA}\\ &= \theta_{G_1 A} \circ F(\phi_A)\end{aligned}$$

for $A \in \mathsf{C}$. $\diamond$

Next is the main observation of this section.

Theorem 9.2.14. *For each small permutative category* $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$*, the data*

$$\big(\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}), (\boxplus, \mathbb{0}, \xi^{\boxplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$

in Definition 9.2.8 form a ring category.

Proof. We need to check the following statements:

(i) $\big(\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}), \boxplus, \mathbb{0}, \xi^{\boxplus}\big)$ is a permutative category.
(ii) $\big(\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}), \otimes, \mathbb{1}\big)$ is a strict monoidal category.
(iii) Each component $\partial^l_{F,F',G}$ is a morphism in $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$, that is, a monoidal natural transformation, and ∂^l is natural with respect to F, F', and G.
(iv) Each component $\partial^r_{F,G,G'}$ is a monoidal natural transformation, and ∂^r is natural with respect to F, G, and G'.
(v) $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ satisfies the seven axioms in Definition 9.1.2.

With two exceptions that we will explain below, the statements (i)–(v) above consist of direct instances of Definition 9.2.8 and the axioms of

- the permutative category C,
- strictly unital symmetric monoidal functors, (9.2.4) and (9.2.5), and
- monoidal natural transformations between such functors, (9.2.6) and (9.2.7).

For example, the 2-by-2 factorization axiom (9.1.12) in $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ means the commutativity of the diagram

(9.2.15)
$$\begin{tikzcd}[column sep=tiny]
& F(GA \oplus G'A) \oplus F'(GA \oplus G'A) \arrow[dr, "\partial^l = 1"] & \\
FGA \oplus FG'A \oplus F'GA \oplus F'G'A \arrow[ur, "F^2 \oplus F'^2"] \arrow[d, "1 \oplus \xi^{\oplus} \oplus 1"'] & & (F \boxplus F')(G \boxplus G')A \\
FGA \oplus F'GA \oplus FG'A \oplus F'G'A \arrow[dr, "\partial^l \oplus \partial^l = 1"'] & & FGA \oplus FG'A \oplus F'GA \oplus F'G'A \arrow[u, "F^2 \oplus F'^2"'] \\
& FGA \oplus F'GA \oplus FG'A \oplus F'G'A \arrow[ur, "1 \oplus \xi^{\oplus} \oplus 1"'] &
\end{tikzcd}$$

in C for objects $F, F', G, G' \in \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ and $A \in \mathsf{C}$. Here we used the definitions (9.2.9) and (9.2.11) to obtain the equalities

$$\begin{aligned}\big(\partial^r_{F \boxplus F', G, G'}\big)_A &= (F \boxplus F')^2_{GA, G'A}\\ &= \big(F^2_{GA,G'A} \oplus F'^2_{GA,G'A}\big)\big(1 \oplus \xi^{\oplus} \oplus 1\big).\end{aligned}$$

The symmetry axiom (1.3.33) in the permutative category $(\mathsf{C}, \xi^{\oplus})$, that is,

$$\xi^{\oplus}\xi^{\oplus} = 1,$$

implies that both composites in (9.2.15) are equal to

$$F^2_{GA,G'A} \oplus F'^2_{GA,G'A}.$$

This is what we mean by *direct instances* above, since the commutativity of the desired diagrams can be seen without factoring them in any way.

The first exception that requires more work is the verification of the naturality of ∂^r in statement (iv) above. This is the assertion that for morphisms

$$\theta : F \longrightarrow F_1, \quad \phi : G \longrightarrow G_1, \quad \text{and} \quad \phi' : G' \longrightarrow G'_1 \in \mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C}),$$

the diagram

$$\begin{tikzcd}
(F \otimes G) \boxplus (F \otimes G') \arrow[r, "\partial^r_{F,G,G'}"] \arrow[d, "(\theta*\phi)\boxplus(\theta*\phi')"'] & F \otimes (G \boxplus G') \arrow[d, "\theta*(\phi\boxplus\phi')"] \\
(F_1 \otimes G_1) \boxplus (F_1 \otimes G'_1) \arrow[r, "\partial^r_{F_1,G_1,G'_1}"] & F_1 \otimes (G_1 \boxplus G'_1)
\end{tikzcd}$$

in $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ is commutative. This, in turn, means that for each object $A \in \mathsf{C}$, the outer diagram

$$\begin{tikzcd}
FGA \oplus FG'A \arrow[r, "F^2"] \arrow[d, "\theta_{GA}\oplus\theta_{G'A}"'] & F(GA \oplus G'A) \arrow[d, "\theta_{GA\oplus G'A}"] \\
F_1GA \oplus F_1G'A \arrow[r, "F_1^2"] \arrow[d, "F_1\phi_A\oplus F_1\phi'_A"'] & F_1(GA \oplus G'A) \arrow[d, "F_1(\phi_A\oplus\phi'_A)"] \\
F_1G_1A \oplus F_1G'_1A \arrow[r, "F_1^2"] & F_1(G_1A \oplus G'_1A)
\end{tikzcd}$$

in C is commutative. The top rectangle is commutative by the compatibility axiom (9.2.7) of θ with the monoidal constraints. The bottom rectangle is commutative by the naturality of the monoidal constraint F_1^2.

The other exception is also about statement (iv). To show that each component

$$(F \otimes G) \boxplus (F \otimes G') \xrightarrow{\partial^r_{F,G,G'}} F \otimes (G \boxplus G')$$

is a monoidal natural transformation, we must show that it is compatible with the monoidal constraints of its (co)domain in the sense of (9.2.7). By (9.2.9), (9.2.11), and (9.2.13), the desired diagram (9.2.7) for $\partial^r_{F,G,G'}$ is the outer diagram in C below for objects $A, B \in \mathsf{C}$.

$$
\begin{tikzcd}[column sep=small]
FGA \oplus FG'A \oplus FGB \oplus FG'B \arrow[rrr, bend left=10, "F^2 \oplus F^2"] \arrow[dr, "1 \oplus F^2 \oplus 1"] \arrow[ddd, "1 \oplus \xi^{\oplus} \oplus 1"'] \arrow[dddr, phantom, "(9.2.5)"] & & & F(GA \oplus G'A) \oplus F(GB \oplus G'B) \arrow[ddd, "F^2"] \\
& \bullet \arrow[dr, "1 \oplus F^2"] \arrow[ddd, "1 \oplus F\xi^{\oplus} \oplus 1"'] \arrow[dddr, phantom, "(\dagger)"] & & \\
& & \bullet \arrow[dr, "F^2"] \arrow[ddd, "1 \oplus F(\xi^{\oplus} \oplus 1)"'] \arrow[ur, phantom, "(\pounds)"] & \\
FGA \oplus FGB \oplus FG'A \oplus FG'B \arrow[dr, "1 \oplus F^2 \oplus 1"] \arrow[ddd, "F^2 \oplus F^2"'] & & & F(GA \oplus G'A \oplus GB \oplus G'B) \arrow[ddd, "F(1 \oplus \xi^{\oplus} \oplus 1)"] \\
& \bullet \arrow[dr, "1 \oplus F^2"] \arrow[dd, phantom, "(\pounds)"] & & \\
& & \bullet \arrow[dr, "F^2"] & \\
F(GA \oplus GB) \oplus F(G'A \oplus G'B) \arrow[rrr, "F^2"] \arrow[d, "F(G^2) \oplus F(G'^2)"'] & & & F(GA \oplus GB \oplus G'A \oplus G'B) \arrow[d, "F(G^2 \oplus G'^2)"] \\
FG(A \oplus B) \oplus FG'(A \oplus B) \arrow[rrr, "F^2"] & & & F\big(G(A \oplus B) \oplus G'(A \oplus B)\big)
\end{tikzcd}
\tag{9.2.16}
$$

In (9.2.16), the middle subdiagram $(\dagger)$ is the diagram

$$
\begin{tikzcd}
FGA \oplus F(G'A \oplus GB) \oplus FG'B \arrow[r, "1 \oplus F^2"] \arrow[d, "1 \oplus F\xi^{\oplus} \oplus 1"'] & FGA \oplus F(G'A \oplus GB \oplus G'B) \arrow[d, "1 \oplus F(\xi^{\oplus} \oplus 1)"] \\
FGA \oplus F(GB \oplus G'A) \oplus FG'B \arrow[r, "1 \oplus F^2"] & FGA \oplus F(GB \oplus G'A \oplus G'B)
\end{tikzcd}
$$

in C. In (9.2.16), $(\dagger)$ and the two unlabeled subdiagrams are commutative by the naturality of F^2. The upper-left subdiagram is commutative by the compatibility (9.2.5) of F^2 with $\xi^{\oplus}$.

Both subdiagrams $(\pounds)$ in (9.2.16) are commutative by the following commutative diagram for objects $A, B, C, D \in \mathsf{C}$.

$$
\begin{tikzcd}
FA \oplus FB \oplus FC \oplus FD \arrow[r, "1 \oplus F^2 \oplus 1"] \arrow[d, "1 \oplus 1 \oplus F^2"'] & FA \oplus F(B \oplus C) \oplus FD \arrow[d, "1 \oplus F^2"] \\
FA \oplus FB \oplus F(C \oplus D) \arrow[r, "1 \oplus F^2"] \arrow[d, "F^2 \oplus 1"'] & FA \oplus F(B \oplus C \oplus D) \arrow[d, "F^2"] \\
F(A \oplus B) \oplus F(C \oplus D) \arrow[r, "F^2"] & F(A \oplus B \oplus C \oplus D)
\end{tikzcd}
\tag{9.2.17}
$$

Here both the top and the bottom rectangles are commutative by the associativity of F^2 in (9.2.5). Therefore, the diagram (9.2.16) is commutative. $\square$

Tight Endomorphism Ring Categories. In Theorem 9.2.14, the endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ is in general *not* a tight ring category because the right factorization morphism $\partial^r_{F,G,G'}$ (9.2.11) has components those of F^2, which is a natural transformation but not a natural isomorphism in general. To obtain a tight variant of the endomorphism ring category, we consider a smaller class of objects as follows.

Definition 9.2.18. Suppose $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ is a small permutative category.

- Define
$$\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$$
as the full subcategory of $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ in Definition 9.2.2 with, as its objects, strictly unital *strong* symmetric monoidal functors $\mathsf{C} \longrightarrow \mathsf{C}$.
- Restrict Definition 9.2.8 to $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ to equip it with ring category data. $\diamond$

Explanation 9.2.19. Consider Definition 9.2.18.

(1) An object in $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ is a symmetric monoidal functor

$$(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{C}$$

with

- unit constraint

$$\mathbb{0} \xrightarrow{F^0} F\mathbb{0}$$

the identity morphism $1_{\mathbb{0}}$ and

- monoidal constraint

$$FA \oplus FB \xrightarrow[\cong]{F^2} F(A \oplus B)$$

a natural isomorphism.

Such symmetric monoidal functors are closed under composition by Definition 1.3.12.

(2) The restriction of Definition 9.2.8 to $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ is well defined because
- $(F \boxplus G)^2_{A,B}$ in (9.2.9) and
- $(FG)^2_{A,B}$ in (9.2.13)

are both isomorphisms when F^2 and G^2 are natural isomorphisms. $\diamond$

Next is the tight variant of Theorem 9.2.14.

Theorem 9.2.20. *For each small permutative category* $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$*, the data*

$$\big(\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C}), (\boxplus, \mathbb{0}, \xi^{\boxplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$

in Definition 9.2.18 form a tight ring category.

Proof. After restricting to $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$, the proof of Theorem 9.2.14 applies here without changes to show that $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ is a ring category. Moreover, we note the following:

- Each component $\partial^l_{F,F',G}$ of the left factorization morphism in (9.2.10) is the identity natural transformation.
- Each component $\partial^r_{F,G,G'}$ of the right factorization morphism in (9.2.11) is a natural isomorphism because F^2 is so.

Therefore, $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ is a tight ring category. $\square$

9.3. Elmendorf-Mandell Bipermutative Categories

In this section, we define bipermutative categories in the sense of Elmendorf-Mandell [**EM06**, Def. 3.6]. Bipermutative categories are the *commutative* ring analogues of ring categories, so their multiplicative structures are permutative. Like ring categories, bipermutative categories have factorization morphisms instead of distributivity morphisms. Therefore, a bipermutative category is, in general, not a symmetric bimonoidal category, or vice versa.

The main observation in this section is Theorem 9.3.7. It identifies *tight* bipermutative categories, which have invertible factorization morphisms, with the subclass of tight symmetric bimonoidal categories with

- strict additive and multiplicative structures and
- identities for the left and right multiplicative zeros.

As a result, both right and left bipermutative categories (Definitions I.2.5.2 and I.2.5.11) are tight bipermutative categories with identity as one factorization morphism; see Corollary 9.3.12. Moreover, the Strictification Theorems I.5.4.6 and I.5.4.7 apply to tight bipermutative categories. Therefore, each tight bipermutative category is adjoint equivalent to one in which the left, or the right, factorization morphism is the identity; see Corollary 9.3.13.

Motivation 9.3.1. Just as a ring category (Definition 9.1.2) is a categorification of a rig using factorization morphisms, bipermutative categories in the sense of Elmendorf-Mandell, which we will define shortly, is a categorification of a commutative rig. The multiplicative structure $\otimes$ is now a permutative category with a braiding $\xi^{\otimes}$. We also need to make sure that the braiding $\xi^{\otimes}$ is compatible with the additive structure in the sense of (9.3.3) and (9.3.4) below. ◇

Definition 9.3.2. A *bipermutative category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big)$$

consisting of

- a ring category
$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$
as in Definition 9.1.2 and
- a permutative category structure $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$, with $\xi^{\otimes}$ called the *multiplicative symmetry*.

These data are required to satisfy the following two axioms for objects $A, B, C \in \mathsf{C}$.

The Zero Symmetry Axiom: There is an equality of morphisms

$$\xi^{\otimes}_{A,\mathbb{0}} = 1_{\mathbb{0}} : A \otimes \mathbb{0} = \mathbb{0} \longrightarrow \mathbb{0} = \mathbb{0} \otimes A. \tag{9.3.3}$$

The Multiplicative Symmetry Factorization Axiom: The diagram

$$\begin{tikzcd}
(A \otimes C) \oplus (B \otimes C) \arrow[r, "\partial^l_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}"'] & (A \oplus B) \otimes C \arrow[d, "\xi^{\otimes}_{A \oplus B, C}"] \\
(C \otimes A) \oplus (C \otimes B) \arrow[r, "\partial^r_{C,A,B}"] & C \otimes (A \oplus B)
\end{tikzcd} \tag{9.3.4}$$

is commutative.

This finishes the definition of a bipermutative category. A bipermutative category is *small*, respectively, *tight*, if the underlying ring category is so. ◇

Explanation 9.3.5. Consider Definition 9.3.2 of a bipermutative category.

(1) A bipermutative category C has two permutative category structures:
 - an additive one $(\oplus, \mathbb{0}, \xi^{\oplus})$ and
 - a multiplicative one $(\otimes, \mathbb{1}, \xi^{\otimes})$.

 This justifies the name *bipermutative* category.
(2) By the symmetry axiom (1.3.33) in the permutative category $(\mathsf{C}, \xi^{\otimes})$, the zero symmetry axiom (9.3.3) is equivalent to the equality

$$\xi^{\otimes}_{\mathbb{0},A} = 1_{\mathbb{0}} : \mathbb{0} \otimes A = \mathbb{0} \longrightarrow \mathbb{0} = A \otimes \mathbb{0}. \tag{9.3.6}$$

of morphisms. ◇

Relationship with Symmetric Bimonoidal Categories. Like ring categories, bipermutative categories have factorization morphisms ∂^l and ∂^r as in (9.1.3), which are natural transformations but not natural isomorphisms in general. So bipermutative categories are different from symmetric bimonoidal categories as in Definition 2.1.1, which have distributivity morphisms δ^l and δ^r that go in the opposite direction as, respectively, ∂^r and ∂^l. Next is the symmetric analogue of Theorem 9.1.15 that identifies *tight* bipermutative categories—that is, those with invertible factorization morphisms ∂^l and ∂^r—with a subclass of tight symmetric bimonoidal categories.

Theorem 9.3.7. *There is a canonical bijective correspondence between*

(1) the class of tight bipermutative categories in Definition 9.3.2 and

(2) the class of tight symmetric bimonoidal categories in Definition 2.1.1 with
- *a permutative category as the additive, respectively, multiplicative, structure, and*
- $\lambda^\bullet = 1$ *and* $\rho^\bullet = 1$.

The correspondence between the factorization morphisms (∂^l, ∂^r) *in (1) and the distributivity morphisms* (δ^l, δ^r) *in (2) is given by the equalities*

$$\delta^l = (\partial^r)^{-1} \quad \textit{and} \quad \delta^r = (\partial^l)^{-1}. \tag{9.3.8}$$

Proof. First suppose C is a tight bipermutative category. By Theorem 9.1.15, the underlying tight ring category of C yields a tight bimonoidal category with permutative additive and multiplicative structures, $\lambda^\bullet = 1$, $\rho^\bullet = 1$, and distributivity morphisms determined by (9.3.8). To see that this is a tight symmetric bimonoidal category, it remains to prove the two Laplaza axioms (2.1.4) and (2.1.18).

- (2.1.4) is equivalent to the multiplicative symmetry factorization axiom (9.3.4).
- Since $\lambda^\bullet = 1$ and $\rho^\bullet = 1$, (2.1.18) is equivalent to the equality $\xi^{\otimes}_{-,\mathbb{0}} = 1_{\mathbb{0}}$, which is the zero symmetry axiom (9.3.3).

Conversely, suppose C is a tight symmetric bimonoidal category as in (2) in the statement of the theorem. By Theorem 9.1.15, C yields a tight ring category with the factorization morphisms

$$\partial^l = (\delta^r)^{-1} \quad \text{and} \quad \partial^r = (\delta^l)^{-1}. \tag{9.3.9}$$

Reusing the previous paragraph, the two bipermutative category axioms (9.3.3) and (9.3.4) are equivalent to, respectively, the axioms (2.1.18) and (2.1.4). □

Example 9.3.10 (Finite Ordinal Categories)**.** The finite ordinal category Σ and its variant Σ' in, respectively, Propositions I.2.4.8 and I.2.4.23 are small tight symmetric bimonoidal categories that satisfy the conditions in Theorem 9.3.7 (2). So each of them is a small tight bipermutative category.

- In Σ, the factorization morphisms are
  - $\partial^l = (\delta^r)^{-1}$ with δ^r in (I.2.4.6) and
  - $\partial^r = (\delta^l)^{-1} = 1$.
- In Σ', the factorization morphisms are
  - $\partial^r = (\delta^l)^{-1}$ with δ^l in (I.2.4.21) and
  - $\partial^l = (\delta^r)^{-1} = 1$. ◇

Example 9.3.11 (Distortion Category)**.** The distortion category $\mathcal{D}$ in Section I.4.2 is a small left bipermutative category, which is, in particular, a small tight symmetric bimonoidal category by Proposition I.2.5.16. Moreover, it satisfies the conditions in Theorem 9.3.7 (2). So $\mathcal{D}$ is a small tight bipermutative category with factorization morphisms

- $\partial^l = (\delta^r)^{-1}$ with δ^r in (I.4.2.24) and
- $\partial^r = (\delta^l)^{-1} = 1$. $\diamond$

Strictification. Recall from Definition I.2.5.2 and Proposition I.2.5.7 that a *right bipermutative category* is precisely a tight symmetric bimonoidal category that satisfies

- the conditions in Theorem 9.3.7 (2) and
- $\delta^r = 1$.

Moreover, the conditions $\lambda^\bullet = 1$ and $\rho^\bullet = 1$ and the axiom (2.1.18) imply $\xi^\otimes_{-,\mathbb{0}} = 1$. By Definition I.2.5.11 and Proposition I.2.5.16, a *left bipermutative category* is such a tight symmetric bimonoidal category but with $\delta^l = 1$ instead of $\delta^r = 1$. The next observation follows from Theorem 9.3.7 and provides examples of tight bipermutative categories.

Corollary 9.3.12. *Each right (respectively, left) bipermutative category becomes a tight bipermutative category with factorization morphisms determined by (9.3.9).*

Corollary 9.3.13. *Each tight bipermutative category, when regarded as a tight symmetric bimonoidal category as in Theorem 9.3.7 (2), is adjoint equivalent to a right (respectively, left) bipermutative category via strong symmetric bimonoidal functors.*

Proof. Apply the Strictification Theorems I.5.4.6 and I.5.4.7. $\square$

In other words, each tight bipermutative category is adjoint equivalent, via strong symmetric bimonoidal functors, to one with $\partial^l = 1$, respectively, $\partial^r = 1$.

9.4. Reduction of Bipermutative Category Axioms

In this section, we observe that in a bipermutative category in Definition 9.3.2, about half of the ring category axioms in Definition 9.1.2 are redundant. One of these redundancies was observed by Elmendorf-Mandell [**EM06**, Fig. 1]; see Note 9.7.4. This reduction of the axioms of a bipermutative category is analogous to the fact that, among the 24 Laplaza axioms in a symmetric bimonoidal category, half of them are formal consequences of the others; see Section I.2.2. The key point is that, in the presence of the multiplicative symmetry $\xi^\otimes$ and the axiom (9.3.4) relating ∂^l and ∂^r, some of the ring category axioms for ∂^l are equivalent to their counterparts for ∂^r. We present these reductions in Lemmas 9.4.2 through 9.4.6 and then summarize them in Theorem 9.4.7.

Convention 9.4.1. Suppose the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^\oplus), (\otimes, \mathbb{1}, \xi^\otimes), (\partial^l, \partial^r)\big)$$

consists of

- a permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^\oplus)$,
- a permutative category $(\mathsf{C}, \otimes, \mathbb{1}, \xi^\otimes)$, and
- natural transformations ∂^l and ∂^r as in (9.1.3)

such that the multiplicative symmetry factorization axiom (9.3.4) holds. $\diamond$

In the proofs of Lemmas 9.4.2 through 9.4.6 below, A, B, C, and D are arbitrary objects in C.

Lemma 9.4.2. *Under Convention 9.4.1, suppose the zero symmetry axiom (9.3.3) also holds. Then*

- *the left zero factorization axioms and*
- *the right zero factorization axioms*

in (9.1.5) are equivalent to each other.

Proof. The left zero factorization axioms say

$$\partial^l_{\mathbb{0},B,C} = 1_{B\otimes C}, \quad \partial^l_{A,\mathbb{0},C} = 1_{A\otimes C}, \quad \text{and} \quad \partial^l_{A,B,\mathbb{0}} = 1_{\mathbb{0}}.$$

The right zero factorization axioms say

$$\partial^r_{\mathbb{0},B,C} = 1_{\mathbb{0}}, \quad \partial^r_{A,\mathbb{0},C} = 1_{A\otimes C}, \quad \text{and} \quad \partial^r_{A,B,\mathbb{0}} = 1_{A\otimes B}.$$

To see that these two sets of axioms are equivalent to each other, consider the commutative diagram

$$\begin{tikzcd}
(A\otimes C)\oplus(B\otimes C) \arrow[r, "\partial^l_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A,C}\oplus\xi^{\otimes}_{B,C}"'] & (A\oplus B)\otimes C \arrow[d, "\xi^{\otimes}_{A\oplus B,C}"] \\
(C\otimes A)\oplus(C\otimes B) \arrow[r, "\partial^r_{C,A,B}"] & C\otimes(A\oplus B)
\end{tikzcd}$$

in C in the multiplicative symmetry factorization axiom (9.3.4).

- If $A = \mathbb{0}$, then the left arrow is $\xi^{\otimes}_{B,C}$ because $\xi^{\otimes}_{\mathbb{0},C} = 1_{\mathbb{0}}$ by (9.3.6), which is equivalent to the zero symmetry axiom (9.3.3). The right arrow is also $\xi^{\otimes}_{B,C}$, which is an isomorphism. So the top arrow $\partial^l_{\mathbb{0},B,C}$ is the identity if and only if the bottom arrow $\partial^r_{C,\mathbb{0},B}$ is the identity.
- If $B = \mathbb{0}$, then the left and the right arrows are both $\xi^{\otimes}_{A,C}$, which is an isomorphism. So the top arrow $\partial^l_{A,\mathbb{0},C}$ is the identity if and only if the bottom arrow $\partial^r_{C,A,\mathbb{0}}$ is the identity.
- If $C = \mathbb{0}$, then the left arrow is

  $$\xi^{\otimes}_{A,\mathbb{0}} \oplus \xi^{\otimes}_{B,\mathbb{0}} = 1_{\mathbb{0}} \oplus 1_{\mathbb{0}} = 1_{\mathbb{0}},$$

  and the right arrow is also

  $$\xi^{\otimes}_{A\oplus B,\mathbb{0}} = 1_{\mathbb{0}}$$

  by the zero symmetry axiom (9.3.3). So the top arrow $\partial^l_{A,B,\mathbb{0}}$ is $1_{\mathbb{0}}$ if and only if the bottom arrow $\partial^r_{\mathbb{0},A,B}$ is $1_{\mathbb{0}}$.

Therefore, the left zero factorization axioms and the right zero factorization axioms are equivalent to each other. □

Lemma 9.4.3. *Under Convention 9.4.1,*

- *the left unit factorization axiom and*
- *the right unit factorization axiom*

in (9.1.6) are equivalent to each other.

Proof. The left and the right unit factorization axioms say, respectively,

$$\partial^l_{A,B,\mathbb{1}} = 1_{A\oplus B} \quad \text{and} \quad \partial^r_{\mathbb{1},B,C} = 1_{B\oplus C}.$$

Consider the commutative diagram (9.3.4) with $C = \mathbb{1}$, which is the diagram in C below.

$$\begin{tikzcd}
(A\otimes\mathbb{1})\oplus(B\otimes\mathbb{1}) \arrow[r, "\partial^l_{A,B,\mathbb{1}}"] \arrow[d, "\xi^{\otimes}_{A,\mathbb{1}}\oplus\xi^{\otimes}_{B,\mathbb{1}}"'] & (A\oplus B)\otimes\mathbb{1} \arrow[d, "\xi^{\otimes}_{A\oplus B,\mathbb{1}}"] \\
(\mathbb{1}\otimes A)\oplus(\mathbb{1}\otimes B) \arrow[r, "\partial^r_{\mathbb{1},A,B}"] & \mathbb{1}\otimes(A\oplus B)
\end{tikzcd}$$

By the unit axiom (1.3.34) in the permutative category $(\mathsf{C},\xi^{\otimes})$, which says

$$\xi^{\otimes}_{-,\mathbb{1}} = 1_{-},$$

both the left and the right arrows in the previous diagram are $1_{A\oplus B}$. Therefore, the top arrow $\partial^l_{A,B,\mathbb{1}}$ is $1_{A\oplus B}$ if and only if the bottom arrow $\partial^r_{\mathbb{1},A,B}$ is $1_{A\oplus B}$. □

Lemma 9.4.4. *Under Convention 9.4.1,*

- *the left symmetry factorization axiom and*
- *the right symmetry factorization axiom*

in (9.1.7) are equivalent to each other.

Proof. This proof adapts the diagram in the proof of Lemma I.2.2.4 by

- turning each arrow around and
- replacing $(\delta^l,\delta^r,\xi^{\oplus}_{?,-},\xi^{\otimes}_{?,-})$ with $(\partial^r,\partial^l,\xi^{\oplus}_{-,?},\xi^{\otimes}_{-,?})$.

So we consider the diagram in C below.

$$\begin{tikzcd}
A(B\oplus C) & & & AB\oplus AC \arrow[lll, "\partial^r"'] \\
& (B\oplus C)A \arrow[ul, "\xi^{\otimes}"] \arrow[r, phantom, "(9.3.4)"] & BA\oplus CA \arrow[l, "\partial^l"'] \arrow[ur, "\xi^{\otimes}\oplus\xi^{\otimes}"'] & \\
& (C\oplus B)A \arrow[u, "\xi^{\oplus}1"'] \arrow[dl, "\xi^{\otimes}"] \arrow[r, phantom, "\text{left (9.1.7)}"] & CA\oplus BA \arrow[l, "\partial^l"'] \arrow[u, "\xi^{\oplus}"] \arrow[dr, "\xi^{\otimes}\oplus\xi^{\otimes}"] & \\
A(C\oplus B) \arrow[uuu, "1\xi^{\oplus}"] \arrow[ur, phantom, "\text{nat}"] & & (9.3.4) & AC\oplus AB \arrow[lll, "\partial^r"'] \arrow[uuu, "\xi^{\oplus}"'] \arrow[ul, phantom, "\text{nat}"]
\end{tikzcd}$$

The middle rectangle is the left symmetry factorization axiom, and the outer rectangle is the right symmetry factorization axiom in (9.1.7). The top and the bottom trapezoids are commutative by (9.3.4). The left and the right trapezoids are commutative by the naturality of, respectively, $\xi^{\otimes}$ and $\xi^{\oplus}$. Since $\xi^{\otimes}$ is a natural isomorphism, the middle rectangle is commutative if and only if the outer rectangle is commutative. □

Lemma 9.4.5. *Under Convention 9.4.1,*

- *the left internal factorization axiom and*
- *the right internal factorization axiom*

in (9.1.8) are equivalent to each other.

Proof. Similar to Lemma 9.4.4, this proof adapts the diagram in the proof of Lemma I.2.2.5 by

- turning each arrow around and
- replacing $(\delta^l, \delta^r, \xi^{\oplus}_{?,-}, \xi^{\otimes}_{?,-})$ with $(\partial^r, \partial^l, \xi^{\oplus}_{-,?}, \xi^{\otimes}_{-,?})$.

So we consider the diagram in C below.

$$\begin{tikzcd}
A(B \oplus C \oplus D) \arrow[ddddd, equal] \arrow[dr, phantom, "(9.3.4)"] & & A(B \oplus C) \oplus AD \arrow[ll, "\partial^r"'] & & & AB \oplus AC \oplus AD \arrow[lll, "\partial^r \oplus 1"'] \arrow[ddddd, equal] \\
& & & A(B \oplus C) \oplus DA \arrow[ul, "1 \oplus \xi^{\otimes}"'] \arrow[dr, phantom, "(9.3.4)"] & AB \oplus AC \oplus DA \arrow[l, "\partial^r \oplus 1"'] \arrow[ur, "1 \oplus \xi^{\otimes}"] & \\
& (B \oplus C \oplus D)A \arrow[uul, "\xi^{\otimes}"] \arrow[d, equal] & (B \oplus C)A \oplus DA \arrow[l, "\partial^l"'] \arrow[uu, "\xi^{\otimes} \oplus \xi^{\otimes}"] \arrow[ur, "\xi^{\otimes} \oplus 1"'] \arrow[d, phantom, "\text{left (9.1.8)}"] & & BA \oplus CA \oplus DA \arrow[ll, "\partial^l \oplus 1"'] \arrow[u, "\xi^{\otimes} \oplus \xi^{\otimes} \oplus 1"'] \arrow[d, equal] & \\
& (B \oplus C \oplus D)A \arrow[ddl, "\xi^{\otimes}"'] \arrow[dr, phantom, "(9.3.4)"] & BA \oplus (C \oplus D)A \arrow[l, "\partial^l"'] \arrow[dd, "\xi^{\otimes} \oplus \xi^{\otimes}"'] \arrow[dr, "\xi^{\otimes} \oplus 1"] & & BA \oplus CA \oplus DA \arrow[ll, "1 \oplus \partial^l"'] \arrow[d, "\xi^{\otimes} \oplus 1"'] & \\
& & & AB \oplus (C \oplus D)A \arrow[dl, "1 \oplus \xi^{\otimes}"] \arrow[dr, phantom, "(9.3.4)"] & AB \oplus CA \oplus DA \arrow[l, "1 \oplus \partial^l"'] \arrow[dr, "1 \oplus \xi^{\otimes} \oplus \xi^{\otimes}"'] & \\
A(B \oplus C \oplus D) & & AB \oplus A(C \oplus D) \arrow[ll, "\partial^r"'] & & & AB \oplus AC \oplus AD \arrow[lll, "1 \oplus \partial^r"']
\end{tikzcd}$$

- The six unlabeled subdiagrams are commutative by the functoriality of $\oplus$ or definition.
- Four subdiagrams are commutative by (9.3.4) as indicated.
- The middle rectangle is the left internal factorization axiom, and the outer diagram is the right internal factorization axiom in (9.1.8).

Since $\xi^{\otimes}$ is a natural isomorphism, the outer diagram is commutative if and only if the middle rectangle is commutative. $\square$

Lemma 9.4.6. *Under Convention 9.4.1, the following two statements hold for the external factorization axioms (9.1.9)–(9.1.11).*

- *(9.1.9) = (9.1.11).*
- *(9.1.11) implies (9.1.10).*

Proof. Similar to Lemmas 9.4.4 and 9.4.5, the two assertions are proved by adapting the diagrams in the proofs of, respectively, Lemmas I.2.2.6 and I.2.2.7 by

- turning each arrow around and
- replacing $(\delta^l, \delta^r, \xi^{\oplus}_{?,-}, \xi^{\otimes}_{?,-})$ with $(\partial^r, \partial^l, \xi^{\oplus}_{-,?}, \xi^{\otimes}_{-,?})$.

For the first assertion, we consider the following diagram in C.

$$
\begin{tikzcd}
AB(C\oplus D) \arrow[ddddd, equal] & & & & ABC\oplus ABD \arrow[llll, "\partial^r"'] \arrow[ddddd, equal] \\
& (C\oplus D)AB \arrow[ul, "\xi^{\otimes}"'] & (9.3.4) & CAB\oplus DAB \arrow[ll, "\partial^l"'] \arrow[ur, "\xi^{\otimes}\oplus\xi^{\otimes}"] & \\
(1.3.28) & (C\oplus D)BA \arrow[u, "1\xi^{\otimes}"'] \arrow[d, equal] & \mathsf{nat} & CBA\oplus DBA \arrow[ll, "\partial^l"'] \arrow[u, "1\xi^{\otimes}\oplus 1\xi^{\otimes}"] \arrow[d, equal] & (1.3.28) \\
& (C\oplus D)BA \arrow[d, "\xi^{\otimes}1"] & (CB\oplus DB)A \arrow[l, "\partial^l 1"'] \arrow[d, "(\xi^{\otimes}\oplus\xi^{\otimes})1"] \arrow[l, phantom, "(9.1.9)", shift left=4ex] & CBA\oplus DBA \arrow[l, "\partial^l"'] \arrow[d, "\xi^{\otimes}1\oplus\xi^{\otimes}1"'] & \\
& B(C\oplus D)A \arrow[dl, "\xi^{\otimes}"] \arrow[u, phantom, "(9.3.4)", shift left=6ex] & (BC\oplus BD)A \arrow[l, "\partial^r 1"'] \arrow[d, "\xi^{\otimes}"] & BCA\oplus BDA \arrow[l, "\partial^l"'] \arrow[dr, "\xi^{\otimes}\oplus\xi^{\otimes}"'] \arrow[ul, phantom, "\mathsf{nat}"] & \\
AB(C\oplus D) & \mathsf{nat} & A(BC\oplus BD) \arrow[ll, "1\partial^r"'] & (9.3.4) & ABC\oplus ABD \arrow[ll, "\partial^r"']
\end{tikzcd}
$$

We will use Proposition 1.3.36, which says that a symmetric monoidal category is precisely a braided monoidal category whose braiding satisfies the symmetry axiom (1.3.33).

- The left and the right trapezoids in the previous diagram are commutative by the third Reidemeister move (1.3.28).
- The three subdiagrams labeled by nat are commutative by the naturality of ∂^l and $\xi^{\otimes}$.
- Three other subdiagrams are commutative by (9.3.4).

Since $\xi^{\otimes}$ is a natural isomorphism, the outer diagram (9.1.11) is commutative if and only if the middle rectangle (9.1.9) is commutative. This proves the first assertion.

For the second assertion, we consider the following diagram in C.

$$
\begin{tikzcd}
A(B\oplus C)D \arrow[dr, "\xi^{\otimes}"] \arrow[ddddd, equal] & & (AB\oplus AC)D \arrow[ll, "\partial^r 1"'] \arrow[d, "\xi^{\otimes}"'] & & ABD\oplus ACD \arrow[ll, "\partial^l"'] \arrow[dl, "\xi^{\otimes}\oplus\xi^{\otimes}"'] \arrow[ddddd, equal] \\
& DA(B\oplus C) \arrow[d, equal] \arrow[ur, phantom, "\mathsf{nat}"] & D(AB\oplus AC) \arrow[l, "1\partial^r"'] & DAB\oplus DAC \arrow[l, "\partial^r"'] \arrow[d, equal] \arrow[ul, phantom, "(9.3.4)"] & \\
& DA(B\oplus C) & (9.1.11) & DAB\oplus DAC \arrow[ll, "\partial^r"'] & \\
(1.3.17) & AD(B\oplus C) \arrow[u, "\xi^{\otimes}1"'] \arrow[d, equal] & \mathsf{nat} & ADB\oplus ADC \arrow[ll, "\partial^r"'] \arrow[u, "\xi^{\otimes}1\oplus\xi^{\otimes}1"] \arrow[d, equal] & (1.3.17) \\
& AD(B\oplus C) & A(DB\oplus DC) \arrow[l, "1\partial^r"'] \arrow[u, phantom, "(9.1.11)"] & ADB\oplus ADC \arrow[l, "\partial^r"'] & \\
A(B\oplus C)D \arrow[ur, "1\xi^{\otimes}"'] & (9.3.4) & A(BD\oplus CD) \arrow[ll, "1\partial^l"'] \arrow[u, "1(\xi^{\otimes}\oplus\xi^{\otimes})"] & \mathsf{nat} & ABD\oplus ACD \arrow[ll, "\partial^r"'] \arrow[ul, "1\xi^{\otimes}\oplus 1\xi^{\otimes}"]
\end{tikzcd}
$$

- The left and the right trapezoids are commutative by the right hexagon axiom (1.3.17) in the permutative category $(\mathsf{C}, \xi^{\otimes})$.
- The three subdiagrams labeled by nat are commutative by the naturality of ∂^r and $\xi^{\otimes}$.
- Two other subdiagrams are commutative by (9.3.4).

Since $\xi^{\otimes}$ is invertible, if (9.1.11) holds, then the outer diagram, which is (9.1.10), is commutative. This proves the second assertion. $\square$

Next is the main observation of this section. It says that about half of the ring category axioms in Definition 9.1.2 are redundant in a bipermutative category in Definition 9.3.2.

Theorem 9.4.7. *Suppose the tuple*

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big)$$

consists of

- *a permutative category* $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$*,*
- *a permutative category* $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$*, and*
- *natural transformations* ∂^l *and* ∂^r *as in (9.1.3).*

Then C *is a bipermutative category if and only if it satisfies*

- *the multiplicative zero axiom (9.1.4),*
- *the left or the right zero factorization axioms (9.1.5),*
- *the left or the right unit factorization axiom (9.1.6),*
- *the left or the right symmetry factorization axiom (9.1.7),*
- *the left or the right internal factorization axiom (9.1.8),*
- *the left (9.1.9) or the right (9.1.11) external factorization axiom,*
- *the 2-by-2 factorization axiom (9.1.12), and*
- *the bipermutative category axioms (9.3.3) and (9.3.4).*

Proof. The *only if* implication follows from Definitions 9.1.2 and 9.3.2. The converse follows from Lemmas 9.4.2 through 9.4.6. $\square$

9.5. Braided Ring Categories

In this section, we first define the braided analogue of a bipermutative category (Definition 9.3.2) by relaxing the multiplicative structure to a braided strict monoidal category. Since the braiding $\xi^{\otimes}$ no longer satisfies the symmetry axiom (1.3.33) in general, we also need to include the analogues of both bipermutative category axioms (9.3.3) and (9.3.4) with each instance of $\xi^{\otimes}_{?,-}$ replaced by $\xi^{\otimes}_{-,?}$ in the opposite direction. This is reminiscent of the fact that, in a braided monoidal category (Definition 1.3.15), there are two hexagon axioms (1.3.17), both of which are equivalent to the hexagon axiom (1.3.35) in a symmetric monoidal category.

There are three main observations in this section.

(1) Theorem 9.5.5, which is the braided analogue of Theorem 9.4.7, says that in a braided ring category, about half of the ring category axioms are redundant.
(2) Theorem 9.5.6 is the braided analogue of Theorems 9.1.15 and 9.3.7. It identifies *tight* braided ring categories with tight braided bimonoidal categories with both monoidal structures strict, $\lambda^{\bullet} = 1$, and $\rho^{\bullet} = 1$.

(3) Corollary 9.5.11 says that each tight braided ring category is adjoint equivalent, via strong braided bimonoidal functors, to one with $\partial^l = 1$ or $\partial^r = 1$.

Definition. Recall from Definition 1.3.15 that a braided monoidal category is *strict* if the underlying monoidal category is strict. Also recall from Definition 9.1.2 that a ring category is *tight* if the factorization morphisms ∂^l and ∂^r are natural isomorphisms.

Definition 9.5.1. A *braided ring category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big)$$

consisting of

- a ring category
$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$
as in Definition 9.1.2 and
- a braided strict monoidal category structure $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$, with $\xi^{\otimes}$ called the *braiding*.

These data are required to satisfy the following axioms for objects $A, B, C \in \mathsf{C}$.

The Zero Braiding Axiom: There are equalities of morphisms as follows.

$$\begin{aligned} \xi^{\otimes}_{A,\mathbb{0}} &= 1_{\mathbb{0}} : A \otimes \mathbb{0} = \mathbb{0} \longrightarrow \mathbb{0} = \mathbb{0} \otimes A \\ \xi^{\otimes}_{\mathbb{0},A} &= 1_{\mathbb{0}} : \mathbb{0} \otimes A = \mathbb{0} \longrightarrow \mathbb{0} = A \otimes \mathbb{0} \end{aligned} \tag{9.5.2}$$

The Braiding Factorization Axiom: The diagram

$$\begin{tikzcd} (A \otimes C) \oplus (B \otimes C) \arrow[r, "\partial^l_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}"'] & (A \oplus B) \otimes C \arrow[d, "\xi^{\otimes}_{A \oplus B, C}"] \\ (C \otimes A) \oplus (C \otimes B) \arrow[r, "\partial^r_{C,A,B}"] \arrow[d, "\xi^{\otimes}_{C,A} \oplus \xi^{\otimes}_{C,B}"'] & C \otimes (A \oplus B) \arrow[d, "\xi^{\otimes}_{C, A \oplus B}"] \\ (A \otimes C) \oplus (B \otimes C) \arrow[r, "\partial^l_{A,B,C}"] & (A \oplus B) \otimes C \end{tikzcd} \tag{9.5.3}$$

is commutative.

This finishes the definition of a braided ring category. A braided ring category is *small*, respectively, *tight*, if the underlying ring category is so. ◇

By Proposition 1.3.36, a symmetric monoidal category is precisely a braided monoidal category whose braiding satisfies the symmetry axiom. Next is the ring category analogue of this fact.

Proposition 9.5.4. *A bipermutative category in Definition 9.3.2 is precisely a braided ring category whose braiding satisfies the symmetry axiom (1.3.33).*

Proof. The bipermutative category axioms (9.3.3) and (9.3.4) are, respectively, the first zero braiding axiom (9.5.2) and the top half of the braiding factorization axiom (9.5.3). In the presence of the symmetry axiom (1.3.33) for the braiding $\xi^{\otimes}$, the two zero braiding axioms,

$$\xi^{\otimes}_{-,\mathbb{0}} = 1 \quad \text{and} \quad \xi^{\otimes}_{\mathbb{0},-} = 1,$$

are equivalent. Moreover, in (9.5.3), the commutativity of the top half for all A, B, C is equivalent to the commutativity of the bottom half for all A, B, C. Therefore, the assertion follows from Proposition 1.3.36. $\square$

Reduction of Axioms. Theorem 9.4.7 shows that in a bipermutative category, about half of the ring category axioms are redundant. Next is the braided analogue.

Theorem 9.5.5. *Suppose the tuple*

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big)$$

consists of

- *a permutative category* $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$,
- *a braided strict monoidal category* $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$, *and*
- *natural transformations* ∂^l *and* ∂^r *as in (9.1.3).*

Then C *is a braided ring category if and only if it satisfies*

- *the multiplicative zero axiom (9.1.4),*
- *the left or the right zero factorization axioms (9.1.5),*
- *the left or the right unit factorization axiom (9.1.6),*
- *the left or the right symmetry factorization axiom (9.1.7),*
- *the left or the right internal factorization axiom (9.1.8),*
- *the left (9.1.9) or the right (9.1.11) external factorization axiom,*
- *the 2-by-2 factorization axiom (9.1.12), and*
- *the braided ring category axioms (9.5.2) and (9.5.3).*

Proof. The *only if* implication follows from Definitions 9.1.2 and 9.5.1. For the converse, we reuse the proofs of Lemmas 9.4.2 through 9.4.6 with the following notes.

- In the proof of Lemma 9.4.2, both
$$\xi^{\otimes}_{\mathbb{0},-} = 1_{\mathbb{0}} \quad \text{and} \quad \xi^{\otimes}_{-,\mathbb{0}} = 1_{\mathbb{0}}$$
are from the zero braiding axiom (9.5.2).
- In the proof of Lemma 9.4.3, the unity property
$$\xi^{\otimes}_{-,\mathbb{1}} = 1$$
is from the left diagram in (1.3.22).
- In the proofs of Lemmas 9.4.4 through 9.4.6, each instance of (9.3.4) is the top half of the braiding factorization axiom (9.5.3).

Therefore, the stated axioms are sufficient to imply that C is a braided ring category. $\square$

Relationship with Braided Bimonoidal Categories. A braided ring category has an underlying ring category, which has factorization morphisms ∂^l and ∂^r that go in the opposite direction as, respectively, the distributivity morphisms δ^r and δ^l in (2.1.31). So a braided ring category is, in general, not a braided bimonoidal category, or vice versa. Next is the braided analogue of Theorems 9.1.15 and 9.3.7 that identifies *tight* braided ring categories—that is, those with invertible factorization morphisms—with a subclass of tight braided bimonoidal categories.

Theorem 9.5.6. *There is a canonical bijective correspondence between*

(1) *the class of tight braided ring categories in Definition 9.5.1 and*
(2) *the class of tight braided bimonoidal categories in Definition 2.1.29 with*

- *a permutative category as the additive structure,*
- *a braided strict monoidal category as the multiplicative structure, and*
- $\lambda^\bullet = 1$ *and* $\rho^\bullet = 1$.

The correspondence between the factorization morphisms (∂^l, ∂^r) *in (1) and the distributivity morphisms* (δ^l, δ^r) *in (2) is given by the equalities*

$$\delta^l = (\partial^r)^{-1} \quad \textit{and} \quad \delta^r = (\partial^l)^{-1}. \tag{9.5.7}$$

Proof. First suppose C is a tight braided ring category. By Theorem 9.1.15, the underlying tight ring category of C yields a tight bimonoidal category with a permutative additive structure, a braided strict monoidal multiplicative structure, $\lambda^\bullet = 1$, $\rho^\bullet = 1$, and distributivity morphisms determined by (9.5.7). To see that this is a tight braided bimonoidal category, it remains to prove (i) the two Laplaza axioms (2.1.4) and (2.1.18) and (ii) their variants (2.1.32) and (2.1.33).

- (2.1.4) and (2.1.32) are equivalent to, respectively, the top and the bottom halves of the braiding factorization axiom (9.5.3).
- Since $\lambda^\bullet = 1$ and $\rho^\bullet = 1$, (2.1.18) and (2.1.33) are equivalent to the equalities
$$\xi^{\otimes}_{-,\mathbb{0}} = 1_{\mathbb{0}} = \xi^{\otimes}_{\mathbb{0},-},$$
which form the zero braiding axiom (9.5.2).

Conversely, suppose C is a tight braided bimonoidal category as in (2) in the statement of the theorem. By Theorem 2.2.1, C satisfies all 24 Laplaza axioms, so it is a tight bimonoidal category. By Theorem 9.1.15, C yields a tight ring category with the factorization morphisms

$$\partial^l = (\delta^r)^{-1} \quad \text{and} \quad \partial^r = (\delta^l)^{-1}. \tag{9.5.8}$$

To see that C is a braided ring category, we reuse the previous paragraph. The zero braiding axiom (9.5.2) is equivalent to the braided bimonoidal category axioms (2.1.18) and (2.1.33). The braiding factorization axiom (9.5.3) is equivalent to the braided bimonoidal category axioms (2.1.4) and (2.1.32). □

Example 9.5.9. By Corollary 5.2.33, the braided distortion category $\mathcal{D}^{\mathrm{br}}$ in Section 5.2 is a small tight braided bimonoidal category. Moreover, it satisfies the conditions in Theorem 9.5.6 (2). So $\mathcal{D}^{\mathrm{br}}$ is a small tight braided ring category with factorization morphisms

- $\partial^l = (\delta^r)^{-1}$ with δ^r in (5.2.26) and
- $\partial^r = (\delta^l)^{-1} = 1$. ◇

Strictification. Recall from Definition 5.1.2 and Proposition 5.1.10 that a *left permbraided category* is precisely a tight braided bimonoidal category that satisfies

- the conditions in Theorem 9.5.6 (2) and
- $\delta^l = 1$.

Moreover, the conditions $\lambda^\bullet = 1$ and $\rho^\bullet = 1$ and the axioms (2.1.18) and (2.1.33) imply

$$\xi^{\otimes}_{-,\mathbb{0}} = 1 \quad \text{and} \quad \xi^{\otimes}_{\mathbb{0},-} = 1.$$

By Definition 5.1.11 and Proposition 5.1.19, a *right permbraided category* is precisely such a tight braided bimonoidal category but with $\delta^r = 1$ instead of $\delta^l = 1$. The next observation follows from Theorem 9.5.6 and provides examples of tight braided ring categories.

Corollary 9.5.10. *Each right (respectively, left) permbraided category becomes a tight braided ring category with factorization morphisms determined by (9.5.8).*

Corollary 9.5.11. *Each tight braided ring category, when regarded as a tight braided bimonoidal category as in Theorem 9.5.6 (2), is adjoint equivalent to a right (respectively, left) permbraided category via strong braided bimonoidal functors.*

Proof. Apply the Strictification Theorems 6.3.6 and 6.3.7. □

In other words, each tight braided ring category is adjoint equivalent, via strong braided bimonoidal functors, to one with $\partial^l = 1$, respectively, $\partial^r = 1$. See also Note 9.7.5.

9.6. Ring Categorical Drinfeld and Symmetric Centers

In this section, we extend the Drinfeld center and the symmetric center to the setting of (braided) ring categories and bipermutative categories. Recall from Theorem 1.4.27 that each monoidal category has a canonically associated *Drinfeld center*, which is a braided monoidal category. Moreover, the *symmetric center* of each braided monoidal category is a symmetric monoidal category. The bimonoidal analogues of these center constructions are in Theorems 4.4.3 and 4.5.3. Both of these theorems will play a role in this section. The following table summaries these center constructions.

category	center	reference
monoidal	braided monoidal	1.4.27
braided monoidal	symmetric monoidal	1.5.3
tight bimonoidal	tight braided bimonoidal	4.4.3
braided bimonoidal	symmetric bimonoidal	4.5.3
tight ring	tight braided ring	9.6.1
braided ring with ∂^l epimorphism	bipermutative	9.6.4

Drinfeld Centers of Tight Ring Categories. For Corollary 9.6.1 below, we use Theorems 9.1.15 and 9.5.6 to regard tight (braided) ring categories (Definitions 9.1.2 and 9.5.1) as tight (braided) bimonoidal categories with

- both the additive and the multiplicative structures strict monoidal,
- $\lambda^\bullet = 1$ and $\rho^\bullet = 1$, and
- $\delta^l = (\partial^r)^{-1}$ and $\delta^r = (\partial^l)^{-1}$.

In Theorem 4.4.3, we observed that the bimonoidal Drinfeld center in Definition 4.1.2 of each tight bimonoidal category is a tight braided bimonoidal category. Next is the (braided) ring category analogue.

Corollary 9.6.1. *The bimonoidal Drinfeld center of each tight ring category is a tight braided ring category.*

Proof. This is the special case of Theorem 4.4.3 for tight ring categories. Indeed, if C is a ring category, then its structure morphisms $\alpha^\oplus$, $\lambda^\oplus$, $\rho^\oplus$, $\alpha^\otimes$, $\lambda^\otimes$, $\rho^\otimes$, $\lambda^\bullet$, and $\rho^\bullet$ are identities. Along with the multiplicative zero axiom (9.1.4) and the zero factorization axiom (9.1.5), we infer that, in its bimonoidal Drinfeld center $\overline{\mathsf{C}}^{\mathrm{bi}}$ in Definition 4.1.2,

- both the additive structure (4.1.6)–(4.1.14) and the multiplicative structure (4.1.5) are strict monoidal, and

- both multiplicative zeros (4.1.15) are identities.

Therefore, $\overline{\mathsf{C}}^{\mathsf{bi}}$ is a tight braided ring category by Theorems 4.4.3 and 9.5.6. □

Symmetric Centers of Braided Ring Categories. In Theorem 4.5.3, we observed that the bimonoidal symmetric center in Definition 4.5.1 of each braided bimonoidal category is a symmetric bimonoidal category. Next is the analogue involving braided ring categories and bipermutative categories.

Definition 9.6.2. For a braided ring category C as in Definition 9.5.1, the *symmetric center of* C is the full subcategory $\mathsf{C}^{\mathsf{sym}}$ consisting of objects $A \in \mathsf{C}$ such that the symmetry axiom

$$\xi^{\otimes}_{-,A}\xi^{\otimes}_{A,-} = 1 : A \otimes - \longrightarrow A \otimes - \tag{9.6.3}$$

holds. ◇

Theorem 9.6.4. *Suppose* C *is a braided ring category in which the left factorization morphism* ∂^l *is a natural epimorphism. Then its symmetric center inherits a bipermutative category structure.*

Proof. We adapt the proof of Theorem 4.5.3 as follows. First we check that the additive structure of C restricts to one on $\mathsf{C}^{\mathsf{sym}}$. The additive zero $\mathbb{0} \in \mathsf{C}$ is in $\mathsf{C}^{\mathsf{sym}}$ because

$$\xi^{\otimes}_{\mathbb{0},-} = 1_{\mathbb{0}} = \xi^{\otimes}_{-,\mathbb{0}}$$

by the zero braiding axiom (9.5.2).

To check that $\mathsf{C}^{\mathsf{sym}}$ is closed under $\oplus$, suppose $A, B \in \mathsf{C}^{\mathsf{sym}}$, so each of them satisfies the symmetry axiom (9.6.3). For each object $C \in \mathsf{C}$, the following diagram in C is commutative.

$$\begin{tikzcd}
(A \oplus B)C & C(A \oplus B) \arrow[l, "\xi^{\otimes}_{C,A\oplus B}"'] \arrow[dl, phantom, "(9.5.3)"] & (A \oplus B)C \arrow[l, "\xi^{\otimes}_{A\oplus B,C}"'] \arrow[dl, phantom, "(9.5.3)"] \\
AC \oplus BC \arrow[u, "\partial^l"] & CA \oplus CB \arrow[l, "\xi^{\otimes}_{C,A} \oplus \xi^{\otimes}_{C,B}"] \arrow[u, "\partial^r"'] & AC \oplus BC \arrow[l, "\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}"'] \arrow[u, "\partial^l"'] \arrow[ll, bend left, "1", "(9.6.3)"']
\end{tikzcd}$$

The above commutative diagram and the assumption that ∂^l is a natural epimorphism imply the symmetry axiom (9.6.3) for $A \oplus B$, that is, the equality

$$\xi^{\otimes}_{C,A\oplus B}\xi^{\otimes}_{A\oplus B,C} = 1 : (A \oplus B)C \longrightarrow (A \oplus B)C,$$

so $A \oplus B \in \mathsf{C}^{\mathsf{sym}}$. Therefore, restricting $(\oplus, \xi^{\oplus})$ to $\mathsf{C}^{\mathsf{sym}}$, the additive structure

$$\left(\mathsf{C}^{\mathsf{sym}}, \oplus, \mathbb{0}, \xi^{\oplus}\right)$$

satisfies all the permutative category axioms as they do in C.

Since the multiplicative structure of C is a braided strict monoidal category, Proposition 1.5.3 shows that the multiplicative structure

$$\left(\mathsf{C}^{\mathsf{sym}}, \otimes, \mathbb{1}, \xi^{\otimes}\right)$$

is a symmetric strict monoidal category, that is, a permutative category. Equipped with the restrictions of the factorization morphisms ∂^l and ∂^r, $\mathsf{C}^{\mathsf{sym}}$ satisfies all the axioms of a braided ring category in Definition 9.5.1 as they do in C. Moreover, since the braiding $\xi^{\otimes}$ in $\mathsf{C}^{\mathsf{sym}}$ satisfies the symmetry axiom (1.3.33) by construction, $\mathsf{C}^{\mathsf{sym}}$ is a bipermutative category by Proposition 9.5.4. □

Remark 9.6.5. Theorem 9.6.4 is *not* a corollary of Theorem 4.5.3. To apply Theorem 4.5.3 to a braided ring category C, we would need to regard C as a braided bimonoidal category using Theorem 9.5.6, which assumes tightness, that is, the invertibility of the factorization morphisms ∂^l and ∂^r. While tightness is sufficient to infer that the symmetric center is a bipermutative category, it is not necessary. The proof of Theorem 9.6.4 shows that we only need the left factorization morphism ∂^l to be a natural epimorphism, instead of the invertibility of both ∂^l and ∂^r. ◇

9.7. Notes

9.7.1 (Ring Categories)**.** The notion of a ring category in Definition 9.1.2 is from [**EM06**, Def. 3.3], which did not explicitly state the unit factorization axiom (9.1.6) and the second identity in (9.1.14). The following table compares the axioms, notation, and terminology in our Definition 9.1.2 of a ring category with the one in [**EM06**].

Definition 9.1.2	[**EM06**, Def. 3.3]
factorization morphisms ∂^l and ∂^r	distributivity maps d_l and d_r
(9.1.4)	axiom (a)
(9.1.8)	axiom (b)
(9.1.7)	axiom (c)
(9.1.9) + (9.1.11)	axiom (d)
(9.1.10)	axiom (e)
(9.1.12)	axiom (f)

We call ∂^l and ∂^r factorization morphisms, instead of *distributivity maps* as in [**EM06**], because these natural transformations are not invertible in general. Moreover, they go in the opposite direction as the distributivity morphisms δ^r and δ^l in a bimonoidal category (Definition 2.1.1). ◇

9.7.2 (Endomorphism Ring Categories)**.** For a small permutative category C, the endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ is an example in [**EM06**, p. 176-177], but we provided all the detail in the proof of Theorem 9.2.14. ◇

9.7.3 (Bipermutative Categories)**.** The notion of a bipermutative category in Definition 9.3.2 is from [**EM06**, Def. 3.6], which did not explicitly state the zero symmetry axiom (9.3.3). Immediately above [**EM06**, Def. 3.6], there is a statement that claims the following:

- Laplaza's symmetric bimonoidal categories are more general than bipermutative categories.
- Symmetric bimonoidal categories can be strictified to equivalent right bipermutative categories.

We emphasize that these claims are only true with the tightness assumption.

- Symmetric bimonoidal categories (Definition 2.1.1) have distributivity natural monomorphisms δ^l and δ^r as in (2.1.3) that go in the opposite direction as the factorization morphisms ∂^r and ∂^l. Since none of them are required to be invertible in general, bipermutative categories are not, in general, symmetric bimonoidal categories. The correct statement is Theorem 9.3.7 that identifies *tight* bipermutative categories with a subclass of *tight* symmetric bimonoidal categories.

- Strictification of symmetric bimonoidal categories to right or left bipermutative categories also requires the tightness assumption. See Theorems I.5.4.6 and I.5.4.7. $\diamond$

9.7.4 (Bipermutative Category Axioms)**.** In a bipermutative category, the redundancy of the middle external factorization axiom (9.1.10) in Lemma 9.4.6 was first observed by Elmendorf-Mandell [**EM06**, Fig. 1]. $\diamond$

9.7.5 (Braided Ring Categories)**.** Braided bimonoidal categories in the sense of Richter [**Ric10**, Def. 5.1] are right permbraided categories in Definition 5.1.11. Equivalently, Richter's braided bimonoidal categories are precisely the braided ring categories in Definition 9.5.1 such that $\partial^l = 1$. So Richter's braided bimonoidal categories appear as the target of the strictification Corollary 9.5.11. Note that our braided bimonoidal categories (Definition 2.1.29) and braided ring categories (Definition 9.5.1) are both strictly more general than Richter's braided bimonoidal categories. $\diamond$

CHAPTER 10

Iterated and E_n-Monoidal Categories

This chapter discusses iterated monoidal categories and their ring analogues, which are called, respectively, n-fold monoidal categories and E_n-monoidal categories. The notion of an n-fold monoidal category simultaneously generalizes strict monoidal categories, braided monoidal categories, and symmetric monoidal categories. Adding a compatible permutative category structure, E_n-monoidal categories simultaneously generalize ring categories, braided ring categories, and bipermutative categories. Theorem III.13.2.1 shows that the categorical operad Mon^n that parametrizes small n-fold monoidal categories is an E_n-operad. The importance of E_n-monoidal categories lies in algebraic K-theory. In Corollary III.13.5.2, we will show that the Elmendorf-Mandell K-theory multifunctor sends small E_n-monoidal categories to symmetric spectra with an E_n structure for $1 \leq n < \infty$. Appendix III.A.2 and Question III.A.4.2 contain open questions related to iterated and E_n-monoidal categories.

Iterated Monoidal Categories. The notion of an n-fold monoidal category is from [**BFSV03**]. An n-fold monoidal category C is equipped with n strict monoidal structures

$$\otimes_1, \ldots, \otimes_n : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C},$$

a common strict monoidal unit $\mathbb{1} \in \mathsf{C}$, and exchange natural transformations

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D)$$

for $1 \leq i < j \leq n$, subject to unity, associativity, and exchange axioms. The $\eta^{i,j}$ are *not* required to be natural isomorphisms. The strictness of $\{\otimes_i\}_{1 \leq i \leq n}$ and $\mathbb{1}$ is assumed for convenience to reduce the amount of data and axioms, especially for E_n-monoidal categories. The strictness conditions can be replaced by suitable associativity isomorphisms; see Note 10.11.2. The strict version, as originally introduced in [**BFSV03**], is already sufficient for the intended algebraic K-theory application in Chapter III.13.

The original motivation and terminology for n-fold monoidal categories came from iterated loop space theory. A loop space of a 1-fold loop space is a 2-fold loop space. A loop space of a 2-fold loop space is a 3-fold loop space, and so forth. Following this line of thinking, one starts with small strict monoidal categories, which correspond to 1-fold loop spaces via the group completions of the classifying spaces. A small 2-fold monoidal category is a monoid in a suitable category of small strict monoidal categories. A small 3-fold monoidal category is a monoid in a suitable category of small 2-fold monoidal categories, and so forth. Small n-fold monoidal categories yield n-fold loop spaces via the group completions of their classifying spaces [**BFSV03**]. Moreover, the categorical operad Mon^n

for small n-fold monoidal categories is operadically equivalent, via the classifying space construction, to the little n-cube operad of Boardman-Vogt [**BV73**] and May [**May72**]. Therefore, Mon^n is an E_n-operad; see Theorem III.13.2.1.

With an appropriate concept of n-fold monoidal functors, we will show that small $(n+1)$-fold monoidal categories are precisely the monoids in the monoidal category of small n-fold monoidal categories. Moreover, braided strict monoidal categories (Definition 1.3.15) are special cases of 2-fold monoidal categories. Permutative categories (Definition 1.3.32) are special cases of n-fold monoidal categories for $n \geq 2$. The following table summarizes the relationships between the various types of small monoidal categories and the iterated loop spaces that they yield via the group completions of their classifying spaces.

	n-fold monoidal categories	loop spaces	references
$n = 1$	strict monoidal categories (10.1.9)	1-fold	[**MS76, Seg74**]
$n = 2$	contain braided s.m.c. (10.1.14)	2-fold	[**Fie∞**]
$2 \leq n < \infty$	contain permutative categories (10.1.21)	n-fold	[**BFSV03, FV03, FSV13**]
$n = \infty$	permutative categories	infinite	[**May72, May74**]

In the $n = 2$ row, *s.m.c.* is short for strict monoidal categories.

E_n**-Monoidal Categories.** Since our intended applications are E_n structures via algebraic K-theory, we should start with permutative categories, which correspond to symmetric spectra under the Segal K-theory functor and the Elmendorf-Mandell K-theory multifunctor. When combined with n-fold monoidal categories, this leads to our notion of an E_n-monoidal category. Conceptually, an E_n-monoidal category is to an n-fold monoidal category as a ring category is to a strict monoidal category. In terms of data, an E_n-monoidal category C consists of

- a permutative category structure $(\oplus, \mathbb{0}, \xi^{\oplus})$,
- an n-fold monoidal category structure $\big(\{\otimes_i\}_{1\leq i\leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1\leq i<j\leq n}\big)$, and
- factorization natural transformations

$$(A \otimes_i C) \oplus (B \otimes_i C) \xrightarrow{\partial^{l,i}_{A,B,C}} (A \oplus B) \otimes_i C$$
$$(A \otimes_i B) \oplus (A \otimes_i C) \xrightarrow{\partial^{r,i}_{A,B,C}} A \otimes_i (B \oplus C)$$

for $1 \leq i \leq n$.

The tuple $(\mathsf{C}, \oplus, \otimes_i, \partial^{l,i}, \partial^{r,i})$ is required to be a ring category (Definition 9.1.2) for each $1 \leq i \leq n$. There are also several axioms that express the compatibility between the exchanges $\eta^{i,j}$ and the factorizations $\partial^{l,i}$ and $\partial^{r,i}$.

The notion of an E_n-monoidal category simultaneously generalizes the three types of ring categories in Chapter 9. An E_1-monoidal category is precisely a ring category. Braided ring categories (Definition 9.5.1) are special cases of E_2-monoidal categories. Bipermutative categories (Definition 9.3.2) are special cases of E_n-monoidal categories for $n \geq 2$.

Connection with Algebraic K-Theory. As proved in [**EM06, EM09**], which we will discuss in Corollaries III.11.3.16 and III.11.6.12, the Elmendorf-Mandell K-theory multifunctor sends

- small ring categories to E_1-symmetric spectra, that is, strict ring symmetric spectra and
- small bipermutative categories to E_∞-symmetric spectra.

In Corollaries III.12.5.3 and III.13.5.2, we will prove the E_n variants, with

- small braided ring categories (Definition 9.5.1) yielding E_2-symmetric spectra and
- small E_n-monoidal categories yielding E_n-symmetric spectra.

For the case $n = 2$, braided ring categories are simpler than E_2-monoidal categories, so they are better as inputs of the K-theory multifunctor.

The result relating small E_n-monoidal categories and E_n-symmetric spectra is a consequence of the existence of the Elmendorf-Mandell K-theory multifunctor and Theorem III.13.4.12. The latter says that an E_n-monoidal structure on a small permutative category C is precisely determined by a Cat-enriched multifunctor

$$F : \mathsf{Mon}^n \longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \text{such that} \quad F(*) = \mathsf{C}.$$

Here Mon^n is the one-object Cat-enriched multicategory for n-fold monoidal categories, and $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-enriched multicategory of small permutative categories. The following table summarizes the relationships between the various types of small ring-like categories and the E_n-symmetric spectra that they yield via the Elmendorf-Mandell K-theory multifunctor.

	categories	symmetric spectra
$n = 1$	ring (9.1.2)	strict ring (III.11.3.16)
$n = 2$	braided ring (9.5.1)	E_2 (III.12.5.3)
$2 \le n < \infty$	E_n-monoidal (10.7.2)	E_n (III.13.5.2)
$n = \infty$	bipermutative (9.3.2)	E_∞ (III.11.6.12)

Organization. A summary of the rest of this chapter follows.

Section 10.1 defines n-fold monoidal categories and studies their relationships with braided monoidal and permutative categories.

- Proposition 10.1.14 proves that braided strict monoidal categories are 2-fold monoidal categories in which
  - the two monoidal products coincide, and
  - the exchange η is a natural isomorphism that satisfies a middle unity property.
- Proposition 10.1.21 proves that permutative categories are n-fold monoidal categories for $n \geq 2$ in which
  - the n monoidal products coincide, and
  - the exchanges $\eta^{i,j}$ are natural isomorphisms that coincide for all $1 \le i < j \le n$ and satisfy a middle unity property and a middle symmetry property.

We emphasize that Definition 10.1.1 of an n-fold monoidal category is *not* an inductive definition. It is defined explicitly for all $n \geq 1$ at the same time.

Section 10.2 contains examples of 2-fold monoidal categories that do not come from braided strict monoidal categories; see Proposition 10.2.8. These 2-fold monoidal categories are constructed from totally ordered monoids, which are monoids equipped with a total ordering that is compatible with the monoid multiplication, such that the monoid unit is also the least element with respect to the ordering. In the associated 2-fold monoidal category, $\otimes_1$ is max with respect to the ordering, and $\otimes_2$ is the monoid multiplication.

Section 10.3 defines n-fold monoidal functors between n-fold monoidal categories and studies their relationships with braided monoidal functors.

- Proposition 10.3.11 shows that, between braided strict monoidal categories, a braided strictly unital monoidal functor is a 2-fold monoidal functor whose two monoidal constraints coincide. The converse holds if the monoidal constraint is an isomorphism.
- Proposition 10.3.15 shows that, between permutative categories, a symmetric strictly unital monoidal functor is an n-fold monoidal functor for $n \geq 2$ whose n monoidal constraints coincide. The converse holds if the monoidal constraint is an isomorphism.
- Lemma 10.3.20 shows that composites of n-fold monoidal functors are well defined.

The category of small n-fold monoidal categories and n-fold monoidal functors is denoted by MCat^n.

Section 10.4 studies monoids in MCat^n.

- Lemma 10.4.2 shows that MCat^n is a monoidal category with the Cartesian product.
- Theorem 10.4.5 shows that monoids in the monoidal category MCat^n are precisely small $(n+1)$-fold monoidal categories.

However, morphisms of monoids in MCat^n are *not* precisely $(n+1)$-fold monoidal functors. Proposition 10.4.13 shows that a morphism of monoids in MCat^n is precisely an $(n+1)$-fold monoidal functor whose last monoidal constraint is the identity.

Section 10.5 studies the free n-fold monoidal category of a small category in Proposition 10.5.9. Theorem 10.5.18 shows that the free n-fold monoidal category of a small category decomposes into smaller pieces involving the categories $\mathsf{Mon}^n(k)$ in Definition 10.5.13. In Definition III.13.1.12, we will use $\{\mathsf{Mon}^n(k)\}_{k\geq 0}$ to form a Cat-enriched operad Mon^n that acts on small n-fold monoidal categories. Moreover, we will see that Mon^n also parametrizes E_n-monoidal category structures on small permutative categories. This will lead to the observation that the Elmendorf-Mandell K-theory of a small E_n-monoidal category is an E_n-symmetric spectrum.

Section 10.6 discusses the Coherence Theorem 10.6.8 for n-fold monoidal categories from [**BFSV03**]. This theorem provides necessary and sufficient conditions for the existence of a morphism $A \longrightarrow B$ in the category $\mathsf{Mon}^n(k)$. Moreover, each nonempty morphism set in $\mathsf{Mon}^n(k)$ has a unique element. As a consequence, in each n-fold monoidal category, each formal diagram built from identity morphisms, the exchanges $\{\eta^{i,j}\}_{i<j}$, the monoidal products $\{\otimes_i\}_{i=1}^n$, and composites is commutative. In Part III.2, Theorem 10.6.8 will be applied in

- Theorem III.13.2.1, which says that the n-fold monoidal category operad Mon^n is an E_n-operad, and
- Theorem III.13.3.3, which is a coherence theorem for Mon^n.

Section 10.7 introduces the notion of an E_n-monoidal category, which combines

- n ring categories
$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes_i, \mathbb{1}), (\partial^{l,i}, \partial^{r,i})\big)$$
for $1 \leq i \leq n$ and
- a compatible n-fold monoidal category
$$\big(\mathsf{C}, \{\otimes_i\}_{1\leq i\leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1\leq i<j\leq n}\big).$$

By definition, an E_1-monoidal category is a ring category. The compatibility between the n ring categories and the n-fold monoidal category is expressed in the axioms (10.7.7)–(10.7.11). The first axiom (10.7.7) says that $\eta^{i,j}$ is the identity of $\mathbb{0}$ if any one of its four input objects is $\mathbb{0}$. The other four axioms, which are called the exchange factorization axioms, express how the exchanges $\eta^{i,j}$ commute with the factorization morphisms $\partial^{l,i}$ and $\partial^{r,i}$.

Section 10.8 shows that the relationship between braided strict monoidal categories and 2-fold monoidal categories extends to the ring setting. More precisely, Theorem 10.8.1 proves that a braided ring category is precisely an E_2-monoidal category in which

- the two ring category structures coincide, and
- the only exchange η is a natural isomorphism that satisfies a middle unity property.

The nontrivial part is to show that the ring category axioms and the braiding factorization axiom (9.5.3) in a braided ring category correspond to the exchange factorization axioms (10.7.8)–(10.7.11) in an E_2-monoidal category.

Section 10.9 is the permutative analogue of Section 10.8. Theorem 10.9.1 shows that a bipermutative category is precisely an E_n-monoidal category for $n \geq 2$ that satisfies the following conditions.

- The n ring category structures coincide.
- All the exchanges $\eta^{i,j}$ are natural isomorphisms that coincide for all $1 \leq i < j \leq n$ and satisfy a middle unity property and a middle symmetry property.

Section 10.10 provides further examples of E_n-monoidal categories by showing that each small category freely generates an E_n-monoidal category. This construction extends the one in Section 10.5 for free n-fold monoidal categories.

Reading Guide.

(1) For n-fold monoidal categories, read Definition 10.1.1 and the statements of Propositions 10.1.14, 10.1.21, and 10.2.8.
(2) For n-fold monoidal functors, read Definition 10.3.1 and the statements of Propositions 10.3.11 and 10.3.15, Lemmas 10.3.20 and 10.4.2, and Theorem 10.4.5.
(3) For free n-fold monoidal categories, read Definitions 10.5.2, 10.5.8, and 10.5.13 and the statements of Proposition 10.5.9 and Theorem 10.5.18.
(4) For the coherence of n-fold monoidal categories, read Definition 10.6.1 and the statement of Theorem 10.6.8.
(5) For E_n-monoidal categories, read Definition 10.7.2 and the statements of Theorems 10.8.1 and 10.9.1 and Proposition 10.10.2.
(6) Go back and read the rest of this chapter.

10.1. Iterated Monoidal Categories

In this section, we define n-fold monoidal categories following [**BFSV03**] and provide some examples. These iterated monoidal categories contain braided strict monoidal categories and permutative categories:

- Proposition 10.1.14 identifies braided strict monoidal categories with a subclass of 2-fold monoidal categories.
- Proposition 10.1.21 identifies permutative categories with a subclass of n-fold monoidal categories for $n \geq 2$.

Proposition 10.2.8 shows that each totally ordered monoid such that the unit is also the least element has a canonical 2-fold monoidal category structure that does not, in general, come from a braided strict monoidal category. With the n-fold monoidal functors in Definition 10.3.1, in Section 10.4 we will see that small n-fold monoidal categories form a monoidal category MCat^n. Moreover, the monoids in MCat^n are precisely small $(n+1)$-fold monoidal categories.

Definition. Recall from Definition 1.3.1 that a monoidal category is *strict* if the associativity isomorphism α, the left unit isomorphism λ, and the right unit isomorphism ρ are identity natural transformations. In this case, they are omitted from the notation.

Definition 10.1.1. For $n \geq 1$, an *n-fold monoidal category* is a tuple

$$\big(\mathsf{C}, \{\otimes_i\}_{1 \leq i \leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \leq i < j \leq n}\big)$$

consisting of the following data.

The Underlying Category: C is a category.
The Unit: $\mathbb{1} \in \mathsf{C}$ is an object, which is called the *unit*
The Multiplicative Structures: For each $1 \leq i \leq n$,

$$(\mathsf{C}, \otimes_i, \mathbb{1})$$

is a strict monoidal category, which is called the *ith monoidal structure*, with $\otimes_i$ called the *ith product*
The Exchanges: For each pair (i, j) with $1 \leq i < j \leq n$,

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D) \tag{10.1.2}$$

is a natural transformation for objects $A, B, C, D \in \mathsf{C}$, which is called the (i, j)*-exchange*

These data are required to satisfy the following equalities and commutative diagrams for objects A, A', A'', B, B', B'', C, C', D, and D' in C. The axioms (10.1.3)–(10.1.6) are defined for $1 \leq i < j \leq n$. The axiom (10.1.7) is defined for $1 \leq i < j < k \leq n$.

The Internal Unity Axiom:

$$\eta^{i,j}_{A,B,\mathbb{1},\mathbb{1}} = 1_{A \otimes_j B} = \eta^{i,j}_{\mathbb{1},\mathbb{1},A,B} \tag{10.1.3}$$

The External Unity Axiom:

$$\eta^{i,j}_{A,\mathbb{1},B,\mathbb{1}} = 1_{A \otimes_i B} = \eta^{i,j}_{\mathbb{1},A,\mathbb{1},B} \tag{10.1.4}$$

The Internal Associativity Axiom:

$$
\begin{tikzcd}[column sep=-2em]
& (A \otimes_j A') \otimes_i (B \otimes_j B') \otimes_i (C \otimes_j C') \arrow[dl, "1_{A \otimes_j A'} \otimes_i \eta^{i,j}_{B,B',C,C'}"'] \arrow[dr, "\eta^{i,j}_{A,A',B,B'} \otimes_i 1_{C \otimes_j C'}"] & \\
(A \otimes_j A') \otimes_i [(B \otimes_i C) \otimes_j (B' \otimes_i C')] \arrow[dr, "\eta^{i,j}_{A,A',B \otimes_i C,B' \otimes_i C'}"'] & & [(A \otimes_i B) \otimes_j (A' \otimes_i B')] \otimes_i (C \otimes_j C') \arrow[dl, "\eta^{i,j}_{A \otimes_i B,A' \otimes_i B',C,C'}"] \\
& (A \otimes_i B \otimes_i C) \otimes_j (A' \otimes_i B' \otimes_i C') &
\end{tikzcd}
\tag{10.1.5}
$$

The External Associativity Axiom:

$$
\begin{tikzcd}[column sep=-2em]
& (A \otimes_j A' \otimes_j A'') \otimes_i (B \otimes_j B' \otimes_j B'') \arrow[dl, "\eta^{i,j}_{A,A' \otimes_j A'',B,B' \otimes_j B''}"'] \arrow[dr, "\eta^{i,j}_{A \otimes_j A',A'',B \otimes_j B',B''}"] & \\
(A \otimes_i B) \otimes_j [(A' \otimes_j A'') \otimes_i (B' \otimes_j B'')] \arrow[dr, "1_{A \otimes_i B} \otimes_j \eta^{i,j}_{A',A'',B',B''}"'] & & [(A \otimes_j A') \otimes_i (B \otimes_j B')] \otimes_j (A'' \otimes_i B'') \arrow[dl, "\eta^{i,j}_{A,A',B,B'} \otimes_j 1_{A'' \otimes_i B''}"] \\
& (A \otimes_i B) \otimes_j (A' \otimes_i B') \otimes_j (A'' \otimes_i B'') &
\end{tikzcd}
\tag{10.1.6}
$$

The Triple Exchange Axiom:

$$
\begin{tikzcd}[column sep=-3em]
& [(A \otimes_k A') \otimes_j (B \otimes_k B')] \otimes_i [(C \otimes_k C') \otimes_j (D \otimes_k D')] \arrow[dl, "\eta^{j,k}_{A,A',B,B'} \otimes_i \eta^{j,k}_{C,C',D,D'}"'] \arrow[dr, "\eta^{i,j}_{A \otimes_k A',B \otimes_k B',C \otimes_k C',D \otimes_k D'}"] & \\
[(A \otimes_j B) \otimes_k (A' \otimes_j B')] \otimes_i [(C \otimes_j D) \otimes_k (C' \otimes_j D')] \arrow[d, "\eta^{i,k}_{A \otimes_j B,A' \otimes_j B',C \otimes_j D,C' \otimes_j D'}"'] & & [(A \otimes_k A') \otimes_i (C \otimes_k C')] \otimes_j [(B \otimes_k B') \otimes_i (D \otimes_k D')] \arrow[d, "\eta^{i,k}_{A,A',C,C'} \otimes_j \eta^{i,k}_{B,B',D,D'}"] \\
[(A \otimes_j B) \otimes_i (C \otimes_j D)] \otimes_k [(A' \otimes_j B') \otimes_i (C' \otimes_j D')] \arrow[dr, "\eta^{i,j}_{A,B,C,D} \otimes_k \eta^{i,j}_{A',B',C',D'}"'] & & [(A \otimes_i C) \otimes_k (A' \otimes_i C')] \otimes_j [(B \otimes_i D) \otimes_k (B' \otimes_i D')] \arrow[dl, "\eta^{j,k}_{A \otimes_i C,A' \otimes_i C',B \otimes_i D,B' \otimes_i D'}"] \\
& [(A \otimes_i C) \otimes_j (B \otimes_i D)] \otimes_k [(A' \otimes_i C') \otimes_j (B' \otimes_i D')] &
\end{tikzcd}
\tag{10.1.7}
$$

This finishes the definition of an n-fold monoidal category. It is *small* if it has a set of objects. $\diamond$

Explanation 10.1.8. Consider Definition 10.1.1 of an n-fold monoidal category.

- The n strict monoidal categories $(\mathsf{C}, \otimes_i, \mathbb{1})$ for $1 \leq i \leq n$ have a common underlying category C and a common monoidal unit $\mathbb{1}$.
- The direction of the (i,j)-exchange $\eta^{i,j}$ in (10.1.2) is reminiscent of the monoidal constraint F^2 of a monoidal functor F in Definition 1.3.7. It goes from $\otimes_j$ on the inside to the outside. Moreover, $\eta^{i,j}$ is *not* required to be a natural isomorphism.
- The triple exchange axiom (10.1.7) has the following symmetry. Each arrow and its counterpart across the center involve the same type of exchange morphisms. For example, both vertical arrows involve $\eta^{i,k}$. $\diamond$

Example 10.1.9. A 1-fold monoidal category is a strict monoidal category $(\mathsf{C}, \otimes, \mathbb{1})$, since there are no (i,j)-exchanges when $n = 1$. $\diamond$

Example 10.1.10. An $(n+1)$-fold monoidal category yields an n-fold monoidal category by forgetting the product $\otimes_{n+1}$ and the natural transformations $\eta^{i,n+1}$ for $1 \leq i < n+1$. $\diamond$

Example 10.1.11. Suppose C is an n-fold monoidal category, and

$$I \subseteq \{1, \ldots, n\}$$

is a nonempty subset with cardinality $|I|$. Then there is an $|I|$-fold monoidal category

$$\big(\mathsf{C}, \{\otimes_i\}_{i \in I}, \mathbb{1}, \{\eta^{i,j}\}_{i<j \in I}\big)$$

that retains the monoidal products $\otimes_i$ and the exchanges $\eta^{i,j}$ with $i, j \in I$. $\diamond$

Example 10.1.12. Given two n-fold monoidal categories C and D, their Cartesian product $\mathsf{C} \times \mathsf{D}$ is an n-fold monoidal category, in which the unit, the multiplicative structures, and the exchanges are all defined componentwise in C and D. $\diamond$

Braided and 2-Fold Monoidal Categories.

Explanation 10.1.13. Unpacking Definition 10.1.1 in the case $n = 2$, a 2-fold monoidal category

$$\big(\mathsf{C}, \otimes_1, \otimes_2, \mathbb{1}, \eta\big)$$

consists of

- a category C,
- a unit object $\mathbb{1} \in \mathsf{C}$,
- two strict monoidal structures $(\mathsf{C}, \otimes_1, \mathbb{1})$ and $(\mathsf{C}, \otimes_2, \mathbb{1})$, and
- a natural transformation

  $$(A \otimes_2 B) \otimes_1 (C \otimes_2 D) \xrightarrow{\eta_{A,B,C,D}} (A \otimes_1 C) \otimes_2 (B \otimes_1 D)$$

  for objects $A, B, C, D \in \mathsf{C}$.

These data are required to satisfy the axioms (10.1.3)–(10.1.6) for $(i, j) = (1, 2)$ with

$$\eta^{1,2} = \eta.$$

The triple exchange axiom (10.1.7), which requires $1 \leq i < j < k \leq n$, does not happen when $n = 2$. $\diamond$

Recall from Definition 1.3.15 that a braided monoidal category is *strict* if the underlying monoidal category is strict. The next observation identifies braided strict monoidal categories with a subclass of 2-fold monoidal categories.

Proposition 10.1.14. *There is a canonical bijective correspondence between*

1. *the class of braided strict monoidal categories and*
2. *the class of 2-fold monoidal categories in Explanation 10.1.13 with*
   - $\otimes_1 = \otimes_2$ *and*
   - η *a natural isomorphism satisfying*

$$\eta_{A,B,\mathbb{1},C} = 1_{A \otimes_1 B \otimes_1 C} = \eta_{A,\mathbb{1},B,C}. \tag{10.1.15}$$

Proof. First suppose $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a braided strict monoidal category with braiding

$$A \otimes B \xrightarrow[\cong]{\xi^{\otimes}_{A,B}} B \otimes A.$$

We define the monoidal structures $\otimes_1$ and $\otimes_2$ and the natural isomorphism η as follows.

$$\begin{gathered} \otimes_1 = \otimes = \otimes_2 \\ \eta_{A,B,C,D} = 1_A \otimes \xi^{\otimes}_{B,C} \otimes 1_D \end{gathered} \tag{10.1.16}$$

We check the 2-fold monoidal category axioms (10.1.3)–(10.1.6) and (10.1.15).

- The condition (10.1.15), the internal unity axiom (10.1.3)

$$\eta_{A,B,\mathbb{1},\mathbb{1}} = 1_{A\otimes B} = \eta_{\mathbb{1},\mathbb{1},A,B},$$

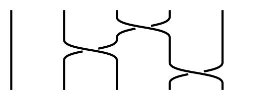
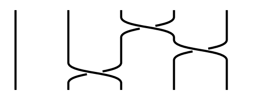
  and the external unity axiom (10.1.4)

$$\eta_{A,\mathbb{1},B,\mathbb{1}} = 1_{A\otimes B} = \eta_{\mathbb{1},A,\mathbb{1},B}$$

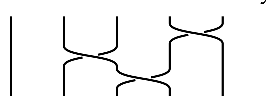
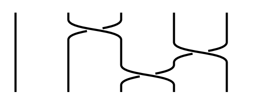
  all follow from the unity properties

$$\xi^{\otimes}_{-,\mathbb{1}} = 1 = \xi^{\otimes}_{\mathbb{1},-}$$

  in a braided strict monoidal category (1.3.22).
- The internal associativity axiom (10.1.5) holds by the Coherence Theorem 1.6.3 in the braided strict monoidal category C. Indeed, the left and right composites in (10.1.5) have, respectively, the left and right braids below, read bottom-to-top, as their underlying braids.

  Since their underlying braids are equal, the two composites in (10.1.5) are equal.
- The external associativity axiom (10.1.6) also holds by Theorem 1.6.3. The left and right composites in (10.1.6) have, respectively, the left and right braids below as their underlying braids.

  Since their underlying braids are equal, the two composites in (10.1.6) are equal.

This shows that

$$(\mathsf{C},\otimes_1,\otimes_2,\mathbb{1},\eta)$$

is a 2-fold monoidal category as in (2) in the statement.

Conversely, suppose

$$(\mathsf{C},\otimes,\otimes,\mathbb{1},\eta)$$

is a 2-fold monoidal category as in (2) in the statement, so $(\mathsf{C},\otimes,\mathbb{1})$ is a strict monoidal category. The component $\xi^{\otimes}_{A,B}$ of the desired braiding $\xi^{\otimes}$ is defined as the composite below.

(10.1.17)
$$\begin{tikzcd} A\otimes B \arrow[r, "\xi^{\otimes}_{A,B}"] \arrow[d, equal] & B\otimes A \arrow[d, equal] \\ \mathbb{1}\otimes A\otimes B\otimes \mathbb{1} \arrow[r, "\eta_{\mathbb{1},A,B,\mathbb{1}}"] & \mathbb{1}\otimes B\otimes A\otimes \mathbb{1} \end{tikzcd}$$

This is a natural isomorphism because η is so. It remains to check the two hexagon axioms (1.3.17) in $(\mathsf{C},\otimes,\mathbb{1},\xi^{\otimes})$. First we need some preliminary equalities.

- With $A' = B = \mathbb{1}$ in the internal associativity axiom (10.1.5),
  - the lower left arrow $\eta_{A,\mathbb{1},C,B'\otimes C'}$ and
  - the upper right arrow $\eta_{A,\mathbb{1},\mathbb{1},B'} \otimes 1$

are both identities by (10.1.15). Switching symbols from (B', C') to (B, D), this yields the equality

$$1_A \otimes \eta_{\mathbb{1},B,C,D} = \eta_{A,B,C,D}. \tag{10.1.18}$$

- With $B' = C = \mathbb{1}$ in the internal associativity axiom (10.1.5),
  - the upper left arrow $1 \otimes \eta_{B,\mathbb{1},\mathbb{1},C'}$ and
  - the lower right arrow $\eta_{A\otimes B,A',\mathbb{1},C'}$

  are both identities by (10.1.15). Switching symbols from (A', B, C') to (B, C, D), this yields the equality

$$\eta_{A,B,C,D} = \eta_{A,B,C,\mathbb{1}} \otimes 1_D. \tag{10.1.19}$$

- (10.1.17), (10.1.18), and (10.1.19) yield the following equalities.

$$1_A \otimes \xi^{\otimes}_{B,C} = \eta_{A,B,C,\mathbb{1}} \tag{10.1.20a}$$

$$\xi^{\otimes}_{A,B} \otimes 1_C = \eta_{\mathbb{1},A,B,C} \tag{10.1.20b}$$

$$1_A \otimes \xi^{\otimes}_{B,C} \otimes 1_D = \eta_{A,B,C,D} \tag{10.1.20c}$$

To obtain the left hexagon axiom (1.3.17), we set $A = B' = C' = \mathbb{1}$ in the internal associativity axiom (10.1.5) to obtain the equality

$$\big(\eta_{\mathbb{1},A',B\otimes C,\mathbb{1}}\big)\big(1_{A'} \otimes \eta_{B,\mathbb{1},C,\mathbb{1}}\big) = \big(\eta_{B,A',C,\mathbb{1}}\big)\big(\eta_{\mathbb{1},A',B,\mathbb{1}} \otimes 1_C\big).$$

By the first external unity axiom (10.1.4), (10.1.17), and (10.1.20a), the previous equality becomes

$$\xi^{\otimes}_{A',B\otimes C} = \big(1_B \otimes \xi^{\otimes}_{A',C}\big)\big(\xi^{\otimes}_{A',B} \otimes 1_C\big),$$

which is the left hexagon axiom.

To obtain the right hexagon axiom (1.3.17), we set $A = B = C' = \mathbb{1}$ in the internal associativity axiom (10.1.5) to obtain the equality

$$\big(\eta_{\mathbb{1},A',C,B'}\big)\big(1_{A'} \otimes \eta_{\mathbb{1},B',C,\mathbb{1}}\big) = \big(\eta_{\mathbb{1},A'\otimes B',C,\mathbb{1}}\big)\big(\eta_{\mathbb{1},A',\mathbb{1},B'} \otimes 1_C\big).$$

By the second external unity axiom (10.1.4), (10.1.17), and (10.1.20b), the previous equality becomes the equality below.

$$\big(\xi^{\otimes}_{A',C} \otimes 1_{B'}\big)\big(1_{A'} \otimes \xi^{\otimes}_{B',C}\big) = \xi^{\otimes}_{A'\otimes B',C}$$

Since this is the right hexagon axiom, $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a braided strict monoidal category.

The above constructions are inverses of each other by (10.1.16), (10.1.17), and (10.1.20c). □

Examples of 2-fold monoidal categories that do not correspond to braided strict monoidal categories will be given in Proposition 10.2.8.

Permutative and n-Fold Monoidal Categories. Recall from Definition 1.3.32 that a permutative category is a symmetric *strict* monoidal category. The next observation identifies permutative categories with a subclass of n-fold monoidal categories in Definition 10.1.1.

Proposition 10.1.21. *For each $n \geq 2$, there is a canonical bijective correspondence between the following two classes.*

(1) *The class of permutative categories.*
(2) *The class of n-fold monoidal categories satisfying the following conditions:*
   - $\otimes_1 = \otimes_2 = \cdots = \otimes_n$.

- *$\eta^{i,j} = \eta^{k,l}$ for all $1 \le i < j \le n$ and $1 \le k < l \le n$.*
- *Each $\eta^{i,j}$ is a natural isomorphism that satisfies the following equalities for $A, B, C, D \in \mathsf{C}$.*

$$\begin{gathered} \eta_{A,B,\mathbb{1},C} = 1_{A\otimes B\otimes C} = \eta_{A,\mathbb{1},B,C} \\ (\eta_{A,C,B,D})(\eta_{A,B,C,D}) = 1_{A\otimes B\otimes C\otimes D} \end{gathered} \tag{10.1.22}$$

Here $\otimes$ is the common value of $\otimes_i$ for $1 \le i \le n$, and η is the common value of $\eta^{i,j}$ for $1 \le i < j \le n$.

Proof. Suppose $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a permutative category. We define $\otimes_i$ for $1 \le i \le n$ and $\eta^{i,j}$ for $1 \le i < j \le n$ as follows.

$$\begin{gathered} \otimes_i = \otimes \\ \eta^{i,j}_{A,B,C,D} = 1_A \otimes \xi^{\otimes}_{B,C} \otimes 1_D \end{gathered} \tag{10.1.23}$$

Then we observe the following.

- The equalities (10.1.22), the internal unity axiom (10.1.3), and the external unity axiom (10.1.4) follow from the symmetry axiom (1.3.33) and the unit axiom (1.3.34) in the permutative category C.
- The other three n-fold monoidal category axioms (10.1.5)–(10.1.7) follow from the Coherence Theorem I.1.3.8 for symmetric monoidal categories.

Conversely, suppose $(\mathsf{C}, \otimes, \mathbb{1}, \eta)$ is an n-fold monoidal category as in (2) in the statement. Since $n \ge 2$, $(\mathsf{C}, \otimes, \otimes, \mathbb{1}, \eta)$ is a 2-fold monoidal category by Example 10.1.10. By Proposition 10.1.14, $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$, with $\xi^{\otimes}$ as in (10.1.17), is a braided strict monoidal category. Since $\xi^{\otimes}$ satisfies the symmetry axiom (1.3.33) by the second equality in (10.1.22), $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a permutative category by Proposition 1.3.36. Moreover, the equality (10.1.20c) is still valid here, and it is proved by the same argument as in the proof of Proposition 10.1.14.

The above constructions are inverses of each other by (10.1.17), (10.1.20c), and (10.1.23). □

10.2. Two-Fold Monoidal Categories From Totally Ordered Monoids

In this section, we give examples of 2-fold monoidal categories from totally ordered monoids that are not coming from braided strict monoidal categories in the sense of Proposition 10.1.14. See also Note 10.11.5 for related discussion. We first recall some relevant definitions.

Definition 10.2.1. Suppose S is a set.

- A *partial ordering* on S is a relation $\le$ on S that satisfies the following three conditions for all $x, y, z \in S$:
  - Reflexivity: $x \le x$.
  - Transitivity: If $x \le y$ and $y \le z$, then $x \le z$.
  - Antisymmetry: If $x \le y$ and $y \le x$, then $x = y$.
- A *partially ordered set* is a set equipped with a partial ordering.
- In a partially ordered set, if $x \le y$ and $x \ne y$, then we write $x < y$.
- In a partially ordered set, a *least element* is an element e such that $e \le x$ for all x.

- A *total ordering* on S is a partial ordering that also satisfies the comparability condition: either
$$x \le y \quad \text{or} \quad y \le x \quad \text{for} \quad x, y \in S.$$
A *totally ordered set* is a set equipped with a total ordering.
- For elements x and y in a totally ordered set $(S, \le)$, define
$$\max(x,y) = \begin{cases} x & \text{if } y \le x \text{ and} \\ y & \text{if } x < y. \end{cases}$$
We call $\max(x,y)$ the *maximum* of x and y. ⋄

Explanation 10.2.2. A partially ordered set $(S, \le)$ is also regarded as a category with object set S and morphism sets
$$S(x,y) = \begin{cases} * & \text{if } x \le y \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$
Reflexivity gives the identity morphisms. Transitivity gives the composition. Unless otherwise specified, a partially ordered set is regarded as a category in this way. Note that each nonempty morphism set has a unique element. ⋄

Recall from Definition 1.3.32 that a permutative category is a *strict* symmetric monoidal category.

Lemma 10.2.3. *Suppose $(S, \le)$ is a totally ordered set with a least element e. Then*
$$(S, \max, e)$$
is a permutative category, with symmetry isomorphism the identity.

Proof. The product max is well defined on morphisms because, if $x \le x'$ and $y \le y'$, then
$$\begin{aligned} x &\le x' \le \max(x', y') \\ y &\le y' \le \max(x', y'). \end{aligned}$$
It follows that
$$\max(x,y) \le \max(x', y').$$
On objects, the product max is associative, symmetric, and unital with respect to e. The last unity property uses the assumption that e is the least element. The associativity of max on morphisms, the functoriality of max, and the permutative category axioms hold because each nonempty morphism set has only one element. □

Definition 10.2.4. A *totally ordered monoid* is a tuple
$$(M, \mu, \mathbb{1}, \le)$$
consisting of

- a monoid $(M, \mu, \mathbb{1})$ in Set (Definition I.1.2.8) and
- a total ordering $\le$ on M

such that
$$x \le y \quad \text{implies} \quad xz \le yz \quad \text{and} \quad zx \le zy \tag{10.2.5}$$
for $x, y, z \in M$, where $\mu(x,y)$ is written as xy. ⋄

Example 10.2.6. Each of the tuples

$$(\mathbb{N}, +, 0, \leq), \quad (\mathbb{R}_{\geq 0}, +, 0, \leq), \quad (\mathbb{N}_{\geq 1}, \times, 1, \leq), \quad \text{and} \quad (\mathbb{R}_{\geq 1}, \times, 1, \leq)$$

is a totally ordered monoid, with $\leq$ the usual ordering of $\mathbb{N}$ or $\mathbb{R}$. The subscript $\geq r$ means the subset of $\mathbb{N}$ or $\mathbb{R}$ greater than or equal to r. ⋄

Lemma 10.2.7. *Suppose $(M, \mu, \mathbb{1}, \leq)$ is a totally ordered monoid. Then the following statements hold for $x, y \in M$.*

(1) *Suppose that the unit $\mathbb{1}$ is also the least element with respect to $\leq$. If $x, y \neq \mathbb{1}$, then $xy \neq \mathbb{1}$.*
(2) *If $x \leq y$ and $x' \leq y'$, then $xx' \leq yy'$.*

Proof. For assertion (1), suppose $x, y \neq \mathbb{1}$. Then

$$\mathbb{1} < x \quad \text{and} \quad \mathbb{1} < y,$$

since $\mathbb{1}$ is the least element. If $xy = \mathbb{1}$, then the axiom (10.2.5), applied to $\mathbb{1} < x$, implies

$$y = \mathbb{1}y \leq xy = \mathbb{1}.$$

Since

$$\mathbb{1} < y \quad \text{and} \quad y \leq \mathbb{1},$$

antisymmetry implies $\mathbb{1} = y$, which is a contradiction. Therefore, $xy \neq \mathbb{1}$, which proves (1). Assertion (2) follows from transitivity and

$$xx' \leq xy' \leq yy',$$

in which each $\leq$ is a result of the axiom (10.2.5). □

Recall from Explanation 10.1.13 that a 2-fold monoidal category has only one exchange $\eta^{1,2} = \eta$. Moreover, the triple exchange axiom (10.1.7) is trivially satisfied when $n = 2$. The following result is [**FSS07**, 4.3]; see also Note 10.11.5.

Proposition 10.2.8. *Suppose $(M, \mu, \mathbb{1}, \leq)$ is a totally ordered monoid such that the unit $\mathbb{1}$ is also the least element with respect to $\leq$. Then the tuple*

$$\big(M, \otimes_1 = \max, \otimes_2 = \mu, \mathbb{1}, \eta\big)$$

is a 2-fold monoidal category, with each component of η the unique morphism.

Proof. We check that the data and axioms in Explanation 10.1.13 for a 2-fold monoidal category hold for the tuple $(M, \max, \mu, \mathbb{1}, \eta)$. Recall that a morphism

$$x \longrightarrow y \in M$$

exists, which must be unique, if and only if $x \leq y$.

The first product. By Lemma 10.2.3, $(M, \max, \mathbb{1})$ is a permutative category, which is, in particular, a strict monoidal category.

The second product. The monoid product μ is well defined on morphisms because, if $x \leq y$ and $x' \leq y'$, then

$$xx' \leq yy'$$

by Lemma 10.2.7 (2). On objects, μ is associative and unital with respect to $\mathbb{1}$ by the monoid axioms. The associativity of μ on morphisms, the functoriality of μ, and the strict monoidal category axioms for the tuple $(M, \mu, \mathbb{1})$ all follow from the fact that each nonempty morphism set has a unique element.

The exchange. For elements $a, b, c, d \in M$, the exchange morphism

$$\max(ab, cd) \xrightarrow{\eta_{a,b,c,d}} \max(a, c)\max(b, d)$$

exists because

$$ab,\ cd \le \max(a, c)\max(b, d) \tag{10.2.9}$$

by Lemma 10.2.7 (2). The naturality of η and the 2-fold monoidal category axioms (10.1.3)–(10.1.6) follow from the fact that each nonempty morphism set has a unique element. □

Example 10.2.10. In each of the totally ordered monoids

$$(\mathbb{N}, +, 0, \le), \quad (\mathbb{R}_{\ge 0}, +, 0, \le), \quad (\mathbb{N}_{\ge 1}, \times, 1, \le), \quad \text{and} \quad (\mathbb{R}_{\ge 1}, \times, 1, \le)$$

in Example 10.2.6, the unit is also the least element. By Proposition 10.2.8, each of them yields a 2-fold monoidal category with

- $\otimes_1 = \max$ with respect to the total ordering $\le$ and
- $\otimes_2$ the monoid product.

Moreover, in each case, $\otimes_1 \ne \otimes_2$. Therefore, these 2-fold monoidal categories do *not* correspond to braided strict monoidal categories in the sense of Proposition 10.1.14. ⋄

10.3. Iterated Monoidal Functors

In this section, we discuss functors between n-fold monoidal categories, which are called n-fold monoidal functors. This notion contains braided and symmetric monoidal functors in the following sense.

- Proposition 10.3.11 shows that, between braided strict monoidal categories, which are also regarded as 2-fold monoidal categories via Proposition 10.1.14, a braided strictly unital monoidal functor is also a 2-fold monoidal functor. The converse holds if the monoidal constraint is a natural isomorphism.
- Proposition 10.3.15 is the symmetric analogue. It involves symmetric strictly unital monoidal functors and n-fold monoidal functors between permutative categories, which are also regarded as n-fold monoidal categories via Proposition 10.1.21.

In Lemma 10.3.20, we check that n-fold monoidal functors are closed under composition. Therefore, there is a category MCat^n with small n-fold monoidal categories as objects and n-fold monoidal functors as morphisms. In Section 10.4, we will extend MCat^n to a monoidal category and observe that its monoids are precisely the small $(n+1)$-fold monoidal categories.

Definition. Recall from Definition 1.3.7 the notion of a monoidal functor between monoidal categories. Also recall n-fold monoidal categories from Definition 10.1.1.

Definition 10.3.1. For n-fold monoidal categories

$$\big(\mathsf{C}, \{\otimes_i\}_{1 \le i \le n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \le i < j \le n}\big) \quad \text{and} \quad \big(\mathsf{D}, \{\otimes_i\}_{1 \le i \le n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \le i < j \le n}\big)$$

with $n \ge 1$, an *n-fold monoidal functor*

$$\big(F, \{F_i^2\}_{i=1}^n\big) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of the following data.

- $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor.
- For each $1 \leq i \leq n$,

$$FA \otimes_i FB \xrightarrow{(F_i^2)_{A,B}} F(A \otimes_i B) \tag{10.3.2}$$

  is a natural transformation for objects $A, B \in \mathsf{C}$, which is called the *ith monoidal constraint*

These data are required to satisfy the following conditions.

Monoidality: For each $1 \leq i \leq n$,

$$(F, F_i^2) : (\mathsf{C}, \otimes_i, \mathbb{1}) \longrightarrow (\mathsf{D}, \otimes_i, \mathbb{1})$$

is a strictly unital monoidal functor.

The Exchange Constraint Axiom: The following diagram in D is commutative for all $A, B, C, D \in \mathsf{C}$ and $1 \leq i < j \leq n$.

$$\begin{tikzcd}[column sep=tiny]
& (FA \otimes_j FB) \otimes_i (FC \otimes_j FD) \arrow[dl, "(F_j^2)_{A,B} \otimes_i (F_j^2)_{C,D}"'] \arrow[dr, "\eta^{i,j}_{FA,FB,FC,FD}"] & \\
F(A \otimes_j B) \otimes_i F(C \otimes_j D) \arrow[d, "(F_i^2)_{A \otimes_j B, C \otimes_j D}"'] & & (FA \otimes_i FC) \otimes_j (FB \otimes_i FD) \arrow[d, "(F_i^2)_{A,C} \otimes_j (F_i^2)_{B,D}"] \\
F\big((A \otimes_j B) \otimes_i (C \otimes_j D)\big) \arrow[dr, "F\eta^{i,j}_{A,B,C,D}"'] & & F(A \otimes_i C) \otimes_j F(B \otimes_i D) \arrow[dl, "(F_j^2)_{A \otimes_i C, B \otimes_i D}"] \\
& F\big((A \otimes_i C) \otimes_j (B \otimes_i D)\big) &
\end{tikzcd} \tag{10.3.3}$$

This finishes the definition of an n-fold monoidal functor. Moreover, an n-fold monoidal functor is *strong* (respectively, *strict*) if each of its n monoidal constraints is a natural isomorphism (respectively, an identity natural transformation). ◇

Explanation 10.3.4. Consider Definition 10.3.1 of an n-fold monoidal functor.

- The exchange constraint axiom (10.3.3) has the following symmetry. Each arrow and its counterpart across the center involve the same type of structure morphisms. For example, the upper left arrow and the lower right arrow both involve the jth monoidal constraint F_j^2.
- Each F_i^2 is a natural transformation but not a natural isomorphism in general.
- The condition that (F, F_i^2) is a strictly unital monoidal functor means that the unit equalities

$$\begin{aligned} F(\mathbb{1}) &= \mathbb{1} \\ (F_i^2)_{\mathbb{1},B} &= 1_{FB} \\ (F_i^2)_{A,\mathbb{1}} &= 1_{FA} \end{aligned} \tag{10.3.5}$$

hold, and the associativity diagram

$$\begin{tikzcd} FA \otimes_i FB \otimes_i FC \arrow[r, "1\otimes_i F_i^2"] \arrow[d, "F_i^2\otimes_i 1"'] & FA \otimes_i F(B \otimes_i C) \arrow[d, "F_i^2"] \\ F(A \otimes_i B) \otimes_i FC \arrow[r, "F_i^2"] & F(A \otimes_i B \otimes_i C) \end{tikzcd} \tag{10.3.6}$$

is commutative for objects $A, B, C \in \mathsf{C}$.

- A *strict* n-fold monoidal functor is a functor that strictly preserves the unit $\mathbb{1}$, the monoidal products $\{\otimes_i\}_{1\le i\le n}$, and the exchanges $\{\eta^{i,j}\}_{1\le i<j\le n}$ by the exchange constraint axiom (10.3.3). ⋄

Example 10.3.7. A 1-fold monoidal functor between 1-fold monoidal categories is precisely a strictly unital monoidal functor between strict monoidal categories. ⋄

Example 10.3.8. Suppose C and D are $(n+1)$-fold monoidal categories. As in Example 10.1.10, suppose C' and D' are the n-fold monoidal categories obtained from, respectively, C and D by forgetting the product $\otimes_{n+1}$ and the natural transformations $\eta^{i,n+1}$ for $1 \le i < n+1$. If

$$\left(F, F_1^2, \ldots, F_n^2, F_{n+1}^2\right) : \mathsf{C} \longrightarrow \mathsf{D}$$

is an $(n+1)$-fold monoidal functor, then

$$\left(F, F_1^2, \ldots, F_n^2\right) : \mathsf{C}' \longrightarrow \mathsf{D}'$$

is an n-fold monoidal functor. ⋄

Example 10.3.9. Suppose

$$I \subseteq \{1, \ldots, n\}$$

is a nonempty subset with cardinality $|I|$, and C and D are n-fold monoidal categories. As in Example 10.1.11, suppose C' an D' are the $|I|$-fold monoidal categories obtained from, respectively, C and D by retaining only the monoidal products $\otimes_i$ and the exchanges $\eta^{i,j}$ with $i, j \in I$. If

$$\left(F, \{F_i^2\}_{i=1}^n\right) : \mathsf{C} \longrightarrow \mathsf{D}$$

is an n-fold monoidal functor, then

$$\left(F, \{F_i^2\}_{i\in I}\right) : \mathsf{C}' \longrightarrow \mathsf{D}'$$

is an $|I|$-fold monoidal functor. ⋄

Example 10.3.10. Suppose

$$F : \mathsf{A} \longrightarrow \mathsf{B} \quad \text{and} \quad G : \mathsf{C} \longrightarrow \mathsf{D}$$

are n-fold monoidal functors. Then their Cartesian product

$$F \times G : \mathsf{A} \times \mathsf{C} \longrightarrow \mathsf{B} \times \mathsf{D}$$

is an n-fold monoidal functor with

- $\mathsf{A} \times \mathsf{C}$ and $\mathsf{B} \times \mathsf{D}$ the Cartesian products of n-fold monoidal categories in Example 10.1.12 and
- monoidal constraints defined componentwise by those of F and G. ⋄

Braided and 2-Fold Monoidal Functors. Recall from Definition 1.3.18 that a braided monoidal functor is a monoidal functor that is also compatible with the braidings, in the sense that the diagram (1.3.19) is commutative. In Proposition 10.3.11 below, we use Proposition 10.1.14 to identify braided strict monoidal categories with a subclass of 2-fold monoidal categories, with the correspondence given by (10.1.16) and (10.1.17). For braided strict monoidal categories, there are two notions of functors, namely, braided monoidal functors and 2-fold monoidal functors in Definition 10.3.1. The next observation identifies these notions of functors with appropriate restrictions.

Proposition 10.3.11. *Suppose* $(\mathsf{C},\otimes,\mathbb{1},\xi^{\otimes})$ *and* $(\mathsf{D},\otimes,\mathbb{1},\xi^{\otimes})$ *are braided strict monoidal categories.*

(1) *Suppose*

$$(F,F^2) : \mathsf{C} \longrightarrow \mathsf{D}$$

is a braided strictly unital monoidal functor. Then

$$(F,F^2,F^2) : \mathsf{C} \longrightarrow \mathsf{D} \tag{10.3.12}$$

is a 2-fold monoidal functor.

(2) *Suppose (10.3.12) is a 2-fold monoidal functor with* F^2 *a natural isomorphism. Then*

$$(F,F^2) : \mathsf{C} \longrightarrow \mathsf{D}$$

is a braided strictly unital monoidal functor.

Proof. For assertion (1), we need to prove the exchange constraint axiom (10.3.3), which asserts the commutativity of the outer diagram below. Since $\otimes_1 = \otimes_2$ in both C and D, we abbreviate each product to concatenation.

$$\begin{tikzcd}
(FA)(FB)(FC)(FD) \arrow[rrr, "1\xi^{\otimes}1"] \arrow[rrr, phantom, shift right=3ex, "(\dagger)"] \arrow[dr, "1F^21"] \arrow[dd, "F^2F^2"'] & & & (FA)(FC)(FB)(FD) \arrow[dl, "1F^21"'] \arrow[dd, "F^2F^2"] \\
& (FA)F(BC)(FD) \arrow[r, "1(F\xi^{\otimes})1"] \arrow[dd, "1F^2"] \arrow[ddr, phantom, "\mathrm{nat}"] & (FA)F(CB)(FD) \arrow[dd, "1F^2"'] & \\
F(AB)F(CD) \arrow[dd, "F^2"'] \arrow[dr, phantom, "(£)"] & & & F(AC)F(BD) \arrow[dd, "F^2"] \arrow[dl, phantom, "(£)"] \\
& (FA)F(BCD) \arrow[r, "1F(\xi^{\otimes}1)"] \arrow[r, phantom, shift right=3ex, "\mathrm{nat}"] \arrow[dl, "F^2"] & (FA)F(CBD) \arrow[dr, "F^2"'] & \\
F(ABCD) \arrow[rrr, "F(1\xi^{\otimes}1)"] & & & F(ACBD)
\end{tikzcd} \tag{10.3.13}$$

- Each of the left and the right trapezoids (£) is commutative by the associativity of F^2 in (10.3.6) twice, as proved in (9.2.17) (with $\oplus$ there replaced by $\otimes_1$ here).
- The middle square and the bottom trapezoid are commutative by the naturality of F^2.
- The top trapezoid (†) is commutative by the compatibility (1.3.19) of F^2 with the braidings.

Therefore, the diagram (10.3.13) is commutative, and (F,F^2,F^2) in (10.3.12) is a 2-fold monoidal functor.

For assertion (2), we assume that (F,F^2,F^2) in (10.3.12) is a 2-fold monoidal functor with F^2 a natural isomorphism. By assumption, (F,F^2) is a strictly unital monoidal functor; that is, it satisfies (10.3.5) and (10.3.6). To see that F^2 is compatible with the braidings in the sense of (1.3.19), we reuse the diagram (10.3.13) and observe the following.

- The outer diagram is commutative by the exchange constraint axiom (10.3.3).
- The four subdiagrams labeled by (£) or nat are commutative, as explained in the previous paragraph.

Since F^2 is assumed to be invertible, it follows that the top trapezoid (†) is commutative. Setting $A = D = \mathbb{1}$ in (†) and using $F(\mathbb{1}) = \mathbb{1}$, we infer that the diagram

$$\begin{tikzcd}
(FB)(FC) \arrow[r, "\xi^{\otimes}"] \arrow[d, "F^2"'] & (FC)(FB) \arrow[d, "F^2"] \\
F(BC) \arrow[r, "F\xi^{\otimes}"] & F(CB)
\end{tikzcd}$$

is commutative. Therefore, the desired diagram (1.3.19) is commutative. This proves that $(F, F^2) : \mathsf{C} \longrightarrow \mathsf{D}$ is a braided strictly unital monoidal functor. $\square$

Remark 10.3.14. In Proposition 10.3.11 (2), the invertibility assumption of F^2 can be replaced by the following weaker assumption:

F^2 and $1_F \otimes_1 F^2$ are natural monomorphisms.

Indeed, with this weaker assumption, to show the commutativity of the subdiagram (†) in (10.3.13), we only need to show that its two composites become equal after post-composition with the monomorphism

$$(FA)F(CB)(FD) \xrightarrow{F^2 \circ (1F^2)} F(ACBD)$$

in (10.3.13). The equality of these two longer composites follows from the commutativity of the outer diagram in (10.3.13) and its four subdiagrams labeled by (£) or nat. The same remark also applies to the symmetric case in Proposition 10.3.15 (2) below. $\diamond$

Symmetric and n-Fold Monoidal Functors. A symmetric monoidal functor in Definition 1.3.32 is a monoidal functor between symmetric monoidal categories that is compatible with the symmetry isomorphisms, in the sense of (1.3.19). In other words, a symmetric monoidal functor has the same definition as a braided monoidal functor. In Proposition 10.3.15 below, we use Proposition 10.1.21 to identify permutative categories with a subclass of n-fold monoidal categories for $n \geq 2$, with the correspondence given by (10.1.17) and (10.1.23). For permutative categories, there are two notions of functors, namely, symmetric monoidal functors and n-fold monoidal functors in Definition 10.3.1. The next observation identifies these notions of functors with appropriate restrictions.

Proposition 10.3.15. *Suppose $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ and $(\mathsf{D}, \otimes, \mathbb{1}, \xi^{\otimes})$ are permutative categories.*

(1) Suppose

$$(F, F^2) : \mathsf{C} \longrightarrow \mathsf{D}$$

is a symmetric strictly unital monoidal functor. Then

$$\big(F, \overbrace{F^2, \ldots, F^2}^{n}\big) : \mathsf{C} \longrightarrow \mathsf{D} \tag{10.3.16}$$

is an n-fold monoidal functor for each $n \geq 2$.

(2) *Suppose (10.3.16) is an n-fold monoidal functor for some $n \geq 2$, with F^2 a natural isomorphism. Then*

$$(F, F^2) : \mathsf{C} \longrightarrow \mathsf{D}$$

is a symmetric strictly unital monoidal functor.

Proof. For assertion (1), (F, F^2) is a braided strictly unital monoidal functor. The diagram (10.3.13) proves that $(F, F^2, \ldots, F^2)$ satisfies the exchange constraint axiom (10.3.3), so it is an n-fold monoidal functor.

For assertion (2), since $n \geq 2$, (F, F^2, F^2) is a 2-fold monoidal functor as in Example 10.3.8. Proposition 10.3.11 (2) shows that (F, F^2) is a symmetric strictly unital monoidal functor. □

Composition of Iterated Monoidal Functors.

Definition 10.3.17. Suppose C, D, and E are n-fold monoidal categories, and

$$\mathsf{C} \xrightarrow{(F, \{F_i^2\}_{i=1}^n)} \mathsf{D} \xrightarrow{(G, \{G_i^2\}_{i=1}^n)} \mathsf{E}$$

are n-fold monoidal functors. Their *composite* is defined as the tuple

$$\big(GF, \{(GF)_i^2\}_{i=1}^n\big) : \mathsf{C} \longrightarrow \mathsf{E}$$

with each $(GF)_i^2$, for $1 \leq i \leq n$, the natural transformation

(10.3.18)
$$\begin{tikzcd} GFA \otimes_i GFB \arrow[r, "(G_i^2)_{FA,FB}"] \arrow[rr, bend left, "((GF)_i^2)_{A,B}"] & G(FA \otimes_i FB) \arrow[r, "G(F_i^2)_{A,B}"] & GF(A \otimes_i B) \end{tikzcd}$$

for objects $A, B \in \mathsf{C}$ ◇

Explanation 10.3.19. In Definition 10.3.17, for each $1 \leq i \leq n$, $\big(GF, (GF)_i^2\big)$ is the composite of the strictly unital monoidal functors

$$(\mathsf{C}, \otimes_i, \mathbb{1}) \xrightarrow{(F, F_i^2)} (\mathsf{D}, \otimes_i, \mathbb{1}) \xrightarrow{(G, G_i^2)} (\mathsf{E}, \otimes_i, \mathbb{1})$$

as in Definition 1.3.12. Therefore, each $\big(GF, (GF)_i^2\big)$ is a strictly unital monoidal functor, and composition of n-fold monoidal functors is strictly associative. Moreover, it has a two-sided strict unit given by the identity functor with identity monoidal constraints. However, we still need to check that n-fold monoidal functors are closed under composition. ◇

Lemma 10.3.20. *The composite of two n-fold monoidal functors is an n-fold monoidal functor. Moreover, composition preserves the strong (respectively, strict) property of n-fold monoidal functors.*

Proof. Using the notations in Definition 10.3.17, for each $1 \leq i \leq n$,

$$\big(GF, (GF)_i^2\big) : (\mathsf{C}, \otimes_i, \mathbb{1}) \longrightarrow (\mathsf{E}, \otimes_i, \mathbb{1})$$

is a strictly unital monoidal functor. It remains to check the exchange constraint axiom (10.3.3) for $\big(GF, \{(GF)_i^2\}_{i=1}^n\big)$, which asserts the commutativity of the outer

diagram below.

$$
\begin{tikzcd}
(GFA \otimes_j GFB) \otimes_i (GFC \otimes_j GFD) \arrow[rrr, "\eta^{i,j}"] \arrow[d, "G_j^2 \otimes_i G_j^2"'] & & & (GFA \otimes_i GFC) \otimes_j (GFB \otimes_i GFD) \arrow[d, "G_i^2 \otimes_j G_i^2"] \\
G(FA \otimes_j FB) \otimes_i G(FC \otimes_j FD) \arrow[d, "G(F_j^2) \otimes_i G(F_j^2)"'] \arrow[dr, "G_i^2"] \arrow[rrr, phantom, "(10.3.3)"] & & & G(FA \otimes_i FC) \otimes_j G(FB \otimes_i FD) \arrow[d, "G(F_i^2) \otimes_j G(F_i^2)"] \arrow[dl, "G_j^2"'] \\
GF(A \otimes_j B) \otimes_i GF(C \otimes_j D) \arrow[d, "G_i^2"'] & \bullet \arrow[r, "G\eta^{i,j}"] \arrow[dl, "G(F_j^2 \otimes_i F_j^2)"] & \bullet \arrow[dr, "G(F_i^2 \otimes_j F_i^2)"'] & GF(A \otimes_i C) \otimes_j GF(B \otimes_i D) \arrow[d, "G_j^2"] \\
G[F(A \otimes_j B) \otimes_i F(C \otimes_j D)] \arrow[d, "G(F_i^2)"'] \arrow[rrr, phantom, "(10.3.3)"] & & & G[F(A \otimes_i C) \otimes_j F(B \otimes_i D)] \arrow[d, "G(F_j^2)"] \\
GF[(A \otimes_j B) \otimes_i (C \otimes_j D)] \arrow[rrr, "GF\eta^{i,j}"] & & & GF[(A \otimes_i C) \otimes_j (B \otimes_i D)]
\end{tikzcd}
$$

- The middle horizontal arrow is

$$G[(FA \otimes_j FB) \otimes_i (FC \otimes_j FD)] \xrightarrow{G\eta^{i,j}} G[(FA \otimes_i FC) \otimes_j (FB \otimes_i FD)].$$

- The left triangle is commutative by the naturality of G_i^2.
- The right triangle is commutative by the naturality of G_j^2.
- The top hexagon is commutative by the axiom (10.3.3) for G.
- The bottom hexagon is obtained from the exchange constraint axiom for F by applying G, so it is commutative.

Therefore, the composite GF also satisfies the exchange constraint axiom, and it is an n-fold monoidal functor. If both F and G are strong (respectively, strict), then so is the composite GF because $\big((GF)_i^2\big)_{A,B}$ in (10.3.18) is an isomorphism (respectively, identity morphism). □

A subcategory is *wide* if it contains all the objects of the larger category.

Definition 10.3.21. For $n \geq 1$,

$$\mathsf{MCat}^n$$

is defined as the category consisting of the following data.

- Its objects are small n-fold monoidal categories in Definition 10.1.1.
- Its morphisms are n-fold monoidal functors in Definition 10.3.1.
- The identity morphism of each small n-fold monoidal category is the identity functor with identity monoidal constraints.
- Composition is as in Definition 10.3.17.

Moreover, $\mathsf{MCat}^n_{\mathsf{sg}}$ (respectively, $\mathsf{MCat}^n_{\mathsf{st}}$) is the wide subcategory of MCat^n with strong (respectively, strict) n-fold monoidal functors as morphisms. ◇

MCat^n, $\mathsf{MCat}^n_{\mathsf{sg}}$, and $\mathsf{MCat}^n_{\mathsf{st}}$ are well-defined categories by Lemma 10.3.20.

Example 10.3.22. Continuing Examples 10.1.9 and 10.3.7, MCat^1 is the category with small strict monoidal categories as objects and strictly unital monoidal functors as morphisms. ◇

We emphasize that MCat^n has *small* n-fold monoidal categories as its objects. The smallness condition is necessary to ensure that, for each pair of small n-fold monoidal categories, there is only a set of n-fold monoidal functors between them.

10.4. Monoids in Iterated Monoidal Categories

In this section, we study monoids in the category MCat^n in Definition 10.3.21.

- Lemma 10.4.2 extends the Cartesian product to the category MCat^n to make it into a monoidal category.
- Theorem 10.4.5 observes that the monoids in MCat^n are precisely the small $(n+1)$-fold monoidal categories in Definition 10.1.1. It follows that k-fold monoids in MCat^n are precisely the small $(n+k)$-fold monoidal categories. This observation provides a conceptual understanding of n-fold monoidal categories as iterated monoids.
- Proposition 10.4.13 observes that monoid morphisms in MCat^n are stricter than $(n+1)$-fold monoidal functors. It may be slightly disappointing that monoid morphisms in MCat^n are not precisely $(n+1)$-fold monoidal functors. In this regard, one may think of n-fold monoidal functors as the correct notions of morphisms that yield the identification of monoids in MCat^n with small $(n+1)$-fold monoidal categories.

Convention 10.4.1. Denote by $(\mathsf{Cat}, \times, \mathbf{1})$ the monoidal category with

- Cat the category of small categories and functors,
- $\times$ the Cartesian product, and
- $\mathbf{1}$ the terminal category with one object $*$ and its identity morphism. $\diamond$

Lemma 10.4.2. *The data*

$$(\mathsf{MCat}^n, \times, \mathbf{1})$$

form a monoidal category for each $n \geq 1$.

Proof. The terminal category $\mathbf{1}$ has a unique n-fold monoidal category structure. The Cartesian products $\times$ of n-fold monoidal categories and n-fold monoidal functors are defined componentwise, as discussed in, respectively, Examples 10.1.12 and 10.3.10. This is a well-defined functor because it is defined componentwise.

The functor parts of the associativity isomorphism α, the left unit isomorphism λ, and the right unit isomorphism ρ are those in the monoidal category $(\mathsf{Cat}, \times, \mathbf{1})$.

- Each component of α is the identity functor with identity monoidal constraints. So it is an n-fold monoidal functor, and α is a natural isomorphism. Moreover, the pentagon axiom (1.3.3) holds.
- Each component of λ, say,

$$\lambda_{\mathsf{C}} : \mathbf{1} \times \mathsf{C} \longrightarrow \mathsf{C},$$

drops the $\mathbf{1}$ component on the left, and has identity monoidal constraints in the C component. So each λ_{C} is an n-fold monoidal functor, and λ is a natural isomorphism. Similarly, ρ is a natural isomorphism. Moreover, the unity axiom (1.3.2) holds.

This shows that $(\mathsf{MCat}^n, \times, \mathbf{1})$ is a monoidal category. $\square$

Monoids in MCat^n. Recall from Definition 1.3.6 that a *monoid* in a monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ consists of an object A and structure morphisms

$$A \otimes A \xrightarrow{\mu} A \xleftarrow{\eta} \mathbb{1},$$

which are called, respectively, the *multiplication* and the *unit*. The unity diagram

$$\begin{tikzcd} \mathbb{1} \otimes A \arrow[r, "\eta\otimes 1"] \arrow[d, "\lambda_A"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{1} \arrow[l, "1\otimes\eta"'] \arrow[d, "\rho_A"] \\ A \arrow[r, equal] & A \arrow[r, equal] & A \end{tikzcd} \tag{10.4.3}$$

and the associativity diagram

$$\begin{tikzcd} (A \otimes A) \otimes A \arrow[r, "\alpha"] \arrow[d, "\mu\otimes 1"'] & A \otimes (A \otimes A) \arrow[r, "1\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[rr, "\mu"] & & A \end{tikzcd} \tag{10.4.4}$$

are required to be commutative.

Theorem 10.4.5. *Monoids in the monoidal category* $(\mathsf{MCat}^n, \times, \mathbf{1})$ *are precisely small* $(n+1)$*-fold monoidal categories.*

Proof. In terms of data, a monoid in MCat^n consists of

- a small n-fold monoidal category
$$\big(\mathsf{C}, \{\otimes_i\}_{1\le i\le n}, \mathbb{1}, \{\eta^{i,j}\}_{1\le i<j\le n}\big)$$
as in Definition 10.1.1,
- a unit n-fold monoidal functor
$$\big(\varepsilon, \{\varepsilon_i^2\}_{1\le i\le n}\big) : \mathbf{1} \longrightarrow \mathsf{C} \tag{10.4.6}$$
as in Definition 10.3.1, and
- a multiplication n-fold monoidal functor
$$\big(\otimes_{n+1}, \{\eta^{i,n+1}\}_{1\le i\le n}\big) : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}. \tag{10.4.7}$$

The unity diagram

$$\begin{tikzcd} \mathbf{1} \times \mathsf{C} \arrow[r, "(\varepsilon,\{\varepsilon_i^2\})\times 1"] \arrow[d, "\cong"'] & \mathsf{C} \times \mathsf{C} \arrow[d, "(\otimes_{n+1},\{\eta^{i,n+1}\})"] & \mathsf{C} \times \mathbf{1} \arrow[l, "1\times(\varepsilon,\{\varepsilon_i^2\})"'] \arrow[d, "\cong"] \\ \mathsf{C} \arrow[r, equal] & \mathsf{C} \arrow[r, equal] & \mathsf{C} \end{tikzcd} \tag{10.4.8}$$

and the associativity diagram

$$\begin{tikzcd} \mathsf{C} \times \mathsf{C} \times \mathsf{C} \arrow[r, "1 \times (\otimes_{n+1}, \{\eta^{i,n+1}\})"] \arrow[d, "(\otimes_{n+1}, \{\eta^{i,n+1}\}) \times 1"'] & \mathsf{C} \times \mathsf{C} \arrow[d, "(\otimes_{n+1}, \{\eta^{i,n+1}\})"] \\ \mathsf{C} \times \mathsf{C} \arrow[r, "(\otimes_{n+1}, \{\eta^{i,n+1}\})"] & \mathsf{C} \end{tikzcd} \tag{10.4.9}$$

in MCat^n are required to be commutative. We will unpack the above data and axioms and observe that they constitute precisely a small $(n+1)$-fold monoidal category

$$\big(\mathsf{C}, \{\otimes_i\}_{1\le i\le n+1}, \mathbb{1}, \{\eta^{i,j}\}_{1\le i<j\le n+1}\big). \tag{10.4.10}$$

First we unpack the data (10.4.6) and (10.4.7). By Definition 10.3.1, the unit n-fold monoidal functor (10.4.6) consists of

- a functor
$$\varepsilon : \mathbf{1} \longrightarrow \mathsf{C},$$
that is, an object $\varepsilon(*) \in \mathsf{C}$, and
- for each $1 \le i \le n$, a morphism
$$\varepsilon(*) \otimes_i \varepsilon(*) \xrightarrow{\varepsilon_i^2} \varepsilon(*) \in \mathsf{C},$$
since $\mathbf{1}$ has only the identity morphism of $*$.

These data satisfy the following conditions.

Monoidality: The unit equalities (10.3.5) are as follows.
$$\varepsilon(*) = \mathbb{1}$$
$$\varepsilon_i^2 = 1_{\mathbb{1}}$$
The associativity diagram (10.3.6) imposes no conditions on
$$\big(\varepsilon = \mathbb{1}, \varepsilon_i^2 = 1_{\mathbb{1}}\big),$$
since it already holds in the strict monoidal category $(\mathsf{C}, \otimes_i, \mathbb{1})$.

The Exchange Constraint Axiom: The commutative diagram (10.3.3) imposes no conditions. Indeed, the left side of that diagram consists of three identity morphisms. The right side consists of $\eta^{i,j}_{\mathbb{1},\mathbb{1},\mathbb{1},\mathbb{1}}$, which is an identity morphism by the internal unity axiom (10.1.3) in C, and two identity morphisms. Therefore, each of the two composites is the identity morphism.

By Definition 10.3.1, the multiplication n-fold monoidal functor (10.4.7) consists of

- a functor
$$\otimes_{n+1} : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$$
and
- for each $1 \le i \le n$, a natural transformation
$$(A \otimes_{n+1} B) \otimes_i (A' \otimes_{n+1} B') \xrightarrow{\eta^{i,n+1}_{(A,B),(A',B')}} (A \otimes_i A') \otimes_{n+1} (B \otimes_i B')$$
for objects $A, B, A', B' \in \mathsf{C}$. We will omit the parentheses in the subscript of $\eta^{i,n+1}$.

These data satisfy the following conditions.

Monoidality: The unit equalities (10.3.5) are as follows.
$$\mathbb{1} \otimes_{n+1} \mathbb{1} = \mathbb{1} \tag{10.4.11a}$$
$$\eta^{i,n+1}_{\mathbb{1},\mathbb{1},A,B} = 1_{A \otimes_{n+1} B} = \eta^{i,n+1}_{A,B,\mathbb{1},\mathbb{1}} \tag{10.4.11b}$$
The condition (10.4.11b) is the internal unity axiom (10.1.3) for $1 \le i < j = n+1$.

The associativity diagram (10.3.6) for $\big(\otimes_{n+1}, \eta^{i,n+1}\big)$ is the internal associativity axiom (10.1.5) for $1 \le i < j = n+1$.

The Exchange Constraint Axiom: The commutative diagram (10.3.3) for the n-fold monoidal functor
$$\big(\otimes_{n+1}, \{\eta^{i,n+1}\}_{1 \le i \le n}\big)$$
is the triple exchange axiom (10.1.7) for $1 \le i < j < k = n+1$.

Next we unpack the axioms (10.4.8) and (10.4.9).

Monoid Unity: The commutativity of the unity diagram (10.4.8) in Cat means the equalities

$$\begin{aligned} \mathbb{1} \otimes_{n+1} A &= A = A \otimes_{n+1} \mathbb{1} \\ 1_{\mathbb{1}} \otimes_{n+1} f &= f = f \otimes_{n+1} 1_{\mathbb{1}} \end{aligned} \tag{10.4.12}$$

for objects A and morphisms $f \in \mathsf{C}$. In other words, the unit $\mathbb{1} \in \mathsf{C}$ is naturally a strict two-sided unit for $\otimes_{n+1}$, as well as $\otimes_i$ for $1 \le i \le n$. Note that the first equality in (10.4.12) subsumes (10.4.11a).

In terms of the monoidal constraints (10.3.2), the commutativity of the unity diagram (10.4.8) means the equalities

$$\eta^{i,n+1}_{\mathbb{1},A,\mathbb{1},B} = 1_{A \otimes_i B} = \eta^{i,n+1}_{A,\mathbb{1},B,\mathbb{1}},$$

which form the external unity axiom (10.1.4) for $1 \le i < j = n+1$.

Monoid Associativity: The commutativity of the associativity diagram (10.4.9) in Cat means that $\otimes_{n+1}$ is strictly associative. So $(\mathsf{C}, \otimes_{n+1}, \mathbb{1})$ is a strict monoidal category.

In terms of the monoidal constraints, the commutativity of (10.4.9) is the external associativity axiom (10.1.6) for $1 \le i < j = n+1$.

Therefore, the data in (10.4.10) form a small $(n+1)$-fold monoidal category.

Conversely, given a small $(n+1)$-fold monoidal category as in (10.4.10), we reuse the argument above to infer that it yields a monoid in MCat^n. □

Monoid Morphisms in MCat^n. In a monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$, recall that a morphism

$$f : (A, \mu, \eta) \longrightarrow (B, \mu, \eta)$$

of monoids is a morphism $f : A \longrightarrow B$ in C such that the compatibility diagrams

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu"'] & B \otimes B \arrow[d, "\mu"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "\eta"] \arrow[d, equal] & A \arrow[d, "f"] \\ \mathbb{1} \arrow[r, "\eta"] & B \end{tikzcd}$$

in C are commutative. The next observation shows that morphisms of monoids in MCat^n are, in general, strictly stronger than $(n+1)$-fold monoidal functors. We use Theorem 10.4.5 to identify monoids in $(\mathsf{MCat}^n, \times, \mathbf{1})$ with small $(n+1)$-fold monoidal categories.

Proposition 10.4.13. *A morphism of monoids in* $(\mathsf{MCat}^n, \times, \mathbf{1})$ *is precisely an* $(n+1)$-*fold monoidal functor*

$$\big(F, \{F^2_i\}_{1 \le i \le n+1}\big)$$

such that F^2_{n+1} *is the identity.*

Proof. Suppose C and D are monoids in MCat^n, that is, small $(n+1)$-fold monoidal categories. A monoid morphism $\mathsf{C} \longrightarrow \mathsf{D}$ consists of a morphism in MCat^n, that is, an n-fold monoidal functor

$$(F, F^2_1, \ldots, F^2_n) : \mathsf{C} \longrightarrow \mathsf{D}$$

such that the compatibility diagrams

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\varepsilon"] \arrow[d, equal] & \mathsf{C} \arrow[d, "(F,\{F_i^2\})"] \\ \mathbf{1} \arrow[r, "\varepsilon"] & \mathsf{D} \end{tikzcd} \tag{10.4.14}$$

and

$$\begin{tikzcd} \mathsf{C}\times\mathsf{C} \arrow[r, "(F,\{F_i^2\})\times(F,\{F_i^2\})"] \arrow[d, "(\otimes_{n+1},\{\eta^{i,n+1}\})"'] & \mathsf{D}\times\mathsf{D} \arrow[d, "(\otimes_{n+1},\{\eta^{i,n+1}\})"] \\ \mathsf{C} \arrow[r, "(F,\{F_i^2\})"] & \mathsf{D} \end{tikzcd} \tag{10.4.15}$$

in MCat^n are commutative.

Considering the equalities of functors and monoidal constraints, the commutative diagram (10.4.14) is equivalent to the following equalities for $1 \le i \le n$.

$$F(\mathbb{1}) = \mathbb{1}$$
$$F(1_{\mathbb{1}}) = 1_{\mathbb{1}} = (F_i^2)_{\mathbb{1},\mathbb{1}}$$

These equalities hold by the functoriality of F and the fact that (F, F_i^2) is a strictly unital monoidal functor (10.3.5). So they impose no restrictions on $(F, \{F_i^2\}_{1\le i\le n})$.

Next, consider the commutative diagram (10.4.15).

- In terms of functors, this commutative diagram means the equality

$$FA \otimes_{n+1} FB = F(A \otimes_{n+1} B)$$

with $A, B \in \mathsf{C}$ both objects or both morphisms. So we may define the $(n+1)$st monoidal constraint (10.3.2)

$$\otimes_{n+1} \circ (F \times F) \xrightarrow{F_{n+1}^2} F \circ \otimes_{n+1}$$

for F as the identity natural transformation. The pair

$$(F, F_{n+1}^2 = 1) : (\mathsf{C}, \otimes_{n+1}, \mathbb{1}) \longrightarrow (\mathsf{D}, \otimes_{n+1}, \mathbb{1})$$

is a strictly unital monoidal functor because the unit equalities (10.3.5) hold and the associativity diagram (10.3.6) is commutative.

- In terms of the n monoidal constraints, the commutativity of (10.4.15) is the exchange constraint axiom (10.3.3) for $1 \le i < j = n+1$, where we use $F_{n+1}^2 = 1$.

Therefore,

$$(F, F_1^2, \ldots, F_n^2, F_{n+1}^2 = 1) : \mathsf{C} \longrightarrow \mathsf{D} \tag{10.4.16}$$

is an $(n+1)$-fold monoidal functor.

Conversely, if (10.4.16) is an $(n+1)$-fold monoidal functor, then the argument above shows that $(F, \{F_i^2\}_{1\le i\le n})$ is a morphism of monoids in MCat^n. $\square$

10.5. Free Iterated Monoidal Categories

Recall from Definition 10.3.1 that $\mathsf{MCat}^n_{\mathsf{st}}$ is the category with small n-fold monoidal categories as objects and *strict* n-fold monoidal functors as morphisms. With Cat denoting the category of small categories and functors, there is a forgetful functor

$$\mathsf{MCat}^n_{\mathsf{st}} \xrightarrow{U} \mathsf{Cat} \tag{10.5.1}$$

that forgets about the n-fold monoidal structure in objects and the identity monoidal constraints in morphisms. In this section, we discuss the left adjoint FMon^n of this forgetful functor, which is the free n-fold monoidal category functor, in Proposition 10.5.9. In Theorem 10.5.18, we observe that the free n-fold monoidal category of a small category splits in Cat into smaller pieces involving the categories $\mathsf{Mon}^n(k)$ in Definition 10.5.13.

Free n-Fold Monoidal Categories. First we define the free n-fold monoidal category of a small category.

Definition 10.5.2. Suppose C is a small category. Define the data of an n-fold monoidal category

$$\mathsf{FMon}^n(\mathsf{C})$$

as follows.

Objects: The objects in $\mathsf{FMon}^n(\mathsf{C})$ are defined as follows.
- Each object in C is also an object in $\mathsf{FMon}^n(\mathsf{C})$.
- $\mathsf{FMon}^n(\mathsf{C})$ is equipped with a *unit* object $\mathbb{1}$ that is not in C.
- Inductively, if X and Y are objects in $\mathsf{FMon}^n(\mathsf{C})$, then so are

$$X \otimes_i Y \quad \text{for} \quad 1 \leq i \leq n.$$

These objects are subject to the relations that, for $1 \leq i \leq n$,
- $\otimes_i$ is strictly associative on objects, and
- $\mathbb{1}$ is the strict two-sided unit for $\otimes_i$.

Morphisms: The morphisms in $\mathsf{FMon}^n(\mathsf{C})$ are defined as follows.
- Each morphism in C is a morphism in $\mathsf{FMon}^n(\mathsf{C})$.
- Each object $A \in \mathsf{FMon}^n(\mathsf{C})$ is equipped with an *identity morphism*

$$A \xrightarrow{1_A} A,$$

which is the identity morphism $1_A \in \mathsf{C}$ if $A \in \mathsf{C}$.
- For objects $A, B, C, D \in \mathsf{FMon}^n(\mathsf{C})$ and $1 \leq i < j \leq n$, $\mathsf{FMon}^n(\mathsf{C})$ is equipped with a morphism

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D), \tag{10.5.3}$$

which is called the (i,j)-*exchange*.
- Inductively, if

$$f : A \longrightarrow B, \quad g : B \longrightarrow C, \quad \text{and} \quad f' : A' \longrightarrow B'$$

are morphisms in $\mathsf{FMon}^n(\mathsf{C})$, then so are

$$A \otimes_i A' \xrightarrow{f \otimes_i f'} B \otimes_i B'$$
$$A \xrightarrow{gf} C$$

for $1 \leq i \leq n$, with gf, which is called the *composite*, the composite in C if $f, g \in \mathsf{C}$.

These morphisms are subject to the relations (i)–(v) below.

(i) Composition is strictly associative and unital with respect to identity morphisms.

(ii) Each $\otimes_i$ preserves identity morphisms and composition.

(iii) Each $\otimes_i$ is strictly associative on morphisms with $1_{\mathbb{1}}$ as the strict two-sided unit.

(iv) For $1 \leq i < j \leq n$ and morphisms

$$f_X : X \longrightarrow X' \in \mathsf{FMon}^n(\mathsf{C}) \quad \text{with} \quad X \in \{A, B, C, D\},$$

the diagram

$$\begin{tikzcd}
(A \otimes_j B) \otimes_i (C \otimes_j D) \arrow[r, "\eta^{i,j}_{A,B,C,D}"] \arrow[d, "(f_A \otimes_j f_B) \otimes_i (f_C \otimes_j f_D)"'] & (A \otimes_i C) \otimes_j (B \otimes_i D) \arrow[d, "(f_A \otimes_i f_C) \otimes_j (f_B \otimes_i f_D)"] \\
(A' \otimes_j B') \otimes_i (C' \otimes_j D') \arrow[r, "\eta^{i,j}_{A',B',C',D'}"] & (A' \otimes_i C') \otimes_j (B' \otimes_i D')
\end{tikzcd} \tag{10.5.4}$$

in $\mathsf{FMon}^n(\mathsf{C})$ is commutative.

(v) The unity, associativity, and exchange axioms (10.1.3)–(10.1.7) hold for objects in $\mathsf{FMon}^n(\mathsf{C})$.

This finishes the definition of $\mathsf{FMon}^n(\mathsf{C})$. ◇

Explanation 10.5.5. Consider Definition 10.5.2.

- An object in $\mathsf{FMon}^n(\mathsf{C})$ is either the unit $\mathbb{1}$ or a finite $\{\otimes_i\}_{i=1}^n$-product of objects in C.
- Relation (ii), which states that $\otimes_i$ preserves identity morphisms and composition, means the equalities

$$1_X \otimes_i 1_Y = 1_{X \otimes_i Y}$$
$$gf \otimes_i g'f' = (g \otimes_i g')(f \otimes_i f')$$

for objects X, Y and morphisms f, g, f', g' in $\mathsf{FMon}^n(\mathsf{C})$, assuming that the composites in the second equality are defined. By these equalities, each morphism in $\mathsf{FMon}^n(\mathsf{C})$ is a finite composite of finite $\{\otimes_i\}_{i=1}^n$-products of morphisms in C and the exchanges $\{\eta^{i,j}\}_{i<j}$ in (10.5.3).
- The strict associativity and unity of each $\otimes_i$ mean the equalities

$$\begin{cases} (X \otimes_i Y) \otimes_i Z = X \otimes_i (Y \otimes_i Z) \\ (f \otimes_i g) \otimes_i h = f \otimes_i (g \otimes_i h) \end{cases}$$

$$\begin{cases} \mathbb{1} \otimes_i X = X = X \otimes_i \mathbb{1} \\ 1_{\mathbb{1}} \otimes_i f = f = f \otimes_i 1_{\mathbb{1}} \end{cases}$$

for objects X, Y, Z and morphisms f, g, h in $\mathsf{FMon}^n(\mathsf{C})$. ◇

Example 10.5.6. In $\mathsf{FMon}^n(\mathsf{C})$, since each $\otimes_i$ is strictly associative, parentheses are not necessary for an iterated product involving $\otimes_i$ for only one i. For example, for objects $A_1, \ldots, A_9 \in \mathsf{C}$, the expression

$$(A_1 \otimes_1 A_2 \otimes_1 A_3) \otimes_3 (A_4 \otimes_2 A_5) \otimes_3 A_6 \otimes_3 \big(A_7 \otimes_1 (A_8 \otimes_3 A_9)\big)$$

is an object in $\mathsf{FMon}^n(\mathsf{C})$, and similarly if the A_i's are morphisms in $\mathsf{FMon}^n(\mathsf{C})$. $\diamond$

Lemma 10.5.7. *For each small category* C, $\mathsf{FMon}^n(\mathsf{C})$ *in Definition 10.5.2 is an* n*-fold monoidal category.*

Proof. The relations (i)–(v) refer to those in Definition 10.5.2.

- $\mathsf{FMon}^n(\mathsf{C})$ is a category by relation (i).
- For $1 \leq i \leq n$,

$$\mathsf{FMon}^n(\mathsf{C}) \times \mathsf{FMon}^n(\mathsf{C}) \xrightarrow{\otimes_i} \mathsf{FMon}^n(\mathsf{C})$$

  is a functor by relation (ii).
- For $1 \leq i \leq n$,

$$\big(\mathsf{FMon}^n(\mathsf{C}), \otimes_i, \mathbb{1}\big)$$

  is a strict monoidal category by the strict associativity and unity of $\otimes_i$ on objects and relation (iii).
- For $1 \leq i < j \leq n$, the exchange $\eta^{i,j}$ in (10.5.3) is a natural transformation by relation (iv).

The n-fold monoidal category axioms hold in $\mathsf{FMon}^n(\mathsf{C})$ by relation (v). $\square$

Free n-Fold Monoidal Category Functor. Next we define what FMon^n does to functors.

Definition 10.5.8. Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor between small categories. Define the data of a functor

$$\mathsf{FMon}^n(F) = \overline{F} : \mathsf{FMon}^n(\mathsf{C}) \longrightarrow \mathsf{FMon}^n(\mathsf{D})$$

as follows.

Objects: The object assignment of $\overline{F}$ is defined as follows.

- The restriction of $\overline{F}$ to $\mathsf{Ob}(\mathsf{C})$ is F.
- $\overline{F}\mathbb{1} = \mathbb{1}$.
- Inductively, if $A, B \in \mathsf{FMon}^n(\mathsf{C})$ are objects with $\overline{F}A, \overline{F}B \in \mathsf{FMon}^n(\mathsf{D})$ already defined, then

$$\overline{F}(A \otimes_i B) = \overline{F}A \otimes_i \overline{F}B \quad \text{for} \quad 1 \leq i \leq n.$$

Morphisms: The morphism assignment of $\overline{F}$ is defined as follows.

- The restriction of $\overline{F}$ to the morphisms in C is F.
- On identity morphisms and the exchanges,

$$\overline{F}1_A = 1_{\overline{F}A}$$
$$\overline{F}\eta^{i,j}_{A,B,C,D} = \eta^{i,j}_{\overline{F}A,\overline{F}B,\overline{F}C,\overline{F}D}$$

  for objects $A, B, C, D \in \mathsf{FMon}^n(\mathsf{C})$ and $1 \leq i < j \leq n$.

- Inductively, if $f, g, f' \in \mathsf{FMon}^n(\mathsf{C})$ are morphisms with gf defined and $\overline{F}f$, $\overline{F}g$, and $\overline{F}f'$ already defined, then

$$\begin{aligned} \overline{F}(f \otimes_i f') &= \overline{F}f \otimes_i \overline{F}f' \\ \overline{F}(gf) &= (\overline{F}g)(\overline{F}f). \end{aligned}$$

This finishes the definition of $\mathsf{FMon}^n(F)$. ⋄

Proposition 10.5.9. *Definitions 10.5.2 and 10.5.8 define a functor*

$$\mathsf{FMon}^n : \mathsf{Cat} \longrightarrow \mathsf{MCat}^n_{\mathsf{st}}$$

that is the left adjoint in the adjunction

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, shift left, "\mathsf{FMon}^n"] & \mathsf{MCat}^n_{\mathsf{st}} \arrow[l, shift left, "U"] \end{tikzcd} \tag{10.5.10}$$

with U the forgetful functor in (10.5.1).

Proof. For a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ between small categories,

$$\mathsf{FMon}^n(F) = \overline{F} : \mathsf{FMon}^n(\mathsf{C}) \longrightarrow \mathsf{FMon}^n(\mathsf{D})$$

in Definition 10.5.8 is a well-defined functor by

- the functoriality of F and
- the fact that $\mathsf{FMon}^n(\mathsf{D})$ also satisfies the relations in Definition 10.5.2.

Equipped with identity monoidal constraints, $\overline{F}$ is a strict n-fold monoidal functor as in Definition 10.3.1 because it strictly preserves each monoidal product $\otimes_i$, the unit $\mathbb{1}$, and each exchange $\eta^{i,j}$. In particular, the exchange constraint axiom (10.3.3) is precisely the strict preservation of $\eta^{i,j}$ by $\overline{F}$. It also follows from Definition 10.5.8 that

$$\mathsf{FMon}^n : \mathsf{Cat} \longrightarrow \mathsf{MCat}^n_{\mathsf{st}}$$

is a functor.

To see that FMon^n is left adjoint to U, suppose

$$G : \mathsf{C} \longrightarrow U\mathsf{D}$$

is a functor with

- C a small category and
- D a small n-fold monoidal category.

We must show that G admits a unique extension to a strict n-fold monoidal functor

$$\overline{G} : \mathsf{FMon}^n(\mathsf{C}) \longrightarrow \mathsf{D}.$$

Since $\overline{G}$ is to extend G, we must define

$$\begin{aligned} \overline{G}A &= GA \\ \overline{G}f &= Gf \end{aligned}$$

for objects A and morphisms f in C. Since $\overline{G}$ is to be a *strict* n-fold monoidal functor, $\overline{G}$ must strictly preserve identity morphisms, composites, the unit $\mathbb{1}$, the monoidal products $\{\otimes_i\}_{i=1}^n$, and the exchanges $\{\eta^{i,j}\}_{1\le i<j\le n}$ by the exchange constraint axiom (10.3.3). So $\overline{G}$ is unique. It is well defined because all the relations that define $\mathsf{FMon}^n(\mathsf{C})$ in Definition 10.5.2 also hold in the n-fold monoidal category D. □

Definition 10.5.11. The functor

$$\mathsf{FMon}^n : \mathsf{Cat} \longrightarrow \mathsf{MCat}^n_{\mathsf{st}}$$

is called the *free n-fold monoidal category functor*. $\diamond$

Example 10.5.12. Suppose S is a set, which is also regarded as a discrete category with only identity morphisms. The morphisms in $\mathsf{FMon}^n(S)$ are generated under $\{\otimes_i\}_{i=1}^n$ and composites by

- the identity morphisms 1_x for $x \in S$ and
- the exchanges

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D)$$

  in (10.5.3) for $A, B, C, D \in \mathsf{FMon}^n(S)$ and $1 \le i < j \le n$.

In particular, $\mathsf{FMon}^n(\varnothing)$ is the terminal category. $\diamond$

Decomposition of Free Iterated Monoidal Categories. Next we discuss a decomposition of the free n-fold monoidal category of a small category that uses the following categories.

Definition 10.5.13. Suppose S is a totally ordered finite set with $k \ge 0$ elements, which is also regarded as a discrete category.

- For $n \ge 1$, define the full subcategory

$$\mathsf{Mon}^n(S) \subset \mathsf{FMon}^n(S) \tag{10.5.14}$$

  in which each object can be written as an iterated $\{\otimes_i\}_{i=1}^n$-product with each element in S occurring precisely once.
- Define a Σ_k-action on $\mathsf{Mon}^n(S)$ by letting each permutation in Σ_k
  - permute the elements in S from the right and
  - change the subscripts in each generating morphism $\eta^{i,j}$ accordingly.

Moreover, define the category

$$\mathsf{Mon}^n(k) = \mathsf{Mon}^n\big(\{1, \ldots, k\}\big)$$

as above with $S = \{1, \ldots, k\}$. $\diamond$

Example 10.5.15. Each of

$$\mathsf{Mon}^n(0) = \{\mathbb{1}\} \quad \text{and} \quad \mathsf{Mon}^n(1) = \{1\}$$

is the terminal category. For $k \ge 1$, $\mathsf{Mon}^n(k)$ is *not* an n-fold monoidal subcategory of $\mathsf{FMon}^n\{1, \ldots, k\}$. $\diamond$

Example 10.5.16. $\mathsf{Mon}^n(2)$ consists of the objects

$$1 \otimes_i 2 \quad \text{and} \quad 2 \otimes_i 1 \quad \text{for} \quad 1 \le i \le n.$$

By the internal and external unity axioms (10.1.3) and (10.1.4), the morphisms in $\mathsf{Mon}^n(2)$ are generated under composites by the exchanges

$$\begin{tikzcd}
\big(\tau(1) \otimes_j \mathbb{1}\big) \otimes_i \big(\mathbb{1} \otimes_j \tau(2)\big) \arrow[d, equal] & \big(\tau(1) \otimes_i \mathbb{1}\big) \otimes_j \big(\mathbb{1} \otimes_i \tau(2)\big) \arrow[d, equal] \\
\tau(1) \otimes_i \tau(2) \arrow[r, "\eta^{i,j}_{\tau(1),\mathbb{1},\mathbb{1},\tau(2)}"] & \tau(1) \otimes_j \tau(2)
\end{tikzcd}$$

$$\begin{tikzcd}
\big(\mathbb{1} \otimes_j \tau(1)\big) \otimes_i \big(\tau(2) \otimes_j \mathbb{1}\big) \arrow[d, equal] & \big(\mathbb{1} \otimes_i \tau(2)\big) \otimes_j \big(\tau(1) \otimes_i \mathbb{1}\big) \arrow[d, equal] \\
\tau(1) \otimes_i \tau(2) \arrow[r, "\eta^{i,j}_{\mathbb{1},\tau(1),\tau(2),\mathbb{1}}"] & \tau(2) \otimes_j \tau(1)
\end{tikzcd}$$

for $1 \leq i < j \leq n$ and $\tau \in \Sigma_2$. The right Σ_2-action on the above exchanges is given by

$$\begin{aligned}
\theta\big(\eta^{i,j}_{\tau(1),\mathbb{1},\mathbb{1},\tau(2)}\big) &= \eta^{i,j}_{\theta\tau(1),\mathbb{1},\mathbb{1},\theta\tau(2)}\\
\theta\big(\eta^{i,j}_{\mathbb{1},\tau(1),\tau(2),\mathbb{1}}\big) &= \eta^{i,j}_{\mathbb{1},\theta\tau(1),\theta\tau(2),\mathbb{1}}
\end{aligned}$$

for $\theta \in \Sigma_2$. $\diamond$

Example 10.5.17. $\mathsf{Mon}^n(3)$ consists of the objects

$$\big(\sigma(1) \otimes_i \sigma(2)\big) \otimes_k \sigma(3) \quad \text{and} \quad \sigma(1) \otimes_i \big(\sigma(2) \otimes_k \sigma(3)\big)$$

for $i, k \in \{1, \ldots, n\}$ and $\sigma \in \Sigma_3$. The morphisms in $\mathsf{Mon}^n(3)$ are generated under composites by the following morphisms for $1 \leq i < j \leq n$, $1 \leq k \leq n$, and $\sigma \in \Sigma_3$.

$$\left\{\begin{array}{l}
\big(\sigma(1) \otimes_i \sigma(2)\big) \otimes_k \sigma(3) \xrightarrow{\eta^{i,j}_{\sigma(1),\mathbb{1},\mathbb{1},\sigma(2)} \otimes_k 1_{\sigma(3)}} \big(\sigma(1) \otimes_j \sigma(2)\big) \otimes_k \sigma(3)\\
\big(\sigma(1) \otimes_i \sigma(2)\big) \otimes_k \sigma(3) \xrightarrow{\eta^{i,j}_{\mathbb{1},\sigma(1),\sigma(2),\mathbb{1}} \otimes_k 1_{\sigma(3)}} \big(\sigma(2) \otimes_j \sigma(1)\big) \otimes_k \sigma(3)\\
\sigma(1) \otimes_k \big(\sigma(2) \otimes_i \sigma(3)\big) \xrightarrow{1_{\sigma(1)} \otimes_k \eta^{i,j}_{\sigma(2),\mathbb{1},\mathbb{1},\sigma(3)}} \sigma(1) \otimes_k \big(\sigma(2) \otimes_j \sigma(3)\big)\\
\sigma(1) \otimes_k \big(\sigma(2) \otimes_i \sigma(3)\big) \xrightarrow{1_{\sigma(1)} \otimes_k \eta^{i,j}_{\mathbb{1},\sigma(2),\sigma(3),\mathbb{1}}} \sigma(1) \otimes_k \big(\sigma(3) \otimes_j \sigma(2)\big)
\end{array}\right.$$

$$\left\{\begin{array}{l}
\big(\sigma(1) \otimes_k \sigma(2)\big) \otimes_i \sigma(3) \xrightarrow{\eta^{i,j}_{(\sigma(1)\otimes_k\sigma(2)),\mathbb{1},\mathbb{1},\sigma(3)}} \big(\sigma(1) \otimes_k \sigma(2)\big) \otimes_j \sigma(3)\\
\big(\sigma(1) \otimes_k \sigma(2)\big) \otimes_i \sigma(3) \xrightarrow{\eta^{i,j}_{\mathbb{1},(\sigma(1)\otimes_k\sigma(2)),\sigma(3),\mathbb{1}}} \sigma(3) \otimes_j \big(\sigma(1) \otimes_k \sigma(2)\big)\\
\sigma(1) \otimes_i \big(\sigma(2) \otimes_k \sigma(3)\big) \xrightarrow{\eta^{i,j}_{\sigma(1),\mathbb{1},\mathbb{1},(\sigma(2)\otimes_k\sigma(3))}} \sigma(1) \otimes_j \big(\sigma(2) \otimes_k \sigma(3)\big)\\
\sigma(1) \otimes_i \big(\sigma(2) \otimes_k \sigma(3)\big) \xrightarrow{\eta^{i,j}_{\mathbb{1},\sigma(1),(\sigma(2)\otimes_k\sigma(3)),\mathbb{1}}} \big(\sigma(2) \otimes_k \sigma(3)\big) \otimes_j \sigma(1)
\end{array}\right.$$

$$\left\{\begin{array}{l}
\big((\sigma(1) \otimes_j \sigma(2)\big) \otimes_i \sigma(3) \xrightarrow{\eta^{i,j}_{\sigma(1),\sigma(2),\sigma(3),\mathbb{1}}} \big(\sigma(1) \otimes_i \sigma(3)\big) \otimes_j \sigma(2)\\
\big((\sigma(1) \otimes_j \sigma(2)\big) \otimes_i \sigma(3) \xrightarrow{\eta^{i,j}_{\sigma(1),\sigma(2),\mathbb{1},\sigma(3)}} \sigma(1) \otimes_j \big(\sigma(2) \otimes_i \sigma(3)\big)\\
\sigma(1) \otimes_i \big(\sigma(2) \otimes_j \sigma(3)\big) \xrightarrow{\eta^{i,j}_{\sigma(1),\mathbb{1},\sigma(2),\sigma(3)}} \big(\sigma(1) \otimes_i \sigma(2)\big) \otimes_j \sigma(3)\\
\sigma(1) \otimes_i \big(\sigma(2) \otimes_j \sigma(3)\big) \xrightarrow{\eta^{i,j}_{\mathbb{1},\sigma(1),\sigma(2),\sigma(3)}} \sigma(2) \otimes_j \big(\sigma(1) \otimes_i \sigma(3)\big)
\end{array}\right.$$

- In the first group, each morphism has the form $\eta^{i,j} \otimes_k 1$ or $1 \otimes_k \eta^{i,j}$.

- In the second group, each morphism has a subscript that is a $\otimes_k$-product.
- In the third group, each morphism has one subscript given by $\mathbb{1}$.

For $\theta \in \Sigma_3$, the right θ-action is given by

$$\begin{aligned}
\theta\big(\eta^{i,j}_{\sigma(1),\mathbb{1},\mathbb{1},\sigma(2)} \otimes_k 1_{\sigma(3)}\big) &= \eta^{i,j}_{\theta\sigma(1),\mathbb{1},\mathbb{1},\theta\sigma(2)} \otimes_k 1_{\theta\sigma(3)}\\
\theta\big(\eta^{i,j}_{\sigma(1)\otimes_k\sigma(2),\mathbb{1},\mathbb{1},\sigma(3)}\big) &= \eta^{i,j}_{\theta\sigma(1)\otimes_k\theta\sigma(2),\mathbb{1},\mathbb{1},\theta\sigma(3)}\\
\theta\big(\eta^{i,j}_{\sigma(1),\sigma(2),\sigma(3),\mathbb{1}}\big) &= \eta^{i,j}_{\theta\sigma(1),\theta\sigma(2),\theta\sigma(3),\mathbb{1}}
\end{aligned}$$

and similarly for the other generating morphisms. $\diamond$

The next result is a decomposition of the free n-fold monoidal category of a small category.

Theorem 10.5.18. *There is a natural isomorphism of categories*

$$\coprod_{k\geq 0} \mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k} \xrightarrow[\cong]{\phi_{\mathsf{C}}} \mathsf{FMon}^n(\mathsf{C}) \tag{10.5.19}$$

for small categories C *that extends the isomorphism*

$$\mathsf{Mon}^n(1) \times \mathsf{C} \cong \mathsf{C},$$

where Σ_k *acts on* $\mathsf{C}^{\times k}$ *by permuting the* k *entries.*

Proof. Pick a small category C and write ϕ for ϕ_{C}. We first define ϕ for $k = 0, 1$.

- For $k = 0$, ϕ sends the unique object and morphism in $\mathsf{Mon}^n(0) \times *$ to, respectively, the unit $\mathbb{1}$ and the identity morphism $1_{\mathbb{1}}$ in $\mathsf{FMon}^n(\mathsf{C})$.
- For $k = 1$, the restriction of ϕ to $\mathsf{Mon}^n(1) \times \mathsf{C}$ is the isomorphism

$$\mathsf{Mon}^n(1) \times \mathsf{C} = \{1\} \times \mathsf{C} \xrightarrow{\cong} \mathsf{C}$$

  followed by the inclusion into $\mathsf{FMon}^n(\mathsf{C})$.

Next we consider the case $k \geq 2$.

Objects. A typical element

$$P(1, \ldots, k) \in \mathsf{Mon}^n(k)$$

is an iterated $\{\otimes_i\}_{i=1}^n$-product with each element in $\{1, \ldots, k\}$ occurring precisely once. For objects $A_i \in \mathsf{C}$ for $1 \leq i \leq k$, define

$$\phi\big(P(1, \ldots, k); \{A_i\}_{i=1}^k\big) = P(A_1, \ldots, A_k) \in \mathsf{FMon}^n(\mathsf{C}), \tag{10.5.20}$$

which is obtained from $P(1, \ldots, k)$ by replacing each $i \in \{1, \ldots, k\}$ with A_i. For example, for $i, k \in \{1, \ldots, n\}$ and $\sigma \in \Sigma_3$,

$$\phi\Big(\big(\sigma(1) \otimes_i \sigma(2)\big) \otimes_k \sigma(3); \{A_i\}_{i=1}^3\Big) = \big(A_{\sigma(1)} \otimes_i A_{\sigma(2)}\big) \otimes_k A_{\sigma(3)}.$$

The formula (10.5.20) is well defined on the objects in $\mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k}$ because

$$\phi\Big(P\big(\theta(1), \ldots, \theta(k)\big); \{A_{\theta^{-1}(i)}\}_{i=1}^k\Big) = P(A_1, \ldots, A_k) \tag{10.5.21}$$

for $\theta \in \Sigma_k$.

Morphisms. A typical morphism

$$f(1, \ldots, k) : P(1, \ldots, k) \longrightarrow Q(1, \ldots, k) \in \mathsf{Mon}^n(k)$$

is a finite composite of iterated $\{\otimes_i\}_{i=1}^n$-product of identity morphisms and exchanges $\{\eta^{i,j}\}_{1\le i<j\le n}$. For morphisms

$$g_i : A_i \longrightarrow B_i \in \mathsf{C} \quad \text{for} \quad 1 \le i \le k,$$

define

$$\phi\big(f(1,\ldots,k);\{g_i\}_{i=1}^k\big) : P(A_1,\ldots,A_k) \longrightarrow Q(B_1,\ldots,B_k) \tag{10.5.22}$$

as either composite in the diagram

$$\begin{tikzcd} P(A_1,\ldots,A_k) \arrow[r, "f(A_1,\ldots,A_k)"] \arrow[d, "P(g_1,\ldots,g_k)"'] & Q(A_1,\ldots,A_k) \arrow[d, "Q(g_1,\ldots,g_k)"] \\ P(B_1,\ldots,B_k) \arrow[r, "f(B_1,\ldots,B_k)"] & Q(B_1,\ldots,B_k) \end{tikzcd} \tag{10.5.23}$$

in $\mathsf{FMon}^n(\mathsf{C})$, which is commutative by repeated applications of (10.5.4). For example, for the first generating morphism in Example 10.5.17, the morphism

$$\phi\big(\eta^{i,j}_{\sigma(1),\mathbb{1},\mathbb{1},\sigma(2)} \otimes_k 1_{\sigma(3)};\{g_i\}_{i=1}^k\big)$$

is the composite

$$\begin{tikzcd} \big(A_{\sigma(1)} \otimes_i A_{\sigma(2)}\big) \otimes_k A_{\sigma(3)} \arrow[r, "\eta^{i,j}_{A_{\sigma(1)},\mathbb{1},\mathbb{1},A_{\sigma(2)}} \otimes_k 1"] & \big(A_{\sigma(1)} \otimes_j A_{\sigma(2)}\big) \otimes_k A_{\sigma(3)} \arrow[d, "(g_{\sigma(1)} \otimes_j g_{\sigma(2)}) \otimes_k g_{\sigma(3)}"'] \\ & \big(B_{\sigma(1)} \otimes_j B_{\sigma(2)}\big) \otimes_k B_{\sigma(3)} \end{tikzcd}$$

in $\mathsf{FMon}^n(\mathsf{C})$. Similar to (10.5.21), the definition (10.5.22) respects the Σ_k-action on $\mathsf{Mon}^n(k)$ and $\mathsf{C}^{\times k}$.

The morphism (10.5.22) is independent of the choice of a decomposition of the morphism $f(1,\ldots,k)$ into a finite composite of iterated $\{\otimes_i\}_{i=1}^n$-product of identity morphisms and exchanges $\{\eta^{i,j}\}_{1\le i<j\le n}$. Indeed, any two such decompositions of $f(1,\ldots,k)$ can be transformed into each other by applying the relations (i)–(v) in Definition 10.5.2 finitely many times. The two corresponding decompositions of $f(A_1,\ldots,A_k) \in \mathsf{FMon}^n(\mathsf{C})$ are equal because $\mathsf{FMon}^n(\mathsf{C})$ also satisfies those five relations.

Functoriality. To see that ϕ is a functor, first note that it preserves identity morphisms because, if $f(1,\ldots,k)$ and each g_i are identity morphisms, then the morphism in (10.5.22) is the composite of two identity morphisms.

To see that ϕ preserves composites, suppose

$$f'(1,\ldots,k) : Q(1,\ldots,k) \longrightarrow R(1,\ldots,k) \in \mathsf{Mon}^n(k) \quad \text{and}$$
$$g'_i : B_i \longrightarrow C_i \in \mathsf{C} \quad \text{for} \quad 1 \le i \le k$$

are morphisms. Consider the diagram

$$\begin{tikzcd} P(A_1,\ldots,A_k) \arrow[r, "f(A_1,\ldots,A_k)"] & Q(A_1,\ldots,A_k) \arrow[r, "f'(A_1,\ldots,A_k)"] \arrow[d, "Q(g_1,\ldots,g_k)"'] & R(A_1,\ldots,A_k) \arrow[d, "R(g_1,\ldots,g_k)"'] \\ & Q(B_1,\ldots,B_k) \arrow[r, "f'(B_1,\ldots,B_k)"] & R(B_1,\ldots,B_k) \arrow[d, "R(g'_1,\ldots,g'_k)"'] \\ & & R(C_1,\ldots,C_k) \end{tikzcd} \tag{10.5.24}$$

in $\mathsf{FMon}^n(\mathsf{C})$.

- The square is commutative by repeated applications of (10.5.4).
- By the functoriality of each $\otimes_i$ in $\mathsf{FMon}^n(\mathsf{C})$, there is an equality

$$R(g_1'g_1, \ldots, g_k'g_k) = R(g_1', \ldots, g_k') \circ R(g_1, \ldots, g_k).$$

  So the top-right composite in (10.5.24) is the morphism

$$\phi\big(f'(1,\ldots,k)f(1,\ldots,k); \{g_i'g_i\}_{i=1}^k\big).$$

- The other composite in (10.5.24) is the morphism

$$\phi\big(f'(1,\ldots,k); \{g_i'\}_{i=1}^k\big) \circ \phi\big(f(1,\ldots,k); \{g_i\}_{i=1}^k\big).$$

This shows that ϕ preserves composites and is a functor.

Naturality. The naturality of ϕ_{C} with respect to the small category C means the commutativity of the diagram

$$\begin{tikzcd} \coprod_{k\geq 0} \mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k} \arrow[r, "\phi_{\mathsf{C}}"] \arrow[d, "\coprod_k \mathrm{Id}\times_{\Sigma_k} F^{\times k}"'] & \mathsf{FMon}^n(\mathsf{C}) \arrow[d, "\mathsf{FMon}^n(F)"] \\ \coprod_{k\geq 0} \mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{D}^{\times k} \arrow[r, "\phi_{\mathsf{D}}"] & \mathsf{FMon}^n(\mathsf{D}) \end{tikzcd} \tag{10.5.25}$$

for a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ between small categories. This diagram is commutative by

- the definition of ϕ, namely, (10.5.20) and (10.5.23), and
- Definition 10.5.8 of $\mathsf{FMon}^n(F)$, with the unit $\mathbb{1}$, identity morphisms, composites, $\{\otimes_i\}_{i=1}^n$, and $\{\eta^{i,j}\}_{1\leq i<j\leq n}$ all strictly preserved.

Indeed, each composite in (10.5.25) sends

- the object $\big(P(1,\ldots,k); \{A_i\}_{i=1}^k\big)$ to $P(FA_1,\ldots,FA_k)$ and
- the morphism $\big(f(1,\ldots,k); \{g_i\}_{i=1}^k\big)$ to the composite

$$\begin{tikzcd} P(FA_1,\ldots,FA_k) \arrow[r, "f(FA_1,\ldots,FA_k)"] & Q(FA_1,\ldots,FA_k) \arrow[d, "Q(Fg_1,\ldots,Fg_k)"] \\ & Q(FB_1,\ldots,FB_k) \end{tikzcd}$$

in $\mathsf{FMon}^n(\mathsf{D})$.

Invertibility. It remains to observe that the functor

$$\coprod_{k\geq 0} \mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k} \xrightarrow{\ \phi\ } \mathsf{FMon}^n(\mathsf{C})$$

is a bijection on objects and morphisms.

- The assignment of ϕ on objects, for $k \in \{0,1\}$ and in (10.5.20) for $k \geq 2$, identifies the objects in its domain and codomain. In each case, the objects consist of the unit $\mathbb{1}$, the objects in C, and the iterated $\{\otimes_i\}_{i=1}^n$-products involving $k \geq 2$ objects in C, with each $\otimes_i$ strictly associative and with $\mathbb{1}$ as the strict two-sided unit.

- Similarly, the assignment of ϕ on morphisms, for $k \in \{0,1\}$ and in (10.5.22) for $k \geq 2$, identifies the generating morphisms and relations in its domain and codomain.

This finishes the proof. □

Example 10.5.26. Taking C as the terminal category $\mathbf{1}$, there are isomorphisms of categories

$$\begin{aligned}\mathsf{FMon}^n(\mathbf{1}) &\cong \coprod_{k\geq 0} \mathsf{Mon}^n(k) \times_{\Sigma_k} \mathbf{1}^{\times k}\\ &\cong \coprod_{k\geq 0} \mathsf{Mon}^n(k)/\Sigma_k.\end{aligned}$$

This is the *free n-fold monoidal category on one object* ◇

Example 10.5.27. For each small n-fold monoidal category C, the counit of the adjunction $\mathsf{FMon}^n \dashv U$ in (10.5.10) is a strict n-fold monoidal functor

$$\mathsf{FMon}^n(U\mathsf{C}) \longrightarrow \mathsf{C}.$$

Combining this with the isomorphism $\phi_{U\mathsf{C}}$ in (10.5.19) and using the notation in (10.5.20)–(10.5.23) yield the following *evaluation functors* θ_k for $k \geq 0$.

$$\begin{array}{ccc}\mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k} & \xrightarrow{\theta_k} & \mathsf{C}\\ \big(P(1,\ldots,k);\{A_i\}_{i=1}^k\big) & \longmapsto & P(A_1,\ldots,A_k)\\ \big(f(1,\ldots,k);\{g_i\}_{i=1}^k\big) & \longmapsto & Q(g_1,\ldots,g_k)\circ f(A_1,\ldots,A_k)\end{array} \tag{10.5.28}$$

The object $P(A_1,\ldots,A_k)$ and the morphisms $Q(g_1,\ldots,g_k)$ and $f(A_1,\ldots,A_k)$ are interpreted in C using its n-fold monoidal structure. ◇

10.6. Coherence of Iterated Monoidal Categories

In this section, we discuss the Coherence Theorem 10.6.8 for n-fold monoidal categories from [**BFSV03**]. As a result of Theorem 10.6.8, in each n-fold monoidal category, each formal diagram built from identity morphisms, the exchanges $\{\eta^{i,j}\}_{i<j}$, the monoidal products $\{\otimes_i\}_{i=1}^n$, and composites is commutative. The following concepts are needed for the coherence theorem.

Definition 10.6.1. Suppose given $a \neq b \in \{1,\ldots,k\}$.

- The *restriction functor*

$$\mathsf{Mon}^n(k) \xrightarrow{R_{a,b}} \mathsf{Mon}^n\big(\{a,b\}\big) \tag{10.6.2}$$

  is defined by

$$R_{a,b}(i) = \begin{cases} i & \text{if } i \in \{a,b\} \text{ and}\\ \mathbb{1} & \text{if } i \in \{1,\ldots,k\} \smallsetminus \{a,b\}\end{cases}$$

  on objects and subscripts of the generating exchanges $\eta^{i,j}$.
- The image of an object or a morphism under $R_{a,b}$ is called its *restriction to* $\{a,b\}$.
- For an object $A \in \mathsf{Mon}^n(k)$, we write

$$a \otimes_i b \in A \quad \text{if} \quad R_{a,b}(A) = a \otimes_i b \in \mathsf{Mon}^n\big(\{a,b\}\big). \tag{10.6.3}$$

This finishes the definition. $\diamond$

Explanation 10.6.4. Similar to Example 10.5.16, the objects in $\mathsf{Mon}^n(\{a,b\})$ are

$$a \otimes_i b \quad \text{and} \quad b \otimes_i a \quad \text{for} \quad i \in \{1,\ldots,n\}.$$

So for each object $A \in \mathsf{Mon}^n(k)$, either

$$a \otimes_i b \in A \quad \text{or} \quad b \otimes_i a \in A$$

for a unique $i \in \{1,\ldots,n\}$. $\diamond$

Example 10.6.5. If

$$A = 2 \otimes_i (3 \otimes_j 1) \in \mathsf{Mon}^n(3),$$

then the following statements hold.

- $2 \otimes_i 1$ is the only object in $\mathsf{Mon}^n(\{1,2\})$ that satisfies $2 \otimes_i 1 \in A$.
- $3 \otimes_j 1$ is the only object in $\mathsf{Mon}^n(\{1,3\})$ that satisfies $3 \otimes_j 1 \in A$.
- $2 \otimes_i 3$ is the only object in $\mathsf{Mon}^n(\{2,3\})$ that satisfies $2 \otimes_i 3 \in A$. $\diamond$

Example 10.6.6. For the morphism

$$(1 \otimes_k 2) \otimes_i 3 \xrightarrow{\eta^{i,j}_{(1\otimes_k 2),\mathbb{1},\mathbb{1},3}} (1 \otimes_k 2) \otimes_j 3 \in \mathsf{Mon}^n(3)$$

with $1 \le i < j \le n$ and $1 \le k \le n$, its restrictions to $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ are, respectively, the following morphisms.

$$\begin{aligned}
1 \otimes_k 2 &\xrightarrow{\eta^{i,j}_{(1\otimes_k 2),\mathbb{1},\mathbb{1},\mathbb{1}} = 1} 1 \otimes_k 2\\
1 \otimes_i 3 &\xrightarrow{\eta^{i,j}_{1,\mathbb{1},\mathbb{1},3}} 1 \otimes_j 3\\
2 \otimes_i 3 &\xrightarrow{\eta^{i,j}_{2,\mathbb{1},\mathbb{1},3}} 2 \otimes_j 3
\end{aligned}$$

The first morphism $\eta^{i,j}_{(1\otimes_k 2),\mathbb{1},\mathbb{1},\mathbb{1}}$ is the identity morphism by the internal unity axiom (10.1.3) in $\mathsf{FMon}^n(\{1,2\})$. $\diamond$

Motivation 10.6.7. Mac Lane's Coherence Theorem I.1.3.3 says that each formal diagram in a monoidal category is commutative. Theorem 10.6.8 below, which is **[BFSV03**, 3.6], is the analogue for n-fold monoidal categories. Among other things, it says that each diagram in each category $\mathsf{Mon}^n(k)$ is commutative. When combined with the evaluation functors in (10.5.28), it follows that each formal diagram, built from the exchanges $\eta^{i,j}$ and identity morphisms, in each n-fold monoidal category is commutative. $\diamond$

Theorem 10.6.8 (n-Fold Monoidal Category Coherence)**.** *Suppose A and B are objects in $\mathsf{Mon}^n(k)$. Then the following two statements hold.*

1. *There is at most one morphism $A \longrightarrow B$.*
2. *There exists a morphism $A \longrightarrow B$ if and only if, for any $a \ne b \in \{1,\ldots,k\}$, $a \otimes_i b \in A$ implies either*
   - *$a \otimes_j b \in B$ for some $j \ge i$ or*
   - *$b \otimes_j a \in B$ for some $j > i$.*

Proof. We refer the reader to **[BFSV03**, 3.6] for the proof. It uses (i) a double induction and (ii) the n-fold monoidal category axioms (10.1.3)–(10.1.7) and the naturality of the exchanges $\eta^{i,j}$. Here we only explain the necessity part in assertion (2). Suppose

- $f : A \longrightarrow B$ is a morphism in $\mathsf{Mon}^n(k)$, and
- $a \otimes_i b \in A$ for some $a \neq b \in \{1, \ldots, k\}$ and $i \in \{1, \ldots, n\}$ as in (10.6.3).

If f is the identity morphism, then $a \otimes_i b \in B$.

If f is not the identity morphism, then its restriction to $\{a, b\}$ is a morphism

$$a \otimes_i b = R_{a,b}(A) \xrightarrow{R_{a,b}(f)} R_{a,b}(B) \in \mathsf{Mon}^n\big(\{a, b\}\big).$$

As in Example 10.5.16 with $\{a, b\}$ in place of $\{1, 2\}$, the morphism $R_{a,b}(f)$ is a composite in $\mathsf{Mon}^n\big(\{a, b\}\big)$ of some exchanges

$$\begin{aligned} \sigma(a) \otimes_k \sigma(b) &\xrightarrow{\eta^{k,l}_{\sigma(a),\mathbb{1},\mathbb{1},\sigma(b)}} \sigma(a) \otimes_l \sigma(b) \\ \sigma(a) \otimes_k \sigma(b) &\xrightarrow{\eta^{k,l}_{\mathbb{1},\sigma(a),\sigma(b),\mathbb{1}}} \sigma(b) \otimes_l \sigma(a) \end{aligned}$$

with $1 \leq k < l \leq n$ and each σ a permutation of $\{a, b\}$. Therefore,

$$R_{a,b}(B) = \sigma(a) \otimes_j \sigma(b)$$

for some $j > i$ and permutation σ of $\{a, b\}$. □

Explanation 10.6.9. By Theorem 10.6.8 (1), each diagram in each category $\mathsf{Mon}^n(k)$ is commutative. Combining this fact with the evaluation functors in (10.5.28), it follows that, in each n-fold monoidal category, each formal diagram built from

- identity morphisms,
- the exchanges $\{\eta^{i,j}\}_{i<j}$,
- the monoidal products $\{\otimes_i\}_{i=1}^n$, and
- composites

is commutative. The n-fold monoidal category axioms (10.1.3)–(10.1.7) are the base cases. ◇

10.7. E_n-Monoidal Categories

In this section, we define E_n-monoidal categories, which generalize both ring categories (Definition 9.1.2) and n-fold monoidal categories (Definition 10.1.1). In Sections 10.8 and 10.9, we will prove the following statements.

- E_2-monoidal categories have braided ring categories (Definition 9.5.1) as special cases, in the same way as 2-fold monoidal categories containing braided strict monoidal categories (Proposition 10.1.14).
- E_n-monoidal categories for $n \geq 2$ have bipermutative categories (Definition 9.3.2) as special cases, in the same way as n-fold monoidal categories containing permutative categories (Proposition 10.1.21).

These E_n-monoidal categories will play an important role in Chapter III.13, where their K-theories are shown to be E_n-symmetric spectra for $n \geq 1$.

Motivation 10.7.1. In Example 10.1.9 and Propositions 10.1.14 and 10.1.21, we saw that n-fold monoidal categories simultaneously generalize

- strict monoidal categories ($n = 1$),
- braided strict monoidal categories ($n = 2$), and
- permutative categories ($n \geq 2$).

Along the same lines, E_n-monoidal categories, which we will define shortly, simultaneously generalize ring categories, bipermutative categories, and braided ring categories in Definitions 9.1.2, 9.3.2, and 9.5.1. To accomplish this, we need an additive structure $\oplus$ and n compatible multiplicative structures $\otimes_1, \ldots, \otimes_n$.

To say more about the compatibility conditions between the additive structure and the multiplicative structures, first recall the following categorical structures.

- A *permutative category* in Definition 1.3.32 is a symmetric strict monoidal category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$. Strictness means that the underlying monoidal category is strict, so the associativity isomorphism, the left unit isomorphism, and the right unit isomorphism are identity natural transformations.
- A *ring category* C in Definition 9.1.2 is equipped with
  - a permutative category structure $(\oplus, \mathbb{0}, \xi^{\oplus})$, which is called the additive structure,
  - a strict monoidal structure $(\otimes, \mathbb{1})$, which is called the multiplicative structure, and
  - factorization natural transformations ∂^l and ∂^r that relate $\oplus$ and $\otimes$.
- An *n-fold monoidal category* C in Definition 10.1.1 is equipped with
  - n strict monoidal structures $\{\otimes_i\}_{1 \leq i \leq n}$ with a common unit $\mathbb{1}$ and
  - exchange natural transformations $\eta^{i,j}$ that relate $\otimes_i$ and $\otimes_j$ for $1 \leq i < j \leq n$.

An E_n-monoidal category C is equipped with

- a permutative category structure $(\oplus, \mathbb{0}, \xi^{\oplus})$,
- an n-fold monoidal category structure
$$\big(\{\otimes_i\}_{1 \leq i \leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \leq i < j \leq n}\big),$$
  and
- factorization morphisms $\{\partial^{l,i}, \partial^{r,i}\}_{1 \leq i \leq n}$

such that
$$(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}, \otimes_i, \mathbb{1}, \partial^{l,i}, \partial^{r,i})$$
is a ring category for each $1 \leq i \leq n$. There are also several compatibility axioms (10.7.7)–(10.7.11) that relate the exchanges $\eta^{i,j}$ and the n ring category structures.

- The zero exchange axiom (10.7.7) says that $\mathbb{0}$ is a strict zero for $\eta^{i,j}$.
- In a typical component $\eta^{i,j}_{A,B,C,D}$, the codomain is
$$(A \otimes_i C) \otimes_j (B \otimes_i D).$$
  In each of the four exchange factorization axioms (10.7.8)–(10.7.11), one of the four codomain factors is changed to a sum, that is,
$$A \oplus A', \quad B \oplus B', \quad C \oplus C', \quad \text{or} \quad D \oplus D'.$$
  Each of these axioms equates two parallel composites with the new codomain. $\diamond$

Definition 10.7.2. For $n \geq 1$, an *E_n-monoidal category* is a tuple

$$\Big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), \{\otimes_i, \partial^{l,i}, \partial^{r,i}\}_{1 \leq i \leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \leq i < j \leq n}\Big)$$

consisting of the following data.

The Ring Category Structures: For each $1 \leq i \leq n$, the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes_i, \mathbb{1}), (\partial^{l,i}, \partial^{r,i})\big) \tag{10.7.3}$$

is a ring category (Definition 9.1.2), with $\oplus$, $\mathbb{0}$, $\xi^{\oplus}$, $\otimes_i$, and $\mathbb{1}$ called, respectively, the *sum*, the *additive zero*, the *additive symmetry*, the *ith product*, and the *unit*. The natural transformations

$$\begin{aligned} (A \otimes_i C) \oplus (B \otimes_i C) &\xrightarrow{\partial^{l,i}_{A,B,C}} (A \oplus B) \otimes_i C \\ (A \otimes_i B) \oplus (A \otimes_i C) &\xrightarrow{\partial^{r,i}_{A,B,C}} A \otimes_i (B \oplus C) \end{aligned} \tag{10.7.4}$$

for objects $A, B, C \in \mathsf{C}$, are called, respectively, the *ith left factorization morphism* and the *ith right factorization morphism*.

The n-Fold Monoidal Structure: The tuple

$$\big(\mathsf{C}, \{\otimes_i\}_{1 \leq i \leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \leq i < j \leq n}\big) \tag{10.7.5}$$

is an n-fold monoidal category (Definition 10.1.1), with (i,j)-*exchange* the natural transformation

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D) \tag{10.7.6}$$

for objects $A, B, C, D \in \mathsf{C}$ and $1 \leq i < j \leq n$.

These data are required to satisfy the following axioms for $1 \leq i < j \leq n$ and objects A, A', B, B', C, C', D, and D' in C.

The Zero Exchange Axiom:

$$\eta^{i,j}_{A,B,C,D} = 1_{\mathbb{0}} \quad \text{if} \quad A,\ B,\ C, \text{ or } D \text{ is } \mathbb{0}. \tag{10.7.7}$$

The Exchange Factorization Axiom: The following four diagrams are commutative. They are called, respectively, EF1, EF2, EF3, and EF4.

$$\begin{tikzcd}[column sep=tiny]
& {[(A \otimes_j B) \otimes_i (C \otimes_j D)] \oplus [(A' \otimes_j B) \otimes_i (C \otimes_j D)]} \arrow[dl, "\eta^{i,j}_{A,B,C,D} \oplus \eta^{i,j}_{A',B,C,D}"'] \arrow[dr, "\partial^{l,i}_{A \otimes_j B, A' \otimes_j B, C \otimes_j D}"] & \\
{[(A \otimes_i C) \otimes_j (B \otimes_i D)] \oplus [(A' \otimes_i C) \otimes_j (B \otimes_i D)]} \arrow[d, "\partial^{l,j}_{A \otimes_i C, A' \otimes_i C, B \otimes_i D}"'] & & {[(A \otimes_j B) \oplus (A' \otimes_j B)] \otimes_i (C \otimes_j D)} \arrow[d, "\partial^{l,j}_{A,A',B} \otimes_i 1"] \\
{[(A \otimes_i C) \oplus (A' \otimes_i C)] \otimes_j (B \otimes_i D)} \arrow[dr, "\partial^{l,i}_{A,A',C} \otimes_j 1"'] & & {[(A \oplus A') \otimes_j B] \otimes_i (C \otimes_j D)} \arrow[dl, "\eta^{i,j}_{A \oplus A',B,C,D}"] \\
& {[(A \oplus A') \otimes_i C] \otimes_j (B \otimes_i D)} &
\end{tikzcd} \tag{10.7.8}$$

$$\begin{tikzcd}[column sep=tiny]
& {[(A \otimes_j B) \otimes_i (C \otimes_j D)] \oplus [(A \otimes_j B') \otimes_i (C \otimes_j D)]} \arrow[dl, "\eta^{i,j}_{A,B,C,D} \oplus \eta^{i,j}_{A,B',C,D}"'] \arrow[dr, "\partial^{l,i}_{A\otimes_j B, A \otimes_j B', C \otimes_j D}"] & \\
{[(A \otimes_i C) \otimes_j (B \otimes_i D)] \oplus [(A \otimes_i C) \otimes_j (B' \otimes_i D)]} \arrow[d, "\partial^{r,j}_{A\otimes_i C, B \otimes_i D, B' \otimes_i D}"'] & & {[(A \otimes_j B) \oplus (A \otimes_j B')] \otimes_i (C \otimes_j D)} \arrow[d, "\partial^{r,j}_{A,B,B'} \otimes_i 1"] \\
(A \otimes_i C) \otimes_j [(B \otimes_i D) \oplus (B' \otimes_i D)] \arrow[dr, "1 \otimes_j \partial^{l,i}_{B,B',D}"'] & & {[A \otimes_j (B \oplus B')] \otimes_i (C \otimes_j D)} \arrow[dl, "\eta^{i,j}_{A,B\oplus B',C,D}"] \\
& (A \otimes_i C) \otimes_j [(B \oplus B') \otimes_i D] &
\end{tikzcd} \tag{10.7.9}$$

$$\begin{tikzcd}[column sep=tiny]
& {[(A \otimes_j B) \otimes_i (C \otimes_j D)] \oplus [(A \otimes_j B) \otimes_i (C' \otimes_j D)]} \arrow[dl, "\eta^{i,j}_{A,B,C,D} \oplus \eta^{i,j}_{A,B,C',D}"'] \arrow[dr, "\partial^{r,i}_{A\otimes_j B, C \otimes_j D, C' \otimes_j D}"] & \\
{[(A \otimes_i C) \otimes_j (B \otimes_i D)] \oplus [(A \otimes_i C') \otimes_j (B \otimes_i D)]} \arrow[d, "\partial^{l,j}_{A\otimes_i C, A \otimes_i C', B \otimes_i D}"'] & & (A \otimes_j B) \otimes_i [(C \otimes_j D) \oplus (C' \otimes_j D)] \arrow[d, "1 \otimes_i \partial^{l,j}_{C,C',D}"] \\
{[(A \otimes_i C) \oplus (A \otimes_i C')] \otimes_j (B \otimes_i D)} \arrow[dr, "\partial^{r,i}_{A,C,C'} \otimes_j 1"'] & & (A \otimes_j B) \otimes_i [(C \oplus C') \otimes_j D] \arrow[dl, "\eta^{i,j}_{A,B,C\oplus C',D}"] \\
& {[A \otimes_i (C \oplus C')] \otimes_j (B \otimes_i D)} &
\end{tikzcd} \tag{10.7.10}$$

$$\begin{tikzcd}[column sep=tiny]
& {[(A \otimes_j B) \otimes_i (C \otimes_j D)] \oplus [(A \otimes_j B) \otimes_i (C \otimes_j D')]} \arrow[dl, "\eta^{i,j}_{A,B,C,D} \oplus \eta^{i,j}_{A,B,C,D'}"'] \arrow[dr, "\partial^{r,i}_{A\otimes_j B, C \otimes_j D, C \otimes_j D'}"] & \\
{[(A \otimes_i C) \otimes_j (B \otimes_i D)] \oplus [(A \otimes_i C) \otimes_j (B \otimes_i D')]} \arrow[d, "\partial^{r,j}_{A\otimes_i C, B \otimes_i D, B \otimes_i D'}"'] & & (A \otimes_j B) \otimes_i [(C \otimes_j D) \oplus (C \otimes_j D')] \arrow[d, "1 \otimes_i \partial^{r,j}_{C,D,D'}"] \\
(A \otimes_i C) \otimes_j [(B \otimes_i D) \oplus (B \otimes_i D')] \arrow[dr, "1 \otimes_j \partial^{r,i}_{B,D,D'}"'] & & (A \otimes_j B) \otimes_i [C \otimes_j (D \oplus D')] \arrow[dl, "\eta^{i,j}_{A,B,C,D\oplus D'}"] \\
& (A \otimes_i C) \otimes_j [B \otimes_i (D \oplus D')] &
\end{tikzcd} \tag{10.7.11}$$

This finishes the definition of an E_n-monoidal category. It is *small* if the category C is small. ◇

Explanation 10.7.12. Consider Definition 10.7.2 of an E_n-monoidal category.

- The n ring category structures in (10.7.3) have a common additive structure $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$, which is a permutative category, and a common unit $\mathbb{1}$.
- In each of the four exchange factorization axioms (10.7.8)–(10.7.11), the hexagon is symmetric across the center in the sense that each arrow and its counterpart across the center involve the same type of structure morphisms. For example, the following symmetry occurs in the hexagon in (10.7.8):
  - The upper left arrow and the lower right arrow involve $\eta^{i,j}$.
  - The lower left arrow and the upper right arrow involve $\partial^{l,i}$.
  - Both vertical arrows involve $\partial^{l,j}$. ◇

Example 10.7.13. An E_1-monoidal category is precisely a ring category as in Definition 9.1.2. ⋄

Example 10.7.14. An E_{n+1}-monoidal category

$$\Big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), \{\otimes_i, \partial^{l,i}, \partial^{r,i}\}_{1\le i\le n+1}, \mathbb{1}, \{\eta^{i,j}\}_{1\le i<j\le n+1}\Big)$$

yields an E_n-monoidal category by forgetting

- the product $\otimes_{n+1}$,
- the factorization morphisms $\partial^{l,n+1}$ and $\partial^{r,n+1}$, and
- the exchange morphisms $\eta^{i,n+1}$ for $1 \le i < n+1$. ⋄

Example 10.7.15. More generally, suppose C is an E_n-monoidal category, and

$$I \subseteq \{1, \ldots, n\}$$

is a nonempty subset with cardinality $|I|$. Then there is an $E_{|I|}$-monoidal category

$$\Big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), \{\otimes_i, \partial^{l,i}, \partial^{r,i}\}_{i\in I}, \mathbb{1}, \{\eta^{i,j}\}_{i<j\in I}\Big)$$

that retains the structure $\otimes_i$, $\partial^{l,i}$, $\partial^{r,i}$, and $\eta^{i,j}$ with indices $i, j \in I$. ⋄

More examples are given in Sections 10.8 through 10.10.

10.8. Braided Ring Categories are E_2-Monoidal Categories

Proposition 10.1.14 identifies braided strict monoidal categories with a subclass of 2-fold monoidal categories. A braided ring category is a braided strict monoidal category with a compatible permutative structure. An E_2-monoidal category is a 2-fold monoidal category with a compatible permutative structure. This section extends Proposition 10.1.14 to braided ring categories and E_2-monoidal categories as follows.

Theorem 10.8.1. *There is a canonical bijective correspondence between*

(1) the class of braided ring categories in Definition 9.5.1 and

(2) the class of E_2-monoidal categories in Definition 10.7.2 with

- $\otimes_1 = \otimes_2$,
- $\partial^{l,1} = \partial^{l,2}$,
- $\partial^{r,1} = \partial^{r,2}$, *and*
- $\eta^{1,2} = \eta$ *a natural isomorphism satisfying*

$$\eta_{A,B,\mathbb{1},C} = 1_{A\otimes_1 B\otimes_1 C} = \eta_{A,\mathbb{1},B,C}. \tag{10.8.2}$$

The correspondence between the braiding $\xi^{\otimes}$ and the exchange η is defined by (10.1.16) and (10.1.17).

Proof. First suppose

$$\Big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\Big) \tag{10.8.3}$$

is a braided ring category as in Definition 9.5.1. We define the data of an E_2-monoidal category as specified in (2) in the statement and (10.1.16):

$$\begin{cases} \otimes_1 = \otimes_2 = \otimes \\ \partial^{l,1} = \partial^{l,2} = \partial^l \\ \partial^{r,1} = \partial^{r,2} = \partial^r \\ \eta_{A,B,C,D} = 1_A \otimes \xi^{\otimes}_{B,C} \otimes 1_D. \end{cases} \tag{10.8.4}$$

By construction, for each $i = 1,2$, the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes_i = \otimes, \mathbb{1}), (\partial^{l,i} = \partial^l, \partial^{r,i} = \partial^r)\big)$$

is a ring category. Moreover, since $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a braided strict monoidal category by definition, Proposition 10.1.14 implies that the tuple

$$\big(\mathsf{C}, \otimes_1 = \otimes, \otimes_2 = \otimes, \mathbb{1}, \eta = 1 \otimes \xi^{\otimes} \otimes 1\big)$$

is a 2-fold monoidal category, with η a natural isomorphism satisfying (10.8.2). Next we check the E_2-monoidal category axioms (10.7.7)–(10.7.11).

With η defined as in (10.8.4), the zero exchange axiom (10.7.7) states that

$$1_A \otimes \xi^{\otimes}_{B,C} \otimes 1_D = 1_{\mathbb{0}} \quad \text{if} \quad A,\ B,\ C, \text{ or } D \text{ is } \mathbb{0}. \tag{10.8.5}$$

- If $A = \mathbb{0}$ or $D = \mathbb{0}$, then (10.8.5) holds by the multiplicative zero axiom (9.1.14) in the ring category C.
- If $B = \mathbb{0}$ or $C = \mathbb{0}$, then

  $$\xi^{\otimes}_{B,C} = 1_{\mathbb{0}}$$

  by the zero braiding axiom (9.5.2) in the braided ring category C. So (10.8.5) holds by the multiplicative zero axiom (9.1.14).

We will abbreviate the strictly associative $\otimes$ using concatenation below, with $\otimes$ taking precedence over $\oplus$ in the absence of clarifying parentheses.

The axioms EF1 (10.7.8) and EF4 (10.7.11) are, respectively, the left and the right outer diagrams below.

$$\begin{tikzcd}
& ABCD \oplus A'BCD \arrow[dl, "1\xi^{\otimes}1 \oplus 1\xi^{\otimes}1"'] \arrow[dr, "\partial^l"] \arrow[ddr, bend left, "\partial^l"'] & \\
ACBD \oplus A'CBD \arrow[d, "\partial^l"'] \arrow[ddr, bend left, "\partial^l"] \arrow[dr, phantom, "(9.1.9)"] & \text{nat} & (AB \oplus A'B)CD \arrow[d, "\partial^l 1"] \arrow[dl, phantom, "(9.1.9)"] \\
(AC \oplus A'C)BD \arrow[dr, "\partial^l 1"'] & & (A \oplus A')BCD \arrow[dl, "1\xi^{\otimes}1"] \\
& (A \oplus A')CBD &
\end{tikzcd}
\qquad
\begin{tikzcd}
& ABCD \oplus ABCD' \arrow[dl, "1\xi^{\otimes}1 \oplus 1\xi^{\otimes}1"'] \arrow[dr, "\partial^r"] \arrow[ddr, bend left, "\partial^r"'] & \\
ACBD \oplus ACBD' \arrow[d, "\partial^r"'] \arrow[ddr, bend left, "\partial^r"] \arrow[dr, phantom, "(9.1.11)"] & \text{nat} & AB(CD \oplus CD') \arrow[d, "1\partial^r"] \arrow[dl, phantom, "(9.1.11)"] \\
AC(BD \oplus BD') \arrow[dr, "1\partial^r"'] & & ABC(D \oplus D') \arrow[dl, "1\xi^{\otimes}1"] \\
& ACB(D \oplus D') &
\end{tikzcd}$$

EF1 on the left is commutative by the left external factorization axiom (9.1.9) and the naturality of ∂^l. Similarly, EF4 on the right is commutative by the right external factorization axiom (9.1.11) and the naturality of ∂^r.

The axiom EF2 (10.7.9) is the outer diagram below.

(10.8.6)
$$
\begin{tikzcd}[column sep=small]
& & ABCD \oplus AB'CD \arrow[dll, "1\xi^{\otimes}1 \oplus 1\xi^{\otimes}1"'] \arrow[drr, "\partial^l"] \arrow[dd, "\partial^r"] & & \\
ACBD \oplus ACB'D \arrow[ddr, "\partial^r"] \arrow[dddd, "\partial^r"'] & & & & (AB \oplus AB')CD \arrow[dd, "\partial^r 1"] \\
& & A(BCD \oplus B'CD) \arrow[dl, "1(\xi^{\otimes}1 \oplus \xi^{\otimes}1)"'] \arrow[dr, "1\partial^l"] \arrow[drr, bend left, "1\partial^l"] & & \\
& A(CBD \oplus CB'D) \arrow[ddl, "1\partial^r"'] \arrow[dr, "1\partial^l"] & & A(BC \oplus B'C)D \arrow[dl, "1(\xi^{\otimes} \oplus \xi^{\otimes})1"'] \arrow[ddr, "1\partial^l 1"] & A(B \oplus B')CD \arrow[dd, equal] \\
& & A(CB \oplus CB')D \arrow[dd, "1\partial^r 1"] & & \\
AC(BD \oplus B'D) \arrow[drr, "1\partial^l"'] & & & & A(B \oplus B')CD \arrow[dll, "1\xi^{\otimes}1"] \\
& & AC(B \oplus B')D & &
\arrow[from=2-1, to=3-3, phantom, "\mathsf{nat}"]
\arrow[from=2-5, to=3-3, phantom, "(9.1.10)"]
\arrow[from=2-1, to=4-2, phantom, "(9.1.11)"]
\arrow[from=3-3, to=5-3, phantom, "\mathsf{nat}"]
\arrow[from=3-3, to=4-5, phantom, "(9.1.9)"]
\arrow[from=4-2, to=6-2, phantom, "(9.1.10)"]
\arrow[from=4-4, to=6-4, phantom, "\text{top (9.5.3)}"]
\end{tikzcd}
$$

- The two subdiagrams labeled by nat are commutative by the naturality of ∂^r and ∂^l.
- Four subdiagrams are commutative by the external factorization axioms (9.1.9)–(9.1.11).
- The lower right subdiagram is commutative by the top half of the braiding factorization axiom (9.5.3).

The axiom EF3 (10.7.10) is the outer diagram below.

(10.8.7)
$$
\begin{tikzcd}[column sep=small]
& & ABCD \oplus ABC'D \arrow[dll, "1\xi^{\otimes}1 \oplus 1\xi^{\otimes}1"'] \arrow[drr, "\partial^r"] \arrow[dd, "\partial^l"] & & \\
ACBD \oplus AC'BD \arrow[ddr, "\partial^l"] \arrow[dddd, "\partial^l"'] & & & & AB(CD \oplus C'D) \arrow[dd, "1\partial^l"] \\
& & (ABC \oplus ABC')D \arrow[dl, "(1\xi^{\otimes} \oplus 1\xi^{\otimes})1"'] \arrow[dr, "\partial^r 1"] \arrow[drr, bend left, "\partial^r 1"] & & \\
& (ACB \oplus AC'B)D \arrow[ddl, "\partial^l 1"'] \arrow[dr, "\partial^r 1"] & & A(BC \oplus BC')D \arrow[dl, "1(\xi^{\otimes} \oplus \xi^{\otimes})1"'] \arrow[ddr, "1\partial^r 1"] & AB(C \oplus C')D \arrow[dd, equal] \\
& & A(CB \oplus C'B)D \arrow[dd, "1\partial^l 1"] & & \\
(AC \oplus AC')BD \arrow[drr, "\partial^r 1"'] & & & & AB(C \oplus C')D \arrow[dll, "1\xi^{\otimes}1"] \\
& & A(C \oplus C')BD & &
\arrow[from=2-1, to=3-3, phantom, "\mathsf{nat}"]
\arrow[from=2-5, to=3-3, phantom, "(9.1.10)"]
\arrow[from=2-1, to=4-2, phantom, "(9.1.9)"]
\arrow[from=3-3, to=5-3, phantom, "\mathsf{nat}"]
\arrow[from=3-3, to=4-5, phantom, "(9.1.11)"]
\arrow[from=4-2, to=6-2, phantom, "(9.1.10)"]
\arrow[from=4-4, to=6-4, phantom, "\text{bottom (9.5.3)}"]
\end{tikzcd}
$$

Six subdiagrams are commutative by the naturality of ∂^l and ∂^r and the external factorization axioms (9.1.9)–(9.1.11) as indicated. The lower right subdiagram is commutative by the bottom half of the braiding factorization axiom (9.5.3). Therefore, (10.8.3) and (10.8.4) define an E_2-monoidal category as in statement (2).

Conversely, suppose C is an E_2-monoidal category as in statement (2). In particular, the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes = \otimes_1, \mathbb{1}), (\partial^l = \partial^{l,1}, \partial^r = \partial^{r,1})\big) \tag{10.8.8}$$

in (10.7.3) is a ring category. Moreover, with the braiding

$$\xi^{\otimes}_{A,B} = \eta_{\mathbb{1},A,B,\mathbb{1}} : AB = \mathbb{1}AB\mathbb{1} \xrightarrow{\cong} \mathbb{1}BA\mathbb{1} = BA \tag{10.8.9}$$

as in (10.1.17), Proposition 10.1.14 implies that

$$\big(\mathsf{C}, \otimes = \otimes_1, \mathbb{1}, \xi^{\otimes} = \eta_{\mathbb{1},-,-,\mathbb{1}}\big)$$

is a braided strict monoidal category. It remains to verify the braided ring category axioms (9.5.2) and (9.5.3). By (10.8.9), the zero braiding axiom (9.5.2) states

$$\eta_{\mathbb{1},A,\mathbb{0},\mathbb{1}} = 1_{\mathbb{0}} = \eta_{\mathbb{1},\mathbb{0},A,\mathbb{1}}.$$

These equalities hold by the zero exchange axiom (10.7.7) in the E_2-monoidal category C.

To verify the top half of the braiding factorization axiom (9.5.3), we reuse the diagram (10.8.6) with $A = D = \mathbb{1}$. Since $(\mathsf{C}, \otimes, \mathbb{1})$ is a strict monoidal category, to check the commutativity of the top half of (9.5.3), it suffices to show that the lower right subdiagram in (10.8.6) with $A = D = \mathbb{1}$ is commutative. Moreover, by the unit factorization axiom (9.1.6) in the ring category (10.8.8), both morphisms

$$\mathbb{1}BC\mathbb{1} \oplus \mathbb{1}B'C\mathbb{1} \xrightarrow{\partial^r_{\mathbb{1},BC\mathbb{1},B'C\mathbb{1}}} \mathbb{1}(BC\mathbb{1} \oplus B'C\mathbb{1}) \xrightarrow{1(\partial^l_{BC,B'C,\mathbb{1}})} \mathbb{1}(BC \oplus B'C)\mathbb{1}$$

in (10.8.6), from the top to the domain of the lower right subdiagram, are identity morphisms. Therefore, the commutativity of the lower right subdiagram in (10.8.6) follows from the commutativity of the other six subdiagrams and the outer diagram, which is commutative by the axiom EF2 (10.7.9) in the E_2-monoidal category C.

The bottom half of the braiding factorization axiom (9.5.3) is obtained from the diagram (10.8.7) with $A = D = \mathbb{1}$, using the procedure in the previous paragraph. Therefore, (10.8.8) and (10.8.9) define a braided ring category.

The constructions above are based on those in Proposition 10.1.14. They are inverse bijections of each other because the additive structure $(\oplus, \mathbb{0}, \xi^{\oplus})$ does not change in either direction. □

10.9. Bipermutative Categories are E_n-Monoidal Categories

Proposition 10.1.21 identifies permutative categories with a subclass of n-fold monoidal categories for $n \geq 2$. A bipermutative category is a permutative category with another compatible permutative structure, and an E_n-monoidal category is an n-fold monoidal category with a compatible permutative structure. This section extends Proposition 10.1.21 to bipermutative categories and E_n-monoidal categories as follows.

Theorem 10.9.1. *For $n \geq 2$, there is a canonical bijective correspondence between the following two classes.*

(1) The class of bipermutative categories in Definition 9.3.2.
(2) The class of E_n-monoidal categories in Definition 10.7.2 that satisfy the following conditions:
- $\otimes_1 = \otimes_2 = \cdots = \otimes_n$.
- $\partial^{l,1} = \partial^{l,2} = \cdots = \partial^{l,n}$.
- $\partial^{r,1} = \partial^{r,2} = \cdots = \partial^{r,n}$.
- $\eta^{i,j} = \eta^{k,l}$ *for all* $1 \leq i < j \leq n$ *and* $1 \leq k < l \leq n$.
- *Each $\eta^{i,j}$ is a natural isomorphism that satisfies the following equalities for $A, B, C, D \in \mathsf{C}$.*

$$\eta_{A,B,\mathbb{1},C} = 1_{A\otimes B\otimes C} = \eta_{A,\mathbb{1},B,C}$$
$$(\eta_{A,C,B,D})(\eta_{A,B,C,D}) = 1_{A\otimes B\otimes C\otimes D} \tag{10.9.2}$$

Here $\otimes$ is the common value of $\otimes_i$ for $1 \leq i \leq n$, and η is the common value of $\eta^{i,j}$ for $1 \leq i < j \leq n$.

The correspondence between the multiplicative symmetry $\xi^{\otimes}$ and the exchange η is defined by (10.1.16) and (10.1.17).

Proof. First suppose

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big) \tag{10.9.3}$$

is a bipermutative category as in Definition 9.3.2. We define the data of an E_n-monoidal category as specified in (2) in the statement and (10.1.16):

$$\begin{cases} \otimes_1 = \otimes_2 = \cdots = \otimes_n = \otimes \\ \partial^{l,1} = \partial^{l,2} = \cdots = \partial^{l,n} = \partial^l \\ \partial^{r,1} = \partial^{r,2} = \cdots = \partial^{r,n} = \partial^r \\ \eta^{i,j}_{A,B,C,D} = 1_A \otimes \xi^{\otimes}_{B,C} \otimes 1_D \quad \text{for} \quad 1 \le i < j \le n. \end{cases} \tag{10.9.4}$$

By construction, for each $1 \le i \le n$, the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes_i = \otimes, \mathbb{1}), (\partial^{l,i} = \partial^l, \partial^{r,i} = \partial^r)\big)$$

is a ring category. Moreover, since $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a permutative category by definition, Proposition 10.1.21 implies that the tuple

$$\big(\mathsf{C}, \{\otimes_i = \otimes\}_{1 \le i \le n}, \mathbb{1}, \{\eta^{i,j} = 1 \otimes \xi^{\otimes} \otimes 1\}_{1 \le i < j \le n}\big)$$

is an n-fold monoidal category, with η a natural isomorphism satisfying (10.9.2). The proof of the E_n-monoidal category axioms (10.7.7)–(10.7.11) uses

- the proof of Theorem 10.8.1 from (10.8.5) to (10.8.7) and
- the fact that C is also a braided ring category by Proposition 9.5.4.

Conversely, suppose C is an E_n-monoidal category as in statement (2). Since $n \ge 2$, by forgetting some structures, Example 10.7.14 implies that

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), \{\otimes_i, \partial^{l,i}, \partial^{r,i}\}_{i=1,2}, \mathbb{1}, \{\eta^{1,2}\}\big)$$

is an E_2-monoidal category. Theorem 10.8.1 implies that

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), \{\otimes_1, \mathbb{1}, \xi^{\otimes} = \eta^{1,2}_{\mathbb{1},-,-,\mathbb{1}}\}, (\partial^{l,1}, \partial^{r,1})\big) \tag{10.9.5}$$

is a braided ring category. Moreover, the braiding $\eta^{1,2}_{\mathbb{1},-,-,\mathbb{1}}$ satisfies the symmetry axiom (1.3.33) by the second equality in (10.9.2). Therefore, (10.9.5) is a bipermutative category by Proposition 9.5.4.

The constructions above are based on those in Proposition 10.1.21. They are inverse bijections of each other because the additive structure $(\oplus, \mathbb{0}, \xi^{\oplus})$ does not change in either direction. $\square$

10.10. Free E_n-Monoidal Categories

In this section, we observe that each small category freely generates an E_n-monoidal category by extending the construction in Section 10.5. This provides further examples of E_n-monoidal categories.

Definition 10.10.1. For a small category C, define the data of an E_n-monoidal category

$$\mathsf{FE}^n(\mathsf{C})$$

as follows.

Objects: The objects in $\mathsf{FE}^n(\mathsf{C})$ are defined as follows.

- Each object in C is also an object in $\mathsf{FE}^n(\mathsf{C})$.
- $\mathsf{FE}^n(\mathsf{C})$ is equipped with two distinguished objects $\mathbb{0}$ and $\mathbb{1}$ that are not in C.
- Inductively, if X and Y are objects in $\mathsf{FE}^n(\mathsf{C})$, then so are

$$X \oplus Y \quad \text{and} \quad X \otimes_i Y \quad \text{for} \quad 1 \le i \le n.$$

These objects are subject to the following relations for $1 \le i \le n$:

- $\oplus$ and $\otimes_i$ are strictly associative on objects.
- $\mathbb{0}$ is the strict two-sided unit for $\oplus$.
- $\mathbb{1}$ is the strict two-sided unit for $\otimes_i$.

Morphisms: The morphisms in $\mathsf{FE}^n(\mathsf{C})$ are defined as follows.

- Each morphism in C is a morphism in $\mathsf{FE}^n(\mathsf{C})$.
- Each object $A \in \mathsf{FE}^n(\mathsf{C})$ is equipped with an *identity morphism*

$$A \xrightarrow{1_A} A,$$

which is the identity morphism $1_A \in \mathsf{C}$ if $A \in \mathsf{C}$.
- $\mathsf{FE}^n(\mathsf{C})$ is equipped with morphisms

$$\begin{aligned} A \oplus B &\xrightarrow{\xi^{\oplus}_{A,B}} B \oplus A \\ (A \otimes_i C) \oplus (B \otimes_i C) &\xrightarrow{\partial^{l,i}_{A,B,C}} (A \oplus B) \otimes_i C \\ (A \otimes_i B) \oplus (A \otimes_i C) &\xrightarrow{\partial^{r,i}_{A,B,C}} A \otimes_i (B \oplus C) \\ (A \otimes_j B) \otimes_i (C \otimes_j D) &\xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D) \end{aligned}$$

for objects $A, B, C, D \in \mathsf{FE}^n(\mathsf{C})$, $1 \le i \le n$, and $1 \le i < j \le n$ for $\eta^{i,j}$.
- Inductively, if

$$f : A \longrightarrow B, \quad g : B \longrightarrow C, \quad \text{and} \quad f' : A' \longrightarrow B'$$

are morphisms in $\mathsf{FE}^n(\mathsf{C})$, then so are

$$\begin{aligned} A \oplus A' &\xrightarrow{f \oplus f'} B \oplus B' \\ A \otimes_i A' &\xrightarrow{f \otimes_i f'} B \otimes_i B' \\ A &\xrightarrow{gf} C \end{aligned}$$

for $1 \le i \le n$, with gf, which is called the *composite*, the composite in C if $f, g \in \mathsf{C}$.

These morphisms are subject to the relations (i)–(ix) below.

(i) Composition is strictly associative and unital with respect to identity morphisms.
(ii) Each $\otimes_i$ and $\oplus$ preserve identity morphisms and composition.
(iii) $\oplus$ is strictly associative on morphisms with $1_{\mathbb{0}}$ as the strict two-sided unit.
(iv) Each $\otimes_i$ is strictly associative on morphisms with $1_{\mathbb{1}}$ as the strict two-sided unit.

(v) For morphisms
$$f_X : X \longrightarrow X' \in \mathsf{FE}^n(\mathsf{C}) \quad \text{with} \quad X \in \{A,B,C,D\},$$
the following diagrams in $\mathsf{FE}^n(\mathsf{C})$ are commutative.

$$\begin{tikzcd}
A \oplus B \arrow[r, "\xi^{\oplus}_{A,B}"] \arrow[d, "f_A \oplus f_B"'] & B \oplus A \arrow[d, "f_B \oplus f_A"] \\
A' \oplus B' \arrow[r, "\xi^{\oplus}_{A',B'}"] & B' \oplus A'
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes_i C) \oplus (B \otimes_i C) \arrow[r, "\partial^{l,i}_{A,B,C}"] \arrow[d, "(f_A \otimes_i f_C) \oplus (f_B \otimes_i f_C)"'] & (A \oplus B) \otimes_i C \arrow[d, "(f_A \oplus f_B) \otimes_i f_C"] \\
(A' \otimes_i C') \oplus (B' \otimes_i C') \arrow[r, "\partial^{l,i}_{A',B',C'}"] & (A' \oplus B') \otimes_i C'
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes_i B) \oplus (A \otimes_i C) \arrow[r, "\partial^{r,i}_{A,B,C}"] \arrow[d, "(f_A \otimes_i f_B) \oplus (f_A \otimes_i f_C)"'] & A \otimes_i (B \oplus C) \arrow[d, "f_A \otimes_i (f_B \oplus f_C)"] \\
(A' \otimes_i B') \oplus (A' \otimes_i C') \arrow[r, "\partial^{r,i}_{A',B',C'}"] & A' \otimes_i (B' \oplus C')
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes_j B) \otimes_i (C \otimes_j D) \arrow[r, "\eta^{i,j}_{A,B,C,D}"] \arrow[d, "(f_A \otimes_j f_B) \otimes_i (f_C \otimes_j f_D)"'] & (A \otimes_i C) \otimes_j (B \otimes_i D) \arrow[d, "(f_A \otimes_i f_C) \otimes_j (f_B \otimes_i f_D)"] \\
(A' \otimes_j B') \otimes_i (C' \otimes_j D') \arrow[r, "\eta^{i,j}_{A',B',C',D'}"] & (A' \otimes_i C') \otimes_j (B' \otimes_i D')
\end{tikzcd}$$

(vi) The tuple
$$\big(\mathsf{FE}^n(\mathsf{C}), \oplus, \mathbb{0}, \xi^{\oplus}\big)$$
satisfies the axioms (1.3.33)–(1.3.35) of a permutative category, with
- (1.3.34) interpreted as $\xi^{\oplus}_{?,\mathbb{0}} = 1_?$ and
- $\alpha = 1$ in (1.3.35).

(vii) For $1 \leq i \leq n$, the tuple
$$\big(\mathsf{FE}^n(\mathsf{C}), (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes_i, \mathbb{1}), (\partial^{l,i}, \partial^{r,i})\big)$$
satisfies the axioms (9.1.4)–(9.1.12) of a ring category.

(viii) The tuple
$$\big(\mathsf{FE}^n(\mathsf{C}), \{\otimes_i\}_{1 \leq i \leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \leq i < j \leq n}\big)$$
satisfies the axioms (10.1.3)–(10.1.7) of an n-fold monoidal category.

(ix) The E_n-monoidal category axioms (10.7.7)–(10.7.11) are satisfied in $\mathsf{FE}^n(\mathsf{C})$.

This finishes the definition of $\mathsf{FE}^n(\mathsf{C})$. ◇

Proposition 10.10.2. *For each small category* C, $\mathsf{FE}^n(\mathsf{C})$ *is an* E_n*-monoidal category.*

Proof. The relations (i)–(ix) refer to those in Definition 10.10.1.
- $\mathsf{FE}^n(\mathsf{C})$ is a category by (i).
- The assignments
$$\oplus, \otimes_i : \mathsf{FE}^n(\mathsf{C}) \times \mathsf{FE}^n(\mathsf{C}) \longrightarrow \mathsf{FE}^n(\mathsf{C})$$
are functors by (ii).

- The triple
$$(\mathsf{FE}^n(\mathsf{C}), \oplus, \mathbb{0})$$
is a strict monoidal category by the strict associativity and unity of $\oplus$ on objects and (iii).
- For $1 \leq i \leq n$, the triple
$$(\mathsf{FE}^n(\mathsf{C}), \otimes_i, \mathbb{1})$$
is a strict monoidal category by the strict associativity and unity of $\otimes_i$ on objects and (iv).
- $\xi^{\oplus}$, $\partial^{l,i}$, $\partial^{r,i}$, and $\eta^{i,j}$ are natural transformations by (v).
- The quadruple
$$\big(\mathsf{FE}^n(\mathsf{C}), \oplus, \mathbb{0}, \xi^{\oplus}\big)$$
is a permutative category by (vi).
- For $1 \leq i \leq n$, the tuple
$$\big(\mathsf{FE}^n(\mathsf{C}), (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes_i, \mathbb{1}), (\partial^{l,i}, \partial^{r,i})\big)$$
is a ring category by (vii).
- The tuple
$$\big(\mathsf{FE}^n(\mathsf{C}), \{\otimes_i\}_{1 \leq i \leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \leq i < j \leq n}\big)$$
is an n-fold monoidal category by (viii).

The E_n-monoidal category axioms hold in $\mathsf{FE}^n(\mathsf{C})$ by (ix). □

10.11. Notes

10.11.1 (Iterated Monoidal Categories and Functors)**.** Definitions 10.1.1 and 10.3.1 of an n-fold monoidal category and functor are [**BFSV03**, Def. 1.7 and 1.8].

- Propositions 10.1.14 and 10.1.21, which identify braided strict monoidal and permutative categories with subclasses of 2-fold and n-fold monoidal categories, are expanded versions of [**BFSV03**, Remarks 1.5 and 1.9].
- Theorem 10.4.5, which states that monoids in MCat^n are precisely small $(n+1)$-fold monoidal categories, is stated in [**BFSV03**, p. 285], but an explicit proof was not given there.
- Theorem 10.5.18, which provides a decomposition of the free n-fold monoidal category, is stated in [**BFSV03**, p. 291].

In addition to Theorem 10.6.8, coherence results for categories equipped with only the exchanges $\eta^{i,j}$ are proved in [**DP12**] ⋄

10.11.2 (Lax Iterated Monoidal Categories)**.** In Definition 10.1.1 of an n-fold monoidal category, the strictness of the monoidal categories $(\mathsf{C}, \otimes_i, \mathbb{1})$ for $1 \leq i \leq n$ is assumed for convenience. It is possible to incorporate associativity and unit isomorphisms into the definition of an n-fold monoidal category. See, for example,

- [**For04**] with associativity isomorphisms and a common strict unit,
- [**FSS07**] with associativity isomorphisms and distinct strict units, and
- [**AM10**, Ch. 6–7] with associativity isomorphisms and distinct units.

Related to 2-fold monoidal categories are the *duoidal categories* in [**BM12**]. See also Appendix III.A.2. ⋄

10.11.3 (Iterated Loop Spaces and E_n Structures)**.** It is shown in [**BFSV03**] that the group completion of the classifying space of a small n-fold monoidal category is an n-fold loop space for each $n \geq 1$. This generalizes the important fact that the group completion of the classifying space of a small strict monoidal category is a loop space. The converse—that each n-fold loop space is the group completion of the classifying space of some small n-fold monoidal category—is proved in [**FV03, FSV13**].

In addition to the work of [**BV73, BFSV03, FV03, FSV13, May72**], there are many other approaches to E_n structures and n-fold loop spaces, including

- Milgram's model of $\Omega^n\Sigma^n X$ [**Mil66**],
- Smith's model of $\Omega^n\Sigma^n X$ [**Smi89**],
- Batanin's n-operads [**Bat08, Bat98, Bat07**],
- Berger's filtration of the Barratt-Eccles operad [**Ber96**],
- Fiedorowicz's E_n-operads [**Fie∞b**], and
- the Fulton-MacPherson E_n-operads [**Fre17, FM94, GJ∞, Sal01**]. ⋄

10.11.4 (E_n-Ring Categories)**.** Our E_n-monoidal categories in Definition 10.7.2 are different from the E_n-ring categories in [**Dun97**, Def. 3.1]. As pointed out in [**EM06**, p. 166], there is a critical error in [**Dun97**, Lemma 2.2(ii)], where ξ is not well defined. ⋄

10.11.5 (Totally Ordered Monoids)**.** For discussion of ordered algebraic structures, the reader is referred to [**Bly05, Ful97, Sch03**]. Proposition 10.2.8, regarding 2-fold monoidal categories from totally ordered monoids, is [**FSS07**, 4.3]. ⋄

Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory

Volume III: From Categories to Structured Ring Spectra

Niles Johnson

Donald Yau

The first author dedicates this book to Nemili, Linus, and Kavya.
The second author dedicates this book to Jacqueline.

Part 1

Enriched Monoidal Categories and Multicategories

CHAPTER 1

Enriched Monoidal Categories

Part 1 is about enriched monoidal categories and enriched multicategories. We will use this material in the following two ways.

- Many of the K-theory constructions presented in Chapters 8, 9, and 10 are enriched, as monoidal functors or as multifunctors. We use the general theory presented here to explain the relevant enrichments of their co/domains and the (multi-)functors themselves.
- The operads that describe E_n-monoidal structure, presented in Chapters 11, 12, and 13, are 1-object enriched multicategories. That point of view is important for the preservation of E_n-monoidal structure by the (enriched) Elmendorf-Mandell K-theory multifunctor.

All of the necessary background on 2-categories and bicategories can be found in Chapter I.6 and **[JY21]**.

This chapter gives introductory definitions and basic results about monoidal structure for enriched categories. We do not assume any prior knowledge of enriched category theory. Throughout this chapter $\mathsf{V} = (\mathsf{V}, \otimes)$ will denote a monidal category and will be assumed either braided or symmetric, as needed. We give a short review of monoidal categories, along with their braided and symmetric variants, in Section 1.1. We then review the basic definitions of V-category, V-functor, and V-natural transformation in Section 1.2.

In Section 1.3, under the assumption that V is braided monoidal, we define the tensor product of enriched categories and show that it defines a monoidal product for the category of V-categories (Theorem 1.3.35). In Section 1.4 we use the tensor product to define monoidal V-categories, V-functors, and V-natural transformations along with braided and symmetric variants.

In Section 1.5 we apply the definitions of Section 1.4 in the case $(\mathsf{V}, \otimes) = (\mathsf{Cat}, \times)$ to obtain definitions of plain, braided, and symmetric monoidal structure for a 2-category. We explain how these definitions are a special, more strict, case of the general definitions for monoidal bicategories from Section I.6.4. In Theorem 1.5.5 we show, for general braided monoidal V, that the 2-category of V-categories is monoidal in the more strict Cat-enriched sense.

1.1. Review of Monoidal Categories

For the reader's convenience we recall definitions and basic properties of monoidal categories, functors, and natural transformations along with the braided and symmetric variants. These definitions along with basic properties will be used throughout this and several subsequent chapters. At the end of the section we include statements of the coherence and strictification theorems, which we will use in a few key places. With the exception of Lemma 1.1.34, all of this material has been covered with more detail previously in Chapters I.1 and II.1.

Monoidal Categories.

Definition 1.1.1. A *monoidal category* is a tuple

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$$

consisting of

- a category C;
- a functor $\otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$, which is called the *monoidal product*;
- an object $\mathbb{1} \in \mathsf{C}$, which is called the *monoidal unit*;
- a natural isomorphism

$$(X \otimes Y) \otimes Z \xrightarrow[\cong]{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z)$$

  for all objects $X, Y, Z \in \mathsf{C}$, which is called the *associativity isomorphism*; and
- natural isomorphisms

$$\mathbb{1} \otimes X \xrightarrow[\cong]{\lambda_X} X \quad \text{and} \quad X \otimes \mathbb{1} \xrightarrow[\cong]{\rho_X} X$$

  for all objects $X \in \mathsf{C}$, which are called the *left unit isomorphism* and the *right unit isomorphism*, respectively.

These data are subject to the following two axioms.

The Unity Axiom: The diagram

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[r, "\alpha_{X,\mathbb{1},Y}"] \arrow[d, "\rho_X \otimes 1_Y"'] & X \otimes (\mathbb{1} \otimes Y) \arrow[d, "1_X \otimes \lambda_Y"] \\
X \otimes Y \arrow[r, equal] & X \otimes Y
\end{tikzcd} \tag{1.1.2}$$

is commutative for all objects $X, Y \in \mathsf{C}$.

The Pentagon Axiom: The pentagon

$$\begin{tikzcd}
& (W \otimes X) \otimes (Y \otimes Z) \arrow[dr, "\alpha_{W,X,Y \otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W \otimes X,Y,Z}"] \arrow[d, "\alpha_{W,X,Y} \otimes 1_Z"'] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X \otimes Y,Z}"] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "1_W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd} \tag{1.1.3}$$

is commutative for all objects $W, X, Y, Z \in \mathsf{C}$.

This finishes the definition of a monoidal category. A monoidal category is *strict* if α, λ, and ρ are identity natural transformations. ◇

In a monoidal category, the equality

$$\lambda_{\mathbb{1}} = \rho_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1} \tag{1.1.4}$$

and the commutative diagrams

$$
\begin{tikzcd}
(\mathbb{1} \otimes X) \otimes Y \arrow[r, "\alpha_{\mathbb{1},X,Y}"] \arrow[d, "\lambda_X \otimes 1_Y"'] & \mathbb{1} \otimes (X \otimes Y) \arrow[d, "\lambda_{X\otimes Y}"] \\
X \otimes Y \arrow[r, equal] & X \otimes Y
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \otimes Y) \otimes \mathbb{1} \arrow[r, "\alpha_{X,Y,\mathbb{1}}"] \arrow[d, "\rho_{X\otimes Y}"'] & X \otimes (Y \otimes \mathbb{1}) \arrow[d, "1_X \otimes \rho_Y"] \\
X \otimes Y \arrow[r, equal] & X \otimes Y
\end{tikzcd}
\tag{1.1.5}
$$

are formal consequences of the monoidal category axioms. These two diagrams are called the *left unity diagram* and the *right unity diagram*, respectively.

Definition 1.1.6. For monoidal categories C and D, a *monoidal functor*

$$(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of

- a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$;
- a natural transformation, which is called the *monoidal constraint*,

$$FX \otimes FY \xrightarrow{F^2} F(X \otimes Y) \in \mathsf{D} \tag{1.1.7}$$

  for objects $X, Y \in \mathsf{C}$; and
- a morphism, which is called the *unit constraint*,

$$\mathbb{1}^{\mathsf{D}} \xrightarrow{F^0} F\mathbb{1}^{\mathsf{C}} \in \mathsf{D}. \tag{1.1.8}$$

These data are required to satisfy the following associativity and unity axioms.

Associativity: The diagram

$$
\begin{tikzcd}
\big(FX \otimes FY\big) \otimes FZ \arrow[r, "\alpha^{\mathsf{D}}"] \arrow[d, "F^2 \otimes 1_{FZ}"'] & FX \otimes \big(FY \otimes FZ\big) \arrow[d, "1_{FX} \otimes F^2"] \\
F(X \otimes Y) \otimes FZ \arrow[d, "F^2"'] & FX \otimes F(Y \otimes Z) \arrow[d, "F^2"] \\
F\big((X \otimes Y) \otimes Z\big) \arrow[r, "F\alpha^{\mathsf{C}}"] & F\big(X \otimes (Y \otimes Z)\big)
\end{tikzcd}
\tag{1.1.9}
$$

is commutative for all objects $X, Y, Z \in \mathsf{C}$.

Unity: The diagrams

$$
\begin{tikzcd}
\mathbb{1}^{\mathsf{D}} \otimes FX \arrow[r, "\lambda^{\mathsf{D}}_{FX}"] \arrow[d, "F^0 \otimes 1_{FX}"'] & FX \\
F\mathbb{1}^{\mathsf{C}} \otimes FX \arrow[r, "F^2"] & F(\mathbb{1}^{\mathsf{C}} \otimes X) \arrow[u, "F\lambda^{\mathsf{C}}_X"']
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes \mathbb{1}^{\mathsf{D}} \arrow[r, "\rho^{\mathsf{D}}_{FX}"] \arrow[d, "1_{FX} \otimes F^0"'] & FX \\
FX \otimes F\mathbb{1}^{\mathsf{C}} \arrow[r, "F^2"] & F(X \otimes \mathbb{1}^{\mathsf{C}}) \arrow[u, "F\rho^{\mathsf{C}}_X"']
\end{tikzcd}
\tag{1.1.10}
$$

are commutative for all objects $X \in \mathsf{C}$. They are called the *left unity diagram* and the *right unity diagram*, respectively.

This finishes the definition of a monoidal functor. A monoidal functor (F, F^2, F^0) is often abbreviated to F.

Moreover, a monoidal functor (F, F^2, F^0) is said to be

- *unital* if F^0 is an isomorphism;
- *strictly unital* if F^0 is the identity morphism;
- *strong* if F^0 and the components of F^2 are isomorphisms; and

- *strict* if F^0 and the components of F^2 are identity morphisms. ⋄

Definition 1.1.11. Suppose

$$\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$$

are monoidal functors. Their *composite*

$$(GF, (GF)^2, (GF)^0) : \mathsf{C} \longrightarrow \mathsf{E}$$

is the monoidal functor with underlying functor GF and the structure morphisms

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "G^0"] \arrow[rr, bend left, "(GF)^0"] & G\mathbb{1} \arrow[r, "G(F^0)"] & GF\mathbb{1}
\end{tikzcd}$$

$$\begin{tikzcd}
GFA \otimes GFB \arrow[r, "G^2"] \arrow[rr, bend left, "(GF)^2"] & G(FA \otimes FB) \arrow[r, "G(F^2)"] & GF(A \otimes B)
\end{tikzcd}$$

for objects $A, B \in \mathsf{C}$. ⋄

Definition 1.1.12. For monoidal functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, a *monoidal natural transformation* $\theta : F \longrightarrow G$ is a natural transformation between the underlying functors such that the diagrams

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\theta_X \otimes \theta_Y"] \arrow[d, "F^2"'] & GX \otimes GY \arrow[d, "G^2"] & \mathbb{1}^{\mathsf{D}} \arrow[r, "F^0"] \arrow[d, equal] & F\mathbb{1}^{\mathsf{C}} \arrow[d, "\theta_{\mathbb{1}^{\mathsf{C}}}"] \\
F(X \otimes Y) \arrow[r, "\theta_{X \otimes Y}"] & G(X \otimes Y) & \mathbb{1}^{\mathsf{D}} \arrow[r, "G^0"] & G\mathbb{1}^{\mathsf{C}}
\end{tikzcd} \tag{1.1.13}$$

are commutative for all objects $X, Y \in \mathsf{C}$. ⋄

Braided Monoidal Categories.

Definition 1.1.14. A *braided monoidal category* is a pair (C, ξ) consisting of the following data.

- $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category.
- ξ is a natural isomorphism

$$X \otimes Y \xrightarrow[\cong]{\xi_{X,Y}} Y \otimes X \tag{1.1.15}$$

  for objects $X, Y \in \mathsf{C}$, which is called the *braiding*.

These data are required to satisfy the *Hexagon Axioms*, stating the commutativity of the following two diagrams, called the *left hexagon diagram* and the *right hexagon diagram*, respectively, for objects $X, Y, Z \in \mathsf{C}$.

$$\begin{tikzcd}
& (Y \otimes X) \otimes Z \arrow[r, "\alpha"] & Y \otimes (X \otimes Z) \arrow[dr, "1_Y \otimes \xi_{X,Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\xi_{X,Y} \otimes 1_Z"] \arrow[dr, "\alpha"'] & & & Y \otimes (Z \otimes X) \\
& X \otimes (Y \otimes Z) \arrow[r, "\xi_{X,YZ}"] & (Y \otimes Z) \otimes X \arrow[ur, "\alpha"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}"] & (X \otimes Z) \otimes Y \arrow[dr, "\xi_{X,Z} \otimes 1_Y"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "1_X \otimes \xi_{Y,Z}"] \arrow[dr, "\alpha^{-1}"'] & & & (Z \otimes X) \otimes Y \\
& (X \otimes Y) \otimes Z \arrow[r, "\xi_{XY,Z}"] & Z \otimes (X \otimes Y) \arrow[ur, "\alpha^{-1}"'] &
\end{tikzcd} \tag{1.1.16}$$

This finishes the definition of a braided monoidal category. A braided monoidal category is *strict* if the underlying monoidal category is strict. $\diamond$

Definition 1.1.17. For braided monoidal categories C and D, a *braided monoidal functor* $(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$ is a monoidal functor between the underlying monoidal categories such that the diagram

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\xi_{FX,FY}", "\cong"'] \arrow[d, "F^2"'] & FY \otimes FX \arrow[d, "F^2"] \\ F(X \otimes Y) \arrow[r, "F\xi_{X,Y}", "\cong"'] & F(Y \otimes X) \end{tikzcd} \tag{1.1.18}$$

is commutative for all objects $X, Y \in \mathsf{C}$. A braided monoidal functor is said to be *strong* (respectively, *strict*, *unital*, or *strictly unital*) if the underlying monoidal functor is so. $\diamond$

Explanation 1.1.19. The two hexagon diagrams in (1.1.16) may be visualized as the braids, read bottom-to-top,

in the braid group B_3, with the braiding ξ interpreted as the generating braid $s_1 \in B_2$. On the left, the two strings labeled by Y and Z cross over the string labeled by X. The two composites along the boundary of the left hexagon diagram (1.1.16) correspond to passing Y and Z over X either one at a time, or both at once. On the right, the string labeled by Z crosses over the two strings labeled by Y and X. The two composites along the boundary of the right hexagon diagram (1.1.16) likewise correspond to the two ways of passing Z over X and Y. $\diamond$

We note several useful consequences of the hexagon axioms (1.1.16); proofs of these are given in Section II.1.3.

In each braided monoidal category (C, ξ), the following two unit diagrams are commutative for all objects $X \in \mathsf{C}$.

$$\begin{tikzcd} X \otimes \mathbb{1} \arrow[r, "\xi_{X,\mathbb{1}}"] \arrow[d, "\rho_X"'] & \mathbb{1} \otimes X \arrow[d, "\lambda_X"] \\ X \arrow[r, equal] & X \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \otimes X \arrow[r, "\xi_{\mathbb{1},X}"] \arrow[d, "\lambda_X"'] & X \otimes \mathbb{1} \arrow[d, "\rho_X"] \\ X \arrow[r, equal] & X \end{tikzcd} \tag{1.1.20}$$

In each braided monoidal category (C, ξ), the equality

$$\xi_{\mathbb{1},\mathbb{1}} = 1_{\mathbb{1} \otimes \mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \longrightarrow \mathbb{1} \otimes \mathbb{1} \tag{1.1.21}$$

holds.

In each braided monoidal category (C,ξ), the following diagram is commutative for all objects $A,B,C \in \mathsf{C}$.

(1.1.22)
$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[dl, "\xi_{A,B}\otimes 1_C"'] & A\otimes(B\otimes C) \arrow[dr, "1_A\otimes\xi_{B,C}"] & \\
(B\otimes A)\otimes C \arrow[dr, "\xi_{B\otimes A,C}"'] & & & A\otimes(C\otimes B) \arrow[dl, "\xi_{A,C\otimes B}"] \\
 & C\otimes(B\otimes A) \arrow[r, "\alpha^{-1}_{C,B,A}"] & (C\otimes B)\otimes A &
\end{tikzcd}$$

Symmetric Monoidal Categories.

Definition 1.1.23. A *symmetric monoidal category* is a monoidal category

$$(\mathsf{C},\otimes,\mathbb{1},\alpha,\lambda,\rho)$$

equipped with a natural isomorphism

$$X\otimes Y \xrightarrow[\cong]{\xi_{X,Y}} Y\otimes X$$

for objects $X,Y \in \mathsf{C}$, which is called the *braiding* or the *symmetry isomorphism*, that satisfies the following axioms.

The Symmetry Axiom: The diagram

(1.1.24)
$$\begin{tikzcd}
X\otimes Y \arrow[r, "\xi_{X,Y}"] \arrow[dr, equal] & Y\otimes X \arrow[d, "\xi_{Y,X}"] \\
 & X\otimes Y
\end{tikzcd}$$

is commutative for all objects $X,Y \in \mathsf{C}$.

The Unit Axiom: The diagram

(1.1.25)
$$\begin{tikzcd}
X\otimes\mathbb{1} \arrow[r, "\xi_{X,\mathbb{1}}"] \arrow[d, "\rho_X"'] & \mathbb{1}\otimes X \arrow[d, "\lambda_X"] \\
X \arrow[r, equal] & X
\end{tikzcd}$$

is commutative for all objects $X \in \mathsf{C}$.

The Hexagon Axiom: The diagram

(1.1.26)
$$\begin{tikzcd}
 & X\otimes(Z\otimes Y) \arrow[r, "1_X\otimes\xi_{Z,Y}"] & X\otimes(Y\otimes Z) \arrow[dr, "\alpha^{-1}"] & \\
(X\otimes Z)\otimes Y \arrow[ur, "\alpha"] \arrow[dr, "\xi_{X\otimes Z,Y}"'] & & & (X\otimes Y)\otimes Z \\
 & Y\otimes(X\otimes Z) \arrow[r, "\alpha^{-1}"] & (Y\otimes X)\otimes Z \arrow[ur, "\xi_{Y,X}\otimes 1_Z"'] &
\end{tikzcd}$$

is commutative for all objects $X,Y,Z \in \mathsf{C}$.

This finishes the definition of a symmetric monoidal category. A *permutative category* is a symmetric monoidal category whose underlying monoidal category is strict.

A *symmetric monoidal functor* between symmetric monoidal categories is a monoidal functor that satisfies (1.1.18). A symmetric monoidal functor is said to be *strong* (respectively, *strict*, *unital*, or *strictly unital*) if the underlying monoidal functor is so.

Proposition II.1.3.36 shows that a symmetric monoidal category is precisely a braided monoidal category whose braiding satisfies the symmetry axiom (1.1.24). ◇

Definition 1.1.27. We let $\mathsf{PermCat}$ denote the 2-category of small permutative categories, symmetric monoidal functors, and monoidal natural transformations. We also use the following locally-full sub 2-categories consisting of the same objects but restricting the 1-cells.

- $\mathsf{PermCat}^{\mathsf{u}}$ has 1-cells given by unital symmetric monoidal functors.
- $\mathsf{PermCat}^{\mathsf{su}}$ has 1-cells given by strictly unital symmetric monoidal functors.
- $\mathsf{PermCat}^{\mathsf{sus}}$ has 1-cells given by strictly unital strong symmetric monoidal functors.

In each case the 2-cells are given by monoidal natural transformations. ◇

Definition 1.1.28. A symmetric monoidal category C is *closed* if, for each object X, the functor

$$- \otimes X : \mathsf{C} \longrightarrow \mathsf{C}$$

admits a right adjoint, which is denoted by $[X, -]$ and is called the *internal hom*. ◇

Coherence and Strictification Theorems. We recall the statements of monoidal coherence and strictification theorems together with the braided and symmetric variants. More detailed discussion has been given in Sections I.1.3 and II.1.6.

We recall from Definitions I.1.3.1 and I.1.3.2 the notions of (normalized) words and canonical maps. We then state the monoidal coherence and strictification theorems.

Definition 1.1.29. A *word* of length $n \geq 0$ is defined inductively as follows.

- The only word of length 0 is the symbol e.
- The only word of length 1 is the symbol $-$.
- If u and v are words of lengths m and n, respectively, then $u \square v$ is a word of length $m + n$.

Moreover, we define the following.

- A *left normalized word* is the word e, $-$, $u \square e$, or $u \square -$, with u a left normalized word.
- A *right normalized word* is the word e, $-$, $e \square u$, or $- \square u$, with u a right normalized word.
- For a monoidal category C, each word w of length n determines a functor $w : \mathsf{C}^n \longrightarrow \mathsf{C}$ by interpreting
  - the length 0 word e as the constant functor at $\mathbb{1}$;
  - the length 1 word $-$ as the identity functor 1_{C}; and
  - $\square$ as the monoidal product in C.

  We also call this functor a *word*. ◇

Definition 1.1.30. For a monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$, *canonical maps* are natural isomorphisms between words of the same length, defined inductively by the following four conditions.

- The identity morphism of $\mathbb{1}$ is a canonical map.
- The identity natural transformation of 1_{C} is a canonical map.
- α, λ, ρ, and their inverses are canonical maps.
- Canonical maps are closed under $\otimes$ and vertical composites. $\diamond$

Theorem 1.1.31 (Mac Lane's Coherence)**.** *Suppose u and v are words $\mathsf{C}^n \longrightarrow \mathsf{C}$ of the same length in a monoidal category C. Then there exists a unique canonical map $u \longrightarrow v$.*

Theorem 1.1.32 (Mac Lane's Strictification)**.** *For each monoidal category C, there exist a strict monoidal category C_{st} and an adjoint equivalence*

$$\begin{tikzcd} \mathsf{C}_{\mathsf{st}} \arrow[r, shift left, "L"] & \mathsf{C} \arrow[l, shift left, "R"] \end{tikzcd}$$

with (i) both L and R strong monoidal functors, and (ii) $LR = 1_{\mathsf{C}}$.

Explanation 1.1.33. We note that the equality $LR = 1_{\mathsf{C}}$ in Theorem 1.1.32 is an equality of monoidal functors, and hence entails that the monoidal and unit constraints $(LR)^2$ and $(LR)^0$, respectively, are identities. The same holds for the Braided and Symmetric Strictification Theorems 1.1.39 and 1.1.42 below. $\diamond$

The following lemma shows that the unit of the adjunction in Theorem 1.1.32 is a monoidal natural transformation. The same conclusion holds for the Braided and Symmetric Strictification Theorems 1.1.39 and 1.1.42 below.

Lemma 1.1.34. *Suppose given an adjoint equivalence of monoidal categories*

$$\begin{tikzcd} \mathsf{D} \arrow[r, shift left, "L"] & \mathsf{C} \arrow[l, shift left, "R"] \end{tikzcd}$$

with (i) both L and R strong monoidal functors, and (ii) $LR = 1_{\mathsf{C}}$. Then the unit

$$\eta : 1_{\mathsf{D}} \longrightarrow RL$$

is a monoidal natural transformation.

Proof. We must show that the following diagrams commute for each $X, Y \in \mathsf{D}$. In the diagram below we let $\varepsilon = 1_{1_{\mathsf{C}}}$ denote the counit of the adjunction.

$$\begin{tikzcd} X \otimes Y \arrow[r, "\eta_X \otimes \eta_Y"] \arrow[d, "1_{X \otimes Y}"'] & RLX \otimes RLY \arrow[d, "(RL)^2"] \\ X \otimes Y \arrow[r, "\eta_{X \otimes Y}"] & RL(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1}^{\mathsf{D}} \arrow[r, "1"] \arrow[d, equal] & \mathbb{1}^{\mathsf{D}} \arrow[d, "\eta_{\mathbb{1}^{\mathsf{D}}}"] \\ \mathbb{1}^{\mathsf{D}} \arrow[r, "(RL)^0"] & RL\mathbb{1}^{\mathsf{D}} \end{tikzcd} \tag{1.1.35}$$

Since (L, R) is an equivalence, it suffices to see that the diagrams commute after applying L. Applying L to the left hand diagram of (1.1.35) yields the outer diagram below, where we recall the monoidal constraint $(RL)^2$ is given by the composite of $R(L^2)$ and R^2 (Definition 1.1.11).

$$\begin{tikzcd}
L(X \otimes Y) \arrow[rrrr, "L(\eta_X \otimes \eta_Y)"] \arrow[dddd, "L1_{X\otimes Y}"'] & & & & L(RLX \otimes RLY) \arrow[dd, "L(R^2)"] \\
& LX \otimes LY \arrow[ul, "L^2"] \arrow[rr, "L\eta_X \otimes L\eta_Y"] \arrow[dr, "1"'] \arrow[dddl, "L^2"'] & & LRLX \otimes LRLY \arrow[ur, "L^2"'] \arrow[dl, "\varepsilon_{LX} \otimes \varepsilon_{LY}"'] \arrow[dr, "(LR)^2"] \arrow[d, phantom, "\star"] & \\
& & LX \otimes LY \arrow[d, "L^2"] & & LR(LX \otimes LY) \arrow[ll, "\varepsilon_{LX\otimes LY}"] \arrow[dd, "LR(L^2)"] \\
& & L(X \otimes Y) & & \\
L(X \otimes Y) \arrow[urr, "1"] \arrow[rrrr, "L(\eta_{X\otimes Y})"] & & & & LRL(X \otimes Y) \arrow[ull, "\varepsilon_{L(X\otimes Y)}"']
\end{tikzcd}$$

Each of the quadrilaterals in the diagram above commutes by naturality of either L^2 or ε. The triangle ☆ commutes because $LR = 1_{\mathsf{C}}$ and in particular the monoidal constraint $(LR)^2$ is the identity. The remaining two middle triangles commute by the triangle identities for the unit and counit of the adjunction. The outer triangle at left commutes by functoriality of L and the triangle at right commutes by definition of $(LR)^2$.

For the right hand diagram of (1.1.35), a similar analysis shows that the diagram commutes after applying L. This analysis uses the triangle diagrams for the adjunction and the condition $(LR)^0 = 1$ implied by $LR = 1_{\mathsf{C}}$. □

The braided coherence and strictification theorems involve the concepts of braided canonical map and underlying braid from Definitions II.1.6.1 and II.1.6.2. In the latter, B_n denotes the nth braid group (Definition II.1.1.1). See Section II.1.1 for further explanations of braids.

Definition 1.1.36. In a braided monoidal category (C, ξ), a *braided canonical map* is a natural isomorphism between permuted words of the same length that has the same definition as a canonical map by also allowing the braiding ξ and its inverse. For a symmetric monoidal category, a braided canonical map is also called a *permuted canonical map*. ◇

Definition 1.1.37. In a braided monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi)$, each braided canonical map ϕ between permuted words of the same length n has an *underlying braid* $\mathsf{br}(\phi) \in B_n$ defined as follows.

- $\mathsf{br}(1_{\mathbb{1}}) = \mathrm{id}_0 \in B_0$.
- The identity natural transformation of 1_{C} has underlying braid $\mathrm{id}_1 \in B_1$.
- The vertical composite $\phi'\phi$ of two braided canonical maps has underlying braid the product $\mathsf{br}(\phi')\mathsf{br}(\phi)$.
- For braided canonical maps ϕ_1 and ϕ_2, the underlying braid of $\phi_1 \otimes \phi_2$ is

$$\mathsf{br}(\phi_1 \otimes \phi_2) = \mathsf{br}(\phi_1) \oplus \mathsf{br}(\phi_2)$$

with $\oplus$ the sum braid (II.1.1.10).

- For permuted words u, v, and w, the associativity isomorphism
$$\alpha_{u,v,w} : (u \otimes v) \otimes w \longrightarrow u \otimes (v \otimes w)$$
has underlying braid
$$\mathsf{br}(\alpha_{u,v,w}) = \mathrm{id} \in B_{|u|+|v|+|w|}.$$
- The unit isomorphisms
$$\lambda : \mathbb{1} \otimes u \longrightarrow u \quad \text{and} \quad \rho : u \otimes \mathbb{1} \longrightarrow u$$
have underlying braids
$$\mathsf{br}(\lambda_u) = \mathsf{br}(\rho_u) = \mathrm{id} \in B_{|u|}.$$
- The braiding
$$\xi_{u,v} : u \otimes v \longrightarrow v \otimes u$$
has underlying braid
$$\mathsf{br}(\xi_{u,v}) = b^{\oplus}_{|u|,|v|} \in B_{|u|+|v|}$$
with the right-hand side the elementary block braid (II.1.2.4).

This finishes the definition of the underlying braid. ◇

Theorem 1.1.38 (Braided Coherence)**.** *In a braided monoidal category* C*, two braided canonical maps with the same (co)domain are equal if their underlying braids are equal.*

Theorem 1.1.39 (Braided Strictification)**.** *For each braided monoidal category* C*, there exist a braided strict monoidal category* C_{st} *and an adjoint equivalence*
$$\mathsf{C}_{\mathsf{st}} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{C}$$
with (i) both L *and* R *strong braided monoidal functors and (ii)* $LR = 1_{\mathsf{C}}$*.*

The symmetric coherence and strictification theorems involve the concepts of permuted word and permuted canonical map from Definition I.1.3.6.

Definition 1.1.40. The symmetric group on n letters is denoted by Σ_n. Suppose C is a monoidal category.

- For a word w of length n and a permutation $\sigma \in \Sigma_n$, the *permuted word* $w\sigma : \mathsf{C}^n \longrightarrow \mathsf{C}$ is the composite functor $w \circ \sigma$, where $\sigma : \mathsf{C}^n \longrightarrow \mathsf{C}^n$ is given by
$$\sigma(x_1, \ldots, x_n) = \big(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}\big)$$
with the x_i's all objects, or all morphisms, in C.
- For a symmetric monoidal category (C, ξ), a *permuted canonical map* is a natural isomorphism between permuted words of the same length, defined as in Definition 1.1.30 by also allowing the symmetry isomorphism ξ. ◇

Theorem 1.1.41 (Symmetric Coherence)**.** *Suppose* $u\sigma$ *and* $v\tau$ *are two permuted words of the same length in a symmetric monoidal category* C*. Then there exists a unique permuted canonical map* $u\sigma \longrightarrow v\tau$*.*

Theorem 1.1.42 (Symmetric Strictification)**.** *For each symmetric monoidal category* C*, there exist a permutative category* C_{st} *and an adjoint equivalence*
$$\mathsf{C}_{\mathsf{st}} \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{C}$$

with (i) both L and R strong symmetric monoidal functors, and (ii) $LR = 1_{\mathsf{C}}$.

We also recall Epstein's Coherence Theorem I.1.3.12 for monoidal and symmetric monoidal functors. This involves the concept of F-iterate from Definition I.1.3.11.

Definition 1.1.43. Suppose $(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$ is a (symmetric) monoidal functor between (symmetric) monoidal categories.

(1) The set of *F-iterates* is the set of functors $\mathsf{C}^n \longrightarrow \mathsf{D}$ for $n \geq 1$ defined inductively by the following two conditions.
- $Fw : \mathsf{C}^n \longrightarrow \mathsf{D}$ is an F-iterate for each (permuted) word $w : \mathsf{C}^n \longrightarrow \mathsf{C}$ of length n.
- If $G : \mathsf{C}^m \longrightarrow \mathsf{D}$ and $H : \mathsf{C}^n \longrightarrow \mathsf{D}$ are F-iterates, then so is the composite

$$\mathsf{C}^{m+n} = \mathsf{C}^m \times \mathsf{C}^n \xrightarrow{G \times H} \mathsf{D} \times \mathsf{D} \xrightarrow{\otimes} \mathsf{D}.$$

(2) The set of *F-coherent maps* is the set of natural transformations between F-iterates defined inductively as follows.
- Suppose θ is a (permuted) canonical map in C that does not involve $\mathbb{1}^{\mathsf{C}}$, λ^{C}, or ρ^{C}. Then $1_F * \theta$ is an F-coherent map.
- The identity natural transformation, α^{D}, their inverses, and ξ^{D} in the symmetric case, applied to F-iterates are F-coherent maps.
- F^2 is an F-coherent map.
- F-coherent maps are closed under vertical composites and $\otimes^{\mathsf{D}}$. $\diamond$

Theorem 1.1.44 (Epstein's Coherence)**.** *Suppose* $F : \mathsf{C} \longrightarrow \mathsf{D}$ *is a (symmetric) monoidal functor between (symmetric) monoidal categories, and* $G, H : \mathsf{C}^n \longrightarrow \mathsf{D}$ *are F-iterates. Then there exists at most one F-coherent map from G to H.*

1.2. Enriched Categories, Functors, and Natural Transformations

In this section we recall some basic definitions regarding categories, functors, and natural transformations that are enriched over a monoidal category $\mathsf{V} = (\mathsf{V}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$. The material in this section does not assume that V has a braiding until the discussion of opposites in Definition 1.2.16 and following.

Definition 1.2.1. Suppose V is a monoidal category. A V-*category* C, also called a *category enriched in* V, consists of:
- a class $\mathsf{Ob}(\mathsf{C})$ of objects in C;
- for each pair of objects X, Y in C, an object $\mathsf{C}(X, Y)$ in V, called the *hom object* with domain X and codomain Y;
- for each triple of objects X, Y, Z in C, a morphism

$$\mathsf{C}(Y, Z) \otimes \mathsf{C}(X, Y) \xrightarrow{m_{X,Y,Z}} \mathsf{C}(X, Z)$$

  in V, called the *composition*; and
- for each object X in C, a morphism

$$\mathbb{1} \xrightarrow{i_X} \mathsf{C}(X, X)$$

  in V, called the *identity* of X.

These data are required to make the *associativity diagram*

$$\begin{tikzcd}
\big(\mathsf{C}(Y,Z)\otimes\mathsf{C}(X,Y)\big)\otimes\mathsf{C}(W,X) \arrow[r, "\alpha"] \arrow[dd, "m\otimes 1"'] & \mathsf{C}(Y,Z)\otimes\big(\mathsf{C}(X,Y)\otimes\mathsf{C}(W,X)\big) \arrow[d, "1\otimes m"] \\
 & \mathsf{C}(Y,Z)\otimes\mathsf{C}(W,Y) \arrow[d, "m"] \\
\mathsf{C}(X,Z)\otimes\mathsf{C}(W,X) \arrow[r, "m"] & \mathsf{C}(W,Z)
\end{tikzcd} \tag{1.2.2}$$

and the *unity diagram*

$$\begin{tikzcd}
\mathbb{1}\otimes\mathsf{C}(X,Y) \arrow[r, "\lambda"] \arrow[d, "i_Y\otimes 1"'] & \mathsf{C}(X,Y) \arrow[d, equal] & \mathsf{C}(X,Y)\otimes\mathbb{1} \arrow[l, "\rho"'] \arrow[d, "1\otimes i_X"] \\
\mathsf{C}(Y,Y)\otimes\mathsf{C}(X,Y) \arrow[r, "m"] & \mathsf{C}(X,Y) & \mathsf{C}(X,Y)\otimes\mathsf{C}(X,X) \arrow[l, "m"']
\end{tikzcd} \tag{1.2.3}$$

commute for objects W, X, Y, Z in C. This finishes the definition of a V-category. A V-category C is *small* if $\mathsf{Ob}(\mathsf{C})$ is a set. $\diamond$

Next we recall functors, natural transformations, and adjunctions in the enriched setting. In the next few definitions, the reader will notice that we recover the plain categorical notions from Section I.1.1 when $\mathsf{V} = \mathsf{Set}$.

Definition 1.2.4. Suppose C and D are V-categories. A V-*functor* $F : \mathsf{C} \longrightarrow \mathsf{D}$ consists of:

- an assignment on objects
$$\mathsf{Ob}(\mathsf{C}) \longrightarrow \mathsf{Ob}(\mathsf{D}), \qquad X \longmapsto FX;$$
and
- for each pair of objects X, Y in C, a morphism
$$\mathsf{C}(X,Y) \xrightarrow{F_{X,Y}} \mathsf{D}\big(FX, FY\big)$$
in V.

These data are required to satisfy the following two axioms.

Composition: For each triple of objects X, Y, Z in C, the diagram

$$\begin{tikzcd}
\mathsf{C}(Y,Z)\otimes\mathsf{C}(X,Y) \arrow[r, "m"] \arrow[d, "F\otimes F"'] & \mathsf{C}(X,Z) \arrow[d, "F"] \\
\mathsf{D}(FY,FZ)\otimes\mathsf{D}(FX,FY) \arrow[r, "m"] & \mathsf{D}(FX,FZ)
\end{tikzcd} \tag{1.2.5}$$

in V is commutative.

Identities: For each object $X \in \mathsf{C}$, the diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "i_X"] \arrow[d, equal] & \mathsf{C}(X,X) \arrow[d, "F"] \\
\mathbb{1} \arrow[r, "i_{FX}"] & \mathsf{D}(FX,FX)
\end{tikzcd} \tag{1.2.6}$$

in V is commutative.

Moreover:

- For V-functors $F : \mathsf{C} \longrightarrow \mathsf{D}$ and $G : \mathsf{D} \longrightarrow \mathsf{E}$, their composition
$$GF : \mathsf{C} \longrightarrow \mathsf{E}$$
is the V-functor defined by composing the assignments on objects and forming the composite
$$(GF)_{X,Y} = G_{FX,FY}F_{X,Y} : \mathsf{C}(X,Y) \longrightarrow \mathsf{E}(GFX, GFY)$$
in V on hom objects.
- The *identity* V*-functor* of C, denoted $1_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{C}$, is given by the identity morphism on $\mathsf{Ob}(\mathsf{C})$ and the identity morphism $1_{\mathsf{C}(X,Y)}$ for objects X, Y in C. ⋄

Definition 1.2.7. Suppose $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ are V-functors between V-categories C and D.

(1) A V*-natural transformation* $\theta : F \longrightarrow G$ consists of a morphism
$$\theta_X : \mathbb{1} \longrightarrow \mathsf{D}(FX, GX)$$
in V, called a *component* of θ, for each object X in C, such that the following diagram commutes for objects X, Y in C.

(1.2.8)
$$\begin{tikzcd}
& \mathbb{1} \otimes \mathsf{C}(X,Y) \arrow[r, "\theta_Y \otimes F"] & \mathsf{D}(FY,GY) \otimes \mathsf{D}(FX,FY) \arrow[dr, "m"] & \\
\mathsf{C}(X,Y) \arrow[ur, "\lambda^{-1}", "\cong"'] \arrow[dr, "\cong", "\rho^{-1}"'] & & & \mathsf{D}(FX,GY) \\
& \mathsf{C}(X,Y) \otimes \mathbb{1} \arrow[r, "G \otimes \theta_X"] & \mathsf{D}(GX,GY) \otimes \mathsf{D}(FX,GX) \arrow[ur, "m"'] &
\end{tikzcd}$$

(2) The *identity* V*-natural transformation of* F, denoted by $1_F : F \longrightarrow F$, is defined by the component
$$(1_F)_X = i_{FX} : \mathbb{1} \longrightarrow \mathsf{D}(FX, FX)$$
for each object X in C. ⋄

We give an alternative characterization of V-naturality in Lemma 2.1.11 below. Just as for ordinary natural transformations, there are two types of compositions for V-natural transformations.

Definition 1.2.9. Suppose $\theta : F \longrightarrow G$ is a V-natural transformation for V-functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$.

(1) Suppose $\phi : G \longrightarrow H$ is another V-natural transformation for a V-functor $H : \mathsf{C} \longrightarrow \mathsf{D}$. The *vertical composition*
$$\phi\theta : F \longrightarrow H$$
is the V-natural transformation whose component $(\phi\theta)_X$ is the composite
$$\begin{tikzcd}
\mathbb{1} \arrow[r, "(\phi\theta)_X"] \arrow[d, "\lambda^{-1}"', "\cong"] & \mathsf{D}(FX,HX) \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\phi_X \otimes \theta_X"] & \mathsf{D}(GX,HX) \otimes \mathsf{D}(FX,GX) \arrow[u, "m"']
\end{tikzcd}$$

in V for each object X in C.

(2) Suppose $\theta' : F' \longrightarrow G'$ is a V-natural transformation for V-functors $F', G' : \mathsf{D} \longrightarrow \mathsf{E}$ with E a V-category. The *horizontal composition*

$$\theta' * \theta : F'F \longrightarrow G'G$$

is the V-natural transformation whose component $(\theta' * \theta)_X$, for an object X in C, is defined as the composite

(1.2.10)
$$\begin{tikzcd}
\mathbb{1} \arrow[r, "(\theta' * \theta)_X"] \arrow[dd, "\lambda^{-1}"', "\cong"] & \mathsf{E}(F'FX, G'GX) \\
& \mathsf{E}(F'GX, G'GX) \otimes \mathsf{E}(F'FX, F'GX) \arrow[u, "m"'] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\theta'_{GX} \otimes \theta_X"] & \mathsf{E}(F'GX, G'GX) \otimes \mathsf{D}(FX, GX) \arrow[u, "1 \otimes F'"']
\end{tikzcd}$$

in V. ◇

Definition 1.2.11. A V-natural transformation $\theta : F \longrightarrow G$ is called a V-*natural isomorphism* if there exists a V-natural transformation $\theta^{-1} : G \longrightarrow F$ such that the equalities

$$\theta^{-1}\theta = 1_F \quad \text{and} \quad \theta\theta^{-1} = 1_G$$

hold. ◇

The following special cases of horizontal composition will be useful in several computations below.

Lemma 1.2.12 (Whiskering). *Suppose given* V*-categories* B, C, D, *and* E; V*-functors* F, G, H, *and* K; *and a* V*-natural transformation* θ *as in the following diagram:*

$$\begin{tikzcd}
\mathsf{B} \arrow[r, "H"] & \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathsf{D} \arrow[r, "K"] & \mathsf{E}
\end{tikzcd}$$

Then for each $Y \in \mathsf{B}$ *and* $X \in \mathsf{C}$ *we have*

$$(\theta * 1_H)_Y = \theta_{HY} \quad \textit{and} \quad (1_K * \theta)_X = (K_{FX,GX}) \circ \theta_X.$$

Proof. For the whiskering of θ with H, consider the following diagram in V, where the top arrow is θ_{HY} and the composite along the remaining boundary is the definition (1.2.10) for $(\theta * 1_H)_Y$.

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "\theta_{HY}"] \arrow[dd, "\lambda^{-1}"'] & & \mathsf{D}(FHY, GHY) \\
& \mathsf{D}(FHY, GHY) \otimes \mathbb{1} \arrow[ur, "\rho"] \arrow[r, "1 \otimes i_{FHY}"] & \mathsf{D}(FHY, GHY) \otimes \mathsf{D}(FHY, FHY) \arrow[u, "m"'] \\
\mathbb{1} \otimes \mathbb{1} \arrow[ur, "\theta_{HY} \otimes 1"] \arrow[urr, "\theta_{HY} \otimes i_{FHY}"'] \arrow[rr, "\theta_{HY} \otimes i_{HY}"] & & \mathsf{D}(FHY, GHY) \otimes \mathsf{C}(HY, HY) \arrow[u, "1 \otimes F"']
\end{tikzcd}$$

The upper left region commutes by naturality of ρ and the equality $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ (1.1.4). The upper right triangle commutes by the right unity diagram (1.2.3). The lower right triangle commutes by functoriality of $\otimes$ and the identity axiom $F \circ i_{HY} = i_{FHY}$ (1.2.6). The remaining triangle commutes by functoriality of $\otimes$.

For the whiskering of θ with K, consider the following diagram in V, where the top arrow is the composite $(K_{FX,GX}) \circ \theta_X$ and the composite around the remaining boundary is (1.2.10) for $(1_K * \theta)_X$.

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "\theta_X"] \arrow[dd, "\lambda^{-1}"'] & & \mathsf{D}(FX,GX) \arrow[r, "K_{FX,GX}"] & \mathsf{E}(KFX,KGX) \\
& \mathbb{1} \otimes \mathsf{E}(KFX,KGX) \arrow[urr, "\lambda"] \arrow[rr, "i_{KGX} \otimes 1"] & & \mathsf{E}(KGX,KGX) \otimes \mathsf{E}(KFX,KGX) \arrow[u, "m"'] \\
\mathbb{1} \otimes \mathbb{1} \arrow[ur, "1 \otimes ((K_{FX,GX}) \circ \theta_X)"'] \arrow[rrr, "i_{KGX} \otimes \theta_X"] & & & \mathsf{E}(KGX,KGX) \otimes \mathsf{D}(FX,GX) \arrow[u, "1 \otimes K_{FX,GX}"']
\end{tikzcd}$$

In the above diagram, the upper left region commutes by naturality of λ, the lower region commutes by functoriality of $\otimes$, and the triangular region commutes by the left unity axiom (1.2.3) for m. □

Definition 1.2.13. Suppose V is a monoidal category. The 2-category V-Cat is defined by the following data.

- The objects are small V-categories, 1-cells are V-functors, and 2-cells are V-natural transformations.
- Horizontal composition of 1-cells is given by composition of V-functors.
- Vertical and horizontal composition of 2-cells are given by those of V-natural transformations.
- Identity V-functors are identity 1-cells.
- Identity V-natural transformations are identity 2-cells.

The axioms of a 2-category follow from the defining axioms above. For example, strict associativity of the vertical composition of 2-cells follows from instances of the enriched composition axiom (1.2.5) and the associativity of composition axiom (1.2.2). ⋄

Recall from Definition I.6.3.9 the notions of internal adjunction and equivalence in a bicategory. The corresponding notions for V-Cat determine the definitions of V-adjunction and V-equivalence. We give detailed definitions for reference.

Definition 1.2.14. Suppose C and D are V-categories, and $L : \mathsf{C} \longrightarrow \mathsf{D}$ and $R : \mathsf{D} \longrightarrow \mathsf{C}$ are V-functors. A V-*adjunction* $L \dashv R$ consists of

- a V-natural transformation $\eta : 1_{\mathsf{C}} \longrightarrow RL$ called the *unit*, and
- a V-natural transformation $\varepsilon : LR \longrightarrow 1_{\mathsf{D}}$ called the *counit*,

such that the following diagrams commute.

$$\begin{tikzcd}
& LRL \arrow[dr, "\varepsilon * 1_L"] & \\
L \arrow[ur, "1_L * \eta"] \arrow[rr, "1_L"] & & L
\end{tikzcd}
\qquad
\begin{tikzcd}
& RLR \arrow[dr, "1_R * \varepsilon"] & \\
R \arrow[ur, "\eta * 1_R"] \arrow[rr, "1_R"] & & R
\end{tikzcd}$$

In this case, L is called the *left adjoint*, and R is called the *right adjoint*. ⋄

Definition 1.2.15. Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a V-functor. We call F a V-*equivalence* if there exist

- a V-functor $F' : \mathsf{D} \longrightarrow \mathsf{C}$ and
- V-natural isomorphisms $\eta : 1_{\mathsf{C}} \xrightarrow{\cong} F'F$ and $\varepsilon : FF' \xrightarrow{\cong} 1_{\mathsf{D}}$.

An *adjoint* V-*equivalence* consists of a V-adjunction $L \dashv R$ such that the unit and counit are V-natural isomorphisms. ◇

Next we define the opposite of a V-category. Unlike the other content in this section, this definition requires that V be braided monoidal

Definition 1.2.16. Suppose $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is a braided monoidal category and suppose that C is a V-category. The *opposite* V-*category*, C^{op}, is defined to have the same objects as C and hom objects

$$\mathsf{C}^{\mathsf{op}}(X, Y) = \mathsf{C}(Y, X).$$

The composition in C^{op} is defined for each triple of objects X, Y, Z in C^{op} as the following composite using the braiding ξ of V and the composition m of C:

$$\begin{tikzcd}
\mathsf{C}^{\mathsf{op}}(Y,Z) \otimes \mathsf{C}^{\mathsf{op}}(X,Y) \arrow[d, equal] & & \mathsf{C}^{\mathsf{op}}(X,Z) \arrow[d, equal] \\
\mathsf{C}(Z,Y) \otimes \mathsf{C}(Y,X) \arrow[r, "\xi"] & \mathsf{C}(Y,X) \otimes \mathsf{C}(Z,Y) \arrow[r, "m"] & \mathsf{C}(Z,X)
\end{tikzcd}$$

The unit of C^{op} is the same as that of C:

$$\mathbb{1} \xrightarrow{i_X} \mathsf{C}(X, X) = \mathsf{C}^{\mathsf{op}}(X, X).$$

We show that these data satisfy the axioms of a V-category in Lemma 1.2.17 below. ◇

Lemma 1.2.17. *In the context of Definition 1.2.16, C^{op} is a V-category.*

Proof. We show that the axioms for C^{op} follow from the corresponding axioms for C together with braiding properties of ξ. First, the associativity diagram (1.2.2) for C^{op} is the outer diagram below for $W, X, Y, Z \in \mathsf{C}$. In this diagram we use the notation

$$\mathsf{C}^{\mathsf{op}}(W, X) = \mathsf{C}(X, W) = \mathsf{C}_{X,W}$$

and likewise for the other hom objects. We also omit tensor symbols to save space.

$$\begin{tikzcd}
& (\mathsf{C}_{Z,Y}\mathsf{C}_{Y,X})\mathsf{C}_{X,W} \arrow[r, "\alpha"] \arrow[dl, "\xi 1"'] & \mathsf{C}_{Z,Y}(\mathsf{C}_{Y,X}\mathsf{C}_{X,W}) \arrow[dr, "1\xi"] & \\
(\mathsf{C}_{Y,X}\mathsf{C}_{Z,Y})\mathsf{C}_{X,W} \arrow[dr, "\xi"] \arrow[dd, "m1"'] & & & \mathsf{C}_{Z,Y}(\mathsf{C}_{X,W}\mathsf{C}_{Y,X}) \arrow[dl, "\xi"'] \arrow[d, "1m"] \\
& \mathsf{C}_{X,W}(\mathsf{C}_{Y,X}\mathsf{C}_{Z,Y}) \arrow[ddl, "1m"'] & (\mathsf{C}_{X,W}\mathsf{C}_{Y,X})\mathsf{C}_{Z,Y} \arrow[l, "\alpha"'] \arrow[dr, "m1"] & \mathsf{C}_{Z,Y}\mathsf{C}_{Y,W} \arrow[d, "\xi"] \\
\mathsf{C}_{Z,X}\mathsf{C}_{X,W} \arrow[d, "\xi"'] & & & \mathsf{C}_{Y,W}\mathsf{C}_{Z,Y} \arrow[d, "m"] \\
\mathsf{C}_{X,W}\mathsf{C}_{Z,X} \arrow[rrr, "m"] & & & \mathsf{C}_{Z,W}
\end{tikzcd}$$

In the above diagram, the upper hexagon commutes by compatibility of ξ with α (1.1.22), the lower pentagon commutes by the associativity diagram (1.2.2) for C, and the other two regions commute by naturality of ξ.

Second, using the same notation, the left and right unity diagrams (1.2.3) for C^{op} are the outer diagrams below.

$$\begin{tikzcd}
\mathbb{1}\mathsf{C}_{Y,X} \arrow[rr, "\lambda"] \arrow[dr, "\xi"'] \arrow[dd, "i1"'] & & \mathsf{C}_{Y,X} \arrow[ddd, equal] & & \mathsf{C}_{Y,X}\mathbb{1} \arrow[ll, "\rho"'] \arrow[dl, "\xi"] \arrow[dd, "i1"] \\
& \mathsf{C}_{Y,X}\mathbb{1} \arrow[ur, "\rho"'] \arrow[dd, "1i"] & & \mathbb{1}\mathsf{C}_{Y,X} \arrow[ul, "\lambda"] \arrow[dd, "1i"'] & \\
\mathsf{C}_{Y,Y}\mathsf{C}_{Y,X} \arrow[dr, "\xi"'] & & & & \mathsf{C}_{Y,X}\mathsf{C}_{X,X} \arrow[dl, "\xi"] \\
& \mathsf{C}_{Y,X}\mathsf{C}_{Y,Y} \arrow[r, "m"] & \mathsf{C}_{Y,X} & \mathsf{C}_{X,X}\mathsf{C}_{Y,X} \arrow[l, "m"'] &
\end{tikzcd}$$

In the above diagram, the top two triangles commute by compatibility of ξ with λ and ρ (1.1.20), the middle two trapezoids commute by the unity diagrams (1.2.3) for C, and the remaining two parallelograms commute by naturality of ξ. □

We will define $(-)^{\mathsf{op}}$ on V-functors and V-natural transformations in Proposition 1.2.19 below. The construction is covariant with respect to horizontal composition, but contravariant with respect to vertical composition of V-natural transformations. We will use the following notation.

Definition 1.2.18. Let $\mathsf{V}\text{-}\mathsf{Cat}^{\mathsf{co}}$ denote the 2-category with the same cells as V-Cat, but the direction of 2-cells reversed. So the category of 1- and 2-cells from C to D is given by the opposite category:

$$\mathsf{V}\text{-}\mathsf{Cat}^{\mathsf{co}}(\mathsf{C},\mathsf{D}) = \mathsf{V}\text{-}\mathsf{Cat}(\mathsf{C},\mathsf{D})^{\mathsf{op}}.$$

Identities and horizontal composition in $\mathsf{V}\text{-}\mathsf{Cat}^{\mathsf{co}}$ are given by the opposite functors of those in V-Cat. ⋄

Proposition 1.2.19. *Suppose* $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ *is a braided monoidal category. Taking opposite* V*-categories defines a 2-functor*

$$(-)^{\mathsf{op}} : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}^{\mathsf{co}}.$$

Proof. Continuing from Lemma 1.2.17, we need to define $(-)^{\mathsf{op}}$ for V-functors and V-natural transformations. Given a V-functor F, we define F^{op} to have the same assignment on objects as F and define $F^{\mathsf{op}}_{X,Y} = F_{Y,X}$ as below.

$$\begin{tikzcd}
\mathsf{C}^{\mathsf{op}}(X,Y) \arrow[r, "F^{\mathsf{op}}_{X,Y}"] \arrow[d, equal] & \mathsf{D}^{\mathsf{op}}(FX,FY) \arrow[d, equal] \\
\mathsf{C}(Y,X) \arrow[r, "F_{Y,X}"'] & \mathsf{D}(FY,FX)
\end{tikzcd}$$

The composition and identity axioms for F^{op} follow from those for F together with naturality of ξ. For example, the diagram below is the composition axiom for F^{op} at $X, Y, Z \in \mathsf{C}$, with tensor symbols omitted for space.

$$\begin{tikzcd}
\mathsf{C}^{\mathsf{op}}(Y,Z)\mathsf{C}^{\mathsf{op}}(X,Y) \arrow[r, "F^{\mathsf{op}}F^{\mathsf{op}}"] \arrow[d, equal] & \mathsf{D}^{\mathsf{op}}(FY,FZ)\mathsf{D}^{\mathsf{op}}(FX,FY) \arrow[d, equal] \\
\mathsf{C}(Z,Y)\mathsf{C}(Y,X) \arrow[r, "FF"'] \arrow[d, "\xi"'] & \mathsf{D}(FZ,FY)\mathsf{D}(FY,FX) \arrow[d, "\xi"] \\
\mathsf{C}(Y,X)\mathsf{C}(Z,Y) \arrow[r, "FF"] \arrow[d, "m"'] & \mathsf{D}(FY,FX)\mathsf{D}(FZ,FY) \arrow[d, "m"] \\
\mathsf{C}(Z,X) \arrow[r, "F"] \arrow[d, equal] & \mathsf{D}(FZ,FX) \arrow[d, equal] \\
\mathsf{C}^{\mathsf{op}}(X,Z) \arrow[r, "F^{\mathsf{op}}"'] & \mathsf{D}^{\mathsf{op}}(FX,FZ)
\end{tikzcd}$$

Since composition of V-functors is given by composition of assignments on objects and composition in V on hom objects, we have $(GF)^{\mathsf{op}} = G^{\mathsf{op}}F^{\mathsf{op}}$ for V-functors

$$\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}.$$

Since the identity of C^{op} has the same components as that of C, we have $(1_{\mathsf{C}})^{\mathsf{op}} = 1_{(\mathsf{C}^{\mathsf{op}})}$.

For V-functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ and a V-natural transformation $\theta : F \longrightarrow G$, the component of $\theta^{\mathsf{op}} : G^{\mathsf{op}} \longrightarrow F^{\mathsf{op}}$ at $X \in \mathsf{C}$ is given by θ_X as below.

$$\begin{tikzcd}
 & \mathsf{D}^{\mathsf{op}}(GX,FX) \arrow[d, equal] \\
\mathbb{1} \arrow[ur, "\theta^{\mathsf{op}}_X"] \arrow[r, "\theta_X"'] & \mathsf{D}(FX,GX)
\end{tikzcd}$$

The V-naturality axiom for θ^{op} follows from that of θ together with naturality of ξ. Preservation of horizontal composition

$$(\theta' * \theta)^{\mathsf{op}} = (\theta')^{\mathsf{op}} * \theta^{\mathsf{op}},$$

for $F', G' : \mathsf{D} \longrightarrow \mathsf{E}$ and $\theta' : F' \longrightarrow G'$, follows from the commutative diagram below for each $X \in \mathsf{C}$. The composite along the right, when composed with λ^{-1}, is $(\theta' * \theta)^{\mathsf{op}}_X = (\theta' * \theta)_X$. The composite along the left, when composed with λ^{-1}, is $((\theta')^{\mathsf{op}} * \theta^{\mathsf{op}})_X$. We omit tensor signs to save space and explain commutativity below.

$$\begin{tikzcd}
\mathsf{E}(F'FX, G'FX)\mathsf{D}(FX, GX) \arrow[ddd, "1G'"'] \arrow[ddddrr, "\xi"] & & \mathbb{1}\mathbb{1} \arrow[ll, "\theta'_{FX}\theta_X"'] \arrow[rr, "\theta'_{GX}\theta_X"] \arrow[dr, "1\theta_X"] \arrow[dd, "\theta_X 1"'] & & \mathsf{E}(F'GX, G'GX)\mathsf{D}(FX, GX) \arrow[ddd, "1F'"] \\
& & & \mathbb{1}\mathsf{D}(FX, GX) \arrow[ur, "\theta'_{GX}1"'] \arrow[dl, "\xi"] & \\
& & \mathsf{D}(FX, GX)\mathbb{1} \arrow[dd, "1\theta'_{FX}"'] & & \\
\mathsf{E}(F'FX, G'FX)\mathsf{E}(G'FX, G'GX) \arrow[dd, "\xi"'] & & & & \mathsf{E}(F'GX, G'GX)\mathsf{E}(F'FX, F'GX) \arrow[dd, "m"] \\
& & \mathsf{D}(FX, GX)\mathsf{E}(F'FX, G'FX) \arrow[dll, "G'1"'] & & \\
\mathsf{E}(G'FX, G'GX)\mathsf{E}(F'FX, G'FX) \arrow[rrrr, "m"] & & & & \mathsf{E}(F'FX, G'GX)
\end{tikzcd}$$

In the above diagram, the top right triangle commutes by functoriality of $\otimes$. The remaining upper/left regions commute by functoriality of $\otimes$ and naturality of ξ, recalling that $\xi_{\mathbb{1},\mathbb{1}} = 1_{\mathbb{1}\mathbb{1}}$ (1.1.21). The lower right region commutes by V-naturality of θ' (1.2.8) at the objects FX and GX, together with the compatibility of ξ with λ and ρ (1.1.20).

Preservation of vertical composition (contravariantly)

$$(\phi\theta)^{\mathsf{op}}_X = (\theta^{\mathsf{op}}\phi^{\mathsf{op}})_X$$

follows simply from naturality of ξ and the definition

$$(\phi\theta)_X = m \circ (\phi_X \otimes \theta_X) \circ \lambda^{-1},$$

again recalling $\xi_{\mathbb{1},\mathbb{1}} = 1_{\mathbb{1}\mathbb{1}}$. □

1.3. The Tensor Product of Enriched Categories

Now we turn to the definition of tensor product for enriched categories. The content of this section, and indeed the rest of this chapter, requires a braiding on the enriching category V. Throughout this section we assume that $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is braided monoidal. We use the braiding to define a monoidal product $\otimes$ on V-Cat.

If, moreover, V is symmetric monoidal, then we can define a braiding for $\otimes$ and show that $(\mathsf{V}\text{-}\mathsf{Cat}, \otimes)$ is a symmetric monoidal category. The following table summarizes the relationships.

(1.3.1)

V	V-Cat
monoidal	exists
braided	monoidal
symmetric	symmetric

See Note 1.6.2 for an expansion of this table via the iterated monoidal structures of Chapter II.10.

It will be convenient for our discussion below to introduce the following notation.

Definition 1.3.2. Let ξ_{mid} denote the natural isomorphism whose component at a quadruple of objects X, Y, Z, and W in V is given by the following composite of ξ with associativity isomorphisms.

$$\begin{tikzcd}
(X \otimes Y) \otimes (Z \otimes W) \arrow[r, "\xi_{\mathrm{mid}}"] \arrow[d, "\alpha^{-1}_{X\otimes Y,Z,W}"'] & (X \otimes Z) \otimes (Y \otimes W) \\
((X \otimes Y) \otimes Z) \otimes W \arrow[d, "\alpha_{X,Y,Z} \otimes 1_W"'] & ((X \otimes Z) \otimes Y) \otimes W \arrow[u, "\alpha_{X\otimes Z,Y,W}"'] \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[r, "(1 \otimes \xi) \otimes 1"] & (X \otimes (Z \otimes Y)) \otimes W \arrow[u, "\alpha^{-1}_{X,Z,Y} \otimes 1_W"']
\end{tikzcd}$$

The isomorphisms ξ_{mid} are called *middle four interchange*. By the Braided Coherence Theorem 1.1.38, ξ_{mid} is the unique braided canonical map in V whose underlying braid is s_2 in B_4. $\diamond$

Definition 1.3.3. Suppose that C and D are V-categories with V braided monoidal. The *tensor product* $\mathsf{C} \otimes \mathsf{D}$ consists of the following.

- Objects are given by pairs: $\mathsf{Ob}(\mathsf{C} \otimes \mathsf{D}) = \mathsf{Ob}\,\mathsf{C} \times \mathsf{Ob}\,\mathsf{D}$. Objects are denoted $X \otimes Y$ for $X \in \mathsf{C}$ and $Y \in \mathsf{D}$.
- For objects $X \otimes Y$ and $X' \otimes Y'$, the hom object with domain $X \otimes Y$ and codomain $X' \otimes Y'$ is given by the monoidal product of hom objects
$$(\mathsf{C} \otimes \mathsf{D})(X \otimes Y, X' \otimes Y') = \mathsf{C}(X, X') \otimes \mathsf{D}(Y, Y').$$
- The composition for $X \otimes Y$, $X' \otimes Y'$, and $X'' \otimes Y''$ is given by the following composite in V, where ξ_{mid} is the middle four interchange of Definition 1.3.2:
$$\begin{tikzcd}
\big(\mathsf{C}(X',X'') \otimes \mathsf{D}(Y',Y'')\big) \otimes \big(\mathsf{C}(X,X') \otimes \mathsf{D}(Y,Y')\big) \arrow[dr, "\xi_{\mathrm{mid}}"'] & & \mathsf{C}(X,X'') \otimes \mathsf{D}(Y,Y'') \\
& \big(\mathsf{C}(X',X'') \otimes \mathsf{C}(X,X')\big) \otimes \big(\mathsf{D}(Y',Y'') \otimes \mathsf{D}(Y,Y')\big) \arrow[ur, "m \otimes m"'] &
\end{tikzcd}$$
- The identity of $X \otimes Y$ is given by the following composite
$$\mathbb{1} \xrightarrow{\lambda^{-1}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{i_X \otimes i_Y} \mathsf{C}(X,X) \otimes \mathsf{D}(Y,Y). \qquad \diamond$$

This defines the data of $\mathsf{C} \otimes \mathsf{D}$. The associativity and unity axioms (1.2.2) and (1.2.3), respectively, follow from those for C and D together with coherence for the braiding of V.

Lemma 1.3.4. *In the context of Definition 1.3.3, $\mathsf{C} \otimes \mathsf{D}$ is a V-category.*

Proof. To verify the associativity diagram (1.2.2), suppose given

$$X_0, X_1, X_2, X_3 \in \mathsf{C} \quad \text{and} \quad Y_0, Y_1, Y_2, Y_3 \in \mathsf{D}.$$

For $i, j \in \{0, 1, 2, 3\}$ we let

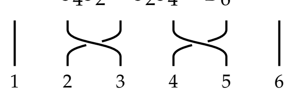
$$\mathsf{C}(X_i, X_j) = X_{ij} \quad \text{and} \quad \mathsf{D}(Y_i, Y_j) = Y_{ij}.$$

Then the associativity diagram for $\mathsf{C} \otimes D$ is the following, in which we omit tensor signs to save space. The arrows labeled s_4, s_2, and s_3 are the unique braided canonical morphisms with the given underlying braids.

$$\begin{tikzcd}
& \big((X_{23}Y_{23})(X_{12}Y_{12})\big)(X_{01}Y_{01}) \arrow[rr, "\alpha"] \arrow[dd, "\xi_{\mathrm{mid}}1"'] & & (X_{23}Y_{23})\big((X_{12}Y_{12})(X_{01}Y_{01})\big) \arrow[dd, "1\xi_{\mathrm{mid}}"] & \\
& & (1) & & \\
& \big((X_{23}X_{12})(Y_{23}Y_{12})\big)(X_{01}Y_{01}) \arrow[ddl, "(mm)1"'] \arrow[dr, "s_4"] & & (X_{23}Y_{23})\big((X_{12}X_{01})(Y_{12}Y_{01})\big) \arrow[dl, "s_2"'] \arrow[ddr, "1(mm)"] & \\
& (2) & \big((X_{23}X_{12})(Y_{23}X_{01})\big)(Y_{12}Y_{01}) \arrow[d, "s_3"] & (3) & \\
(X_{13}Y_{13})(X_{01}Y_{01}) \arrow[dd, "\xi_{\mathrm{mid}}"'] & & \big((X_{23}X_{12})X_{01}\big)\big((Y_{23}Y_{12})Y_{01}\big) \arrow[ddll, "(m1)(m1)"'] \arrow[dr, "\alpha\alpha"] & & (X_{23}Y_{23})(X_{02}Y_{02}) \arrow[dd, "\xi_{\mathrm{mid}}"] \\
& & \star & \big(X_{23}(X_{12}X_{01})\big)\big(Y_{23}(Y_{12}Y_{01})\big) \arrow[dr, "(1m)(1m)"'] & \\
(X_{13}X_{01})(Y_{13}Y_{01}) \arrow[rr, "mm"'] & & X_{03}Y_{03} & & (X_{23}X_{02})(Y_{23}Y_{02}) \arrow[ll, "mm"]
\end{tikzcd}$$

The three numbered regions commute by the Braided Coherence Theorem 1.1.38. The composites around region (1) have the following underlying braid.

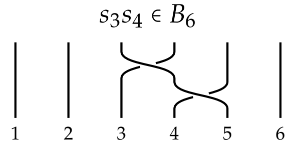
$$s_4s_2 = s_2s_4 \in B_6$$



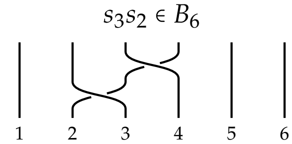
(1)

Composites around regions (2) and (3) have the following underlying braids and are equal by Theorem 1.1.38 together with naturality of the braided canonical maps.

$$s_3s_4 \in B_6 \qquad\qquad s_3s_2 \in B_6$$



(2)



(3)

The remaining region ☆ commutes by associativity in C and D.

For the left unity diagram (1.2.3) we have the following, using the notation as above.

$$\begin{tikzcd}
\mathbb{1}(X_{01}Y_{01}) \arrow[rrr, "\lambda"] \arrow[d, "\lambda^{-1}(11)"'] & & & X_{01}Y_{01} \arrow[dd, equal] \\
(\mathbb{1}\mathbb{1})(X_{01}Y_{01}) \arrow[r, "\xi_{\mathrm{mid}}"] \arrow[d, "(ii)(11)"'] & (\mathbb{1}X_{01})(\mathbb{1}Y_{01}) \arrow[urr, "\lambda\lambda"'] \arrow[d, "(i1)(i1)"'] & \star & \\
(X_{11}Y_{11})(X_{01}Y_{01}) \arrow[r, "\xi_{\mathrm{mid}}"] & (X_{11}X_{01})(Y_{11}Y_{01}) \arrow[rr, "mm"] & & X_{01}Y_{01}
\end{tikzcd}$$

In the above diagram, the two unlabeled regions commute by naturality and unity properties of the braiding ξ. The underlying braids of composites around their boundaries are identities. The remaining region labeled ☆ commutes by the left unity in C and D. The right unity diagram for $\mathsf{C} \otimes \mathsf{D}$ is similar. □

Definition 1.3.5. For a braided monoidal category V, the tensor product of V-categories extends to V-functors and V-natural transformations as follows. Suppose that $F, F' : \mathsf{C} \longrightarrow \mathsf{C}'$ and $G, G' : \mathsf{D} \longrightarrow \mathsf{D}'$ are V-functors. Suppose, moreover, that $\theta : F \longrightarrow F'$ and $\omega : G \longrightarrow G'$ are V-natural transformations.

(1) The V-functor $F \otimes G$ is defined on objects by the Cartesian product, so

$$(F \otimes G)(X \otimes Y) = FX \otimes GY.$$

It is defined on hom objects by the monoidal product in V, so for $X, X' \in \mathsf{C}$ and $Y, Y' \in \mathsf{D}$ we have

$$(F \otimes G)_{X \otimes Y, X' \otimes Y'} = (F_{X,X'}) \otimes (G_{Y,Y'}).$$

The composition and identity axioms of Definition 1.2.4 follow from those of F and G separately together with the naturality of the braiding ξ in V.

(2) The V-natural transformation $\theta \otimes \omega$ has components for each $X \otimes Y \in \mathsf{C} \otimes \mathsf{D}$ given by composing the inverse of a unit morphism with θ_X and ω_Y to form the following composite:

$$\mathbb{1} \xrightarrow{\lambda^{-1}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\theta_X \otimes \omega_Y} \mathsf{C}'(FX, F'X) \otimes \mathsf{D}'(GY, G'Y).$$

The enriched naturality axiom (1.2.8) for $\theta \otimes \omega$ follow from those of θ and ω separately, naturality of the brading ξ, and the unit axiom (1.1.25) for ξ with unit morphisms. ⋄

Proposition 1.3.6. *The tensor product is a 2-functor*

$$\otimes : \mathsf{V}\text{-}\mathsf{Cat} \times \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}.$$

Proof. We verify functoriality with respect to horizontal composition of V-natural transformations. Functoriality with respect to horizontal composition of 1-cells and vertical composition of 2-cells are similar. Preservation of unit 1- and 2-cells is verified directly from Definition 1.3.5.

Now suppose given V-functors and V-natural transformations as in the diagram below.

$$\begin{tikzcd} \mathsf{C} \otimes \mathsf{D} \arrow[r, bend left, "F \otimes G"] \arrow[r, bend right, "H \otimes K"'] \arrow[r, phantom, "\Downarrow \theta \otimes \omega"] & \mathsf{C}' \otimes \mathsf{D}' \arrow[r, bend left, "F' \otimes G'"] \arrow[r, bend right, "H' \otimes K'"'] \arrow[r, phantom, "\Downarrow \theta' \otimes \omega'"] & \mathsf{C}'' \otimes \mathsf{D}'' \end{tikzcd}$$

We will show that the components of $(\theta' \otimes \omega) * (\theta \otimes \omega)$ and $(\theta' * \theta) \otimes (\omega' * \omega)$ at an object $X \otimes Y \in \mathsf{C} \otimes \mathsf{D}$ are equal. To simplify the diagrams, we will use the following notation:

$$\begin{aligned} \mathsf{C}'(FX, HX) &= X_{00} & \mathsf{D}'(GY, KY) &= Y_{00} \\ \mathsf{C}''(F'FX, F'HX) &= X_{01} & \mathsf{D}''(G'GY, G'KY) &= Y_{01} \\ \mathsf{C}''(F'HX, H'HX) &= X_{12} & \mathsf{D}''(G'KY, K'KY) &= Y_{12} \\ \mathsf{C}''(F'FX, H'HX) &= X_{02} & \mathsf{D}''(G'GY, K'KY) &= Y_{02}. \end{aligned}$$

For example, in this notation the component of $\theta' * \theta$ at X is the composite

$$\mathbb{1} \xrightarrow{\lambda^{-1}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\theta'_{HX} \otimes \theta_X} X_{12} \otimes X_{00} \xrightarrow{1 \otimes F'} X_{12} \otimes X_{01} \xrightarrow{m} X_{02}.$$

In the diagram below, the composite along the left edge is $(\theta' \otimes \omega') * (\theta \otimes \omega)$ while the composite along the right edge is $(\theta' * \theta) \otimes (\omega' * \omega)$.

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\lambda^{-1}"] & \mathbb{1}^{\otimes 2} \arrow[r, "\lambda^{-1} \otimes \lambda^{-1}"] & (\mathbb{1}^{\otimes 2}) \otimes (\mathbb{1}^{\otimes 2}) \arrow[dl, "(\theta'_{HX} \otimes \omega'_{KY}) \otimes (\theta_X \otimes \omega_Y)"'] \arrow[dr, "(\theta'_{HX} \otimes \theta_X) \otimes (\omega'_{KY} \otimes \omega_Y)"] & & \\ & (X_{12} \otimes Y_{12}) \otimes (X_{00} \otimes Y_{00}) \arrow[rr, "\xi_{\mathrm{mid}}"] \arrow[d, "(1 \otimes 1) \otimes (F' \otimes G')"'] & & (X_{12} \otimes X_{00}) \otimes (Y_{12} \otimes Y_{00}) \arrow[d, "(1 \otimes F') \otimes (1 \otimes G')"] & \\ & (X_{12} \otimes Y_{12}) \otimes (X_{01} \otimes Y_{01}) \arrow[rr, "\xi_{\mathrm{mid}}"] & & (X_{12} \otimes X_{01}) \otimes (Y_{12} \otimes Y_{01}) \arrow[r, "m \otimes m"] & X_{02} \otimes Y_{02} \end{tikzcd}$$

To see that each subdiagram above commutes, recall that $\xi_{\mathbb{1},\mathbb{1}}$ is the identity on $\mathbb{1} \otimes \mathbb{1}$ (1.1.21). Then both subdiagrams above commute by naturality of ξ. □

We now turn to the unitors, associator, and braiding for $(\mathsf{V}\text{-}\mathsf{Cat}, \otimes)$. Here we simply define the data and show what basic properties they satisfy. We will use these to define monoidal V-categories for general braided monoidal V and then observe, in Theorem 1.5.5, that V-Cat is a braided monoidal Cat-category and is symmetric if V is symmetric.

Definition 1.3.7. The *unit* V*-category*, $\mathbb{I}$, is the one-object V-category whose unique hom object is the unit, $\mathbb{1}$, of V. The composition and identity structure morphisms are given, respectively, by the unit and identity morphisms of $\mathbb{1}$. ◇

Definition 1.3.8. The *left unitor*, $\ell^{\otimes}$, and *right unitor*, $r^{\otimes}$, are 2-natural isomorphisms

$$\begin{tikzcd} & \mathsf{V\text{-}Cat}^2 \arrow[dr, "\otimes"] & \\ \mathsf{V\text{-}Cat} \arrow[ur, "\mathbb{I}\times 1"] \arrow[rr, "1"'] & \arrow[u, phantom, "\Downarrow \ell^{\otimes}"] & \mathsf{V\text{-}Cat} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathsf{V\text{-}Cat}^2 \arrow[dr, "\otimes"] & \\ \mathsf{V\text{-}Cat} \arrow[ur, "1\times \mathbb{I}"] \arrow[rr, "1"'] & \arrow[u, phantom, "\Downarrow r^{\otimes}"] & \mathsf{V\text{-}Cat} \end{tikzcd}$$

in $\mathsf{2Cat}(\mathsf{V\text{-}Cat}, \mathsf{V\text{-}Cat})$ defined as follows. The unitor components at a V-category C,

$$\mathbb{I}\otimes \mathsf{C} \xrightarrow{\ell^{\otimes}_{\mathsf{C}}} \mathsf{C} \xleftarrow{r^{\otimes}_{\mathsf{C}}} \mathsf{C}\otimes\mathbb{I},$$

are given on objects by the unitors for the Cartesian product and are given on hom objects by the unit isomorphisms

$$\mathbb{1}\otimes \mathsf{C}(X,X') \xrightarrow{\lambda} \mathsf{C}(X,X') \xleftarrow{\rho} \mathsf{C}(X,X')\otimes\mathbb{1}$$

with X and X' objects of C. We show that the components of $\ell^{\otimes}_{\mathsf{C}}$ and $r^{\otimes}_{\mathsf{C}}$ are V-functors in Lemma 1.3.9 below. The naturality of λ and ρ with respect to morphisms in V implies that $\ell^{\otimes}$ and $r^{\otimes}$ are 2-natural with respect to V-functors and V-natural transformations. ◇

Lemma 1.3.9. *In the context of Definition 1.3.8, the components* $\ell^{\otimes}_{\mathsf{C}}$ *and* $r^{\otimes}_{\mathsf{C}}$ *are* V-*functors.*

Proof. For $\ell^{\otimes}_{\mathsf{C}}$, the composition axiom (1.2.5) is commutativity of the outer diagram below for each triple $X, X', X'' \in \mathsf{C}$, where we use the notation

$$\mathsf{C}(X,X') = X_{01} \qquad \mathsf{C}(X',X'') = X_{12} \qquad \mathsf{C}(X,X'') = X_{02}.$$

$$\begin{tikzcd}
(\mathbb{1}\otimes X_{12})\otimes(\mathbb{1}\otimes X_{01}) \arrow[rr, "\lambda\otimes\lambda"] \arrow[d, "\xi_{\mathrm{mid}}"'] & \arrow[d, phantom, "\star"] & X_{12}\otimes X_{01} \arrow[dd, "m"] \\
(\mathbb{1}\otimes\mathbb{1})\otimes(X_{12}\otimes X_{01}) \arrow[r, "\lambda\otimes(1\otimes 1)"] \arrow[d, "\lambda\otimes m"'] & \mathbb{1}\otimes(X_{12}\otimes X_{01}) \arrow[ur, "\lambda"] \arrow[dl, "1\otimes m"] & \\
\mathbb{1}\otimes X_{02} \arrow[rr, "\lambda"] & & X_{02}
\end{tikzcd}$$

The underlying braids of the two composites around the trapezoid labeled ☆ are both $\mathrm{id}_2 \in B_2$ and therefore ☆ commutes by the Braided Coherence Theorem 1.1.38. The remaining regions commute by functoriality of $\otimes$ and naturality of λ.

The identity axiom (1.2.6) is commutativity of the following diagram for $X \in \mathsf{C}$, which follows from naturality of λ.

$$\begin{tikzcd} & & \mathbb{1} \otimes \mathsf{C}(X,X) \arrow[d, "\lambda"] \\ & \mathbb{1}\otimes\mathbb{1} \arrow[ur, "1 \otimes i_X"] & \\ \mathbb{1} \arrow[ur, "\lambda^{-1}"] \arrow[rr, "i_X"'] & & \mathsf{C}(X,Y) \end{tikzcd}$$

The corresponding axioms for $r^{\otimes}_{\mathsf{C}}$ are similar, making use of the right unit isomorphism ρ. $\square$

Definition 1.3.10. The *associator*, $a^{\otimes}$, is a 2-natural isomorphism

$$\begin{tikzcd} \mathsf{V\text{-}Cat}^3 \arrow[r, "\otimes \times 1"] \arrow[d, "1 \times \otimes"'] \arrow[dr, phantom, "a^{\otimes}\swarrow"] & \mathsf{V\text{-}Cat}^2 \arrow[d, "\otimes"] \\ \mathsf{V\text{-}Cat}^2 \arrow[r, "\otimes"'] & \mathsf{V\text{-}Cat} \end{tikzcd}$$

in $\mathsf{2Cat}(\mathsf{V\text{-}Cat}^3, \mathsf{V\text{-}Cat})$ defined as follows. For C, D, E in $\mathsf{V\text{-}Cat}$, the associator component

$$a^{\otimes}_{\mathsf{C},\mathsf{D},\mathsf{E}} : (\mathsf{C} \otimes \mathsf{D}) \otimes \mathsf{E} \longrightarrow \mathsf{C} \otimes (\mathsf{D} \otimes \mathsf{E})$$

is given on objects by the associativity isomorphism of the Cartesian product and is given on hom objects by the associativity isomorphism

$$\alpha : \big(\mathsf{C}(X,X') \otimes \mathsf{D}(Y,Y')\big) \otimes \mathsf{E}(Z,Z') \longrightarrow \mathsf{C}(X,X') \otimes \big(\mathsf{D}(Y,Y') \otimes \mathsf{E}(Z,Z')\big)$$

for objects $X, X' \in \mathsf{C}$, $Y, Y' \in \mathsf{D}$, and $Z, Z' \in \mathsf{E}$. The braided monoidal axioms for $(\mathsf{V}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi)$ imply that each component $a^{\otimes}_{\mathsf{C},\mathsf{D},\mathsf{E}}$ is a V-functor. Naturality of α with respect to morphisms in V implies that $a^{\otimes}$ is 2-natural with respect to V-functors and V-natural transformations. $\diamond$

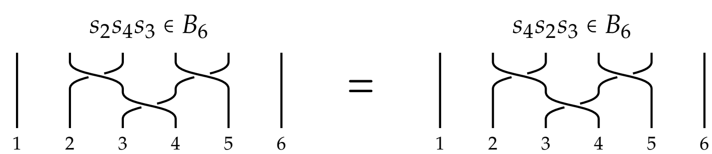
Lemma 1.3.11. *In the context of Definition 1.3.10 the associator components* $a^{\otimes}_{\mathsf{C},\mathsf{D},\mathsf{E}}$ *are* V-*functorial.*

Proof. The composition axiom (1.2.5) for $a^{\otimes}_{\mathsf{C},\mathsf{D},\mathsf{E}}$ is commutativity of the outer diagram below for objects $X, X', X'' \in \mathsf{C}$; $Y, Y', Y'' \in \mathsf{D}$; and $Z, Z', Z'' \in \mathsf{E}$. In the below diagram we use the following notation and omit the tensor signs to save space.

$$\begin{array}{lll} \mathsf{C}(X,X') = X_{01} & \mathsf{C}(X',X'') = X_{12} & \mathsf{C}(X,X'') = X_{02} \\ \mathsf{D}(Y,Y') = Y_{01} & \mathsf{D}(Y',Y'') = Y_{12} & \mathsf{D}(Y,Y'') = Y_{02} \\ \mathsf{E}(Z,Z') = Z_{01} & \mathsf{E}(Z',Z'') = Z_{12} & \mathsf{E}(Z,Z'') = Z_{02} \end{array}$$

$$\begin{tikzcd}
\big((X_{12}Y_{12})Z_{12}\big)\big((X_{01}Y_{01})Z_{01}\big) \arrow[r, "\alpha\alpha"] \arrow[d, "\xi_{\mathrm{mid}}"'] & \big(X_{12}(Y_{12}Z_{12})\big)\big(X_{01}(Y_{01}Z_{01})\big) \arrow[d, "\xi_{\mathrm{mid}}"] \\
\big((X_{12}Y_{12})(X_{01}Y_{01})\big)(Z_{12}Z_{01}) \arrow[d, "\xi_{\mathrm{mid}}(11)"'] \arrow[r, phantom, "\star"] & (X_{12}X_{01})\big((Y_{12}Z_{12})(Y_{01}Z_{01})\big) \arrow[d, "(11)\xi_{\mathrm{mid}}"] \\
\big((X_{12}X_{01})(Y_{12}Y_{01})\big)(Z_{12}Z_{01}) \arrow[r, "\alpha"] \arrow[d, "(mm)m"'] & (X_{12}X_{01})\big((Y_{12}Y_{01})(Z_{12}Z_{01})\big) \arrow[d, "m(mm)"] \\
(X_{02}Y_{02})Z_{02} \arrow[r, "\alpha"] & X_{02}(Y_{02}Z_{02})
\end{tikzcd}$$

Commutativity of the region labeled ☆ follows from the Braided Coherence Theorem 1.1.38 because the underlying braids of the boundary composites, shown below, are equal. The remaining rectangle commutes by naturality of α.



The identity axiom (1.2.6) for $a^{\otimes}_{\mathsf{C},\mathsf{D},\mathsf{E}}$ is similar to that of the unitors, following from unity properties of V (1.1.5) and naturality of α. $\square$

Explanation 1.3.12 (2-Naturality of the Associator and Unitors)**.** The 2-naturality of $a^{\otimes}$ means that the following diagram commutes for V-functors

$$F : \mathsf{C} \longrightarrow \mathsf{C}', \quad G : \mathsf{D} \longrightarrow \mathsf{D}', \quad \text{and} \quad H : \mathsf{E} \longrightarrow \mathsf{E}'.$$

$$\begin{tikzcd}
(\mathsf{C} \otimes \mathsf{D}) \otimes \mathsf{E} \arrow[r, "(F \otimes G) \otimes H"] \arrow[d, "a^{\otimes}"'] & (\mathsf{C}' \otimes \mathsf{D}') \otimes \mathsf{E}' \arrow[d, "a^{\otimes}"] \\
\mathsf{C} \otimes (\mathsf{D} \otimes \mathsf{E}) \arrow[r, "F \otimes (G \otimes H)"] & \mathsf{C}' \otimes (\mathsf{D}' \otimes \mathsf{E}')
\end{tikzcd} \tag{1.3.13}$$

The 2-naturality of $\ell^{\otimes}$ and $r^{\otimes}$ yield similar commutative diagrams. $\diamond$

The unitors $\ell^{\otimes}$ and $r^{\otimes}$ as well as the associator $a^{\otimes}$ satisfy the following two important properties generalizing those of monoidal categories.

Lemma 1.3.14 (Middle Unity for $\ell^{\otimes}$ and $r^{\otimes}$)**.** *The following diagram commutes for each pair of* V*-categories* C *and* D*.*

Middle Unity Diagram:

$$\begin{tikzcd}(\mathsf{C} \otimes \mathbb{I}) \otimes \mathsf{D} \arrow[r, "a^{\otimes}_{\mathsf{C},\mathbb{I},\mathsf{D}}"] \arrow[d, "r^{\otimes}_{\mathsf{C}} \otimes 1_{\mathsf{D}}"'] & \mathsf{C} \otimes (\mathbb{I} \otimes \mathsf{D}) \arrow[d, "1_{\mathsf{C}} \otimes \ell^{\otimes}_{\mathsf{D}}"] \\ \mathsf{C} \otimes \mathsf{D} \arrow[r, "1_{\mathsf{C}\otimes\mathsf{D}}"] & \mathsf{C} \otimes \mathsf{D}\end{tikzcd} \tag{1.3.15}$$

Proof. To verify that the diagram of V-functors commutes, we first observe that it commutes on objects by the middle unity axiom for the Cartesian product. For objects $X, X' \in \mathsf{C}$ and $Y, Y' \in \mathsf{D}$ the middle unity diagram on hom objects is the following.

$$\begin{tikzcd}(\mathsf{C}(X,X') \otimes \mathbb{1}) \otimes \mathsf{D}(Y,Y') \arrow[r, "\alpha"] \arrow[d, "\rho\otimes 1"'] & \mathsf{C}(X,X') \otimes (\mathbb{1} \otimes \mathsf{D}(Y,Y')) \arrow[d, "1\otimes\lambda"] \\ \mathsf{C}(X,X') \otimes \mathsf{D}(Y,Y') \arrow[r, "1"] & \mathsf{C}(X,X') \otimes \mathsf{D}(Y,Y')\end{tikzcd} \tag{1.3.16}$$

This diagram in V commutes by the middle unity axiom for λ and ρ. □

Lemma 1.3.17 (Pentagon Axiom for $a^{\otimes}$)**.** *The following diagram commutes for each tuple of* V*-categories* A, B, C, *and* D.

Pentagon Diagram:

$$\begin{tikzcd}[column sep=small] & (\mathsf{A} \otimes \mathsf{B}) \otimes (\mathsf{C} \otimes \mathsf{D}) \arrow[dr, "a^{\otimes}_{\mathsf{A},\mathsf{B},\mathsf{C}\otimes\mathsf{D}}"] & \\ \big((\mathsf{A} \otimes \mathsf{B}) \otimes \mathsf{C}\big) \otimes \mathsf{D} \arrow[ur, "a^{\otimes}_{\mathsf{A}\otimes\mathsf{B},\mathsf{C},\mathsf{D}}"] \arrow[d, "a^{\otimes}_{\mathsf{A},\mathsf{B},\mathsf{C}} \otimes 1_{\mathsf{D}}"'] & & \mathsf{A} \otimes \big(\mathsf{B} \otimes (\mathsf{C} \otimes \mathsf{D})\big) \\ \big(\mathsf{A} \otimes (\mathsf{B} \otimes \mathsf{C})\big) \otimes \mathsf{D} \arrow[rr, "a^{\otimes}_{\mathsf{A},\mathsf{B}\otimes\mathsf{C},\mathsf{D}}"] & & \mathsf{A} \otimes \big((\mathsf{B} \otimes \mathsf{C}) \otimes \mathsf{D}\big) \arrow[u, "1_{\mathsf{A}} \otimes a^{\otimes}_{\mathsf{B},\mathsf{C},\mathsf{D}}"']\end{tikzcd} \tag{1.3.18}$$

Proof. Our argument that (1.3.18) commutes is similar to the argument of Lemma 1.3.14. First we observe that the diagram commutes on objects by the pentagon axiom for the Cartesian product. Next, we observe that commutativity of the diagram on hom objects is equivalent to the pentagon axiom for those hom objects and the associator in V. □

We also have the following result generalizing the additional unity properties (1.1.4) and (1.1.5).

Lemma 1.3.19. *The following equality in* V-Cat *holds:*

$$\ell^{\otimes}_{\mathbb{I}} = r^{\otimes}_{\mathbb{I}} : \mathbb{I} \otimes \mathbb{I} \longrightarrow \mathbb{I}. \tag{1.3.20}$$

In addition, the following left unity diagram *and* right unity diagram *commute for each pair of* V*-categories* C *and* D.

$$\begin{tikzcd}[column sep=small] (\mathbb{I} \otimes \mathsf{C}) \otimes \mathsf{D} \arrow[rr, "a^{\otimes}_{\mathbb{I},\mathsf{C},\mathsf{D}}"] \arrow[dr, "\ell^{\otimes}_{\mathsf{C}} \otimes 1_{\mathsf{D}}"'] & & \mathbb{I} \otimes (\mathsf{C} \otimes \mathsf{D}) \arrow[dl, "\ell^{\otimes}_{\mathsf{C}\otimes\mathsf{D}}"] \\ & \mathsf{C} \otimes \mathsf{D} & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] (\mathsf{C} \otimes \mathsf{D}) \otimes \mathbb{I} \arrow[rr, "a^{\otimes}_{\mathsf{C},\mathsf{D},\mathbb{I}}"] \arrow[dr, "r^{\otimes}_{\mathsf{C}\otimes\mathsf{D}}"'] & & \mathsf{C} \otimes (\mathsf{D} \otimes \mathbb{I}) \arrow[dl, "1_{\mathsf{C}} \otimes r^{\otimes}_{\mathsf{D}}"] \\ & \mathsf{C} \otimes \mathsf{D} & \end{tikzcd} \tag{1.3.21}$$

Proof. We sketch two separate proofs of this result.

(1) Repeat the arguments from Remark I.1.2.5 that establish (1.1.4) and (1.1.5) from the axioms of a monoidal category, using the diagrams (1.3.15) and (1.3.18) instead.
(2) Argue as in Lemmas 1.3.14 and 1.3.17 that the result follows from (1.1.4) and (1.1.5) on objects (for the Cartesian product) and on hom objects (for the monoidal product in V). ☐

Now we turn to the braiding for the tensor product of V-categories. For simplicity we require that V be symmetric monoidal, although more subtle hypotheses in terms of iterated monoidal structures can be used to obtain more subtle results. See Note 1.6.2 for further comments and references.

Definition 1.3.22. Suppose that (V, ξ) is a symmetric monoidal category. The *braiding*, $\beta^{\otimes}$, is a 2-natural isomorphism

$$\begin{tikzcd} \mathsf{V}\text{-}\mathsf{Cat}^2 \arrow[rr, "\otimes"] \arrow[dr, "\tau"'] & \arrow[d, phantom, "\Downarrow\beta^{\otimes}"] & \mathsf{V}\text{-}\mathsf{Cat} \\ & \mathsf{V}\text{-}\mathsf{Cat}^2 \arrow[ur, "\otimes"'] & \end{tikzcd}$$

in $2\mathsf{Cat}(\mathsf{V}\text{-}\mathsf{Cat}^2, \mathsf{V}\text{-}\mathsf{Cat})$ defined as follows. For C and D in V-Cat, the braiding component

$$\beta^{\otimes}_{\mathsf{C},\mathsf{D}} : \mathsf{C} \otimes \mathsf{D} \longrightarrow \mathsf{D} \otimes \mathsf{C}$$

is given on objects by the braiding of the Cartesian product and is given on hom objects by the braiding in V

$$\xi : \mathsf{C}(X, X') \otimes \mathsf{D}(Y, Y') \longrightarrow \mathsf{D}(Y, Y') \otimes \mathsf{C}(X, X')$$

for objects $X, X' \in \mathsf{C}$ and $Y, Y' \in \mathsf{D}$. The symmetric monoidal axioms for $(\mathsf{V}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi)$ imply that each component $\beta^{\otimes}_{\mathsf{C},\mathsf{D}}$ is a V-functor (see Lemma 1.3.23). Naturality of ξ with respect to morphisms in V implies that $\beta^{\otimes}$ is 2-natural with respect to V-functors and V-natural transformations. ◇

Lemma 1.3.23. *In the context of Definition 1.3.22 the braiding components* $\beta^{\otimes}_{\mathsf{C},\mathsf{D}}$ *are* V*-functorial.*

Proof. The composition axiom (1.2.5) for $\beta^{\otimes}_{\mathsf{C},\mathsf{D}}$ is commutativity of the outer diagram below for objects $X, X', X'' \in \mathsf{C}$ and $Y, Y', Y'' \in \mathsf{D}$. In the below diagram we use the following notation and omit the tensor signs to save space.

$$\begin{array}{lll} \mathsf{C}(X, X') = X_{01} & \mathsf{C}(X', X'') = X_{12} & \mathsf{C}(X, X'') = X_{02} \\ \mathsf{D}(Y, Y') = Y_{01} & \mathsf{D}(Y', Y'') = Y_{12} & \mathsf{D}(Y, Y'') = Y_{02} \end{array}$$

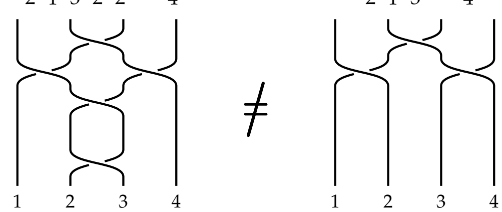
$$\begin{tikzcd}
(X_{12}Y_{12})(X_{01}Y_{01}) \arrow[rr, "\xi\xi"] \arrow[d, "\xi_{\mathrm{mid}}"'] \arrow[drr, phantom, "☆"] & & (Y_{12}X_{12})(Y_{01}X_{01}) \arrow[d, "\xi_{\mathrm{mid}}"] \\
(X_{12}X_{01})(Y_{12}Y_{01}) \arrow[rr, "\xi"] \arrow[d, "mm"'] & & (Y_{12}Y_{01})(X_{12}X_{01}) \arrow[d, "mm"] \\
X_{02}Y_{02} \arrow[rr, "\xi"] & & Y_{02}X_{02}
\end{tikzcd} \tag{1.3.24}$$

In the above diagram, the middle horizontal arrow ξ interchanges $X_{12}X_{01}$ and $Y_{12}Y_{01}$. The composites around the region labeled ☆ have the same underlying permutation and therefore ☆ commutes by the Symmetric Coherence Theorem 1.1.41. The lower rectangle commutes by naturality of ξ.

Checking the identity axiom (1.2.6) for $\beta^{\otimes}_{\mathsf{C},\mathsf{D}}$ is similar to that of the unitors and associator, following from symmetric unity properties of V and naturality of ξ. □

Explanation 1.3.25. In (1.3.24) above, the underlying braids for the two composites around the region labeled ☆ are the following distinct elements of B_4.

$s_2s_1s_3s_2s_2 \in B_4$ $\qquad$ $s_2s_1s_3 \in B_4$



Therefore, if V is merely braided monoidal then the data of $\beta^{\otimes}_{\mathsf{C},\mathsf{D}}$ generally do not form a V-functor. ◇

Explanation 1.3.26 (2-Naturality of the Braiding)**.** The 2-naturality of $\beta^{\otimes}$ means that the following diagram commutes for V-functors

$$F : \mathsf{C} \longrightarrow \mathsf{C}' \quad \text{and} \quad G : \mathsf{D} \longrightarrow \mathsf{D}'.$$

$$\begin{tikzcd}
\mathsf{C} \otimes \mathsf{D} \arrow[r, "F \otimes G"] \arrow[d, "\beta^{\otimes}_{\mathsf{C},\mathsf{D}}"'] & \mathsf{C}' \otimes \mathsf{D}' \arrow[d, "\beta^{\otimes}_{\mathsf{C}',\mathsf{D}'}"] \\
\mathsf{D} \otimes \mathsf{C} \arrow[r, "G \otimes F"] & \mathsf{D}' \otimes \mathsf{C}'
\end{tikzcd} \tag{1.3.27}$$

◇

Next we discuss axioms satisfied by $\beta^{\otimes}$.

Convention 1.3.28. Since Definition 1.3.22 assumes that V is symmetric monoidal, that assumption is in effect any time $\beta^{\otimes}$ appears below. ◇

Lemma 1.3.29 (Hexagon Axioms for $\beta^{\otimes}$)**.** *Suppose* V *is symmetric monoidal. The following diagrams commute for each triple of* V*-categories* A, B, *and* C, *with* $a^{-\otimes}$ *denoting the inverse of* $a^{\otimes}$.

Left Hexagon Diagram:

$$
\begin{tikzcd}
& (B \otimes A) \otimes C \arrow[r, "a^{\otimes}_{B,A,C}"] & B \otimes (A \otimes C) \arrow[dr, "1_B \otimes \beta^{\otimes}_{A,C}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\beta^{\otimes}_{A,B} \otimes 1_C"] \arrow[dr, "a^{\otimes}_{A,B,C}"'] & & & B \otimes (C \otimes A) \\
& A \otimes (B \otimes C) \arrow[r, "\beta^{\otimes}_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[ur, "a^{\otimes}_{B,C,A}"'] &
\end{tikzcd}
\tag{1.3.30}
$$

Right Hexagon Diagram:

$$
\begin{tikzcd}
& A \otimes (C \otimes B) \arrow[r, "a^{-\otimes}_{A,C,B}"] & (A \otimes C) \otimes B \arrow[dr, "\beta^{\otimes}_{A,C} \otimes 1_B"] & \\
A \otimes (B \otimes C) \arrow[ur, "1_A \otimes \beta^{\otimes}_{B,C}"] \arrow[dr, "a^{-\otimes}_{A,B,C}"'] & & & (C \otimes A) \otimes B \\
& (A \otimes B) \otimes C \arrow[r, "\beta^{\otimes}_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[ur, "a^{-\otimes}_{C,A,B}"'] &
\end{tikzcd}
\tag{1.3.31}
$$

Proof. Our arguments that (1.3.30) and (1.3.31) commute are similar to the arguments of Lemmas 1.3.14 and 1.3.17. Commutativity on objects follows from the hexagon diagrams for the braiding of the Cartesian product, and commutativity on hom objects follows from the hexagon axioms for the braiding of V. $\square$

Lemma 1.3.32 (Symmetry Axiom for $\beta^{\otimes}$)**.** *Suppose* V *is symmetric monoidal. Then the following diagram commutes for each pair of* V*-categories* C *and* D*.*

Symmetry Diagram:

$$
\begin{tikzcd}
C \otimes D \arrow[rr, "1"] \arrow[dr, bend right, "\beta^{\otimes}_{C,D}"'] & & C \otimes D \\
& D \otimes C \arrow[ur, bend right, "\beta^{\otimes}_{D,C}"'] &
\end{tikzcd}
\tag{1.3.33}
$$

Proof. Similarly to the proofs of Lemmas 1.3.14, 1.3.17, and 1.3.29, commutativity of (1.3.33) follows on objects by the symmetry of the Cartesian product and follows on hom objects by the symmetry axiom for ξ. $\square$

Lemma 1.3.34 (Unit Properties for $\beta^{\otimes}$)**.** *Suppose* V *is symmetric monoidal. The following diagrams commute for each* V*-category* C*.*

$$
\begin{tikzcd}
A \otimes \mathbb{I} \arrow[rr, "\beta^{\otimes}_{A,\mathbb{I}}"] \arrow[dr, "r^{\otimes}_A"'] & & \mathbb{I} \otimes A \arrow[dl, "\ell^{\otimes}_A"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I} \otimes C \arrow[rr, "\beta^{\otimes}_{\mathbb{I},C}"] \arrow[dr, "\ell^{\otimes}_C"'] & & C \otimes \mathbb{I} \arrow[dl, "r^{\otimes}_C"] \\
& C &
\end{tikzcd}
$$

Proof. Commutativity on objects follows from the corresponding properties of the Cartesian product. On hom objects, commutativity follows from the unit diagrams (1.1.20) in V. □

Restricting to the underlying 1-category of V-Cat, the preceding definitions and results of this section prove the following. In fact they prove a stronger Cat-enriched result that we state as Theorem 1.5.5 below, after giving the relevant background in Sections 1.4 and 1.5.

Theorem 1.3.35. *Suppose $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is a braided monoidal category. Equipped with the data above, $(\mathsf{V}\text{-}\mathsf{Cat}, \otimes, \mathbb{I}, a^{\otimes}, \ell^{\otimes}, r^{\otimes})$ is a monoidal category. If V is symmetric monoidal then so is $(\mathsf{V}\text{-}\mathsf{Cat}, \beta^{\otimes})$.*

In Explanation 1.3.25 above we explained that the braiding $\beta^{\otimes}$ cannot be defined as a V-functor when V is merely braided monoidal. However, the same assignment on objects can be extended to a V-functor using the opposite V-category construction from Definition 1.2.16. We will make use of this in Example 1.4.27 below.

Definition 1.3.36. Suppose (V, ξ) is a braided monoidal category and suppose $\mathsf{C}, \mathsf{D} \in \mathsf{V}\text{-}\mathsf{Cat}$. Define a V-functor

$$\gamma_{\mathsf{C},\mathsf{D}} : \mathsf{C}^{\mathsf{op}} \otimes \mathsf{D}^{\mathsf{op}} \longrightarrow (\mathsf{D} \otimes \mathsf{C})^{\mathsf{op}}$$

with the assignment

$$X \otimes Y \longmapsto Y \otimes X$$

on objects $X \otimes Y \in \mathsf{C}^{\mathsf{op}} \otimes \mathsf{D}^{\mathsf{op}}$. On hom objects we define $(\gamma_{\mathsf{C},\mathsf{D}})_{X\otimes Y, X'\otimes Y'}$ via the braiding ξ below.

$$\begin{tikzcd}
(\mathsf{C}^{\mathsf{op}} \otimes \mathsf{D}^{\mathsf{op}})(X \otimes Y, X' \otimes Y') \arrow[d, equal] & (\mathsf{D} \otimes \mathsf{C})^{\mathsf{op}}(Y \otimes X, Y' \otimes X') \arrow[d, equal] \\
\mathsf{C}^{\mathsf{op}}(X, X') \otimes \mathsf{D}^{\mathsf{op}}(Y, Y') \arrow[d, equal] & (\mathsf{D} \otimes \mathsf{C})(Y' \otimes X', Y \otimes X) \arrow[d, equal] \\
\mathsf{C}(X', X) \otimes \mathsf{D}(Y', Y) \arrow[r, "\xi"] & \mathsf{D}(Y', Y) \otimes \mathsf{C}(X', X)
\end{tikzcd}$$

We verify the V-functor axioms for $\gamma_{\mathsf{C},\mathsf{D}}$ in Lemma 1.3.37. When clear from context we will omit the subscripts C and D. ◇

Lemma 1.3.37. *In the context of Definition 1.3.36,*

$$\gamma : \mathsf{C}^{\mathsf{op}} \otimes \mathsf{D}^{\mathsf{op}} \longrightarrow (\mathsf{D} \otimes \mathsf{C})^{\mathsf{op}}$$

is a V-functor.

Proof. To verify the composition axiom (1.2.5) for γ, suppose given

$$X_0, X_1, X_2 \in \mathsf{C} \quad \text{and} \quad Y_0, Y_1, Y_2 \in \mathsf{D}.$$

For $i, j \in \{0, 1, 2, 3\}$ we let

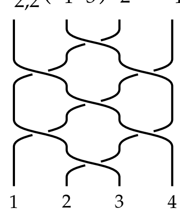
$$\mathsf{C}^{\mathsf{op}}(X_i, X_j) = \overline{X}_{ij} = X_{ji} = \mathsf{C}(X_j, X_i) \quad \text{and} \quad \mathsf{D}^{\mathsf{op}}(Y_i, Y_j) = \overline{Y}_{ij} = Y_{ji} = \mathsf{D}(Y_j, Y_i).$$

Then the composition axiom for γ is the outer diagram below, where we omit tensors to save space.

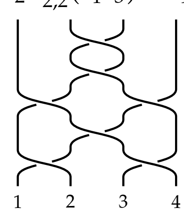
$$\begin{tikzcd}
(\overline{X}_{12}\overline{Y}_{12})(\overline{X}_{01}\overline{Y}_{01}) \arrow[rr, "\xi\xi", "s_1s_3"'] \arrow[d, "\xi_{\mathrm{mid}}"', "s_2"] & & (\overline{Y}_{12}\overline{X}_{12})(\overline{Y}_{01}\overline{X}_{01}) \arrow[d, equal] \\
(\overline{X}_{12}\overline{X}_{01})(\overline{Y}_{12}\overline{Y}_{01}) \arrow[d, equal] & & (Y_{21}X_{21})(Y_{10}X_{10}) \arrow[d, "b^{\oplus}_{2,2} = s_2s_1s_3s_2"', "\xi"] \\
(X_{21}X_{10})(Y_{21}Y_{10}) \arrow[d, "\xi\xi"', "s_1s_3"] \arrow[rr, phantom, "\star"] & & (Y_{10}X_{10})(Y_{21}X_{21}) \arrow[d, "\xi_{\mathrm{mid}}"] \\
(X_{10}X_{21})(Y_{10}Y_{21}) \arrow[rr, "\xi", "b^{\oplus}_{2,2} = s_2s_1s_3s_2"'] \arrow[d, "mm"'] & & (Y_{10}Y_{21})(X_{10}X_{21}) \arrow[d, "mm"] \\
X_{20}Y_{20} \arrow[rr, "\xi"] & & Y_{20}X_{20}
\end{tikzcd}$$

In the diagram above, the vertical composite along the left is the composition for $\mathsf{C}^{\mathsf{op}} \otimes \mathsf{D}^{\mathsf{op}}$, beginning with ξ_{mid} as in Definition 1.3.3. The vertical composite along the right is the composition in $(\mathsf{D} \otimes \mathsf{C})^{\mathsf{op}}$, beginning with the block braid ξ as in Definition 1.2.16. The lower region commutes by naturality of ξ and the upper region $\star$ commutes by the Braided Coherence Theorem 1.1.38: each morphism is labeled with its underlying braid, and the two boundary composites are the following.

$b^{\oplus}_{2,2}(s_1s_3)s_2 \in B_4$ $\qquad\qquad$ $s_2\, b^{\oplus}_{2,2}(s_1s_3) \in B_4$



$=$



Using the same notation, the identity axiom (1.2.6) for γ is the following.

$$\begin{tikzcd}
& & \overline{X}_{00}\overline{Y}_{00} \arrow[d, equal] \\
& & X_{00}Y_{00} \arrow[d, "\xi"] \\
\mathbb{1} \arrow[r, "\lambda^{-1}"] & \mathbb{1}\mathbb{1} \arrow[uur, "ii"] \arrow[ddr, "ii"'] & \\
& & Y_{00}X_{00} \arrow[d, equal] \\
& & \overline{Y}_{00}\overline{X}_{00}
\end{tikzcd}$$

Recalling that $\xi_{\mathbb{1},\mathbb{1}} = 1_{\mathbb{1}\mathbb{1}}$ (1.1.21), the diagram above commutes by naturality of ξ. $\square$

Explanation 1.3.38. The one-object V-category $\mathbb{I}$ has

$$\mathbb{I}^{\mathsf{op}}(*,*) = \mathbb{I}(*,*) = \mathbb{1}.$$

Since $\xi_{\mathbb{1},\mathbb{1}}$, $\rho_{\mathbb{1}}$, and $\lambda_{\mathbb{1}}$ are identities, the composition laws for $\mathbb{I}$ and its opposite are equal. Therefore, we have

$$\mathbb{I}^{\mathsf{op}} = \mathbb{I}$$

as V-categories. $\diamond$

For future reference, we will need the following properties of γ.

Lemma 1.3.39. *Suppose given* V*-categories*

$$\mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{C}_0, \mathsf{C}_1, \mathsf{D}_0, \text{ and } \mathsf{D}_1$$

along with V*-functors*

$$\mathsf{C}_0 \xrightarrow{F} \mathsf{C}_1 \quad \textit{and} \quad \mathsf{D}_0 \xrightarrow{G} \mathsf{D}_1.$$

Then following diagrams of V*-categories and* V*-functors commute.*

(1.3.40)
$$\begin{tikzcd}
\mathsf{C}_0^{\mathsf{op}} \otimes \mathsf{D}_0^{\mathsf{op}} \arrow[r, "F^{\mathsf{op}} \otimes G^{\mathsf{op}}"] \arrow[d, "\gamma_{\mathsf{C}_0,\mathsf{D}_0}"'] & \mathsf{C}_1^{\mathsf{op}} \otimes \mathsf{D}_1^{\mathsf{op}} \arrow[d, "\gamma_{\mathsf{C}_1,\mathsf{D}_1}"] \\
(\mathsf{D}_0 \otimes \mathsf{C}_0)^{\mathsf{op}} \arrow[r, "(G \otimes F)^{\mathsf{op}}"] & (\mathsf{D}_1 \otimes \mathsf{C}_1)^{\mathsf{op}}
\end{tikzcd}$$

(1.3.41)
$$\begin{tikzcd}
(\mathsf{C}^{\mathsf{op}} \otimes \mathsf{D}^{\mathsf{op}}) \otimes \mathsf{E}^{\mathsf{op}} \arrow[r, "a^{\otimes}"] \arrow[d, "\gamma_{\mathsf{C},\mathsf{D}} \otimes 1"'] & \mathsf{C}^{\mathsf{op}} \otimes (\mathsf{D}^{\mathsf{op}} \otimes \mathsf{E}^{\mathsf{op}}) \arrow[d, "1 \otimes \gamma_{\mathsf{D},\mathsf{E}}"] \\
(\mathsf{D} \otimes \mathsf{C})^{\mathsf{op}} \otimes \mathsf{E}^{\mathsf{op}} \arrow[d, "\gamma_{\mathsf{D}\otimes\mathsf{C},\mathsf{E}}"'] & \mathsf{C}^{\mathsf{op}} \otimes (\mathsf{E} \otimes \mathsf{D})^{\mathsf{op}} \arrow[d, "\gamma_{\mathsf{C},\mathsf{E}\otimes\mathsf{D}}"] \\
(\mathsf{E} \otimes (\mathsf{D} \otimes \mathsf{C}))^{\mathsf{op}} & ((\mathsf{E} \otimes \mathsf{D}) \otimes \mathsf{C})^{\mathsf{op}} \arrow[l, "(a^{\otimes})^{\mathsf{op}}"']
\end{tikzcd}$$

(1.3.42)
$$\begin{tikzcd}
\mathsf{C}^{\mathsf{op}} \otimes \mathbb{I}^{\mathsf{op}} \arrow[r, "\gamma_{\mathsf{C},\mathbb{I}}"] \arrow[d, equal] & (\mathbb{I} \otimes \mathsf{C})^{\mathsf{op}} \arrow[d, "(\ell^{\otimes})^{\mathsf{op}}"] \\
\mathsf{C}^{\mathsf{op}} \otimes \mathbb{I} \arrow[r, "r^{\otimes}"'] & \mathsf{C}^{\mathsf{op}}
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathbb{I}^{\mathsf{op}} \otimes \mathsf{C}^{\mathsf{op}} \arrow[r, "\gamma_{\mathbb{I},\mathsf{C}}"] \arrow[d, equal] & (\mathsf{C} \otimes \mathbb{I})^{\mathsf{op}} \arrow[d, "(r^{\otimes})^{\mathsf{op}}"] \\
\mathbb{I} \otimes \mathsf{C}^{\mathsf{op}} \arrow[r, "\ell^{\otimes}"'] & \mathsf{C}^{\mathsf{op}}
\end{tikzcd}$$

Proof. For each of the diagrams, one can check commutativity on objects directly because $(-)^{\mathsf{op}}$ is the identity on objects. On hom objects, one unpacks the various opposites involved and finds that the relevant diagram commutes by the corresponding property of ξ. We give the unpacked diagrams below.

For $X, X' \in \mathsf{C}_0$ and $Y, Y' \in \mathsf{D}_0$, the diagram (1.3.40) at the hom object

$$\mathsf{C}_0^{\mathsf{op}}(X, X') \otimes \mathsf{D}_0^{\mathsf{op}}(Y, Y')$$

is the following. It commutes by naturality of ξ.

$$\begin{tikzcd}
\mathsf{C}_0(X',X) \otimes \mathsf{D}_0(Y',Y) \arrow[r, "F \otimes G"] \arrow[d, "\xi"'] & \mathsf{C}_1(FX',FX) \otimes \mathsf{D}_1(GY',GY) \arrow[d, "\xi"] \\
\mathsf{D}_0(Y',Y) \otimes \mathsf{C}_0(X',X) \arrow[r, "G \otimes F"] & \mathsf{D}_1(GY',GY) \otimes \mathsf{C}_1(FX',FX)
\end{tikzcd}$$

For $X, X' \in \mathsf{C}$, $Y, Y' \in \mathsf{D}$, and $Z, Z' \in \mathsf{E}$, the diagram (1.3.41) at the hom object

$$(\mathsf{C}^{\mathsf{op}}(X, X') \otimes \mathsf{D}^{\mathsf{op}}(Y, Y')) \otimes \mathsf{E}^{\mathsf{op}}(Z, Z')$$

is the following. It commutes by the hexagon (1.1.22) for ξ and α.

$$\begin{tikzcd}
\big(\mathsf{C}(X',X) \otimes \mathsf{D}(Y',Y)\big) \otimes \mathsf{E}(Z',Z) \arrow[r, "\alpha"] \arrow[d, "\xi \otimes 1"'] & \mathsf{C}(X',X) \otimes \big(\mathsf{D}(Y',Y) \otimes \mathsf{E}(Z',Z)\big) \arrow[d, "1 \otimes \xi"] \\
\big(\mathsf{D}(Y',Y) \otimes \mathsf{C}(X',X)\big) \otimes \mathsf{E}(Z',Z) \arrow[d, "\xi"'] & \mathsf{C}(X',X) \otimes \big(\mathsf{E}(Z',Z) \otimes \mathsf{D}(Y',Y)\big) \arrow[d, "\xi"] \\
\mathsf{E}(Z',Z) \otimes \big(\mathsf{D}(Y',Y) \otimes \mathsf{C}(X',X)\big) & \big(\mathsf{E}(Z',Z) \otimes \mathsf{D}(Y',Y)\big) \otimes \mathsf{C}(X',X) \arrow[l, "\alpha"']
\end{tikzcd}$$

For $X, X' \in \mathsf{C}$, the diagrams (1.3.42) at the hom object $\mathsf{C}^{\mathsf{op}}(X, X')$ are the following. They commute by the unity properties (1.1.20) for ξ.

$$\begin{tikzcd}
\mathsf{C}(X',X) \otimes \mathbb{1} \arrow[rr, "\xi"] \arrow[dr, "\rho"'] & & \mathbb{1} \otimes \mathsf{C}(X',X) \arrow[dl, "\lambda"] \\
& \mathsf{C}(X',X) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes \mathsf{C}(X',X) \arrow[rr, "\xi"] \arrow[dr, "\lambda"'] & & \mathsf{C}(X',X) \otimes \mathbb{1} \arrow[dl, "\rho"] \\
& \mathsf{C}(X',X) &
\end{tikzcd}$$

This completes the proof. $\square$

Explanation 1.3.43. One can define an opposite monoidal structure on $\mathsf{V}\text{-}\mathsf{Cat}$ with monoidal product $\otimes^{\tau}$ given by twisting with the braiding τ of $(\mathsf{Cat}, \times)$. Then Lemma 1.3.39 shows that γ provides a monoidal constraint for $(-)^{\mathsf{op}}$, making it a monoidal functor (of 1-categories) from $(\mathsf{V}\text{-}\mathsf{Cat}, \otimes)$ to $(\mathsf{V}\text{-}\mathsf{Cat}, \otimes^{\tau})$. Following Theorem 1.3.35, one can show that γ is 2-natural and one expects these claims will extend to make $(-)^{\mathsf{op}}$ monoidal as a 2-functor to $\mathsf{V}\text{-}\mathsf{Cat}^{\mathsf{co}}$ (with twisted monoidal product). Further development of these points is beyond our current scope. $\diamond$

1.4. Monoidal Enriched Categories

In this section we use the monoidal structure of Section 1.3 to define V-enriched monoidal structures. As in Section 1.3, we assume throughout this section that $(\mathsf{V}, \otimes, \xi)$ is a braided monoidal category. We define monoidal V-categories, V-functors, and V-natural transformations and, under the further assumption that V is symmetric monoidal, we define braided and symmetric variants.

In the definitions below, we will write the tensor product of V-categories as juxtaposition. We will denote the associator, unitors, and braiding of V-Cat with superscripts: $a^{\otimes}$, $\ell^{\otimes}$, $r^{\otimes}$, and $\beta^{\otimes}$. We will use superscripts with a minus sign, $-\otimes$, to denote their inverses.

Motivation 1.4.1 (Monoidal Categories)**.** For a monoidal category C, the associativity and unit isomorphisms are natural transformations described in Definition 1.1.1 via their components. The axioms for these data are given in terms of commuting diagrams of components. They can be rewritten equivalently as equalities of certain composites of natural transformations—that is, as equalities of pasting diagrams in Cat.

This equivalent form of data (as 2-cells) and axioms (as equalities of certain pasting diagrams) yields a definition that can be studied in any monoidal 2-category A. The definitions and results in this section are obtained by applying such a point of view to the 2-category A = V-Cat with its tensor product. We encourage the reader to observe, for each of the definitions below, how the corresponding diagrams in Cat would define the monoidal structures discussed in Section 1.1. The theory of pseudoalgebras over 2-monads yields an even more general approach to this theory, and we give some references in Note 1.6.3.

However, to minimize the necessary background, our presentation below does not depend on this general theory. Indeed, we will *apply* the material in this section to *define* the 2-dimensional monoidal structure on V-Cat in Section 1.5. Therefore, to avoid circularity, in this section we neither use nor mention general results about 2-dimensional monoidal structure. A reader familiar with the more general theory may recognize many of these results as special cases of more general ones. A reader unfamiliar with the more general theory may take these results as motivating examples. ◇

Definition 1.4.2. A *monoidal* V*-category* is a tuple

$$(\mathsf{K}, \boxtimes, \mathrm{I}^{\boxtimes}, a^{\boxtimes}, \ell^{\boxtimes}, r^{\boxtimes})$$

consisting of the following data.

Base V-category: It has a V-category K called the *base* V*-category*.

Monoidal Composition: It has a V-functor

$$\mathsf{K} \otimes \mathsf{K} \xrightarrow{\boxtimes} \mathsf{K}$$

called the *monoidal composition*.

Monoidal Identity: It has a V-functor

$$\mathbb{I} \xrightarrow{\mathrm{I}^{\boxtimes}} \mathsf{K}$$

called the *monoidal identity*. The image of the unique object in $\mathbb{I}$ is also denoted $\mathrm{I}^{\boxtimes}$ and called the *identity object*.

Monoidal Unitors: It has V-natural isomorphisms

(1.4.3)
$$\begin{tikzcd}
& \mathbb{I}\mathsf{K} \arrow[r, "\mathrm{I}^{\boxtimes}1_{\mathsf{K}}"] \arrow[d, phantom, "\Downarrow\ell^{\boxtimes}"] & \mathsf{K}^2 \arrow[dr, "\boxtimes"] & \\
\mathsf{K} \arrow[ur, "\ell^{-\otimes}"] \arrow[rrr, "1_{\mathsf{K}}"'] & & & \mathsf{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathsf{K}\mathbb{I} \arrow[r, "1_{\mathsf{K}}\mathrm{I}^{\boxtimes}"] \arrow[d, phantom, "\Downarrow r^{\boxtimes}"] & \mathsf{K}^2 \arrow[dr, "\boxtimes"] & \\
\mathsf{K} \arrow[ur, "r^{-\otimes}"] \arrow[rrr, "1_{\mathsf{K}}"'] & & & \mathsf{K}
\end{tikzcd}$$

in $\mathsf{V}\text{-}\mathsf{Cat}(\mathsf{K},\mathsf{K})$, called the *left monoidal unitor* and the *right monoidal unitor*, respectively. Their components at an object $X \in \mathsf{K}$ are, respectively,

$$\mathbb{1} \xrightarrow{\ell_X^{\boxtimes}} \mathsf{K}(\mathrm{I}^{\boxtimes} \boxtimes X, X) \quad \text{and} \quad \mathbb{1} \xrightarrow{r_X^{\boxtimes}} \mathsf{K}(X \boxtimes \mathrm{I}^{\boxtimes}, X).$$

Monoidal Associator: It has a V-natural isomorphism

(1.4.4)
$$\begin{tikzcd}
(\mathsf{K}^2)\mathsf{K} \arrow[r, "\boxtimes 1_{\mathsf{K}}"] \arrow[d, "a^{\otimes}"'] \arrow[dr, phantom, "a^{\boxtimes}\Swarrow"] & \mathsf{K}^2 \arrow[dd, "\boxtimes"] \\
\mathsf{K}(\mathsf{K}^2) \arrow[d, "1_{\mathsf{K}}\boxtimes"'] & \\
\mathsf{K}^2 \arrow[r, "\boxtimes"'] & \mathsf{K}
\end{tikzcd}$$

in $\mathsf{V}\text{-}\mathsf{Cat}((\mathsf{K}^2)\mathsf{K},\mathsf{K})$, called the *monoidal associator*. Its component at a triple of objects $X, Y, Z \in \mathsf{K}$ is a morphism in V

$$\mathbb{1} \xrightarrow{a_{X,Y,Z}^{\boxtimes}} \mathsf{K}\big((X \boxtimes Y) \boxtimes Z, X \boxtimes (Y \boxtimes Z)\big).$$

These data are required to satisfy the following two axioms, where we write 1 for the identity V-functor.

Unity Axiom: The composites of the following two *middle unity pasting diagrams* are equal.

(1.4.5)

$$\begin{tikzcd}
& (\mathsf{K}\mathbb{I})\mathsf{K} \arrow[r, "(1\mathrm{I}^{\boxtimes})1"] \arrow[d, "a^{\otimes}"] \arrow[dl, phantom, "\star"] & (\mathsf{K}^2)\mathsf{K} \arrow[r, "\boxtimes 1"] \arrow[d, "a^{\otimes}"] \arrow[dr, phantom, "a^{\boxtimes}\Swarrow"] & \mathsf{K}^2 \arrow[dd, "\boxtimes"] \\
\mathsf{K}^2 \arrow[ur, "r^{-\otimes}1"] \arrow[r, "1\ell^{-\otimes}"'] \arrow[drr, bend right, "1^2"'] & \mathsf{K}(\mathbb{I}\mathsf{K}) \arrow[r, "1(\mathrm{I}^{\boxtimes}1)"] \arrow[dr, phantom, "1\ell^{\boxtimes}\Swarrow"] & \mathsf{K}(\mathsf{K}^2) \arrow[d, "1\boxtimes"] & \\
& & \mathsf{K}^2 \arrow[r, "\boxtimes"] & \mathsf{K}
\end{tikzcd}$$

$$\parallel$$

$$\begin{tikzcd}
& (\mathsf{K}\mathbb{I})\mathsf{K} \arrow[r, "(1\mathrm{I}^{\boxtimes})1"] \arrow[dr, phantom, "r^{\boxtimes}1\Swarrow"] & (\mathsf{K}^2)\mathsf{K} \arrow[r, "\boxtimes 1"] & \mathsf{K}^2 \arrow[dd, "\boxtimes"] \\
\mathsf{K}^2 \arrow[ur, "r^{-\otimes}1"] \arrow[urrr, bend right, "1^2"'] \arrow[drr, bend right, "1^2"'] & & & \\
& & \mathsf{K}^2 \arrow[r, "\boxtimes"] & \mathsf{K}
\end{tikzcd}$$

In the first diagram above, the unlabeled rectangle commutes by naturality of $a^{\otimes}$ (1.3.13) and the triangle labeled $\star$ commutes by the middle unity for $\ell^{\otimes}$ and $r^{\otimes}$ (1.3.15).

Pentagon Axiom: The composites of the following two *pentagon pasting diagrams* are equal.

(1.4.6)

$$\begin{tikzcd}
 & (K(K^2))K \arrow[dl, "a^{\otimes}"'] & & & ((K^2)K)K \arrow[lll, "a^{\otimes}1"'] \arrow[dr, "(\boxtimes 1)1"] \arrow[ddll, "a^{\otimes}"'] & \\
K((K^2)K) \arrow[dd, "1a^{\otimes}"'] \arrow[rr, phantom, "\star"] & & & & & (K^2)K \arrow[dll, "a^{\otimes}"'] \arrow[dd, "\boxtimes 1"] \\
 & & (K^2)(K^2) \arrow[r, "\boxtimes 1^2"] \arrow[dll, "a^{\otimes}"'] \arrow[d, "1^2\boxtimes"] & K(K^2) \arrow[d, "1\boxtimes"] \arrow[dr, phantom, "a^{\boxtimes}\Swarrow"] & & \\
K(K(K^2)) \arrow[dr, "1(1\boxtimes)"'] & & (K^2)K \arrow[r, "\boxtimes 1"] \arrow[dl, "a^{\otimes}"'] \arrow[dr, phantom, "a^{\boxtimes}\Swarrow"] & K^2 \arrow[d, "\boxtimes"] & & K^2 \arrow[dll, "\boxtimes"] \\
 & K(K^2) \arrow[r, "1\boxtimes"'] & K^2 \arrow[r, "\boxtimes"'] & K & &
\end{tikzcd}$$

$$\|$$

$$\begin{tikzcd}
 & (K(K^2))K \arrow[dl, "a^{\otimes}"'] \arrow[dr, "(1\boxtimes)1"] & & ((K^2)K)K \arrow[ll, "a^{\otimes}1"'] \arrow[dr, "(\boxtimes 1)1"] & \\
K((K^2)K) \arrow[dr, "1(\boxtimes 1)"] \arrow[dd, "1a^{\otimes}"'] & & (K^2)K \arrow[dl, "a^{\otimes}"'] \arrow[ddrr, "\boxtimes 1"] \arrow[r, phantom, "a^{\boxtimes}1\Swarrow"] & & (K^2)K \arrow[dd, "\boxtimes 1"] \\
 & K(K^2) \arrow[ddr, "1\boxtimes"] \arrow[r, phantom, "a^{\boxtimes}\Swarrow"] & & & \\
K(K(K^2)) \arrow[dr, "1(1\boxtimes)"'] \arrow[r, phantom, "1a^{\boxtimes}\Swarrow"] & & & & K^2 \arrow[dl, "\boxtimes"] \\
 & K(K^2) \arrow[r, "1\boxtimes"'] & K^2 \arrow[r, "\boxtimes"'] & K &
\end{tikzcd}$$

The central square in the first diagram above commutes by 2-functoriality of $\otimes$ in each variable (Proposition 1.3.6). The other unmarked quadrilaterals in the two diagrams above commute by 2-naturality of $a^{\otimes}$ (1.3.13). The pentagon labeled $\star$ commutes by the pentagon axiom for $a^{\otimes}$ (1.3.18).

This finishes the definition of a monoidal V-category. We say that a monoidal V-category is *strict* if the monoidal unitors $\ell^{\boxtimes}$ and $r^{\boxtimes}$ along with the monoidal associator $a^{\boxtimes}$ are identity V-natural transformations. $\diamond$

Explanation 1.4.7 (Interchange for a Monoidal V-category)**.** In the context of Definition 1.4.2, the composition axiom for V-functoriality of $\boxtimes$ (Definition 1.2.4) implies that, for all objects

$$X, X', Y, Y', Z, Z' \in K,$$

the following *enriched interchange* diagram in V commutes.

$$
\begin{tikzcd}
& \mathsf{K}(Y \boxtimes Y', Z \boxtimes Z') \otimes \mathsf{K}(X \boxtimes X', Y \boxtimes Y') \arrow[dr, "m"] & \\
\Big(\mathsf{K}(Y,Z) \otimes \mathsf{K}(Y',Z')\Big) \otimes \Big(\mathsf{K}(X,Y) \otimes \mathsf{K}(X',Y')\Big) \arrow[ur, "\boxtimes \otimes \boxtimes"] \arrow[d, "\xi_{\mathrm{mid}}"'] & & \mathsf{K}(X \boxtimes X', Z \boxtimes Z') \\
\Big(\mathsf{K}(Y,Z) \otimes \mathsf{K}(X,Y)\Big) \otimes \Big(\mathsf{K}(Y',Z') \otimes \mathsf{K}(X',Y')\Big) \arrow[dr, "m \otimes m"'] & & \\
& \mathsf{K}(X,Z) \otimes \mathsf{K}(X',Z') \arrow[uur, "\boxtimes"'] &
\end{tikzcd}
$$

◇

Explanation 1.4.8 (Monoidal V-Category Axiom Components)**.** Each of the axioms in Definition 1.4.2 is an equality of composites of V-natural transformations. In each case one can express the relevant equality via components. For example, the Pentagon Axiom (1.4.6) is equivalent to the following equality of components for each $((X \otimes Y) \otimes Z) \otimes W \in ((\mathsf{K}^2)\mathsf{K})\mathsf{K}$.

$$
\begin{tikzcd}
& & \begin{array}{l} \mathsf{K}\big((X \boxtimes Y) \boxtimes (Z \boxtimes W)\,,\, X \boxtimes (Y \boxtimes (Z \boxtimes W))\big) \\ \otimes \mathsf{K}\big(((X \boxtimes Y) \boxtimes Z) \boxtimes W\,,\, (X \boxtimes Y) \boxtimes (Z \boxtimes W)\big) \end{array} \arrow[dd] \\
\mathbb{1} \otimes \mathbb{1} \arrow[urr, "a^{\boxtimes} \otimes a^{\boxtimes}"] & & \\
\mathbb{1} \arrow[u, "\lambda^{-1}"'] \arrow[d, "\lambda^{-1} \circ \lambda^{-1}"] & & \mathsf{K}\big(((X \boxtimes Y) \boxtimes Z) \boxtimes W\,,\, X \boxtimes (Y \boxtimes (Z \boxtimes W))\big) \\
\mathbb{1} \otimes (\mathbb{1} \otimes \mathbb{1}) \arrow[drr, "(1 \boxtimes a^{\boxtimes}) \otimes (a^{\boxtimes} \otimes (a^{\boxtimes} \boxtimes 1))"'] & & \\
& & \begin{array}{l} \mathsf{K}\big(X \boxtimes ((Y \boxtimes Z) \boxtimes W)\,,\, X \boxtimes (Y \boxtimes (Z \boxtimes W))\big) \\ \otimes \Big(\mathsf{K}\big((X \boxtimes (Y \boxtimes Z)) \boxtimes W\,,\, X \boxtimes ((Y \boxtimes Z) \boxtimes W)\big) \\ \otimes \mathsf{K}\big(((X \boxtimes Y) \boxtimes Z) \boxtimes W\,,\, (X \boxtimes (Y \boxtimes Z)) \boxtimes W\big)\Big) \end{array} \arrow[uu]
\end{tikzcd}
$$

◇

For the definition of braided monoidal V-category, we will need a mate of $a^{\boxtimes}$ defined similarly to the mates of a pentagonator discussed just before Lemma I.6.5.2.

Definition 1.4.9. Suppose K is a monoidal V-category. We let $a_1^{\boxtimes}$ denote the mate of $a^{\boxtimes}$ given by the inverse of $a^{\otimes}$, as shown below.

$$\begin{tikzcd}
\mathsf{K}(\mathsf{K}^2) \arrow[r, "a^{-\otimes}"] \arrow[d, "1_{\mathsf{K}}\boxtimes"'] & (\mathsf{K}^2)\mathsf{K} \arrow[r, "\boxtimes 1_{\mathsf{K}}"] \arrow[d, phantom, "a_1^{\boxtimes}\Swarrow"] & \mathsf{K}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K}^2 \arrow[rr, "\boxtimes"'] & & \mathsf{K}
\end{tikzcd}$$

We let $a_1^{-\boxtimes}$ denote the inverse of $a_1^{\boxtimes}$. ⋄

Recall from Convention 1.3.28 that we assume V to be symmetric monoidal whenever using the braiding $\beta^{\otimes}$ of $(\mathsf{V}\text{-}\mathsf{Cat},\otimes)$.

Definition 1.4.10. Suppose V is a symmetric monoidal category. A *braided monoidal* V*-category* is a pair

$$(\mathsf{K},\beta^{\boxtimes})$$

consisting of the following data.

- K is a monoidal V-category as in Definition 1.4.2.
- $\beta^{\boxtimes}$ is a V-natural isomorphism

$$\begin{tikzcd}
\mathsf{K}^2 \arrow[rr, "\boxtimes"] \arrow[dr, bend right, "\beta^{\otimes}"'] & \arrow[d, phantom, "\Downarrow\beta^{\boxtimes}"] & \mathsf{K} \\
& \mathsf{K}^2 \arrow[ur, bend right, "\boxtimes"'] &
\end{tikzcd}$$

in $\mathsf{V}\text{-}\mathsf{Cat}(\mathsf{K}^2,\mathsf{K})$ called the *braiding* of K.

These data are required to satisfy the following two axioms.

Left Hexagon Axiom: The composites of the following two *left hexagon pasting diagrams* are equal.

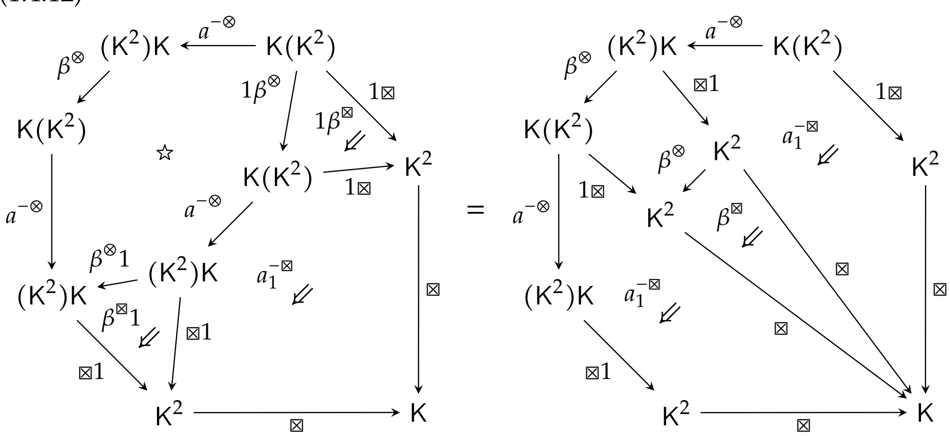
(1.4.11)

$$\begin{tikzcd}
& \mathsf{K}(\mathsf{K}^2) \arrow[dl, "\beta^{\otimes}"'] & & (\mathsf{K}^2)\mathsf{K} \arrow[ll, "a^{\otimes}"'] \arrow[dl, "\beta^{\otimes}1"'] \arrow[dr, "\boxtimes 1"] \arrow[d, phantom, "\beta^{\boxtimes}1\Swarrow"] & \\
(\mathsf{K}^2)\mathsf{K} \arrow[dd, "a^{\otimes}"'] \arrow[r, phantom, "\star"] & & (\mathsf{K}^2)\mathsf{K} \arrow[rr, "\boxtimes 1"'] \arrow[dl, "a^{\otimes}"'] & & \mathsf{K}^2 \arrow[ddd, "\boxtimes"] \\
& & & & \\
\mathsf{K}(\mathsf{K}^2) \arrow[dr, "1\boxtimes"'] & \mathsf{K}(\mathsf{K}^2) \arrow[l, "1\beta^{\otimes}"'] \arrow[d, "1\boxtimes"] \arrow[rr, phantom, "a^{\boxtimes}\Swarrow"] \arrow[dl, phantom, "1\beta^{\boxtimes}\Swarrow"] & & & \\
& \mathsf{K}^2 \arrow[rrr, "\boxtimes"'] & & & \mathsf{K}
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{K}(\mathsf{K}^2) \arrow[dl, "\beta^{\otimes}"'] \arrow[dr, "1\boxtimes"] & & (\mathsf{K}^2)\mathsf{K} \arrow[ll, "a^{\otimes}"'] \arrow[dr, "\boxtimes 1"] \arrow[d, phantom, "a^{\boxtimes}\Swarrow"] & \\
(\mathsf{K}^2)\mathsf{K} \arrow[dd, "a^{\otimes}"'] \arrow[dr, "\boxtimes 1"'] & & \mathsf{K}^2 \arrow[dl, "\beta^{\otimes}"] \arrow[dddrr, "\boxtimes"] \arrow[d, phantom, "\beta^{\boxtimes}\Swarrow"] & & \mathsf{K}^2 \arrow[ddd, "\boxtimes"] \\
& \mathsf{K}^2 \arrow[ddrrr, "\boxtimes"'] & & & \\
\mathsf{K}(\mathsf{K}^2) \arrow[dr, "1\boxtimes"'] \arrow[r, phantom, "a^{\boxtimes}\Swarrow"] & & & & \\
& \mathsf{K}^2 \arrow[rrr, "\boxtimes"'] & & & \mathsf{K}
\end{tikzcd}$$

In the diagrams above, the unlabeled quadrilateral commutes by 2-naturality of $\beta^{\otimes}$ (1.3.27) and the hexagon labeled ☆ commutes by the left hexagon axiom for $\beta^{\otimes}$ (1.3.30).

Right Hexagon Axiom: The composites of the following two *right hexagon pasting diagrams* are equal.

(1.4.12)



In the diagrams above, the unlabeled quadrilateral commutes by 2-naturality of $\beta^{\otimes}$ (1.3.27) and the hexagon labeled ☆ commutes by the right hexagon axiom for $\beta^{\otimes}$ (1.3.31). The 2-cell isomorphism $a_1^{-\boxtimes}$ is the inverse of $a_1^{\boxtimes}$ described in Definition 1.4.9.

This finishes the definition of a braided monoidal V-category. ⋄

Definition 1.4.13. Suppose V is a symmetric monoidal category. A *symmetric monoidal* V*-category* is a braided monoidal V-category

$$(\mathsf{K}, \beta^{\boxtimes})$$

that satisfies the following axiom.

Symmetry Axiom: The composites of the following two *symmetry pasting diagrams* are equal.

(1.4.14)

$$\begin{tikzcd}
\mathsf{K}^2 \arrow[rr, "\boxtimes"] \arrow[dr, "\beta^{\otimes}"'] \arrow[ddr, bend right, "1"'] \arrow[drr, phantom, "\Downarrow\beta^{\boxtimes}"] & & \mathsf{K} \\
\arrow[r, phantom, "☆"] & \mathsf{K}^2 \arrow[ur, "\boxtimes"'] \arrow[d, "\beta^{\otimes}"'] \arrow[r, phantom, "\Downarrow\beta^{\boxtimes}"] & {} \\
& \mathsf{K}^2 \arrow[uur, bend right, "\boxtimes"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{K}^2 \arrow[rr, "\boxtimes"] \arrow[dr, bend right, "1"'] & & \mathsf{K} \\
& \mathsf{K}^2 \arrow[ur, bend right, "\boxtimes"'] &
\end{tikzcd}$$

The right hand diagram above indicates the identity V-natural transformation. In the left hand diagram, the triangle labeled ☆ commutes by the symmetry axiom for $\beta^{\otimes}$ (1.3.33). When this axiom holds, $\beta^{\boxtimes}$ is also called the *symmetry* of K. ⋄

Next we give the definitions of monoidal V-functors and V-natural transformations. These require only that V is braided monoidal. We will use the following two mates.

Definition 1.4.15. Suppose K is a monoidal V-category. We let $\ell_1^{\boxtimes}$ and $r_1^{\boxtimes}$ denote the mates of $\ell^{\boxtimes}$ and $r^{\boxtimes}$ given, respectively, by using $\ell^{\otimes}$ and $r^{\otimes}$ in place of their inverses, as shown below.

(1.4.16)
$$
\begin{tikzcd}
 & \mathbb{I}\mathsf{K} \arrow[r, "\mathrm{I}^{\boxtimes}1_{\mathsf{K}}"] \arrow[dl, "\ell^{\otimes}"'] \arrow[dr, phantom, "\Downarrow \ell_1^{\boxtimes}"] & \mathsf{K}^2 \arrow[dr, "\boxtimes"] & \\
\mathsf{K} \arrow[rrr, "1_{\mathsf{K}}"'] & & & \mathsf{K}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathsf{K}\mathbb{I} \arrow[r, "1_{\mathsf{K}}\mathrm{I}^{\boxtimes}"] \arrow[dl, "r^{\otimes}"'] \arrow[dr, phantom, "\Downarrow r_1^{\boxtimes}"] & \mathsf{K}^2 \arrow[dr, "\boxtimes"] & \\
\mathsf{K} \arrow[rrr, "1_{\mathsf{K}}"'] & & & \mathsf{K}
\end{tikzcd}
$$

◇

Definition 1.4.17. Suppose K and L are monoidal V-categories. A *monoidal* V-*functor*

$$(F, F^2, F^0) : \mathsf{K} \longrightarrow \mathsf{L}$$

consists of

- a V-functor $F : \mathsf{K} \longrightarrow \mathsf{L}$;
- a V-natural transformation

$$
\begin{tikzcd}
\mathsf{K}^2 \arrow[r, "F \otimes F"] \arrow[d, "\boxtimes"'] \arrow[dr, phantom, "F^2 \swArrow"] & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K} \arrow[r, "F"] & \mathsf{L}
\end{tikzcd}
$$

  called the *monoidal constraint*; and
- a V-natural transformation

$$
\begin{tikzcd}
 & \mathbb{I} \arrow[dl, "\mathrm{I}^{\boxtimes}"'] \arrow[dr, "\mathrm{I}^{\boxtimes}"] \arrow[d, phantom, "F^0 \swArrow"] & \\
\mathsf{K} \arrow[rr, "F"] & {} & \mathsf{L}
\end{tikzcd}
$$

  called the *unit constraint*.

These data are required to satisfy the following associativity and unity axioms.

Associativity: The composites of the following two *associativity pasting diagrams* are equal.

$$
\begin{tikzcd}
 & (\mathsf{K}^2)\mathsf{K} \arrow[r, "(FF)F"] \arrow[dl, "a^{\otimes}"'] \arrow[dr, "\boxtimes 1"] & (\mathsf{L})^2\mathsf{L} \arrow[dr, "\boxtimes 1"] \arrow[d, phantom, "F^2 1 \swArrow"] & \\
\mathsf{K}(\mathsf{K}^2) \arrow[d, "1\boxtimes"'] \arrow[dr, phantom, "a^{\boxtimes} \swArrow"] & & \mathsf{K}^2 \arrow[r, "FF"] \arrow[d, "\boxtimes"] \arrow[dr, phantom, "F^2 \swArrow"] & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K}^2 \arrow[rr, "\boxtimes"'] & & \mathsf{K} \arrow[r, "F"'] & \mathsf{L}
\end{tikzcd}
=
\begin{tikzcd}
 & (\mathsf{K}^2)\mathsf{K} \arrow[r, "(FF)F"] \arrow[dl, "a^{\otimes}"'] & (\mathsf{L})^2\mathsf{L} \arrow[dr, "\boxtimes 1"] \arrow[dl, "a^{\otimes}"] & \\
\mathsf{K}(\mathsf{K}^2) \arrow[r, "F(FF)"'] \arrow[d, "1\boxtimes"'] \arrow[dr, phantom, "1F^2 \swArrow"] & \mathsf{L}(\mathsf{L}^2) \arrow[d, "1\boxtimes"] \arrow[r, phantom, "a^{\boxtimes} \swArrow"] & {} & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K}^2 \arrow[r, "FF"] \arrow[dr, "\boxtimes"'] & \mathsf{L}^2 \arrow[rr, "\boxtimes"] \arrow[d, phantom, "F^2 \swArrow"] & & \mathsf{L} \\
 & \mathsf{K} \arrow[urr, "F"'] & &
\end{tikzcd}
$$

In the right hand diagram above, the unlabeled parallelogram commutes by naturality of $a^{\otimes}$.

Left Unity: The composites of the following two *left unity pasting diagrams* are equal.

$$
\begin{tikzcd}
\mathbb{I}\mathsf{K} \arrow[rr, "\mathrm{I}^{\boxtimes}1"] \arrow[dr, "\mathrm{I}^{\boxtimes}1"'] \arrow[dd, "\ell^{\otimes}"'] \arrow[drr, phantom, "F^0 1\Downarrow"] & & \mathsf{LK} \arrow[r, "1F"] & \mathsf{L}^2 \arrow[dd, "\boxtimes"] \arrow[ddll, phantom, "F^2 \Swarrow"] \\
& \mathsf{K}^2 \arrow[ur, "F1"'] \arrow[dl, "\boxtimes"] \arrow[l, phantom, "\overset{\ell_1^{\boxtimes}}{\Leftarrow}"] & & \\
\mathsf{K} \arrow[rrr, "F"] & & & \mathsf{L}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathbb{I}\mathsf{K} \arrow[rr, "\mathrm{I}^{\boxtimes}1"] \arrow[drr, "1F"'] \arrow[dd, "\ell^{\otimes}"'] & & \mathsf{LK} \arrow[r, "1F"] & \mathsf{L}^2 \arrow[dd, "\boxtimes"] \\
& & \mathbb{I}\mathsf{L} \arrow[ur, "\mathrm{I}^{\boxtimes}1"] \arrow[dr, "\ell^{\otimes}"'] \arrow[r, phantom, "\ell_1^{\boxtimes} \Swarrow"] & \\
\mathsf{K} \arrow[rrr, "F"] & & & \mathsf{L}
\end{tikzcd}
$$

In the right hand diagram above, the lower unlabeled quadrilateral commutes by naturality of $\ell^{\otimes}$ and the upper unlabeled region commutes by 2-functorality of $\otimes$. The 2-cell isomorphisms labeled $\ell_1^{\boxtimes}$ are each the mate of $\ell^{\boxtimes}$ described in Definition 1.4.15.

Right Unity: The composites of the following two *right unity pasting diagrams* are equal.

$$
\begin{tikzcd}
\mathsf{K}\mathbb{I} \arrow[rr, "1\mathrm{I}^{\boxtimes}"] \arrow[dr, "1\mathrm{I}^{\boxtimes}"'] \arrow[dd, "r^{\otimes}"'] \arrow[drr, phantom, "1F^0\Downarrow"] & & \mathsf{KL} \arrow[r, "F1"] & \mathsf{L}^2 \arrow[dd, "\boxtimes"] \arrow[ddll, phantom, "F^2 \Swarrow"] \\
& \mathsf{K}^2 \arrow[ur, "1F"'] \arrow[dl, "\boxtimes"] \arrow[l, phantom, "\overset{r_1^{\boxtimes}}{\Leftarrow}"] & & \\
\mathsf{K} \arrow[rrr, "F"] & & & \mathsf{L}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathsf{K}\mathbb{I} \arrow[rr, "1\mathrm{I}^{\boxtimes}"] \arrow[drr, "F1"'] \arrow[dd, "r^{\otimes}"'] & & \mathsf{KL} \arrow[r, "F1"] & \mathsf{L}^2 \arrow[dd, "\boxtimes"] \\
& & \mathsf{L}\mathbb{I} \arrow[ur, "1\mathrm{I}^{\boxtimes}"] \arrow[dr, "r^{\otimes}"'] \arrow[r, phantom, "r_1^{\boxtimes} \Swarrow"] & \\
\mathsf{K} \arrow[rrr, "F"] & & & \mathsf{L}
\end{tikzcd}
$$

In the right hand diagram above, the lower unlabeled quadrilateral commutes by naturality of $r^{\otimes}$ and the upper unlabeled region commutes by 2-functorality of $\otimes$. The 2-cell isomorphisms labeled $r_1^{\boxtimes}$ are each the mate of $r^{\boxtimes}$ described in Definition 1.4.15.

This finishes the definition of a monoidal V-functor. Moreover, we have the following additional variants.

- A *unital monoidal* V*-functor* is one for which F^0 is invertible.
- A *strictly unital monoidal* V*-functor* is one for which F^0 is an identity.
- A *strong monoidal* V*-functor* is one for which both F^0 and F^2 are invertible.
- A *strict monoidal* V*-functor* is one for which both F^0 and F^2 are identities.

$\diamond$

Definition 1.4.18. Suppose K and L are braided monoidal V-categories, with V symmetric monoidal. A *braided monoidal* V-functor

$$(F, F^2, F^0) : \mathsf{K} \longrightarrow \mathsf{L}$$

is a monoidal V-functor that satisfies the following axiom.

Braid Axiom: The composites of the following two *braiding pasting diagrams* are equal.

$$\begin{tikzcd}
& \mathsf{K}^2 \arrow[r, "FF"] \arrow[dl, "\beta^{\otimes}"'] & \mathsf{L}^2 \arrow[dl, "\beta^{\otimes}"'] \arrow[dd, "\boxtimes"] \\
\mathsf{K}^2 \arrow[r, "FF"] \arrow[dr, "\boxtimes"'] & \mathsf{L}^2 \arrow[dr, "\boxtimes"] \arrow[r, phantom, "\overset{\beta^{\boxtimes}}{\Leftarrow}"] \arrow[d, phantom, "F^2 \Swarrow"'] & {} \\
& \mathsf{K} \arrow[r, "F"'] & \mathsf{L}
\end{tikzcd}
=
\begin{tikzcd}
& \mathsf{K}^2 \arrow[r, "FF"] \arrow[dl, "\beta^{\otimes}"'] \arrow[dd, "\boxtimes"] \arrow[ddr, phantom, "F^2 \Swarrow"] & \mathsf{L}^2 \arrow[dd, "\boxtimes"] \\
\mathsf{K}^2 \arrow[dr, "\boxtimes"'] \arrow[r, phantom, "\overset{\beta^{\boxtimes}}{\Leftarrow}"] & {} & \\
& \mathsf{K} \arrow[r, "F"'] & \mathsf{L}
\end{tikzcd}$$

In the left hand diagram above, the unlabeled quadrilateral commutes by naturality of $\beta^{\otimes}$. This finishes the definition of a braided monoidal V-functor.

If K and L are symmetric monoidal V-categories then we say that F is a *symmetric monoidal* V*-functor*. ◇

Definition 1.4.19. For a pair of composable monoidal V-functors

$$\mathsf{K} \xrightarrow{F} \mathsf{L} \xrightarrow{P} \mathsf{M},$$

the *composite* PF is a monoidal V-functor with monoidal and unit constraints given, respectively, by the composites of the following pasting diagrams.

$$\begin{tikzcd}
\mathsf{K}^2 \arrow[r, "F \otimes F"] \arrow[d, "\boxtimes"'] \arrow[dr, phantom, "F^2 \Swarrow"] & \mathsf{L}^2 \arrow[r, "P \otimes P"] \arrow[d, "\boxtimes"] \arrow[dr, phantom, "P^2 \Swarrow"] & \mathsf{M}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K} \arrow[r, "F"'] & \mathsf{L} \arrow[r, "P"'] & \mathsf{M}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathbb{I} \arrow[dl, "I^{\boxtimes}"'] \arrow[dr, "I^{\boxtimes}"] \arrow[dd, "I^{\boxtimes}"] & \\
\mathsf{K} \arrow[dr, "F"'] \arrow[r, phantom, "\overset{F^0}{\Leftarrow}"] & {} \arrow[r, phantom, "\overset{P^0}{\Leftarrow}"] & \mathsf{M} \\
& \mathsf{L} \arrow[ur, "P"'] &
\end{tikzcd}$$

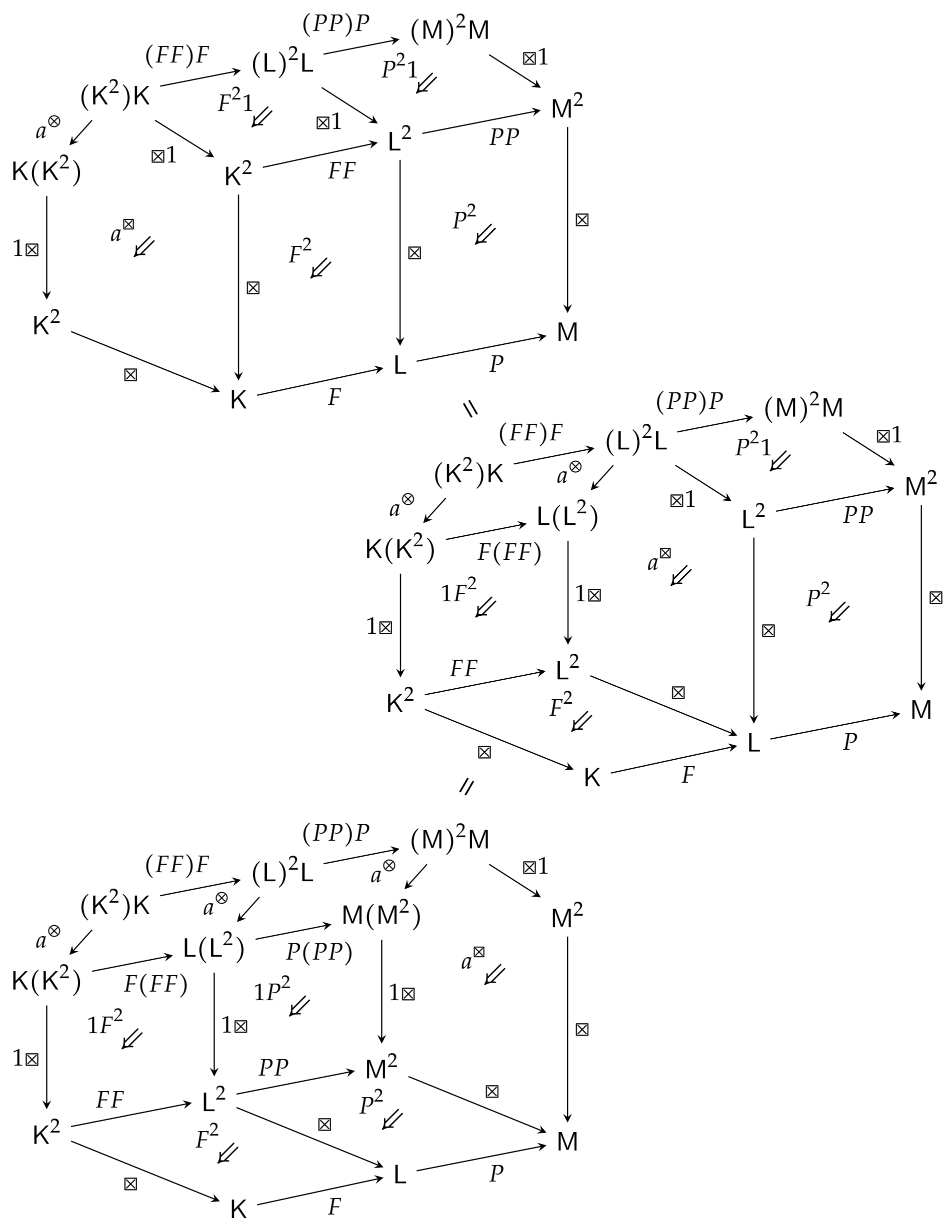
In Proposition 1.4.21 below we verify that (braided) monoidal V-functors are closed under composition. ◇

Lemma 1.4.20. *Suppose* K *is a monoidal* V*-category. The identity* V*-functor* 1_{K} *is monoidal as a* V*-functor. If* K *is braided monoidal (with* V *symmetric) then so is* 1_{K}*.*

Proof. We take the monoidal and unit constraints of 1_{K} to be identity V-natural transformations. Then the associativity and unity axioms of Definition 1.4.17 are tautologies because horizontal and vertical composition of V-natural transformations are strictly unital. The same holds for the braid axiom of Definition 1.4.18 if K is braided monoidal. □

Proposition 1.4.21. *In the context of Definition 1.4.19, the composite* PF *is a monoidal* V*-functor. If* F *and* P *are braided monoidal* V*-functors (with* V *symmetric monoidal), then so is* PF*. Moreover, composition of monoidal* V*-functors is strictly associative and unital.*

Proof. We verify the associativity axiom of Definition 1.4.17 via the equalities of pasting diagrams below; the unity and (in the braided case) braid axioms are similar.



The equalities above follow from the associativity axioms for F and P. This proves the associativity axiom for PF because, by 2-functoriality of $\otimes$ (Proposition 1.3.6), we have the following:

$$\begin{tikzcd}
(\mathsf{K}^2\mathsf{K}) \arrow[r, "(FF)F"] \arrow[d, "\boxtimes 1"'] \arrow[dr, phantom, "F^2 1 \swarrow"] & (\mathsf{L}^2)\mathsf{L} \arrow[r, "(PP)P"] \arrow[d, "\boxtimes 1"'] \arrow[dr, phantom, "P^2 1 \swarrow"] & (\mathsf{M}^2)\mathsf{M} \arrow[d, "\boxtimes 1"] \\
\mathsf{K}^2 \arrow[r, "FF"'] & \mathsf{L}^2 \arrow[r, "PP"'] & \mathsf{M}^2
\end{tikzcd}
=
\begin{tikzcd}
(\mathsf{K}^2\mathsf{K}) \arrow[r, "(FF)F"] \arrow[d, "\boxtimes 1"'] \arrow[drr, phantom, "(PF)^2 1 \swarrow"] & (\mathsf{L}^2)\mathsf{L} \arrow[r, "(PP)P"] & (\mathsf{M}^2)\mathsf{M} \arrow[d, "\boxtimes 1"] \\
\mathsf{K}^2 \arrow[r, "FF"'] & \mathsf{L}^2 \arrow[r, "PP"'] & \mathsf{M}^2
\end{tikzcd}$$

along with a similar equality for $1(PF)^2$.

The assertion that composition of (braided) monoidal V-functors is strictly associative and unital follows from the corresponding statements for underlying V-functors together with strictness of vertical and horizontal composition for V-natural transformations. For example, strictly unital composition implies that the monoidal and unit constraints of $1_{\mathsf{L}} \circ F$ and $F1_{\mathsf{K}}$ are equal to those of F. Similarly, the monoidal and unit constraints of a three-fold composite are given by the three-fold pasting of monoidal and unit constraints. □

Definition 1.4.22. Suppose K and L are monoidal V-categories, and suppose that $F, G : \mathsf{K} \longrightarrow \mathsf{L}$ are monoidal V-functors. A *monoidal* V*-natural transformation*

$$\theta : F \longrightarrow G$$

is a V-natural transformation of underlying V-functors that satisfies the following two additional axioms.

Monoidal Naturality: The composites of the following two *monoidal naturality pasting diagrams* are equal.

$$\begin{tikzcd}
\mathsf{K}^2 \arrow[r, bend left, "F \otimes F"] \arrow[d, "\boxtimes"'] \arrow[dr, phantom, "F^2 \swarrow"] & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\theta\Downarrow"] & \mathsf{L}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{K}^2 \arrow[r, bend left, "F \otimes F"] \arrow[r, bend right, "G \otimes G"'] \arrow[r, phantom, "\theta\otimes\theta\Downarrow"] \arrow[d, "\boxtimes"'] \arrow[dr, phantom, "G^2 \swarrow"] & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K} \arrow[r, bend right, "G"'] & \mathsf{L}
\end{tikzcd}$$

Unit Naturality: The composites of the following two *unit naturality pasting diagrams* are equal.

$$\begin{tikzcd}
& \mathbb{I} \arrow[dl, "I^{\boxtimes}"'] \arrow[dr, "I^{\boxtimes}"] \arrow[d, phantom, "F^0 \swarrow"] & \\
\mathsf{K} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\theta\Downarrow"] & & \mathsf{L}
\end{tikzcd}
=
\begin{tikzcd}
& \mathbb{I} \arrow[dl, "I^{\boxtimes}"'] \arrow[dr, "I^{\boxtimes}"] \arrow[d, phantom, "G^0 \swarrow"] & \\
\mathsf{K} \arrow[rr, bend right, "G"'] & & \mathsf{L}
\end{tikzcd}$$

This finishes the definition of a monoidal V-natural transformation. Verification that monoidal V-natural transformations are closed under vertical and horizontal composition is given in Proposition 1.4.24 below. Then identities and composites of monoidal V-natural transformations are defined via underlying V-natural transformations. ⋄

Explanation 1.4.23 (Components of monoidal V-natural transformations)**.** Recalling the definitions of horizontal and vertical composition for V-natural transformations (Definition 1.2.9), the monoidal naturality axiom of Definition 1.4.22 is equivalent to commutativity of the following diagram in V for each pair of objects X and Y in K.

$$\begin{tikzcd}[column sep=tiny]
& \mathsf{L}\big(F(X \boxtimes Y), G(X \boxtimes Y)\big) \otimes \mathsf{L}\big(FX \boxtimes FY, F(X \boxtimes Y)\big) \arrow[dr, "m"] & \\
\mathbb{1} \otimes \mathbb{1} \arrow[ur, "\theta_{X \boxtimes Y} \otimes F^2_{X,Y}"] \arrow[d, "G^2_{X,Y} \otimes (\theta \otimes \theta)_{X \otimes Y}"'] & & \mathsf{L}\big(FX \boxtimes FY, G(X \boxtimes Y)\big) \\
\mathsf{L}\big(GX \boxtimes GY, G(X \boxtimes Y)\big) \otimes \mathsf{L}\big(FX, GX\big) \otimes \mathsf{L}\big(FY, GY\big) \arrow[r, "1 \otimes \boxtimes"'] & \mathsf{L}\big(GX \boxtimes GY, G(X \boxtimes Y)\big) \otimes \mathsf{L}\big(FX \boxtimes FY, GX \boxtimes GY\big) \arrow[ur, "m"'] &
\end{tikzcd}$$

In the diagram above, $(\theta \otimes \theta)_{X \otimes Y}$ is the component of

$$\theta \otimes \theta : F \otimes F \longrightarrow G \otimes G$$

at $X \otimes Y$. Following Definition 1.3.5, it is defined to be

$$\mathbb{1} \xrightarrow{\lambda^{-1}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\theta_X \otimes \theta_Y} \mathsf{L}\big(FX, GX\big) \otimes \mathsf{L}\big(FY, GY\big)$$

Similarly, the unit naturality axiom of Definition 1.4.22 is equivalent to commutativity of the following diagram in V, with $\mathrm{I} = \mathrm{I}^{\boxtimes}$ denoting the unit objects of K and L.

$$\begin{tikzcd}
& & \mathsf{L}(F\mathrm{I}, G\mathrm{I}) \otimes \mathsf{L}(\mathrm{I}, F\mathrm{I}) \arrow[dr, "m"] & \\
& \mathbb{1} \otimes \mathbb{1} \arrow[ur, "\theta_{\mathrm{I}} \otimes F^0"] & & \\
\mathbb{1} \arrow[ur, "\lambda^{-1}"] \arrow[rrr, "G^0"] & & & \mathsf{L}(\mathrm{I}, G\mathrm{I})
\end{tikzcd}$$

◇

Proposition 1.4.24. *Monoidal* V*-natural transformations are closed under vertical and horizontal composition of underlying* V*-natural transformations.*

Proof. This result follows from 2-functoriality of $\otimes$ (Proposition 1.3.6). We will demonstrate for vertical composition; horizontal composition is similar. Suppose that

$$F, G, H : \mathsf{K} \longrightarrow \mathsf{L}$$

are monoidal V-functors between monoidal V-categories. Suppose that

$$\theta : F \longrightarrow G \quad \text{and} \quad \phi : G \longrightarrow H$$

are monoidal V-natural transformations. Then monoidal naturality of θ and ϕ gives the first two equalities of pasting diagrams below. The third follows by 2-functoriality of $\otimes$; we write $\nu = \phi\theta$ to conserve space.

$$\begin{tikzcd}
\mathsf{K}^2 \arrow[r, bend left, "F\otimes F"] \arrow[d, "\boxtimes"'] \arrow[dr, phantom, "F^2 \Swarrow"] & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K} \arrow[r, bend left, "F"] \arrow[r, "G"'] \arrow[r, bend right, "H"'] \arrow[r, phantom, shift left=2, "\Downarrow\theta"] \arrow[r, phantom, shift right=2, "\Downarrow\phi"] & \mathsf{L}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{K}^2 \arrow[r, bend left, "F\otimes F"] \arrow[r, "G\otimes G"'] \arrow[r, phantom, shift left=2, "\theta\otimes\theta\Downarrow"] \arrow[d, "\boxtimes"'] \arrow[dr, phantom, "G^2 \Swarrow"] & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K} \arrow[r, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, shift right=2, "\Downarrow\phi"] & \mathsf{L}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{K}^2 \arrow[r, bend left, "F\otimes F"] \arrow[r, "G\otimes G"'] \arrow[r, bend right, "H\otimes H"'] \arrow[r, phantom, shift left=2, "\theta\otimes\theta\Downarrow"] \arrow[r, phantom, shift right=2, "\phi\otimes\phi\Downarrow"] \arrow[d, "\boxtimes"'] \arrow[dr, phantom, "H^2 \Swarrow"] & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K} \arrow[r, bend right, "H"'] & \mathsf{L}
\end{tikzcd}
=
\begin{tikzcd}
\mathsf{K}^2 \arrow[r, bend left, "F\otimes F"] \arrow[r, bend right, "H\otimes H"'] \arrow[r, phantom, "\nu\otimes\nu\Downarrow"] \arrow[d, "\boxtimes"'] \arrow[dr, phantom, "H^2 \Swarrow"] & \mathsf{L}^2 \arrow[d, "\boxtimes"] \\
\mathsf{K} \arrow[r, bend right, "H"'] & \mathsf{L}
\end{tikzcd}$$

The equalities above show that $\phi\theta$ satisfies the monoidal naturality condition of Definition 1.4.22. The unit naturality condition is similar. $\square$

The results of Propositions 1.4.21 and 1.4.24 show that (braided) monoidal V-functors and monoidal V-natural transformations satisfy the composition laws of a 2-category.

Definition 1.4.25. Suppose $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is a braided monoidal category. For items (2) and (3) suppose that V is symmetric. We define the following 2-categories of small monoidal V-categories.

(1) Let V-MCat denote the 2-category whose objects are small monoidal V-categories, 1-cells are monoidal V-functors, and 2-cells are monoidal V-natural transformations.
(2) Let V-BMCat denote the 2-category whose objects are small braided monoidal V-categories, 1-cells are braided monoidal V-functors, and 2-cells are monoidal V-natural transformations.
(3) Let V-SMCat denote the 2-category whose objects are small symmetric monoidal V-categories, 1-cells are symmetric monoidal V-functors, and 2-cells are monoidal V-natural transformations.

These are related by 2-functors where the first recognizes symmetric structure as a special case of the braided analogue, the second forgets the braiding data, and the third forgets all monoidal data:

$$\text{V-SMCat} \longrightarrow \text{V-BMCat} \longrightarrow \text{V-MCat} \longrightarrow \text{V-Cat}. \qquad \diamond$$

Definition 1.4.26. A *monoidal* V*-adjunction* between monoidal V-categories K and L consists of a V-adjunction

$$\begin{tikzcd}
\mathsf{K} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{L} \arrow[l, bend left, "G"]
\end{tikzcd}$$

such that F and G are monoidal V-functors and the unit and counit are monoidal V-natural transformations. If K and L are small, this is an internal adjunction in V-MCat. A *monoidal adjoint* V*-equivalence* consists of a monoidal V-adjunction such that the unit and counit are monoidal V-natural isomorphisms. For V symmetric monoidal, a monoidal V-adjunction or V-equivalence is called *braided*, respectively *symmetric*, if F and G are braided, respectively symmetric, monoidal as V-functors. ◇

Next we describe a rotated multiplication on the opposite of a monoidal V-category. Recall from Definition 1.3.36 the V-functor

$$\mathsf{C}^{\mathsf{op}} \otimes \mathsf{D}^{\mathsf{op}} \xrightarrow{\gamma} (\mathsf{D} \otimes \mathsf{C})^{\mathsf{op}}$$

for V-categories C and D. Recall from Proposition 1.2.19 that $(-)^{\mathsf{op}}$ is a 2-functor on V-Cat that is covariant with respect to horizontal composition of 1- and 2-cells, but contravariant with respect to vertical composition of 2-cells.

Example 1.4.27 (Rotated Multiplication)**.** Suppose $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is a braided monoidal category and suppose $\mathsf{K} = (\mathsf{K}, \boxtimes, \mathrm{I}, a, \ell, r)$ is a monoidal V-category. The *rotation of* K is defined by the following data.

- The underlying V-category is the opposite, $\overline{\mathsf{K}} = \mathsf{K}^{\mathsf{op}}$.
- The monoidal composition

$$\overline{\boxtimes} : \overline{\mathsf{K}} \otimes \overline{\mathsf{K}} \longrightarrow \overline{\mathsf{K}}$$

  is defined to be the composite

$$\mathsf{K}^{\mathsf{op}} \otimes \mathsf{K}^{\mathsf{op}} \xrightarrow{\gamma} (\mathsf{K} \otimes \mathsf{K})^{\mathsf{op}} \xrightarrow{\boxtimes^{\mathsf{op}}} \mathsf{K}^{\mathsf{op}}$$

  using γ (Definition 1.3.36) and the opposite V-functor of $\boxtimes$. Therefore, for objects $X, X', Y, Y' \in \mathsf{K}$ we have $X \overline{\boxtimes} Y = Y \boxtimes X$. The morphism on hom objects is given by the following composite of ξ and $\boxtimes$.

$$\begin{array}{ccccc} \overline{\mathsf{K}}(X,X') \otimes \overline{\mathsf{K}}(Y,Y') & & & & \overline{\mathsf{K}}(X \overline{\boxtimes} Y, X' \overline{\boxtimes} Y') \\ \| & & & & \| \\ \mathsf{K}(X',X) \otimes \mathsf{K}(Y',Y) & \xrightarrow{\xi} & \mathsf{K}(Y',Y) \otimes \mathsf{K}(X',X) & \xrightarrow{\boxtimes} & \mathsf{K}(Y' \boxtimes X', Y \boxtimes X) \end{array}$$

- The monoidal identity is

$$\overline{\mathrm{I}} = \mathrm{I}^{\mathsf{op}} : \mathbb{I} = \mathbb{I}^{\mathsf{op}} \longrightarrow \mathsf{K}^{\mathsf{op}}.$$

- The monoidal unitors are given by

$$\overline{\ell} = r^{-1} : \mathbb{1} \longrightarrow \mathsf{K}(X, X \boxtimes \mathrm{I}) = \overline{\mathsf{K}}(\mathrm{I} \overline{\boxtimes} X, X)$$

  and

$$\overline{r} = \ell^{-1} : \mathbb{1} \longrightarrow \mathsf{K}(X, \mathrm{I} \boxtimes X) = \overline{\mathsf{K}}(X \overline{\boxtimes} \mathrm{I}, X)$$

  for $X \in \mathsf{K}$.
- The monoidal associator component at $X, Y, Z \in \overline{\mathsf{K}}$ is given by

$$\overline{a}_{X,Y,Z} = a_{Z,Y,X} : \mathbb{1} \longrightarrow \mathsf{K}\big((Z \boxtimes Y) \boxtimes X, Z \boxtimes (Y \boxtimes X)\big) = \overline{\mathsf{K}}\big((X \overline{\boxtimes} Y) \overline{\boxtimes} Z, X \overline{\boxtimes} (Y \overline{\boxtimes} Z)\big)$$

  for $X, Y, Z \in \mathsf{K}$.

To verify that the unitors and associator have the required sources and targets, we use Lemma 1.3.39 to see that the unlabeled regions in each of the following diagrams commute. Recall from Proposition 1.2.19 that $(-)^{\mathsf{op}}$ reverses directions of V-natural transformations. The components of $\overline{r}$, $\overline{\ell}$, and $\overline{a}$ given above are those of the V-natural transformations indicated below.

(1.4.28)

$$
\begin{tikzcd}
\overline{\mathsf{K}}\mathbb{I} \arrow[rr, "1_{\overline{\mathsf{K}}}\mathrm{I}"] \arrow[dr, "\gamma"] & & \overline{\mathsf{K}}^2 \arrow[d, "\gamma"] & \\
& (\mathbb{I}\mathsf{K})^{\mathsf{op}} \arrow[r, "(\mathrm{I}1_{\mathsf{K}})^{\mathsf{op}}"] \arrow[dr, phantom, "\Downarrow(\ell^{\mathsf{op}})^{-1}"] & (\mathsf{K}^2)^{\mathsf{op}} \arrow[dr, "\boxtimes^{\mathsf{op}}"] & \\
\overline{\mathsf{K}} \arrow[uu, "r_{\overline{\mathsf{K}}}^{-\otimes}"] \arrow[ur, "(\ell_{\mathsf{K}}^{-\otimes})^{\mathsf{op}}"] \arrow[rrr, "(1_{\mathsf{K}})^{\mathsf{op}} = 1_{\overline{\mathsf{K}}}"'] & & & \overline{\mathsf{K}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{I}\overline{\mathsf{K}} \arrow[rr, "\mathrm{I}1_{\overline{\mathsf{K}}}"] \arrow[dr, "\gamma"] & & \overline{\mathsf{K}}^2 \arrow[d, "\gamma"] & \\
& (\mathsf{K}\mathbb{I})^{\mathsf{op}} \arrow[r, "(1_{\mathsf{K}}\mathrm{I})^{\mathsf{op}}"] \arrow[dr, phantom, "\Downarrow(r^{\mathsf{op}})^{-1}"] & (\mathsf{K}^2)^{\mathsf{op}} \arrow[dr, "\boxtimes^{\mathsf{op}}"] & \\
\overline{\mathsf{K}} \arrow[uu, "\ell_{\overline{\mathsf{K}}}^{-\otimes}"] \arrow[ur, "(r_{\mathsf{K}}^{-\otimes})^{\mathsf{op}}"] \arrow[rrr, "(1_{\mathsf{K}})^{\mathsf{op}} = 1_{\overline{\mathsf{K}}}"'] & & & \overline{\mathsf{K}}
\end{tikzcd}
$$

(1.4.29)

$$
\begin{tikzcd}
& \overline{\mathsf{K}}\,\overline{\mathsf{K}}^2 \arrow[r, "1\gamma"] & \overline{\mathsf{K}}(\mathsf{K}^2)^{\mathsf{op}} \arrow[r, "1\boxtimes^{\mathsf{op}}"] \arrow[dd, "\gamma"] & \overline{\mathsf{K}}^2 \arrow[dd, "\gamma"] \\
\overline{\mathsf{K}}^2\,\overline{\mathsf{K}} \arrow[ur, "a^{\otimes}"] \arrow[dd, "\gamma1"'] & & & \\
& & ((\mathsf{K}^2)\mathsf{K})^{\mathsf{op}} \arrow[r, "(\boxtimes 1_{\mathsf{K}})^{\mathsf{op}}"] \arrow[dl, "(a^{\otimes})^{\mathsf{op}}"'] & (\mathsf{K}^2)^{\mathsf{op}} \arrow[dd, "\boxtimes^{\mathsf{op}}"] \\
(\mathsf{K}^2)^{\mathsf{op}}\,\overline{\mathsf{K}} \arrow[r, "\gamma"] \arrow[d, "\boxtimes^{\mathsf{op}}1"'] & (\mathsf{K}(\mathsf{K}^2))^{\mathsf{op}} \arrow[d, "(1_{\mathsf{K}}\boxtimes)^{\mathsf{op}}"'] & \arrow[ur, phantom, "\Nearrow(a^{\boxtimes})^{\mathsf{op}}"] & \\
\overline{\mathsf{K}}^2 \arrow[r, "\gamma"] & (\mathsf{K}^2)^{\mathsf{op}} \arrow[rr, "\boxtimes^{\mathsf{op}}"] & & \overline{\mathsf{K}}
\end{tikzcd}
$$

One can use either of two approaches to verify that these data satisfy the unity and associativity axioms of Definition 1.4.2. In one approach, use (2-) naturality of γ to verify that the relevant pasting diagrams reduce to whiskerings of those for K. In the other, consider components as in Explanation 1.4.8 to see that the required equalities for $\overline{\mathsf{K}}$ reduce to corresponding equalities for K. For example, the pentagon axiom for $\bar{a}$ is, for each quadruple of objects $X, Y, Z, W \in \overline{\mathsf{K}}$, an equality of components

$$
\begin{aligned}
\mathbb{1} \longrightarrow\ & \overline{\mathsf{K}}\Big(\big((X \,\overline{\boxtimes}\, Y) \,\overline{\boxtimes}\, Z\big) \,\overline{\boxtimes}\, W,\ X \,\overline{\boxtimes}\, \big(Y \,\overline{\boxtimes}\, (Z \,\overline{\boxtimes}\, W)\big)\Big) \\
&= \mathsf{K}\Big(\big((W \boxtimes Z) \boxtimes Y\big) \boxtimes X,\ W \boxtimes \big(Z \boxtimes (Y \boxtimes X)\big)\Big)
\end{aligned}
$$

The required equality follows from the corresponding equality for a at the quadruple W, Z, Y, X. Similarly, the unity axiom for $\overline{\mathsf{K}}$ at $X, Y \in \overline{\mathsf{K}}$ holds by the unity axiom for K at Y, X. ◇

1.5. Cat-Monoidal 2-Categories

The case $\mathsf{V} = \mathsf{Cat}$, the category of small categories, warrants special attention because the definition of monoidal Cat-category gives an important notion of strict monoidal bicategory. Since the notion of (locally small) 2-category is the same as that of a Cat-enriched category, and each 2-category is a bicategory, the term "monoidal 2-category" might be understood to mean either of the following:

- a monoidal V-category in the case $\mathsf{V} = \mathsf{Cat}$, or
- a monoidal bicategory whose underlying bicategory is a 2-category.

These are distinct notions and therefore to avoid ambiguity we introduce the following terms for the Cat-enriched case.

Definition 1.5.1. Suppose $(\mathsf{V}, \otimes, \mathbb{1}) = (\mathsf{Cat}, \times, \mathbf{1})$ in the context of Definitions 1.4.2, 1.4.10, and 1.4.13. We introduce the following special terminology.

- We say that K is a Cat*-monoidal 2-category* if K is a monoidal Cat-category in the sense of Definition 1.4.2.
- We say that K is a *braided* Cat*-monoidal 2-category* if K is a braided monoidal Cat-category in the sense of Definition 1.4.10.
- We say that K is a *symmetric* Cat*-monoidal 2-category* if K is a symmetric monoidal Cat-category in the sense of Definition 1.4.13. ◇

Next we relate the definition of Cat-monoidal 2-category, along with the braided and symmetric variants, to the 1-categorical analogues.

Explanation 1.5.2 (Relation to Monoidal 1-Categories)**.** Suppose that $(\mathsf{K}, \otimes, \mathbb{I}, a, \ell, r)$ is a Cat-monoidal 2-category.

(1) The underlying 1-categorical data of $(\mathsf{K}, \otimes, \mathbb{I}, a, \ell, r)$ form a monoidal 1-category. The axioms for a monoidal 1-category are obtained by taking components of the diagrams (1.4.5) and (1.4.6) above.
(2) In particular, the components of a, ℓ, and r satisfy the equality

$$\ell_{\mathbb{I}} = r_{\mathbb{I}} : \mathbb{I} \otimes \mathbb{I} \longrightarrow \mathbb{I}$$

from (1.1.4) along with the left and right unity diagrams of (1.1.5).
(3) If (K, β) is braided Cat-monoidal, then the underlying 1-category is braided monoidal.
(4) If (K, β) is symmetric Cat-monoidal, then the underlying 1-category is symmetric monoidal. ◇

Next we relate the definition of Cat-monoidal 2-category, along with the braided and symmetric variants, to the definitions of plain, braided, and symmetric monoidal bicategory.

Explanation 1.5.3 (Relation to Monoidal Bicategories)**.** Suppose K is a Cat-monoidal 2-category.

(1) The data of K are those of a monoidal bicategory whose base bicategory is a 2-category and for which the pentagonator π and 2-unitors λ, μ, and ρ are all identities. Commutativity of the pentagon diagram (1.4.6) is equivalent to triviality of π; commutativity of the middle unity diagram (1.4.5) is equivalent to triviality of μ; and the triviality of λ, respectively ρ, is equvialent to commutativity of the left, respectively right, unity diagram from (1.1.5). The non-Abelian 4-cocycle condition (I.6.4.2) along with the

two normalization axioms (I.6.4.3) and (I.6.4.4) then consist entirely of identity 2-cells because $\otimes$ is 2-functorial and a, ℓ, and r are 2-natural.

(2) If, moreover, K is braided Cat-monoidal, then the corresponding bicategory is braided monoidal with hexagonators $R_{-|--}$ and $R_{--|-}$ being identities. The four axioms of Definition I.6.5.3 are equalities of identity 2-cells.

(3) If, moreover, K is symmetric Cat-monoidal, then the corresponding bicategory is symmetric monoidal with syllepsis ν being an identity. The two syllepsis axioms of Definition I.6.5.7 as well as the triple braid axiom of Definition I.6.5.9 are equalities of identity 2-cells. ◇

Example 1.5.4. Equipped with the Cartesian product, $(\mathsf{Cat}, \times, \mathbf{1})$ is a symmetric Cat-monoidal 2-category. The associator, unitors, and braiding are those of the Cartesian product, and are 2-natural by construction. ◇

Theorem 1.5.5. *Suppose* $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ *is a braided monoidal category. Then the tensor product of* V*-categories makes* $(\mathsf{V}\text{-}\mathsf{Cat}, \otimes, \mathbb{I})$ *a* Cat*-monoidal 2-category. If* V *is symmetric monoidal, then so is* $(\mathsf{V}\text{-}\mathsf{Cat}, \beta^{\otimes})$.

Proof. All of the relevant data and axioms are described in Section 1.3 above. The data are the following.

- The monoidal composition is the tensor product for V-categories; see Definitions 1.3.3 and 1.3.5. It is shown to be a 2-functor in Proposition 1.3.6.
- The identity object is the unit V-category; see Definition 1.3.7.
- The monoidal unitors are $\ell^{\otimes}$ and $r^{\otimes}$; see Definition 1.3.8.
- The monoidal associator is $a^{\otimes}$; see Definition 1.3.10.
- If V is symmetric, the braiding is $\beta^{\otimes}$; see Definition 1.3.22.

The axioms are checked as follows.

- The unity axiom (1.4.5) is verified on components by Lemma 1.3.14.
- The pentagon axiom (1.4.6) is verified on components by Lemma 1.3.17.
- If V is symmetric, the hexagon axioms (1.4.11) and (1.4.12) are verified on components by Lemma 1.3.29.
- If V is symmetric, the symmetry axiom (1.4.14) is verified on components by Lemma 1.3.32. □

Restricting to the underlying 1-categorical data recovers the 1-categorical monoidal structure stated in Theorem 1.3.35.

1.6. Notes

1.6.1 (Enriched Categories)**.** The standard reference for the theory of enriched categories is [**Kel05**]. Many of the basic definitions are developed in [**EK66**], which is itself based on preliminary work referenced there. ◇

1.6.2 (Iterated Monoidal Structure on V and $\mathsf{V}\text{-}\mathsf{Cat}$)**.** Forcey [**For04**] defines enrichment over V when V has an iterated monoidal structure described in Chapter II.10. The main result [**For04**, Theorem1] shows that if V is n-fold monoidal then $\mathsf{V}\text{-}\mathsf{Cat}$ is $(n-1)$-fold monoidal. The cases $n = 1$ and $n = 2$, along with the relevant Strictification Theorems 1.1.32 and 1.1.39, imply the statements in the top two rows of (1.3.1). Recall from Proposition II.10.1.21 that the case $n = \infty$ corresponds to V being permutative. This explains why our table in (1.3.1) collapses at the symmetric case. ◇

1.6.3 (Enriched Monoidal Categories via 2-Categorical Algebra)**.** Day and Street [**DS97**] develop a theory of pseudomonoids in a Gray monoid and apply it to give definitions of monoidal V-categories (along with the braided and symmetric variants) that correspond, via the tricategorical strictification of [**GPS95, Gur13**], to those of Section 1.4. This approach via pseudomonoids or more general 2-monad theory also appears in the thesis of Cruttwell [**Cru09**]. We have chosen to give more direct definitions to limit the 2-categorical background required for this work. ◇

1.6.4 (Enriched Monoidal Categories for Strict V)**.** For V strict monoidal, work of Kong and Zheng [**KZ18**, Definitions 2.3 and 2.4] gives definitions of (braided) monoidal V-categories in terms of the underlying categories. Theorem 2.5.1 below, which to our knowledge has not previously appeared in the literature, shows that the definitions of [**KZ18**] are special cases of the definitions given here.

Work of Morrison and Penneys [**MP19**] also gives a definition of strict monoidal V-category in the case that V is strict. In that work the authors pose the question of defining non-strict monoidal V-categories and proving coherence theorems. The definitions in this chapter together with the results of Chapter 2 provide positive answers to those questions.

The following table, where V denotes the enriching braided monoidal category, summarizes the various levels of strictness for [**MP19**], [**KZ18**] (where V is denoted B), and this chapter.

	[**MP19**]	[**KZ18**]	1.4
V	strict	strict	non-strict
monoidal V-categories	strict	non-strict	non-strict

◇

CHAPTER 2

Change of Enrichment

This chapter discusses change of enrichment along a monoidal, respectively braided monoidal, respectively symmetric monoidal functor

$$U : \mathsf{V} \longrightarrow \mathsf{W}.$$

Sections 2.1 through 2.4 describe the general 2-functoriality properties for change of enrichment along U. They show, moreover, that it is compatible with relevant monoidal structures under further assumptions that U, V, and W are braided or symmetric monoidal. The results are summarized in the statements of Theorems 2.2.7, 2.3.9, and 2.4.10.

We have two distinct purposes for covering this material. The first is for Sections 2.5 and 2.6, where change of enrichment along the corepresented functor

$$\mathsf{V}(\mathbb{1}, -) : \mathsf{V} \longrightarrow \mathsf{Set}$$

allows us to lift the coherence and strictification theory for ordinary monoidal categories, along with their braided and symmetric variants, to coherence and strictification results for monoidal V-categories. Similar results hold for the corresponding braided and symmetric variants.

Our second purpose for covering this material is for the development of various enriched monoidal categories and functors for our K-theory applications in Part 2. We will apply change of enrichment to develop new enriched monoidal structures from existing ones.

2.1. Change of Enriching Categories

In this section we discuss change of enriching category via a monoidal functor. Throughout this section we assume that

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a monoidal functor between monoidal categories. Braidings are not required.

Definition 2.1.1. Suppose $(\mathsf{V}, \otimes, \mathbb{1})$ and $(\mathsf{W}, \otimes, \mathbb{1})$ are monoidal categories and

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a monoidal functor. Then there is an induced *change of enrichment* 2-functor

$$(-)_U : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{W}\text{-}\mathsf{Cat}$$

given as follows.

- For a small V-category C, the W-category C_U has the same objects and has hom objects
$$\mathsf{C}_U(X, Y) = U(\mathsf{C}(X, Y)).$$
The composition and identity of C_U are given by applying U to those of C and composing the monoidal and unit constraints of U, respectively.

- For a V-functor $F : \mathsf{C} \longrightarrow \mathsf{D}$, the W-functor F_U is the same assignment on objects and is the induced morphism on hom objects

$$F_U = U(F_{X,Y}) : U(\mathsf{C}(X,Y)) \longrightarrow U(\mathsf{D}(FX,FY)).$$

- For a V-natural transformation $\theta : F \longrightarrow G$, the W-natural transformation θ_U has components given by the following composite in W:

$$\mathbb{1} \xrightarrow{U^0} U\mathbb{1} \xrightarrow{U\theta_X} U(\mathsf{D}(FX,GX)). \qquad \diamond$$

Proposition 2.1.2. *In the context of Definition 2.1.1, $(-)_U$ is a 2-functor.*

Proof. Verifying that $(-)_U$ gives the requisite assignments on 0-, 1-, and 2-cells consists of verifying that each of C_U, F_U, and θ_U satisfy the axioms for W-category, W-functor, and W-natural transformation, respectively. We will check the first, and the other two are similar.

The composition in C_U is defined by the following composite for objects X, Y, and Z in C:

(2.1.3)

$$U(\mathsf{C}(Y,Z)) \otimes U(\mathsf{C}(X,Y)) \xrightarrow{U^2} U(\mathsf{C}(Y,Z) \otimes \mathsf{C}(X,Y)) \xrightarrow{Um} U(\mathsf{C}(X,Z)).$$

The associativity diagram (1.2.2) for this composition is the outer diagram below, where we write $\mathsf{C}_{A,B}$ for the object $\mathsf{C}(A,B)$ to save space.

$$\begin{tikzcd}
& (U\mathsf{C}_{Y,Z} \otimes U\mathsf{C}_{X,Y}) \otimes U\mathsf{C}_{W,X} \arrow[rr, "\alpha"] \arrow[dl, "U^2 \otimes 1"'] & & U\mathsf{C}_{Y,Z} \otimes (U\mathsf{C}_{X,Y} \otimes U\mathsf{C}_{W,X}) \arrow[dr, "1 \otimes U^2"] & \\
U(\mathsf{C}_{Y,Z} \otimes \mathsf{C}_{X,Y}) \otimes U\mathsf{C}_{W,X} \arrow[dr, "U^2"] \arrow[dd, "Um \otimes 1"'] & & & & U\mathsf{C}_{Y,Z} \otimes U(\mathsf{C}_{X,Y} \otimes \mathsf{C}_{W,X}) \arrow[dl, "U^2"'] \arrow[dd, "1 \otimes Um"] \\
& U\big((\mathsf{C}_{Y,Z} \otimes \mathsf{C}_{X,Y}) \otimes \mathsf{C}_{W,X}\big) \arrow[rr, "U\alpha"] \arrow[dd, "U(m \otimes 1)"] & & U\big(\mathsf{C}_{Y,Z} \otimes (\mathsf{C}_{X,Y} \otimes \mathsf{C}_{W,X})\big) \arrow[dd, "U(1 \otimes m)"'] & \\
U\mathsf{C}_{X,Z} \otimes U\mathsf{C}_{W,X} \arrow[dr, "U^2"'] & & & & U\mathsf{C}_{Y,Z} \otimes U\mathsf{C}_{W,Y} \arrow[dl, "U^2"] \\
& U(\mathsf{C}_{X,Z} \otimes \mathsf{C}_{W,X}) \arrow[dr, "Um"'] & & U(\mathsf{C}_{Y,Z} \otimes \mathsf{C}_{W,Y}) \arrow[dl, "Um"] & \\
& & U\mathsf{C}_{W,Z} & &
\end{tikzcd}$$

The upper hexagon commutes by the associativity axiom (1.1.9) for U. The lower pentagon commutes by applying U to the associativity diagram (1.2.2) for composition in C. Each of the two quadrilaterals commutes because U preserves identity morphisms and U^2 is natural.

The identity of an object X in C_U is defined by the following composite:

$$\mathbb{1} \xrightarrow{U^0} U\mathbb{1} \xrightarrow{U(i_X)} U(\mathsf{C}(X,X)). \tag{2.1.4}$$

The left side of the unity diagram (1.2.3) for this composition is the outer diagram below; we again abbreviate $\mathsf{C}(A,B)$ as $\mathsf{C}_{A,B}$.

$$\begin{tikzcd}
\mathbb{1} \otimes U\mathsf{C}_{X,Y} \arrow[rr, "\lambda"] \arrow[d, "U^0 \otimes 1"'] & & U\mathsf{C}_{X,Y} \arrow[dd, "1"] \\
U\mathbb{1} \otimes U\mathsf{C}_{X,Y} \arrow[r, "U^2"] \arrow[d, "Ui_Y \otimes 1"'] & U(\mathbb{1} \otimes \mathsf{C}_{X,Y}) \arrow[ur, "U\lambda"] \arrow[d, "U(i_Y \otimes 1)"] & \\
U\mathsf{C}_{Y,Y} \otimes U\mathsf{C}_{X,Y} \arrow[r, "U^2"] & U(\mathsf{C}_{Y,Y} \otimes \mathsf{C}_{X,Y}) \arrow[r, "Um"] & U\mathsf{C}_{X,Y}
\end{tikzcd}$$

The upper quadrilateral commutes by the left unity axiom (1.1.10) for U. The lower left square commutes because U preserves identities and U^2 is natural. The quadrilateral at right commutes by applying U to the left side of the unity diagram (1.2.3) for composition in C. The right side of the unity diagram for composition in C_U is similar.

To verify that $(-)_U$ is a 2-functor, we must verify that it preserves identity 1- and 2-cells, horizontal composition of 1- and 2-cells, and vertical composition of 2-cells. The identity V-functor 1_{C} is given by the identity on objects and identity morphisms on each hom object $\mathsf{C}(X,Y)$. Therefore $(1_{\mathsf{C}})_U = 1_{\mathsf{C}_U}$ because U preserves identity morphisms. For V-functors

$$\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E},$$

recall from Definition 1.2.4 that the composite GF is given by the composite on objects of C and

$$(GF)_{X,Y} = G_{FX,FY}F_{X,Y}$$

for $X, Y \in \mathsf{C}$. Therefore,

$$(GF)_{U;X,Y} = (G_U \circ F_U)_{X,Y}$$

because U preserves composition. The computations showing that $(-)_U$ preserves horizontal and vertical composition of 2-cells are similar and make use of the monoidal compatibility axioms (1.1.9) and (1.1.10) for U. □

Lemma 2.1.5. *The corepresented functor*

$$\mathsf{V}(\mathbb{1}, -) : \mathsf{V} \longrightarrow \mathsf{Set}$$

is a monoidal functor with respect to the monoidal product in V *and the Cartesian product in* Set*. If* V *is braided, respectively symmetric, monoidal then* $\mathsf{V}(\mathbb{1}, -)$ *is a braided, respectively symmetric, monoidal functor.*

Proof. The monoidal constraint for $\mathsf{V}(\mathbb{1}, -)$ is given by functoriality of the monoidal product and composition with a unitor:

$$\mathsf{V}(\mathbb{1}, X) \times \mathsf{V}(\mathbb{1}, Y) \xrightarrow{- \otimes -} \mathsf{V}(\mathbb{1} \otimes \mathbb{1}, X \otimes Y) \xrightarrow{(\lambda^{-1})^*} \mathsf{V}(\mathbb{1}, X \otimes Y).$$

The unit constraint for $\mathsf{V}(\mathbb{1},-)$ is given by the identity morphism of $\mathbb{1}$, where $*$ denotes the terminal set:

$$* \xrightarrow{1_{\mathbb{1}}} \mathsf{V}(\mathbb{1},\mathbb{1}).$$

The associativity axiom (1.1.9) follows from naturality of α: for morphisms

$$f \in \mathsf{V}(\mathbb{1},X), \quad g \in \mathsf{V}(\mathbb{1},Y), \quad \text{and} \quad h \in \mathsf{V}(\mathbb{1},Z),$$

the following diagram commutes by naturality of α and compatibility with unit isomorphisms (1.1.5).

$$\begin{tikzcd}
& (\mathbb{1}\otimes\mathbb{1})\otimes\mathbb{1} \arrow[r, "(f\otimes g)\otimes h"] \arrow[d, "\alpha"'] & (X\otimes Y)\otimes Z \arrow[d, "\alpha"] \\
\mathbb{1} \arrow[ur, "(\lambda^{-1}\otimes 1)\circ\lambda^{-1}"] \arrow[r, "\lambda^{-1}\circ\lambda^{-1}"'] & \mathbb{1}\otimes(\mathbb{1}\otimes\mathbb{1}) \arrow[r, "f\otimes(g\otimes h)"'] & X\otimes(Y\otimes Z)
\end{tikzcd}$$

Combining this with the functoriality of $\otimes$ verifies the associativity axiom.

The left and right unity axioms (1.1.10) follow likewise from the naturality of the unitors in V. For example, the left unity axiom follows from commutativity of

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "f"] \arrow[d, "\lambda^{-1}"'] & X \\
\mathbb{1}\otimes\mathbb{1} \arrow[r, "1\otimes f"] & \mathbb{1}\otimes X \arrow[u, "\lambda"']
\end{tikzcd}$$

for each $f \in \mathsf{V}(\mathbb{1},X)$. The right unity axiom follows similarly, and uses the equality $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ (1.1.4). This completes the proof that $\mathsf{V}(\mathbb{1},-)$ is a monoidal functor.

If V is braided or symmetric monoidal, then similar analysis shows that the braid compatibility axiom (1.1.18) follows from naturality of ξ and the equality $\xi_{\mathbb{1},\mathbb{1}} = 1_{\mathbb{1}\otimes\mathbb{1}}$ (1.1.21). □

We apply Lemma 2.1.5 together with Proposition 2.1.2 to define the underlying category of a V-category.

Definition 2.1.6. Suppose C is a V-category. The *underlying category of* C is denoted C_0 and defined as follows. The objects of C_0 are those of C, and the morphism sets are defined by

$$\mathsf{C}_0(X,Y) = \mathsf{V}(\mathbb{1},\mathsf{C}(X,Y)). \qquad \diamond$$

Lemma 2.1.7. *The 2-functor*

$$(-)_0 : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Cat}$$

is injective on 2-cells. That is, two parallel V*-natural transformations* θ *and* ω *are equal if and only if their underlying natural transformations* θ_0 *and* ω_0 *are equal.*

Proof. Suppose $F,G : \mathsf{C} \longrightarrow \mathsf{D}$ are V-functors and $\theta,\omega : F \longrightarrow G$ are V-natural transformations. The component of θ_0 at an object X is defined to be the image of the following composite, where $*$ denotes the terminal set:

$$* \xrightarrow{1_{\mathbb{1}}} \mathsf{V}(\mathbb{1},\mathbb{1}) \xrightarrow{\mathsf{V}(\mathbb{1},\theta_X)} \mathsf{V}(\mathbb{1},\mathsf{D}(FX,GX)).$$

This morphism of sets sends the unique element of $*$ to the composite $\theta_X \circ 1_{\mathbb{1}} \in \mathsf{V}(\mathbb{1}, \mathsf{D}(FX, GX))$. Similarly, we have $(\omega_0)_X = \omega_X \circ 1_{\mathbb{1}}$. The result then follows because $\theta_X = \omega_X$ if and only if $\theta_X \circ 1_{\mathbb{1}} = \omega_X \circ 1_{\mathbb{1}}$. □

Next we define underlying represented, respectively corepresented, functors on the underlying category of a V-category D. We use them to give an alternative condition for V-naturality in Lemma 2.1.11 below.

Definition 2.1.8. Suppose D is a V-category and

$$\theta : \mathbb{1} \longrightarrow \mathsf{D}(P, Q)$$

is a morphism in V. Then for each $X \in \mathsf{D}$ we let $\mathsf{D}(X, \theta)$ denote the following composite in V.

(2.1.9)
$$\begin{tikzcd}
\mathsf{D}(X,P) \arrow[r, "\mathsf{D}(X,\theta)"] \arrow[d, "\lambda^{-1}"'] & \mathsf{D}(X,Q) \\
\mathbb{1} \otimes \mathsf{D}(X,P) \arrow[r, "\theta \otimes 1"] & \mathsf{D}(P,Q) \otimes \mathsf{D}(X,P) \arrow[u, "m"']
\end{tikzcd}$$

Similarly, for each $Y \in \mathsf{D}$ we let $\mathsf{D}(\theta, Y)$ denote the following composite in V.

(2.1.10)
$$\begin{tikzcd}
\mathsf{D}(Q,Y) \arrow[r, "\mathsf{D}(\theta,Y)"] \arrow[d, "\rho^{-1}"'] & \mathsf{D}(P,Y) \\
\mathsf{D}(Q,Y) \otimes \mathbb{1} \arrow[r, "1 \otimes \theta"] & \mathsf{D}(Q,Y) \otimes \mathsf{D}(P,Q) \arrow[u, "m"']
\end{tikzcd}$$

Recalling the definition of the identity (2.1.4) and composition (2.1.3) in D_0, it follows directly from the axioms for unity (1.2.3) and associativity of composition (1.2.2) that the constructions above define functors

$$\mathsf{D}(X, -) : \mathsf{D}_0 \longrightarrow \mathsf{V} \quad \text{and} \quad \mathsf{D}(-, Y) : \mathsf{D}_0^{\mathsf{op}} \longrightarrow \mathsf{V}$$

called the *underlying corepresented functor* and *underlying represented functor*, respectively. ◇

Lemma 2.1.11. *Suppose $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ are V-functors between V-categories C and D. Suppose that*

$$\theta_X : \mathbb{1} \longrightarrow \mathsf{D}(FX, GX)$$

is a collection of morphisms in V for $X \in \mathsf{C}$. The morphisms θ_X are the components of a V-natural transformation θ if and only if the following diagram commutes for each pair of objects X and Y in C.

(2.1.12)
$$\begin{tikzcd}
\mathsf{C}(X,Y) \arrow[r, "F"] \arrow[d, "G"'] & \mathsf{D}(FX,FY) \arrow[d, "\mathsf{D}(FX,\theta_Y)"] \\
\mathsf{D}(GX,GY) \arrow[r, "\mathsf{D}(\theta_X,GY)"] & \mathsf{D}(FX,GY)
\end{tikzcd}$$

Proof. Consider the following diagram in V for each $X, Y \in \mathsf{C}$.

$$\begin{tikzcd}
 & & \mathbb{1} \otimes \mathsf{D}(FX,FY) \arrow[dr, "\theta_Y \otimes 1"] & & \\
 & \mathsf{D}(FX,FY) \arrow[ur, "\lambda^{-1}"] & & \mathsf{D}(FY,GY) \otimes \mathsf{D}(FX,FY) \arrow[ddr, "m"] & \\
 & & \mathbb{1} \otimes \mathsf{C}(X,Y) \arrow[uu, "1 \otimes F"] \arrow[ur, "\theta_Y \otimes F"'] & & \\
\mathsf{C}(X,Y) \arrow[uur, "F"] \arrow[urr, "\lambda^{-1}"] \arrow[drr, "\rho^{-1}"'] \arrow[ddr, "G"'] & & & & \mathsf{D}(FX,GY) \\
 & & \mathsf{C}(X,Y) \otimes \mathbb{1} \arrow[dr, "G \otimes \theta_X"] \arrow[dd, "G \otimes 1"'] & & \\
 & \mathsf{D}(GX,GY) \arrow[dr, "\rho^{-1}"'] & & \mathsf{D}(GX,GY) \otimes \mathsf{D}(FX,GX) \arrow[uur, "m"'] & \\
 & & \mathsf{D}(GX,GY) \otimes \mathbb{1} \arrow[ur, "1 \otimes \theta_X"'] & &
\end{tikzcd}$$

Commutativity of the innermost region in the above diagram is the naturality condition (1.2.8) of Definition 1.2.7 above. On the other hand, by the definitions of $\mathsf{D}(FX, \theta_Y)$ and $\mathsf{D}(\theta_X, GY)$ in Definition 2.1.8, the outermost composites in the above diagram are precisely the morphisms in (2.1.12).

The two quadrilaterals and two triangles in the above diagram commute by naturality of the unitors and functoriality of the monoidal product

$$\otimes : \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V},$$

respectively. Therefore, the innermost region in the above diagram commutes if and only if the two composites along the boundary are equal. This proves the result. $\square$

2.2. 2-Functoriality of Change of Enrichment

In this section we show that the assignment from monoidal functors

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

to 2-functors

$$(-)_U : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{W}\text{-}\mathsf{Cat}$$

is 2-functorial with respect to monoidal functors and monoidal natural transformations. Throughout this section we work with general monoidal categories, monoidal functors, and monoidal natural transformations; braidings are not assumed.

Next we define the 2-natural transformation, between change of enrichment 2-functors, that is induced by a monoidal natural transformation.

Definition 2.2.1. Suppose V and W are moniodal categories, $U, T : \mathsf{V} \longrightarrow \mathsf{W}$ are moniodal functors, and $\mu : U \longrightarrow T$ is a monoidal natural transformation. For each V-category C let

$$\mathsf{C}_\mu : \mathsf{C}_U \longrightarrow \mathsf{C}_T$$

denote the W-functor that is the identity on objects and whose morphism on hom objects is

$$U\mathsf{C}(X,Y) \xrightarrow{\mu_{\mathsf{C}(X,Y)}} T\mathsf{C}(X,Y),$$

the component of μ at $\mathsf{C}(X,Y)$. The composition and identity axioms of Definition 1.2.4 for C_μ follow from the monoidal axioms (1.1.13) for μ and naturality of μ. ◇

Proposition 2.2.2. *In the context of Definition 2.2.1, the components* C_μ *are 2-natural with respect to* V*-functors and* V*-natural transformations.*

Proof. Suppose given V-categories C and D together with V-functors F and G and a V-natural transformation θ as in the diagram below:

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathsf{D}. \end{tikzcd}$$

Consider the following two diagrams in W-Cat.

(2.2.3)
$$\begin{tikzcd} \mathsf{C}_U \arrow[r, bend left, "F_U"] \arrow[r, bend right, "G_U"'] \arrow[r, phantom, "\Downarrow\theta_U"] \arrow[d, "\mathsf{C}_\mu"'] & \mathsf{D}_U \arrow[d, "\mathsf{D}_\mu"] \\ \mathsf{C}_T \arrow[r, bend right, "G_T"'] & \mathsf{D}_T \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C}_U \arrow[r, bend left, "F_U"] \arrow[d, "\mathsf{C}_\mu"'] & \mathsf{D}_U \arrow[d, "\mathsf{D}_\mu"] \\ \mathsf{C}_T \arrow[r, bend left, "F_T"] \arrow[r, bend right, "G_T"'] \arrow[r, phantom, "\Downarrow\theta_T"] & \mathsf{D}_T \end{tikzcd}$$

The quadrilateral region of each diagram commutes by naturality of μ. For example the diagram involving G becomes the following naturality square for each hom object $\mathsf{C}(X,Y)$.

$$\begin{tikzcd} U\mathsf{C}(X,Y) \arrow[r, "UG"] \arrow[d, "\mu"'] & U\mathsf{D}(GX,GY) \arrow[d, "\mu"] \\ T\mathsf{C}(X,Y) \arrow[r, "TG"] & T\mathsf{D}(GX,GY) \end{tikzcd}$$

To compare the two whiskerings indicated in the two diagrams of (2.2.3), we use the Whiskering Lemma 1.2.12. For each $X \in \mathsf{C}$, the component $(1_{\mathsf{D}_\mu} * \theta_U)_X$ is given, via Lemma 1.2.12, by the upper right composite in the diagram below. Similarly, the component $(\theta_T * 1_{\mathsf{C}_\mu})_X$ is given by the lower composite.

$$\begin{tikzcd} & U\mathbb{1} \arrow[r, "U(\theta_X)"] \arrow[d, "\mu"] & U\mathsf{D}(FX,GX) \arrow[d, "\mu"] \\ \mathbb{1} \arrow[ur, "U^0"] \arrow[dr, "T^0"'] & T\mathbb{1} \arrow[r, "T(\theta_X)"] & T\mathsf{D}(FX,GX) \end{tikzcd}$$

In the above diagram, the triangle commutes by the unit condition (1.1.13) for μ and the square commutes by naturality of μ. Thus the two whiskerings in (2.2.3) are equal. This finishes the verification that $(-)_\mu$ is 2-natural. □

Recall from Definition 1.1.11 the monoidal and unit constraints of a composite of monoidal functors

$$\mathsf{V}_1 \xrightarrow{U_1} \mathsf{V}_2 \xrightarrow{U_2} \mathsf{V}_3.$$

are given, respectively by

$$(U_2U_1)^2 = (U_2(U_1^2)) \circ U_2^2 \quad \text{and} \quad (U_2U_1)^0 = (U_2(U_1^0)) \circ U_2^0.$$

Proposition 2.2.4. *Suppose given monoidal categories V_1, V_2, and V_3 together with monoidal functors*

$$\mathsf{V}_1 \xrightarrow{U_1} \mathsf{V}_2 \xrightarrow{U_2} \mathsf{V}_3.$$

Then change of enrichment $(-)_{(U_2U_1)}$ along the composite U_2U_1 is equal as a 2-functor to the composite

$$\mathsf{V}_1\text{-}\mathsf{Cat} \xrightarrow{(-)_{U_1}} \mathsf{V}_2\text{-}\mathsf{Cat} \xrightarrow{(-)_{U_2}} \mathsf{V}_3\text{-}\mathsf{Cat}.$$

Proof. First we describe

$$((-)_{U_1})_{U_2} \quad \text{and} \quad (-)_{(U_2U_1)}$$

on objects of V_1-Cat. For a V_1-category C, the objects of the V_3-categories $(\mathsf{C}_{U_1})_{U_2}$ and $\mathsf{C}_{(U_2U_1)}$ are the same as those of C. Moreover we have an equality of hom objects

$$(\mathsf{C}_{U_1})_{U_2}(X,Y) = U_2(\mathsf{C}_{U_1}(X,Y)) = U_2U_1\mathsf{C}(X,Y) = \mathsf{C}_{(U_2U_1)}(X,Y).$$

To finish checking that $(\mathsf{C}_{U_1})_{U_2}$ and $\mathsf{C}_{(U_2U_1)}$ are equal as V_3-categories, we need to verify that the composition and identity morphisms are equal.

For objects $X, Y, Z \in \mathsf{C}$, the composition morphism for $(\mathsf{C}_{U_1})_{U_2}$ is given by the composite

$$\begin{tikzcd}
(\mathsf{C}_{U_1})_{U_2}(Y,Z) \otimes (\mathsf{C}_{U_1})_{U_2}(X,Y) \arrow[d, "U_2^2"] & \\
U_2\big(\mathsf{C}_{U_1}(Y,Z) \otimes \mathsf{C}_{U_1}(X,Y)\big) \arrow[d, "U_2(U_1^2)"] & \\
U_2U_1\big(\mathsf{C}(Y,Z) \otimes \mathsf{C}(X,Y)\big) \arrow[r, "U_2U_1m"] & U_2U_1\mathsf{C}(X,Z)
\end{tikzcd} \tag{2.2.5}$$

because $(U_1m) \circ U_1^2$ is the composition for C_{U_1} and U_2 is functorial. But since the monoidal constraint of the composite U_2U_1 is precisely $(U_2(U_1^2)) \circ U_2^2$, (2.2.5) also gives the composition morphism for $\mathsf{C}_{(U_2U_1)}$. A similar computation using the unit constraint for (U_2U_1) shows that the identity morphisms of $(\mathsf{C}_{U_1})_{U_2}$ and $\mathsf{C}_{(U_2U_1)}$ are equal. This shows that

$$((-)_{U_1})_{U_2} \quad \text{and} \quad (-)_{(U_2U_1)}$$

give the same assignments on objects of V_1-Cat.

Next we check assignments on 1-cells. For a V_1-functor

$$F : \mathsf{C} \longrightarrow \mathsf{D}$$

the morphisms on hom objects for both $(F_{U_1})_{U_2}$ and $F_{(U_2 U_1)}$ are given by

$$U_2 U_1 \mathsf{C}(X,Y) \xrightarrow{U_2 U_1 (F_{X,Y})} U_2 U_1 \mathsf{D}(FX, FY)$$

for $X, Y \in \mathsf{C}$. Therefore, $(F_{U_1})_{U_2} = F_{(U_2 U_1)}$.

Finally we check assignments on 2-cells. For a V_1-natural transformation $\theta : F \longrightarrow G$, each of $(\theta_{U_1})_{U_2}$ and $\theta_{(U_2 U_1)}$ the component at $X \in \mathsf{C}$ given by the composite

$$\mathbb{1} \xrightarrow{U_2^0} U_2 \mathbb{1} \xrightarrow{U_2(U_1^0)} U_2 U_1 \mathbb{1} \xrightarrow{U_2 U_1(\theta_X)} U_2 U_1 \mathsf{D}(FX, GX)$$

because U_2 is functorial and the unit constraint of the composite $U_2 U_1$ is precisely $(U_2(U_1^0)) \circ U_2^0$. This shows that $(\theta_{U_1})_{U_2} = \theta_{(U_2 U_1)}$ and completes the proof. □

Recall from Example I.6.1.14 that MCat denotes the 2-category of small monoidal categories, monoidal functors, and monoidal natural transformations.

Motivation 2.2.6 (2-Functoriality and Size)**.** The preceding results of this section can be summarized as a 2-functoriality statement where

$$\begin{aligned} \mathsf{V} &\longmapsto \mathsf{V}\text{-}\mathsf{Cat}, \\ U &\longmapsto (-)_U, \quad \text{and} \\ \mu &\longmapsto (-)_\mu \end{aligned}$$

are the assignments on 0-, 1-, and 2-cells of a 2-functor E out of MCat. However, a nontrivial set-theoretic consideration is necessary. Although we define each $\mathsf{V}\text{-}\mathsf{Cat}$ as a 2-category consisting of small V-categories, $\mathsf{V}\text{-}\mathsf{Cat}$ itself is generally not small. For this reason, the codomain of E must be enlarged. Subtleties of this sort motivated Eilenberg and Mac Lane to introduce the category of "large" categories in the original category theory paper [**EML45**], and that is one resolution available here too.

A similar but technically different approach can be given as follows. The definitions of small monoidal categories, small V-categories, and all other uses of small in this work, implicitly assume a chosen model for set theory, called a *universe* $\mathcal{U}$. The term *set* is reassigned to mean *member of* $\mathcal{U}$, and a category or V-category C called *small* if $\mathsf{Ob}\,\mathsf{C} \in \mathcal{U}$. Let us temporarily call this condition $\mathcal{U}$-small. Grothendieck's *Axiom of Universes* (see Note 2.7.3) implies (by choosing a larger inaccessible cardinal) the existence of a Grothendieck universe $\mathcal{U}'$ such that, for each $\mathcal{U}$-small monoidal category V, and for $\mathsf{V}\text{-}\mathsf{Cat}$ denoting the 2-category of $\mathcal{U}$-small V-categories, we have

$$\mathsf{Ob}(\mathsf{V}\text{-}\mathsf{Cat}) \in \mathcal{U}'.$$

Thus, if $2\mathsf{Cat}'$ denotes the 2-category of all $\mathcal{U}'$-small 2-categories, then $\mathsf{V}\text{-}\mathsf{Cat}$ is an object of $2\mathsf{Cat}'$ for each $\mathcal{U}$-small monoidal category V. All of our previous discussion of $2\mathsf{Cat}$ applies to $2\mathsf{Cat}'$ in the (Grothendieck) universe $\mathcal{U}'$.

We point out that this is only one convenient way of resolving the size subtlety. This one, the Eilenberg-Mac Lane use of "large", and a number of other more nuanced approaches, are outlined in [**Shu∞b**]. We state the next result with a hypothesis about $\mathcal{U}'$ that may follow from any of such approaches. ⋄

Theorem 2.2.7. *Suppose $\mathcal{U}'$ is a universe of sets containing* $\mathsf{Ob}(\mathsf{V}\text{-}\mathsf{Cat})$ *for each small monoidal category* V*. Let* $2\mathsf{Cat}'$ *denote the 2-category of 2-categories that are small with* *respect to $\mathcal{U}'$. Then change of enrichment provides a 2-functor*

$$E : \mathsf{MCat} \longrightarrow \mathsf{2Cat}'$$

given by

- $E\mathsf{V} = \mathsf{V}\text{-}\mathsf{Cat}$
- $EU = (-)_U$
- $E\mu = (-)_\mu$

for a small monoidal category V, monoidal functor U, and monoidal natural transformation μ, respectively.

Proof. By hypothesis on $\mathcal{U}'$, together with Propositions 2.1.2 and 2.2.2, E is a valid assignment on objects, 1-cells, and 2-cells. Now we verify that E is 2-functorial. To verify that E preserves identities, consider 1_{V}. Inspection of the definition shows that $(-)_{1_{\mathsf{V}}}$ is the identity V-functor. Similarly for a monoidal functor U the 2-natural transformation

$$(-)_{1_U} : (-)_U \longrightarrow (-)_U$$

has all identity components and is therefore the identity.

Proposition 2.2.4 shows that E preserves horizontal composition of 1-cells. Now we verify that E preserves horizontal and vertical composition of 2-cells. Suppose given monoidal categories, functors, and natural transformations as in the following diagram:

$$\begin{tikzcd} \mathsf{V}_1 \arrow[r, bend left, "U_1"] \arrow[r, bend right, "T_1"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathsf{V}_2 \arrow[r, bend left, "U_2"] \arrow[r, bend right, "T_2"'] \arrow[r, phantom, "\Downarrow\pi"] & \mathsf{V}_3. \end{tikzcd}$$

To verify

$$(E\pi) * (E\mu) = E(\pi * \mu),$$

it suffices to verify that the components at each V_1-category C are equal. Unwinding the definitions, these components are V_3-functors that are identities on objects and whose morphisms on hom objects $\mathsf{C}(X,Y)$ are given by either of the two composites around the square below—the two composites are equal because they are the two equal expressions for the component of $(\pi * \mu)$ at the object $\mathsf{C}(X,Y)$ in V_1.

$$\begin{tikzcd} U_2U_1\mathsf{C}(X,Y) \arrow[r, "\pi * U_1"] \arrow[d, "U_2 * \mu"'] & T_2U_1\mathsf{C}(X,Y) \arrow[d, "T_2 * \mu"] \\ U_2T_1\mathsf{C}(X,Y) \arrow[r, "\pi * T_1"] & T_2T_1\mathsf{C}(X,Y) \end{tikzcd}$$

Lastly we check that E preserves vertical composition of monoidal natural transformations. Suppose given monoidal functors

$$S, T, U : \mathsf{V} \longrightarrow \mathsf{W}$$

and a composable pair of monoidal natural transformations

$$U \xrightarrow{\mu} T \xrightarrow{\nu} S.$$

For each $\mathsf{C} \in \mathsf{V}\text{-}\mathsf{Cat}$, the W-functors

$$\mathsf{C}_U \xrightarrow{\mathsf{C}_\mu} \mathsf{C}_T \xrightarrow{\mathsf{C}_\nu} \mathsf{C}_S \quad \text{and} \quad \mathsf{C}_U \xrightarrow{\mathsf{C}_{(\nu\mu)}} \mathsf{C}_S$$

are identities on objects and have the same morphisms on hom objects

$$U\mathsf{C}(X,Y) \xrightarrow{\mu} T\mathsf{C}(X,Y) \xrightarrow{\nu} S\mathsf{C}(X,Y)$$

because the component of $\nu\mu$ at $\mathsf{C}(X,Y) \in \mathsf{V}$ is the composite of the corresponding components of ν and μ. This completes the proof that E is a 2-functor. □

Explanation 2.2.8. Even though Theorem 2.2.7 is stated for small monoidal categories V, the arguments that change of enrichment E preserves identities and composition of monoidal functors and/or monoidal natural transformations are independent of size. For a statement involving some specific monoidal categories, such as the adjunction result stated in Corollary 2.2.10 below, one can make a size-independent conclusion in one of two ways.

If Grothendieck's *Axiom of Universes* is assumed (see Note 2.7.3), then one can use Theorem 2.2.7 by choosing a sufficiently large universe in which the given monoidal categories are small. In that case, as noted in Motivation 2.2.6, the Axiom of Universes also implies the existence of $\mathcal{U}'$ in the statement of Theorem 2.2.7. In terms of cardinality, these are obtained by taking successively larger inaccessible cardinals.

Alternatively, one can use the preceding results and the arguments given in the proof of Theorem 2.2.7 to show directly that the relevant identities and composites are preserved. This approach is less concise, but avoids the set-theoretic considerations and the assumption of Grothendieck's axiom. ⋄

Definition 2.2.9. A *monoidal adjunction* between monoidal categories V and W consists of an adjunction

$$\begin{tikzcd} \mathsf{W} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathsf{V} \arrow[l, bend left, "U"] \end{tikzcd}$$

such that T and U are monoidal functors and the unit and counit are monoidal natural transformations. If V and W are small, this is an internal adjunction in MCat. A *monoidal adjoint equivalence* consists of a monoidal adjunction such that the unit and counit are monoidal natural isomorphisms. ⋄

Recall, as a special case of Definition I.6.3.9, an adjunction of 2-categories consists of 2-functors F and G

$$\begin{tikzcd} \mathsf{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{B} \arrow[l, bend left, "G"] \end{tikzcd}$$

together with 2-natural unit and counit satisfying triangle identities. If A and B are small then this is an internal adjunction in $\mathsf{2Cat}$.

Corollary 2.2.10. *Suppose given a monoidal adjunction*

$$\begin{tikzcd} \mathsf{W} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathsf{V}. \arrow[l, bend left, "U"] \end{tikzcd}$$

Then change of enrichment induces an adjunction of 2-categories

$$\begin{tikzcd} \mathsf{W}\text{-}\mathsf{Cat} \arrow[r, bend left, "(-)_T"] \arrow[r, phantom, "\perp"] & \mathsf{V}\text{-}\mathsf{Cat}. \arrow[l, bend left, "(-)_U"] \end{tikzcd}$$

If $T \dashv U$ is a monoidal equivalence, then $(-)_T \dashv (-)_U$ is an equivalence of 2-categories.

Proof. For small monoidal categories V and W, the result follows from the statement of Theorem 2.2.7. As discussed in Explanation 2.2.8, one can deduce the result for general V and W by either:

- enlarging to a Grothendieck universe that contains $\mathsf{Ob}\,\mathsf{V}$ and $\mathsf{Ob}\,\mathsf{W}$ (under the assumption of Grothendieck's Axiom of Universes) or
- applying the proofs of Proposition 2.2.4 and Theorem 2.2.7 to T, U, the unit, and the counit of the adjunction. □

The Monoidal Strictification Theorem 1.1.32 is a monoidal equivalence by Lemma 1.1.34 and therefore yields an important special case of Corollary 2.2.10.

Corollary 2.2.11. *Suppose* V *is a monoidal category and*

$$\begin{tikzcd} \mathsf{V}_{\mathsf{st}} \arrow[r, shift left, "L"] & \mathsf{V} \arrow[l, shift left, "R"] \end{tikzcd}$$

is a monoidal equivalence with V_{st} *strict monoidal. Then there is an equivalence of 2-categories*

$$\begin{tikzcd} (\mathsf{V}_{\mathsf{st}})\text{-}\mathsf{Cat} \arrow[r, shift left, "(-)_L"] & \mathsf{V}\text{-}\mathsf{Cat}. \arrow[l, shift left, "(-)_R"] \end{tikzcd}$$

2.3. Preservation of Enriched Tensor

In this section we discuss change of enrichment along a braided monoidal functor. Throughout, we suppose that $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ and $\mathsf{W} = (\mathsf{W}, \otimes, \xi)$ are braided monoidal categories and

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a braided monoidal functor. Recall from Definition 2.1.1 the change of enrichment 2-functor induced by U and denoted

$$(-)_U : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{W}\text{-}\mathsf{Cat}.$$

First we identify how change of enrichment is compatible with the enriched tensor product.

Definition 2.3.1. For V-categories C and D, we define

$$(-)^2_U : \mathsf{C}_U \otimes \mathsf{D}_U \longrightarrow (\mathsf{C} \otimes \mathsf{D})_U$$

to be the identity on objects and given on hom objects by the morphism in W

$$\begin{tikzcd} (\mathsf{C}_U \otimes \mathsf{D}_U)(X \otimes Y, X' \otimes Y') \arrow[d, phantom, "\shortparallel"] & (\mathsf{C} \otimes \mathsf{D})_U(X \otimes Y, X' \otimes Y') \arrow[d, phantom, "\shortparallel"] \\ \mathsf{C}_U(X, X') \otimes \mathsf{D}_U(Y, Y') \arrow[r, "U^2"] & (\mathsf{C}(X, X') \otimes \mathsf{D}(Y, Y'))_U \end{tikzcd}$$

for $X, X' \in \mathsf{C}$ and $Y, Y' \in \mathsf{D}$. We define

$$(-)^0_U : \mathbb{I} \longrightarrow \mathbb{I}_U$$

as the identity on the unique object and

$$\mathbb{1} \xrightarrow{U^0} U\mathbb{1}$$

on the unique hom object. ⋄

Lemma 2.3.2 (W-Functoriality and 2-Naturality of $(-)^2_U$)**.** *The assignment $(-)^2_U$ defines a W-functor. The definition of $(-)^2_U$ is 2-natural with respect to*

- V-*functors* $\mathsf{C} \longrightarrow \mathsf{C}'$ *and* $\mathsf{D} \longrightarrow \mathsf{D}'$*; and*
- V-*natural transformations thereof.*

Proof. The composition and identity axioms of Definition 1.2.4 require that the following diagrams, respectively, commute for each $X \otimes Y$, $X' \otimes Y'$, and $X'' \otimes Y''$ in $\mathsf{C} \otimes \mathsf{D}$. In the first diagram m_1 denotes the composition in $\mathsf{C}_U \otimes \mathsf{D}_U$ and m_2 denotes the composition in $(\mathsf{C} \otimes \mathsf{D})_U$. In the second diagram i_1 denotes the unit of $\mathsf{C}_U \otimes \mathsf{D}_U$ and i_2 denotes the unit of $(\mathsf{C} \otimes \mathsf{D})_U$.

$$\begin{tikzcd}
(\mathsf{C}_U \otimes \mathsf{D}_U)(X' \otimes Y', X'' \otimes Y'') \otimes (\mathsf{C}_U \otimes \mathsf{D}_U)(X \otimes Y, X' \otimes Y') \arrow[d, "U^2 \otimes U^2"'] \arrow[dr, "m_1"] & \\
& (\mathsf{C}_U \otimes \mathsf{D}_U)(X \otimes Y, X'' \otimes Y'') \arrow[dd, "U^2"] \\
(\mathsf{C} \otimes \mathsf{D})_U(X' \otimes Y', X'' \otimes Y'') \otimes (\mathsf{C} \otimes \mathsf{D})_U(X \otimes Y, X' \otimes Y') \arrow[dr, "m_2"'] & \\
& (\mathsf{C} \otimes \mathsf{D})_U(X \otimes Y, X'' \otimes Y'')
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "i_1"] \arrow[d, equal] & (\mathsf{C}_U \otimes \mathsf{D}_U)(X \otimes Y, X \otimes Y) \arrow[d, "U^2"] \\
\mathbb{1} \arrow[r, "i_2"] & (\mathsf{C} \otimes \mathsf{D})_U(X \otimes Y, X \otimes Y)
\end{tikzcd}$$

Unpacking the definitions of m_1 and m_2 via Definition 1.3.3 and (2.1.3) shows that the first diagram commutes by the associativity axiom (1.1.9) and the braid axiom (1.1.18) for U. Similarly, unpacking the definitions of i_1 and i_2 via (2.1.4) and Definition 1.3.3 shows that the second diagram commutes by the unity axioms (1.1.10) for U.

To check naturality of $(-)^2_U$, consider W-functors and W-natural transformations as in the diagrams below.
(2.3.3)

$$\begin{tikzcd}
\mathsf{C}_U \otimes \mathsf{D}_U \arrow[rr, bend left, "F_U \otimes G_U"] \arrow[rr, bend right, "F'_U \otimes G'_U"'] \arrow[rr, phantom, "\Downarrow \theta_U \otimes \omega_U"] \arrow[d, "(-)^2_U"'] & & \mathsf{C}'_U \otimes \mathsf{D}'_U \arrow[d, "(-)^2_U"] \\
(\mathsf{C} \otimes \mathsf{D})_U \arrow[rr, bend right, "(F' \otimes G')_U"'] & & (\mathsf{C}' \otimes \mathsf{D}')_U
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{C}_U \otimes \mathsf{D}_U \arrow[rr, bend left, "F_U \otimes G_U"] \arrow[d, "(-)^2_U"'] & & \mathsf{C}'_U \otimes \mathsf{D}'_U \arrow[d, "(-)^2_U"] \\
(\mathsf{C} \otimes \mathsf{D})_U \arrow[rr, bend left, "(F \otimes G)_U"] \arrow[rr, bend right, "(F' \otimes G')_U"'] \arrow[rr, phantom, "\Downarrow (\theta \otimes \omega)_U"] & & (\mathsf{C}' \otimes \mathsf{D}')_U
\end{tikzcd}$$

Naturality of $(-)^2_U$ with respect to V-functors, i.e., commutativity of the empty region in each diagram, follows by checking the induced morphisms on hom objects and using naturality of U^2 with respect to morphisms in V. Naturality of $(-)^2_U$ with respect to V-natural transformations, i.e., equality of the indicated whiskerings with $(-)^2_U$, follows by checking components via the Whiskering Lemma 1.2.12. Equality of the relevant components follows from naturality of U^2, functoriality of $\otimes$, naturality of λ in V, and the left unity axiom (1.1.10) for U. $\square$

Lemma 2.3.4 (Associativity Axiom for $(-)_U$)**.** *The following diagram commutes for each triple of* V*-categories* B, C, *and* D*.*

Associativity Diagram:

$$\begin{tikzcd}
(\mathsf{B}_U \otimes \mathsf{C}_U) \otimes \mathsf{D}_U \arrow[r, "a^{\otimes}"] \arrow[d, "(-)^2_U \otimes 1"'] & \mathsf{B}_U \otimes (\mathsf{C}_U \otimes \mathsf{D}_U) \arrow[d, "1 \otimes (-)^2_U"] \\
(\mathsf{B} \otimes \mathsf{C})_U \otimes \mathsf{D}_U \arrow[d, "(-)^2_U"'] & \mathsf{B}_U \otimes (\mathsf{C} \otimes \mathsf{D})_U \arrow[d, "(-)^2_U"] \\
((\mathsf{B} \otimes \mathsf{C}) \otimes \mathsf{D})_U \arrow[r, "(a^{\otimes})_U"] & (\mathsf{B} \otimes (\mathsf{C} \otimes \mathsf{D}))_U
\end{tikzcd}$$

Proof. Commutativity of the diagram on objects $(W \otimes X) \otimes Y \in (\mathsf{B} \otimes \mathsf{C}) \otimes \mathsf{D}$ is immediate because $(-)^2_U$ is the identity on objects. For hom objects

$$\big(\mathsf{B}_U(W, W') \otimes \mathsf{C}_U(X, X')\big) \otimes \mathsf{D}_U(Y, Y'),$$

commutativity follows from the definition of $a^{\otimes}$ (Definition 1.3.10) and the associativity axiom (1.1.9) for U. $\square$

Lemma 2.3.5 (Unity Axiom for $(-)_U$)**.** *The following two diagrams commute for each* V*-category* C*.*

Unity Diagrams:

$$\begin{tikzcd}
\mathbb{I} \otimes \mathsf{C}_U \arrow[r, "\ell^{\otimes}"] \arrow[d, "(-)^0_U \otimes 1"'] & \mathsf{C}_U \\
\mathbb{I}_U \otimes \mathsf{C}_U \arrow[r, "(-)^2_U"] & (\mathbb{I} \otimes \mathsf{C})_U \arrow[u, "(\ell^{\otimes})_U"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{C}_U \otimes \mathbb{I} \arrow[r, "r^{\otimes}"] \arrow[d, "1 \otimes (-)^0_U"'] & \mathsf{C}_U \\
\mathsf{C}_U \otimes \mathbb{I}_U \arrow[r, "(-)^2_U"] & (\mathsf{C} \otimes \mathbb{I})_U \arrow[u, "(r^{\otimes})_U"']
\end{tikzcd}$$

Proof. Similar to the proof of Lemma 2.3.4, commutativity on objects is immediate and commutativity on hom objects follows from the definitions of $\ell^{\otimes}$ and $r^{\otimes}$ (Definition 1.3.8) together with the unity axioms (1.1.10) for U. □

Lemma 2.3.6 (Braid Axiom for $(-)_U$)**.** *Suppose* V *is symmetric monoidal. The following diagram commutes for each pair of* V*-categories* C *and* D.

Braid Diagram:

$$\begin{tikzcd}
\mathsf{C}_U \otimes \mathsf{D}_U \arrow[r, "\beta^{\otimes}"] \arrow[d, "(-)^2_U"'] & \mathsf{D}_U \otimes \mathsf{C}_U \arrow[d, "(-)^2_U"] \\
(\mathsf{C} \otimes \mathsf{D})_U \arrow[r, "\beta^{\otimes}_U"] & (\mathsf{D} \otimes \mathsf{C})_U
\end{tikzcd}$$

Proof. Similarly to the proofs of Lemmas 2.3.4 and 2.3.5, commutativity on objects is immediate and commutativity on hom objects follows from the definition of $\beta^{\otimes}$ (Definition 1.3.22) together with the braid axiom (1.1.18) for U. □

Lemmas 2.3.4 through 2.3.6 together verify the axioms of Definitions 1.1.6 and 1.1.17, therefore proving the following result.

Theorem 2.3.7. *Suppose*

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a braided monoidal functor between braided monoidal categories. Then the change of enrichment

$$(-)_U : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{W}\text{-}\mathsf{Cat}$$

is a Cat*-monoidal 2-functor with respect to the enriched tensor products. Moreover, if* U *is a symmetric monoidal functor between symmetric monoidal categories, then* $(-)_U$ *is symmetric* Cat*-monoidal.*

Theorem 2.3.7 is the first step toward extending Theorem 2.2.7 to the braided and symmetric cases. We will show in Theorem 2.3.9 that change of enrichment provides a 2-functor from braided monoidal categories to Cat-monoidal 2-categories. As discussed in Motivation 2.2.6, this gives a convenient packaging of the relevant statements about preserving units and composition, but at the cost of some set-theoretic subtlety. We will use a similar approach as in Theorem 2.2.7, but applied to the second two of the 2-categories defined as follows.

Definition 2.3.8.

(1) We let BMCat denote the 2-category consisting of
- small braided monoidal categories as objects,
- braided monoidal functors as 1-cells, and
- monoidal natural transformations as 2-cells.

(2) We let SMCat denote the 2-category consisting of
  - small symmetric monoidal categories as objects,
  - symmetric monoidal functors as 1-cells, and
  - monoidal natural transformations as 2-cells.

(3) We let CM2Cat denote the 2-category consisting of
  - small Cat-monoidal 2-categories as objects,
  - Cat-monoidal 2-functors as 1-cells, and
  - Cat-monoidal 2-natural transformations as 2-cells.

(4) We let SCM2Cat denote the 2-category consisting of
  - small symmetric Cat-monoidal 2-categories as objects,
  - symmetric Cat-monoidal 2-functors as 1-cells, and
  - symmetric Cat-monoidal 2-natural transformations as 2-cells.

For verification that these form 2-categories, SMCat is described in Example I.6.1.15. Units and compositions of braided monoidal functors and natural transformations are given by their underlying monoidal counterparts, so the 2-category axioms for BMCat follow from those of MCat (Example I.6.1.14). The 2-categories CM2Cat and SCM2Cat are special cases of V-MCat, respectively V-SMCat in Definition 1.4.25 with V = Cat. ◇

Theorem 2.3.9. *Suppose $\mathcal{U}'$ is a universe of sets containing* $\mathsf{Ob}(\mathsf{V}\text{-}\mathsf{Cat})$ *for each small monoidal category* V. *Let* $\mathsf{CM2Cat}'$ *and* $\mathsf{SCM2Cat}'$ *denote the 2-categories of* Cat*-monoidal, respectively symmetric* Cat*-monoidal, 2-categories that are small with respect to $\mathcal{U}'$. Then change of enrichment provides a 2-functor*

$$E : \mathsf{BMCat} \longrightarrow \mathsf{CM2Cat}'$$

given by

- $E\mathsf{V} = \mathsf{V}\text{-}\mathsf{Cat}$
- $EU = (-)_U$
- $E\mu = (-)_\mu$

for a small braided monoidal category V*, braided monoidal functor U, and monoidal natural transformation μ, respectively. Restricted along the inclusion*

$$\mathsf{SMCat} \longrightarrow \mathsf{BMCat},$$

E takes values in $\mathsf{SCM2Cat}'$.

Proof. Theorem 2.3.7 shows that the assignment on 1-cells is valid; the assignments on 0- and 2-cells are those of Theorem 2.2.7. Since identities and vertical or horizontal composites of 2-cells in BMCat, respectively $\mathsf{CM2Cat}'$ are those of MCat, respectively $\mathsf{2Cat}'$, it follows from Theorem 2.2.7 that these are preserved in the braided case.

Given braided monoidal functors

$$\mathsf{V}_1 \xrightarrow{U_1} \mathsf{V}_2 \xrightarrow{U_2} \mathsf{V}_3,$$

Proposition 2.2.4 verifies that

$$((-)_{U_1})_{U_2} \quad \text{and} \quad (-)_{(U_2U_1)}$$

are equal as 2-functors. To complete the proof of the first assertion, we verify that the monoidal and unit constraints are equal. These constraints are 2-natural transformations whose components, at a pair of V_1-categories C and D, are V_3-functors

$$\mathsf{C}_{(U_2U_1)} \otimes \mathsf{D}_{(U_2U_1)} = (\mathsf{C}_{U_1})_{U_2} \otimes (\mathsf{D}_{U_1})_{U_2} \longrightarrow ((\mathsf{C} \otimes \mathsf{D})_{U_1})_{U_2} = (\mathsf{C} \otimes \mathsf{D})_{(U_2U_1)}.$$

Recalling Definition 2.3.1, these V_3-functors are identities on objects. Therefore, it remains only to verify they are equal on hom objects.

By Definition 1.4.19, the monoidal and unit constraints of the composite $((-)_{U_1})_{U_2}$ are given by the composite 2-natural transformations of the following pasting diagrams, where $\mathbf{1}$ denotes the terminal 2-category.

(2.3.10)
$$\begin{tikzcd}
(\mathsf{V}_1\text{-}\mathsf{Cat})^2 \arrow[r, "(-)_{U_1} \otimes (-)_{U_1}"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Swarrow (-)^2_{U_1}"] & (\mathsf{V}_2\text{-}\mathsf{Cat})^2 \arrow[r, "(-)_{U_2} \otimes (-)_{U_2}"] \arrow[d, "\otimes"] \arrow[dr, phantom, "\Swarrow (-)^2_{U_2}"] & (\mathsf{V}_3\text{-}\mathsf{Cat})^2 \arrow[d, "\otimes"] \\
\mathsf{V}_1\text{-}\mathsf{Cat} \arrow[r, "(-)_{U_1}"] & \mathsf{V}_2\text{-}\mathsf{Cat} \arrow[r, "(-)_{U_2}"] & \mathsf{V}_3\text{-}\mathsf{Cat}
\end{tikzcd}$$

(2.3.11)
$$\begin{tikzcd}
 & \mathbf{1} \arrow[dl, "\mathbb{I}"'] \arrow[dr, "\mathbb{I}"] \arrow[d, "\mathbb{I}"] & \\
\mathsf{V}_1\text{-}\mathsf{Cat} \arrow[r, phantom, "\Leftarrow (-)^0_{U_1}"] \arrow[r, "(-)_{U_1}"'] & \mathsf{V}_2\text{-}\mathsf{Cat} \arrow[r, "(-)_{U_2}"'] \arrow[r, phantom, "\Swarrow (-)^0_{U_2}"] & \mathsf{V}_3\text{-}\mathsf{Cat}
\end{tikzcd}$$

For $X, X' \in \mathsf{C}$ and $Y, Y' \in \mathsf{D}$ the morphism on hom objects given by (2.3.10) (at C, D) is the composite below.

$$\begin{tikzcd}
(\mathsf{C}_{U_1})_{U_2}(X,X') \otimes (\mathsf{D}_{U_1})_{U_2}(Y,Y') \arrow[d, "U_2^2"'] & \\
U_2\big((\mathsf{C}_{U_1})(X,X') \otimes (\mathsf{D}_{U_1})(Y,Y')\big) \arrow[r, "U_2(U_1^2)"] & U_2U_1\big(\mathsf{C}(X,X') \otimes \mathsf{D}(Y,Y')\big)
\end{tikzcd}$$

This composite is, by Definition 1.1.11, the monoidal constraint of the composite U_2U_1, and therefore is the morphism on hom objects given by $(-)^2_{U_2U_1}$ at C, D. A similar unpacking shows that the unit constraints agree, and this finishes the proof that

$$E : \mathsf{BMCat} \longrightarrow \mathsf{CM2Cat}'$$

is a 2-functor.

Recalling Proposition 1.4.21, the composite of symmetric monoidal enriched functors is again symmetric monoidal. Therefore the second assertion, regarding the restriction of E to SMCat, follows from the first assertion and the symmetric cases of Theorems 1.5.5 and 2.3.7. □

Explanation 2.3.12. As in Explanation 2.2.8, we note that one can apply the arguments in the proof of Theorem 2.3.9 to specific cases of interest without assumptions such as the Axiom of Universes. For example, one can do so for the proofs of Corollaries 2.2.10 and 2.2.11 below. ⋄

Definition 2.3.13. A *braided monoidal adjunction* between braided monoidal categories V and W consists of an adjunction

$$\begin{tikzcd} \mathsf{W} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathsf{V} \arrow[l, bend left, "U"] \end{tikzcd}$$

such that T and U are braided monoidal functors and the unit and counit are monoidal natural transformations. If V and W are small, this is an internal adjunction in BMCat. A *braided monoidal adjoint equivalence* consists of a braided monoidal adjunction such that the unit and counit are monoidal natural isomorphisms.

A braided monoidal adjunction, respectively equivalence, between symmetric monoidal categories is called a *symmetric monoidal adjunction*, respectively *symmetric monoidal equivalence*. ⋄

Corollary 2.3.14. *Suppose given a braided monoidal adjunction*

$$\begin{tikzcd} \mathsf{W} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathsf{V}. \arrow[l, bend left, "U"] \end{tikzcd}$$

Then change of enrichment induces an adjunction of Cat*-monoidal 2-categories*

$$\begin{tikzcd} \mathsf{W}\text{-}\mathsf{Cat} \arrow[r, bend left, "(-)_T"] \arrow[r, phantom, "\perp"] & \mathsf{V}\text{-}\mathsf{Cat}. \arrow[l, bend left, "(-)_U"] \end{tikzcd}$$

If $T \dashv U$ is a braided monoidal equivalence, then $(-)_T \dashv (-)_U$ is a Cat*-monoidal equivalence of 2-categories.*

If $T \dashv U$ is a symmetric monoidal adjunction between symmetric monoidal categories, then $(-)_T \dashv (-)_U$ is a symmetric Cat*-monoidal adjunction.*

The Braided Strictification Theorem 1.1.39 is a braided monoidal equivalence by Lemma 1.1.34 and therefore yields an important special case of Corollary 2.3.14.

Corollary 2.3.15. *Suppose* V *is a braided monoidal category and*

$$\begin{tikzcd} \mathsf{V}_{\mathsf{st}} \arrow[r, shift left, "L"] & \mathsf{V} \arrow[l, shift left, "R"] \end{tikzcd}$$

is a braided monoidal equivalence with V_{st} *strict braided monoidal. Then there is a* Cat*-monoidal equivalence of 2-categories*

$$\begin{tikzcd} (\mathsf{V}_{\mathsf{st}})\text{-}\mathsf{Cat} \arrow[r, shift left, "(-)_L"] & \mathsf{V}\text{-}\mathsf{Cat}. \arrow[l, shift left, "(-)_R"] \end{tikzcd}$$

The Symmetric Strictification Theorem 1.1.42 is a symmetric monoidal equivalence by Lemma 1.1.34 and therefore yields another important special case of Corollary 2.3.14.

Corollary 2.3.16. *Suppose* V *is a symmetric monoidal category and*

$$\begin{tikzcd} \mathsf{V}_{\mathsf{st}} \arrow[r, shift left, "L"] & \mathsf{V} \arrow[l, shift left, "R"] \end{tikzcd}$$

is a symmetric monoidal equivalence with V_{st} *strict symmetric monoidal. Then there is a symmetric* Cat*-monoidal equivalence of 2-categories*

$$(\mathsf{V}_{\mathsf{st}})\text{-}\mathsf{Cat} \underset{(-)_R}{\overset{(-)_L}{\rightleftarrows}} \mathsf{V}\text{-}\mathsf{Cat}.$$

2.4. Preservation of Enriched Monoidal Structure

In Theorem 2.3.7 above we showed that change of enrichment along a braided monoidal functor

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

provides a Cat-monoidal 2-functor

$$(-)_U : (\mathsf{V}\text{-}\mathsf{Cat}, \otimes) \longrightarrow (\mathsf{W}\text{-}\mathsf{Cat}, \otimes).$$

In this section we show that $(-)_U$ preserves enriched monoidal structure on each monoidal V-category, V-functor, and V-natural transformation. The main result is that $(-)_U$ restricts to 2-functors indicated in the following diagram (with V, W, and U assumed to be symmetric monoidal for the definitions of enriched braided or symmetric monoidal structures). See Theorem 2.4.10 and Explanation 2.4.15.

$$\begin{tikzcd}
\mathsf{V}\text{-}\mathsf{SMCat} \arrow[r] \arrow[d, "(-)_U"] & \mathsf{V}\text{-}\mathsf{BMCat} \arrow[r] \arrow[d, "(-)_U"] & \mathsf{V}\text{-}\mathsf{MCat} \arrow[r] \arrow[d, "(-)_U"] & \mathsf{V}\text{-}\mathsf{Cat} \arrow[d, "(-)_U"] \\
\mathsf{W}\text{-}\mathsf{SMCat} \arrow[r] & \mathsf{W}\text{-}\mathsf{BMCat} \arrow[r] & \mathsf{W}\text{-}\mathsf{MCat} \arrow[r] & \mathsf{W}\text{-}\mathsf{Cat}
\end{tikzcd}$$

Again throughout this section V and W are braided monoidal categories and $U : \mathsf{V} \longrightarrow \mathsf{W}$ is a braided monoidal functor. We will make use of the following details and conventions.

Convention 2.4.1 (Notation for Monoidal Data)**.** We denote the monoidal products in V-Cat and W-Cat via juxtaposition, and use exponents such as K^2 to denote KK. We also omit superscripts $\otimes$ and let $(\mathbb{I}, a, \ell, r, \beta)$ denote the braided monoidal data of either V-Cat or W-Cat, with context clarifying which we mean. In most cases the data of V-Cat will appear only when considering their images under $(-)_U$, and therefore will appear as $(\mathbb{I}_U, a_U, \ell_U, r_U, \beta_U)$. ◇

Explanation 2.4.2 (2-Functoriality of $(-)_U$)**.** In Definition 2.4.3 below we will use the following three consequences of Proposition 2.1.2, that change of enrichment, $(-)_U$, is a 2-functor.

(1) Change of enrichment strictly preserves identities, so for a V-category K we have $(1_{\mathsf{K}})_U = 1_{(\mathsf{K}_U)}$. We will use 1_U to denote this identity W-functor.
(2) Change of enrichment strictly preserves inverses, so for a V-functor F we have $(F^{-1})_U = (F_U)^{-1}$. We will use this for the images of the unitors, ℓ and r, denoting the inverses of ℓ_U and r_U by ℓ_U^{-1} and r_U^{-1}, respectively.
(3) Change of enrichment strictly preserves horizontal composition. Therefore, if θ is a V-natural transformation whose domain is a composite GF, then the domain of θ_U is $G_U F_U = (GF)_U$. We will apply this to the V-natural transformations in the data of a (braided) monoidal V-category. ◇

Definition 2.4.3. Suppose $\mathsf{K} = (\mathsf{K}, \boxtimes, \mathrm{I}^{\boxtimes}, a^{\boxtimes}, \ell^{\boxtimes}, r^{\boxtimes})$ is a monoidal V-category. Define a monoidal structure

$$(\mathsf{K}_U, \boxtimes', \mathrm{I}', a', \ell', r'),$$

with base W-category K_U and the following additional data. In these definitions we use Convention 2.4.1 for the monoidal data of V-Cat and W-Cat and implicitly use the 2-functoriality of $(-)_U$ as noted in Explanation 2.4.2.

Monoidal Composition: The monoidal composition $\boxtimes'$ is defined to be the composite

$$(\mathsf{K}_U)^2 \xrightarrow{(-)^2_U} (\mathsf{K}^2)_U \xrightarrow{\boxtimes_U} \mathsf{K}_U.$$

Monoidal Identity: The monoidal identity I' is defined to be the composite

$$\mathbb{I} \xrightarrow{(-)^0_U} \mathbb{I}_U \xrightarrow{\mathrm{I}^{\boxtimes}_U} \mathsf{K}_U.$$

Monoidal Unitors: The left unitor, ℓ' is equal to $\ell^{\boxtimes}$ composed with an identity 2-cell as shown in the pasting diagram below. The triangular regions commute by definition, the unlabeled quadrilateral commutes by naturality of $(-)^2_U$ and the quadrilateral labeled ☆ commutes by the left unity axiom for $(-)_U$, Lemma 2.3.5.

$$\begin{tikzcd}
 & \mathbb{I}\mathsf{K}_U \arrow[rrr, "\mathrm{I}'1"] \arrow[dr, "(-)^0_U 1"'] \arrow[dd, phantom, "☆"] & & & (\mathsf{K}_U)^2 \arrow[dl, "(-)^2_U"'] \arrow[dddr, "\boxtimes'"] & \\
 & & \mathbb{I}_U\mathsf{K}_U \arrow[urr, "\mathrm{I}_U 1"'] \arrow[d, "(-)^2_U"'] & & & \\
 & {} & (\mathbb{I}\mathsf{K})_U \arrow[r, "(\mathrm{I}1)_U"] \arrow[r, phantom, "\Downarrow(\ell^{\boxtimes})_U"', shift right=3] & (\mathsf{K}^2)_U \arrow[drr, "\boxtimes_U"] & & \\
\mathsf{K}_U \arrow[uuur, "\ell^{-1}"] \arrow[urr, "\ell_U^{-1}"] \arrow[rrrrr, "1"'] & & & & & \mathsf{K}_U
\end{tikzcd} \tag{2.4.4}$$

The right unitor, r' is equal to $r^{\boxtimes}$ composed with an identity 2-cell as shown in the pasting diagram below. The triangular regions commute by definition, the unlabeled quadrilateral commutes by naturality of $(-)^2_U$ and the quadrilateral labeled ☆ commutes by the right unity axiom for $(-)_U$, Lemma 2.3.5.

$$\begin{tikzcd}
 & \mathsf{K}_U\mathbb{I} \arrow[rrr, "1\mathrm{I}'"] \arrow[dr, "1(-)^0_U"'] \arrow[dd, phantom, "☆"] & & & (\mathsf{K}_U)^2 \arrow[dl, "(-)^2_U"'] \arrow[dddr, "\boxtimes'"] & \\
 & & \mathsf{K}_U\mathbb{I}_U \arrow[urr, "1\mathrm{I}_U"'] \arrow[d, "(-)^2_U"'] & & & \\
 & {} & (\mathsf{K}\mathbb{I})_U \arrow[r, "(1\mathrm{I})_U"] \arrow[r, phantom, "\Downarrow(r^{\boxtimes})_U"', shift right=3] & (\mathsf{K}^2)_U \arrow[drr, "\boxtimes_U"] & & \\
\mathsf{K}_U \arrow[uuur, "r^{-1}"] \arrow[urr, "r_U^{-1}"] \arrow[rrrrr, "1"'] & & & & & \mathsf{K}_U
\end{tikzcd} \tag{2.4.5}$$

Monoidal Associator: The monoidal associator a' is equal to a whiskering of $a^{\boxtimes}$ composed with identity 2-cells as shown in the pasting diagram below. The triangular regions commute by definition, the unlabeled quadrilaterals commute by naturality of $(-)^2_U$, and the hexagon labeled ☆ commutes by the associativity axiom for $(-)_U$, Lemma 2.3.4.

$$
\begin{tikzcd}
 & & ((\mathsf{K}_U)^2)\mathsf{K}_U \arrow[rrr, "\boxtimes' 1"] \arrow[dr, "(-)^2_U 1"'] \arrow[ddll, "a"'] & & & (\mathsf{K}_U)^2 \arrow[ddl, "(-)^2_U"'] \arrow[ddddd, "\boxtimes'"] \\
 & & & (\mathsf{K}^2)_U\mathsf{K}_U \arrow[urr, "\boxtimes_U 1"'] \arrow[dd, "(-)^2_U"'] & & \\
\mathsf{K}_U((\mathsf{K}_U)^2) \arrow[dr, "1(-)^2_U"] \arrow[ddd, "1\boxtimes'"'] & & \star & & & \\
 & & & ((\mathsf{K}^2)\mathsf{K})_U \arrow[r, "(\boxtimes 1)_U"'] \arrow[dl, "a_U"] & (\mathsf{K}^2)_U \arrow[ddr, "\boxtimes_U"] & \\
 & \mathsf{K}_U(\mathsf{K}^2)_U \arrow[r, "(-)^2_U"] \arrow[ddl, "1\boxtimes_U"] & (\mathsf{K}(\mathsf{K}^2))_U \arrow[d, "(1\boxtimes)_U"] & (a^{\boxtimes})_U \Swarrow & & \\
 & & (\mathsf{K}^2)_U \arrow[drrr, "\boxtimes_U"] & & & \\
(\mathsf{K}_U)^2 \arrow[urr, "(-)^2_U"] \arrow[rrrrr, "\boxtimes'"'] & & & & & \mathsf{K}_U
\end{tikzcd}
\tag{2.4.6}
$$

If, moreover, K is braided monoidal with braiding $\beta^{\boxtimes}$ (when V is assumed symmetric), we make the following definition.

Braiding: The braiding β' is equal to a whiskering of $\beta^{\boxtimes}$ composed with identity 2-cells as shown in the pasting diagram below. The triangular regions commute by definition and the quadrilateral labeled ☆ commutes by the braid axiom for $(-)_U$, Lemma 2.3.6.

$$
\begin{tikzcd}
(\mathsf{K}_U)^2 \arrow[rrr, "\boxtimes"] \arrow[dr, "(-)^2_U"'] \arrow[dddrr, bend right, "\beta"'] & & & \mathsf{K}_U \\
 & (\mathsf{K}^2)_U \arrow[urr, "\boxtimes_U"'] \arrow[dr, bend left, "\beta_U"'] \arrow[r, phantom, "\Downarrow\beta^{\boxtimes}"] & {} & \\
\star & & (\mathsf{K}^2)_U \arrow[uur, "\boxtimes_U"'] & \\
 & & (\mathsf{K}_U)^2 \arrow[u, "(-)^2_U"] \arrow[uuur, bend right, "\boxtimes'"'] &
\end{tikzcd}
\tag{2.4.7}
$$

This finishes the definition of the data for monoidal structure on K_U. We show that these data satisfy the relevant axioms in Theorem 2.4.10 below. ⋄

Definition 2.4.8. Suppose $(F, F^2, F^0) : \mathsf{K} \longrightarrow \mathsf{L}$ is a monoidal functor between monoidal V-categories K and L. When K_U and L_U are given the monoidal W-category structure of Definition 2.4.3, then we define a monoidal functor

$$(F_U, (F^2_U)', (F^0_U)') : \mathsf{K}_U \longrightarrow \mathsf{L}_U$$

with underlying W-functor F_U and the following monoidal and unit constraints. As above we use Convention 2.4.1 for the monoidal data of V-Cat and W-Cat.

Monoidal Constraint: The monoidal constraint $(F^2_U)'$ is defined to be the wiskering of F^2_U shown below, where the upper square commutes by naturality of $(-)^2_U$.

$$\begin{tikzcd}
\mathsf{K}^2_U \arrow[r, "(F_U)(F_U)"] \arrow[d, "(-)^2_U"'] & \mathsf{L}^2_U \arrow[d, "(-)^2_U"] \\
(\mathsf{K}^2)_U \arrow[r, "(FF)_U"] \arrow[d, "\boxtimes_U"'] \arrow[dr, phantom, "\overset{F^2_U}{\swArrow}"] & (\mathsf{L}^2)_U \arrow[d, "\boxtimes_U"] \\
\mathsf{K}_U \arrow[r, "F_U"] & \mathsf{L}_U
\end{tikzcd}$$

Unit Constraint: The unit constraint $(F^0_U)'$ is defined to be the whiskering of F^0_U shown below.

$$\begin{tikzcd}
 & \mathbb{I} \arrow[d, "(-)^0_U"] & \\
 & \mathbb{I}_U \arrow[dl, "\mathrm{I}_U"'] \arrow[dr, "\mathrm{I}_U"] \arrow[d, phantom, "\overset{F^0_U}{\swArrow}"] & \\
\mathsf{K}_U \arrow[rr, "F_U"] & & \mathsf{L}_U
\end{tikzcd}$$

This finishes the definition of $(F_U, (F^2_U)', (F^0_U)')$. We address the relevant axioms in Theorem 2.4.10 below. $\diamond$

Lemma 2.4.9. *Suppose*

$$\mathsf{K} \xrightarrow{F} \mathsf{L} \xrightarrow{P} \mathsf{M}$$

is a composable pair of monoidal V*-functors. Then*

$$(PF)_U = (P_U)(F_U).$$

Proof. The 2-functoriality of $(-)_U$ implies that the asserted equality holds for underlying V-functors; we need to check the monoidal and unit constraints are equal. Recalling Definition 1.4.19, the monoidal and unit constraints of PF are given by pasting those of F and P. Thus the monoidal and unit constraints for $(PF)_U$ are given by the pastings below.

$$
\begin{tikzcd}
\mathsf{K}_U^2 \arrow[r, "(F_U)(F_U)"] \arrow[d, "(-)_U^2"'] & \mathsf{L}_U^2 \arrow[r, "(P_U)(P_U)"] \arrow[d, "(-)_U^2"] & \mathsf{M}_U^2 \arrow[d, "(-)_U^2"] \\
(\mathsf{K}^2)_U \arrow[r, "(FF)_U"] \arrow[d, "\boxtimes_U"'] \arrow[dr, phantom, "F_U^2 \Swarrow"] & (\mathsf{L}^2)_U \arrow[r, "(PP)_U"] \arrow[d, "\boxtimes_U"] \arrow[dr, phantom, "P_U^2 \Swarrow"] & (\mathsf{M}^2)_U \arrow[d, "\boxtimes_U"] \\
\mathsf{K}_U \arrow[r, "F_U"] & \mathsf{L}_U \arrow[r, "P_U"] & \mathsf{M}_U
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& \mathbb{I} \arrow[d, "(-)_U^0"] & \\
& \mathbb{I}_U \arrow[dl, "\mathrm{I}_U"'] \arrow[dr, "\mathrm{I}_U"] \arrow[dd, "\mathrm{I}_U"] & \\
\mathsf{K}_U \arrow[dr, "F_U"'] \arrow[r, phantom, "F_U^0 \Leftarrow"] & & \mathsf{M}_U \arrow[l, phantom, "P_U^0 \Leftarrow"'] \\
& \mathsf{L}_U \arrow[ur, "P_U"'] &
\end{tikzcd}
$$

Since (as in any 2-category) horizontal composition of V-natural transformations distributes over vertical composition, the indicated composites are precisely the monoidal and unit constraints of $(P_U)(F_U)$. This completes the proof. $\square$

Theorem 2.4.10. *Suppose*

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a braided monoidal functor between braided monoidal categories. In the context of Definitions 2.4.3 and 2.4.8 above, we have the following results. For items (1) and (2), the braided and symmetric monoidal cases assume moreover that U, V, and W are symmetric monoidal.

(1) *If K is a monoidal, respectively braided monoidal, respectively symmetric monoidal, V-category, then K_U is a monoidal, respectively braided monoidal, respectively symmetric monoidal, W-category.*

(2) *If $F : \mathsf{K} \longrightarrow \mathsf{L}$ is a monoidal, respectively braided monoidal, respectively symmetric monoidal, V-functor between monoidal, respectively braided monoidal, respectively symmetric monoidal, V-categories, then*

$$F_U : \mathsf{K}_U \longrightarrow \mathsf{L}_U$$

is a monoidal, respectively braided monoidal, respectively symmetric monoidal, W-functor.

(3) *If $\theta : F \longrightarrow G$ is a monoidal V-natural transformation between monoidal V-functors F and G, then*

$$\theta_U : F_U \longrightarrow G_U$$

is a monoidal W-natural transformation.

Proof. Each of the three assertions follows by verifying the relevant axioms from Section 1.4. To ease readability of the computations below, we denote the whiskerings of a W-natural transformation θ with the W-functor $(-)_U^2$ as

$$\theta * (-)_U^2 \quad \text{and} \quad (-)_U^2 * \theta$$

instead of our usual convention using the identity on $(-)_U^2$

$$\theta * 1_{(-)_U^2} \quad \text{and} \quad 1_{(-)_U^2} * \theta.$$

For assertion (1), we begin with the middle unity axiom (1.4.5). Below we show the pasting of a' and $1\ell'$.

(2.4.11)

$$\begin{tikzcd}
(\mathsf{K}_U\mathbb{I})\mathsf{K}_U \arrow[rr, "(1(-)^0_U)1"] \arrow[dddr, "a"] & & (\mathsf{K}_U\mathbb{I}_U)\mathsf{K}_U \arrow[r, "(1\mathrm{I}_U)1"] \arrow[dd, "a"] & (\mathsf{K}_U)^2\mathsf{K}_U \arrow[r, "(-)^2_U 1"] \arrow[dd, "a"] & (\mathsf{K}^2)_U\mathsf{K}_U \arrow[r, "\boxtimes_U 1"] \arrow[d, "(-)^2_U"'] & (\mathsf{K}_U)^2 \arrow[dd, "(-)^2_U"] \\
 & & & & (\mathsf{K}^2\mathsf{K})_U \arrow[d, "a_U"'] \arrow[dr, "(\boxtimes 1)_U"] & \\
 & & \mathsf{K}_U(\mathbb{I}_U\mathsf{K}_U) \arrow[r, "1(\mathrm{I}_U 1)"] \arrow[d, "1(-)^2_U"'] & \mathsf{K}_U(\mathsf{K}_U)^2 \arrow[d, "1(-)^2_U"'] & (\mathsf{K}\mathsf{K}^2)_U \arrow[dd, "(1\boxtimes)_U"'] \arrow[ddr, phantom, "(a^{\boxtimes})_U \Swarrow"] & (\mathsf{K}^2)_U \arrow[dd, "\boxtimes_U"] \\
 & \mathsf{K}_U(\mathbb{I}\mathsf{K}_U) \arrow[ur, "1((-)^0_U 1)"] & \mathsf{K}_U(\mathbb{I}\mathsf{K})_U \arrow[r, "1(\mathrm{I}1)_U"] \arrow[dr, phantom, "1(\ell^{\boxtimes})_U \Swarrow"] & \mathsf{K}_U(\mathsf{K}^2)_U \arrow[ur, "(-)^2_U"] \arrow[d, "1\boxtimes_U"] & & \\
(\mathsf{K}_U)^2 \arrow[uuuu, "r^{-1}1"] \arrow[ur, "1\ell^{-1}"'] \arrow[urr, "1\ell_U^{-1}"] \arrow[rrr, "1^2"'] & & & (\mathsf{K}_U)^2 \arrow[r, "(-)^2_U"'] & (\mathsf{K}^2)_U \arrow[r, "\boxtimes_U"'] & \mathsf{K}_U
\end{tikzcd}$$

The 2-naturality of $(-)^2_U$ gives the equality

$$(-)^2_U * [1(\ell^{\boxtimes})_U] = (1\ell^{\boxtimes})_U * (-)^2_U,$$

and we use this together with several of the other basic axioms for monoidal structure to show that the composite of (2.4.11) is equal to that of (2.4.12).

(2.4.12)

$$\begin{tikzcd}
(\mathsf{K}_U\mathbb{I})\mathsf{K}_U \arrow[r, "(1(-)^0_U)1"] & (\mathsf{K}_U\mathbb{I}_U)\mathsf{K}_U \arrow[r, "(1\mathrm{I}_U)1"] \arrow[d, "(-)^2_U 1"'] & (\mathsf{K}_U)^2\mathsf{K}_U \arrow[r, "(-)^2_U 1"] & (\mathsf{K}^2)_U\mathsf{K}_U \arrow[r, "\boxtimes_U 1"] \arrow[d, "(-)^2_U"'] & (\mathsf{K}_U)^2 \arrow[dd, "(-)^2_U"] \\
 & (\mathsf{K}\mathbb{I})_U\mathsf{K}_U \arrow[r, "(-)^2_U"] & ((\mathsf{K}\mathbb{I})\mathsf{K})_U \arrow[r, "((1\mathrm{I})1)_U"] \arrow[d, "a_U"] & (\mathsf{K}^2\mathsf{K})_U \arrow[d, "a_U"'] \arrow[dr, "(\boxtimes 1)_U"] & \\
 & & (\mathsf{K}(\mathbb{I}\mathsf{K}))_U \arrow[r, "(1(\mathrm{I}1))_U"] \arrow[dr, phantom, "(1\ell^{\boxtimes})_U \Swarrow"] & (\mathsf{K}\mathsf{K}^2)_U \arrow[dd, "(1\boxtimes)_U"'] \arrow[ddr, phantom, "(a^{\boxtimes})_U \Swarrow"] & (\mathsf{K}^2)_U \arrow[dd, "\boxtimes_U"] \\
 & (\mathsf{K}^2)_U \arrow[uur, "(r^{-1}1)_U"] \arrow[ur, "(1\ell^{-1})_U"'] \arrow[drr, "1"] & & & \\
(\mathsf{K}_U)^2 \arrow[uuuu, "r^{-1}1"] \arrow[uuur, "r_U^{-1}1"] \arrow[ur, "(-)^2_U"'] \arrow[rr, "1^2"'] & & (\mathsf{K}_U)^2 \arrow[r, "(-)^2_U"'] & (\mathsf{K}^2)_U \arrow[r, "\boxtimes_U"'] & \mathsf{K}_U
\end{tikzcd}$$

Next we apply $(-)_U$ to the middle unity axiom (1.4.5) for K and use the resulting equality to show that the composite of (2.4.12) is equal to that of (2.4.13).

(2.4.13)

$$\begin{tikzcd}
(\mathsf{K}_U\mathbb{I})\mathsf{K}_U \arrow[r, "(1(-)^0_U)1"] & (\mathsf{K}_U\mathbb{I}_U)\mathsf{K}_U \arrow[r, "(1\mathrm{I}_U)1"] \arrow[d, "(-)^2_U 1"'] & (\mathsf{K}_U)^2\mathsf{K}_U \arrow[r, "(-)^2_U 1"] & (\mathsf{K}^2)_U\mathsf{K}_U \arrow[r, "\boxtimes_U 1"] \arrow[d, "(-)^2_U"'] & (\mathsf{K}_U)^2 \arrow[dd, "(-)^2_U"] \\
& (\mathsf{K}\mathbb{I})_U\mathsf{K}_U \arrow[r, "(-)^2_U"] & ((\mathsf{K}\mathbb{I})\mathsf{K})_U \arrow[r, "((1\mathrm{I})1)_U"] \arrow[dr, phantom, "(r^{\boxtimes}1)_U\Downarrow"] & (\mathsf{K}^2\mathsf{K})_U \arrow[dr, "(\boxtimes 1)_U"] & \\
& & & & (\mathsf{K}^2)_U \arrow[dd, "\boxtimes_U"] \\
& (\mathsf{K}^2)_U \arrow[uur, "(r^{-1}1)_U"] \arrow[urrr, "(1^2)_U"'] \arrow[drr, "1"] & & & \\
(\mathsf{K}_U)^2 \arrow[uuuu, "r^{-1}1"] \arrow[uuur, "r_U^{-1}1"] \arrow[ur, "(-)^2_U"'] \arrow[rr, "1^2"'] & & (\mathsf{K}_U)^2 \arrow[r, "(-)^2_U"'] & (\mathsf{K}^2)_U \arrow[r, "\boxtimes_U"'] & \mathsf{K}_U
\end{tikzcd}$$

Finally, we again use 2-naturality of $(-)^2_U$, giving

$$(r^{\boxtimes}1_U) * (-)^2_U = (-)^2_U * [(r^{\boxtimes}_U 1)],$$

to conclude that the composite of (2.4.13) is equal to that of (2.4.14) below. This last composite is equal to the whiskering of $r'1$ required for the second half of (1.4.5) for K_U.

(2.4.14)

$$\begin{tikzcd}
(\mathsf{K}_U\mathbb{I})\mathsf{K}_U \arrow[r, "(1(-)^0_U)1"] & (\mathsf{K}_U\mathbb{I}_U)\mathsf{K}_U \arrow[r, "(1\mathrm{I}_U)1"] \arrow[d, "(-)^2_U 1"'] & (\mathsf{K}_U)^2\mathsf{K}_U \arrow[r, "(-)^2_U 1"] & (\mathsf{K}^2)_U\mathsf{K}_U \arrow[r, "\boxtimes_U 1"] & (\mathsf{K}_U)^2 \arrow[dd, "(-)^2_U"] \\
& (\mathsf{K}\mathbb{I})_U\mathsf{K}_U \arrow[urr, "(1\mathrm{I})_U 1"'] \arrow[r, phantom, "(r^{\boxtimes})_U 1 \Downarrow"] & & & \\
& & & & (\mathsf{K}^2)_U \arrow[dd, "\boxtimes_U"] \\
& & & & \\
(\mathsf{K}_U)^2 \arrow[uuuu, "r^{-1}1"] \arrow[uuur, "r_U^{-1}1"] \arrow[uuuurrrr, "1^2"'] \arrow[rr, "1^2"'] & & (\mathsf{K}_U)^2 \arrow[r, "(-)^2_U"'] & (\mathsf{K}^2)_U \arrow[r, "\boxtimes_U"'] & \mathsf{K}_U
\end{tikzcd}$$

Each of the remaining axioms to be verified for item (1) is an equality of two composites of pasting diagrams, and in each diagram the only nontrivial 2-cells are given by applying $(-)_U$ to corresponding 2-cells of K. The equality then follows as above, using defining properties of (braided/symmetric) monoidal categories and functors, together with the image under $(-)_U$ of the relevant axiom for K.

For example, the pentagon axiom for K_U follows directly from the definitions and the pentagon axiom for K. If K is symmetric, the symmetry axiom for K_U follows likewise from that of K. Similarly, in the case that K is braided, the two hexagon axioms require equalities such as

$$(-)^2_U * (\beta^{\boxtimes}_U 1) = (\beta^{\boxtimes}1)_U * (-)^2_U,$$

that follow from the 2-naturality of $(-)^2_U$ as above. In addition, for the right hexagon axiom (1.4.12) one uses 2-functoriality of $(-)_U$ to see that the mate of a' described in Definition 1.4.9 is equal to an appropriate whiskering of $(a_1^{\boxtimes})_U$. Thus (again using 2-functoriality of $(-)_U$) the inverse $(a_1')^{-1}$ is obtained by whiskering $a_1^{-\boxtimes}$.

For assertion (2) the axioms of Definitions 1.4.17 and 1.4.18 are similar but simpler, following from naturality of $(-)^2_U$ and the corresponding axioms for F. For assertion (3) the axioms of Definition 1.4.22 are immediate from the definitions. □

Explanation 2.4.15. Using the notation of Definition 1.4.25, Theorem 2.4.10 proves that $(-)_U$ restricts to 2-functors indicated in the diagram below, with V, W, and U assumed symmetric for the definitions of V-BMCat, W-BMCat, V-SMCat, and W-SMCat.

$$\begin{tikzcd}
\mathsf{V}\text{-}\mathsf{SMCat} \arrow[r] \arrow[d, "(-)_U"] & \mathsf{V}\text{-}\mathsf{BMCat} \arrow[r] \arrow[d, "(-)_U"] & \mathsf{V}\text{-}\mathsf{MCat} \arrow[r] \arrow[d, "(-)_U"] & \mathsf{V}\text{-}\mathsf{Cat} \arrow[d, "(-)_U"] \\
\mathsf{W}\text{-}\mathsf{SMCat} \arrow[r] & \mathsf{W}\text{-}\mathsf{BMCat} \arrow[r] & \mathsf{W}\text{-}\mathsf{MCat} \arrow[r] & \mathsf{W}\text{-}\mathsf{Cat}
\end{tikzcd} \tag{2.4.16}$$

By Lemma 2.4.9, $(-)_U$ preserves composition of monoidal V-functors. By Proposition 1.4.21, composites of braided and symmetric monoidal enriched functors are given by composites of the underlying monoidal enriched functors. By Proposition 1.4.24 composites of monoidal enriched natural transformations are given by composites of the underlying enriched natural transformations. Therefore 2-functoriality of $(-)_U$ on V-Cat implies 2-functoriality of each restriction. ⋄

When V is braided monoidal then the corepresented functor $U = \mathsf{V}(\mathbb{1}, -)$ is braided monoidal by Lemma 2.1.5. It is, furthermore, symmetric if V is symmetric. Therefore, by Theorem 2.4.10 we have the following.

Corollary 2.4.17. *Suppose V is a braided monoidal category. Then the following statements hold. For items (1) and (2), the braided and symmetric monoidal cases assume moreover that V is symmetric monoidal.*

(1) *Suppose K is a monoidal, respectively braided monoidal, respectively symmetric monoidal, V-category. Then the underlying category K_0 is a monoidal, respectively braided monoidal, respectively symmetric monoidal, category.*

(2) *Suppose that $F : \mathsf{K} \longrightarrow \mathsf{L}$ is a monoidal, respectively braided monoidal, respectively symmetric monoidal, V-functor between monoidal, respectively braided monoidal, respectively symmetric monoidal, V-categories. Then the underlying functor*

$$F_0 : \mathsf{K}_0 \longrightarrow \mathsf{L}_0$$

is monoidal, respectively braided monoidal, respectively symmetric monoidal.

(3) *Suppose that $\theta : F \longrightarrow G$ is a monoidal V-natural transformation between monoidal V-functors. Then the underlying natural transformation*

$$\theta_0 : F_0 \longrightarrow G_0$$

is a monoidal natural transformation.

The proof of Theorem 2.4.10 has a technical feature that will be useful to state explicitly. We will use this in the proof of Theorem 2.5.1 below.

Explanation 2.4.18. For each of the axioms (1.4.5), (1.4.6), (1.4.11), (1.4.12), and (1.4.14), let θ denote the composite of the top and/or left pasting diagram for K and ω denote the composite of the bottom and/or right pasting diagram for K. So each axiom is an equality $\theta = \omega$.

Let θ' and ω' denote top/left and bottom/right composites, respectively, in the corresponding axioms for K_U. The monoidal constraint of $(-)_U$ is $(-)^2_U$ and we let $(-)^3_U$ denote the two equal composites

$$(\mathsf{K}_U)^3 \longrightarrow (\mathsf{K}^3)_U$$

given by iterating $(-)^2_U$.

With this notation, the argument for each axiom in the proof of Theorem 2.4.10 consists of two steps:

(1) First, $\theta' = \theta_U * (-)^n_U$ and $\omega' = \omega_U * (-)^n_U$, where $n = 2$ or 3.
(2) Second, $\theta_U * (-)^n_U = \omega_U * (-)^n_U$ by the corresponding axiom for K.

This implies then that $\theta' = \omega'$.

For example, the diagrams (2.4.11) and (2.4.14) above are the pasting diagrams for θ' and ω', respecitevly, in the middle unity axiom. On the other hand, diagrams (2.4.12) and (2.4.13) are simply the pasting diagrams for $\theta_U * (-)^2_U$ and $\omega_U * (-)^2_U$, respectively, with several additional identity 2-cells.

Now we point out that this argument can be used in reverse, under the assumption that $(-)_U$ is faithful on 2-cells. Indeed, if $\theta' = \omega'$, then by item (1) above we conclude $\theta_U * (-)^n_U = \omega_U * (-)^n_U$. Since $(-)^2_U$ is an identity on objects, and hence so is $(-)^3_U$, we apply the Whiskering Lemma 1.2.12 to conclude that the components of θ_U are equal to those of $\theta_U * (-)^n_U$. Similarly, the components of ω_U are equal to those of $\omega_U * (-)^n_U$. Therefore, $\theta_U = \omega_U$. Assuming that $(-)_U$ is injective on 2-cells, then we conclude $\theta = \omega$. ◇

2.5. Coherence of Enriched Monoidal Categories

In this section we extend the coherence results for categories that are monoidal, respectively braided monoidal, respectively symmetric monoidal, to V-enriched counterparts. Throughout we assume $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is a braided monoidal category. Recall from Convention 1.3.28 that we assume V to be symmetric monoidal whenever making use of the braiding $\beta^{\otimes}$ for $(\mathsf{V}\text{-}\mathsf{Cat}, \otimes)$. We begin with the following reverse of Corollary 2.4.17, and then use it to lift monoidal coherence results from Cat to $\mathsf{V}\text{-}\mathsf{Cat}$.

Theorem 2.5.1. *Suppose that $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is a braided monoidal category and consider the change of enrichment*

$$(-)_0 : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Cat}.$$

For the braided and symmetric monoidal cases in the statements below, suppose that V is symmetric monoidal.

(1) Suppose K is a V-category.
(a) Suppose $(\boxtimes, \mathrm{I}, a, \ell, r)$ is the data of a monoidal V-category structure for K and let $(\boxtimes', \mathrm{I}', a', \ell', r')$ denote the functors and natural transformations given by change of enrichment. Then

$$(\mathsf{K}, \boxtimes, \mathrm{I}, a, \ell, r)$$

is a monoidal V*-category if and only if*

$$(\mathsf{K}_0, \boxtimes', \mathrm{I}', a', \ell', r')$$

is a monoidal category.

(b) *Suppose, moreover, that* β *is the data of a braiding for a braided monoidal* V*-category structure on* K *and let* β' *denote the natural transformation given by change of enrichment. Then*

$$(\mathsf{K}, \boxtimes, \mathrm{I}, a, \ell, r, \beta)$$

is a braided, respectively symmetric, monoidal V*-category if and only if*

$$(\mathsf{K}_0, \boxtimes', \mathrm{I}', a', \ell', r', \beta')$$

is a braided, respectively symmetric, monoidal category.

(2) *Suppose* K *and* L *are monoidal, respectively braided monoidal, respectively symmetric monoidal,* V*-categories and suppose*

$$F : \mathsf{K} \longrightarrow \mathsf{L}$$

is a V*-functor. Suppose given* V*-natural transformations* (F^2, F^0) *as in the data for* F *to be a monoidal, respectively braided monoidal, respectively symmetric monoidal,* V*-functor. Let* $((F^2)', (F^0)')$ *denote the natural transformations given by change of enrichment. Then*

$$(F, F^2, F^0) : \mathsf{K} \longrightarrow \mathsf{L}$$

satisfies the axioms of a monoidal, respectively braided monoidal, respectively symmetric monoidal, V*-functor if and only if*

$$(F_0, (F^2)', (F^0)') : \mathsf{K}_0 \longrightarrow \mathsf{L}_0$$

satisfies the axioms of a monoidal, respectively braided monoidal, respectively symmetric monoidal, functor.

(3) *Suppose* F *and* G *are monoidal* V*-functors. A* V*-natural transformation*

$$\theta : F \longrightarrow G$$

is monoidal V*-natural if and only if the underlying data*

$$\theta_0 : F_0 \longrightarrow G_0$$

is a monoidal natural transformation.

Proof. We discuss assertion (1) in the symmetric monoidal case; the same argument applies to the other cases and the other two assertions. One implication is provided by Corollary 2.4.17: if $(\mathsf{K}, \boxtimes, \mathrm{I}, a, \ell, r, \beta)$ is symmetric monoidal in the V-enriched sense, then $(\mathsf{K}_0, \boxtimes', \mathrm{I}', a', \ell', r', \beta')$ satisfies the axioms in Cat for a symmetric monoidal category.

For the reverse implication, suppose that $(\mathsf{K}_0, \boxtimes', \mathrm{I}', a', \ell', r', \beta')$ satisfies the appropriate symmetric monoidal axioms. We then use the technical argument outlined at the end of Explanation 2.4.18 as follows. For each axiom of Definitions 1.4.2, 1.4.10, and 1.4.13, we apply $(-)_0$ to the two sides of the required equality and whisker with a monoidal constraint $(-)^2_U$ or $(-)^3_U$ to obtain two natural transformations that are equal in Cat by the corresponding axiom for K_0. Using Lemma 1.2.12 to check the components of the whiskerings, and noting that each of $(-)^n_U$ is the identity on objects, the images under $(-)_0$ are equal in Cat. We noted in Lemma 2.1.7 above that $(-)_0$ is injective on V-natural transformations and therefore each axiom of Definition 1.4.2 holds in V-Cat. □

As a further corollary of Theorem 2.5.1 and the 2-cell injectivity result of Lemma 2.1.7, we can extend the Coherence Theorems 1.1.31, 1.1.38, and 1.1.41 to give a coherence result (Theorem 2.5.6 below) for monoidal, braided monoidal, and symmetric monoidal V-categories (with V symmetric monoidal in the latter cases). First we extend the definitions of (*permuted*) *word* and (*permuted*) *canonical map*.

Definition 2.5.2. Suppose V is a braided monoidal category (respectively symmetric monoidal in the braided and symmetric cases below) and suppose K is a monoidal V-category. Extending Definitions 1.1.29 and 1.1.30, we say that a V-functor

$$w : \mathsf{K}^{\otimes n} \longrightarrow \mathsf{K}$$

is a V-*word* if the underlying functor w_0 composed with the monoidal constraint for $(-)_0$ is a word

$$\mathsf{K}_0^n \cong (\mathsf{K}^n)_0 \xrightarrow{w_0} \mathsf{K}_0.$$

We say that a V-natural isomorphism between V-words of the same length is a *canonical* V-*map* if its underlying natural transformation induces a canonical map.

Moreover, if K is braided, respectively symmetric, monoidal, we extend Definitions 1.1.36 and 1.1.40 to say that $w\sigma$ is a *permuted* V-*word* if the following composite of σ, the monoidal constraint, and w_0 is a permuted word

$$\mathsf{K}_0^n \xrightarrow{\sigma} \mathsf{K}_0^n \cong (\mathsf{K}^n)_0 \xrightarrow{w_0} \mathsf{K}_0.$$

We call this composite the *underlying permuted word*. We say that a V-natural isomorphism is a *braided canonical* V-*map*, respectively *permuted canonical* V-*map* if its underlying natural isomorphism induces a braided, respectively permuted canonical map. Extending Definition 1.1.37, the *underlying braid* of a braided canonical V-map is the underlying braid of its underlying braided canonical map. ⋄

Definition 2.5.3. Suppose K is a monoidal V-category and suppose $X = (X_1, \ldots, X_n)$ is a sequence of objects $X_i \in \mathsf{K}$. The *left normalized product* is defined to be

$$(\cdots((X_1 \boxtimes X_2) \boxtimes X_3)\cdots) \boxtimes X_n. \tag{2.5.4}$$

The *right normalized product* is defined to be

$$X_1 \boxtimes (X_2 \boxtimes (X_3 \boxtimes (\cdots(X_{n-1} \boxtimes X_n)\cdots))). \tag{2.5.5}$$

The left and right normalized products of the empty sequence are both defined to be the monoidal identity object $\mathrm{I} \in \mathsf{K}$. ⋄

The next result is a coherence theorem for monoidal, braided monoidal, and symmetric monoidal V-categories.

Theorem 2.5.6 (Enriched Monoidal Coherence)**.** *Suppose* V *is braided monoidal (respectively symmetric monoidal in the braided and symmetric cases below) and suppose* K *is a monoidal* V*-category.*

(1) If $u, v : \mathsf{K}^{\otimes n} \longrightarrow \mathsf{K}$ are V*-words of the same length then there exists a unique canonical* V*-map $u \longrightarrow v$.*
(2) Suppose, moreover, that K *is braided monoidal and $u\sigma$ and $v\tau$ are permuted* V*-words. If two braided canonical* V*-maps $u\sigma \longrightarrow v\tau$ have the same underlying braid, then they are equal.*

(3) *Suppose, moreover, that* K *is symmetric monoidal. If* $u\sigma$ *and* $v\tau$ *are permuted* V-*words of the same length then there exists a unique permuted canonical* V-*map* $u\sigma \longrightarrow v\tau$.

Proof. The existence statements follow from the existence statements of Theorems 1.1.31 and 1.1.41 because, by Theorem 2.5.1, the data of the V-enriched monoidal structure corresponds to the data of the underlying monoidal structure. Thus we have V-natural transformations whose underlying natural transformations compose to give any canonical map. The uniqueness statements follow from Lemma 2.1.7 together with the uniqueness statements of Theorems 1.1.31, 1.1.38, and 1.1.41. □

Next we describe an enriched analogue of Epstein's Coherence Theorem 1.1.44 for (symmetric) monoidal functors.

Definition 2.5.7. Suppose $F : \mathsf{K} \longrightarrow \mathsf{L}$ is either

- a monoidal V-functor between monoidal V-categories, with V braided monoidal, or
- a symmetric monoidal V-functor between symmetric monoidal V-categories, with V symmetric monoidal.

Recall $F_0 : \mathsf{K}_0 \longrightarrow \mathsf{L}_0$ denotes the underlying (symmetric) monoidal functor of F.

(1) A V-functor
$$G : \mathsf{K}^{\otimes n} \longrightarrow \mathsf{L}$$
is an *F-iterate* if the composite
$$(\mathsf{K}_0^n) \xrightarrow{(-)_0^n} (\mathsf{K}^{\otimes n})_0 \xrightarrow{G_0} \mathsf{L}_0$$
is an F_0-iterate in the sense of Definition 1.1.43 (1).

(2) A V-natural transformation
$$\theta : G \longrightarrow H$$
between parallel F-iterates $\mathsf{K}^{\otimes n} \longrightarrow \mathsf{L}$ is an *F-coherent map* if
$$\theta_0 * 1_{(-)_0^n} : G_0 \circ (-)_0^n \longrightarrow H_0 \circ (-)_0^n$$
is an F_0-coherent map in the sense of Definition 1.1.43 (2). ◇

Theorem 2.5.8 (Enriched Epstein's Coherence)**.** *In the context of Definition 2.5.7, there exists at most one F-coherent map between any two parallel F-iterates $G, H : \mathsf{K}^{\otimes n} \longrightarrow \mathsf{L}$.*

Proof. If θ and ω are F-coherent maps $G \longrightarrow H$, then Epstein's Theorem 1.1.44 implies that we have an equality of F_0-coherent maps
$$\theta_0 * 1_{(-)_0^n} = \omega_0 * 1_{(-)_0^n}.$$
Since $(-)_0^n$ is the identity on objects, the first equality in the Whiskering Lemma 1.2.12 implies that $\theta_0 = \omega_0$. Then the result follows by injectivity of $(-)_0$ on 2-cells (Lemma 2.1.7). □

2.6. Strictification of Enriched Monoidal Categories

Recall from Definition 1.4.2 that a monoidal V-category is called strict if the unitors and associator are identity V-natural transformations. We now use the Enriched Monoidal Coherence Theorem 2.5.6 to extend Mac Lane's Strictification Theorem 1.1.32 to the V-enriched setting. Recall from Definition 1.4.26 that a monoidal adjoint V-equivalence consists of a V-adjoint pair of monoidal V-functors such that the unit and counit are monoidal V-natural isomorphisms.

Theorem 2.6.1 (Enriched Monoidal Strictification)**.** *Suppose* V *is a braided monoidal category and suppose* K *is a monoidal* V*-category. Then there exist*

- *a canonical strict monoidal* V*-category* K_{st} *and*
- *a canonical monoidal adjoint* V*-equivalence*

$$\mathsf{K}_{\mathsf{st}} \underset{i}{\overset{(-)'}{\rightleftarrows}} \mathsf{K}$$

with $(-)'$ *unital strong monoidal,* i *strong monoidal, and* $(-)' \circ i = 1_{\mathsf{K}}$*.*

Proof. The argument here follows the argument of Mac Lane's Strictification Theorem 1.1.32, given in [**ML98**, XI.3 Theorem 1], but uses the Coherence Theorem 2.5.6 in place of the plain monoidal Coherence Theorem 1.1.31.

First we define K_{st} as a V-category. We denote the V-category structure of K with superscripts, such as i^{K} and m^{K}, and use unadorned notation for the structure in K_{st} to be defined. The objects of K_{st} are finite, possibly empty, sequences of objects in K. For such a sequence $X = (X_1, \ldots, X_n)$ we define X' in K to be the left normalized product (2.5.4). In particular, for $X = ()$ the empty tuple we define $()' = \mathrm{I}^{\mathsf{K}}$, the monoidal unit of K. Given two sequences of objects $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_m)$ with $X_i, Y_j \in \mathsf{K}$ we define the hom object of K_{st} as

$$\mathsf{K}_{\mathsf{st}}(X, Y) = \mathsf{K}(X', Y').$$

The identity morphisms and composition in K_{st},

$$\mathbb{1} \xrightarrow{i_X} \mathsf{K}_{\mathsf{st}}(X, X) = \mathsf{K}(X', X')$$

and

$$\mathsf{K}(Y', Z') \otimes \mathsf{K}(X', Y') = \mathsf{K}_{\mathsf{st}}(Y, Z) \otimes \mathsf{K}_{\mathsf{st}}(X, Y) \xrightarrow{m_{X,Y,Z}} \mathsf{K}_{\mathsf{st}}(X, Z) = \mathsf{K}(X', Z'),$$

are likewise defined via the corresponding structure in K:

$$i_X = i^{\mathsf{K}}_{X'} \quad \text{and} \quad m_{X,Y,Z} = m^{\mathsf{K}}_{X',Y',Z'}.$$

This makes K_{st} into a V-category with $(-)'$ a V-functor.

Next we define the monoidal composition

$$\boxtimes : \mathsf{K}_{\mathsf{st}} \otimes \mathsf{K}_{\mathsf{st}} \longrightarrow \mathsf{K}_{\mathsf{st}}.$$

The product on objects is given by concatenation of sequences, with the empty sequence as the monoidal unit. Thus the monoidal composition is strictly associative and unital on objects. To define $\boxtimes$ on hom objects, suppose $A, X, B, Y \in \mathsf{K}_{\mathsf{st}}$ are sequences of objects of K. Then we have

$$\mathsf{K}_{\mathsf{st}}(A, X) \otimes \mathsf{K}_{\mathsf{st}}(B, Y) = \mathsf{K}(A', X') \otimes \mathsf{K}(B', Y')$$

and

$$\mathsf{K}_{\mathsf{st}}(A \boxtimes B, X \boxtimes Y) = \mathsf{K}((A \boxtimes B)', (X \boxtimes Y)').$$

By the Coherence Theorem 2.5.6 (1) there is a unique canonical V-map that, on hom objects, induces

$$\mathsf{K}(A' \boxtimes^{\mathsf{K}} B', X' \boxtimes^{\mathsf{K}} Y') \longrightarrow \mathsf{K}((A \boxtimes B)', (X \boxtimes Y)')$$

where $\boxtimes^{\mathsf{K}}$ denotes the monoidal composition in K. Then we define

$$\boxtimes : \mathsf{K}_{\mathsf{st}}(A, X) \otimes \mathsf{K}_{\mathsf{st}}(B, Y) \longrightarrow \mathsf{K}_{\mathsf{st}}(A \boxtimes B, X \boxtimes Y)$$

as the composite of this canonical V-map with the monoidal composition in K:

$$\mathsf{K}(A', X') \otimes \mathsf{K}(B', Y') \xrightarrow{\boxtimes^{\mathsf{K}}} \mathsf{K}(A' \boxtimes^{\mathsf{K}} B', X' \boxtimes^{\mathsf{K}} Y') \longrightarrow \mathsf{K}((A \boxtimes B)', (X \boxtimes Y)').$$

For the monoidal axioms of K_{st}, the composites around the boundaries of the unity diagrams (1.4.3) and the associativity diagram (1.4.4) are equal as V-functors by the Coherence Theorem 2.5.6 (1) for K. Therefore, we may define the unitors and associator of K_{st} as identity V-natural transformations. The unity and pentagon axioms for K_{st} then hold because each 2-cell is an identity.

We take the unit constraint $((-)')^0$ to be the identity on

$$()' = \mathrm{I}^{\mathsf{K}}$$

and we take the monoidal constraint $((-)')^2$ to be the V-natural transformation with components

$$\mathbb{1} \longrightarrow \mathsf{K}(X' \boxtimes Y', (X \boxtimes Y)')$$

given by the unique canonical V-maps. To verify that $(-)'$ is monoidal as a V-functor, each axiom of Definition 1.4.17 asserts an equality between two parallel canonical V-maps of K. So they are equal by the uniqueness part of Theorem 2.5.6 (1). Therefore, $(-)'$ is a unital strong monoidal V-functor.

The V-functor

$$i : \mathsf{K} \longrightarrow \mathsf{K}_{\mathsf{st}}$$

is defined by sending each object of K to the length 1 sequence and by the identity on hom objects. Components of its unit and monoidal constraints

$$\mathbb{1} \xrightarrow{i^0} \mathsf{K}_{\mathsf{st}}((), (\mathrm{I}^{\mathsf{K}})) \quad \text{and} \quad \mathbb{1} \xrightarrow{i^2} \mathsf{K}_{\mathsf{st}}((P) \boxtimes (Q), (P \boxtimes^{\mathsf{K}} Q)),$$

with $P, Q \in \mathsf{K}$, are given by the identities of I^{K} and $P \boxtimes^{\mathsf{K}} Q$, respectively. To see that i is a strong monoidal V-functor, one can again appeal to Theorem 2.5.6 (1) for the axioms of Definition 1.4.17, or one can make a direct argument as in Explanation 2.6.2 below.

We then have $(iP)' = P$ for $P \in \mathsf{K}$ and unit of the adjunction $(-)' \dashv i$ given by components

$$\mathbb{1} \xrightarrow{i^{\mathsf{K}}_{X'}} \mathsf{K}(X', X') = \mathsf{K}_{\mathsf{st}}(X, i(X'))$$

for $X \in \mathsf{K}_{\mathsf{st}}$. Thus $(-)' \dashv i$ defines a strong monoidal adjoint V-equivalence. □

Explanation 2.6.2. In the proof of Theorem 2.6.1, one can also give the following direct argument that the V-functor

$$i : \mathsf{K} \longrightarrow \mathsf{K}_{\mathsf{st}}$$

is monoidal.

Associativity: Since i^2 is componentwise an identity morphism and since $\boxtimes$ on hom objects involves a canonical V-map, both composites are equal to a component of the monoidal associator a in K. On the left pasting diagram, this component of a comes from the last 2-cell,

$$1_i * a.$$

On the right pasting diagram, this component of a comes from the middle 2-cell

$$1_{\boxtimes} * (1 \otimes i^2) * 1_{a^{\otimes}}.$$

Left Unity: Both pasting diagrams are equal to an identity morphism in K. The right pasting diagram is the identity because ℓ in K_{st} is the identity. On the left pasting diagram, the first 2-cell

$$1_{\boxtimes} * 1_{(1\otimes i)} * (i^0 \otimes 1)$$

is given by a component of $(\ell^{\mathsf{K}})^{-1}$ in K and the component of

$$i^2 * 1_{\mathrm{I}\otimes 1}$$

is an identity. The component of the last 2-cell

$$1_i * \ell_1$$

is a component of ℓ^{K} in K. The composite of these is an identity in K.

Right Unity: Similarly, both pasting diagrams are equal to an identity morphism in K. The right pasting diagram is an identity because K_{st} is strict, and the left pasting diagram has identity components given by composing r^{-1} with r.

One way of explaining the asymmetry between the associativity axiom (where both sides are given by generally nontrivial components) and the unity axioms (where both sides are given by identities) is the following. Even though the components of i^0 and i^2 are given by identities in K, they are not identities in K_{st}. Therefore, their whiskerings appearing in the axioms may have non-identity components. Such whiskerings appear on both sides of the associativity axiom, but only on the left side of each unity axiom. $\diamond$

Next we turn to the braided case, extending the Braided Strictification Theorem 1.1.39 to braided monoidal V-categories.

Theorem 2.6.3 (Enriched Braided Strictification)**.** *Suppose V is a symmetric monoidal category and suppose K is a braided monoidal V-category. Then there exist*

- *a canonical braided strict monoidal V-category K_{st} and*
- *a canonical braided monoidal adjoint V-equivalence*

$$\begin{tikzcd} \mathsf{K}_{\mathsf{st}} \arrow[r, shift left, "(-)'"] & \mathsf{K} \arrow[l, shift left, "i"] \end{tikzcd}$$

with $(-)'$ unital strong braided monoidal, i strong braided monoidal, and $(-)' \circ i = 1_{\mathsf{K}}$.

Proof. The definition of K_{st} and the monoidal adjoint V-equivalence

$$\begin{tikzcd} \mathsf{K}_{\mathsf{st}} \arrow[r, shift left, "(-)'"] & \mathsf{K} \arrow[l, shift left, "i"] \end{tikzcd}$$

are given as in Theorem 2.6.1. Now using the braiding in K, define the braiding in K_{st} to have components

$$\mathbb{1} \xrightarrow{\beta} \mathsf{K}_{\mathsf{st}}(X \boxtimes Y, Y \boxtimes X) = \mathsf{K}((X \boxtimes Y)', (Y \boxtimes X)')$$

at $X, Y \in \mathsf{K}_{\mathsf{st}}$ given by the composite of the braiding for K and the two canonical V-maps from Theorem 2.5.6 (1) having components

$$\mathbb{1} \longrightarrow \mathsf{K}((X \boxtimes Y)', X' \boxtimes^{\mathsf{K}} Y') \quad \text{and} \quad \mathbb{1} \longrightarrow \mathsf{K}(Y' \boxtimes^{\mathsf{K}} X', (Y \boxtimes X)').$$

For each of the hexagon axioms in K_{st}, (1.4.11) and (1.4.12), both sides have the same underlying braid described in Explanation II.1.2.19. Therefore, the axioms hold by Theorem 2.5.6 (2). Therefore, K_{st} is a braided monoidal V-category.

Now to verify that $(-)'$ satisfies the braid axiom in Definition 1.4.18, note that on both sides each component

$$\mathbb{1} \longrightarrow \mathsf{K}(X' \boxtimes^{\mathsf{K}} Y', (Y \boxtimes X)')$$

has underlying braid given by the same elementary block braid (see Definition II.1.2.3, with m and n given by the lengths of the sequences X and Y). Therefore, the two are equal by Theorem 2.5.6 (2) for K. This shows that $(-)'$ is a strong braided monoidal V-functor.

For i, the underlying braid on both sides of the braid axiom is the generating braid s_1 in B_2. Therefore, i is a strong braided monoidal V-functor. □

Finally we turn to the symmetric case, extending the Symmetric Strictification Theorem 1.1.42 to symmetric monoidal V-categories.

Theorem 2.6.4 (Enriched Symmetric Strictification)**.** *Suppose V is a symmetric monoidal category and suppose K is a symmetric monoidal V-category. Then there exist*

- *a canonical symmetric strict monoidal V-category K_{st} and*
- *a canonical symmetric monoidal adjoint V-equivalence*

$$\mathsf{K}_{\mathsf{st}} \underset{i}{\overset{(-)'}{\rightleftarrows}} \mathsf{K}$$

with $(-)'$ unital strong symmetric monoidal, i strong symmetric monoidal, and $(-)' \circ i = 1_{\mathsf{K}}$.

Proof. The definition of K_{st} and the adjoint V-equivalence

$$\mathsf{K}_{\mathsf{st}} \underset{i}{\overset{(-)'}{\rightleftarrows}} \mathsf{K}$$

are given as in Theorems 2.6.1 and 2.6.3. The symmetry axiom of Definition 1.4.13 for K_{st} follows from the uniqueness part of Theorem 2.5.6 (3) for K. □

2.7. Notes

2.7.1 (Literature on Change of Enrichment)**.** Change of enrichment along a monoidal functor is discussed in **[EK66]**. The thesis of Cruttwell **[Cru09]** provides some further development: Theorem 2.2.7 above appears as **[Cru09**, Theorem 4.3.2] and Theorem 2.4.10 (1) above appears as **[Cru09**, Theorem 5.7.1]. ⋄

2.7.2 (Coherence and Strictification)**.** Kong and Zheng give an enriched strictification theorem [**KZ18**, Proposition 2.4], under the assumption that V is strict. Theorem 2.6.1 generalizes to the case that V is not necessarily strict.

The general theory of pseudoalgebras over 2-monads gives another approach to coherence and strictification results. Starting points for this theory are [**BKP89**] and [**Lac02**]. The thesis of Houston [**Hou07**] discusses pseudomonoids specifically and explains general methods for translating results about monoidal categories to corresponding results for pseudomonoids. ◇

2.7.3 (Axiom of Universes)**.** In Motivation 2.2.6 and Explanation 2.2.8, and in the proofs of Corollaries 2.2.10 and 2.3.14, we refer to Grothendieck's *Axiom of Universes*. This axiom asserts that every set belongs to some Grothendieck universe. See [**JY21**, Section 1.1] for further discussion. The results in this chapter do not depend on the Axiom of Universes, but it gives a convenient way of identifying and precisely navigating the size subtleties that arise in the statements of Theorems 2.2.7 and 2.3.9 and their applications. The survey [**Shu∞b**] describes several other potential approaches. ◇

CHAPTER 3

Self-Enrichment and Enriched Yoneda

Throughout this chapter we assume that V is a symmetric monoidal closed category. The material has three separate but closely related points of focus.

Canonical Self-Enrichment. In Section 3.1 we explain how a symmetric monoidal closed category V is canonically enriched over itself. We denote the resulting V-category as $\underline{\mathsf{V}}$. In Section 3.2 we describe represented V-functors and several consequences of the closed structure on V that will be useful throughout the chapter. In Section 3.3 we apply Theorem 2.5.1 to show that $\underline{\mathsf{V}}$ is symmetric monoidal as a V-category.

Enriched Co/Ends and the Enriched Yoneda Lemma. In Section 3.4 we give a preliminary version of the enriched Yoneda Lemma that we call the Enriched Yoneda Bijection 3.4.12 because it asserts a bijection of certain underlying sets. The Yoneda V-functor is described in Section 3.5 and requires background material on V-enriched coends and ends given there. Then in Section 3.6 we give additional necessary background and prove the full V-Yoneda Lemma 3.6.9.

Enriched Symmetric Monoidal Diagram Categories. In Section 3.7 we apply the previous material to categories of enriched symmetric monoidal functors from a small V-category $\mathcal{D}$ to $\underline{\mathsf{V}}$. Using the V-Yoneda Lemma 3.6.9 we prove an equivalent statement known as the V-Yoneda Density Theorem 3.7.8. With this we show that the category of symmetric monoidal V-functors from $\mathcal{D}$ to $\underline{\mathsf{V}}$ is symmetric monoidal and closed. In Section 3.8 we use change of enrichment to show that the category of enriched diagrams to $\underline{\mathsf{V}}$ is symmetric monoidal as a V-category. The additional section Section 3.9 describes tensor and cotensor structures for enriched categories. The material there will be applied to certain diagram categories in Chapters 4, 7, and 9.

3.1. Self-Enriched Categories

Recall from Definition 1.1.28 that a symmetric monoidal category V is closed if, for each object $X \in \mathsf{V}$, the endofunctor $- \otimes X$ has a right adjoint $[X,-] : \mathsf{V} \longrightarrow \mathsf{V}$.

Definition 3.1.1. Suppose V is a symmetric monoidal closed category. For each $X \in \mathsf{V}$ the *evaluation at* X, denoted eval, is the counit

$$[X,-] \otimes X \xrightarrow{\mathsf{eval}} \mathrm{Id}_{\mathsf{V}}.$$

The *coevaluation at* X, denoted coeval, is the unit

$$\mathrm{Id}_{\mathsf{V}} \xrightarrow{\mathsf{coeval}} [X, - \otimes X]. \qquad \diamond$$

Explanation 3.1.2. For reference below, we note two particular consequences of the closed structure on V. Both follow from examining the relevant adjunctions.

(1) The following diagram of sets commutes for each $A, C \in \mathsf{V}$, where the unlabeled isomorphisms are those of the closed structure.

$$\begin{tikzcd}
\mathsf{V}(\mathbb{1}, [A,C]) \arrow[r, "\lambda^*"] \arrow[d, "{[\lambda, C]_*}"'] & \mathsf{V}(\mathbb{1} \otimes \mathbb{1}, [A,C]) \arrow[d, "\cong"] \\
\mathsf{V}(\mathbb{1}, [\mathbb{1} \otimes A, C]) \arrow[r, "\cong"] & \mathsf{V}(\mathbb{1}, [\mathbb{1}, [A,C]])
\end{tikzcd} \tag{3.1.3}$$

(2) The following diagram in V commutes for each $P, Q, A, C \in \mathsf{V}$, where each of the isomorphisms is given by the closed structure of V.

$$\begin{tikzcd}
& {[P, [Q \otimes A, C]]} \arrow[dr, "\cong"] & \\
{[P, [Q, [A,C]]]} \arrow[ur, "\cong"] \arrow[d, "\cong"] & & {[P \otimes (Q \otimes A), C]} \\
{[P \otimes Q, [A,C]]} \arrow[rr, "\cong"] & & {[(P \otimes Q) \otimes A, C]} \arrow[u, "\cong"']
\end{tikzcd} \tag{3.1.4}$$

$\diamond$

Definition 3.1.5. Suppose V is a symmetric monoidal closed category. We let $\underline{\mathsf{V}}$ denote V equipped with the *canonical self-enrichment* defined as follows.

- The objects are those of V.
- For each pair of objects $X, Y \in \underline{\mathsf{V}}$, the hom object with domain X and codomain Y is $\underline{\mathsf{V}}(X,Y) = [X,Y]$.
- For each triple of objects $X, Y, Z \in \underline{\mathsf{V}}$, the composition

$$m_{X,Y,Z} : [Y,Z] \otimes [X,Y] \longrightarrow [X,Z]$$

is defined as the adjoint to the following composite in V.

$$\begin{tikzcd}
([Y,Z] \otimes [X,Y]) \otimes X \arrow[d, "\alpha"'] & Z \\
{[Y,Z] \otimes ([X,Y] \otimes X)} \arrow[r, "1 \otimes \mathsf{eval}"] & {[Y,Z] \otimes Y} \arrow[u, "\mathsf{eval}"']
\end{tikzcd} \tag{3.1.6}$$

- For each object $X \in \underline{\mathsf{V}}$, the identity

$$i_X : \mathbb{1} \longrightarrow [X,X]$$

is defined as the adjoint to the left unit isomorphism in V

$$\mathbb{1} \otimes X \xrightarrow{\lambda} X.$$

Proposition 3.1.11 below shows that these satisfy the axioms of Definition 1.2.1. $\diamond$

Explanation 3.1.7. The multiplication and identity morphisms of Definition 3.1.5 are defined via adjoints. Recall that for an adjuction $F \dashv G$, the adjoints of morphisms

$$FX \xrightarrow{f} Y \quad \text{and} \quad X \xrightarrow{g} GY$$

are the following composites with the unit, respectively counit, of the adjunction:

$$X \longrightarrow GFX \xrightarrow{Gf} GY \quad \text{and} \quad FX \xrightarrow{Fg} FGY \longrightarrow Y.$$

In the context of Definition 3.1.5, this means that the composition of hom objects is given by the following composite.

$$\begin{tikzcd}[column sep=small]
{[Y,Z]\otimes[X,Y]} \arrow[dr, "\mathsf{coeval}"'] & & {[X,[Y,Z]\otimes([X,Y]\otimes X)]} \arrow[dr, "{[X,1\otimes\mathsf{eval}]}"] & & {[X,Y]} \\
& {[X,([Y,Z]\otimes[X,Y])\otimes X]} \arrow[ur, "{[X,\alpha]}"] & & {[X,[Y,Z]\otimes Y]} \arrow[ur, "{[X,\mathsf{eval}]}"'] &
\end{tikzcd}$$

Similarly, the identity is defined by the following composite:

$$\mathbb{1} \xrightarrow{\mathsf{coeval}} [X, \mathbb{1} \otimes X] \xrightarrow{[X,\lambda]} [X,X].$$

Moreover, m and i are defined as adjoints and adjunction is a bijection, so taking adjoints of m and i, respectively, results in the following two commutative diagrams.

$$\begin{tikzcd}
\big([Y,Z]\otimes[X,Y]\big)\otimes X \arrow[r, "m\otimes 1"] \arrow[d, "\alpha"'] & {[X,Z]\otimes X} \arrow[dd, "\mathsf{eval}"] \\
{[Y,Z]\otimes\big([X,Y]\otimes X\big)} \arrow[d, "1\otimes\mathsf{eval}"'] & \\
{[Y,Z]\otimes Y} \arrow[r, "\mathsf{eval}"] & Z
\end{tikzcd} \tag{3.1.8}$$

$$\begin{tikzcd}
\mathbb{1}\otimes X \arrow[r, "i\otimes 1"] \arrow[dr, "\lambda"'] & {[X,X]\otimes X} \arrow[d, "\mathsf{eval}"] \\
& X
\end{tikzcd} \tag{3.1.9}$$

We also note, for a morphism $\psi : P' \longrightarrow P$ in V, we have the following commutative diagram expressing a compatibility of the composition with the induced morphism $[\psi, Q] : [P, Q] \longrightarrow [P', Q]$ for $P, P', Q, R \in \mathsf{V}$.

(3.1.10)
$$
\begin{tikzcd}
{[Q,R] \otimes [P,Q]} \arrow[r, "1 \otimes {[\psi,Q]}"] \arrow[d, "m"'] & {[Q,R] \otimes [P',Q]} \arrow[d, "m"] \\
{[P,R]} \arrow[r, "{[\psi,R]}"] & {[P',R]}
\end{tikzcd}
$$

Commutativity of the above diagram follows by taking adjoints and using the observations above. ⋄

Proposition 3.1.11. *Suppose V is a symmetric monoidal closed category. Then the closed structure gives $\underline{\mathsf{V}}$ the structure of a V-category.*

Proof. The data of the canonical self-enrichment is given in Definition 3.1.5. We verify commutativity of the associativity and unity diagrams of Definition 1.2.1 by verifying that their adjoints commute.

To take the adjoint of the associativity diagram (1.2.2) we apply $- \otimes W$ and then compose with the evaluation

$$\mathsf{eval} : [W, Z] \otimes W \longrightarrow Z.$$

Doing so yields the outermost composites to Z in the diagram below.

$$
\begin{tikzcd}[column sep=small]
& \big(([Y,Z] \otimes [X,Y]) \otimes [W,X]\big) \otimes W \arrow[rr, "\alpha \otimes 1"] \arrow[d, "\alpha"] \arrow[ddddddd, bend right=60, "(m \otimes 1) \otimes 1"'] & & \big([Y,Z] \otimes ([X,Y] \otimes [W,X])\big) \otimes W \arrow[d, "\alpha"'] \arrow[ddddddd, bend left=60, "(1 \otimes m) \otimes 1"] & \\
& ([Y,Z] \otimes [X,Y]) \otimes ([W,X] \otimes W) \arrow[dr, "\alpha"] \arrow[dd, "(1 \otimes 1) \otimes \mathsf{eval}"] \arrow[ddddd, bend right=50, "m \otimes (1 \otimes 1)"'] & & {[Y,Z]} \otimes \big(([X,Y] \otimes [W,X]) \otimes W\big) \arrow[dl, "1 \otimes \alpha"'] \arrow[ddddd, bend left=50, "1 \otimes (m \otimes 1)"] & \\
& & {[Y,Z]} \otimes \big([X,Y] \otimes ([W,X] \otimes W)\big) \arrow[dr, "1 \otimes (1 \otimes \mathsf{eval})"] \arrow[r, phantom, "\star"] & {} & \\
& ([Y,Z] \otimes [X,Y]) \otimes X \arrow[rr, "\alpha"] \arrow[d, "m \otimes 1"] \arrow[dr, phantom, "\star"] & & {[Y,Z]} \otimes ([X,Y] \otimes X) \arrow[d, "1 \otimes \mathsf{eval}"] & \\
& {[X,Z]} \otimes X \arrow[dr, "\mathsf{eval}"] & {} & {[Y,Z]} \otimes Y \arrow[dl, "\mathsf{eval}"'] & \\
& & Z & & \\
& {[X,Z]} \otimes ([W,X] \otimes W) \arrow[uu, "1 \otimes \mathsf{eval}"] \arrow[dr, phantom, "\star"] & & {[Y,Z]} \otimes ([W,Y] \otimes W) \arrow[uu, "1 \otimes \mathsf{eval}"'] \arrow[dl, phantom, "\star"] & \\
& ([X,Z] \otimes [W,X]) \otimes W \arrow[u, "\alpha"] \arrow[r, "m \otimes 1"'] & {[W,Z]} \otimes W \arrow[uu, "\mathsf{eval}"'] & ([Y,Z] \otimes [W,Y]) \otimes W \arrow[u, "\alpha"'] \arrow[l, "m \otimes 1"] &
\end{tikzcd}
$$

In the diagram above, commutativity of the regions marked ☆ follows from (3.1.8). Commutativity of the remaining subdiagrams follows from naturality of α

(three times), the pentagon axiom (1.1.3), and the equality

$$(m\otimes 1)\circ\big((1\otimes 1)\otimes \mathsf{eval}\big) = m\otimes \mathsf{eval} = (1\otimes \mathsf{eval})\circ\big(m\otimes(1\otimes 1)\big).$$

For the unity diagram (1.2.3), we apply $-\otimes X$ and then compose with the evaluation

$$\mathsf{eval} : [X,Y]\otimes X \longrightarrow Y.$$

Doing so yields the outermost composites to Y on both sides of the diagram below.

$$\begin{tikzcd}
\big(\mathbb{1}\otimes[X,Y]\big)\otimes X \arrow[rrr, "\lambda\otimes 1"] \arrow[dr, "\alpha"] \arrow[ddddddd, "(i\otimes 1)\otimes 1"'] & & & {[X,Y]\otimes X} \arrow[ddd, "\mathsf{eval}"] & & & \big([X,Y]\otimes\mathbb{1}\big)\otimes X \arrow[lll, "\rho\otimes 1"'] \arrow[dl, "\alpha"] \arrow[ddddddd, "(1\otimes i)\otimes 1"] \\
& \mathbb{1}\otimes\big([X,Y]\otimes X\big) \arrow[urr, "\lambda"'] \arrow[dr, "1\otimes\mathsf{eval}"] \arrow[ddddd, "i\otimes(1\otimes 1)"'] & & & & & \\
& & & & & {[X,Y]\otimes(\mathbb{1}\otimes X)} \arrow[uull, "1\otimes\lambda"'] \arrow[ddd, "1\otimes(i\otimes 1)"] \arrow[d, phantom, "\heartsuit"] & \\
& & \mathbb{1}\otimes Y \arrow[r, "\lambda"] \arrow[d, "i\otimes 1"] \arrow[dr, phantom, "\heartsuit"] & Y \arrow[d, equal] & & & \\
& & {[Y,Y]\otimes Y} \arrow[r, "\mathsf{eval}"] & Y & & & \\
& & & & & {[X,Y]\otimes\big([X,X]\otimes X\big)} \arrow[uuuuull, "1\otimes\mathsf{eval}"] \arrow[ll, phantom, "\star"] & \\
& {[Y,Y]\otimes\big([X,Y]\otimes X\big)} \arrow[uur, "1\otimes\mathsf{eval}"] \arrow[drr, phantom, "\star"] & & & & & \\
\big([Y,Y]\otimes[X,Y]\big)\otimes X \arrow[ur, "\alpha"] \arrow[rrr, "m\otimes 1"] & & & {[X,Y]\otimes X} \arrow[uuu, "\mathsf{eval}"'] & & & \big([X,Y]\otimes[X,X]\big)\otimes X \arrow[lll, "m\otimes 1"'] \arrow[uul, "\alpha"']
\end{tikzcd}$$

In the diagram above, the unlabeled arrows are composites of an associator with an evaluation. The regions marked ☆ commute by (3.1.8) and the regions marked ♡ commute by (3.1.9). The remaining regions commute by the unity properties (1.1.5) and (1.1.2), naturality of λ, naturality of α (twice), and the equality

$$(1\otimes\mathsf{eval})\circ(i\otimes(1\otimes 1)) = i\otimes\mathsf{eval} = (i\otimes 1)\circ(1\otimes\mathsf{eval}). \qquad \square$$

3.2. Represented Enriched Functors

In this section we describe corepresented and represented V-functors taking values in $\underline{\mathsf{V}}$. In Lemma 3.2.12 we give a characterization of V-natural transformations between V-functors with codomain $\underline{\mathsf{V}}$.

Definition 3.2.1. Suppose C is a V-category and suppose $X\in\mathsf{C}$. Then $\mathcal{Y}^X = \mathsf{C}(X,-)$ denotes the *corepresented* V*-functor*

$$\mathcal{Y}^X : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$$

whose assignment on objects is $W \longmapsto \mathsf{C}(X,W)$ and whose morphism of hom objects

$$(\mathcal{Y}^X)_{Z,W} : \mathsf{C}(Z,W) \longrightarrow [\mathcal{Y}^X Z, \mathcal{Y}^X W]$$

is defined as the adjoint to composition in C:

$$\mathsf{C}(Z,W)\otimes\mathcal{Y}^X Z = \mathsf{C}(Z,W)\otimes\mathsf{C}(X,Z) \xrightarrow{m} \mathsf{C}(X,W) = \mathcal{Y}^X W$$

for each $Z,W\in\mathsf{C}$. The composition and identity axioms of Definition 1.2.4 follow from the associativity and unity axioms (Definition 1.2.1) of C together with the definitions of composition and identity morphisms in $\underline{\mathsf{V}}$. ⋄

For the dual definition, recall from Definition 1.2.16 the opposite of an enriched category.

Definition 3.2.2. Suppose C is a V-category and suppose $Y \in \mathsf{C}$. Then $\mathcal{Y}_Y = \mathsf{C}(-,Y)$ denotes the *represented* V*-functor*

$$\mathcal{Y}_Y : \mathsf{C}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}$$

whose assignment on objects is $Z \longmapsto \mathsf{C}(Z,Y)$ and whose morphism of hom objects

$$(\mathcal{Y}_Y)_{Z,W} : \mathsf{C}^{\mathsf{op}}(Z,W) = \mathsf{C}(W,Z) \longrightarrow [\mathcal{Y}_Y Z, \mathcal{Y}_Y W]$$

is the adjoint to composition in C^{op}:

$$\mathsf{C}(W,Z) \otimes \mathcal{Y}_Y Z = \mathsf{C}(W,Z) \otimes \mathsf{C}(Z,Y) \xrightarrow{\xi} \mathsf{C}(Z,Y) \otimes \mathsf{C}(W,Z) \xrightarrow{m} \mathsf{C}(W,Y) = \mathcal{Y}_Y W.$$

Verifying V-functoriality of $\mathcal{Y}_Y$ is similar to that of $\mathcal{Y}^X$. $\diamond$

Definition 3.2.3. Suppose V is a symmetric monoidal closed category and suppose that $\theta : P \longrightarrow Q$ is a morphism in V. Let $\theta^\perp$ denote the morphism

$$\mathbb{1} \xrightarrow{\theta^\perp} [P,Q]$$

adjoint to $\theta \circ \lambda$. The correspondence $\theta \leftrightarrow \theta^\perp$ provides an isomorphism of categories between V and $(\underline{\mathsf{V}})_0$, the underlying category of the self-enrichment of V. We call this the *canonical underlying isomorphism* of V. $\diamond$

Explanation 3.2.4. To see that the correspondence $\theta \leftrightarrow \theta^\perp$ in Definition 3.2.3 preserves composition, suppose given a composable pair of morphisms in V

$$P \xrightarrow{\theta} Q \xrightarrow{\omega} R.$$

Then the equality

$$(\omega\theta)^\perp = (\omega^\perp)(\theta^\perp) \tag{3.2.5}$$

is commutativity of the following diagram in V.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "(\omega\theta)^\perp"] \arrow[d, "\lambda^{-1}"'] & {[P,R]} \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "(\omega^\perp) \otimes (\theta^\perp)"] & {[Q,R] \otimes [P,Q]} \arrow[u, "m"']
\end{tikzcd}$$

The equality (3.2.5) then follows by considering the adjoints of the composites above, using functoriality of $\otimes$, naturality of λ, and the definition of the composition m. $\diamond$

Recall from Definition 2.1.8 with $\mathsf{D} = \underline{\mathsf{V}}$ we have morphisms

$$\underline{\mathsf{V}}(X,\theta^\perp) = [X,\theta^\perp] : [X,P] \longrightarrow [X,Q] \quad \text{and} \quad [Q,Y] \xrightarrow{[\theta^\perp, Y]} [P,Y]. \tag{3.2.6}$$

In Lemmas 3.2.7 and 3.2.8 we show that these morphisms have a simple description via θ.

Lemma 3.2.7. *Suppose $\theta : P \longrightarrow Q$ is a morphism in V and X is an object of V. The adjoint of*

$$[X, \theta^{\perp}] : [X, P] \longrightarrow [X, Q]$$

is

$$[X, P] \otimes X \xrightarrow{\mathsf{eval}} P \xrightarrow{\theta} Q.$$

Proof. The result follows by commutativity of the diagram below. Recalling Definition 2.1.8, the adjoint to $[X, \theta^{\perp}]$ is the composite along the left, bottom, and right of the diagram.

$$\begin{tikzcd}
{[X,P] \otimes X} \arrow[rr, "\mathsf{eval}"] \arrow[dddd, "\lambda^{-1} \otimes 1"'] \arrow[dddr, "\lambda^{-1}"] & & P \arrow[rrr, "\theta"] \arrow[d, "\lambda^{-1}"] & & & Q \\
& & \mathbb{1} \otimes P \arrow[rr, "\theta^{\perp} \otimes 1"] & & {[P,Q] \otimes P} \arrow[ur, "\mathsf{eval}"] & \\
& & & & & {[X,Q] \otimes X} \arrow[uu, "\mathsf{eval}"'] \\
& \mathbb{1} \otimes ([X,P] \otimes X) \arrow[uur, "1 \otimes \mathsf{eval}"] \arrow[rrr, "\theta^{\perp} \otimes (1 \otimes 1)"] & & & {[P,Q] \otimes ([X,P] \otimes X)} \arrow[uu, "1 \otimes \mathsf{eval}"'] & \\
(\mathbb{1} \otimes [X,P]) \otimes X \arrow[ur, "\alpha"] \arrow[rrrrr, "(\theta^{\perp} \otimes 1) \otimes 1"] & & & & & ([P,Q] \otimes [X,P]) \otimes X \arrow[ul, "\alpha"'] \arrow[uu, "m \otimes 1"']
\end{tikzcd}$$

In the above diagram, the upper-right trapezoid commutes by definition of $\theta^{\perp}$ and the rightmost trapezoid commutes by (3.1.8). The other regions commute by naturality of λ and α, functoriality of $\otimes$, and the left unity diagram (1.1.5). $\square$

Lemma 3.2.8. *Suppose $\theta : P \longrightarrow Q$ is a morphism in V and Y is an object of V. The adjoint of*

$$[\theta^{\perp}, Y] : [Q, Y] \longrightarrow [P, Y]$$

is

$$[Q, Y] \otimes P \xrightarrow{1 \otimes \theta} [Q, Y] \otimes Q \xrightarrow{\mathsf{eval}} Y.$$

Proof. The result follows by commutativity of the diagram below. Recalling Definition 2.1.8, the adjoint to $[\theta^{\perp}, Y]$ is the composite along the left, bottom, and right of the diagram.

$$\begin{tikzcd}
{[Q,Y]\otimes P} \arrow[rr, "1\otimes\theta"] \arrow[ddr, "1\otimes\lambda^{-1}"] \arrow[ddd, "\rho^{-1}\otimes 1"'] & & {[Q,Y]\otimes Q} \arrow[r, "\mathsf{eval}"] & Y \\
 & & & {[P,Y]\otimes P} \arrow[u, "\mathsf{eval}"'] \\
 & {[Q,Y]\otimes(\mathbb{1}\otimes P)} \arrow[r, "1\otimes(\theta^{\perp}\otimes 1)"] & {[Q,Y]\otimes([P,Q]\otimes P)} \arrow[uu, "1\otimes\mathsf{eval}"'] & \\
{([Q,Y]\otimes\mathbb{1})\otimes P} \arrow[ur, "\alpha"] \arrow[rrr, "(1\otimes\theta^{\perp})\otimes 1"] & & & {([Q,Y]\otimes[P,Q])\otimes P} \arrow[ul, "\alpha"] \arrow[uu, "m\otimes 1"']
\end{tikzcd}$$

In the above diagram, the upper trapezoid commutes by definition of $\theta^{\perp}$ and the rightmost trapezoid commutes by (3.1.8). The other regions commute by naturality of α and the middle unity diagram (1.1.2). □

Explanation 3.2.9. The statements of Lemmas 3.2.7 and 3.2.8 will be useful in our work below, but they also allow us to explain the relationship between three subtly different versions of the represented and corepresented functors. Recall we use the notation $\underline{\mathsf{V}}$ to denote V equipped with its self-enrichment as in Definition 3.1.5, and reserve the notation V for the ordinary symmetric monoidal closed category as in Section 1.1.

For $X, Y \in \mathsf{V}$ we have:

(1) the ordinary functors

$$[X,-] : \mathsf{V} \longrightarrow \mathsf{V} \quad \text{and} \quad [-,Y] : \mathsf{V}^{\mathsf{op}} \longrightarrow \mathsf{V}$$

defined via the closed structure on V,

(2) the V-enriched functors

$$\mathcal{Y}^X : \underline{\mathsf{V}} \longrightarrow \underline{\mathsf{V}} \quad \text{and} \quad \mathcal{Y}^Y : \underline{\mathsf{V}}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}$$

of Definitions 3.2.1 and 3.2.2 with $\mathsf{C} = \underline{\mathsf{V}}$, and

(3) the ordinary functors

$$[X,(-)^{\perp}] : (\underline{\mathsf{V}})_0 \longrightarrow \mathsf{V} \quad \text{and} \quad [(-)^{\perp},Y] : (\underline{\mathsf{V}})_0^{\mathsf{op}} \longrightarrow \mathsf{V}$$

defined in (3.2.6) as a special case of Definition 2.1.8.

The functors (1) and (3) correspond under the canonical underlying isomorphism of categories $(\underline{\mathsf{V}})_0 \cong \mathsf{V}$ given by the identity on objects and the correspondence $\theta \leftrightarrow \theta^{\perp}$ from Definition 3.2.3. Indeed for a morphism $\theta : P \longrightarrow Q$ in V we have equalities

$$[X,\theta] = [X,\theta^{\perp}] \quad \text{and} \quad [\theta,Y] = [\theta^{\perp},Y]$$

because, by Lemma 3.2.7 and Lemma 3.2.8, respectively, the corresponding adjoints are equal.

To compare with (2), we consider the induced functor on $(\underline{\mathsf{V}})_0$. For $\theta : P \longrightarrow Q$ in V the morphism $(\mathcal{Y}^X)_0\theta^{\perp}$ is defined to be the composite

$$\mathbb{1} \xrightarrow{\theta^{\perp}} [P,Q] \xrightarrow{\mathcal{Y}^X} [[X,P],[X,Q]].$$

Using the definition of $\mathcal{Y}^X$ and $[X, \theta^\perp]$, one verifies that the adjoint of $(\mathcal{Y}^X)_0\theta^\perp$ is equal to the composite

$$\mathbb{1} \otimes [X,P] \xrightarrow{\lambda} [X,P] \xrightarrow{[X,\theta^\perp]} [X,Q].$$

With a similar calculation for $(\mathcal{Y}_Y)_0\theta^\perp$ we have

$$(\mathcal{Y}^X)_0\theta^\perp = [X,\theta^\perp]^\perp = [X,\theta]^\perp \quad \text{and} \quad (\mathcal{Y}_Y)_0\theta^\perp = [\theta^\perp, Y]^\perp = [\theta, Y]^\perp. \qquad \diamond$$

In our work below we will use the following characterization of V-naturality for the special case of V-functors with codomain $\underline{\mathsf{V}}$.

Lemma 3.2.10. *Suppose that* C *is a* V*-category and that* $F, G : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ *are* V*-functors. Suppose that*

$$\theta_X : FX \longrightarrow GX$$

is a collection of morphisms in V *for* $X \in \mathsf{C}$*. The morphisms*

$$\theta_X^\perp : \mathbb{1} \longrightarrow [FX, GX]$$

are the components of a V*-natural transformation* $\theta^\perp$ *if and only if the following diagram commutes for each pair of objects* X *and* Y *in* C*.*

$$\begin{tikzcd}
\mathsf{C}(X,Y) \otimes FX \arrow[r, "F \otimes 1"] \arrow[d, "1 \otimes \theta_X"'] & {[FX, FY] \otimes FX} \arrow[r, "\mathsf{eval}"] & FY \arrow[d, "\theta_Y"] \\
\mathsf{C}(X,Y) \otimes GX \arrow[r, "G \otimes 1"] & {[GX, GY] \otimes GX} \arrow[r, "\mathsf{eval}"] & GY
\end{tikzcd} \tag{3.2.11}$$

Proof. Beginning with the characterization of V-naturality in Lemma 2.1.11, with $\mathsf{D} = \underline{\mathsf{V}}$, we apply $- \otimes FX$ to the diagram (2.1.12) and compose with evaluation. Then Lemmas 3.2.7 and 3.2.8, together with the equality

$$(1 \otimes \theta_X) \circ (G \otimes 1) = G \otimes \theta_X = (G \otimes 1) \circ (1 \otimes \theta_X),$$

show that the adjoints to $[\theta_X^\perp, GY]$ and $[FX, \theta_Y^\perp]$ are the composites with $1 \otimes \theta_X$ and θ_Y indicated in (3.2.11). This shows that the two composites around the boundary of (3.2.11) are the adjoints of the two composites around (2.1.12). Therefore, for each $X, Y \in \mathsf{C}$, (2.1.12) commutes if and only if (3.2.11) commutes. $\square$

The following immediate applications of Lemma 3.2.10 will be useful in our work below.

Lemma 3.2.12. *Suppose, in the context of Lemma 3.2.10, that each* θ_X *is an isomorphism in* V

$$\theta_X : FX \xrightarrow{\cong} GX.$$

Then the components $\theta_X^\perp$ *are* V*-natural if and only if the inverse components* $(\theta_X^{-1})^\perp$ *are* V*-natural.*

Proof. For each $X, Y \in \mathsf{C}$, the vertical composites in the diagram below are identities.

$$\begin{tikzcd}
\mathsf{C}(X,Y) \otimes FX \arrow[r, "F \otimes 1"] \arrow[d, "1 \otimes \theta_X"'] & {[FX,FY] \otimes FX} \arrow[r, "\mathsf{eval}"] & FY \arrow[d, "\theta_Y"] \\
\mathsf{C}(X,Y) \otimes GX \arrow[r, "G \otimes 1"] \arrow[d, "1 \otimes \theta_X^{-1}"'] & {[GX,GY] \otimes GX} \arrow[r, "\mathsf{eval}"] & GY \arrow[d, "\theta_Y^{-1}"] \\
\mathsf{C}(X,Y) \otimes FX \arrow[r, "F \otimes 1"] & {[FX,FY] \otimes FX} \arrow[r, "\mathsf{eval}"] & FY
\end{tikzcd}$$

Therefore, the upper half commutes if and only if the lower half commutes. Hence the result follows from Lemma 3.2.10. $\square$

Lemma 3.2.13. *Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a V-functor between V-categories. Then the morphisms*

$$\mathsf{C}(X,Y) \xrightarrow{F_{X,Y}} \mathsf{D}(FX,FY)$$

for $X, Y \in \mathsf{C}$ determine V-natural transformations

$$\mathsf{C}(-,W) \xrightarrow{(F_{-,W})^{\perp}} \mathsf{D}(F(-),FW) \quad \textit{and} \quad \mathsf{C}(Z,-) \xrightarrow{(F_{Z,-})^{\perp}} \mathsf{D}(FZ,F(-))$$

for $Z, W \in \mathsf{C}$.

Proof. We apply Lemma 3.2.10 to show that both of the V-naturality conditions follow from the composition axiom for F. For each $W \in \mathsf{C}$ consider the V-functors

$$\mathsf{C}(-,W) : \mathsf{C}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}} \quad \text{and} \quad \mathsf{D}(F(-),FW) : \mathsf{C}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}.$$

The diagram (3.2.11) for $F_{-,W}$ is the outer diagram below, for $X, Y \in \mathsf{C}$, where $m' = m \circ \xi$ is the composition in C^{op}, respectively D^{op}.

$$\begin{tikzcd}[column sep=small]
 & {[\mathsf{C}(Y,W),\mathsf{C}(X,W)] \otimes \mathsf{C}(Y,W)} \arrow[drr, "\mathsf{eval}"] & & \\
\mathsf{C}(X,Y) \otimes \mathsf{C}(Y,W) \arrow[ur, "\mathsf{C}(-,W) \otimes 1"] \arrow[rr, "\xi"'] \arrow[ddd, "1 \otimes F_{Y,W}"'] & & \mathsf{C}(Y,W) \otimes \mathsf{C}(X,Y) \arrow[r, "m"'] \arrow[dd, "F_{Y,W} \otimes F_{X,Y}"] & \mathsf{C}(X,W) \arrow[ddd, "F_{X,W}"] \\
 & & & \\
 & \mathsf{D}(FX,FY) \otimes \mathsf{D}(FY,FW) \arrow[r, "\xi"] & \mathsf{D}(FY,FW) \otimes \mathsf{D}(FX,FY) \arrow[dr, "m"] & \\
\mathsf{C}(X,Y) \otimes \mathsf{D}(FY,FW) \arrow[ur, "F \otimes 1"] \arrow[dr, "\mathsf{D}(F(-),FW) \otimes 1"'] & & & \mathsf{D}(FX,FW) \\
 & {[\mathsf{D}(FY,FW),\mathsf{D}(FX,FW)] \otimes \mathsf{D}(FY,FW)} \arrow[urr, "\mathsf{eval}"'] & &
\end{tikzcd}$$

In the above diagram, the upper and lower regions commute by definitions of $\mathsf{C}(-,W)$ and $\mathsf{D}(F(-),FW)$, respectively. The pentagonal region at left commutes by naturality of ξ, and the remaining trapezoid is the composition axiom. Therefore $(F_{-,W})^{\perp}$ is V-natural. The argument for V-naturality of $(F_{Z,-})^{\perp}$ is similar but does not require the symmetry isomorphism ξ. □

In our work below we will need one further result proved using Lemma 3.2.10. For the following, recall from Definition 1.2.14 a V-adjunction $L \dashv R$ is an adjoint pair of 1-cells in V-Cat.

Proposition 3.2.14. *A pair of* V*-functors*

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "L"] & \mathsf{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

is a V*-adjoint pair with* $L \dashv R$ *if and only if there are isomorphisms in* V

$$\mathsf{D}(LX,Y) \cong \mathsf{C}(X,RY)$$

that are V*-natural with respect to* $X \in \mathsf{C}$ *and* $Y \in \mathsf{D}$.

Proof. For one implication, suppose given unit and counit V-natural transformations

$$\eta : 1_{\mathsf{C}} \longrightarrow RL \quad \text{and} \quad \varepsilon : LR \longrightarrow 1_{\mathsf{D}}$$

making $L \dashv R$ a V-adjunction. For each $X \in \mathsf{C}$ and $Y \in \mathsf{D}$ we define η^* and ε_* to be the following composites in V, with tensor symbols omitted from the diagrams.

$$\begin{tikzcd} \mathsf{C}(RLX,RY) \arrow[r, "\eta^*"] \arrow[d, "\rho^{-1}"'] & \mathsf{C}(X,RY) \\ \mathsf{C}(RLX,RY)\mathbb{1} \arrow[d, "1\eta_X"'] & \\ \mathsf{C}(RLX,RY)\mathsf{C}(X,RLX) \arrow[uur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{D}(LX,LRY) \arrow[r, "\varepsilon_*"] \arrow[d, "\lambda^{-1}"'] & \mathsf{D}(LX,Y) \\ \mathbb{1}\mathsf{D}(LX,LRY) \arrow[d, "\varepsilon_Y 1"'] & \\ \mathsf{D}(LRY,Y)\mathsf{D}(LX,LRY) \arrow[uur, "m"'] & \end{tikzcd}$$

Then we define $\zeta_{X,Y}$ and $\upsilon_{X,Y}$ via the following composites, respectively.

$$\begin{tikzcd} \mathsf{D}(LX,Y) \arrow[r, "\zeta_{X,Y}"] \arrow[d, "R_{LX,Y}"'] & \mathsf{C}(X,RY) \\ \mathsf{C}(RLX,RY) \arrow[ur, "\eta^*"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{C}(X,RY) \arrow[r, "\upsilon_{X,Y}"] \arrow[d, "L_{X,RY}"'] & \mathsf{D}(LX,Y) \\ \mathsf{D}(LX,LRY) \arrow[ur, "\varepsilon_*"'] & \end{tikzcd}$$

To verify that $\zeta^{\perp}$ and $\upsilon^{\perp}$ are V-natural in each variable, first observe by Lemma 3.2.10 that the morphisms $R_{LX,Y}$ and $L_{X,RY}$ define V-natural $R^{\perp}$ and $L^{\perp}$. Then V-naturality of η and ε implies that the composites determining ζ and υ determine V-natural transformations.

The triangle identities of Definition 1.2.14 for η and ε imply that ζ and υ are mutually inverse. For example, the following commutative diagram shows that

$v\zeta$ is equal to the composite $(L\eta^*)\varepsilon^*$, where ε^* is defined dually to ε_*.

$$\begin{tikzcd}
\mathsf{D}(LX,Y) \arrow[r, "R"] \arrow[ddr, "\varepsilon^*"'] & \mathsf{C}(RLX,RY) \arrow[r, "\eta^*"] \arrow[d, "L"] & \mathsf{C}(X,RY) \arrow[d, "L"] \\
& \mathsf{D}(LRLX,LRY) \arrow[r, "L\eta^*"] \arrow[d, "\varepsilon_*"] & \mathsf{D}(LX,LRY) \arrow[d, "\varepsilon_*"] \\
& \mathsf{D}(LRLX,Y) \arrow[r, "L\eta^*"] & \mathsf{D}(LX,Y)
\end{tikzcd}$$

In the above diagram, the left inner region commutes by the V-naturality condition (1.2.8) for ε. The other regions commute by V-functoriality of L and associativity of composition in D. Then the left triangle identity implies $(L\eta^*)\varepsilon^* = 1$. A similar calculation using V-naturality of η and the right triangle identity shows that the reverse composite ζv is also the identity. This finishes the proof of the first implication.

For the reverse implication, suppose given isomorphisms in V

$$\mathsf{D}(LX,Y) \xrightarrow[\cong]{\zeta_{X,Y}} \mathsf{C}(X,RY)$$

such that $\zeta^{\perp}$ is V-natural with respect to $X \in \mathsf{C}$ and $Y \in \mathsf{D}$. By Lemma 3.2.10, V-naturality of $\zeta^{\perp}$ is equivalent to commutativity of the following diagram in V for each $X, X' \in \mathsf{C}$ and $Y, Y' \in \mathsf{D}$, again with tensor symbols omitted.

$$\begin{tikzcd}
\big(\mathsf{C}^{\mathsf{op}}(X,X')\mathsf{D}(Y,Y')\big)\mathsf{D}(LX,Y) \arrow[ddr, "(L^{\mathsf{op}}1)1"] \arrow[ddddd, "(11)\zeta_{X,Y}"'] & & & \mathsf{D}(LX',Y') \arrow[ddddd, "\zeta_{X',Y'}"] \\
& & \mathsf{D}^{\mathsf{op}}(LX,LX')\mathsf{D}(LX,Y') \arrow[ur, "m\,\xi"] & \\
& \big(\mathsf{D}^{\mathsf{op}}(LX,LX')\mathsf{D}(Y,Y')\big)\mathsf{D}(LX,Y) \arrow[ur, "(1m)\alpha"] & & \\
& \big(\mathsf{C}^{\mathsf{op}}(X,X')\mathsf{C}(RY,RY')\big)\mathsf{C}(X,RY) \arrow[dr, "(1m)\alpha"] & & \\
& & \mathsf{C}^{\mathsf{op}}(X,X')\mathsf{C}(X,RY') \arrow[dr, "m\,\xi"] & \\
\big(\mathsf{C}^{\mathsf{op}}(X,X')\mathsf{D}(Y,Y')\big)\mathsf{C}(X,RY) \arrow[uur, "(1R)1"] & & & \mathsf{C}(X',RY')
\end{tikzcd}$$

In the above diagram, the upper and lower composites are those that determine the corresponding morphisms in (3.2.11) for $\mathsf{D}(L(-),-)$ and $\mathsf{C}(-,R(-))$.

This V-naturality condition implies that the following two diagrams commute for each $X, X' \in \mathsf{C}$ and $Y, Y' \in \mathsf{D}$, again with tensor symbols omitted. The first follows from the special case $X = X'$ and the second follows from the special case $Y = Y'$.

(3.2.15)

$$\begin{tikzcd}
\mathsf{D}(Y,Y')\mathsf{D}(LX',Y) \arrow[rr, "m"] \arrow[dd, "1\zeta_{X',Y}"'] & & \mathsf{D}(LX',Y') \arrow[dd, "\zeta_{X',Y'}"] \\
& \mathsf{C}(RY,RY')\mathsf{C}(X',RY) \arrow[dr, "m"] & \\
\mathsf{D}(Y,Y')\mathsf{C}(X',RY) \arrow[ur, "R_{Y,Y'}1"'] & & \mathsf{C}(X',RY')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{D}(LX,Y)\mathsf{C}(X',X) \arrow[dr, "1L_{X',X}"] \arrow[dd, "\zeta_{X,Y}1"'] & & \mathsf{D}(LX',Y) \arrow[dd, "\zeta_{X',Y}"] \\
& \mathsf{D}(LX,Y)\mathsf{D}(LX',LX) \arrow[ur, "m"'] & \\
\mathsf{C}(X,RY)\mathsf{C}(X',X) \arrow[rr, "m"] & & \mathsf{C}(X',RY)
\end{tikzcd}$$

Now we define components of a unit η and counit ε via the following composites, respectively.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\eta_X"] \arrow[d, "i_{LX}"'] & \mathsf{C}(X,RLX) \\
\mathsf{D}(LX,LX) \arrow[ur, "\zeta_{X,LX}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "\varepsilon_Y"] \arrow[d, "i_{RY}"'] & \mathsf{D}(LRY,Y) \\
\mathsf{C}(RY,RY) \arrow[ur, "\zeta^{-1}_{RY,Y}"'] &
\end{tikzcd}$$

We use (3.2.15) to show that η and ε are V-natural. The V-naturality condition (1.2.8) for η at $X, X' \in \mathsf{C}$ and $Y, Y' \in \mathsf{D}$ is the outer diagram below, with tensor symbols omitted.

$$\begin{tikzcd}
& \mathbb{1}\mathsf{C}(X',X) \arrow[rr, "\eta_{X'}1"] \arrow[dr, "i1"'] & & \mathsf{C}(X,RLX)\mathsf{C}(X',X) \arrow[dddr, "m"] & \\
& & \mathsf{D}(LX,LX)\mathsf{C}(X',X) \arrow[ur, "\zeta 1"'] \arrow[d, "1L"] & & \\
\mathbb{1}\mathsf{C}(X',X) \arrow[uur, "11"] \arrow[rr, "iL"] & & \mathsf{D}(LX,LX)\mathsf{D}(LX',LX) \arrow[d, "m"] & & \\
\mathsf{C}(X',X) \arrow[u, "\lambda^{-1}", "\cong"'] \arrow[rr, "L"] \arrow[d, "\rho^{-1}"', "\cong"] & & \mathsf{D}(LX',LX) \arrow[rr, "\zeta"] & & \mathsf{C}(X',RLX) \\
\mathsf{C}(X',X)\mathbb{1} \arrow[rr, "Li"'] \arrow[ddr, "(RL)_{X',X}1"'] & & \mathsf{D}(LX',LX)\mathsf{D}(LX',LX') \arrow[u, "m"'] \arrow[d, "R1"] & & \\
& & \mathsf{D}(RLX',RLX)\mathsf{D}(LX',LX') \arrow[dr, "1\zeta"] & & \\
& \mathsf{C}(RLX',RLX)\mathbb{1} \arrow[ur, "1i"] \arrow[rr, "1\eta_{X'}"'] & & \mathsf{C}(RLX',RLX)\mathsf{C}(X',RLX') \arrow[uuur, "m"'] &
\end{tikzcd}$$

In the above diagram, the lower right region commutes by the first diagram of (3.2.15), with $Y = LX'$ and $Y' = LX$. The upper right region commutes by the second diagram of (3.2.15), with $Y = LX$. The other regions of the above diagram commute by definition of η, unity properties, and V-functoriality of L and R. The argument for V-naturality of ε is similar, again reducing to special cases of (3.2.15).

The triangle identities for η and ε follow from the definitions—notably the invertibility of each $\zeta_{X,Y}$. One uses the formula for composition of natural transformations (Definition 1.2.9) and again reduces to special cases of (3.2.15) to show that each of the relevant components is an identity. □

3.3. Self-Enriched Symmetric Monoidal Categories

In this section we show that the canonical self-enrichment of a symmetric monoidal closed category V extends to an enrichment of the data $\otimes$, $\mathbb{1}$, α, λ, ρ, and ξ over V. In Theorem 3.3.2 we prove that $\underline{\mathsf{V}}$ is a symmetric monoidal V-category.

Definition 3.3.1. We extend the monoidal product and monoidal unit of V to V-functors

$$\boxtimes : \underline{\mathsf{V}} \otimes \underline{\mathsf{V}} \longrightarrow \underline{\mathsf{V}} \quad \text{and} \quad \mathrm{I} : \mathbb{I} \longrightarrow \underline{\mathsf{V}}$$

with the same assignments on objects, so

$$X \boxtimes Y = X \otimes Y \quad \text{and} \quad \mathrm{I}* = \mathbb{1}.$$

We use the notation $\boxtimes$ and I to help distinguish between the different roles that $\otimes$ and $\mathbb{1}$ play in this definition.

For a pair of objects $(X,X'),(Y,Y') \in \underline{\mathsf{V}} \otimes \underline{\mathsf{V}}$, the morphism

$$\boxtimes_{(X,X'),(Y,Y')} : [X,Y] \otimes [X',Y'] \longrightarrow [X \boxtimes X', Y \boxtimes Y']$$

is defined as the adjoint to the following composite, where ξ_{mid} is the middle four interchange of Definition 1.3.2.

$$\begin{tikzcd}
([X,Y] \otimes [X',Y']) \otimes (X \otimes X') \arrow[r, "\xi_{\mathrm{mid}}"] & ([X,Y] \otimes X) \otimes ([X',Y'] \otimes X') \arrow[d, "\mathrm{eval} \otimes \mathrm{eval}"] \\
 & Y \otimes Y'
\end{tikzcd}$$

The morphism

$$\mathbb{1} \longrightarrow [\mathbb{1},\mathbb{1}]$$

is defined as the adjoint to

$$\mathbb{1} \otimes \mathbb{1} \xrightarrow{\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}} \mathbb{1}.$$

For both $\boxtimes$ and I, the composition and identity axioms of Definition 1.2.4 follow by verifying that the relevant adjoints are equal. ◇

Theorem 3.3.2. *Suppose* $\mathsf{V} = (\mathsf{V}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi)$ *is a symmetric monoidal closed category. Then* $\underline{\mathsf{V}}$ *is symmetric monoidal as a* V*-category.*

Proof. Definition 3.3.1 extends $\otimes$ and $\mathbb{1}$ to V-functors denoted $\boxtimes$ and I, respectively. Using the notation of Definition 3.2.3, the natural transformations α, λ, ρ, and ξ define the following adjoint components for $X,Y,Z \in \mathsf{V}$.

$$\begin{aligned}
\alpha^{\perp}_{X,Y,Z} &: \mathbb{1} \longrightarrow [(X \boxtimes Y) \boxtimes Z, X \boxtimes (Y \boxtimes Z)] \\
\lambda^{\perp}_{X} &: \mathbb{1} \longrightarrow [\mathbb{1} \boxtimes X, X] \\
\rho^{\perp}_{X} &: \mathbb{1} \longrightarrow [X \boxtimes \mathbb{1}, X] \\
\xi^{\perp}_{X,Y} &: \mathbb{1} \longrightarrow [X \boxtimes Y, Y \boxtimes X].
\end{aligned}$$

We use Lemma 3.2.10 to verify V-naturality for each collection of components. We then verify that these satisfy the axioms of Definitions 1.4.2, 1.4.10, and 1.4.13. As in Section 1.4, we will write the monoidal product in V-Cat as juxtaposition throughout this proof.

For $\alpha^{\perp}$, we have $\mathsf{C} = (\mathsf{V}^2)\mathsf{V}$, with $F = \boxtimes \circ (\boxtimes 1)$ and $G = \boxtimes \circ (1 \boxtimes)$. Recall from Definition 3.3.1 that

$$\boxtimes : [X,Y] \otimes [X',Y'] \longrightarrow [X \boxtimes X', Y \boxtimes Y']$$

is adjoint to a composite of ξ_{mid} with $\mathsf{eval} \otimes \mathsf{eval}$. Using functoriality of $\otimes$, the morphisms of hom objects defined by F are adjoint to

$$\begin{tikzcd}
(([X,Y] \otimes [X',Y']) \otimes [X'',Y'']) \otimes ((X \otimes X') \otimes X'') \arrow[d] \\
(([X,Y] \otimes X) \otimes ([X',Y'] \otimes X')) \otimes ([X'',Y''] \otimes X'') \arrow[d, "(\mathsf{eval}^{\otimes 2}) \otimes \mathsf{eval}"] \\
(Y \otimes Y') \otimes Y''
\end{tikzcd}$$

where the first unlabeled morphism is a permutation of terms given by α and ξ. Similarly, the morphisms of hom objects defined by G are adjoint to

$$\begin{tikzcd}
([X,Y] \otimes ([X',Y'] \otimes [X'',Y''])) \otimes (X \otimes (X' \otimes X'')) \arrow[d] \\
([X,Y] \otimes X) \otimes (([X',Y'] \otimes X') \otimes ([X'',Y''] \otimes X'')) \arrow[d, "\mathsf{eval} \otimes (\mathsf{eval}^{\otimes 2})"] \\
Y \otimes (Y' \otimes Y'')
\end{tikzcd}$$

where the first unlabeled morphism is a permutation of terms given by α and ξ. Now we apply Lemma 3.2.10. Since the morphisms along the top and bottom of (3.2.11) are the adjoints of F and G, respectively, then $\alpha^{\perp}$ is V-natural if and only if the boundary of the following diagram commutes, where the two unlabeled morphisms are permutations of terms given by α and ξ.

$$\begin{tikzcd}
(([X,Y] \otimes [X',Y']) \otimes [X'',Y'']) \otimes ((X \otimes X') \otimes X'') \arrow[dd, "\alpha \otimes \alpha"'] \arrow[dr] & & (Y \otimes Y') \otimes Y'' \arrow[dd, "\alpha"] \\
& (([X,Y] \otimes X) \otimes ([X',Y'] \otimes X')) \otimes ([X'',Y''] \otimes X'') \arrow[ur, "(\mathsf{eval}^{\otimes 2}) \otimes \mathsf{eval}"] \arrow[dd, bend left, "\alpha"] & \\
([X,Y] \otimes ([X',Y'] \otimes [X'',Y''])) \otimes (X \otimes (X' \otimes X'')) \arrow[dr] & & Y \otimes (Y' \otimes Y'') \\
& ([X,Y] \otimes X) \otimes (([X',Y'] \otimes X') \otimes ([X'',Y''] \otimes X'')) \arrow[ur, "\mathsf{eval} \otimes (\mathsf{eval}^{\otimes 2})"'] &
\end{tikzcd}$$

The left half of the above diagram commutes by the Symmetric Coherence Theorem 1.1.41. The right half commutes by naturality of α.

Verifying V-naturality for $\lambda^{\perp}$, $\rho^{\perp}$, and $\xi^{\perp}$ follows the same structure: In each case the relevant functors F and G are defined via their adjoints, which involve canonical morphisms and evaluation. In each case, the corresponding version of (3.2.11) has one half that commutes by coherence, and another half that commutes by naturality.

The result then follows by Theorem 2.5.1: the symmetric monoidal axioms for $\underline{\mathsf{V}}$ as a V-category hold if and only if the corresponding axioms hold for V as a plain category. □

Definition 3.3.3. Suppose V and W are symmetric monoidal closed categories and suppose $U : \mathsf{V} \longrightarrow \mathsf{W}$ is a symmetric monoidal functor. For each $X, Y \in \mathsf{V}$, let

$$U_{X,Y} : U[X,Y] \longrightarrow [UX, UY]$$

be the morphism in W adjoint to the composite

$$U[X,Y] \otimes UX \xrightarrow{U^2} U([X,Y] \otimes X) \xrightarrow{U\mathsf{eval}} UY.$$

The *standard enrichment* of U is the symmetric monoidal W-functor

$$U : \underline{\mathsf{V}}_U \longrightarrow \underline{\mathsf{W}}$$

defined by the same assignment on objects as the underlying functor U and defined on hom objects by the morphisms $U_{X,Y}$. This finishes the definition of U as an assignment on objects and a collection of morphisms between hom objects. We verify that these define a symmetric monoidal W-functor in Proposition 3.3.4. ◇

Proposition 3.3.4. *In the context of Definition 3.3.3, the standard enrichment of U is a symmetric monoidal W-functor.*

Proof. This proof has three parts. The first is to check the composition and identity axioms of Definition 1.2.4 to verify that U is a W-functor. Each of these is verified by checking that its adjoint commutes. For the composition axiom, with $X, Y, Z \in \mathsf{V}$, taking adjoints and using the definitions of $U_{X,Z}$ and the composition in $\underline{\mathsf{W}}$ yields the outer composites in the following diagram. In this diagram, and the other diagrams in this proof, we abbreviate the monoidal products in V and W as concatenation and use subscripts A_U to denote UA for $A \in \mathsf{V}$.

$$\begin{tikzcd}
([Y,Z]_U[X,Y]_U)X_U \arrow[rr, "U^2 1"] \arrow[dr, "(1U^2)\circ\alpha"] \arrow[dddd, "(U_{Y,Z}U_{X,Y})1"'] & & ([Y,Z][X,Y])_U X_U \arrow[rr, "m_U 1"] \arrow[dr, "\alpha_U \circ U^2"] & & {[X,Z]_U X_U} \arrow[dd, "U^2"] \\
& {[Y,Z]_U([X,Y]X)_U} \arrow[rr, "U^2"] \arrow[dd, "1\mathsf{eval}_U"] & & ([Y,Z]([X,Y]X))_U \arrow[dd, "(1\mathsf{eval})_U"] & \\
& & & & ([X,Z]X)_U \arrow[dd, "\mathsf{eval}_U"] \\
& {[Y,Z]_U Y_U} \arrow[rr, "U^2"] \arrow[dr, "U_{Y,Z}1"] & & ([Y,Z]Y)_U \arrow[dr, "\mathsf{eval}_U"] & \\
([Y_U,Z_U][X_U,Y_U])X_U \arrow[rr, "(1\mathsf{eval})\circ\alpha"] & & {[Y_U,Z_U]Y_U} \arrow[rr, "\mathsf{eval}"] & & Z_U
\end{tikzcd}$$

Commutativity of the subdiagrams above follows from the associativity axiom (1.1.9) for U; naturality of U^2; and the definitions of $U_{X,Y}$, $U_{Y,Z}$, and the composition in $\underline{V}$. A similar adjoint diagram verifies the identity axiom for U.

For the second part of this proof, we use Lemma 3.2.10 to verify that the adjoints $(U^0)^\perp$ and $(U^2)^\perp$ are W-natural. These provide the enriched monoidal and unit constraints for U.

The component of $(U^2)^\perp$ at an object $X \otimes X' \in \mathsf{V} \otimes \mathsf{V}$ is a morphism in W

$$(U^2)^\perp_{X\otimes X'} : \mathbb{1} \longrightarrow [(UX) \otimes (UX'), U(X \otimes X')].$$

For each $X \otimes X'$ and $Y \otimes Y'$ in $\mathsf{V} \otimes \mathsf{V}$, the relevant version of (3.2.11) is the following diagram.

$$\begin{tikzcd}[column sep=small]
([X,Y]_U[X',Y']_U)(X_UX'_U) \arrow[d, "1U^2"'] \arrow[r] & {[X_UX'_U, Y_UY'_U](X_UX'_U)} \arrow[r, "\mathrm{eval}"] & Y_UY'_U \arrow[d, "U^2"] \\
([X,Y]_U[X',Y']_U)(XX')_U \arrow[r] & {[(XX')_U, (YY')_U](XX')_U} \arrow[r, "\mathrm{eval}"] & (YY')_U
\end{tikzcd} \tag{3.3.5}$$

In the above diagram, the top unlabeled morphism is the identity on $(X_UX'_U)$ and the following composite on the other factor:

$$[X,Y]_U[X',Y']_U \xrightarrow{U_{X,Y}U_{X',Y'}} [X_U, Y_U][X'_U, Y'_U] \xrightarrow{\boxtimes} [X_UX'_U, Y_UY'_U].$$

The bottom unlabeled morphism in (3.3.5) is the identity on $(XX')_U$ and the following composite on the other factor:

$$\begin{tikzcd}
{[X,Y]_U[X',Y']_U} \arrow[dr, "U^2"] & & & {[(XX')_U, (YY')_U].} \\
& ([X,Y][X'Y'])_U \arrow[r, "\boxtimes_U"] & {[XX', YY']_U} \arrow[ur, "U_{XX',YY'}"] &
\end{tikzcd}$$

Commutativity of (3.3.5) follows from Epstein's Coherence Theorem 1.1.44 together with the naturality of U^2.

The component of $(U^0)^\perp$ at the unique object of $\mathbb{I}$ is a morphism in W

$$\mathbb{1} \longrightarrow [\mathbb{1}, \mathbb{1}_U],$$

where $\mathbb{1}$ denotes the monoidal unit of either V or W, distinguished by context. Recall from Definition 3.3.1 that the unit for self-enriched monoidal structure is determined by the adjoint of $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$. Let

$$\widetilde{\lambda} : \mathbb{1} \longrightarrow [\mathbb{1}, \mathbb{1}]$$

denote this adjoint of $\lambda_{\mathbb{1}}$ in both V and W. With this notation, and recalling the definition of the monoidal identity for $\underline{V}_U$ from Definition 2.4.3, the version of

(3.2.11) for W-naturality of $(U^0)^{\perp}$ is commutativity of the outer diagram below.

$$\begin{tikzcd}[column sep=small]
& & & {[\mathbb{1},\mathbb{1}]\mathbb{1}} \arrow[dr, "\mathsf{eval}"] & \\
\mathbb{1}\mathbb{1} \arrow[urrr, "\widetilde{\lambda}1"] \arrow[rrrr, "\lambda"'] \arrow[d, "1U^0"'] & & & & \mathbb{1} \arrow[d, "U^0"] \\
\mathbb{1}\mathbb{1}_U \arrow[rrrr, bend left, "\lambda"] \arrow[dr, "U^0 1"'] & & (\mathbb{1}\mathbb{1})_U \arrow[rr, "\lambda_U"] \arrow[dr, "(\widetilde{\lambda}1)_U"'] & & \mathbb{1}_U \\
& \mathbb{1}_U\mathbb{1}_U \arrow[ur, "U^2"] \arrow[dr, "(\widetilde{\lambda})_U 1"'] & & \big([\mathbb{1},\mathbb{1}]\mathbb{1}\big)_U \arrow[ur, "\mathsf{eval}_U"] & \\
& & {[\mathbb{1},\mathbb{1}]_U\mathbb{1}_U} \arrow[ur, "U^2"'] \arrow[dr, "U_{\mathbb{1},\mathbb{1}}1"'] & & \\
& & & {[\mathbb{1}_U,\mathbb{1}_U]\mathbb{1}_U} \arrow[uuur, "\mathsf{eval}"'] &
\end{tikzcd}$$

In the above diagram, the two triangular regions involving $\widetilde{\lambda}$ commute by definition. The upper central region commutes by naturality of λ. The two lower quadrilateral regions commute by naturality of U^2 and definition of $U_{\mathbb{1},\mathbb{1}}$, respectively. The remaining region, involving U^0, U^2, and λ, is the left unity diagram (1.1.10) for U.

The third and final part of this proof consists of checking the symmetric monoidal axioms of Definitions 1.4.17 and 1.4.18. This is similar to the verification of axioms in the proof of Theorem 3.3.2. We check the axioms on components, and in each case taking adjoints yields the corresponding axiom for (U, U^2, U^0).

To illustrate, the associativity axiom of Definition 1.4.17 is the following equality of components for each

$$(X \otimes Y) \otimes Z \in (\mathsf{V}^{\otimes 2}) \otimes \mathsf{V}.$$

$$\begin{tikzcd}
\mathbb{1}(\mathbb{1}\mathbb{1}) \arrow[r, "\alpha_U^{\perp}\left((U^2)^{\perp} ((U^2)^{\perp}1) \right)"] & \begin{array}{c} \big[((XY)Z)_U\,,\,(X(YZ))_U\big] \\ \Big(\big[(XY)_U Z_U\,,\,((XY)Z)_U\big] \\ \big[(X_U Y_U)Z_U\,,\,(XY)_U Z_U\big]\Big) \end{array} \arrow[d] \\
\mathbb{1} \arrow[u, "\lambda^{-1}\circ\lambda^{-1}"'] \arrow[d, "\lambda^{-1}\circ\lambda^{-1}"] & \big[(X_U Y_U)Z_U\,,\,(X(YZ))_U\big] \\
\mathbb{1}(\mathbb{1}\mathbb{1}) \arrow[r, "(U^2)^{\perp}\left((1(U^2)^{\perp})\,\alpha^{\perp} \right)"'] & \begin{array}{c} \big[X_U(YZ)_U\,,\,(X(YZ))_U\big] \\ \Big(\big[X_U(Y_U Z_U)\,,\,X_U(YZ)_U\big] \\ \big[(X_U Y_U)Z_U\,,\,X_U(Y_U Z_U)\big]\Big) \end{array} \arrow[u]
\end{tikzcd}$$

The adjoints of the two composites in the above diagram are two morphisms

$$(X_U Y_U)Z_U \longrightarrow (X(YZ))_U,$$

and it follows from the definitions of $\alpha^\perp$ and $(U^2)^\perp$ that they are the two composites in the associativity axiom (1.1.9) for U. The two unity axioms of Definition 1.4.17 and the braid axiom of Definition 1.4.18 follow in the same way. $\square$

3.4. Enriched Yoneda Bijection

In this section we prove the V-Yoneda Bijection 3.4.12. This is a preliminary version that will be used in the proof of the full V-Yoneda Lemma 3.6.9. We assume throughout that V is symmetric monoidal closed and that C is a small V-category. Recall from Definitions 3.2.1 and 3.2.2 the corepresented and represented V-functors

$$\mathcal{Y}^X = \mathsf{C}(X,-) : \mathsf{C} \longrightarrow \underline{\mathsf{V}} \quad \text{and} \quad \mathcal{Y}_Y = \mathsf{C}(-,Y) : \mathsf{C}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}$$

for $X, Y \in \mathsf{C}$.

Definition 3.4.1. Suppose $W \in \mathsf{V}$. We let κ_W denote the composite shown below.

$$\begin{tikzcd} {[\mathbb{1}, W]} \arrow[rr, "\kappa_W"] \arrow[dr, "\rho_{[\mathbb{1},W]}"'] & & W \\ & {[\mathbb{1}, W] \otimes \mathbb{1}} \arrow[ur, "\mathsf{eval}"'] & \end{tikzcd}$$

Using the triangle identities for the closed monoidal structure of V, one verifies that κ_W is an isomorphism with inverse given by

$$W \xrightarrow{\mathsf{coeval}} [\mathbb{1}, W \otimes \mathbb{1}] \xrightarrow{[\mathbb{1},\rho]} [\mathbb{1}, W].$$

When clear from context we will often omit the subscript W. $\diamond$

Explanation 3.4.2. Checking adjoints shows that the following two diagrams involving κ commute for $Z, W \in \mathsf{V}$. The first uses unity properties in V, and the second follows from the first along with additional unity properties, naturality of the monoidal structure morphisms, and the commutativity of (3.1.8) relating the enriched composition m with the counit eval in V.

$$(3.4.3) \quad \begin{tikzcd} {[\mathbb{1}, Z] \otimes \mathbb{1}} \arrow[r, "\kappa_Z \otimes 1"] \arrow[dr, "\mathsf{eval}"'] & Z \otimes \mathbb{1} \arrow[d, "\rho_Z"] \\ & Z \end{tikzcd} \qquad \begin{tikzcd} {[Z, W] \otimes [\mathbb{1}, Z]} \arrow[r, "1 \otimes \kappa_Z"] \arrow[d, "m"'] & {[Z, W] \otimes Z} \arrow[d, "\mathsf{eval}"] \\ {[\mathbb{1}, W]} \arrow[r, "\kappa_W"] & W \end{tikzcd}$$

$\diamond$

Next we record some notation to be used in our discussion of the V-enriched Yoneda lemmas.

Definition 3.4.4. Suppose C is a small V-category and $F, G : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ are V-functors.

- We let V-nat(F, G) denote the collection of V-natural transformations $F \longrightarrow G$. We note that V-nat(F, G) is a set because the components of a V-natural transformation are indexed by objects of C.

- For objects $X, Y \in \mathsf{V}$, recall that $\mathsf{V}(X, Y)$ denotes the *set* of morphisms $X \longrightarrow Y$ in the category V, while $\underline{\mathsf{V}}(X, Y) = [X, Y]$ denotes the V-object of morphisms for the standard self-enrichment of V.
- Recall that $\mathcal{Y}^X = \mathsf{C}(X, -)$ denotes the corepresented V-functor $\mathsf{C} \longrightarrow \underline{\mathsf{V}}$ for $X \in \mathsf{C}$ (Definition 3.2.1).
- Given a morphism $\eta \in \mathsf{V}(\mathbb{1}, FX)$ for $X \in \mathsf{C}$, define $\overline{\eta}_Y$ for each $Y \in \mathsf{V}$ as the composite in the following diagram.

$$\begin{tikzcd} \mathsf{C}(X,Y) \arrow[r, "\overline{\eta}_Y"] \arrow[d, "F"'] & FY \\ {[FX, FY]} \arrow[r, "{[\eta, FY]}"] & {[\mathbb{1}, FY]} \arrow[u, "\kappa_{FY}"'] \end{tikzcd} \tag{3.4.5}$$

Using the notation of Definition 3.2.3, let

$$\widetilde{\eta}_Y = (\overline{\eta}_Y)^{\perp} : \mathbb{1} \longrightarrow [\mathsf{C}(X, Y), FY]. \tag{3.4.6}$$

By Lemma 3.2.10, using the V-functoriality of F and $\mathcal{Y}^X$, one verifies that the components $\widetilde{\eta}_Y$ define a V-natural transformation

$$\widetilde{\eta} : \mathcal{Y}^X \longrightarrow F.$$

- Given $X \in \mathsf{C}$ and a V-natural transformation $\theta : \mathcal{Y}^X \longrightarrow F$ we define $\theta_0 : \mathbb{1} \longrightarrow FX$ as the composite in the following diagram.

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\theta_0"] \arrow[d, "\theta_X"'] & FX \\ {[\mathsf{C}(X,X), FX]} \arrow[r, "{[i_X, FX]}"] & {[\mathbb{1}, FX]} \arrow[u, "\kappa_{FX}"'] \end{tikzcd} \tag{3.4.7}$$

◇

Explanation 3.4.8. Our proof of Theorem 3.4.12 will rely on several observations, recorded as commutativity of the diagrams below. In each case one verifies the equality by taking adjoints and using

- basic properties of the closed monoidal structure on V (Definition 1.1.1, (1.1.4), and (1.1.5)),
- definitions and basic properties of the canonical self-enrichment (Definition 3.1.5 and Explanation 3.1.7),
- definition of the corepresented functor $\mathcal{Y}^X$ (Definition 3.2.1), and
- properties of the isomorphisms κ (Explanation 3.4.2).

Thus the following diagrams commute for $X, Y \in \mathsf{C}$.

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\eta"] \arrow[d, "i_{FX}"'] & FX \\ {[FX, FX]} \arrow[r, "{[\eta, FX]}"] & {[\mathbb{1}, FX]} \arrow[u, "\kappa_{FX}"'] \end{tikzcd} \tag{3.4.9}$$

(3.4.10)
$$\begin{tikzcd}
\mathcal{Y}^X Y \otimes \mathcal{Y}^X X \arrow[d, "\mathcal{Y}^X \otimes 1"'] \arrow[dr, "m"] & \\
{[\mathcal{Y}^X X, \mathcal{Y}^X Y] \otimes \mathcal{Y}^X X} \arrow[r, "\mathsf{eval}"'] & \mathcal{Y}^X Y
\end{tikzcd}$$

(3.4.11)
$$\begin{tikzcd}
{[\mathcal{Y}^X Y, W] \otimes \mathsf{C}(X,Y)} \arrow[rr, "\mathsf{eval}"] \arrow[d, "1 \otimes \mathcal{Y}^X"'] & & W \\
{[\mathcal{Y}^X Y, W] \otimes [\mathcal{Y}^X X, \mathcal{Y}^X Y]} \arrow[r, "m"] & {[\mathcal{Y}^X X, W]} \arrow[r, "{[i_X, W]}"] & {[\mathbb{1}, W]} \arrow[u, "\kappa_W"']
\end{tikzcd}$$

$\diamond$

Theorem 3.4.12 (V-Yoneda Bijection)**.** *Suppose* V *is a symmetric monoidal closed category and* C *is a small* V*-category. For each* V*-functor* $F : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ *there is a bijection of sets*

(3.4.13)
$$\mathsf{V}\text{-nat}(\mathcal{Y}^X, F) \cong \mathsf{V}(\mathbb{1}, FX)$$

induced by the assignments

$$\theta \longmapsto \theta_0 \quad \text{and} \quad \eta \longmapsto \widetilde{\eta}$$

of Definition 3.4.4 for $\theta \in \mathsf{V}\text{-nat}(\mathcal{Y}^X, F)$ *and* $\eta \in \mathsf{V}(\mathbb{1}, FX)$*. Dually, for each* V*-functor* $G : \mathsf{C}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}$ *there is a bijection of sets*

(3.4.14)
$$\mathsf{V}\text{-nat}(\mathcal{Y}_Y, G) \cong \mathsf{V}(\mathbb{1}, GY).$$

Proof. For the second assertion, we note that

$$\mathcal{Y}_Y = \mathsf{C}(-, Y) = \mathsf{C}^{\mathsf{op}}(Y, -)$$

as V-functors. Therefore, the second follows from applying the first to C^{op}.

To prove the first assertion, suppose given $\eta \in \mathsf{V}(\mathbb{1}, FX)$. Then $\widetilde{\eta}_0$ is the composite

$$\mathbb{1} \xrightarrow{\widetilde{\eta}_X} [\mathsf{C}(X,X), FX] \xrightarrow{[i_X, FX]} [\mathbb{1}, FX] \xrightarrow{\kappa} FX$$

and $\widetilde{\eta}_X = (\overline{\eta}_X)^{\perp}$. To see that $\widetilde{\eta}_0 = \eta$, we first use naturality of ρ to obtain commutative rectangles along the top of the diagram below.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\widetilde{\eta}_X"] \arrow[d, "\rho^{-1}"'] & {[\mathcal{Y}^X X, FX]} \arrow[rr, "{[i_X, FX]}"] \arrow[d, "\rho^{-1}"'] & & {[\mathbb{1}, FX]} \arrow[r, "\kappa"] \arrow[d, "\rho^{-1}"'] & FX \arrow[d, "\rho^{-1}"] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\widetilde{\eta}_X \otimes 1"] \arrow[dr, "1 \otimes i_X"] \arrow[ddd, "\lambda"'] & {[\mathcal{Y}^X X, FX] \otimes \mathbb{1}} \arrow[rr, "{[i_X, FX] \otimes 1}"] \arrow[dr, "1 \otimes i_X"] & & {[\mathbb{1}, FX] \otimes \mathbb{1}} \arrow[r, "\kappa \otimes 1"] \arrow[ddr, "\mathsf{eval}"] & FX \otimes \mathbb{1} \arrow[dd, "\rho"] \\
& \mathbb{1} \otimes \mathsf{C}(X, X) \arrow[r, "\widetilde{\eta}_X \otimes 1"] \arrow[d, "\lambda"] & {[\mathcal{Y}^X X, FX] \otimes \mathsf{C}(X, X)} \arrow[drr, "\mathsf{eval}"] & & \\
& \mathsf{C}(X, X) \arrow[r, "F"] & {[FX, FX]} \arrow[r, "{[\eta, FX]}"] & {[\mathbb{1}, FX]} \arrow[r, "\kappa"] & FX \\
\mathbb{1} \arrow[ur, "i_X"] \arrow[urr, "i_{FX}"] \arrow[urrrr, "\eta"] & & & &
\end{tikzcd}$$

In the above diagram, the left vertical composite is the identity because $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ (1.1.4). Then we obtain the remaining commutative regions using the following, respectively: naturality of λ, functoriality of $\otimes$, the adjoint of $[i_X, FX]$, the triangle (3.4.3), the definition of $\widetilde{\eta}_X = (\overline{\eta}_X)^{\perp}$, the identity axiom for F (Definition 1.2.4), and (3.4.9).

Next, suppose given $\theta \in \mathsf{V}\text{-nat}(\mathcal{Y}^X, F)$. Then $\widetilde{(\theta_0)}$ is a V-natural transformation whose component at $Y \in \mathsf{C}$ is $\overline{(\theta_0)}_Y^{\perp}$ with $\overline{(\theta_0)}_Y$ given by the composite

$$\mathsf{C}(X, Y) \xrightarrow{F} [FX, FY] \xrightarrow{[\theta_0, FY]} [\mathbb{1}, FY] \xrightarrow{\kappa} FY.$$

To see that $\widetilde{(\theta_0)}_Y = \theta_Y$ for each $Y \in \mathsf{C}$, we use the diagram below to show that the adjoint of θ_Y, along the top edge, is equal to $\overline{(\theta_0)}_Y \circ \lambda$, along the left and bottom edges.

$$\begin{tikzcd}
\mathbb{1} \otimes \mathsf{C}(X,Y) \arrow[rr, "\theta_Y \otimes 1"] \arrow[dddddr, "\lambda"] \arrow[ddddddddd, "\lambda"'] & & {[\mathcal{Y}^X Y, FY] \otimes \mathsf{C}(X,Y)} \arrow[rrr, "\mathsf{eval}"] \arrow[d, "1 \otimes \mathcal{Y}^X"'] & & & FY \arrow[dddddddd, "\rho^{-1}"] \\
& & {[\mathcal{Y}^X Y, FY] \otimes [\mathcal{Y}^X X, \mathcal{Y}^X Y]} \arrow[dd, "m"'] & & & \\
& & & {[\mathbb{1}, FY]} \arrow[uurr, "\kappa"] & {[FX,FY] \otimes FX} \arrow[uur, "\mathsf{eval}"'] & \\
& & {[\mathcal{Y}^X X, FY]} \arrow[ur, "{[i_X, FY]}"] & & & \\
& & & {[FX,FY] \otimes [\mathbb{1}, FX]} \arrow[uu, "m"] \arrow[uur, "1 \otimes \kappa"'] & & \\
& \mathsf{C}(X,Y) \arrow[ddd, "\rho^{-1}"'] & & & & \\
& & {[FX,FY] \otimes [\mathcal{Y}^X X, FX]} \arrow[uuu, "m"] \arrow[uur, "1 \otimes {[i_X, FX]}"'] & & & \\
& & {[FX,FY] \otimes \mathbb{1}} \arrow[u, "1 \otimes \theta_X"] \arrow[rr, "{[\theta_0, FY] \otimes 1}"] \arrow[dd, "\rho"] & & {[\mathbb{1}, FY] \otimes \mathbb{1}} \arrow[uuuuuuur, "\mathsf{eval}"] \arrow[ddl, "\rho"'] \arrow[dr, "\kappa \otimes 1"] & \\
& \mathsf{C}(X,Y) \otimes \mathbb{1} \arrow[ur, "F \otimes 1"] \arrow[dl, "\rho"] & & & & FY \otimes \mathbb{1} \arrow[d, "\rho"] \\
\mathsf{C}(X,Y) \arrow[rr, "F"] & & {[FX, FY]} \arrow[r, "{[\theta_0, FY]}"] & {[\mathbb{1}, FY]} \arrow[rr, "\kappa"] & & FY
\end{tikzcd}$$

In the above diagram, each of the quadrilaterals along the bottom commutes by naturality of ρ and the trapezoid at left commutes by invertibility of ρ. Commutativity of the remaing regions is a consequence of the following, respectively: the V-naturality of θ (1.2.8), compatibility of $\mathcal{Y}^X$ with m (3.4.11), compatibility of m with morphisms in V (3.1.10), the square (3.4.3), the definition of θ_0, and the triangle (3.4.3). $\square$

The following application of Theorem 3.4.12 will be useful in Section 3.9 below.

Proposition 3.4.15. *Suppose* D *and* C *are small* V*-categories and suppose given* V*-functors*

$$\begin{tikzcd} \mathsf{D} \arrow[r, bend left, "F"] & \mathsf{C}. \arrow[l, bend left, "G"] \end{tikzcd}$$

Then the underlying functors form an adjoint pair $F_0 \dashv G_0$ *if and only if* $F \dashv G$ *is a* V*-adjoint pair.*

Proof. One implication follows directly from 2-functoriality of

$$(-)_0 : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Cat},$$

proved in Proposition 2.1.2 and Lemma 2.1.5.

For the reverse implication, suppose given V-functors F and G such that $F_0 \dashv G_0$ is an adjoint pair. Then we have unit and counit with components

$$\eta_X \in \mathsf{D}_0(X, GFX) = \mathsf{V}(\mathbb{1}, \mathsf{D}(X, GFX)) \quad \text{and}$$
$$\varepsilon_Y \in \mathsf{C}_0(FGY, Y) = \mathsf{V}(\mathbb{1}, \mathsf{C}(FGY, Y))$$

for each $X \in \mathsf{D}$ and $Y \in \mathsf{C}$. By the Yoneda Bijection 3.4.12 we have

$$\mathsf{V}\text{-nat}(\mathsf{C}(FX,-),\mathsf{D}(X,G(-))) \cong \mathsf{V}(\mathbb{1},\mathsf{D}(X,GFX)) \quad \text{and}$$
$$\mathsf{V}\text{-nat}(\mathsf{D}(-,GY),\mathsf{C}(F(-),Y) \cong \mathsf{V}(\mathbb{1},\mathsf{C}(FGY,Y)).$$

Recalling the proof of Theorem 3.4.12, the morphisms η_X and ε_Y therefore induce components

(3.4.16) $$\mathsf{C}(FX,Y) \xrightarrow{\overline{\eta}_{X,Y}} \mathsf{D}(X,GY) \quad \text{and} \quad \mathsf{D}(X,GY) \xrightarrow{\overline{\varepsilon}_{X,Y}} \mathsf{C}(FX,Y)$$

such that the $\overline{\eta}^{\perp}$ is V-natural with respect to $Y \in \mathsf{V}$ for each $X \in \mathsf{V}$ and $\overline{\varepsilon}^{\perp}$ is V-natural with respect to $X \in \mathsf{V}$ for each $Y \in \mathsf{V}$. The triangle identities for $F_0 \dashv G_0$ imply that these components are mutually inverse isomorphisms in V.

For V-naturality of each morphism with respect to the other variable, we recall from Lemma 3.2.12 that a collection of isomorphisms in V determines a V-natural transformation if and only if its collection of inverse morphisms does so. Therefore, the V-naturality of $\overline{\eta}^{\perp}$ with respect to Y implies that of $\overline{\varepsilon}^{\perp}$ and the V-naturality of $\overline{\varepsilon}^{\perp}$ with respect to X implies that of $\overline{\eta}^{\perp}$. Lastly, by Proposition 3.2.14 the V-naturality of isomorphisms (3.4.16) implies that $F \dashv G$ is a V-adjunction. □

3.5. Enriched Ends and Internal Mapping Objects

The statement of the full V-enriched Yoneda Lemma 3.6.9 requires the following V-enriched notion of coends and ends generalizing the plain categorical notion from Definition I.1.1.16. Proposition 3.5.5 shows that all V-coends, respectively V-ends, exist if V is cocomplete, respectively complete.

Definition 3.5.1. Suppose A is a V-category and suppose $F : \mathsf{A}^{\mathsf{op}} \otimes \mathsf{A} \longrightarrow \underline{\mathsf{V}}$ is a V-functor.

(1) A V-*cowedge* of F is a pair (X,ζ) consisting of
- an object $X \in \underline{\mathsf{V}}$ and
- morphisms $\zeta_a : F(a,a) \longrightarrow X$ in V for $a \in \mathsf{A}$

such that the following diagram in V commutes for all objects $a, a' \in \mathsf{A}$.

(3.5.2)
$$\begin{tikzcd}
\mathsf{A}(a,a') \arrow[r, "F(-,a)"] \arrow[d, "F(a',-)"'] & {[F(a',a),F(a,a)]} \arrow[d, "{[F(a',a),\zeta_a]}"] \\
{[F(a',a),F(a',a')]} \arrow[r, "{[F(a',a),\zeta_{a'}]}"] & {[F(a',a),X]}
\end{tikzcd}$$

(2) A V-*coend* of F is an initial V-cowedge and is denoted by the pair

$$\left(\int^{a\in\mathsf{A}} F(a,a),\omega\right)$$

or simply by $\int^{a\in\mathsf{A}} F(a,a)$ with ω implicit.

A V-*end* of F is the dual notion given as follows.

(3) A V-*wedge* of F is a pair (Y,δ) consisting of
- an object $Y \in \underline{\mathsf{V}}$ and
- morphisms $\delta_a : Y \longrightarrow F(a,a)$ in V for $a \in \mathsf{A}$

such that the following diagram in V commutes for all objects $a, a' \in \mathsf{A}$.

$$\begin{tikzcd} \mathsf{A}(a,a') \arrow[r, "F(a,-)"] \arrow[d, "F(-,a')"'] & {[F(a,a), F(a,a')]} \arrow[d, "{[\delta_a, F(a,a')]}"] \\ {[F(a',a'), F(a,a')]} \arrow[r, "{[\delta_{a'}, F(a,a')]}"] & {[Y, F(a,a')]} \end{tikzcd} \tag{3.5.3}$$

(4) A V-*end* of F is a terminal V-wedge and is denoted by the pair

$$\Big(\int_{a\in\mathsf{A}} F(a,a), \sigma\Big)$$

or simply by $\int_{a\in\mathsf{A}} F(a,a)$ with σ implicit.

When clear from context, we will omit the ambient category from the superscript or subscript. ⋄

Explanation 3.5.4 (Universal Properties of Enriched Coends and Ends)**.** The definition of a V-coend for F as an initial V-cowedge means that it satisfies the following universal property. Given any V-cowedge with components ζ_a as shown below, there exists a unique morphism in V such that the following diagram commutes for all $a \in \mathsf{A}$.

$$\begin{tikzcd} F(a,a) \arrow[d, "\omega_a"'] \arrow[dr, "\zeta_a"] & \\ \int^a F(a,a) \arrow[r, dashed, "\exists!"] & X \end{tikzcd}$$

Dually, a V-end for F satisfies the following universal property. Given any V-wedge with components δ_a as shown below, there exists a unique morphism in V such that the following diagram commutes for all $a \in \mathsf{A}$.

$$\begin{tikzcd} Y \arrow[r, dashed, "\exists!"] \arrow[dr, "\delta_a"'] & \int_a F(a,a) \arrow[d, "\sigma_a"] \\ & F(a,a) \end{tikzcd}$$

⋄

Proposition 3.5.5. *Suppose given a* V*-functor*

$$F : \mathsf{A}^{\mathsf{op}} \otimes \mathsf{A} \longrightarrow \underline{\mathsf{V}}$$

as in Definition 3.5.1. If V *is cocomplete and* A *is small, then the* V*-coend of* F *exists and is given by a coequalizer in* V

$$\int^{a\in\mathsf{A}} F(a,a) \cong \mathsf{coeq}\Big(\coprod_{a,a'\in\mathsf{A}} \mathsf{A}(a,a') \otimes F(a',a) \rightrightarrows \coprod_{a\in\mathsf{A}} F(a,a)\Big). \tag{3.5.6}$$

Dually, if V *is complete and* A *is small, then the* V*-end of* F *exists and is given by an equalizer in* V

$$\int_{a\in\mathsf{A}} F(a,a) \cong \mathsf{eq}\Big(\prod_{a\in\mathsf{A}} F(a,a) \rightrightarrows \prod_{a,a'\in\mathsf{A}} [\mathsf{A}(a,a'),F(a,a')]\Big). \tag{3.5.7}$$

Proof. The symmetric monoidal closed structure of V gives isomorphisms

$$\mathsf{V}(P,[Q,R]) \cong \mathsf{V}(P\otimes Q,R) \xrightarrow[\cong]{\xi^*} \mathsf{V}(Q\otimes P,R) \cong \mathsf{V}(Q,[P,R]) \tag{3.5.8}$$

for each $P,Q,R \in \mathsf{V}$. Using the first bijection of (3.5.8), the V-cowedge condition (3.5.2) for components ζ_a is equivalent to commutativity of the following diagram for each $a,a' \in \mathsf{A}$.

$$\begin{tikzcd}
\mathsf{A}(a,a')\otimes F(a',a) \arrow[r] \arrow[d] & F(a,a) \arrow[d, "\zeta_a"] \\
F(a',a') \arrow[r, "\zeta_{a'}"] & X
\end{tikzcd}$$

In the above diagram, the unlabeled horizontal, respectively vertical, morphism corresponds to $F(-,a)$, respectively $F(a',-)$ under the first bijection of (3.5.8). Therefore the coend of F can be expressed as the indicated coequalizer.

Dually, using the composite bijection of (3.5.8), the V-wedge condition (3.5.3) for components δ_a is equivalent to commutativity of the following diagram for each $a,a' \in \mathsf{A}$.

$$\begin{tikzcd}
Y \arrow[r, "\delta_a"] \arrow[d, "\delta_{a'}"'] & F(a,a) \arrow[d] \\
F(a',a') \arrow[r] & {[\mathsf{A}(a,a'),F(a,a')]}
\end{tikzcd}$$

In the above diagram, the unlabeled vertical, respectively horizontal, morphism corresponds to $F(a,-)$, respectively $F(-,a')$ under the bijection (3.5.8). □

Explanation 3.5.9. In the context of Definition 3.5.1, a V-coend $\int^a F(a,a)$ is called a *coend in* $\underline{\mathsf{V}}$ if, for each $b \in \mathsf{V}$, the morphisms $[\omega_a,b]$ induce an isomorphism

$$\Big[\int^a F(a,a),b\Big] \xrightarrow{\cong} \int_a \Big[F(a,a),b\Big]. \tag{3.5.10}$$

Similarly, a V-end $\int_a F(a,a)$ is called an *end in* $\underline{\mathsf{V}}$ if, for each $b \in \mathsf{V}$, the morphisms $[b,\sigma_a]$ induce an isomorphism

$$\Big[b,\int_a F(a,a)\Big] \xrightarrow{\cong} \int_a \Big[b,F(a,a)\Big]. \tag{3.5.11}$$

If, as we assume throughout this chapter, V is symmetric monoidal closed, then (3.5.10) and (3.5.11) are necessarily isomorphisms and therefore every V-end, respectively V-coend, is an end in $\underline{\mathsf{V}}$, respectively coend in $\underline{\mathsf{V}}$. Therefore, we will not have need for this alternate terminology. ◇

Definition 3.5.12. Suppose $F, G : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ are V-functors with V complete. We define the *mapping object* as the V-end

$$\mathsf{Map}(F,G) = \int_{Y \in \mathsf{C}} \big[FY, GY\big] \tag{3.5.13}$$

with V-wedge components

$$\mathsf{Map}(F,G) \xrightarrow{\sigma_Z} [FZ, GZ]$$

for each $Z \in \mathsf{C}$. ◇

Lemma 3.5.14. *In the context of Definition 3.5.12, the underlying set*

$$\mathsf{V}\big(\mathbb{1}, \mathsf{Map}(F,G)\big)$$

is naturally isomorphic to the set of V*-natural transformations* $F \longrightarrow G$.

Proof. By universality of the end, each morphism in V from $\mathbb{1}$ to $\mathsf{Map}(F,G)$ is uniquely determined by a V-wedge with components

$$\mathbb{1} \xrightarrow{\theta_X} [FX, GX]$$

such that the following diagram commutes for each $X, Y \in \mathsf{C}$.

$$\begin{tikzcd}
\mathsf{C}(X,Y) \arrow[r, "{[FX,-]}"] \arrow[d, "{[-,GY]}"'] & \big[[FX,GX],[FX,GY]\big] \arrow[d, "{[\theta_X,[FX,GY]]}"] \\
\big[[FY,GY],[FX,GY]\big] \arrow[r, "{[\theta_Y,[FX,GY]]}"] & \big[\mathbb{1},[FX,GY]\big]
\end{tikzcd}$$

Taking adjoints and unpacking the definitions of the morphisms involved, commutativity of the diagram above is equivalent to commutativity of the defining diagram (1.2.8) for V-naturality of θ. □

Definition 3.5.15. Suppose $\theta : P \longrightarrow [Q, R]$ is a morphism in V. The *transform* of θ is

$$\theta^{\#} : Q \longrightarrow [P, R]$$

defined as the adjoint of

$$Q \otimes P \xrightarrow{\xi} P \otimes Q \xrightarrow{\theta \otimes 1} [Q,R] \otimes Q \xrightarrow{\mathsf{eval}} R. \quad ◇$$

The following result will be useful in our discussion of the Enriched Yoneda Lemma 3.6.9 below.

Lemma 3.5.16. *Suppose* $F : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ *is a* V*-functor, suppose* $X \in \mathsf{C}$*, and suppose*

$$\theta_Y : \mathcal{Y}^X Y \longrightarrow [Q, FY]$$

is a morphism in V *for each* $Y \in \mathsf{C}$*. The transforms*

$$\theta_Y^{\#} : Q \longrightarrow [\mathcal{Y}^X Y, FY]$$

form a wedge (Q, θ) *if and only if the morphisms*

$$\theta_Y^{\perp} : \mathbb{1} \longrightarrow [\mathcal{Y}^X Y, [Q, FY]]$$

adjoint to $\theta_Y \circ \lambda$ *are components of a* V*-natural transformation* $\mathcal{Y}^X \longrightarrow [Q, F(-)]$.

Proof. The V-naturality condition for $\theta_Y^{\perp}$ given in Lemma 3.2.10 is that the diagram below commutes for each $Y \in \mathsf{C}$.

(3.5.17)

$$\begin{tikzcd}
\mathsf{C}(X,Y) \otimes \mathcal{Y}^X X \arrow[r, "\mathcal{Y}^X \otimes 1"] \arrow[d, "1 \otimes \theta_X"'] & {[\mathcal{Y}^X X, \mathcal{Y}^X Y] \otimes \mathcal{Y}^X X} \arrow[r, "\mathsf{eval}"] & \mathcal{Y}^X Y \arrow[d, "\theta_Y"] \\
\mathsf{C}(X,Y) \otimes [Q, FX] \arrow[r, "{[Q,F(-)] \otimes 1}"] & {\big[[Q,FX],[Q,FY]\big] \otimes [Q,FX]} \arrow[r, "\mathsf{eval}"] & {[Q,FY]}
\end{tikzcd}$$

To explain the V-wedge condition (3.5.3) for $\theta^{\#}$, we let $\overline{\theta}_Y$ denote the adjoint of θ_Y, given by the following composite.

$$\begin{tikzcd}
(\mathcal{Y}^X Y) \otimes Q \arrow[rr, "\overline{\theta}"] \arrow[dr, "\theta_Y \otimes 1"'] & & FY \\
& {[Q,FY] \otimes Q} \arrow[ur, "\mathsf{eval}"'] &
\end{tikzcd}$$

The V-wedge condition (3.5.3) for $\theta^{\#}$ is that the outer diagram below commutes for each $Y \in \mathsf{C}$. In this diagram the unlabeled isomorphisms are given by the monoidal closed structure of V.

(3.5.18)

$$\begin{tikzcd}
\mathsf{C}(X,Y) \arrow[rr, "\mathcal{Y}^X"] \arrow[ddd, "F"'] & & {[\mathcal{Y}^X X, \mathcal{Y}^X Y]} \arrow[rr, "{[-,FY]}"] \arrow[dd, "- \otimes Q"'] & & {\big[[\mathcal{Y}^X Y, FY],[\mathcal{Y}^X X, FY]\big]} \arrow[dl, "\cong"'] \arrow[ddddd, "{[\theta^{\#}_Y,[\mathcal{Y}^X,FY]]}"] \\
& & & {\big[[\mathcal{Y}^X Y, FY] \otimes \mathcal{Y}^X X, FY\big]} \arrow[ddd, "{[\theta^{\#}_Y \otimes 1, FY]}"] & \\
& \star & {[(\mathcal{Y}^X X) \otimes Q, (\mathcal{Y}^X Y) \otimes Q]} \arrow[d, "{[(\mathcal{Y}^X X) \otimes Q, \overline{\theta}_Y]}"] & & \\
{[FX,FY]} \arrow[rr, "{[\overline{\theta}_X, FY]}"] \arrow[dr, "{[\mathsf{eval},1]}"] \arrow[dd, "{[\mathcal{Y}^X X, -]}"'] & & {[(\mathcal{Y}^X X) \otimes Q, FY]} \arrow[dr, "{[\xi, FY]}"] & & \\
& {\big[[\mathcal{Y}^X X, FX] \otimes \mathcal{Y}^X X, FY\big]} \arrow[rr, "{[\theta^{\#}_X \otimes 1, FY]}"'] & & {[Q \otimes \mathcal{Y}^X X, FY]} & \\
{\big[[\mathcal{Y}^X X, FX],[\mathcal{Y}^X X, FY]\big]} \arrow[ur, "\cong"] \arrow[rrrr, "{[\theta^{\#}_X,[\mathcal{Y}^X,FY]]}"'] & & & & {\big[Q,[\mathcal{Y}^X X, FY]\big]} \arrow[ul, "\cong"']
\end{tikzcd}$$

In the above diagram, each region except the one labeled ☆ commutes as a consequence of the symmetric monoidal closed structure of V and the definitions of the relevant morphisms. Therefore $\theta^{\#}$ satisfies the V-wedge condition if and only if the region ☆ commutes.

The adjoints of the two composites around the boundary of ☆ are given by the composites around the boundary of the diagram below. In this below diagram, each region except the one labeled ♡ commutes as a consequence of the symmetric monoidal closed structure of V and the definitions of the relevant morphisms. Therefore, the composites around the boundary of the diagram below are equal if and only if the composites around the boundary of the region ♡ are equal.

$$
\begin{tikzcd}
& & \big[(\mathcal{Y}^X X)\otimes Q,(\mathcal{Y}^X Y)\otimes Q\big]\otimes\big((\mathcal{Y}^X X)\otimes Q\big) \arrow[drr, "{[(\mathcal{Y}^X X)\otimes Q,\overline{\theta}_Y]\otimes(1\otimes 1)}"] \arrow[ddr, "\mathsf{eval}"'] & & \\
\mathsf{C}(X,Y)\otimes\big((\mathcal{Y}^X X)\otimes Q\big) \arrow[urr, "\big((\mathcal{Y}^X(-))\otimes Q\big)\otimes(1\otimes 1)"] \arrow[dddddd, "F\otimes(1\otimes 1)"'] & & & & \big[(\mathcal{Y}^X X)\otimes Q,FY\big]\otimes\big((\mathcal{Y}^X X)\otimes Q\big) \arrow[dddd, "\mathsf{eval}"] \\
& (\mathsf{C}(X,Y)\otimes\mathcal{Y}^X X)\otimes Q \arrow[ul, "\alpha"] \arrow[rr, "m\otimes 1"] \arrow[d, "(1\otimes\theta_X)\otimes 1"'] \arrow[dr, phantom, "\heartsuit"] & & (\mathcal{Y}^X Y)\otimes Q \arrow[dddr, "\overline{\theta}_Y"] & \\
& (\mathsf{C}(X,Y)\otimes[Q,FX])\otimes Q \arrow[dr, "([Q,F(-)]\otimes 1)\otimes 1"] \arrow[dd, "(F\otimes 1)\otimes 1"'] & & & \\
& & \big(\big[[Q,FX],[Q,FX]\big]\otimes[Q,FX]\big)\otimes Q \arrow[dr, "\mathsf{eval}\otimes 1"] & & \\
& ([FX,FY]\otimes[Q,FX])\otimes Q \arrow[rr, "m\otimes 1"] \arrow[dr, "\alpha"] & & [Q,FX]\otimes Q \arrow[r, "\mathsf{eval}"] & FY \\
& & [FX,FY]\otimes([Q,FX]\otimes Q) \arrow[d, "1\otimes\mathsf{eval}"] & & \\
[FX,FY]\otimes\big((\mathcal{Y}^X X)\otimes Q\big) \arrow[rr, "1\otimes\overline{\theta}"] & & [FX,FY]\otimes FX \arrow[uurr, "\mathsf{eval}"'] & &
\end{tikzcd}
$$

Noting that the composite along the top of (3.5.17) is the composition m in C, the composites around the boundary of the region labeled ♡ in the diagram above are the adjoints of the composites around the boundary of (3.5.17). Thus for each $Y \in \mathsf{C}$ the V-naturality diagram (3.5.17) commutes if and only if the wedge diagram (3.5.18) does so. □

Now we show that the mapping objects give a V-category of V-functors $\mathsf{C} \longrightarrow \underline{\mathsf{V}}$.

Definition 3.5.19. Suppose C is a small V-category with V complete. We define the *mapping* V-*category* C-V to have objects given by V-functors $\mathsf{C} \longrightarrow \underline{\mathsf{V}}$ and hom objects

$$\mathsf{C}\text{-}\mathsf{V}(F,G) = \mathsf{Map}(F,G)$$

for V-functors F and G.

For each $F \in \mathsf{C}\text{-}\mathsf{V}$, the identity i_F is the universal arrow induced by $\theta_Z = i_{FZ}$ for each $Z \in \mathsf{C}$.

(3.5.20)
$$\begin{tikzcd} \mathbb{1} \arrow[r, "i_F"] \arrow[dr, "i_{FZ}"'] & \mathsf{Map}(F,F) \arrow[d, "\sigma_Z"] \\ & {[FZ,FZ]} \end{tikzcd}$$

For $F,G,H \in \mathsf{C}\text{-}\mathsf{V}$, the composition morphism $m = m^{\mathsf{Map}}$ is the universal arrow induced by the composite $m \circ (\sigma \otimes \sigma)$ below.

(3.5.21)
$$\begin{tikzcd} \mathsf{Map}(G,H) \otimes \mathsf{Map}(F,G) \arrow[r, "m^{\mathsf{Map}}"] \arrow[d, "\sigma_Z \otimes \sigma_Z"'] & \mathsf{Map}(F,H) \arrow[d, "\sigma_Z"] \\ {[GZ,HZ] \otimes [FZ,GZ]} \arrow[r, "m"] & {[FZ,HZ]} \end{tikzcd}$$

In Lemma 3.5.24 we show that the unity and associativity axioms hold for $\mathsf{C}\text{-}\mathsf{V}$, making it a V-category. Lemma 3.5.14 above shows that the underlying category of $\mathsf{C}\text{-}\mathsf{V}$ is isomorphic to the plain category $\mathsf{V}\text{-}\mathsf{Cat}(\mathsf{C},\underline{\mathsf{V}})$. $\diamond$

Explanation 3.5.22. In (3.5.21) above, the adjoint of $\sigma_Z \circ m^{\mathsf{Map}}$ is equal to the lower left composite below.

(3.5.23)
$$\begin{tikzcd} \big(\mathsf{Map}(G,H) \otimes \mathsf{Map}(F,G)\big) \otimes FZ \arrow[d, "(\sigma_Z \otimes \sigma_Z) \otimes 1"'] & \\ \big([GZ,HZ] \otimes [FZ,GZ]\big) \otimes FZ \arrow[r, "\mathsf{eval} \circ (m \otimes 1)"] \arrow[d, "\alpha"'] & HZ \\ {[GZ,HZ] \otimes \big([FZ,GZ] \otimes FZ\big)} \arrow[r, "1 \otimes \mathsf{eval}"'] & {[GZ,HZ] \otimes GZ} \arrow[u, "\mathsf{eval}"'] \end{tikzcd}$$

The rectangle commutes by definition of the composition m in $\underline{\mathsf{V}}$ via (3.1.6). $\diamond$

Lemma 3.5.24. *In the context of Definition 3.5.19,* $\mathsf{C}\text{-}\mathsf{V}$ *is a* V*-category.*

Proof. We begin with the left unity diagram (1.2.3). By the universal property of V-ends, it suffices to show that the following diagram commutes for each $Z \in \mathsf{C}$. Here and throughout the rest of this proof we abbreviate $\mathsf{Map}(-,-)$ as $\mathsf{M}(-,-)$ and omit the tensor symbols in V.

$$\begin{tikzcd} \mathbb{1}\mathsf{M}(F,G) \arrow[r, "\lambda"] \arrow[d, "i_G 1"'] & \mathsf{M}(F,G) \arrow[dr, "\sigma_Z"] & \\ & & {[FZ,GZ]} \\ \mathsf{M}(G,G)\mathsf{M}(F,G) \arrow[r, "m^{\mathsf{Map}}"] & \mathsf{M}(F,G) \arrow[ur, "\sigma_Z"'] & \end{tikzcd}$$

To see that the above diagram commutes, we verify that the following adjoint diagram commutes, where we have used (3.5.23) for the adjoint of $\sigma_Z \circ m^{\mathsf{Map}}$.

$$
\begin{tikzcd}
\big(\mathbb{1}\mathsf{M}(F,G)(FZ)\big) \arrow[rr, "\lambda 1"] \arrow[dr, "(1\sigma_Z)1"] \arrow[dd, "(i_G 1)1"'] & & \mathsf{M}(F,G)(FZ) \arrow[dr, "\sigma_Z 1"] & \\
& \big(\mathbb{1}[FZ,GZ]\big)(FZ) \arrow[rr, "\lambda 1"] \arrow[dd, "(i_{GZ}1)1"] & & {[FZ,GZ](FZ)} \arrow[d, "\mathsf{eval}"] \\
\big(\mathsf{M}(G,G)\mathsf{M}(F,G)\big)(FZ) \arrow[dr, "(\sigma_Z\sigma_Z)1"'] & & & GZ \\
& \big([GZ,GZ][FZ,GZ]\big)(FZ) \arrow[uurr, "m1"] & &
\end{tikzcd}
$$

In the above diagram, the parallelogram at left commutes by the definition of i_G (3.5.20), the upper parallelogram commutes by naturality of λ, and the remaining triangle commutes by the left unity diagram for $\underline{\mathsf{V}}$. This verifies that the left unity diagram for C-V commutes. The right unity diagram for C-V is similar, using the right unity of $\underline{\mathsf{V}}$.

Now we turn to the associativity diagram (1.2.2) for m^{Map}. Again using universality of V-ends, it suffices to show that the following diagram commutes for $E, F, G, H \in \mathsf{C}$-$\mathsf{V}$.

$$
\begin{tikzcd}
\big(\mathsf{M}(G,H)\mathsf{M}(F,G)\big)\mathsf{M}(E,F) \arrow[rr, "\alpha"] \arrow[d, "m^{\mathsf{Map}} 1"'] & & \mathsf{M}(G,H)\big(\mathsf{M}(F,G)\mathsf{M}(E,F)\big) \arrow[d, "1m^{\mathsf{Map}}"] \\
\mathsf{M}(F,H)\mathsf{M}(E,F) \arrow[dd, "m^{\mathsf{Map}}"'] & & \mathsf{M}(G,H)\mathsf{M}(E,G) \arrow[d, "m^{\mathsf{Map}}"] \\
& & \mathsf{M}(E,H) \arrow[d, "\sigma_Z"] \\
\mathsf{M}(E,H) \arrow[rr, "\sigma_Z"] & & {[EZ,HZ]}
\end{tikzcd}
$$

To see that the above diagram commutes, we verify that the following adjoint diagram commutes, where we have used (3.5.23) twice for adjoints of $\sigma_Z \circ m^{\mathsf{Map}}$ and abbreviated

$$
L = \Big(\big([GZ,HZ][FZ,GZ]\big)[EZ,FZ]\Big)(EZ)
$$

and

$$
R = \Big([GZ,HZ]\big([FZ,GZ][EZ,FZ]\big)\Big)(EZ).
$$

$$\begin{tikzcd}[column sep=small]
& \Big(\big(\mathsf{M}(G,H)\mathsf{M}(F,G)\big)\mathsf{M}(E,F)\Big)(EZ) \arrow[rrr, "\alpha 1"] \arrow[dl, "(m^{\mathsf{Map}}1)1"'] \arrow[dr, "((\sigma_Z\sigma_Z)\sigma_Z)1"] &&& \Big(\mathsf{M}(G,H)\big(\mathsf{M}(F,G)\mathsf{M}(E,F)\big)\Big)(EZ) \arrow[dl, "(\sigma_Z(\sigma_Z\sigma_Z))1"'] \arrow[dr, "(1m^{\mathsf{Map}})1"] & \\
\big(\mathsf{M}(F,H)\mathsf{M}(E,F)\big)(EZ) \arrow[dr, "(\sigma_Z\sigma_Z)1"'] && L \arrow[r, "\alpha 1"'] \arrow[dl, "(m1)1"] & R \arrow[dr, "(1m)1"'] && \big(\mathsf{M}(G,H)\mathsf{M}(E,G)\big)(EZ) \arrow[dl, "(\sigma_Z\sigma_Z)1"] \\
& \big([FZ,HZ][EZ,FZ]\big)(EZ) \arrow[dr, "m1"] &&& \big([GZ,HZ][EZ,GZ]\big)(EZ) \arrow[dll, "m1"'] & \\
&& {[EZ,HZ](EZ)} \arrow[r, "\mathsf{eval}"] & HZ &&
\end{tikzcd}$$

In the above diagram, the quadrilaterals at left and right commute by definition of m^{Map}. The upper middle region commutes by naturality of α and the lower middle region commutes by associativity for m in $\underline{\mathsf{V}}$. This completes the proof that $\mathsf{C}\text{-}\mathsf{V}$ is a V-category. □

Definition 3.5.25 (Yoneda V-functor)**.** Suppose C is a small V-category with V complete. Define the *Yoneda* V*-functor*

$$\mathcal{Y}^{(-)} : \mathsf{C} \longrightarrow (\mathsf{C}\text{-}\mathsf{V})^{\mathsf{op}}$$

by

$$X \longmapsto \mathcal{Y}^X$$

on objects and the following universal arrow on hom objects:

$$\begin{tikzcd}
\mathsf{C}(X,Y) \arrow[r, "(\mathcal{Y}^{(-)})_{X,Y}"] \arrow[dr, "(\mathcal{Y}_Z)_{X,Y}"'] & \mathsf{Map}(\mathcal{Y}^Y, \mathcal{Y}^X) \arrow[d, "\sigma_Z"] \\
& {[\mathsf{C}(Y,Z), \mathsf{C}(X,Z)]}
\end{tikzcd} \tag{3.5.26}$$

We show in Lemma 3.5.29 that the identity and composition axioms hold for $\mathcal{Y}^{(-)}$, making it a V-functor. We show in Corollary 3.6.11 that $\mathcal{Y}^{(-)}$ induces isomorphisms on hom objects. For this reason, $\mathcal{Y}^{(-)}$ is also called the V-*Yoneda embedding*. ◇

Explanation 3.5.27. In the context of Definition 3.5.25 above, the adjoint of $\sigma_Z \circ (\mathcal{Y}^{(-)})_{X,Y}$ is the adjoint of $(\mathcal{Y}_Z)_{X,Y}$. Recalling Definition 3.2.2, it is the below composite of the symmetry ξ in V and the composition m of C:

$$\mathsf{C}(X,Y) \otimes \mathsf{C}(Y,Z) \xrightarrow{\xi} \mathsf{C}(Y,Z) \otimes \mathsf{C}(X,Y) \xrightarrow{m} \mathsf{C}(X,Z). \tag{3.5.28}$$ ◇

Lemma 3.5.29. *In the context of Definition 3.5.25, $\mathcal{Y}^{(-)}$ is a* V*-functor.*

Proof. Throughout this proof we will omit the tensor symbols in V and we will use the following abbreviations for $X, Y \in \mathsf{C}$:

$$\mathsf{M} = \mathsf{Map} \quad \text{and} \quad \mathsf{C}_{X,Y} = \mathsf{C}(X,Y).$$

Now we begin with the identity axiom (1.2.6). By universality of V-ends, it suffices to show that the following diagram commutes for $Y, Z \in \mathsf{C}$.

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, "i_Y"] \arrow[dr, "i_{\mathcal{Y}^Y}"] \arrow[ddr, "i_{\mathsf{C}_{Y,Z}}"'] & & \mathsf{C}_{Y,Y} \arrow[dl, "\mathcal{Y}^{(-)}"'] \arrow[ddl, "\mathcal{Y}_Z"] \\
& \mathsf{M}(\mathcal{Y}^Y, \mathcal{Y}^Y) \arrow[d, "\sigma_Z"] & \\
& {[\mathsf{C}_{Y,Z}, \mathsf{C}_{Y,Z}]} &
\end{tikzcd}$$

Taking adjoints, commutativity of the above diagram is equivalent to that of the outer diagram below. We have used (3.5.28) for the adjoint of $\mathcal{Y}_Z$.

$$\begin{tikzcd}
\mathbb{1}\mathsf{C}_{Y,Z} \arrow[r, "i_Y 1"] \arrow[dr, "\xi"] \arrow[ddr, "\lambda"'] & \mathsf{C}_{Y,Y}\mathsf{C}_{Y,Z} \arrow[dr, "\xi"] & \\
& \mathsf{C}_{Y,Z}\mathbb{1} \arrow[r, "1 i_Y"] \arrow[d, "\rho"] & \mathsf{C}_{Y,Z}\mathsf{C}_{Y,Y} \arrow[dl, "m"] \\
& \mathsf{C}_{Y,Z} &
\end{tikzcd}$$

In the above diagram, the upper parallelogram commutes by naturality of ξ. The triangle at left commutes by symmetry of units (1.1.20) and the triangle at right commutes by the right unity axiom for composition m in C.

Now we turn to the composition axiom (1.2.5). By universality of V-ends, it suffices to verify that the following diagram commutes for $W, X, Y, Z \in \mathsf{C}$. In the below diagram, $(m^{\mathsf{Map}})'$ denotes the composition in $(\mathsf{C}\text{-}\mathsf{V})^{\mathsf{op}}$ and is defined via the lower triangle.

$$\begin{tikzcd}
\mathsf{C}_{Y,Z}\mathsf{C}_{X,Y} \arrow[r, "m"] \arrow[d, "\mathcal{Y}^{(-)}\mathcal{Y}^{(-)}"'] & \mathsf{C}_{X,Z} \arrow[d, "\mathcal{Y}^{(-)}"] \arrow[dr, "\mathcal{Y}_W"] & \\
\mathsf{M}(\mathcal{Y}^Z, \mathcal{Y}^Y)\mathsf{M}(\mathcal{Y}^Y, \mathcal{Y}^X) \arrow[r, "(m^{\mathsf{Map}})'"] \arrow[d, "\xi"'] & \mathsf{M}(\mathcal{Y}^Z, \mathcal{Y}^X) \arrow[r, "\sigma_W"] & {[\mathcal{Y}^Z W, \mathcal{Y}^X W]} \\
\mathsf{M}(\mathcal{Y}^Y, \mathcal{Y}^X)\mathsf{M}(\mathcal{Y}^Z, \mathcal{Y}^Y) \arrow[ur, "m^{\mathsf{Map}}"'] & &
\end{tikzcd}$$

Taking adjoints, commutativity of the diagram above is equivalent to that of the outer diagram below. We have used (3.5.28) for the adjoint of $\mathcal{Y}_Z$ and the lower composite of (3.5.23) for the adjoint of $\sigma_W \circ m^{\mathsf{Map}}$.

(3.5.30)

$$\begin{tikzcd}
 & & (\mathsf{C}_{Y,Z}\mathsf{C}_{X,Y})\mathsf{C}_{Z,W} \arrow[dll, "(\mathcal{Y}^{(-)}\mathcal{Y}^{(-)})1"'] \arrow[dd, "\xi 1"] \arrow[dr, "\xi"] \arrow[rrr, "m1"] & & & \mathsf{C}_{X,Z}\mathsf{C}_{Z,W} \arrow[dd, "\xi"] \\
\big(\mathsf{M}(\mathcal{Y}^Z,\mathcal{Y}^Y)\mathsf{M}(\mathcal{Y}^Y,\mathcal{Y}^X)\big)\mathsf{C}_{Z,W} \arrow[ddd, "\xi 1"'] & & & \mathsf{C}_{Z,W}(\mathsf{C}_{Y,Z}\mathsf{C}_{X,Y}) \arrow[drr, "1m"] \arrow[dd, "\alpha^{-1}"] & & \\
 & & (\mathsf{C}_{X,Y}\mathsf{C}_{Y,Z})\mathsf{C}_{Z,W} \arrow[ddll, "(\mathcal{Y}^{(-)}\mathcal{Y}^{(-)})1"'] \arrow[dd, "\alpha"] & & & \mathsf{C}_{Z,W}\mathsf{C}_{X,Z} \arrow[ddddd, "m"] \\
 & & & (\mathsf{C}_{Z,W}\mathsf{C}_{Y,Z})\mathsf{C}_{X,Y} \arrow[dd, "\xi"'] \arrow[dr, "m1"] & & \\
\big(\mathsf{M}(\mathcal{Y}^Y,\mathcal{Y}^X)\mathsf{M}(\mathcal{Y}^Z,\mathcal{Y}^Y)\big)\mathsf{C}_{Z,W} \arrow[ddddd, "(\sigma_W\sigma_W)1"'] & & \mathsf{C}_{X,Y}(\mathsf{C}_{Y,Z}\mathsf{C}_{Z,W}) \arrow[dr, "1\xi"] \arrow[ddd, "\mathcal{Y}^{(-)}(\mathcal{Y}^{(-)}1)"'] & & \mathsf{C}_{Y,W}\mathsf{C}_{X,Y} \arrow[dd, "\xi"'] \arrow[dr, "1"] & \\
 & & & \mathsf{C}_{X,Y}(\mathsf{C}_{Z,W}\mathsf{C}_{Y,Z}) \arrow[dr, "1m"'] & & \mathsf{C}_{Y,W}\mathsf{C}_{X,Y} \arrow[dd, "m"] \\
 & & & & \mathsf{C}_{X,Y}\mathsf{C}_{Y,W} \arrow[ur, "\xi"'] & \\
 & & \mathsf{M}(\mathcal{Y}^Y,\mathcal{Y}^X)\big(\mathsf{M}(\mathcal{Y}^Z,\mathcal{Y}^Y)\mathsf{C}_{Z,W}\big) \arrow[ddr, "\sigma_W(\sigma_W 1)"'] & & & \mathsf{C}_{X,W} \\
 & & & & {[\mathsf{C}_{Y,W},\mathsf{C}_{X,W}]\mathsf{C}_{Y,W}} \arrow[ur, "\mathsf{eval}"'] & \\
\big([\mathsf{C}_{Y,W},\mathsf{C}_{X,W}][\mathsf{C}_{Z,W},\mathsf{C}_{Y,W}]\big)\mathsf{C}_{Z,W} \arrow[rrr, "\alpha"] & & & {[\mathsf{C}_{Y,W},\mathsf{C}_{X,W}]\big([\mathsf{C}_{Z,W},\mathsf{C}_{Y,W}]\mathsf{C}_{Z,W}\big)} \arrow[ur, "1\mathsf{eval}"'] & &
\end{tikzcd}$$

In the above diagram, the four unlabeled regions commute by naturality of α (once) and ξ (three times). The two regions labeled ♡ commute, respectively, by the definition of $\mathcal{Y}^{(-)}$ and by associativity of the composition m in C. The two regions labeled ☆ commute by the symmetry axiom (1.1.24) for ξ and the Symmetric Coherence Theorem 1.1.41 for V. □

Explanation 3.5.31. We note that the composition axiom for $\mathcal{Y}^{(-)}$ depends heavily on the underlying assumption that V is symmetric monoidal. The two regions marked ☆ in (3.5.30) do not commute when V is merely braided monoidal. ◇

Definition 3.5.32. Suppose $F : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ is a V-functor with V complete. Define

$$\mathsf{Map}(-,F) : (\mathsf{C}\text{-}\mathsf{V})^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}$$

by

$$G \longmapsto \mathsf{Map}(G,F) = \int_{Z\in\mathsf{C}} [GZ,FZ]$$

on objects and the following universal arrow on hom objects:

$$
\begin{tikzcd}
\mathsf{Map}(G,H) \arrow[rr, "{\big(\mathsf{Map}(-,F)\big)_{G,H}}"] \arrow[d, "\sigma_W"'] & & {[\mathsf{Map}(H,F),\mathsf{Map}(G,F)]} \arrow[d, "{[\mathsf{Map}(H,F),\sigma_W]}"] \\
{[GW,HW]} \arrow[dr, "{[-,FW]}"'] & & {[\mathsf{Map}(H,F),[GW,FW]]} \\
& {[[HW,FW],[GW,FW]]} \arrow[ur, "{[\sigma_W,[GW,FW]]}"'] &
\end{tikzcd}
\tag{3.5.33}
$$

As we describe in Explanation 3.5.34 below, $\mathsf{Map}(-,F)$ is the represented V-functor $\mathcal{Y}_F$ for the V-category C-V. ⋄

Explanation 3.5.34 (V-Functoriality of $\mathsf{Map}(-,F)$)**.** In the context of Definition 3.5.32 above, the adjoint of (3.5.33) is given by the following diagram, where the unlabeled top arrow is the adjoint of $\big(\mathsf{Map}(-,F)\big)_{G,H}$.

$$
\begin{tikzcd}
\mathsf{Map}(G,H)\otimes\mathsf{Map}(H,F) \arrow[rr] \arrow[d, "\sigma_W\otimes\sigma_W"'] & & \mathsf{Map}(G,F) \arrow[dr, "\sigma_W"] & \\
& & & {[GW,FW]} \\
{[GW,HW]\otimes[HW,FW]} \arrow[rr, "\xi"] & & {[HW,FW]\otimes[GW,HW]} \arrow[ur, "m"'] &
\end{tikzcd}
\tag{3.5.35}
$$

Recalling (3.5.21), this shows that $\big(\mathsf{Map}(-,F)\big)_{G,H}$ is adjoint to

$$\big(m^{\mathsf{Map}}\big)' = m^{\mathsf{Map}} \circ \xi,$$

the composition in $(\mathsf{C}\text{-}\mathsf{V})^{\mathsf{op}}$. Therefore, $\mathsf{Map}(-,F)$ is the V-functor represented by F. ⋄

Now Lemma 3.5.29 and V-functoriality of $\mathsf{Map}(-,F)$ give the following.

Proposition 3.5.36. *For each* V*-functor* $F : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$,

$$\mathsf{Map}(\mathcal{Y}^{(-)},F) : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$$

is a V*-functor.*

Explanation 3.5.37. The composite $\mathsf{Map}(\mathcal{Y}^{(-)},F)$ in Proposition 3.5.36 is given on objects by the assignment

$$X \longmapsto \mathsf{Map}(\mathcal{Y}^X,F) = \int_{Z\in\mathsf{C}} [\mathcal{Y}^X Z, FZ]$$

for $X \in \mathsf{C}$. The morphism on hom objects $\mathsf{C}(X,Y)$ is the top horizontal morphism in the following diagram, induced by the other morphisms.

$$\begin{tikzcd}
\mathsf{C}(X,Y) \arrow[rrrr, "\mathsf{Map}(\mathcal{Y}^{(-)},F)"] \arrow[dr, "\mathcal{Y}^{(-)}"] \arrow[dd, "\mathsf{C}(-,Z)"'] & & & & {[\mathsf{Map}(\mathcal{Y}^X,F),\mathsf{Map}(\mathcal{Y}^Y,F)]} \arrow[dd, "{[\mathsf{Map}(\mathcal{Y}^X,F),\sigma_Z]}"] \\
& \mathsf{Map}(\mathcal{Y}^Y,\mathcal{Y}^X) \arrow[urrr, "\mathsf{Map}(-,F)"'] \arrow[dl, "\sigma_Z"] & & & \\
{[\mathcal{Y}^Y Z,\mathcal{Y}^X Z]} \arrow[dr, "{[-,FZ]}"'] & & & & \big[\mathsf{Map}(\mathcal{Y}^X,F),[\mathcal{Y}^Y Z,FZ]\big] \\
& \big[[\mathcal{Y}^X Z,FZ],[\mathcal{Y}^Y Z,FZ]\big] \arrow[urrr, "{[\sigma_Z,[\mathcal{Y}^Y Z,FZ]]}"'] & & &
\end{tikzcd} \tag{3.5.38}$$

In the above diagram, the upper triangular region is the definition of the composite V-functor on hom objects, and the remaining two regions are the definitions (3.5.26) and (3.5.33) of $\mathcal{Y}^{(-)}$ and $\mathsf{Map}(-,F)$, respectively. $\diamond$

3.6. Enriched Yoneda Lemma

Throughout this section we assume V is a symmetric monoidal closed category that is complete. The main purpose of this section is to prove the V-Yoneda Lemma 3.6.9.

Definition 3.6.1. For each V-functor $F : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ and each $X \in \mathsf{C}$ let

$$FX \xrightarrow{\phi_X} \mathsf{Map}(\mathcal{Y}^X,F)$$

denote the morphism in V induced by morphisms

$$FX \xrightarrow{F^{\#}_{X,Z}} [\mathcal{Y}^X Z, FZ]$$

that are the transforms of

$$\mathcal{Y}^X Z = \mathsf{C}(X,Z) \xrightarrow{F_{X,Z}} [FX, FZ].$$

We show in Lemma 3.6.5 below that the ϕ_X determine a V-natural transformation

$$\phi^{\perp} : F \longrightarrow \mathsf{Map}(\mathcal{Y}^{(-)},F). \qquad \diamond$$

Explanation 3.6.2 (The morphisms ϕ_X)**.** The composition axiom for F (1.2.5) together with Lemma 3.2.10 shows that the morphisms

$$F^{\perp}_{X,Z} : \mathbb{1} \longrightarrow [\mathsf{C}(X,Z),[FX,FZ]]$$

determine a V-natural transformation $\mathsf{C}(-,-) \longrightarrow [F(-),F(-)]$. In particular, for fixed X, Lemma 3.5.16 shows that the morphisms $F^{\#}_{X,Z}$ for $Z \in \mathsf{C}$ satisfy the V-wedge condition (3.5.3), thus inducing the morphisms

$$FX \xrightarrow{\phi_X} \int_{Z \in \mathsf{C}} \big[\mathcal{Y}^X Z, FZ\big] = \mathsf{Map}(\mathcal{Y}^X,F)$$

in V. $\diamond$

We will use the following result with $H = \mathcal{Y}_Z$ to show that each $F^{\#}_{-,Z}$ determines a V-natural transformation. Then we use that result to show that $\phi^{\perp}$ is V-natural.

Lemma 3.6.3. *Suppose $F : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ and $H : \mathsf{C}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}$ are V-functors and suppose $Q \in \mathsf{V}$. A collection of morphisms*

$$\theta_X : FX \longrightarrow [HX, Q]$$

for $X \in \mathsf{C}$ determines a V-natural transformation $\theta^{\perp}$ if and only if the transforms

$$\theta^{\#}_X : HX \longrightarrow [FX, Q]$$

determine a V-natural transformation $(\theta^{\#})^{\perp}$.

Proof. We apply Lemma 3.2.10. Throughout this proof we omit tensor symbols and use $[-, Q]$ to denote the V-enriched functor from $\underline{\mathsf{V}}^{\mathsf{op}}$ to $\underline{\mathsf{V}}$ represented by Q. The diagram (3.2.11) for $\theta^{\perp}$ is the following, for $X, Y \in \mathsf{C}$.

$$\begin{tikzcd}
\mathsf{C}(X,Y)FX \arrow[rr, "F1"] \arrow[ddd, "1\theta_X"'] & & {[FX,FY]FX} \arrow[r, "\mathsf{eval}"] & FY \arrow[ddd, "\theta_Y"] \\
& {[HY,HX][HX,Q]} \arrow[drr, phantom, ""] \arrow[ddr, "{[-,Q]1}"] & & \\
& & & \\
\mathsf{C}(X,Y)[HX,Q] \arrow[uur, "H1"] \arrow[rr, "{[H(-),Q]1}"] & & {\big[[HX,Q],[HY,Q]\big][HX,Q]} \arrow[r, "\mathsf{eval}"] & {[HY,Q]}
\end{tikzcd}$$

The diagram (3.2.11) for $(\theta^{\#})^{\perp}$ is the following, for $X, Y \in \mathsf{C}$.

$$\begin{tikzcd}
\mathsf{C}(X,Y)HY \arrow[rr, "H1"] \arrow[ddd, "1\theta^{\#}_Y"'] & & {[HY,HX]HY} \arrow[r, "\mathsf{eval}"] & HX \arrow[ddd, "\theta^{\#}_X"] \\
& {[FX,FY][FY,Q]} \arrow[ddr, "{[-,Q]1}"] & & \\
& & & \\
\mathsf{C}(X,Y)[FY,Q] \arrow[uur, "F"] \arrow[rr, "{[F(-),Q]1}"] & & {\big[[FY,Q],[FX,Q]\big][FY,Q]} \arrow[r, "\mathsf{eval}"] & {[FX,Q]}
\end{tikzcd}$$

Now let

$$\overline{\theta}_X : FX \otimes HX \longrightarrow Q$$

denote the morphism adjoint to θ_X for each $X \in \mathsf{C}$, so that $\overline{\theta}_X \xi$ is the adjoint of $\theta^{\#}_X$. Then taking adjoints and using naturality of ξ and α together with the definition of the represented V-functors $[-, Q]$ (Definition 3.2.2), we see that each of the diagrams above commutes if and only if the following diagram commutes.

$$\begin{tikzcd}
\big(\mathsf{C}(X,Y)FX\big)HY \arrow[r, "(F1)1"] \arrow[dd] & \big([FX,FY]FX\big)HY \arrow[r, "\mathsf{eval}1"] & (FY)(HY) \arrow[dr, "\overline{\theta}_Y"] & \\
& & & Q \\
\big(\mathsf{C}(X,Y)HY\big)FX \arrow[r, "(H1)1"] & \big([HY,HX]HY\big)FX \arrow[r, "\mathsf{eval}1"] & (HX)(FX) \arrow[ur, "\overline{\theta}_X \xi"'] &
\end{tikzcd}$$

Therefore, $\theta^{\perp}$ is V-natural if and only if $(\theta^{\#})^{\perp}$ is V-natural. $\square$

Lemma 3.6.4. *For each* V*-functor* $F : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ *and each* $Z \in \mathsf{C}$*, the transforms* $F^{\#}_{X,Z}$ *determine a* V*-natural transformation*

$$(F^{\#}_{-,Z})^{\perp} : F \longrightarrow [\mathcal{Y}_Z, FZ]$$

Proof. By Lemma 3.2.13, the composition axiom for F implies that $F_{-,Z}$ determines a V-natural transformation

$$F^{\perp}_{-,Z} : \mathcal{Y}_Z \longrightarrow [F(-), FZ].$$

Then the result follows from Lemma 3.6.3. □

Lemma 3.6.5. *In the context of Definition 3.6.1, the morphisms*

$$\phi_X : FX \longrightarrow \mathsf{Map}(\mathcal{Y}^X, F)$$

for $X \in \mathsf{C}$ *are components of a* V*-natural transformation*

$$\phi^{\perp} : F \longrightarrow \mathsf{Map}(\mathcal{Y}^{(-)}, F).$$

Proof. We apply Lemma 3.2.10. We omit tensor symbols and abbreviate $\mathsf{M} = \mathsf{Map}$ in this proof. The diagram (3.2.11) for ϕ is the upper region labeled ☆ in the diagram below. By universality of V-ends it suffices to show that the composites around ☆ are equal after composed with σ_Z for each $Z \in \mathsf{C}$.

$$\begin{tikzcd}
\mathsf{C}(X,Y)FX \arrow[rr, "F1"] \arrow[ddd, "1\phi_X"'] & & {[FX,FY]FX} \arrow[rr, "\mathsf{eval}"] & & FY \arrow[ddd, "\phi_Y"] \\
& & \star & & \\
& & {[\mathsf{M}(\mathcal{Y}^X,F),\mathsf{M}(\mathcal{Y}^Y,F)]\mathsf{M}(\mathcal{Y}^X,F)} \arrow[drr, "\mathsf{eval}"] & & \\
\mathsf{C}(X,Y)\mathsf{M}(\mathcal{Y}^X,F) \arrow[urr, "{[\mathsf{M}(\mathcal{Y}^{(-)},F)]1}"] \arrow[rr, "\mathcal{Y}^{(-)}1"'] \arrow[dd, "1\sigma_Z"'] & & \mathsf{M}(\mathcal{Y}^Y,\mathcal{Y}^X)\mathsf{M}(\mathcal{Y}^X,F) \arrow[u, "\mathsf{M}(-,F)1"'] \arrow[rr, "(m^{\mathsf{Map}})'"'] \arrow[dd, "\sigma_Z\sigma_Z"] & & \mathsf{M}(\mathcal{Y}^Y,F) \arrow[dd, "\sigma_Z"] \\
& \mathsf{M}(\mathcal{Y}^Y,\mathcal{Y}^X)[\mathcal{Y}^X Z,FZ] \arrow[dr, "\sigma_Z 1"] & & & \\
\mathsf{C}(X,Y)[\mathcal{Y}^X Z,FZ] \arrow[ur, "\mathcal{Y}^{(-)}1"] \arrow[rr, "(\mathcal{Y}^{(-)}Z)1"] \arrow[drr, "{[\mathcal{Y}^{(-)}Z,FZ]1}"'] & & {[\mathcal{Y}^Y Z,\mathcal{Y}^X Z][\mathcal{Y}^X Z,FZ]} \arrow[rr, "m'"] \arrow[d, "{[-,FZ]1}"] & & {[\mathcal{Y}^Y Z,FZ]} \\
& & {\big[[\mathcal{Y}^X Z,FZ],[\mathcal{Y}^Y Z,FZ]\big][\mathcal{Y}^X Z,FZ]} \arrow[urr, "\mathsf{eval}"'] & &
\end{tikzcd}$$

In the above diagram, the two triangular regions below ☆ commute by definition, where $(m^{\mathsf{Map}})' = m^{\mathsf{Map}} \circ \xi$ is the composition in $(\mathsf{C}\text{-}\mathsf{V})^{\mathsf{op}}$. We also let $m' = m \circ \xi$. The remaining regions, from left to right and top to bottom, commute by functoriality of $\otimes$, definition of m^{Map} (3.5.21) and naturality of ξ, definition of $\mathcal{Y}^{(-)}$ on hom objects (3.5.26), and definition of $[\mathcal{Y}^{(-)}Z, FZ]$.

Lastly, by Definition 3.6.1 the components of ϕ are determined by the equalities

$$F^{\#}_{X,Z} = \sigma_Z \circ \phi_X \quad \text{and} \quad F^{\#}_{Y,Z} = \sigma_Z \circ \phi_Y.$$

Therefore, the outer diagram above commutes by naturality of $(F^{\#}_{-,Z})^{\perp}$, proved in Lemma 3.6.4. This implies that the composites around ☆ are equal after composing with each σ_Z, and thus completes the proof that $\phi^{\perp}$ is V-natural. □

Lemma 3.6.6. *Suppose that f, g, and h are morphisms in V as shown at left below. Then the diagram at left commutes if and only if the diagram at right commutes.*

(3.6.7)
$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & {[C,Z]} \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "h^{\#}"] \arrow[dr, "g^{\#}"'] & & {[A,Z]} \\ & {[B,Z]} \arrow[ur, "{[f,Z]}"'] & \end{tikzcd}$$

Proof. By taking adjoints one verifies that each of the two traingles in (3.6.7) commutes if and only if the triangle below commutes, where the unlabeled arrows are the adjoints of g and h, respectively.

$$\begin{tikzcd} B \otimes C \arrow[dr] & \\ & Z \\ A \otimes C \arrow[uu, "f \otimes C"] \arrow[ur] & \end{tikzcd}$$

□

Lemma 3.6.8. *Suppose $\eta : \mathbb{1} \longrightarrow [Q, A]$ is a morphism in V and let η' denote the composite*

$$Q \xrightarrow{\eta^{\#}} [\mathbb{1}, A] \xrightarrow[\cong]{\kappa} A.$$

Then the following diagram in V commutes.

$$\begin{tikzcd} {[A,B]} \arrow[r, "{[\eta',B]}"] \arrow[d, "{[Q,-]}"'] & {[Q,B]} \\ {[[Q,A],[Q,B]]} \arrow[r, "{[\eta,[Q,B]]}"] & {[\mathbb{1},[Q,B]]} \arrow[u, "\kappa"', "\cong"] \end{tikzcd}$$

Proof. Taking adjoints, the diagram in the statement commutes if and only if the outer diagram below commutes.

$$\begin{tikzcd}
{[A,B]\otimes\mathbb{1}} \arrow[rr, "{[\eta',B]\otimes 1}"] \arrow[dr, "1\otimes\eta"] \arrow[dd, "{[Q,-]\otimes 1}"'] & & {[Q,B]\otimes\mathbb{1}} \arrow[dr, "\rho"] & \\
& {[A,B]\otimes[Q,A]} \arrow[rr, "m"] \arrow[d, "{[Q,-]\otimes 1}"] & & {[Q,B]} \\
{[[Q,A],[Q,B]]\otimes\mathbb{1}} \arrow[r, "1\otimes\eta"] & {[[Q,A],[Q,B]]\otimes[Q,A]} \arrow[urr, "\mathsf{eval}"'] & &
\end{tikzcd}$$

In the above diagram, commutativity of the region at left follows from functoriality of $\otimes$. Commutativity of the lower right triangle follows from the definition of $[Q,-]$. Commutativity of the remaining region follows from another adjoint diagram argument comparing the corresponding morphisms

$$([A,B]\otimes\mathbb{1})\otimes Q \longrightarrow B. \qquad \square$$

Now we use the V-Yoneda Bijection 3.4.12 to prove the following.

Theorem 3.6.9 (V-Yoneda Lemma)**.** *Suppose V is a symmetric monoidal closed category that is complete. Suppose C is a small V-category and $F:\mathsf{C}\longrightarrow\underline{\mathsf{V}}$ is a V-functor. Then*

$$FX \xrightarrow{\phi_X} \mathsf{Map}(\mathcal{Y}^X, F)$$

is an isomorphism for each $X\in\mathsf{C}$. These provide the components of a V-natural isomorphism

$$\phi^\perp : F \longrightarrow \mathsf{Map}(\mathcal{Y}^{(-)}, F).$$

Proof. The definition of ϕ is given in Definition 3.6.1 above. The V-naturality of $\phi^\perp$ is proved in Lemma 3.6.5 above. Now we show that each ϕ_X is an isomorphism in V by showing that FX satisfies the universal property of the end $\mathsf{Map}(\mathcal{Y}^X, F)$.

To do so, suppose that

$$Q \xrightarrow{\theta_Z^\#} [\mathcal{Y}^X Z, FZ]$$

is a collection of morphisms for $Z\in\mathsf{V}$ that are transforms of

$$\mathcal{Y}^X Z \xrightarrow{\theta_Z} [Q, FZ].$$

By Lemma 3.5.16 the transforms $\theta^\#$ satisfy the V-wedge condition (3.5.3) if and only if $\theta^\perp$ are the components of a V-natural transformation $\mathcal{Y}^X \longrightarrow [Q, F(-)]$. In this case, the V-Yoneda Bijection of Theorem 3.4.12 implies that there is a unique morphism in V

$$\eta : \mathbb{1} \longrightarrow [Q, FX]$$

such that the outer square of the diagram below commutes for each $Y \in \mathsf{C}$.

$$\begin{tikzcd}
\mathcal{Y}^X Y \arrow[rrr, "\theta_Y"] \arrow[dr, "F"'] \arrow[dd, "{[Q,F(-)]}"'] & & & {[Q,FY]} \\
& {[FX,FY]} \arrow[urr, "{[\eta',FY]}"'] \arrow[dl, "{[Q,-]}"] & & \\
{[[Q,FX],[Q,FY]]} \arrow[rrr, "{[\eta,[Q,FY]]}"] & & & {[\mathbb{1},[Q,FY]]} \arrow[uu, "\kappa"', "\cong"]
\end{tikzcd}$$

In the above diagram, the lower right quadrilateral commutes by Lemma 3.6.8 and therefore the upper triangular region also commutes. By Lemma 3.6.6 the commutativity of that upper triangular region is equivalent to commutativity of the triangle below.

$$\begin{tikzcd}
& FX \arrow[dr, "\phi_Y"] & \\
Q \arrow[ur, "\eta'"] \arrow[rr, "\theta^\#_Y"] & & {[\mathcal{Y}^X Y, FY]}
\end{tikzcd} \tag{3.6.10}$$

This argument shows that uniqueness of η' making (3.6.10) commute is equivalent to uniquess of η in the V-Yoneda Bijection of Theorem 3.4.12. Therefore, (FX, ϕ) satisfies the universal property of the end defining $\mathsf{Map}(\mathcal{Y}^X, F)$. □

Corollary 3.6.11 (V-Yoneda Embedding Theorem)**.** *The Yoneda* V*-functor induces isomorphisms in* V

$$\mathsf{C}(X,Y) \xrightarrow{(\mathcal{Y}^{(-)})_{X,Y}} \mathsf{Map}(\mathcal{Y}^Y, \mathcal{Y}^X)$$

for $X, Y \in \mathsf{C}$.

Proof. Taking $F = \mathcal{Y}^X$ in Theorem 3.6.9 (interchanging X and Y), we have

$$\phi_Y = \mathcal{Y}^{(-)}_{X,Y}$$

because the transform

$$(\mathcal{Y}^X)^\#_{Y,Z} : \mathcal{Y}^X Y \longrightarrow [\mathcal{Y}^Y Z, \mathcal{Y}^X Z]$$

in Definition 3.6.1 is the same as $(\mathcal{Y}_Z)_{X,Y}$ in (3.5.26). Therefore, the result follows from Theorem 3.6.9. □

Definition 3.6.12. Suppose $F, G : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ are V-functors with V complete and suppose

$$\theta : F \longrightarrow G$$

is a V-natural transformation. We define

$$\mathsf{Map}(\mathcal{Y}^{(-)}, F) \xrightarrow{\mathsf{Map}(\mathcal{Y}^{(-)}, \theta)} \mathsf{Map}(\mathcal{Y}^{(-)}, G)$$

to be the V-natural transformation with component $\mathsf{Map}(\mathcal{Y}^X, \theta)$ at $X \in \mathsf{C}$ determined by the following diagram for $Z \in \mathsf{C}$.

$$\begin{tikzcd}
\mathsf{Map}(\mathcal{Y}^X, F) \arrow[rr, "\mathsf{Map}(\mathcal{Y}^X,\theta)"] \arrow[d, "\sigma_Z"'] & & \mathsf{Map}(\mathcal{Y}^X, G) \arrow[d, "\sigma_Z"] \\
{[\mathcal{Y}^X Z, FZ]} \arrow[d, "{[\mathcal{Y}^X Z, \lambda^{-1}]}"'] & & {[\mathcal{Y}^X Z, GZ]} \\
{[\mathcal{Y}^X Z, \mathbb{1} \otimes (FZ)]} \arrow[r, "{[\mathcal{Y}^X Z, \theta_Z 1]}"] & {\big[\mathcal{Y}^X Z, [FZ, GZ] \otimes (FZ)\big]} \arrow[ur, "{[\mathcal{Y}^X Z, \mathsf{eval}]}"'] &
\end{tikzcd} \tag{3.6.13}$$

That is, we define

$$\mathsf{Map}(\mathcal{Y}^{(-)}, \theta)_X = \big(\mathsf{Map}(\mathcal{Y}^X, \theta)\big)^{\perp}.$$

We show in Lemma 3.6.14 that $\mathsf{Map}(\mathcal{Y}^{(-)}, \theta)$ is V-natural. ◇

Lemma 3.6.14. *In the context of Definition 3.6.12,*

(1) $\mathsf{Map}(\mathcal{Y}^{(-)}, \theta)$ *is a* V*-natural transformation and*
(2) the following diagram of V*-natural transformations is commutative, where* $\phi^{\perp}$ *denotes the* V*-natural transformations of Definition 3.6.1 for both* F *and* G*, respectively.*

$$\begin{tikzcd}
F \arrow[r, "\phi^{\perp}"] \arrow[d, "\theta"'] & \mathsf{Map}(\mathcal{Y}^{(-)}, F) \arrow[d, "\mathsf{Map}(\mathcal{Y}^{(-)},\theta)"] \\
G \arrow[r, "\phi^{\perp}"] & \mathsf{Map}(\mathcal{Y}^{(-)}, G)
\end{tikzcd} \tag{3.6.15}$$

Proof. It suffices, for both assertions in the statement, to show that (3.6.15) is componentwise commutative. Then V-naturality of

$$\mathsf{Map}(\mathcal{Y}^{(-)}, \theta) = (\phi^{\perp})\,\theta\,(\phi^{\perp})^{-1}$$

follows from V-naturality of θ and of each $\phi^{\perp}$, proved in Lemma 3.6.5.

For each $X \in \mathsf{C}$, let ω_X denote the composite

$$FX \xrightarrow{\lambda^{-1}} \mathbb{1} \otimes FX \xrightarrow{\theta_X \otimes 1} [FX, GX] \otimes FX \xrightarrow{\mathsf{eval}} GX,$$

so that $\theta_X = \omega_X^{\perp}$. Then by functoriality of $(-)^{\perp}$ (3.2.5) we have

$$(\phi_X^{\perp})\,\theta_X\,(\phi_X^{\perp})^{-1} = (\phi_X \omega_X \phi_X^{-1})^{\perp}.$$

Therefore, using the universality of V-ends, the component of (3.6.15) at X commutes if and only if the following diagram commutes for each $Z \in \mathsf{C}$. Here and throughout the rest of this proof we omit tensor symbols. The triangles below determine the morphisms ϕ_X (Definition 3.6.1)

$$\begin{tikzcd}
& \mathsf{Map}(\mathcal{Y}^X, F) \arrow[dr, "\sigma_Z"] & \\
FX \arrow[ur, "\phi_X"] \arrow[rr, "F^{\#}_{X,Z}"'] \arrow[d, "\omega_X"'] & & {[\mathcal{Y}^X Z, FZ]} \arrow[d, "{[\mathcal{Y}^X Z, \omega_Z]}"] \\
GX \arrow[rr, "G^{\#}_{X,Z}"] \arrow[dr, "\phi_X"'] & & {[\mathcal{Y}^X Z, GZ]} \\
& \mathsf{Map}(\mathcal{Y}^X, G) \arrow[ur, "\sigma_Z"'] &
\end{tikzcd}$$

By taking adjoints of the composites in the inner rectangle above—using Definition 3.5.15 for the adjoints of the transforms $F^{\#}$ and $G^{\#}$—we obtain the rectangle (3.2.11) that determines V-naturality of $\omega^{\perp} = \theta$. Therefore, V-naturality of θ implies that the above diagram commutes for each $Z \in \mathsf{C}$. Hence (3.6.15) commutes componentwise. $\square$

Explanation 3.6.16. The commutativity of (3.6.15) also proves that the assignment

$$\theta \longmapsto \mathsf{Map}(\mathcal{Y}^{(-)}, \theta)$$

is functorial with respect to identities and vertical composites. Therefore, we have a functor of categories

$$\mathsf{V}\text{-}\mathsf{Cat}(\mathsf{C}, \underline{\mathsf{V}}) \xrightarrow{\mathsf{Map}(\mathcal{Y}^{(-)}, -)} \mathsf{V}\text{-}\mathsf{Cat}(\mathsf{C}, \underline{\mathsf{V}}).$$

The V-Yoneda Lemma 3.6.9 together with (3.6.15) imply that this functor is naturally isomorphic to the identity functor. $\diamond$

3.7. Symmetric Monoidal Diagram Categories

Throughout this section we let V denote a symmetric monoidal closed category that is complete and cocomplete. We let $\mathcal{D}$ denote a small symmetric monoidal V-category. The purpose of this section is to describe a symmetric monoidal closed structure on the category of V-functors from $\mathcal{D}$ to $\underline{\mathsf{V}}$. Throughout this section, recall from Theorem 3.3.2 that the monoidal data of V are V-natural. These, along with the V-natural monoidal structure of $\mathcal{D}$, will be used as part of the V-natural monoidal structure on the V-functor category.

Definition 3.7.1. Suppose C is a V-category and $\mathcal{D}$ is a small V-category. A V-functor $\mathcal{D} \longrightarrow \mathsf{C}$ is called a *$\mathcal{D}$-shaped diagram* in C. The category of V-functors and V-natural transformations is denoted $\mathcal{D}$-C. In the case $\mathsf{C} = \underline{\mathsf{V}}$ with its canonical self-enrichment, we will write

$$\mathcal{D}\text{-}\mathsf{V} = \mathsf{V}\text{-}\mathsf{Cat}(\mathcal{D}, \underline{\mathsf{V}}).$$

In this context we will often use lowercase letters $a, b, c, \ldots \in \mathcal{D}$ for objects of $\mathcal{D}$, uppercase letters $X, Y, Z, \ldots \in \mathcal{D}\text{-}\mathsf{C}$ for $\mathcal{D}$-shaped diagrams, and subscripts X_a for the value of a diagram X at an object a. $\diamond$

Explanation 3.7.2. In Definition 3.5.19 and Lemma 3.5.24, with $\mathcal{D} = \mathsf{C}$, we used the notation $\mathcal{D}$-V for the V-enriched category of $\mathcal{D}$-shaped diagrams in $\underline{\mathsf{V}}$ with hom objects given by the mapping objects $\mathsf{Map}(-,-)$ (Definition 3.5.12). By Lemma 3.5.14, the underlying category of this V-enriched category is isomorphic to the category of V-functors and V-natural transformations denoted $\mathcal{D}$-V in Definition 3.7.1. Throughout this section we will let $\mathcal{D}$-V denote the ordinary category and define a symmetric monoidal closed structure for it. In Explanation 3.8.3 we will relate this back to the V-enriched structure of Definition 3.5.19 and Lemma 3.5.24. ⋄

Definition 3.7.3. Suppose that $(\mathsf{V}, \otimes)$ is a symmetric monoidal closed category that is complete and cocomplete. Suppose that $\mathcal{D}$ is small symmetric monoidal V-category with product denoted $\boxdot$. Suppose given $\mathcal{D}$-shaped diagrams $X, Y \in \mathcal{D}$-V. The *Day convolution product* of X and Y is denoted $X \otimes Y$ and is defined as the V-coend

$$X \otimes Y = \int^{(a,b)\in\mathcal{D}\otimes\mathcal{D}} \mathcal{D}(a \boxdot b, -) \otimes \big((X_a) \otimes (Y_b)\big). \tag{3.7.4}$$

The *hom diagram* from X to Y is denoted $\mathsf{Hom}_{\mathcal{D}}(X, Y)$ and is defined as the V-end

$$\mathsf{Hom}_{\mathcal{D}}(X, Y) = \int_{c\in\mathcal{D}} \Big[\int^{b\in\mathcal{D}} \mathcal{D}(- \boxdot b, c) \otimes X_b, Y_c \Big] \tag{3.7.5}$$

$$\cong \int_{(b,c)\in\mathcal{D}\otimes\mathcal{D}} \Big[\mathcal{D}(- \boxdot b, c) \otimes X_b, Y_c \Big] \tag{3.7.6}$$

with the isomorphism following from (3.5.10). The assumption that V is complete and cocomplete implies that the objects

$$(X \otimes Y)_a \in \mathsf{V} \quad \text{and} \quad \mathsf{Hom}_{\mathcal{D}}(X, Y)_a \in \mathsf{V}$$

exist for all diagrams X and Y in $\mathcal{D}$-V and all $a \in \mathcal{D}$. The V-functoriality of

$$X \otimes Y : \mathcal{D} \longrightarrow \underline{\mathsf{V}} \quad \text{and} \quad \mathsf{Hom}_{\mathcal{D}}(X, Y) : \mathcal{D} \longrightarrow \underline{\mathsf{V}}$$

follows from universality of the (co)ends along with V-functoriality of the monoidal products and internal homs in $\underline{\mathsf{V}}$ and $\mathcal{D}$. ⋄

In Proposition 3.7.19 below we give the definitions of $\otimes$ and $\mathsf{Hom}_{\mathcal{D}}$ on morphisms of $\mathcal{D}$-V—that is, V-natural transformations—and show that they are functors

$$\otimes : \mathcal{D}\text{-}\mathsf{V} \times \mathcal{D}\text{-}\mathsf{V} \longrightarrow \mathcal{D}\text{-}\mathsf{V} \quad \text{and} \quad \mathsf{Hom}_{\mathcal{D}} : \mathcal{D}\text{-}\mathsf{V}^{\mathsf{op}} \times \mathcal{D}\text{-}\mathsf{V} \longrightarrow \mathcal{D}\text{-}\mathsf{V}.$$

We show that $\otimes$ and $\mathsf{Hom}_{\mathcal{D}}$ are adjoints in Proposition 3.7.21. Finally in Theorem 3.7.22 we show that these give a symmetric monoidal closed structure on $\mathcal{D}$-V.

To express the associativity and unit isomorphisms for Day convolution, we will need some introductory definitions and results related to the V-Yoneda Lemma 3.6.9. For $X \in \mathcal{D}$-V, it follows from V-naturality of the morphisms

$$\mathcal{D}(x, s) \xrightarrow{X_{x,s}} [X_x, X_s].$$

and Lemma 3.5.16 that the adjoints

$$\mathcal{D}(x, s) \otimes X_x \xrightarrow{\overline{X}_{x,s}} X_s$$

satisfy the V-cowedge condition (3.5.2).

Definition 3.7.7. For each V-functor $X : \mathcal{D} \longrightarrow \underline{\mathsf{V}}$ and each $s \in \mathcal{D}$, let

$$\int^{x\in\mathcal{D}} \mathcal{D}(x, s) \otimes X_x \xrightarrow{\psi_s} X_s$$

denote the morphism induced by the adjoints of

$$\mathcal{D}(x,s) \xrightarrow{X_{x,s}} [X_x, X_s]. \qquad \diamond$$

Theorem 3.7.8 (V-Yoneda Density)**.** *Suppose $\mathcal{D}$ is a small* V*-category and*

$$X : \mathcal{D} \longrightarrow \underline{\mathsf{V}}$$

is a V*-functor. Then for each $s \in \mathcal{D}$*

$$\int^x \mathcal{D}(x,s) \otimes X_x \xrightarrow{\psi_s} X_s$$

is an isomorphism. These provide components of a V*-natural isomorphism*

$$\int^x \mathcal{D}(x,-) \otimes X_x \xrightarrow{\psi^\perp} X.$$

Proof. The V-naturality of $\psi^\perp$ follows via Lemma 3.2.10 from V-naturality of the morphisms $(X_{x,s})^\perp$. To prove that each ψ_s is an isomorphism in V, it suffices to prove that the induced morphisms

$$[X_s, Z] \xrightarrow{[\psi_s, Z]} \Big[\int^{x \in \mathcal{D}} \mathcal{D}(x,s) \otimes X_x, Z \Big]$$

are isomorphisms for each $Z \in \mathsf{V}$. First by (3.5.10) we have

$$\Big[\int^{x \in \mathcal{D}} \mathcal{D}(x,s) \otimes X_x, Z \Big] \cong \int_{x \in \mathcal{D}} \Big[\mathcal{D}(x,s) \otimes X_x, Z \Big]$$

and then using the closed structure we have

$$\Big[\mathcal{D}(x,s) \otimes X_x, Z \Big] \cong \Big[\mathcal{D}(x,s), [X_x, Z] \Big]$$

for each $x \in \mathcal{D}$. The composite morphisms

$$[X_s, Z] \longrightarrow \int_{x \in \mathcal{D}} \Big[\mathcal{D}(x,s), [X_x, Z] \Big] = \int_{x \in \mathcal{D}^{\mathsf{op}}} \Big[\mathcal{D}^{\mathsf{op}}(s,x), [X_x, Z] \Big]$$

are those induced by the structure morphisms of the V-functor

$$[X_{(-)}, Z] : \mathcal{D}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}.$$

Therefore, the V-Yoneda Lemma 3.6.9 shows that these morphisms, and hence also the morphisms $[\psi_s, Z]$, are isomorphisms for each $Z \in \mathsf{V}$. $\square$

Explanation 3.7.9. In the context of Theorem 3.7.8, the V-Yoneda Lemma 3.6.9 can be written in the following equivalent form. For each $s \in \mathcal{D}$

$$X_s \xrightarrow{\phi_s} \int_x [\mathcal{D}(s,x), X_x]$$

is an isomorphism. These provide components of a V-natural isomorphism

$$X \xrightarrow{\phi^\perp} \int_x [\mathcal{D}(-,x), X_x] \tag{3.7.10}$$

$\diamond$

As an immediate application of Theorem 3.7.8, we have the following.

Explanation 3.7.11. In the context of Definition 3.7.3 we have a sequence of V-natural isomorphisms given by the isomorphism (3.5.10), interchanging order of ends, the symmetric monoidal closed structure of V, the isomorphism (3.5.11), and the V-Yoneda Density Theorem 3.7.8.

$$
\begin{aligned}
\mathsf{Hom}_{\mathcal{D}}(X,Y) &= \int_c \Big[\int^b \mathcal{D}(-\boxdot b, c) \otimes X_b, Y_c \Big] \\
&\cong \int_c \int_b \Big[\mathcal{D}(-\boxdot b, c) \otimes X_b, Y_c \Big] \\
&\cong \int_b \int_c \Big[X_b, [\mathcal{D}(-\boxdot b, c), Y_c] \Big] \\
&\cong \int_b \Big[X_b, \int_c [\mathcal{D}(-\boxdot b, c), Y_c] \Big] \\
&\cong \int_{b\in\mathcal{D}} \Big[X_b, Y_{-\boxdot b} \Big]. \qquad (3.7.12)
\end{aligned}
$$

With this latter formulation one can see more directly that we have

$$
\mathsf{Map}(X,Y) \cong \mathsf{Hom}_{\mathcal{D}}(X,Y)_e \qquad (3.7.13)
$$

where $e \in \mathcal{D}$ is the monoidal unit. ◇

We will use Theorem 3.7.8 as stated, along with two special cases defined as follows.

Definition 3.7.14. For each triple $a, b, c \in \mathcal{D}$, let

$$
\int^x \mathcal{D}(x \boxdot b, c) \otimes \mathcal{D}(a, x) \xrightarrow[\cong]{\gamma_{a,c;b}} \mathcal{D}(a \boxdot b, c) \qquad (3.7.15)
$$

denote the morphisms induced by

$$
\mathcal{D}(x \boxdot b, c) \otimes \mathcal{D}(a, x) \xrightarrow{1 \otimes (-\boxdot b)} \mathcal{D}(x \boxdot b, c) \otimes \mathcal{D}(a \boxdot b, x \boxdot b) \xrightarrow{m} \mathcal{D}(a \boxdot b, c).
$$

Similarly, let

$$
\int^x \mathcal{D}(b \boxdot x, c) \otimes \mathcal{D}(a, x) \xrightarrow[\cong]{\gamma_{b;a,c}} \mathcal{D}(b \boxdot a, c) \qquad (3.7.16)
$$

denote the morphisms induced by

$$
\mathcal{D}(b \boxdot x, c) \otimes \mathcal{D}(a, x) \xrightarrow{1 \otimes (b \boxdot -)} \mathcal{D}(b \boxdot x, c) \otimes \mathcal{D}(b \boxdot a, b \boxdot x) \xrightarrow{m} \mathcal{D}(b \boxdot a, c).
$$

Applying Theorem 3.7.8 with $X_s = \mathcal{D}(s \boxdot b, c)$ and $X_s = \mathcal{D}(b \boxdot s, c)$, respectively, together with an appropriate symmetry isomorphism ξ shows that the $\gamma_{a,c;b}$ and $\gamma_{b;a,c}$ are isomorphisms. ◇

Using the isomorphisms given above, we define the following V-natural isomorphisms of diagrams. These will define the components of the associativity and unit isomorphisms for the Day convolution product in Theorem 3.7.22.

Definition 3.7.17. Suppose $\mathcal{D}$ is a small symmetric monoidal V-category with monoidal unit e. Suppose X, Y, and Z are V-functors $\mathcal{D} \longrightarrow \underline{\mathsf{V}}$. Define the *unit diagram* J to be the V-functor

$$
J = \mathcal{Y}^e = \mathcal{D}(e, -) : \mathcal{D} \longrightarrow \underline{\mathsf{V}}. \qquad (3.7.18)
$$

Define V-natural isomorphisms

$$\begin{aligned}\alpha_{X,Y,Z} &: (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)\\ \lambda_X &: J \otimes X \longrightarrow X\\ \rho_X &: X \otimes J \longrightarrow X\\ \xi_{X,Y} &: X \otimes Y \longrightarrow Y \otimes X\end{aligned}$$

as follows.

Associativity: The *associativity isomorphsm* $\alpha_{X,Y,Z}$ is the V-natural isomorphism with components determined by the following composite, explained below, for each $s \in \mathcal{D}$:

$$\begin{aligned}[(X \otimes Y) \otimes Z]_s &= \int^{x,r} \mathcal{D}(x \boxdot r, s) \otimes (X \otimes Y)_x \otimes Z_r\\ &= \int^{x,r} \mathcal{D}(x \boxdot r, s) \otimes \Big(\Big(\int^{p,q} \mathcal{D}(p \boxdot q, x) \otimes (X_p \otimes Y_q)\Big) \otimes Z_r\Big)\\ &\cong \int^{p,q,r} \Big(\int^{x} \mathcal{D}(x \boxdot r, s) \otimes \mathcal{D}(p \boxdot q, x)\Big) \otimes \Big((X_p \otimes Y_q) \otimes Z_r\Big)\\ &\cong \int^{p,q,r} \Big(\int^{x} \mathcal{D}(p \boxdot x, s) \otimes \mathcal{D}(q \boxdot r, x)\Big) \otimes \Big(X_p \otimes (Y_q \otimes Z_r)\Big)\\ &\cong \int^{p,x} (\mathcal{D}(p \boxdot x, s) \otimes X_p) \otimes \Big(\int^{q,r} \mathcal{D}(q \boxdot r, x) \otimes (Y_q \otimes Z_r)\Big)\\ &= [X \otimes (Y \otimes Z)]_s.\end{aligned}$$

The first isomorphism above is given by associativity isomorphisms in V along with interchanging order of coends. The middle isomorphism is given by associativity

$$(X_p \otimes Y_q) \otimes Z_r \cong X_p \otimes (Y_q \otimes Z_r)$$

in V together with the morphisms $\gamma_{(p \boxdot q),s;r}$ and $\gamma^{-1}_{p;(q \boxdot r),s}$ from Definition 3.7.14 as well as the isomorphism

$$\mathcal{D}(p \boxdot (q \boxdot r), s) \cong \mathcal{D}((p \boxdot q) \boxdot r, s)$$

induced by associativity in $\mathcal{D}$. The third isomorphism is similar to the first, but also uses a symmetry isomorphism in V to permute X_p with $\mathcal{D}(q \boxdot r, x)$.

Left Unit: The *left unit isomorphism* λ_X is the V-natural isomorphism with components determined by the following composite of isomorphisms, explained below, for each $s \in \mathcal{D}$:

$$\begin{aligned}(J \otimes X)_s &= \int^{p,q} \mathcal{D}(p \boxdot q, s) \otimes (\mathcal{D}(e, p) \otimes X_q)\\ &\cong \int^{q} \Big(\int^{p} \mathcal{D}(p \boxdot q, s) \otimes \mathcal{D}(e, p)\Big) \otimes X_q\\ &\cong \int^{q} \mathcal{D}(q, s) \otimes X_q\\ &\cong X_s.\end{aligned}$$

The first isomorphism is given by associativity in V and interchanging order of limits. The second isomorphism is given by $\gamma_{e,s;q}$ (3.7.15) and the left unit isomorphism of $\mathcal{D}$. The final isomorphism is the V-Yoneda Density Theorem Theorem 3.7.8 for X.

Right Unit: The *right unit isomorphism* ρ_X is the V-natural isomorphism with components determined by the following composite for each $s \in \mathcal{D}$ defined similarly to those for λ_X:

$$\begin{aligned}(X \otimes J)_s &= \int^{q,p} \mathcal{D}(q \boxdot p, s) \otimes (X_q \otimes \mathcal{D}(e,p)) \\ &\cong \int^q \Big(\int^p \mathcal{D}(q \boxdot p, s) \otimes \mathcal{D}(e,p) \Big) \otimes X_q \\ &\cong \int^q \mathcal{D}(q,s) \otimes X_q \\ &\cong X_s.\end{aligned}$$

The only significant difference between ρ_X and λ_X is that the first isomorphism above also requires use of the symmetry isomorphism in V

$$X_q \otimes \mathcal{D}(e,p) \xrightarrow[\cong]{\xi} \mathcal{D}(e,p) \otimes X_q.$$

Symmetry: The *symmetry isomorphism* $\xi_{X,Y}$ is the V-natural isomorphism with components determined by the following isomorphism for each $s \in \mathcal{D}$.

$$\begin{aligned}(X \otimes Y)_s &= \int^{p,q} \mathcal{D}(p \boxdot q, s) \otimes (X_p \otimes Y_q) \\ &\cong \int^{q,p} \mathcal{D}(q \boxdot p, s) \otimes (Y_q \otimes X_p) \\ &= (Y \otimes X)_s\end{aligned}$$

This isomorphism is induced by interchanging order of limits along with the symmetry isomorphisms

$$X_p \otimes Y_q \cong Y_q \otimes X_p \quad \text{and} \quad \mathcal{D}(p \boxdot q, s) \cong \mathcal{D}(q \boxdot p, s)$$

of V and $\mathcal{D}$, respectively. ◇

Now we give a series of results aiming to show that $\mathcal{D}$-V has a symmetric monoidal closed structure.

Proposition 3.7.19. *The convolution product and hom diagram of Definition 3.7.3 are functorial with respect to* V*-natural transformations.*

Proof. Suppose given V-natural transformations

$$\theta : X \longrightarrow X' \quad \text{and} \quad \eta : Y \longrightarrow Y'$$

for $X, X', Y, Y' \in \mathcal{D}$-V. Then

$$\theta \otimes \eta : X \otimes Y \longrightarrow X' \otimes Y'$$

has components $(\theta \otimes \eta)_c$ for $c \in \mathcal{D}$ defined as

$$\int^{a,b} \mathcal{D}(a \boxdot b, c) \otimes (X_a \otimes Y_b) \xrightarrow{\int^{a,b} 1 \otimes (\theta_a \otimes \eta_b)} \int^{a,b} \mathcal{D}(a \boxdot b, c) \otimes (X'_a \otimes Y'_b).$$

Similarly,

$$\mathsf{Hom}_{\mathcal{D}}(\theta, \eta) : \mathsf{Hom}_{\mathcal{D}}(X', Y) \longrightarrow \mathsf{Hom}_{\mathcal{D}}(X, Y')$$

has components $\mathsf{Hom}_{\mathcal{D}}(\theta,\eta)_a$ for $a \in \mathcal{D}$ defined, via (3.7.12), as

$$\mathsf{Hom}_{\mathcal{D}}(X', Y)_a \cong \int_b [X'_b, Y_{a \boxdot b}] \xrightarrow{\int_b [\theta_b, \eta_{a \boxdot b}]} \int_b [X_b, Y'_{a \boxdot b}] \cong \mathsf{Hom}_{\mathcal{D}}(X, Y')_a.$$

To verify that these components $(\theta \otimes \eta)_c$ and $\mathsf{Hom}_{\mathcal{D}}(\theta,\eta)_a$ define V-natural transformations, we use the alternative characterization given in Lemma 2.1.11. For $\theta \otimes \eta$, this means we must verify the following version of (2.1.12) commutes for each pair of objects $c, c' \in \mathcal{D}$.

$$\begin{tikzcd}
\mathcal{D}(c,c') \arrow[r, "(X \otimes Y)_{c,c'}"] \arrow[d, "(X' \otimes Y')_{c,c'}"'] & {[(X \otimes Y)_c, (X \otimes Y)_{c'}]} \arrow[d, "{[(X \otimes Y)_c, (\theta \otimes \eta)_{c'}]}"] \\
{[(X' \otimes Y')_c, (X' \otimes Y')_{c'}]} \arrow[r, "{[(\theta \otimes \eta)_c, (X' \otimes Y')_{c'}]}"] & {[(X \otimes Y)_c, (X' \otimes Y')_{c'}]}
\end{tikzcd}$$

The adjoint of the above diagram is the outer diagram below. The unlabeled morphisms are given by associators in V and universality of the coends; the morphisms m_* are induced by the composition of $\mathcal{D}$. To save space the monoidal products $\boxdot$ and $\otimes$ are denoted by juxtaposition.

(3.7.20)
$$\begin{tikzcd}
& \int^{a,b} \Big(\mathcal{D}(c,c')\mathcal{D}(ab,c)\Big)\big(X_a Y_b\big) \arrow[dr, "m_*"] \arrow[dd, "\int^{a,b}(11)(\theta_a \eta_b)"] & \\
\mathcal{D}(c,c')\Big(\int^{a,b} \mathcal{D}(ab,c)\big(X_a Y_b\big)\Big) \arrow[ur] \arrow[dd, "1\big(\int^{a,b} 1(\theta_a \eta_b)\big)"'] & & \int^{a,b} \mathcal{D}(ab,c')\big(X_a Y_b\big) \arrow[dd, "\int^{a,b} 1(\theta_a \eta_b)"] \\
& \int^{a,b} \Big(\mathcal{D}(c,c')\mathcal{D}(ab,c)\Big)\big(X'_a Y'_b\big) \arrow[dr, "m_*"] & \\
\mathcal{D}(c,c')\Big(\int^{a,b} \mathcal{D}(ab,c)\big(X'_a Y'_b\big)\Big) \arrow[ur] & & \int^{a,b} \mathcal{D}(ab,c')\big(X'_a Y'_b\big)
\end{tikzcd}$$

The above diagram commutes by functoriality of $\otimes$ and naturality of the monoidal structure and coend morphisms. This shows that $\theta \otimes \eta$ is V-natural. The functoriality of $\otimes$ with respect to identities and composition of V-natural transformations follows from functoriality of $\otimes$ and universality of the coends.

A similar analysis shows that $\mathsf{Hom}_{\mathcal{D}}(\theta,\eta)$ is V-natural: the adjoint of the diagram (2.1.12) for $\mathsf{Hom}_{\mathcal{D}}$ is the following, explained below.

$$\begin{tikzcd}
 & \int_b \mathcal{D}(a,a')\big[X'_b, Y_{ab}\big] \arrow[dr, "\phi"] \arrow[dd, "\int_b 1[\theta_b,\eta_{ab}]"] & \\
\mathcal{D}(a,a')\Big(\int_b \big[X'_b, Y_{ab}\big]\Big) \arrow[ur] \arrow[dd, "1\left(\int_b[\theta_b,\eta_{ab}]\right)"'] & & \int_b \big[X'_b, Y_{a'b}\big] \arrow[dd, "\int^{a,b}[\theta_b,\eta_{a'b}]"] \\
 & \int_b \mathcal{D}(a,a')\big[X_b, Y'_{ab}\big] \arrow[dr, "\phi"] & \\
\mathcal{D}(a,a')\Big(\int_b \big[X_b, Y'_{ab}\big]\Big) \arrow[ur] & & \int_b \big[X_b, Y'_{a'b}\big]
\end{tikzcd}$$

In the above diagram, the unlabeled morphisms are given by universality of the ends and the morphisms labeled ϕ are induced by the composites

$$\begin{tikzcd}
\mathcal{D}(a,a')\big[X'_b, Y_{ab}\big] \arrow[dr, "((-)b)\,1"] & & & \big[X'_b, Y_{a'b}\big] \\
 & \mathcal{D}(ab,a'b)\big[X'_b, Y_{ab}\big] \arrow[r, "Y\,1"] & \big[Y_{ab}, Y_{a'b}\big]\big[X'_b, Y_{ab}\big] \arrow[ur, "m"] &
\end{tikzcd}$$

for each $b \in \mathcal{D}$. Commutativity of (3.7.20), and hence V-naturality of $\mathsf{Hom}_{\mathcal{D}}(\theta,\eta)$, follows as for $\theta \otimes \eta$. Then functoriality of $\mathsf{Hom}_{\mathcal{D}}$ with respect to θ (contravariant) and η (covariant) follows from functoriality of $[-,-]$ and universality of the ends. $\square$

Proposition 3.7.21. *The convolution product and hom diagram of Definition 3.7.3 are adjoint: for each $Y \in \mathcal{D}$-V there is an adjunction*

$$- \otimes Y \dashv \mathsf{Hom}_{\mathcal{D}}(Y,-).$$

Proof. We will show the following isomorphism in V for each triple of diagrams $X, Y, Z \in \mathcal{D}$-V:

$$\mathsf{Map}\big(X \otimes Y, Z\big) \cong \mathsf{Map}\big(X, \mathsf{Hom}_{\mathcal{D}}(Y,Z)\big).$$

The isomorphism of underlying sets of V-natural transformations then follows from Lemma 3.5.14.

Beginning with the definitions of Map (3.5.13) and $\otimes$ (3.7.4) we have the following sequence of isomorphisms, explained below:

$$\begin{aligned}
\mathsf{Map}\Big(X \otimes Y, Z\Big) &= \int_c \Big[(X \otimes Y)_c\,,\, Z_c\Big] = \int_c \Big[\int^{a,b} \mathcal{D}(a \boxdot b, c) \otimes (X_a \otimes Y_b)\,,\, Z_c\Big] \\
&\cong \int_{c,a} \Big[X_a \otimes \Big(\int^b \mathcal{D}(a \boxdot b, c) \otimes Y_b\Big)\,,\, Z_c\Big] \\
&\cong \int_{a,c} \Big[X_a\,,\, \Big[\int^b \mathcal{D}(a \boxdot b, c) \otimes Y_b\,,\, Z_c\Big]\Big] \\
&\cong \int_a \Big[X_a\,,\, \int_c \Big[\int^b \mathcal{D}(a \boxdot b, c) \otimes Y_b\,,\, Z_c\Big]\Big] \\
&= \int_a \Big[X_a\,,\, \mathsf{Hom}_{\mathcal{D}}(Y, Z)_a\Big] = \mathsf{Map}\Big(X, \mathsf{Hom}_{\mathcal{D}}(Y, Z)\Big).
\end{aligned}$$

After using the definitions in the first line, the middle three lines are given by:

(1) the isomorphism (3.5.10) and then associativity and symmetry in V followed by commuting $X_a \otimes -$ with coends;
(2) the closed structure of V and interchanging order of ends; and
(3) the isomorphism (3.5.11).

The final line is given by definition of the hom diagram (3.7.5) and of the mapping object. □

Theorem 3.7.22 (**[Day70]**). *Suppose* V *is a symmetric monoidal closed category that is complete and cocomplete. Suppose $\mathcal{D}$ is a small symmetric monoidal* V*-category. Then $\mathcal{D}$-*V *is a symmetric monoidal closed category with the following data.*

- *The monoidal product is given by the Day convolution product (3.7.4).*
- *The closed structure is given by the hom diagram (3.7.5).*
- *The monoidal unit is given by the diagram $J = \mathcal{Y}^e$ (3.7.18), where e denotes the monoidal unit of $\mathcal{D}$.*
- *The unit, associativity, and symmetry isomorphisms are given by the components in Definition 3.7.17.*

Proof. Functoriality of the monoidal product and internal hom are discussed in Proposition 3.7.19. The symmetric monoidal axioms of Definitions 1.1.1 and 1.1.23 follow from

- the universal properties of (co)ends,
- symmetric monoidal axioms for $\mathcal{D}$ and V, and
- functoriality of the monoidal structures for $\mathcal{D}$ and V.

As an example, we prove the pentagon axiom (1.1.3) for $\mathcal{D}$-V. For $X, Y, Z, W \in \mathcal{D}$-V we compute below each of the five vertices of the pentagon. In each case, first we use the definition of $\otimes$ (3.7.4), second we interchange coends and use associativity along with (in all but the first case) symmetry in V, and third we use isomorphisms γ from Definition 3.7.14. To save space, we denote $\boxdot$ and $\otimes$ by juxtaposition.

(3.7.23)
$$\begin{aligned}\Big[\big((XY)Z\big)W\Big]_t &= \int^{x,s} \mathcal{D}(xs,t)\bigg(\int^{y,r} \mathcal{D}(yr,x)\bigg(\bigg(\int^{p,q} \mathcal{D}(pq,y)(X_pY_q)\bigg)Z_r\bigg)W_s\bigg)\\ &\cong \int^{p,q,r,s}\bigg(\int^{x,y} \mathcal{D}(xs,t)\Big(\mathcal{D}(yr,x)\mathcal{D}(pq,y)\Big)\bigg)\Big(\big((X_pY_q)Z_r\big)W_s\Big)\\ &\cong \int^{p,q,r,s} \mathcal{D}\Big(\big((pq)r\big)s,t\Big)\Big(\big((X_pY_q)Z_r\big)W_s\Big)\end{aligned}$$

(3.7.24)
$$\begin{aligned}\Big[(XY)(ZW)\Big]_t &= \int^{x,y} \mathcal{D}(xy,t)\bigg(\bigg(\int^{p,q} \mathcal{D}(pq,x)(X_pY_q)\bigg)\bigg(\int^{r,s} \mathcal{D}(rs,y)(Z_rW_s)\bigg)\bigg)\\ &\cong \int^{p,q,r,s}\bigg(\int^{x,y} \mathcal{D}(xy,t)\Big(\mathcal{D}(pq,x)\mathcal{D}(rs,y)\Big)\bigg)\Big((X_pY_q)(Z_rW_s)\Big)\\ &\cong \int^{p,q,r,s} \mathcal{D}\Big((pq)(rs),t\Big)\Big((X_pY_q)(Z_rW_s)\Big)\end{aligned}$$

(3.7.25)
$$\begin{aligned}\Big[X\big(Y(ZW)\big)\Big]_t &= \int^{p,x} \mathcal{D}(px,t)\bigg(X_p\bigg(\int^{q,y} \mathcal{D}(qy,x)\bigg(Y_q\bigg(\int^{r,s} \mathcal{D}(rs,y)(Z_rW_s)\bigg)\bigg)\bigg)\bigg)\\ &\cong \int^{p,q,r,s}\bigg(\int^{x,y} \mathcal{D}(px,t)\Big(\mathcal{D}(qy,x)\mathcal{D}(rs,y)\Big)\bigg)\Big(X_p\Big(Y_q(Z_rW_s)\Big)\Big)\\ &\cong \int^{p,q,r,s} \mathcal{D}\Big(p\big(q(rs)\big),t\Big)\Big(X_p\Big(Y_q(Z_rW_s)\Big)\Big)\end{aligned}$$

(3.7.26)
$$\begin{aligned}\Big[\big(X(YZ)\big)W\Big]_t &= \int^{x,s} \mathcal{D}(xs,t)\bigg(\bigg(\int^{p,y} \mathcal{D}(py,x)\bigg(X_p\bigg(\int^{q,r} \mathcal{D}(qr,y)(Y_qZ_r)\bigg)\bigg)\bigg)W_s\bigg)\\ &\cong \int^{p,q,r,s}\bigg(\int^{x,y} \mathcal{D}(xs,t)\Big(\mathcal{D}(py,x)\mathcal{D}(qr,y)\Big)\bigg)\Big(\Big(X_p(Y_qZ_r)\Big)W_s\Big)\\ &\cong \int^{p,q,r,s} \mathcal{D}\Big(\big(p(qr)\big)s,t\Big)\Big(\Big(X_p(Y_qZ_r)\Big)W_s\Big)\end{aligned}$$

(3.7.27)
$$\begin{aligned}\Big[X\big((YZ)W\big)\Big]_t &= \int^{p,x} \mathcal{D}(px,t)\bigg(X_p\bigg(\int^{y,s} \mathcal{D}(ys,x)\bigg(\bigg(\int^{q,r} \mathcal{D}(qr,y)(Y_qZ_r)\bigg)W_s\bigg)\bigg)\bigg)\\ &\cong \int^{p,q,r,s}\bigg(\int^{x,y} \mathcal{D}(px,t)\Big(\mathcal{D}(ys,x)\mathcal{D}(qr,y)\Big)\bigg)\Big(X_p\Big((Y_qZ_r)W_s\Big)\Big)\\ &\cong \int^{p,q,r,s} \mathcal{D}\Big(p\big((qr)s\big),t\Big)\Big(X_p\Big((Y_qZ_r)W_s\Big)\Big)\end{aligned}$$

Then, by the universal property of coends, the pentagon axiom for each quadruple $X,Y,Z,W \in \mathcal{D}\text{-}\mathsf{V}$ follows from:

(1) the pentagon axiom for $\mathcal{D}$ (involving the components at p, q, r, and s),
(2) the pentagon axiom for V (involving the components at X_p, Y_q, Z_r, and W_s), and
(3) the functoriality of $\otimes$ in V.

The other symmetric monoidal category axioms for $\mathcal{D}$-V are proved in similar ways. The closed structure is provided by the $\otimes$-$\mathrm{Hom}_{\mathcal{D}}$ adjunction are given in Proposition 3.7.21. $\square$

Recall the concept of a symmetric monoidal V-functor from Definitions 1.4.17 and 1.4.18. We end this section with the following change-of-shape theorem.

Theorem 3.7.28. *Suppose V is a symmetric monoidal closed category that is complete and cocomplete. Suppose $F : \mathcal{D} \longrightarrow \mathcal{E}$ is a symmetric monoidal V-functor between small symmetric monoidal V-categories. Then precomposition with F is part of a symmetric monoidal functor*

$$\mathcal{E}\text{-}\mathsf{V} \xrightarrow{F^*} \mathcal{D}\text{-}\mathsf{V}.$$

Proof. Since F is a V-functor, precomposition with it yields a well-defined V-functor, $F^* = - \circ F$. The unit constraint of F^* is the following composite of V-natural transformations, with e and e' the monoidal units in, respectively, $\mathcal{D}$ and $\mathcal{E}$.

$$\begin{tikzcd}
J^{\mathcal{D}} = \mathcal{D}(e,-) \arrow[rr, "(F^*)^0"] \arrow[dr, "F"'] & & \mathcal{E}\big(e', F(-)\big) = J^{\mathcal{E}} \circ F \\
& \mathcal{E}\big(Fe, F(-)\big) \arrow[ur, "(F^0)^*"'] &
\end{tikzcd}$$

Here $(F^0)^*$ is induced by the monoidal constraint F^0 of the monoidal V-functor (F, F^2, F^0).

The monoidal constraint $(F^*)^2$ of F^* is defined as follows. For $X, Y \in \mathcal{E}$-V, the (X,Y) component of $(F^*)^2$ is defined by the following commutative diagram in $\mathcal{D}$-V, with the monoidal products in $\mathcal{D}$, $\mathcal{E}$, and V denoted by juxtaposition.

$$\begin{tikzcd}
F^*X \otimes F^*Y \arrow[rr, "(F^*)^2_{X,Y}"] \arrow[d, equal] & & F^*(X \otimes Y) \arrow[d, equal] \\
\int^{(a,b) \in \mathcal{D}\otimes\mathcal{D}} \mathcal{D}(ab,-)(X_{Fa}Y_{Fb}) & & \int^{(c,d) \in \mathcal{E}\otimes\mathcal{E}} \mathcal{E}\big(cd, F(-)\big)(X_cY_d) \\
\mathcal{D}(ab,-)(X_{Fa}Y_{Fb}) \arrow[u, "\omega_{a,b}"] \arrow[dr, "F \otimes 1"'] & & \mathcal{E}\big((Fa)(Fb), F(-)\big)(X_{Fa}Y_{Fb}) \arrow[u, "\omega_{Fa,Fb}"'] \\
& \mathcal{E}\big(F(ab), F(-)\big)(X_{Fa}Y_{Fb}) \arrow[ur, "(F^2_{a,b})^* \otimes 1"'] &
\end{tikzcd}$$

The two vertical morphisms ω are V-coend structure morphisms. The naturality of $(F^*)^2$ follows from the V-naturality of F^2. The axioms in Definitions 1.1.6 and 1.1.17 for $\big(F^*, (F^*)^2, (F^*)^0\big)$ to be a symmetric monoidal functor follow from the corresponding axioms for the symmetric monoidal V-functor (F, F^2, F^0) in Definitions 1.4.17 and 1.4.18 $\square$

3.8. Enriched Diagram Categories

Throughout this section we continue to let V denote a symmetric monoidal closed category that is complete and cocomplete.

Lemma 3.8.1. *In the context of Definition 3.7.17, evaluation at e defines a symmetric monoidal functor*

$$\mathsf{ev}_e : \mathcal{D}\text{-}\mathsf{V} \longrightarrow \mathsf{V}.$$

Proof. For $X, Y \in \mathcal{D}\text{-}\mathsf{V}$ we define $\mathsf{ev}_e X = X_e$ and for a natural transformation of V-functors we let ev_e take the component at e. The monoidal constraint

$$X_e \otimes Y_e \xrightarrow{\mathsf{ev}_e^2} (X \otimes Y)_e = \int^{p,q} \mathcal{D}(p \boxdot q, e) \otimes \big(X_p \otimes Y_q\big)$$

is given by the following composite with the structure morphism $\omega_{(e,e)}$ for the coend:

$$X_e \otimes Y_e \longrightarrow \mathcal{D}(e \boxdot e, e) \otimes \big(X_e \otimes Y_e\big) \xrightarrow{\omega_{(e,e)}} \int^{p,q} \mathcal{D}(p \boxdot q, e) \otimes \big(X_p \otimes Y_q\big). \tag{3.8.2}$$

The unlabeled morphism above is given by the inverse of the left unitor and tensoring with the composite

$$\mathbb{1} \xrightarrow{i_e} \mathcal{D}(e,e) \xrightarrow{\cong} \mathcal{D}(e \boxdot e, e)$$

induced by the identity and unitors for e in $\mathcal{D}$.

Each of the symmetric monoidal axioms for ev_e follows from universality of the coends and a corresponding commutativity of a diagram of components at e. For example, the symmetry axiom follows from commutativity of

$$\begin{tikzcd}
X_e \otimes Y_e \arrow[r, "\xi"] \arrow[d] & Y_e \otimes X_e \arrow[d] \\
\mathcal{D}(e \boxdot e, e) \otimes \big(X_e \otimes Y_e\big) \arrow[r, "\cong"] & \mathcal{D}(e \boxdot e, e) \otimes \big(Y_e \otimes X_e\big)
\end{tikzcd}$$

In the above diagram, the vertical morphisms are given as in (3.8.2) and the unlabeled horizontal morphism is induced by the symmetry on $e \boxdot e$ (which is the identity) and on $X_e \otimes Y_e$. The diagram therefore commutes by naturality of the left unit isomorphism. The associativity and unity axioms for ev_e follow similarly. $\square$

For a symmetric monoidal functor $U : \mathsf{V} \longrightarrow \mathsf{W}$, recall the change of enrichment 2-functor $(-)_U : \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathsf{W}\text{-}\mathsf{Cat}$ from Definition 2.1.1.

Explanation 3.8.3 (Various Enrichments for $\mathcal{D}$-V)**.** We now have three interpretations of the notation $\mathcal{D}$-V:

(1) In the notation of Definition 3.7.1 and Theorem 3.7.22, $\mathcal{D}$-V denotes the symmetric monoidal closed category of V-functors and V-natural transformations from $\mathcal{D}$ to $\underline{\mathsf{V}}$.

(2) Via the canonical self-enrichment, $\mathcal{D}$-V is symmetric monoidal as a $\mathcal{D}$-V-category with hom objects given by (3.7.12)

$$\mathsf{Hom}(X,Y) \cong \int_{b \in \mathcal{D}} [X_b, Y_{-\boxdot b}]$$

for $X, Y \in \mathcal{D}\text{-}\mathsf{V}$.

(3) In the notation of Definition 3.5.19 and Lemma 3.5.24, $\mathcal{D}$-V denotes the V-enriched category with

$$\mathcal{D}\text{-}\mathsf{V}(X, Y) = \mathsf{Map}(X, Y).$$

We discussed items (1) and (3) in Explanation 3.7.2 above: by Lemma 3.5.14, (1) is the underlying category of (3). Now we observe, by (3.7.13), that (3) is obtained from (2) by change of enrichment along the symmetric monoidal functor ev_e. This last observation shows that (3) is symmetric monoidal as a V-category by Theorem 2.4.10. ◇

Next we show that change of enrichment along a symmetric monoidal functor $U : \mathsf{V} \longrightarrow \mathsf{W}$ defines a symmetric monoidal functor on diagram categories. Recall from Proposition 3.3.4 that the standard enrichment of U defines a symmetric monoidal W-functor $\mathsf{V}_U \longrightarrow \mathsf{W}$.

Proposition 3.8.4. *Suppose V and W are complete and cocomplete symmetric monoidal closed categories. Suppose $U : \mathsf{V} \longrightarrow \mathsf{W}$ is a symmetric monoidal functor. Then change of enrichment and composition with the standard enrichment of U defines a symmetric monoidal functor*

$$U_* : \mathcal{D}\text{-}\mathsf{V} \longrightarrow \mathcal{D}_U\text{-}\mathsf{W}$$

Proof. The underlying functor U_* is given by

$$\mathsf{V}\text{-}\mathsf{Cat}(\mathcal{D}, \mathsf{V}) \xrightarrow{(-)_U} \mathsf{W}\text{-}\mathsf{Cat}(\mathcal{D}_U, \mathsf{V}_U) \xrightarrow{U \circ -} \mathsf{W}\text{-}\mathsf{Cat}(\mathcal{D}_U, \mathsf{W}).$$

For $X \in \mathcal{D}\text{-}\mathsf{V}$, the W-enriched diagram $U_* X$ sends $p \in \mathcal{D}_U$ to $(U_* X)_p = U(X_p)$. In our work below we will omit parentheses from such expressions. For $p, q \in \mathcal{D}$ the morphism on hom objects

$$(U_* X)_{p,q} : U\mathcal{D}(p, q) \longrightarrow [UX_p, UX_q]$$

is the morphism in W adjoint to the following composite:

$$\begin{tikzcd}
& U(\mathcal{D}(p,q) \otimes X_p) \arrow[r, "U(X_{p,q} \otimes 1)"] & U([X_p, X_q] \otimes X_p) \arrow[dr, "U\mathsf{eval}"] & \\
U\mathcal{D}(p,q) \otimes UX_p \arrow[ur, "U^2"] & & & UX_q.
\end{tikzcd}$$

For $X, Y \in \mathcal{D}\text{-}\mathsf{V}$ the monoidal constraint

$$(U_* X) \otimes (U_* Y) \xrightarrow{U_*^2} U_*(X \otimes Y)$$

is given by components for each $s \in \mathcal{D}$ via the following composite. The first morphism below uses the monoidal constraint of U (twice) and the second uses the universal property of the coend.

$$\begin{tikzcd}
\big((U_*X)\otimes(U_*Y)\big)_s \arrow[rr, "U_*^2"] \arrow[d, equal] & & U_*(X\otimes Y)_s \arrow[d, equal] \\
\int^{p,q} \mathcal{D}_U(p\boxdot q,s)\otimes\big((UX_p)\otimes(UY_q)\big) \arrow[dr] & & U\big(\int^{p,q}\mathcal{D}(p\boxdot q,s)\otimes(X_p\otimes Y_q)\big) \\
& \int^{p,q} U\big(\mathcal{D}(p\boxdot q,s)\otimes(X_p\otimes Y_q)\big) \arrow[ur] &
\end{tikzcd}$$

We take the unit constraint U_*^0 to be the identity because

$$U_*J = U\mathcal{D}(e,-)$$

is the monoidal unit of $\mathcal{D}_U$-W.

The symmetric monoidal functor axioms for U_* follow from those of U together with universality of coends. We illustrate this with the left unity axiom (1.1.10). The right unity, associativity, and symmetry axioms for U_* follow similarly.

The left unity axiom for U_* requires that the following diagram commute for each $s\in\mathcal{D}$.

$$\begin{tikzcd}
\big(UJ\otimes U_*X\big)_s \arrow[r, "\lambda_{U_X X}"] \arrow[d, "1"'] & U_*X_s \\
\int^{p,q} U\mathcal{D}(p\boxdot q,s)\otimes\big(U\mathcal{D}(e,p)\otimes UX_q\big) \arrow[r, "U_*^2"] & U\big(\int^{p,q}\mathcal{D}(p\boxdot q,s)\otimes(\mathcal{D}(e,p)\otimes X_q)\big) \arrow[u, "U_*\lambda_X"']
\end{tikzcd}$$

Commutativity of the above diagram follows from commutativity of the following two diagrams for $p,q,s\in\mathcal{D}$, which we explain below.

$$\begin{tikzcd}
U\mathcal{D}(p\boxdot q,s)\otimes U\mathcal{D}(e,p) \arrow[r, "\gamma"] \arrow[d, "U^2"'] & U\mathcal{D}(e\boxdot q,s) \\
U\big(\mathcal{D}(p\boxdot q,s)\otimes\mathcal{D}(e,p)\big) \arrow[ur, "U\gamma"'] &
\end{tikzcd} \tag{3.8.5}$$

$$\begin{tikzcd}
\int^{q} U\mathcal{D}(q,s)\otimes UX_q \arrow[r, "\cong"] \arrow[d] & UX_s \\
\int^{q} U\big(\mathcal{D}(q,s)\otimes X_q\big) \arrow[r, "\exists!"] & U\big(\int^{q}\mathcal{D}(q,s)\otimes X_q\big) \arrow[u, "\cong"']
\end{tikzcd} \tag{3.8.6}$$

The diagram (3.8.5) compares the morphisms γ of Definition 3.7.14 for $\mathcal{D}_U$ and $\mathcal{D}$. Commutativity follows from the definition of composition in $\mathcal{D}_U$ via U^2. The diagram (3.8.6) compares the isomorphisms of the V-Yoneda Density Theorem 3.7.8

for UX and X. Commutativity follows from universality of the relevant coends. $\square$

One family of important special cases is those arising from ordinary symmetric monoidal categories $\mathcal{D}$ via the following change of enrichment.

Definition 3.8.7. Let

$$F_{\mathbb{1}} : (\mathsf{Set}, \times, *) \longrightarrow (\mathsf{V}, \otimes, \mathbb{1}) \tag{3.8.8}$$

denote the strictly unital strong symmetric monoidal functor that sends a set X to $\coprod_X \mathbb{1}$. The unit constraint is $1_{\mathbb{1}}$. The monoidal constraint

$$FX \otimes FY \xrightarrow{F^2_{X,Y}} F(X \times Y)$$

is the following composite

$$\Big(\coprod_X \mathbb{1}\Big) \otimes \Big(\coprod_Y \mathbb{1}\Big) \xrightarrow{\cong} \coprod_{X\times Y} (\mathbb{1} \otimes \mathbb{1}) \xrightarrow{\cong} \coprod_{X\times Y} \mathbb{1}.$$

In the above composite, the first morphism is the canonical isomorphism commuting $\otimes$ with small coproducts (since V is symmetric monoidal closed) and the second isomorphism is given by the left unit isomorphism in V. $\diamond$

Definition 3.8.9. Suppose $(\mathcal{D}, \boxdot)$ is a small symmetric monoidal category. The *unitary enrichment* of $\mathcal{D}$ over V is denoted $\mathcal{D}_{\mathbb{1}}$ and is defined as the change of enrichment for $\mathcal{D}$, in the sense of Definition 2.1.1 and Proposition 2.1.2, along the functor $F_{\mathbb{1}}$ of (3.8.8) above.

So $\mathcal{D}_{\mathbb{1}}$ has the same objects as $\mathcal{D}$ and hom objects given by the coproduct in V

$$\mathcal{D}_{\mathbb{1}}(b,c) = \coprod_{p \in \mathcal{D}(b,c)} \mathbb{1}. \tag{3.8.10}$$

By Theorem 2.4.10, $\mathcal{D}_{\mathbb{1}}$ is a symmetric monoidal V-category. $\diamond$

Recall from Lemma 2.1.5 the corepresented functor

$$\mathsf{V}(\mathbb{1}, -) : \mathsf{V} \longrightarrow \mathsf{Set}$$

is symmetric monoidal.

Lemma 3.8.11. *In the context of Definition 3.8.7, the corepresented functor*

$$\mathsf{V}(\mathbb{1}, -) : \mathsf{V} \longrightarrow \mathsf{Set}$$

is right adjoint to $F_{\mathbb{1}}$.

Proof. The unit has component at a set X

$$\eta_X : X \longrightarrow \mathsf{V}\Big(\mathbb{1}, \coprod_X \mathbb{1}\Big)$$

which sends an element $i \in X$ to

$$\kappa_i : \mathbb{1} \longrightarrow \coprod_X \mathbb{1},$$

the canonical inclusion at summand i. The counit has component at an object A of V

$$\varepsilon_A : F_{\mathbb{1}}\big(\mathsf{V}(\mathbb{1}, A)\big) = \coprod_{\mathsf{V}(\mathbb{1},A)} \mathbb{1} \longrightarrow A$$

induced, for each summand $f \in \mathsf{V}(\mathbb{1}, A)$, by f.

The right triangle identity A in V is the composite

$$\mathsf{V}(\mathbb{1}, A) \longrightarrow \mathsf{V}\big(\mathbb{1}, \coprod_{\mathsf{V}(\mathbb{1},A)} \mathbb{1}\big) \longrightarrow \mathsf{V}(\mathbb{1}, A).$$

This composite is the identity because each $f \in \mathsf{V}(\mathbb{1}, A)$ is sent to the composite

$$\mathbb{1} \xrightarrow{\kappa_f} \coprod_{\mathsf{V}(\mathbb{1},A)} \mathbb{1} \xrightarrow{\varepsilon_A} A$$

that, by definition of ε, is f. The left triangle identity for X in Set is the composite

$$\coprod_X \mathbb{1} \longrightarrow \coprod_{\mathsf{V}(\mathbb{1}, \coprod_X \mathbb{1})} \mathbb{1} \longrightarrow \coprod_X \mathbb{1}.$$

This composite is the identity because the summand $\mathbb{1}$ at $i \in X$ maps via the identity on $\mathbb{1}$ to the summand at κ_i in the middle term and, by definition of ε, this summand maps via κ_i to $\coprod_X \mathbb{1}$. □

Proposition 3.8.12. *In the context of Definition 3.8.9, there is an isomorphism of categories*

(3.8.13) $$\mathsf{Cat}(\mathcal{D}, \mathsf{V}) \cong \mathsf{V}\text{-}\mathsf{Cat}(\mathcal{D}_{\mathbb{1}}, \underline{\mathsf{V}}) = (\mathcal{D}_{\mathbb{1}})\text{-}\mathsf{V}.$$

Proof. This is the isomorphism on hom categories of the change of enrichment adjunction induced by

$$F_{\mathbb{1}} \dashv \mathsf{V}(\mathbb{1}, -)$$

of Lemma 3.8.11. More explicitly, a functor of categories

$$X : \mathcal{D} \longrightarrow \mathsf{V}$$

corresponds to a V-functor

$$X_{\mathbb{1}} : \mathcal{D}_{\mathbb{1}} \longrightarrow \underline{\mathsf{V}}$$

with the same assignment on objects and with

$$\mathcal{D}_{\mathbb{1}}(a,b) = \coprod_{p \in \mathcal{D}(a,b)} \mathbb{1} \xrightarrow{(X_{\mathbb{1}})_{a,b}} [X_a, X_b]$$

given by the bijection

$$\mathsf{V}(\mathbb{1}, [X_a, X_b]) \cong \mathsf{V}(X_a, X_b)$$

for each pair of objects a and b in $\mathcal{D}$. The bijection of natural and V-natural transformations is similar. □

Example 3.8.14 (Diagrams in Cat)**.** For this example, suppose

$$(\mathsf{V}, \otimes, \mathbb{1}) = (\mathsf{Cat}, \times, \mathbf{1}),$$

the category of small categories with the Cartesian product and terminal category as its unit. Suppose $\mathcal{D}$ is an ordinary symmetric monoidal 1-category, regarded as a symmetric Cat-monoidal 2-category with identity 2-cells. Then $\mathcal{D}$ has the unitary enrichment over Cat. Suppose given diagrams

$$X, Y : \mathcal{D} \longrightarrow \mathsf{Cat}.$$

We have the following simplification in this case, for objects b and c in $\mathcal{D}$:

(3.8.15) $$\big[\mathcal{D}(b,c), [X_b, Y_c]\big] \cong \Big[\coprod_{p \in \mathcal{D}(b,c)} X_b, Y_c\Big] \cong \prod_{p \in \mathcal{D}(b,c)} [X_b, Y_c]$$

where the coproduct and product are indexed over morphisms in $\mathcal{D}$ and $[-,-]$ denotes the category of functors and natural transformations—the closed structure of Cat.

Recall from (3.7.13) we have

$$\mathsf{Hom}_{\mathcal{D}}(X, Y)_e \cong \mathsf{Map}(X, Y).$$

Using the simplification (3.8.15) and the equalizer formula (3.5.7), we have

$$\mathsf{Map}(X, Y) \cong \mathsf{eq}\Big(\prod_b [X_b, Y_b] \rightrightarrows \prod_{p:b \to c} [X_b, Y_c]\Big). \tag{3.8.16}$$

The two arrows in the equalizer are given, for each component $p \in \mathcal{D}(b, c)$, by pre- and post-composition with X_p and Y_p. Therefore a tuple of morphisms

$$\alpha_b : X_b \longrightarrow Y_b$$

determines an object of $\mathsf{Map}(X, Y)$ if and only if the following diagram in Cat commutes for each morphism p.

$$\begin{tikzcd} X_b \arrow[r, "X_p"] \arrow[d, "\alpha_b"'] & X_c \arrow[d, "\alpha_c"] \\ Y_b \arrow[r, "Y_p"] & Y_c \end{tikzcd} \tag{3.8.17}$$

This is precisely the 1-cell naturality condition (I.6.2.24) for α to define a 2-natural transformation from X to Y. The 2-cell naturality condition (I.6.2.25) is trivially satisfied because $\mathcal{D}$ has only identity 2-cells. Similarly, the morphisms in $\mathsf{Map}(X, Y)$ correspond to tuples of natural transformations, Γ_b, such that the whiskerings with X_p or Y_p are equal. This is precisely the modification axiom (I.6.3.2) for Γ to be a modification between 2-natural transformations. ⋄

3.9. Tensored and Cotensored Enriched Categories

In this section we define and prove basic results about tensor and cotensor structures.

Definition 3.9.1 (Tensored and Cotensored)**.** Suppose V is a symmetric monoidal closed category and suppose C is a V-category.

Tensored: We say that C is *tensored over* V if C is equipped, for each $X \in$ C and $A \in$ V, with an object $X \otimes A \in$ C together with isomorphisms in V

$$\mathsf{C}(X \otimes A, Y) \cong [A, \mathsf{C}(X, Y)]$$

for all $Y \in$ C.

Cotensored: We say that C is *cotensored over* V if C is equipped, for each $X \in$ C and $A \in$ V, with an object $X^A \in$ C together with isomorphisms in V

$$\mathsf{C}(Y, X^A) \cong [A, \mathsf{C}(Y, X)]$$

for all $Y \in$ C. ⋄

Example 3.9.2. The closed structure for V makes the self-enriched V-category $\underline{\mathsf{V}}$ tensored and cotensored over V with tensor given by the monoidal product and cotensor given by the closed structure. Isomorphisms

$$[X \otimes Y, Z] \cong [X, [Y, Z]]$$

for X, Y, and Z in V follow from the Yoneda Lemma, the closed structure, and the associativity in V. Indeed, for $W \in \mathsf{V}$ we have

$$\begin{aligned}\mathsf{V}(W,[X\otimes Y,Z]) \cong \mathsf{V}(W\otimes(X\otimes Y),Z) \xrightarrow[\cong]{\alpha^*} & \mathsf{V}((W\otimes X)\otimes Y,Z)\\ \cong & \mathsf{V}(W\otimes X,[Y,Z])\\ \cong & \mathsf{V}(W,[X,[Y,Z]]).\end{aligned}$$

Then the tensor and cotensor isomorphisms of Definition 3.9.1 follow from the symmetry of V. ◇

Although Definition 3.9.1 does not assert any particular compatibility of the tensor/cotensor isomorphisms, the following result uses Proposition 3.4.15 to show that the tensor and cotensor determine V-adjunctions.

Proposition 3.9.3. *In the context of Definition 3.9.1, for each X and Y in V there are unique extensions of the tensor and cotensor to V-functors*

$$\underline{\mathsf{V}} \xrightarrow{X\otimes -} \mathsf{C} \quad \textit{and} \quad \underline{\mathsf{V}} \xrightarrow{Y^{(-)}} \mathsf{C}^{\mathsf{op}},$$

respectively, such that the following are V-adjunctions

$$(X\otimes -) \dashv \mathcal{Y}^X \quad \textit{and} \quad Y^{(-)} \dashv \mathcal{Y}_Y.$$

Proof. We first note that, by the Yoneda Lemma for ordinary categories, the tensors and cotensors on underlying categories extend uniquely to bifunctors

$$\mathsf{C}_0\times\mathsf{V} \xrightarrow{-\otimes-} \mathsf{C}_0 \quad \text{and} \quad \mathsf{C}_0\times\mathsf{V}^{\mathsf{op}} \xrightarrow{(-)^{(-)}} \mathsf{C}_0$$

such that for each $X, Y \in \mathsf{V}$ there are adjunctions

$$\begin{tikzcd} \mathsf{V} \arrow[r, bend left, "X\otimes -"] \arrow[r, phantom, "\perp"] & \mathsf{C}_0 \arrow[l, bend left, "\mathsf{C}(X,-)"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathsf{V} \arrow[r, bend left, "Y^{(-)}"] \arrow[r, phantom, "\perp"] & \mathsf{C}_0^{\mathsf{op}} \arrow[l, bend left, "\mathsf{C}(-,Y)"] \end{tikzcd}$$

of underlying categories. It suffices to show that the underlying functors $X\otimes -$ and $Y^{(-)}$ can be extended to V-functors. Then the result follows from Proposition 3.4.15.

We will give the argument for $X\otimes -$; the argument for $Y^{(-)}$ is similar and formally dual. To begin, we first observe the following details for $X\otimes -$. Both follow from the details noted in Explanation 3.1.2. Here and throughout the rest of the proof we omit tensor symbols to save space.

For each $Y \in \mathsf{C}$ and $P, A \in \mathsf{V}$, the tensor and closed structures give isomorphisms in V

$$\mathsf{C}(X(PA),Y) \cong [PA,\mathsf{C}(X,Y)] \cong [P,[A,\mathsf{C}(X,Y)]] \cong [P,\mathsf{C}(XA,Y)] \cong \mathsf{C}((XA)P,Y).$$

The induced bijection of morphisms in C_0

$$\mathsf{C}_0(X(PA),Y) \cong \mathsf{C}_0((XA)P,Y) \tag{3.9.4}$$

is natural with respect to $Y \in \mathsf{C}_0$ and $P, A \in \mathsf{V}$ by naturality of the underlying adjunction. Therefore, by the Yoneda Lemma for ordinary categories, the bijection (3.9.4) is induced by an isomorphism $w = w_{P,A}$ in C_0

$$(XA)P \xrightarrow[\cong]{w_{P,A}} X(PA)$$

that is natural with respect to $P, A \in \mathsf{V}$. Unwinding the relevant adjunctions and using $C = \mathsf{C}(X,Y)$ in (3.1.4), the following diagram in C_0 commutes for each $P, Q, A \in \mathsf{V}$.

(3.9.5)
$$\begin{tikzcd}
& (X(QA))P \arrow[dr, "w_{P,QA}"] & \\
((XA)Q)P \arrow[ur, "w_{Q,A} \otimes P"] \arrow[d, "w_{P,Q}"'] & & X(P(QA)) \\
(XA)(PQ) \arrow[r, "w_{PQ,A}"] & X((PQ)A) \arrow[ur, "X\alpha_{P,Q,A}"'] &
\end{tikzcd}$$

Similarly, there is a morphism

$$j_X \in \mathsf{C}_0(X\mathbb{1}, X)$$

corresponding, under the following isomorphisms of the closed and tensor structures, to $1_X \in \mathsf{C}_0(X,X)$:

$$\mathsf{V}(\mathbb{1}, \mathsf{C}(X,X)) \xrightarrow{\lambda^*} \mathsf{V}(\mathbb{1}\mathbb{1}, \mathsf{C}(X,X)) \xrightarrow{\cong} \mathsf{V}(\mathbb{1}, [\mathbb{1}, \mathsf{C}(X,X)]) \xrightarrow{\cong} \mathsf{V}(\mathbb{1}, \mathsf{C}(X\mathbb{1}, X)).$$

Unwinding the relevant adjunctions and using $C = \mathsf{C}(X,Y)$ in (3.1.3), the following diagram in C_0 commutes for each $A \in \mathsf{V}$.

(3.9.6)
$$\begin{tikzcd}
(XA)\mathbb{1} \arrow[d, "w_{\mathbb{1},A}"'] \arrow[dr, "j_{XA}"] & \\
X(\mathbb{1}A) \arrow[r, "X\lambda"'] & XA
\end{tikzcd}$$

Now we give the definition of $X \otimes -$ on hom objects. For each $A, A', P \in \mathsf{V}$ we have

(3.9.7)
$$\begin{aligned}
\mathsf{V}(P, [A,A']) \cong \mathsf{V}(PA, A') \longrightarrow \mathsf{V}(\mathbb{1}, \mathsf{C}(X(PA), XA')) &\cong \mathsf{V}(\mathbb{1}, \mathsf{C}((XA)P, XA')) \\
&\cong \mathsf{V}(\mathbb{1}, [P, \mathsf{C}(XA, XA')]) \cong \mathsf{V}(P, \mathsf{C}(XA, XA'))
\end{aligned}$$

where:

- the first and last isomorphisms are given by the closed structure and unit isomorphism of V,
- the arrow on the first row is given by the underlying functor $X \otimes -$,
- the second isomorphism in the first row is given by precomposition with $w_{P,A}$, and
- the first isomorphism on the second row is given by the tensor structure of C.

Taking $P = [A, A']$, the identity $1_{[A,A']}$ corresponds to a morphism in V that we take as the definition of $X \otimes -$ on hom objects:

$$[A, A'] \xrightarrow{(X\otimes -)_{A,A'}} \mathsf{C}(XA, XA').$$

Now we verify the axioms of Definition 1.2.4 for $X \otimes -$.

The composition axiom (1.2.5) follows from commutativity of the below outer diagram for each $P, Q, A, A', A'' \in \mathsf{V}$. Following the diagram we explain commutativity of each region.

$$\begin{tikzcd}
\mathsf{V}(P,[A',A''])\times\mathsf{V}(Q,[A,A']) \arrow[dr, "\otimes"] \arrow[dd, "\cong"'] & & \mathsf{V}(PQ,[A,A'']) \arrow[d, "\cong"] \\
& \mathsf{V}(PQ,[A',A''][A,A']) \arrow[ur, "m_*"] \arrow[d, phantom, "(1)"] & \mathsf{V}((PQ)A,A'') \arrow[d, "(\alpha^{-1})^*", "\cong"'] \\
\mathsf{V}(PA',A'')\times\mathsf{V}(QA,A') \arrow[dr, "1\times(P\otimes -)"] \arrow[dd, "(X\otimes -)\times(X\otimes -)"'] & {} & \mathsf{V}(P(QA),A'') \arrow[dd, "X\otimes -"] \\
& \mathsf{V}(PA',A'')\times\mathsf{V}(P(QA),PA') \arrow[ur, "\circ"] \arrow[d, phantom, "(2)"] & \\
\mathsf{C}_0(X(PA'),XA'')\times\mathsf{C}_0(X(QA),XA') \arrow[dr, "w^*\times(-\otimes P)"] \arrow[dd, "w^*\times w^*"'] & {} & \mathsf{C}_0(X(P(QA)),XA'') \arrow[d, "w^*"] \\
& \mathsf{C}_0((XA')P,XA'')\times\mathsf{C}_0((X(QA))P,(XA')P) \arrow[ur, "(w^{-1})^*\circ"] \arrow[d, phantom, "(3)"] & \mathsf{C}_0((X(QA))P,XA'') \arrow[d, "(w^*)P"] \\
\mathsf{C}_0((XA')P,XA'')\times\mathsf{C}_0((XA)Q,XA') \arrow[dr, "1\times(-\otimes P)"] \arrow[dd, "\cong"'] & {} & \mathsf{C}_0(((XA)Q)P,XA'') \arrow[d, "(w^{-1})_*"] \\
& \mathsf{C}_0((XA')P,XA'')\times\mathsf{C}_0(((XA)Q)P,(XA')P) \arrow[ur, "\circ"] \arrow[d, phantom, "(4)"] & \mathsf{C}_0((XA)(PQ),XA'') \arrow[d, "\cong"] \\
\mathsf{V}(P,\mathsf{C}(XA',XA''))\times\mathsf{V}(Q,\mathsf{C}(XA,XA')) \arrow[rr, "m_*\otimes"] & {} & \mathsf{V}(PQ,\mathsf{C}(XA,XA''))
\end{tikzcd}$$

Now we explain each of the regions in the above diagram.

(1) The unlabeled vertical isomorphisms are given by the closed structure of V. The region commutes by functoriality of the monoidal product in V.

(2) The lower right morphism is given by composition in C_0 and precomposition with w^{-1}. Unwinding the definitions, commutativity of this region follows from the naturality noted in (3.9.4) for morphisms $QA \longrightarrow A'$ in V.

(3) Unwinding the definitions, commutativity of this diagram follows from underlying functoriality of $- \otimes P$ with respect to morphisms in C_0.

(4) The unlabeled vertical isomorphisms are from the closed structure and unit isomorphisms of V. The lower horizontal morphism is the monoidal

product of V followed by the composition in C. Commutativity of this diagram follows the same argument as (1), using underlying functoriality of $- \otimes P$.

Lastly, we observe that the vertical composite along the right is equal to $(X \otimes -)_{A,A''}$ by the compatibility of w with α noted in (3.9.5) above.

The identity axiom (1.2.6) for $(X \otimes -)$ follows from commutativity of the diagram below by evaluating at $i_A \in \mathsf{V}(\mathbb{1}, [A,A])$, as we explain below.

$$\begin{tikzcd}
\mathsf{V}(\mathbb{1},[A,A]) \arrow[r, "\cong"] \arrow[ddr, "\cong"'] & \mathsf{V}(\mathbb{1}A, A) \arrow[r, "X\otimes -"] & \mathsf{C}_0(X(\mathbb{1}A), XA) \arrow[dr, "w^*"] & \\
 & & & \mathsf{C}_0((XA)\mathbb{1}, XA) \arrow[d, "\cong"] \\
 & \mathsf{V}(A,A) \arrow[uu, "\lambda^*"'] \arrow[r, "X\otimes -"'] & \mathsf{C}_0(XA, XA) \arrow[uu, "X\lambda^*"] \arrow[ur, "j^*"] \arrow[r, equal] & \mathsf{V}(\mathbb{1}, \mathsf{C}(XA,XA))
\end{tikzcd}$$

In the above diagram, the top and right composite is (3.9.7) with $A' = A$ and $P = \mathbb{1}$, and therefore the image of i_A under this composite is

$$\mathbb{1} \xrightarrow{i_A} [A,A] \xrightarrow{(X\otimes -)_{A,A}} \mathsf{C}(XA, XA).$$

The left and bottom composite is i_{XA} because the underlying functor $X \otimes -$ preserves identity morphisms. The middle rectangle commutes by underlying functoriality of $X \otimes -$. The triangle involving $X\lambda$, w, and j commutes by (3.9.6) above. The remaining triangles commute by definition. This completes the proof that $X \otimes -$ extends to a V-functor determined by the underlying tensor adjunction. $\square$

Recall from Definition 2.1.1 that for V-category C and a monoidal functor

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

we have the W-category C_U with hom objects $\mathsf{C}_U(X,Y) = U\mathsf{C}(X,Y)$ for $X, Y \in \mathsf{C}$. The following result generalizes the change of enrichment for change of tensors and cotensors as well.

Theorem 3.9.8. *Suppose V and W are symmetric monoidal closed categories and suppose*

$$\begin{tikzcd}
\mathsf{W} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{V} \arrow[l, bend left, "U"]
\end{tikzcd}$$

is an adjunction of monoidal functors, with F^2 invertible. If C is enriched, tensored, and cotensored over V, then C_U is enriched, tensored, and cotensored over W with

- *$X \otimes A = X \otimes FA$ and*
- *$X^A = X^{FA}$*

for $X \in \mathsf{C}$ and $A \in \mathsf{W}$.

Proof. The assertion that C_U is a W-category is proved in Proposition 2.1.2. To show that C_U is tensored over W it suffices, by the Yoneda Lemma for ordinary categories, to show, for each $X, Y \in \mathsf{C}$ and $A \in \mathsf{V}$, there are bijections of sets

$$\mathsf{W}(B, \mathsf{C}_U(X \otimes FA, Y)) \cong \mathsf{W}(B, [A, \mathsf{C}_U(X,Y)])$$

that are natural with respect to $B \in \mathsf{W}$. This we show by the following sequence of natural bijections:

$$\begin{aligned}
\mathsf{W}(B, \mathsf{C}_U(X \otimes FA, Y)) &= \mathsf{W}(B, U\mathsf{C}(X \otimes FA, Y)) && \\
&\cong \mathsf{V}(FB, \mathsf{C}(X \otimes FA, Y)) && \text{since } F \dashv U, \\
&\cong \mathsf{V}(FB, [FA, \mathsf{C}(X, Y)]) && \text{since C tensored over V}, \\
&\cong \mathsf{V}(FB \otimes FA, \mathsf{C}(X, Y)) && \text{since V closed monoidal}, \\
&\cong \mathsf{V}(F(B \otimes A), \mathsf{C}(X, Y)) && \text{since } F^2 \text{ invertible}, \\
&\cong \mathsf{W}(B \otimes A, \mathsf{C}_U(X, Y)) && \text{since } F \dashv U, \\
&\cong \mathsf{W}(B, [A, \mathsf{C}_U(X, Y)]) && \text{since W closed monoidal}.
\end{aligned}$$

A dual argument shows that C_U is cotensored over W. We have the following sequence of natural bijections:

$$\begin{aligned}
\mathsf{W}(B, \mathsf{C}_U(X, Y^{FA})) &= \mathsf{W}(B, U\mathsf{C}(X, Y^{FA})) && \\
&\cong \mathsf{V}(FB, \mathsf{C}(X, Y^{FA})) && \text{since } F \dashv U, \\
&\cong \mathsf{V}(FB, [FA, \mathsf{C}(X, Y)]) && \text{since C cotensored over V}, \\
&\cong \mathsf{V}(FB \otimes FA, \mathsf{C}(X, Y)) && \text{since V closed monoidal}, \\
&\cong \mathsf{V}(F(B \otimes A), \mathsf{C}(X, Y)) && \text{since } F^2 \text{ invertible}, \\
&\cong \mathsf{W}(B \otimes A, U\mathsf{C}(X, Y)) && \text{since } F \dashv U, \\
&\cong \mathsf{W}(B, [A, U\mathsf{C}(X, Y)]) && \text{since W closed monoidal}.
\end{aligned}$$

This completes the proof. □

Example 3.9.9. In the context of Theorem 3.9.8, the special case $\mathsf{C} = \underline{\mathsf{V}}$ shows that $\underline{\mathsf{V}}_U$ is enriched, tensored, and cotensored over W. ◇

One of our important applications of Theorem 3.9.8 will be in the following context.

- Suppose V is a complete and cocomplete symmetric monoidal closed category.
- Suppose $\mathcal{D}$ is a small symmetric monoidal V-category with monoidal unit e.

In this context, recalling Definition 3.7.1 and Theorem 3.7.22, we have a symmetric monoidal closed category $\mathcal{D}$-V with monoidal product given by Day convolution and monoidal unit given by the V-functor

$$J = \mathcal{Y}^e = \mathcal{D}(e, -) : \mathcal{D} \longrightarrow \underline{\mathsf{V}}.$$

In this context, by Lemma 3.8.1, there is a symmetric monoidal functor

$$\mathsf{ev}_e : \mathcal{D}\text{-}\mathsf{V} \longrightarrow \mathsf{V}$$

given by evaluation at e. Now we define a left adjoint for ev_e.

Definition 3.9.10. Suppose $(\mathsf{V}, \otimes, \mathbb{1})$ is a complete and cocomplete symmetric monoidal closed category. Suppose $(\mathcal{D}, \boxdot, e)$ is a small symmetric monoidal V-category. Define a strong symmetric monoidal functor

$$L_e : \mathsf{V} \longrightarrow \mathcal{D}\text{-}\mathsf{V}$$

as follows. For each object A of V, let

$$L_e A = A \otimes J,$$

the composite of V-functors

$$\mathcal{D} \xrightarrow{J} \underline{\mathsf{V}} \xrightarrow{A \otimes -} \underline{\mathsf{V}}. \tag{3.9.11}$$

For morphisms $A \longrightarrow B$ in V we have corresponding V-natural transformations

$$A \otimes - \longrightarrow B \otimes -.$$

Whiskering with J defines L_e on morphisms. Functoriality of L_e follows because whiskering is functorial with respect to vertical composition of 2-cells.

Recalling the symmetric monoidal data for $\underline{\mathsf{V}}$ from Definition 3.3.1 and Theorem 3.3.2, the monoidal unit for $\underline{\mathsf{V}}$ is given by $\mathbb{1}$ and provides a V-natural isomorphism

$$1_{\underline{\mathsf{V}}} \cong \mathbb{1} \otimes -$$

which then provides

$$L_e^0 : J \xrightarrow{\cong} \mathbb{1} \otimes J = L_e \mathbb{1}.$$

The monoidal constraint L_e^2 is given by the composite of V-natural isomorphisms indicated by the following pasting diagram involving

- functoriality of the tensor product of V-categories with respect to composition of V-functors (Proposition 1.3.6),
- the unit isomorphism λ for J, from Definition 3.7.17, and
- the unique permuted canonical V-map provided by the Enriched Monoidal Coherence Theorem 2.5.6.

$$\begin{tikzcd}
\mathcal{D} \otimes \mathcal{D} \arrow[rr, "(L_e A) \otimes (L_e B)"] \arrow[dr, "J \otimes J"] \arrow[dd, "\boxdot"'] & & \underline{\mathsf{V}} \otimes \underline{\mathsf{V}} \arrow[dd, "\otimes"] \\
\arrow[r, phantom, "\lambda \Swarrow"] & \underline{\mathsf{V}} \otimes \underline{\mathsf{V}} \arrow[ur, "(A \otimes -) \otimes (B \otimes -)"'] \arrow[d, "\otimes"] \arrow[r, phantom, "\Swarrow"] & \\
\mathcal{D} \arrow[r, "J"] & \underline{\mathsf{V}} \arrow[r, "(A \otimes B) \otimes -"] & \underline{\mathsf{V}}
\end{tikzcd}$$

◇

Lemma 3.9.12. *In the context of Definition 3.9.10, L_e is a strong symmetric monoidal functor and provides an adjunction*

$$\begin{tikzcd} \mathsf{V} \arrow[r, bend left, "L_e"] \arrow[r, phantom, "\perp"] & \mathcal{D}\text{-}\mathsf{V} \arrow[l, bend left, "\mathsf{ev}_e"] \end{tikzcd} \tag{3.9.13}$$

where ev_e is evaluation at the monoidal unit e, as in Lemma 3.8.1.

Proof. The associativity axiom (1.1.9) and unity axiom (1.1.10) for L_e hold by Enriched Monoidal Coherence Theorem 2.5.6 for $\underline{\mathsf{V}}$ and ordinary Monoidal Coherence Theorem 1.1.31 for $\mathcal{D}$-V. The symmetry axiom (1.1.18) for L_e follows similarly, using the symmetric case of Theorem 2.5.6.

For A in V we have unit components η_A given by the composite

$$A \xrightarrow[\cong]{\rho^{-1}} A \otimes \mathbb{1} \xrightarrow{1 \otimes 1_e} A \otimes \mathcal{D}(e,e) = (\mathsf{ev}_e L_e)A.$$

Naturality of η follows from naturality of ρ and functoriality of the monoidal product in V.

For $X \in \mathcal{D}$-V we have counit components

$$\varepsilon_X : (L_e \mathsf{ev}_e)X = X_e \otimes J \longrightarrow X$$

defined, as V-natural transformations, via components

$$\mathbb{1} \longrightarrow [\mathcal{D}(e,a),[X_e,X_a]] \xrightarrow{\cong} [X_e \otimes \mathcal{D}(e,a), X_a]$$

where the first morphism is the adjoint of $X_{e,a}$ and the second uses the symmetry and closed monoidal structure of V. The V-naturality of each component ε_X follows from functoriality of X. The naturality of ε with respect to morphisms (V-natural transformations) θ from X to Y in $\mathcal{D}$-V is the commutativity, for each a in $\mathcal{D}$, of the following diagram in V.

$$\begin{tikzcd}
 & {[X_a,Y_a] \otimes [X_e \otimes \mathcal{D}(e,a), X_a]} \arrow[dr, "m"] & \\
\mathbb{1} \otimes \mathbb{1} \arrow[ur, "\theta_a \otimes (\varepsilon_X)_a"] \arrow[dr, "(\varepsilon_Y)_a \otimes ((L_e \mathsf{ev}_e)\theta)_a"'] & & {[X_e \otimes \mathcal{D}(e,a), Y_a]} \\
 & {[Y_e \otimes \mathcal{D}(e,a), Y_a] \otimes [X_e \otimes \mathcal{D}(e,a), Y_e \otimes \mathcal{D}(e,a)]} \arrow[ur, "m"'] &
\end{tikzcd}$$

Taking adjoints, commutativity of the diagram above follows from V-naturality of θ.

The left triangle identity for A in V is the composite of V-natural transformations

$$A \otimes J \xrightarrow{\eta_A \otimes J} (A \otimes \mathcal{D}(e,e)) \otimes J \xrightarrow{\varepsilon} A \otimes J \tag{3.9.14}$$

whose component at each a in $\mathcal{D}$ is adjoint to the composite

$$A \otimes \mathcal{D}(e,a) \longrightarrow (A \otimes \mathcal{D}(e,e)) \otimes \mathcal{D}(e,a) \longrightarrow A \otimes \mathcal{D}(e,a)$$

given by composition with 1_e. Therefore (3.9.14) is the identity. The right triangle identity for X in $\mathcal{D}$-V is the composite

$$X_e \xrightarrow{\eta_{X_e}} X_e \otimes \mathcal{D}(e,e) \xrightarrow{(\varepsilon_X)_e} X_e$$

also given by composition with 1_e. This completes the proof. □

Applying Theorem 3.9.8 to the adjunction of (3.9.13) then gives the following.

Corollary 3.9.15. *Suppose* V *is a complete and cocomplete symmetric monoidal closed category. Suppose* $\mathcal{D}$ *is a small symmetric monoidal* V*-category with monoidal unit* e*. Then* $\mathcal{D}$-V *is symmetric monoidal closed and, by changing enrichment along* ev_e*, is enriched, tensored, and cotensored over* V*.*

Explanation 3.9.16. In the context of Corollary 3.9.15, one can give a separate, more direct, explanation for the tensoring and cotensoring of $\mathcal{D}$-V over V. Note that the following explanation does not depend on the monoidal structure of $\mathcal{D}$. Recall from Explanation 3.8.3 that we regard $\mathcal{D}$-V as a V-enriched category with hom objects

$$\mathsf{Map}(X,Y) = \int_b [X_b, Y_b]$$

for X and Y in $\mathcal{D}$-V. Then for A in V we define the tensor and cotensor via composition of V-functors

$$X \otimes A : \mathcal{D} \xrightarrow{X} \underline{\mathsf{V}} \xrightarrow{-\otimes A} \underline{\mathsf{V}}$$

and

$$Y^A : \mathcal{D} \xrightarrow{Y} \underline{\mathsf{V}} \xrightarrow{[A,-]} \underline{\mathsf{V}}$$

generalizing the definition of L_e from Definition 3.9.10.

Then for the isomorphisms making $\mathcal{D}$-V tensored over V we have the following, using the closed structure of V and (3.5.11)

$$\begin{aligned}
\mathsf{Map}(X \otimes A, Y) &= \int_b [(X \otimes A)_b, Y_b] \\
&= \int_b [X_b \otimes A, Y_b] \\
&\cong \int_b \big[A, [X_b, Y_b]\big] \\
&\cong \Big[A, \int_b [X_b, Y_b]\Big] \\
&= [A, \mathsf{Map}(X,Y)].
\end{aligned}$$

Similarly, for the cotensor isomorphisms we have

$$\begin{aligned}
\mathsf{Map}(X, Y^A) &= \int_b [X_b, (Y^A)_b] \\
&= \int_b [X_b, [A, Y_b]] \\
&\cong \int_b [A, [X_b, Y_b]] \\
&\cong [A, \mathsf{Map}(X,Y)].
\end{aligned}$$

$\diamond$

3.10. Notes

3.10.1 (Self-Enrichment)**.** Kelly shows a version of Proposition 3.1.11 in **[Kel05**, Section 1.6]. Our approach is more elementary and direct than that of **[Kel05]**, which is in turn a simplification of general coherence methods of **[KML71a, KML71b, KML72]**. $\diamond$

3.10.2 (Enriched Yoneda Results)**.** The V-Yoneda Bijection of Theorem 3.4.12 is often called the *Weak Yoneda Lemma*. The V-Yoneda Density Theorem 3.7.8 is often called simply the Density Theorem or the Yoneda Lemma. It is equivalent to the V-Yoneda Lemma 3.6.9. For further discussion of these results we refer the reader to [**DK69**], [**Kel05**, Sections 1.9, 2.4, and 5.1], [**Bor94b**, Section 6.3], [**Lor21**, Section 4.3.2]. ⋄

3.10.3 (Simplifications for Enriched Co/Ends in V)**.** Our treatment of V-coends and V-ends in Section 3.5 involves several simplifications that do not hold for the more general theory of enriched co/ends.

- When V is symmetric monoidal closed, V-enriched co/limits in $\underline{V}$ (in the conical, unweighted sense) are canonically isomorphic to those in the underlying category V_0. Thus co/completeness of V_0 in the unenriched sense implies enriched co/completeness of V (with respect to conical co/limits). See [**DK69, Kel05**] for further explanation.
- The generalization of the isomorphisms (3.5.10) and (3.5.11) provides the more useful notion of co/end in a general V-category C. See [**DK69, Day70**].
- Our existence result Proposition 3.5.5 is a special case of that given in [**DK69**].
- The theory of V-enriched *weighted co/limits* extends the theory of conical co/limits and is defined in terms of certain co/ends. These more general constructions are necessary for general V-enriched category theory, and we refer the reader to [**Kel05, Lor21**] for further explanation. ⋄

3.10.4 (Enriched Diagram Categories)**.** Our discussion of enriched diagram categories in Section 3.7 follows [**DK69, Day70**]. A different sketch of the main ideas around Day's convolution product, under slightly simpler assumptions, is given in [**Ric20**]. The thesis of Corner [**Cor16**] gives a review of Day convolution and an extension to monoidal bicategories. Day-Street [**DS97**] gives a more general treatment of the convolution product and hom diagram as part of a more general theory of V-*modules*. ⋄

3.10.5 (Enriched Tensors and Cotensors)**.** For further discussion of enriched tensors and cotensors we refer the reader to [**Kel69, Bor94b, Kel05, Rie14**]. In the literature, the term *copowered*, respectively *powered*, is sometimes used in place of the term tensored, respectively cotensored.

Our Definition 3.9.1 differs slightly from the literature in that the tensor and cotensor isomorphisms are not assumed to be V-natural. Thus our conclusion in Proposition 3.9.3 does not have all the features of an *enriched 2-variable adjunction* as in [**Rie14**, Remark 3.7.4] or [**Shu∞a**]. Theorem 3.9.8 and Corollary 3.9.15, which will be used in our work below, only establish tensor and cotensor structures in the sense of Definition 3.9.1. ⋄

CHAPTER 4

Pointed Objects, Smash Products, and Pointed Homs

Throughout this chapter we suppose C is a category with a terminal object T and discuss objects X of C that are pointed by a morphism

$$\iota^X : T \longrightarrow X.$$

The category of pointed objects and morphisms, C_*, is the category under T. With the exception of some preliminary definitions in Section 4.1, we assume throughout that C is complete and cocomplete with a symmetric monoidal closed structure

$$(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi, \mathsf{Hom}).$$

In Section 4.1 we define the smash product of pointed objects and show that it induces a symmetric monoidal product on C_*. In Section 4.2 we define the pointed hom and show that it is adjoint to the smash product. The following examples of pointed categories will be essential parts for our algebraic K-theory work in Part 2.

- With $\mathsf{C} = \mathsf{Cat}$ and $T = \mathbf{1}$, the terminal category, Cat_* is the category of pointed categories and pointed functors.
- With $\mathsf{C} = \mathsf{PermCat}$ and $T = \mathbf{1}$ with its trivial monoidal product, every permutative category has a canonical basepoint given by its monoidal unit. For symmetric monoidal functors, preserving this basepoint is precisely the strictly unital condition.
- In Section 5.6 we describe the Boardman-Vogt tensor product for multicategories and its associated smash product.
- With $\mathsf{C} = \mathsf{Set}$ and $T = *$, the one-point set, Set_* is the category of pointed sets and pointed functions.
- In Section 7.1 we describe the category of pointed simplicial sets, sSet_*. Symmetric spectra, defined in Section 7.4, are certain sequences of pointed simplicial sets.

In many—but not all—of these applications, $\otimes$ is the Cartesian product in C and the monoidal unit is a terminal object so we have $\mathbb{1} = T$. The important exception is $\mathsf{C} = \mathsf{Multicat}$, the category of small multicategories discussed in Chapters 5 and 6.

In Section 4.3 below we discuss categories of pointed diagrams. Theorem 4.3.37 gives a summary statement. The development of K-theory for small permutative categories, given in Chapters 8, 9, and 10, will make use of certain pointed diagram categories and their symmetric monoidal closed structure.

4.1. Smash Products

We begin by defining categories of pointed objects. We show that completeness and cocompleteness are inherited from the ambient category C, and then we go on to define the smash product of pointed objects.

Definition 4.1.1. Suppose C is a category with terminal object T. We let C_* denote the category under T. Its objects are morphisms

$$\iota^X : T \longrightarrow X \in \mathsf{C},$$

which are called *pointed objects* with ι^X the *basepoint*. The morphisms of C_*, which are called *pointed morphisms*, are those morphisms in C that preserve the structure morphisms ι. ⋄

Definition 4.1.2. Suppose C is a category with a terminal object T and with coproducts. If X is an object of C, then $X_+ = X \coprod T$ denotes the pointed object given by the structure morphism of the coproduct

$$T \longrightarrow X \coprod T.$$

This defines a functor $\mathsf{C} \longrightarrow \mathsf{C}_*$ called *adjoining a disjoint basepoint*. ⋄

Proposition 4.1.3. *In the context of Definition 4.1.2, there is an adjunction of categories*

$$\mathsf{C} \underset{U}{\overset{(-)_+}{\rightleftarrows}} \mathsf{C}_*$$

with left adjoint $(-)_+$ given by adjoining a disjoint basepoint and right adjoint U given by forgetting basepoints.

Proof. The unit of the adjunction is the coproduct summand inclusion away from T. The component of the counit at a pointed object Y is

$$(UY)_+ \longrightarrow Y \quad \text{for} \quad Y \in \mathsf{C}_*$$

given by the identity on Y and the unique pointed morphism from the disjoint basepoint to Y. The triangle identities are the following composites for $X \in \mathsf{C}$ and $Y \in \mathsf{C}_*$:

$$X_+ \longrightarrow (U(X_+))_+ \longrightarrow X_+ \quad \text{and} \quad UY \longrightarrow U((UY)_+) \longrightarrow UY. \qquad \square$$

Definition 4.1.4. Suppose C is a category with terminal object T. For pointed objects X and Y the *wedge product* $X \vee Y$ is the pushout in C of the span

$$X \xleftarrow{\iota^X} T \xrightarrow{\iota^Y} Y.$$

Equivalently, it is the coequalizer of the two induced morphisms

$$\begin{tikzcd} T \arrow[r, shift left, "\iota^X"] \arrow[r, shift right, "\iota^Y"'] & X \coprod Y \arrow[r, dashed] & X \vee Y. \end{tikzcd}$$

The wedge product is sometimes also called the *wedge sum*. ⋄

Theorem 4.1.5. *Suppose* C *is a category with terminal object T. If* C *is complete and cocomplete, then so is* C_*.

Proof. The coproduct of pointed objects is given by the wedge product. Coequalizers of pointed objects are given by coequalizers in C equipped with the induced basepoint. Similarly, products and equalizers in C_* are given in C and equipped with the basepoints induced by these constructions. $\square$

Definition 4.1.6 (Smash Product)**.** Suppose that $\mathsf{C} = (\mathsf{C}, \otimes, \mathbb{1})$ is a monoidal category with terminal object T and with coproducts. Then the *smash product* $X \wedge Y$ is the following pushout in C.

$$\begin{tikzcd}
(X \otimes T) \coprod (T \otimes Y) \arrow[r, "(1_X \otimes \iota^Y) \coprod (\iota^X \otimes 1_Y)"] \arrow[d] & X \otimes Y \arrow[d] \\
T \arrow[r] & X \wedge Y
\end{tikzcd} \tag{4.1.7}$$

The *smash unit* E is defined by adjoining a disjoint basepoint to the unit of C:

$$E = \mathbb{1}_+ = \mathbb{1} \coprod T.$$

In Theorem 4.1.8 we show that $(\mathsf{C}_*, \wedge, E)$ is a symmetric monoidal category when $(\mathsf{C}, \otimes)$ is symmetric monoidal closed. In Section 4.2 we show, furthermore, that under these hypotheses $\wedge$ also has an adjoint. $\diamond$

Beginning now, and continuing through Section 4.2, we suppose that $(\mathsf{C}, \otimes, \mathsf{Hom})$ is a complete and cocomplete symmetric monoidal closed category with terminal object T.

Theorem 4.1.8. *Suppose*

$$\mathsf{C} = (\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi, \mathsf{Hom})$$

is a symmetric monoidal closed category that is complete and cocomplete with terminal object T. Then $(\mathsf{C}_*, \wedge, E)$ is a complete and cocomplete symmetric monoidal category.

Proof. Theorem 4.1.5 shows that C_* is complete and cocomplete. Now we show that the smash product defines a symmetric monoidal product with unit

$$E = \mathbb{1}_+ = \mathbb{1} \coprod T.$$

Because C is symmetric monoidal closed, the functors

$$X \otimes - : \mathsf{C} \longrightarrow \mathsf{C} \quad \text{and} \quad - \otimes Z : \mathsf{C} \longrightarrow \mathsf{C}$$

for $X, Z \in \mathsf{C}$ are left adjoints. Throughout this proof we will use the fact that these left adjoints preserve small colimits.

To construct the associativity isomorphism $\alpha^\wedge$ for the smash product, suppose (X, ι^X), (Y, ι^Y), and (Z, ι^Z) are pointed objects. We write

$$X \otimes Y \xrightarrow{\pi_{X,Y}} X \wedge Y \in \mathsf{C}$$

for the right vertical morphism in the pushout (4.1.7). Omitting $\otimes$ to save space, first observe that there is a pushout

$$\begin{tikzcd}
(XY)T \coprod (XT)Z \coprod (TY)Z \arrow[r] \arrow[dd] & (XY)Z \arrow[d, "\pi_{X,Y} 1"] \\
 & (X \wedge Y)Z \arrow[d, "\pi_{X \wedge Y, Z}"] \\
T \arrow[r] & (X \wedge Y) \wedge Z
\end{tikzcd} \tag{4.1.9}$$

in C with the top horizontal arrow induced by the basepoints of X, Y, and Z. In more detail, consider the following diagram in C.

(4.1.10)
$$\begin{tikzcd}
(XY)T \coprod (XT)Z \coprod (TY)Z \arrow[r] \arrow[d, "\omega"'] & (XY)Z \arrow[d, "\pi_{X,Y}1"] \\
(X \wedge Y)T \coprod TZ \arrow[r] \arrow[d] & (X \wedge Y)Z \arrow[d, "\pi_{X\wedge Y,Z}"] \\
T \arrow[r] & (X \wedge Y) \wedge Z
\end{tikzcd}$$

- The bottom square in (4.1.10) is a pushout by the definition (4.1.7) of the smash product.
- The top square in (4.1.10) is obtained by first applying $- \otimes Z$ to the pushout (4.1.7) defining $X \wedge Y$. This yields the pushout

(4.1.11)
$$\begin{tikzcd}
(XT)Z \coprod (TY)Z \arrow[r] \arrow[d, "\tau"'] & (XY)Z \arrow[d, "\pi_{X,Y}1"] \\
TZ \arrow[r] & (X \wedge Y)Z
\end{tikzcd}$$

  because $- \otimes Z$ is a left adjoint, which preserves all small colimits. The top left vertical arrow in (4.1.10) is the coproduct

$$\omega = \left(\pi_{X,Y} \otimes 1_T\right) \coprod \tau.$$

  Since (4.1.11) is a pushout, so is the top square in (4.1.10) by the functoriality of $\otimes$ in C.

Therefore, the outer diagram in (4.1.10), which is (4.1.9), is a pushout.

A similar argument using $X \otimes -$ yields the following pushout in C with the right normalized bracketing:

(4.1.12)
$$\begin{tikzcd}
X(YT) \coprod X(TZ) \coprod T(YZ) \arrow[r] \arrow[dd] & X(YZ) \arrow[d, "1\pi_{Y,Z}"] \\
 & X(Y \wedge Z) \arrow[d, "\pi_{X,Y\wedge Z}"] \\
T \arrow[r] & X \wedge (Y \wedge Z)
\end{tikzcd}$$

Now we consider the following morphism from the pushout (4.1.9), shown in the back square below, to the pushout (4.1.12), shown in the front square below.

(4.1.13)
$$\begin{tikzcd}
(XY)T \coprod (XT)Z \coprod (TY)Z \arrow[rr] \arrow[dd] \arrow[dr, "\alpha \amalg \alpha \amalg \alpha"] & & (XY)Z \arrow[dd] \arrow[dr, "\alpha"] & \\
 & X(YT) \coprod X(TZ) \coprod T(YZ) \arrow[rr] \arrow[dd] & & X(YZ) \arrow[dd] \\
T \arrow[rr] \arrow[dr, equal] & & (X \wedge Y) \wedge Z \arrow[dr, dashed, "\exists !", "\alpha^{\wedge}_{X,Y,Z}"'] & \\
 & T \arrow[rr] & & X \wedge (Y \wedge Z)
\end{tikzcd}$$

The left face in (4.1.13) commutes because T is a terminal object. The top face commutes by the naturality of α. The universal property of pushouts implies that there exists a unique dotted morphism $\alpha^{\wedge}_{X,Y,Z}$ that makes the entire cube commutative. Since each α is an isomorphism, so is $\alpha^{\wedge}_{X,Y,Z}$. We define this as the associativity isomorphism $\alpha^{\wedge}$ for the pointed objects X, Y, and Z. The naturality of $\alpha^{\wedge}$ follows from the naturality of α and the universal property of pushouts.

The pentagon axiom (1.1.3) for $\alpha^{\wedge}$ follows from that of α and the universal property of pushouts. More precisely, for pointed objects W, X, Y, and Z, we write

$$\begin{aligned} V_l &= ((WX)Y)T \amalg ((WX)T)Z \amalg ((WT)Y)Z \amalg ((TX)Y)Z \\ V_r &= W(X(YT)) \amalg W(X(TZ)) \amalg W(T(YZ)) \amalg T(X(YZ)) \end{aligned}$$

and consider the following cube.

(4.1.14)
$$\begin{tikzcd}
V_l \arrow[rr] \arrow[dr] \arrow[dd] & & ((WX)Y)Z \arrow[dr] \arrow[dd] & \\
& V_r \arrow[rr] \arrow[dd] & & W(X(YZ)) \arrow[dd] \\
T \arrow[rr] \arrow[dr, equal] & & ((W \wedge X) \wedge Y) \wedge Z \arrow[dr, dashed, "\exists !"] & \\
& T \arrow[rr] & & W \wedge (X \wedge (Y \wedge Z))
\end{tikzcd}$$

In the top face in (4.1.14), the morphism $V_l \longrightarrow V_r$ from the back pushout to the front pushout is a coproduct of four morphisms, each being either composite in the pentagon diagram (1.1.3) in C. The top right morphism

$$((W \otimes X) \otimes Y) \otimes Z \longrightarrow W \otimes (X \otimes (Y \otimes Z))$$

is either composite in the pentagon diagram in C for W, X, Y, and Z. The universal property of pushouts implies that there is a unique dotted arrow that renders the cube commutative, which proves the pentagon axiom for $\alpha^{\wedge}$.

To define the left unit isomorphism $\lambda^{\wedge}$ for the smash product, for a pointed object X, consider the following diagram in C.

(4.1.15)
$$\begin{tikzcd}
(\mathbb{1} \amalg T)T \amalg TX \arrow[r] \arrow[d] & (\mathbb{1} \amalg T)X \cong \mathbb{1}X \amalg TX \arrow[d] \arrow[ddr, bend left, "\ell"] & \\
T \arrow[r] \arrow[drr, bend right, "\iota^X"'] & (\mathbb{1} \amalg T) \wedge X \arrow[dr, dashed, "\exists ! \lambda^{\wedge}_X"'] & \\
& & X
\end{tikzcd}$$

In (4.1.15), the top left square is a pushout by the definition (4.1.7) of the smash product. The morphism ℓ is induced by the left unit isomorphism

$$\mathbb{1} \otimes X \xrightarrow{\lambda_X} X$$

in C and the composite

$$T \otimes X \longrightarrow T \xrightarrow{\iota^X} X.$$

The outer diagram in (4.1.15) is commutative, so there is a unique dotted arrow $\lambda^\wedge_X$ that renders the entire diagram commutative. Its naturality follows from that of λ and the universal property of pushouts. Since λ is an isomorphism, $\lambda^\wedge_X$ is an isomorphism, with inverse the composite

$$X \xrightarrow{\lambda^{-1}} \mathbb{1} \otimes X \longrightarrow (\mathbb{1} \coprod T) \otimes X \longrightarrow (\mathbb{1} \coprod T) \wedge X.$$

The right unit isomorphism $\rho^\wedge$ for the smash product is defined similarly by the right unit isomorphism ρ in C and the pushout (4.1.7) that defines $\wedge$.

With the smash unit

$$E = \mathbb{1} \coprod T,$$

the unity axiom (1.1.2) for the smash product asserts the commutativity of the following triangle in C for pointed objects X and Y.

$$\begin{tikzcd}
(X \wedge E) \wedge Y \arrow[rr, "\alpha^\wedge_{X,E,Y}"] \arrow[dr, "\rho^\wedge_X \wedge 1_Y"'] & & X \wedge (E \wedge Y) \arrow[dl, "1_X \wedge \lambda^\wedge_Y"] \\
& X \wedge Y &
\end{tikzcd} \tag{4.1.16}$$

Similar to the above proof of the pentagon axiom, the proof of the unity axiom for the smash product is a cube argument that reduces to the unity axiom for $\otimes$. More precisely, we consider the following morphism of spans in C, with the $\otimes$ symbol omitted to save space.

$$\begin{tikzcd}
(XE)T \coprod (XT)Y \coprod (TE)Y \arrow[rr] \arrow[d] \arrow[dr, "f"] & & (XE)Y \arrow[dr, "g"] & \\
T \arrow[dr, equal] & XT \coprod TY \arrow[rr] \arrow[d] & & XY \\
& T & &
\end{tikzcd} \tag{4.1.17}$$

Using the fact that $- \otimes E$ commutes with small colimits, the morphisms f and g in (4.1.17) are defined by the following morphisms, with t the unique morphism to the terminal object T and ρ the right unit isomorphism in C.

$$\begin{aligned}
(XE)T \cong (X\mathbb{1})T \coprod (XT)T &\xrightarrow{(\rho 1_T, (\iota^X \circ t) 1_T)} XT \\
(XT)Y &\xrightarrow{t 1_Y} TY \\
(TE)Y &\xrightarrow{t 1_Y} TY \\
(XE)Y \cong (X\mathbb{1})Y \coprod (XT)Y &\xrightarrow{(\rho 1_Y, (\iota^X \circ t) 1_Y)} XY
\end{aligned} \tag{4.1.18}$$

Since (4.1.17) is commutative, there is a unique induced morphism from the back pushout to the front pushout, namely,

$$(X \wedge E) \wedge Y \xrightarrow{\rho^\wedge_X \wedge 1_Y} X \wedge Y$$

in the triangle (4.1.16).

On the other hand, the composite

$$(X \wedge E) \wedge Y \xrightarrow{(1_X \wedge \lambda^{\wedge}_Y) \circ \alpha^{\wedge}_{X,E,Y}} X \wedge Y$$

in (4.1.16) is induced by the following morphisms, instead of those in (4.1.18), from the back span to the front span in (4.1.17).

$$\begin{gathered}
(XE)T \xrightarrow{\alpha} X(ET) \xrightarrow{1_X t} XT \\
(XT)Y \xrightarrow{\alpha} X(TY) \xrightarrow{1_X t} XT \\
(T\mathbb{1})Y \xrightarrow{\alpha} T(\mathbb{1}Y) \xrightarrow{1_T \lambda} TY \\
(TT)Y \xrightarrow{\alpha} T(TY) \xrightarrow{1_T(\iota^Y \circ t)} TY \\
(X\mathbb{1})Y \xrightarrow{\alpha} X(\mathbb{1}Y) \xrightarrow{1_X \lambda} XY \\
(XT)Y \xrightarrow{\alpha} X(TY) \xrightarrow{1_X(\iota^Y \circ t)} XY
\end{gathered}$$

To see that the two induced morphisms in (4.1.16) are equal, by the universal property of the back pushout in (4.1.17), we must check that the two composites

$$\begin{tikzcd}
(XE)Y \cong (X\mathbb{1})Y \coprod (XT)Y \arrow[r, shift left, "{(\rho_X 1_Y, (\iota^X \circ t)1_Y)}"] \arrow[r, shift right, "{(1_X\lambda_Y, 1_X(\iota^Y \circ t)) \circ (\alpha \amalg \alpha)}"'] & XY \arrow[d, "\pi_{X,Y}"] \\
& X \wedge Y
\end{tikzcd} \tag{4.1.19}$$

are equal. On the summand $(X\mathbb{1})Y$, the equality

$$\rho_X \otimes 1_Y = (1_X \otimes \lambda_Y) \circ \alpha : (X \otimes \mathbb{1}) \otimes Y \longrightarrow X \otimes Y$$

holds by the unity axiom in C. On the other summand $(XT)Y$, the two composites in (4.1.19) form the outer diagram below.

$$\begin{tikzcd}
(XT)Y \arrow[rr, "\alpha"] \arrow[d, "t1_Y"'] & & X(TY) \arrow[d, "1_X t"] \\
TY \arrow[r, "t"] \arrow[d, "\iota^X 1_Y"'] & T \arrow[d] & XT \arrow[l, "t"'] \arrow[d, "1_X \iota^Y"] \\
XY \arrow[r, "\pi_{X,Y}"] & X \wedge Y & XY \arrow[l, "\pi_{X,Y}"']
\end{tikzcd}$$

In the above diagram, the top pentagon commutes because T is a terminal object. The two lower parallelograms commute by the pushout definition of $X \wedge Y$. This proves the unity axiom for the smash product.

The symmetry isomorphism

$$X \wedge Y \xrightarrow{\xi^{\wedge}_{X,Y}} Y \wedge X$$

for the smash product is defined as the unique induced morphism from the pushout (4.1.7) defining $X \wedge Y$ to the pushout defining $Y \wedge X$, using the symmetry isomorphism $\xi_{X,Y}$ in C. Similar to the previous two paragraphs, the symmetry axiom (1.1.24) and the unit axiom (1.1.25) for $\wedge$ follow from those in C. In the presence of the symmetry axiom, the hexagon axiom (1.1.26) for $\wedge$ is equivalent to the left hexagon axiom in (1.1.16), which asserts the commutativity of the following diagram for pointed objects X, Y, and Z.

(4.1.20)
$$\begin{tikzcd}
& (Y \wedge X) \wedge Z \arrow[r, "\alpha^{\wedge}"] & Y \wedge (X \wedge Z) \arrow[dr, "1 \wedge \xi^{\wedge}"] & \\
(X \wedge Y) \wedge Z \arrow[ur, "\xi^{\wedge} \wedge 1"] \arrow[dr, "\alpha^{\wedge}"'] & & & Y \wedge (Z \wedge X) \\
& X \wedge (Y \wedge Z) \arrow[r, "\xi^{\wedge}"] & (Y \wedge Z) \wedge X \arrow[ur, "\alpha^{\wedge}"'] &
\end{tikzcd}$$

To prove the commutativity of (4.1.20), we again use a cube argument and consider the following morphisms of spans in C.

(4.1.21)
$$\begin{tikzcd}
(XY)T \sqcup (XT)Z \sqcup (TY)Z \arrow[rr] \arrow[dr, "\xi 1 \sqcup \xi 1 \sqcup \xi 1"] \arrow[dd, bend right] & & (XY)Z \arrow[dr, "\xi 1"] & & & \\
& (YX)T \sqcup (YT)Z \sqcup (TX)Z \arrow[rr] \arrow[dr, "\alpha \sqcup \alpha \sqcup \alpha"] \arrow[dd, bend right] & & (YX)Z \arrow[dr, "\alpha"] & & \\
T \arrow[dr, equal] & & Y(XT) \sqcup Y(TZ) \sqcup T(XZ) \arrow[rr] \arrow[dr, "1\xi \sqcup 1\xi \sqcup 1\xi"] \arrow[dd, bend right] & & Y(XZ) \arrow[dr, "1\xi"] & \\
& T \arrow[dr, equal] & & Y(ZT) \sqcup Y(TX) \sqcup T(ZX) \arrow[rr] \arrow[dd, bend right] & & Y(ZX) \\
& & T \arrow[dr, equal] & & & \\
& & & T & &
\end{tikzcd}$$

In (4.1.21), the top middle face is commutative by the naturality of α. The other two top faces are commutative by the naturality of ξ. So the diagram (4.1.21) is commutative. The unique induced morphism from the back pushout to the front pushout is the top composite in the left hexagon diagram (4.1.20).

Moreover, the left hexagon axiom in C is an equality

$$1\xi \circ \alpha \circ \xi 1 = \alpha \circ \xi \circ \alpha, \tag{4.1.22}$$

with the left-hand side the composite of the indicated morphisms in (4.1.21). Similarly, the left hexagon axiom in C yields the equality

$$\begin{aligned}
&(1\xi \sqcup 1\xi \sqcup 1\xi) \circ (\alpha \sqcup \alpha \sqcup \alpha) \circ (\xi 1 \sqcup \xi 1 \sqcup \xi 1) \\
&= (\alpha \sqcup \alpha \sqcup \alpha) \circ (\xi \sqcup \xi \sqcup \xi) \circ (\alpha \sqcup \alpha \sqcup \alpha),
\end{aligned} \tag{4.1.23}$$

with the top line the composite of the indicated morphisms in (4.1.21). Using

- the right-hand side of (4.1.22) and
- the bottom line in (4.1.23)

in the diagram (4.1.21), the unique induced morphism from the back pushout to the front pushout is the bottom composite in the left hexagon diagram (4.1.20). Therefore, (4.1.20) is commutative. $\square$

4.2. Pointed Homs

We remind the reader that $\mathsf{C} = (\mathsf{C}, \otimes, \mathsf{Hom})$ is assumed to be a complete and cocomplete symmetric monoidal closed category with terminal object T. In Section 4.1 we defined the smash product and showed that it defines a symmetric monoidal product $\wedge$ on C_*, with unit $E = \mathbb{1} \coprod T$. Now we define a closed structure for $(\mathsf{C}_*, \wedge, E)$

Definition 4.2.1. For pointed objects X and Y in C_*, we define the *pointed hom* as the following pullback in C.

(4.2.2)
$$\begin{tikzcd} \mathsf{Hom}_*(X,Y) \arrow[r] \arrow[d] & T \arrow[d] \\ \mathsf{Hom}(X,Y) \arrow[r] & \mathsf{Hom}(T,Y) \end{tikzcd}$$

The composite

$$T \cong \mathsf{Hom}(X,T) \longrightarrow \mathsf{Hom}(X,Y) \longrightarrow \mathsf{Hom}(T,Y)$$

induced by the structure morphisms for X and Y is equal to the vertical morphism in the diagram, and therefore induces a canonical structure morphism $T \longrightarrow \mathsf{Hom}_*(X,Y)$ making $\mathsf{Hom}_*(X,Y)$ a pointed object. ◇

Theorem 4.2.3. *Suppose*

$$\mathsf{C} = (\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi, \mathsf{Hom})$$

is a symmetric monoidal closed category that is complete and cocomplete with terminal object T. Then

$$(\mathsf{C}_*, \wedge, E, \mathsf{Hom}_*)$$

is a complete and cocomplete symmetric monoidal closed category.

Proof. Theorem 4.1.5 shows that C_* is complete and cocomplete. Theorem 4.1.8 shows that $(\mathsf{C}_*, \wedge)$ is symmetric monoidal with unit E. Now we show that Hom_* is adjoint to $\wedge$.

We will show

$$\mathsf{Hom}_*(X \wedge Y, Z) \cong \mathsf{Hom}_*(X, \mathsf{Hom}_*(Y,Z))$$

by showing that the right hand side satisfies the universal property of the pullback defining the left hand side. To do so, first note that the functors

$$\mathsf{Hom}(X,-) : \mathsf{C} \longrightarrow \mathsf{C} \quad \text{and} \quad \mathsf{Hom}(-,Z) : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{C}$$

both preserve small limits by the $\otimes$-Hom adjunction. That is, $\mathsf{Hom}(X,-)$ preserves small limits in C, and $\mathsf{Hom}(-,Z)$ converts small colimits in C (limits in C^{op}) to limits in C.

In the diagram below, we apply $\mathsf{Hom}(-,Z)$ to the pushout diagram (4.1.7) defining $X \wedge Y$. This yields a pullback in C shown at right. Next we observe that the definition of $\mathsf{Hom}_*(X \wedge Y, Z)$ makes it a pullback shown at left.

(4.2.4)

$$\begin{tikzcd}
\mathsf{Hom}_*(X \wedge Y, Z) \arrow[r] \arrow[d] & \mathsf{Hom}(X \wedge Y, Z) \arrow[r] \arrow[d] & \mathsf{Hom}(X \otimes Y, Z) \arrow[d] \\
T \arrow[r] & \mathsf{Hom}(T, Z) \arrow[r] & \mathsf{Hom}((X \otimes T) \coprod (T \otimes Y), Z)
\end{tikzcd}$$

We will show that $\mathsf{Hom}_*(X, \mathsf{Hom}_*(Y, Z))$ is also a pullback of the outer rectangle, via a sequence of isomorphisms that are natural in X, Y, and Z.

To begin, consider the following diagram in C. The horizontal morphisms are given by the unique morphisms to T or by pulling back along the basepoint of X. The other morphisms are described below.

(4.2.5)

$$\begin{tikzcd}
\mathsf{Hom}_*(X, \mathsf{Hom}_*(Y, Z)) \arrow[rr] \arrow[d] & & T \arrow[d] & \\
\mathsf{Hom}(X, \mathsf{Hom}_*(Y, Z)) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{Hom}(T, \mathsf{Hom}_*(Y, Z)) \arrow[dr] \arrow[dd] & \\
& T \arrow[rr] \arrow[dd] & & T \arrow[dd] \\
\mathsf{Hom}(X, \mathsf{Hom}(Y, Z)) \arrow[rr] \arrow[dr] & & \mathsf{Hom}(T, \mathsf{Hom}(Y, Z)) \arrow[dr] & \\
& \mathsf{Hom}(X, \mathsf{Hom}(T, Z)) \arrow[rr] & & \mathsf{Hom}(T, \mathsf{Hom}(T, Z))
\end{tikzcd}$$

The upper square of (4.2.5) is the pullback defining $\mathsf{Hom}_*(X, \mathsf{Hom}_*(Y, Z))$, and the other two squares involving a single vertex T (that is, the left vertical and right vertical squares) are given by applying $\mathsf{Hom}(X, -)$, respectively $\mathsf{Hom}(T, -)$ to the pullback squares defining $\mathsf{Hom}_*(Y, Z)$. Thus, because $\mathsf{Hom}(X, -)$ and $\mathsf{Hom}(T, -)$ are right adjoints, each of the squares involving a single vertex T is a pullback.

Now we use the closed structure of C to replace four lower vertices of (4.2.5) with objects of the form $\mathsf{Hom}(- \otimes -, Z)$ and recognize the right vertical composite

$$T \longrightarrow \mathsf{Hom}(T, \mathsf{Hom}_*(Y, Z)) \longrightarrow \mathsf{Hom}(T \otimes Y, Z)$$

as equal to the composite

$$T \cong \mathsf{Hom}(T \otimes Y, T) \longrightarrow \mathsf{Hom}(T \otimes Y, Z).$$

Using this equality and rearranging (4.2.5), we have $\mathsf{Hom}_*(X, \mathsf{Hom}_*(Y, Z))$ as a limit in the following diagram in C.

$$\begin{tikzcd}
\mathsf{Hom}_*(X, \mathsf{Hom}_*(Y, Z)) \arrow[rr] \arrow[d] & & T \arrow[dr] \arrow[dd] & \\
\mathsf{Hom}(X \otimes Y, Z) \arrow[rrr] \arrow[drr] & & & \mathsf{Hom}(T \otimes Y, Z) \arrow[d] \\
& & \mathsf{Hom}(X \otimes T, Z) \arrow[r] & \mathsf{Hom}(T \otimes T, Z)
\end{tikzcd}$$

The vertex $\mathsf{Hom}(T \otimes T, Z)$ is redundant, and we replace the two arrows out of T, respectively $\mathsf{Hom}(X \otimes Y, Z)$, with the corresponding arrow into

$$\mathsf{Hom}(X \otimes T, Z) \times \mathsf{Hom}(T \otimes Y, Z) \cong \mathsf{Hom}\big((X \otimes T) \coprod (T \otimes Y), Z \big).$$

This exhibits $\mathsf{Hom}_*(X, \mathsf{Hom}_*(Y, Z))$ as the pullback below, which is the outer pullback of (4.2.4).

$$\begin{tikzcd}
\mathsf{Hom}_*(X, \mathsf{Hom}_*(Y, Z)) \arrow[r] \arrow[d] & T \arrow[d] \\
\mathsf{Hom}(X \otimes Y, Z) \arrow[r] & \mathsf{Hom}((X \otimes T) \coprod (T \otimes Y), Z)
\end{tikzcd}$$

Therefore

$$\mathsf{Hom}_*(X, \mathsf{Hom}_*(Y, Z)) \cong \mathsf{Hom}_*(X \wedge Y, Z),$$

naturally in X, Y, and Z. □

4.3. Pointed Diagram Categories

In this section we suppose that

$$\mathsf{V} = (\mathsf{V}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, [-,-], \mathrm{T})$$

is a complete and cocomplete symmetric monoidal closed category with chosen terminal object T, sometimes denoted T^{V}. Recalling the smash product and internal hom constructions of Sections 4.1 and 4.2, we have a symmetric monoidal closed category

$$(\mathsf{V}_*, \wedge, E, [-,-]_*)$$

where $E = \mathbb{1} \coprod \mathrm{T}$ is the monoidal unit and $[-,-]_*$ denotes the pointed hom of Definition 4.2.1.

Convention 4.3.1. To avoid ambiguity with our notation for diagram categories, throughout this section we write

$$\mathsf{VCat} \quad \text{and} \quad \mathsf{V}_*\mathsf{Cat} \tag{4.3.2}$$

for the 2-categories of V-enriched, respectively V_*-enriched, categories, functors and natural transformations. ◇

Definition 4.3.3. Suppose $(\mathcal{D}, \boxdot, e)$ is a symmetric monoidal category. We say that T is a *null object* for $\mathcal{D}$ if

- T is both initial and terminal in $\mathcal{D}$, and
- there are natural isomorphisms

$$a \boxdot \mathrm{T} \cong \mathrm{T} \cong \mathrm{T} \boxdot a$$

for objects a in $\mathcal{D}$. ◇

We will use the following notation throughout this section.

Definition 4.3.4. Let $(\mathsf{Set}_*, \wedge, \underline{1}, *)$ denote the category of pointed finite sets with its smash product $\wedge$, unit $\underline{1} = \{0, 1\}$, with basepoint element 0, and terminal 1-point set $*$. ◇

The main result of this section, Theorem 4.3.19, gives a comparison between pointed diagrams from $(\mathcal{D}, \mathrm{T}^{\mathcal{D}})$ to V_* with enriched diagrams out of a certain V_*-enriched category $\widehat{\mathcal{D}}$ described in Definition 4.3.11. To prepare for this, we first give the following general comparison between

- pointed functors between categories whose basepoints are both initial and terminal, and
- Set_*-enriched functors between such categories.

Lemma 4.3.5. *Suppose $(\mathsf{B}, \mathrm{T}^{\mathsf{B}})$ and $(\mathsf{C}, \mathrm{T}^{\mathsf{C}})$ are pointed categories in which the basepoints T^{B} and T^{C} are both initial and terminal.*

(1) Taking zero morphisms as basepoints of their hom sets, both B and C are enriched categories over Set_*.
(2) There is an equivalence of categories

$$\mathsf{Cat}_*((\mathsf{B}, \mathrm{T}^{\mathsf{B}}), (\mathsf{C}, \mathrm{T}^{\mathsf{C}})) \simeq (\mathsf{Set}_*\mathsf{Cat})(\mathsf{B}, \mathsf{C}), \tag{4.3.6}$$

where the right hand side denotes the category of Set_*-enriched functors and natural transformations.

Proof. For the first assertion, the composition and identities of B and C are Set_*-enriched because composition with a zero morphism is zero. For the second assertion, first observe that a pointed functor

$$H : (\mathsf{B}, \mathrm{T}^{\mathsf{B}}) \longrightarrow (\mathsf{C}, \mathrm{T}^{\mathsf{C}})$$

preserves the zero morphisms and is therefore Set_*-enriched. Pointed natural transformations are likewise Set_*-enriched. This provides a functor

$$\mathsf{Cat}_*((\mathsf{B}, \mathrm{T}^{\mathsf{B}}), (\mathsf{C}, \mathrm{T}^{\mathsf{C}})) \longrightarrow (\mathsf{Set}_*\mathsf{Cat})(\mathsf{B}, \mathsf{C})$$

For the reverse direction, the forgetful functor

$$U : \mathsf{Set}_* \longrightarrow \mathsf{Set},$$

which is the identity on objects and morphisms, is symmetric monoidal with unit constraint the inclusion of the one point set at the non-basepoint element of $\underline{1}$, the monoidal unit for Set_*. The monoidal constraint

$$(UX) \times (UY) = X \times Y \xrightarrow{U^2_{X,Y}} X \wedge Y$$

for pointed sets X and Y is given by the structure morphism of the pushout (4.1.7) defining $X \wedge Y$. Changing enrichment along U provides a forgetful functor

$$(\mathsf{Set}_*\mathsf{Cat})(\mathsf{B}, \mathsf{C}) \longrightarrow \mathsf{Cat}(\mathsf{B}, \mathsf{C}).$$

from the category of Set_*-enriched functors and natural transformations to the underlying category of ordinary Set-enriched functors and natural transformations.

We show that Set_*-enriched functors preserve initial and terminal objects up to unique isomorphism, and then use these isomorphisms to provide strictly pointed functors. To make this explicit, let $0_{Z,W}$ denote the zero morphism

$$Z \longrightarrow \mathrm{T} \longrightarrow W$$

factoring through the respective basepoint object for each pair of objects Z and W objects in either B or C. Then we note the following.

- If F is a Set_*-enriched functor from B to C, then preserving the basepoint of each hom set means
$$F(0_{X,Y}) = 0_{FX,FY}$$
for each pair of objects X and Y in B.
- For any object Z of B or C, we have
$$0_{Z,Z} = 1_Z$$
if and only if $Z \cong \mathrm{T}$, the respective basepoint object. This isomorphism is necessarily unique.
- By the identity axiom (1.2.6) for a Set_*-enriched functor $F : \mathsf{B} \longrightarrow \mathsf{C}$, we have
$$1_{F(\mathrm{T}^{\mathsf{B}})} = F(1_{\mathrm{T}^{\mathsf{B}}}).$$

Combining these observations, we have

$$1_{F(\mathrm{T}^{\mathsf{B}})} = F(1_{\mathrm{T}^{\mathsf{B}}}) = F(0_{\mathrm{T}^{\mathsf{B}},\mathrm{T}^{\mathsf{B}}}) = 0_{F(\mathrm{T}^{\mathsf{B}}),F(\mathrm{T}^{\mathsf{B}})}$$

and, therefore, a unique isomorphism

$$F(\mathrm{T}^{\mathsf{B}}) \cong \mathrm{T}^{\mathsf{C}}. \tag{4.3.7}$$

Since T^{B} and T^{C} are both initial and terminal, we can change the definition of F on objects to

$$F'X = \begin{cases} FX & \text{if} \quad X \neq \mathrm{T}^{\mathsf{B}} \\ \mathrm{T}^{\mathsf{C}} & \text{if} \quad X = \mathrm{T}^{\mathsf{B}}. \end{cases}$$

and define F' on morphisms via F and composition with (4.3.7) for the (necessarily unique) morphisms to or from T^{B} in B. The functoriality of F and uniqueness of (4.3.7) ensure that F' remains functorial. We likewise replace a Set_*-enriched natural transformation

$$\alpha : F \longrightarrow G$$

with a pointed natural transformation

$$\alpha' : F' \longrightarrow G'$$

by changing the component at T^{B} to be the identity of T^{C}. Naturality of α' follows from that of α and the uniqueness of (4.3.7).

For a pointed functor

$$H : (\mathsf{B}, \mathrm{T}^{\mathsf{B}}) \longrightarrow (\mathsf{C}, \mathrm{T}^{\mathsf{C}}),$$

we have $H' = H$. For a general Set_*-enriched functor F as above, there is a natural isomorphism

$$\varepsilon : F' \xrightarrow{\cong} F$$

whose only potential non-identity component is the isomorphism (4.3.7). Therefore we have $\varepsilon' = 1_{F'}$. This proves the triangle identities for the second assertion. □

Recall from Definition 3.8.9 the unitary enrichment of a symmetric monoidal category over V. Now we introduce a pointed variant over V_*.

Definition 4.3.8. For a pointed set X, we let $X^\flat$ denote the *punctured set*

$$X^\flat = X \backslash \{*\}$$

consisting of all elements of X except its basepoint. The symbol $\flat$ is the musical flat symbol, and $X^\flat$ may be read as *X-flat* or *X-punctured*, with the mnemonic that a punctured tire is flat. ◇

Definition 4.3.9. Define a pointed and strictly unital strong symmetric monoidal functor

$$\widehat{F} \colon (\mathsf{Set}_*, \wedge, \underline{1}, *) \longrightarrow (\mathsf{V}_*, \wedge, E, \mathrm{T}) \tag{4.3.10}$$

by

$$\widehat{F}X = \bigvee_{i \in X^\flat} E \quad \text{for} \quad X \in \mathsf{Set}_*$$

where we take T for the wedge over an empty indexing set. For a morphism of pointed sets $f \colon X \longrightarrow Y$, the induced morphism

$$\bigvee_{i \in X^\flat} E \longrightarrow \bigvee_{i \in Y^\flat} E$$

has summand at $i \in X^\flat$ given by the identity 1_E if $f(i)$ is not the basepoint of Y (and, therefore, $f(i) \in Y^\flat$) or by the zero morphism to the target (factoring through the basepoint of the wedge) if $f(i) = *$.

The unit constraint $\widehat{F}^0$ is 1_E. The monoidal constraint

$$\widehat{F}X \wedge \widehat{F}Y \xrightarrow{\widehat{F}^2_{X,Y}} \widehat{F}(X \wedge Y)$$

is given by the following composite

$$\Big(\bigvee_{i \in X^\flat} E\Big) \wedge \Big(\bigvee_{j \in Y^\flat} E\Big) \xrightarrow{\cong} \bigvee_{(i,j) \in X^\flat \times Y^\flat} E \wedge E \xrightarrow{\cong} \bigvee_{(i,j) \in (X \wedge Y)^\flat} E.$$

In the above composite, the first morphism is the canonical isomorphism commuting $\wedge$ with small coproducts (since V_* is symmetric monoidal closed) and the second morphism is given by the bijection of indexing sets

$$X^\flat \times Y^\flat \cong (X \wedge Y)^\flat$$

and the unit isomorphisms of E. ◇

Definition 4.3.11. Suppose $(\mathcal{D}, \boxdot, e, \mathrm{T}^{\mathcal{D}})$ is a small symmetric monoidal category with null object $\mathrm{T}^{\mathcal{D}}$ and with its canonical enrichment over Set_* as in Lemma 4.3.5 (1).

Suppose T^{V} is a chosen terminal object of V. The *pointed unitary enrichment* of $\mathcal{D}$ over V_* is denoted $\widehat{\mathcal{D}}$ and is defined as the change of enrichment for $\mathcal{D}$, in the sense of Definition 2.1.1 and Proposition 2.1.2, along the functor $\widehat{F}$ of (4.3.10) above.

So $\widehat{\mathcal{D}}$ has the same objects as $\mathcal{D}$ and has hom objects given by the coproduct in V_*

$$\widehat{\mathcal{D}}(a,b) = \bigvee_{p \in (\mathcal{D}(a,b))^\flat} E \tag{4.3.12}$$

where we take T^{V} for the empty wedge. By Theorem 2.4.10, $\widehat{\mathcal{D}}$ is a symmetric monoidal V_*-category. ◇

Explanation 4.3.13. Recalling the unitary enrichment of $\mathcal{D}$ over E, the monoidal unit of V_*, Proposition 3.8.12 provides an isomorphism of categories

$$\mathsf{Cat}(\mathcal{D}, \mathsf{V}_*) \cong \mathsf{V}_*\mathsf{Cat}(\mathcal{D}_E, \underline{\mathsf{V}_*}), \tag{4.3.14}$$

where $\mathsf{V}_*\mathsf{Cat}$ is the 2-category of V_*-categories, V_*-functors, and V_*-natural transformations. When $\mathcal{D}$ has a null object as its basepoint, as in Definition 4.3.11, then there is also a V_*-functor

$$\mathcal{D}_E \longrightarrow \widehat{\mathcal{D}} \tag{4.3.15}$$

induced, as in Definition 2.2.1, by a monoidal natural transformation

$$F_E \longrightarrow \widehat{F}$$

whose component at a pointed set X is the morphism

$$\bigvee_{i \in X} E \longrightarrow \bigvee_{i \in X^\flat} E$$

given by the identity 1_E for each non-basepoint summand and by the zero morphism for the basepoint summand. Restriction along the V_*-functor (4.3.15) provides a 2-functor

$$\mathsf{V}_*\mathsf{Cat}(\widehat{\mathcal{D}}, \underline{\mathsf{V}_*}) \longrightarrow \mathsf{V}_*\mathsf{Cat}(\mathcal{D}_E, \underline{\mathsf{V}_*}). \tag{4.3.16}$$

We show in Theorem 4.3.19 below that this corresponds under the isomorphism (4.3.14) to the forgetful inclusion of pointed functors among all functors. ◇

Example 4.3.17. In the case $\mathsf{V}_* = \mathsf{Set}_*$, the enrichment $\widehat{\mathcal{D}}$ of Definition 4.3.11 is the canonical Set_*-enrichment described in Lemma 4.3.5 (1). ◇

For the remainder of this section we will have the following standing assumptions.

Convention 4.3.18 (Context for Pointed Diagram Categories)**.**

(1) Suppose $(\mathsf{V}, \otimes, \mathbb{1}, \mathrm{T}^{\mathsf{V}})$ is a complete and cocomplete symmetric monoidal closed category with chosen terminal object T^{V}.
(2) Then by Theorem 4.2.3 $(\mathsf{V}_*, \wedge, E, [-,-]_*)$ is a complete and cocomplete symmetric monoidal closed category with monoidal smash product $\wedge$, monoidal unit
$$E = \mathbb{1} \coprod \mathrm{T}^{\mathsf{V}}$$
and pointed hom $[-,-]_*$.
(3) Suppose $(\mathcal{D}, \boxdot, e, \mathrm{T}^{\mathcal{D}})$ is a small pointed symmetric monoidal category with chosen null object $\mathrm{T}^{\mathcal{D}}$ as basepoint. ◇

Theorem 4.3.19. *In the context of Convention 4.3.18 there is an equivalence of categories*

$$\mathsf{Cat}_*((\mathcal{D}, \mathrm{T}^{\mathcal{D}}), (\mathsf{V}_*, \mathrm{T}^{\mathsf{V}})) \xrightarrow{\simeq} \mathsf{V}_*\mathsf{Cat}(\widehat{\mathcal{D}}, \underline{\mathsf{V}_*}). \tag{4.3.20}$$

Moreover, this equivalence commutes with the isomorphism (4.3.14) in the sense of the following commutative diagram, where the vertical functors are given by the forgetful inclusion and by (4.3.16).

$$
\begin{tikzcd}
\mathsf{Cat}_*((\mathcal{D},\mathrm{T}^{\mathcal{D}}),(\mathsf{V}_*,\mathrm{T}^{\mathsf{V}})) \arrow[r, "\simeq"] \arrow[d] & \mathsf{V}_*\mathsf{Cat}(\widehat{\mathcal{D}},\underline{\mathsf{V}_*}) \arrow[d] \\
\mathsf{Cat}(\mathcal{D},\mathsf{V}_*) \arrow[r, "\cong"] & \mathsf{V}_*\mathsf{Cat}(\mathcal{D}_E,\underline{\mathsf{V}_*})
\end{tikzcd}
\tag{4.3.21}
$$

Proof. As noted in the proof of Proposition 3.8.12, the lower isomorphism of (4.3.21) is the isomorphism on hom categories for the change of enrichment adjunction induced by

$$F_E \dashv \mathsf{V}_*(E,-).$$

The right adjoint $\mathsf{V}_*(E,-)$ factors through Set_* as

$$\mathsf{V}_* \xrightarrow{\widehat{U}} \mathsf{Set}_* \longrightarrow \mathsf{Set}$$

where the second functor forgets the basepoint and, for each A in V_* we let $\widehat{U}A$ denote $\mathsf{V}_*(E,A)$ equipped with the zero basepoint.

As in Lemma 3.8.11, $\widehat{U}$ is right adjoint to $\widehat{F}$ with natural isomorphisms

$$\mathsf{V}_*\Big(\bigvee_{X^\flat} E, A\Big) \cong \prod_{X^\flat} \mathsf{V}_*(E,A) \cong \mathsf{Set}_*(X,\mathsf{V}_*(E,A))$$

for each A in V_*. Thus we have the following serially-commuting diagram of adjunctions, where the vertical functors are given by the forgetful inclusions.

$$
\begin{tikzcd}
\mathsf{Set}_* \arrow[r, bend left, "\widehat{F}"] \arrow[r, phantom, "\perp"] \arrow[d] & \mathsf{V}_* \arrow[l, bend left, "\widehat{U}"] \arrow[d] \\
\mathsf{Set} \arrow[r, bend left, "F_E"] \arrow[r, phantom, "\perp"] & \mathsf{V} \arrow[l, bend left, "\mathsf{V}_*(E,-)"]
\end{tikzcd}
$$

In the above diagram, the lower adjunction induces the isomorphism (4.3.14) shown along the bottom edge of (4.3.21). The upper adjunction induces an isomorphism

$$(\mathsf{Set}_*\mathsf{Cat})(\mathcal{D},\mathsf{V}_*) \cong \mathsf{V}_*\mathsf{Cat}(\widehat{\mathcal{D}},\underline{\mathsf{V}_*}) \tag{4.3.22}$$

between V_*-enriched diagrams and Set_*-enriched diagrams. We compose this isomorphism with the equivalence of Lemma 4.3.5 (2) to obtain (4.3.20). The diagram (4.3.21) commutes by commutativity of the adjunctions above. □

Continuing in the context of Convention 4.3.18, the Day convolution and hom diagram for enriched diagrams from $\widehat{\mathcal{D}}$ to $\underline{\mathsf{V}_*}$ provide a symmetric monoidal closed structure for pointed diagrams from $(\mathcal{D},\mathrm{T}^{\mathcal{D}})$ to $(\mathsf{V}_*,\mathrm{T}^{\mathsf{V}})$. We record these here for later reference, using $\widehat{\mathcal{D}} \wedge \widehat{\mathcal{D}}$ to denote the monoidal product as V_*-categories, called the tensor product in Definition 1.3.3.

Definition 4.3.23. In the context of Convention 4.3.18, suppose given pointed diagrams

$$X, Y \in \mathsf{Cat}_*((\mathcal{D}, \mathrm{T}^{\mathcal{D}}), (\mathsf{V}_*, \mathrm{T}^{\mathsf{V}})).$$

We will use the notation

$$\mathcal{D}^\flat(x,y) = \big(\mathcal{D}(x,y)\big)^\flat \quad \text{for} \quad x, y \in \mathcal{D}.$$

The Day convolution product for X and Y is the V_*-coend

$$X \wedge Y = \int^{(a,b) \in \widehat{\mathcal{D}} \wedge \widehat{\mathcal{D}}} \bigvee_{\mathcal{D}^\flat(a \boxdot b, -)} X_a \wedge Y_b \tag{4.3.24}$$

where we remind the reader that an empty wedge is T^{V} by choice in the definition of $\widehat{F}$ (Definition 4.3.9) and $\widehat{\mathcal{D}}$ (4.3.12). If the input object is $\mathrm{T}^{\mathcal{D}}$, we choose T^{V} for the coend. This definition is naturally isomorphic to the Day convolution (3.7.4) for $\widehat{\mathcal{D}}$ and V_*, via the isomorphisms

$$\widehat{\mathcal{D}}(a \boxdot b, c) \wedge (X_a \wedge Y_b) = \bigvee_{\mathcal{D}^\flat(a \boxdot b, c)} E \wedge (X_a \wedge Y_b) \cong \bigvee_{\mathcal{D}^\flat(a \boxdot b, c)} X_a \wedge Y_b$$

induced by left unit isomorphisms in V_*, for a, b, and c in $\mathcal{D}$.

The hom diagram from X to Y is the V_*-end

$$\mathsf{Hom}_{\mathcal{D}_*}(X, Y) = \int_{(b,c) \in \widehat{\mathcal{D}} \wedge \widehat{\mathcal{D}}} \Big[\bigvee_{\mathcal{D}^\flat(- \boxdot b; c)} X_b, Y_c \Big]_* \tag{4.3.25}$$

$$\cong \int_{b \in \widehat{\mathcal{D}}} [X_b, Y_{- \boxdot b}]_* \tag{4.3.26}$$

where the isomorphism is from (3.7.12). We again recall that an empty wedge is T^{V} by choice in the definition of $\widehat{F}$ (Definition 4.3.9). When the input object is $\mathrm{T}^{\mathcal{D}}$, we choose T^{V} for the end. This definition is naturally isomorphic to the hom diagram (3.7.5) for $\widehat{\mathcal{D}}$ and V_*, via the isomorphisms

$$\Big[\widehat{\mathcal{D}}(a \boxdot b, c) \wedge X_b, Y_c\Big]_* = \Big[\bigvee_{\mathcal{D}^\flat(a \boxdot b, c)} E \wedge X_b, Y_c \Big]_* \cong \Big[\bigvee_{\mathcal{D}^\flat(a \boxdot b, c)} X_b, Y_c \Big]_*$$

induced by the left unit isomorphisms in V_*, for a, b, and c in $\mathcal{D}$.

The mapping object from X to Y is the V_*-end

$$\mathsf{Map}_{\mathcal{D}_*}(X, Y) = \int_{b \in \widehat{\mathcal{D}}} [X_b, Y_b]_* \cong \mathsf{Hom}_{\mathcal{D}_*}(X, Y)_e \tag{4.3.27}$$

where the isomorphism follows from (3.7.12) and the unit isomorphisms in $\mathcal{D}$.

The monoidal unit diagram is the V_*-functor

$$J(-) = \bigvee_{\mathcal{D}^\flat(e, -)} E : \widehat{\mathcal{D}} \longrightarrow \underline{\mathsf{V}_*} \tag{4.3.28}$$

where, as above, we take the empty wedge to be T^{V}. ◇

Explanation 4.3.29. The Day convolution, hom diagram, and monoidal unit defined in Definition 4.3.23 are all defined as V_*-enriched functors

$$\widehat{\mathcal{D}} \longrightarrow \underline{\mathsf{V}_*}.$$

Equivalently, using the isomorphism (4.3.22) from the proof of Theorem 4.3.19, these are Set_*-enriched functors

$$\mathcal{D} \longrightarrow \mathsf{V}_*.$$

Our choices of initial and terminal object when the inputs are $\mathrm{T}^{\mathcal{D}}$ make each of these a pointed functor

$$(\mathcal{D}, \mathrm{T}^{\mathcal{D}}) \longrightarrow (\mathsf{V}_*, \mathrm{T}^{\mathsf{V}}).$$

Moreover, the associativity, unit, and symmetry isomorphisms of Definition 3.7.17 are all pointed natural transformations with these definitions. We will use the notation of Definition 4.3.23 for any of these three contexts. ◇

Combining Definition 4.3.23 and Explanation 4.3.29, we have the following corollary of Theorems 3.7.22 and 4.3.19.

Corollary 4.3.30. *In the context of Convention 4.3.18, the Day convolution, monoidal unit, and hom diagram for pointed functors*

$$(\mathcal{D}, \mathrm{T}^{\mathcal{D}}) \longrightarrow (\mathsf{V}_*, \mathrm{T}^{\mathsf{V}})$$

provide a symmetric monoidal closed structure for

$$\mathsf{Cat}_*((\mathcal{D}, \mathrm{T}^{\mathcal{D}}), (\mathsf{V}_*, \mathrm{T}^{\mathsf{V}})).$$

This makes the functors in the equivalence (4.3.20) into strong symmetric monoidal functors.

Recall from Definition 3.7.1 we have the notation

$$(\widehat{\mathcal{D}})\text{-}(\mathsf{V}_*) = \mathsf{V}_*\mathsf{Cat}(\widehat{\mathcal{D}}, \underline{\mathsf{V}_*}) \tag{4.3.31}$$

for the category of enriched $\widehat{\mathcal{D}}$-shaped diagrams in the self-enrichment of V_*.

Definition 4.3.32. In the context of Convention 4.3.18, we let

$$\mathcal{D}_*\text{-}\mathsf{V} = \mathsf{Cat}_*\big((\mathcal{D}, \mathrm{T}^{\mathcal{D}}), (\mathsf{V}_*, \mathrm{T}^{\mathsf{V}})\big) \tag{4.3.33}$$ ◇

Explanation 4.3.34. In the notation of Definition 4.3.32, Theorem 4.3.19 gives an equivalence of categories

$$\mathcal{D}_*\text{-}\mathsf{V} \simeq (\widehat{\mathcal{D}})\text{-}(\mathsf{V}_*) \tag{4.3.35}$$

and Definition 4.3.23, together with Explanation 4.3.29, uses this equivalence to define

$$(\mathcal{D}_*\text{-}\mathsf{V}, \wedge, J, \mathsf{Hom}_{\mathcal{D}_*})$$

as a symmetric monoidal closed category such that (4.3.35) is an equivalence via strong symmetric monoidal functors. ◇

As an application of Lemmas 3.8.1 and 3.9.12 together with the equivalence of categories from Theorem 4.3.19, evaluation at the monoidal unit of $\widehat{\mathcal{D}}$ defines a symmetric monoidal functor ev_e with left adjoint we again denote L_e. The left adjoint

$$L_e : \mathsf{V}_* \rightleftarrows \mathcal{D}_*\text{-}\mathsf{V} : \mathsf{ev}_e \tag{4.3.36}$$

is strong symmetric monoidal by Lemma 3.9.12 and Corollary 4.3.30.

Theorem 4.3.37. *Suppose* $(\mathsf{V}, \otimes, \mathbb{1}, \mathrm{T}^{\mathsf{V}})$ *is a complete and cocomplete symmetric monoidal closed category with chosen terminal object* T^{V}*. Suppose* $(\mathcal{D}, \boxdot, e, \mathrm{T}^{\mathcal{D}})$ *is a small symmetric monoidal category with chosen null object* $\mathrm{T}^{\mathcal{D}}$*. Then the category of pointed diagrams*

$$\mathcal{D}_*\text{-}\mathsf{V} = \mathsf{Cat}_*\big((\mathcal{D}, \mathrm{T}^{\mathcal{D}}), (\mathsf{V}_*, \mathrm{T}^{\mathsf{V}})\big)$$

is a complete and cocomplete symmetric monoidal closed category with

- *monoidal product given by the Day convolution* $\wedge$*,*
- *internal hom given by* $\mathsf{Hom}_{\mathcal{D}_*}$*, and*
- *monoidal unit* J*.*

Moreover, the adjunction (L_e, ev_e) *makes the pointed diagram category enriched, tensored, and cotensored over* V_* *with mapping objects given by* $\mathsf{Map}_{\mathcal{D}_*}$*.*

Proof. Limits and colimits of pointed diagrams are computed objectwise, with functoriality determined by the universal properties. The symmetric monoidal closed structure is from Definition 4.3.23 and Explanation 4.3.29 together with Theorems 3.7.22 and 4.3.19. The tensor and cotensor over V_* follows from the adjunction (4.3.36) and Corollary 3.9.15 or from Explanation 3.9.16 and the equivalence of Theorem 4.3.19 directly. □

4.4. Notes

4.4.1 (Smash and Pointed Hom)**.** Our proof of the $\wedge$-Hom_* adjunction in Theorem 4.2.3 follows that of [**EM09**, Lemma 4.20]. An alternate proof of Theorem 4.1.8, using the Yoneda Lemma and the $\wedge$-Hom_* adjunction from the proof of Theorem 4.2.3 instead of our direct pushout arguments, is sketched in the proofs of [**EM09**, Theorem 4.11 and Lemma 4.20]. ◇

4.4.2 (Choices for Pointed Convolution, Hom, and Monoidal Unit Diagrams)**.** In Definition 4.3.23 we emphasize choosing T^{V}, instead of some other initial and terminal object, for the empty wedges and constant co/ends when the input object is $\mathrm{T}^{\mathcal{D}}$. These choices make the convolution, hom, and monoidal unit diagrams automatically pointed diagrams.

Without these conventions, the respective diagrams will still be enriched in Set_* and one can alternatively apply the strictification $(-)'$ defined in the proof of Lemma 4.3.5 to obtain pointed convolution, hom, and monoidal unit diagrams. ◇

CHAPTER 5

Multicategories

In this chapter and the next we recall parts of the theory of multicategories. We will use this in Chapters 8, 9, and 10 to define the K-theory functors K^{Se} (Segal) and K^{EM} (Elmendorf-Mandell) that construct symmetric spectra from small permutative categories. We will show that K^{EM} preserves Cat-enrichment, as defined in Chapter 6, and then use this in Part 2, Chapters 11, 12, and 13, to transport E_n-monoidal structures on small permutative categories to E_n-monoidal structures on the corresponding symmetric spectra.

In Section 5.1 we give the basic definitions and define the 2-category of small multicategories, denoted $\mathsf{Multicat}$. In Section 5.2 we discuss the Cartesian product of multicategories. Then we define pointed multicategories in Section 5.3 and explain the canonical inclusions of permutative categories as (pointed) multicategories. Our K-theory constructions in Part 2 will make use of these inclusions.

The remaining sections are devoted to showing that $\mathsf{Multicat}$ is symmetric monoidal closed. In Section 5.4 we review the general theory of limits and colimits for algebras over monads. We apply this in Section 5.5 to show that $\mathsf{Multicat}$ is complete and cocomplete. Then in Sections 5.6 and 5.7 we use certain colimits and limits to develop the tensor product and internal hom for $\mathsf{Multicat}$. Applying the general theory of Chapter 4, these descend to the category of small pointed multicategories, giving a smash product and pointed hom.

5.1. The 2-Category of Multicategories

We begin with some notation and then give the definition of multicategory.

Definition 5.1.1. Suppose C is a class.

(1) Denote by
$$\mathsf{Prof}(C) = \coprod_{n \geq 0} C^{\times n}$$
the class of finite ordered sequences of elements in C. An element in $\mathsf{Prof}(C)$ is called a *C-profile*.

(2) A typical C-profile of length $n = \mathrm{len}\langle c \rangle$ is denoted by $\langle c \rangle = (c_1, \ldots, c_n) \in C^{\times n}$ or by $\langle c_i \rangle_i$ to indicate the indexing variable. The empty C-profile is denoted by $\langle \rangle$.

(3) We let $\oplus$ denote the concatenation of profiles, and note that $\oplus$ is an associative binary operation with unit given by the empty tuple $\langle \rangle$.

(4) An element in $\mathsf{Prof}(C) \times C$ is denoted as $\big(\langle c \rangle ; c'\big)$ with $c' \in C$ and $\langle c \rangle \in \mathsf{Prof}(C)$. $\diamond$

Now we come to the definition of multicategory. In the literature, what we call a multicategory in the next definition is sometimes called a *symmetric multicategory* or a *colored operad*, with the term multicategory reserved for the non-symmetric version. Since we only consider the version with symmetric group action, we simply call them multicategories.

Definition 5.1.2. A *multicategory* $(\mathsf{M},\gamma,\mathbb{1})$ consists of the following data.

- M is equipped with a class $\mathsf{Ob}\,\mathsf{M}$ of *objects*. We abbreviate $\mathsf{Prof}(\mathsf{Ob}\,\mathsf{M})$ as $\mathsf{Prof}(\mathsf{M})$.
- For $c' \in \mathsf{Ob}\,\mathsf{M}$ and $\langle c\rangle = (c_1,\ldots,c_n) \in \mathsf{Prof}(\mathsf{M})$, M is equipped with a set
$$\mathsf{M}\big(\langle c\rangle\,;c'\big) = \mathsf{M}\big(c_1,\ldots,c_n\,;c'\big)$$
of *n-ary operations* with *input profile* $\langle c\rangle$ and *output* c'.
- For $\big(\langle c\rangle\,;c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ as above and a permutation $\sigma \in \Sigma_n$, M is equipped with a bijection
$$\mathsf{M}\big(\langle c\rangle\,;c'\big) \xrightarrow[\cong]{\sigma} \mathsf{M}\big(\langle c\rangle\sigma\,;c'\big),$$
called the *right action* or the *symmetric group action*, in which
$$\langle c\rangle\sigma = \big(c_{\sigma(1)},\ldots,c_{\sigma(n)}\big)$$
is the right permutation of $\langle c\rangle$ by σ.
- For $c \in \mathsf{Ob}\,\mathsf{M}$, M is equipped with an element
$$\mathbb{1}_c \in \mathsf{M}\big(c\,;c\big),$$
called the *c-colored unit*.
- For $c'' \in \mathsf{Ob}\,\mathsf{M}$, $\langle c'\rangle = (c'_1,\ldots,c'_n) \in \mathsf{Prof}(\mathsf{M})$, and $\langle c_j\rangle = (c_{j,1},\ldots,c_{j,k_j}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1,\ldots,n\}$, let $\langle c\rangle = \oplus_j\langle c_j\rangle \in \mathsf{Prof}(\mathsf{M})$ be the concatenation of the $\langle c_j\rangle$. Then M is equipped with a map
$$\mathsf{M}\big(\langle c'\rangle\,;c''\big) \times \prod_{j=1}^{n} \mathsf{M}\big(\langle c_j\rangle\,;c'_j\big) \xrightarrow{\gamma} \mathsf{M}\big(\langle c\rangle\,;c''\big) \tag{5.1.3}$$
called the *composition*.

These data are required to satisfy the following axioms.

Symmetric Group Action: For $\big(\langle c\rangle\,;c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $n = \mathrm{len}\langle c\rangle$ and $\sigma,\tau \in \Sigma_n$, the diagram
$$\begin{tikzcd}
\mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[r, "\sigma"] \arrow[dr, "\sigma\tau"'] & \mathsf{M}\big(\langle c\rangle\sigma\,;c'\big) \arrow[d, "\tau"] \\
& \mathsf{M}\big(\langle c\rangle\sigma\tau\,;c'\big)
\end{tikzcd}$$
is commutative. Moreover, the identity permutation in Σ_n acts as the identity map on $\mathsf{M}\big(\langle c\rangle\,;c'\big)$.

Associativity: Suppose given
- $c''' \in \mathsf{Ob}\,\mathsf{M}$,
- $\langle c''\rangle = (c''_1,\ldots,c''_n) \in \mathsf{Prof}(\mathsf{M})$,
- $\langle c'_j\rangle = (c'_{j,1},\ldots,c'_{j,k_j}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1,\ldots,n\}$, and
- $\langle c_{j,i}\rangle = (c_{j,i,1},\ldots,c_{j,i,\ell_{j,i}}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1,\ldots,n\}$ and each $i \in \{1,\ldots,k_j\}$,

such that $k_j = \operatorname{len}\langle c'_j\rangle > 0$ for at least one j. For each j, let $\langle c_j\rangle = \oplus_{i=1}^{k_j}\langle c_{j,i}\rangle$ denote the concatenation of the $\langle c_{j,i}\rangle$. Let $\langle c\rangle = \oplus_{j=1}^{n}\langle c_j\rangle$ denote the concatenation of the $\langle c_j\rangle$. Let $\langle c'\rangle = \oplus_{j=1}^{n}\langle c'_j\rangle$ denote the concatenation of the $\langle c'_j\rangle$.

Then the *associativity diagram* below commutes.

(5.1.4)

$$\begin{tikzcd}[column sep=small]
& \mathsf{M}\big(\langle c'\rangle\,;c'''\big) \times \prod\limits_{j=1}^{n}\Big[\prod\limits_{i=1}^{k_j}\mathsf{M}\big(\langle c_{j,i}\rangle\,;c'_{j,i}\big)\Big] \arrow[ddr, "\gamma"] & \\
\mathsf{M}\big(\langle c''\rangle\,;c'''\big) \times \Big[\prod\limits_{j=1}^{n}\mathsf{M}\big(\langle c'_j\rangle\,;c''_j\big)\Big] \times \prod\limits_{j=1}^{n}\Big[\prod\limits_{i=1}^{k_j}\mathsf{M}\big(\langle c_{j,i}\rangle\,;c'_{j,i}\big)\Big] \arrow[ur, "{(\gamma,1)}"] \arrow[d, "\text{permute}"', "\cong"] & & \\
\mathsf{M}\big(\langle c''\rangle\,;c'''\big) \times \prod\limits_{j=1}^{n}\Big[\mathsf{M}\big(\langle c'_j\rangle\,;c''_j\big) \times \prod\limits_{i=1}^{k_j}\mathsf{M}\big(\langle c_{j,i}\rangle\,;c'_{j,i}\big)\Big] \arrow[dr, "{(1,\prod_j\gamma)}"'] & & \mathsf{M}\big(\langle c\rangle\,;c'''\big) \\
& \mathsf{M}\big(\langle c''\rangle\,;c'''\big) \times \prod\limits_{j=1}^{n}\mathsf{M}\big(\langle c_j\rangle\,;c''_j\big) \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Unity: Suppose $c' \in \operatorname{Ob}\mathsf{M}$.

(1) If $\langle c\rangle = (c_1,\ldots,c_n) \in \mathsf{Prof}(\mathsf{M})$ has length $n \geq 1$, then the following *right unity diagram* is commutative. Here $\{*\}$ is the one-point set, and $\{*\}^n$ is its n-fold product.

$$\begin{tikzcd}
\mathsf{M}\big(\langle c\rangle\,;c'\big) \times \{*\}^n \arrow[r, "\cong"] \arrow[d, "{(1,\prod 1_{c_j})}"'] & \mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[d, "1"] \\
\mathsf{M}\big(\langle c\rangle\,;c'\big) \times \prod\limits_{j=1}^{n}\mathsf{M}\big(c_j\,;c_j\big) \arrow[r, "\gamma"] & \mathsf{M}\big(\langle c\rangle\,;c'\big)
\end{tikzcd} \tag{5.1.5}$$

(2) For any $\langle c\rangle \in \mathsf{Prof}(\mathsf{M})$, the *left unity diagram* below is commutative.

$$\begin{tikzcd}
\{*\} \times \mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[r, "\cong"] \arrow[d, "{(1_{c'},1)}"'] & \mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[d, "1"] \\
\mathsf{M}\big(c'\,;c'\big) \times \mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[r, "\gamma"] & \mathsf{M}\big(\langle c\rangle\,;c'\big)
\end{tikzcd} \tag{5.1.6}$$

Equivariance: Suppose that in the definition of γ (5.1.3), $\operatorname{len}\langle c_j\rangle = k_j \geq 0$.

(1) For each $\sigma \in \Sigma_n$, the following *top equivariance diagram* is commutative.

$$\begin{tikzcd}
\mathsf{M}\big(\langle c'\rangle\,;\,c''\big) \times \prod_{j=1}^{n} \mathsf{M}\big(\langle c_j\rangle\,;\,c'_j\big) \arrow[r, "(\sigma,\sigma^{-1})"] \arrow[d, "\gamma"'] & \mathsf{M}\big(\langle c'\rangle\sigma\,;\,c''\big) \times \prod_{j=1}^{n} \mathsf{M}\big(\langle c_{\sigma(j)}\rangle\,;\,c'_{\sigma(j)}\big) \arrow[d, "\gamma"] \\
\mathsf{M}\big(\langle c_1\rangle,\ldots,\langle c_n\rangle\,;\,c''\big) \arrow[r, "\sigma\langle k_{\sigma(1)},\ldots,k_{\sigma(n)}\rangle"] & \mathsf{M}\big(\langle c_{\sigma(1)}\rangle,\ldots,\langle c_{\sigma(n)}\rangle\,;\,c''\big)
\end{tikzcd} \tag{5.1.7}$$

Here $\sigma\langle k_{\sigma(1)},\ldots,k_{\sigma(n)}\rangle \in \Sigma_{k_1+\cdots+k_n}$ is right action of the block permutation (II.1.1.19) that permutes the n consecutive blocks of lengths $k_{\sigma(1)}, \ldots, k_{\sigma(n)}$ as σ permutes $\{1,\ldots,n\}$, leaving the relative order within each block unchanged.

(2) Given permutations $\tau_j \in \Sigma_{k_j}$ for $1 \le j \le n$, the following *bottom equivariance diagram* is commutative.

$$\begin{tikzcd}
\mathsf{M}\big(\langle c'\rangle\,;\,c''\big) \times \prod_{j=1}^{n} \mathsf{M}\big(\langle c_j\rangle\,;\,c'_j\big) \arrow[r, "(1,\Pi\tau_j)"] \arrow[d, "\gamma"'] & \mathsf{M}\big(\langle c'\rangle\,;\,c''\big) \times \prod_{j=1}^{n} \mathsf{M}\big(\langle c_j\rangle\tau_j\,;\,c'_j\big) \arrow[d, "\gamma"] \\
\mathsf{M}\big(\langle c_1\rangle,\ldots,\langle c_n\rangle\,;\,c''\big) \arrow[r, "\tau_1\times\cdots\times\tau_n"] & \mathsf{M}\big(\langle c_1\rangle\tau_1,\ldots,\langle c_n\rangle\tau_n\,;\,c''\big)
\end{tikzcd} \tag{5.1.8}$$

Here $\tau_1 \times \cdots \times \tau_n \in \Sigma_{k_1+\cdots+k_n}$ is the block sum (II.1.1.8) given by the image of $(\tau_1,\ldots,\tau_n)$ under the canonical inclusion

$$\Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \longrightarrow \Sigma_{k_1+\cdots+k_n}.$$

This finishes the definition of a multicategory.

Moreover:

- A multicategory with only one object is an *operad*.
- If M is an operad, then its set of n-ary operations is denoted by M_n.
- A multicategory is *small* if its class of objects is a set.
- We will denote the image of an operation $\phi \in \mathsf{M}\big(\langle c\rangle\,;\,c'\big)$ under the right action of a permutation σ as $\phi \cdot \sigma \in \mathsf{M}\big(\langle c\rangle\sigma\,;\,c'\big)$.
- We will denote the composition of operations $\phi' \in \mathsf{M}\big(\langle c'\rangle\,;\,c''\big)$ and $\phi_j \in \mathsf{M}\big(\langle c_j\rangle\,;\,c'_j\big)$ with juxtaposition or $\circ$, as in the following:

$$\gamma(\phi',(\phi_1,\ldots,\phi_n)) = \phi' \circ (\phi_1,\ldots,\phi_n) = (\phi')\,(\phi_1,\ldots,\phi_n). \qquad \diamond$$

Example 5.1.9 (Unary Multicategories)**.** Suppose M is a category. Then there is a multicategory M^u whose object class is $\mathsf{Ob}\,\mathsf{M}$ and whose operations are given by

$$\mathsf{M}^u\big(\langle c\rangle\,;\,c'\big) = \begin{cases} \mathsf{M}(c,c') & \text{if } \langle c\rangle = c \in \mathsf{Ob}\,\mathsf{M}, \\ \varnothing & \text{otherwise.} \end{cases}$$

In other words, M^u only has unary operations, which are the morphisms in M. The colored units and composition in M^u are given by those in M. There are no non-trivial symmetric group actions, and the multicategory axioms reduce to the axioms of a category.

Conversely, if M is a multicategory then there is an underlying category, $r\mathsf{M}$ consisting of the same objects and morphisms given by the unary operations of M. ◇

Example 5.1.10 (Endomorphism Operad)**.** Suppose X is an object of a permutative category $(\mathsf{C}, \oplus, e)$. Then there is a multicategory $\mathsf{End}(X)$ called the *endomorphism operad* with a single object and with n-ary operations

$$\mathsf{End}(X)_n = \mathsf{C}(X^{\oplus n}, X).$$

One detail of this example that will be important for our work below is the composition with nullary operations $\mathsf{End}(X)_0 = \mathsf{C}(e, X)$. For example, the composition

$$\mathsf{End}(X)_2 \times \big(\mathsf{End}(X)_0 \times \mathsf{End}(X)_1\big) \longrightarrow \mathsf{End}(X)_1$$

is given by the following composite for $f_i \in \mathsf{End}(X)_i$:

$$X \longrightarrow e \oplus X \xrightarrow{f_0 \oplus f_1} X \oplus X \xrightarrow{f_2} X.$$

The unlabeled morphism is given by (the inverse of) a unit isomorphism, which is an identity because C is permutative. We will recall this observation in the proof of Proposition 8.5.4 below. ◇

Example 5.1.11 (Endomorphism Multicategory)**.** Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category. Then Example 5.1.10 generalizes to define the *endomorphism multicategory* $\mathsf{End}(\mathsf{C})$ with object set given by $\mathsf{Ob}\,\mathsf{C}$ and with

$$\mathsf{End}(\mathsf{C})\big(\langle X \rangle; Y\big) = \mathsf{C}(X_1 \oplus \cdots \oplus X_n, Y)$$

for $Y \in \mathsf{Ob}\,\mathsf{C}$ and $\langle X \rangle = (X_1, \cdots, X_n) \in (\mathsf{Ob}\,\mathsf{C})^{\times n}$. We note as in Example 5.1.10 that compositions involving the empty tuple are defined via the (strict) unit isomorphisms of C. ◇

Definition 5.1.12. A *multifunctor* $F : \mathsf{M} \longrightarrow \mathsf{N}$ between multicategories M and N consists of the following data:

- an assignment
$$F : \mathsf{Ob}\,\mathsf{M} \longrightarrow \mathsf{Ob}\,\mathsf{N},$$
where $\mathsf{Ob}\,\mathsf{M}$ and $\mathsf{Ob}\,\mathsf{N}$ are the classes of objects of M and N, respectively, and
- for each $\big(\langle c \rangle; c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $\langle c \rangle = (c_1, \ldots, c_n)$, a function
$$F : \mathsf{M}\big(\langle c \rangle; c'\big) \longrightarrow \mathsf{N}\big(F\langle c \rangle; Fc'\big),$$
where $F\langle c \rangle = (Fc_1, \ldots, Fc_n)$.

These data are required to preserve the symmetric group action, the colored units, and the composition in the following sense.

Symmetric Group Action: For each $\big(\langle c \rangle; c'\big)$ as above and each permutation $\sigma \in \Sigma_n$, the following diagram is commutative.

$$\begin{tikzcd}
\mathsf{M}\big(\langle c \rangle; c'\big) \arrow[r, "F"] \arrow[d, "\sigma"', "\cong"] & \mathsf{N}\big(F\langle c \rangle; Fc'\big) \arrow[d, "\sigma"', "\cong"] \\
\mathsf{M}\big(\langle c \rangle\sigma; c'\big) \arrow[r, "F"] & \mathsf{N}\big(F\langle c \rangle\sigma; c'\big)
\end{tikzcd} \tag{5.1.13}$$

Units: For each $c \in \mathsf{Ob}\,\mathsf{M}$, the following equality holds:

$$F1_c = 1_{Fc} \in \mathsf{N}\big(Fc\,;Fc\big). \tag{5.1.14}$$

Composition: For c'', $\langle c'_j \rangle$, and $\langle c \rangle = \oplus_j \langle c_j \rangle$ as in the definition of γ (5.1.3), the following diagram is commutative.

$$\begin{tikzcd}
\mathsf{M}\big(\langle c'\rangle\,;c''\big) \times \prod_{j=1}^{n} \mathsf{M}\big(\langle c_j\rangle\,;c'_j\big) \arrow[r, "(F,\prod F)"] \arrow[d, "\gamma"'] & \mathsf{N}\big(F\langle c'\rangle\,;Fc''\big) \times \prod_{j=1}^{n} \mathsf{N}\big(F\langle c_j\rangle\,;Fc'_j\big) \arrow[d, "\gamma"] \\
\mathsf{M}\big(\langle c\rangle\,;c''\big) \arrow[r, "F"] & \mathsf{N}\big(F\langle c\rangle\,;Fc''\big)
\end{tikzcd} \tag{5.1.15}$$

This finishes the definition of a multifunctor.

Moreover:

(1) For another multifunctor $G : \mathsf{N} \longrightarrow \mathsf{P}$ between multicategories, where P has object class $\mathsf{Ob}\,\mathsf{P}$, the *composition* $GF : \mathsf{M} \longrightarrow \mathsf{P}$ is the multifunctor defined by composing the assignments on objects

$$\mathsf{Ob}\,\mathsf{M} \xrightarrow{F} \mathsf{Ob}\,\mathsf{N} \xrightarrow{G} \mathsf{Ob}\,\mathsf{P}$$

and the functions on n-ary operations

$$\mathsf{M}\big(\langle c\rangle\,;c'\big) \xrightarrow{F} \mathsf{N}\big(F\langle c\rangle\,;Fc'\big) \xrightarrow{G} \mathsf{P}\big(GF\langle c\rangle\,;GFc'\big).$$

(2) The *identity multifunctor* $1_{\mathsf{M}} : \mathsf{M} \longrightarrow \mathsf{M}$ is defined by the identity assignment on objects and the identity function on n-ary operations.

(3) An *operad morphism* is a multifunctor between two multicategories with one object. ◇

Lemma 5.1.16. *Composition of multifunctors is well defined, associative, and unital with respect to the identity multifunctors.*

Proof. For multifunctors $F : \mathsf{M} \longrightarrow \mathsf{N}$ and $G : \mathsf{N} \longrightarrow \mathsf{P}$, GF preserves the compositions in M and P because the following diagram is commutative.

$$\begin{tikzcd}
\mathsf{M}\big(\langle c'\rangle\,;c''\big) \times \prod_{j=1}^{n} \mathsf{M}\big(\langle c_j\rangle\,;c'_j\big) \arrow[r, "\gamma"] \arrow[d, "(F,\prod F)"'] \arrow[dd, bend right, "(GF,\prod GF)"'] & \mathsf{M}\big(\langle c\rangle\,;c''\big) \arrow[d, "F"] \arrow[dd, bend left, "GF"] \\
\mathsf{N}\big(F\langle c'\rangle\,;Fc''\big) \times \prod_{j=1}^{n} \mathsf{N}\big(F\langle c_j\rangle\,;Fc'_j\big) \arrow[r, "\gamma"] \arrow[d, "(G,\prod G)"'] & \mathsf{N}\big(F\langle c\rangle\,;Fc''\big) \arrow[d, "G"] \\
\mathsf{P}\big(GF\langle c'\rangle\,;GFc''\big) \times \prod_{j=1}^{n} \mathsf{P}\big(GF\langle c_j\rangle\,;GFc'_j\big) \arrow[r, "\gamma"] & \mathsf{P}\big(GF\langle c\rangle\,;GFc''\big)
\end{tikzcd}$$

Similarly, GF preserves the symmetric group actions and the colored units in M and P, so it is a multifunctor. Composition of multifunctors is associative and unital because composition of functions is associative and unital. □

Natural transformations and their horizontal and vertical compositions also have direct generalizations to multicategories.

Definition 5.1.17. Suppose $F, G : \mathsf{M} \longrightarrow \mathsf{N}$ are multifunctors as in Definition 5.1.12. A *multinatural transformation* $\alpha : F \longrightarrow G$ consists of unary operations

$$\alpha_c \in \mathsf{N}\big(Fc\,;Gc\big) \quad \text{for} \quad c \in \mathsf{Ob}\,\mathsf{M}$$

such that, for each n-ary operation $p \in \mathsf{M}\big(\langle c\rangle\,;c'\big)$ with $\langle c\rangle = (c_1, \ldots, c_n)$, the following *naturality condition* holds, with composition taken in N:

$$(Gp) \circ \big(\alpha_{c_1}, \ldots, \alpha_{c_n}\big) = \alpha_{c'} \circ (Fp) \in \mathsf{N}\big(F\langle c\rangle\,;Gc'\big).$$

- Each α_c is called a *component* of α.
- The *identity multinatural transformation* $1_F : F \longrightarrow F$ has components

$$(1_F)_c = 1_{Fc} \in \mathsf{N}\big(Fc\,;Fc\big) \quad \text{for} \quad c \in \mathsf{Ob}\,\mathsf{M}. \qquad \diamond$$

Definition 5.1.18. Suppose $\alpha : F \longrightarrow G$ is a multinatural transformation between multifunctors as in Definition 5.1.17.

(1) Suppose $\beta : G \longrightarrow H$ is a multinatural transformation for a multifunctor $H : \mathsf{M} \longrightarrow \mathsf{N}$. The *vertical composition*

$$\beta\alpha : F \longrightarrow H$$

is the multinatural transformation with components given by composition in N

$$(\beta\alpha)_c = \beta_c \circ \alpha_c \in \mathsf{N}\big(Fc\,;Hc\big) \quad \text{for} \quad c \in \mathsf{Ob}\,\mathsf{M}.$$

(2) Suppose $\alpha' : F' \longrightarrow G'$ is a multinatural transformation for multifunctors $F', G' : \mathsf{N} \longrightarrow \mathsf{P}$. The *horizontal composition*

$$\alpha' * \alpha : F'F \longrightarrow G'G$$

is the multinatural transformation with components given by composition in P

$$(\alpha' * \alpha)_c = \alpha'_{Gc} \circ (F'\alpha_c) = (G'\alpha_c) \circ (\alpha'_{Fc}) \in \mathsf{P}\big(F'Fc\,;G'Gc\big)$$

for each object $c \in \mathsf{Ob}\,\mathsf{M}$, in which the second equality follows from the naturality of α'. $\diamond$

Example 5.1.19 (Functors and Natural Transformations)**.** A functor $F : \mathsf{M} \longrightarrow \mathsf{N}$ between categories is also a multifunctor when M and N are regarded as multicategories with only unary operations as in Example 5.1.9. Similarly, a natural transformation $\alpha : F \longrightarrow F'$ between functors is a multinatural transformation when F and F' are regarded as multifunctors. $\diamond$

Theorem 5.1.20. *There is a 2-category* $\mathsf{Multicat}$ *consisting of the following data.*

- *Its objects are small multicategories.*
- *For small multicategories* M *and* N, *the hom category* $\mathsf{Multicat}(\mathsf{M}, \mathsf{N})$ *has:*
  - *multifunctors* $\mathsf{M} \longrightarrow \mathsf{N}$ *as 1-cells;*
  - *multinatural transformations between such multifunctors as 2-cells;*
  - *vertical composition as composition; and*
  - *identity multinatural transformations as identity 2-cells.*
- *The identity 1-cell* 1_{M} *is the identity multifunctor* 1_{M}.
- *Horizontal composition of 1-cells is the composition of multifunctors.*
- *Horizontal composition of 2-cells is that of multinatural transformations.*

Definition 5.1.21. The *initial operad* I consists of a single object $*$ and a single unary operation given by the unit 1_* on its single object. $\diamond$

Explanation 5.1.22. For $n \neq 1$, the n-ary operation sets I_n are empty. Since multifunctors preserve identity operations, I is indeed initial among operads. More generally, for a multicategory M, a multifunctor $\mathsf{I} \longrightarrow \mathsf{M}$ consists of a choice of object in M. Multinatural transformations between multifunctors

$$F : \mathsf{I} \longrightarrow \mathsf{M} \quad \text{and} \quad G : \mathsf{I} \longrightarrow \mathsf{M}$$

consist of unary operations $F* \longrightarrow G*$. Thus, for small M, $\mathsf{Multicat}(\mathsf{I}, \mathsf{M})$ is isomorphic to the underlying category of objects and unary operations in M. ◇

Definition 5.1.23. We will use the notation $\mathsf{Multicat}$ both for the 2-category of small multicategories and for its underlying 1-category. ◇

Recall from Definition I.6.3.9 the notions of internal adjunction and equivalence in a bicategory or 2-category.

Definition 5.1.24. An *equivalence*, respectively *adjoint equivalence*, of multicategories is an internal equivalence, respectively adjoint equivalence, in the 2-category of multicategories. ◇

Explanation 5.1.25. Suppose M and N are multicategories. A multifunctor

$$F : \mathsf{M} \longrightarrow \mathsf{N}$$

is an equivalence if there is a multifunctor $G : \mathsf{N} \longrightarrow \mathsf{M}$ together with isomorphisms of multifunctors

$$\eta : 1_{\mathsf{M}} \longrightarrow GF \quad \text{and} \quad \varepsilon : FG \longrightarrow 1_{\mathsf{N}}.$$

These data are an adjoint equivalence if they satisfy the addional triangle axioms of an internal adjunction in $\mathsf{Multicat}$. ◇

5.2. The Cartesian Structure on Multicategories

Now we discuss the Cartesian product for multicategories.

Definition 5.2.1. The *terminal multicategory* T consists of a single object and a single n-ary operation ι_n for each $n \geq 0$. ◇

Explanation 5.2.2 (The Commutative Operad)**.** The terminal multicategory T is also denoted Com and is known as the *commutative operad*. For a permutative category C, a multifunctor $P : \mathsf{T} \longrightarrow \mathsf{End}(\mathsf{C})$ is precisely a commutative monoid structure on the object $P*$. Its unit is given by the nullary operation $P\iota_0$, its multiplication is given by the binary operation $P\iota_2$, and the axioms (associativity and symmetry) follow from the composition and symmetric group actions on $\mathsf{T} = \mathsf{Com}$ being trivial.

For a general multicategory M, a multifunctor $P : \mathsf{T} \longrightarrow \mathsf{M}$ is determined by a choice of object $P*$ and a choice of n-ary operation

$$P\iota_n \in \mathsf{M}\big(\underbrace{(P*, \ldots, P*)}_{n \text{ terms}}; P*\big)$$

for each $n \geq 0$, subject to the following compatibility conditions.

Unity: The equality $P\iota_1 = 1_{P*}$ holds.
Composition: The set of operations $P\iota_n$ is closed under composition in M.
Symmetry: Each $P\iota_n$ is fixed by the right symmetric group action. ◇

Definition 5.2.3. Suppose M and N are multicategories. Their *Cartesian product* is the multicategory $\mathsf{M} \times \mathsf{N}$ whose objects are given by pairs (c, d) with $c \in \mathsf{M}$ and $d \in \mathsf{N}$ and whose n-ary operations are given by the Cartesian product of sets

$$(\mathsf{M} \times \mathsf{N})\big(\langle c, d\rangle\,;\,(c', d')\big) = \mathsf{M}\big(\langle c\rangle\,;\,c'\big) \times \mathsf{N}\big(\langle d\rangle\,;\,d'\big).$$

for $\langle c, d\rangle = ((c_1, d_1), \ldots, (c_n, d_n)) \in \mathsf{Prof}(\mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N})$. The other data of $\mathsf{M} \times \mathsf{N}$ is specified as follows.

- The right action of a permutation σ is given by the diagonal
$$\mathsf{M}\big(\langle c\rangle\,;\,c'\big) \times \mathsf{N}\big(\langle d\rangle\,;\,d'\big) \xrightarrow{(\sigma,\sigma)} \mathsf{M}\big(\langle c\rangle\sigma\,;\,c'\big) \times \mathsf{N}\big(\langle d\rangle\sigma\,;\,d'\big)$$
- The (c, d)-colored unit is the pair $(1_c, 1_d)$.
- The composition is given by permuting factors and then applying the product of the compositions in M and N:
$$\begin{tikzcd}[column sep=tiny] \Big[\mathsf{M}\big(\langle c'\rangle\,;\,c''\big) \times \mathsf{N}\big(\langle d'\rangle\,;\,d''\big)\Big] \times \prod_{j=1}^{n}\Big[\mathsf{M}\big(\langle c_j\rangle\,;\,c'_j\big) \times \mathsf{N}\big(\langle d_j\rangle\,;\,d'_j\big)\Big] \arrow[d] \\ \Big[\mathsf{M}\big(\langle c'\rangle\,;\,c''\big) \times \prod_{j=1}^{n}\mathsf{M}\big(\langle c_j\rangle\,;\,c'_j\big)\Big] \times \Big[\mathsf{N}\big(\langle d'\rangle\,;\,d''\big) \times \prod_{j=1}^{n}\mathsf{N}\big(\langle d_j\rangle\,;\,d'_j\big)\Big] \arrow[d] \\ \mathsf{M}\big(\langle c\rangle\,;\,c''\big) \times \mathsf{N}\big(\langle d\rangle\,;\,d''\big) \end{tikzcd}$$
  where $\langle c\rangle = \oplus_j \langle c_j\rangle$ and $\langle d\rangle = \oplus_j \langle d_j\rangle$ are the concatenations of the $\langle c_j\rangle$ and $\langle d_j\rangle$, respectively.

The multicategory axioms for these data follow directly from the axioms for M and N together with the axioms for $\times$ as a symmetric monoidal product on Set. $\diamond$

Definition 5.2.4. For multifunctors

$$F : \mathsf{M} \longrightarrow \mathsf{M}' \quad \text{and} \quad G : \mathsf{N} \longrightarrow \mathsf{N}',$$

the Cartesian product $F \times G$ is given by the pairwise assignment on objects and the product of the corresponding morphisms on sets of operations:

$$\mathsf{M}\big(\langle c\rangle\,;\,c'\big) \times \mathsf{N}\big(\langle d\rangle\,;\,d'\big) \longrightarrow \mathsf{M}'\big(F\langle c\rangle\,;\,Fc'\big) \times \mathsf{N}'\big(G\langle d\rangle\,;\,Gd'\big).$$

As above, the multifunctor axioms for $F \times G$ follow from those of F and G together with the axioms for $\times$ as a symmetric monoidal product on Set. $\diamond$

Proposition 5.2.5. *The category of small multicategories is symmetric monoidal with respect to the Cartesian product. The unit for this product is the terminal multicategory* T.

Proof. Since T has a single object, and each set of n-ary operations is a singleton, the unit isomorphisms

$$\mathsf{T} \times \mathsf{M} \cong \mathsf{M} \cong \mathsf{M} \times \mathsf{T}$$

for each small multicategory M follow from those for the singleton in Set. Likewise, the symmetry for the Cartesian product of multicategories is given by the symmetry of sets. The axioms making $\times$ a symmetric monoidal product in $\mathsf{Multicat}$ follow from those in Set. For example, the hexagon axiom (I.1.2.22) follows from commutativity of the following diagram in Set for each triple of small multicategories L, M, and N and each triple of inputs and outputs

$$\begin{aligned}
\big(\langle b\rangle\,;b'\big) &\in \mathsf{Prof}(\mathsf{L}) \times \mathsf{Ob}\,\mathsf{L},\\
\big(\langle c\rangle\,;c'\big) &\in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}, \text{ and}\\
\big(\langle d\rangle\,;d'\big) &\in \mathsf{Prof}(\mathsf{N}) \times \mathsf{Ob}\,\mathsf{N}.
\end{aligned}$$

$$\begin{tikzcd}[column sep=tiny]
& \mathsf{L}\big(\langle b\rangle;b'\big)\times\big[\mathsf{M}\big(\langle c\rangle;c'\big)\times\mathsf{N}\big(\langle d\rangle;d'\big)\big] \arrow[dr, "1\times\xi"] & \\
\big[\mathsf{L}\big(\langle b\rangle;b'\big)\times\mathsf{M}\big(\langle c\rangle;c'\big)\big]\times\mathsf{N}\big(\langle d\rangle;d'\big) \arrow[ur, "\cong"] \arrow[d, "\xi"'] & & \mathsf{L}\big(\langle b\rangle;b'\big)\times\big[\mathsf{N}\big(\langle d\rangle;d'\big)\times\mathsf{M}\big(\langle c\rangle;c'\big)\big] \arrow[d, "\cong"] \\
\mathsf{N}\big(\langle d\rangle;d'\big)\times\big[\mathsf{L}\big(\langle b\rangle;b'\big)\times\mathsf{M}\big(\langle c\rangle;c'\big)\big] \arrow[dr, "\cong"'] & & \big[\mathsf{L}\big(\langle b\rangle;b'\big)\times\mathsf{N}\big(\langle d\rangle;d'\big)\big]\times\mathsf{M}\big(\langle c\rangle;c'\big) \\
& \big[\mathsf{N}\big(\langle d\rangle;d'\big)\times\mathsf{L}\big(\langle b\rangle;b'\big)\big]\times\mathsf{M}\big(\langle c\rangle;c'\big) \arrow[ur, "\xi\times 1"'] &
\end{tikzcd}$$

The other axioms are proved in the same manner. □

5.3. Permutative Categories as Pointed Multicategories

In this section we describe pointed multicategories and the canonical basepoint for the multicategory $\mathsf{End}(\mathsf{C})$ associated to a permutative category C in Example 5.1.11. Then we go on to discuss 2-functoriality of $\mathsf{End}(-)$. The multifunctors induced by symmetric monoidal functors do not generally preserve the canonical basepoints. However, we show below that *strictly unital* symmetric monoidal functors yield pointed multifunctors.

The following is a special case of the more general Definition 4.1.1.

Definition 5.3.1. A *pointed multicategory* $(\mathsf{M}, *^{\mathsf{M}}, \iota^{\mathsf{M}})$ consists of a multicategory M together with a multifunctor $\mathsf{T} \longrightarrow \mathsf{M}$ determined by an object $*^{\mathsf{M}}$ and operations ι_n^{M} described in Explanation 5.2.2. Typically we will omit the superscripts when M is clear from context.

A *pointed multifunctor*

$$F : (\mathsf{M}, *^{\mathsf{M}}, \iota^{\mathsf{M}}) \longrightarrow (\mathsf{N}, *^{\mathsf{N}}, \iota^{\mathsf{N}})$$

is a multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$ that commutes with the multifunctors $\mathsf{T} \longrightarrow \mathsf{M}$ and $\mathsf{T} \longrightarrow \mathsf{N}$. A *pointed multinatural transformation* between pointed multifunctors F and G is a multinatural transformation $\alpha : F \longrightarrow G$ such that the component $\alpha_{*^{\mathsf{M}}} : F(*^{\mathsf{M}}) \longrightarrow G(*^{\mathsf{M}})$ is the colored unit on $*^{\mathsf{N}}$. With these definitions, composites of pointed multifunctors and multinatural transformations are again pointed. ◇

Definition 5.3.2. We let $\mathsf{Multicat}_*$ denote the 2-category of pointed small multicategories, pointed multifunctors, and pointed multinatural transformations. The compositions and identity cells of $\mathsf{Multicat}_*$ are defined as in $\mathsf{Multicat}$ (see Theorem 5.1.20). $\diamond$

Explanation 5.3.3. As noted in the introduction to Chapter 4, we will be interested in a smash product for pointed multicategories, but not the one obtained from the Cartesian product. We will develop a tensor product for multicategories in Section 5.6 below, and its associated smash product will be the one of interest in the applications of Part 2. $\diamond$

Definition 5.3.4. Suppose $(\mathsf{C}, \oplus, e)$ is a permutative category. The *canonical basepoint* of $\mathsf{End}(\mathsf{C})$ is given by the unit object e, with n-ary basepoint operations $\iota_n = 1_e$ for all $n \geq 0$. $\diamond$

Explanation 5.3.5 (Strictly Unital Symmetric Monoidal Functors)**.** Recall from Definitions 1.1.6 and 1.1.23 that a symmetric monoidal functor

$$F : \mathsf{C} \longrightarrow \mathsf{D}$$

has structure morphisms

$$F^0 : \mathbb{1}^{\mathsf{C}} \longrightarrow F\mathbb{1}^{\mathsf{D}} \quad \text{and}$$
$$F^2_{X,Y} : FX \oplus FY \longrightarrow F(X \oplus Y)$$

satisfying axioms for associativity (1.1.9), unity (1.1.10), and compatibility with the symmetry (1.1.18). We say that F is *strictly unital* if F^0 is the identity morphism. If C and D are permutative and F is strictly unital, then the unity axioms imply that $F^2_{X,Y}$ is an identity whenever X or Y is the monoidal unit of C. Recall from Definition 1.1.27 that

$$\mathsf{PermCat}^{\mathsf{su}}$$

denotes the 2-category consisting of permutative categories, strictly unital symmetric monoidal functors, and monoidal natural transformations. $\diamond$

Proposition 5.3.6. *The construction* $\mathsf{End}(\mathsf{C})$ *of Example 5.1.11 defines a 2-functor*

$$\mathsf{End} : \mathsf{PermCat} \longrightarrow \mathsf{Multicat}$$

from the 2-category of small permutative categories to the 2-category of small multicategories.

Proof. Given a symmetric monoidal functor $F : \mathsf{C} \longrightarrow \mathsf{D}$, we apply F componentwise on tuples of objects in $\mathsf{End}(\mathsf{C})$ to define an assignment $\mathsf{Prof}(\mathsf{C}) \longrightarrow \mathsf{Prof}(\mathsf{D})$. Given an operation

$$f \in \mathsf{End}(\mathsf{C})\big(\langle X \rangle\,;\, X'\big) = \mathsf{C}(\oplus_i X_i, X'),$$

we define $(\mathsf{End}(F))f$ by the composite

$$\oplus_i FX_i \longrightarrow F(\oplus_i X_i) \xrightarrow{Ff} FX'.$$

where the unlabeled morphism is given by iterating F^2 if $\langle X \rangle$ is nonempty, and is given by F^0 if $\langle X \rangle = \langle \rangle$. The associativity and unity axoms for F imply that this definition of $\mathsf{End}(F)$ is compatible with the composition of operations in $\mathsf{End}(\mathsf{C})$ and $\mathsf{End}(\mathsf{D})$. The symmetry axiom ensures that $\mathsf{End}(F)$ commutes with the symmetric group actions on $\mathsf{End}(\mathsf{C})$ and $\mathsf{End}(\mathsf{D})$. Likewise, the components of a monoidal natural transformation $\alpha : F \longrightarrow F'$ satisfy the naturality axiom for a multinatural transformation from $\mathsf{End}(F)$ to $\mathsf{End}(F')$.

To verify that End is 2-functorial, first note that $\mathsf{End}(1_{\mathsf{C}}) = 1_{\mathsf{End}(\mathsf{C})}$ by checking that these multifunctors agree on objects and operations. Next, for a composable pair of symmetric monoidal functors, G and F, we have

$$\mathsf{End}(GF) = \mathsf{End}(F) \circ \mathsf{End}(G)$$

because $(GF)^2 = G(F^2) \circ G^2$. □

Proposition 5.3.7. *The 2-functor*

$$\mathsf{End} : \mathsf{PermCat} \longrightarrow \mathsf{Multicat}$$

is bijective on 1-cells and 2-cells.

Proof. The proof of Proposition 5.3.6 gives the construction of a multifunctor $\mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$ from a symmetric monoidal functor $\mathsf{C} \longrightarrow \mathsf{D}$. Here we discuss an inverse construction.

For each $\langle X \rangle \in \mathsf{Prof}(\mathsf{C})$, we let $\iota_{\langle X \rangle} \in \mathsf{End}(\mathsf{C})\big(\langle X \rangle ; \oplus_i X_i\big)$ denote the operation corresponding to the identity $1_{\oplus_i X_i}$. Given a multifunctor

$$G : \mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$$

we define the underlying functor of

$$F : \mathsf{C} \longrightarrow \mathsf{D}$$

by restricting G to objects and unary operations. Then we define

$$F^2_{X,Y} = G\iota_{(X,Y)} \quad \text{and} \quad F^0 = G\iota_{\langle\rangle}.$$

Naturality of F^2 with respect to morphisms in C, the associativity axiom (I.1.2.14), and the unity axioms (I.1.2.15) follow from the composition axiom (5.1.15) for G together with the following equalities in $\mathsf{End}(\mathsf{C})$ for $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ in C:

(1) (functoriality of F^2)

$$\iota_{(X',Y')} \circ (f, g) = (f \oplus g) \circ \iota_{(X,Y)};$$

(2) (associativity axiom)

$$\iota_{(X,Y\oplus Z)} \circ \big(1_X, \iota_{(Y,Z)}\big) = \iota_{(X,Y,Z)} = \iota_{(X\oplus Y,Z)} \circ \big(\iota_{(X,Y)}, 1_Z\big); \quad \text{and}$$

(3) (unity axioms)

$$\iota_{(e,X)} \circ (\iota_{\langle\rangle}, 1_X) = 1_X \quad \text{and} \quad \iota_{(X,e)} \circ (1_X, \iota_{\langle\rangle}) = 1_X.$$

The symmetry axiom (I.1.2.26) for F holds by the equivariance axiom (5.1.13) for G.

To see that this construction determines a bijection between monoidal functors $\mathsf{C} \longrightarrow \mathsf{D}$ and multifunctors $\mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$, observe first by the associativity of composition that the values of G on $\iota_{\langle X \rangle}$ for profiles $\langle X \rangle$ of length $n > 2$ are determined by its values on $\iota_{\langle\rangle}$ and $\iota_{(X,Y)}$. Next observe that composition with $\iota_{\langle X \rangle}$

induces a bijection on the top row of the following diagram for any $\langle X\rangle \in \mathsf{Prof}(\mathsf{C})$ and any $Y \in \mathsf{C}$.

$$\begin{tikzcd}
\mathsf{End}(\mathsf{C})\big(\oplus_i X_i\,;Y\big) \arrow[r, "(-)\circ\iota_{\langle X\rangle}", "\cong"'] \arrow[d, "G"'] & \mathsf{End}(\mathsf{C})\big(\langle X\rangle\,;Y\big) \arrow[d, "G"] \\
\mathsf{End}(\mathsf{D})\big(G(\oplus_i X_i)\,;GY\big) \arrow[r, "(-)\circ(G\iota_{\langle X\rangle})"] & \mathsf{End}(\mathsf{D})\big(G\langle X\rangle\,;GY\big)
\end{tikzcd}$$

This shows that the values of G on arbitrary n-ary operations are determined by its values on unary operations and on the operations $\iota_{\langle X\rangle}$. Thus G is determined by the data of F constructed above.

Now we turn to the correspondence between monoidal natural transformations of symmetric monoidal functors $\mathsf{C} \longrightarrow \mathsf{D}$ and multinatural transformations of multifunctors $\mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$. Each type of transformation is determined by its components, which are unary operations. As noted in the proof of Proposition 5.3.6, the components of a monoidal natural transformation $F \longrightarrow F'$ already satisfy the additional axioms for multinatural transformations

$$\mathsf{End}(F) \longrightarrow \mathsf{End}(F'),$$

and thus End is bijective on 2-cells. $\square$

Now we discuss preservation of the canonical basepoints.

Lemma 5.3.8. *Suppose C and D are permutative categories.*

- *Suppose $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a symmetric monoidal functor. The multifunctor*

$$\mathsf{End}(F) : \mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$$

  preserves the canonical basepoints if and only if F is strictly unital.
- *Suppose $\alpha : F \longrightarrow F'$ is a monoidal natural transformation between strictly unital symmetric monoidal functors $F, F' : \mathsf{C} \longrightarrow \mathsf{D}$. Then $\mathsf{End}(\alpha)$ is a pointed multinatural transformation.*

Proof. Given a symmetric monoidal functor $F : \mathsf{C} \longrightarrow \mathsf{D}$, let

$$G = \mathsf{End}(F) : \mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$$

denote the corresponding multifunctor. Since F and G are the same assignments on objects, each strictly preserves the unit if and only if the other does. Moreover, the proof of Proposition 5.3.7 shows that $F^0 = G\iota_{\langle\rangle}$ and $F^2_{X,Y} = G\iota_{\langle X,Y\rangle}$. Thus we conclude that F is strictly unital if and only if G is a pointed multifunctor.

Given a monoidal natural transformation $\alpha : F \longrightarrow F'$ between strictly unital symmetric monoidal functors

$$F, F' : \mathsf{C} \longrightarrow \mathsf{D},$$

the unit condition (1.1.13) for α implies that $\alpha_e = 1_e$. Therefore, $\mathsf{End}(\alpha)$ is a pointed multinatural transformation. $\square$

Using Lemma 5.3.8, we have the following corollary of Propositions 5.3.6 and 5.3.7.

Corollary 5.3.9. *The endomorphism construction* End *restricts to a 2-functor*

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}_*$$

that is bijective on 1-cells and 2-cells.

5.4. Limits and Colimits of Monadic Algebras

This section reviews the general theory of limits and colimits for algebras over monads. We will apply this in Section 5.5 to show that Multicat, the category of small multicategories and multifunctors, is complete and cocomplete.

Definition 5.4.1. A *monad* on a category B is a triple (T, μ, η) in which

- $T : \mathsf{B} \longrightarrow \mathsf{B}$ is a functor and
- $\mu : T^2 \longrightarrow T$, called the *multiplication*, and $\eta : 1_{\mathsf{M}} \longrightarrow T$, called the *unit*, are natural transformations

such that the associativity and unity diagrams

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathsf{M}} \circ T \arrow[r, "\eta T"] \arrow[d, equal] & T^2 \arrow[d, "\mu"'] & T \circ 1_{\mathsf{M}} \arrow[l, "T\eta"'] \arrow[d, equal] \\ T \arrow[r, equal] & T \arrow[r, equal] & T \end{tikzcd}$$

are commutative. We often abbreviate such a monad to T. ◇

Definition 5.4.2. Suppose (T, μ, η) is a monad on a category B.

(1) A *T-algebra* is a pair (X, θ) consisting of
   - an object X in B and
   - a morphism $\theta : TX \longrightarrow X$, called the *structure morphism*,

   such that the associativity and unity diagrams

(5.4.3)
$$\begin{tikzcd} T^2X \arrow[r, "T\theta"] \arrow[d, "\mu_X"'] & TX \arrow[d, "\theta"] \\ TX \arrow[r, "\theta"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "\theta"] \\ & X \end{tikzcd}$$

   are commutative.

(2) A *morphism of T-algebras*

$$f : (X, \theta^X) \longrightarrow (Y, \theta^Y)$$

   is a morphism $f : X \longrightarrow Y$ in B such that the diagram

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\theta^X"'] & TY \arrow[d, "\theta^Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

   is commutative. Identities and composition of T-algebra morphisms are defined in B.

(3) The category of T-algebras is denoted by $\mathsf{Alg}(T)$. ◇

Example 5.4.4. Suppose given an adjunction

$$\begin{tikzcd} \mathsf{B} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathsf{A} \arrow[l, bend left, "U"] \end{tikzcd}$$

with unit $\eta : 1_{\mathsf{B}} \longrightarrow UL$ and counit $\varepsilon : LU \longrightarrow 1_{\mathsf{A}}$. The composite functor UL is a monad on B, with unit and composition given, respectively, by

$$\eta : 1_{\mathsf{B}} \longrightarrow UL \quad \text{and} \quad U\varepsilon L : ULUL \longrightarrow UL,$$

where $U\varepsilon L = 1_U * \varepsilon * 1_L$. One can verify that the triangle identities (I.1.1.11) for $L \dashv U$ imply that the associativity and unity diagrams of Definition 5.4.1 commute for $T = UL$.

For any $A \in \mathsf{A}$, there is a canonical UL-algebra structure on UA given by $U\varepsilon = 1_U * \varepsilon$:

$$(UL)(UA) = (ULU)A \xrightarrow{U\varepsilon_A} UA.$$

Naturality of the counit implies that $Uf : UA \longrightarrow UA'$ is a morphism of UL-algebras for each $f : A \longrightarrow A'$ in A. Thus U induces a canonical functor

$$\mathsf{A} \longrightarrow \mathsf{Alg}(UL). \tag{5.4.5}$$

$\diamond$

Definition 5.4.6. In the context of Example 5.4.4, we say that $L \dashv U$ is a *monadic adjunction* if the canonical functor (5.4.5) induced by U is an equivalence of categories

$$\mathsf{A} \xrightarrow{\simeq} \mathsf{Alg}(UL). \tag{5.4.7}$$

Moreover, we say that the adjunction $L \dashv U$ is *strictly* monadic if (5.4.7) is an isomorphism of categories.

We say that $U : \mathsf{A} \longrightarrow \mathsf{B}$ is a *monadic functor* if it has a left adjoint L and the pair (L, U) is a monadic adjunction. Similarly, we say that U is *strictly monadic* if it has a left adjoint L such that (L, U) is a strictly monadic adjunction. $\diamond$

Explanation 5.4.8 (Limits of Algebras)**.** For any monad T on a category B, the forgetful functor $\mathsf{Alg}(T) \longrightarrow \mathsf{B}$ creates limits in $\mathsf{Alg}(T)$. That is, given a diagram of T-algebras, one proves that, upon forgetting to B, if the limit of the underlying diagram exists then it has a canonical T-algebra structure and is indeed the limit of the given diagram in $\mathsf{Alg}(T)$. $\diamond$

In many applications of interest one can also obtain colimits in $\mathsf{Alg}(T)$ from those in the underlying category, although the construction is more complicated. We recall several standard definitions and results from the literature and apply them below to the category of small multicategories.

To fix terminology, we begin with the following reformulation of colimits from Definition I.1.1.13.

Definition 5.4.9 (Cocones and Colimits)**.** Suppose that $F : \mathsf{D} \longrightarrow \mathsf{C}$ is a functor of categories. A *cocone* under F is an object $W \in \mathsf{C}$ together with a natural transformation $c : F \longrightarrow \Delta_W$, where Δ_W denotes the constant functor at W, given by the composite in the diagram below.

$$\begin{tikzcd} \mathsf{D} \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "c \Swarrow"] & \mathsf{C} \\ * \arrow[ur, "W"'] & \end{tikzcd}$$

A morphism of cocones $(W, c) \longrightarrow (W', c')$ consists of a morphism $W \longrightarrow W'$ in C whose whiskering with c is equal to c'. A *colimit* for F is an initial object in the category of cocones under F. $\diamond$

Definition 5.4.10. Suppose A is a category and $f,g \in \mathsf{A}(X,Y)$ is a parallel pair of morphisms.

(1) A *coequalizer* of f and g consists of a pair (Z,h)

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"'] & Z \end{tikzcd}$$

that is a colimit of the diagram formed by f and g.

(2) A *split coequalizer* of f and g consists of an object $Z \in \mathsf{A}$ together with morphisms h, s, and t

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[l, bend right, "t"'] \arrow[r, "h"'] & Z, \arrow[l, bend right, "s"'] \end{tikzcd}$$

such that

$$hf = hg, \quad hs = 1_Z, \quad gt = 1_Y, \quad \text{and} \quad ft = sh.$$

We call these the *splitting conditions*.

(3) An *absolute coequalizer* for (f,g) consists of an object and morphism pair (Z,h) such that (FZ,Fh) is a coequalizer of (Ff,Fg) for any functor F with domain A. $\diamond$

Explanation 5.4.11. The splitting conditions in the definition of split coequalizer imply that Z is indeed a coequalizer of f and g. Moreover, since these conditions are preserved by any functor, a split coequalizer is an absolute coequalizer. $\diamond$

Definition 5.4.12. Suppose $U : \mathsf{A} \longrightarrow \mathsf{B}$ is a functor. We say that U *strictly creates coequalizers* for a parallel pair of morphisms (f,g) in A if, for each coequalizer (W,k) of (Uf,Ug) in B,

$$\begin{tikzcd} UX \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UY \arrow[r, "k"'] & W, \end{tikzcd}$$

there is a unique object and morphism pair (Z,h) in A such that

$$(UZ,Uh) = (W,k)$$

and (Z,h) is a coequalizer of f and g in A. $\diamond$

Theorem 5.4.13 (Beck's Precise Tripleability)**.** *For an adjunction*

$$\begin{tikzcd} \mathsf{B} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathsf{A} \arrow[l, bend left, "U"] \end{tikzcd}$$

the following statements are equivalent.

(1) The adjunction $L \dashv U$ is strictly monadic.

(2) U strictly creates coequalizers for parallel pairs (f,g) in A for which (Uf,Ug) has an absolute coequalizer in B.

(3) U strictly creates coequalizers for parallel pairs (f,g) in A for which (Uf,Ug) has a split coequalizer in B.

Proof. We provide a sketch proof, following that of [**ML98**, VI.7 Theorem 1]. We refer the reader there for complete details, with the warning that [**ML98**] omits the term "strictly" from our (2) and (3) but the conditions are identical. See Note 5.8.2 for further explanation of this point.

To show that (1) implies (2), let $T = UL$ and suppose given a parallel pair of morphisms in $\mathsf{A} \cong \mathsf{Alg}(T)$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

such that (Uf, Ug) has an absolute coequalizer (W, k) in B. We must show two things.

- There is a unique T-algebra structure on W. Letting Z denote the corresponding T-algebra, we have $Z = UW$.
- The pair (Z, k) is a coequalizer of f and g in $\mathsf{A} \cong \mathsf{Alg}(T)$.

Applying T to the coequalizer diagram for W, one has the following diagram in B, where the solid vertical arrows are the T-algebra structure morphisms for UX and UY.

$$\begin{tikzcd} TUX \arrow[r, shift left, "TUf"] \arrow[r, shift right, "TUg"'] \arrow[d] & TUY \arrow[r, "Tk"] \arrow[d] & TW \arrow[d, dashed, "\theta"] \\ UX \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UY \arrow[r, "k"] & W \end{tikzcd} \tag{5.4.14}$$

The dashed arrow θ is uniquely determined because, by hypothesis, (W, k) is an absolute coequalizer and therefore (TW, Tk) is a coequalizer of (TUf, TUg).

Next one shows that θ satisfies the associativity and unity diagrams (5.4.3) of a T-algebra structure morphism. The associativity diagram follows from using the absolute coequalizer condition in a second instance to conclude that (T^2W, T^2k) is a coequalizer of (T^2Uf, T^2Ug). This implies that the two composites $T^2W \longrightarrow W$ determined by θ and μ are equal. The unity diagram follows similarly by considering the analogue of (5.4.14) with the units for UX and UY.

Thus the absolute coequalizer condition determines a unique T-algebra structure on $W = UZ$ such that k is a morphism of T-algebras. One uses the absolute coequalizer condition again to argue that Z has the universal property of a coequalizer of f and g in $\mathsf{A} \cong \mathsf{Alg}(T)$. This concludes the argument that (1) implies (2).

It is immediate that (2) implies (3) because each split coequalizer is an example of an absolute coequalizer. To show that (3) implies (1), we begin with the following key observation.

For any monad (T, μ, η) and T-algebra (X, θ), the multiplication and unit structure morphisms make the following diagram a coequalizer in $\mathsf{Alg}(T)$ that becomes a split coequalizer after forgetting down to the underlying category on which T acts.

(5.4.15)
$$\begin{tikzcd} T^2X \arrow[r, shift left, "T\theta"] \arrow[r, shift right, "\mu_X"'] & TX \arrow[l, bend right, "\eta_{TX}"'] \arrow[r, "\theta"] & X \arrow[l, bend right, "\eta_X"'] \end{tikzcd}$$

The arrows $T\theta$, μ_X, and θ are morphisms of T-algebras, but the splittings are merely morphisms in the underlying category.

For the argument that (3) implies (1), again let $T = UL$ and note that for any object $A \in \mathsf{A}$ we have a diagram

(5.4.16)
$$\begin{tikzcd} LULUA \arrow[r, shift left, "\varepsilon_{LUA}"] \arrow[r, shift right, "LU\varepsilon_A"'] & LUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd}$$

given by the counit ε of the adjunction $L \dashv U$. Applying U yields a split coequalizer as an instance of (5.4.15) with $X = UA$. The assumption (3) implies therefore that each object, respectively morphism, in A is uniquely determined as a coequalizer, respectively morphism of coequalizers, in (5.4.16). The desired isomorphism in (1) follows by further developing this observation. □

The second major result that we will need from the theory of monad algebras is Theorem 5.4.18 below, concerning filtered colimits.

Definition 5.4.17 (Filtered Colimits)**.** We say that a category J is *filtered* if every finite diagram

$$\mathsf{N} \longrightarrow \mathsf{J}$$

has a cocone.

(1) A *filtered colimit* is a colimit over a filtered diagram category.
(2) We say that a monad $T : \mathsf{B} \longrightarrow \mathsf{B}$ *preserves filitered colimits* if, whenever $Z = \mathsf{colim}_{\mathsf{J}} X$ for a filtered diagram

$$\mathsf{J} \xrightarrow{X} \mathsf{B},$$

then applying T yields a colimit, TZ, of the composite diagram

$$\mathsf{J} \xrightarrow{TX} \mathsf{B}. \qquad \diamond$$

Theorem 5.4.18. *Suppose T is a monad on a complete and cocomplete category B. If T preserves filtered colimits, then the category of T-algebras, $\mathsf{Alg}(T)$, is complete and cocomplete.*

Explanation 5.4.19. Theorem 5.4.18 appears in many introductory texts. For example, [**Rie16**, Theorem 5.6.12] gives a complete proof following [**Bor94b**, Proposition 4.3.6]. The hypothesis that T preserves filtered colimits is not required for the construction of limits, but is required in order to show that $\mathsf{Alg}(T)$ has colimits. Since neither coequalizers nor coproducts are filtered colimits, the existence of colimits is not an immediate consequence of the hypotheses.

One can construct coproducts of algebras via certain coequalizers of free algebras, so the majority of the work in proving cocompleteness for $\mathsf{Alg}(T)$ is the construction of coequalizers. To construct a coequalizer of algebra morphisms, one constructs a certain sequence of coequalizers in the underlying category, none of which are necessarily T-algebras. However, because sequential colimits are filtered, their colimit is preserved by T and one uses this to define T-algebra structure morphisms. The hypothesis that T preserves filtered colimits is used several more times while verifying the algebra axioms of Definition 5.4.2. $\diamond$

For our application of Theorems 5.4.13 and 5.4.18 below, we will use the following observations about finite products.

Lemma 5.4.20.

(1) *Suppose* C *is a category with products and suppose*

$$\begin{tikzcd} X_i \arrow[r, shift left] \arrow[r, shift right] & Y_i \arrow[l, bend right] \arrow[r] & Z_i \arrow[l, bend right] \end{tikzcd}$$

is a split coequalizer in C *for each* i *in an indexing set* I. *Then the product*

$$\begin{tikzcd} \prod_{i\in I} X_i \arrow[r, shift left] \arrow[r, shift right] & \prod_{i\in I} Y_i \arrow[l, bend right] \arrow[r] & \prod_{i\in I} Z_i \arrow[l, bend right] \end{tikzcd}$$

is a split coequalizer in C.

(2) *Suppose* J *is a small filtered category,* n *is a natural number, and*

$$F_i : \mathsf{J} \longrightarrow \mathsf{Set}$$

is a diagram with colimit Z_i *for each* $i \in \{1, \ldots, n\}$. *Then the product* $\prod_{i=1}^n Z_i$ *is a colimit for* $\prod_{i=1}^n F_i$.

Proof. Statement (1) holds because the splittings are preserved and therefore imply that the resulting diagram is a coequalizer (see Explanation 5.4.11). For statement (2), the key observation is that the colimit for a diagram of sets

$$\mathsf{D} \xrightarrow{X} \mathsf{Set}$$

is given by the quotient of $\coprod_{d\in\mathsf{D}} X(d)$ subject to the relation that, for each $x \in X(d)$ and $x' \in X(d')$ we have $x \sim x'$ if and only if there are morphisms in D

$$h : d \longrightarrow t \quad \text{and} \quad h' : d' \longrightarrow t$$

such that

$$(Xh)(x) = (Xh')(x')$$

in the set $X(t)$. We call d, respectively d', the *witness* for x, respectively x'. We call (h, h') the *witness* for the equivalence $x \sim x'$.

In the case of $Z_i = \mathsf{colim}_{\mathsf{J}} F_i$, we have the canonical function

$$\mathsf{colim}_{\mathsf{J}} \textstyle\prod_{i=1}^n F_i \longrightarrow \prod_{i=1}^n Z_i$$

that sends each equivalence class of tuples to the tuple of corresponding equivalence classes. This is well defined because an equivalence of tuples

$$(z_i)_{i=1}^n \sim (z_i')_{i=1}^n$$

implies a componentwise equivalence $z_i \sim z_i'$ for each i.

We show that the canonical function is bijective. To check surjectivity, suppose given $z_i \in Z_i$, each with witness d_i. Because J is filtered, it has a cocone t for the set of objects

$$d_1, \ldots, d_n \in \mathsf{D}.$$

Let

$$h_i : d_i \longrightarrow t$$

denote the morphisms of this cocone, and let

$$\overline{z}_i = (F_i h_i)(z_i),$$

so $z_i \sim \overline{z}_i$ for each i, witnessed by $(h_i, 1_j)$. Since each $\overline{z}_i \in F_i(t)$, we have

$$(\overline{z}_i)_{i=1}^n \in \mathsf{colim}_{\mathsf{J}} \textstyle\prod_{i=1}^n F_i$$

such that the image of this tuple under the canonical function is equivalent to $(z_i)_{i=1}^n$. This shows that the canonical function is surjective.

To verify that the canonical function is injective, suppose given (z_i) and (z_i') such that $z_i \sim z_i'$ for each i, with witnesses

$$h_i : d_i \longrightarrow t_i \quad \text{and} \quad h_i' : d_i' \longrightarrow t_i.$$

Because J is filtered, there is a cocone s for the finite subdiagram spanned by the h_i and h_i'. Taking the product of composites

$$d_i \longrightarrow t_i \longrightarrow s, \quad \text{respectively} \quad d_i' \longrightarrow t_i \longrightarrow s,$$

over $i \in \{1, \ldots, n\}$ provides a witness for the equivalence of tuples

$$(z_i)_{i=1}^n \sim (z_i')_{i=1}^n \quad \text{in} \quad \mathsf{colim}_{\mathsf{J}} \prod_{i=1}^n F_i.$$

Therefore, the canonical function is injective. □

5.5. Limits and Colimits of Multicategories

We will use Theorems 5.4.13 and 5.4.18 to show that $\mathsf{Multicat}$ is complete and cocomplete. The underlying category A for our application will be the category of small multigraphs, which we now describe.

Definition 5.5.1. A *multigraph* X consists of a class $\mathsf{Vt}X$ of vertices, together with a set $X(\langle c \rangle ; c')$ for each tuple of vertices $\langle c \rangle = (c_1, \ldots, c_n)$ and c'. We refer to the elements of $X(\langle c \rangle ; c')$ as *multiedges*, with *source* $\langle c \rangle$ and *target* c'. As with multicategories, we let $\mathsf{Prof}(X)$ denote $\mathsf{Prof}(\mathsf{Vt}X)$.

A morphism of multigraphs

$$f : X \longrightarrow Y$$

consists of a function on vertices

$$f : \mathsf{Vt}X \longrightarrow \mathsf{Vt}Y$$

together with a function on multiedges

$$f : X(\langle x \rangle ; x') \longrightarrow Y(f\langle x \rangle ; f(x'))$$

for each $(\langle x \rangle ; x') \in \mathsf{Prof}(X) \times \mathsf{Vt}X$, with $f\langle x \rangle$ being the tuple whose jth entry is $f(x_j)$. This finishes the definition of a multigraph.

Moreover:

- A multigraph is *small* if its class of vertices is a set.
- The collection of small multigraphs and their morphisms form a category, denoted MGraph.
- The category of multigraphs with a fixed set of vertices C and whose morphisms are the identity on vertices is denoted $\mathsf{MGraph}^{\mathsf{C}}$. ◇

Explanation 5.5.2. Note that MGraph^C is isomorphic to the category of functors and natural transformations

$$\mathsf{Set}^{\mathsf{Prof}(C)\times C},$$

where $\mathsf{Prof}(C) \times C$ is the discrete category with object set $\mathsf{Prof}(C) \times C$. ⋄

Definition 5.5.3. Let $\mathsf{Multicat}^C$ denote the subcategory of $\mathsf{Multicat}$ consisting of those multicategories with object set C and those multifunctors that are the identity on objects. Let

$$U^C : \mathsf{Multicat}^C \longrightarrow \mathsf{MGraph}^C$$

denote the functor that takes a multicategory M to its underlying multigraph. The vertices of $U\mathsf{M}$ are the objects of M and the multiedges of $U\mathsf{M}$ are the operations of M. A multifunctor F that is the identity on object set C defines an assignment on sets of operations, and this gives a morphism UF of underlying multigraphs. ⋄

Theorem 5.5.4 (**[Yau16**, Theorem 20.3.22]**).** *For a fixed set of objects C, the forgetful functor $U^C : \mathsf{Multicat}^C \longrightarrow \mathsf{MGraph}^C$ has a left adjoint L^C.*

Explanation 5.5.5. We will not need the precise definition of the left adjoint L^C in our discussion below. However, the following information will be useful.

For $X \in \mathsf{MGraph}^C$, the set of operations $L^C X(\langle c\rangle ; c')$ is generated under formal composition and permutation by the multiedges of X. To make this precise, **[Yau16]** indexes such composites by certain rooted trees (planar and labeled by C). Then

$$L^C X\big(\langle c\rangle ; c'\big) = \coprod_{[T]} \prod_{v} X\big(\langle c_v\rangle ; c'_v\big),$$

where the coproduct is over isomorphism classes of trees, the product is over the internal vertices v of a class representative T, and $\big(\langle c_v\rangle ; c'_v\big)$ is a labeling of v by objects of C. Thus each element of $L^C X\big(\langle c\rangle ; c'\big)$ is a tuple of composible multiedges of X, whose individual components have profile $\big(\langle c_v\rangle ; c'_v\big)$ and whose underlying multigraph has the shape of some planar tree T.

For our purposes, the crucial feature of L^C is that the number of internal vertices v in each tree T is finite. Therefore, $L^C X\big(\langle c\rangle ; c'\big)$ is a coproduct of finite products in Set. For further details and proofs, we refer the reader to **[Yau16]**. ⋄

Definition 5.5.6. Suppose $f_0 : C \longrightarrow D$ is a function between sets. If X is a multigraph with vertex set D, we define $f_0^* X$ to be the multigraph whose vertices are the elements of C and whose multiedges are given by applying f_0:

$$(f_0^* X)\big(\langle x\rangle ; x'\big) = X\big(f_0\langle x\rangle ; f_0(x')\big). \tag{5.5.7}$$

Then f_0 determines a morphism of multigraphs, using the same notation,

$$f_0^* X \xrightarrow{f_0} X$$

given by f_0 on vertices and by the identity on multiedges. If M is a multicategory with object set C, we define $f_0^*\mathsf{M}$ by applying f_0^* to the underlying multigraph of M. Defining symmetric group actions, units, and composition in $f_0^*\mathsf{M}$ via the equalities (5.5.7), f_0 induces a multifunctor

$$f_0^*\mathsf{M} \xrightarrow{f_0} \mathsf{M}.$$

⋄

Explanation 5.5.8. Suppose $f : X \longrightarrow Y$ is a morphism of multigraphs. Using the notation of Definition 5.5.6, f factors uniquely as a composite

$$X \xrightarrow{f_1} f_0^* Y \xrightarrow{f_0} Y$$

where

- $f_0 : \mathsf{Vt}X \longrightarrow \mathsf{Vt}Y$ is the function on vertices determined by f and
- $f_1 : X \longrightarrow f_0^* Y$ is a multigraph morphism in $\mathsf{MGraph}^{\mathsf{Vt}X}$.

If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ is a composable pair of multigraph morphisms, the factorization of their composite is given by

$$(gf)_0 = g_0 f_0 \quad \text{and} \quad (gf)_1 = (f_0^* g_1) \circ f_1$$

as in the following commutative diagram.

$$\begin{tikzcd}
 & & Y \arrow[dr, "g_1"] & & \\
X \arrow[r, "f_1"] & f_0^* Y \arrow[ur, "f_0"] \arrow[dr, "f_0^* g_1"'] & & g_0^* Z \arrow[r, "g_0"] & Z \\
 & & f_0^* g_0^* Z \arrow[ur, "f_0"'] & &
\end{tikzcd}$$

Similarly, if $F : \mathsf{M} \longrightarrow \mathsf{N}$ is a multifunctor, F factors uniquely as a composite

$$\mathsf{M} \xrightarrow{F_1} F_0^* \mathsf{N} \xrightarrow{F_0} \mathsf{N}$$

where

- $F_0 : \mathsf{Ob}\,\mathsf{M} \longrightarrow \mathsf{Ob}\,\mathsf{N}$ is the function on objects determined by F and
- $F_1 : \mathsf{M} \longrightarrow F_0^* \mathsf{N}$ is a multifunctor in $\mathsf{Multicat}^{\mathsf{Ob}\,\mathsf{M}}$. ⋄

We will use these factorizations to extend the adjunction of Theorem 5.5.4 as follows.

Proposition 5.5.9. *The forgetful functor* $U : \mathsf{Multicat} \longrightarrow \mathsf{MGraph}$ *has a left adjoint* L *given on each subcategory* MGraph^C *by the adjoint* L^C.

Proof. Each multigraph X has a set of vertices C, so we define $LX = L^C X$. For a multigraph morphism $f : X \longrightarrow Y$ where Y has vertex set $\mathsf{Vt}Y = D$, we factor $f = f_0 f_1$ as in Explanation 5.5.8 and define $L(f_0 f_1)$ to be the composite

$$L^C X \xrightarrow{L^C f_1} L^C(f_0^* Y) \xrightarrow{(f_0)'} L^D Y$$

where $(f_0)'$ is the multifunctor given by f_0 on objects and induced by the identity on sets of multiedges.

The units and counits of each adjunction $L^C \dashv U^C$ from Theorem 5.5.4 provide components of a unit and counit for $L \dashv U$. Naturality with respect to general morphisms in $\mathsf{Multicat}$ and MGraph follows from naturality with respect to each fixed object set C and the uniqueness of the decompositions in Explanation 5.5.8. The triangle identities for $L \dashv U$ follow from the triangle identities for each fixed object set C. □

Explanation 5.5.10 (Products of Multicategories)**.** In Definition 5.2.3 we defined the Cartesian product of multicategories. We can now observe that this Cartesian product is created by taking the product of underlying multigraphs and verifying that the additional structure and axioms of a multicategory hold componentwise. This construction is a special case of the more general construction for limits of monad algebras mentioned in Explanation 5.4.8. $\diamond$

Theorem 5.5.11. *The adjunction*

$$\begin{tikzcd} \mathsf{MGraph} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathsf{Multicat} \arrow[l, bend left, "U"] \end{tikzcd}$$

is strictly monadic.

Proof. We will use Theorem 5.4.13. Therefore, it suffices to show that U strictly creates coequalizers of parallel pairs $F, G : \mathsf{Multicat}(\mathsf{M}, \mathsf{N})$ such that (UF, UG) has a split coequalizer in MGraph.

Suppose (F, G) is such a pair, and suppose

$$\begin{tikzcd} U\mathsf{M} \arrow[r, shift left, "UF"] \arrow[r, shift right, "UG"'] & U\mathsf{N} \arrow[l, bend right, "R"'] \arrow[r, "K"'] & W \arrow[l, bend right, "S"'] \end{tikzcd}$$

is a split coequalizer of multigraphs. Being a split coequalizer implies that each set of multiedges in W with input profile $\langle c\rangle \in \mathsf{Prof}(W)$ and output c' is a split coequalizer

$$\begin{tikzcd} U\mathsf{M}\big(RS\langle c\rangle\,;RSc'\big) \arrow[r, shift left, "UF"] \arrow[r, shift right, "UG"'] & U\mathsf{N}\big(S\langle c\rangle\,;Sc'\big) \arrow[l, bend right, "R"'] \arrow[r, "K"'] & W\big(\langle c\rangle\,;c'\big). \arrow[l, bend right, "S"'] \end{tikzcd}$$

For each $c \in \mathsf{Vt}W$ we define the c-colored unit in W to be the image under K of the edge in $U\mathsf{N}$ corresponding to the (Sc)-colored unit in N.

Now we recall, from Lemma 5.4.20 (1), that split coequalizers commute with finite products. Thus the left- and right-hand columns of the diagram below are split coequalizers, where the labels at left indicate a product of morphisms given by the indicated morphism of multigraphs. The solid horizontal arrows are induced by the composition in M and N, respectively, and therefore commute with the arrows labeled UF and UG (although they do not necessarily commute with the sections S and T). Thus there is a unique induced function between the coequalizers, drawn as the dashed arrow.

$$\begin{tikzcd}
U\mathsf{M}\big(RS\langle c'\rangle\,;RSc''\big)\times\prod_i U\mathsf{M}\big(RS\langle c\rangle_i\,;RSc'_i\big) \arrow[r] \arrow[d, shift right=1, "UG"'] \arrow[d, shift left=1, "UF"] & U\mathsf{M}\big(RS\langle c\rangle\,;RSc''\big) \arrow[d, shift right=1, "UG"'] \arrow[d, shift left=1, "UF"] \\
U\mathsf{N}\big(S\langle c'\rangle\,;Sc''\big)\times\prod_i U\mathsf{N}\big(S\langle c\rangle_i\,;Sc'_i\big) \arrow[r] \arrow[u, bend right, "R"'] \arrow[d, "K"'] & U\mathsf{N}\big(S\langle c\rangle\,;Sc''\big) \arrow[u, bend right, "R"'] \arrow[d, "K"'] \\
W\big(\langle c'\rangle\,;c''\big)\times\prod_i W\big(\langle c\rangle_i\,;c'_i\big) \arrow[r, dashed, "\exists!"] \arrow[u, bend right, "S"'] & W\big(\langle c\rangle\,;c''\big). \arrow[u, bend right, "S"']
\end{tikzcd} \tag{5.5.12}$$

This defines the composition in W. Similarly, we define the symmetric group actions in W as the unique morphism of coequalizers shown below.

$$\begin{tikzcd}
U\mathsf{M}\big(RS\langle c\rangle\,;RSc'\big) \arrow[r, "U\sigma"] \arrow[d, shift right=1, "UG"'] \arrow[d, shift left=1, "UF"] & U\mathsf{M}\big(RS\langle c\rangle\sigma\,;RSc'\big) \arrow[d, shift right=1, "UG"'] \arrow[d, shift left=1, "UF"] \\
U\mathsf{N}\big(S\langle c\rangle\,;Sc'\big) \arrow[r, "U\sigma"] \arrow[u, bend right, "R"'] \arrow[d, "K"'] & U\mathsf{N}\big(S\langle c\rangle\sigma\,;Sc'\big) \arrow[u, bend right, "R"'] \arrow[d, "K"'] \\
W\big(\langle c\rangle\,;c'\big). \arrow[r, dashed, "\exists!"] \arrow[u, bend right, "S"'] & W\big(\langle c\rangle\sigma\,;c'\big). \arrow[u, bend right, "S"']
\end{tikzcd} \tag{5.5.13}$$

Each of the axioms in the definition of a multicategory (Definition 5.1.2) is a commuting diagram whose objects are finite products of sets of operations and whose morphisms are a finite product of the following types:

- compositions,
- permutations,
- identities, and
- morphisms from singleton sets picking out unit operations.

Therefore, the morphisms in each diagram commute with those given by UF, respectively UG, on components.

Again using the fact that split coequalizers commute with finite products, together with functoriality of coequalizers with respect to natural transformations of diagrams, we see that each axiom holds for W and therefore W is a multicategory. For the remainder of this proof we will write UW for the underlying multigraph of the multicategory W.

Now we show that W is a coequalizer in $\mathsf{Multicat}$. Suppose that P is another cocone for F and G. Since UW is the coequalizer of UF and UG, then there is a unique induced morphism of multigraphs $UW \longrightarrow U\mathsf{P}$. A similar argument based on split coequalizers commuting with finite products shows that this morphism satisfies the axioms of a multifunctor $W \longrightarrow \mathsf{P}$. It is uniquely determined by its underlying function of multigraphs, and therefore W is the coequalizer of F and G in $\mathsf{Multicat}$.

The choice of c-colored units in W is the unique one such that K preserves units. Moreover, the uniqueness of the dashed arrow in (5.5.12) shows that the composition in W is uniquely determined by the coequalizer condition. Similarly, the uniqueness of the dashed arrow in (5.5.13) shows that the symmetric group actions in W are uniquely determined. Therefore, our argument above shows that the split coequalizer W has a unique multicategory structure induced by that of M and N via split coequalizers. This verifies that U strictly creates coequalizers for (F, G), and therefore verifies condition (3) of Theorem 5.4.13. □

Theorem 5.5.14. *The category* $\mathsf{Multicat}$ *is complete and cocomplete.*

Proof. We have shown that $\mathsf{Multicat}$ is monadic over MGraph in Theorem 5.5.11. As noted in Explanation 5.4.8, limits of algebras are created in the underlying category.

To show that $\mathsf{Multicat}$ is cocomplete, we apply Theorem 5.4.18. Thus it suffices to verify that the monad $T = UL$ of Theorem 5.5.11 preserves filtered colimits. This follows because, for each multigraph W and each pair $\big(\langle c\rangle\,;\,c'\big) \in \mathsf{Prof}(W) \times \mathsf{Vt}W$, the set of multiedges $ULW\big(\langle c\rangle\,;\,c'\big)$ is given by a coproduct of finite products of sets (see Explanation 5.5.5). Since coproducts commute with arbitrary colimits and we recall, from Lemma 5.4.20 (2), that filtered colimits are preserved by finite products, it follows that T preserves filtered colimits. □

Colimits of multicategories are generally difficult to describe explicitly. However, the following feature is relatively straightforward and will be useful in our further work.

Proposition 5.5.15. *Suppose* D *is a small category and* $X : \mathsf{D} \longrightarrow \mathsf{Multicat}$ *is a diagram of multicategories. Then the set of objects of* $\mathsf{colim}\, X$ *is given by the colimit of objects:*

$$\mathsf{Ob}(\mathsf{colim}\, X) \cong \mathsf{colim}(\mathsf{Ob}\, X).$$

Proof. Suppose W is a colimit of X in $\mathsf{Multicat}$ and let T be the colimit of $\mathsf{Ob}\, X$ in Set. For each $d \in \mathsf{D}$, let

$$F_d : Xd \longrightarrow \mathsf{W} \quad \text{and} \quad g_d : \mathsf{Ob}(Xd) \longrightarrow T$$

be the structure morphisms to the respective colimits. We show that the canonical map of sets

$$T \longrightarrow \mathsf{Ob}\,\mathsf{W} \tag{5.5.16}$$

is a bijection.

To see that (5.5.16) is surjective, suppose a is an object of W. Then there is some $d \in \mathsf{D}$ and object $x \in \mathsf{Ob}(Xd)$ such that $F_d(x) = a$. Then $g_d(x)$ is an element of T that maps to a under (5.5.16).

To see that (5.5.16) is injective, suppose given $z, z' \in T$ such that both z and z' map to $a \in \mathsf{Ob}\,\mathsf{W}$ under (5.5.16). There are $d, d' \in \mathsf{D}$ together with

$$x \in \mathsf{Ob}(Xd) \quad \text{and} \quad x' \in \mathsf{Ob}(Xd')$$

such that $g_d(x) = z$ and $g_{d'}(x') = z'$. If z and z' both map to the same object $a \in \mathsf{Ob}(W)$ then there must be some $d'' \in \mathsf{D}$ together with morphisms

$$d \longrightarrow d'' \longleftarrow d'$$

such that the objects $x \in \mathsf{Ob}(Xd)$ and $x' \in \mathsf{Ob}(Xd')$ map to the same object of Xd''. But this implies that $z = z'$ and therefore (5.5.16) is injective. □

5.6. Tensor and Smash Products of Multicategories

In this section we define monoidal products for multicategories (and pointed multicategories) that are part of a symmetric monoidal closed structure.

Definition 5.6.1. Given profiles $\langle c\rangle \in \mathsf{Prof}(C)$ and $\langle d\rangle \in \mathsf{Prof}(D)$ with $m = \mathrm{len}\langle c\rangle$ and $n = \mathrm{len}\langle d\rangle$, we define

$$\begin{aligned}
\langle c\rangle \times d_j &= \langle (c_i, d_j)\rangle_i = ((c_1, d_j), (c_2, d_j), \ldots),\\
c_i \times \langle d\rangle &= \langle (c_i, d_j)\rangle_j = ((c_i, d_1), (c_i, d_2), \ldots),\\
\langle c\rangle \otimes \langle d\rangle &= \langle \langle (c_i, d_j)\rangle_i\rangle_j, \quad \text{and}\\
\langle c\rangle \otimes^{\mathsf{t}} \langle d\rangle &= \langle \langle (c_i, d_j)\rangle_j\rangle_i.
\end{aligned}$$

Let $\xi^{\otimes} = \xi^{\otimes}_{m,n}$ denote the permutation $\langle c\rangle \otimes \langle d\rangle \leftrightarrow \langle c\rangle \otimes^{\mathsf{t}} \langle d\rangle$ induced by changing order of indexing. ◇

Explanation 5.6.2. In the context of Definition 5.6.1, we note the following relationships to earlier material.

- Viewing the tuples $\langle c\rangle$ and $\langle d\rangle$ as elements of the distortion category (Definition I.4.2.1), the product $\langle c\rangle \otimes \langle d\rangle$ is that of (I.4.2.15) with $(-,-)$ in place of $+$.
- Taking $m = \mathrm{len}\langle c\rangle$ and $n = \mathrm{len}\langle d\rangle$ as objects of the finite ordinal category Σ (Definition I.2.4.1), the permutation $\xi^{\otimes}_{m,n}$ is induced by the multiplicative symmetry (I.2.4.5) between mn and nm. Explanation I.2.4.7 gives a geometric description via the transpose of an $n \times m$ matrix. ◇

Definition 5.6.3. For multigraphs X and Y with vertex classes C and D, respectively, we define a multigraph $X \,\&\, Y$ with vertex class $C \times D$ as follows. Given

$$\langle c, d\rangle = ((c_1, d_1), \ldots, (c_n, d_n)) \in \mathsf{Prof}(C \times D) \quad \text{and} \quad (c', d') \in C \times D,$$

the set of multiedges with source $\langle c, d\rangle$ and target (c', d') is given by the following coproduct over pairs $\langle c''\rangle$, $\langle d''\rangle$ such that $\langle c''\rangle \otimes \langle d''\rangle = \langle c, d\rangle$:

$$(X \,\&\, Y)\big(\langle c, d\rangle\,;\,(c', d')\big) = \coprod_{\langle c''\rangle \otimes \langle d''\rangle = \langle c, d\rangle} X\big(\langle c''\rangle\,;\,c'\big) \times Y\big(\langle d''\rangle\,;\,d'\big). \tag{5.6.4}$$ ◇

Explanation 5.6.5. If the indexing set of the coproduct (5.6.4) is empty, then

$$(X \,\&\, Y)\big(\langle c, d\rangle\,;\,(c', d')\big)$$

is empty. For an example where the indexing set has more than a single element, suppose given $c \in C$ and $d \in D$. Then the pair $((c,d),(c,d))$ is equal to both

$$(c) \otimes (d,d) \quad \text{and} \quad (c,c) \otimes (d). \qquad \diamond$$

Explanation 5.6.6 (Definition via Kan Extensions)**.** Recall the concept of Kan extensions from Definition I.1.1.18 and Explanation I.1.1.19. Viewing multigraphs as functors

$$X : \mathsf{Prof}(X) \times \mathsf{Vt}X \longrightarrow \mathsf{Set},$$

the product $X \,\&\, Y$ is the left Kan extension along the composite

$$\begin{tikzcd}
\mathsf{Prof}(X) \times \mathsf{Vt}X \times \mathsf{Prof}(Y) \times \mathsf{Vt}Y \arrow[dr, "\cong"] & & \\
& (\mathsf{Prof}(X) \times \mathsf{Prof}(Y)) \times (\mathsf{Vt}X \times \mathsf{Vt}Y) \arrow[dr, "\otimes \times 1 \times 1"] & \\
& & \mathsf{Prof}(X \times Y) \times (\mathsf{Vt}X \times \mathsf{Vt}Y)
\end{tikzcd} \tag{5.6.7}$$

that swaps $\mathsf{Vt}X$ with $\mathsf{Prof}(Y)$ and then takes the tensor product of profiles. That is, letting $X \boxtimes Y$ denote the external product

$$(\langle c\rangle, c', \langle d\rangle, d') \longmapsto X\big(\langle c\rangle\,;\, c'\big) \times Y\big(\langle d\rangle\,;\, d'\big),$$

one verifies that $X \,\&\, Y$ is initial among left extensions as in the diagram below, where the unlabeled arrow is the composite (5.6.7) and the double arrow denotes a natural transformation of functors.

$$\begin{tikzcd}
\mathsf{Prof}(X) \times \mathsf{Vt}X \times \mathsf{Prof}(Y) \times \mathsf{Vt}Y \arrow[rr, "X \boxtimes Y"] \arrow[dr] & \arrow[d, phantom, "\Downarrow"] & \mathsf{Set} \\
& \mathsf{Prof}(X \times Y) \times \mathsf{Vt}X \times \mathsf{Vt}Y \arrow[ur, dashed, "X \,\&\, Y"'] &
\end{tikzcd}$$

Since $\mathsf{Prof}(X) \times \mathsf{Vt}X \times \mathsf{Prof}(Y) \times \mathsf{Vt}Y$ is discrete, the coend formula for left Kan extension (Explanation I.1.1.19) reduces to the coproduct (5.6.4). $\diamond$

Definition 5.6.8. Suppose M and N are small multicategories. We define the *sharp product* $\mathsf{M} \# \mathsf{N}$ as the following pushout along morphisms induced by the inclusions

$$\mathsf{Ob}\,\mathsf{M} \hookrightarrow \mathsf{M} \quad \text{and} \quad \mathsf{Ob}\,\mathsf{N} \hookrightarrow \mathsf{N}.$$

$$\begin{tikzcd}
\mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N} \arrow[r] \arrow[d] & \coprod_{d \in \mathsf{Ob}\,\mathsf{N}} \mathsf{M} \arrow[d] \\
\coprod_{c \in \mathsf{Ob}\,\mathsf{M}} \mathsf{N} \arrow[r] & \mathsf{M} \# \mathsf{N}
\end{tikzcd}$$

$\diamond$

Explanation 5.6.9 (Unpacking the Sharp Product)**.** Restricting Definition 5.6.8 to objects and recalling Proposition 5.5.15, we see that

$$\mathsf{Ob}(\mathsf{M} \# \mathsf{N}) \cong \mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N}.$$

The operations of $\mathsf{M} \# \mathsf{N}$ are generated by operations of the form

$$\phi \times d \in \mathsf{M} \times \{d\} \quad \text{and} \quad c \times \psi \in \{c\} \times \mathsf{N}$$

subject to the following symmetry and compatibility axioms determined by the pushout.

(1) For $(c,d) \in \mathsf{M} \# \mathsf{N}$, we have

$$1_c \times d = 1_{(c,d)} = c \times 1_d.$$

(2) For operations $\phi, \phi_1, \ldots, \phi_n$ in M such that the composite below is defined, we have

$$(\phi \times d) \circ (\phi_1 \times d, \ldots, \phi_n \times d) = (\phi \circ (\phi_1, \ldots, \phi_n)) \times d.$$

(3) For $\sigma \in \Sigma_n$, we have

$$(\phi \times d) \cdot \sigma = (\phi \cdot \sigma) \times d.$$

(4) For operations $\psi, \psi_1, \ldots, \psi_m$ in N such that the composite below is defined, we have

$$(c \times \psi) \circ (c \times \psi_1, \ldots, c \times \psi_m) = c \times (\psi \circ (\psi_1, \ldots, \psi_m)).$$

(5) For $\sigma \in \Sigma_m$, we have

$$(c \times \psi) \cdot \sigma = c \times (\psi \cdot \sigma).$$

The first three conditions imply that, for any $d \in \mathsf{N}$, the assignment $\mathsf{M} \longrightarrow \mathsf{M} \# \mathsf{N}$ given by $\phi \longmapsto \phi \times d$ is a multifunctor. Likewise, the first, fourth, and fifth conditions imply that $\psi \longmapsto c \times \psi$ is a multifunctor $\mathsf{N} \longrightarrow \mathsf{M} \# \mathsf{N}$ for any $c \in \mathsf{M}$. Taken together, the conditions are equivalent to the requirement that a multifunctor

$$F : \mathsf{M} \# \mathsf{N} \longrightarrow \mathsf{P}$$

consists of an assignment on objects,

$$F(c,d) \in \operatorname{Ob} \mathsf{P}$$

for $(c,d) \in \operatorname{Ob} \mathsf{M} \times \operatorname{Ob} \mathsf{N}$ such that each

$$F(c,-) : \mathsf{N} \longrightarrow \mathsf{P} \quad \text{and} \quad F(-,d) : \mathsf{M} \longrightarrow \mathsf{P}$$

is a multifunctor. ◇

Definition 5.6.10. Suppose given small multicategories M and N along with operations

$$\phi \in \mathsf{M}\big(\langle c \rangle\,;\, c'\big) \quad \text{and} \quad \psi \in \mathsf{N}\big(\langle d \rangle\,;\, d'\big).$$

We define

$$\begin{aligned}
\phi \times \langle d \rangle &= \langle \phi \times d_j \rangle_j \in \prod_j \mathsf{M}\big(\langle c \rangle\,;\, c'\big) \times \{d_j\} \\
\langle c \rangle \times \psi &= \langle c_i \times \psi \rangle_i \in \prod_i \{c_i\} \times \mathsf{N}\big(\langle d \rangle\,;\, d'\big) \\
\phi \otimes \psi &= (c' \times \psi) \circ (\phi \times \langle d \rangle) \in (\mathsf{M} \# \mathsf{N})\big(\langle c \rangle \otimes \langle d \rangle\,;\, (c',d')\big) \\
\phi \otimes^{\mathsf{t}} \psi &= (\phi \times d') \circ (\langle c \rangle \times \psi) \in (\mathsf{M} \# \mathsf{N})\big(\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle\,;\, (c',d')\big).
\end{aligned}$$

Let $\xi^{\otimes}$ denote the bijection

$$(\mathsf{M} \# \mathsf{N})\big(\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle\,;\, (c',d')\big) \xrightarrow{\cong} (\mathsf{M} \# \mathsf{N})\big(\langle c \rangle \otimes \langle d \rangle\,;\, (c',d')\big)$$

induced by the permutation $\xi^{\otimes}$ that interchanges order of indexing. ◇

Definition 5.6.11 (Boardman-Vogt Tensor Product of Multicategories)**.** For small multicategories M and N, the tensor products of Definition 5.6.10 give two canonical morphisms of multigraphs

$$(UM) \,\&\, (UN) \underset{\xi^{\otimes} \circ \otimes^{\mathsf{t}}}{\overset{\otimes}{\rightrightarrows}} U(\mathsf{M} \# \mathsf{N}). \tag{5.6.12}$$

Taking adjoints, we have two morphisms in $\mathsf{Multicat}$, and we define $\mathsf{M} \otimes \mathsf{N}$ to be their coequalizer.

$$L((U\mathsf{M}) \,\&\, (U\mathsf{N})) \rightrightarrows \mathsf{M} \# \mathsf{N} \dashrightarrow \mathsf{M} \otimes \mathsf{N}$$

For an object $(c,d) \in \mathsf{M} \# \mathsf{N}$, we let $c \otimes d$ denote its image in $\mathsf{M} \otimes \mathsf{N}$. ◇

Explanation 5.6.13 (Definition via Kan Extensions)**.** Continuing from Explanation 5.6.6, the tensor products of Definition 5.6.10 give two different left extensions shown below.

$$\begin{tikzcd}
\mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M} \times \mathsf{Prof}(\mathsf{N}) \times \mathsf{Ob}\,\mathsf{N} \arrow[rr, "(U\mathsf{M}) \boxtimes (U\mathsf{N})"] \arrow[dr] \arrow[r, phantom, shift right=6ex, "\otimes\Downarrow"] & & \mathsf{Set} \\
& \mathsf{Prof}(\mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N}) \times \mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N} \arrow[ur, swap, "U(\mathsf{M} \# \mathsf{N})"] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M} \times \mathsf{Prof}(\mathsf{N}) \times \mathsf{Ob}\,\mathsf{N} \arrow[rr, "(U\mathsf{M}) \boxtimes (U\mathsf{N})"] \arrow[dr] \arrow[r, phantom, shift right=6ex, "\xi^{\otimes} \circ \otimes^{\mathsf{t}}\Downarrow"] & & \mathsf{Set} \\
& \mathsf{Prof}(\mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N}) \times \mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N} \arrow[ur, swap, "U(\mathsf{M} \# \mathsf{N})"] &
\end{tikzcd}$$

The canonical morphisms of (5.6.12) are the corresponding universal morphisms out of the left Kan extension $(U\mathsf{M}) \,\&\, (U\mathsf{N})$. ◇

Explanation 5.6.14 (Unpacking the Tensor Product)**.** Restricting Definition 5.6.11 to objects, we see that

$$\mathsf{Ob}(\mathsf{M} \otimes \mathsf{N}) \cong \mathsf{Ob}(\mathsf{M} \# \mathsf{N}) \cong \mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N}.$$

Continuing from our unpacking of the sharp product in Explanation 5.6.9, the operations of $\mathsf{M} \otimes \mathsf{N}$ are generated by

$$\phi \otimes d \in \mathsf{M}\big(\langle c \rangle ; c'\big) \times \{d\} \quad \text{and} \quad c \otimes \psi \in \{c\} \times \mathsf{N}\big(\langle d \rangle ; d'\big)$$

subject to the relations of $\mathsf{M} \# \mathsf{N}$ along with one additional *interchange relation*

$$\phi \otimes \psi = (\phi \otimes^{\mathsf{t}} \psi) \cdot \xi^{\otimes}. \tag{5.6.15}$$

If we draw an operation as an arrow from its input profile to its output object, the interchange relation means that the two composites

$$\begin{tikzcd}
\langle c\rangle \otimes \langle d\rangle \arrow[r, "\phi \otimes \langle d\rangle"] & c' \times \langle d\rangle \arrow[dr, "c' \otimes \psi"] & \\
& & (c', d') \\
\langle c\rangle \otimes^{\mathsf{t}} \langle d\rangle \arrow[r, "\langle c\rangle \otimes \psi"'] & \langle c\rangle \times d' \arrow[ur, "\phi \otimes d'"'] &
\end{tikzcd}$$

correspond under the bijection

$$\xi^{\otimes} : (\mathsf{M} \otimes \mathsf{N})\big(\langle c\rangle \otimes^{\mathsf{t}} \langle d\rangle\, ;\, c' \otimes d'\big) \xrightarrow{\cong} (\mathsf{M} \otimes \mathsf{N})\big(\langle c\rangle \otimes \langle d\rangle\, ;\, c' \otimes d'\big).$$

A multifunctor

$$F : \mathsf{M} \otimes \mathsf{N} \longrightarrow \mathsf{P}$$

consists of an assignment on objects $F(c,d) \in \mathsf{Ob}\,\mathsf{P}$ for $(c,d) \in \mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N}$ such that each $F(c,-)$ and $F(-,d)$ is a multifunctor and such that we have

$$F(\phi \otimes \psi) = F(\phi \otimes^{\mathsf{t}} \psi) \cdot \xi^{\otimes} \tag{5.6.16}$$

for each $\phi \in \mathsf{M}\big(\langle c\rangle\, ;\, c'\big)$ and $\psi \in \mathsf{N}\big(\langle d\rangle\, ;\, d'\big)$. ⋄

Definition 5.6.17. For small multicategories M and N, let

$$\beta : \mathsf{M} \,\#\, \mathsf{N} \longrightarrow \mathsf{N} \,\#\, \mathsf{M}$$

denote the multifunctor given on objects by $\beta(c,d) = (d,c)$ and on generating operations by

$$\beta(\phi \times d) = d \times \phi \quad \text{and} \quad \beta(c \times \psi) = \psi \times c.$$

Let

$$\beta : \mathsf{M} \otimes \mathsf{N} \longrightarrow \mathsf{N} \otimes \mathsf{M}$$

denote the induced multifunctor on tensor products. ⋄

Recall from Definition 5.1.21 the initial operad I has a single object $*$ and single operation 1_*.

Theorem 5.6.18. *The tensor product*

$$\otimes : \mathsf{Multicat} \times \mathsf{Multicat} \longrightarrow \mathsf{Multicat}$$

is a symmetric monoidal product on the category of small multicategories, with unit given by the initial operad I and symmetry given by β.

Proof. Observe first that $(\#, \mathsf{I}, \beta)$ is a symmetric monoidal product on $\mathsf{Multicat}$:

- Functoriality of # follows from functoriality of the pushout in Definition 5.6.8.
- The associativity and unit isomorphisms follow from Explanation 5.6.9. Multifunctors with domain I are simply a choice of object, and therefore the unit isomorphisms

  $$\mathsf{M} \,\#\, \mathsf{I} \xrightarrow{\cong} \mathsf{M} \xleftarrow{\cong} \mathsf{I} \,\#\, \mathsf{M}$$

  are determined by the identities on M. Likewise we have

  $$(\mathsf{L} \,\#\, \mathsf{M}) \,\#\, \mathsf{N} \xrightarrow{\cong} \mathsf{L} \,\#\, (\mathsf{M} \,\#\, \mathsf{N})$$

  determined by the identity multifunctors in each variable.

- The universal property of pushouts implies that these satisfy the axioms of a symmetric monoidal category listed in Definitions 1.1.1 and 1.1.23.

Similarly, functoriality of the tensor product follows from functoriality of the coequalizer in Definition 5.6.11. The associativity, unit, and symmetry isomorphisms for the tensor product are induced by those for the sharp product, and satisfy the corresponding axioms by the universal product of coequalizers. □

Motivation 5.6.19 (2-functoriality of the Tensor Product)**.** We have not yet discussed 2-functoriality of the tensor product with respect to multinatural transformations. We will describe this in Section 6.4 and show that it provides a symmetric Cat-monoidal structure as in Definition 1.5.1. ◇

Recall from Definition 5.3.1 a pointed multicategory consists of a multicategory M together with a multifunctor $\mathsf{T} \longrightarrow \mathsf{M}$ determined by an object $*^{\mathsf{M}}$ and operations ι^{M}. The following are special cases of the smash and wedge products defined in Section 4.1.

Definition 5.6.20. Given small pointed multicategories $(\mathsf{M}, *^{\mathsf{M}}, \iota^{\mathsf{M}})$ and $(\mathsf{N}, *^{\mathsf{N}}, \iota^{\mathsf{N}})$, we define the *smash product* as the following pushout in Multicat.

(5.6.21)
$$\begin{tikzcd}
(\mathsf{M} \otimes \mathsf{T}) \coprod (\mathsf{T} \otimes \mathsf{N}) \arrow[r] \arrow[d] & \mathsf{M} \otimes \mathsf{N} \arrow[d] \\
\mathsf{T} \arrow[r] & \mathsf{M} \wedge \mathsf{N}
\end{tikzcd}$$

The *smash unit* multicategory is

(5.6.22)
$$\mathsf{S} = \mathsf{I}_+ = \mathsf{I} \coprod \mathsf{T}. \quad ◇$$

Definition 5.6.23. Given small pointed multicategories $(\mathsf{M}, *^{\mathsf{M}}, \iota^{\mathsf{M}})$ and $(\mathsf{N}, *^{\mathsf{N}}, \iota^{\mathsf{N}})$, we define the *wedge product* as the coequalizer in Multicat of the two structure morphisms

$$\begin{tikzcd}
\mathsf{T} \arrow[r, shift left, "*^{\mathsf{M}}"] \arrow[r, shift right, "*^{\mathsf{N}}"'] & \mathsf{M} \coprod \mathsf{N} \arrow[r, dashed] & \mathsf{M} \vee \mathsf{N}
\end{tikzcd}$$

The wedge product is sometimes also called the *wedge sum*. ◇

In Theorem 5.7.22 below we will show, as a special case of Theorem 4.2.3, that the smash product is part of a symmetric monoidal closed structure on $\mathsf{Multicat}_*$.

5.7. The Internal Hom for Multicategories

In this section we describe internal hom objects for Multicat and $\mathsf{Multicat}_*$. We will use the following notation for a tuple of multifunctors $\langle F \rangle$.

Definition 5.7.1. Suppose given multicategories M and N together with a tuple of multifunctors $\langle F \rangle = (F_1, \ldots, F_m)$ where each

$$F_i : \mathsf{M} \longrightarrow \mathsf{N} \quad \text{for} \quad 1 \le i \le m.$$

Then we use the following notation.

- For $c \in \operatorname{Ob}\mathsf{M}$, let $\langle F \rangle c = \langle F_i c \rangle_i$.
- For $\langle c \rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\operatorname{Ob}\mathsf{M})$, let
$$\langle Fc \rangle = \langle \langle F_i c_j \rangle_i \rangle_j \quad \text{and} \quad \langle Fc \rangle^{\mathsf{t}} = \langle \langle F_i c_j \rangle_j \rangle_i.$$

- For $\phi \in \mathsf{M}\big(\langle c\rangle\,;c'\big)$, let
$$\langle F\rangle\phi = \langle F_i\phi\rangle_i \in \prod_i \mathsf{N}\big(F_i\langle c\rangle\,;F_ic'\big).$$

$\diamond$

Definition 5.7.2. Suppose M and N are small multicategories. The *internal hom multicategory* $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ is defined as follows. The objects of $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ are multifunctors $\mathsf{M} \longrightarrow \mathsf{N}$. The operations
$$\alpha : \langle F\rangle \longrightarrow G$$
in $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ are called *transformations* and are given by components
$$\alpha_c \in \mathsf{N}\big(\langle F\rangle c\,;Gc\big)$$
for each $c \in \mathsf{Ob}\,\mathsf{M}$ such that, for each operation $\phi \in \mathsf{M}\big(\langle c\rangle\,;c'\big)$, the following *naturality condition* holds, with $\alpha_{\langle c\rangle} = \langle\alpha_{c_j}\rangle_j$, $\mathrm{len}\langle F\rangle = m$, and $\mathrm{len}\langle c\rangle = n$:
$$(G\phi)\circ\alpha_{\langle c\rangle} = (\alpha_{c'}\circ\langle F\rangle\phi)\cdot\xi^{\otimes}_{m,n}. \tag{5.7.3}$$

The unit operation $1_F : F \longrightarrow F$ is given by the identity multinatural transformation whose component at c is 1_{Fc}. The composition and symmetric group actions on operations of $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ are given componentwise by those in N.

We describe (5.7.3) further in Explanation 5.7.4. In Lemma 5.7.6 we show that (5.7.3) is closed under composition. Using this, we complete the proof that $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ is a multicategory in Proposition 5.7.11. $\diamond$

Explanation 5.7.4 (Operations in $\mathsf{Hom}(\mathsf{M},\mathsf{N})$)**.** If we draw an operation as an arrow from its input profile to its output object, the equality (5.7.3) means that the two composites
$$\begin{tikzcd}
\langle Fc\rangle \arrow[r, "\alpha_{\langle c\rangle}"] & G\langle c\rangle \arrow[dr, "G\phi"] & \\
 & & Gc' \\
\langle Fc\rangle^{\mathsf{t}} \arrow[r, "\langle F\rangle\phi"'] & \langle F\rangle c' \arrow[ur, "\alpha_{c'}"'] &
\end{tikzcd}$$
correspond under the bijection
$$\xi^{\otimes} : \mathsf{N}\big(\langle Fc\rangle^{\mathsf{t}}\,;Gc'\big) \xrightarrow{\cong} \mathsf{N}\big(\langle Fc\rangle\,;Gc'\big).$$
$\diamond$

Explanation 5.7.5 (Category of Unary Operations in $\mathsf{Hom}(\mathsf{M},\mathsf{N})$)**.** In the case that $\langle F\rangle$ is a singleton, then $\xi^{\otimes}$ is an identity and the equality (5.7.3) reduces to the naturality condition of Definition 5.1.17. Therefore, the set of unary operations $\mathsf{Hom}(\mathsf{M},\mathsf{N})(F,G)$ is precisely the set of multinatural transformations from F to G. Thus the underlying category of objects and unary operations of $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ is $\mathsf{Multicat}(\mathsf{M},\mathsf{N})$. $\diamond$

Lemma 5.7.6. *In the context of Definition 5.7.2, the naturality condition (5.7.3) is closed under composition.*

Proof. Suppose given multifunctors
$$E_{i,k}, F_i, G : \mathsf{M} \longrightarrow \mathsf{N} \quad\text{for}\quad 1 \le i \le m \quad\text{and}\quad 1 \le k \le p_i.$$
Let
$$\langle F\rangle = \langle F_i\rangle_i, \qquad \langle E_i\rangle = \langle E_{i,k}\rangle_k, \qquad \langle E\rangle = \oplus_i\langle E_i\rangle = \langle\langle E_{i,k}\rangle_k\rangle_i,$$

$$\text{and} \quad p = p_1 + \cdots + p_m.$$

So $\langle F \rangle$ has length m, each $\langle E_i \rangle$ has length p_i, and $\langle E \rangle$ has length $p = \sum_i p_i$. Then suppose given transformations

$$\alpha \in \mathsf{Hom}(\mathsf{M},\mathsf{N})\big(\langle F \rangle \,;\, G\big) \quad \text{and} \quad \omega_i \in \mathsf{Hom}(\mathsf{M},\mathsf{N})\big(\langle E_i \rangle \,;\, F_i\big).$$

We must show that the composite $\alpha \circ \langle \omega \rangle$ satisfies the naturality condition (5.7.3). That is, we must show

$$(G\phi) \circ \big(\alpha \circ \langle \omega \rangle\big)_{\langle c \rangle} = \big((\alpha \circ \langle \omega \rangle)_{c'} \circ \langle E \rangle \phi\big) \cdot \xi^{\otimes}_{p,n} \tag{5.7.7}$$

for each $\langle c \rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\mathsf{M})$ and each operation $\phi \in \mathsf{M}\big(\langle c \rangle \,;\, c'\big)$. For this purpose, we introduce the following notation.

- As in Definition 5.7.1, we let

$$\langle F \rangle \phi = \langle F_i \phi \rangle_i \in \prod_{i=1}^{m} \mathsf{N}\big(F_i \langle c \rangle \,;\, F_i c'\big)$$

  for $\phi \in \mathsf{M}\big(\langle c \rangle \,;\, c'\big)$.
- For $\langle c \rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\mathsf{M})$ we let

$$\langle \omega_c \rangle = \langle \langle (\omega_i)_{c_j} \rangle_i \rangle_j \quad \text{and} \quad \langle \omega_c \rangle^{\mathsf{t}} = \langle \langle (\omega_i)_{c_j} \rangle_j \rangle_i$$

- As described in Explanation 5.6.2, we use subscripts $\xi^{\otimes}_{m,n}$ to indicate the component of $\xi^{\otimes}$ (I.2.4.5) at mn. So, for example, we have

$$\langle \omega_c \rangle^{\mathsf{t}} \cdot \xi^{\otimes}_{m,n} = \langle \omega_c \rangle. \tag{5.7.8}$$

- We write $\langle p \rangle n$ for the tuple of indices

$$\oplus_{j=1}^{n} \langle p \rangle = (\underbrace{p_1, \ldots, p_m}_{1}, \ldots, \underbrace{p_1, \ldots, p_m}_{n}).$$

  So $\langle p \rangle n$ has length mn.
- We use the notation $\xi^{\otimes}_{m,n}\langle \langle p \rangle n \rangle$ for the block permutation (II.1.1.19) induced by $\xi^{\otimes}_{m,n}$ on mn subintervals whose lengths are given by $\langle p \rangle n$. So $\xi^{\otimes}_{m,n}\langle \langle p \rangle n \rangle$ permutes the subintervals as $\xi^{\otimes}_{m,n}$ permutes the elements of mn, but leaves the relative order within each block unchanged.

The desired equality (5.7.7) is given by the following computation. In this computation we use the naturality conditions (5.7.3) for α and ω, together with the multicategory axioms for N and the following equality relating (block) transposition permutations:

$$\Big(\oplus_{i=1}^{m} \xi^{\otimes}_{p_i,n} \Big) \cdot \xi^{\otimes}_{m,n}\langle \langle p \rangle n \rangle = \xi^{\otimes}_{(p_1+\cdots+p_m),n} = \xi^{\otimes}_{p,n} \tag{5.7.9}$$

We explain the equality (5.7.9) in Explanation 5.7.10 below. Now we give the computation showing (5.7.7).

$$\begin{aligned}
\big(G\phi\big) \circ \big(\alpha \circ \langle\omega\rangle\big)_{\langle c\rangle} &= \big(G\phi\big) \circ \big\langle \alpha_{c_j} \circ \langle\omega\rangle_{c_j} \big\rangle_j \\
&\qquad \text{by definition,} \\
&= \big((G\phi) \circ \alpha_{\langle c\rangle}\big) \circ \langle \omega_c \rangle \\
&\qquad \text{by associativity,} \\
&= \big((\alpha_{c'} \circ \langle F\rangle\phi) \cdot \xi^{\otimes}_{m,n}\big) \circ \langle \omega_c \rangle \\
&\qquad \text{by naturality (5.7.3) for } \alpha, \\
&= \Big(\big(\alpha_{c'} \circ \langle F\rangle\phi\big) \circ \langle \omega_c\rangle^{\mathsf{t}}\Big) \cdot \xi^{\otimes}_{m,n}\langle\langle p\rangle n\rangle \\
&\qquad \text{by (5.7.8) and top equivariance,} \\
&= \Big(\alpha_{c'} \circ \big\langle F_i\phi \circ (\omega_i)_{\langle c\rangle}\big\rangle_i\Big) \cdot \xi^{\otimes}_{m,n}\langle\langle p\rangle n\rangle \\
&\qquad \text{by associativity,} \\
&= \Big(\alpha_{c'} \circ \big\langle \big((\omega_i)_{c'} \circ \langle E_i\rangle\phi\big) \cdot \xi^{\otimes}_{p_i,n}\big\rangle_i\Big) \cdot \xi^{\otimes}_{m,n}\langle\langle p\rangle n\rangle \\
&\qquad \text{by naturality (5.7.3) for } \omega, \\
&= \Big(\alpha_{c'} \circ \big\langle (\omega_i)_{c'} \circ \langle E_i\rangle\phi\big\rangle_i\Big) \cdot \Big(\Big(\oplus_{i=1}^m \xi^{\otimes}_{p_i,n}\Big) \cdot \xi^{\otimes}_{m,n}\langle\langle p\rangle n\rangle\Big) \\
&\qquad \text{by bottom equivariance and group action,} \\
&= \Big(\alpha_{c'} \circ \big\langle (\omega_i)_{c'} \circ \langle E_i\rangle\phi\big\rangle_i\Big) \cdot \xi^{\otimes}_{p,n} \\
&\qquad \text{by the hexagon equality (5.7.9),} \\
&= \Big(\big(\alpha \circ \langle\omega\rangle\big)_{c'} \circ \big\langle\langle E_i\rangle\phi\big\rangle_i\Big) \cdot \xi^{\otimes}_{p,n} \\
&\qquad \text{by associativity,} \\
&= \Big(\big(\alpha \circ \langle\omega\rangle\big)_{c'} \circ \langle E\rangle\phi\Big) \cdot \xi^{\otimes}_{p,n} \\
&\qquad \text{by definition.}
\end{aligned}$$

Therefore, $\alpha \circ \langle\omega\rangle$ is a transformation. □

Explanation 5.7.10 (Block Transposition Relation)**.** Here we explain the equality (5.7.9),

$$\Big(\oplus_{i=1}^m \xi^{\otimes}_{p_i,n}\Big) \cdot \xi^{\otimes}_{m,n}\langle\langle p\rangle n\rangle = \xi^{\otimes}_{(p_1+\cdots+p_m),n} = \xi^{\otimes}_{p,n},$$

used in the proof of Lemma 5.7.6 above. We first give explicit formulas showing the two sides are equal. Then we give a second, more geometric argument, using the interpretation of $\xi^{\otimes}$ as matrix transposition.

Using the formulas of Definition I.2.4.1 and (II.1.1.19), both sides of (5.7.9) send

$$A = (j-1)p + p_1 + \cdots + p_{i-1} + k$$

to

$$B = n(p_1 + \cdots + p_{i-1} + k - 1) + j$$

for

$$i \in \{1,\ldots,m\}, \qquad j \in \{1,\ldots,n\}, \quad \text{and} \quad k \in \{1,\ldots,p_i\}.$$

The block permutation $\xi^{\otimes}_{m,n}\langle\langle p\rangle n\rangle$ sends A to

$$A' = n(p_1 + \cdots + p_{i-1}) + (j-1)p_i + k.$$

The block sum

$$\oplus_{i=1}^{m}\xi^{\otimes}_{p_i,n}$$

then sends A' to B.

Alternatively, we also have the following geometric interpretation of (5.7.9) in terms of matrix transposition. A special case of this argument, where all of the p_i are equal, is used in Explanation I.2.4.14 to explain the hexagon axiom for Σ. Suppose

$$P_i = [\bullet,\ldots,\bullet] \quad \text{for} \quad 1 \le j \le n \quad \text{and} \quad 1 \le i \le m$$

is a $1 \times p_i$ matrix with p_i objects, and consider the $n \times p$ matrix

$$M = \begin{bmatrix} P_{11} & \cdots & P_{1m} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nm} \end{bmatrix}$$

with each P_{ji} a copy of P_i for $1 \le j \le n$. Then the left hand side of (5.7.9) is the composite of permutations along the top and right of the following diagram, while the right hand side is the other permutation, with $(-)^T$ denoting transpose of matrices.

$$\begin{tikzcd}
\begin{bmatrix} P_{11} & \cdots & P_{1m} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nm} \end{bmatrix} \arrow[r, "\xi^{\otimes}_{m,n}\langle\langle p\rangle n\rangle"] \arrow[dr, "\xi^{\otimes}_{p,n}"'] & \begin{bmatrix} P_{11} & \cdots & P_{n1} \\ \vdots & \ddots & \vdots \\ P_{1m} & \cdots & P_{nm} \end{bmatrix} = \begin{bmatrix} p_1 \otimes n \\ \vdots \\ p_m \otimes n \end{bmatrix} \arrow[d, "\oplus_{i=1}^{m}\xi^{\otimes}_{p_i,n}"] \\
& \begin{bmatrix} P^T_{11} & \cdots & P^T_{n1} \\ \vdots & \ddots & \vdots \\ P^T_{1m} & \cdots & P^T_{nm} \end{bmatrix} = \begin{bmatrix} n \otimes p_1 \\ \vdots \\ n \otimes p_m \end{bmatrix}
\end{tikzcd}$$

The two equalities in the above diagram are instances of the convention for products in Σ, as described in Explanation I.2.4.7. For example, each $p_i \otimes n$ is viewed either as a row of $p_i n$ objects arranged into n intervals, each with p_i objects, or as an $n \times p_i$ matrix. $\diamond$

Proposition 5.7.11. *In the context of Definition 5.7.2, $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ is a multicategory.*

Proof. Lemma 5.7.6 shows that the naturality condition (5.7.3) is closed under composition. Closure of the naturality condition under permutations follows from the top and bottom equivariance axioms for composition in N.

The four multicategory axioms for $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ follow from those of N. For example, let $\mathsf{H} = \mathsf{Hom}(\mathsf{M},\mathsf{N})$ and suppose given

- $F''' \in \mathsf{H}$,
- $\langle F''\rangle \in \mathsf{Prof}(\mathsf{H})$ of length n,
- $\langle F'_j\rangle \in \mathsf{Prof}(\mathsf{H})$ of length k_j for each $j \in \{1,\ldots,n\}$, and
- $\langle F_{j,i}\rangle \in \mathsf{Prof}(\mathsf{H})$ of length $\ell_{j,i}$ for each $i \in \{1,\ldots,k_j\}$.

Then, using the notation conventions for concatenation from (5.1.4), the associativity diagram

$$\begin{tikzcd}[column sep=small]
& \mathsf{H}\big(\langle F'\rangle\,;\,F'''\big)\times\prod_{j=1}^{n}\Big[\prod_{i=1}^{k_j}\mathsf{H}\big(\langle F_{j,i}\rangle\,;\,F'_{j,i}\big)\Big] \arrow[dr, "\gamma"] & \\
\mathsf{H}\big(\langle F''\rangle\,;\,F'''\big)\times\Big[\prod_{j=1}^{n}\mathsf{H}\big(\langle F'_j\rangle\,;\,F''_j\big)\Big]\times\prod_{j=1}^{n}\Big[\prod_{i=1}^{k_j}\mathsf{H}\big(\langle F_{j,i}\rangle\,;\,F'_{j,i}\big)\Big] \arrow[ur, "{(\gamma,1)}"] \arrow[d, "\text{permute}"', "\cong"] & & \mathsf{H}\big(\langle F\rangle\,;\,F'''\big) \\
\mathsf{H}\big(\langle F''\rangle\,;\,F'''\big)\times\prod_{j=1}^{n}\Big[\mathsf{H}\big(\langle F'_j\rangle\,;\,F''_j\big)\times\prod_{i=1}^{k_j}\mathsf{H}\big(\langle F_{j,i}\rangle\,;\,F'_{j,i}\big)\Big] \arrow[dr, "{(1,\prod_j\gamma)}"'] & & \\
& \mathsf{H}\big(\langle F''\rangle\,;\,F'''\big)\times\prod_{j=1}^{n}\mathsf{H}\big(\langle F_j\rangle\,;\,F''_j\big) \arrow[uur, "\gamma"'] &
\end{tikzcd}$$

commutes because, for

- $\alpha''\in\mathsf{H}\big(\langle F''\rangle\,;\,F'''\big)$,
- $\alpha'_j\in\mathsf{H}\big(\langle F'_j\rangle\,;\,F''_j\big)$ for $j\in\{1,\ldots,n\}$, and
- $\alpha_{j,i}\in\mathsf{H}\big(\langle F_{j,i}\rangle\,;\,F'_{j,i}\big)$ for $j\in\{1,\ldots,n\}$ and $i\in\{1,\ldots,k_j\}$,

the two composites around the diagram, which are

$$(\alpha''\circ\langle\alpha'_j\rangle)\circ\langle\langle\alpha_{j,i}\rangle_i\rangle_j \quad\text{and}\quad \alpha''\circ\langle\alpha'_j\circ\langle\alpha_{j,i}\rangle_i\rangle_j,$$

are given componentwise by the corresponding two composites around the following diagram for each $c\in\mathsf{M}$, and are therefore equal.

$$\begin{tikzcd}[column sep=small]
& \mathsf{N}\big(\langle F'\rangle c\,;\,F'''c\big)\times\prod_{j=1}^{n}\Big[\prod_{i=1}^{k_j}\mathsf{N}\big(\langle F_{j,i}\rangle c\,;\,F'_{j,i}c\big)\Big] \arrow[dr, "\gamma"] & \\
\mathsf{N}\big(\langle F''\rangle c\,;\,F'''c\big)\times\Big[\prod_{j=1}^{n}\mathsf{N}\big(\langle F'_j\rangle c\,;\,F''_jc\big)\Big]\times\prod_{j=1}^{n}\Big[\prod_{i=1}^{k_j}\mathsf{N}\big(\langle F_{j,i}\rangle c\,;\,F'_{j,i}c\big)\Big] \arrow[ur, "{(\gamma,1)}"] \arrow[d, "\text{permute}"', "\cong"] & & \mathsf{N}\big(\langle F\rangle c\,;\,F'''c\big) \\
\mathsf{N}\big(\langle F''\rangle c\,;\,F'''c\big)\times\prod_{j=1}^{n}\Big[\mathsf{N}\big(\langle F'_j\rangle c\,;\,F''_jc\big)\times\prod_{i=1}^{k_j}\mathsf{N}\big(\langle F_{j,i}\rangle c\,;\,F'_{j,i}c\big)\Big] \arrow[dr, "{(1,\prod_j\gamma)}"'] & & \\
& \mathsf{N}\big(\langle F''\rangle c\,;\,F'''c\big)\times\prod_{j=1}^{n}\mathsf{N}\big(\langle F_j\rangle c\,;\,F''_jc\big) \arrow[uur, "\gamma"'] &
\end{tikzcd}$$

The other axioms of Definition 5.1.2 are verified componentwise in the same way. □

In the special case $\mathsf{N} = \mathsf{End}(\mathsf{C})$ for a permutative category C, the following result gives a description of $\mathsf{Hom}(\mathsf{M}, \mathsf{N})$ arising from the monoidal product on C.

Lemma 5.7.12. *Suppose* M *is a small multicategory and* $(\mathsf{C}, \oplus, e, \xi)$ *is a small permutative category.*

(1) Taking the pointwise monoidal product of multifunctors induces a permutative structure on $\mathsf{Multicat}(\mathsf{M}, \mathsf{End}(\mathsf{C}))$.

(2) There is an isomorphism of multicategories

$$\mathsf{End}\big(\mathsf{Multicat}(\mathsf{M}, \mathsf{End}(\mathsf{C}))\big) \cong \mathsf{Hom}(\mathsf{M}, \mathsf{End}(\mathsf{C})).$$

Proof. First we explain the pointwise monoidal product. Suppose given multifunctors

$$F, F' \in \mathsf{Multicat}(\mathsf{M}, \mathsf{End}(\mathsf{C})).$$

For each $c \in \mathsf{M}$ and $\phi \in \mathsf{M}\big(\langle c \rangle ; c'\big)$, let $\xi_\oplus$ denote the shuffle isomorphism

$$\bigoplus_{c_i \in \langle c \rangle} (Fc_i \oplus F'c_i) \xrightarrow{\xi_\oplus} \Big(\bigoplus_{c_i \in \langle c \rangle} Fc_i\Big) \oplus \Big(\bigoplus_{c_i \in \langle c \rangle} F'c_i\Big)$$

induced by the permutation $\xi^{\otimes}_{2,\mathrm{len}\langle c \rangle}$ (I.2.4.5) in the finite ordinal category. Then we define

$$(F \oplus F')c = (Fc) \oplus (F'c) \quad \text{and} \quad (F \oplus F')\phi = ((F\phi) \oplus (F'\phi)) \circ \xi_\oplus.$$

For multinatural transformations

$$\alpha : F \longrightarrow G \quad \text{and} \quad \alpha' : F' \longrightarrow G'$$

we define

$$(\alpha \oplus \alpha')_c = \alpha_c \oplus \alpha'_c.$$

The monoidal unit is the constant multifunctor at $e \in \mathsf{C}$, and the permutative structure for C implies that this is a strict symmetric monoidal product for

$$\mathsf{Multicat}(\mathsf{M}, \mathsf{End}(\mathsf{C})).$$

Second we verify the asserted isomorphism. To begin, observe that the objects and unary operations on both sides are the same by definition. For operations

$$\eta \in \mathsf{Hom}(\mathsf{M}, \mathsf{End}(\mathsf{C}))\big(\langle F \rangle ; F'\big),$$

we have, by definition,

$$\eta_c : \langle F \rangle c = \bigoplus_{F_i \in \langle F \rangle} F_i c \longrightarrow F'c$$

and in this context the equality (5.7.3) is equivalent to commutativity of the following diagram in C for each $\phi \in \mathsf{M}(\langle c \rangle ; c')$.

$$\begin{tikzcd}
& \bigoplus_i \bigoplus_j F_i c_j \arrow[r, "\langle F \rangle \phi"] \arrow[dl, "\xi"', "\cong"] & \bigoplus_i F_i c' \arrow[dd, "\eta_{c'}"] \\
\bigoplus_j \bigoplus_i F_i c_j \arrow[d, "\eta_{\langle c \rangle}"'] & & \\
\bigoplus_j F' c_j \arrow[rr, "F'\phi"] & & F' c'
\end{tikzcd}$$

Inverting the shuffle isomorphism, this is precisely the condition in Definition 5.1.17 for η to define a multinatural transformation

$$\oplus_i F_i \longrightarrow F.$$

Thus the data of an operation in $\mathsf{Hom}(\mathsf{M}, \mathsf{End}(\mathsf{C}))$ is precisely the same as that of an operation in

$$\mathsf{End}\big(\mathsf{Multicat}(\mathsf{M}, \mathsf{End}(\mathsf{C}))\big).$$

This correspondence preserves compositions and symmetric group actions because these are induced by the permutative structure of C. $\square$

Proposition 5.7.13. *Given small multicategories* M, N, *and* P, *there are isomorphisms of categories*

$$\mathsf{Multicat}(\mathsf{M}, \mathsf{Hom}(\mathsf{N}, \mathsf{P})) \cong \mathsf{Multicat}(\mathsf{M} \otimes \mathsf{N}, \mathsf{P}) \cong \mathsf{Multicat}(\mathsf{N}, \mathsf{Hom}(\mathsf{M}, \mathsf{P}))$$

that are natural in M, N, *and* P.

Proof. We discuss the first isomorphism; the second follows from the first by applying the symmetry β. We define a functor

$$\Theta : \mathsf{Multicat}(\mathsf{M}, \mathsf{Hom}(\mathsf{N}, \mathsf{P})) \longrightarrow \mathsf{Multicat}(\mathsf{M} \otimes \mathsf{N}, \mathsf{P})$$

as follows. Given

$$H : \mathsf{M} \longrightarrow \mathsf{Hom}(\mathsf{N}, \mathsf{P})$$

we first define an assignment

$$\Theta H : \mathsf{M} \otimes \mathsf{N} \longrightarrow \mathsf{P}$$

beginning with $\mathsf{M} \times \mathsf{N}$ and then descending to $\mathsf{M} \# \mathsf{N}$ and $\mathsf{M} \otimes \mathsf{N}$.

- For $(c, d) \in \mathsf{M} \times \mathsf{N}$, let
$$(\Theta H)(c, d) = (Hc)d.$$
- For $\phi \in \mathsf{M}(\langle c \rangle ; c')$ and $\psi \in \mathsf{N}(\langle d \rangle ; d')$, let
$$(\Theta H)(\phi \times d) = (H\phi)_d \quad \text{and} \quad (\Theta H)(c \times \psi) = (Hc)\psi.$$

Now by definition
$$(\Theta H)(c,-) = (Hc)(-)$$
is a multifunctor from N to P for each object c in M. Next we observe that
$$(\Theta H)(-,d) : \mathsf{M} \longrightarrow \mathsf{P}$$
is a multifunctor for each $d \in \mathsf{N}$. The basic reason for this is that the composition of operations in $\mathsf{Hom}(\mathsf{N},\mathsf{P})$ is defined componentwise. For example, given composable operations ϕ' and $\langle\phi\rangle$ in M, we have and equality of operations in $\mathsf{Hom}(\mathsf{N},\mathsf{P})$,
$$H(\phi') \circ H\langle\phi\rangle = H(\phi' \circ \langle\phi\rangle),$$
because H is a multifunctor. Now because composition of operations in $\mathsf{Hom}(\mathsf{N},\mathsf{P})$ is defined componentwise, we have the following for each $d \in \mathsf{N}$:
$$\begin{aligned}(\Theta H)(\phi' \times d) \circ \langle(\Theta H)(\phi_i \times d)\rangle_i &= H(\phi')_d \circ (H\langle\phi\rangle)_d \\ &= (H(\phi' \circ \langle\phi\rangle))_d \\ &= (\Theta H)((\phi' \circ \langle\phi\rangle) \times d).\end{aligned}$$
The rest of the multifunctor axioms for $(\Theta H)(-,d)$ are verified similarly.

Therefore, recalling Explanation 5.6.9, ΘH gives a well-defined multifunctor on $\mathsf{M} \# \mathsf{N}$. To verify that this assignment descends to a unique multifunctor on the tensor product $\mathsf{M} \otimes \mathsf{N}$, we recall from Explanation 5.6.14 that it suffices to show
$$(\Theta H)(\phi \otimes \psi) = (\Theta H)((\phi \otimes^{\mathsf{t}} \psi) \cdot \xi^{\otimes})$$
for all operations
$$\phi \in \mathsf{M}\big(\langle c\rangle\,;c'\big) \quad \text{and} \quad \psi \in \mathsf{N}\big(\langle d\rangle\,;d'\big).$$
Using the decompositions of Definition 5.6.10 and the definition of ΘH, we must verify
$$((Hc')\psi) \circ \langle(H\phi)_{d_j}\rangle_j = \big((H\phi)_{d'} \circ \langle(Hc_i)\rangle_i\psi\big) \cdot \xi^{\otimes}.$$
This equality follows directly from the defining equality (5.7.3) at the operation
$$\psi \in \mathsf{N}\big(\langle d\rangle\,;d'\big)$$
with
$$\alpha = (H\phi), \quad F_i = Hc_i, \quad \text{and} \quad G = Hc'.$$
Therefore
$$\Theta H : \mathsf{M} \otimes \mathsf{N} \longrightarrow \mathsf{P}$$
is a well-defined multifunctor for each
$$H \in \mathsf{Multicat}(\mathsf{M}, \mathsf{Hom}(\mathsf{N},\mathsf{P})).$$

For a multinatural transformation
$$\eta \in \mathsf{Multicat}(\mathsf{M}, \mathsf{Hom}(\mathsf{N},\mathsf{P}))(H,H'),$$
we define
$$\Theta\eta \in \mathsf{Multicat}(\mathsf{M} \otimes \mathsf{N}, \mathsf{P})(\Theta H, \Theta H')$$
via components
$$(\Theta\eta)_{c \otimes d} = (\eta_c)_d.$$
One verifies the analogue of equality (5.7.3) for $\Theta\eta$ on generating operations $\phi \otimes d$ and $c \otimes \psi$ using, respectively, componentwise composition of operations in

$\mathsf{Hom}(\mathsf{N},\mathsf{P})$ at the object d and the analogue of (5.7.3) for each η_c at the operation ψ. This shows that Θ defines a function

$$\mathsf{Multicat}(\mathsf{M},\mathsf{Hom}(\mathsf{N},\mathsf{P}))(H,H') \xrightarrow{\Theta} \mathsf{Multicat}(\mathsf{M}\otimes\mathsf{N},\mathsf{P})(\Theta H,\Theta H')$$

for each pair

$$H,H' \in \mathsf{Multicat}(\mathsf{M},\mathsf{Hom}(\mathsf{N},\mathsf{P})).$$

Finally, one verifies that Θ preserves identities and composition because these are determined componentwise:

$$(\Theta 1_H)_{c\otimes d} = ((1_H)_c)_d \quad \text{and}$$
$$(\Theta(\eta'\eta))_{c\otimes d} = ((\eta'\eta)_c)_d = (\eta'_c)_d \circ (\eta_c)_d) = ((\Theta\eta')\circ(\Theta\eta))_{c\otimes d}$$

for composable multinatural transformations η' and η and for each $c\otimes d \in \mathsf{M}\otimes\mathsf{N}$. This finishes the definition of a functor

$$\Theta : \mathsf{Multicat}(\mathsf{M},\mathsf{Hom}(\mathsf{N},\mathsf{P})) \longrightarrow \mathsf{Multicat}(\mathsf{M}\otimes\mathsf{N},\mathsf{P}).$$

Next we define a functor

$$\Psi : \mathsf{Multicat}(\mathsf{M}\otimes\mathsf{N},\mathsf{P}) \longrightarrow \mathsf{Multicat}(\mathsf{M},\mathsf{Hom}(\mathsf{N},\mathsf{P}))$$

as follows. Given a multifunctor $K : \mathsf{M}\otimes\mathsf{N} \longrightarrow \mathsf{P}$ we first define

$$(\Psi K)c = K(c\otimes(-)) : \mathsf{N} \longrightarrow \mathsf{P}$$

and

$$(\Psi K)\phi = K(\phi\otimes -) \in \mathsf{Hom}(\mathsf{N},\mathsf{P})\big((\Psi K)\langle c\rangle\,;(\Psi K)c'\big)$$

for

$$\big(\langle c\rangle\,;c'\big) \in \mathsf{Prof}(\mathsf{Ob}\,\mathsf{M})\times\mathsf{Ob}\,\mathsf{M} \quad \text{and} \quad \phi \in \mathsf{M}\big(\langle c\rangle\,;c'\big).$$

Given a multinatural transformation $\omega \in \mathsf{Multicat}(\mathsf{M}\otimes\mathsf{N},\mathsf{P})(K,K')$, we define

$$(\Psi\omega)_c = \omega_{c\otimes -} : K(c\otimes -) \longrightarrow K'(c\otimes -).$$

Checking the definitions, one verifies the following.

(1) $(\Psi K)c$ is a multifunctor $\mathsf{N} \longrightarrow \mathsf{P}$ for each $c \in \mathsf{M}$ and $(\Psi K)\phi$ is an operation of $\mathsf{Hom}(\mathsf{N},\mathsf{P})$ for each $\phi \in \mathsf{M}\big(\langle c\rangle\,;c'\big)$. Therefore, ΨK defines a function on objects and operations of M taking values in $\mathsf{Hom}(\mathsf{N},\mathsf{P})$.

(2) ΨK satisfies the axioms of a multifunctor

$$\Psi K : \mathsf{M} \longrightarrow \mathsf{Hom}(\mathsf{N},\mathsf{P})$$

for each $K \in \mathsf{Multicat}(\mathsf{M}\otimes\mathsf{N},\mathsf{P})$.

(3) $\Psi\omega$ satisfies the axioms of an operation $\Psi K \longrightarrow \Psi K'$ for each

$$\omega \in \mathsf{Multicat}(\mathsf{M}\otimes\mathsf{N},\mathsf{P})(K,K').$$

Therefore, Ψ defines a function on the objects and morphisms

$$\mathsf{Multicat}(\mathsf{M}\otimes\mathsf{N},\mathsf{P}) \longrightarrow \mathsf{Multicat}(\mathsf{M},\mathsf{Hom}(\mathsf{N},\mathsf{P})).$$

(4) Ψ is functorial.

Lastly, one verifies that Θ and Ψ define an inverse bijection on objects and morphisms and, therefore, define an isomorphism of categories. Naturality of the isomorphism with respect to multifunctors of M, N, and P follows from the definitions of Θ and Ψ. $\square$

The isomorphism of Proposition 5.7.13 shows that the internal Hom provides a closed structure for $(\mathsf{Multicat}, \otimes, \beta)$. Combining Proposition 5.7.13 with Theorems 5.5.14 and 5.6.18 we have the following.

Theorem 5.7.14. *The category of small multicategories* Multicat *is complete, cocomplete, symmetric monoidal, and closed. The monoidal product is given by the tensor product of Definition 5.6.11 and the closed structure is given by the internal hom of Definition 5.7.2.*

Recalling Example 3.9.2, the symmetric monoidal closed structure for Multicat also makes it tensored and cotensored over itself. Thus we record the following internal $\otimes$-Hom adjunction.

Lemma 5.7.15. *In the context of Proposition 5.7.13, there are isomorphisms of multicategories*

$$\mathsf{Hom}(\mathsf{M}, \mathsf{Hom}(\mathsf{N}, \mathsf{P})) \cong \mathsf{Hom}(\mathsf{M} \otimes \mathsf{N}, \mathsf{P}) \cong \mathsf{Hom}(\mathsf{N}, \mathsf{Hom}(\mathsf{M}, \mathsf{P})) \tag{5.7.16}$$

that are natural in M, N, *and* P.

Restricting to pointed multicategories, we now define a pointed internal hom. This is a special case of Definition 4.2.1.

Definition 5.7.17. Suppose M and N are small pointed multicategories. We define the *pointed hom* multicategory as the following pullback in Multicat.

$$\begin{tikzcd} \mathsf{Hom}_*(\mathsf{M},\mathsf{N}) \arrow[r] \arrow[d] & \mathsf{T} \arrow[d] \\ \mathsf{Hom}(\mathsf{M},\mathsf{N}) \arrow[r] & \mathsf{Hom}(\mathsf{T},\mathsf{N}) \end{tikzcd} \tag{5.7.18}$$

The composite

$$\mathsf{T} \cong \mathsf{Hom}(\mathsf{M}, \mathsf{T}) \longrightarrow \mathsf{Hom}(\mathsf{M}, \mathsf{N}) \longrightarrow \mathsf{Hom}(\mathsf{T}, \mathsf{N})$$

induced by the structure morphisms for M and N is equal to the vertical morphism in the diagram, and therefore induces a canonical structure morphism $\mathsf{T} \longrightarrow \mathsf{Hom}_*(\mathsf{M}, \mathsf{N})$ making $\mathsf{Hom}_*(\mathsf{M}, \mathsf{N})$ a pointed multicategory. ⋄

Explanation 5.7.19. The objects and operations of $\mathsf{Hom}_*(\mathsf{M}, \mathsf{N})$ can be identified as follows. The objects are those multifunctors $\mathsf{M} \longrightarrow \mathsf{N}$ that commute with the structure morphisms from T. The n-ary operations are those operations η in $\mathsf{Hom}(\mathsf{M}, \mathsf{N})$, between pointed multifunctors, such that

$$\eta_{*^{\mathsf{M}}} = \iota_n^{\mathsf{N}},$$

with $*^{\mathsf{M}}$ the basepoint of M. Specializing Explanation 5.7.5, we see that the underlying category of objects and unary operations of $\mathsf{Hom}_*(\mathsf{M}, \mathsf{N})$ is $\mathsf{Multicat}_*(\mathsf{M}, \mathsf{N})$. That is, we have an isomorphism of categories

$$\mathsf{Multicat}(\mathsf{I}, \mathsf{Hom}_*(\mathsf{M}, \mathsf{N})) \cong \mathsf{Multicat}_*(\mathsf{M}, \mathsf{N}). \tag{5.7.20}$$

⋄

The following pointed version of Lemma 5.7.12 will be used in the proof of Proposition 10.6.7.

Lemma 5.7.21. *Suppose* M *is a small pointed multicategory and* $(\mathsf{C}, \oplus, e, \xi)$ *is a small permutative category.*

(1) *The pointwise monoidal product on* $\mathsf{Multicat}(\mathsf{M}, \mathsf{End}(\mathsf{C}))$ *restricts to make*

$$\mathsf{Multicat}_*(\mathsf{M}, \mathsf{End}(\mathsf{C}))$$

a permutative category.

(2) *There is an isomorphism of multicategories*

$$\mathsf{End}\big(\mathsf{Multicat}_*(\mathsf{M}, \mathsf{End}(\mathsf{C}))\big) \cong \mathsf{Hom}_*(\mathsf{M}, \mathsf{End}(\mathsf{C})).$$

Proof. First, if

$$F, F' \in \mathsf{Multicat}_*(\mathsf{M}, \mathsf{End}(\mathsf{C}))$$

then so is $F \oplus F'$, because C is permutative. Likewise, if α and α' are pointed multinatural transformations between pointed multifunctors, then so is $\alpha \oplus \alpha'$. The pointwise monoidal unit, which is the constant multifunctor at the unit $e \in \mathsf{C}$, is a pointed multifunctor. The symmetry isomorphism of

$$\mathsf{Multicat}(\mathsf{M}, \mathsf{End}(\mathsf{C})),$$

induced by that of C, is pointed because C is permutative.

Second, we observe that the correspondence of operations established in the proof of Lemma 5.7.12 restricts to a correspondence between pointed operations. Using Explanation 5.7.19, we see that n-ary operations

$$\eta \in \mathsf{Hom}_*(\mathsf{M}, \mathsf{End}(\mathsf{C}))\big(\langle F \rangle ; F\big)$$

have components

$$\eta_c : \langle F \rangle c = \bigoplus_{F_i \in \langle F \rangle} F_i c \longrightarrow F' c$$

such that

$$\eta_{*\mathsf{M}} = \iota_n = 1_e : \bigoplus_{F_i \in \langle F \rangle} e \longrightarrow e.$$

Since this is precisely the condition for n-ary operations of

$$\mathsf{End}(\mathsf{Multicat}_*(\mathsf{M}, \mathsf{End}(\mathsf{C}))),$$

the result follows. $\square$

Now applying Theorem 4.2.3 to the complete and cocomplete symmetric monoidal closed category

$$(\mathsf{Multicat}, \otimes, \mathsf{I}, \mathsf{Hom})$$

with terminal object T, we have the following.

Theorem 5.7.22. *The category of pointed small multicategories* $\mathsf{Multicat}_*$ *is complete, cocomplete, symmetric monoidal, and closed. The monoidal product is given by the smash product of Definition 5.6.20 with monoidal unit*

$$\mathsf{S} = \mathsf{I} \coprod \mathsf{T}.$$

The closed structure is given by the pointed hom of Definition 5.7.17.

Recall from Corollary 5.3.9 that taking endomorphism multicategories provides a 2-functor

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{End}} \mathsf{Multicat}_*$$

from permutative categories and strictly unital functors to pointed multicategories and pointed multifunctors. The smash product of multicategories does not restrict along this 2-functor, as we now show.

Proposition 5.7.23. *The symmetric monoidal structure on* $\mathsf{Multicat}_*$ *in Theorem 5.7.22 does not restrict to a symmetric monoidal structure on* $\mathsf{PermCat}^{\mathsf{su}}$ *along the 2-functor* End *in Corollary 5.3.9.*

Proof. The monoidal unit in $\mathsf{Multicat}_*$ is $\mathsf{S} = \mathsf{I} \coprod \mathsf{T}$ with object set $\{1, 0\}$, where 1 denotes the unique object of I and 0 denotes the unique object of T, the basepoint of S. The entries of this 2-object multicategory are given by

$$\mathsf{S}\big(\varepsilon_1, \ldots, \varepsilon_n\,;\,\varepsilon\big) = \begin{cases} \mathsf{T}\big(0, \ldots, 0\,;\,0\big) = * & \text{if } \varepsilon_1 = \cdots = \varepsilon_n = \varepsilon = 0,\\ \mathsf{I}\big(1\,;\,1\big) = * & \text{if } \varepsilon_1 = \varepsilon = 1 \text{ with } n = 1, \text{ and}\\ \varnothing & \text{otherwise.}\end{cases}$$

If the symmetric monoidal structure on $\mathsf{Multicat}_*$ restricts to one on $\mathsf{PermCat}^{\mathsf{su}}$ along End, then

$$\mathsf{S} = \mathsf{End}(\mathsf{J})$$

for some permutative category $(\mathsf{J}, \oplus, 0)$ with object set $\{0, 1\}$. But then there are equalities

$$\begin{aligned}\varnothing = \mathsf{S}\big(0, 1\,;\,1\big) &= \mathsf{End}(\mathsf{J})\big(0, 1\,;\,1\big)\\ &= \mathsf{J}(0 \oplus 1; 1) = \mathsf{J}(1; 1) \neq \varnothing,\end{aligned}$$

which cannot happen. □

5.8. Notes

5.8.1 (Multicategories)**.** What we call a multicategory is also called a *symmetric multicategory*, with the plain term *multicategory* reserved for the non-symmetric definition. The terms *operad*, a *symmetric operad*, and a *colored operad* are also common. The book **[Yau16]** is a gentle introduction to multicategories.

Historically, multicategories without symmetric group actions were introduced by Lambek **[Lam69]**. May **[May72]** introduced the term *operad* for a one-object multicategory. The tensor product was introduced by Boardman and Vogt **[BV73]** and proved to give a symmetric monoidal closed structure on the category of small multicategories and multifunctors by Moerdijk and Toen **[MT10]**. Homotopy theory of multicategories is discussed in **[MT10, WY18]**. Applications of multicategories outside of pure mathematics can be found in **[Yau18, Yau20]**. ⋄

5.8.2 (Creating Coequalizers)**.** In the context of Definition 5.4.12 we caution that some authors, notably **[ML98]**, write that *U creates coequalizers* for what we have called *strictly* creating coequalizers in Definition 5.4.12. We have included the extra adjective because a more general condition, without the adjective "strictly", is often used in the literature and is equivalent to the comparison (5.4.7)

$$\mathsf{A} \longrightarrow \mathsf{Alg}(UL)$$

being a mere equivalence of categories. See, for example, **[Rie16**, Section 5] for further development. ⋄

5.8.3 (Monadicity Theorems)**.** The original reference for tripleability/monadicity theorems is the thesis of Beck **[Bec67]**. Our proof of Theorem 5.4.13 sketches that of **[ML98**, VI.7 Theorem 1], which follows **[Bec67]**. Variations and generalizations of Theorem 5.4.13 are given in **[Bec67]** and the later literature. See **[ML98**, Section VI.7] and **[Rie16**, Section 5.5] for further discussion and references. ⋄

5.8.4 (Finite Products and Filtered Colimits)**.** Our proof of Lemma 5.4.20 (2), that finite products commute with filtered colimits in Set, is an adaptation of a more general argument in [**Rie16**, Theorem 3.8.9] showing that general finite limits commute with filtered colimits in Set. The argument is special to the category Set, as it depends on the construction of colimits there. ⋄

5.8.5 (Symmetric Monoidal Closed Structure)**.** Our presentation of the tensor product in Definition 5.6.11 follows that of Elmendorf-Mandell [**EM09**] and is equivalent to the tensor product introduced by Boardman-Vogt [**BV73**]. We refer the reader to [**MT10**] for additional development.

Our presentations of the internal hom and its pointed variant also follow [**EM09**]. A more general properad analogue of Proposition 5.7.13 and Theorem 5.7.14 is given by Hackney-Robertson-Yau in [**HRY15**, Theorem 4.30 and Corollary 4.31]. ⋄

5.8.6 (Closed Symmetric Multicategories)**.** The category $\mathsf{Multicat}$ of small multicategories is closed symmetric monoidal by Theorem 5.7.14. This implies that $\mathsf{Multicat}$ is a *closed symmetric multicategory* in the sense of [**Zak18**, 1.2]. The internal hom objects are the small multicategories

$$\mathsf{Hom}(\mathsf{A}_1 \otimes \cdots \otimes \mathsf{A}_n; \mathsf{B})$$

for $n \geq 0$ and small multicategories $\mathsf{A}_1, \ldots, \mathsf{A}_n, \mathsf{B}$. The evaluation morphism is the second isomorphism in Proposition 5.7.13 with

$$\begin{aligned} \mathsf{M} &= \mathsf{A}_1 \otimes \cdots \otimes \mathsf{A}_n \\ \mathsf{N} &= \mathsf{Hom}(\mathsf{A}_1 \otimes \cdots \otimes \mathsf{A}_n; \mathsf{B}) \\ \mathsf{P} &= \mathsf{B}. \end{aligned}$$

This notion of a closed symmetric multicategory is the symmetric analogue of a *closed multicategory* in the sense of [**Lam69**, p. 106] and [**Man12**, 3.6], which considered multicategories without symmetric group actions. The same remarks also apply to the category of pointed small multicategories $\mathsf{Multicat}_*$, which is closed symmetric monoidal by Theorem 5.7.22 with the pointed hom in Definition 5.7.17. ⋄

CHAPTER 6

Enriched Multicategories

In this chapter we define multicategories enriched over a symmetric monoidal category $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$. To simplify the presentation, we give the definitions under the assumption that V is permutative. These provide the general definitions via strictification, and we describe that process in Explanation 6.1.19 below. This will be important because in our two applications of interest V is symmetric monoidal but not permutative. The first is $\mathsf{V} = \mathsf{Cat}$, and will be discussed below. The second uses $\mathsf{V} = \mathsf{Cat}_*$, the category of small pointed categories, and $\mathsf{V} = \mathsf{sSet}_*$, the category of pointed simplicial sets. The latter will be discussed in Chapter 7.

In Section 6.1 we define the 2-category of V-enriched multicategories, multifunctors, and multinatural transformations. In Section 6.2 we discuss change of enrichment along a symmetric monoidal functor. In Section 6.3 we consider a symmetric monoidal V-category K, as defined in Section 1.4, and show that $\mathsf{End}(\mathsf{K})$ is V-enriched as a multicategory.

In Sections 6.4 through 6.6 we apply the preceding general theory to the case $\mathsf{V} = \mathsf{Cat}$ and $\mathsf{K} = \mathsf{Multicat}_*$. This begins in Section 6.4 where we extend Theorem 5.7.22 to show that $\mathsf{Multicat}_*$ is a symmetric Cat-monoidal 2-category and therefore a Cat-enriched multicategory.

Our main reason for considering the Cat-enriched multicategory structure on $\mathsf{Multicat}_*$, rather than the Cat-monoidal structure from which it arises, is the following. As noted in Proposition 5.7.23, the symmetric monoidal structure of $\mathsf{Multicat}_*$ does not restrict along the 2-functor

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{End}} \mathsf{Multicat}_*$$

of Section 5.3. In contrast, the multicategory structure *does* restrict along inclusions such as End.

We describe this induced multicategory structure in Section 6.5. The objects in this multicategory are small permutative categories. Each category

$$\mathsf{PermCat}^{\mathsf{su}}\left(\langle \mathsf{C} \rangle ; \mathsf{D}\right)$$

has multilinear functors as objects (Definition 6.5.4) and multilinear transformations as morphisms (Definition 6.5.11).

The Cat-enriched multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$ is of fundamental importance for our applications in Part 2 because it encodes the algebraic structures of interest for our K-theory multifunctors. It is the domain of the Elmendorf-Mandell K-theory multifunctor defined in Chapter 10. Therefore, in Section 6.6 we give a second, self-contained and detailed proof that $\mathsf{PermCat}^{\mathsf{su}}$ is a Cat-enriched multicategory. As we explain there, the multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$ is a generalization of the 2-category of small symmetric monoidal categories, symmetric monoidal functors, and monoidal natural transformations. Keeping this connection in mind, the proofs that (i) the various parts of $\mathsf{PermCat}^{\mathsf{su}}$ are well defined and (ii) the Cat-enriched multicategory axioms are satisfied, are not conceptually difficult. However, they require a nontrivial amount of notation to keep track of the many lists of objects. See also Note 6.7.2. By reversing the linearity constraints, the detailed proofs in Section 6.6 can also be used for colax multilinear functors and colax multilinear transformations; see Definition 10.7.24 for $\mathsf{PermCat}^{\mathsf{su}}_{\mathsf{co}}$.

6.1. Enriched Multicategories

Throughout this section we suppose that $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is a permutative category. Recall from Definition 1.1.23 this means that the associativity and unit isomorphisms of V are identities. However in some of the axioms it will be useful to explicitly write the units

$$\lambda : \mathbb{1} \otimes X \longrightarrow X \quad \text{and} \quad \rho : X \otimes \mathbb{1} \longrightarrow X$$

even though they are identities.

Definition 6.1.1. Suppose V is a permutative category. A V-*enriched multicategory* $(\mathsf{M}, \gamma, 1)$ consists of the following data.

- M is equipped with a class $\mathsf{Ob}\,\mathsf{M}$ of *objects*. We write $\mathsf{Prof}(\mathsf{M})$ for $\mathsf{Prof}(\mathsf{Ob}\,\mathsf{M})$.
- For $c' \in \mathsf{Ob}\,\mathsf{M}$ and $\langle c \rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\mathsf{M})$, M is equipped with an object of V
$$\mathsf{M}\big(\langle c \rangle\,; c'\big) = \mathsf{M}\big(c_1, \ldots, c_n\,; c'\big) \in \mathsf{V}$$
called the *n-ary operation object* with *input profile* $\langle c \rangle$ and *output* c'.
- For $\big(\langle c \rangle\,; c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ as above and a permutation $\sigma \in \Sigma_n$, M is equipped with an isomorphism in V
$$\mathsf{M}\big(\langle c \rangle\,; c'\big) \xrightarrow[\cong]{\sigma} \mathsf{M}\big(\langle c \rangle\sigma\,; c'\big),$$
called the *right action* or the *symmetric group action*, in which
$$\langle c \rangle\sigma = \big(c_{\sigma(1)}, \ldots, c_{\sigma(n)}\big)$$
is the right permutation of $\langle c \rangle$ by σ.
- For $c \in \mathsf{Ob}\,\mathsf{M}$, M is equipped with a morphism
$$1_c : \mathbb{1} \longrightarrow \mathsf{M}\big(c\,; c\big),$$
called the *c-colored unit*.
- For $c'' \in \mathsf{Ob}\,\mathsf{M}$, $\langle c' \rangle = (c'_1, \ldots, c'_n) \in \mathsf{Prof}(\mathsf{M})$, and $\langle c_j \rangle = (c_{j,1}, \ldots, c_{j,k_j}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1, \ldots, n\}$, let $\langle c \rangle = \oplus_j \langle c_j \rangle \in \mathsf{Prof}(\mathsf{M})$ be the concatenation of the $\langle c_j \rangle$. Then M is equipped with a morphism in V
$$\mathsf{M}\big(\langle c' \rangle\,; c''\big) \otimes \bigotimes_{j=1}^{n} \mathsf{M}\big(\langle c_j \rangle\,; c'_j\big) \xrightarrow{\gamma} \mathsf{M}\big(\langle c \rangle\,; c''\big) \tag{6.1.2}$$
called the *composition*.

These data are required to satisfy the following axioms.

Symmetric Group Action: For $(\langle c\rangle; c') \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $n = \mathrm{len}\langle c\rangle$ and $\sigma, \tau \in \Sigma_n$, the following diagram in V commutes.

$$\begin{tikzcd}
\mathsf{M}(\langle c\rangle; c') \arrow[r, "\sigma"] \arrow[dr, "\sigma\tau"'] & \mathsf{M}(\langle c\rangle\sigma; c') \arrow[d, "\tau"] \\
& \mathsf{M}(\langle c\rangle\sigma\tau; c')
\end{tikzcd}$$

Moreover, the identity permutation in Σ_n acts as the identity morphism of $\mathsf{M}(\langle c\rangle; c')$.

Associativity: Suppose given

- $c''' \in \mathsf{Ob}\,\mathsf{M}$,
- $\langle c''\rangle = (c''_1, \ldots, c''_n) \in \mathsf{Prof}(\mathsf{M})$,
- $\langle c'_j\rangle = (c'_{j,1}, \ldots, c'_{j,k_j}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1, \ldots, n\}$, and
- $\langle c_{j,i}\rangle = (c_{j,i,1}, \ldots, c_{j,i,\ell_{j,i}}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1, \ldots, n\}$ and each $i \in \{1, \ldots, k_j\}$,

such that $k_j = \mathrm{len}\langle c'_j\rangle > 0$ for at least one j. For each j, let $\langle c_j\rangle = \oplus_{i=1}^{k_j}\langle c_{j,i}\rangle$ denote the concatenation of the $\langle c_{j,i}\rangle$. Let $\langle c\rangle = \oplus_{j=1}^{n}\langle c_j\rangle$ denote the concatenation of the $\langle c_j\rangle$. Let $\langle c'\rangle = \oplus_{j=1}^{n}\langle c'_j\rangle$ denote the concatenation of the $\langle c'_j\rangle$.

Then the *associativity diagram* below commutes.

(6.1.3)

$$\begin{tikzcd}
& \mathsf{M}(\langle c'\rangle; c''') \otimes \bigotimes_{j=1}^{n}\Big[\bigotimes_{i=1}^{k_j} \mathsf{M}(\langle c_{j,i}\rangle; c'_{j,i})\Big] \arrow[dr, "\gamma"] & \\
\mathsf{M}(\langle c''\rangle; c''') \otimes \Big[\bigotimes_{j=1}^{n} \mathsf{M}(\langle c'_j\rangle; c''_j)\Big] \otimes \bigotimes_{j=1}^{n}\Big[\bigotimes_{i=1}^{k_j} \mathsf{M}(\langle c_{j,i}\rangle; c'_{j,i})\Big] \arrow[ur, "{(\gamma,1)}"] \arrow[d, "\text{permute}"', "\cong"] & & \mathsf{M}(\langle c\rangle; c''') \\
\mathsf{M}(\langle c''\rangle; c''') \otimes \bigotimes_{j=1}^{n}\Big[\mathsf{M}(\langle c'_j\rangle; c''_j) \otimes \bigotimes_{i=1}^{k_j} \mathsf{M}(\langle c_{j,i}\rangle; c'_{j,i})\Big] \arrow[dr, "{(1,\otimes_j \gamma)}"'] & & \\
& \mathsf{M}(\langle c''\rangle; c''') \otimes \bigotimes_{j=1}^{n} \mathsf{M}(\langle c_j\rangle; c''_j) \arrow[uur, "\gamma"'] &
\end{tikzcd}$$

Unity: Suppose $c' \in \mathsf{Ob}\,\mathsf{M}$.

(1) If $\langle c\rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\mathsf{M})$ has length $n \geq 1$, then the following *right unity diagram* is commutative. Here $\mathbb{1}^n$ is the n-fold monoidal product of $\mathbb{1}$ with itself and the unlabeled morphism is given by a

composite of (strict) units in V.

$$\begin{tikzcd} \mathsf{M}\big(\langle c\rangle\,;c'\big)\otimes\mathbb{1}^n \arrow[r] \arrow[d, "1\otimes(\otimes_j 1_{c_j})"'] & \mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[d, "1"] \\ \mathsf{M}\big(\langle c\rangle\,;c'\big)\otimes\bigotimes_{j=1}^n \mathsf{M}\big(c_j\,;c_j\big) \arrow[r, "\gamma"] & \mathsf{M}\big(\langle c\rangle\,;c'\big) \end{tikzcd} \tag{6.1.4}$$

(2) For any $\langle c\rangle \in \mathsf{Prof}(\mathsf{M})$, the *left unity diagram* below is commutative, where the unlabeled morphism is given by a (strict) unit in V.

$$\begin{tikzcd} \mathbb{1}\otimes\mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[r] \arrow[d, "1_{c'}\otimes 1"'] & \mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[d, "1"] \\ \mathsf{M}\big(c'\,;c'\big)\otimes\mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[r, "\gamma"] & \mathsf{M}\big(\langle c\rangle\,;c'\big) \end{tikzcd} \tag{6.1.5}$$

Equivariance: Suppose that in the definition of γ (6.1.2), $\mathrm{len}\langle c_j\rangle = k_j \geq 0$.

(1) For each $\sigma \in \Sigma_n$, the following *top equivariance diagram* is commutative.

(6.1.6)
$$\begin{tikzcd} \mathsf{M}\big(\langle c'\rangle\,;c''\big)\otimes\bigotimes_{j=1}^n \mathsf{M}\big(\langle c_j\rangle\,;c_j'\big) \arrow[r, "(\sigma,\sigma^{-1})"] \arrow[d, "\gamma"'] & \mathsf{M}\big(\langle c'\rangle\sigma\,;c''\big)\otimes\bigotimes_{j=1}^n \mathsf{M}\big(\langle c_{\sigma(j)}\rangle\,;c_{\sigma(j)}'\big) \arrow[d, "\gamma"] \\ \mathsf{M}\big(\langle c_1\rangle,\ldots,\langle c_n\rangle\,;c''\big) \arrow[r, "\sigma\langle k_{\sigma(1)},\ldots,k_{\sigma(n)}\rangle"] & \mathsf{M}\big(\langle c_{\sigma(1)}\rangle,\ldots,\langle c_{\sigma(n)}\rangle\,;c''\big) \end{tikzcd}$$

Here $\sigma\langle k_{\sigma(1)},\ldots,k_{\sigma(n)}\rangle \in \Sigma_{k_1+\cdots+k_n}$ is right action of the block permutation (II.1.1.19) that permutes the n consecutive blocks of lengths $k_{\sigma(1)}$, …, $k_{\sigma(n)}$ as σ permutes $\{1,\ldots,n\}$, leaving the relative order within each block unchanged.

(2) Given permutations $\tau_j \in \Sigma_{k_j}$ for $1 \leq j \leq n$, the following *bottom equivariance diagram* is commutative.

$$\begin{tikzcd} \mathsf{M}\big(\langle c'\rangle\,;c''\big)\otimes\bigotimes_{j=1}^n \mathsf{M}\big(\langle c_j\rangle\,;c_j'\big) \arrow[r, "(1,\otimes_j\tau_j)"] \arrow[d, "\gamma"'] & \mathsf{M}\big(\langle c'\rangle\,;c''\big)\otimes\bigotimes_{j=1}^n \mathsf{M}\big(\langle c_j\rangle\tau_j\,;c_j'\big) \arrow[d, "\gamma"] \\ \mathsf{M}\big(\langle c_1\rangle,\ldots,\langle c_n\rangle\,;c''\big) \arrow[r, "\tau_1\times\cdots\times\tau_n"] & \mathsf{M}\big(\langle c_1\rangle\tau_1,\ldots,\langle c_n\rangle\tau_n\,;c''\big) \end{tikzcd} \tag{6.1.7}$$

Here the block sum $\tau_1\times\cdots\times\tau_n \in \Sigma_{k_1+\cdots+k_n}$ (II.1.1.8) is the image of $(\tau_1,\ldots,\tau_n)$ under the canonical inclusion

$$\Sigma_{k_1}\times\cdots\times\Sigma_{k_n} \longrightarrow \Sigma_{k_1+\cdots+k_n}.$$

This finishes the definition of a V-enriched multicategory. A V-enriched multicategory is *small* if its class of objects is a set. ◇

Definition 6.1.8. A V-enriched multicategory with only one object is called a V-*enriched operad*. If M is a V-enriched operad, then its object of n-ary operations is denoted by $\mathsf{M}_n \in \mathsf{V}$. ◇

Example 6.1.9. Suppose M is a V-enriched multicategory and c is an object of M. Then $\mathsf{End}(c)$ is the V-enriched operad consisting of the single object c and n-ary operation object

$$\mathsf{End}(c)_n = \mathsf{M}\big(\langle c\rangle\, ;\, c\big) \in \mathsf{V},$$

where $\langle c\rangle$ denotes the constant n-tuple at c. The symmetric group action, unit, and composition of $\mathsf{End}(c)$ are given by those of M. ⋄

Definition 6.1.10. A V-*enriched multifunctor* $F : \mathsf{M} \longrightarrow \mathsf{N}$ between V-enriched multicategories M and N consists of the following data:

- an assignment
$$F : \mathsf{Ob}\,\mathsf{M} \longrightarrow \mathsf{Ob}\,\mathsf{N},$$
where $\mathsf{Ob}\,\mathsf{M}$ and $\mathsf{Ob}\,\mathsf{N}$ are the classes of objects of M and N, respectively, and
- for each $\big(\langle c\rangle\, ;\, c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $\langle c\rangle = (c_1, \ldots, c_n)$, a morphism in V
$$F : \mathsf{M}\big(\langle c\rangle\, ;\, c'\big) \longrightarrow \mathsf{N}\big(F\langle c\rangle\, ;\, Fc'\big),$$
where $F\langle c\rangle = (Fc_1, \ldots, Fc_n)$.

These data are required to preserve the symmetric group action, the colored units, and the composition in the following sense.

Symmetric Group Action: For each $\big(\langle c\rangle\, ;\, c'\big)$ as above and each permutation $\sigma \in \Sigma_n$, the following diagram in V is commutative.

$$\begin{tikzcd}
\mathsf{M}\big(\langle c\rangle\, ;\, c'\big) \arrow[r, "F"] \arrow[d, "\sigma"', "\cong"] & \mathsf{N}\big(F\langle c\rangle\, ;\, Fc'\big) \arrow[d, "\sigma"', "\cong"] \\
\mathsf{M}\big(\langle c\rangle\sigma\, ;\, c'\big) \arrow[r, "F"] & \mathsf{N}\big(F\langle c\rangle\sigma\, ;\, c'\big)
\end{tikzcd} \tag{6.1.11}$$

Units: For each $c \in \mathsf{Ob}\,\mathsf{M}$, the following diagram in V is commutative.

$$\begin{tikzcd}
& \mathsf{M}\big(c\, ;\, c\big) \arrow[d, "F"] \\
\mathbb{1} \arrow[ur, "1_c"] \arrow[r, "1_{Fc}"'] & \mathsf{N}\big(Fc\, ;\, Fc\big)
\end{tikzcd} \tag{6.1.12}$$

Composition: For c'', $\langle c'_j\rangle$, and $\langle c\rangle = \oplus_j \langle c_j\rangle$ as in the definition of γ (5.1.3), the following diagram in V is commutative.

$$\begin{tikzcd}
\mathsf{M}\big(\langle c'\rangle\, ;\, c''\big) \otimes \bigotimes_{j=1}^{n} \mathsf{M}\big(\langle c_j\rangle\, ;\, c'_j\big) \arrow[r, "{(F,\otimes_j F)}"] \arrow[d, "\gamma"'] & \mathsf{N}\big(F\langle c'\rangle\, ;\, Fc''\big) \otimes \bigotimes_{j=1}^{n} \mathsf{N}\big(F\langle c_j\rangle\, ;\, Fc'_j\big) \arrow[d, "\gamma"] \\
\mathsf{M}\big(\langle c\rangle\, ;\, c''\big) \arrow[r, "F"] & \mathsf{N}\big(F\langle c\rangle\, ;\, Fc''\big)
\end{tikzcd} \tag{6.1.13}$$

This finishes the definition of a V-enriched multifunctor.

Moreover:

(1) For another V-enriched multifunctor $G : \mathsf{N} \longrightarrow \mathsf{P}$ between V-enriched multicategories, where P has object class $\mathsf{Ob}\,\mathsf{P}$, the *composition* GF :

$\mathsf{M} \longrightarrow \mathsf{P}$ is the V-enriched multifunctor defined by composing the assignments on objects

$$\operatorname{Ob}\mathsf{M} \xrightarrow{F} \operatorname{Ob}\mathsf{N} \xrightarrow{G} \operatorname{Ob}\mathsf{P}$$

and the morphisms on n-ary operations

$$\mathsf{M}\big(\langle c\rangle\,;c'\big) \xrightarrow{F} \mathsf{N}\big(F\langle c\rangle\,;Fc'\big) \xrightarrow{G} \mathsf{P}\big(GF\langle c\rangle\,;GFc'\big).$$

(2) The *identity* V-*enriched multifunctor* $1_{\mathsf{M}} : \mathsf{M} \longrightarrow \mathsf{M}$ is defined by the identity assignment on objects and the identity morphism on n-ary operations.

(3) A V-*enriched operad morphism* is a V-enriched multifunctor between two V-enriched multicategories with one object.

The proof of Lemma 5.1.16 generalizes to the V-enriched case, with $\otimes$ in place of $\times$, and shows that composition of V-enriched multifunctors is well defined, associative, and unital with respect to identity multifunctors. $\diamond$

Definition 6.1.14. Suppose P is a V-enriched operad and M is a V-enriched multicategory. A P-*algebra* in M is a pair

$$(c,\theta)$$

consisting of an object c in M and a V-enriched multifunctor

$$\theta : \mathsf{P} \longrightarrow \mathsf{M}$$

that sends the single object of P to c. Equivalently, θ is a V-enriched operad morphism

$$\theta : \mathsf{P} \longrightarrow \mathsf{End}(c). \qquad \diamond$$

Definition 6.1.15. Suppose $F, G : \mathsf{M} \longrightarrow \mathsf{N}$ are V-enriched multifunctors as in Definition 6.1.10. A V-*enriched multinatural transformation* $\alpha : F \longrightarrow G$ consists of morphisms in V

$$\alpha_c : \mathbb{1} \longrightarrow \mathsf{N}\big(Fc\,;Gc\big) \quad \text{for} \quad c \in \operatorname{Ob}\mathsf{M}$$

such that the following V-*naturality diagram* in V commutes for each $\big(\langle c\rangle\,;c'\big) \in \operatorname{Prof}(\mathsf{M}) \times \operatorname{Ob}\mathsf{M}$ with $\langle c\rangle = (c_1,\ldots,c_n)$.

$$\begin{tikzcd}
& \mathbb{1} \otimes \mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[r, "\alpha_{c'} \otimes F"] & \mathsf{N}\big(Fc'\,;Gc'\big) \otimes \mathsf{N}\big(F\langle c\rangle\,;Fc'\big) \arrow[dr, "\gamma"] & \\
\mathsf{M}\big(\langle c\rangle\,;c'\big) \arrow[ur, "\lambda^{-1}", "1"'] \arrow[dr, "1"] & & & \mathsf{N}\big(F\langle c\rangle\,;Gc'\big) \\
& \mathsf{M}\big(\langle c\rangle\,;c'\big) \otimes \bigotimes_{j=1}^{n} \mathbb{1} \arrow[r, "G \otimes \bigotimes_{j=1}^{n} \alpha_{c_j}"] & \mathsf{N}\big(G\langle c\rangle\,;Gc'\big) \otimes \bigotimes_{j=1}^{n} \mathsf{N}\big(Fc_j\,;Gc_j\big) \arrow[ur, "\gamma"'] &
\end{tikzcd} \tag{6.1.16}$$

This finishes the definition of a V-enriched multinatural transformation.

- Each α_c is called a *component* of α.

- The *identity* V*-enriched multinatural transformation* $1_F : F \longrightarrow F$ has components

$$(1_F)_c = 1_{Fc} \quad \text{for} \quad c \in \mathrm{Ob}\,\mathsf{M}. \qquad \diamond$$

Definition 6.1.17. Suppose $\alpha : F \longrightarrow G$ is a V-enriched multinatural transformation between V-enriched multifunctors as in Definition 6.1.15.

(1) Suppose $\beta : G \longrightarrow H$ is a V-enriched multinatural transformation for a V-enriched multifunctor $H : \mathsf{M} \longrightarrow \mathsf{N}$. The *vertical composition*

$$\beta\alpha : F \longrightarrow H$$

is the V-enriched multinatural transformation with components at $c \in \mathrm{Ob}\,\mathsf{M}$ given by the following composites in V.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "(\beta\alpha)_c"] \arrow[d, "\lambda^{-1}"', "1"] & \mathsf{N}(Fc\,;Hc) \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\beta_c \otimes \alpha_c"] & \mathsf{N}(Gc\,;Hc) \otimes \mathsf{N}(Fc\,;Gc) \arrow[u, "\gamma"']
\end{tikzcd}$$

(2) Suppose $\alpha' : F' \longrightarrow G'$ is a V-enriched multinatural transformation for V-enriched multifunctors $F', G' : \mathsf{N} \longrightarrow \mathsf{P}$. The *horizontal composition*

$$\alpha' * \alpha : F'F \longrightarrow G'G$$

is the V-enriched multinatural transformation with components at $c \in \mathrm{Ob}\,\mathsf{M}$ given by the following composites in V.

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "(\alpha' * \alpha)_c"] \arrow[dd, "\lambda^{-1}"', "1"] & \mathsf{P}(F'Fc\,;G'Gc) \\
& \mathsf{P}(F'Gc\,;G'Gc) \otimes \mathsf{P}(F'Fc\,;F'Gc) \arrow[u, "\gamma"'] \\
\mathbb{1} \otimes \mathbb{1} \arrow[r, "\alpha'_{Gc} \otimes \alpha_c"] & \mathsf{P}(F'Gc\,;G'Gc) \otimes \mathsf{N}(Fc\,;Gc) \arrow[u, "1 \otimes F'"']
\end{tikzcd}$$

$\diamond$

Theorem 6.1.18. *There is a 2-category* V-Multicat *consisting of the following data.*

- *Its objects are small* V*-enriched multicategories.*
- *For small* V*-enriched multicategories* M *and* N*, the hom category* V-Multicat(M, N) *has:*
  - V*-enriched multifunctors* $\mathsf{M} \longrightarrow \mathsf{N}$ *as 1-cells;*
  - V*-enriched multinatural transformations as 2-cells;*
  - *vertical composition as composition; and*
  - *identity* V*-enriched multinatural transformations as identity 2-cells.*
- *The identity 1-cell* 1_{M} *is the identity* V*-enriched multifunctor* 1_{M}*.*
- *Horizontal composition of 1-cells is the composition of* V*-enriched multifunctors.*
- *Horizontal composition of 2-cells is that of* V*-enriched multinatural transformations.*

Explanation 6.1.19 (Enrichment over Non-Strict V)**.** We extend the definitions in this and subsequent sections to general symmetric monoidal V by using the same data and then, in each axiom, choosing an association of iterated monoidal products and inserting associativity isomorphisms so that the new diagrams correspond to the given ones under the symmetric strictification

$$\mathsf{V} \longrightarrow \mathsf{V}_{\mathsf{st}}$$

of Theorem 1.1.42. Because strictification is an equivalence, the strict diagrams commute if and only if their preimages in V commute. ◇

6.2. Change of Enriching Categories

In this section we discuss change of enrichment along a symmetric monoidal functor

$$U : \mathsf{V} \longrightarrow \mathsf{W}.$$

We continue the assumption that V and W are permutative categories. The definitions and results of this section are extended to general V and W via strictification as in Explanation 6.1.19, replacing U with the composite

$$\mathsf{V}_{\mathsf{st}} \xrightarrow{R} \mathsf{V} \xrightarrow{U} \mathsf{W} \xrightarrow{L} \mathsf{W}_{\mathsf{st}}.$$

Definition 6.2.1. Suppose $(\mathsf{V}, \otimes, \mathbb{1})$ and $(\mathsf{W}, \otimes, \mathbb{1})$ are permutative categories and

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a symmetric monoidal functor. For a V-enriched multicategory M we define a W-enriched multicategory M_U with the following data.

- The objects of M_U are those of M.
- The n-ary operation objects are given by

  $$\mathsf{M}_U\big(\langle c\rangle\,;c'\big) = U\mathsf{M}\big(\langle c\rangle\,;c'\big)$$

  for $c' \in \mathsf{Ob}\,\mathsf{M}$ and $\langle c\rangle \in \mathsf{Prof}(\mathsf{M})$.
- The symmetric group action is given by

  $$U\mathsf{M}\big(\langle c\rangle\,;c'\big) \xrightarrow{U\sigma} U\mathsf{M}\big(\langle c\rangle\sigma\,;c'\big)$$

  for $\sigma \in \Sigma_n$ where n is the length of $\langle c\rangle$.
- The c-colored unit for $c \in \mathsf{Ob}\,\mathsf{M}$ is given by

  $$\mathbb{1} \xrightarrow{U^0} U\mathbb{1} \xrightarrow{U1_c} U\mathsf{M}\big(c\,;c\big).$$

- The composition for M_U is given as follows for $c'' \in \mathsf{Ob}\,\mathsf{M}$, $\langle c'\rangle \in \mathsf{Prof}(\mathsf{M})$, and $\langle c_j\rangle \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1,\ldots,n\}$. Let $\langle c\rangle = \oplus_j \langle c_j\rangle \in \mathsf{Prof}(\mathsf{M})$ be the concatenation of the $\langle c_j\rangle$. Then the composition of M_U is given by

$$\begin{tikzcd}
U\mathsf{M}\big(\langle c'\rangle\,;c''\big) \otimes \bigotimes_{j=1}^{n} U\mathsf{M}\big(\langle c_j\rangle\,;c'_j\big) \arrow[d] & \\
U\Big(\mathsf{M}\big(\langle c'\rangle\,;c''\big) \otimes \bigotimes_{j=1}^{n} \mathsf{M}\big(\langle c_j\rangle\,;c'_j\big)\Big) \arrow[r, "U\gamma"] & U\mathsf{M}\big(\langle c\rangle\,;c''\big)
\end{tikzcd} \tag{6.2.2}$$

where the vertical morphism is the U-coherent map given by iterates of U^2 and is unique by Epstien's Coherence Theorem 1.1.44. $\diamond$

Proposition 6.2.3. *In the context of Definition 6.2.1, M_U is a W-enriched multicategory.*

Proof. Each of the axioms of Definition 6.1.1 for M_U follows by applying U to the corresponding axiom for M and then using naturality of U^2 and symmetric monoidal axioms of U. $\square$

Definition 6.2.4. Suppose $(\mathsf{V}, \otimes, \mathbb{1})$ and $(\mathsf{W}, \otimes, \mathbb{1})$ are permutative categories and

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a symmetric monoidal functor. Suppose M and N are V-enriched multicategories and

$$F : \mathsf{M} \longrightarrow \mathsf{N}$$

is a V-enriched multifunctor. We define a W-enriched multifunctor

$$F_U : \mathsf{M}_U \longrightarrow \mathsf{N}_U$$

with the following data.

- The assignment on objects is given by that of F.
- The morphism on operation objects is given by

$$UF : U\mathsf{M}\big(\langle c \rangle ; c'\big) \longrightarrow U\mathsf{N}\big(F\langle c \rangle ; Fc'\big)$$

for $c \in \operatorname{Ob}\mathsf{M}$ and $\langle c \rangle \in \operatorname{Prof}(\mathsf{M})$. $\diamond$

Proposition 6.2.5. *In the context of Definition 6.2.4, F_U is a W-enriched multifunctor.*

Proof. The symmetric group action diagram (6.1.11) and unit diagram (6.1.12) for F_U both commute by functoriality of U and the corresponding diagrams for F. For the composition diagram (6.1.13), first note that U^2, and hence also the U-coherent maps defining composition in M_U and N_U (6.2.2), are natural with respect to morphisms in V. Then the composition diagram for F_U commutes by applying U to the corresponding diagram for F, using functoriality of U and naturality of the U-coherent maps in (6.2.2). $\square$

Definition 6.2.6. Suppose $(\mathsf{V}, \otimes, \mathbb{1})$ and $(\mathsf{W}, \otimes, \mathbb{1})$ are permutative categories and

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a symmetric monoidal functor. Suppose M and N are V-enriched multicategories and

$$\alpha : F \longrightarrow G : \mathsf{M} \longrightarrow \mathsf{N}$$

are V-enriched multifunctors and a V-enriched multinatural transformation, respectively. We define a W-enriched multinatural transformation

$$\alpha_U : F_U \longrightarrow G_U$$

with components

$$\mathbb{1} \xrightarrow{U^0} U\mathbb{1} \xrightarrow{U\alpha} U\mathsf{N}\big(Fc ; Gc\big). \quad \diamond$$

Proposition 6.2.7. *In the context of Definition 6.2.6, α_U is a W-enriched multinatural transformation.*

Proof. Verification that the W-naturality diagram (6.1.16) holds for α_U is similar to the proof of Proposition 6.2.5 for F_U. We apply U to the V-naturality diagram for α and then use functoriality of U and naturality of U^0 and of the U-coherent maps appearing in (6.2.2) for M_U and N_U. □

Definition 6.2.8. Suppose V and W are permutative categories and

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

is a symmetric monoidal functor. Then there is a *change of enrichment* 2-functor

$$(-)_U : \mathsf{V}\text{-}\mathsf{Multicat} \longrightarrow \mathsf{W}\text{-}\mathsf{Multicat}$$

given by Definitions 6.2.1, 6.2.4, and 6.2.6. ◇

Proposition 6.2.9. *In the context of Definition 6.2.8,*

$$(-)_U : \mathsf{V}\text{-}\mathsf{Multicat} \longrightarrow \mathsf{W}\text{-}\mathsf{Multicat}$$

is a 2-functor.

Proof. Change of enrichment preserves identities and composition of multifunctors by functoriality of U. Preservation of identity multinatural transformations holds by the definition of identity operations via composition with U^0. Horizontal and vertical composition of multinatural transformations are preserved by functoriality of U and naturality of U^0 and U^2. □

6.3. Enriched Endomorphism Multicategories

The two main types of enriched multicategories we will consider are enriched operads and those arising from enriched symmetric monoidal categories. We describe the latter in this section.

Again throughout this section we suppose that $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ is a permutative category. However, as outlined in Explanation 6.1.19, we will implicitly extend the definitions here to general symmetric monoidal V via strictification.

Convention 6.3.1 (Left Normalized Iterated Products)**.** Suppose K is a monoidal V-category and $\langle X \rangle = (X_1, \ldots, X_n)$ is a tuple of objects of K. Recall from Definition 2.5.3 the *left normalized product* is defined to be

$$\boxtimes\langle X \rangle = (\cdots((X_1 \boxtimes X_2) \boxtimes X_3)\cdots) \boxtimes X_n.$$

For tuples of objects $\langle X_j \rangle = (X_{j,1}, \ldots, X_{j,k_j})$ with $j \in \{1, \ldots, n\}$, let $\langle X \rangle$ denote their concatenation. By the coherence for monoidal V-categories, Theorem 2.5.6, there is a unique canonical V-map giving an isomorphism

$$\boxtimes\langle X \rangle \xrightarrow{\cong} \mathop{\boxtimes}_{j=1}^{n} \Big(\mathop{\boxtimes}_{i=1}^{k_j} X_{j,i} \Big).$$

We call this the *normalization map* and write

$$\mathsf{K}\Big(\mathop{\boxtimes}_{j=1}^{n} \Big(\mathop{\boxtimes}_{i=1}^{k_j} X_{j,i} \Big), Y\Big) \xrightarrow[\mathsf{norm}]{\cong} \mathsf{K}(\boxtimes\langle X \rangle, Y)$$

for the induced morphism on hom objects. ◇

Example 6.3.2. In the context of Convention 6.3.1, if $n = 3$ and $(k_1, k_2, k_3) = (2, 2, 1)$, then the normalization map at left below is given by an (inverse) associator. If $n = 3$ and $(k_1, k_2, k_3) = (2, 0, 1)$, then the normalization map at right below is given by a right unitor.

$$\begin{tikzcd}
((X_{1,1} \boxtimes X_{1,2}) \boxtimes (X_{2,1} \boxtimes X_{2,2})) \boxtimes X_{3,1} \arrow[d, "\mathsf{norm}"', "\cong"] \\
(((X_{1,1} \boxtimes X_{1,2}) \boxtimes X_{2,1}) \boxtimes X_{2,2}) \boxtimes X_{3,1}
\end{tikzcd}
\qquad
\begin{tikzcd}
((X_{1,1} \boxtimes X_{1,2}) \boxtimes \mathrm{I}) \boxtimes X_{3,1} \arrow[d, "\mathsf{norm}"', "\cong"] \\
(X_{1,1} \boxtimes X_{1,2}) \boxtimes X_{3,1}
\end{tikzcd}$$

◇

Definition 6.3.3 (Enriched Endomorphism Multicategory)**.** Suppose that V is a permutative category and K is a symmetric monoidal V-category. The V-*enriched endomorphism multicategory* of K, denoted $\mathsf{End}(\mathsf{K})$, is a V-enriched multicategory defined as follows.

- The objects are those of K.
- For an object $X' \in \mathsf{K}$ and a tuple $\langle X \rangle \in \mathsf{Prof}(\mathsf{K})$, we define the V-object of operations

$$\mathsf{End}(\mathsf{K})\big(\langle X \rangle ; X'\big) = \mathsf{K}(\boxtimes\langle X \rangle, X'),$$

  where $\boxtimes\langle X \rangle$ denotes the left normalized product.
- For $\langle X \rangle$ and X' as above and a permutation $\sigma \in \Sigma_n$, the right action of σ is defined as the following composite.

$$\begin{tikzcd}
\mathsf{K}(\boxtimes\langle X \rangle, X') \arrow[rr, "\sigma"] \arrow[d, "\rho^{-1}"'] & & \mathsf{K}(\boxtimes\langle X \rangle\sigma, X') \\
\mathsf{K}(\boxtimes\langle X \rangle, X') \otimes \mathbb{1} \arrow[r, "1 \otimes \beta_\sigma"] & \mathsf{K}(\boxtimes\langle X \rangle, X') \otimes \mathsf{K}(\boxtimes\langle X \rangle\sigma, \boxtimes\langle X \rangle) \arrow[ur, "m"'] &
\end{tikzcd} \tag{6.3.4}$$

  In the above diagram, β_σ denotes the V-natural isomorphism that permutes coordinates according to the permutation σ. The existence and uniqueness of such a V-natural isomorphism follows from the symmetric form of coherence, Theorem 2.5.6.
- For each $X \in \mathsf{K}$ the unit of $\mathsf{End}(\mathsf{K})$ is given by that of K,

$$1_X : \mathbb{1} \longrightarrow \mathsf{K}(X, X) = \mathsf{End}(\mathsf{K})\big(X ; X\big).$$

- The composition γ in $\mathsf{End}(\mathsf{K})$ is defined as the following composite for tuples of objects X'', $\langle X' \rangle = (X'_1, \ldots, X'_n)$, and $\langle X_j \rangle = (X_{j,1}, \ldots, X_{j,k_j})$ with

$j \in \{1, \ldots, n\}$ and with $\langle X \rangle$ being the concatenation of the $\langle X_j \rangle$.

$$\begin{tikzcd}
\mathsf{K}(\boxtimes\langle X'\rangle, X'') \otimes \bigotimes_{j=1}^{n} \mathsf{K}(\boxtimes_{i=1}^{k_j} X_{j,i}, X'_j) \arrow[r, "\gamma"] \arrow[d, "1 \otimes \boxtimes_{j=1}^{n-1}"'] & \mathsf{K}(\boxtimes\langle X\rangle, X'') \\
\mathsf{K}(\boxtimes\langle X'\rangle, X'') \otimes \mathsf{K}(\boxtimes_{j=1}^{n} \boxtimes_{i=1}^{k_j} X_{j,i}, \boxtimes\langle X'\rangle) \arrow[r, "m"] & \mathsf{K}(\boxtimes_{j=1}^{n} \boxtimes_{i=1}^{k_j} X_{j,i}, X'') \arrow[u, "\mathsf{norm}", "\cong"']
\end{tikzcd} \tag{6.3.5}$$

◇

Proposition 6.3.6. *Suppose that* V *is a permutative category and* K *is a symmetric monoidal* V*-category. Then* $\mathsf{End}(\mathsf{K})$ *is a* V*-enriched multicategory.*

Proof. We will verify commutativity of the associativity diagram (6.1.3) for $\mathsf{End}(\mathsf{K})$. The other axioms are similar. We use the following notation for W, Z_j, Y_{ji}, and X_{jik} objects of K with $j \in \{1, \ldots, n\}$, $i \in 1, \ldots, m_j$, and $k \in \{1, \ldots, \ell_{ji}\}$.

$$\begin{aligned}
\mathsf{E}^{ZW} &= \mathsf{K}(\boxtimes_{j=1}^{n} Z_j, W) & \mathsf{E}^{YZ} &= \mathsf{K}(\boxtimes_{j=1}^{n} \boxtimes_{i=1}^{m_j} Y_{ji}, \boxtimes_{j=1}^{n} Z_j) \\
\mathsf{E}^{YZ}_j &= \mathsf{K}(\boxtimes_{i=1}^{m_j} Y_{ji}, Z_j) & \mathsf{E}^{YW} &= \mathsf{K}(\boxtimes_{j=1}^{n} \boxtimes_{i=1}^{m_j} Y_{ji}, W) \\
\mathsf{E}^{XY}_{ji} &= \mathsf{K}(\boxtimes_{k=1}^{\ell_{ji}} X_{jik}, Y_{ji}) & \mathsf{E}^{XY} &= \mathsf{K}(\boxtimes_{j=1}^{n} \boxtimes_{i=1}^{m_j} \boxtimes_{k=1}^{\ell_{ji}} X_{jik}, \boxtimes_{j=1}^{n} \boxtimes_{i=1}^{m_j} Y_{ji}) \\
\mathsf{E}^{XY}_j &= \mathsf{K}(\boxtimes_{i=1}^{m_j} \boxtimes_{k=1}^{\ell_{ji}} X_{jik}, \boxtimes_{i=1}^{m_j} Y_{ji}) & \mathsf{E}^{XZ}_j &= \mathsf{K}(\boxtimes_{i=1}^{m_j} \boxtimes_{k=1}^{\ell_{ji}} X_{jik}, Z_j) \\
\mathsf{E}^{XZ} &= \mathsf{K}(\boxtimes_{j=1}^{n} \boxtimes_{i=1}^{m_j} \boxtimes_{k=1}^{\ell_{ji}} X_{jik}, \boxtimes_{j=1}^{n} Z_j) & \mathsf{E}^{XW} &= \mathsf{K}(\boxtimes_{j=1}^{n} \boxtimes_{i=1}^{m_j} \boxtimes_{k=1}^{\ell_{ji}} X_{jik}, W)
\end{aligned}$$

Thus, for example, we have

$$\mathsf{End}(\mathsf{K})\big(\langle Z \rangle ; W\big) = \mathsf{K}(\boxtimes\langle Z\rangle, W) = \mathsf{E}^{ZW}.$$

Moreover, we use the following notation.

- The concatenation of the $\langle Y_j \rangle$ is denoted $\langle Y \rangle$ and
$$\mathsf{E}^{YW}_{\mathsf{norm}} = \mathsf{End}(\mathsf{K})(\boxtimes\langle Y\rangle, W).$$
- For each j, the concatenation of the $\langle X_{ji} \rangle$ is denoted $\langle X_j \rangle$ and
$$\mathsf{E}^{XZ}_{j\,\mathsf{norm}} = \mathsf{End}(\mathsf{K})(\boxtimes\langle X_j\rangle, Z_j).$$
- The concatenation of the $\langle X_j \rangle$ is denoted $\langle X \rangle$ and
$$\mathsf{E}^{XW}_{\mathsf{norm}} = \mathsf{End}(\mathsf{K})(\boxtimes\langle X\rangle, W).$$

The associativity diagram for $\mathsf{End}(\mathsf{K})$ is the outer diagram below, where the left and right vertical composites are, by definition, $\gamma \otimes 1$ and $1 \otimes \bigotimes_j \gamma$, respectively.

$$
\begin{tikzcd}
\mathsf{E}^{ZW} \otimes \bigotimes\limits_{j=1}^{n} \mathsf{E}_j^{YZ} \otimes \bigotimes\limits_{j=1}^{n} \Bigl[\bigotimes\limits_{i=1}^{k_j} \mathsf{E}_{j\,i}^{XY}\Bigr] \arrow[rrrr, "\cong", "\text{permute}"'] \arrow[dd, "1 \otimes \boxtimes \otimes 1"] \arrow[drr, "1 \otimes 1 \otimes \bigotimes_j \boxtimes"] & & & & \mathsf{E}^{ZW} \otimes \bigotimes\limits_{j=1}^{n} \Bigl[\mathsf{E}_j^{YZ} \otimes \bigotimes\limits_{i=1}^{k_j} \mathsf{E}_{j\,i}^{XY}\Bigr] \arrow[dd, "1 \otimes \bigotimes_j {[1 \otimes \boxtimes]}"'] \\
& & \mathsf{E}^{ZW} \otimes \bigotimes\limits_{j=1}^{n} \mathsf{E}_j^{YZ} \otimes \bigotimes\limits_{j=1}^{n} \mathsf{E}_j^{XY} \arrow[drr, "\cong", "\text{permute}"'] \arrow[dd, "1 \otimes \boxtimes \otimes \boxtimes"'] \arrow[urr, phantom, "(1)"] & & \\
\mathsf{E}^{ZW} \otimes \mathsf{E}^{YZ} \otimes \bigotimes\limits_{j=1}^{n} \Bigl[\bigotimes\limits_{i=1}^{k_j} \mathsf{E}_{j\,i}^{XY}\Bigr] \arrow[dd, "m \otimes 1"] \arrow[drr, "1 \otimes 1 \otimes \boxtimes"] \arrow[urr, phantom, "(2)"] & & & & \mathsf{E}^{ZW} \otimes \bigotimes\limits_{j=1}^{n} \Bigl[\mathsf{E}_j^{YZ} \otimes \mathsf{E}_j^{XY}\Bigr] \arrow[dd, "1 \otimes \bigotimes_j m"'] \\
& & \mathsf{E}^{ZW} \otimes \mathsf{E}^{YZ} \otimes \mathsf{E}^{XY} \arrow[ddl, "m \otimes 1"'] \arrow[ddr, "1 \otimes m"] \arrow[urr, phantom, "(3)"] \arrow[dd, phantom, "(4)"] & & \\
\mathsf{E}^{YW} \otimes \bigotimes\limits_{j=1}^{n} \Bigl[\bigotimes\limits_{i=1}^{k_j} \mathsf{E}_{j\,i}^{XY}\Bigr] \arrow[dr, "1 \otimes \boxtimes"] \arrow[ddd, "\cong"', "\mathsf{norm} \otimes 1"] \arrow[urr, phantom, "(2)"] & & & & \mathsf{E}^{ZW} \otimes \bigotimes\limits_{j=1}^{n} \mathsf{E}_j^{XZ} \arrow[dl, "1 \otimes \boxtimes"'] \arrow[ddd, "1 \otimes \mathsf{norm}"', "\cong"] \\
& \mathsf{E}^{YW} \otimes \mathsf{E}^{XY} \arrow[dr, "m"] \arrow[ddl, phantom, "(5)"] & & \mathsf{E}^{ZW} \otimes \mathsf{E}^{XZ} \arrow[dl, "m"'] \arrow[ddr, phantom, "(5)"] & \\
& & \mathsf{E}^{XW} \arrow[d, "\cong"', "\mathsf{norm}"] & & \\
\mathsf{E}^{YW}_{\mathsf{norm}} \otimes \bigotimes\limits_{j=1}^{n} \Bigl[\bigotimes\limits_{i=1}^{k_j} \mathsf{E}_{j\,i}^{XY}\Bigr] \arrow[rr, "\gamma"] & & \mathsf{E}^{XW}_{\mathsf{norm}} & & \mathsf{E}^{ZW} \otimes \bigotimes\limits_{j=1}^{n} \mathsf{E}^{XZ}_{j\,\mathsf{norm}} \arrow[ll, "\gamma"']
\end{tikzcd}
$$

Commutativity of each of the inner regions is verified as follows.

(1) Use naturality of the symmetry, ξ, in V.
(2) Use functoriality of the monoidal product, $\otimes$.
(3) Use enriched functoriality of $\boxtimes$ (interchange; see Explanation 1.4.7).
(4) Use associativity of the composition m in K (1.2.2).
(5) Use functoriality of $\otimes$ and naturality of m with respect to morphisms in V. □

The following is a special case of Example 6.1.9.

Definition 6.3.7 (Enriched Endomorphism Operad)**.** Suppose V is a permutative category and K is a symmetric monoidal V-category. For each object $X \in \mathsf{K}$, the *endomorphism operad* of X is the V-enriched multicategory with a single object X and with n-ary operations

$$\mathsf{End}(X)_n = \mathsf{End}(\mathsf{K})\big(\langle X\rangle\,;\,X\big)$$

where $\langle X\rangle$ is the length-n tuple whose entries are all the single object X. Symmetric group actions, unit, and compositions are defined as in $\mathsf{End}(\mathsf{K})$. ⋄

Definition 6.3.8. Suppose that V is a permutative category, K and L are symmetric monoidal V-categories, and

$$F : \mathsf{K} \longrightarrow \mathsf{L}$$

is a symmetric monoidal V-functor. We define a V-enriched multifunctor

$$\mathsf{End}(F) : \mathsf{End}(\mathsf{K}) \longrightarrow \mathsf{End}(\mathsf{L})$$

with assignment on objects given by F and morphisms on operation objects

$$\mathsf{End}(F) : \mathsf{End}(\mathsf{K})\big(\langle X\rangle\,;\,X'\big) \longrightarrow \mathsf{End}(\mathsf{L})\big(F\langle X\rangle\,;\,X'\big) \tag{6.3.9}$$

defined as follows. For each $n \geq 1$, there is an F-coherent map with components

$$F^n : \mathbb{1} \longrightarrow \mathsf{L}\big(\,\boxtimes(F\langle X\rangle)\,,\,F(\boxtimes\langle X\rangle)\,\big)$$

provided by iterated application of F^2, where $\langle X\rangle$ has length n. This is unique by Enriched Epstein's Coherence Theorem 2.5.8. Then (6.3.9) is given by applying F to mapping objects

$$F : \mathsf{K}(\boxtimes\langle X\rangle, X') \longrightarrow \mathsf{L}(F(\boxtimes\langle X\rangle), FX')$$

and then precomposing with the components of F^n. ⋄

Proposition 6.3.10. *In the context of Definition 6.3.8,*

$$\mathsf{End}(F) : \mathsf{End}(\mathsf{K}) \longrightarrow \mathsf{End}(\mathsf{L})$$

is a V*-enriched multifunctor.*

Proof. The unit condition (6.1.12) is given by the identity axiom (1.2.6) for F. We apply the coherence Theorem 2.5.8 to show that the diagrams for preservation of the symmetric group action (6.1.11) and composition (6.1.13) commute.

By definition, the morphisms given by $\mathsf{End}(F)$ on operation objects are F-coherent maps. The permutation action (6.3.4) and composition (6.3.5) of $\mathsf{End}(\mathsf{K})$ and $\mathsf{End}(\mathsf{L})$ are given by composites of the monoidal product $\boxtimes$, symmetry β, and the normalization maps of Convention 6.3.1. All of these are (permuted) canonical V-maps and hence also F-coherent maps. This shows that (6.1.11) and (6.1.13) for $\mathsf{End}(F)$ consist entirely of F-coherent maps. Therefore their respective composites are equal by Theorem 2.5.8. □

6.4. The Multicategory of Small Multicategories

Now we use the theory above to show that the 2-categories $\mathsf{Multicat}$ and $\mathsf{Multicat}_*$ have the structure of Cat-enriched multicategories. Although Cat is not a permutative category, we understand the definition of Cat-enriched multicategory via strictification, as outlined in Explanation 6.1.19.

We first show that each of $(\mathsf{Multicat}, \otimes)$ and $(\mathsf{Multicat}_*, \wedge)$ is symmetric Cat-monoidal in the sense of Definition 1.5.1. Then we apply Definition 6.3.7 and Proposition 6.3.6 to obtain Cat-enriched multicategory structures.

To show that $(\mathsf{Multicat}, \otimes)$ is symmetric Cat-monoidal, we must first extend the Boardman-Vogt tensor product of Definition 5.6.11 to multinatural transformations.

Definition 6.4.1. Suppose that

$$F, F' : \mathsf{M} \longrightarrow \mathsf{M}' \quad \text{and} \quad G, G' : \mathsf{N} \longrightarrow \mathsf{N}'$$

are multifunctors and suppose moreover that

$$\theta : F \longrightarrow F' \quad \text{and} \quad \omega : G \longrightarrow G'$$

are multinatural transformations. Define a multinatural transformation

$$\theta \otimes \omega : F \otimes G \longrightarrow F' \otimes G'$$

via components

$$(\theta \otimes \omega)_{(c,d)} = \theta_c \times \omega_d$$

for $c \in \mathsf{M}$ and $d \in \mathsf{N}$. $\diamond$

Proposition 6.4.2. *In the context of Definition 6.4.1, $\theta \otimes \omega$ is a multinatural transformation $F \otimes G \longrightarrow F' \otimes G'$.*

Proof. We verify the naturality condition of Definition 5.1.17 for generating operations $\phi \otimes d$ and $c \otimes \psi$ with

$$\phi \in \mathsf{M}\big(\langle c \rangle ; c'\big) \quad \text{and} \quad \psi \in \mathsf{N}\big(\langle d \rangle ; d'\big).$$

Let $(\theta \otimes \omega)_{\langle c \rangle \otimes d}$ denote the tuple of unary operations $\theta_{c_i} \times \omega_d$ indexed by the objects of $\langle c \rangle \otimes d$. Similarly let $(\theta \otimes \omega)_{c \otimes \langle d \rangle}$ denote the tuple indexed by $c \otimes \langle d \rangle$. The equalities

$$\begin{aligned}
\big(F'\phi \otimes G'd\big) \circ \big((\theta \otimes \omega)_{\langle c \rangle \otimes d}\big) &= \big(\theta_{c'} \times \omega_d\big) \circ \big(F\phi \otimes Gd\big) \quad \text{and} \\
\big(F'c \otimes G'\psi\big) \circ \big((\theta \otimes \omega)_{c \otimes \langle d \rangle}\big) &= \big(\theta_c \times \omega_{d'}\big) \circ \big(Fc \otimes G\psi\big)
\end{aligned}$$

follow from the naturality conditions for θ and ω, respectively, together with the unit conditions

$$1_{G'd}\omega_d = \omega_d 1_{Gd} \quad \text{and} \quad 1_{F'c}\theta_c = \theta_c 1_{Fc}.$$

$\square$

Theorem 6.4.3. *The tensor product of small multicategories is a 2-functor*

$$\mathsf{Multicat} \times \mathsf{Multicat} \xrightarrow{\otimes} \mathsf{Multicat}$$

and $(\mathsf{Multicat}, \otimes)$ *is a symmetric* Cat*-monoidal 2-category.*

Proof. The 2-functoriality of $\otimes$ follows because the units and composition of multinatural transformations are given componentwise Recalling Definition 6.4.1, the components of $\theta \otimes \omega$ are given pairwise by those of θ and ω. Therefore, $\otimes$ is 2-functorial.

To show that $\mathsf{Multicat}$ is symmetric Cat-monoidal, we will apply Theorem 2.5.1. So we need to continue showing that the data of the symmetric monoidal structure on $\mathsf{Multicat}$ extend to 2-functors and 2-natural transformations.

To that end, we first note that the unit

$$* \xrightarrow{\mathsf{I}} \mathsf{Multicat}$$

extends to a 2-functor by sending the unique 2-cell of the terminal 2-category to the identity multinatural transformation on the identity multifunctor 1_{I}. Next we show that the symmetry β is 2-natural. Suppose given multifunctors

$$F, F' : \mathsf{M} \longrightarrow \mathsf{N} \quad \text{and} \quad G, G' : \mathsf{M}' \longrightarrow \mathsf{N}'$$

together with multinatural transformations

$$\theta : F \longrightarrow F' \quad \text{and} \quad \omega : G \longrightarrow G'.$$

The whiskering of $\omega \otimes \theta$ with $\beta_{\mathsf{M},\mathsf{M}'}$,

$$\begin{tikzcd} \mathsf{M} \otimes \mathsf{M}' \arrow[r, "\beta"] & \mathsf{M}' \otimes \mathsf{M} \arrow[r, bend left, "G \otimes F"] \arrow[r, bend right, "G' \otimes F'"'] \arrow[r, phantom, "\Downarrow \omega \otimes \theta"] & \mathsf{N}' \otimes \mathsf{N}, \end{tikzcd}$$

has component at an object $c \otimes c' \in \mathsf{M} \otimes \mathsf{M}'$ given by

$$(\omega \otimes \theta)_{\beta(c \otimes c')} = \omega_{c'} \otimes \theta_c.$$

On the other hand, the whiskering of $\beta_{\mathsf{N},\mathsf{N}'}$ with $\theta \otimes \omega$,

$$\begin{tikzcd} \mathsf{M} \otimes \mathsf{M}' \arrow[r, bend left, "F \otimes G"] \arrow[r, bend right, "F' \otimes G'"'] \arrow[r, phantom, "\Downarrow \theta \otimes \omega"] & \mathsf{N} \otimes \mathsf{N}' \arrow[r, "\beta"] & \mathsf{N}' \otimes \mathsf{N}, \end{tikzcd}$$

has component at an object $c \otimes c' \in \mathsf{M} \otimes \mathsf{M}'$ given by

$$\beta(\theta_c \otimes \omega_{c'}) = \omega_{c'} \otimes \theta_c.$$

Therefore, β is 2-natural.

Similar arguments for the unit and associativity isomorphisms show that these are 2-natural as well. Ultimately their 2-naturality follows from the 2-naturality of the corresponding data for the Cartesian product, pushouts, and coequalizers. Since the data of the symmetric monoidal structure on $(\mathsf{Multicat}, \otimes)$ are the underlying functors and natural transformations of Cat-enriched data, Theorem 2.5.1 shows that $(\mathsf{Multicat}, \otimes)$ is symmetric Cat-monoidal. □

The proof of Theorem 6.4.3 descends to the smash product of pointed multicategories and gives the following.

Theorem 6.4.4. *The smash of small multicategories is a 2-functor*

$$\mathsf{Multicat}_* \times \mathsf{Multicat}_* \xrightarrow{\wedge} \mathsf{Multicat}_*$$

and $(\mathsf{Multicat}_*, \wedge)$ *is a symmetric* Cat*-monoidal 2-category.*

By applying Proposition 6.3.6 to $(\mathsf{Multicat}, \otimes)$ and $(\mathsf{Multicat}_*, \wedge)$, we have the following.

Corollary 6.4.5. *Each of* $\mathsf{Multicat}$ *and* $\mathsf{Multicat}_*$ *has the structure of* Cat*-enriched multicategory induced by the products* $\otimes$ *and* $\wedge$*, respectively.*

Convention 6.4.6 (Multicategories $\mathsf{Multicat}$ and $\mathsf{Multicat}_*$)**.** Instead of the overly-cumbersome $\mathsf{End}(\mathsf{Multicat})$ and $\mathsf{End}(\mathsf{Multicat}_*)$, we will continue to use the notation $\mathsf{Multicat}$ and $\mathsf{Multicat}_*$ even when regarding these as Cat-enriched multicategories. ◇

6.5. Permutative Categories and Multilinearity

Recall from Definition 1.1.27 that $\mathsf{PermCat}^{\mathsf{su}}$ denotes the 2-category of small permutative categories, strictly unital symmetric monoidal functors, and monoidal natural transformations. Moreover, recall from Corollary 5.3.9 that taking endomorphism multicategories provides a 2-functor

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}_*$$

that is bijective on 1- and 2-cells.

Therefore, we have a multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$ given by restricting that of $\mathsf{Multicat}_*$. Likewise, the Cat-enriched multicategory structure on $\mathsf{Multicat}_*$ restricts to such a structure for $\mathsf{PermCat}^{\mathsf{su}}$. This is in contrast to Proposition 5.7.23, which shows that the symmetric monoidal structure of $\mathsf{Multicat}_*$ does not restrict along End.

Definition 6.5.1. We define a Cat-enriched multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$ as the sub-multicategory of $\mathsf{Multicat}_*$ given by the image of End. Recalling Corollary 5.3.9, this is the full sub-multicategory whose objects are given by $\mathsf{End}(\mathsf{C})$ for permutative categories C. As an extension of Convention 6.4.6, we continue to use the notation $\mathsf{PermCat}^{\mathsf{su}}$ for the Cat-enriched multicategory. ◇

The Cat-enriched multicategory structure of $\mathsf{PermCat}^{\mathsf{su}}$ is fundamental to our K-theory applications in Part 2 because it is the domain of the Elmendorf-Mandell K-theory multifunctor defined in Chapter 10. Therefore, in Section 6.6 we give a second, more direct proof that $\mathsf{PermCat}^{\mathsf{su}}$ is a Cat-enriched multicategory.

In the remainder of this section we give more detailed descriptions of the categories of operations

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle \,;\, \mathsf{D}\big)$$

for permutative categories $\mathsf{C}_1, \ldots, \mathsf{C}_n$, and D. Such a category is, by definition, the category of multifunctors

$$F : \wedge_{i=1}^{n} \mathsf{End}(\mathsf{C}_i) \longrightarrow \mathsf{End}(\mathsf{D}) \tag{6.5.2}$$

and multinatural transformations between them.

Definition 6.5.3. Suppose $\langle X\rangle \in \prod_j \mathsf{C}_j$ is a tuple of objects.

- For an object $X_i' \in \mathsf{C}_i$ with $i \in \{1, \ldots, n\}$, we let $\langle X \circ_i X_i'\rangle$ denote the tuple whose jth entry is X_j for $j \neq i$, and whose ith entry is X_i'.
- Similarly, for objects $X_i' \in \mathsf{C}_i$ and $X_k' \in \mathsf{C}_k$ with $i, k \in \{1, \ldots, n\}$ and $i \neq k$, we let $\langle X \circ_i X_i' \circ_k X_k'\rangle$ denote the tuple whose jth entry is that of $\langle X \circ_i X_i'\rangle$ for $j \neq k$ and whose kth entry is X_k'. ◇

Definition 6.5.4 (Multilinear Functors)**.** Suppose $\mathsf{C}_1, \ldots, \mathsf{C}_n$, and D are permutative categories. An *n-linear functor* from $\langle \mathsf{C}\rangle$ to D is a functor

$$F : \mathsf{C}_1 \times \cdots \times \mathsf{C}_n \longrightarrow \mathsf{D}$$

together with, for each $i \in \{1, \ldots, n\}$, a natural transformation F_i^2 called the *ith linearity constraint* with components

$$F_i^2 : F\langle X \circ_i X_i\rangle \oplus F\langle X \circ_i X_i'\rangle \longrightarrow F\langle X \circ_i (X_i \oplus X_i')\rangle$$

for $\langle X\rangle \in \prod_j \mathsf{C}_j$ and $X_i' \in \mathsf{C}_i$. These data satisfy the following axioms.

Unity: If any $X_j = e$, the unit of C_j, then $F\langle X\rangle = e$, the unit in D. Moreover, $F\langle f \circ_j 1_e\rangle = 1_e$ for any morphisms $f_i \in \mathsf{C}_i$ for $i \neq j$.

Constraint Unity: If any $X_j = e$ or if $X_i' = e$, then F_i^2 is an identity morphism.

Constraint Associativity: The following diagram commutes for each $i \in \{1, \ldots, n\}$ and $\langle X \rangle \in \prod_j \mathsf{C}_j$, with $X_i', X_i'' \in \mathsf{C}_i$.

$$\begin{tikzcd}
F\langle X \circ_i X_i \rangle \oplus F\langle X \circ_i X_i' \rangle \oplus F\langle X \circ_i X_i'' \rangle \arrow[r, "1 \oplus F_i^2"] \arrow[d, "F_i^2 \oplus 1"'] & F\langle X \circ_i X_i \rangle \oplus F\langle X \circ_i (X_i' \oplus X_i'') \rangle \arrow[d, "F_i^2"] \\
F\langle X \circ_i (X_i \oplus X_i') \rangle \oplus F\langle X \circ_i X_i'' \rangle \arrow[r, "F_i^2"] & F\langle X \circ_i (X_i \oplus X_i' \oplus X_i'') \rangle
\end{tikzcd} \tag{6.5.5}$$

Constraint Symmetry: The following diagram commutes for each $i \in \{1, \ldots, n\}$ and $\langle X \rangle \in \prod_j \mathsf{C}_j$, with $X_i' \in \mathsf{C}_i$.

$$\begin{tikzcd}
F\langle X \circ_i X_i \rangle \oplus F\langle X \circ_i X_i' \rangle \arrow[r, "F_i^2"] \arrow[d, "\xi"'] & F\langle X \circ_i (X_i \oplus X_i') \rangle \arrow[d, "F\langle 1 \circ_i \xi \rangle"] \\
F\langle X \circ_i X_i' \rangle \oplus F\langle X \circ_i X_i \rangle \arrow[r, "F_i^2"] & F\langle X \circ_i (X_i' \oplus X_i) \rangle
\end{tikzcd} \tag{6.5.6}$$

Constraint 2-By-2: The following diagram commutes for each

$$i \neq k \in \{1, \ldots, n\}, \quad \langle X \rangle \in \textstyle\prod_j \mathsf{C}_j, \quad X_i' \in \mathsf{C}_i, \quad \text{and} \quad X_k' \in \mathsf{C}_k.$$

$$\begin{tikzcd}[column sep=small]
& F\langle X \circ_i (X_i \oplus X_i') \circ_k X_k \rangle \oplus F\langle X \circ_i (X_i \oplus X_i') \circ_k X_k' \rangle \arrow[dr, "F_k^2"] & \\
\begin{array}{c} F\langle X \circ_i X_i \circ_k X_k \rangle \oplus F\langle X \circ_i X_i' \circ_k X_k \rangle \\ \oplus F\langle X \circ_i X_i \circ_k X_k' \rangle \oplus F\langle X \circ_i X_i' \circ_k X_k' \rangle \end{array} \arrow[ur, "F_i^2 \oplus F_i^2"] \arrow[d, "1 \oplus \xi \oplus 1"'] & & F\langle X \circ_i (X_i \oplus X_i') \circ_k (X_k \oplus X_k') \rangle \\
\begin{array}{c} F\langle X \circ_i X_i \circ_k X_k \rangle \oplus F\langle X \circ_i X_i \circ_k X_k' \rangle \\ \oplus F\langle X \circ_i X_i' \circ_k X_k \rangle \oplus F\langle X \circ_i X_i' \circ_k X_k' \rangle \end{array} \arrow[dr, "F_k^2 \oplus F_k^2"'] & & \\
& F\langle X \circ_i X_i \circ_k (X_k \oplus X_k') \rangle \oplus F\langle X \circ_i X_i' \circ_k (X_k \oplus X_k') \rangle \arrow[uur, "F_i^2"'] &
\end{tikzcd} \tag{6.5.7}$$

A 0-*linear functor* is a choice of object in D, regarded as a functor

$$F : \mathbf{1} \longrightarrow \mathsf{D}$$

from the empty product. We say that F is a *multilinear functor* if it is n-linear for some $n \geq 0$. In Proposition 6.5.10 below we will show that the operations of $\mathsf{PermCat}^{\mathsf{su}}$ are precisely multilinear functors. $\diamond$

Example 6.5.8. In the context of Definition 6.5.4 with $n = 1$, comparing the axioms above with those of Definition 1.1.6 shows that a 1-linear functor is precisely a strictly unital symmetric monoidal functor. $\diamond$

Example 6.5.9 (Ring Category Product)**.** Recall from Definition II.9.1.2 the definition of ring category. If

$$\mathsf{C} = \big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$

is a ring category, then

$$F = \otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$$

is a 2-linear functor with $F_1^2 = \partial^l$ and $F_2^2 = \partial^r$. The following table lists the correspondence between axioms of Definition 6.5.4 for $n = 2$ and those of Definition II.9.1.2.

2-Linear Functor	**Ring Category Product**
Unity	Multiplicative Zero
Constraint Unity	Zero Factorization
Constraint Associativity	Internal Factorization
Constraint Symmetry	Symmetry Factorization
Constraint 2-By-2	2-By-2 Factorization

We explain further details of this correspondence when, in the proof of Theorem 11.2.16, we use it along with 1- and 3-linearity to show that ring category structure on C is encoded by action of the Cat-enriched associative operad.

Theorem 11.2.16 is, in turn, used in Theorems 11.5.5, 12.4.5, and 13.4.12 to establish similar correspondences between certain operad actions and, respectively, bipermutative, braided ring, and E_n-monoidal structures on C. In the proof of Theorem 13.4.12, 4-linearity is used to describe the exchanges $\eta^{k,l}$ (Definition 13.4.1). $\diamond$

Proposition 6.5.10. *Suppose $\mathsf{C}_1, \ldots, \mathsf{C}_n$ and D are small permutative categories. The objects of*

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big) = \mathsf{Multicat}_*\big(\bigwedge_j \mathsf{End}(\mathsf{C}_j), \mathsf{End}(\mathsf{D})\big)$$

are precisely the multilinear functors $\langle \mathsf{C} \rangle \longrightarrow \mathsf{D}$.

Proof. Recall from the proofs of Proposition 5.3.7 and Lemma 5.3.8 that, in the correspondence between strictly unital symmetric monoidal functors and pointed multifunctors

$$F : \mathsf{C} \longrightarrow \mathsf{D} \quad \text{and} \quad G : \mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D}),$$

respectively, the strictly unital aspect of F corresponds to preservation of basepoints by G, the monoidal constraints $F^2_{X,Y}$ correspond to the values of G on $\iota_{(X,Y)}$, and the symmetric monoidal axioms for F correspond to the multifunctoriality axioms for G. Applying this to Definition 6.5.4, we can describe a multilinear functor

$$F : \langle \mathsf{C} \rangle \longrightarrow \mathsf{D}$$

as a functor on underlying objects and unary operations of $\wedge_j \mathsf{End}(\mathsf{C}_j)$ such that

- F is strictly unital in each variable;
- F is a symmetric monoidal functor when restricted to each C_i; and
- F satisfies a 2-by-2 interchange condition for the monoidal structures in separate variables.

Recalling our discussion of multifunctors out of a tensor product in Explanation 5.6.14, F corresponds to an assignment on objects

$$\widetilde{F} : \prod_i \mathsf{Ob}\,\mathsf{End}(\mathsf{C}_i) \longrightarrow \mathsf{Ob}\,\mathsf{End}(\mathsf{D})$$

such that

- $\widetilde{F}$ is a pointed multifunctor in each variable separately, because in each variable F is strictly unital symmetric monoidal;
- $\widetilde{F}$ satisfies the interchange relation (5.6.15), by the constraint 2-by-2 axiom for F; and
- $\widetilde{F}$ descends to the smash product because it is pointed in each variable.

This proves that multilinear functors $\langle \mathsf{C} \rangle \longrightarrow \mathsf{D}$ correspond to multifunctors

$$\bigwedge_j \mathsf{End}(\mathsf{C}_j) \longrightarrow \mathsf{End}(\mathsf{D}). \qquad \square$$

One can also give a description similar to that of Definition 6.5.4 for the morphisms of

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big).$$

Definition 6.5.11 (Multilinear Transformations)**.** Suppose $\mathsf{C}_1, \ldots, \mathsf{C}_n$, and D are permutative categories. Suppose

$$F, F' : \langle \mathsf{C} \rangle \longrightarrow \mathsf{D}$$

are n-linear functors. An *n-linear transformation* is a natural transformation of underlying functors

$$\alpha : F \longrightarrow F'$$

that satisfies the following two *multilinearity conditions*.

(1) The diagram

$$\begin{tikzcd}
F\langle X \circ_i X_i \rangle \oplus F\langle X \circ_i X_i' \rangle \arrow[r, "F_i^2"] \arrow[d, "\alpha \oplus \alpha"'] & F\langle X \circ_i (X_i \oplus X_i') \rangle \arrow[d, "\alpha"] \\
F'\langle X \circ_i X_i \rangle \oplus F'\langle X \circ_i X_i' \rangle \arrow[r, "(F')_i^2"] & F'\langle X \circ_i (X_i \oplus X_i') \rangle
\end{tikzcd} \tag{6.5.12}$$

commutes for each $i \in \{1, \ldots, n\}$ and $\langle X \rangle \in \prod_j \mathsf{C}_j$ with $X_i' \in \mathsf{C}_i$.

(2) The component of α at a tuple $\langle X \rangle$ is an identity if any $X_i = e$.

As we explain in the proof of Proposition 6.5.13 below, the two multilinearity conditions make α a monoidal natural transformation in each variable separately. We say that α is a *multilinear transformation* if it is n-linear for some $n \geq 0$. $\diamond$

Proposition 6.5.13. *Suppose $\mathsf{C}_1, \ldots, \mathsf{C}_n$ and D are small permutative categories. Suppose that F and F' are multilinear functors, regarded as objects of*

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big) = \mathsf{Multicat}_*\big(\bigwedge_j \mathsf{End}(\mathsf{C}_j), \mathsf{End}(\mathsf{D})\big).$$

The morphisms between F and F' in $\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle\,;\mathsf{D}\big)$ *are precisely the multilinear transformations* $F \longrightarrow F'$.

Proof. By definition, the morphisms in $\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle\,;\mathsf{D}\big)$ are the pointed multinatural transformations between multifunctors

$$\bigwedge_j \mathsf{End}(\mathsf{C}_j) \longrightarrow \mathsf{End}(\mathsf{D}).$$

Recalling the proofs of Proposition 5.3.7 and Lemma 5.3.8, such a multinatural transformation corresponds to a natural transformation of multilinear functors,

$$\alpha : F \longrightarrow F',$$

that is monoidal in each variable separately. Since F and F' are strictly unital in each variable, the two conditions of Definition 1.1.12 for monoidal naturality are, for each variable separately, precisely the two multinaturality conditions of Definition 6.5.11. $\square$

6.6. The Multicategory of Small Permutative Categories

In this section, we provide a direct proof that $\mathsf{PermCat}^{\mathsf{su}}$ is a Cat-enriched multicategory. The fact that $\mathsf{PermCat}^{\mathsf{su}}$ is a Cat-enriched multicategory is crucial in Corollaries 11.3.16, 11.6.12, 12.5.3, and 13.5.2, where $\mathsf{PermCat}^{\mathsf{su}}$ appears as the domain of the Elmendorf-Mandell K-theory multifunctor. See also Note 6.7.2.

Definitions. The objects in $\mathsf{PermCat}^{\mathsf{su}}$ are small permutative categories. For small permutative categories $\mathsf{C}_1, \ldots, \mathsf{C}_n$, and D, the category

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle\,;\mathsf{D}\big) = \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}_1,\ldots,\mathsf{C}_n\rangle\,;\mathsf{D}\big)$$

is defined as follows.

- Its objects are n-linear functors (Definition 6.5.4)

$$\mathsf{C}_1 \times \cdots \times \mathsf{C}_n \longrightarrow \mathsf{D}.$$

- Its morphisms are multilinear transformations (Definition 6.5.11).
- Identity morphisms are identity natural transformations.
- Categorical composition is the vertical composition of natural transformations (Definition I.1.1.8).

Units. For each small permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$, the C-colored unit in the category $\mathsf{PermCat}^{\mathsf{su}}\big(\mathsf{C}\,;\mathsf{C}\big)$ is the identity functor $1_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{C}$ with the identity linearity constraint

$$1_{A\oplus B} : A \oplus B \longrightarrow A \oplus B \quad \text{for} \quad A, B \in \mathsf{C}. \tag{6.6.1}$$

Equivariance. With $\langle \mathsf{C}\rangle = (\mathsf{C}_1, \ldots, \mathsf{C}_n)$ and $\sigma \in \Sigma_n$, the symmetric group action

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle\,;\mathsf{D}\big) \xrightarrow[\cong]{\sigma} \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle\sigma\,;\mathsf{D}\big) \tag{6.6.2}$$

sends an n-linear functor

$$\big(F, \{F_j^2\}_{1\le j\le n}\big) : \mathsf{C}_1 \times \cdots \times \mathsf{C}_n \longrightarrow \mathsf{D}$$

to the composite functor

$$\begin{tikzcd}
\mathsf{C}_{\sigma(1)} \times \cdots \times \mathsf{C}_{\sigma(n)} \arrow[rr, "F^\sigma"] \arrow[dr, "\sigma"', "\cong"] & & \mathsf{D} \\
& \mathsf{C}_1 \times \cdots \times \mathsf{C}_n \arrow[ur, "F"'] &
\end{tikzcd}$$

with σ permuting the entries from the left. For $1 \leq j \leq n$, the jth linearity constraint of F^σ is $F^2_{\sigma(j)}$ applied to appropriately permuted sequences of objects. In other words, for objects

- $\langle A \rangle = (A_1, \ldots, A_n) \in \prod_{i=1}^n \mathsf{C}_{\sigma(i)}$ and
- $A'_j \in \mathsf{C}_{\sigma(j)}$,

with the $\circ_j$ notation in Definition 6.5.3, the jth linearity constraint $(F^\sigma)^2_j$ is the following composite in D.

$$\begin{tikzcd}
F^\sigma\langle A\rangle \oplus F^\sigma\langle A \circ_j A'_j\rangle \arrow[r, "(F^\sigma)^2_j"] \arrow[d, equal] & F^\sigma\langle A \circ_j (A_j \oplus A'_j)\rangle \arrow[d, equal] \\
F\big(\sigma\langle A\rangle\big) \oplus F\big(\sigma\langle A\rangle \circ_{\sigma(j)} A'_j\big) \arrow[r, "F^2_{\sigma(j)}"] & F\big(\sigma\langle A\rangle \circ_{\sigma(j)} (A_j \oplus A'_j)\big)
\end{tikzcd} \tag{6.6.3}$$

The symmetric group action sends a multilinear transformation $\alpha : F \longrightarrow G$ between n-linear functors $F, G \in \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle ; \mathsf{D}\big)$ to the horizontal composite (Definition I.1.1.8)

$$\alpha^\sigma = \alpha * 1_\sigma : F^\sigma \longrightarrow G^\sigma. \tag{6.6.4}$$

Composition of multilinear functors. For the multicategorical composition γ, suppose

- $\langle \mathsf{B}_j\rangle = (\mathsf{B}_{j,1}, \ldots, \mathsf{B}_{j,k_j})$ is a k_j-tuple of small permutative categories for each $1 \leq j \leq n$ and
- $\langle \mathsf{B}\rangle = \oplus_{j=1}^n \langle \mathsf{B}_j\rangle$ is their concatenation.

For multilinear functors

- $\big(F, \{F^2_j\}_{1\leq j\leq n}\big) \in \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle ; \mathsf{D}\big)$ and
- $\big(H_j, \{H^2_{j,i}\}_{1\leq i\leq k_j}\big) \in \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{B}_j\rangle ; \mathsf{C}_j\big)$ for $1 \leq j \leq n$,

the composition

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle ; \mathsf{D}\big) \times \prod_{j=1}^n \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{B}_j\rangle ; \mathsf{C}_j\big) \xrightarrow{\gamma} \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{B}\rangle ; \mathsf{D}\big) \tag{6.6.5}$$

is defined by

$$\begin{aligned}
&\gamma\Big(\big(F, \{F^2_j\}_{1\leq j\leq n}\big), \Big(\big(H_j, \{H^2_{j,i}\}_{1\leq i\leq k_j}\big)\Big)_{1\leq j\leq n}\Big) \\
&= \Big(F \circ \textstyle\prod_j H_j, \Big\{\big(F \circ \prod_j H_j\big)^2_l\Big\}_{1\leq l\leq k_1+\cdots+k_n}\Big).
\end{aligned} \tag{6.6.6}$$

Below we will abbreviate the object in (6.6.6) to $\gamma(F, (H_j))$.

In (6.6.6), $F \circ \prod_j H_j$ is the composite functor

$$\prod_{j=1}^{n} \prod_{i=1}^{k_j} \mathsf{B}_{j,i} \xrightarrow{\prod_j H_j} \prod_{j=1}^{n} \mathsf{C}_j \xrightarrow{F} \mathsf{D}.$$

To explain its linearity constraints, consider

- objects $W_{j,i} \in \mathsf{B}_{j,i}$ for $1 \leq j \leq n$ and $1 \leq i \leq k_j$,
- an object $W'_{j,i} \in \mathsf{B}_{j,i}$ for a particular choice of (j,i) with $l = k_1 + \cdots + k_{j-1} + i$,
- $\langle W_j \rangle = (W_{j,1}, \ldots, W_{j,k_j}) \in \prod_{i=1}^{k_j} \mathsf{B}_{j,i}$,
- $\langle W \rangle = \oplus_{j=1}^{n} \langle W_j \rangle \in \prod_{j=1}^{n} \prod_{i=1}^{k_j} \mathsf{B}_{j,i}$ their concatenation, and
- $\langle HW \rangle = \big(H_1\langle W_1 \rangle, \ldots, H_n\langle W_n \rangle\big) \in \prod_{j=1}^{n} \mathsf{C}_j$.

Note that

$$\begin{aligned} \langle W_j \circ_i W'_{j,i} \rangle &= \big(\overbrace{W_{j,1}, \ldots, W_{j,i-1}}^{\text{empty if } i=1}, W'_{j,i}, \overbrace{W_{j,i+1}, \ldots, W_{j,k_j}}^{\text{empty if } i=k_j} \big) \\ \langle W_j \circ_i (W_{j,i} \oplus W'_{j,i}) \rangle &= \big(\underbrace{W_{j,1}, \ldots, W_{j,i-1}}_{\text{empty if } i=1}, W_{j,i} \oplus W'_{j,i}, \underbrace{W_{j,i+1}, \ldots, W_{j,k_j}}_{\text{empty if } i=k_j} \big). \end{aligned} \tag{6.6.7}$$

The lth linearity constraint $\big(F \circ \prod_j H_j\big)_l^2$ in (6.6.6) is defined as the following composite in D.

$$\begin{tikzcd}[column sep=small]
& F\langle HW \rangle \oplus F\langle HW \circ_j H_j\langle W_j \circ_i W'_{j,i} \rangle\rangle \arrow[ddr, "F_j^2"] & \\
(F \circ \prod_j H_j)\langle W \rangle \oplus (F \circ \prod_j H_j)\langle W \circ_l W'_{j,i} \rangle \arrow[ur, equal] \arrow[dd, "(F \circ \prod_j H_j)_l^2"'] & & \\
& & F\langle HW \circ_j (H_j\langle W_j \rangle \oplus H_j\langle W_j \circ_i W'_{j,i} \rangle)\rangle \arrow[ddl, "F\langle 1 \circ_j H_{j,i}^2 \rangle"] \\
(F \circ \prod_j H_j)\langle W \circ_l (W_{j,i} \oplus W'_{j,i}) \rangle \arrow[dr, equal] & & \\
& F\langle HW \circ_j H_j\langle W_j \circ_i (W_{j,i} \oplus W'_{j,i}) \rangle\rangle &
\end{tikzcd} \tag{6.6.8}$$

Note that if $n = 1 = k_1$, then $\big(FH, (FH)^2\big)$ in (6.6.6) is precisely the composite of strictly unital symmetric monoidal functors in Definition II.1.3.12.

Composition of multilinear transformations. For multilinear transformations

- $\alpha : F \longrightarrow F'$ in $\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle; \mathsf{D}\big)$ and
- $\beta_j : H_j \longrightarrow H'_j$ in $\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{B}_j \rangle; \mathsf{C}_j\big)$ for each $1 \leq j \leq n$, with $\langle \beta \rangle = (\beta_1, \ldots, \beta_n)$,

the multicategorical composition is defined as the horizontal composite

$$\gamma\big(\alpha, \langle \beta \rangle\big) = \alpha * \textstyle\prod_{j=1}^{n} \beta_j : F \circ \prod_j H_j \longrightarrow F' \circ \prod_j H'_j \tag{6.6.9}$$

of natural transformations (Definition I.1.1.8).

Proofs. By Theorem 6.4.4, Convention 6.4.6, and Definition 6.5.1, the definitions above define a Cat-enriched multicategory $\mathsf{PermCat}^{\mathsf{su}}$. The rest of this section contains an alternative direct proof of this fact in several steps.

Lemma 6.6.10. *$\gamma(F,(H_j))$ in (6.6.6) satisfies the constraint 2-by-2 axiom (6.5.7).*

Proof. There are two cases. For the first case, suppose $p \neq q \in \{1,\ldots,n\}$, $1 \le a \le k_p$, and $1 \le b \le k_q$. Consider objects

- $\langle W_j \rangle = (W_{j,1},\ldots,W_{j,k_j}) \in \prod_{i=1}^{k_j} \mathsf{B}_{j,i}$ for $1 \le j \le n$,
- $W'_{p,a} \in \mathsf{B}_{p,a}$, $\langle W'_p \rangle = \langle W_p \circ_a W'_{p,a} \rangle \in \prod_{i=1}^{k_p} \mathsf{B}_{p,i}$,
- $W'_{q,b} \in \mathsf{B}_{q,b}$, $\langle W'_q \rangle = \langle W_q \circ_b W'_{q,b} \rangle \in \prod_{i=1}^{k_q} \mathsf{B}_{q,i}$,
- $\langle W \rangle = \oplus_{j=1}^{n} \langle W_j \rangle \in \prod_{j=1}^{n} \prod_{i=1}^{k_j} \mathsf{B}_{j,i}$,
- $X_j = H_j\langle W_j \rangle \in \mathsf{C}_j$ for $1 \le j \le n$,
- $X'_p = H_p\langle W'_p \rangle$, $X'_q = H_q\langle W'_q \rangle$,
- $\langle X \rangle = (X_1,\ldots,X_n) \in \prod_{j=1}^{n} \mathsf{C}_j$,
- $Y_p = H_p\langle W_p \circ_a (W_{p,a} \oplus W'_{p,a}) \rangle \in \mathsf{C}_p$,
- $Y_q = H_q\langle W_q \circ_b (W_{q,b} \oplus W'_{q,b}) \rangle \in \mathsf{C}_q$, and
- $U = F\langle X \circ_p (X_p \oplus X'_p) \circ_q (X_q \oplus X'_q) \rangle \in \mathsf{D}$.

By the definition (6.6.8) of the linearity constraints $(F \circ \prod_j H_j)^2_l$, the constraint 2-by-2 diagram (6.5.7) for $\gamma(F,(H_j))$ for the above objects is the outer diagram below.

$$
\begin{tikzcd}
& F\langle X \circ_p (X_p \oplus X'_p) \rangle \oplus F\langle X \circ_p (X_p \oplus X'_p) \circ_q X'_q \rangle \arrow[dr, "F\langle 1 \circ_p H^2_{p,a} \rangle \oplus F\langle 1 \circ_p H^2_{p,a} \rangle"] \arrow[ddd, "F^2_q"] & \\
& & F\langle X \circ_p Y_p \rangle \oplus F\langle X \circ_p Y_p \circ_q X'_q \rangle \arrow[dd, "F^2_q"] \arrow[from=2-3, to=4-2, phantom, "F^2_q \ \mathsf{nat}"] \\
F\langle X \rangle \oplus F\langle X \circ_p X'_p \rangle \oplus F\langle X \circ_q X'_q \rangle \oplus F\langle X \circ_p X'_p \circ_q X'_q \rangle \arrow[uur, "F^2_p \oplus F^2_p"] \arrow[dd, "1 \oplus \xi^{\oplus} \oplus 1"'] & & \\
& U \arrow[r, "F\langle 1 \circ_p H^2_{p,a} \rangle"] \arrow[dd, equal] \arrow[l, phantom, "(6.5.7)"] & F\langle X \circ_p Y_p \circ_q (X_q \oplus X'_q) \rangle \arrow[dd, "F\langle 1 \circ_q H^2_{q,b} \rangle"] \\
F\langle X \rangle \oplus F\langle X \circ_q X'_q \rangle \oplus F\langle X \circ_p X'_p \rangle \oplus F\langle X \circ_p X'_p \circ_q X'_q \rangle \arrow[dd, "F^2_q \oplus F^2_q"'] & & \\
& U \arrow[r, "F\langle 1 \circ_p H^2_{p,a} \circ_q H^2_{q,b} \rangle"] \arrow[ddr, "F\langle 1 \circ_q H^2_{q,b} \rangle"] \arrow[ddd, phantom, "F^2_p \ \mathsf{nat}"] & F\langle X \circ_p Y_p \circ_q Y_q \rangle \\
F\langle X \circ_q (X_q \oplus X'_q) \rangle \oplus F\langle X \circ_p X'_p \circ_q (X_q \oplus X'_q) \rangle \arrow[ur, "F^2_p"] \arrow[ddr, "F\langle 1 \circ_q H^2_{q,b} \rangle \oplus F\langle 1 \circ_q H^2_{q,b} \rangle"'] & & \\
& & F\langle X \circ_p (X_p \oplus X'_p) \circ_q Y_q \rangle \arrow[uu, "F\langle 1 \circ_p H^2_{p,a} \rangle"'] \\
& F\langle X \circ_q Y_q \rangle \oplus F\langle X \circ_p X'_p \circ_q Y_q \rangle \arrow[ur, "F^2_p"'] &
\end{tikzcd}
$$

- The upper left subdiagram is commutative by the constraint 2-by-2 axiom (6.5.7) for F.
- The top right and the bottom subdiagrams are commutative by the naturality of, respectively, the linearity constraints F^2_q and F^2_p.

- The two unlabeled subdiagrams are commutative by the functoriality of F.

This proves the first case of the constraint 2-by-2 axiom.

For the other case of the constraint 2-by-2 axiom (6.5.7) for $\gamma(F,(H_j))$ in (6.6.6), suppose $p = q \in \{1,\ldots,n\}$ and $a \neq b \in \{1,\ldots,k_p\}$. Consider objects

- $\langle W_j\rangle \in \prod_i \mathsf{B}_{j,i}$ and $\langle W\rangle = \oplus_{j=1}^n \langle W_j\rangle \in \prod_j \prod_i \mathsf{B}_{j,i}$ as above,
- $W'_{p,a} \in \mathsf{B}_{p,a}$, $\langle W_p^a\rangle = \langle W_p \circ_a W'_{p,a}\rangle$,
- $W'_{p,b} \in \mathsf{B}_{p,b}$, $\langle W_p^b\rangle = \langle W_p \circ_b W'_{p,b}\rangle$,
- $\langle W_p^{a,b}\rangle = \langle W_p \circ_a W'_{p,a} \circ_b W'_{p,b}\rangle$,
- $X_j = H_j\langle W_j\rangle \in \mathsf{C}_j$, $\langle X\rangle = (X_1,\ldots,X_n) \in \prod_j \mathsf{C}_j$,
- $X_p^a = H_p\langle W_p^a\rangle$, $X_p^b = H_p\langle W_p^b\rangle$, $X_p^{a,b} = H_p\langle W_p^{a,b}\rangle$,
- $Y_a = H_p\langle W_p \circ_a (W_{p,a} \oplus W'_{p,a})\rangle$,
- $Y_b = H_p\langle W_p \circ_b (W_{p,b} \oplus W'_{p,b})\rangle$,
- $Z_a = H_p\langle W_p \circ_a (W_{p,a} \oplus W'_{p,a}) \circ_b W'_{p,b}\rangle$,
- $Z_b = H_p\langle W_p \circ_a W'_{p,a} \circ_b (W_{p,b} \oplus W'_{p,b})\rangle$,
- $Z = H_p\langle W_p \circ_a (W_{p,a} \oplus W'_{p,a}) \circ_b (W_{p,b} \oplus W'_{p,b})\rangle$,
- $U_1 = F\langle X \circ_p (X_p \oplus X_p^a \oplus X_p^b \oplus X_p^{a,b})\rangle$,
- $U_2 = F\langle X\rangle \oplus F\langle X \circ_p (X_p^a \oplus X_p^b \oplus X_p^{a,b})\rangle$,
- $U_3 = F\langle X\rangle \oplus F\langle X \circ_p (X_p^a \oplus X_p^b)\rangle \oplus F\langle X \circ_p X_p^{a,b}\rangle$,
- $U_4 = F\langle X\rangle \oplus F\langle X \circ_p (X_p^b \oplus X_p^a)\rangle \oplus F\langle X \circ_p X_p^{a,b}\rangle$,
- $U_5 = F\langle X\rangle \oplus F\langle X \circ_p (X_p^b \oplus X_p^a \oplus X_p^{a,b})\rangle$, and
- $U_6 = F\langle X \circ_p (X_p \oplus X_p^b \oplus X_p^a \oplus X_p^{a,b})\rangle$.

The objects X_p^a, X_p^b, $X_p^{a,b}$, Y_a, Y_b, Z_a, Z_b, and Z are in C_p, and each $U_?$ is in D. The constraint 2-by-2 diagram (6.5.7) for $\gamma(F,(H_j))$ for the above objects is the outer

diagram below.

$$
\begin{tikzcd}[column sep=small]
& F\langle X \circ_p (X_p \oplus X_p^a)\rangle \oplus F\langle X \circ_p (X_p^b \oplus X_p^{a,b})\rangle \arrow[dr, bend left, "F\langle 1 \circ_p H^2_{p,a}\rangle \oplus F\langle 1 \circ_p H^2_{p,a}\rangle"] \arrow[dd, "F_p^2"'] & \\
& & F\langle X \circ_p Y_a\rangle \oplus F\langle X \circ_p Z_a\rangle \arrow[dd, "F_p^2"] \\
& U_1 \arrow[dr, "F\langle 1 \circ_p (H^2_{p,a} \oplus H^2_{p,a})\rangle"] \arrow[ddddd, bend left=60, "F\langle 1 \circ_p (1 \oplus \xi^{\oplus} \oplus 1)\rangle"] & \\
F\langle X\rangle \oplus F\langle X \circ_p X_p^a\rangle \oplus F\langle X \circ_p X_p^b\rangle \oplus F\langle X \circ_p X_p^{a,b}\rangle \arrow[uuur, bend left, "F_p^2 \oplus F_p^2"] \arrow[dr, "1 \oplus F_p^2 \oplus 1"'] \arrow[dd, "1 \oplus \xi^{\oplus} \oplus 1"'] \arrow[ur, phantom, "(6.5.5)"] & U_2 \arrow[u, "F_p^2"'] \arrow[ddd, bend left=60, "f"] & F\langle X \circ_p (Y_a \oplus Z_a)\rangle \arrow[ddd, "F\langle 1 \circ_p H^2_{p,b}\rangle"] \\
\arrow[r, phantom, "(6.5.6)"] & U_3 \arrow[u, "1 \oplus F_p^2"'] \arrow[d, "1 \oplus F\langle 1 \circ_p \xi^{\oplus}\rangle \oplus 1"'] & \\
F\langle X\rangle \oplus F\langle X \circ_p X_p^b\rangle \oplus F\langle X \circ_p X_p^a\rangle \oplus F\langle X \circ_p X_p^{a,b}\rangle \arrow[r, "1 \oplus F_p^2 \oplus 1"] \arrow[dd, "F_p^2 \oplus F_p^2"'] & U_4 \arrow[d, "1 \oplus F_p^2"'] & \\
\arrow[r, phantom, "(6.5.5)"] & U_5 \arrow[d, "F_p^2"'] & F\langle X \circ_p Z\rangle \\
F\langle X \circ_p (X_p \oplus X_p^b)\rangle \oplus F\langle X \circ_p (X_p^a \oplus X_p^{a,b})\rangle \arrow[r, "F_p^2"] \arrow[ddr, bend right, "F\langle 1 \circ_p H^2_{p,b}\rangle \oplus F\langle 1 \circ_p H^2_{p,b}\rangle"'] & U_6 \arrow[dr, bend right, "F\langle 1 \circ_p (H^2_{p,b} \oplus H^2_{p,b})\rangle"'] \arrow[r, phantom, "(6.5.7)"] & \\
& & F\langle X \circ_p (Y_b \oplus Z_b)\rangle \arrow[uu, "F\langle 1 \circ_p H^2_{p,a}\rangle"'] \\
& F\langle X \circ_p Y_b\rangle \oplus F\langle X \circ_p Z_b\rangle \arrow[ur, bend right, "F_p^2"'] &
\end{tikzcd}
$$

- The morphism $f : U_2 \longrightarrow U_5$ is
$$1 \oplus F\langle 1 \circ_p (\xi^{\oplus} \oplus 1)\rangle.$$
- The four unlabeled subdiagrams are commutative by the naturality of F_p^2.
- Each of the two subdiagrams labeled by (6.5.5) is commutative by the pth linearity constraint associativity axiom (6.5.5) for F twice, as in (II.9.2.17) with F_p^2 in place of F^2.
- The middle left trapezoid is commutative by the pth linearity constraint symmetry axiom (6.5.6) for F.
- The remaining subdiagram is obtained from the constraint 2-by-2 diagram (6.5.7) for H_p by applying $F\langle 1 \circ_p -\rangle$, so it is commutative.

This proves the second case of the constraint 2-by-2 axiom (6.5.7) for $\gamma(F, (H_j))$ in (6.6.6). $\square$

Lemma 6.6.11. *$\gamma(F, (H_j))$ in (6.6.6) is a k-linear functor in* $\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{B}\rangle\,;\,\mathsf{D}\big)$ *with* $k = k_1 + \cdots + k_n$.

Proof. The naturality of the lth linearity constraint $\big(F \circ \prod_j H_j\big)_l^2$ in (6.6.8) follows from the functoriality of F and the naturality of F_j^2 and $H_{j,i}^2$. Next we check that $\gamma(F, (H_j))$ satisfies the axioms in Definition 6.5.4 for a k-linear functor from $\langle \mathsf{B}\rangle$ to D. Its constraint 2-by-2 axiom (6.5.7) is verified in Lemma 6.6.10. Its unity axiom and constraint unity axiom follow from those of F and the H_j's.

For the constraint associativity axiom (6.5.5), consider the objects in (6.6.7) and

- $W''_{j,i} \in \mathsf{B}_{j,i}$,
- $\langle W_j^1\rangle = \langle W_j \circ_i W'_{j,i}\rangle$,

- $\langle W_j^2\rangle = \langle W_j \circ_i W''_{j,i}\rangle$,
- $\langle W_j^{01}\rangle = \langle W_j \circ_i (W_{j,i} \oplus W'_{j,i})\rangle$,
- $\langle W_j^{12}\rangle = \langle W_j \circ_i (W'_{j,i} \oplus W''_{j,i})\rangle$,
- $\langle W_j^{012}\rangle = \langle W_j \circ_i (W_{j,i} \oplus W'_{j,i} \oplus W''_{j,i})\rangle$,
- $\langle HW_j\rangle = H_j\langle W_j\rangle$,
- $\langle HW_j^?\rangle = H_j\langle W_j^?\rangle \in \mathsf{C}_j$ for $? \in \{1,2,01,12,012\}$,
- $\langle HW^?\rangle = \langle HW \circ_j \langle HW_j^?\rangle\rangle \in \mathsf{D}$,
- $\langle HW_j\rangle^{0,1} = \langle HW_j\rangle \oplus \langle HW_j^1\rangle$,
- $\langle HW_j\rangle^{1,2} = \langle HW_j^1\rangle \oplus \langle HW_j^2\rangle$,
- $\langle HW_j\rangle^{01,2} = \langle HW_j^{01}\rangle \oplus \langle HW_j^2\rangle$,
- $\langle HW_j\rangle^{0,12} = \langle HW_j\rangle \oplus \langle HW_j^{12}\rangle$, and
- $\langle HW_j\rangle^{0,1,2} = \langle HW_j\rangle \oplus \langle HW_j^1\rangle \oplus \langle HW_j^2\rangle$.

The constraint associativity axiom (6.5.5) for $\gamma(F,(H_j))$ in (6.6.6) for the objects above is the outer diagram in D below.

$$\begin{tikzcd}
& & F\langle HW\rangle \oplus F\langle HW^1\rangle \oplus F\langle HW^2\rangle \arrow[dll, "F_j^2 \oplus 1"'] \arrow[drr, "1 \oplus F_j^2"] & & \\
F\langle HW \circ_j \langle HW_j\rangle^{0,1}\rangle \oplus F\langle HW^2\rangle \arrow[d, "F\langle 1 \circ_j H^2_{j,i}\rangle \oplus 1"'] \arrow[drr, "F_j^2"] & & (6.5.5) & & F\langle HW\rangle \oplus F\langle HW \circ_j \langle HW_j\rangle^{1,2}\rangle \arrow[dll, "F_j^2"'] \arrow[d, "1 \oplus F\langle 1 \circ_j H^2_{j,i}\rangle"] \\
F\langle HW^{01}\rangle \oplus F\langle HW^2\rangle \arrow[d, "F_j^2"'] & & F\langle HW \circ_j \langle HW_j\rangle^{0,1,2}\rangle \arrow[dll, bend left, "F\langle 1 \circ_j (H^2_{j,i} \oplus 1)\rangle"'] \arrow[drr, bend right, "F\langle 1 \circ_j (1 \oplus H^2_{j,i})\rangle"] & & F\langle HW\rangle \oplus F\langle HW^{12}\rangle \arrow[d, "F_j^2"] \\
F\langle HW \circ_j \langle HW_j\rangle^{01,2}\rangle \arrow[drr, "F\langle 1 \circ_j H^2_{j,i}\rangle"'] & & (6.5.5) & & F\langle HW \circ_j \langle HW_j\rangle^{0,12}\rangle \arrow[dll, "F\langle 1 \circ_j H^2_{j,i}\rangle"] \\
& & F\langle HW^{012}\rangle & &
\end{tikzcd}$$

- The top diamond is commutative by the jth linearity constraint associativity axiom (6.5.5) for F.
- The bottom diamond is obtained from the ith linearity constraint associativity diagram (6.5.5) for H_j by applying $F\langle 1 \circ_j -\rangle$, so it is commutative.
- The left and right triangles are commutative by the naturality of F_j^2.

This proves the constraint associativity axiom (6.5.5) for $\gamma(F,(H_j))$.

The constraint symmetry axiom (6.5.6) for $\gamma(F,(H_j))$ is the outer diagram in D below.

$$\begin{tikzcd}
& F\langle HW\rangle \oplus F\langle HW^1\rangle \arrow[dl, "\xi^{\oplus}"'] \arrow[dr, "F_j^2"] & \\
F\langle HW^1\rangle \oplus F\langle HW\rangle \arrow[d, "F_j^2"'] & & F\langle HW \circ_j (\langle HW_j\rangle \oplus \langle HW_j^1\rangle)\rangle \arrow[d, "F\langle 1 \circ_j H^2_{j,i}\rangle"] \arrow[dll, "F\langle 1 \circ_j \xi^{\oplus}\rangle"'] \\
F\langle HW \circ_j (\langle HW_j^1\rangle \oplus \langle HW_j\rangle)\rangle \arrow[dr, "F\langle 1 \circ_j H^2_{j,i}\rangle"'] & & F\langle HW^{01}\rangle \arrow[dl, "F\langle 1 \circ_j H_j\langle 1 \circ_i \xi^{\oplus}\rangle\rangle"] \\
& F\langle HW \circ_j H_j\langle W_j \circ_i (W'_{j,i} \oplus W_{j,i})\rangle\rangle &
\end{tikzcd}$$

The top subdiagram is commutative by the jth linearity constraint symmetry axiom (6.5.6) for F. The bottom subdiagram is obtained from the ith linearity constraint symmetry diagram (6.5.6) for H_j by applying $F\langle 1 \circ_j -\rangle$, so it is commutative. This finishes the proof that $\gamma(F,(H_j))$ in (6.6.6) is a k-linear functor. $\square$

Lemma 6.6.12. *γ in (6.6.5) is a functor.*

Proof. Lemma 6.6.11 shows that γ is well defined on multilinear functors.

To see that the horizontal composite natural transformation

$$\gamma(\alpha,\langle\beta\rangle) = \alpha * \textstyle\prod_j \beta_j : F \circ \prod_j H_j \longrightarrow F' \circ \prod_j H'_j$$

in (6.6.9) is a multilinear transformation, we check the two conditions in Definition 6.5.11. The second condition, which says that the component of $\gamma(\alpha,\langle\beta\rangle)$ at a tuple with at least one entry $\mathbb{0}$ is the identity morphism $1_{\mathbb{0}}$, follows from the corresponding property for α and the β_j's.

The other condition is the commutativity of the diagram (6.5.12) for $\gamma(\alpha,\langle\beta\rangle)$. To prove this, consider the objects in (6.6.7) and in the proof of Lemma 6.6.11. We extend those notations to H' and β. For example, we have the morphisms

- $\beta_{j,\langle W_j\rangle} : \langle HW_j\rangle = H_j\langle W_j\rangle \longrightarrow H'_j\langle W_j\rangle = \langle H'W_j\rangle$,
- $\beta = \beta_{\langle W\rangle} = \big(\beta_{1,\langle W_1\rangle},\ldots,\beta_{n,\langle W_n\rangle}\big) : \langle HW\rangle \longrightarrow \langle H'W\rangle$,
- $\beta^1 = \beta_{\langle W^1\rangle} = \langle \beta_{\langle W\rangle} \circ_j \beta_{j,\langle W_j^1\rangle}\rangle : \langle HW^1\rangle \longrightarrow \langle H'W^1\rangle$, and
- $\beta_{\langle W^{01}\rangle} = \langle \beta_{\langle W\rangle} \circ_j \beta_{j,\langle W_j^{01}\rangle}\rangle : \langle HW^{01}\rangle \longrightarrow \langle H'W^{01}\rangle$.

The diagram (6.5.12) for $\gamma(\alpha, \langle\beta\rangle)$ is the outer diagram in D below.

$$\begin{tikzcd}[column sep=small]
& & F\langle HW\rangle \oplus F\langle HW^1\rangle \arrow[dll, "F\beta \oplus F\beta^1"'] \arrow[d, "\alpha\oplus\alpha"] \arrow[drr, "F_j^2"] & & \\
F\langle H'W\rangle \oplus F\langle H'W^1\rangle \arrow[d, "\alpha\oplus\alpha"'] & & F'\langle HW\rangle \oplus F'\langle HW^1\rangle \arrow[dll, "F'\beta\oplus F'\beta^1"'] \arrow[d, "F_j'^2"] & (6.5.12) & F\langle HW \circ_j \langle HW_j\rangle^{0,1}\rangle \arrow[dll, "\alpha"'] \arrow[ddll, "F\varepsilon"] \arrow[d, "F\langle 1\circ_j H^2_{j,i}\rangle"] \\
F'\langle H'W\rangle \oplus F'\langle H'W^1\rangle \arrow[d, "F_j'^2"'] & \mathsf{nat} & F'\langle HW \circ_j \langle HW_j\rangle^{0,1}\rangle \arrow[dll, "F'\varepsilon"'] & (6.5.12) & F\langle HW^{01}\rangle \arrow[d, "F\beta_{\langle W^{01}\rangle}"] \\
F'\langle H'W \circ_j \langle H'W_j\rangle^{0,1}\rangle \arrow[drr, "F'\langle 1\circ_j \kappa\rangle"'] & & F\langle H'W \circ_j \langle H'W_j\rangle^{0,1}\rangle \arrow[ll, "\alpha"'] \arrow[rr, "F\langle 1\circ_j\kappa\rangle"] & & F\langle H'W^{01}\rangle \arrow[dll, "\alpha"] \\
& & F'\langle H'W^{01}\rangle & &
\end{tikzcd}$$

- The morphism ε is
$$\langle \beta_{\langle W\rangle} \circ_j (\beta_{j,\langle W_j\rangle} \oplus \beta_{j,\langle W_j^1\rangle})\rangle.$$
- The morphism
$$\kappa : \langle H'W_j\rangle^{0,1} \longrightarrow \langle H'W_j^{01}\rangle$$
is the ith linearity constraint of H_j'.
- The three unlabeled subdiagrams are commutative by the naturality of α.
- The subdiagram labeled by nat is commutative by the naturality of $F_j'^2$.
- The upper right subdiagram is commutative by (6.5.12) for α.
- The remaining subdiagram is obtained from the diagram (6.5.12) for β by applying $F\langle \beta_{\langle W\rangle} \circ_j -\rangle$, so it is commutative.

This shows that γ is well defined on multilinear transformations.

Moreover, γ preserves identity multilinear transformations because horizontal composition preserves identity natural transformations. Finally, γ preserves the categorical composition in its (co)domain by the following facts:

- The categorical composition in each category $\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle\,;\mathsf{D}\big)$ is vertical composition of natural transformations.
- On multilinear transformations, γ is defined by horizontal composition of natural transformations.
- Natural transformations satisfy the middle-four exchange property
$$(\theta'\theta) * (\phi'\phi) = (\theta' * \phi')(\theta * \phi)$$
with respect to vertical composition and horizontal composition, which are denoted by, respectively, concatenation and $*$.

This finishes the proof that γ is a functor. □

Theorem 6.6.13. $\mathsf{PermCat}^{\mathsf{su}}$ *is a* Cat*-enriched multicategory.*

Proof. We check the axioms in Definition 6.1.1.

- The symmetric group action (6.6.2) is a well-defined right Σ_n-action because it is given by pre-composition with $\sigma \in \Sigma_n$ or 1_σ.

- The equivariance axioms (6.1.6) and (6.1.7) also follow from the fact that the symmetric group action (6.6.2) in $\mathsf{PermCat}^{\mathsf{su}}$ is defined by precomposition with a permutation.
- The unity axioms (6.1.4) and (6.1.5) follow from the fact that, for each small permutative category C, the C-colored unit in $\mathsf{PermCat}^{\mathsf{su}}$ is the identity functor 1_{C} with the identity linearity constraint (6.6.1).

For the associativity axiom (6.1.3) on objects, consider multilinear functors

- $\big(F, \{F_j^2\}_{1 \le j \le n}\big) \in \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle\,; \mathsf{D}\big)$,
- $\big(H_j, \{H_{j,i}^2\}_{1 \le i \le k_j}\big) \in \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{B}_j \rangle\,; \mathsf{C}_j\big)$ for $1 \le j \le n$, and
- $\big(G_{j,i}, \{G_{j,i,h}^2\}_{1 \le h \le l_{j,i}}\big) \in \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{A}_{j,i} \rangle\,; \mathsf{B}_{j,i}\big)$ with

$$\langle \mathsf{A}_{j,i} \rangle = \big(\mathsf{A}_{j,i,1}, \ldots, \mathsf{A}_{j,i,l_{j,i}}\big)$$

an $l_{j,i}$-tuple of small permutative categories for $1 \le j \le n$ and $1 \le i \le k_j$.

Then each composite in the associativity diagram (6.1.3) sends the tuple

$$\begin{gathered}\Big((F, \{F_j^2\}), \big((H_j, \{H_{j,i}^2\})\big)_{1 \le j \le n}, \Big(\big((G_{j,i}, \{G_{j,i,h}^2\})\big)_{1 \le i \le k_j}\Big)_{1 \le j \le n}\Big) \\ \in \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle\,; \mathsf{D}\big) \times \prod_{j=1}^{n} \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{B}_j \rangle\,; \mathsf{C}_j\big) \times \prod_{j=1}^{n} \prod_{i=1}^{k_j} \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{A}_{j,i} \rangle\,; \mathsf{B}_{j,i}\big)\end{gathered} \tag{6.6.14}$$

to the composite functor

$$\prod_{j=1}^{n} \prod_{i=1}^{k_j} \prod_{h=1}^{l_{j,i}} \mathsf{A}_{j,i,h} \xrightarrow{\prod_j \prod_i G_{j,i}} \prod_{j=1}^{n} \prod_{i=1}^{k_j} \mathsf{B}_{j,i} \xrightarrow{\prod_j H_j} \prod_{j=1}^{n} \mathsf{C}_j \xrightarrow{F} \mathsf{D}.$$

To see that their corresponding linearity constraints are equal, for $1 \le j \le n$, $1 \le i \le k_j$, and $1 \le h \le l_{j,i}$, consider

- $l_j = l_{j,1} + \cdots + l_{j,k_j}$,
- objects $V_{j,i,h} \in \mathsf{A}_{j,i,h}$,
- for a particular choice of (j,i,h), an object $V'_{j,i,h} \in \mathsf{A}_{j,i,h}$ with $p = l_1 + \cdots + l_{j-1}$ and $q = l_{j,1} + \cdots + l_{j,i-1}$,
- $\langle V_{j,i} \rangle = (V_{j,i,1}, \ldots, V_{j,i,l_{j,i}}) \in \prod_{h=1}^{l_{j,i}} \mathsf{A}_{j,i,h}$,
- $\langle V'_{j,i} \rangle = \langle V_{j,i} \circ_h V'_{j,i,h} \rangle$,
- $\langle V''_{j,i} \rangle = \langle V_{j,i} \circ_h (V_{j,i,h} \oplus V'_{j,i,h}) \rangle$,
- $\langle V_j \rangle = \oplus_{i=1}^{k_j} \langle V_{j,i} \rangle \in \prod_{i=1}^{k_j} \prod_{h=1}^{l_{j,i}} \mathsf{A}_{j,i,h}$,
- $\langle V'_j \rangle = \langle V_j \circ_{q+h} V'_{j,i,h} \rangle$,
- $\langle V \rangle = \oplus_{j=1}^{n} \langle V_j \rangle \in \prod_{j=1}^{n} \prod_{i=1}^{k_j} \prod_{h=1}^{l_{j,i}} \mathsf{A}_{j,i,h}$,
- $\langle V' \rangle = \langle V \circ_{p+q+h} V'_{j,i,h} \rangle$,
- $\langle G_j V_j \rangle = \big(G_{j,1}\langle V_{j,1} \rangle, \ldots, G_{j,k_j}\langle V_{j,k_j} \rangle\big) \in \prod_{i=1}^{k_j} \mathsf{B}_{j,i}$,
- $\langle G_j V'_j \rangle = \langle G_j V_j \circ_i G_{j,i}\langle V'_{j,i} \rangle \rangle$,
- $\langle HGV \rangle = \big(H_1\langle G_1 V_1 \rangle, \ldots, H_n\langle G_n V_n \rangle\big) \in \prod_{j=1}^{n} \mathsf{C}_j$, and
- $\langle HGV' \rangle = \langle HGV \circ_j H_j\langle G_j V'_j \rangle \rangle$.

By the definition (6.6.8) and the functoriality of F, each composite in the associativity diagram (6.1.3), when applied to the multilinear functors in (6.6.14) and the objects above, has $(p+q+h)$th linearity constraint the following composite in D.

$$\begin{tikzcd}
F\langle HGV\rangle \oplus F\langle HGV'\rangle \arrow[d, "F_j^2"'] & F\langle HGV \circ_j H_j\langle G_jV_j \circ_i G_{j,i}\langle V''_{j,i}\rangle\rangle\rangle \\
F\langle HGV \circ_j (H_j\langle G_jV_j\rangle \oplus H_j\langle G_jV'_j\rangle)\rangle \arrow[d, "F\langle 1\circ_j H^2_{j,i}\rangle"'] & \\
F\langle HGV \circ_j H_j\langle G_jV_j \circ_i (G_{j,i}\langle V_{j,i}\rangle \oplus G_{j,i}\langle V'_{j,i}\rangle)\rangle\rangle \arrow[uur, bend right, "F\langle 1\circ_j H_j\langle 1\circ_i G^2_{j,i,h}\rangle\rangle"'] &
\end{tikzcd}$$

This proves that γ in (6.6.5) satisfies the associativity axiom (6.1.3) on multilinear functors.

Since multilinear transformations are natural transformations with additional properties, but not additional data, the associativity of γ on multilinear transformations follows from the associativity of horizontal composition of natural transformations. $\square$

6.7. Notes

6.7.1 (Enriched Multicategories)**.** For further development of the theory of enriched multicategories, we refer the reader to [**YJ15, Yau20**]. $\diamond$

6.7.2 (Small Permutative Categories)**.** In Section 6.6, we proved in detail that $\mathsf{PermCat}^{\mathsf{su}}$ is a Cat-enriched multicategory. This fact was stated in [**EM06**, 1.1], but a detailed proof was not given there. Our proof in Section 6.6 that $\mathsf{PermCat}^{\mathsf{su}}$ is a Cat-enriched multicategory is a generalization of the proof that small symmetric monoidal categories, symmetric monoidal functors, and monoidal natural transformations form a 2-category. There are two main differences between the cases of $\mathsf{PermCat}^{\mathsf{su}}$ and small symmetric monoidal categories:

- The constraint 2-by-2 axiom (6.5.7) only happens in multilinear functors. So Lemma 6.6.10, which proves the constraint 2-by-2 axiom for $\gamma(F,(H_j))$, does not have an analogue for symmetric monoidal functors.
- The notation for $\mathsf{PermCat}^{\mathsf{su}}$ is much more complicated because it involves lists of objects instead of one object. $\diamond$

Part 2

Algebraic *K*-Theory

CHAPTER 7

Homotopy Theory Background

In Part 2 we develop the E_n-monoidal K-theory spectra associated to small E_n-monoidal categories. In this chapter we review the parts of homotopy theory that will be necessary for the constructions and their applications.

Convention 7.0.1 (K-theory and J-theory)**.** Throughout Part 2, and throughout this entire work, we use the umbrella term *K-theory* for any construction taking values in a category of spectra. This is the standard usage in the literature and is motivated by Quillen's definition of the higher algebraic K-groups of a commutative ring R as the homotopy groups of an associated spectrum KR **[Qui73, Gra76]**. See Note 10.8.10 for further discussion of this point.

The K-theory constructions we develop will all take values in the category of symmetric spectra and will all be either:

- functors, such as K^{Se} (Definition 8.5.1),
- (enriched) symmetric monoidal functors, such as $\mathsf{K}^{\mathcal{F}}$ (Definition 8.2.5) and $\mathsf{K}^{\mathcal{G}}$ (Definition 9.3.14), or
- (enriched) multifunctors, such as K^{EM} (Definition 10.3.32).

We follow **[EM06, EM09]** in using the umbrella term *J-theory* for constructions that take values in certain pointed diagrams of small categories. The two cases of interest for us will be Γ-categories (Definition 8.1.17) and $\mathcal{G}_*$-categories (Definition 9.2.1). The motivation for this terminology is that construction of the respective J-theory is a key step in both the Segal and Elmendorf-Mandell K-theory constructions. See also Note 8.6.1. $\diamond$

Now we describe the material in this chapter.

Organization. In Section 7.1 we review simplicial objects in a general category, focusing on the special case of simplicial sets. In Section 7.2 we describe the nerve functor from small categories to simplicial sets. This functor provides the fundamental connection between categorical algebra and homotopy theory through which all of our applications factor.

Section 7.3 introduces a certain diagram category of pointed simplicial sets known as symmetric sequences. These are sequences of pointed simplicial sets with compatible symmetric group actions. Symmetric spectra are defined in Section 7.4 as left modules over the symmetric sequence formed by the simplicial spheres, S^n. The Segal and Elmendorf-Mandell K-theory functors defined in Chapters 8, 9, and 10 will take values in the category of symmetric spectra.

In Section 7.5 we show that the category of symmetric spectra is complete and cocomplete. In Section 7.6 we describe its smash product and internal hom. The operad actions that define E_n-symmetric spectra for $1 \leq n \leq \infty$ in Chapters 11, 12, and 13 are defined in this symmetric monoidal closed structure.

In Section 7.7 we review the definitions of Quillen model categories, which provide a general framework for homotopy theory. In Section 7.8 we describe several important examples, including model structures for pointed simplicial sets and for symmetric spectra.

7.1. Simplicial Objects

Simplicial sets, which we define below, give a combinatorial approach to working with topological spaces. We will also use simplicial categories, and therefore begin with a presentation of simplicial objects in general categories.

Definition 7.1.1. For natural numbers $n \geq 0$, the totally ordered set $\{0 < 1 < \cdots < n\}$ is denoted $\underline{n}$. We let Δ denote the category whose objects are $\underline{n}$ for all $n \geq 0$ and whose morphisms are order-preserving functions.

The morphisms of Δ are generated by

$$\begin{aligned} d^i &: \underline{n-1} \longrightarrow \underline{n}, \quad 0 \leq i \leq n, \quad \text{and} \\ s^i &: \underline{n+1} \longrightarrow \underline{n}, \quad 0 \leq i \leq n, \end{aligned}$$

known as the *coface* and *codegeneracy* morphisms, respectively. The coface d^i is the unique order-preserving injection whose image does not contain $i \in \underline{n}$. The codegeneracy s^i is the unique order-preserving surjection such that the preimage of $i \in \underline{n}$ is a two-element set.

The coface and codegeneracy morphisms are subject to the following *cosimplicial identities*.

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \quad \text{if} \quad i < j \\ s^j d^i &= d^i s^{j-1} \quad \text{if} \quad i < j \\ s^j d^j &= 1 = s^j d^{j+1} \\ s^j d^i &= d^{i-1} s^j \quad \text{if} \quad i > j+1 \\ s^j s^i &= s^i s^{j+1} \quad \text{if} \quad i \leq j \end{aligned}$$

◇

The cofaces, codegeneracies, and cosimplicial identities are a generating set of morphisms and relations for Δ.

Explanation 7.1.2. In the context of Definition 7.1.1, the coface and codegeneracy maps are given by the following formulas. For $0 \leq i \leq n$,

$$d^i(k) = \begin{cases} k, & \text{if} \quad k < i \\ k+1, & \text{if} \quad k \geq i \end{cases} \quad \text{and} \quad s^i(k) = \begin{cases} k, & \text{if} \quad k \leq i \\ k-1, & \text{if} \quad k > i. \end{cases}$$

◇

Explanation 7.1.3. It will sometimes be useful to regard the totally ordered set $\underline{n}$ as a category with a unique morphism $i \longrightarrow j$ for each $i \leq j$ in $\underline{n}$. Because the morphisms in Δ are order-preserving, they yield functors between the corresponding categories. ◇

Definition 7.1.4. Suppose C is a category. A *simplicial* C*-object* is a functor

$$X : \Delta^{\mathsf{op}} \longrightarrow \mathsf{C}.$$

We let $X_n = X\underline{n}$ and follow the usual convention of letting d_i, respectively s_i, denote Xd^i, respectively Xs^i. These are called *face* and *degeneracy* morphisms. When necessary for clarity, we write d_i^X and s_i^X to indicate the functor X. Morphisms $f : X \longrightarrow Y$ of simplicial objects are given by natural transformations, and we let f_n denote the compoent of f at $\underline{n}$. The category of simplicial C-objects is denoted sC. ⋄

Our two most frequent uses of this notion will be the cases $\mathsf{C} = \mathsf{Set}$ or $\mathsf{C} = \mathsf{Cat}$, yielding the category of simplicial sets, sSet, and the category of simplicial small categories, sCat, respectively.

Example 7.1.5 (One-Point Simplicial Set)**.** The *one-point simplicial set* is denoted $*$ and is defined by the constant functor at the one-point set $*$. ⋄

Explanation 7.1.6. The face and degeneracy morphisms are subject to the following *simplicial identities*.

$$\begin{aligned}
d_i d_j &= d_{j-1} d_i \quad \text{if} \quad i < j \\
d_i s_j &= s_{j-1} d_i \quad \text{if} \quad i < j \\
d_j s_j &= 1 = d_{j+1} s_j \\
d_i s_j &= s_j d_{i-1} \quad \text{if} \quad i > j+1 \\
s_i s_j &= s_{j+1} s_i \quad \text{if} \quad i \leq j
\end{aligned}$$

⋄

Explanation 7.1.7 (Bisimplicial Sets)**.** In the case $\mathsf{C} = \mathsf{sSet}$, we call simplicial sSet-objects *bisimplicial sets*, and we implicitly identify $\mathsf{s}(\mathsf{sSet})$ with the category of functors and natural transformations $\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \longrightarrow \mathsf{Set}$. Composing with the diagonal on Δ^{op} gives a functor from bisimplicial sets to simplicial sets. ⋄

Definition 7.1.8. For a simplicial set X, the elements of X_n are called *n-simplices*. Those which are in the image of a degeneracy s_i are called *degenerate* n-simplices. Often we focus only on the nondegenerate simplices. ⋄

The following definitions provide important basic examples of simplicial sets.

Definition 7.1.9. The *standard n-simplex*, $\Delta^n \in \mathsf{sSet}$, is the simplicial set defined by the represented functor $\Delta(-, \underline{n})$. The *fundamental simplex* $\iota_n = 1_{\underline{n}}$ is the identity morphism in Δ^n_n. The *boundary*, $\partial\Delta^n$, is the smallest simplicial subset of Δ^n containing $d_i \iota_n$ for all $0 \leq i \leq n$.

The *k-horn* $\Lambda^n_k \subset \Delta^n$ is the simplicial subset generated by all the faces $d_i \iota_n$ except for the kth face, $d_k \iota_n$. ⋄

Explanation 7.1.10. The standard n-simplex may be visualized as follows. Its 0-simplices are the elements of $\underline{n}$, which may be thought of as vertices. Its nondegenerate 1-simplices are pairs $i < j$ in $\underline{n}$, which may be thought of as edges from vertex i to vertex j. Its nondegenerate 2-simplices are triples $i < j < k$ in $\underline{n}$, which may be thought of as oriented triangles. Diagrams for $n = 2$ and $n = 3$ are drawn below.

$$\begin{tikzcd}
& 1 \arrow[dr, "{(1,2)}"] & \\
0 \arrow[ur, "{(0,1)}"] \arrow[rr, "{(0,2)}"'] & & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dr, "{(1,3)}"] \arrow[dd, "{(1,2)}"] & \\
0 \arrow[ur, "{(0,1)}"] \arrow[rr, dashed, "{(0,3)}"] \arrow[dr, "{(0,2)}"'] & & 3 \\
& 2 \arrow[ur, "{(2,3)}"'] &
\end{tikzcd}$$

There are no nondegenerate simplices of Δ^n above dimension n. Each Δ^n has a *long spine* consisting of n 1-simplices $(i, i+1)$ and a *short spine* consisting of the 1-simplex $(0, n)$.

The kth horn Λ^n_k consists of all faces that include the kth 0-simplex. For example, Λ^2_1 consists of the 1-simplices

$$0 \xrightarrow{(0,1)} 1 \xrightarrow{(1,2)} 2$$

and Λ^3_1 consists of three triangular faces, omitting the face $(0,2,3)$. ⋄

Definition 7.1.11. Since $\Delta^n = \Delta(-,\underline{n})$ is covariant in $\underline{n}$, we have a functor

$$\Delta^? : \Delta \longrightarrow \mathsf{sSet}$$

whose value at $\underline{n} \in \Delta$ is Δ^n. ⋄

Definition 7.1.12. The *simplicial circle* is denoted S^1 and is defined to be the quotient $\Delta^1/\partial\Delta^1$. For $0 \le j \le n+1$, let $h_j \in \Delta(\underline{n},\underline{1})$ denote the function

$$h_j(i) = \begin{cases} 0, & \text{if } i < j \\ 1, & \text{if } i \ge j. \end{cases}$$

Then the n-simplices of S^1 form an $(n+1)$-element set given by

$$S^1_n = \Delta(\underline{n},\underline{1})/(h_{n+1} = h_0).$$

Identifying S^1_n with $\underline{n}$ via $h_j \leftrightarrow j$ for $0 \le j \le n$, then the face $d_i : S^1_n \longrightarrow S^1_{n-1}$ is the unique order-preserving surjection such that $d_i(i+1) = i$ and $s_i : S^1_n \longrightarrow S^1_{n+1}$ is the unique order-preserving injection whose image skips $i+1$. Thus all of the faces and degeneracies preserve the elements $0 \leftrightarrow h_0$, and we take this as a simplicial basepoint of S^1. The only nondegenerate simplex is $1 \leftrightarrow h_1 \in S^1_1$. ⋄

Definition 7.1.13. We let Top denote the category of compactly generated weak Hausdorff spaces and continuous functions. For each $n \ge 0$ we let $\triangle^n$ denote the *topological n-simplex*

$$\triangle^n = \left\{(t_0,\ldots,t_n) \mid 0 \le t_i \le 1,\ \textstyle\sum_i t_i = 1\right\} \subset \mathbb{R}^{(n+1)}.$$ ⋄

Definition 7.1.14. Given a simplicial set X, the *geometric realization* $|X|$ is the topological space defined as the coequalizer

$$\operatorname{coeq}\left(\coprod_{\underline{n} \,\to\, \underline{m} \in \Delta} X_m \times \triangle^n \rightrightarrows \coprod_{\underline{n}\in\Delta} X_n \times \triangle^n \right).$$

Geometric realization defines a functor

$$|-| : \mathsf{sSet} \longrightarrow \mathsf{Top}$$

and has a right adjoint given by the *total singular complex*

$$\mathsf{Sing} : \mathsf{Top} \longrightarrow \mathsf{sSet}.$$ ⋄

Explanation 7.1.15. The coequalizer in the definition of geometric realization is the coend

$$\int^{\underline{n}\in\Delta} \mathsf{sSet}(\Delta^n, X) \times \triangle^n \cong \int^{\underline{n}\in\Delta} X_n \times \triangle^n.$$ ⋄

Example 7.1.16 (Simplices and Circles)**.** The geometric realization of the standard n-simplex Δ^n is the topological n-simplex, $\triangle^n$. The geometric realization of the simplicial circle is a topological circle. ⋄

As a diagram category, sC inherits limits and colimits from C. We record several of the key consequences here, recalling from the following.

- Tensors and cotensors are discussed in Section 3.9.
- Mapping objects in diagram categories are defined in Definition 3.5.19

Proposition 7.1.17. *Suppose that* C *has all limits and colimits.*

(1) *The category* C *is tensored and cotensored over* Set, *with*

$$X \otimes S = \coprod_S X \quad \text{and} \quad X^S = \prod_S X$$

for $X \in \mathsf{C}$ *and* $S \in \mathsf{Set}$.

(2) *Limits and colimits in* sC *exist and are computed objectwise.*

(3) *If* C *is monoidal (resp. braided monoidal, resp. symmetric monoidal), then so is* sC, *with the monoidal product computed objectwise. The monoidal unit of* sC *is the constant simplicial object at the monoidal unit of* C.

(4) *If* $\mathsf{C} = (\mathsf{C}, \otimes, \mathbb{1}, [-,-])$ *is symmetric monoidal closed, then* sC *is enriched, tensored, and cotensored over* C *with*

$$\begin{aligned} \mathsf{Map}(X,Y) &= \int_{\underline{n} \in \Delta^{\mathsf{op}}} [X_n, Y_n] \\ (X \otimes A)_n &= X_n \otimes A \\ (X^A)_n &= [A, X_n] \end{aligned}$$

where $n \geq 0$, X *and* Y *are objects of* sC, *and* A *is an object of* C.

Proof.

(1) The tensor and cotensor over Set are provided by the isomorphisms

$$\mathsf{C}(\textstyle\coprod_S X, Y) \cong \prod_S \mathsf{C}(X,Y) \cong \mathsf{Set}(S, \mathsf{C}(X,Y))$$
and
$$\mathsf{C}(Y, \textstyle\prod_S X) \cong \prod_S \mathsf{C}(X,Y) \cong \mathsf{Set}(S, \mathsf{C}(X,Y))$$

for a set S and objects X, Y of C.

(2) For a small category $\mathcal{D}$ and a functor

$$F : \mathcal{D} \longrightarrow \mathsf{sC},$$

the definitions

$$(\mathsf{lim}_{\mathcal{D}} F)_n = \mathsf{lim}_{\mathcal{D}}(F_n) \quad \text{and} \quad (\mathsf{colim}_{\mathcal{D}} F)_n = \mathsf{colim}_{\mathcal{D}}(F_n)$$

define simplicial objects in C by universality of the limits and colimits.

(3) If C has monoidal product $\otimes$, the monoidal product of simplicial objects X and Y is given by

$$(X \otimes Y)_n = X_n \otimes Y_n.$$

This simplicial object is the composite of

$$\Delta^{\mathsf{op}} \xrightarrow{\text{diag}} \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \xrightarrow{X \times Y} \mathsf{C} \times \mathsf{C} \xrightarrow{\otimes} \mathsf{C}$$

where diag denotes the diagonal functor. The monoidal, respectively braided monoidal, respectively symmetric monoidal structure and axioms follow from those of $(\mathsf{C}, \otimes)$.

(4) When C is symmetric monoidal closed, then sC is a special case of the mapping diagram category denoted (Δ^{op})-C in Definition 3.5.19. This provides a C-enrichment by Lemma 3.5.24. The tensor and cotensor isomorphisms

$$\mathsf{Map}(X \otimes A, Y) \cong [A, \mathsf{Map}(X, Y)]$$
and
$$\mathsf{Map}(X, Y^A) \cong [A, \mathsf{Map}(X, Y)]$$

are given by commuting ends with $[-,-]$ and by the closed structure of C, as in Explanation 3.9.16. □

Explanation 7.1.18. When $(\mathsf{C}, \otimes) = (\mathsf{Set}, \times)$, the equalizer formula (3.5.7) shows that

$$\mathsf{Map}(X, Y) = \mathsf{sSet}(X, Y)$$

is the set of natural transformations, that is, simplicial set morphisms, from X to Y. See Example 3.8.14 for a discussion of the case $(\mathsf{C}, \otimes) = (\mathsf{Cat}, \times)$, where $\mathsf{Map}(X, Y)$ is the category whose objects are 2-natural transformations and whose morphisms are modifications. ⋄

Proposition 7.1.19. *The category of simplicial sets is symmetric monoidal closed with product given by the Cartesian product*

$$(X \times Y)_n = X_n \times Y_n$$

and internal hom given by

$$\mathsf{Hom}(X, Y)_n = \mathsf{Map}(X \times \Delta^n, Y). \tag{7.1.20}$$

Proof. The symmetric monoidal structure is that of Proposition 7.1.17 (3). Writing $[-,-]$ for $\mathsf{Set}(-,-)$ and recalling

$$\Delta^n_m = \Delta(\underline{m}, \underline{n}) = \Delta^{\mathsf{op}}(\underline{n}, \underline{m})$$

from Definition 7.1.9, the closed monoidal adjunction is given as follows:

$$\begin{aligned}
\mathsf{Map}(X, \mathsf{Hom}(Y, Z)) &= \int_{\underline{n} \in \Delta^{\mathsf{op}}} \big[X_n, \mathsf{Hom}(Y, Z)_n\big] = \int_{\underline{n} \in \Delta^{\mathsf{op}}} \big[X_n, \mathsf{Map}(Y \times \Delta^n, Z)\big] \\
&= \int_{\underline{n} \in \Delta^{\mathsf{op}}} \Big[X_n, \int_{\underline{m} \in \Delta^{\mathsf{op}}} [Y_m \times \Delta^n_m, Z_m]\Big] \\
&\cong \int_{\underline{m} \in \Delta^{\mathsf{op}}} \int_{\underline{n} \in \Delta^{\mathsf{op}}} \big[\Delta^n_m \times X_n, [Y_m, Z_m]\big] \\
&\cong \int_{\underline{m} \in \Delta^{\mathsf{op}}} \Big[\int^{\underline{n} \in \Delta^{\mathsf{op}}} \Delta^{\mathsf{op}}(\underline{n}, \underline{m}) \times X_n, [Y_m, Z_m]\Big] \\
(\star) \qquad &\cong \int_{\underline{m} \in \Delta^{\mathsf{op}}} \big[X_m, [Y_m, Z_m]\big] \\
&\cong \mathsf{Map}(X \times Y, Z).
\end{aligned}$$

In the above computation, the isomorphism labeled $\star$ is an application of the V-Yoneda Density Theorem 3.7.8 with $\mathsf{V} = \mathsf{Set}$. The other isomorphisms follow from definitions, moving co/ends in or out of hom objects (Explanation 3.5.9), commuting ends, and the closed structure for Set. □

In Chapter 4 we discussed the general theory of pointed objects, smash products, and pointed homs.

Definition 7.1.21. If C has a terminal object, $*$, then the constant simplicial C-object at $*$ is terminal in sC. Applying Definition 4.1.1 to sC, we have the category of pointed simplicial objects in C, denoted sC_*. ◇

Explanation 7.1.22. Since the basepoint of sC is constant at $*$, and morphisms of simplicial objects are defined levelwise, the category of pointed simplicial objects in C is naturally isomorphic to the category of simplicial objects in C_*. Thus the notation sC_* can be read as either $\mathsf{s}(\mathsf{C}_*)$ or $(\mathsf{sC})_*$. ◇

In the important special case $\mathsf{C} = \mathsf{Set}$, we have the following.

Definition 7.1.23 (Pointed Simplicial Sets)**.** The category of *pointed simplicial sets*, sSet_*, is the category of functors

$$\Delta^{\mathsf{op}} \longrightarrow \mathsf{Set}_*$$

and pointed natural transformations between them. It is symmetric monoidal closed with smash product $X \wedge Y$ given levelwise

$$(X \wedge Y)_n = X_n \wedge Y_n$$

for $X, Y \in \mathsf{sSet}_*$ and with internal hom object defined levelwise by

$$\mathsf{Hom}_{\mathsf{sSet}_*}(X, Y)_n = \mathsf{sSet}_*(X \wedge \Delta^n_+, Y).$$

The closed monoidal adjunction is given by a computation similar to that of Proposition 7.1.19 but with $(\mathsf{V}, \otimes) = (\mathsf{Set}_*, \wedge)$. ◇

Explanation 7.1.24. In Definitions 4.1.6 and 4.2.1 we give general constructions of smash product and pointed hom in a complete and cocomplete symmetric monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, T)$ with terminal object T. Using functoriality of pushouts and pullbacks, and commuting ends with pullbacks, the smash product and internal hom defined in Definition 7.1.23 are equivalent descriptions of those given by Definitions 4.1.6 and 4.2.1 for $(\mathsf{sSet}, \times, *, *)$. ◇

7.2. Simplicial Homotopy and Nerve

In this section we define simplicial homotopy and the nerve functor from small categories to simplicial sets.

Definition 7.2.1 (Simplicial Homotopy)**.** Suppose

$$f, g : X \longrightarrow Y$$

are morphisms of simplicial sets X and Y. A *simplicial homotopy* from f to g is a morphism of simplicial sets, H, such that the following diagram commutes.

$$\begin{tikzcd}
X \times \Delta^0 \arrow[r, "\cong"] \arrow[d, "1 \times d^1"'] & X \arrow[dr, "f"] & \\
X \times \Delta^1 \arrow[rr, "H"] & & Y \\
X \times \Delta^0 \arrow[r, "\cong"] \arrow[u, "1 \times d^0"] & X \arrow[ur, "g"'] &
\end{tikzcd}$$

In the above diagram, the unlabeled isomorphisms are the projection isomorphism

$$X \times \Delta^0 \cong X.$$

This finishes the definition of a simplicial homotopy. We say that a morphism

$$f : X \longrightarrow Y$$

is a *simplicial homotopy equivalence* if there is a morphism

$$f' : Y \longrightarrow X$$

together with simplicial homotopies from 1_X to $f'f$ and from ff' to 1_Y. ◇

Explanation 7.2.2. We note that simplicial homotopy does not provide an equivalence relation without additional fibrancy conditions on X. We refer the reader to [**GJ09**, Section 1.6] for further development of the homotopy theory of simplicial sets. ◇

Definition 7.2.3. Suppose that C is a small category. For each totally ordered set $\underline{p}$, regarded as a category, we let $N_p\mathsf{C}$ denote the set of functors $\underline{p} \longrightarrow \mathsf{C}$. This is the set of objects of the functor category $\mathsf{C}^{\underline{p}}$. Then

$$N_{(-)}\mathsf{C} = \mathsf{Ob}\left(\mathsf{C}^{(-)}\right) : \Delta^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

is a simplicial set called the *nerve* of C. Thus the nerve defines a functor

$$N : \mathsf{Cat} \longrightarrow \mathsf{sSet}.$$

The nerve has a left adjoint

$$h : \mathsf{sSet} \longrightarrow \mathsf{Cat}$$

that takes a simplicial set X to a category hX defined as follows. The objects of hX are the elements of X_0. The morphisms of hX are generated by the 1-simplices $\sigma \in X_1$, where each σ is regarded as a morphism

$$\sigma : d_1\sigma \longrightarrow d_0\sigma.$$

These morphisms are subject to the relation

$$(d_0\sigma) \circ (d_2\sigma) \sim d_1\sigma$$

for all 2-simplices σ. ◇

Explanation 7.2.4. The set of 0-simplices $N_0\mathsf{C}$ is given by the set of objects of C. For a 1-simplex $f : u \longrightarrow v$ in $N_1\mathsf{C}$,

$$d_0(f) = v \quad \text{and} \quad d_1(f) = u.$$

For $p > 1$, the set of p-simplices $N_p\mathsf{C}$ is given by p-tuples $(f_1, \ldots, f_p)$ of composable morphisms

$$\bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \bullet \cdots \bullet \xrightarrow{f_p} \bullet.$$

The faces d_i are given by composing f_i with f_{i+1} for $0 < i < p$. For $i = 0$ or $i = p$, respectively, d_i is given by deleting f_1 or f_p, respectively. The morphisms f_i can be thought of as the long spine of the simplex $(f_1, \ldots, f_p)$, and their composites form its other 1-dimensional faces. The degeneracy s_i is given by inserting an identity morphism at position i. The unity and associativity axioms for composition (Definition I.1.1.1) correspond to the simplicial identities $d_id_{i+1} = d_id_i$ and $d_js_j = 1 = d_{j+1}s_j$. The other simplicial identities listed in Explanation 7.1.6 are a matter of reindexing. ◇

Proposition 7.2.5. *Suppose* C *and* D *are small categories.*

(1) *A natural transformation between functors*

$$F, G : \mathsf{C} \longrightarrow \mathsf{D}$$

induces a simplicial homotopy from NF to NG.

(2) *An adjunction*

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "L"] & \mathsf{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

with $L \dashv R$ induces a simplicial homotopy equivalence on nerves:

$$NL : N\mathsf{C} \longrightarrow N\mathsf{D}.$$

Proof. The second assertion follows from the first. To prove the first assertion, let $\mathbb{I}$ denote a category with two objects a single non-identity morphism. Let 0, respectively 1, denote the source, respectively target, of the nonidentity morphism. Let

$$i_0, i_1 : \mathbb{1} \longrightarrow \mathbb{I}$$

denote the two functors from the terminal category to $\mathbb{I}$, sending the unique object to 0 and 1, respectively. Then a natural transformation

$$\alpha : F \longrightarrow G$$

provides a functor, also denoted α, such that the following diagram commutes in Cat, where the unlabeled isomorphisms are given by the projection away from **1**.

$$\begin{tikzcd} \mathsf{C} \times \mathbf{1} \arrow[r, "\cong"] \arrow[d, "1 \times i_0"'] & \mathsf{C} \arrow[dr, "F"] & \\ \mathsf{C} \times \mathbb{I} \arrow[rr, "\alpha"] & & \mathsf{D} \\ \mathsf{C} \times \mathbf{1} \arrow[r, "\cong"] \arrow[u, "1 \times i_1"] & \mathsf{C} \arrow[ur, "G"'] & \end{tikzcd}$$

In the above diagram, the functor α sends a pair of morphisms

$$(a \xrightarrow{f} b\,,\, 0 \longrightarrow 1) \quad \text{in} \quad \mathsf{C} \times \mathbb{I}$$

to either of the two composites in the following naturality diagram for α.

$$\begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, "\alpha_a"'] & Fb \arrow[d, "\alpha_b"] \\ Ga \arrow[r, "Gf"] & Gb \end{tikzcd}$$

Since the nerve functor N is a right adjoint, it commutes with the Cartesian product. Taking nerve of the diagram above therefore provides a simplicial homotopy from NF to NG. □

Example 7.2.6 (Bar Construction)**.** Suppose that G is a group, and let ΣG denote the 1-object category with morphisms given by G. Then the nerve $N(\Sigma G)$ is the *simplicial bar construction* on G. It has a single 0-simplex, and the set of p-simplices for $p > 0$ is given by p-tuples of elements of G. The faces d_i for $0 < i < p$ are given by multiplying group elements, while d_0 and d_p are given by deleting the first and last element of the tuple, respectively. The degeneracies are given by inserting identities in the appropriate positions. ◇

Example 7.2.7 (Nerve of a Simplicial Category)**.** Suppose A is a simplicial category. Composing with the nerve, we have a simplicial object in sSet:

$$\Delta^{\mathsf{op}} \xrightarrow{A} \mathsf{Cat} \xrightarrow{N} \mathsf{sSet}.$$

This yields a bisimplicial set, and restriction to the diagonal of $\Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}}$ yields a simplicial set

$$\overline{N}A : \Delta^{\mathsf{op}} \longrightarrow \mathsf{Set}$$

given by $(\overline{N}A)_p = N_p A_p$ for $\underline{p} \in \Delta^{\mathsf{op}}$. ◇

Example 7.2.8 (Classifying Spaces)**.** For a small category C, the geometric realization of the nerve $N\mathsf{C}$ is called the *classifying space* of C and denoted $B\mathsf{C} = |N\mathsf{C}|$. If $\mathsf{C} = \Sigma G$ is a 1-object groupoid, then $B\Sigma G$ is an Eilenberg-Mac Lane space of type $(G, 1)$. It is usually denoted BG and called the classifying space of G (instead of ΣG). ◇

Proposition 7.2.9. *The nerve is a strong symmetric monoidal functor*

$$N : (\mathsf{Cat}, \times, \mathbf{1}) \longrightarrow (\mathsf{sSet}, \times, *).$$

Proof. Recall from Definition 7.2.3 that N is a right adjoint. Therefore, N commutes with small limits and, in particular, preserves terminal objects and commutes with Cartesian products. □

7.3. Symmetric Sequences of Pointed Simplicial Sets

This section describes symmetric sequences in sSet_*. These are used to describe symmetric spectra in Section 7.4. Recall from Definition I.2.4.1 the finite ordinal category Σ whose objects are natural numbers and morphisms are permutations.

Definition 7.3.1. A *symmetric sequence* $X = \{X_k\}_{k \geq 0}$ is a functor

$$X : \Sigma \longrightarrow \mathsf{sSet}_*.$$

Thus each $X_k = Xk$ is an object of sSet_* and is equipped with an action of the symmetric group

$$\Sigma_k \longrightarrow \mathsf{sSet}_*(X_k, X_k)_0.$$

The category of symmetric sequences is the category of functors $\Sigma \longrightarrow \mathsf{sSet}_*$ and natural transformations between them, denoted sSet_*^{Σ}. ◇

Explanation 7.3.2. A symmetric sequence consists of pointed simplicial sets X_k for $k \geq 0$ together with a basepoint-preserving action of the symmetric group Σ_k on each X_k. A morphism $X \longrightarrow Y$ of symmetric sequences consists of Σ_k-equivariant and basepoint-preserving morphisms $X_k \longrightarrow Y_k$ for each $k \geq 0$. ◇

We will use the additive structure of Σ below along with the unitary enrichment Definition 3.8.9 over sSet_*. Recall the Day convolution and hom diagram from Definition 3.7.3. The following is a special case of the convolution product (3.7.4) for $\mathsf{V} = \mathsf{sSet}_*$ and $\mathcal{D} = \Sigma$.

Definition 7.3.3. Suppose that X and Y are two symmetric sequences. The *Day convolution*, $X \square Y$, is a symmetric sequence given by

$$(X \square Y)_k = \bigvee_{p+q=k} \Sigma_k \times_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q.$$

The fibre product is taken with respect to the inclusion

$$i_{p,q} : \Sigma_p \times \Sigma_q \hookrightarrow \Sigma_{(p+q)}$$

where Σ_p permutes the first p elements and Σ_q permutes the last q elements.

The unit object I is the symmetric sequence given by $I_0 = S^0$ and $I_k = *$ for $k > 0$. The unit isomorphisms

$$\lambda : I \square X \longrightarrow X \quad \text{and} \quad \rho : X \square I \longrightarrow X$$

are induced by the isomorphisms

$$\Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} (* \wedge X_q) \cong * \quad \text{and} \quad \Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} (X_p \wedge *) \cong *$$

along with the unit isomorphisms for S^0 in sSet_*. The associativity isomorphism

$$\alpha : (X \square Y) \square Z \xrightarrow{\cong} X \square (Y \square Z)$$

is given on summands by the composite isomorphisms shown below, where the first and third are the universal isomorphisms for iterated pullbacks and the second is induced by the associativity in sSet_*.

$$\begin{array}{c}
\Sigma_{(p+q+r)} \times_{\Sigma_{(p+q)} \times \Sigma_r} ((\Sigma_{p+q} \times_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q) \wedge Z_r) \\
\big\downarrow \cong \\
\Sigma_{(p+q+r)} \times_{\Sigma_p \times \Sigma_q \times \Sigma_r} ((X_p \wedge Y_q) \wedge Z_r) \\
\big\downarrow \cong \\
\Sigma_{(p+q+r)} \times_{\Sigma_p \times \Sigma_q \times \Sigma_r} (X_p \wedge (Y_q \wedge Z_r)) \\
\big\downarrow \cong \\
\Sigma_{(p+q+r)} \times_{\Sigma_p \times \Sigma_{(q+r)}} (X_p \wedge (\Sigma_{q+r} \times_{\Sigma_q \times \Sigma_r} Y_q \wedge Z_r))
\end{array}$$

There is a symmetry isomorphism

$$\xi : X \square Y \longrightarrow Y \square X$$

given on summands by the isomorphism

$$\xi_{p,q} : \Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \xrightarrow{\cong} \Sigma_{(q+p)} \times_{\Sigma_q \times \Sigma_p} Y_q \wedge X_p$$

induced by the symmetry on sSet_* and the block permutation $\chi_{p,q} \in \Sigma_{(p+q)}$ that swaps the first p and last q elements. Naturality of the symmetry, associativity, and unit isomorphisms follows by naturality of the canonical morphisms together with naturality of the corresponding data in sSet_*. ⋄

Definition 7.3.4. For each $n \in \Sigma$, let $\Sigma[n]$ denote the symmetric sequence given by the corepresented functor

$$\Sigma[n] = \Sigma(\underline{n}, -) : \Sigma \longrightarrow \mathsf{sSet}_*,$$

with $\Sigma(\underline{n}, \underline{m})$ being the one point simplicial set for $m \neq n$ and the constant simplicial set Σ_n with basepoint the identity permutation for $m = n$. ⋄

Definition 7.3.5. Suppose X and Y are symmetric sequences and A is a pointed simplicial set.

(1) The *simplicial tensor* is the symmetric sequence $X \wedge A$ defined for each $k \geq 0$ as
$$(X \wedge A)_k = X_k \wedge A$$
with the diagonal action of Σ_k acting trivially on A. As a functor out of Σ, it is the composite
$$\Sigma \xrightarrow{X} \mathsf{sSet}_* \xrightarrow{(-)\wedge A} \mathsf{sSet}_*.$$
(2) The *simplicial cotensor* is the symmetric sequence X^A defined for each $k \geq 0$ as
$$(X^A)_k = (X_k)^A$$
with the pointwise action of Σ_k. As a functor out of Σ, it is the composite
$$\Sigma \xrightarrow{X} \mathsf{sSet}_* \xrightarrow{(-)^A} \mathsf{sSet}_*.$$
(3) The *symmetric mapping object* is the pointed simplicial set
$$\mathsf{Map}_\Sigma(X,Y) = \mathsf{sSet}_*^\Sigma(X \wedge \Delta^?_+, Y),$$
where $\Delta^?_+$ is given by adjoining a disjoint basepoint to $\Delta^?$ (see Definition 7.1.11).

(4) The *symmetric hom object* is the symmetric sequence
$$\mathsf{Hom}_\Sigma(X,Y) = \mathsf{Map}_\Sigma(X \,\square\, \Sigma[-]_+, Y)$$
where $\Sigma[-]_+$ is given by adjoining a disjoint basepoint to $\Sigma[-]$.

For the closed monoidal adjunction, natural isomorphisms
$$\mathsf{Map}_\Sigma(X, \mathsf{Hom}_\Sigma(Y,Z)) \cong \mathsf{Map}_\Sigma(X \,\square\, Y, Z)$$
are given similarly to those of Proposition 7.1.19, using the V-Yoneda Density Theorem 3.7.8 with $\mathsf{V} = \mathsf{sSet}_*$. $\diamond$

Definition 7.3.6. For each $n \geq 0$
$$i_n : \mathsf{sSet}_* \longrightarrow \mathsf{sSet}_*^\Sigma$$
be the functor that sends a pointed simplicial set X to the symmetric sequence whose nth term is $X \wedge (\Sigma_n)_+$ and whose other terms are the terminal simplicial set $*$. A morphism of pointed simplicial sets $f : A \longrightarrow B$ induces a morphism of symmetric sequences $i_n A \longrightarrow i_n B$ given at level n by $f \wedge \mathrm{id}$.

Let
$$\mathsf{ev}_n : \mathsf{sSet}_*^\Sigma \longrightarrow \mathsf{sSet}_*$$
be the functor that sends a symmetric sequence X to the pointed simplicial set X_n and that sends a morphism of symmetric sequences to its component at n. For each $n \geq 0$ these are an adjoint pair of functors
$$\begin{tikzcd} \mathsf{sSet}_* \arrow[r, bend left, "i_n"] \arrow[r, phantom, "\perp"] & \mathsf{sSet}_*^\Sigma. \arrow[l, bend left, "\mathsf{ev}_n"] \end{tikzcd}$$
It follows from the definition of the Day convolution $\square$ that i_0 is strong symmetric monoidal and ev_0 is symmetric monoidal. $\diamond$

The following result is a special case of Theorem 3.7.22 combined with Theorem 3.9.8. Additional discussion of this application appears in [**HSS00, Hov01, Ric20**].

Theorem 7.3.7. *Equipped with the Day convolution product and symmetric hom, the category of symmetric sequences,* sSet_*^Σ*, is a symmetric monoidal closed category with*

- *monoidal product given by Day convolution of Definition 7.3.3 and*
- *internal hom given by the symmetric hom objects of Definition 7.3.5.*

Moreover, the adjunction (i_0, ev_0) *makes* sSet_*^Σ *enriched, tensored, and cotensored over* sSet_**. The* sSet_* *enrichment is given by the symmetric mapping objects of Definition 7.3.5.*

Definition 7.3.8. For $X \in \mathsf{sSet}_*$ and $k > 0$ we let $X^{\wedge k}$ denote the k-fold smash product defined iteratively by $X^{\wedge k+1} = X \wedge X^{\wedge k}$. We call this the *right normalized* convention for smash powers. The empty smash product $X^{\wedge 0}$ is the unit, S^0.

The symmetric group action of Σ_p on $X^{\wedge p}$ is given by the unique coherence isomorphism in sSet_* (from Theorem 1.1.41) that permutes the p copies of X. We call this the *action by permuting factors*. ◇

Recall from Definition I.1.2.8 the concept of a monoid in a monoidal category.

Proposition 7.3.9. *Suppose* $X \in \mathsf{sSet}_*$ *and let* $\mathsf{Sym}(X)$ *be the symmetric sequence given by* $\mathsf{Sym}(X)_k = X^{\wedge k}$ *with* Σ_k *acting by permuting factors. Then* $\mathsf{Sym}(X)$ *is a commutative monoid in the category of symmetric sequences.*

Proof. The multiplication morphism

$$\mu : \mathsf{Sym}(X) \,\square\, \mathsf{Sym}(X) \longrightarrow \mathsf{Sym}(X)$$

is defined on summands

$$\mu^\Sigma_{p,q} : \Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} X^{\wedge p} \wedge X^{\wedge q} \longrightarrow X^{\wedge (p+q)}$$

by the associativity in sSet_* and the action of Σ_{p+q} permuting factors. The monoid associativity and unity axioms follow from the Symmetric Coherence Theorem Theorem 1.1.41 for sSet_*.

To see that $\mathsf{Sym}(X)$ is a commutative monoid, we have the following commuting diagram for each summand of $(\mathsf{Sym}(X) \,\square\, \mathsf{Sym}(X))_{(p+q)}$.

$$\begin{tikzcd}
\Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} X^{\wedge p} \wedge X^{\wedge q} \arrow[rr, "\xi_{p,q}"] \arrow[dr, "\mu^\Sigma_{p,q}"'] & & \Sigma_{(q+p)} \times_{\Sigma_q \times \Sigma_p} X^{\wedge q} \wedge X^{\wedge p} \arrow[dl, "\mu^\Sigma_{q,p}"] \\
& X^{\wedge (p+q)} &
\end{tikzcd}$$

□

7.4. Symmetric Spectra

Our model for the stable homotopy category will be the category of symmetric spectra. It has a symmetric monoidal smash product induced by the convolution product on symmetric sequences.

We begin with the following important special case of Definition 7.3.8.

Definition 7.4.1. For $p \geq 0$, the *simplicial p-sphere* is the p-fold smash product

$$S^p = (S^1)^{\wedge p},$$

using the right normalized convention of Definition 7.3.8. The *symmetric sphere* is the symmetric sequence $S = \mathsf{Sym}(S^1)$. ◇

Definition 7.4.2. Suppose $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category, and (A, μ, η) is a monoid in C. A *left A-module*, also called a left module over A, is a pair

$$(M, \theta)$$

consisting of

- an object $M \in \mathsf{C}$ and
- a *structure morphism* $\theta : A \otimes M \longrightarrow M$

such that the following associativity and unity diagrams commute.

$$\begin{tikzcd}
(A \otimes A) \otimes M \arrow[r, "\alpha"] \arrow[dd, "\mu\otimes 1"'] & A \otimes (A \otimes M) \arrow[d, "1\otimes\theta"] \\
 & A \otimes M \arrow[d, "\theta"] \\
A \otimes M \arrow[r, "\theta"] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \otimes M \arrow[r, "\lambda"] \arrow[d, "\eta\otimes 1"'] & M \arrow[d, equal] \\
A \otimes M \arrow[r, "\theta"] & M
\end{tikzcd}$$

A *morphism*

$$f : (M, \theta^M) \longrightarrow (N, \theta^N)$$

of left A-modules is a morphism $f : M \longrightarrow N \in \mathsf{C}$ such that the diagram

$$\begin{tikzcd}
A \otimes M \arrow[r, "\theta^M"] \arrow[d, "1\otimes f"'] & M \arrow[d, "f"] \\
A \otimes N \arrow[r, "\theta^N"] & N
\end{tikzcd}$$

commutes. The identity morphism of a left A-module (M, θ) is $1_M \in \mathsf{C}$, and composition is defined in C. The above data define the category of left A-modules. ◇

Example 7.4.3. In the context of Definition 7.4.2, (A, μ) is a left A-module. ◇

Explanation 7.4.4 (Right Modules)**.** In the context of Definition 7.4.2, there is a similar definition of *right A-module* N, with structure morphism

$$N \otimes A \longrightarrow N.$$

If $(\mathsf{C}, \otimes)$ is symmetric monoidal and A is a commutative monoid in C, then the category of right A-modules is isomorphic to the category of left A-modules via the symmetry of C. ◇

Definition 7.4.5. The category of *symmetric spectra*, SymSp, is the category consisting of left modules over the symmetric sphere and left module morphisms. ◇

Proposition 7.5.5 below shows that SymSp is complete and cocomplete. Theorem 7.6.15 below shows that SymSp is symmetric monoidal closed.

Explanation 7.4.6. We have the following explicit description of symmetric spectra. An action

$$\rho : S \,\square\, X \longrightarrow X$$

has summands for each $k = p + q \geq 0$

$$\rho^{\Sigma}_{p,q} : \Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} S^p \wedge X_q \longrightarrow X_{p+q}.$$

The action ρ is determined by morphisms

$$\rho_{p,q} : S^p \wedge X_q \longrightarrow X_{p+q}$$

for $p, q \geq 0$ with the following properties.

Unity: The morphism

$$\rho_{0,q} : S^0 \wedge X_q \longrightarrow X_q \tag{7.4.7}$$

is the unit isomorphism for the smash product.

Associativity: The following diagram commutes for each r, p, and q, where the unlabeled vertical morphism is the smash product of the associativity isomorphisms $S^r \wedge S^p \cong S^{r+p}$ with the identity on X_q.

$$\begin{tikzcd}
(S^r \wedge S^p) \wedge X_q \arrow[rr, "a"] \arrow[d] & & S^r \wedge (S^p \wedge X_q) \arrow[d, "1 \wedge \rho_{p,q}"] \\
S^{r+p} \wedge X_q \arrow[dr, "\rho_{r+p,q}"'] & & S^r \wedge X_{p+q} \arrow[dl, "\rho_{r,p+q}"] \\
& X_{r+p+q} &
\end{tikzcd} \tag{7.4.8}$$

Equivariance: Each $\rho_{p,q}$ is equivariant with respect to the $(\Sigma_p \times \Sigma_q)$-action on the source and the inclusion

$$i_{p,q} : \Sigma_p \times \Sigma_q \hookrightarrow \Sigma_{(p+q)}$$

where Σ_p permutes the first p elements and Σ_q permutes the last q elements.

An S-module morphism $X \longrightarrow X'$ consists of a sequence of morphisms

$$f_k : X_k \longrightarrow X'_k$$

that are equivariant with respect to the action of Σ_k on X_k and X'_k and that commute with the structure morphisms $\rho_{p,q}$. ◇

Basic Examples of Symmetric Spectra.

Example 7.4.9 (The Sphere Spectrum)**.** By Example 7.4.3, the symmetric sphere S with its monoid multiplication is a symmetric spectrum. ◇

Example 7.4.10 (Suspension Spectra)**.** Suppose K is a pointed simplicial set. The *suspension spectrum* $\Sigma^\infty K$ is the symmetric spectrum with

$$(\Sigma^\infty K)_n = S^n \wedge K \in \mathsf{sSet}_* \quad \text{for} \quad n \geq 0.$$

The Σ_n-action on $S^n \wedge K$ is given by permuting the n smash factors in

$$S^n = (S^1)^{\wedge n}.$$

For $p, q \geq 0$, the left S-module structure morphism $\rho_{p,q}$ is the composite

$$\begin{tikzcd}
S^p \wedge (S^q \wedge K) \arrow[rr, "\rho_{p,q}"] \arrow[dr, "\cong"'] & & S^{p+q} \wedge K \\
& (S^p \wedge S^q) \wedge K \arrow[ur, "\cong"'] &
\end{tikzcd}$$

of associativity isomorphisms if $p, q > 0$. It is

- the left unit isomorphism $\lambda_{S^q \wedge K}$ if $p = 0$ and
- $1_{S^p} \wedge \lambda_K$ if $q = 0$.

The fact that $\rho_{p,q}$ is well defined and the unity, associativity, and equivariance conditions in Explanation 7.4.6 follow from the Symmetric Coherence Theorem 1.1.41 for sSet_*. ◇

Example 7.4.11 (Eilenberg-Mac Lane Spectra)**.** Suppose R is a ring. The *Eilenberg-Mac Lane spectrum* HR is the symmetric spectrum with

$$(HR)_n = R \otimes S^n \in \mathsf{sSet}_* \quad \text{for} \quad n \geq 0.$$

For $k \geq 0$, its set of k-simplices is the free left R-module

$$(R \otimes S^n)_k = \bigoplus_{(\underline{k}^n)^\flat} R$$

with

$$\underline{k}^n \cong \underline{k}^{\wedge n} = (S^n)_k$$

the set of k-simplices in the n-sphere S^n. The basepoint of $(R \otimes S^n)_k$ is the element 0. The simplicial structure morphisms in $(HR)_n$ are induced by those in S^n.

To describe the symmetric sequence structure on HR, we represent a typical direct sum generator in $(HR)_n$ in the form

$$r_{i_1,\ldots,i_n} \qquad \text{with } r \in R \text{ and each } i_j \in \underline{k}^\flat.$$

The Σ_n-action on $(HR)_n$ is given on these generators by

$$\sigma \cdot r_{i_1,\ldots,i_n} = r_{i_{\sigma^{-1}(1)},\ldots,i_{\sigma^{-1}(n)}} \quad \text{for} \quad \sigma \in \Sigma_n.$$

The left S-module structure morphisms on HR,

$$S^p \wedge (R \otimes S^q) \xrightarrow{\rho_{p,q}} R \otimes S^{p+q} \quad \text{for} \quad p, q \geq 0,$$

are defined on k-simplices by

$$\begin{array}{ccc} \underline{k}^{\wedge p} \wedge \Big(\bigoplus_{(\underline{k}^q)^\flat} R \Big) & \xrightarrow{(\rho_{p,q})_k} & \bigoplus_{(\underline{k}^{p+q})^\flat} R \\ \big((h_1, \ldots, h_p), r_{i_1,\ldots,i_q}\big) & \longmapsto & r_{h_1,\ldots,h_p,i_1,\ldots,i_q} \end{array}$$

for $k \geq 0$ and $h_1, \ldots, h_p, i_1, \ldots, i_q \in \underline{k}^\flat$. Moreover, since $(\rho_{p,q})_k$ preserves the basepoint, we have

$$(\rho_{p,q})_k(0, r) = 0.$$

The unity, associativity, and equivariance conditions in Explanation 7.4.6 hold because they do for direct sum generators. ◇

Chapters 11, 12, and 13 have further examples of symmetric spectra with extra structure arising from categorical data. We will revisit Examples 7.4.9 through 7.4.11 in

- Examples 11.3.13 through 11.3.15 in the context of strict ring symmetric spectra and
- Examples 11.6.7, 11.6.9, and 11.6.10 in the context of E_∞-symmetric spectra.

7.5. Limits and Colimits of Symmetric Spectra

The category of symmetric spectra is defined as the category of left S-modules, in the sense of Definition 7.4.2, over the symmetric sphere S. Recall the concepts of a monad and an algebra over a monad in Definitions 5.4.1 and 5.4.2. Using Proposition 7.5.1 below, we can also regard SymSp as the category of algebras over the monad $S \,\square\, -$ in the category sSet_*^{Σ} of symmetric sequences.

Modules as Monadic Algebras.

Proposition 7.5.1. *Suppose $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ is a monoidal category, and (A, μ, η) is a monoid in C.*

(1) *There is a monad*

$$(A \otimes -, \pi, \nu)$$

on C with multiplication π and unit ν defined as the following composites for each object $X \in \mathsf{C}$.

$$\begin{tikzcd} A \otimes (A \otimes X) \arrow[rr, "\pi_X"] \arrow[dr, "\alpha^{-1}"'] & & A \otimes X \\ & (A \otimes A) \otimes X \arrow[ur, "\mu \otimes 1"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\nu_X"] \arrow[dr, "\lambda^{-1}"'] & & A \otimes X \\ & \mathbb{1} \otimes X \arrow[ur, "\eta \otimes 1"'] & \end{tikzcd}$$

(2) *There is a canonical isomorphism between*
- *the category of left A-modules in Definition 7.4.2 and*
- *the category of algebras over the monad $(A \otimes -, \pi, \nu)$.*

Proof. For assertion (1), we check the commutativity of the associativity and unity diagrams in Definition 5.4.1 for $(A \otimes -, \pi, \nu)$. Since each diagram asserts the equality of parallel natural transformations, it suffices to check each diagram at an object $X \in \mathsf{C}$. The monad associativity diagram for $(A \otimes -, \pi, \nu)$ at an object X is the outer diagram in C below, where all the tensor symbols are omitted to save space.

$$\begin{tikzcd} A(A(AX)) \arrow[r, "1\alpha^{-1}"] \arrow[dd, "\alpha^{-1}"'] & A((AA)X) \arrow[r, "1(\mu 1)"] \arrow[d, "\alpha^{-1}"] & A(AX) \arrow[d, "\alpha^{-1}"] \\ & (A(AA))X \arrow[r, "(1\mu)1"] \arrow[d, "\alpha^{-1}1"] & (AA)X \arrow[dd, "\mu 1"] \\ (AA)(AX) \arrow[r, "\alpha^{-1}"] \arrow[d, "\mu 1"'] & ((AA)A)X \arrow[d, "(\mu 1)1"] & \\ A(AX) \arrow[r, "\alpha^{-1}"] & (AA)X \arrow[r, "\mu 1"] & AX \end{tikzcd}$$

- The upper left rectangle is commutative by the pentagon axiom (1.1.3) in C.
- The lower right rectangle is commutative by the associativity axiom of the monoid (A, μ, η).
- The lower left and upper right rectangles are commutative by the naturality of α.

The monad unity diagram for $(A \otimes -, \pi, \nu)$ at an object X is the outer diagram in C below.

$$\begin{tikzcd}
A(AX) \arrow[dd, "\alpha^{-1}"'] & \mathbb{1}(AX) \arrow[l, "\eta 1"'] \arrow[d, "\alpha^{-1}"] & AX \arrow[l, "\lambda^{-1}"'] \arrow[r, "1\lambda^{-1}"] \arrow[dd, equal] & A(\mathbb{1}X) \arrow[r, "1(\eta 1)"] \arrow[d, "\alpha^{-1}"] & A(AX) \arrow[dd, "\alpha^{-1}"] \\
& (\mathbb{1}A)X \arrow[dl, "(\eta 1)1"'] \arrow[dr, "\lambda 1"] & & (A\mathbb{1})X \arrow[dl, "\rho 1"'] \arrow[dr, "(1\eta)1"] & \\
(AA)X \arrow[rr, "\mu 1"] & & AX & & (AA)X \arrow[ll, "\mu 1"']
\end{tikzcd}$$

In the left half of the diagram above,

- the left and right trapezoids are commutative by, respectively, the naturality of α and the left unity property (1.1.5) in C, and
- the bottom triangle is commutative by the left unity axiom in the monoid (A, μ, η).

Similarly, the right half of the previous diagram is commutative by the naturality of α, the unity axiom (1.1.2) in C, and the right unity axiom in the monoid (A, μ, η). This proves assertion (1).

For assertion (2), the associativity and unity diagrams for a left A-module in Definition 7.4.2 are precisely those for an $(A \otimes -)$-algebra in (5.4.3), after inverting the natural isomorphisms α and λ. Morphisms, identity morphisms, and composition of left A-modules also correspond to those of $(A \otimes -)$-algebras. $\square$

Proposition 7.5.1 applies to the symmetric sphere in Definition 7.4.1 and yields the following description of the category SymSp of symmetric spectra.

Corollary 7.5.2. *There is a canonical isomorphism of categories*

$$\mathsf{SymSp} \cong \mathsf{Alg}(S \,\square\, -),$$

with S the symmetric sphere in the symmetric monoidal category sSet_*^{Σ}.

Limits and Colimits of Modules. The next observation is a variant of Theorem 5.4.18 that applies to left modules, with different assumptions and a much simpler proof. We will use it with the symmetric sphere.

Proposition 7.5.3. *Suppose (A, μ, η) is a monoid in a complete and cocomplete monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ such that*

$$A \otimes - : \mathsf{C} \longrightarrow \mathsf{C}$$

preserves small colimits. Then for the monad $A \otimes -$ in Proposition 7.5.1, its category of algebras, $\mathsf{Alg}(A \otimes -)$, has all small limits and colimits, which are, furthermore, preserved by the forgetful functor

$$U : \mathsf{Alg}(A \otimes -) \longrightarrow \mathsf{C}.$$

Proof. For a functor F as in

$$\mathsf{D} \xrightarrow{F} \mathsf{Alg}(A \otimes -) \xrightarrow{U} \mathsf{C}$$

with D a small category, the composite $UF : \mathsf{D} \longrightarrow \mathsf{C}$ has a colimit in C. The natural morphism

$$\mathsf{colim}\,(A \otimes UF) \xrightarrow[\cong]{\omega} A \otimes (\mathsf{colim}\, UF)$$

is an isomorphism by assumption. We equip the object $\mathsf{colim}\, UF \in \mathsf{C}$ with the structure morphism ϕ below.

$$\begin{tikzcd} A \otimes (\mathsf{colim}\, UF) \arrow[rr, "\phi"] \arrow[dr, "\cong"'] & & \mathsf{colim}\, UF \\ & \mathsf{colim}\,(A \otimes UF) \arrow[ur, "\mathsf{colim}_{d \in \mathsf{D}} \theta_d"'] & \end{tikzcd}$$

For each $d \in \mathsf{D}$,

$$A \otimes (UFd) \xrightarrow{\theta_d} UFd$$

is the $(A \otimes -)$-algebra structure morphism of Fd. By

- the universal property of colimits and
- the assumption that $A \otimes -$ preserves small colimits,

the associativity and unity axioms (5.4.3) for the pair

$$\big(\mathsf{colim}\, UF, \phi\big)$$

to be an $(A \otimes -)$-algebra reduce to those for each Fd. The fact that it is a colimit of F follows from the definition of the isomorphism ω.

Similarly, a limit of F in $\mathsf{Alg}(A \otimes -)$ consists of the object $\lim UF \in \mathsf{C}$ and structure morphism φ defined by the commutative diagrams

$$\begin{tikzcd} A \otimes (\lim UF) \arrow[r, "\varphi"] \arrow[d, "1 \otimes p_d"'] & \lim UF \arrow[d, "p_d"] \\ A \otimes (UFd) \arrow[r, "\theta_d"] & UFd \end{tikzcd}$$

in C for $d \in \mathsf{D}$. Here

$$\lim UF \xrightarrow{p_d} UFd$$

is a structure morphism of the limit. □

Explanation 7.5.4. In Proposition 7.5.3, the condition that $A \otimes -$ preserves small colimits is true if C is symmetric monoidal closed. For example, this is the case when

- A is the symmetric sphere S in Definition 7.4.1 and
- C is the category of symmetric sequences sSet_*^{Σ} in Definition 7.3.1. ◇

Now we apply the results above to SymSp.

Proposition 7.5.5. *The category of symmetric spectra,* SymSp, *is complete and cocomplete.*

Proof. By Proposition 7.1.17, sSet_*^{Σ}, on which the monad $S \,\square\, -$ acts, is complete, cocomplete, and symmetric monoidal closed. We finish the proof using Corollary 7.5.2, Proposition 7.5.3, and Explanation 7.5.4. □

7.6. Smash Products, Internal Hom, and (Co)tensored Structure of Symmetric Spectra

Proposition 7.5.5 shows that the category of symmetric spectra is complete and cocomplete. Therefore we have the following constructions of monoidal product and internal hom. Since S is a commutative monoid in sSet_*^Σ, these provide a symmetric monoidal closed structure for SymSp.

Smash Product.

Definition 7.6.1. The *smash product* of symmetric spectra X and Y is denoted

$$X \wedge Y = X \,\square_S\, Y$$

and is given by the coequalizer in sSet_*^Σ

$$\begin{tikzcd} (X \,\square\, S) \,\square\, Y \arrow[r, shift left, "(1 \,\square\, \rho^Y) \circ a"] \arrow[r, shift right, "\overline{\rho}^X \,\square\, 1"'] & X \,\square\, Y \arrow[r] & X \,\square_S\, Y \end{tikzcd}$$

where

$$\rho^Y : S \,\square\, Y \longrightarrow Y$$

is the left action of S on Y and $\overline{\rho}^X$ is the right action of S on X defined as the composite

$$\overline{\rho}^X = \rho^X \circ \xi : X \,\square\, S \longrightarrow S \,\square\, X \longrightarrow X$$

of the symmetry ξ with ρ^X, the left action of S on X. The left S-module structure of $X \,\square_S\, Y$ is given by the left action of S on X. ⋄

The following observation is useful for constructing morphisms out of a smash product of symmetric spectra. We will use it in Proposition 11.3.2 to describe strict ring symmetric spectra.

Proposition 7.6.2. *Suppose (X, ρ^X), (Y, ρ^Y), and (Z, ρ^Z) are symmetric spectra. Then a morphism*

$$X \,\square_S\, Y \xrightarrow{\overline{f}} Z \in \mathsf{SymSp}$$

determines and is uniquely determined by a morphism

$$X \,\square\, Y \xrightarrow{f} Z \in \mathsf{sSet}_*^\Sigma$$

such that the diagram

$$\begin{tikzcd} (X \,\square\, S) \,\square\, Y \arrow[r, "\xi \square 1", "\cong"'] \arrow[d, "a"', "\cong"] & (S \,\square\, X) \,\square\, Y \arrow[r, "a", "\cong"'] \arrow[d, "\rho^X \square 1"'] & S \,\square\, (X \,\square\, Y) \arrow[dd, "1 \square f"] \\ X \,\square\, (S \,\square\, Y) \arrow[d, "1 \square \rho^Y"'] & X \,\square\, Y \arrow[d, "f"'] & \\ X \,\square\, Y \arrow[r, "f"] & Z & S \,\square\, Z \arrow[l, "\rho^Z"'] \end{tikzcd} \tag{7.6.3}$$

in sSet_*^Σ commutes.

Proof. Given a morphism $\overline{f} : X \,\square_S\, Y \longrightarrow Z$ of symmetric spectra, the morphism f is defined as the composite

$$X \,\square\, Y \longrightarrow X \,\square_S\, Y \xrightarrow{\;\overline{f}\;} Z \in \mathsf{sSet}_*^{\Sigma}.$$

- The left half of the diagram (7.6.3) follows from the coequalizer that defines the smash product $X \,\square_S\, Y$ in Definition 7.6.1.
- The right half of (7.6.3) expresses the compatibility of $\overline{f}$ with the left S-action.

This argument can also be used in reverse. □

Explanation 7.6.4. Using Definition 7.3.3 of the Day convolution and Explanation 7.4.6, we can further unpack the morphism $f : X \,\square\, Y \longrightarrow Z$ in Proposition 7.6.2 as the family of $(\Sigma_p \times \Sigma_q)$-equivariant morphisms

$$X_p \wedge Y_q \xrightarrow{\;f_{p,q}\;} Z_{p+q} \in \mathsf{sSet}_* \quad \text{for} \quad p, q \geq 0.$$

The right half of the diagram (7.6.3) is equivalent to the diagram

(7.6.5)
$$\begin{tikzcd}
(S^n \wedge X_p) \wedge Y_q \arrow[r, "a", "\cong"'] \arrow[d, "\rho^X_{n,p} \wedge 1"'] & S^n \wedge (X_p \wedge Y_q) \arrow[dd, "1 \wedge f_{p,q}"] \\
X_{n+p} \wedge Y_q \arrow[d, "f_{n+p,q}"'] & \\
Z_{n+p+q} & S^n \wedge Z_{p+q} \arrow[l, "\rho^Z_{n,p+q}"']
\end{tikzcd}$$

in sSet_* for $n, p, q \geq 0$. With the abbreviation

(7.6.6)
$$\Sigma_n \times \Sigma_p = \Sigma_{n,p},$$

the left half of the diagram (7.6.3) is equivalent to the diagram

(7.6.7)
$$\begin{tikzcd}[column sep=tiny]
\Sigma_{(p+n+q)} \times_{\Sigma_{p+n,q}} \Big(\Sigma_{(p+n)} \times_{\Sigma_{p,n}} X_p \wedge S^n\Big) \wedge Y_q \arrow[dd, "a"', "\cong"] \arrow[dr, "\xi_{p,n} \wedge 1"] & \\
& \Sigma_{(p+n+q)} \times_{\Sigma_{n+p,q}} \Big(\Sigma_{(n+p)} \times_{\Sigma_{n,p}} S^n \wedge X_p\Big) \wedge Y_q \arrow[dd, "\rho^X_{n,p} \wedge 1"] \\
\Sigma_{(p+n+q)} \times_{\Sigma_{p,n+q}} X_p \wedge \Big(\Sigma_{(n+q)} \times_{\Sigma_{n,q}} S^n \wedge Y_q\Big) \arrow[dd, "1 \wedge \rho^Y_{n,q}"'] & \\
& \Sigma_{(p+n+q)} \times_{\Sigma_{n+p,q}} X_{n+p} \wedge Y_q \arrow[d, "f_{n+p,q}"] \\
\Sigma_{(p+n+q)} \times_{\Sigma_{p,n+q}} X_p \wedge Y_{n+q} \arrow[r, "f_{p,n+q}"] & Z_{n+p+q}
\end{tikzcd}$$

in sSet_* for $n, p, q \geq 0$. ◇

Internal Hom.

Definition 7.6.8. The *internal hom* for symmetric spectra X and Y is given by the equalizer in sSet_*^{Σ}

$$\mathsf{Hom}_S(X,Y) \longrightarrow \mathsf{Hom}_{\Sigma}(X,Y) \rightrightarrows \mathsf{Hom}_{\Sigma}(S \Box X, Y) \tag{7.6.9}$$

where the two parallel arrows are adjoint to the two composites in the diagram below, determined by the S-module structures of X and Y denoted ρ^X and ρ^Y, respectively.

$$\begin{tikzcd}
\mathsf{Hom}_{\Sigma}(X,Y) \Box (S \Box X) \arrow[r] \arrow[d, "1 \Box \rho^X"'] & S \Box Y \arrow[d, "\rho^Y"] \\
\mathsf{Hom}_{\Sigma}(X,Y) \Box X \arrow[r] & Y
\end{tikzcd}$$

The upper horizontal morphism is the composite of associativity and symmetry

$$\mathsf{Hom}_{\Sigma}(X,Y) \Box (S \Box X) \longrightarrow S \Box (\mathsf{Hom}_{\Sigma}(X,Y) \Box X)$$

together with evaluation. The lower horizontal morphism is evaluation. The left S-module structure of $\mathsf{Hom}_S(X,Y)$ is induced by that of Y. $\diamond$

Proposition 7.6.10. *Suppose (A,ρ^A), (X,ρ^X), and (Y,ρ^Y) are symmetric spectra. Then a morphism*

$$A \xrightarrow{\overline{f}} \mathsf{Hom}_S(X,Y) \in \mathsf{SymSp}$$

determines and is uniquely determined by a morphism

$$A \Box X \xrightarrow{f} Y \in \mathsf{sSet}_*^{\Sigma}$$

such that the diagram

$$\begin{tikzcd}
A \Box (S \Box X) \arrow[r, "\cong"] \arrow[dd, "1\Box\rho^X"'] & S \Box (A \Box X) \arrow[r, "a^{-1}"] \arrow[d, "1\Box f"'] & (S \Box A) \Box X \arrow[dd, "\rho^A\Box 1"] \\
 & S \Box Y \arrow[d, "\rho^Y"'] & \\
A \Box X \arrow[r, "f"] & Y & A \Box X \arrow[l, "f"']
\end{tikzcd} \tag{7.6.11}$$

commutes, with $\cong$ the unique coherence isomorphism that swaps A and S.

Proof. Given a morphism $\overline{f} \in \mathsf{SymSp}$ as stated above, the morphism f is the adjoint of the composite

$$A \xrightarrow{\overline{f}} \mathsf{Hom}_S(X,Y) \longrightarrow \mathsf{Hom}_{\Sigma}(X,Y) \in \mathsf{sSet}_*^{\Sigma}$$

under the $(\Box, \mathsf{Hom}_{\Sigma})$-adjunction in Theorem 7.3.7.

- The left half of the diagram (7.6.11) follows from the equalizer that defines the internal hom Hom_S in (7.6.9).
- The right half of (7.6.11) expresses the compatibility of $\overline{f} \in \mathsf{sSet}_*^{\Sigma}$ with the left S-action.

This argument can also be used in reverse. $\square$

Explanation 7.6.12. Similar to Explanation 7.6.4, the morphism $f : A \,\square\, X \longrightarrow Y$ in Proposition 7.6.10 is equivalent to the family of $(\Sigma_p \times \Sigma_q)$-equivariant morphisms

$$A_p \wedge X_q \xrightarrow{f_{p,q}} Y_{p+q} \in \mathsf{sSet}_* \quad \text{for} \quad p, q \geq 0.$$

The right half of the diagram (7.6.11) is equivalent to the diagram

$$\begin{tikzcd}
S^p \wedge (A_n \wedge X_q) \arrow[r, "a^{-1}", "\cong"'] \arrow[d, "1 \wedge f_{n,q}"'] & (S^p \wedge A_n) \wedge X_q \arrow[dd, "\rho^A_{p,n} \wedge 1"] \\
S^p \wedge Y_{n+q} \arrow[d, "\rho^Y_{p,n+q}"'] & \\
Y_{p+n+q} & A_{p+n} \wedge X_q \arrow[l, "f_{p+n,q}"']
\end{tikzcd}$$

in sSet_* for $n, p, q \geq 0$. With the abbreviation in (7.6.6), the left half of the diagram (7.6.11) is equivalent to the diagram

$$\begin{tikzcd}
& \Sigma_{(p+n+q)} \times_{\Sigma_{p,n+q}} S^p \wedge \left(\Sigma_{n+q} \times_{\Sigma_{n,q}} A_n \wedge X_q\right) \arrow[dd, "1 \wedge f_{n,q}"] \\
\Sigma_{(n+p+q)} \times_{\Sigma_{n,p+q}} A_n \wedge \left(\Sigma_{(p+q)} \times_{\Sigma_{p,q}} S^p \wedge X_q\right) \arrow[ur, "a(\xi_{n,p} \wedge 1)a^{-1}"] \arrow[dd, "1 \wedge \rho^X_{p,q}"'] & \\
& \Sigma_{(p+n+q)} \times_{\Sigma_{p,n+q}} S^p \wedge Y_{n+q} \arrow[dd, "\rho^Y_{p,n+q}"] \\
\Sigma_{(n+p+q)} \times_{\Sigma_{n,p+q}} A_n \wedge X_{p+q} \arrow[dr, "f_{n,p+q}"] & \\
& Y_{n+p+q}
\end{tikzcd}$$

in sSet_* for $n, p, q \geq 0$. $\diamond$

Simplicial Tensored and Cotensored Structure. Next we note, for each $n \geq 0$, there is a level-n adjunction with pointed simplicial sets.

Definition 7.6.13. For each $n \geq 0$ we compose the adjunction (i_n, ev_n) of Definition 7.3.6 with the free-forgetful adjunction for left S-modules to obtain an adjunction

$$\begin{tikzcd}
\mathsf{sSet}_* \arrow[r, shift left=1.5, "F_n"] \arrow[r, phantom, "\perp"] & \mathsf{SymSp} \arrow[l, shift left=1.5, "\mathsf{ev}_n"]
\end{tikzcd}$$

where $\mathsf{ev}_n X = X_n$ for a symmetric spectrum X and $F_n A = S \,\square\, i_n A$ for a pointed simplicial set A. With respect to the smash product $\wedge = \square_S$, the left adjoint F_0 is strong symmetric monoidal and the right adjoint ev_0 is symmetric monoidal. $\diamond$

Explanation 7.6.14. Suppose $K \in \mathsf{sSet}_*$ and $n, k \geq 0$. Then $F_n K$ is the symmetric spectrum with

$$\begin{aligned}(F_n K)_k &= (S \,\square\, i_n K)_k \\ &= \bigvee_{p+q=k} \Sigma_k \times_{\Sigma_p \times \Sigma_q} S^p \wedge (i_n K)_q \\ &= \begin{cases} * & \text{if } k < n \text{ and} \\ \Sigma_k \times_{\Sigma_{k-n} \times \Sigma_n} S^{k-n} \wedge \big(K \wedge (\Sigma_n)_+\big) & \text{if } k \geq n. \end{cases}\end{aligned}$$

The left S-action

$$S^p \wedge (F_n K)_q \xrightarrow{\rho_{p,q}} (F_n K)_{p+q} \quad \text{for} \quad p, q \geq 0$$

is the basepoint inclusion if $q < n$. If $q \geq n$, then $\rho_{p,q}$ is induced by the canonical isomorphism

$$S^p \wedge S^{q-n} \xrightarrow[\cong]{\mu^S_{p,q-n}} S^{p+q-n}.$$

In particular, we have

$$F_0 K = \Sigma^\infty K,$$

the suspension spectrum of K in Example 7.4.10. $\diamond$

An adjunction between $\wedge = \square_S$ and Hom_S follows formally from their constructions as coequalizers and equalizers, respectively. Thus we have the following result, with the second half following from Theorem 3.9.8.

Theorem 7.6.15. *The category of symmetric spectra,* SymSp, *is a symmetric monoidal closed category with*

- *monoidal product given by* $\wedge = \square_S$,
- *internal hom given by* Hom_S, *and*
- *monoidal unit the symmetric sphere* S.

Moreover, the adjunction (F_0, ev_0) *makes* SymSp *enriched, tensored, and cotensored over* sSet_*.

Applying Theorems 2.4.10 and 3.3.2 with the change of enrichment by ev_0, we have the following.

Corollary 7.6.16. *The symmetric monoidal structure* $(\mathsf{SymSp}, \wedge, S)$ *is* sSet_**-enriched.*

Explanation 7.6.17. Suppose $X \in \mathsf{SymSp}$ and $K \in \mathsf{sSet}_*$. Then the tensored and cotensored structures in Theorem 7.6.15 are given by

$$\begin{aligned} X \wedge K &= X \,\square_S\, \Sigma^\infty K \\ X^K &= \mathsf{Hom}_S(\Sigma^\infty K, X), \end{aligned}$$

where we used the fact $F_0 = \Sigma^\infty$ in Explanation 7.6.14. $\diamond$

Definition 7.6.18. Because the equalizer (7.6.9) is computed levelwise, we define Map_S as the following special case for $X, Y \in \mathsf{SymSp}$:

$$\mathsf{Map}_S(X, Y) = \mathsf{Hom}_S(X, Y)_0$$

is the equalizer

$$\mathsf{Map}_S(X, Y) \longrightarrow \mathsf{Map}_\Sigma(X, Y) \rightrightarrows \mathsf{Map}_\Sigma(S \,\square\, X, Y).$$

Therefore, from Definition 7.3.5 (3) we have

$$\mathsf{Map}_S(X,Y) \cong \mathsf{SymSp}(X \wedge \Delta^{?}_{+}, Y) \tag{7.6.19}$$

as pointed simplicial sets. ◇

Explanation 7.6.20 (Underlying Symmetric Sequence of Hom Object)**.** Recall from Definition 7.6.13 the adjunction (F_n, ev_n) and from Definition 7.3.4 the symmetric sequence $\Sigma[n] = \Sigma(\underline{n}, -)$ for each $n \geq 0$. We have $F_n S^0 = S \,\Box\, \Sigma[n]_+$ and therefore, for each $n \geq 0$ and each $W \in \mathsf{SymSp}$, a composite of natural isomorphisms of pointed simplicial sets

$$\mathsf{Map}_S(S \,\Box\, \Sigma[n]_+, W) \cong \mathsf{Map}_{\mathsf{sSet}_*}(S^0, \mathsf{ev}_n W) \cong W_n.$$

Since $\Sigma[-]_+$ is a functor from Σ^{op} to sSet_*^{Σ}, these isomorphisms are equivariant with respect to the induced action of Σ_n and we have an isomorphism of symmetric sequences

$$\mathsf{Map}_S(S \,\Box\, \Sigma[-]_+, W) \cong W.$$

Applying this to $W = \mathsf{Hom}_S(X,Y)$ for symmetric spectra X and Y, the $\wedge$-Hom_S adjunction gives an isomorphism of symmetric sequences

$$\mathsf{Hom}_S(X,Y) \cong \mathsf{Map}_S(S \,\Box\, \Sigma[-]_+, \mathsf{Hom}_S(X,Y)) \cong \mathsf{Map}_S(X \wedge (S \,\Box\, \Sigma[-]_+), Y).$$

Combining this with (7.6.19) we have

$$\mathsf{Hom}_S(X,Y) \cong \mathsf{SymSp}((X \wedge (S \,\Box\, \Sigma[-]_+)) \wedge \Delta^{?}_{+}, Y). \tag{7.6.21}$$ ◇

Recall from Definition 6.3.7 the enriched endomorphism operad of an object X in a symmetric monoidal V-category with V permutative. The category of pointed simplicial sets is symmetric monoidal closed, but is not strictly associative or unital. Nevertheless, as noted in Explanation 6.1.19, we extend Definitions 6.1.1 and 6.3.3 to the sSet_*-enriched case via the Symmetric Strictification Theorem 1.1.42. Then we have the following.

Definition 7.6.22. Suppose X is a symmetric spectrum. The *endomorphism simplicial operad* of X is the sSet_*-enriched operad $\mathsf{End}(X)$ whose nth term for $n \geq 0$ is

$$\mathsf{End}(X)_n = \mathsf{Map}_S(X^{\wedge n}, X) = \mathsf{SymSp}(X^{\wedge n} \wedge \Delta^{?}_{+}, X).$$ ◇

7.7. Quillen Model Categories

Definition 7.7.1. Suppose M is a category.

(1) For morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow D$ in M, we write $f \boxslash g$ if for each solid-arrow commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

in M, there is a morphism $B \longrightarrow C$ that makes the entire diagram commutative.

(2) For a class $\mathcal{A}$ of morphisms in M, we define the classes of morphisms

$$^{\boxslash}\mathcal{A} = \{f \in \mathsf{M} \mid f \boxslash a \text{ for all } a \in \mathcal{A}\},$$
$$\mathcal{A}^{\boxslash} = \{g \in \mathsf{M} \mid a \boxslash g \text{ for all } a \in \mathcal{A}\}.$$

(3) We say that a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in M *functorially factors* M if each morphism h in M has a functorial factorization $h = gf$ such that $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

(4) A *weak factorization system* in M is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in M such that
  (i) $(\mathcal{L}, \mathcal{R})$ functorially factors M,
  (ii) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$, and
  (iii) $\mathcal{R} = \mathcal{L}^{\boxslash}$. ◇

Explanation 7.7.2. The lifting properties imply a number of closure properties. Among them are the following (see, e.g., [**MP12**, Proposition 14.1.8]):

- Each of $\mathcal{L}$ and $\mathcal{R}$ contains all isomorphisms.
- Each of $\mathcal{L}$ and $\mathcal{R}$ is closed under composition and retracts.
- The left class, $\mathcal{L}$, is closed under pushouts and coproducts.
- The right class, $\mathcal{R}$, is closed under pullbacks and products. ◇

Definition 7.7.3. A *model category* is a complete and cocomplete category M equipped with three classes of morphisms $(\mathcal{W}, \mathcal{C}, \mathcal{F})$, called *weak equivalences*, *cofibrations*, and *fibrations*, that satisfy the following two axioms:

2**-out-of-**3**:** For any morphisms f and g in M such that the composition gf is defined, if any two of the three morphisms f, g, and gf are in $\mathcal{W}$, then so is the third.

WFS: $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

This finishes the definition of a model category. We have the following additional concepts in a model category M.

(1) An *acyclic (co)fibration* is a morphism that is both a (co)fibration and a weak equivalence.
(2) An object $A \in \mathsf{M}$ is *cofibrant* if the unique morphism $\varnothing \longrightarrow A$ from the initial object is a cofibration.
(3) An object $A \in \mathsf{M}$ if *fibrant* if the unique morphism $A \longrightarrow *$ to the terminal object is a fibration.
(4) A *cofibrant replacement* for $A \in \mathsf{M}$ is an object A^c obtained by the factorization of $\varnothing \longrightarrow A$ as a cofibration followed by an acyclic fibration:

$$\varnothing \longrightarrow A^c \xrightarrow{\sim} A.$$

(5) A *fibrant replacement* for $A \in \mathsf{M}$ is an object A^f obtained by the factorization of $A \longrightarrow *$ as an acyclic cofibration followed by a fibration:

$$A \xrightarrow{\sim} A^f \longrightarrow *.$$

(6) A *cylinder object* for $A \in \mathsf{M}$ is an object $\mathsf{Cyl}(A) \in \mathsf{M}$ obtained by factoring the fold morphism $A \coprod A \longrightarrow A$ as a cofibration followed by an acyclic fibration:

$$A \coprod A \longrightarrow \mathsf{Cyl}(A) \xrightarrow{\sim} A.$$

(7) A *path object* for $A \in \mathsf{M}$ is an object $\mathsf{Path}(A) \in \mathsf{M}$ obtained by factoring the diagonal $A \longrightarrow A \times A$ into an acyclic cofibration followed by a fibration:

$$A \xrightarrow{\sim} \mathsf{Path}(A) \longrightarrow A \times A.$$

(8) Functoriality of the weak factorization system $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$, respectively $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$, implies that the cofibrant replacements and cylinder objects, respectively fibrant replacements and path objects, are functorial.

(9) If A is fibrant, then A^c is also fibrant. Likewise, if B is cofibrant then B^f is also cofibrant. We let $(-)^{cf} = ((-)^c)^f$ and $(-)^{fc} = ((-)^f)^c$ denote the composites. The lifting properties imply there is a canonical weak equivalence $A^{cf} \xrightarrow{\sim} A^{fc}$ for each $A \in \mathsf{M}$. ◇

Definition 7.7.4. Suppose that $f, g : A \longrightarrow B$ are morphisms in a model category M.

- A *left homotopy* from f to g is a morphism $h : \mathsf{Cyl}(A) \longrightarrow B$ such that the following diagram in M commutes.

$$\begin{tikzcd}
A \arrow[dr, "i_0"'] \arrow[rrr, bend left, "f"] & & & \\
& A \coprod A \arrow[r] & \mathsf{Cyl}(A) \arrow[r, "h"] & B \\
A \arrow[ur, "i_1"] \arrow[rrr, bend right, "g"'] & & &
\end{tikzcd}$$

- A *right homotopy* from f to g is a morphism $h : A \longrightarrow \mathsf{Path}(B)$ such that the following diagram in M commutes.

$$\begin{tikzcd}
& & & B \\
A \arrow[r, "h"] \arrow[urrr, bend left, "f"] \arrow[drrr, bend right, "g"'] & \mathsf{Path}(B) \arrow[r] & B \times B \arrow[ur, "p_1"'] \arrow[dr, "p_2"] & \\
& & & B
\end{tikzcd}$$

◇

Proposition 7.7.5 ([**MP12**, Proposition 14.3.11])**.** *Suppose $f, g : A \longrightarrow B$ are morphisms in a model category M.*

- *If A is cofibrant and f is left homotopic to g, then f is right homotopic to g.*
- *If B is fibrant and f is right homotopic to g, then f is left homotopic to g.*

Definition 7.7.6. Suppose $f, g : A \longrightarrow B$ are morphisms in a model category M, and suppose that A is cofibrant and B is fibrant. We say that f is *homotopic* to g if f is left, or equivalently right, homotopic to g. When each of A and B is both fibrant and cofibrant, we say that f is a *homotopy equivalence* if there is a morphism $f' : B \longrightarrow A$ such that the composites $f'f$ and ff' are homotopic to the respective identities. ◇

Definition 7.7.7. Let M_{cf} denote the full subcategory of M whose objects are both fibrant and cofibrant. For objects $A, B \in \mathsf{M}_{cf}$, let $[A, B]$ denote the set of homotopy equivalence classes of morphisms $A \longrightarrow B$. ◇

Theorem 7.7.8 ([**MP12**, Theorem 14.3.15])**.** *Suppose that $f : A \longrightarrow B$ is a morphism in a model category M, and suppose that each of A and B is both fibrant and cofibrant. Then f is a homotopy equivalence if and only if it is a weak equivalence.*

Theorem 7.7.8 is the key result that justifies the following definition.

Definition 7.7.9. Suppose that M is a model category. The *homotopy category*, $Ho\mathsf{M}$ is defined to have the same objects as M and morphism sets

$$Ho\mathsf{M}(A,B) = [A^{cf}, B^{cf}].$$

By functoriality of factorizations, there is a canonical functor

$$\gamma : \mathsf{M} \longrightarrow Ho\mathsf{M}$$

that is the identity on objects and sends a morphism $f : A \longrightarrow B$ to the homotopy class of f^{cf}. $\diamond$

Explanation 7.7.10. One can show that there is a natural homotopy equivalence $(A^c)^f \xrightarrow{\sim} (A^f)^c$, and thus $Ho\mathsf{M}$ does not depend on which order one takes fibrant and cofibrant replacements. See [**MP12**, Section 14.4]. $\diamond$

Definition 7.7.11. Suppose that C is a category and $\mathcal{W}$ is a class of morphisms in C. We say that a functor $\lambda : \mathsf{C} \longrightarrow \mathsf{C}'$ is a *localization* of C at $\mathcal{W}$ if the following two conditions hold.

(1) For each morphism $f \in \mathsf{W}$, the morphism λf is an isomorphism.
(2) If $\kappa : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor as in the solid-arrow diagram below such that κf is an isomorphism for each $f \in \mathcal{W}$, then there is a unique functor $\overline{\kappa}$ such that $\overline{\kappa}\lambda = \kappa$.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & \mathsf{D} \\ \mathsf{C}' \arrow[ur, dashed, "\exists!\,\overline{\kappa}"'] & \end{tikzcd}$$

When such a localization exists, it is denoted $\mathsf{C}[\mathcal{W}^{-1}]$. $\diamond$

Theorem 7.7.12 ([**MP12**, Theorem 14.4.7])**.** *Suppose M is a model category with weak equivalences $\mathcal{W}$. The functor $\gamma : \mathsf{M} \longrightarrow Ho\mathsf{M}$ is a localization of M at $\mathcal{W}$.*

Explanation 7.7.13. In the context of Definition 7.7.11, one can always define morphisms in $\mathsf{C}[\mathcal{W}^{-1}]$ as a quotient of a free graph. However, in general this produces has a proper class of morphisms between two objects instead of a set. The machinery of model categories guarantees that the morphisms in $Ho\mathsf{M}$ between two objects do form a set, and thus $Ho\mathsf{M}$ is a category in the same Grothendieck universe as M. $\diamond$

Now we turn to functors between model categories, and the conditions under which we have an induced functor on homotopy categories. Certainly functors that preserve weak equivalences will induce functors on homotopy categories. However one has similar results under more general hypotheses, and these are needed in most applications of interest.

Definition 7.7.14. Suppose that $F : \mathsf{M} \longrightarrow \mathsf{N}$ is a functor of model categories. A *left functor* of F is a pair (L, λ) consisting of a functor

$$L : Ho\mathsf{M} \longrightarrow Ho\mathsf{N}$$

and a natural transformation λ as in the diagram below, where γ_{M} and γ_{N} are the canonical localizations.

$$\begin{tikzcd} \mathsf{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{M}}"'] \arrow[dr, phantom, "\lambda\nearrow"] & \mathsf{N} \arrow[d, "\gamma_{\mathsf{N}}"] \\ Ho\mathsf{M} \arrow[r, "L"'] & Ho\mathsf{N} \end{tikzcd}$$

A morphism of left functors $(L, \lambda) \longrightarrow (L, \lambda')$ is a natural transformation η : $L \longrightarrow L'$ such that the following equality of pasting diagrams holds.

$$\begin{tikzcd} \mathsf{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{M}}"'] \arrow[dr, phantom, "\lambda'\nearrow"] & \mathsf{N} \arrow[d, "\gamma_{\mathsf{N}}"] \\ Ho\mathsf{M} \arrow[r, bend left, "L'"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\eta\Uparrow"] & Ho\mathsf{N} \end{tikzcd} = \begin{tikzcd} \mathsf{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{M}}"'] \arrow[dr, phantom, "\lambda\nearrow"] & \mathsf{N} \arrow[d, "\gamma_{\mathsf{N}}"] \\ Ho\mathsf{M} \arrow[r, bend right, "L"'] & Ho\mathsf{N} \end{tikzcd}$$

A *left derived functor* for F is a terminal object $(\mathbb{L}F, \tilde{\lambda})$ among left functors of F. ⋄

Definition 7.7.15. Suppose that $F : \mathsf{M} \longrightarrow \mathsf{N}$ is a functor of model categories. A *right functor* of F is a pair (R, ρ) consisting of a functor

$$R : Ho\mathsf{M} \longrightarrow Ho\mathsf{N}$$

and a natural transformation ρ as in the diagram below, where γ_{M} and γ_{N} are the canonical localizations.

$$\begin{tikzcd} \mathsf{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{M}}"'] \arrow[dr, phantom, "\rho\swarrow"] & \mathsf{N} \arrow[d, "\gamma_{\mathsf{N}}"] \\ Ho\mathsf{M} \arrow[r, "R"'] & Ho\mathsf{N} \end{tikzcd}$$

A morphism of right functors $(R, \rho) \longrightarrow (R, \rho')$ is a natural transformation η : $R \longrightarrow R'$ such that the following equality of pasting diagrams holds.

$$\begin{tikzcd} \mathsf{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{M}}"'] \arrow[dr, phantom, "\rho\swarrow"] & \mathsf{N} \arrow[d, "\gamma_{\mathsf{N}}"] \\ Ho\mathsf{M} \arrow[r, bend left, "R"] \arrow[r, bend right, "R'"'] \arrow[r, phantom, "\eta\Downarrow"] & Ho\mathsf{N} \end{tikzcd} = \begin{tikzcd} \mathsf{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathsf{M}}"'] \arrow[dr, phantom, "\rho'\swarrow"] & \mathsf{N} \arrow[d, "\gamma_{\mathsf{N}}"] \\ Ho\mathsf{M} \arrow[r, bend right, "R'"'] & Ho\mathsf{N} \end{tikzcd}$$

A *right derived functor* for F is an initial object $(\mathbb{R}F, \tilde{\rho})$ among right functors of F. ⋄

Explanation 7.7.16. Since initial and terminal objects are unique up to unique isomorphism, it is usual to refer to *the* left and right derived functors of F. ⋄

The following result gives a set of conditions that often occur in applictions and under which one has left and right derived functors.

Proposition 7.7.17 ([**MP12**, Propositions 16.1.3 and 16.1.4]). *Suppose that $F : \mathsf{M} \longrightarrow \mathsf{N}$ is a functor of model categories.*

(1) If F takes acyclic cofibrations between cofibrant objects to weak equivalences, then $F \circ (-)^c$ carries weak equivalences in M to isomorphisms in $Ho\mathsf{N}$. The induced functor
$$\mathbb{L}F : Ho\mathsf{M} \longrightarrow Ho\mathsf{N}$$
is the left derived functor of F.

(2) If F takes acyclic fibrations between fibrant objects to weak equivalences, then $F \circ (-)^f$ carries weak equivalences in M to isomorphisms in $Ho\mathsf{N}$. The induced functor
$$\mathbb{R}F : Ho\mathsf{M} \longrightarrow Ho\mathsf{N}$$
is the right derived functor of F.

The next result records a useful set of equivalent conditions, and follows from an application of adjoints and the lifting properties of weak factorization systems.

Lemma 7.7.18. *Suppose that M and N are model categories and suppose that (F, U) is an adjunction*
$$F : \mathsf{M} \underset{}{\overset{}{\rightleftarrows}} \mathsf{N} : U. \quad (\perp)$$

Then the following conditions are equivalent.

(1) F preserves cofibrations and U preserves fibrations.
(2) F preserves cofibrations and acyclic cofibrations.
(3) U preserves fibrations and acyclic fibrations.
(4) F preserves acyclic cofibrations and U preserves acyclic fibrations.

Definition 7.7.19. We say that an adjoint pair of functors (F, U) is a *Quillen adjunction* if the equivalent conditions in Lemma 7.7.18 are satisfied. We say that (F, U) is a *Quillen equivalence* if it is a Quillen adjunction with the following property. For any cofibrant $X \in \mathsf{M}$ and fibrant $Y \in \mathsf{N}$, a morphism $FX \longrightarrow Y$ is a weak equivalence in N if and only if its adjoint $X \longrightarrow UY$ is a weak equivalence in M. In this case, we denote the adjunction with $\simeq_Q$, as in the following.
$$F : \mathsf{M} \underset{\simeq_Q}{\rightleftarrows} \mathsf{N} : U.$$

◇

The definitions of Quillen adjunction and Quillen equivalence are designed to induce the corresponding structures on homotopy categories, as the following result explains.

Proposition 7.7.20 ([**MP12**, Proposition 16.2.2]). *Suppose that (F, U) is a Quillen adjunction. Then the derived functors $\mathbb{L}F$ and $\mathbb{R}U$ exist and form an adjoint pair of functors between $Ho\mathsf{M}$ and $Ho\mathsf{N}$. If (F, U) is a Quillen equivalence, then $(\mathbb{L}F, \mathbb{R}U)$ is an adjoint equivalence.*

Now we discuss cofibrantly generated model categories. The concept of a small object A in the following definition essentially means that a morphism from A to the codomain of a sufficiently long composition must factor through some stage. We regard an ordinal as a category with a unique morphism $i \longrightarrow j$ if and only if $i \leq j$. See [**Pin14, SV02**] for further discussion of cardinals and ordinals.

Definition 7.7.21. Suppose C is a cocomplete category, and $\mathcal{I}$ is a collection of morphisms in C.

(1) For an ordinal α, an *α-sequence* in C is a colimit-preserving functor

$$X : \alpha \longrightarrow \mathsf{C}.$$

The induced morphism

$$X_0 \longrightarrow \mathsf{colim}_{\beta<\alpha} X_\beta$$

is called a *transfinite composition*.

(2) For an ordinal α and an α-sequence X, if each morphism $X_\beta \longrightarrow X_{\beta+1}$ belongs to $\mathcal{I}$ for each ordinal β satisfying $\beta+1<\alpha$, then we call the above morphism a *transfinite composition of morphisms in $\mathcal{I}$*.

(3) A *relative $\mathcal{I}$-cell complex* is a transfinite composition of pushouts of morphisms in $\mathcal{I}$. The collection of relative $\mathcal{I}$-cell complexes is denoted by $\mathsf{Cell}(\mathcal{I})$.

(4) For an object A in C and a cardinal κ, we say that A is *κ-small relative to $\mathcal{I}$* if for
- each regular cardinal $\alpha \geq \kappa$ and
- each α-sequence X in C with each morphism $X_\beta \longrightarrow X_{\beta+1}$ in $\mathcal{I}$ for each ordinal β satisfying $\beta+1<\alpha$,

the induced morphism of sets

$$\mathsf{colim}_{\beta<\alpha}\mathsf{C}(A, X_\beta) \longrightarrow \mathsf{C}\Big(A, \mathsf{colim}_{\beta<\alpha} X_\beta\Big)$$

is a bijection.

(5) We say that A is *small relative to $\mathcal{I}$* if it is κ-small relative to $\mathcal{I}$ for some cardinal κ.

(6) We say that $\mathcal{I}$ *permits the small object argument* if the domain of each morphism in $\mathcal{I}$ is small relative to $\mathsf{Cell}(\mathcal{I})$.

$\diamond$

Definition 7.7.22. A model category $(\mathsf{M}, \mathcal{W}, \mathcal{C}, \mathcal{F})$ is *cofibrantly generated* if it is equipped with two sets $\mathcal{I}$ and $\mathcal{J}$ of morphisms such that the following three statements hold:

(i) Both $\mathcal{I}$ and $\mathcal{J}$ permit the small object argument.
(ii) $\mathcal{F} = \mathcal{J}^{\boxslash}$.
(iii) $\mathcal{F} \cap \mathcal{W} = \mathcal{I}^{\boxslash}$.

In this case, the morphisms in $\mathcal{I}$ are called *generating cofibrations* and the morphisms in $\mathcal{J}$ are called *generating acyclic cofibrations*. $\diamond$

In particular, in a cofibrantly generated model category, fibrations are detected by the set $\mathcal{J}$, and acyclic fibrations are detected by the set $\mathcal{I}$. The cofibrations and acyclic cofibrations are defined by lifting against acyclic fibrations and fibrations, respectively. These include the morphisms of $\mathcal{I}$, respectively $\mathcal{J}$, but generally others as well.

Definition 7.7.23. Suppose $(\mathsf{M}, \wedge, S^0)$ is a symmetric monoidal category. The *pushout product* of morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow D$ is the morphism $f \square g$

induced by the universal property of the pushout in the following diagram.

$$\begin{tikzcd}
A \wedge C \arrow[r, "A \wedge g"] \arrow[d, "f \wedge C"'] & A \wedge D \arrow[d] \arrow[ddr, bend left, "f \wedge D"] & \\
B \wedge C \arrow[r] \arrow[drr, bend right, "B \wedge g"'] & (B \wedge C) \cup_{A \wedge C} (A \wedge D) \arrow[dr, dashed, "f \square g"] & \\
& & B \wedge D
\end{tikzcd}$$

◇

Definition 7.7.24. Suppose $(\mathsf{M}, \wedge, S^0)$ is a symmetric monoidal closed category and that M has a model structure. We say that $(\mathsf{M}, \wedge, S^0)$ is a *monoidal model category* if the following axiom holds.

Pushout product axiom: If f and g are cofibrations, then the pushout product $f \square g$ is a cofibration and is acyclic if either f or g is acyclic. ◇

7.8. Examples of Quillen Model Categories

Example 7.8.1 (Small Categories **[JT91, Rez∞]**)**.** The category Cat of small categories and functors has a *standard model structure* given as follows.

Fibrations: A functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a *fibration* if for each
- object $X \in \mathsf{C}$ and
- isomorphism $f : FX \xrightarrow{\cong} Y \in \mathsf{D}$,

there exists an isomorphism $g : X \xrightarrow{\cong} Z \in \mathsf{C}$ such that

$$FZ = Y \quad \text{and} \quad Fg = f.$$

Cofibrations: A functor is a *cofibration* if it is injective on objects.

Weak equivalences: A functor is a *weak equivalence* if it is an equivalence of categories.

This is a monoidal model category with respect to the Cartesian product. It is cofibrantly generated with the following data.

Generating acyclic cofibration: $\mathcal{J}$ contains the single functor

$$\{0\} \xrightarrow{\quad t \quad} J$$

that sends the object 0 in the domain to 0 in the codomain $J = \{0 \xrightarrow{\cong} 1\}$, which has two objects and a single isomorphism between them.

Generating cofibrations: The set

$$\mathcal{I} = \{u, v, w\}$$

contains the following three functors. The functor

$$\varnothing \xrightarrow{\quad u \quad} \{0\}$$

is the unique functor. Suppose
- $V_0 = \{0, 1\}$ is a discrete category with two objects, and

- $V = \{0 \longrightarrow 1\}$ contains two objects and one non-identity morphism between them.

The functor
$$V_0 \xrightarrow{v} V$$
is the identity function on objects. Suppose $W = \{0 \rightrightarrows 1\}$ contains two objects and a parallel pair of non-identity morphisms. The functor
$$W \xrightarrow{w} V$$
is the identity on objects and sends both non-identity morphisms in W to the unique non-identity morphism in V.

A functor is a fibration if and only if it has the right lifting property with respect to $\mathcal{J} = \{t\}$. A functor is an acyclic fibration if and only if it has the right lifting property with respect to $\mathcal{I} = \{u, v, w\}$. ◇

Example 7.8.2 (Chain Complexes [**Hov99**, Section 2.3])**.** Suppose that R is a commutative ring, and $\mathsf{Ch}(R)$ the category of chain complexes over R. The *projective model structure* on $\mathsf{Ch}(R)$ consists of the following.

Weak equivalences: $\mathcal{W}$ is the set of homology isomorphisms.
Fibrations: $\mathcal{F}$ is the set of levelwise surjections.

The cofibrations are defined by lifting against the acyclic fibrations. This model structure is also known as the *standard model structure*

There is a second model structure on $\mathsf{Ch}(R)$ that has the same weak equivalences (homology isomorphisms), but whose cofibrations are levelwise injections and whose fibrations are defined by lifting against acyclic cofibrations. Although these model structures are distinct, the identity functor is a Quillen equivalence between them. ◇

Example 7.8.3 (Simplicial Sets [**MP12**, Section 17.5])**.** The standard model structure on sSet is a cofibrantly generated monoidal model structure, given as follows.

Generating cofibrations: $\mathcal{C}$ is the set of boundary inclusions
$$\partial\Delta^n \longrightarrow \Delta^n.$$

Generating acyclic cofibrations: $\mathcal{J}$ is the set of horn inclusions
$$\Lambda^n_k \longrightarrow \Delta^n.$$

Fibrations: Morphisms in $\mathcal{F} = \mathcal{J}^{\boxslash}$ are called *Kan fibrations*, and the fibrant objects are called *Kan complexes*.

Weak equivalences: $\mathcal{W}$ is the set of morphsisms $f : X \longrightarrow Y$ such that
$$f^* : [Y, Z] \longrightarrow [X, Z]$$
is a bijection for all Kan complexes Z, where $[X, Z]$ denotes left homotopy equivalence classes of morphisms with respect to the cylinder object $\mathsf{Cyl}(X) = X \times \Delta^1$. ◇

Explanation 7.8.4. In the context of Example 7.8.3, the cofibrations in sSet are levelwise inclusions. In particular, every simplicial set is cofibrant. A simplicial homotopy equivalence (Definition 7.2.1) is a left homotopy in the sense of Definition 7.7.4. Therefore, every simplicial homotopy equivalence is a weak equivalence in sSet. By Theorem 7.7.8, the weak equivalences between Kan complexes are precisely the simplicial homotopy equivalences. ◇

Example 7.8.5 (Topological Spaces [**MP12**, Section 17.2]). The *Quillen model structure* on Top is a cofibrantly generated monoidal model structure defined as follows.

Generating Cofibrations: $\mathcal{I}$ is the set of inclusions
$$S^{n-1} \longrightarrow D^n$$
of the topological $(n-1)$-sphere as the boundary of the n-disk for $n \geq 0$, where S^{-1} denotes the empty set.

Generating Acyclic Cofibrations: $\mathcal{J}$ is the set of inclusions
$$D^n \cong D^n \times \{0\} \longrightarrow D^n \times I$$
where I is the unit interval. ⋄

Theorem 7.8.6. *With the model structures defined in Examples 7.8.3 and 7.8.5, the geometric realization and total singular complex functors define a Quillen equivalence.*
$$|-| : \mathsf{Top} \underset{}{\overset{}{\rightleftarrows}}_{\simeq_Q} \mathsf{sSet} : \mathsf{Sing}$$

The category of symmetric spectra, SymSp, has a model structure called the *stable model structure* that we now explain.

Definition 7.8.7. A morphism $f = \{f_k\}_{k\geq 0}$ of symmetric spectra is a *level equivalence* if each f_k is a weak equivalence of pointed simplicial sets. It is a *level fibration* if each f_k is a Kan fibration. ⋄

Explanation 7.8.8. The weak equivalences of symmetric spectra are known as *stable equivalences* and are detected in cohomology. They include the level equivalences, but are more general. ⋄

Theorem 7.8.9 ([**HSS00**, Theorem 3.4.4, Corollary 5.3.8]). *The category of symmetric spectra is a monoidal model category with weak equivalences given by the stable equivalences and fibrations given by the level fibrations.*

Explanation 7.8.10. The model structure on symmetric spectra is Quillen equivalent to other modern models of stable homotopy theory, including orthogonal spectra and S-modules. See [**MMSS01**] for further details. ⋄

7.9. Notes

7.9.1 (Simplicial Objects). See [**Fri12**] for an introduction to simplicial sets. For further details and development of simplicial objects, see [**Cur71, GZ67, GJ09, May92, Rie14, Ric20**]. Discussion of limits and colimits in diagram categories can be found in [**Rie16**], and the resulting enrichment properties are in [**Kel05**]. ⋄

7.9.2 (Nerve and Geometric Realization). The nerve and geometric realization functors described in Definitions 7.1.14 and 7.2.3, respectively, together with their adjoints, arise from the theory of Kan extensions and are discussed in [**Rie14**]. Additional references include [**Ric20, MP12**]. ⋄

7.9.3 (Compactly Generated Topological Spaces). With the usual product topology, the category of all topological spaces fails to be symmetric monoidal closed with respect to the Cartesian product. Moreover, the induced smash product fails to be associative. Restricting to compactly generated weak Hausdorff spaces, as we do in Definition 7.1.13, avoids the point-set pathologies of the larger category. Familiar classes of spaces, including CW complexes, compact spaces, locally compact spaces, topological manifolds, and metrizable spaces are all weak

Hausdorff and compactly generated. See [**Rie14**, Section 6.1] for definitions and a detailed overview. For further reference see [**Bor94b**, Section 7.1], [**May99**, Section 5], and [**MS06**, Section 1.7]. Further references include [**Bro64, GZ67, Kel55, May72, Ste67**]. In the literature, these spaces are sometimes called *k-spaces* and *Kelley spaces*. $\diamond$

7.9.4 (Symmetric Spectra)**.** The category of symmetric spectra was developed in Hovey-Shipley-Smith [**HSS00**]. See [**HSS00**, Proposition 1.2.10] for completeness and cocompleteness as in Proposition 7.5.5. See [**HSS00**, Theorem 2.2.10] for the monoidal product and internal hom as in Theorem 7.6.15. For further development, we also refer the reader to [**MMSS01, Ric20**] and [**Hov01**]. The latter of those discusses symmetric spectra in a general pointed bicomplete category C.

Symmetric spectra are one of several prominent models for stable homotopy theory with a symmetric monoidal product. Others include the S-modules of Elmendorf-Kriz-Mandell-May [**EKMM97**] and the orthogonal spectra of Mandell-May-Schwede-Shipley [**MMSS01**]. All of these, and others, are defined and compared in [**MMSS01**]. $\diamond$

7.9.5 (Sequential Spectra)**.** The earliest and simplest model for stable homotopy theory is the category of *sequential spectra*, $\mathsf{Sp}^{\mathbb{N}}$ introduced by Lima [**Lim59**]. The objects are sequences of pointed simplicial sets together with structure morphisms

$$X_n \in \mathsf{sSet}_* \quad \text{and} \quad \sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}, \quad \text{for} \quad n \geq 0.$$

The morphisms of sequential spectra are sequences of morphisms in sSet_* commuting with the structure morphisms σ. One motivating interest in sequential spectra is that they represent generalized cohomology theories of spaces (see, for example, [**May99**]).

One can describe sequential spectra as modules over the sequential sphere spectrum $\{S^n\}$. However, the construction of a smash product as in Section 7.4 fails because the sequential sphere spectrum is not a commutative monoid. See [**HSS00**, Section 2.3] for further details on this point. The more sophisticated categories of spectra, including symmetric spectra, were motivated by interest in symmetric monoidal constructions in stable homotopy theory.

The homotopy groups of a sequential spectrum X are given by a limit of the simplicial homotopy groups

$$\pi_n X = \lim_{k \to \infty} \pi_{n+k} X_k$$

that, for each n, is guaranteed to stabilize by the Freudenthal Suspension Theorem. Weak equivalences of sequential spectra are morphisms that induce isomorphisms on π_n for all n, and these are part of a Quillen model structure described in [**BF78**]. The essential requirement for any category of spectra is that its homotopy category be equivalent to the homotopy category of sequential spectra.

There is a forgetful functor

$$U : \mathsf{SymSp} \longrightarrow \mathsf{Sp}^{\mathbb{N}}$$

given by forgetting the symmetric group actions. If f is a weak equivalence of symmetric spectra (i.e. a stable equivalence), then Uf is a weak equivalence (i.e. a π_*-isomorphism) of sequential spectra. However the converse generally does not hold. Nevertheless, [**HSS00**] shows that there is a Quillen equivalence between the (model) category of symmetric spectra and that of sequential spectra. We refer the reader there for detailed definitions of the model structures and proof of the Quillen equivalence. ⋄

7.9.6 (Model Categories)**.** The concept of a model category is originally due to Quillen [**Qui67**]. Our presentation generally follows [**MP12**]. Many additional examples of model categories can be found in [**WY18, WY20**]. The first of these, [**WY18**], gives a general approach to (semi-)model structures on the category of algebras over a colored operad in a monoidal model category. For additional references, see [**DS95, Hov99, Hir03, Rie11**]. ⋄

CHAPTER 8

Segal K-Theory of Permutative Categories

In this chapter we discuss the Segal K-theory spectra associated to small permutative categories. This construction, an extension of that given by Segal **[Seg74]**, takes values in symmetric spectra. The construction has three components.

(1) There is a functor
$$\mathsf{K}^{\mathcal{F}} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$$
where Γ-sSet is a certain category of (pointed) diagrams of simplicial sets. We define Γ-sSet in Section 8.1 and $\mathsf{K}^{\mathcal{F}}$ in Section 8.2.

(2) Composition with the nerve defines a functor of (pointed) diagram categories
$$N_* : \Gamma\text{-}\mathsf{Cat} \longrightarrow \Gamma\text{-}\mathsf{sSet}$$
given in Definition 8.1.18.

(3) There is a functor
$$\mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \Gamma\text{-}\mathsf{Cat}$$
that constructs Γ-categories from small permutative categories. The definition of J^{Se} is in Section 8.5.

The Segal K-theory functor, K^{Se}, is the composite
$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{J}^{\mathsf{Se}}} \Gamma\text{-}\mathsf{Cat} \xrightarrow{N_*} \Gamma\text{-}\mathsf{sSet} \xrightarrow{\mathsf{K}^{\mathcal{F}}} \mathsf{SymSp}.$$

In the treatment below, we give two different but equivalent constructions of the functor J^{Se}. The first, denoted $\mathsf{C}^{\mathcal{F}}$ for a small permutative category C, is in Section 8.3. It is more straightforward and consists of systems of objects indexed by certain pairs of disjoint finite sets. This is the construction originally given by Segal. There are three variants of this construction, each with respective benefits, and all give level-equivalent symmetric spectra.

The second construction is the one we denote J^{Se}. Its construction uses certain diagram multicategories that we call *partition multicategories*. These package the combinatorial structure given by partitions of finite sets and make the categorical properties of J^{Se} more apparent. For example, we describe the 2-functoriality of J^{Se} in these terms. Moreover, the approach via partition multicategories is more readily comparable to our constructions of Elmendorf-Mandell K-theory, denoted K^{EM}, in Chapter 10. We describe the partition multicategories in Section 8.4 and then define J^{Se} and K^{Se} in Section 8.5.

The following table summarizes Segal K-theory and the main constructions in this chapter. The upper portion concerns the definition of Segal K-theory and the constructions $(-)^{\mathcal{F}}$ due to Segal. The lower portion concerns details related to the construction of J^{Se} via partition multicategories.

Segal K-theory	$\mathsf{K}^{\mathsf{Se}} = \mathsf{K}^{\mathcal{F}} N_* \mathsf{J}^{\mathsf{Se}}$	(8.5.1)
indexing category	$\mathcal{F}$	(8.1.1)
K-theory	$\mathsf{K}^{\mathcal{F}} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$	(8.2.5)
nerve	$N_* : \Gamma\text{-}\mathsf{Cat} \longrightarrow \Gamma\text{-}\mathsf{sSet}$	(8.1.18)
Segal Γ-categories	$(-)^{\mathcal{F}}, (-)^{\mathcal{F}}_{\mathsf{lax}}, (-)^{\mathcal{F}}_{\mathsf{co}}$	(8.3.14)
comparison $(-)^{\mathcal{F}}, (-)^{\mathcal{F}}_{\mathsf{lax}}, (-)^{\mathcal{F}}_{\mathsf{co}}$	8.3.21, 8.5.2	
Segal J-theory	$\mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \Gamma\text{-}\mathsf{Cat}$	
definition	$\mathsf{J}^{\mathsf{Se}} = \mathsf{J}^{\mathcal{M}} \circ \mathsf{End}$	(8.5.1)
endomorphism multicategories	$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}_*$	(5.1.11, 5.3.9)
partition multicategories	$\mathcal{M} : \mathcal{F}^{\mathsf{op}} \longrightarrow \mathsf{Multicat}_*$	(8.4.1, 8.4.7)
partition J-theory	$\mathsf{J}^{\mathcal{M}} : \mathsf{Multicat}_* \longrightarrow \Gamma\text{-}\mathsf{Cat}$	(8.4.10)
Segal J-theory and Γ-categories	$\mathsf{J}^{\mathsf{Se}} \cong (-)^{\mathcal{F}}_{\mathsf{lax}}$	(8.4.8)
level equivalence	$\mathsf{K}^{\mathsf{Se}} \xrightarrow{\sim} \mathsf{K}^{\mathsf{EM}}$	(10.6.10)

Reading Guide.

(1) For the essential definitions, read Definitions 8.1.1, 8.1.8, and 8.1.16 through 8.1.18, together with the statement of Proposition 8.2.6.

(2) For Segal's construction alone, it suffices to read the material from (1) and then Definitions 8.3.1, 8.3.9, 8.3.12, and 8.3.14 followed by the statement of Proposition 8.3.13.

(3) For our second, equivalent construction, the material in Section 8.3 may be skipped. After (1), read Definition 8.4.1 and the statement of Proposition 8.4.7. Then read Definitions 8.4.10 and 8.5.1. After that, the items in (2) may be read as a concrete simplification of this material.

(4) For the material on partition multicategories in Section 8.4, the reader will want to be familiar with the basic definitions of multicategories and multifunctors from Section 5.1. However, the additional material in Chapter 5 will not play a major role in this chapter.

(5) Go back and read the rest of this chapter.

8.1. Categories of Γ-Objects

In this section we introduce Γ-objects and their morphisms. The two important cases of interest for us will be Γ-simplicial sets and, later, Γ-categories.

Definition 8.1.1. Let FinSet_* denote the category whose objects are pointed finite sets and whose morphisms are basepoint-preserving functions. Let $\mathcal{F}$ denote the full subcategory of FinSet_* whose objects are pointed finite sets $\underline{n} = \{0, \ldots, n\}$ for natural numbers $n \geq 0$. The pointed finite set $\underline{0}$ is an initial and terminal object for both FinSet_* and $\mathcal{F}$, providing a basepoint such that the inclusion

$$\mathcal{F} \longrightarrow \mathsf{FinSet}_*$$

is a pointed functor. $\diamond$

Explanation 8.1.2. Note that FinSet_* is not small, just as the category of all (not-necessarily-pointed) finite sets is not small. However, $\mathcal{F}$ is a small skeleton for FinSet_*. The inclusion

$$\mathcal{F} \longrightarrow \mathsf{FinSet}_*$$

is fully faithful by definition and is essentially surjective because each pointed finite a is (non-uniquely) isomorphic to $\underline{n-1}$, where $n = |a|$ is the cardinality of a. Therefore the inclusion is an equivalence of categories. $\diamond$

Definition 8.1.3. We make the following notational conventions for pointed finite sets $a \in \mathsf{FinSet}_*$.

- Unless otherwise specified, the basepoint of a general pointed finite set a is denoted $*$.
- The basepoint of $\underline{n}$ is the element 0.
- We let $a^\flat$ denote the *punctured* finite set $a \backslash \{*\}$, obtained by removing the basepoint.
- We let $|a|$ denote the cardinality of a. $\diamond$

Explanation 8.1.4. For a natural number n, we have

$$\underline{n}^\flat = \{1, \ldots, n\}$$

and $|\underline{n}| = n + 1$. The notation $(-)^\flat$ was introduced in Definition 4.3.8, and $a^\flat$ may be read as *a-flat* or *a-punctured*. $\diamond$

Definition 8.1.5. Suppose $\underline{m}$ and $\underline{n}$ are objects of $\mathcal{F}$. The *lexicographic order* for $\underline{m} \wedge \underline{n}$ is the bijection

$$L : \underline{m} \wedge \underline{n} \cong \underline{mn}$$

given by

$$L(x,y) = \begin{cases} 0 & \text{if } x = 0 \quad \text{or} \quad y = 0, \\ n(x-1) + y & \text{if } x > 0 \quad \text{and} \quad y > 0. \end{cases}$$

We will also use L to denote the composite

$$\bigvee_{\underline{m}^\flat} \underline{n} \cong \underline{m} \wedge \underline{n} \cong \underline{mn},$$

where the first bijection identifies the wedge summand indexed by $i \in \underline{m}^\flat$ with the elements $(i,y) \in \underline{m} \wedge \underline{n}$. $\diamond$

Explanation 8.1.6. Using the lexicographic order, the smash product of pointed finite sets induces a monoidal product on $\mathcal{F}$ that we also denote $\wedge$. Elementary algebra with the formula for L shows that this product is strictly associative and unital with strict monoidal unit $\underline{1}$, so $(\mathcal{F}, \wedge, \underline{1})$ is a permutative category. Note that its symmetry is generally not trivial, and is induced by that of FinSet_*

$$\underline{m} \wedge \underline{n} \cong \underline{n} \wedge \underline{m}.$$

When we take the smash products of objects in $\mathcal{F}$ below, we will use the lexicographic order unless otherwise stated. $\diamond$

Explanation 8.1.7. One may visualize the lexicographical order

$$L : \underline{m} \wedge \underline{n} \cong \underline{mn}$$

as an $(m+1) \times (n+1)$ matrix, with $\underline{m}$ and $\underline{n}$ indexing, respectively, the rows and the columns. Each entry in the first row and the first column yields $0 \in \underline{mn}$ under L.

For the other entries, one counts from left to right in each row, from top to bottom. With this matrix interpretation, the symmetry isomorphism

$$\underline{m} \wedge \underline{n} \cong \underline{n} \wedge \underline{m}$$

corresponds to taking the transpose.

The associativity of $\wedge$ corresponds to a 3-dimensional matrix. In terms of algebra, the associativity of $\wedge$ means the commutativity of the following diagram for all m, n, and p.

$$\begin{tikzcd}
(\underline{m} \wedge \underline{n}) \wedge \underline{p} \arrow[r, "L \wedge 1"] \arrow[d, "\cong"'] & \underline{mn} \wedge \underline{p} \arrow[dd, "L"] \\
\underline{m} \wedge (\underline{n} \wedge \underline{p}) \arrow[d, "1 \wedge L"'] & \\
\underline{m} \wedge \underline{np} \arrow[r, "L"] & \underline{mnp}
\end{tikzcd}$$

To check its commutativity, consider elements $x \in \underline{m}$, $y \in \underline{n}$, and $z \in \underline{p}$. If one of x, y, or z is 0, then each of the two composites in the previous diagram sends (x, y, z) to 0. If $x, y, z > 0$, then both composites send (x, y, z) to

$$np(x-1) + p(y-1) + z = p\big(n(x-1) + y - 1\big) + z$$

in $\underline{mnp}$. ◇

Definition 8.1.8. Suppose $(\mathsf{C}, *)$ is a pointed category with $*$ terminal in C. A Γ-*object* in C is a pointed functor

$$X : (\mathcal{F}, \underline{0}) \longrightarrow (\mathsf{C}, *).$$

The category of Γ-objects in C, denoted Γ-C, is

$$\mathsf{Cat}_*\big((\mathcal{F}, \underline{0}), (\mathsf{C}, *)\big),$$

the category of pointed functors from $(\mathcal{F}, \underline{0})$ to $(\mathsf{C}, *)$ and pointed natural transformations. ◇

Explanation 8.1.9 (Notation Γ)**.** The category Γ-C would be denoted $\mathcal{F}_*$-C in the notation of Definition 4.3.32. The usage of Γ is a historical artifact of Segal's construction [**Seg74**], which defined a category Γ that is isomorphic to $\mathcal{F}^{\mathsf{op}}$ and then defined Γ-objects as pointed functors out of Γ^{op}. The usage of Γ in this context is a fixture of the literature. ◇

Explanation 8.1.10 (Categories of Γ-objects)**.** Some authors work with diagrams on FinSet instead of $\mathcal{F}$, as doing so avoids choices of specific finite sets $\underline{n}$. We define Γ-objects as functors out of $\mathcal{F}$, the skeletal replacement of FinSet, so that the collection of Γ-objects forms a category. Moreover, we implicitly use the canonical ordering on each $\underline{n}$ in constructions that are indexed over the elements of $\underline{n}$. ◇

Explanation 8.1.11 (Canonial Basepoints for Γ-Objects)**.** In the context of Definition 8.1.8, recall from Definition 4.1.1 that C_* denotes the category of objects and morphisms under $*$ in C. Because $\underline{0}$ is also initial in $\mathcal{F}$, there is a canonical isomorphism of categories

$$\Gamma\text{-}\mathsf{C} \cong \Gamma\text{-}(\mathsf{C}_*). \tag{8.1.12}$$

For each Γ-object X, the unique morphisms

$$\underline{0} \longrightarrow \underline{n}$$

induce canonical basepoints

$$* = X\underline{0} \longrightarrow X\underline{n}.$$

By functoriality, each Γ-object in C thereby determines a unique Γ-object in C_*, and conversely. Morphisms of Γ-objects (pointed natural transformations between pointed functors) necessarily preserve the canonical basepoints, and thus we have a bijection between the morphisms of Γ-C and those of Γ-(C_*). ◇

Recall the notion of model category from Definition 7.7.3.

Definition 8.1.13. In the context of Definition 8.1.8, suppose furthermore that C is a model category. A *levelwise weak equivalence* of Γ-objects in C is a morphism f such that each $f_{\underline{n}}$ is a weak equivalence in the model structure for C. ◇

Definition 8.1.14. Suppose X is a Γ-object in C and suppose C has products. For $n > 0$, the nth *Segal map* is the morphism

$$p_n : X\underline{n} \longrightarrow \prod_{i \in \underline{n}^\flat} X\underline{1}$$

induced by the maps $\delta_i : \underline{n} \longrightarrow \underline{1}$ with $\delta_i(j) = 0$ for $j \neq i$ and $\delta_i(i) = 1$. The 0th Segal map, p_0, is the identity on the terminal object $*$.

Suppose, moreover, that C is a model category. We say that X is *special* if the Segal maps p_n are weak equivalences for all natural numbers $n \geq 0$. ◇

Explanation 8.1.15. In the context of Definition 8.1.14, the Segal map p_0 is an identity by definition and the Segal map p_1 is an identity by functoriality of X. So the first nontrivial Segal map is

$$p_2 : X\underline{2} \longrightarrow X\underline{1} \times X\underline{1}. \qquad ◇$$

The two types of Γ-objects we will see most frequently below are the following. Because we will make frequent use of the canonical basepoints for Γ-objects, we take these as part of the definitions.

Definition 8.1.16 (Γ-Simplicial Sets)**.** A Γ-*simplicial set* is a Γ-object in sSet_*. We let

$$\Gamma\text{-}\mathsf{sSet}$$

denote the category of Γ-objects and morphisms in sSet_*. By (8.1.12) this is canonically isomorphic to the category of Γ-objects and morphisms in sSet. ◇

Recall from Definition I.6.3.1 the notion of modification between transformations.

Definition 8.1.17 (Γ-Categories)**.** A Γ-*category* is a Γ-object in Cat_*. We let

$$\Gamma\text{-}\mathsf{Cat}$$

denote the category of Γ-objects and morphisms in Cat_*. By (8.1.12) this is canonically isomorphic to the category of Γ-objects and morphisms in Cat.

We extend Γ-Cat to a 2-category by taking *pointed modifications*, that is, modifications θ between pointed natural transformations such that the component at $\underline{0}$ is the identity natural transformation on the identity functor of $\mathbf{1}$. We will use the notation Γ-Cat for both this 2-category and its underlying 1-category. ◇

Recall the nerve functor N from Definition 7.2.3.

Definition 8.1.18. We let

$$N_* : \Gamma\text{-Cat} \longrightarrow \Gamma\text{-sSet}$$

denote the functor induced by composition with the nerve functor N. For a Γ-category X, the Γ-simplicial set $N_* X$ is the composite

$$\mathcal{F} \xrightarrow{X} \mathsf{Cat} \xrightarrow{N} \mathsf{sSet}.$$

For a morphism of Γ-categories f, the morphism $N_* f$ is the whiskering of f (as a natural transformation) with N. ◇

8.2. Symmetric Spectra from Γ-Simplicial Sets

Now we define the K-theory of Γ-simplicial sets as a functor

$$\mathsf{K}^{\mathcal{F}} : \Gamma\text{-sSet} \longrightarrow \mathsf{SymSp}.$$

Recall S^p denotes the p-fold smash product of the simplicial circle, S^1 (Definition 7.4.1). In Definition 7.1.12 the set of n-simplices $(S^1)_n$ is identified with $\underline{n}$. We will also need the following identification of the n-simplices $(S^p)_n$ for each p and n.

Definition 8.2.1 ($\mathcal{F}$-Sphere)**.** Taking the lexicographic ordering (Definition 8.1.5) defines a bijection

$$(S^p)_n = \underline{n}^{\wedge p} \cong \underline{n^p}$$

for $p > 0$. Let $\overline{S}^0 = \underline{1}$, the constant simplicial set, and for $p > 0$ let $\overline{S}^p$ be the pointed simplicial set whose n-simplicies are $\underline{n^p}$ and such that $S^p \cong \overline{S}^p$ is an isomorphism of simplicial sets. Let $\overline{S}$ be the symmetric sequence determined by these isomorphisms, so that $S \cong \overline{S}$ is an isomorphism of symmetric spectra. We refer to $\overline{S}$ as the *$\mathcal{F}$-sphere*. ◇

The structure morphisms for the symmetric spectra we construct will be defined using the following morphisms.

Definition 8.2.2. Suppose X is a Γ-simplicial set. For each $\underline{m}$ and $\underline{n}$ in $\mathcal{F}$, and for each $i \in \underline{m}^\flat$, let

$$h_i : \underline{n} \longrightarrow \bigvee_{\underline{m}^\flat} \underline{n} \cong \underline{mn}.$$

be the composite of the structure morphism indexed by the inclusion $\{i\} \subset \underline{m}^\flat$ and the isomorphism given by lexicographic ordering (Definition 8.1.5). Applying X, we define

$$\eta_{m,n,i} = Xh_i : X\underline{n} \longrightarrow X\underline{mn}.$$

Taking the wedge sum, we have

$$\eta_{m,n} : \underline{m} \wedge X\underline{n} \cong \bigvee_{i \in \underline{m}^\flat} X\underline{n} \longrightarrow X\underline{mn}. \tag{8.2.3}$$

These morphisms are natural with respect to morphisms of $\underline{m}$ and $\underline{n}$ in $\mathcal{F}$. ◇

Explanation 8.2.4. The map $h_i : \underline{n} \longrightarrow \underline{mn}$ in Definition 8.2.2 is given by

$$h_i(j) = \begin{cases} 0 & \text{if } j = 0 \text{ and} \\ n(i-1) + j & \text{if } j \in \underline{n}^\flat. \end{cases}$$

Using the matrix interpretation of $\underline{m} \wedge \underline{n}$ in Explanation 8.1.7, h_i corresponds to the inclusion of the row indexed by $i \in \underline{m}^\flat$. ◇

In the following definition we will construct pointed bisimplicial sets as functors

$$\Delta^{\mathsf{op}} \longrightarrow \mathsf{sSet}_*$$

and take the diagonal as discussed in Explanation 7.1.7 to obtain a pointed simplicial set.

Definition 8.2.5. Suppose X is a Γ-simplicial set. The *K-theory* of X is a symmetric spectrum $\mathsf{K}^{\mathcal{F}}X$ defined as follows.

For each natural number $k \geq 0$, let $(\mathsf{K}^{\mathcal{F}}X)k$ be the pointed simplicial set obtained by taking the diagonal of the pointed bisimplicial set $X \circ \overline{S}^k$:

$$\Delta^{\mathsf{op}} \xrightarrow{\overline{S}^k} \mathcal{F} \xrightarrow{X} \mathsf{sSet}_*.$$

The components $\eta_{m,n}$ defined in (8.2.3) assemble to give a natural transformation in the diagram below.

$$\begin{tikzcd}
 & & \mathcal{F} \times \mathcal{F} \arrow[r, "1 \times X"] \arrow[d, "\wedge"] \arrow[dr, phantom, "\swArrow\eta"] & \mathcal{F} \times \mathsf{sSet}_* \arrow[d, "\wedge"] \\
 & \Delta^{\mathsf{op}} \times \Delta^{\mathsf{op}} \arrow[ur, "\overline{S}^p \times \overline{S}^q"] & & \\
\Delta^{\mathsf{op}} \arrow[ur, "\mathrm{diag}"] \arrow[rr, "\overline{S}^{p+q}"'] & & \mathcal{F} \arrow[r, "X"'] & \mathsf{sSet}_*
\end{tikzcd}$$

Taking the diagonal of these pointed bisimplicial sets and using the isomorphsm $S \cong \overline{S}$, we obtain morphisms in sSet_*

$$\rho_{p,q} : S^p \wedge (\mathsf{K}^{\mathcal{F}}X)q \xrightarrow{\cong} \overline{S}^p \wedge (\mathsf{K}^{\mathcal{F}}X)q \longrightarrow (\mathsf{K}^{\mathcal{F}}X)(p+q).$$

In Proposition 8.2.6 we show that these data define a symmetric spectrum $\mathsf{K}^{\mathcal{F}}X$ for each Γ-simplicial set X and moreover that $\mathsf{K}^{\mathcal{F}}$ determines a functor

$$\mathsf{K}^{\mathcal{F}} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}.$$

In Explanation 8.2.7 we give an alternate description of each $(\mathsf{K}^{\mathcal{F}}X)k$ in terms of its n-simplices. $\diamond$

Proposition 8.2.6. *The data* $\{(\mathsf{K}^{\mathcal{F}}X)k,\ \rho_{p,q} \mid k,p,q \geq 0\}$ *of Definition 8.2.5 define a functor*

$$\mathsf{K}^{\mathcal{F}} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}.$$

Proof. We will use the description in Explanation 7.4.6 to verify that each pair $(\mathsf{K}^{\mathcal{F}}X, \rho)$ is a symmetric spectrum. The unity condition (7.4.7) follows because $\overline{S}^0$ is the constant simplicial set $\underline{1}$ and each $\eta_{1,n}$ is the identity. The associativity condition (7.4.8) holds because associativity of the lexicographic smash product L (Definition 8.1.5) implies that the following diagram for components of η commutes for each $\underline{\ell}$, $\underline{m}$, and $\underline{n}$ in $\mathcal{F}$.

$$\begin{tikzcd}
\underline{\ell} \wedge \underline{m} \wedge X\underline{n} \arrow[r, "1 \wedge \eta_{m,n}"] \arrow[d, "L \wedge 1"'] & \underline{\ell} \wedge X(\underline{mn}) \arrow[d, "\eta_{\ell,mn}"] \\
\underline{\ell m} \wedge X\underline{n} \arrow[r, "\eta_{\ell m,n}"'] & X(\underline{\ell mn})
\end{tikzcd}$$

The action of Σ_k on $(\mathsf{K}^{\mathcal{F}}X)k$ is given by permuting the factors of $\overline{S}^k$. The equivariance of $\rho_{p,q}$ with respect to the standard inclusion $i_{p,q} : \Sigma_p \times \Sigma_q \hookrightarrow \Sigma_{p+q}$ then follows from the definition of the Σ_p action on the smash powers $S^p \cong \overline{S}^p$ (see Definitions 7.3.8 and 7.4.1).

Since the definition of η is natural with respect to morphisms of Γ-simplicial sets (i.e., natural transformations of functors), so are the structure morphisms $\rho_{p,q}$. This shows that $\mathsf{K}^{\mathcal{F}}$ takes values in the category of symmetric spectra and morphisms thereof. Functoriality follows because each step in the construction of $(\mathsf{K}^{\mathcal{F}}X)k$ is functorial. □

Explanation 8.2.7 (Simplices of $\mathsf{K}^{\mathcal{F}}X$)**.** Here we give a more explicit definition of the simplices of $(\mathsf{K}^{\mathcal{F}}X)k$ from Definition 8.2.5. This is the description often given in the literature. For each $\underline{n}$, evaluating X at $\overline{S}^k_n \in \mathcal{F}$ gives a simplicial set

$$\big(\underline{m} \longmapsto (X(\overline{S}^k_n))_m\big).$$

The pointed simplicial set $(\mathsf{K}^{\mathcal{F}}X)k$ is given by its diagonal:

$$(\mathsf{K}^{\mathcal{F}}X)k = X(\overline{S}^k) = \big(\underline{n} \longmapsto (X(\overline{S}^k_n))_n\big).$$

The structure morphisms

$$\Sigma X(\overline{S}^{k-1}) \longrightarrow X(\overline{S}^k)$$

are given, for each $\underline{n}$, by

$$\overline{S}^1_n \wedge (X(\overline{S}^{k-1}_n))_n \cong \bigvee_{i \in (\overline{S}^1_n)^\flat} (X(\overline{S}^{k-1}_n))_n \longrightarrow (X(\overline{S}^k_n))_n$$

induced by the inclusions

$$h_i : \overline{S}^{k-1}_n \cong \{0, i\} \wedge \overline{S}^{k-1}_n \longrightarrow \overline{S}^1_n \wedge \overline{S}^{k-1}_n \cong \overline{S}^k_n$$

for $i \in (\overline{S}^1_n)^\flat$. ◇

8.3. Γ-Categories from Permutative Categories

In this section we define Segal's construction of Γ-categories from small permutative categories. There are three variants of this construction, and they all yield equivalent K-theory spectra by Theorems 8.3.21 and 8.5.2.

Definition 8.3.1. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category and $\underline{n}$ is an object of $\mathcal{F}$. An *$\underline{n}$-system* in C is a pair

$$(C, \rho) = \{C_s, \rho_{s,t}\}$$

consisting of

- a system of objects, with $C_s \in \mathsf{C}$ for each basepoint-free subset $s \subset \underline{n}^\flat$, and
- a system of morphisms, with

  $$\rho_{s,t} : C_s \oplus C_t \longrightarrow C_{s \cup t}$$

  for each pair of disjoint basepoint-free subsets:

  $$s, t \subset \underline{n}^\flat \quad \text{and} \quad s \cap t = \varnothing.$$

We call $\rho_{s,t}$ the (s,t)-*gluing morphism*. These data are subject to the following axioms.

Object Unity: For the empty subset we have

$$C_\varnothing = e. \tag{8.3.2}$$

Gluing Unity: The $(\varnothing, t)$- and $(s, \varnothing)$-gluing morphisms are identities:

$$\rho_{\varnothing,t} = 1_{C_t} \quad \text{and} \quad \rho_{s,\varnothing} = 1_{C_s}. \tag{8.3.3}$$

Gluing Symmetry: The following diagram commutes for each pair of disjoint basepoint-free subsets s and t, where ξ denotes the symmetry isomorphism of C.

$$\begin{tikzcd} C_s \oplus C_t \arrow[r, "\rho_{s,t}"] \arrow[d, "\xi"'] & C_{s\cup t} \arrow[d, "1"] \\ C_t \oplus C_s \arrow[r, "\rho_{t,s}"] & C_{t\cup s} \end{tikzcd} \tag{8.3.4}$$

Gluing Associativity: The following diagram commutes for each triple of pairwise disjoint basepoint-free subsets s, t, and u.

$$\begin{tikzcd} C_s \oplus C_t \oplus C_u \arrow[r, "\rho_{s,t} \oplus 1"] \arrow[d, "1 \oplus \rho_{t,u}"'] & C_{s\cup t} \oplus C_u \arrow[d, "\rho_{s\cup t,u}"] \\ C_s \oplus C_{t\cup u} \arrow[r, "\rho_{s,t\cup u}"] & C_{s\cup t\cup u} \end{tikzcd} \tag{8.3.5}$$

This finishes the definition of an $\underline{n}$-system in C. We also have the following variant definitions and terms.

- When helpful for clarity, we also call an $\underline{n}$-system a *lax $\underline{n}$-system*.
- A *strong $\underline{n}$-system* is an $\underline{n}$-system for which all of the gluing morphisms $\rho_{s,t}$ are isomorphisms.
- A *colax $\underline{n}$-system* in C is an $\underline{n}$-system in C^{op}. That is, the direction of each (s,t)-gluing morphism is reversed but the same axioms (with reversed arrows $\rho_{s,t}$) are satisfied. ◇

Definition 8.3.6. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category and suppose given $\underline{n}$-systems

$$\{C_s, \rho_{s,t}\} \quad \text{and} \quad \{C'_s, \rho'_{s,t}\}$$

for $\underline{n}$ in $\mathcal{F}$. A *morphism of $\underline{n}$-systems*, denoted

$$\{\alpha_s\} : \{C_s, \rho_{s,t}\} \longrightarrow \{C'_s, \rho'_{s,t}\},$$

consists of component morphisms

$$\alpha_s : C_s \longrightarrow C'_s,$$

for each basepoint-free subset $s \subset \underline{n}^\flat$. These components are subject to the following axioms.

Unitary: For the empty subset we have

$$\alpha_\varnothing = 1_e. \tag{8.3.7}$$

Gluing Compatibility: The following diagram commutes for each pair of disjoint basepoint-free subsets s and t.

$$\begin{tikzcd} C_s \oplus C_t \arrow[r, "\rho_{s,t}"] \arrow[d, "\alpha_s \oplus \alpha_t"'] & C_{s\cup t} \arrow[d, "\alpha_{s\cup t}"] \\ C'_s \oplus C'_t \arrow[r, "\rho'_{s,t}"] & C'_{s\cup t} \end{tikzcd} \tag{8.3.8}$$

This finishes the definition of a morphism of $\underline{n}$-systems. We also have the following variant definitions.

- A morphism of strong $\underline{n}$-systems is defined as above, with $\{C_s, \rho_{s,t}\}$ and $\{C'_s, \rho'_{s,t}\}$ strong $\underline{n}$-systems.
- A morphism of colax $\underline{n}$-systems is defined as above, with $\{C_s, \rho_{s,t}\}$ and $\{C'_s, \rho'_{s,t}\}$ colax $\underline{n}$-systems, but the gluing compatibility axiom (8.3.8) has the variance of ρ reversed as in the following diagram.

$$\begin{tikzcd} C_s \oplus C_t \arrow[d, "\alpha_s \oplus \alpha_t"'] & C_{s\cup t} \arrow[l, "\rho_{s,t}"'] \arrow[d, "\alpha_{s\cup t}"] \\ C'_s \oplus C'_t & C'_{s\cup t} \arrow[l, "\rho'_{s,t}"'] \end{tikzcd}$$

The identity morphism for a lax, strong, or colax $\underline{n}$-system $\{C_s, \rho_{s,t}\}$ consists of identities

$$\alpha_s = 1_{C_s} \quad \text{for} \quad s \subset \underline{n}^\flat.$$

Composition of morphisms is defined componentwise. This composition is associative and unital because the composition in C is so. ◇

Definition 8.3.9. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category and suppose given $\underline{n}$ in $\mathcal{F}$. Define a pointed category $\mathsf{C}^{\mathcal{F}}\underline{n}$ as follows.

- The objects of $\mathsf{C}^{\mathcal{F}}\underline{n}$ are the strong $\underline{n}$-systems in C.
- The morphisms of $\mathsf{C}^{\mathcal{F}}\underline{n}$ are the morphisms of strong $\underline{n}$-systems.
- The basepoint of $\mathsf{C}^{\mathcal{F}}\underline{n}$ is the constant $\underline{n}$-system with

$$C_s = e \quad \text{and} \quad \rho_{s,t} = 1_e$$

  for all disjoint basepoint-free subsets:

$$s, t \subset \underline{n}^\flat \quad \text{and} \quad s \cap t = \varnothing.$$

This finishes the definition of $\mathsf{C}^{\mathcal{F}}\underline{n}$. We also have the following variant definitions and notation.

- To emphasize that the objects of $\mathsf{C}^{\mathcal{F}}\underline{n}$ are the strong $\underline{n}$-systems, we also use the notation

$$\mathsf{C}^{\mathcal{F}}_{\cong}\underline{n} = \mathsf{C}^{\mathcal{F}}\underline{n}.$$

- We let

$$\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\underline{n}$$

  denote the category of lax $\underline{n}$-systems and morphisms of such. The basepoint is that of $\mathsf{C}^{\mathcal{F}}_{\cong}\underline{n}$.

- We let
$$\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}\underline{n}$$
denote the category of colax $\underline{n}$-systems and morphisms of such. The base-point is that of $\mathsf{C}^{\mathcal{F}}_{\cong}\underline{n}$. ◇

In the following examples, we describe explicitly the categories $\mathsf{C}^{\mathcal{F}}\underline{n}$ for $n = 0, 1, 2$.

Example 8.3.10. There are canonical isomorphisms
$$\mathsf{C}^{\mathcal{F}}\underline{0} \cong \mathbf{1}$$
$$\mathsf{C}^{\mathcal{F}}\underline{1} \cong \mathsf{C}$$
with $\mathbf{1}$ the terminal category. ◇

Example 8.3.11. Since
$$2^{\underline{2}^{\flat}} = \big\{\varnothing, \{1\}, \{2\}, \{1,2\}\big\},$$
up to a canonical isomorphism, $\mathsf{C}^{\mathcal{F}}\underline{2}$ may be described as follows. An object in $\mathsf{C}^{\mathcal{F}}\underline{2}$ is a quadruple
$$\Big(X_1, X_2, X_{12}, \rho^X : X_1 \oplus X_2 \xrightarrow{\cong} X_{12}\Big)$$
consisting of

- three objects $X_1, X_2, X_{12} \in \mathsf{C}$ and
- an isomorphism $\rho^X \in \mathsf{C}$ as indicated.

A morphism
$$\big(X_1, X_2, X_{12}, \rho^X\big) \xrightarrow{\alpha = (\alpha_1, \alpha_2, \alpha_{12})} \big(Y_1, Y_2, Y_{12}, \rho^Y\big) \in \mathsf{C}^{\mathcal{F}}\underline{2}$$
consists of morphisms
$$X_a \xrightarrow{\alpha_a} Y_a \in \mathsf{C} \quad \text{for} \quad a \in \{1, 2, 12\}$$
such that the diagram
$$\begin{tikzcd} X_1 \oplus X_2 \arrow[r, "\rho^X"] \arrow[d, "\alpha_1 \oplus \alpha_2"'] & X_{12} \arrow[d, "\alpha_{12}"] \\ Y_1 \oplus Y_2 \arrow[r, "\rho^Y"] & Y_{12} \end{tikzcd}$$
commutes.

- The identity morphism is given by
$$1_{(X_1, X_2, X_{12}, \rho^X)} = (1_{X_1}, 1_{X_2}, 1_{X_{12}}).$$
- Composition is defined entrywise in C.

This finishes the description of $\mathsf{C}^{\mathcal{F}}\underline{2}$. ◇

Now we define $\mathsf{C}^{\mathcal{F}}$ on the morphisms of $\mathcal{F}$.

Definition 8.3.12. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category and suppose given a morphism $\underline{n}$ in $\mathcal{F}$
$$\psi : \underline{n} \longrightarrow \underline{m}.$$
Define a pointed functor
$$\mathsf{C}^{\mathcal{F}}\psi : \mathsf{C}^{\mathcal{F}}\underline{n} \longrightarrow \mathsf{C}^{\mathcal{F}}\underline{m}$$
as follows.

- Given an $\underline{n}$-system $\{C_s, \rho_{s,t}\}$, define an $\underline{m}$-system $\{C_r^\psi, \rho_{r,q}^\psi\}$ with
$$C_r^\psi = C_{\psi^{-1}(r)} \quad \text{and}$$
$$\rho_{r,q}^\psi = \rho_{\psi^{-1}(r),\psi^{-1}(q)}$$
for disjoint basepoint-free subsets:
$$r, q \subset \underline{m}^\flat \quad \text{and} \quad r \cap q = \varnothing.$$
- Given a morphism of $\underline{n}$-systems $\{\alpha_s\}$, define a morphism of $\underline{m}$-systems $\{\alpha_r^\psi\}$ with
$$\alpha_r^\psi = \alpha_{\psi^{-1}(r)} \quad \text{for} \quad r \subset \underline{m}^\flat.$$

These definitions give well-defined $\underline{m}$-systems and morphisms thereof because taking inverse images of a basepoint-preserving function preserves the properties of being basepoint-free and disjoint. If C_s, respectively $\rho_{s,t}$, is constant at e, respectively 1_e, then so is C_r^ψ, respectively $\rho_{r,q}^\psi$.

Define the pointed functor $\mathsf{C}^{\mathcal{F}}\psi$ by the following assignment on objects and morphisms:
$$\{C_s, \rho_{s,t}\} \longmapsto \{C_r^\psi, \rho_{r,q}^\psi\} \quad \text{and}$$
$$\{\alpha_s\} \longmapsto \{\alpha_r^\psi\}.$$
Functoriality of $\mathsf{C}^{\mathcal{F}}\psi$ follows because composition and identities in $\mathsf{C}^{\mathcal{F}}\underline{n}$ are defined componentwise.

This finishes the definition of $\mathsf{C}^{\mathcal{F}}\psi$. We also have the following variant definitions and notation extending that of Definition 8.3.9.

- For emphasis or clarity we sometimes also use the notation
$$\mathsf{C}^{\mathcal{F}}_{\cong}\psi = \mathsf{C}^{\mathcal{F}}\psi.$$
- We let
$$\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\psi : \mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\underline{n} \longrightarrow \mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\underline{m}$$
and
$$\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}\psi : \mathsf{C}^{\mathcal{F}}_{\mathrm{co}}\underline{n} \longrightarrow \mathsf{C}^{\mathcal{F}}_{\mathrm{co}}\underline{m}$$
be the pointed functors given by the same definitions as above for lax, respectively colax, $\underline{n}$-systems and morphisms. ⋄

Taken together, Definitions 8.3.1, 8.3.6, 8.3.9, and 8.3.12 give assignments on objects and morphisms
$$\mathsf{C}^{\mathcal{F}} = \mathsf{C}^{\mathcal{F}}_{\cong} : \mathcal{F} \longrightarrow \mathsf{Cat}_*,$$
$$\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}} : \mathcal{F} \longrightarrow \mathsf{Cat}_*, \quad \text{and}$$
$$\mathsf{C}^{\mathcal{F}}_{\mathrm{co}} : \mathcal{F} \longrightarrow \mathsf{Cat}_*.$$

Now we show that these are Γ-categories.

Proposition 8.3.13. *Suppose C is a small permutative category. Each of the assignments*
$$\mathsf{C}^{\mathcal{F}} = \mathsf{C}^{\mathcal{F}}_{\cong}, \quad \mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}, \quad \textit{and} \quad \mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$$
defines a Γ-category.

Proof. First note that if $\psi = 1_{\underline{n}}$ then the definition of $\mathsf{C}^{\mathcal{F}}\psi$ will be the identity functor. Functoriality with respect to a composable pair ψ and ϕ follows because $(\psi\phi)^{-1} = \phi^{-1}\psi^{-1}$. The same argument applies to show that $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$ and $\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$ are functorial. □

Definition 8.3.14. Suppose C is a small permutative category. We call

- $\mathsf{C}^{\mathcal{F}} = \mathsf{C}^{\mathcal{F}}_{\cong}$ the *strong Segal Γ-category* of C,
- $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$ the *lax Segal Γ-category* of C, and
- $\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$ the *colax Segal Γ-category* of C. ⋄

Now we relate the three Segal Γ-category constructions.

Definition 8.3.15. Suppose $(\mathsf{C}, \oplus, e, \xi)$ is a small permutative category and

$$\nu \in \{\cong, \mathrm{lax}, \mathrm{co}\}$$

is one of the variant subscripts for $\mathsf{C}^{\mathcal{F}}$. For each $\underline{n}$ in $\mathcal{F}$, let $(e, \ldots, e)$ be the basepoint of C^n and define pointed functors

$$P_\nu : \mathsf{C}^{\mathcal{F}}_\nu \underline{n} \longrightarrow \mathsf{C}^n \quad \text{and} \quad Q_\nu : \mathsf{C}^n \longrightarrow \mathsf{C}^{\mathcal{F}}_\nu \underline{n}$$

as follows. For objects $\{C_s, \rho_{s,t}\}$ and morphisms $\{\alpha_s\}$ in $\mathsf{C}^{\mathcal{F}}_\nu \underline{n}$, define

$$P_\nu\{C_s, \rho_{s,t}\} = (C_{\{1\}}, \ldots, C_{\{n\}}) \quad \text{and}$$
$$P_\nu\{\alpha_s\} = (\alpha_{\{1\}}, \ldots, \alpha_{\{n\}}).$$

For objects $(C_1, \ldots, C_n)$ and morphisms $(\alpha_1, \ldots, \alpha_n)$ in C^n, define

$$Q_\nu(C_1, \ldots, C_n) = \begin{cases} \{\oplus_{i\in s} C_i \;,\; \xi_{s,t}\}, & \text{if} \quad \nu \in \{\cong, \mathrm{lax}\} \\ \{\oplus_{i\in s} C_i \;,\; \xi^{-1}_{s,t}\}, & \text{if} \quad \nu = \mathrm{co}, \end{cases}$$

and

$$Q_\nu(\alpha_1, \ldots, \alpha_n) = \{\oplus_{i\in s} \alpha_i\}$$

where

$$\xi_{s,t} : \Big(\bigoplus_{i\in s} C_i\Big) \oplus \Big(\bigoplus_{i\in t} C_i\Big) \longrightarrow \bigoplus_{i\in(s\cup t)} C_i$$

is the unique morphism given by permuting summands with the symmetry of C. Functoriality of P_ν and Q_ν follows because composition and identities in each of the categories $\mathsf{C}^{\mathcal{F}}_\nu \underline{n}$ and C^n are given componentwise. Each of P_ν and Q_ν is a pointed functor by definition of the basepoints in $\mathsf{C}^{\mathcal{F}}_\nu \underline{n}$ and the assumption that e is a strict unit for C. ⋄

Proposition 8.3.16. *In the context of Definition 8.3.15,*

- $(P_{\cong}, Q_{\cong})$ *is an adjoint equivalence,*
- P_{lax} *is a right adjoint, with left adjoint* Q_{lax}*, and*
- P_{co} *is a left adjoint, with right adjoint* Q_{co}*.*

Proof. For each variant

$$\nu \in \{\cong, \mathrm{lax}, \mathrm{co}\},$$

the composite

$$\mathsf{C}^n \xrightarrow{Q_\nu} \mathsf{C}^{\mathcal{F}}_\nu \underline{n} \xrightarrow{P_\nu} \mathsf{C}^n$$

is the identity functor. The other composite

$$\mathsf{C}^{\mathcal{F}}_\nu \underline{n} \xrightarrow{P_\nu} \mathsf{C}^n \xrightarrow{Q_\nu} \mathsf{C}^{\mathcal{F}}_\nu \underline{n}$$

is given on objects and morphisms of $\mathsf{C}^{\mathcal{F}}_{\nu}\underline{n}$ by the following assignments:

$$\{C_s, \rho_{s,t}\} \longmapsto \begin{cases} \{\oplus_{i\in s} C_{\{i\}}\ ,\ \xi_{s,t}\} & \text{for} \quad \nu\ = \cong,\ \nu = \text{lax} \\ \{\oplus_{i\in s} C_{\{i\}}\ ,\ \xi_{s,t}^{-1}\} & \text{for} \quad \nu = \text{co} \end{cases}$$

and

$$\{\alpha_s\} \longmapsto \{\oplus_{i\in s} \alpha_{\{i\}}\}.$$

If the variant ν is either $\cong$ or lax, then the gluing morphisms of each $\underline{n}$-system provide morphisms

$$\widehat{\rho}_s : \bigoplus_{i\in s} C_{\{i\}} \longrightarrow C_s \tag{8.3.17}$$

for each basepoint-free subset $s \subset \underline{n}^{\flat}$. These are uniquely determined by the gluing associativity condition (8.3.5). Using gluing associativity again together with gluing symmetry (8.3.4), the morphisms $\widehat{\rho}_s$ satisfy the gluing compatibility condition (8.3.8) with the morphisms $\xi_{s,t}$ and, therefore, define a morphism

$$\{\widehat{\rho}_s\} : \{\oplus_{i\in s} C_{\{i\}}\ ,\ \xi_{s,t}\} \longrightarrow \{C_s, \rho_{s,t}\}. \tag{8.3.18}$$

If the variant ν is co, the colax gluing morphisms of a colax $\underline{n}$-system provide morphisms $\widehat{\rho}_s^{\,\text{co}}$ that go in the opposite direction of (8.3.17). Then we have a morphism of colax $\underline{n}$-systems

$$\{\widehat{\rho}_s^{\,\text{co}}\} : \{C_s, \rho_{s,t}\} \longrightarrow \{\oplus_{i\in s} C_{\{i\}}, \xi_{s,t}^{-1}\}.$$

In any of the three cases for ν, the morphisms (8.3.18) are natural with respect to morphisms $\{\alpha_s\}$ by the gluing compatibility (8.3.8) for $\{\alpha_s\}$ with the morphisms $\rho_{s,t}$. Thus we have a natural transformation γ with components

$$\gamma_{\{C_s,\rho_{s,t}\}} = \{\widehat{\rho}_s\}$$

that provides either a counit (if ν is either $\cong$ or lax) or a unit (if ν is co). Since the gluing morphisms for the basepoint of $\mathsf{C}^{\mathcal{F}}_{\nu}\underline{n}$ are identities, the component of γ at the basepoint is its identity.

In any of the three cases for ν, one verifies the triangle identities with the following computations.

- For $\{C_s, \rho_{s,t}\}$ in $\mathsf{C}^{\mathcal{F}}_{\nu}\underline{n}$,
$$P_\nu \gamma_{\{C_s,\rho_{s,t}\}} = P_\nu\{\widehat{\rho}_s\} = (\widehat{\rho}_{\{1\}}, \ldots, \widehat{\rho}_{\{n\}}) = 1_{(C_{\{1\}},\ldots,C_{\{n\}})}.$$
- For $(C_1, \ldots, C_n)$ in C^n,
$$\begin{aligned} \gamma_{Q_\nu(C_1,\ldots,C_n)} &= \begin{cases} \{\widehat{\xi}_s\} & \text{if} \quad \nu\ = \cong,\ \nu = \text{lax}, \\ \{\widehat{\xi^{-1}}{}_s^{\text{co}}\} & \text{if} \quad \nu = \text{co}, \end{cases} \\ &= \left\{1_{\left(\oplus_{i\in s} C_{\{i\}}\right)}\right\}. \end{aligned}$$

The assertion that $(P_{\cong}, Q_{\cong})$ is an adjoint equivalence follows because the components of γ are given by the gluing morphisms $\rho_{s,t}$. $\square$

Observe that each strong $\underline{n}$ system is lax and, by taking inverses of the gluing morphisms, also colax. Thus we have forgetful functors, given by the identities on morphisms

$$\mathsf{C}^{\mathcal{F}}_{\text{co}}\underline{n} \longleftarrow \mathsf{C}^{\mathcal{F}}_{\cong}\underline{n} \longrightarrow \mathsf{C}^{\mathcal{F}}_{\text{lax}}\underline{n} \tag{8.3.19}$$

for each $\underline{n}$ in $\mathcal{F}$. These are natural with respect to morphisms in $\mathcal{F}$ and therefore we have the following.

Definition 8.3.20. Suppose C is a small permutative category. The *levelwise inclusions*

$$\mathsf{C}^{\mathcal{F}}_{\mathrm{co}} \longleftarrow \mathsf{C}^{\mathcal{F}}_{\cong} \longrightarrow \mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$$

are the morphisms of Γ-categories induced by the forgetful functors (8.3.19). ⋄

Recall from Definition 8.1.18 that

$$N_* : \Gamma\text{-}\mathsf{Cat} \longrightarrow \Gamma\text{-}\mathsf{sSet}$$

denotes the functor induced by composition with the nerve functor N.

Theorem 8.3.21. *Suppose C is a small permutative category. Each of the Γ-simplicial sets*

$$N_*\mathsf{C}^{\mathcal{F}}_{\cong}, \quad N_*\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}, \quad \textit{and} \quad N_*\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$$

is special. Moreover, the levelwise inclusions

$$\mathsf{C}^{\mathcal{F}}_{\mathrm{co}} \longleftarrow \mathsf{C}^{\mathcal{F}}_{\cong} \longrightarrow \mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$$

induce levelwise weak equivalences of Γ-simplicial sets.

Proof. For each variant

$$\nu \in \{\cong, \mathrm{lax}, \mathrm{co}\}$$

we have an isomorphism of categories

$$\mathsf{C}^{\mathcal{F}}_{\nu}\underline{1} \cong \mathsf{C}.$$

Since the nerve functor is a right adjoint, we have an isomorphism of simplicial sets

$$N(\mathsf{C}^n) \cong (N\mathsf{C})^n$$

for each $n \geq 0$. Recalling Proposition 7.2.5, an adjunction of categories induces a simplicial homotopy equivalence on nerves. Therefore the adjunctions of Proposition 8.3.16 provide simplicial homotopy equivalences in the composite below

$$N\mathsf{C}^{\mathcal{F}}_{\nu}\underline{n} \xrightarrow{\simeq} N(\mathsf{C}^n) \cong (N\mathsf{C})^n \cong (N\mathsf{C}^{\mathcal{F}}_{\nu}\underline{1})^n$$

for each $\underline{n}$ in $\mathcal{F}$. We recall from Explanation 7.8.4 that each simplicial homotopy equivalence is a weak equivalence. This completes the proof that each $N\mathsf{C}^{\mathcal{F}}_{\nu}$ is special.

For the second assertion we note, moreover, that the levelwise inclusions commute with the functors

$$Q_\nu : \mathsf{C}^n \longrightarrow \mathsf{C}^{\mathcal{F}}_{\nu}\underline{n}$$

for each variant ν and each $\underline{n}$ in $\mathcal{F}$. Therefore the levelwise inclusions are weak equivalences by the 2-out-of-3 property of Definition 7.7.3. □

8.4. Partition Multicategories

For each small permutative category C and each object $\underline{n}$ of $\mathcal{F}$, the objects of the category $\mathsf{C}^{\mathcal{F}}\underline{n}$ are systems indexed by disjoint pairs of basepoint-free subsets of $\underline{n}$; the morphisms are similarly indexed. The partition multicategories we introduce in this section provide an abstract approach to such indexed systems. In Proposition 8.4.8 we show that the Γ-categories constructed via partition multicategories are isomorphic to those constructed via $(-)^{\mathcal{F}}_{\mathsf{lax}}$.

Our approach via partition multicategories will make the categorical properties of the Segal J-theory and K-theory constructions in Section 8.5 more transparent. Moreover, our development of Elmendorf-Mandell K-theory in Chapter 10 depends on a generalization of the material here.

Definition 8.4.1. Suppose a is a pointed finite set. We define the *partition multicategory*, $\mathcal{M}a$, as follows. The object set of $\mathcal{M}a$ is $2^{a^\flat}$, the set of basepoint-free subsets of a. For an n-tuple of subsets

$$\langle s\rangle = (s_1, \ldots, s_n) \in \mathsf{Prof}(2^{a^\flat})$$

and a subset $t \in 2^{a^\flat}$, we define the set of operations, $(\mathcal{M}a)\big(\langle s\rangle\,;t\big)$, to be a 1-element set if $\langle s\rangle$ is a partition of t, and empty otherwise. If $\langle s\rangle$ is a partition of t, we let $\iota_{\langle s\rangle}$ denote the single operation in $(\mathcal{M}a)\big(\langle s\rangle\,;t\big)$.

Thus

$$(\mathcal{M}a)\big(\langle s\rangle\,;t\big) = \begin{cases} \{\iota_{\langle s\rangle}\} & \text{if } s_i \cap s_j = \varnothing \text{ for all } i \neq j \text{ and } t = \coprod_i s_i, \\ \varnothing & \text{otherwise.} \end{cases}$$

The empty set $\varnothing \in 2^{a^\flat}$ provides a basepoint for $\mathcal{M}a$ with the unique operations in

$$(\mathcal{M}a)\big(\underbrace{(\varnothing, \ldots, \varnothing)}_{n \text{ terms}}\,;\varnothing\big) \quad \text{for} \quad n \geq 0.$$

This defines the objects, operations, and basepoint of $\mathcal{M}a$. In Proposition 8.4.2 we show that these data satisfy the axioms of a pointed multicategory. ◇

Proposition 8.4.2. *In the context of Definition 8.4.1, $\mathcal{M}a$ is a pointed multicategory.*

Proof. For unit operations we have

$$1_t = \iota_t \in (\mathcal{M}a)\big(t\,;t\big).$$

For symmetric group actions, suppose $\langle s\rangle \in \mathsf{Prof}(2^{a^\flat})$ has length n. If $(\mathcal{M}a)\big(\langle s\rangle\,;t\big)$ is nonempty (and therefore a singleton) then so is

$$(\mathcal{M}a)\big(\langle s\rangle\sigma\,;t\big) \quad \text{for} \quad \sigma \in \Sigma_n.$$

Therefore, the symmetric group actions

$$(\mathcal{M}a)\big(\langle s\rangle\,;t\big) \xrightarrow[\cong]{\sigma} (\mathcal{M}a)\big(\langle s\rangle\sigma\,;t\big) \quad \text{for} \quad \sigma \in \Sigma_n$$

are uniquely determined. For composition, suppose $\langle s\rangle \in \mathsf{Prof}(2^{a^\flat})$ has length n and suppose given

$$\langle r_i\rangle \in \mathsf{Prof}(2^{a^\flat}) \quad \text{for} \quad i \in \{1, \ldots, n\}.$$

Let $\langle r\rangle$ denote the concatenation of the $\langle r_i\rangle$. If

$$(\mathcal{M}a)\big(\langle s\rangle\,;\,t\big)\times\prod_{i=1}^{n}(\mathcal{M}a)\big(\langle r_i\rangle\,;\,s_i\big)$$

is nonempty, then so is

$$(\mathcal{M}a)\big(\langle r\rangle\,;\,t\big).$$

Therefore, the composition

$$(\mathcal{M}a)\big(\langle s\rangle\,;\,t\big)\times\prod_{i=1}^{n}(\mathcal{M}a)\big(\langle r_i\rangle\,;\,s_i\big)\longrightarrow(\mathcal{M}a)\big(\langle r\rangle\,;\,t\big)$$

is uniquely determined. Each of the axioms in Definition 5.1.2 consists of diagrams where either the common domain of the two composites is empty and there is nothing to verify, or the common domain is a singleton and therefore so is each of the other operation sets in the diagram. □

Explanation 8.4.3. For a pointed finite set a, let C denote the discrete permutative category whose objects are those of the commutative monoid

$$(2^{a^\flat},\cup,\varnothing).$$

Then the objects of $\mathcal{M}a$ are those of $\mathsf{End}(\mathsf{C})$ but the operations are only those whose input profiles consist of pairwise disjoint subsets. ⋄

In the following examples, we describe explicitly the partition multicategories $\mathcal{M}\underline{n}$ for $n=0,1,2$.

Example 8.4.4. The partition multicategory $\mathcal{M}\underline{0}$ has object set

$$2^{\underline{0}^\flat}=\{\varnothing\}.$$

Each set of operations has a single element,

$$\mathcal{M}\underline{0}\big(\langle\varnothing\rangle\,;\,\varnothing\big)=\{\iota^n\},$$

where $\langle\varnothing\rangle=(\varnothing,\ldots,\varnothing)$ contains n copies of the empty set. Its multicategory structure is given as follows.

- The $\varnothing$-colored unit is $\iota^1\in\mathcal{M}\underline{0}\big(\varnothing\,;\,\varnothing\big)$.
- Each ι^n is fixed by the right Σ_n-action.
- The composition is given by

  $$\iota^n\circ\big(\iota^{k_1},\ldots,\iota^{k_n}\big)=\iota^{k_1+\cdots+k_n}$$

  for $n\geq 1$ and $k_1,\ldots,k_n\geq 0$.

Therefore, there is a canonical isomorphism

$$\mathcal{M}\underline{0}\cong\mathsf{T}$$

with the terminal multicategory T (Definition 5.2.1). ⋄

Example 8.4.5. The partition multicategory $\mathcal{M}\underline{1}$ has object set

$$2^{\underline{1}^\flat}=\big\{\varnothing,\{1\}\big\}.$$

Its nonempty sets of operations are

$$\mathcal{M}\underline{1}\big(\langle\varnothing\rangle\,;\,\varnothing\big)=\{\iota^n\}$$
$$\mathcal{M}\underline{1}\big((\varnothing,\ldots,\{1\},\ldots,\varnothing)\,;\,\{1\}\big)=\{\pi^n_j\}$$

for $n \geq 0$ and $1 \leq j \leq n$. In the definition of π_j^n above,

$$(\varnothing, \ldots, \{1\}, \ldots, \varnothing)$$

has length n with $\{1\}$ in the jth entry and $\varnothing$ in other entries. The multicategory structure involving only ι^n for $n \geq 0$ is as in $\mathcal{M}\underline{0}$ in Example 8.4.4.

The rest of the multicategory structure in $\mathcal{M}\underline{1}$ is given as follows.

- The $\{1\}$-colored unit is
$$\pi_1^1 \in \mathcal{M}\underline{1}\big(\{1\}\,;\{1\}\big).$$
- The right Σ_n-action is given by
$$\pi_j^n \cdot \sigma = \pi_{\sigma^{-1}(j)}^n \quad \text{for} \quad \sigma \in \Sigma_n.$$

The composition involving π_j^n is given by

$$\pi_j^n \circ \Big(\iota^{k_1}, \ldots, \iota^{k_{j-1}}, \pi_p^{k_j}, \iota^{k_{j+1}}, \ldots, \iota^{k_n}\Big) = \pi_{k_1+\cdots+k_{j-1}+p}^{k_1+\cdots+k_n}$$

for $1 \leq p \leq k_j$. This finishes the description of $\mathcal{M}\underline{1}$. ◇

Example 8.4.6. The partition multicategory $\mathcal{M}\underline{2}$ has object set

$$2^{\underline{2}^\flat} = \big\{\varnothing, \{1\}, \{2\}, \{1,2\}\big\}.$$

Its nonempty sets of operations are

$$\begin{aligned}
\mathcal{M}\underline{2}\big(\langle\varnothing\rangle\,;\varnothing\big) &= \{\iota^n\}\\
\mathcal{M}\underline{2}\big((\varnothing, \ldots, \{1\}, \ldots, \varnothing)\,;\{1\}\big) &= \{\pi_j^n\}\\
\mathcal{M}\underline{2}\big((\varnothing, \ldots, \{2\}, \ldots, \varnothing)\,;\{2\}\big) &= \{\tau_j^n\}\\
\mathcal{M}\underline{2}\big((\varnothing, \ldots, \{1\}, \ldots, \{2\}, \ldots, \varnothing)\,;\{1,2\}\big) &= \{\theta_{i,j}^n\}
\end{aligned}$$

for $n \geq 0$, $1 \leq j \leq n$ in the second and third lines, and $1 \leq i \neq j \leq n$ in the last line.

- ι^n and π_j^n are as in $\mathcal{M}\underline{1}$ in Example 8.4.5.
- In the definition of τ_j^n,
$$(\varnothing, \ldots, \{2\}, \ldots, \varnothing)$$
has length n with $\{2\}$ in the jth entry and $\varnothing$ in other entries.
- In the definition of $\theta_{i,j}^n$, $\{1\}$ and $\{2\}$ are in, respectively, the ith and the jth entries in
$$(\varnothing, \ldots, \{1\}, \ldots, \{2\}, \ldots, \varnothing),$$
with both $i < j$ and $i > j$ possible. All other entries are $\varnothing$.

The multicategory structure in $\mathcal{M}\underline{2}$ involving only ι^n and π_j^n is as in $\mathcal{M}\underline{1}$ in Example 8.4.5. The multicategory structure involving only ι^n and τ_j^n is also as in $\mathcal{M}\underline{1}$, with τ_j^n playing the role of π_j^n.

The rest of the multicategory structure in $\mathcal{M}\underline{2}$ is given as follows.

- The $\{1,2\}$-colored unit is
$$\theta_{1,2}^2 \in \mathcal{M}\underline{2}\big((\{1\}, \{2\})\,;\{1,2\}\big).$$
- The right Σ_n-action is given by
$$\theta_{i,j}^n \cdot \sigma = \theta_{\sigma^{-1}(i),\sigma^{-1}(j)}^n \quad \text{for} \quad \sigma \in \Sigma_n.$$

The composition involving $\theta^n_{i,j}$ is given by

$$\theta^n_{i,j} \circ \big(\iota^{k_1}, \ldots, \pi^{k_i}_p, \ldots, \tau^{k_j}_q, \ldots, \iota^{k_n}\big) = \theta^{k_1+\cdots+k_n}_{k_1+\cdots+k_{i-1}+p,\, k_1+\cdots+k_{j-1}+q}$$

for $1 \le i \ne j \le n$, $1 \le p \le k_i$, and $1 \le q \le k_j$. For $1 \le r \le n$, the rth entry in

$$\big(\iota^{k_1}, \ldots, \pi^{k_i}_p, \ldots, \tau^{k_j}_q, \ldots, \iota^{k_n}\big)$$

is

$$\begin{cases} \iota^{k_r} & \text{if } r \ne i, j, \\ \pi^{k_i}_p & \text{if } r = i, \text{ and} \\ \tau^{k_j}_q & \text{if } r = j. \end{cases}$$

This finishes the description of $\mathcal{M}\underline{2}$. ◇

Proposition 8.4.7. *The assignment*

$$\underline{n} \longmapsto \mathcal{M}\underline{n}$$

is the assignment on objects of a pointed functor

$$\mathcal{M} : (\mathcal{F}^{\mathsf{op}}, \underline{0}) \longrightarrow (\mathsf{Multicat}_*, \mathsf{T})$$

taking values in the category of small pointed multicategories.

Proof. Suppose $\psi : \underline{n} \longrightarrow \underline{m}$ is a morphism of pointed finite sets. Since ψ preserves the basepoint, taking inverse images provides a function from subsets of $\underline{m}^\flat$ to subsets of $\underline{n}^\flat$:

$$2^{\underline{m}^\flat} \longrightarrow 2^{\underline{n}^\flat}, \quad u \longmapsto \psi^{-1}(u).$$

We denote this function $\tilde{\psi}$. Since taking inverse images preserves disjunction, $\tilde{\psi}$ defines a pointed multifunctor

$$\tilde{\psi} : \mathcal{M}\underline{m} \longrightarrow \mathcal{M}\underline{n}$$

that is an identity if ψ is an identity. For composable functions ϕ and ψ in $\mathcal{F}$, we have $(\phi\psi)^{-1} = \psi^{-1}\phi^{-1}$ and thus $\mathcal{M}$ is (contravariantly) functorial. □

We use the partition multicategories to give a reformulation of the lax Segal Γ-categories associated to a small permutative category. This reformulation will be used in our comparison with the more general Elmendorf-Mandell K-theory in Chapter 10. More immediately, we will use this reformulation to verify functoriality of the lax Segal Γ-categories as the input permutative category C varies.

Proposition 8.4.8. *Suppose C is a small permutative category. For each $\underline{n}$ in $\mathcal{F}$ there is an isomorphism of pointed categories*

$$\mathsf{Multicat}_*(\mathcal{M}\underline{n}, \mathsf{End}(\mathsf{C})) \cong \mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\underline{n}.$$

These isomorphisms are natural with respect to morphisms in $\mathcal{F}$.

Proof. The fundamental reason for this result is that the operations $\iota_{\langle c \rangle}$ of length $n > 2$ can be decomposed as composites of binary operations $\iota_{(s,t)}$. This implies that pointed multifunctors from $\mathcal{M}\underline{n}$ to $\mathsf{End}(\mathsf{C})$ are determined by their values on binary operations, in the following two specific ways.

Suppose given a pointed multifunctor

$$F : \mathcal{M}\underline{n} \longrightarrow \mathsf{End}(\mathsf{C}).$$

(1) For a tuple $\langle c\rangle$ of length $n > 2$, the composition axiom (5.1.15) implies that $F\iota_{\langle c\rangle}$ is determined by its value on the operations $\iota_{(s,t)}$ for disjoint pairs $s,t \in 2^{\underline{n}^b}$. In particular, for pairwise disjoint subsets $s,t,u \in 2^{\underline{n}^b}$ the ternary operation $\iota_{(s,t,u)}$ is equal to both of the following composites:

$$\iota_{(s\cup t,u)} \circ \big(\iota_{(s,t)}, \iota_u\big) = \iota_{(s,t,u)} = \iota_{(s,t\cup u)} \circ \big(\iota_s, \iota_{(t,u)}\big).$$

Therefore the images of these composites under F are equal.

(2) Preservation of the symmetric group action (5.1.13) holds if and only if each $F\iota_{(s,t)}$ commutes with the transposition of s and t as in the following diagram.

$$\begin{tikzcd} Ft \oplus Fs \arrow[rr, "\xi"] \arrow[dr, "F\iota_{(t,s)}"'] & & Fs \oplus Ft \arrow[dl, "F\iota_{(s,t)}"] \\ & F(s\cup t) & \end{tikzcd}$$

Moreover, the composite

$$(\mathcal{M}\underline{n})\big((\varnothing,t)\,;\,t\big) \times \big((\mathcal{M}\underline{n})\big(\langle\rangle\,;\,\varnothing\big) \times (\mathcal{M}\underline{n})\big(t\,;\,t\big)\big) \longrightarrow (\mathcal{M}\underline{n})\big(t\,;\,t\big)$$

takes the unique triple of operations in the source to the unique operation in the target, that is the identity on t. Therefore, after applying F, we have the following commuting diagram in C.

$$\begin{tikzcd} & e \oplus Ft \arrow[r, "F\iota_{\langle\rangle} \oplus 1_{Ft}"] & e \oplus Ft \arrow[dr, "F\iota_{(\varnothing,t)}"] & \\ Ft \arrow[ur, "1"] \arrow[rrr, "1_{Ft}"] & & & Ft \end{tikzcd}$$

If F is a pointed multifunctor then $F\iota_{\langle\rangle} = 1_e$ and therefore $F\iota_{(\varnothing,t)} = 1_{Ft}$.

These observations show that the data of an object in $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\underline{n}$ is equivalent to the data of a pointed multifunctor $\mathcal{M}\underline{n} \longrightarrow \mathsf{End}(\mathsf{C})$. The object unity axiom (8.3.2) and gluing unity axiom (8.3.3) correspond to the requirement that F be pointed. The gluing symmetry axiom (8.3.4) corresponds to the requirement that F preserves symmetry as in (2) above. The gluing associativity axiom (8.3.5) corresponds to the composition axiom for F as in (1) above. Therefore, the correspondence

$$F \leftrightarrow \{(Fs, F\iota_{(s,t)})\}$$

establishes a bijection on objects between $\mathsf{Multicat}_*(\mathcal{M}\underline{n}, \mathsf{End}(\mathsf{C}))$ and $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\underline{n}$.

We have two similar observations for pointed multinatural transformations $\theta : F \longrightarrow F'$ between pointed multifunctors

$$F, F' : \mathcal{M}\underline{n} \longrightarrow \mathsf{End}(\mathsf{C}).$$

(1) The naturality condition of Definition 5.1.17 holds if and only if θ is natural with respect to the pairwise operations $\iota_{(s,t)}$, as in the following diagram.

$$\begin{tikzcd} Fs \oplus Ft \arrow[r, "F\iota_{(s,t)}"] \arrow[d, "\theta_s \oplus \theta_t"'] & F(s \cup t) \arrow[d, "\theta_{s \cup t}"] \\ F's \oplus F't \arrow[r, "F'\iota_{(s,t)}"] & F'(s \cup t) \end{tikzcd}$$

(2) Because F, F', and θ are pointed, the basepoint component $\theta_\varnothing$ is the identity 1_e.

Thus the correspondence

$$\theta \leftrightarrow \{\theta_s\}$$

establishes a bijection between the morphisms of $\mathsf{Multicat}_*(\mathcal{M}\underline{n}, \mathsf{End}(\mathsf{C}))$ and those of $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\underline{n}$. The pointed condition for θ corresponds to the unity condition (8.3.7) and the naturality condition for θ corresponds to the gluing compatibility axiom (8.3.8).

To verify naturality of these isomorphisms with respect to a morphism of pointed finite sets

$$\psi : \underline{n} \longrightarrow \underline{m} \quad \text{in} \quad \mathcal{F},$$

recall from Proposition 8.4.7 that $\mathcal{M}\psi = \tilde{\psi}$ is given by the function

$$2^{\underline{m}^\flat} \longrightarrow 2^{\underline{n}^\flat}$$

that takes inverse images of subsets. Then the functor

$$\mathsf{Multicat}_*(\mathcal{M}\underline{n}, \mathsf{End}(\mathsf{C})) \longrightarrow \mathsf{Multicat}_*(\mathcal{M}\underline{m}, \mathsf{End}(\mathsf{C}))$$

induced by $\mathcal{M}\psi$ is given on multifunctors F and multinatural transformations θ by

$$(\tilde{\psi}^* F)s = F(\psi^{-1}(s)) \quad \text{and} \quad (\tilde{\psi}^*\theta)_s = \theta_{\psi^{-1}(s)}.$$

Restricted to binary operations, this is precisely Definition 8.3.12 for $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\psi$. □

Explanation 8.4.9. Variants of Proposition 8.4.8 can be given for the strong and colax Segal Γ-category constructions. However we focus on the lax case because it is the one we will use to compare with the Elmendorf-Mandell construction in Chapter 10. ◇

Definition 8.4.10. Let $\mathsf{J}^{\mathcal{M}}$ denote the 2-functor

$$\mathsf{J}^{\mathcal{M}} = \mathsf{Multicat}_*(\mathcal{M}, -) : \mathsf{Multicat}_* \longrightarrow \Gamma\text{-}\mathsf{Cat}.$$

For each small pointed multicategory P, the Γ-category $\mathsf{J}^{\mathcal{M}}\mathsf{P}$ is

$$\mathsf{J}^{\mathcal{M}}\mathsf{P} = \mathsf{Multicat}_*(\mathcal{M}(-), \mathsf{P}) : \mathcal{F} \longrightarrow \mathsf{Cat}.$$

We call this the *partition J-theory* of multicategories, or the *$\mathcal{M}$-partition J-theory* to emphasize the dependence on $\mathcal{M}$. ◇

Explanation 8.4.11. In the context of Definition 8.4.10, suppose given small pointed multicategories, pointed multifunctors and a pointed multinatural transformation:

$$H, H' : \mathsf{P} \longrightarrow \mathsf{P}' \quad \text{and} \quad \theta : H \longrightarrow H' \quad \text{in} \quad \mathsf{Multicat}_*.$$

Then
$$(\mathsf{J}^{\mathcal{M}} H)_{\underline{n}}, (\mathsf{J}^{\mathcal{M}} H')_{\underline{n}} : (\mathsf{J}^{\mathcal{M}}\mathsf{P})\underline{n} \longrightarrow (\mathsf{J}^{\mathcal{M}}\mathsf{P}')\underline{n}$$
are given by composition with H and H', denoted H_* and H'_*, respectively. The component of $\mathsf{J}^{\mathcal{M}}\theta$ at $\underline{n}$ is given by whiskering with θ:
$$\begin{tikzcd}
\mathsf{Multicat}_*(\mathcal{M}\underline{n}, \mathsf{P}) \arrow[r, bend left, "H_*"] \arrow[r, bend right, "H'_*"'] \arrow[r, phantom, "\Downarrow \theta * (1_-)"] & \mathsf{Multicat}_*(\mathcal{M}\underline{n}, \mathsf{P}').
\end{tikzcd}$$
◇

8.5. Segal J-Theory and K-Theory

Recall from Example 5.1.11 the endomorphism multicategory $\mathsf{End}(\mathsf{C})$ for a permutative category C. Recall from Corollary 5.3.9 that this construction provides a 2-functor
$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}_*.$$
From Definition 8.4.10 we have the 2-functor
$$\mathsf{J}^{\mathcal{M}} = \mathsf{Multicat}_*(\mathcal{M}, -) : \mathsf{Multicat}_* \longrightarrow \Gamma\text{-}\mathsf{Cat}.$$

Definition 8.5.1 (Segal J-theory and K-theory)**.** Suppose C is a small permutative category. The *Segal J-theory* of C is the Γ-category
$$\mathsf{J}^{\mathsf{Se}}\mathsf{C} = \mathsf{Multicat}_*(\mathcal{M}(-), \mathsf{End}(\mathsf{C})) : \mathcal{F} \longrightarrow \mathsf{Cat}_*.$$
As a 2-functor, J^{Se} is the composite
$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{End}} \mathsf{Multicat}_* \xrightarrow{\mathsf{J}^{\mathcal{M}}} \Gamma\text{-}\mathsf{Cat}.$$

Taking levelwise nerves, $N_*\mathsf{J}^{\mathsf{Se}}\mathsf{C}$ is a Γ-simplicial set. We define the *Segal K-theory* of C to be the symmetric spectrum
$$\mathsf{K}^{\mathsf{Se}}\mathsf{C} = \mathsf{K}^{\mathcal{F}}(N_*\mathsf{J}^{\mathsf{Se}}\mathsf{C})$$
given by applying the K-theory construction of Definition 8.2.5 to $N_*\mathsf{J}^{\mathsf{Se}}\mathsf{C}$. As a functor, K^{Se} is the composite
$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{End}} \mathsf{Multicat}_* \xrightarrow{\mathsf{J}^{\mathcal{M}}} \Gamma\text{-}\mathsf{Cat} \xrightarrow{N_*} \Gamma\text{-}\mathsf{sSet} \xrightarrow{\mathsf{K}^{\mathcal{F}}} \mathsf{SymSp}.$$
◇

Recall from Section 8.3 the three Γ-simplicial sets $\mathsf{C}^{\mathcal{F}}_{\nu}$ associated to a small permutative category C, where ν is one of the variants
$$\nu \in \{\cong, \mathrm{lax}, \mathrm{co}\}.$$
The three Γ-simplicial sets are levelwise weakly equivalent by Theorem 8.3.21. Therefore, by Proposition 8.4.8 we have the following.

Theorem 8.5.2. *For each small permutative category C and each variant*
$$\nu \in \{\cong, \mathrm{lax}, \mathrm{co}\},$$
there is a level equivalence of symmetric spectra
$$\mathsf{K}^{\mathcal{F}}(N_*\mathsf{C}^{\mathcal{F}}_{\nu}) \simeq \mathsf{K}^{\mathsf{Se}}\mathsf{C}.$$

Recall from Lemma 5.7.21 that for any small pointed multicategory M, the category

$$\mathsf{Multicat}_*(\mathsf{M}, \mathsf{End}(\mathsf{C}))$$

has a permutative structure given by the pointwise monoidal product. We will compare this with the following.

Definition 8.5.3. Suppose a is a pointed finite set, regarded as a discrete category, and $(\mathsf{C}, \oplus, e)$ is a small permutative category. The *pointwise monoidal product* on

$$\mathsf{Cat}_*(a, \mathsf{C}),$$

the category of pointed functors and natural transformations, is given as follows. For pointed functors

$$F, F' : a \longrightarrow \mathsf{C},$$

define a pointed functor $F \oplus F'$ by its assignment on elements:

$$(F \oplus F')x = (Fx) \oplus (F'x) \quad \text{for} \quad x \in a.$$

For pointed natural transformations

$$\theta : F \longrightarrow G \quad \text{and} \quad \theta' : F' \longrightarrow G',$$

define a pointed natural transformation $\theta \oplus \theta'$ with components

$$(\theta \oplus \theta')_x = \theta_x \oplus \theta'_x \quad \text{for} \quad x \in a.$$

The monoidal unit is the constant functor at $e \in \mathsf{C}$. The associativity and unit isomorphisms are identities because C is permutative. The symmetry is given elementwise by that of C and the hexagon axiom (1.1.26) follows from that of C. $\diamond$

The next result is a reformulation of the adjunction Proposition 8.3.16 for $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$. We repeat the proof with additional details because this result will be used in Proposition 10.6.7 for our comparison with the J-theory of Elmendorf-Mandell.

Proposition 8.5.4. *Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category, equipped with basepoint e. Suppose a is a pointed finite set. There are strictly unital strong symmetric monoidal functors*

$$\mathsf{Cat}_*(a, \mathsf{C}) \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{Multicat}_*(\mathcal{M}a, \mathsf{End}(\mathsf{C}))$$

and a strong symmetric monoidal adjunction with the following defined below:

- *left adjoint $L = (-)^{\oplus}$,*
- *right adjoint $R = \overline{(-)}$,*
- *unit $\eta : \mathrm{Id} \longrightarrow RL$ the identity, and*
- *counit $\varepsilon : LR \longrightarrow \mathrm{Id}$.*

If, moreover, C is a groupoid, then ε is a natural isomorphism and (L, R) is a strong symmetric monoidal equivalence.

Proof. Suppose F is a pointed multifunctor

$$F : \mathcal{M}a \longrightarrow \mathsf{End}(\mathsf{C}).$$

Let $*$ denote the basepoint of a. Restricting to singleton subsets of $a^\flat$ yields an assignment

$$RF = \overline{F} : a \longrightarrow \mathrm{Ob}\,\mathsf{C}$$
$$(x \in a) \longmapsto \begin{cases} e & \text{if} \quad x = * \\ F\{x\} & \text{otherwise.} \end{cases}$$

If F is constant at $e \in \mathsf{C}$, then so is $\overline{F}$.

A pointed multinatural transformation $\theta : F \longrightarrow F'$ has, in particular, unary operations

$$\theta_x : F\{x\} \longrightarrow F'\{x\}$$

for each $x \in a^\flat$. These give a natural transformation

$$R\theta = \overline{\theta} : \overline{F} \longrightarrow \overline{F}'$$

with components

$$\overline{\theta}_x = \begin{cases} 1_e & \text{if} \quad x = * \\ \theta_x & \text{otherwise.} \end{cases}$$

If $\eta : G \longrightarrow G'$ is another pointed multinatural transformation between pointed multifunctors

$$G, G' : \mathcal{M}a \longrightarrow \mathsf{End}(\mathsf{C}),$$

we have

$$\overline{F \oplus G} = \overline{F} \oplus \overline{G} \quad \text{and} \quad \overline{\theta \oplus \eta} = \overline{\theta} \oplus \overline{\eta}.$$

This defines a strict symmetric monoidal functor

$$R = \overline{(-)} : \mathsf{Multicat}_*(\mathcal{M}a, \mathsf{End}(\mathsf{C})) \longrightarrow \mathsf{Cat}_*(a, \mathsf{C}).$$

For a reverse construction, we first fix a total order on $a^\flat$ for the remainder of this proof. Suppose P is an assignment

$$P : a \longrightarrow \mathsf{C}$$

with $P* = e$. We define a pointed multifunctor

$$LP = P^\oplus : \mathcal{M}a \longrightarrow \mathsf{End}(\mathsf{C})$$

by sending a subset $s \in 2^{a^\flat}$ to the sum

$$P^\oplus s = \bigoplus_{x \in s} Px$$

in C, where the order of summation is given by the chosen total ordering on a. If $\langle c \rangle$ is a partition of s, define $P^\oplus$ on the single element $\iota_{\langle c \rangle}$ of $\mathcal{M}a\big(\langle c \rangle ; s\big)$ to be the unique coherence isomorphism

$$\bigoplus_{c_i} \bigoplus_{x \in c_i} Px \longrightarrow \bigoplus_{x \in s} Px$$

in C that permutes the summands Px, given by the Symmetric Coherence Theorem 1.1.41. If any c_i is empty, we implicitly compose with the (strict) unit isomorphisms of C, as noted in Examples 5.1.10 and 5.1.11. The multifunctor axioms for $P^\oplus$ follow from Theorem 1.1.41 applied to C. Note that $P^\oplus \varnothing = e$ because the empty sum in C is the unit object. Similarly, $P^\oplus \iota_{\langle\rangle}$ is the identity on the empty sum and therefore $P^\oplus$ is a pointed multifunctor.

A pointed natural transformation

$$\omega : P \longrightarrow P', \quad \text{for} \quad P, P' \in \mathsf{Cat}_*(a, \mathsf{C}),$$

has components

$$\omega_x : Px \longrightarrow P'x, \quad \text{for} \quad x \in a,$$

and these assemble to give

$$(L\omega)_s = \omega_s^{\oplus} = \bigoplus_{x \in s} \omega_x, \quad \text{for} \quad s \subset a^{\flat}.$$

The naturality axiom for $\omega_s^{\oplus}$ follows from the naturality of the associativity and unit isomorphisms in C.

Now we turn to the data and axioms making $L = (-)^{\oplus}$ a symmetric monoidal functor. Given another assignment $Q : a \longrightarrow \mathsf{C}$, we define

$$P^{\oplus} \oplus Q^{\oplus} \xrightarrow{\cong} (P \oplus Q)^{\oplus}$$

to have component at $s \subset a^{\flat}$ given by the shuffle permutation

$$\Big(\bigoplus_{x \in s} Px\Big) \oplus \Big(\bigoplus_{x \in s} Qx\Big) \xrightarrow{\cong} \bigoplus_{x \in s}(Px \oplus Qx). \tag{8.5.5}$$

To see that these components give a well-defined isomorphism in

$$\mathsf{Multicat}_*(\mathcal{M}a, \mathsf{End}(\mathsf{C})),$$

note that the values of $P^{\oplus}$ and $Q^{\oplus}$ on the operations $\iota_{\langle c \rangle}$ are also given by permutations of summands and, therefore, the components (8.5.5) satisfy the naturality condition of Definition 5.1.17 by the Symmetric Coherence Theorem 1.1.41. Naturality of the symmetry in C implies that the structure isomorphism for $(-)^{\oplus}$ is natural with respect to morphisms in $\mathsf{Cat}_*(\mathcal{M}a, \mathsf{C})$.

In both $\mathsf{Multicat}(\mathcal{M}a, \mathsf{End}(\mathsf{C}))$ and $\mathsf{Cat}_*(a, \mathsf{C})$, the unit for the pointwise monoidal product is the constant functor at the unit $e \in \mathsf{C}$. Thus $(-)^{\oplus}$ strictly preserves the unit. The symmetric monoidal functor axioms for $(-)^{\oplus}$ also follow from Theorem 1.1.41. This completes the definition of strictly unital symmetric monoidal functors

$$\begin{aligned} R = \overline{(-)} &: \mathsf{Multicat}_*(\mathcal{M}a, \mathsf{End}(\mathsf{C})) \longrightarrow \mathsf{Cat}_*(a, \mathsf{C}) \quad \text{and} \\ L = (-)^{\oplus} &: \mathsf{Cat}_*(a, \mathsf{C}) \longrightarrow \mathsf{Multicat}_*(\mathcal{M}a, \mathsf{End}(\mathsf{C})). \end{aligned}$$

Now we discuss the unit η and counit ε of this adjunction. First, we have $\overline{(-)^{\oplus}} = 1$ because restricting to singletons gives equalities

$$\overline{P^{\oplus}} = P \quad \text{and} \quad \overline{\omega^{\oplus}} = \omega$$

for each $P \in \mathsf{Cat}_*(a, \mathsf{C})$ and each $\omega \in \mathsf{Cat}_*(a, \mathsf{C})(P, P')$. Therefore we take η to be the identity. For the other composite, we define a multinatural transformation with components

$$\varepsilon_F : \overline{F}^{\oplus} \longrightarrow F$$

as follows. For each subset $s \in 2^{a^{\flat}}$, let $\langle s^{\bullet} \rangle$ denote the tuple formed by all singleton subsets of s, ordered by the chosen ordering on a, and let $F\langle s^{\bullet} \rangle$ denote the corresponding tuple given by applying F to each of the singletons $\{x\} \in \langle s^{\bullet} \rangle$. If s is empty then the tuple of singletons is the empty tuple. The unary operations in $\mathsf{End}(\mathsf{C})$ are morphisms in C, and we let $\varepsilon_{F;s}$ be the morphism

$$\varepsilon_{F;s} : \overline{F}^{\oplus} s = \bigoplus_{x \in s} F\{x\} \longrightarrow Fs$$

given by applying F to $\iota_{\langle s^\bullet \rangle}$, the single element of $\mathcal{M}a\big(\langle s^\bullet \rangle\,; s\big)$. If $s = \varnothing$, then

$$\varepsilon_{F;s} = F\iota_{\langle\rangle} = 1_e$$

because F is pointed.

The naturality condition of Definition 5.1.17 for these components is the following equality:

$$(F\iota_{\langle c \rangle}) \circ \big(\bigoplus_{c_i \in \langle c \rangle} \varepsilon_{F;c_i} \big) = \varepsilon_{F;s} \circ \big(\overline{F}^{\oplus} \iota_{\langle c \rangle}\big). \tag{8.5.6}$$

Since

$$\varepsilon_{F;s} = F\iota_{\langle s^\bullet \rangle} \quad \text{and} \quad \varepsilon_{F;c_i} = F\iota_{\langle c_i^\bullet \rangle} \quad \text{for} \quad s \in 2^{a^\flat}, c_i \in \langle c \rangle,$$

the naturality condition is equivalent to commutativity in C of the outer rectangle (8.5.7) below. In that diagram, each $\langle c_i^\bullet \rangle$ denotes the tuple of singletons in c_i, with the order inherited from a; the tuple $\langle c^\bullet \rangle$ is the concatenation of the $\langle c_i^\bullet \rangle$, in the order determined by $\langle c \rangle$; and s is the union of the subsets c_i, with the order determined as a subset of $a^\flat$.

$$\begin{tikzcd}
\bigoplus\limits_{c_i \in \langle c \rangle} \bigoplus\limits_{x \in c_i} F\{x\} \arrow[rr, "\bigoplus\limits_{c_i \in \langle c \rangle} F\iota_{\langle c_i^\bullet \rangle}"] \arrow[d, "\overline{F}^{\oplus} \iota_{\langle c \rangle}"'] \arrow[drr, "F\iota_{\langle c^\bullet \rangle}"'] && \bigoplus\limits_{c_i \in \langle c \rangle} Fc_i \arrow[d, "F\iota_{\langle c \rangle}"] \\
\bigoplus\limits_{x \in s} F\{x\} \arrow[rr, "F\iota_{\langle s^\bullet \rangle}"'] && Fs
\end{tikzcd} \tag{8.5.7}$$

The upper-right triangle commutes because the composition in $\mathcal{M}a$ gives

$$\iota_{\langle c \rangle} \circ \big(\iota_{\langle c^\bullet{}_1 \rangle}, \ldots, \iota_{\langle c^\bullet{}_n \rangle}\big) = \iota_{\langle c^\bullet \rangle}$$

and F preserves composition. The morphism $\overline{F}^{\oplus} \iota_{\langle c \rangle}$ is, by definition, given by permuting terms. Therefore the equivariance condition (5.1.13) for F implies that the lower-left triangle commutes. This concludes the definition of the component

$$\varepsilon_F : \overline{F}^{\oplus} \longrightarrow F$$

and the proof that it is a multinatural transformation.

For a multinatural transformation $\theta : F \longrightarrow F'$, the naturality of θ at the operation $\iota_{\langle s^\bullet \rangle}$ ensures that the following diagram commutes for each $s \subset a^\flat$.

$$\begin{tikzcd}
\bigoplus\limits_{x \in s} F\{x\} \arrow[r, "\overline{\theta}_s^{\oplus}"] \arrow[d, "\varepsilon_{F;s}"'] & \bigoplus\limits_{x \in s} F'\{x\} \arrow[d, "\varepsilon_{F';s}"] \\
Fs \arrow[r, "\theta_s"'] & F's
\end{tikzcd}$$

Therefore, the components of ε at F and F' commute with θ and $\overline{\theta}^{\oplus}$ because the relevant composite multinatural transformations have the same components at s for each $s \in \mathcal{M}a$. Hence

$$\varepsilon : LR = \overline{(\ )}^{\oplus} \longrightarrow \mathrm{Id}$$

is a natural transformation of functors.

Since η is the identity, the triangle identities for this adjunction reduce to the equalities

$$\varepsilon_{P^{\oplus}} = 1_{P^{\oplus}} \quad \text{and} \quad \overline{\varepsilon_F} = 1_{\overline{F}}$$

for P and F as above. Both of these equalities follow directly from the definitions.

Verifying that ε is monoidal natural requires commutativity of the following rectangle for pointed multifunctors F and G and subsets $s \subset a^{\flat}$.

$$\begin{tikzcd}
\Big(\bigoplus_{x\in s} F\{x\}\Big) \oplus \Big(\bigoplus_{x\in s} G\{x\}\Big) \arrow[r, "F\iota_{\langle s^\bullet\rangle} \oplus G\iota_{\langle s^\bullet\rangle}"] \arrow[d, "\cong"'] & Fs \oplus Gs \arrow[d, "1"] \\
\Big(\bigoplus_{x\in s} (F\oplus G)\{x\}\Big) \arrow[r, "(F\oplus G)\iota_{\langle s^\bullet\rangle}"] & (F\oplus G)s
\end{tikzcd}$$

Commutativity of this diagram follows by definition of $(F \oplus G)$ on operations, as given in the proof of Lemma 5.7.12. Therefore, the functors $R = \overline{(\ )}$ and $L = (\ )^{\oplus}$ determine a symmetric monoidal adjunction. Since the components of ε are given by morphisms in C, this adjunction is an equivalence if C is a groupoid. $\square$

8.6. Notes

8.6.1 (Segal K-Theory)**.** The K-theory functor for small permutative categories first developed by Segal **[Seg74]** uses the Γ-simplicial set $\mathsf{C}^{\mathcal{F}}$ of Section 8.3 instead of the approach via the partition multicategories $\mathcal{M}\underline{n}$ given in Section 8.4. The K-theory construction via $\mathsf{C}^{\mathcal{F}}$ is the standard one. See, for example, **[May78, EM06, Man10]**.

Our definitions of J^{Se}, and K^{Se} in Section 10.3, in terms of the partition multicategories $\mathcal{M}a$ are equivalent to those of **[EM09]** using the *terminal parameter multicategory for modules*, denoted E there. Our justifications for attaching Segal's name to these constructions are the isomorphisms of Proposition 8.4.8 and the level equivalences of Theorem 8.5.2. In the literature, including **[GJO17b, Man10]**, the name *Segal K-theory* is sometimes used for J^{Se}. $\diamond$

8.6.2 (Variants of Segal K-theory)**.** Our discussion of the lax and colax variants for $\mathsf{C}^{\mathcal{F}}$, together with the proof that they result in levelwise equivalent Γ-simplicial sets (Theorem 8.3.21) follows the sketch given in **[Man10**, Section 3]. See **[GJO17a]** for a general approach to equivalences of homotopy theories between categories of strict, strong, and lax algebra morphisms for a monad. Examples given there include categories of small symmetric monoidal categories, Γ-categories, and n-fold monoidal categories. $\diamond$

8.6.3 (2-Functoriality of $\mathsf{C}^{\mathcal{F}}_{v}$)**.** One can show the 2-functoriality of $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$ directly, without explicit use of the isomorphism in Proposition 8.4.8. There are similar 2-functoriality results for $\mathsf{C}^{\mathcal{F}}_{\cong}$ and $\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$. However, the natural domain of definition for $\mathsf{C}^{\mathcal{F}}_{\cong}$ is

$$\mathsf{PermCat}^{\mathsf{sus}},$$

whose morphisms are given by strictly unital *strong* monoidal functors (Definition 1.1.27). Likewise, the natural domain of definition for $\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$ consists of small permutative categories and *colax symmetric monoidal* functors, also known as *oplax symmetric monoidal* functors or *lax symmetric comonoidal* functors. These are defined with the same axioms as symmetric monoidal functors but their monoidal and unit constraints, (1.1.7) and (1.1.8) respectively, have the opposite variance. Strictly unital colax symmetric monoidal functors are the colax 1-linear functors in Definition 10.7.24. These are used in $\mathsf{PermCat}^{\mathsf{su}}_{\mathsf{co}}$ and Proposition 10.7.27. ⋄

8.6.4 (Γ-Simplicial Sets and Connective Spectra)**.** In Note 7.9.5 we recall the earlier and simpler notion of sequential spectra. A sequential spectrum is called *connective* if its homotopy groups π_n are trivial for $n < 0$. We let $\mathsf{Sp}^{\mathbb{N}}_{\geq 0}$ denote the category of connective sequential spectra. A sequential spectrum X is a *weak Ω-spectrum* if the adjoint structure morphisms

$$X_n \longrightarrow \Omega X_{n+1}$$

are weak equivalences of pointed simplicial sets, where Ω denotes the pointed simplicial loop functor $\mathsf{Hom}_{\mathsf{sSet}_*}(S^1, -)$.

When introducing Γ-simplicial sets in **[Seg74]**, Segal shows that the spectrum associated to a special Γ-simplicial set is a weak Ω-spectrum. Therefore by Theorem 8.3.21 each spectrum $\mathsf{K}^{\mathsf{Se}}\mathsf{C}$, with C a small permutative category, is a weak Ω-spectrum.

Segal **[Seg74]** also develops homotopy theory for Γ-simplicial sets and its relation to that of connective spectra. Bousfield-Friedlander **[BF78]** extend this to model structures for both Γ-sSet and $\mathsf{Sp}^{\mathbb{N}}_{\geq 0}$, with the weak equivalences of Γ-sSet being created by Segal's K-theory functor $\mathsf{K}^{\mathcal{F}}$ and with the fibrant objects being the weak Ω-spectra. They go on to show that there is a Quillen equivalence of model categories

$$\Gamma\text{-}\mathsf{sSet} \simeq_Q \mathsf{Sp}^{\mathbb{N}}_{\geq 0}. \tag{8.6.5}$$

⋄

8.6.6 (Permutative Categories and Connective Spectra)**.** Continuing from the previous discussion in Note 8.6.4, one can define weak equivalences of small permutative categories to be the strictly unital symmetric monoidal functors that induce weak equivalences of Γ-simplicial sets (that is, those functors that induce π_*-isomorphisms of associated sequential spectra). It follows from work of Thomason **[Tho95]**, for the more general category of small symmetric monoidal categories and all symmetric monoidal functors, that when one inverts the respective weak equivalences there is an equivalence of homotopy categories

$$\mathit{Ho}(\mathsf{PermCat}^{\mathsf{su}}) \simeq \mathit{Ho}(\Gamma\text{-}\mathsf{sSet}). \tag{8.6.7}$$

Combining (8.6.7) with (8.6.5) means that, up to weak equivalence, all connective spectra can be obtained from the Segal K-theory of small permutative categories. In **[Man10]**, Mandell gives a second proof of (8.6.7) by defining a functor

$$P = \mathcal{P} \circ \mathcal{S} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{PermCat},$$

via a certain Grothendieck construction, and showing it is inverse to $N_* \mathsf{J}^{\mathsf{Se}}$. This work is extended to 2-dimensional category theory, for symmetric monoidal 2-categories and Γ-2-categories, in **[GJO17b]**. ⋄

CHAPTER 9

Categories of $\mathcal{G}_*$-Objects

This is the first of two chapters generalizing the Segal K-theory functor

$$\mathsf{K}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp},$$

from Chapter 8, to a simplicially-enriched multifunctor

$$\mathsf{K}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp},$$

due to Elmendorf-Mandell [**EM06, EM09**]. The material in this chapter concerns the replacement of $\mathcal{F}$ and its associated K-theory

$$\Gamma\text{-}\mathsf{Cat} \xrightarrow{N_*} \Gamma\text{-}\mathsf{sSet} \xrightarrow{\mathsf{K}^{\mathcal{F}}} \mathsf{SymSp}$$

with another diagram category $\mathcal{G}$ and an associated K-theory functor

$$\mathcal{G}_*\text{-}\mathsf{Cat} \xrightarrow{N_*} \mathcal{G}_*\text{-}\mathsf{sSet} \xrightarrow{\mathsf{K}^{\mathcal{G}}} \mathsf{SymSp}.$$

The corresponding J-theory, J^{EM}, is given in Chapter 10.

Beyond simply replacing $\mathcal{F}$ and $\mathsf{K}^{\mathcal{F}}$, this chapter also describes the relevant enriched monoidal structures. These are essential for our applications to K-theory of E_n-monoidal categories.

(1) Theorem 9.2.15 shows that $\mathcal{G}_*$-Cat and $\mathcal{G}_*$-sSet are symmetric monoidal closed categories and are enriched, as symmetric monoidal categories, over Cat_* and sSet_*, respectively.

(2) Theorem 9.2.19 shows that the nerve functor gives $\mathcal{G}_*$-Cat an enrichment over sSet_* and that

$$N_* : \mathcal{G}_*\text{-}\mathsf{Cat} \longrightarrow \mathcal{G}_*\text{-}\mathsf{sSet}$$

is symmetric monoidal as a sSet_*-enriched functor.

(3) Theorem 9.4.9 shows that

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$$

is symmetric monoidal as a sSet_*-enriched functor.

Organization. In Section 9.1 we define the category $\mathcal{G}$, which replaces $\mathcal{F}$. The objects of $\mathcal{G}$ consist of certain tuples of objects from $\mathcal{F}$, and $\mathcal{G}$ has a permutative structure given by concatenation of tuples. Taking the smash product of pointed finite sets, with lexicographic ordering, provides a functor

$$\wedge : \mathcal{G} \longrightarrow \mathcal{F} \tag{9.0.1}$$

described in Definition 9.1.15.

In Section 9.2 we show that $\mathcal{G}_*$-objects in C form a symmetric monoidal closed category, where C is a complete and cocomplete symmetric monoidal closed category (Theorem 9.2.15). Our two examples of interest are $\mathsf{C} = \mathsf{sSet}_*$ and $\mathsf{C} = \mathsf{Cat}_*$.

- Our method of proof uses the general theory for (pointed) diagram categories developed in Section 4.3.
- We also apply the general theory of enriched monoidal categories and functors from Chapters 2 and 3 to show that N_* is enriched symmetric monoidal (Theorem 9.2.19).
- Of potentially independent interest, our method of proof also shows that Γ-C is a symmetric monoidal closed category (Explanation 9.2.18), with product induced by the smash product of pointed finite sets in $\mathcal{F}$. See Note 9.5.5 for further generalization.

In Section 9.3 we define the associated K-theory functor

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp},$$

which replaces $\mathsf{K}^{\mathcal{F}}$. We also show, as Proposition 9.3.16, that $\mathsf{K}^{\mathcal{G}}$ agrees with $\mathsf{K}^{\mathcal{F}}$ along the functor

$$\Gamma\text{-}\mathsf{sSet} \longrightarrow \mathcal{G}_*\text{-}\mathsf{sSet}$$

induced by the smash product of pointed finite sets (9.0.1).

In Section 9.4 we show that $\mathsf{K}^{\mathcal{G}}$ is a sSet_*-enriched symmetric monoidal functor (Theorem 9.4.9). We show, as a result of the compatibility with $\mathsf{K}^{\mathcal{F}}$, that $\mathsf{K}^{\mathcal{F}}$ is also symmetric monoidal as a sSet_*-enriched functor (Theorem 9.4.18).

9.1. The Category $\mathcal{G}$

Recall from Definition 8.1.1 that $\mathcal{F}$ denotes the category whose objects are pointed finite sets $\underline{n} = \{0, \ldots, n\}$ for $n \geq 0$ and whose morphisms are basepoint-preserving functions. In this section we define another category, $\mathcal{G}$, whose objects are tuples of objects from $\mathcal{F}$, subject to certain identifications.

Motivation 9.1.1 (Smash Products of Pointed Finite Sets)**.** Recall, as described in Explanation 8.1.6, that $(\mathcal{F}, \wedge, \underline{1})$ is a permutative category. Here, $\wedge$ denotes the smash product of pointed finite sets implicitly composed with the lexicographic order isomorphism

$$\underline{m} \wedge \underline{n} \cong \underline{mn}.$$

The germ of the idea for a K-theory functor that preserves multiplicative structure is that it ought to be encoded via the smash product in $\mathcal{F}$. Working this out with $\mathcal{F}$ alone, one encounters a crucial obstruction that we clarify in Note 10.8.6.

The diagram category $\mathcal{G}$, to be described in Definition 9.1.7 below, resolves the difficulty. Although presented differently, the germ of the idea remains the same. With that in mind, one might conceive of the objects and morphisms of $\mathcal{G}$ as something like "pre-smash-products" of pointed finite sets and their morphisms. We take tuples of objects of $\mathcal{F}$, but with certain technical identifications corresponding to the following properties of the smash product.

- The 1-point set $\underline{0}$ is a null object for $(\mathcal{F}, \wedge, \underline{1})$, in the sense of Definition 4.3.3: $\underline{0}$ is both initial and terminal, and smash product of any object with $\underline{0}$ results in $\underline{0}$. Thus, for a q-tuple of objects $(\underline{n}_1, \ldots, \underline{n}_q)$, we will have

  $$\bigwedge_j \underline{n}_j = \underline{0}$$

  if any $n_j = 0$.

- The unit for $\wedge$ is $\underline{1}$. Thus, for a q-tuple and an r-tuple,
$$(\underline{n}_1, \ldots, \underline{n}_q) \quad \text{and} \quad (\underline{m}_1, \ldots, \underline{m}_r),$$
we will have
$$\bigwedge_j \underline{n}_j = \bigwedge_k \underline{m}_k$$
whenever the tuples differ by insertion or deletion of $\underline{1}$ in various positions. There will of course be other circumstances where the two are equal, depending on the prime factors of the numbers involved, but here we focus on equalities that are a result of extraneous multiplication by 1.
- The product $\wedge$ is symmetric monoidal, but not strictly so. Thus we will have isomorphisms, but generally not equalities, given by permuting the terms of a smash product.
- For each $q > 0$ there is at least one q-tuple whose smash product is $\underline{0}$. But there is no such tuple for $q = 0$; the smash product over the empty tuple is $\underline{1}$. We will need an extra object $*$ to account for this, and it will be identified with all q-tuples for $q > 0$ whose smash products are $\underline{0}$. $\diamond$

With the observations of Motivation 9.1.1 in mind, we give the definition of $\mathcal{G}$ in terms of the following preliminary definitions. These will account for insertions terms $\underline{1}$, permutation of terms, and identifications of tuples whose smash products are $\underline{0}$ in $\mathcal{F}$.

Definition 9.1.2. Let Inj denote the category whose objects are *unpointed finite sets*
$$\overline{p} = \underline{p}^\flat = \begin{cases} \{1, \ldots, p\} & \text{if} \quad p > 0, \\ \varnothing & \text{if} \quad p = 0, \end{cases}$$
for each natural number $p \geq 0$, and whose morphisms are injections
$$f : \overline{q} \hookrightarrow \overline{p}. \qquad \diamond$$

Definition 9.1.3. For each unpointed finite set $\overline{q}$ with $q > 0$, define
$$\mathcal{F}^{(q)} = \mathcal{F}^{\wedge q},$$
the q-fold smash power of pointed categories, where $\mathcal{F}$ has basepoint $\underline{0}$. We denote the objects of $\mathcal{F}^{(q)}$ as
$$\langle \underline{n} \rangle = (\underline{n}_1, \ldots, \underline{n}_q) \quad \text{for} \quad \underline{n}_i \in \mathcal{F}$$
where $\langle \underline{n} \rangle$ is identified with the basepoint of $\mathcal{F}^{(q)}$ if any $\underline{n}_i = \underline{0}$. We denote the morphisms of $\mathcal{F}^{(q)}$ as
$$\langle \psi \rangle = (\psi_1, \ldots, \psi_q)$$
for morphisms ψ_i in $\mathcal{F}$. A tuple $\langle \psi \rangle$ is a zero morphism if any ψ_i factors through $\underline{0}$ in $\mathcal{F}$.

For $q = 0$, define
$$\mathsf{Ob}\, \mathcal{F}^{(0)} = \{*, \langle\rangle\}$$
where $*$ is the basepoint object and $\langle\rangle$ is the empty tuple. Define the morphisms of $\mathcal{F}^{(0)}$ such that $*$ is both initial and terminal and the only nonzero morphisms are identities. That is,

- there are unique morphisms
$$\langle\rangle \longrightarrow * \longrightarrow \langle\rangle,$$

- the only endomorphism of $*$ is its identity, and
- the endomorphisms of $\langle\rangle$ consist of its identity and a zero morphism given by the unique composite through $*$.

We note that $\mathcal{F}^{(0)}$ is distinct from the smash product over the empty indexing set in the category of small pointed categories. We give an alternate description and further details in Note 9.5.2, but they will not be necessary for our further work. ◇

Definition 9.1.4. Suppose

$$f : \overline{q} \hookrightarrow \overline{p} \quad \text{in} \quad \mathsf{Inj}$$

is an injection of unpointed finite sets. We define a functor

$$f_* : \mathcal{F}^q \longrightarrow \mathcal{F}^p$$

called the *reindexing injection* as follows. For each q-tuple of pointed finite sets

$$\langle \underline{n} \rangle = (\underline{n}_1, \ldots, \underline{n}_q) \in \mathcal{F}^q,$$

let $\underline{n}_\varnothing$ denote the pointed finite set $\underline{1}$ and define $f_*\langle \underline{n} \rangle$ to be the p-tuple whose jth entry is $\underline{n}_{f^{-1}(j)}$. For each q-tuple of pointed functions

$$\langle \psi \rangle = (\psi_1, \ldots, \psi_q) \in \mathcal{F}^q,$$

let $\psi_\varnothing$ denote the identity $1_{\underline{1}}$ and define $f_*\langle \psi \rangle$ to be the p-tuple whose jth entry is $\psi_{f^{-1}(j)}$.

If any $\underline{n}_i = \underline{0}$, then the $f(i)$th coordinate of $f_*\langle n \rangle$ is $\underline{0}$. Therefore, f_* descends to give pointed functors

$$f_* : \mathcal{F}^{(q)} \longrightarrow \mathcal{F}^{(p)}.$$

For $q = 0$ and f the unique injection

$$\varnothing \hookrightarrow \overline{p},$$

we have

$$f_* : \mathcal{F}^{(0)} \longrightarrow \mathcal{F}^{(p)} \quad \text{with} \quad f_*\langle\rangle = (\underbrace{\underline{1}, \ldots, \underline{1}}_{p \text{ terms}}).$$ ◇

Explanation 9.1.5. For an injection $f : \overline{q} \longrightarrow \overline{p}$, the entries of $f_*\langle \underline{n} \rangle$ are reindexed by f, with $\underline{1}$ inserted for each index not in the image of f. Equivalently, f_* takes the ith coordinate of $\mathcal{F}^q$ to the $f(i)$th coordinate of $\mathcal{F}^p$, and has the value $\underline{1}$ in each coordinate of $\mathcal{F}^p$ not in the image of f. ◇

Example 9.1.6. Consider the injection

$$f : \overline{2} \longrightarrow \overline{3}, \qquad (1 \longmapsto 3;\ 2 \longmapsto 1)$$

Then

$$f_*(\underline{n}_1, \underline{n}_2) = (\underline{n}_2, \underline{1}, \underline{n}_1).$$

◇

Definition 9.1.7. We define a small pointed category $\mathcal{G}$ as follows.

Objects: The set of objects is the wedge of pointed sets

$$\mathsf{Ob}\,\mathcal{G} = \bigvee_{q \geq 0} \mathsf{Ob}\left(\mathcal{F}^{(q)}\right).$$

The distinct objects of $\mathcal{G}$ consist of a basepoint object $*$ together with q-tuples of pointed finite sets

$$\langle \underline{n} \rangle = (\underline{n}_1, \ldots, \underline{n}_q)$$

where $q \geq 0$ and each $\underline{n}_i \in \mathcal{F}$ with $n_i > 0$. Each q-tuple $\langle \underline{n} \rangle$ for which some $n_i = 0$ is identified with the basepoint object of $\mathcal{F}^{(q)}$ and thus also with the basepoint $*$ of the wedge.

Morphisms: The basepoint $*$ is both initial and terminal in $\mathcal{G}$. The set of morphisms from a q-tuple $\langle \underline{n} \rangle$ to a p-tuple $\langle \underline{m} \rangle$ is

$$\begin{aligned} \mathcal{G}(\langle \underline{n} \rangle, \langle \underline{m} \rangle) &= \bigvee_{f \in \mathsf{Inj}(\overline{q}, \overline{p})} \Big(\mathcal{F}^{(p)}\big(f_* \langle \underline{n} \rangle, \langle \underline{m} \rangle\big) \Big) \\ &= \bigvee_{f \in \mathsf{Inj}(\overline{q}, \overline{p})} \Big(\bigwedge_{j=1}^{p} \mathcal{F}\big(\underline{n}_{f^{-1}(j)}, \underline{m}_j\big) \Big) \end{aligned} \tag{9.1.8}$$

where $\underline{n}_\varnothing = \underline{1}$ as in Definition 9.1.4. For $p = 0$, the empty smash product of pointed sets is the unit for the smash product—a two-element set. In this case we must also have $q = 0$, and

$$\mathcal{G}(\langle\rangle, \langle\rangle) = \mathcal{F}^{(0)}(\langle\rangle, \langle\rangle)$$

is a two-element set consisting of a zero morphism and an identity morphism.

In (9.1.8) for $p > 0$ we denote a morphism by a pair $(f, \langle \psi \rangle)$, where

$$f : \overline{q} \hookrightarrow \overline{p} \quad \text{in} \quad \mathsf{Inj}$$

and

$$\langle \psi \rangle : f_* \langle \underline{n} \rangle \longrightarrow \langle \underline{m} \rangle$$

is a morphism in $\mathcal{F}^{(p)}$. Thus each

$$\psi_j : \underline{n}_{f^{-1}(j)} \longrightarrow \underline{m}_j$$

is a morphism in $\mathcal{F}$. A morphism $(f, \langle \psi \rangle)$ is identified with the zero morphism

$$\langle \underline{n} \rangle \longrightarrow * \longrightarrow \langle \underline{m} \rangle$$

if there exists a component morphism

$$\psi_j : \underline{n}_{f^{-1}(j)} \longrightarrow \underline{m}_j$$

that is a zero morphism in $\mathcal{F}$, factoring through $\underline{0}$.

Identities: The identity on a q-tuple $\langle \underline{n} \rangle$ is given by the pair $(1_{\overline{q}}, 1_{\langle \underline{n} \rangle})$.

Composition: The composite of morphisms

$$\langle \underline{n} \rangle \xrightarrow{(f, \langle \psi \rangle)} \langle \underline{m} \rangle \xrightarrow{(g, \langle \phi \rangle)} \langle \underline{\ell} \rangle$$

is given by the pair $(gf, \langle \phi \rangle \circ g_* \langle \psi \rangle)$.

This finishes the definition of $\mathcal{G}$. The composition in $\mathcal{G}$ is associative and unital since $(gf)_* = g_* \circ f_*$ for composable injections f and g. Moreover, we note the following.

- We call q the *length* of a q-tuple $\langle \underline{n} \rangle$.
- The *zero morphism* from $\langle \underline{n} \rangle$ to $\langle \underline{m} \rangle$ is the unique morphism factoring through the basepoint $*$.

- Each morphism set $\mathcal{G}(\langle \underline{n} \rangle, \langle \underline{m} \rangle)$ is a pointed set with basepoint the zero morphism.
- For readability, we sometimes use a semicolon and write $\mathcal{G}\big(\langle \underline{n} \rangle\,;\,\langle \underline{m} \rangle\big)$ for the set of morphisms in $\mathcal{G}$ from $\langle \underline{n} \rangle$ to $\langle \underline{m} \rangle$.

In Note 9.5.3 we give an alternate description of $\mathcal{G}$ as a Grothendieck construction, but that description will not be necessary for our further work. ◇

Explanation 9.1.9. We discuss some special cases of Definition 9.1.7.

(1) Each of the following objects in $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ are identified with the basepoint in $\mathcal{G}$, for any $\underline{n}$ in $\mathcal{F}$.
$$* = (\underline{0}) = (\underline{0}, \underline{n}) = (\underline{n}, \underline{0}).$$
(2) Let d^2 and s^0 denote the following morphisms in $\mathcal{F}$, respectively:
$$\begin{aligned} d^2 &: \underline{1} \longrightarrow \underline{2} && (1 \longmapsto 1)\\ s^0 &: \underline{2} \longrightarrow \underline{1} && (1 \longmapsto 0; 2 \longmapsto 1). \end{aligned}$$
Then neither d^2 nor s^0 is a zero morphism, but the composite
$$\underline{1} \xrightarrow{d^2} \underline{2} \xrightarrow{s^0} \underline{1}$$
is a zero morphism in $\mathcal{F}$. The composite in $\mathcal{G}$
$$(\underline{1}, \underline{n}, \underline{m}) \xrightarrow{(1_{\overline{3}}\,,\,(d^2, 1_{\underline{n}}, 1_{\underline{m}}))} (\underline{2}, \underline{n}, \underline{m}) \xrightarrow{(1_{\overline{3}}\,,\,(s^0, 1_{\underline{n}}, 1_{\underline{m}}))} (\underline{1}, \underline{n}, \underline{m})$$
is a zero morphism for any $\underline{n}$ and $\underline{m}$ in $\mathcal{F}$.

(3) Suppose $q = 2$, $p = 3$, and $\langle \underline{\ell} \rangle$ has length 4. Suppose f and g are the following injections, respectively:
$$\begin{aligned} f &: \overline{2} \longrightarrow \overline{3}, && (1 \longmapsto 3;\ 2 \longmapsto 1)\\ g &: \overline{3} \longrightarrow \overline{4}, && (1 \longmapsto 4;\ 2 \longmapsto 1;\ 3 \longmapsto 2). \end{aligned}$$
Then the second coordinate of the composite
$$\langle \underline{n} \rangle \xrightarrow{(f,\langle \psi \rangle)} \langle \underline{m} \rangle \xrightarrow{(g,\langle \phi \rangle)} \langle \underline{\ell} \rangle$$
is the following morphism in $\mathcal{F}^{(4)}$:
$$(\underline{1}\,,\, \underline{n}_1\,,\, \underline{1}\,,\, \underline{n}_2) \xrightarrow{(\psi_2, \psi_3, 1_{\underline{1}}, \psi_1)} (\underline{m}_2\,,\, \underline{m}_3\,,\, \underline{1}\,,\, \underline{m}_1) \xrightarrow{(\phi_1, \phi_2, \phi_3, \phi_4)} (\underline{\ell}_1\,,\, \underline{\ell}_2\,,\, \underline{\ell}_3\,,\, \underline{\ell}_4).$$
(4) In the case $q = 0$, we have the empty tuple $\langle\rangle$ of length zero. The only nonzero morphism from $\langle\rangle$ to $\langle\rangle$ is the identity. If $\langle \underline{m} \rangle$ has length $p > 0$, then a morphism
$$\langle\rangle \longrightarrow \langle \underline{m} \rangle$$
is determined by $(i_p, \langle \psi \rangle)$ where
$$i_p : \overline{0} = \varnothing \hookrightarrow \overline{p}$$
is the unique inclusion and
$$\langle \psi \rangle : \langle \underline{1} \rangle \longrightarrow \langle \underline{m} \rangle \quad \text{in} \quad \mathcal{F}^{(p)}$$

where $\langle \underline{1} \rangle$ is the constant p-tuple at $\underline{1}$. Such a morphism factors uniquely as the composite

$$\langle\rangle \xrightarrow{(i_p, 1_{\langle \underline{1} \rangle})} \langle \underline{1} \rangle \xrightarrow{(1, \langle \psi \rangle)} \langle \underline{m} \rangle. \tag{9.1.10}$$ ◇

Explanation 9.1.11 ($\mathcal{G}$ as a Small Skeleton)**.** Recall, from Explanation 8.1.2, that $\mathcal{F}$ is a small skeletion for FinSet_*. The category $\mathcal{G}$ is a small skeleton for a similarly-constructed category whose objects are finite tuples of pointed finite sets and whose morphisms are given by tuples of morphisms along with injections of indexing sets. We work with diagrams on $\mathcal{G}$ for the same reasons mentioned in Explanation 8.1.10: The collection of $\mathcal{G}_*$-objects in a pointed category C will form a category because $\mathcal{G}$ is small. Moreover each $\underline{n}_i$ has a canonical total ordering that will be used in the constructions below. ◇

We now discuss a permutative structure on $\mathcal{G}$ given by concatenation of tuples. Recall from Definition 4.3.3 an object $\mathtt{T}$ of a symmetric monoidal category is called a null object if it is null for the monoidal product and is initial and terminal.

Definition 9.1.12. The *concatenation product*

$$\mathcal{G} \times \mathcal{G} \xrightarrow{\oplus} \mathcal{G}$$

is defined as follows. The concatenation isomorphism of tuples

$$\mathcal{F}^q \times \mathcal{F}^{q'} \cong \mathcal{F}^{q+q'}$$

descends to

$$\mathcal{F}^{(q)} \wedge \mathcal{F}^{(q')} \xrightarrow{\oplus} \mathcal{F}^{(q+q')}$$

for each $q, q' \geq 0$. This defines

$$\langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle \quad \text{and} \quad \langle \psi \rangle \oplus \langle \psi' \rangle$$

for tuples of pointed finite sets and pointed functions, respectively. Since $\oplus$ is a pointed functor, the basepoint $*$ is a null object for this product in the sense of Definition 4.3.3.

Since $*$ is null, the product of any morphism with a morphism from, respectively to, the basepoint is uniquely determined. Given morphisms

$$(f, \langle \psi \rangle) \in \mathcal{G}(\langle \underline{n} \rangle, \langle \underline{m} \rangle) \quad \text{and} \quad (f', \langle \psi' \rangle) \in \mathcal{G}(\langle \underline{n'} \rangle, \langle \underline{m'} \rangle),$$

we first define the concatenation of injections

$$f : \overline{q} \hookrightarrow \overline{p} \quad \text{and} \quad f' : \overline{q'} \hookrightarrow \overline{p'}$$

to be the injection

$$(f \oplus f')(i) = \begin{cases} f(i) & \text{for} \quad i \in \{1, \ldots, q\} \\ p + f'(i - q) & \text{for} \quad i \in \{q+1, \ldots, q+q'\}. \end{cases}$$

Then the concatenation product of $(f, \langle \psi \rangle)$ and $(f', \langle \psi' \rangle)$ is given by the pair

$$(f, \langle \psi \rangle) \oplus (f', \langle \psi' \rangle) = (f \oplus f', \langle \psi \rangle \oplus \langle \psi' \rangle).$$

For checking functoriality, the essential observation is that for morphisms

$$\begin{aligned}(f,\langle\psi\rangle) &\in \mathcal{G}(\langle\underline{n}\rangle,\langle\underline{m}\rangle),\\ (f',\langle\psi'\rangle) &\in \mathcal{G}(\langle\underline{n'}\rangle,\langle\underline{m'}\rangle),\\ (g,\langle\phi\rangle) &\in \mathcal{G}(\langle\underline{m}\rangle,\langle\underline{\ell}\rangle), \quad \text{and}\\ (g',\langle\phi'\rangle) &\in \mathcal{G}(\langle\underline{m'}\rangle,\langle\underline{\ell'}\rangle),\end{aligned}$$

we have

$$(g\oplus g')_*(\langle\psi\rangle\oplus\langle\psi'\rangle) = (g_*\langle\psi\rangle)\oplus(g'_*\langle\psi'\rangle).$$

The concatentation product is strictly associative, and the empty tuple $\langle\rangle$ provides a strict unit. This concludes the definition and verification that $(\mathcal{G},\oplus,\langle\rangle)$ is a strict monoidal category.

The symmetry components

$$\xi_{\langle\underline{n}\rangle,\langle\underline{n'}\rangle} : \langle\underline{n}\rangle\oplus\langle\underline{n'}\rangle \longrightarrow \langle\underline{n'}\rangle\oplus\langle\underline{n}\rangle \tag{9.1.13}$$

are given by pairs $(\gamma_{q,q'},1)$ where the first entry

$$\gamma_{q,q'} : \overline{q+q'} \longrightarrow \overline{q'+q}$$

is the block permutation swapping the first q elements with the last q' elements. This is the permutation denoted $\tau\langle q,q'\rangle$ in (II.1.2.2). The second entry is the identity on

$$(\gamma_{q,q'})_*\big(\langle\underline{n}\rangle\oplus\langle\underline{n'}\rangle\big) = \langle\underline{n'}\rangle\oplus\langle\underline{n}\rangle.$$

We take the components of ξ involving the basepoint $*$ to be the identity on $*$. The components of ξ involving the empty tuple $\langle\rangle$ are identities because the block permutations $\gamma_{0,q'}$ and $\gamma_{q,0}$ are identities.

We verify that the components $\xi_{\langle\underline{n}\rangle,\langle\underline{n'}\rangle}$ are natural and that $(\mathcal{G},\oplus,\langle\rangle,\xi)$ is a permutative category in Proposition 9.1.14. ⋄

Proposition 9.1.14. *The concatenation product*

$$\mathcal{G}\times\mathcal{G} \xrightarrow{\oplus} \mathcal{G}$$

makes

$$(\mathcal{G},\oplus,\langle\rangle,\xi,*)$$

a permutative category with null object $*$.

Proof. The descriptions in Definition 9.1.12 show that $(\mathcal{G},\oplus,\langle\rangle)$ is a strict monoidal category with null object $*$. The components ξ are defined there, and now we verify that they are natural with respect to morphisms in $\mathcal{G}$. There is nothing to check for monoidal products involving $*$ because all such products are equal to $*$ and the relevant components are identities. Next, for each pair of morphisms

$$\begin{aligned}(f,\langle\psi\rangle) &\in \mathcal{G}(\langle\underline{n}\rangle,\langle\underline{m}\rangle) \quad \text{and}\\ (f',\langle\psi'\rangle) &\in \mathcal{G}(\langle\underline{n'}\rangle,\langle\underline{m'}\rangle)\end{aligned}$$

we verify that the following diagram commutes in $\mathcal{G}$.

$$\begin{tikzcd}
\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle \arrow[r, "\xi_{\langle \underline{n} \rangle, \langle \underline{n}' \rangle}"] \arrow[d, "(f,\langle\psi\rangle) \oplus (f',\langle\psi'\rangle)"'] & \langle \underline{n}' \rangle \oplus \langle \underline{n} \rangle \arrow[d, "(f',\langle\psi'\rangle) \oplus (f,\langle\psi\rangle)"] \\
\langle \underline{m} \rangle \oplus \langle \underline{m}' \rangle \arrow[r, "\xi_{\langle \underline{m} \rangle, \langle \underline{m}' \rangle}"] & \langle \underline{m}' \rangle \oplus \langle \underline{m}' \rangle
\end{tikzcd}$$

Commutativity for the first components follows because the block permutations γ are natural with respect to concatenation of injections f and f'. Commutativity for the second components follows because the second component of ξ is an identity and $\gamma_*(\langle\psi\rangle \oplus \langle\psi'\rangle) = \langle\psi'\rangle \oplus \langle\psi\rangle$. The symmetry axiom (1.1.24) follows from the equality of permutations

$$\gamma_{q',q}\gamma_{q,q'} = 1_{\overline{q+q'}}.$$

The unit axiom (1.1.25) for ξ holds by definition. The hexagon axiom (1.1.26) follows from the equality of block permutations shown in the following commutative diagram.

$$\begin{tikzcd}
& \overline{q+q''+q'} \\
\overline{q+q'+q''} \arrow[ur, "1_{\overline{q}} \oplus \gamma_{q',q''}"] \arrow[dr, "\gamma_{q+q',q''}"'] & \\
& \overline{q''+q+q'} \arrow[uu, "\gamma_{q'',q} \oplus 1_{\overline{q'}}"']
\end{tikzcd}$$

This completes the proof that $(\mathcal{G}, \oplus, \langle\rangle, \xi)$ is a permutative category. $\square$

Now we compare the category $\mathcal{G}$ and its concatenation product to $\mathcal{F}$ and its smash product.

Definition 9.1.15. The smash product of pointed finite sets defines a strict symmetric monoidal pointed functor

$$\wedge : (\mathcal{G}, \oplus, \langle\rangle, *) \longrightarrow (\mathcal{F}, \wedge, \underline{1}, \underline{0})$$

whose value at a tuple of pointed finite sets $\langle \underline{n} \rangle$ is

$$\wedge \langle \underline{n} \rangle = \underline{n_1 \cdots n_q}$$

where $\langle \underline{n} \rangle$ has length $q > 0$. We define

$$\wedge\langle\rangle = \underline{1} \quad \text{and} \quad \wedge * = \underline{0}.$$

To define $\wedge$ on morphisms in $\mathcal{G}$, suppose given

$$(f, \langle\psi\rangle) : \langle \underline{n} \rangle \longrightarrow \langle \underline{m} \rangle$$

with $\langle \underline{n} \rangle \in \mathcal{F}^{(q)}$ and $\langle \underline{m} \rangle \in \mathcal{F}^{(p)}$. Recall from Definition 8.1.5 the lexicographic ordering bijections on FinSet_*. The value of $\wedge$ on $(f, \langle\psi\rangle)$ is given by composing the lexicographic ordering bijections

$$\bigwedge_{k=1}^{q} \underline{n}_k \cong \underline{\textstyle\prod_k n_k} \quad \text{and} \quad \bigwedge_{j=1}^{p} \underline{m}_j \cong \underline{\textstyle\prod_j m_j}$$

with the composite

$$\bigwedge_{k=1}^{q} \underline{n}_k \xrightarrow[\text{permute}]{\cong} \bigwedge_{\substack{j\in\overline{p},\\ f^{-1}(j)\neq\varnothing}} \underline{n}_{f^{-1}(j)} \xrightarrow{\cong} \bigwedge_{j=1}^{p} \underline{n}_{f^{-1}(j)} \xrightarrow{\wedge_j \psi_j} \bigwedge_{j=1}^{p} \underline{m}_j \tag{9.1.16}$$

consisting of:

(1) the permutation induced by f, followed by
(2) unit isomorphisms for the entries of $f_*\langle \underline{n}\rangle$ not indexed by f, and then by
(3) the smash product of morphisms $\psi_j \in \langle\psi\rangle$.

If any $\underline{n}_i = \underline{0}$, then (9.1.16) is the zero morphism to $\wedge_j \underline{m}_j$.

If $(f, \langle\psi\rangle) = (1, \langle 1\rangle)$ is an identity morphism in $\mathcal{G}$, then the morphism (9.1.16) is the composite of identity morphisms. That $\wedge$ preserves composition follows from functoriality of the smash product on $\mathcal{F}$ along with naturality of its unit and symmetry isomorphisms. The monoidal constraint of $\wedge$ is the identity because $\wedge$ for $\mathcal{F}$ implicitly composes with the lexicographic isomorphisms. $\diamond$

Example 9.1.17. Consider the injection

$$f : \overline{2} \longrightarrow \overline{3}, \qquad (1 \longmapsto 3;\ 2 \longmapsto 1)$$

and suppose that

$$(f, \langle\psi\rangle) : (\underline{n}_1, \underline{n}_2) \longrightarrow (\underline{m}_1, \underline{m}_2, \underline{m}_3)$$

is a morphism in $\mathcal{G}$. Then $\wedge(f, \langle\psi\rangle)$ is the composite

$$\underline{n_1 n_2} \cong \underline{n}_1 \wedge \underline{n}_2 \xrightarrow{\cong} \underline{n}_2 \wedge \underline{n}_1 \xrightarrow{\cong} \underline{n}_2 \wedge \underline{1} \wedge \underline{n}_1 \xrightarrow{\wedge_j \psi_j} \underline{m}_1 \wedge \underline{m}_2 \wedge \underline{m}_3 \cong \underline{m_1 m_2 m_3}.$$

$\diamond$

9.2. Symmetric Monoidal Closed Structure for $\mathcal{G}_*$-Objects

In this section we describe $\mathcal{G}_*$-objects and symmetric monoidal closed structures for categories of such. The material here is an application of the general theory of enriched (pointed) diagram categories in Section 4.3, which itself depends on the preceding material in Part 1. However, for the applications here, the reader need only be familiar with the definitions and statements of results that we reference.

Definition 9.2.1. A *$\mathcal{G}_*$-object* in a pointed category $(\mathsf{C}, *)$ is a pointed functor

$$X : (\mathcal{G}, *) \longrightarrow (\mathsf{C}, *).$$

The category of $\mathcal{G}_*$-objects in C, denoted

$$\mathcal{G}_*\text{-}\mathsf{C} = \mathsf{Cat}_*\big((\mathcal{G}, *), (\mathsf{C}, *)\big) \tag{9.2.2}$$

is the category of pointed functors and natural transformations. $\diamond$

Explanation 9.2.3 (Canonical Basepoints for $\mathcal{G}_*$-Objects)**.** In the context of Definition 9.2.1, there is a canonical isomorphism of categories

$$\mathcal{G}_*\text{-}\mathsf{C} \cong \mathcal{G}_*\text{-}(\mathsf{C}_*). \tag{9.2.4}$$

For each $\mathcal{G}_*$-object X, the unique morphisms

$$* \longrightarrow \langle \underline{n}\rangle$$

induce canonical basepoints

$$* = X* \longrightarrow X\langle \underline{n}\rangle$$

for each $\langle \underline{n} \rangle$ in $\mathcal{G}$. By functoriality, each $\mathcal{G}_*$-object in C thereby determines a unique $\mathcal{G}_*$-object in C_*, and conversely. Morphisms of $\mathcal{G}_*$-objects (pointed natural transformations between pointed functors) necessarily preserve these basepoints, and thus we have a bijection between the morphisms of $\mathcal{G}_*$-C and those of $\mathcal{G}_*$-(C_*).

The theory of pointed diagrams from Section 4.3 is formulated for $\mathcal{G}_*$-(C_*), and so we will use (9.2.4) implicitly to apply it below. We've given the definition of $\mathcal{G}_*$-C as in (9.2.2) so that it is not necessary to explicitly note the canonical basepoints in our further work below. $\diamond$

Recall from Definition 4.3.23 and Corollary 4.3.30 the Day convolution, hom diagram, and monoidal unit for pointed diagrams out of a small symmetric monoidal category with a null object as its basepoint. Applying this theory to $\mathcal{G}$, we have the following preliminary definitions.

Definition 9.2.5. Suppose $(\mathsf{C}, \otimes, \mathbb{1}, *)$ is a symmetric monoidal closed category that is complete and cocomplete with chosen terminal object $*$. In the notation of Section 4.3, taking

$$(\mathcal{D}, \boxdot, e, \mathrm{T}^{\mathcal{D}}) = (\mathcal{G}, \oplus, \langle\rangle, *),$$

we have the following.

(1) For $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ in $\mathcal{G}$, we use the notation

$$\mathcal{G}^\flat\big(\langle \underline{a} \rangle ; \langle \underline{b} \rangle\big) = \big(\mathcal{G}(\langle \underline{a} \rangle, \langle \underline{b} \rangle)\big)^\flat \tag{9.2.6}$$

for the subset of nonzero morphisms.

(2) From Definition 4.3.11, we let $\widehat{\mathcal{G}}$ denote $\mathcal{G}$ equipped with the pointed unitary enrichment over C_* defined by

$$\widehat{\mathcal{G}}\big(\langle \underline{a} \rangle, \langle \underline{b} \rangle\big) = \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle ; \langle \underline{b} \rangle)} E \tag{9.2.7}$$

where $E = \mathbb{1} \coprod *$ is the monoidal unit of $(\mathsf{C}_*, \wedge)$. The empty wedge is the terminal object $*$ of C.

(3) The monoidal product of C_*-categories, called the tensor product in Definition 1.3.3, is denoted $\wedge$ and, in particular, $\widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}$ has hom objects given by smash products. $\diamond$

The following specializes Definition 4.3.23 and the adjunction (4.3.36) to $\mathcal{G}_*$-C.

Definition 9.2.8. Suppose $(\mathsf{C}, \otimes, \mathbb{1}, *)$ is a symmetric monoidal closed category that is complete and cocomplete with chosen terminal object $*$. Suppose given $\mathcal{G}_*$-objects X and Y. We define a mapping object $\mathsf{Map}_*(X, Y)$ in C_* and $\mathcal{G}_*$-objects

$$X \wedge Y, \quad \mathsf{Hom}_*(X, Y), \quad \text{and} \quad J$$

as follows. Each of $X \wedge Y$, $\mathsf{Hom}_*(X, Y)$, and J are pointed diagrams, so they send the basepoint $*$ of $\mathcal{G}$ to that of C.

- The Day convolution product of X and Y is

$$X \wedge Y = \int^{(\langle \underline{a} \rangle, \langle \underline{b} \rangle) \in \widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{b} \rangle ; -)} \big(X\langle \underline{a} \rangle \wedge Y\langle \underline{b} \rangle\big). \tag{9.2.9}$$

- The hom diagram for X and Y is

$$\mathsf{Hom}_*(X,Y) = \int_{(\langle \underline{b}\rangle,\langle \underline{c}\rangle)\in \widehat{\mathcal{G}}\wedge\widehat{\mathcal{G}}} \prod_{\mathcal{G}^\flat(-\oplus\langle \underline{b}\rangle\,;\,\langle \underline{c}\rangle)} \big[X\langle \underline{b}\rangle, Y\langle \underline{c}\rangle\big]_* \tag{9.2.10}$$

$$\cong \int_{\langle \underline{b}\rangle\in\widehat{\mathcal{G}}} [X\langle \underline{b}\rangle, Y(-\oplus\langle \underline{b}\rangle)]_* \tag{9.2.11}$$

where $[-,-]_*$ denotes the pointed internal hom of $(\mathsf{C}_*,\wedge)$.

- The mapping object for X and Y is

$$\mathsf{Map}_*(X,Y) = \int_{\langle \underline{b}\rangle\in\widehat{\mathcal{G}}} [X\langle \underline{b}\rangle, Y\langle \underline{b}\rangle]_* = \big(\mathsf{Hom}_*(X,Y)\big)\langle\rangle, \tag{9.2.12}$$

where the second equality holds because $\langle\rangle$ is a strict unit in $\mathcal{G}$.

- The monoidal unit diagram is

$$J = \widehat{\mathcal{G}}\big(\langle\rangle,-\big) = \bigvee_{\mathcal{G}^\flat(\langle\rangle\,;\,-)} E. \tag{9.2.13}$$

These define a symmetric monoidal closed structure for $\mathcal{G}_*$-C by Corollary 4.3.30 and the isomorphism (9.2.4).

Moreover, evaluation at the empty tuple $\langle\rangle$ defines a symmetric monoidal functor $\mathsf{ev}_{\langle\rangle}$ that has a strong symmetric monoidal left adjoint $L_{\langle\rangle}$:

$$L_{\langle\rangle} : \mathsf{C}_* \rightleftarrows \mathcal{G}_*\text{-}\mathsf{C} : \mathsf{ev}_{\langle\rangle}. \tag{9.2.14}$$

The left adjoint $L_{\langle\rangle}$ is defined for $A \in \mathsf{C}_*$ by composition

$$A\wedge J : \widehat{\mathcal{G}} \xrightarrow{J} \mathsf{C}_* \xrightarrow{A\wedge -} \mathsf{C}_*. \qquad \diamond$$

Applying Theorem 4.3.37 in this case, we have the following.

Theorem 9.2.15. *Suppose $(\mathsf{C},\otimes,\mathbb{1},*)$ is a symmetric monoidal closed category that is complete and cocomplete with chosen terminal object $*$. Then $\mathcal{G}_*$-C is a complete and cocomplete symmetric monoidal closed category with*

- *monoidal product given by the Day convolution $\wedge$ (9.2.9),*
- *internal hom given by Hom_* (9.2.10), and*
- *monoidal unit given by J (9.2.13).*

Moreover, the adjunction $(L_{\langle\rangle},\mathsf{ev}_{\langle\rangle})$ makes $\mathcal{G}_*$-C enriched, tensored, and cotensored over C_* with mapping objects given by Map_*.

Our applications of Theorem 9.2.15 will have $\mathsf{C} = \mathsf{sSet}$, so $\mathsf{C}_* = \mathsf{sSet}_*$ and $\mathsf{C} = \mathsf{Cat}$, so $\mathsf{C}_* = \mathsf{Cat}_*$.

Explanation 9.2.16 (Mapping Objects for Diagrams in Cat_*)**.** Example 3.8.14 describes the following. Suppose $\mathcal{D}$ is a small symmetric monoidal category with unitary enrichment $\widehat{\mathcal{D}}$ over Cat. If X and Y are diagrams from $\widehat{\mathcal{D}}$ to Cat then, by the equalizer formula (3.5.7), the mapping object

$$\mathsf{Map}(X,Y) = \int_{b\in\widehat{\mathcal{D}}} \mathsf{Cat}(X_b,Y_b)$$

is the category of 2-natural transformations and modifications between X and Y regarded as 2-functors.

Similarly, now suppose $\widehat{\mathcal{D}}$ is the pointed unitary enrichment (Definition 4.3.11) over $(\mathsf{Cat}_*, \wedge)$. If X and Y are diagrams from $\widehat{\mathcal{D}}$ to Cat_*, then the objects of the pointed category

$$\mathsf{Map}_*(X,Y) = \int_{b \in \widehat{\mathcal{D}}} \mathsf{Cat}_*(X_b, Y_b)$$

are 2-natural transformations α such that each component

$$\alpha_b : X_b \longrightarrow Y_b$$

is a pointed functor. The basepoint of $\mathsf{Map}_*(X,Y)$ has each component at $b \in \mathcal{D}$ given by the constant functor to the basepoint of Y_b. The morphisms of $\mathsf{Map}_*(X,Y)$ from α to α' are given by modifications Θ such that each component

$$\Theta_b : \alpha_b \longrightarrow \alpha'_b$$

is a pointed natural transformation. That is, for each $b \in \mathcal{D}$ the component of Θ at the basepoint T of X_b is the identity morphism of $\alpha_b(\mathrm{T}) = \alpha'_b(\mathrm{T})$. ◇

Explanation 9.2.17 (Mapping Objects for Diagrams in sSet_*)**.** Here we give a similar explanation to that of Explanation 9.2.16. Suppose that $\mathcal{D}$ is a small symmetric monoidal category and suppose that $\widehat{\mathcal{D}}$ is its strictly unitary enrichment over $(\mathsf{sSet}_*, \wedge)$. Recall from Proposition 7.1.19 that the internal hom for pointed simplicial sets has set of n-simplices given by

$$\mathsf{Hom}^{\mathsf{sSet}_*}(A,B)_n = \mathsf{sSet}_*(A \wedge \Delta^n_+, B) = \int_{\underline{m} \in \Delta^{\mathsf{op}}} \mathsf{Set}_*(A_m \wedge \Delta^n_m, B_m).$$

for pointed simplicial sets A and B.

Now if X and Y are diagrams from $\widehat{\mathcal{D}}$ to sSet_*, we write $X_{b,n}$ for the n-simplices of X_b with $b \in \mathcal{D}$. A similar application of the equalizer formula (3.5.7) shows that n-simplices of the mapping object $\mathsf{Map}_*(X,Y)$

$$\begin{aligned}\mathsf{Map}_*(X,Y)_n &= \int_{b \in \widehat{\mathcal{D}}} \mathsf{sSet}_*(X_{b,-} \wedge \Delta^n_+, Y_{b,-}) \\ &= \int_{b \in \widehat{\mathcal{D}}} \int_{\underline{m} \in \Delta^{\mathsf{op}}} \mathsf{Set}_*(X_{b,m} \wedge \Delta^n_m, Y_{b,m})\end{aligned}$$

are given by natural transformations between functors

$$X \wedge \Delta^n, Y : \mathcal{D} \times \Delta^{\mathsf{op}} \longrightarrow \mathsf{Set}_*.$$

Equivalently, $\mathsf{Map}_*(X,Y)$ is isomorphic to the internal hom for simplicial objects of $\mathcal{D}_*$-Set. ◇

Explanation 9.2.18 (Symmetric Monoidal Closed Structure for Γ-Objects)**.** One can also apply the theory of Section 4.3 to $\mathcal{F}$, taking

$$(\mathcal{D}, \boxdot, e, \mathrm{T}^{\mathcal{D}}) = (\mathcal{F}, \wedge, \underline{1}, \underline{0}).$$

With C as in Definition 9.2.8, and with $(\mathcal{G}, \oplus, \langle\rangle)$ replaced by $(\mathcal{F}, \wedge, \underline{1})$, the Day convolution, hom diagram, and monoidal unit for

$$\mathcal{F}_*\text{-}\mathsf{C} = \Gamma\text{-}\mathsf{C}$$

are defined via formulas similar to those of Definition 9.2.8. By Corollary 4.3.30 this defines a symmetric monoidal closed structure for Γ-C. In Theorem 9.4.18 below we will see that the functor

$$\mathsf{K}^{\mathcal{F}} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$$

of Proposition 8.2.6 is symmetric monoidal as a sSet_*-functor. However one encounters difficulties when trying to develop symmetric monoidal structure for

$$\mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \Gamma\text{-}\mathsf{Cat}.$$

See Note 10.8.6 for additional remarks on this point. $\diamond$

As with Γ-categories and Γ-simplicial sets, we will apply the nerve functor N (Definition 7.2.3) to obtain $\mathcal{G}_*$-simplicial sets from $\mathcal{G}_*$-categories. Now we show that N is compatible with the symmetric monoidal structures discussed above. We will use several of the general results from Chapters 2 and 3 related to change of enrichments, self-enrichments, and enriched diagram categories.

Recall from Proposition 7.2.9 that the nerve is a strong symmetric monoidal functor

$$N : (\mathsf{Cat}, \times, \mathbf{1}) \longrightarrow (\mathsf{sSet}_*, \times, *).$$

Theorem 9.2.19. *The nerve functor N induces the following.*

(1) There is a sSet_*-enrichment of $\mathcal{G}_*$-Cat with hom objects given by

$$N\big(\mathsf{Map}_*(X,Y)\big)$$

for $\mathcal{G}_*$-categories X and Y.

(2) Nerve induces a symmetric monoidal sSet_*-functor

$$N_* : \mathcal{G}_*\text{-}\mathsf{Cat} \longrightarrow \mathcal{G}_*\text{-}\mathsf{sSet}$$

with the following data.

- *For a $\mathcal{G}_*$-category X, N_*X is the $\mathcal{G}_*$-simplicial set given by the composite*

$$\mathcal{G} \xrightarrow{X} \mathsf{Cat}_* \xrightarrow{N} \mathsf{sSet}_*.$$

- *For $\mathcal{G}_*$-categories X and Y, the morphism on hom objects*

$$N_* : N\big(\mathsf{Map}_*(X,Y)\big) \longrightarrow \mathsf{Map}_*(N_*X, N_*Y)$$

sends a 0-simplex $\alpha : X \longrightarrow Y$ in the source to the morphism of simplicial sets $N_*\alpha$ given by the whiskering of α with N

$$\begin{tikzcd} \mathcal{G} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{Cat}_* \arrow[r, "N"] & \mathsf{sSet}_*. \end{tikzcd} \tag{9.2.20}$$

Proof. Recall from Definition 9.2.5 that $\widehat{\mathcal{G}}$ denotes the pointed unitary enrichment (9.2.7) of $\mathcal{G}$ over $(\mathsf{C}_*, \wedge, E)$, for a complete and cocomplete symmetric monoidal closed category $(\mathsf{C}, \otimes, \mathbb{1}, *)$ with terminal object $*$ and

$$E = \mathbb{1} \coprod *.$$

The symmetric monoidal closed structure on $\mathcal{G}_*$-C, described in Theorem 9.2.15, is given via the equivalence (4.3.35)

$$\mathcal{G}_*\text{-}\mathsf{C} \simeq \widehat{\mathcal{G}}\text{-}(\mathsf{C}_*) = \mathsf{C}_*\mathsf{Cat}(\widehat{\mathcal{G}}, \underline{\mathsf{C}_*}).$$

The right hand side above denotes the category of C_*-enriched functors and natural transformations from $\widehat{\mathcal{G}}$ to $\underline{\mathsf{C}_*}$, where the latter is the canonical self-enrichment of C_* (Definition 3.1.5).

Our two cases of interest are

$$(\mathsf{C}, \otimes, \mathbb{1}, *) = (\mathsf{Cat}, \times, \mathbf{1}, \mathbf{1})$$
$$(\mathsf{C}, \otimes, \mathbb{1}, *) = (\mathsf{sSet}, \times, *, *).$$

We take $N\mathbf{1} = *$ so that N preserves the monoidal unit and terminal object strictly. Therefore, with $\widehat{\mathcal{G}}$ denoting the pointed unitary enrichment for Cat_*, the change of enrichment $\widehat{\mathcal{G}}_N$ is the pointed unitary enrichment of $\mathcal{G}$ for sSet_*.

Since N is strong symmetric monoidal (Proposition 7.2.9), change of enrichment and composition with N defines a symmetric monoidal functor

$$N_* : \widehat{\mathcal{G}}\text{-}(\mathsf{Cat}_*) \longrightarrow (\widehat{\mathcal{G}}_N)\text{-}(\mathsf{sSet}_*) = \widehat{\mathcal{G}}\text{-}(\mathsf{sSet}_*).$$

by Proposition 3.8.4. On the right hand side we have used the observation above that $\widehat{\mathcal{G}}_N$ is the pointed unitary enrichment of $\mathcal{G}$ for sSet_*.

Next, we take the canonical self-enrichments of both $\widehat{\mathcal{G}}\text{-}(\mathsf{Cat}_*)$ and $\widehat{\mathcal{G}}\text{-}(\mathsf{sSet}_*)$, using the same notation for readability. The standard enrichment of $U = N_*$ (Definition 3.3.3) is a symmetric monoidal $\underline{\mathsf{W}}$-functor

$$U = N_* : (\widehat{\mathcal{G}}\text{-}(\mathsf{Cat}_*))_U \longrightarrow \widehat{\mathcal{G}}\text{-}(\mathsf{sSet}_*) \tag{9.2.21}$$

with $\mathsf{W} = \widehat{\mathcal{G}}\text{-}(\mathsf{sSet}_*)$. For a pair of $\mathcal{G}_*$-categories X and Y, the morphism on hom objects is

$$N_*(\mathsf{Hom}_*(X,Y)) \longrightarrow \mathsf{Hom}_*(N_*X, N_*Y) \tag{9.2.22}$$

given by adjunction, the monoidal structure of N_*, and evaluation (see Definition 3.3.3).

Now we use the evaluation at the monoidal unit $\langle\rangle$ in $\mathcal{G}$. By Lemma 3.8.1, $\mathsf{ev}_{\langle\rangle}$ is a symmetric monoidal functor

$$\widehat{\mathcal{G}}\text{-}\mathsf{sSet} \longrightarrow \mathsf{sSet}.$$

Changing enrichment of $U = N_*$ along $\mathsf{ev}_{\langle\rangle}$, as in Definition 2.1.1, makes

$$(U)_{\mathsf{ev}_{\langle\rangle}} = (N_*)_{\mathsf{ev}_{\langle\rangle}} : \big((\widehat{\mathcal{G}}\text{-}\mathsf{Cat}_*)_U\big)_{\mathsf{ev}_{\langle\rangle}} \longrightarrow \big(\widehat{\mathcal{G}}\text{-}\mathsf{sSet}_*\big)_{\mathsf{ev}_{\langle\rangle}} \tag{9.2.23}$$

a symmetric monoidal sSet_*-functor.

For ease of notation we usually write N_* for the change of enrichment $(N_*)_{\mathsf{ev}\langle\rangle}$. The domain of (9.2.23) has hom objects given as follows for $\mathcal{G}_*$-categories X and Y:

$$\mathsf{ev}_{\langle\rangle}\Big(N_*\big(\mathsf{Hom}_*(X,Y)\big)\Big) = N\big(\mathsf{Hom}_*(X,Y)_{\langle\rangle}\big) \cong N\big(\mathsf{Map}_*(X,Y)\big).$$

This gives the sSet_*-enrichment of $\mathcal{G}_*$-Cat. By definition (Theorem 9.2.15) the codomain of (9.2.23) is the sSet_*-enrichment of $\mathcal{G}_*$-sSet, with hom objects also denoted Map_*. This finishes the proof that N_* defines a symmetric monoidal sSet_*-functor.

For $\mathcal{G}_*$-categories X and Y, the morphism of simplicial sets

$$N\big(\mathsf{Map}_*(X,Y)\big) \longrightarrow \mathsf{Map}_*(N_*X, N_*Y) \tag{9.2.24}$$

is given by evaluating (9.2.22) at the empty tuple, $\langle\rangle$. Checking components at objects of the categories $X\langle \underline{m}\rangle$, for $\langle \underline{m}\rangle$ in $\mathcal{G}$, confirms that (9.2.24) is given on a 0-simplex α by the whiskering (9.2.20) as asserted. □

9.3. Symmetric Spectra from $\mathcal{G}_*$-Simplicial Sets

In this section we construct a symmetric spectrum $\mathsf{K}^{\mathcal{G}}X$ associated to a $\mathcal{G}_*$-simplicial set X. We begin with necessary preliminary definitions and then give the construction in Definition 9.3.14. We prove functoriality of $\mathsf{K}^{\mathcal{G}}$ and compare it with $\mathsf{K}^{\mathcal{F}}$ in Proposition 9.3.16.

Recall from Definition 9.1.15 the smash product of pointed finite sets defines a functor

$$\wedge : \mathcal{G} \longrightarrow \mathcal{F}.$$

Definition 9.3.1. For each natural number p, we have the *inclusion of length-p tuples*

$$i_p : \mathcal{F}^{(p)} \longrightarrow \mathcal{G} \tag{9.3.2}$$

given by the identity on objects and by

$$\langle \psi \rangle \longmapsto (1_{\overline{p}}, \langle \psi \rangle).$$

on morphisms. ◇

Explanation 9.3.3. For $p > 0$, the composite

$$\mathcal{F}^{(p)} \xrightarrow{i_p} \mathcal{G} \xrightarrow{\wedge} \mathcal{F}$$

is the p-fold smash product of pointed finite sets and pointed morphisms. For $p = 0$, the composite sends $*$ to $\underline{0}$ and $\langle\rangle$ to $\underline{1}$. ◇

Definition 9.3.4. For each $p > 0$ we define a functor from the p-fold Cartesian product

$$c_p : \mathcal{F}^p \longrightarrow \mathcal{F}^{(p)} \tag{9.3.5}$$

by iterating the canonical projection

$$c : \mathsf{A} \times \mathsf{B} \longrightarrow \mathsf{A} \wedge \mathsf{B} \tag{9.3.6}$$

in the definition of the smash product (4.1.7) for pointed categories A and B with $\mathsf{C} = \mathsf{Cat}$. For $p = 0$ we define

$$c_0 : \mathcal{F}^0 = \mathbf{1} \longrightarrow \mathcal{F}^{(0)} \tag{9.3.7}$$

by sending the unique object of $\mathbf{1}$ to the empty tuple $\langle\rangle$. We note that c_0 is *not* a pointed functor. ◇

Now we define a functor for $\mathcal{G}_*$-simplicial sets that restricts to that of Definition 8.2.5 along the functor

$$\Gamma\text{-}\mathsf{sSet} \xrightarrow{\wedge^*} \mathcal{G}_*\text{-}\mathsf{sSet} \tag{9.3.8}$$

induced by the smash product $\wedge : \mathcal{G} \longrightarrow \mathcal{F}$ of Definition 9.1.15. In the following discussion, recall from Definition 8.2.1 the $\mathcal{F}$-sphere $\overline{S}$ is a symmetric spectrum with n-simplices of the pth term given by

$$\overline{S}^p_n = \underline{n^p}.$$

Under the composite

$$\mathcal{F}^{(p)} \xrightarrow{i_p} \mathcal{G} \xrightarrow{\wedge} \mathcal{F}$$

we have

$$\underbrace{(\overline{S}^1_n, \ldots, \overline{S}^1_n)}_{p} \longmapsto \overline{S}^p_n.$$

In particular, for $p = 0$, the empty tuple $\langle\rangle$ is sent to $\overline{S}^0_n$.

The structure morphisms for the symmetric spectra we construct will be defined using the following morphisms.

Definition 9.3.9. Suppose X is a $\mathcal{G}_*$-simplicial set. For each $\underline{m} \in \mathcal{F}$ and each q-tuple $\langle \underline{n} \rangle \in \mathcal{G}$ with $q \geq 0$ we combine the inclusion i_1 with the concatenation $\oplus$ in $\mathcal{G}$ to form the $(q+1)$-tuple

$$(i_1 \underline{m}) \oplus \langle \underline{n} \rangle = (\underline{m}\,, \underline{n}_1\,, \ldots, \underline{n}_q).$$

For each $x \in \underline{m}^\flat$, let

$$h_x : \underline{1} \longrightarrow \underline{m}$$

be the pointed function that sends 1 to x. Let

$$f_1 : \overline{q} \longrightarrow \overline{q+1}$$

be given by $f_1(i) = i+1$. Then for each $x \in \underline{m}^\flat$ we have a morphism in $\mathcal{G}$

$$(f_1, (i_1 h_x) \oplus 1_{\langle \underline{n} \rangle}) : \langle \underline{n} \rangle \longrightarrow (i_1 \underline{m}) \oplus \langle \underline{n} \rangle.$$

Applying X to this morphism, we have a morphism in sSet_*

$$\eta_{\underline{m}, \langle \underline{n} \rangle, x} = X(f_1, (i_1 h_x) \oplus 1_{\langle \underline{n} \rangle}) : X\langle \underline{n} \rangle \longrightarrow X((i_1 \underline{m}) \oplus \langle \underline{n} \rangle).$$

Taking the wedge, we have

$$\eta_{\underline{m}, \langle \underline{n} \rangle} : \underline{m} \wedge X\langle \underline{n} \rangle \cong \bigvee_{x \in \underline{m}^\flat} X\langle \underline{n} \rangle \longrightarrow X((i_1 \underline{m}) \oplus \langle \underline{n} \rangle). \tag{9.3.10}$$

If $m = 0$, then $(i_1 \underline{0}) \oplus \langle \underline{n} \rangle$ has first entry $\underline{0}$, so it is the null object $*$ in $\mathcal{G}$. Since $X(*) = *$, $\eta_{0, \langle \underline{n} \rangle}$ is the identity morphism 1_*.

If $\langle \underline{m} \rangle \in \mathcal{F}^{(p)}$ is a p-tuple of pointed finite sets with $p > 1$, we define

$$\eta_{\langle \underline{m} \rangle, \langle \underline{n} \rangle} : \underline{m}_1 \wedge (\underline{m}_2 \wedge (\cdots(\underline{m}_p \wedge X\langle \underline{n} \rangle)\cdots)) \longrightarrow X((i_p \langle \underline{m} \rangle) \oplus \langle \underline{n} \rangle) \tag{9.3.11}$$

inductively as the composite

$$\begin{tikzcd}[column sep=-3em] & \underline{m}_1 \wedge X((i_{p-1}\langle \underline{m}' \rangle) \oplus \langle \underline{n} \rangle) \arrow[dr, "\eta_{\underline{m}_1, \langle \underline{n}' \rangle}"] & \\ \underline{m}_1 \wedge (\underline{m}_2 \wedge (\cdots(\underline{m}_p \wedge X\langle \underline{n} \rangle)\cdots)) \arrow[ur, "1_{\underline{m}_1} \wedge \eta_{\langle \underline{m}' \rangle, \langle \underline{n} \rangle}"] & & X((i_p \langle \underline{m} \rangle) \oplus \langle \underline{n} \rangle) \end{tikzcd}$$

where

$$\langle \underline{m}' \rangle = (\underline{m}_2, \ldots, \underline{m}_p) \quad \text{and} \quad \langle \underline{n}' \rangle = (i_{p-1} \langle \underline{m}' \rangle) \oplus \langle \underline{n} \rangle.$$

Similarly to (9.3.10), if any $m_i = 0$, then

$$(i_p \langle \underline{m} \rangle) \oplus \langle \underline{n} \rangle$$

has an entry $\underline{0}$, so it is $*$ in $\mathcal{G}$. Since $X(*) = *$, (9.3.11) is 1_*.

These morphisms are natural with respect to $\langle \underline{m} \rangle \in \mathcal{F}^{(p)}$ and $\langle \underline{n} \rangle \in \mathcal{G}$. For $p = 0$, we define

$$\eta_{\langle\rangle, \langle \underline{n} \rangle} = 1 : X\langle \underline{n} \rangle \longrightarrow X\langle \underline{n} \rangle \tag{9.3.12}$$

to be the identity of $X\langle \underline{n}\rangle$. For the basepoint $* \in \mathcal{G}$, we recall that $*$ is a null object with respect to the concatenation in $\mathcal{G}$ and, for p-tuples $\langle \underline{m}\rangle$ with $p \geq 0$, define

$$\eta_{\langle \underline{m}\rangle,*} = 1_*, \tag{9.3.13}$$

the identity on the initial and terminal pointed simplicial set. ⋄

Generalizing the constructions of Definition 8.2.5, we will construct pointed bisimplicial sets as functors

$$\Delta^{\mathsf{op}} \longrightarrow \mathsf{sSet}_*$$

and take the diagonal as discussed in Explanation 7.1.7 to obtain a pointed simplicial set.

Definition 9.3.14. Suppose X is a $\mathcal{G}_*$-simplicial set. We define a pointed simplicial set $(\mathsf{K}^{\mathcal{G}}X)k$ for each natural number $k \geq 0$ and structure morphisms

$$\rho_{p,q} : S^p \wedge (\mathsf{K}^{\mathcal{G}}X)q \longrightarrow (\mathsf{K}^{\mathcal{G}}X)(p+q)$$

for natural numbers $p, q \geq 0$. In Proposition 9.3.16 below we will show that these data define a functor

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}.$$

For each natural number $k \geq 0$, let $(\mathsf{K}^{\mathcal{G}}X)k$ be the pointed simplicial set obtained by taking the diagonal of the pointed bisimplicial set given by the following composite:

$$\Delta^{\mathsf{op}} \xrightarrow{\mathrm{diag}} (\Delta^{\mathsf{op}})^k \xrightarrow{(\overline{S}^1)^k} \mathcal{F}^k \xrightarrow{c_k} \mathcal{F}^{(k)} \xrightarrow{i_k} \mathcal{G} \xrightarrow{X} \mathsf{sSet}_*.$$

In this composite, the first functor is the diagonal, the second is the k-fold Cartesian product of $\overline{S}^1$, the third is the canonical functor to the smash product (9.3.5), and the fourth is the inclusion of length k-tuples (9.3.2).

The components of η defined in (9.3.11), (9.3.12), and (9.3.13) assemble to give a natural transformation in the following diagram, whose terms we explain below.
(9.3.15)

$$\begin{tikzcd}
 & & \mathcal{F}^p \times \mathcal{F}^q \arrow[dr, "c_p \times c_q"] \arrow[ddd, "\cong"] & & \mathcal{F}^{(p)} \times \mathcal{G} \arrow[dd, "i_p \times 1"'] \arrow[dr, "1 \times X"] & \\
 & (\Delta^{\mathsf{op}})^p \times (\Delta^{\mathsf{op}})^q \arrow[ur, "(\overline{S}^1)^p \times (\overline{S}^1)^q"] \arrow[ddd, "\cong"] & & \mathcal{F}^{(p)} \times \mathcal{F}^{(q)} \arrow[ur, "1 \times i_q"] \arrow[dd, "c"] & & \mathcal{F}^{(p)} \times \mathsf{sSet}_* \arrow[dd, "c"] \\
 & & & & \mathcal{G} \times \mathcal{G} \arrow[d, "\oplus"'] \arrow[r, phantom, "\Swarrow\eta"] & \\
\Delta^{\mathsf{op}} \arrow[uur, "\mathrm{diag}"] \arrow[dr, "\mathrm{diag}"'] & & \mathcal{F}^{p+q} \arrow[dr, "c_{p+q}"'] & \mathcal{F}^{(p)} \wedge \mathcal{F}^{(q)} \arrow[d, "\cong"] & \mathcal{G} \arrow[dr, "X"'] & \mathcal{F}^{(p)} \wedge \mathsf{sSet}_* \arrow[d, "\wedge"] \\
 & (\Delta^{\mathsf{op}})^{p+q} \arrow[ur, "(\overline{S}^1)^{p+q}"'] & & \mathcal{F}^{(p+q)} \arrow[ur, "i_{p+q}"] & & \mathsf{sSet}_*
\end{tikzcd}$$

In the diagram (9.3.15) above:

- The unnamed vertical isomorphisms are given by associativity isomorphisms (for $p, q > 0$) or unit isomorphisms (for $p = 0$ or $q = 0$) of the Cartesian or smash products.
- The vertical morphisms c are given by the canonical functors to the smash product (9.3.6).
- The vertical morphism labeled $\wedge$ is the iterated levelwise smash product of pointed finite sets with simplicial sets:

$$\mathcal{F}^{(p)} \wedge \mathsf{sSet}_* \xrightarrow{(1_{\mathcal{F}^{(p-1)}}, \wedge)} \mathcal{F}^{(p-1)} \wedge \mathsf{sSet}_* \cdots \longrightarrow \mathcal{F} \wedge \mathsf{sSet}_* \xrightarrow{\wedge} \mathsf{sSet}_*.$$

- The natural transformation η has components $\eta_{\langle \underline{m} \rangle, \langle \underline{n} \rangle}$ given in (9.3.11).
- The unlabeled parallelogram and triangular regions commute.

The upper, respectively lower, composite around the boundary of (9.3.15) is the pointed bisimplicial set

$$\overline{S}^p \wedge X(\underbrace{\overline{S}^1, \ldots, \overline{S}^1}_{q}), \quad \text{respectively} \quad X(\underbrace{\overline{S}^1, \ldots, \overline{S}^1}_{p+q}),$$

where we have made use of the equality $\overline{S}^p = (\overline{S}^1)^{\wedge p}$ in the diagram category $\mathsf{Cat}(\Delta^{\mathsf{op}}, \mathcal{F})$. Taking diagonals and using the isomorphism $S \cong \overline{S}$, we have morphisms in sSet_*

$$\rho_{p,q} : S^p \wedge (\mathsf{K}^{\mathcal{G}} X)q \xrightarrow{\cong} \overline{S}^p \wedge (\mathsf{K}^{\mathcal{G}} X)q \longrightarrow (\mathsf{K}^{\mathcal{G}} X)(p+q).$$

This completes the definition of $\rho_{p,q}$. $\diamond$

As an example of the construction $\mathsf{K}^{\mathcal{G}}$, Lemma 9.4.3 below shows that $\mathsf{K}^{\mathcal{G}} J$ is the symmetric sphere spectrum, where J is the monoidal unit of $\mathcal{G}_*$-sSet from (9.2.13).

Proposition 9.3.16. *The data $\{(\mathsf{K}^{\mathcal{G}} X)k,\ \rho_{p,q} \mid k, p, q \geq 0\}$ of Definition 9.3.14 define a functor*

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$$

such that the following diagram commutes.

$$\begin{tikzcd} \Gamma\text{-}\mathsf{sSet} \arrow[d, "\wedge^*"'] \arrow[dr, "\mathsf{K}^{\mathcal{F}}"] & \\ \mathcal{G}_*\text{-}\mathsf{sSet} \arrow[r, "\mathsf{K}^{\mathcal{G}}"'] & \mathsf{SymSp} \end{tikzcd} \tag{9.3.17}$$

Proof. We first show that the construction $(\mathsf{K}^{\mathcal{G}} X, \rho)$ of Definition 9.3.14 is a symmetric spectrum. For this, we use the description of symmetric spectra from Explanation 7.4.6. The unity condition (7.4.7) holds because each $\eta_{\langle\rangle, \langle \underline{n} \rangle}$ is an identity. The associativity condition (7.4.8) for ρ follows from the inductive definition of η. The equivariance condition follows because the action of Σ_k on $(\mathsf{K}^{\mathcal{G}} X)k$ is given by permuting the coordinates of $(\overline{S}^1)^k$.

The components of η defined in (9.3.11), (9.3.12), and (9.3.13) are natural with respect to morphisms of $\mathcal{G}_*$-simplicial sets (that is, natural transformations of functors) and hence the structure morphisms $\rho_{p,q}$ are too. This shows that $\mathsf{K}^{\mathcal{G}}$ takes values in the category of symmetric spectra and morphisms thereof. Functoriality of $\mathsf{K}^{\mathcal{G}}$ with respect to morphisms of $\mathcal{G}_*$-simplicial sets follows because each step in the construction of $(\mathsf{K}^{\mathcal{G}} X)k$ is functorial.

Commutativity of the diagram (9.3.17) follows by checking the definitions of $(\mathsf{K}^{\mathcal{F}} X)k$ and $\rho_{p,q}$. The key fact is that our definitions of

$$\wedge : \mathcal{G} \longrightarrow \mathcal{F} \quad \text{and} \quad \overline{S}^k \in \mathsf{Cat}(\Delta^{\mathsf{op}}, \mathcal{F})$$

ensure that we have, for each n, an equality

$$\wedge(\underbrace{\overline{S}^1_n, \ldots, \overline{S}^1_n}_{k}) = \overline{S}^k_n. \qquad \square$$

9.4. $\mathsf{K}^{\mathcal{G}}$ is Symmetric Monoidal

Recall from Corollary 7.6.16 that the category of symmetric spectra is symmetric monoidal as a sSet_*-category. Applying Theorem 9.2.15 in the case

$$(\mathsf{C}, \otimes, \mathbb{1}, *) = (\mathsf{sSet}, \times, \mathbf{1}, \mathbf{1})$$

shows that $\mathcal{G}_*$-sSet is enriched over sSet_* and is symmetric monoidal as a sSet_*-category. In this section, we show that the functor $\mathsf{K}^{\mathcal{G}}$ in Proposition 9.3.16 is a symmetric monoidal sSet_*-functor (Definition 1.4.18).

- Lemma 9.4.3 shows that $\mathsf{K}^{\mathcal{G}}$ sends the monoidal unit in $\mathcal{G}_*$-sSet to the symmetric sphere, which is the monoidal unit in SymSp.
- Lemma 9.4.6 computes $\mathsf{K}^{\mathcal{G}}$ of a triple product in $\mathcal{G}_*$-sSet.
- Theorem 9.4.9 is the main result of this section, which uses both Lemmas 9.4.3 and 9.4.6.
- Theorem 9.4.18 shows that the other two functors in (9.3.17), $\mathsf{K}^{\mathcal{F}}$ and $\wedge^*$, are also symmetric monoidal sSet_*-functors. See Note 9.5.5 for further generalization.

Monoidal Units. First recall the following structure.

- $(\mathsf{sSet}_*, \wedge, S^0)$ is the symmetric monoidal closed category of pointed simplicial sets (Definition 7.1.23).
- The category $\mathcal{G}$ is a permutative category (Proposition 9.1.14) with
  - the concatenation product $\oplus$ and
  - the empty tuple $\langle\rangle$ as the monoidal unit.
- From (9.2.6) we use

  $$\mathcal{G}^\flat(\langle \underline{a} \rangle ; \langle \underline{b} \rangle) = \big(\mathcal{G}(\langle \underline{a} \rangle, \langle \underline{b} \rangle)\big)^\flat$$

  to denote the set of nonzero morphisms from $\langle \underline{a} \rangle$ to $\langle \underline{b} \rangle$ in $\mathcal{G}$.
- In the symmetric monoidal closed category $\mathcal{G}_*$-sSet, the monoidal unit (9.2.13) is defined by

(9.4.1)
$$J\langle \underline{m} \rangle = \bigvee_{\mathcal{G}^\flat(\langle\rangle ; \langle \underline{m} \rangle)} S^0 \quad \text{for} \quad \langle \underline{m} \rangle \in \mathcal{G} \quad \text{and} \quad J* = *.$$

  The rest of the symmetric monoidal structure in $\mathcal{G}_*$-sSet is induced by the Day convolution and hom diagrams as described in Definition 9.2.8.

- For $X \in \mathcal{G}_*\text{-sSet}$ and $m, p \geq 0$, we write

$$(\mathsf{K}^{\mathcal{G}} X)_{p,m} = X\big(\underline{m}^{\oplus p}\big)_m \tag{9.4.2}$$

for the pointed set of m-simplices in the pointed simplicial set $(\mathsf{K}^{\mathcal{G}} X)_p$ (Definition 9.3.14).

The following preliminary observation says that the symmetric sphere (Definitions 7.4.1 and 8.2.1) is the K-theory of the monoidal unit $J \in \mathcal{G}_*\text{-sSet}$. It is both an example of the construction $\mathsf{K}^{\mathcal{G}}$ and an important part of Theorem 9.4.9 below, which asserts that $\mathsf{K}^{\mathcal{G}}$ is a symmetric monoidal sSet_*-functor.

Lemma 9.4.3. *There is a canonical isomorphism*

$$S \cong \mathsf{K}^{\mathcal{G}} J \tag{9.4.4}$$

of symmetric spectra with

- *S the symmetric sphere,*
- *$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-sSet} \longrightarrow \mathsf{SymSp}$ the functor in Proposition 9.3.16, and*
- *$J \in \mathcal{G}_*\text{-sSet}$ the monoidal unit.*

Proof. We compute the symmetric spectrum $\mathsf{K}^{\mathcal{G}} J$ and show that it is canonically isomorphic to S. By Explanation 9.1.9 (4), a morphism

$$\langle\rangle \longrightarrow \langle \underline{m} \rangle = \big(\underline{m_1}, \ldots, \underline{m_p}\big) \in \mathcal{G}$$

consists of the unique injection

$$\varnothing = \overline{0} \longrightarrow \overline{p} = \{1, \ldots, p\}$$

together with pointed functions

$$\underline{1} \xrightarrow{\psi_i} \underline{m_i} \quad \text{for} \quad 1 \leq i \leq p.$$

Applied to (9.4.1), this implies that

$$J\langle \underline{m} \rangle = \bigvee_{\mathcal{G}^\flat(\langle\rangle;\langle \underline{m}\rangle)} S^0 \cong \bigwedge_{i=1}^{p} \underline{m_i} \in \mathsf{sSet}_*. \tag{9.4.5}$$

For $p, n \geq 0$, the pointed simplicial set $(\mathsf{K}^{\mathcal{G}} J)_p$ has, as its set of n-simplices, the pointed set

$$\begin{aligned}(\mathsf{K}^{\mathcal{G}} J)_{p,n} &= J\big(\underline{n}^{\oplus p}\big)_n \\ &\cong \underline{n}^{\wedge p} \\ &= (S^p)_n.\end{aligned}$$

- The first equality above is from (9.4.2).
- The isomorphism is by (9.4.5).
- The last equality is from Definitions 7.1.12 and 7.1.23.

Fixing p and varying n, the pointed simplicial set structure of $(\mathsf{K}^{\mathcal{G}} J)_p$ comes from the $\underline{n}$ variable as in Definition 7.1.12. So the above isomorphisms yield an isomorphism of pointed simplicial sets

$$(\mathsf{K}^{\mathcal{G}} J)_p \cong S^p.$$

The Σ_p-action on $(\mathsf{K}^{\mathcal{G}} J)_{p,n}$ comes from permuting the p copies of $\underline{n}$ in $\underline{n}^{\oplus p}$. So the Σ_p-action on $(\mathsf{K}^{\mathcal{G}} J)_p$ corresponds to that on S^p under the above isomorphism.

By Definition 9.3.9, for $q \geq 0$, the symmetric spectrum structure morphism

$$S^1 \wedge (\mathsf{K}^{\mathcal{G}} J)_q \longrightarrow (\mathsf{K}^{\mathcal{G}} J)_{1+q}$$

is induced by the following composite for $\langle \underline{n} \rangle = (\underline{n_1}, \ldots, \underline{n_q})$, $m \geq 0$, $x \in \underline{m}^\flat$, and $h_x : \underline{1} \longrightarrow \underline{m}$ the pointed function with $h_x(1) = x$.

$$\begin{tikzcd}[column sep=small]
J\langle \underline{n} \rangle \arrow[rr, "\eta_{\underline{m}, \langle \underline{n} \rangle, x}"] \arrow[d, equal] & & J\big(i_1 \underline{m} \oplus \langle \underline{n} \rangle\big) \arrow[d, equal] \\
\underline{n_1} \wedge \cdots \wedge \underline{n_q} \arrow[dr, "\cong"'] & & \underline{m} \wedge \underline{n_1} \wedge \cdots \wedge \underline{n_q} \\
& \{0,1\} \wedge \underline{n_1} \wedge \cdots \wedge \underline{n_q} \arrow[ur, "h_x \wedge 1"'] &
\end{tikzcd}$$

So the S^1-action on $\mathsf{K}^{\mathcal{G}} J$ corresponds to the structure morphism

$$S^1 \wedge S^q \xrightarrow{\cong} S^{1+q}$$

on the symmetric sphere. By the associativity condition (7.4.8) in Explanation 7.4.6, for $p, q \geq 0$, the symmetric spectrum structure morphism

$$S^p \wedge (\mathsf{K}^{\mathcal{G}} J)_q \longrightarrow (\mathsf{K}^{\mathcal{G}} J)_{p+q}$$

also corresponds to that for the symmetric sphere. $\square$

***K*-Theory of Triple Products.** To facilitate the proof of Theorem 9.4.9, the next result contains some preliminary computation. Recall from Theorem 9.2.15 with

$$(\mathsf{C}, \otimes, \mathbb{1}, *) = (\mathsf{sSet}, \times, \mathbf{1}, \mathbf{1})$$

that $\mathcal{G}_*$-sSet is a symmetric monoidal sSet_*-category with

- monoidal product $\wedge$ in (9.2.9) induced by Day convolution and
- J in (9.4.1) as the monoidal unit.

We use the notation (9.4.2) below. Moreover, for indexing of coends we write

$$\widehat{\mathcal{G}}^2 = \widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}$$

and

$$\widehat{\mathcal{G}}^3 = \widehat{\mathcal{G}} \wedge (\widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}) \quad \text{or} \quad (\widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}) \wedge \widehat{\mathcal{G}}$$

with context determining the latter.

Lemma 9.4.6. *For $X, Y, Z \in \mathcal{G}_*$-sSet and $m, p \geq 0$, there are isomorphisms of pointed sets*

$$\begin{aligned}
&\mathsf{K}^{\mathcal{G}}\big(X \wedge (Y \wedge Z)\big)_{p,m} \\
&\quad \cong \int^{(\langle \underline{a} \rangle, \langle \underline{b} \rangle, \langle \underline{c} \rangle) \in \widehat{\mathcal{G}}^3} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{b} \rangle \oplus \langle \underline{c} \rangle; \underline{m}^{\oplus p})} X\langle \underline{a} \rangle_m \wedge \big(Y\langle \underline{b} \rangle_m \wedge Z\langle \underline{c} \rangle_m\big) \\
&\mathsf{K}^{\mathcal{G}}\big((X \wedge Y) \wedge Z\big)_{p,m} \\
&\quad \cong \int^{(\langle \underline{a} \rangle, \langle \underline{b} \rangle, \langle \underline{c} \rangle) \in \widehat{\mathcal{G}}^3} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{b} \rangle \oplus \langle \underline{c} \rangle; \underline{m}^{\oplus p})} \big(X\langle \underline{a} \rangle_m \wedge Y\langle \underline{b} \rangle_m\big) \wedge Z\langle \underline{c} \rangle_m
\end{aligned} \tag{9.4.7}$$

that are natural in X, Y, and Z.

Proof. Applying (9.4.2) to $X \wedge Y$, there are pointed sets as follows.

$$\begin{aligned}\mathsf{K}^{\mathcal{G}}(X \wedge Y)_{p,m} &= (X \wedge Y)\big(\underline{m}^{\oplus p}\big)_m \\ &= \int^{(\langle \underline{a} \rangle, \langle \underline{b} \rangle) \in \widetilde{\mathcal{G}}^2} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{b} \rangle; \underline{m}^{\oplus p})} X\langle \underline{a} \rangle_m \wedge Y\langle \underline{b} \rangle_m\end{aligned} \tag{9.4.8}$$

Using (9.4.8) twice, the first isomorphism in (9.4.7) follows from the following computation.

$$\begin{aligned}&\mathsf{K}^{\mathcal{G}}\big(X \wedge (Y \wedge Z)\big)_{p,m} \\ &= \int^{\langle \underline{a} \rangle, \langle \underline{d} \rangle} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{d} \rangle; \underline{m}^{\oplus p})} X\langle \underline{a} \rangle_m \wedge (Y \wedge Z)\langle \underline{d} \rangle_m \\ &= \int^{\langle \underline{a} \rangle, \langle \underline{d} \rangle} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{d} \rangle; \underline{m}^{\oplus p})} X\langle \underline{a} \rangle_m \wedge \Big[\int^{\langle \underline{b} \rangle, \langle \underline{c} \rangle} \bigvee_{\mathcal{G}^\flat(\langle \underline{b} \rangle \oplus \langle \underline{c} \rangle; \langle \underline{d} \rangle)} Y\langle \underline{b} \rangle_m \wedge Z\langle \underline{c} \rangle_m\Big] \\ &\cong \int^{\langle \underline{a} \rangle, \langle \underline{b} \rangle, \langle \underline{c} \rangle} \Big[\bigvee_{\int^{\langle \underline{d} \rangle} \mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{d} \rangle; \underline{m}^{\oplus p}) \times \mathcal{G}^\flat(\langle \underline{b} \rangle \oplus \langle \underline{c} \rangle; \langle \underline{d} \rangle)} X\langle \underline{a} \rangle_m \wedge \big(Y\langle \underline{b} \rangle_m \wedge Z\langle \underline{c} \rangle_m\big)\Big] \\ &\cong \int^{\langle \underline{a} \rangle, \langle \underline{b} \rangle, \langle \underline{c} \rangle} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{b} \rangle \oplus \langle \underline{c} \rangle; \underline{m}^{\oplus p})} X\langle \underline{a} \rangle_m \wedge \big(Y\langle \underline{b} \rangle_m \wedge Z\langle \underline{c} \rangle_m\big)\end{aligned}$$

- The first isomorphism above uses the fact that $X\langle \underline{a} \rangle_m \wedge -$ is a left adjoint, which commutes with small colimits. Then it switches the outer wedge and the coend $\int^{\langle \underline{b} \rangle, \langle \underline{c} \rangle}$, which is possible because they are both colimits. Next it switches the order of the coends from
$$\int^{\langle \underline{a} \rangle, \langle \underline{d} \rangle} \int^{\langle \underline{b} \rangle, \langle \underline{c} \rangle} \quad \text{to} \quad \int^{\langle \underline{a} \rangle, \langle \underline{b} \rangle, \langle \underline{c} \rangle} \int^{\langle \underline{d} \rangle}.$$
Then it combines the two wedges and brings $\int^{\langle \underline{d} \rangle}$ down to the index of the combined wedge, which is the only place where $\langle \underline{d} \rangle$ appears.
- The last isomorphism uses the isomorphism (3.7.16) on the wedge index.

Each step in each of the isomorphisms above is natural in X, Y, and Z. This proves the first isomorphism in (9.4.7). The second isomorphism in (9.4.7) is proved similarly using (i) the commutativity of $- \wedge Z\langle \underline{c} \rangle_m$ with small colimits and (ii) the isomorphism (3.7.15). □

Main Result. For the following result, first recall from

- Definitions 1.4.17 and 1.4.18 a symmetric monoidal V-functor, which is, furthermore, *unital* if the unit constraint is invertible, and
- Theorem 7.6.15 that the category SymSp is a symmetric monoidal sSet_*-category with
  - the smash product $\Box_S$ (Definition 7.6.1) as the monoidal product and
  - the symmetric sphere S as the monoidal unit.

Theorem 9.4.9. *The functor*

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$$

in Proposition 9.3.16 extends to a unital symmetric monoidal sSet_**-functor.*

Proof. Recall that the monoidal product in $\mathcal{G}_*$-sSet is defined by that of the enriched diagram category $(\widehat{\mathcal{G}})$-(sSet_*), where $\widehat{\mathcal{G}}$ has the pointed unitary sSet_* enrichment (9.2.7). Therefore, by Theorem 2.5.1 (2), it suffices to extend $\mathsf{K}^{\mathcal{G}}$ to a unital symmetric monoidal functor. This consists of the following steps:

(1) We first define the unit and monoidal constraints for $\mathsf{K}^{\mathcal{G}}$.
(2) Then we check that they are well defined.
(3) Then we show that they satisfy the axioms of a symmetric monoidal functor.

The unit constraint. The canonical isomorphism

$$S \xrightarrow[\cong]{(\mathsf{K}^{\mathcal{G}})^0} \mathsf{K}^{\mathcal{G}} J$$

in (9.4.4) is the unit constraint for $\mathsf{K}^{\mathcal{G}}$.

The monoidal constraint. By Proposition 7.6.2 and Explanation 7.6.4, for $X, Y \in \mathcal{G}_*$-sSet, the (X, Y) component of the monoidal constraint

$$\mathsf{K}^{\mathcal{G}} X \,\square_S\, \mathsf{K}^{\mathcal{G}} Y \xrightarrow{(\mathsf{K}^{\mathcal{G}})^2_{X,Y}} \mathsf{K}^{\mathcal{G}}(X \wedge Y) \in \mathsf{SymSp} \tag{9.4.10}$$

is uniquely determined by a family of $(\Sigma_p \times \Sigma_q)$-equivariant morphisms

$$(\mathsf{K}^{\mathcal{G}} X)_p \wedge (\mathsf{K}^{\mathcal{G}} Y)_q \xrightarrow{(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q}} \mathsf{K}^{\mathcal{G}}(X \wedge Y)_{p+q} \in \mathsf{sSet}_* \tag{9.4.11}$$

for $p, q \geq 0$ such that the diagrams (7.6.5) and (7.6.7) are commutative. Using (9.4.8), at level $m \geq 0$, the morphism $(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q,m}$ in (9.4.11) is defined as the following composite of pointed functions.

(9.4.12)

$$\begin{tikzcd}
X\big(\underline{m}^{\oplus p}\big)_m \wedge Y\big(\underline{m}^{\oplus q}\big)_m \arrow[r, "\iota"] \arrow[dr, bend right, "(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q,m}"'] & \displaystyle\bigvee_{\mathcal{G}^\flat(\underline{m}^{\oplus p} \oplus \underline{m}^{\oplus q};\, \underline{m}^{\oplus(p+q)})} X\big(\underline{m}^{\oplus p}\big)_m \wedge Y\big(\underline{m}^{\oplus q}\big)_m \arrow[d, "\omega_{(\underline{m}^{\oplus p},\, \underline{m}^{\oplus q})}"] \\
& \displaystyle\int^{(\langle \underline{a} \rangle, \langle \underline{b} \rangle) \in \widehat{\mathcal{G}}^2} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \oplus \langle \underline{b} \rangle;\, \underline{m}^{\oplus(p+q)})} X\langle \underline{a} \rangle_m \wedge Y\langle \underline{b} \rangle_m
\end{tikzcd}$$

- ι is the wedge summand inclusion for the identity morphism of
$$\underline{m}^{\oplus p} \oplus \underline{m}^{\oplus q} = \underline{m}^{\oplus(p+q)} \in \mathcal{G}.$$
- $\omega_{(\underline{m}^{\oplus p},\, \underline{m}^{\oplus q})}$ is the coend structure morphism (Definition 3.5.1) for
$$\langle \underline{a} \rangle = \underline{m}^{\oplus p} \quad \text{and} \quad \langle \underline{b} \rangle = \underline{m}^{\oplus q}.$$

As m varies, the collection of morphisms in (9.4.12) is a morphism $(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q}$ of pointed simplicial sets. Its naturality with respect to X and Y follows from the functoriality of $\wedge$ and the naturality of coends. Next we check equivariance and the commutativity of the diagrams (7.6.5) and (7.6.7).

Equivariance. To show that the morphism $(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q}$ is $(\Sigma_p \times \Sigma_q)$-equivariant, it suffices to show that, for permutations $(\sigma, \tau) \in \Sigma_p \times \Sigma_q$, the following diagram of pointed functions is commutative, where most instances of $\oplus$ are omitted to save space.

$$\begin{tikzcd}
X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[rr, "\iota"] \arrow[dr, "\iota"] \arrow[dd, "X(\sigma)_m \wedge Y(\tau)_m"'] & & \bigvee\limits_{\mathcal{G}^\flat(\underline{m}^p\underline{m}^q\,;\,\underline{m}^{p+q})} X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[dd, "\omega"] \\
& \bigvee\limits_{\mathcal{G}^\flat(\underline{m}^p\underline{m}^q\,;\,\underline{m}^{p+q})} X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[dddl, "\bigvee X(\sigma)_m \wedge Y(\tau)_m"'] \arrow[dd, "(\sigma\oplus\tau)^*"'] & \\
X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[dd, "\iota"'] & & \int^{\langle \underline{a}\rangle, \langle \underline{b}\rangle} \bigvee\limits_{\mathcal{G}^\flat(\langle \underline{a}\rangle\langle \underline{b}\rangle\,;\,\underline{m}^{p+q})} X\langle \underline{a}\rangle_m \wedge Y\langle \underline{b}\rangle_m \arrow[dd, "\int (\sigma\oplus\tau)_*"] \\
& \bigvee\limits_{\mathcal{G}^\flat(\underline{m}^p\underline{m}^q\,;\,\underline{m}^{p+q})} X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[dr, "\omega"] & \\
\bigvee\limits_{\mathcal{G}^\flat(\underline{m}^p\underline{m}^q\,;\,\underline{m}^{p+q})} X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[rr, "\omega"] & & \int^{\langle \underline{a}\rangle, \langle \underline{b}\rangle} \bigvee\limits_{\mathcal{G}^\flat(\langle \underline{a}\rangle\langle \underline{b}\rangle\,;\,\underline{m}^{p+q})} X\langle \underline{a}\rangle_m \wedge Y\langle \underline{b}\rangle_m
\end{tikzcd}$$

- In the above diagram, ι and ω are as in (9.4.12).
- In the morphism $X(\sigma)_m \wedge Y(\tau)_m$,

$$\sigma = (\sigma, \{1\}) : \underline{m}^{\oplus p} \longrightarrow \underline{m}^{\oplus p} \in \mathcal{G}$$

  consists of the permutation

$$\sigma : \overline{p} \longrightarrow \overline{p}$$

  and the entrywise identity function

$$\{1\} : \sigma_* \underline{m}^{\oplus p} \longrightarrow \underline{m}^{\oplus p}.$$

  The morphism

$$\tau = \big(\tau, \{1\}\big) : \underline{m}^{\oplus q} \longrightarrow \underline{m}^{\oplus q} \in \mathcal{G}$$

  is defined similarly.
- The morphism $(\sigma \oplus \tau)^*$ is the precomposition with

$$\sigma \oplus \tau : \underline{m}^{\oplus p} \oplus \underline{m}^{\oplus q} \longrightarrow \underline{m}^{\oplus p} \oplus \underline{m}^{\oplus q} \in \mathcal{G}$$

  in the wedge index. The morphism $(\sigma \oplus \tau)_*$ is the post-composition with $\sigma \oplus \tau$ in the wedge index.
- The two subdiagrams containing the left vertical composite and the right vertical composite are commutative by definition. The quadrilateral containing the bottom horizontal arrow is commutative by the definition of the coend in the lower right corner.

The commutativity of the above diagram proves that the morphism $(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q}$ in (9.4.11) is $(\Sigma_p \times \Sigma_q)$-equivariant.

Compatibility with S-action. To prove the commutativity of the diagram (7.6.5) for $(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q}$, by the associativity condition in Explanation 7.4.6 of the structure morphisms in symmetric spectra, it suffices to consider the case $n = 1$. At level

$m \geq 0$ with $n = 1$, (7.6.5) is the following diagram of pointed functions, with α an associativity isomorphism and ρ the S^1-action.

$$\begin{tikzcd}
\big((S^1)_m \wedge (\mathsf{K}^{\mathcal{G}}X)_{p,m}\big) \wedge (\mathsf{K}^{\mathcal{G}}Y)_{q,m} \arrow[r, "\alpha"] \arrow[d, "\rho\wedge 1"'] & (S^1)_m \wedge \big((\mathsf{K}^{\mathcal{G}}X)_{p,m} \wedge (\mathsf{K}^{\mathcal{G}}Y)_{q,m}\big) \arrow[dd, "1\wedge(\mathsf{K}^{\mathcal{G}})^2"] \\
(\mathsf{K}^{\mathcal{G}}X)_{1+p,m} \wedge (\mathsf{K}^{\mathcal{G}}Y)_{q,m} \arrow[d, "(\mathsf{K}^{\mathcal{G}})^2"'] & \\
\mathsf{K}^{\mathcal{G}}(X\wedge Y)_{1+p+q,m} & (S^1)_m \wedge \mathsf{K}^{\mathcal{G}}(X\wedge Y)_{p+q,m} \arrow[l, "\rho"']
\end{tikzcd} \tag{9.4.13}$$

To show that this diagram is commutative, first recall the following from Definition 9.3.9.

- For each $x \in \underline{m}^\flat$, $h_x : \underline{1} \longrightarrow \underline{m}$ is the pointed function with
$$h_x(1) = x.$$
- For $p \geq 0$, $f_1 : \overline{p} \longrightarrow \overline{1+p}$ is the pointed function given by
$$f_1(j) = j+1 \quad \text{for} \quad j \in \overline{p} = \{1, \ldots, p\}.$$
- We write
$$i_1^x = \big(f_1, (i_1 h_x) \oplus 1_{\underline{m}^{\oplus p}}\big) : \underline{m}^{\oplus p} \longrightarrow \underline{m}^{\oplus(1+p)} \in \mathcal{G},$$
so there is an equality
$$X(i_1^x) = \eta_{\underline{m},\underline{m}^{\oplus p},x} : X\big(\underline{m}^{\oplus p}\big) \longrightarrow X\big(\underline{m}^{\oplus(1+p)}\big) \in \mathsf{sSet}_*.$$

The S^1-action (9.3.10) on $\mathsf{K}^{\mathcal{G}}X$ is defined by these morphisms $X(i_1^x)$ for $x \in \underline{m}^\flat$.

Using (9.4.8), to show that (9.4.13) is commutative, it suffices to show that the outer diagram below is commutative for $x \in \underline{m}^\flat$, with the symbol $\oplus$ for concatenation omitted to save space.

$$\begin{tikzcd}[column sep=small]
X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[rr, "\iota"] \arrow[d, "X(i_1^x)_m\wedge 1"'] \arrow[dr, "\iota"] & & \displaystyle\bigvee_{\mathcal{G}^\flat(\underline{m}^p\underline{m}^q;\,\underline{m}^{p+q})} X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[ddl, bend left, "(i_1^x)_*"'] \arrow[ddd, "\omega_{(\underline{m}^p,\underline{m}^q)}"] \\
X(\underline{m}^{1+p})_m \wedge Y(\underline{m}^q)_m \arrow[d, "\iota"'] & \displaystyle\bigvee_{\mathcal{G}^\flat(\underline{m}^{1+p}\underline{m}^q;\,\underline{m}^{1+p+q})} X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[dl, "\bigvee X(i_1^x)_m\wedge 1"'] \arrow[d, "(i_1^x)^*"] & \\
\displaystyle\bigvee_{\mathcal{G}^\flat(\underline{m}^{1+p}\underline{m}^q;\,\underline{m}^{1+p+q})} X(\underline{m}^{1+p})_m \wedge Y(\underline{m}^q)_m \arrow[d, "\omega_{(\underline{m}^{1+p},\underline{m}^q)}"'] & \displaystyle\bigvee_{\mathcal{G}^\flat(\underline{m}^p\underline{m}^q;\,\underline{m}^{1+p+q})} X(\underline{m}^p)_m \wedge Y(\underline{m}^q)_m \arrow[dl, "\omega_{(\underline{m}^p,\underline{m}^q)}"'] & \\
\displaystyle\int^{\langle\underline{c}\rangle,\langle\underline{d}\rangle} \bigvee_{\mathcal{G}^\flat(\langle\underline{c}\rangle\langle\underline{d}\rangle;\,\underline{m}^{1+p+q})} X\langle\underline{c}\rangle_m \wedge Y\langle\underline{d}\rangle_m & & \displaystyle\int^{\langle\underline{a}\rangle,\langle\underline{b}\rangle} \bigvee_{\mathcal{G}^\flat(\langle\underline{a}\rangle\langle\underline{b}\rangle;\,\underline{m}^{p+q})} X\langle\underline{a}\rangle_m \wedge Y\langle\underline{b}\rangle_m \arrow[ll, "(i_1^x)_*"']
\end{tikzcd}$$

- In the above diagram, each ι is a wedge summand inclusion corresponding to an identity morphism in the wedge index. The upper left triangle and the upper right subdiagram are commutative by definition.
- The lower left parallelogram is commutative by the definition of the coend in the lower left corner.

- The lower right subdiagram is commutative by the naturality of coends.

This proves the commutativity of the diagram (9.4.13), which, in turn, implies the commutativity of (7.6.5) for $(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q}$. A similar argument shows that (7.6.7) is commutative. This finishes the construction of the monoidal constraint $(\mathsf{K}^{\mathcal{G}})^2$.

Next we check the symmetric monoidal functor axioms in Definitions 1.1.6 and 1.1.17 for $(\mathsf{K}^{\mathcal{G}}, (\mathsf{K}^{\mathcal{G}})^2, (\mathsf{K}^{\mathcal{G}})^0)$.

Unity. The left unity axiom in (1.1.10) asserts the commutativity of the following diagram in SymSp.

$$\begin{tikzcd}
S \,\square_S\, \mathsf{K}^{\mathcal{G}} X \arrow[r, "\lambda"] \arrow[d, "(\mathsf{K}^{\mathcal{G}})^0 \square_S 1"'] & \mathsf{K}^{\mathcal{G}} X \\
\mathsf{K}^{\mathcal{G}} J \,\square_S\, \mathsf{K}^{\mathcal{G}} X \arrow[r, "(\mathsf{K}^{\mathcal{G}})^2"] & \mathsf{K}^{\mathcal{G}}(J \wedge X) \arrow[u, "\mathsf{K}^{\mathcal{G}}\lambda"']
\end{tikzcd} \tag{9.4.14}$$

To show that (9.4.14) is commutative, it suffices to show that the following diagram of pointed functions is commutative for $p, m \geq 0$.

$$\begin{tikzcd}
(S^1)_m \wedge (\mathsf{K}^{\mathcal{G}} X)_{p,m} \arrow[r, "\lambda"] \arrow[d, "(\mathsf{K}^{\mathcal{G}})^0 \wedge 1"'] & (\mathsf{K}^{\mathcal{G}} X)_{1+p,m} \\
(\mathsf{K}^{\mathcal{G}} J)_{1,m} \wedge (\mathsf{K}^{\mathcal{G}} X)_{p,m} \arrow[r, "(\mathsf{K}^{\mathcal{G}})^2_{J,X;1,p,m}"] & \mathsf{K}^{\mathcal{G}}(J \wedge X)_{1+p,m} \arrow[u, "\mathsf{K}^{\mathcal{G}}\lambda"']
\end{tikzcd} \tag{9.4.15}$$

It suffices to show that the diagram (9.4.15) is commutative when restricted to each $x \in \underline{m}^\flat \subset (S^1)_m$. By the definitions

- of the canonical isomorphism (9.4.4)
$$(\mathsf{K}^{\mathcal{G}})^0 : S \xrightarrow{\cong} \mathsf{K}^{\mathcal{G}} J,$$
- of $(\mathsf{K}^{\mathcal{G}})^2$ in (9.4.12), and
- of the left unit isomorphism
$$\lambda : J \wedge X \xrightarrow{\cong} X$$
induced by that of the Day convolution product in Definition 3.7.17,

the diagram (9.4.15) restricted to $x \in \underline{m}^\flat$ is the outer diagram below, where the symbol $\oplus$ is omitted to save space.

$$\begin{tikzcd}
X(\underline{m}^p)_m \arrow[rr, "X(i_1^x)"] \arrow[d, "\cong"'] & & X(\underline{m}^{1+p})_m \\
\{0,1\} \wedge X(\underline{m}^p)_m \arrow[d, "h_x \wedge 1"'] & X(\underline{m}^{1+p})_m \arrow[ur, "1"] \arrow[d, "\iota"] & \displaystyle\int^{\langle \underline{b} \rangle} \bigvee_{\mathcal{G}^\flat(\langle \underline{b} \rangle; \underline{m}^{1+p})} X\langle \underline{b} \rangle_m \arrow[u, "\cong"'] \\
\underline{m} \wedge X(\underline{m}^p)_m \arrow[ur, "\eta_{\underline{m},\underline{m}^p}"] \arrow[d, "\iota"'] & \displaystyle\bigvee_{\mathcal{G}^\flat(\underline{m}^{1+p}; \underline{m}^{1+p})} X(\underline{m}^{1+p})_m \arrow[ur, "\omega_{\underline{m}^{1+p}}"] & \displaystyle\int^{\langle \underline{b} \rangle} \Big[\int^{\langle \underline{a} \rangle} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \langle \underline{b} \rangle; \underline{m}^{1+p})} J\langle \underline{a} \rangle_m \Big] \wedge X\langle \underline{b} \rangle_m \arrow[u, "\cong"'] \\
\displaystyle\bigvee_{\mathcal{G}^\flat(\underline{m}\underline{m}^p; \underline{m}^{1+p})} \underline{m} \wedge X(\underline{m}^p)_m \arrow[ur, "\bigvee \eta_{\underline{m},\underline{m}^p}"] \arrow[rr, "\omega_{(\underline{m},\underline{m}^p)}"] & & \displaystyle\int^{\langle \underline{a} \rangle, \langle \underline{b} \rangle} \bigvee_{\mathcal{G}^\flat(\langle \underline{a} \rangle \langle \underline{b} \rangle; \underline{m}^{1+p})} J\langle \underline{a} \rangle_m \wedge X\langle \underline{b} \rangle_m \arrow[u, "\cong"']
\end{tikzcd}$$

- Each ι is a wedge summand inclusion corresponding to an identity morphism in the wedge index. Each ω is a coend structure morphism.
- The morphism $\eta_{\underline{m},\underline{m}^p}$ is the one in (9.3.10) at level m. The upper left subdiagram is commutative by the definition of $\eta_{\underline{m},\underline{m}^p}$.
- The lower left parallelogram is commutative by the naturality of wedges.
- The upper right parallelogram is commutative by the definition of the upper right vertical isomorphism.
- The lower right subdiagram is commutative by the definitions of the two vertical isomorphisms.

This proves the left unity axiom in (1.1.10). The right unity axiom is proved by a similar argument.

Associativity. The associativity axiom (1.1.9) for $\mathsf{K}^{\mathcal{G}}$ asserts the commutativity of the following diagram in SymSp, where the bottom α is the associativity isomorphism for the Day convolution product in Definition 3.7.17.

$$\begin{tikzcd}
(\mathsf{K}^{\mathcal{G}}X \square_S \mathsf{K}^{\mathcal{G}}Y) \square_S \mathsf{K}^{\mathcal{G}}Z \arrow[r, "\alpha"] \arrow[d, "(\mathsf{K}^{\mathcal{G}})^2 \square_S 1"'] & \mathsf{K}^{\mathcal{G}}X \square_S (\mathsf{K}^{\mathcal{G}}Y \square_S \mathsf{K}^{\mathcal{G}}Z) \arrow[d, "1 \square_S (\mathsf{K}^{\mathcal{G}})^2"] \\
\mathsf{K}^{\mathcal{G}}(X \wedge Y) \square_S \mathsf{K}^{\mathcal{G}}Z \arrow[d, "(\mathsf{K}^{\mathcal{G}})^2"'] & \mathsf{K}^{\mathcal{G}}X \square_S \mathsf{K}^{\mathcal{G}}(Y \wedge Z) \arrow[d, "(\mathsf{K}^{\mathcal{G}})^2"] \\
\mathsf{K}^{\mathcal{G}}\big((X \wedge Y) \wedge Z\big) \arrow[r, "\mathsf{K}^{\mathcal{G}}\alpha"] & \mathsf{K}^{\mathcal{G}}\big(X \wedge (Y \wedge Z)\big)
\end{tikzcd} \tag{9.4.16}$$

By (9.4.7) and the definition (9.4.12) of $(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q,m}$, to show that (9.4.16) is commutative, it suffices to show that, for $m, p, q, r \geq 0$, the following diagram of pointed functions is commutative.

$$\begin{tikzcd}
\big(X(\underline{m}^{\oplus p})_m \wedge Y(\underline{m}^{\oplus q})_m\big) \wedge Z(\underline{m}^{\oplus r})_m \arrow[r, "\alpha"] \arrow[d, "\iota"'] & X(\underline{m}^{\oplus p})_m \wedge \big(Y(\underline{m}^{\oplus q})_m \wedge Z(\underline{m}^{\oplus r})_m\big) \arrow[d, "\iota"] \\
\bigvee \big(X(\underline{m}^{\oplus p})_m \wedge Y(\underline{m}^{\oplus q})_m\big) \wedge Z(\underline{m}^{\oplus r})_m \arrow[d, "\omega"'] & \bigvee X(\underline{m}^{\oplus p})_m \wedge \big(Y(\underline{m}^{\oplus q})_m \wedge Z(\underline{m}^{\oplus r})_m\big) \arrow[d, "\omega"] \\
\int \bigvee \big(X\langle \underline{a}\rangle_m \wedge Y\langle \underline{b}\rangle_m\big) \wedge Z\langle \underline{c}\rangle_m \arrow[r, "\int \bigvee \alpha"] & \int \bigvee X\langle \underline{a}\rangle_m \wedge \big(Y\langle \underline{b}\rangle_m \wedge Z\langle \underline{c}\rangle_m\big)
\end{tikzcd}$$

- In the above diagram, α is an associativity isomorphism.
- In the middle level, each wedge is indexed by the pointed set

$$\mathcal{G}^\flat\big(\underline{m}^{\oplus p} \oplus \underline{m}^{\oplus q} \oplus \underline{m}^{\oplus r}\,;\, \underline{m}^{\oplus(p+q+r)}\big).$$

  Each ι is the wedge summand inclusion corresponding to the identity morphism of $\underline{m}^{\oplus(p+q+r)}$.
- In the bottom level, each wedge is indexed by the pointed set

$$\mathcal{G}^\flat\big(\langle \underline{a}\rangle \oplus \langle \underline{b}\rangle \oplus \langle \underline{c}\rangle\,;\, \underline{m}^{\oplus(p+q+r)}\big).$$

  Each coend is indexed by

$$\big(\langle \underline{a}\rangle, \langle \underline{b}\rangle, \langle \underline{c}\rangle\big) \in \widehat{\mathcal{G}}^3.$$

Each ω is the coend structure morphism for

$$\langle \underline{a} \rangle = \underline{m}^{\oplus p}, \quad \langle \underline{b} \rangle = \underline{m}^{\oplus q}, \quad \text{and} \quad \langle \underline{c} \rangle = \underline{m}^{\oplus r}.$$

The above diagram is commutative by the definition of the bottom horizontal morphism $\int \bigvee \alpha$.

Symmetry. The compatibility axiom (1.1.18) for $\mathsf{K}^{\mathcal{G}}$ asserts the commutativity of the following diagram in SymSp, where the bottom ξ is the symmetry isomorphism for the Day convolution product in Definition 3.7.17.

$$\begin{tikzcd} \mathsf{K}^{\mathcal{G}} X \mathbin{\Box_S} \mathsf{K}^{\mathcal{G}} Y \arrow[r, "\xi"] \arrow[d, "(\mathsf{K}^{\mathcal{G}})^2"'] & \mathsf{K}^{\mathcal{G}} Y \mathbin{\Box_S} \mathsf{K}^{\mathcal{G}} X \arrow[d, "(\mathsf{K}^{\mathcal{G}})^2"] \\ \mathsf{K}^{\mathcal{G}}(X \wedge Y) \arrow[r, "\mathsf{K}^{\mathcal{G}}\xi"] & \mathsf{K}^{\mathcal{G}}(Y \wedge X) \end{tikzcd} \tag{9.4.17}$$

Similar to the proof of the associativity axiom (9.4.16) above, the commutativity of (9.4.17) follows from (9.4.8), the definition (9.4.12) of $(\mathsf{K}^{\mathcal{G}})^2_{X,Y;p,q,m}$, and the definition of $(\mathsf{K}^{\mathcal{G}}\xi)_{p,m}$. □

Theorem 9.4.18. *The functors*

$$\begin{tikzcd} \Gamma\text{-}\mathsf{sSet} \arrow[r, "\mathsf{K}^{\mathcal{F}}"] \arrow[d, "\wedge^*"'] & \mathsf{SymSp} \\ \mathcal{G}_*\text{-}\mathsf{sSet} & \end{tikzcd}$$

in (9.3.17) are symmetric monoidal sSet_**-functors.*

Proof. Since

$$\mathsf{K}^{\mathcal{F}} = \mathsf{K}^{\mathcal{G}} \circ \wedge^*,$$

by Theorem 9.4.9 it suffices to show that $\wedge^*$ is a symmetric monoidal sSet_*-functor. Since each of $\mathcal{F}$ and $\mathcal{G}$ is given the pointed unitary enrichment over sSet_* (Definition 4.3.11), as in the first paragraph of the proof of Theorem 9.4.9, it suffices to show that $\wedge^*$ is a symmetric monoidal functor. By Theorem 3.7.28, it suffices to show that

$$\widehat{\mathcal{G}} \xrightarrow{\wedge} \widehat{\mathcal{F}}$$

is a symmetric monoidal sSet_*-functor. Once again, by the pointed unitary enrichment, it remains to observe that

$$\mathcal{G} \xrightarrow{\wedge} \mathcal{F}$$

in Definition 9.1.15 is a strict symmetric monoidal functor that preserves the null objects. □

9.5. Notes

9.5.1 (References)**.** The category $\mathcal{G}$ in Definition 9.1.7 and the associated K-theory functor $\mathsf{K}^{\mathcal{G}}$ in Definition 9.3.14 were first developed in [**EM06, EM09**]. Theorem 9.2.15, which follows from the more general Theorem 4.3.37, is stated as [**EM09**, Theorem 5.6]. A similar construction of symmetric spectra from certain diagram categories is given in [**BO20**] (see Note 9.5.3 below). ⋄

9.5.2 ($\mathcal{F}^{(0)}$)**.** The category of small pointed categories and pointed functors, Cat_*, has a symmetric monoidal smash product whose unit E is a discrete category with two objects. So the category $\mathcal{F}^{(0)}$ of Definition 9.1.3 is not the monoidal unit of $(\mathsf{Cat}_*, \wedge)$ and, therefore, not the smash product over the empty indexing set in $(\mathsf{Cat}_*, \wedge)$. However, $\mathcal{F}^{(0)}$ is the monoidal unit for the smash product in Cat_0, the category of pointed small categories whose basepoints are initial and terminal, together with pointed functors. All of the other smash powers $\mathcal{F}^{(q)}$ for $q > 0$ are the same whether taken in Cat_* or Cat_0. Further discussion of Cat_0 as a module category is given in [**EM09**, Section 5], and this perspective is used in the definition of $\mathcal{G}$ as a Grothendieck construction in Note 9.5.3. $\diamond$

9.5.3 (Alternate Description of $\mathcal{G}$)**.** For those readers familiar with Grothendieck constructions (see e.g., [**BW12**, Chapter 12] or [**JY21**, Chapter 10]), the category $\mathcal{G}$ of Definition 9.1.7 can be described similarly.

Recall from Definition 9.1.2 that Inj denotes the category whose objects are unpointed finite sets $\overline{q} = \{1, \ldots, q\}$ for $q \geq 0$ and whose morphisms are injections. As in Note 9.5.2, let Cat_0 denote the category of small pointed categories whose basepoints are both initial and terminal, together with pointed functors. Then, as described further in Note 9.5.2,

$$\mathcal{F}^{(-)} : \mathsf{Inj} \longrightarrow \mathsf{Cat}_0$$

defines a functor that sends an object $\overline{p}$ to the p-fold smash power in Cat_0 and sends an injection $f : \overline{p} \hookrightarrow \overline{q}$ to the functor f_* defined in Definition 9.1.7. Then $\mathcal{G}$ is the Grothendieck construction

$$\int_{\mathsf{Inj}} \mathcal{F}^{(-)}$$

formed in Cat_0. This construction is similar to the usual Grothendieck construction, but uses the coproduct (wedge) and smash product in Cat_0 instead of the disjoint union and Cartesian product in Cat. This is the definition given in [**EM09**, Section 5], denoted $\mathcal{G}_*$ there.

Alternatively, one can make a Grothendieck construction in $(\mathsf{Cat}, \times)$

$$\mathcal{H} = \int_{\mathsf{Inj}} \mathcal{F}^{\times(-)},$$

where $\mathcal{F}^{\times(q)}$ denotes the q-fold Cartesian product. Then $\mathcal{G}$ can be defined as a quotient of $\mathcal{H}$, identifying all tuples $\langle \underline{n} \rangle$ having an entry $\underline{0}$ to a basepoint $*$ and all morphisms factoring through such tuples to the zero morphism in $\mathcal{G}_*$. This description is given in [**EM06**]; see Note 9.5.4 for an explanation of the notation change. Bohmann and Osorno, in [**BO20**, Section 5], use a similar Grothendieck construction, denoted $\mathcal{E}$, over Cartesian products of Δ^{op} in Cat. They then define $\mathcal{E}_*$-objects as diagrams that factor through a quotient similar to that defining $\mathcal{G}$ from $\mathcal{H}$. $\diamond$

9.5.4 (Notation Warning for $\mathcal{G}_*$)**.** We warn the reader comparing with the work of Elmendorf-Mandell [**EM06, EM09**] that the category we denote $\mathcal{G}$ in Definition 9.1.7 is denoted $\mathcal{G}_*$ in those references. Moreover, those references use $\mathcal{G}$ to denote what we call $\mathcal{H}$ in Note 9.5.3. We have made this change of notation because the category we call $\mathcal{G}$ is the one that parallels most closely the skeleton of finite sets, $\mathcal{F}$. Moreover, then we can reserve the notation $\mathcal{G}_*$ for the pointed diagram categories $\mathcal{G}_*$-Cat and $\mathcal{G}_*$-sSet. See Note 10.8.5 for further comparison of notation between [**EM06, EM09**] and this work. $\diamond$

9.5.5 (Generality of $\mathsf{K}^{\mathcal{G}}$)**.** In Theorem 9.4.9, we showed that the K-theory construction

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$$

is unital symmetric monoidal as a sSet_*-functor. This theorem holds more generally with sSet_* replaced by the category sC_* of pointed simplicial objects in a complete, cocomplete, and Cartesian closed category C.

- The relevant structure on $\mathcal{G}_*$-C, the category of $\mathcal{G}_*$-objects in C, is still given by Theorem 9.2.15.
- Instead of symmetric spectra based on pointed simplicial sets (Explanation 7.4.6), one considers symmetric spectra based on sC_* as defined in, for example, [**EM06**, 7.1].

The constructions and proof of Theorem 9.4.9 translate to this more general setting essentially verbatim. This more general form of the theorem implies [**EM06**, 7.4], which says that $\mathsf{K}^{\mathcal{G}}$, denoted by I there, is an enriched multifunctor. $\diamond$

CHAPTER 10

Elmendorf-Mandell K-Theory of Permutative Categories

This is the second of two chapters generalizing the Segal K-theory functor

$$\mathsf{K}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp},$$

from Chapter 8, to a simplicially-enriched multifunctor

$$\mathsf{K}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp},$$

due to Elmendorf-Mandell **[EM06, EM09]**. Chapter 9 describes the sSet_*-enriched symmetric monoidal categories and functors

$$\mathcal{G}_*\text{-}\mathsf{Cat} \xrightarrow{N_*} \mathcal{G}_*\text{-}\mathsf{sSet} \xrightarrow{\mathsf{K}^{\mathcal{G}}} \mathsf{SymSp}.$$

In this chapter we describe $\mathcal{G}_*$-diagrammatic replacements of the partition multicategory $\mathcal{M}$ and its associated J-theory,

$$\begin{tikzcd} \mathsf{PermCat}^{\mathsf{su}} \arrow[r, "\mathsf{End}"] \arrow[rr, bend left, "\mathsf{J}^{\mathsf{Se}}"] & \mathsf{Multicat}_* \arrow[r, "\mathsf{J}^{\mathcal{M}}"] & \Gamma\text{-}\mathsf{Cat} \end{tikzcd}$$

with a partition multicategory for tuples, $\mathcal{T}$, and an associated J-theory

$$\begin{tikzcd} \mathsf{PermCat}^{\mathsf{su}} \arrow[r, "\mathsf{End}"] \arrow[rr, bend left, "\mathsf{J}^{\mathsf{EM}}"] & \mathsf{Mod}^{\mathcal{M}\underline{1}} \arrow[r, "\mathsf{J}^{\mathcal{T}}"] & \mathcal{G}_*\text{-}\mathsf{Cat}. \end{tikzcd}$$

It is important to note the change in the codomain of End (and the domain of $\mathsf{J}^{\mathcal{T}}$) from $\mathsf{Multicat}_*$ to $\mathsf{Mod}^{\mathcal{M}\underline{1}}$—the category of left modules over $\mathcal{M}\underline{1}$ (Example 8.4.5). We discuss the reasons for this change in the Literature subsection below. For now we only need to note that End factors through $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ (Lemma 10.2.14).

Theorem 10.6.10 shows that the two constructions

$$\mathsf{K}^{\mathsf{Se}} = \mathsf{K}^{\mathcal{F}} N_* \mathsf{J}^{\mathsf{Se}} \quad \text{and} \quad \mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathsf{EM}}$$

are naturally level-equivalent as functors from $\mathsf{PermCat}^{\mathsf{su}}$ to SymSp. But, as with $\mathcal{G}$ and $\mathsf{K}^{\mathcal{G}}$ in Chapter 9, we do more than simply replacing $\mathsf{J}^{\mathcal{M}}$ with $\mathsf{J}^{\mathcal{T}}$. We also describe the relevant enriched monoidal structures.

(1) The smash product of small pointed multicategories restricts to a symmetric monoidal product for $\mathsf{Mod}^{\mathcal{M}\underline{1}}$, with monoidal unit $\mathcal{M}\underline{1}$. With respect to this structure, $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is a Cat_*-enriched symmetric monoidal category (Definition 10.1.36).

(2) Theorem 10.3.17 shows that

$$\mathsf{J}^{\mathcal{T}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}$$

is symmetric monoidal as a Cat_*-enriched functor.

(3) Lemma 10.2.14 shows that

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

is a Cat-enriched multifunctor.

(4) Changing enrichments to sSet via the nerve, the Elmendorf-Mandell K-theory, K^{EM}, is the composite sSet-enriched multifunctor

$$\begin{tikzcd}
\mathsf{PermCat}^{\mathsf{su}} \arrow[r, "\mathsf{End}"] \arrow[rr, bend left, "\mathsf{J}^{\mathsf{EM}}"] & \mathsf{Mod}^{\mathcal{M}\underline{1}} \arrow[r, "\mathsf{J}^{\mathcal{T}}"] & \mathcal{G}_*\text{-}\mathsf{Cat} \arrow[r, "N_*"] & \mathcal{G}_*\text{-}\mathsf{sSet} \arrow[r, "\mathsf{K}^{\mathcal{G}}"] & \mathsf{SymSp}.
\end{tikzcd}$$

As a consequence of the enriched multifunctoriality, K^{EM} preserves operad actions. We state this result as Theorem 10.3.33 and apply it in Chapters 11, 12, and 13.

The following table summarizes the Segal and Elmendorf-Mandell K-theory constructions, extending the table given in the introduction to Chapter 8. Each N_* is induced by composition with the nerve functor

$$N : \mathsf{Cat} \longrightarrow \mathsf{sSet},$$

from Definition 7.2.3.

	$\mathsf{K}^{\mathsf{Se}} = \mathsf{K}^{\mathcal{F}} N_* \mathsf{J}^{\mathsf{Se}}$ (8.5.1)	$\mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathsf{EM}}$ (10.3.32)
indexing category	$\mathcal{F}$ (8.1.1)	$\mathcal{G}$ (9.1.7)
K-theory	$\mathsf{K}^{\mathcal{F}} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$ (8.2.5)	$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$ (9.3.14)
nerve	$N_* : \Gamma\text{-}\mathsf{Cat} \longrightarrow \Gamma\text{-}\mathsf{sSet}$ (8.1.18)	$N_* : \mathcal{G}_*\text{-}\mathsf{Cat} \longrightarrow \mathcal{G}_*\text{-}\mathsf{sSet}$ (9.2.19)
$\Gamma/\mathcal{G}_*$-categories	$(-)^{\mathcal{F}}, (-)^{\mathcal{F}}_{\mathsf{lax}}, (-)^{\mathcal{F}}_{\mathsf{co}}$ (8.3.14)	$(-)^{\mathcal{G}}_{\cong}, (-)^{\mathcal{G}}, (-)^{\mathcal{G}}_{\mathsf{co}}$ (10.4.20)
variants equiv.	(8.5.2)	(10.7.16, 10.7.19)
J-theory	$\mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \Gamma\text{-}\mathsf{Cat}$	$\mathsf{J}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}$
definition	$\mathsf{J}^{\mathsf{Se}} = \mathsf{J}^{\mathcal{M}} \circ \mathsf{End}$ (8.5.1)	$\mathsf{J}^{\mathsf{EM}} = \mathsf{J}^{\mathcal{T}} \circ \mathsf{End}$ (10.3.25)
End	$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}} \hookrightarrow \mathsf{Multicat}_*$ (5.1.11, 5.3.9, 10.2.14)	
partitions	$\mathcal{M} : \mathcal{F}^{\mathsf{op}} \longrightarrow \mathsf{Multicat}_*$ (8.4.1)	$\mathcal{T} : \mathcal{G}^{\mathsf{op}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$ (10.3.1)
partition J-theory	$\mathsf{J}^{\mathcal{M}} : \mathsf{Multicat}_* \longrightarrow \Gamma\text{-}\mathsf{Cat}$ (8.4.10)	$\mathsf{J}^{\mathcal{T}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}$ (10.3.9)
equiv. desc. J	$\mathsf{J}^{\mathsf{Se}} \cong (-)^{\mathcal{F}}_{\mathsf{lax}}$ (8.4.8)	$\mathsf{J}^{\mathsf{EM}} \cong (-)^{\mathcal{G}}$ (10.5.1)
level equiv.	$\mathsf{K}^{\mathsf{Se}} \xrightarrow{\sim} \mathsf{K}^{\mathsf{EM}}$ (10.6.10)	

Organization. Sections 10.1 and 10.2 discuss smash products of the partition multicategories $\mathcal{M}a$ from Section 8.4. It follows from Proposition 10.1.6 that $\mathcal{M}\underline{1}$ is a commutative monoid with respect to this product. Important properties of $\mathcal{M}\underline{1}$-modules are discussed in the remainder of Section 10.1 and in Section 10.2.

Section 10.3 defines a partition multifunctor for $\mathcal{G}$ via smash products of the partition multicategories $\mathcal{M}\underline{n}$. This is denoted $\mathcal{T}$ and defines a partition J-theory

$$\mathsf{J}^{\mathcal{T}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}.$$

The main result of this section is Theorem 10.3.17, which shows that $\mathsf{J}^{\mathcal{T}}$ is symmetric monoidal as a Cat_*-enriched functor. Combining this with the previous material from Chapter 9, we define the Elmendorf-Mandell J- and K-theory multifunctors in Definitions 10.3.25 and 10.3.32, respectively.

Section 10.4 defines $\mathcal{G}_*$-categories $\mathsf{C}^{\mathcal{G}}$, including lax, strong, and colax variants (Definition 10.4.20), that generalize the Segal Γ-categories $\mathsf{C}^{\mathcal{F}}$ from Section 8.3. Section 10.5 shows that the constructions $\mathsf{J}^{\mathsf{EM}}\mathsf{C}$ and $\mathsf{C}^{\mathcal{G}} = \mathsf{C}^{\mathcal{G}}_{\mathsf{lax}}$ give naturally isomorphic $\mathcal{G}_*$-categories (Proposition 10.5.1). Definition 10.5.3 uses the strong and colax variants of $(-)^{\mathcal{G}}$ to define strong and colax Elmendorf-Mandell K-theory, denoted $\mathsf{K}^{\mathsf{EM}}_{\cong}$ and $\mathsf{K}^{\mathsf{EM}}_{\mathsf{co}}$.

Section 10.6 is devoted to proving that the Segal and Elmendorf-Mandell K-theory symmetric spectra are naturally level-equivalent constructions. The main result is Theorem 10.6.10, and depends on several preliminary comparisons for partition multicategories.

Section 10.7 compares the K-theory symmetric spectra given by the three variant constructions $\mathsf{C}^{\mathcal{G}} = \mathsf{C}^{\mathcal{G}}_{\mathsf{lax}}$, $\mathsf{C}^{\mathcal{G}}_{\cong}$, and $\mathsf{C}^{\mathcal{G}}_{\mathsf{co}}$. The main results are Theorems 10.7.16 and 10.7.19, which show that the resulting symmetric spectra are level equivalent.

Reading Guide. The material in this chapter depends on the smash product of pointed multicategories from Section 5.6 as well as the material on enriched multicategories from Chapter 6. A familiarity with the general definitions and main results will be useful. Throughout this chapter we give references to particular definitions and results from earlier material as needed.

(1) For $\mathcal{M}\underline{1}$-modules, read Definition 10.1.1 and the statement of Proposition 10.1.6, followed by Explanations 10.1.10 and 10.1.11. Then read Definition 10.1.12 and the important properties of $\mathcal{M}\underline{1}$-modules given in the statement of Proposition 10.1.28. Finally, read Definition 10.1.36 followed by Proposition 10.2.7, Definition 10.2.13, and the statement of Lemma 10.2.14.
(2) For the partition J-theory $\mathsf{J}^{\mathcal{T}}$, read Definition 10.3.1, Explanation 10.3.2, and the statement of Theorem 10.3.17.
(3) For J^{EM} and K^{EM}, read Definition 10.3.25 and the following text up through Definition 10.3.32. Read both the statement and the short proof of Theorem 10.3.33. Then read the statement of Theorem 10.6.10 for the level equivalence between K^{Se} and K^{EM}.
(4) For the reader who is only interested in the further applications to E_n-monoidal structures, in Chapters 11, 12, and 13, the remaining content of this chapter is not necessary. Go back and finish reading Sections 10.1 through 10.3 and 10.6. Then go on to the subsequent chapters.
(5) For the reader who is interested in the $\mathcal{G}_*$-categories $\mathsf{C}^{\mathcal{G}}$ and variants, read all of Section 10.4, the statement of Proposition 10.5.1, and Definition 10.5.3. Then read the statements of Theorems 10.7.16 and 10.7.19.
(6) Go back and read the rest of this chapter.

The detailed proofs and variant definitions in this chapter and the previous one have several additional purposes.

Cat**-Enriched Multifunctors.** In the definitions of K^{Se} and K^{EM} via the respective composites

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{End}} \mathsf{Multicat}_* \xrightarrow{\mathsf{J}^{\mathcal{M}}} \Gamma\text{-}\mathsf{Cat} \xrightarrow{N_*} \Gamma\text{-}\mathsf{sSet} \xrightarrow{\mathsf{K}^{\mathcal{F}}} \mathsf{SymSp}$$

and

$$\mathsf{PermCat^{su}} \xrightarrow{\mathsf{End}} \mathsf{Mod}^{\mathcal{M}\underline{1}} \xrightarrow{\mathsf{J}^{\mathcal{T}}} \mathcal{G}_*\text{-}\mathsf{Cat} \xrightarrow{N_*} \mathcal{G}_*\text{-}\mathsf{sSet} \xrightarrow{\mathsf{K}^{\mathcal{G}}} \mathsf{SymSp},$$

we see that almost all of the categories and functors are symmetric monoidal. The only exceptions are the following.

- There is not a symmetric monoidal structure on $\mathsf{PermCat^{su}}$ inherited from either $\mathsf{Multicat}_*$ (Proposition 5.7.23), or the proper subcategory $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ (Proposition 10.2.17). It is merely a Cat-enriched multicategory.
- As a consequence of the previous item, End is merely a Cat-enriched multifunctor. This is why $\mathsf{K^{EM}}$ is defined as a multifunctor instead of a symmetric monoidal functor.
- The monoidal constraint for $\mathcal{M}$ is not an isomorphism, and thus $\mathsf{J}^{\mathcal{M}}$ is not a symmetric monoidal functor (see Explanation 10.1.10 and Note 10.8.6). This is why, for our applications to K-theory of E_n-monoidal categories, we use $\mathsf{K^{EM}}$ instead of $\mathsf{K^{Se}}$.

Literature. We alluded above to the importance of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ as the domain of $\mathsf{J}^{\mathcal{T}}$. This is because there are subtle errors in [**EM09**] and subsequent literature regarding the domain of $\mathsf{J}^{\mathcal{T}}$. In our treatment we note the following.

(1) $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is a proper subcategory of $\mathsf{Multicat}_*$ (Examples 10.2.15 and 10.2.16). Notably, it does not contain the unit pointed multicategory, S.
(2) $\mathsf{J}^{\mathcal{T}}$ does *not* extend, as a monoidal functor, to all of $\mathsf{Multicat}_*$ (see Question A.5.1).

This final item is contrary to the statement of [**EM09**, Theorem 1.3] and some other assertions there. We give a careful account with corrections in Note 10.8.2. All of the incorrect statements are corrected by restricting $\mathsf{J}^{\mathcal{T}}$ to $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. The essential difficulty is the *unit constraint* for $\mathsf{J}^{\mathcal{T}}$, and we discuss it further in Note 10.8.3 and Question A.5.1.

We point out that the important structure result for $\mathcal{M}\underline{1}$-modules, Proposition 10.1.28, is also given in [**EM09**]. See Note 10.8.1 for further details about that result.

Lax versus Strong K-Theory. The lax construction $\mathsf{K^{EM}} = \mathsf{K^{EM}_{lax}}$ has a significant difference from the strong construction $\mathsf{K^{EM}_{\cong}}$ that is not present for $\mathsf{K^{Se}_{lax}}$ and $\mathsf{K^{Se}_{\cong}}$. First, in similarity with the case of Segal K-theory, Theorem 10.7.16 shows that there is a level equivalence

$$\mathsf{K^{EM}_{\cong}C} \longrightarrow \mathsf{K^{EM}_{lax}C}$$

for each small permutative category C. As noted in Explanation 10.7.23, the domain of definition for $\mathsf{K^{EM}_{\cong}}$ is the subcategory $\mathsf{PermCat^{sus}}$ consisting of strictly unital *strong* symmetric monoidal functors. All of these statements are similarly true for $\mathsf{K^{Se}_{lax}}$ and $\mathsf{K^{Se}_{\cong}}$.

Moreover, one obtains an equivalence of homotopy categories between $\mathsf{PermCat^{su}}$ and $\mathsf{PermCat^{sus}}$ upon inverting those morphisms that induce weak-equivalences on Segal K-theory spectra. These are discussed, for example, in [**Man10**, Theorem 3.9] and [**GJO17a**, Theorem 2.15]. These equivalences are obtained from general (2-)monadic strictification theory, and they imply that, for the purposes of Segal K-theory, there is no homotopy-theoretic difference between $\mathsf{PermCat^{su}}$ and $\mathsf{PermCat^{sus}}$.

However, such strictification functors generally do *not* preserve additional structure such as the multicategory structure present for small permutative categories. General monoidal strictification theory, such as that of [**WY19**, Theorems 4.4 and 4.6], does not apply because $\mathsf{PermCat}^{\mathsf{su}}$ does not have a monoidal structure.

Therefore, for the purposes of E_n-monoidal structures on Elmendorf-Mandell K-theory spectra, the difference between the domains of definition $\mathsf{PermCat}^{\mathsf{su}}$ and $\mathsf{PermCat}^{\mathsf{sus}}$ is a significant one. For example (see Explanation 10.7.23), small ring categories are monoids in $\mathsf{PermCat}^{\mathsf{su}}$. But monoids in $\mathsf{PermCat}^{\mathsf{sus}}$ are small ring categories with *invertible* factorization morphisms. A general small ring category C can be regarded as an object of $\mathsf{PermCat}^{\mathsf{sus}}$ by forgetting structure, but then one cannot detect the monoid structure on $\mathsf{K}^{\mathsf{Se}}_{\cong}\mathsf{C}$. The non-invertible factorization morphisms have been forgotten. Similar issues arise for other E_n-algebras in $\mathsf{PermCat}^{\mathsf{su}}$ and we discuss them in Question A.5.7.

10.1. The Partition Product

Recall the partition multicategory $\mathcal{M}$ of Definition 8.4.1, whose basepoint is the empty set

$$\varnothing \in 2^{a^\flat}.$$

Examples 8.4.4 through 8.4.6 give explicit descriptions of $\mathcal{M}\underline{0}$, $\mathcal{M}\underline{1}$, and $\mathcal{M}\underline{2}$. Recall from Definition 5.6.20 and Theorem 5.7.22 the smash product of pointed multicategories. In this section we record several details related to smash products of partition multicategories.

Definition 10.1.1. For a pair of finite pointed sets a and b, the Cartesian product of subsets induces a multifunctor

$$\prod_{a,b} : \mathcal{M}a \wedge \mathcal{M}b \longrightarrow \mathcal{M}(a \wedge b) \tag{10.1.2}$$

called the *partition product*. The assignment on objects is given by the Cartesian product of subsets, noting that

$$s \times t \subset (a^\flat \times b^\flat) \cong (a \wedge b)^\flat \quad \text{for} \quad s \subset a^\flat \quad \text{and} \quad t \subset b^\flat.$$

Recalling Explanation 5.6.14, the generating operations of

$$\mathcal{M}a \otimes \mathcal{M}b$$

are of the form

$$\iota_{\langle s \rangle} \otimes t \in \mathcal{M}a\big(\langle s \rangle\, ;\, s'\big) \times \{t\} \quad \text{and} \quad s \otimes \iota_{\langle t \rangle} \in \{s\} \times \mathcal{M}b\big(\langle t \rangle\, ;\, t'\big)$$

where

- $s \subset a^\flat$, $t \subset b^\flat$,
- $\langle s \rangle$ is a partition of s' in $\mathcal{M}a$, and
- $\langle t \rangle$ is a partition of t' in $\mathcal{M}b$.

In this context we define partitions of $s' \times t$ and $s \times t'$ by, respectively,

$$\langle s \rangle \times t = \langle s_i \times t \rangle_i \quad \text{and} \quad s \times \langle t \rangle = \langle s \times t_j \rangle_j.$$

Then we define $\prod_{a,b}$ on generating operations by

$$\iota_{\langle s \rangle} \otimes t \longmapsto \iota_{\langle s \rangle \times t} \quad \text{and} \quad s \otimes \iota_{\langle t \rangle} \longmapsto \iota_{s \times \langle t \rangle}.$$

This assignment preserves the interchange relation of the tensor product as in (5.6.16) because the two composites

$$
\begin{tikzcd}
\langle s\rangle \otimes \langle t\rangle \arrow[r, "\iota_{\langle s\rangle} \otimes \langle t\rangle"] & s' \times \langle t\rangle \arrow[dr, "s' \otimes \iota_{\langle t\rangle}"] & \\
& & (s', t') \\
\langle s\rangle \otimes^{\mathsf{t}} \langle t\rangle \arrow[r, "\langle s\rangle \otimes \iota_{\langle t\rangle}"'] & \langle s\rangle \times t' \arrow[ur, "\iota_{\langle s\rangle} \otimes t'"'] &
\end{tikzcd}
\tag{10.1.3}
$$

are sent to

$$\iota_{\langle s\rangle \otimes \langle t\rangle} \quad \text{and} \quad \iota_{\langle s\rangle \otimes^{\mathsf{t}} \langle t\rangle},$$

respectively, and these are interchanged by the action of the transposition reindexing denoted $\xi^{\otimes}$ in Definition 5.6.10 and Explanation 5.6.14.

The definition of $\prod_{a,b}$ descends to the smash product of $\mathcal{M}a$ and $\mathcal{M}b$ because the Cartesian product of any set with the empty set is again empty. Therefore, each generating operation of the form

$$\iota_{\langle s\rangle} \times \varnothing \quad \text{or} \quad \varnothing \times \iota_{\langle t\rangle}$$

is sent to a partition of the empty set, the basepoint of $\mathcal{M}(a \wedge b)$.

Each of the axioms in Definition 5.1.12 for $F = \prod_{a,b}$ consists of diagrams where either the common domain of the two composites is empty and there is nothing to verify, or the common domain is a singleton and, therefore, so is each of the other operation sets in the diagram. $\diamond$

Lemma 10.1.4. *The partition product components $\prod_{a,b}$ are natural with respect to morphisms of pointed finite sets.*

Proof. Suppose given pointed finite sets and morphisms

$$\psi : a \longrightarrow c \quad \text{and} \quad \phi : b \longrightarrow d.$$

Recalling Proposition 8.4.7, we have

$$\tilde{\psi} = \psi^{-1} : 2^{c^b} \longrightarrow 2^{a^b} \quad \text{and} \quad \tilde{\phi} = \phi^{-1} : 2^{d^b} \longrightarrow 2^{b^b}.$$

The following diagram of sets commutes, where the horizontal morphisms are given by Cartesian products of subsets.

$$
\begin{tikzcd}
2^{c^b} \times 2^{d^b} \arrow[r, "- \times -"] \arrow[d, "\tilde{\psi} \times \tilde{\phi}"'] & 2^{(c \wedge d)^b} \arrow[d, "\widetilde{\psi \wedge \phi}"] \\
2^{a^b} \times 2^{b^b} \arrow[r, "- \times -"] & 2^{(a \wedge b)^b}
\end{tikzcd}
$$

Therefore, the following diagram of multicategories commutes on objects.

$$
\begin{tikzcd}
\mathcal{M}c \times \mathcal{M}d \arrow[r, "\prod"] \arrow[d, "\tilde{\psi} \times \tilde{\phi}"'] & \mathcal{M}(c \wedge d) \arrow[d, "\widetilde{\psi \wedge \phi}"] \\
\mathcal{M}a \times \mathcal{M}b \arrow[r, "\prod"] & \mathcal{M}(a \wedge b)
\end{tikzcd}
\tag{10.1.5}
$$

Commutativity of the above diagram on generating operations of the form

$$\iota_{\langle s\rangle} \wedge t \quad \text{or} \quad s \wedge \iota_{\langle t\rangle}$$

follows, as for objects, because taking inverse images commutes with taking Cartesian products. This finishes the proof of naturality for $\prod$. $\square$

Proposition 10.1.6. *The partition multicategory $\mathcal{M}$ defines a symmetric monoidal functor*

$$(\mathcal{F}^{\mathsf{op}}, \wedge, \underline{1}) \longrightarrow (\mathsf{Multicat}_*, \wedge, \mathsf{S}).$$

Proof. The unit constraint

$$\mathcal{M}^0 : \mathsf{S} \longrightarrow \mathcal{M}\underline{1} \tag{10.1.7}$$

is the pointed multifunctor determined by sending the unique object of I to $\{1\} \subset \underline{1}^\flat$. The monoidal constraint is given by the composite of the partition product $\prod$ with the lexicographic isomorphisms

$$\mathcal{M}^2_{\underline{m},\underline{n}} : \mathcal{M}\underline{m} \wedge \mathcal{M}\underline{n} \xrightarrow{\prod_{\underline{m},\underline{n}}} \mathcal{M}(\underline{m} \wedge \underline{n}) \cong \mathcal{M}(\underline{mn}). \tag{10.1.8}$$

Naturality of $\mathcal{M}^2$ is given by that of $\prod$ (Lemma 10.1.4).

The associativity axiom (1.1.9) for $\mathcal{M}^2 = \prod$ is that the following diagram commutes for each triple of objects $\underline{\ell}$, $\underline{m}$, and $\underline{n}$ in $\mathcal{F}$.

$$\begin{tikzcd}
(\mathcal{M}\underline{\ell} \wedge \mathcal{M}\underline{m}) \wedge \mathcal{M}\underline{n} \arrow[r, "\alpha"] \arrow[d, "\prod \wedge 1"'] & \mathcal{M}\underline{\ell} \wedge (\mathcal{M}\underline{m} \wedge \mathcal{M}\underline{n}) \arrow[d, "1 \wedge \prod"] \\
\mathcal{M}(\underline{\ell} \wedge \underline{m}) \wedge \mathcal{M}\underline{n} \arrow[d, "\prod"] & \mathcal{M}\underline{\ell} \wedge \mathcal{M}(\underline{m} \wedge \underline{n}) \arrow[d, "\prod"] \\
\mathcal{M}((\underline{\ell} \wedge \underline{m}) \wedge \underline{n}) \arrow[r, "\mathcal{M}(\alpha^{-1})"] & \mathcal{M}(\underline{\ell} \wedge (\underline{m} \wedge \underline{n}))
\end{tikzcd} \tag{10.1.9}$$

Commutativity of the above diagram uses associativity of the Cartesian product for subsets. Generating operations of the form

$$(\iota_{\langle s\rangle} \otimes t) \otimes v, \quad (s \otimes \iota_{\langle t\rangle}) \otimes v, \quad \text{and} \quad ((s,t)) \otimes \iota_{\langle v\rangle}$$

are sent, via either composite of (10.1.9), to

$$\iota_{\langle s\rangle \times (t \times v)}, \quad \iota_{s \times (\langle t\rangle \times v)}, \quad \text{and} \quad \iota_{s \times (t \times \langle v\rangle)},$$

respectively.

The left unity axiom (1.1.10) follows because

$$\textstyle\prod_{\underline{1},\underline{n}} : \mathcal{M}\underline{1} \wedge \mathcal{M}\underline{n} \longrightarrow \mathcal{M}\underline{n}$$

sends the object $(\{1\}, s)$ to $1 \times s \cong s \subset \underline{n}^\flat$. The right unity axiom is similar. The braiding axiom (1.1.18) follows by symmetry of the Cartesian product. $\square$

Explanation 10.1.10. The monoidal constraint $\mathcal{M}^2$ determined by $\prod$ is generally far from an *isomorphism* of pointed multicategories, and thus $\mathcal{M}$ is not *strong* as a monoidal functor. This is the main reason for introducing the category $\mathcal{G}$ and the construction $\mathcal{T}$ in Definition 10.3.1 below. $\diamond$

Explanation 10.1.11 ($\mathcal{M}\underline{1}$-modules)**.** Since $\underline{1}$ is the monoidal unit in $\mathcal{F}$, it follows from Proposition 10.1.6 that $\mathcal{M}\underline{1}$ is a commutative monoid in $\mathsf{Multicat}_*$. The unit morphism

$$\eta = \mathcal{M}^0 : \mathsf{S} \longrightarrow \mathcal{M}\underline{1}$$

is the unit constraint (10.1.7). The multiplication morphism is given by

$$\pi = \textstyle\prod_{\underline{1},\underline{1}} : \mathcal{M}\underline{1} \wedge \mathcal{M}\underline{1} \longrightarrow \mathcal{M}\underline{1}.$$

For each pointed finite set b, the partition product multifunctors

$$\begin{aligned} \textstyle\prod_{\underline{1},b} &: \mathcal{M}\underline{1} \wedge \mathcal{M}b \longrightarrow \mathcal{M}(\underline{1} \wedge b) \cong \mathcal{M}b \\ \textstyle\prod_{b,\underline{1}} &: \mathcal{M}b \wedge \mathcal{M}\underline{1} \longrightarrow \mathcal{M}(b \wedge \underline{1}) \cong \mathcal{M}b \end{aligned}$$

make $\mathcal{M}b$ a bimodule over $\mathcal{M}\underline{1}$. ◇

Definition 10.1.12. The 2-category of left $\mathcal{M}\underline{1}$-modules, denoted $\mathsf{Mod}^{\mathcal{M}\underline{1}}$, has objects and 1-cells given by modules and module morphisms as in Proposition 7.5.1. For $\mathcal{M}\underline{1}$-modules N and N' together with $\mathcal{M}\underline{1}$-module morphisms

$$F, F' : \mathsf{N} \longrightarrow \mathsf{N}',$$

the set of $\mathcal{M}\underline{1}$-module 2-cells consists of pointed multinatural transformations

$$\theta : F \longrightarrow F'$$

such that the whiskerings indicated in the following diagram are equal, where μ denotes the module structure for both N and N'.

(10.1.13)
$$\begin{tikzcd} \mathcal{M}\underline{1} \wedge \mathsf{N} \arrow[rr, bend left, "1F"] \arrow[rr, bend right, "1F'"'] \arrow[rr, phantom, "1\theta\Downarrow"] \arrow[d, "\mu"'] & & \mathcal{M}\underline{1} \wedge \mathsf{N}' \arrow[d, "\mu"] \\ \mathsf{N} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "F'"'] \arrow[rr, phantom, "\theta\Downarrow"] & & \mathsf{N}' \end{tikzcd}$$

Identities and compositions in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ are given by those of $\mathsf{Multicat}_*$.

We will use the same notation for the underlying 1-category of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. In Proposition 10.1.28 below we show that $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is a full sub-2-category of $\mathsf{Multicat}_*$, with the same monoidal product. This defines a Cat_*-enriched symmetric monoidal structure on $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ in Definition 10.1.36. ◇

Corollary 10.1.14. *The category of $\mathcal{M}\underline{1}$-modules is complete and cocomplete.*

Proof. The result follows from Proposition 7.5.1 (2) together with Proposition 7.5.3 and Explanation 7.5.4. □

We will use the following special cases below.

Lemma 10.1.15. *For each pointed finite set b, the partition products for $\underline{1}$ and b are isomorphisms*

(10.1.16)
$$\begin{aligned} \textstyle\prod_{\underline{1},b} &: \mathcal{M}\underline{1} \wedge \mathcal{M}b \xrightarrow{\cong} \mathcal{M}(\underline{1} \wedge b) \cong \mathcal{M}b \\ \textstyle\prod_{b,\underline{1}} &: \mathcal{M}b \wedge \mathcal{M}\underline{1} \xrightarrow{\cong} \mathcal{M}(b \wedge \underline{1}) \cong \mathcal{M}b. \end{aligned}$$

Proof. We will show that $\prod_{\underline{1},b}$ is an isomorphism. The assertion for $\prod_{b,\underline{1}}$ then follows because $\mathcal{M}$ is symmetric monoidal (Proposition 10.1.6). If $b = \underline{0}$, there is nothing to prove because both sides of (10.1.16) are isomorphic to the terminal multicategory, T. For the remainder of the proof we assume that b has more than one element.

For the remainder of this proof we write $\prod$ for $\prod_{\underline{1},b}$. We recall and extend the notation of Example 8.4.5 as follows for $t \subset b^\flat$.

- Let
$$(t)_j^n = (\varnothing, \ldots, t, \ldots, \varnothing)$$
denote the n-tuple with all entries $\varnothing$ except for t in position j.
- Let $(\varnothing)^n$ denote the n-tuple with all entries $\varnothing$.
- For $b = \underline{1}$, let $(1)_j^n = (\{1\})_j^n$.
- The nonempty sets of operations in $\mathcal{M}\underline{1}$ are singletons
$$\mathcal{M}\underline{1}\big((\varnothing)^n\,;\varnothing\big) = \{\iota^n\}$$
$$\mathcal{M}\underline{1}\big((1)_j^n\,;\{1\}\big) = \{\pi_j^n\}.$$
We will also write $\pi_j^n(1) = \pi_j^n$.
- Define operations ι^n and $\pi_j^n(t)$ in $\mathcal{M}b$ as
$$\mathcal{M}b\big((\varnothing)^n\,;\varnothing\big) = \{\iota^n\}$$
$$\mathcal{M}b\big((t)_j^n\,;t\big) = \{\pi_j^n(t)\}.$$
- We use $\wedge$ instead of $\otimes$ to denote operations in $\mathcal{M}\underline{1} \wedge \mathcal{M}b$.

The definition of $\prod$ via Cartesian products of subsets shows that it is bijective on objects. The nontrival generating operations of $\mathcal{M}\underline{1} \wedge \mathcal{M}b$, and their images under $\prod$, are

$$\begin{aligned} &\{1\} \wedge \iota_{\langle u\rangle} \longmapsto \iota_{\langle u\rangle} \quad \text{for} \quad \langle u\rangle \text{ a partition of } u' \subset b^\flat \\ &\quad\text{and} \\ &\pi_j^n(1) \wedge t \longmapsto \pi_j^n(t) \quad \text{for} \quad t \subset b^\flat. \end{aligned}$$

This shows that $\prod$ is surjective on operations. To show that $\prod$ is injective on operations, we need to show

$$\pi_j^n(1) \wedge t = \{1\} \wedge \pi_j^n(t) \quad \text{for} \quad 1 \le j \le n \quad \text{and} \quad t \subset b^\flat. \tag{10.1.17}$$

Using the interchange relation (10.1.3) in $\mathcal{M}\underline{1} \otimes \mathcal{M}b$ with

$$\begin{aligned} \langle s\rangle &= (1)_j^n & s' &= \{1\} & \iota_{\langle s\rangle} &= \pi_j^n(1) \\ \langle t\rangle &= (t)_j^n & t' &= t & \iota_{\langle t\rangle} &= \pi_j^n(t), \end{aligned}$$

we see that the composites below are identified under the action of $\xi^\otimes$.

$$\begin{tikzcd} (1)_j^n \otimes (t)_j^n \arrow[r, "\pi_j^n(1)\otimes(t)_j^n"] & \{1\}\times(t)_j^n \arrow[dr, "\{1\}\otimes\pi_j^n(t)"] & \\ & & (\{1\},t) \\ (1)_j^n \otimes^{\mathsf{t}} (t)_j^n \arrow[r, "(1)_j^n\otimes\pi_j^n(t)"'] & (1)_j^n \times t \arrow[ur, "\pi_j^n(1)\otimes t"'] & \end{tikzcd} \tag{10.1.18}$$

Descending to the smash product $\mathcal{M}\underline{1} \wedge \mathcal{M}b$, each of the leftmost tuples of objects is sent to the n^2-tuple whose only non-basepoint entry is $(\{1\}, t)$ in position

$$n(j-1) + j \in \overline{n \cdot n} \quad \leftrightarrow \quad (j,j) \in \overline{n} \times \overline{n},$$

with the correspondence given by the lexicographic ordering (Definition 8.1.5). The other entries are basepoints, using the equalities

$$(\{1\}, \varnothing) = (\varnothing, \varnothing) = (\varnothing, t)$$

in the smash product. Likewise, when descending to $\mathcal{M}\underline{1} \wedge \mathcal{M}b$, the upper middle and lower middle objects are both sent to the n-tuple whose only non-basepoint entry is $(\{1\}, t)$. Letting

$$(1,t)^{n^2}_{n(j-1)+j} \quad \text{and} \quad (1,t)^n_j$$

denote these tuples, (10.1.18) becomes the following in $\mathcal{M}\underline{1} \wedge \mathcal{M}b$, where we use $\wedge$ in place of $\otimes$ and $\times$.

$$\begin{tikzcd}
(1)^n_j \wedge (t)^n_j \arrow[r, "\pi^n_j(1) \wedge (t)^n_j"] \arrow[d, equal] & \{1\} \wedge (t)^n_j \arrow[dr, "\{1\} \wedge \pi^n_j(t)"] \arrow[d, equal] & \\
(1,t)^{n^2}_{n(j-1)+j} \arrow[d, equal] & (1,t)^n_j \arrow[d, equal] & (\{1\}, t) \\
(1)^n_j \wedge^{\mathsf{t}} (t)^n_j \arrow[r, "(1)^n_j \wedge \pi^n_j(t)"'] & (1)^n_j \wedge t \arrow[ur, "\pi^n_j(1) \wedge t"'] &
\end{tikzcd} \tag{10.1.19}$$

The two composites in (10.1.19) are equal by the interchange relation. For (10.1.17), we need to show that the two rightmost operations are equal.

To continue on, we make the following simplifications and notation.

- By equivariance in $\mathcal{M}\underline{1} \wedge \mathcal{M}b$, it suffices to prove (10.1.17) in the case $j = 1$.
- To avoid confusion with some additional notation below, we let $s = \{1\}$, $(s)^n_j = (1)^n_j$, and $\pi^n_j(s) = \pi^n_j(1)$.
- We let $\mathsf{P} = \mathcal{M}\underline{1} \wedge \mathcal{M}b$.
- To denote the basepoint of P, we use
$$* = (\varnothing, \varnothing) = (s, \varnothing) = (\varnothing, t).$$
- We let $(*)^n$ denote the n-tuple in P whose entries are all basepoints.
- The operations $\iota^n \wedge t$, $\iota^n \wedge \varnothing$, $s \wedge \iota^n$, and $\varnothing \wedge \iota^n$ are basepoint operations and hence are all equal in P. We let ι^n_{P} denote this operation, so
$$\iota^n_{\mathsf{P}} = \iota^n \wedge t = \iota^n \wedge \varnothing = s \wedge \iota^n = \varnothing \wedge \iota^n \in \mathsf{P}\big((*)^n; *\big). \tag{10.1.20}$$
- We let
$$\gamma\big(\phi; \psi_1, \ldots, \psi_m\big) = \phi \circ \langle \psi \rangle$$
denote composition of an m-ary operation ϕ with an m-tuple of operations $\langle \psi \rangle$ in P.
- We use $\gamma^{\mathcal{M}\underline{1}}$ and $\gamma^{\mathcal{M}b}$ similarly to denote composition in the indicated multicategories.

Now we show (10.1.17) in the case $j = 1$ and $n = 2$. We will use each of the following computations.

- In $\mathsf{P}\big((s,t)\,;\,(s,t)\big)$ we have

$$\gamma\big(\pi_1^2(s) \wedge t\,;\, 1_{(s,t)}\,,\, \iota^0 \wedge t\big) = \gamma^{\mathcal{M}\underline{1}}\big(\pi_1^2(s)\,;\, 1_s\,,\, \iota^0\big) \wedge t = 1_s \wedge t = 1_{(s,t)} \tag{10.1.21}$$

because $- \wedge t$ is a multifunctor from $\mathcal{M}\underline{1}$ to P.
- In $\mathsf{P}\big((\varnothing,\varnothing)\,;\,(s,\varnothing)\big) = \mathsf{P}\big(*\,;\,*\big)$ we have

$$\gamma\big(\pi_1^2(s) \wedge \varnothing\,;\, s \wedge \iota^0\,,\, 1_{(\varnothing,\varnothing)}\big) = \gamma\big(\iota_{\mathsf{P}}^2\,;\, \iota_{\mathsf{P}}^0\,,\, \iota_{\mathsf{P}}^1\big) = 1_* \tag{10.1.22}$$

because $\pi_1^2(s) \wedge \varnothing$ is equal to ι_{P}^2.
- In $\mathsf{P}\big((s,t)\,;\,(s,t)\big)$ we have

$$\gamma\big(s \wedge \pi_1^2(t)\,;\, 1_{(s,t)}\,,\, s \wedge \iota^0\big) = s \wedge \gamma^{\mathcal{M}b}\big(\pi_1^2(t)\,;\, 1_t\,,\, \iota^0\big) = s \wedge 1_t = 1_{(s,t)} \tag{10.1.23}$$

because $s \wedge -$ is a multifunctor from $\mathcal{M}b$ to P.
- In $\mathsf{P}\big((\varnothing,\varnothing)\,;\,(\varnothing,t)\big) = \mathsf{P}\big(*\,;\,*\big)$ we have

$$\gamma\big(\varnothing \wedge \pi_1^2(t)\,;\, \iota^0 \wedge t, 1_{(\varnothing,\varnothing)}\big) = \gamma\big(\iota_{\mathsf{P}}^2\,;\, \iota_{\mathsf{P}}^0, \iota_{\mathsf{P}}^1\big) = \iota_{\mathsf{P}}^1 = 1_* \tag{10.1.24}$$

because $\varnothing \wedge \pi_1^2(t)$ is equal to ι_{P}^2.

Next, we apply the above computations:

$$\begin{aligned}
s \wedge \pi_1^2(t) &= \gamma\big(s \wedge \pi_1^2(t)\,;\, 1_{(s,t)}\,,\, 1_*\big)\\
&\qquad\text{by right unity (5.1.5) in } \mathsf{P}\\
&= \gamma\Big(s \wedge \pi_1^2(t)\,;\, \gamma\big(\pi_1^2(s) \wedge t\,;\, 1_{(s,t)}\,,\, \iota^0 \wedge t\big)\,,\, \gamma\big(\pi_1^2(s) \wedge \varnothing\,;\, s \wedge \iota^0\,,\, 1_{(\varnothing,\varnothing)}\big)\Big)\\
&\qquad\text{by (10.1.21) and (10.1.22)}\\
&= \gamma\Big(\gamma\big(s \wedge \pi_1^2(t)\,;\, \pi_1^2(s) \wedge t\,,\, \pi_1^2(s) \wedge \varnothing\big)\,;\, 1_{(s,t)}\,,\, \iota^0 \wedge t\,,\, s \wedge \iota^0\,,\, 1_{(\varnothing,\varnothing)}\Big)\\
&\qquad\text{by associativity (5.1.4) in } \mathsf{P}\\
&= \gamma\Big(\gamma\big(\pi_1^2(s) \wedge t\,;\, s \wedge \pi_1^2(t)\,,\, \varnothing \wedge \pi_1^2(t)\big)\,;\, 1_{(s,t)}\,,\, s \wedge \iota^0\,,\, \iota^0 \wedge t\,,\, 1_{(\varnothing,\varnothing)}\Big)\\
&\qquad\text{by interchange (10.1.19) and } \iota_{\mathsf{P}}^0 \text{ equalities (10.1.20)}\\
&= \gamma\Big(\pi_1^2(s) \wedge t\,;\, \gamma\big(s \wedge \pi_1^2(t)\,;\, 1_{(s,t)}\,,\, s \wedge \iota^0\big)\,,\, \gamma\big(\varnothing \wedge \pi_1^2(t)\,;\, \iota^0 \wedge t\,,\, 1_{(\varnothing,\varnothing)}\big)\Big)\\
&\qquad\text{by associativity (5.1.4) in } \mathsf{P}\\
&= \gamma\big(\pi_1^2(s) \wedge t\,;\, 1_{(s,t)}\,,\, 1_*\big)\\
&\qquad\text{by (10.1.23) and (10.1.24)}\\
&= \pi_1^2(s) \wedge t\\
&\qquad\text{by right unity (5.1.5) in } \mathsf{P}.
\end{aligned}$$

This proves (10.1.17) in the case $j = 1$ and $n = 2$.

For $j = 1$ and $n \geq 3$ we note

$$\pi_1^n(s) = \gamma^{\mathcal{M}\underline{1}}\big(\pi_1^2(s)\,;\, \pi_1^2(s)\,,\, \iota^{n-2}\big) \quad \text{in} \quad \mathcal{M}\underline{1}\big(s, \underbrace{\varnothing, \ldots, \varnothing}_{(n-1)}\,;\, s\big) \tag{10.1.25}$$

and

$$\pi_1^n(t) = \gamma^{\mathcal{M}b}\big(\pi_1^2(t)\,;\, \pi_1^2(t)\,,\, \iota^{n-2}\big) \quad \text{in} \quad \mathcal{M}b\big(t, \underbrace{\varnothing, \ldots, \varnothing}_{(n-1)}\,;\, t\big). \tag{10.1.26}$$

Then, using the composites above and (10.1.17) for $n = 2$ we have

$$\begin{aligned}
s \wedge \pi_1^n(t) &= s \wedge \gamma^{\mathcal{M}b}\big(\pi_1^2(t)\,;\, \pi_1^2(t)\,,\, \iota^{n-2}\big) \\
&\qquad \text{by (10.1.26)} \\
&= \gamma\big(s \wedge \pi_1^2(t)\,;\, s \wedge \pi_1^2(t)\,,\, s \wedge \iota^{n-2}\big) \\
&\qquad \text{by multifunctoriality of } s \wedge - \\
&= \gamma\big(\pi_1^2(s) \wedge t\,;\, \pi_1^2(s) \wedge t\,,\, \iota^{n-2} \wedge t\big) \\
&\qquad \text{by the case } n = 2 \text{ and } \iota_{\mathsf{P}}^n \text{ equalities (10.1.20)} \\
&= \gamma^{\mathcal{M}\underline{1}}\big(\pi_1^2(s)\,;\, \pi_1^2(s)\,,\, \iota^{n-2}\big) \wedge t \\
&\qquad \text{by multifunctoriality of } - \wedge t \\
&= \pi_1^n(s) \wedge t \\
&\qquad \text{by (10.1.25).}
\end{aligned}$$

This proves (10.1.17) in the case $j = 1$ and $n \geq 3$. The case for general j follows by equivariance. This finishes the proof that $\prod_{\underline{1},b}$ is injective on operations, and thus finishes the proof that $\prod_{\underline{1},b}$ is an isomorphism of pointed multicategories. □

As a special case of Lemma 10.1.15, the multiplication for $\mathcal{M}\underline{1}$ is an isomorphism

$$\mathcal{M}\underline{1} \wedge \mathcal{M}\underline{1} \xrightarrow{\cong} \mathcal{M}\underline{1}. \tag{10.1.27}$$

This implies the following. See Note 10.8.4 for discussion of more general cases.

Proposition 10.1.28.

(1) Suppose N *is a left* $\mathcal{M}\underline{1}$*-module in* $\mathsf{Multicat}_*$*. Then the structure morphism*

$$\mu : \mathcal{M}\underline{1} \wedge \mathsf{N} \longrightarrow \mathsf{N}$$

is an isomorphism with inverse given by the unit

$$\mathsf{N} \xrightarrow[\cong]{\lambda^{-1}} \mathsf{S} \wedge \mathsf{N} \xrightarrow{\eta \wedge 1} \mathcal{M}\underline{1} \wedge \mathsf{N}$$

where λ *is the left unit isomorphism for* $\wedge$ *and* $\eta = \mathcal{M}^0$ *is the unit for* $\mathcal{M}\underline{1}$*.*

(2) Similarly, the structure morphism for a right module is an isomorphism with inverse given by $(1 \wedge \eta) \circ \rho^{-1}$*, where* ρ *is the right unit isomorphism for* $\wedge$*.*

(3) Each small pointed multicategory N *admits at most one left* $\mathcal{M}\underline{1}$*-module structure and one right* $\mathcal{M}\underline{1}$*-module structure.*

(4) The 2-category of left, respectively right, $\mathcal{M}\underline{1}$*-modules is a full sub-2-category of* $\mathsf{Multicat}_*$*.*

(5) For a left $\mathcal{M}\underline{1}$*-module* $(\mathsf{N}, \mu^{\mathsf{N}})$ *and a right* $\mathcal{M}\underline{1}$*-module* $(\mathsf{P}, \mu^{\mathsf{P}})$*, the two morphisms in* $\mathsf{Multicat}_*$

$$(\mathsf{P} \wedge \mathcal{M}\underline{1}) \wedge \mathsf{N} \underset{(1 \wedge \mu^{\mathsf{N}}) \circ \alpha}{\overset{\mu^{\mathsf{P}} \wedge 1}{\rightrightarrows}} \mathsf{P} \wedge \mathsf{N} \tag{10.1.29}$$

are equal and so the canonical morphism to the coequalizer

$$\mathsf{P} \wedge \mathsf{N} \longrightarrow \mathsf{P} \wedge_{\mathcal{M}\underline{1}} \mathsf{N}$$

is an isomorphism in $\mathsf{Multicat}_*$*.*

Proof. Let

$$\eta = \mathcal{M}^0 : \mathsf{S} \longrightarrow \mathcal{M}\underline{1} \quad \text{and} \quad \pi = \textstyle\prod_{\underline{1},\underline{1}} : \mathcal{M}\underline{1} \wedge \mathcal{M}\underline{1} \longrightarrow \mathcal{M}\underline{1}$$

denote the unit and multiplication morphisms, respectively, for $\mathcal{M}\underline{1}$. As stated above, π is an isomorphism by Lemma 10.1.15 with $b = \underline{1}$. The left and right unit diagrams for η and π are the following

$$\begin{tikzcd}
\mathsf{S} \wedge \mathcal{M}\underline{1} \arrow[r, "\eta\wedge 1"] \arrow[dr, "\lambda"'] & \mathcal{M}\underline{1} \wedge \mathcal{M}\underline{1} \arrow[d, "\pi", "\cong"'] & \mathcal{M}\underline{1} \wedge \mathsf{S} \arrow[l, "1\wedge\eta"'] \arrow[dl, "\rho"] \\
& \mathcal{M}\underline{1} &
\end{tikzcd} \tag{10.1.30}$$

We note, therefore, that $(\eta \wedge 1)$ and $(1 \wedge \eta)$ in the above diagram are both isomorphisms.

To show (1), suppose N is a left $\mathcal{M}\underline{1}$-module. Applying $(- \wedge \mathsf{N})$ to (10.1.30) and using the inverse of $(1 \wedge \eta)$, we obtain the upper region of the following diagram. In this diagram we write

$$\mathsf{A} = \mathcal{M}\underline{1}$$

and α denotes the associativity isomorphism for $\wedge$. Here and below we write $\wedge$ as juxtaposition to save space.

$$\begin{tikzcd}
& & \mathsf{AN} \arrow[dr, "(\rho 1)^{-1}"] & \\
& (\mathsf{SA})\mathsf{N} \arrow[ur, "\lambda 1"] \arrow[r, "(\eta 1)1"] \arrow[d, "\alpha"] & (\mathsf{AA})\mathsf{N} \arrow[r, "((1\eta)1)^{-1}"] \arrow[d, "\alpha"] & (\mathsf{AS})\mathsf{N} \arrow[d, "\alpha"] \\
\mathsf{AN} \arrow[ur, "(\lambda 1)^{-1}"] \arrow[r, "(\lambda_{\mathsf{AN}})^{-1}"'] \arrow[d, "\mu"'] & \mathsf{S}(\mathsf{AN}) \arrow[r, "\eta(11)"] \arrow[d, "1\mu"] & \mathsf{A}(\mathsf{AN}) \arrow[d, "1\mu"'] & \mathsf{A}(\mathsf{SN}) \arrow[l, "(1(\eta 1))"'] \arrow[dl, "1\lambda"] \\
\mathsf{N} \arrow[r, "\lambda^{-1}"] & \mathsf{SN} \arrow[r, "\eta 1"] & \mathsf{AN} &
\end{tikzcd} \tag{10.1.31}$$

In the above diagram, the uppermost region commutes by (10.1.30). The triangle and two squares on the middle row commute by (1.1.5) and naturality of α. On the bottom row, the leftmost square commutes by naturality of λ and the remaining square commutes by functoriality of $\wedge$. The remaining triangle commutes by the unit condition for left module structure on N.

Now the composite in (10.1.31) from the leftmost vertex AN around the top and right to the bottom right vertex AN is an identity by invertibility of λ and the middle unity axiom (1.1.2). This shows that the composite

$$\mathsf{AN} \xrightarrow{\mu} \mathsf{N} \xrightarrow{\lambda^{-1}} \mathsf{SN} \xrightarrow{\eta 1} \mathsf{AN}$$

is the identity. The reverse composite

$$\mathsf{N} \xrightarrow{\lambda^{-1}} \mathsf{SN} \xrightarrow{\eta 1} \mathsf{AN} \xrightarrow{\mu} \mathsf{N}$$

is the identity by the left unit condition for N. This completes the proof of (1). The corresponding calculation for right module structure shows (2).

The remaining parts of this proof follow from (1) and (2). For assertion (3), suppose

$$\mu, \mu' : \mathcal{M}\underline{1} \wedge \mathsf{N} \longrightarrow \mathsf{N}$$

are the multiplication morphisms for two left module structures on N. Then by (1) both μ and μ' are isomorphisms with inverse given by the composite

$$(\eta \wedge 1) \circ \lambda^{-1} : N \longrightarrow \mathcal{M}\underline{1} \wedge N.$$

Therefore $\mu = \mu'$. This proves the uniqueness of left module structures. A similar argument using (2) proves the uniqueness of right module structures.

Now we show that assertion (4) for left modules follows from (1). A similar argument shows that the assertion for right modules follows from (2). By (3) the forgetful 2-functor U from left $\mathcal{M}\underline{1}$-modules to pointed multicategories is injective on objects. The forgetful 2-functor U is faithful on 1- and 2-cells because each left-module multifunctor and multinatural transformation is determined by its underlying data in $\mathsf{Multicat}_*$. Thus the 2-category of left $\mathcal{M}\underline{1}$-modules is identified with a sub-2-category of $\mathsf{Multicat}_*$.

Now suppose that N and N' are left $\mathcal{M}\underline{1}$-modules and suppose that

$$F : \mathsf{N} \longrightarrow \mathsf{N}' \quad \text{in} \quad \mathsf{Multicat}_*$$

is a pointed multifunctor. Naturality of λ and η implies that the following outer diagram commutes.

(10.1.32)
$$\begin{tikzcd}
\mathsf{AN} \arrow[r, "1F"] \arrow[d, bend left, "\mu"] & \mathsf{AN}' \arrow[d, bend right, "\mu"'] \\
\mathsf{N} \arrow[r, "F"] \arrow[u, bend left, "(\eta 1)\circ\lambda^{-1}"] & \mathsf{N}' \arrow[u, bend right, "(\eta 1)\circ\lambda^{-1}"']
\end{tikzcd}$$

By (1), the two morphisms in each column are inverses and so the innermost diagram also commutes. Thus F is an module morphism for $\mathsf{A} = \mathcal{M}\underline{1}$. The same argument for pointed multinatural transformations

$$\theta : F \longrightarrow F'$$

shows that the whiskerings indicated in the diagram below are equal, and hence θ is an module 2-cell for $\mathsf{A} = \mathcal{M}\underline{1}$.

(10.1.33)
$$\begin{tikzcd}
\mathsf{AN} \arrow[r, bend left, "1F"] \arrow[r, bend right, "1F'"'] \arrow[r, phantom, "1\theta\Downarrow"] \arrow[d, "\mu"'] & \mathsf{AN}' \arrow[d, "\mu"] \\
\mathsf{N} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\theta\Downarrow"] & \mathsf{N}'
\end{tikzcd}$$

Now we show that assertion (5) follows from (1) and (2). Again using naturality of α and middle unity, we have

$$\big((1\eta)1\big) \circ (\rho^{-1}1) = \Big(\alpha^{-1} \circ \big(1(\eta 1)\big) \circ (1\lambda^{-1})\Big).$$

By (1) and (2), this provides a common inverse for the two morphisms in (10.1.29). $\square$

Explanation 10.1.34. Lemma 10.1.15 for $b \neq \underline{1}$ follows from Proposition 10.1.28. However, the case $b = \underline{1}$ is required for the proof of Proposition 10.1.28. $\diamond$

The unit

$$\eta = \mathcal{M}^0 : \mathsf{S} \longrightarrow \mathcal{M}\underline{1}$$

is not an isomorphism. However, if N has a left $\mathcal{M}\underline{1}$-module structure, then $(\eta \wedge 1_{\mathsf{N}})$ is an isomorphism by Proposition 10.1.28 (1). Similarly, by Proposition 10.1.28 (2), $(1_{\mathsf{N}} \wedge \eta)$ is an isomorphism if N has a right $\mathcal{M}\underline{1}$-module structure. In Proposition 10.2.1 we show that the converse statements hold.

Explanation 10.1.35 (Sub 2-Category of $\mathcal{M}\underline{1}$-Modules)**.** For a commutative monoid A in a symmetric monoidal category C, the category of left A-modules is generally *not* a subcategory of C because a given object of C may admit several distinct A-module structures. The multiplication and unit are additional data of an A-module. For $\mathsf{A} = \mathcal{M}\underline{1}$, Proposition 10.1.28 shows that

- an $\mathcal{M}\underline{1}$-module structure on N is unique if it exists and
- pointed multifunctors and pointed multinatural transformations are automatically compatible with such module structures.

Thus $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is identified with a full sub-2-category of $\mathsf{Multicat}_*$.

Part (5) of Proposition 10.1.28 shows that the smash product of $\mathcal{M}\underline{1}$-modules is naturally isomorphic to the underlying smash product of pointed multicategories. In Definition 10.1.36 we use this to define the enriched symmetric monoidal structure of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. $\diamond$

Recall the descriptions of $(\mathsf{Multicat}, \otimes)$ and $(\mathsf{Multicat}_*, \wedge)$ as symmetric Cat-monoidal 2-categories from Theorems 6.4.3 and 6.4.4, respectively. By Proposition 10.1.28, the same verification of 2-naturality for the symmetric monoidal data applies to the sub-2-category of left $\mathcal{M}\underline{1}$-modules in $\mathsf{Multicat}_*$. We extend Definition 10.1.12 as follows.

Definition 10.1.36. Let

$$\left(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \wedge, \mathcal{M}\underline{1}\right)$$

denote the full Cat_*-enriched subcategory of left $\mathcal{M}\underline{1}$-modules in $\mathsf{Multicat}_*$, with hom categories

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}(\mathsf{N}, \mathsf{N}') = \mathsf{Multicat}_*(\mathsf{N}, \mathsf{N}').$$

By Proposition 10.1.28, the smash product and symmetry of $(\mathsf{Multicat}_*, \wedge)$ restrict to $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. With $\mathcal{M}\underline{1}$ as monoidal unit and unit isomorphisms given by the module structure and symmetry, these data make $(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \wedge)$ a symmetric monoidal Cat_*-category. Each of the axioms of Definition 1.4.13 for $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ follows from the corresponding axioms for $\mathsf{Multicat}_*$ because the forgetful inclusion

$$\mathsf{Mod}^{\mathcal{M}\underline{1}} \hookrightarrow \mathsf{Multicat}_* \tag{10.1.37}$$

is faithful on 2-cells. With this structure, the inclusion (10.1.37) is a symmetric monoidal Cat_*-functor whose monoidal constraint is the identity and whose unit constraint is given by

$$\eta = \mathcal{M}^0 : \mathsf{S} \longrightarrow \mathcal{M}\underline{1}. \qquad \diamond$$

10.2. Characterization of $\mathcal{M}\underline{1}$-Modules

As noted in Explanation 10.1.11, $\mathcal{M}\underline{n}$ is a bimodule over $\mathcal{M}\underline{1}$ for each $\underline{n}$ in $\mathcal{F}$. In Propositions 10.2.1 and 10.2.7 we give characterizations of $\mathcal{M}\underline{1}$-module structure. We use these to discuss further examples.

Proposition 10.2.1. *A pointed multicategory* N *is a left* $\mathcal{M}\underline{1}$*-module if and only if*

$$\mathsf{S} \wedge \mathsf{N} \xrightarrow{\eta \wedge 1} \mathcal{M}\underline{1} \wedge \mathsf{N}$$

is an isomorphism. Similarly, N *is a right* $\mathcal{M}\underline{1}$*-module if and only if* $1 \wedge \eta$ *is an isomorphism.*

Proof. We show that the statement for left modules follows from part (1) of Proposition 10.1.28 and the unit properties of the smash product $\wedge$. The statement for right modules follows likewise from Proposition 10.1.28 (2).

If N is a left $\mathcal{M}\underline{1}$-module, then $\eta \wedge 1$ is an isomorphism by Proposition 10.1.28 (1). For the converse, suppose that $\eta \wedge 1$ is an isomorphism and define a multiplication μ as the composite

$$\begin{tikzcd}
\mathcal{M}\underline{1} \wedge \mathsf{N} \arrow[rr, "\mu"] \arrow[dr, "(\eta \wedge 1)^{-1}"'] & & \mathsf{N} \\
& \mathsf{S} \wedge \mathsf{N} \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

Commutativity of the unity diagram of Definition 7.4.2 for μ is equivalent to that of the above diagram defining μ.

Commutativity of the associativity diagram for μ follows from commutativity of the outer diagram below, where we write $\wedge$ as juxtaposition, $\mathsf{A} = \mathcal{M}\underline{1}$, and $\pi = \Pi_{\underline{1},\underline{1}}$ is the monoid multiplication for $\mathcal{M}\underline{1}$.

$$\begin{tikzcd}
(\mathsf{AA})\mathsf{N} \arrow[rr, "\alpha"] \arrow[dr, "(1\eta)^{-1}1"] \arrow[ddr, bend right, "\pi 1"'] & & \mathsf{A}(\mathsf{AN}) \arrow[dl, "1(\eta 1)^{-1}"'] \arrow[ddl, bend left, "1\mu"] \\
& (\mathsf{AS})\mathsf{N} \arrow[r, "\alpha"] \arrow[d, "\rho 1"'] & \mathsf{A}(\mathsf{SN}) \arrow[dl, "1\lambda"] \\
& \mathsf{AN} &
\end{tikzcd}$$

In the above diagram, the upper trapezoid commutes by naturality of the associativity isomorphism α. The inner triangle commutes by the unity axiom (1.1.2) for $\wedge$. The region at left commutes by the right unit axiom for π and the region at right commutes by the definition of μ. Therefore the outer diagram commutes. $\square$

By Proposition 10.2.1, one way to check that an object P in $\mathsf{Multicat}_*$ is *not* a left $\mathcal{M}\underline{1}$-module is to check that $\eta \wedge 1_{\mathsf{P}}$ is not an isomorphism, and likewise for right $\mathcal{M}\underline{1}$-modules.

Example 10.2.2. Consider the monoidal unit (5.6.22)

$$\mathsf{S} = \mathsf{I} \coprod \mathsf{T}.$$

By naturality of the right unitor, $(\eta \wedge 1_{\mathsf{S}})$ is an isomorphism if and only if η is so. Since η is not an isomorphism, then neither is $(\eta \wedge 1_{\mathsf{S}})$. Therefore, by Proposition 10.2.1, S does not have a left $\mathcal{M}\underline{1}$-module structure. Likewise S does not have a right module structure over $\mathcal{M}\underline{1}$. See Example 10.2.16 for a separate argument regarding the non-existence of $\mathcal{M}\underline{1}$-module structure for S. ◇

Uniqueness of the module structures in Proposition 10.1.28 implies the following.

Corollary 10.2.3. *Each pointed multicategory* P *is a left* $\mathcal{M}\underline{1}$*-module if and only if it is a right* $\mathcal{M}\underline{1}$*-module. In this case, the left and right module actions commute with the symmetry of* $\wedge$ *as in the following diagram.*

$$\begin{tikzcd}
\mathcal{M}\underline{1} \wedge \mathsf{P} \arrow[rr, "\beta"] \arrow[dr] & & \mathsf{P} \wedge \mathcal{M}\underline{1} \arrow[dl] \\
& \mathsf{P} &
\end{tikzcd}$$

Proof. By naturality of the symmetry isomorphism for $\wedge$, we have that $(\eta \wedge 1_{\mathsf{P}})$ is an isomorphism if and only if $(1_{\mathsf{P}} \wedge \eta)$ is so. By Propositions 10.1.28 and 10.2.1 these determine unique left- and right-module structures. □

In the discussion below we use the following notation from Example 8.4.5:

$$\mathcal{M}\underline{1}\big(\langle\varnothing\rangle\,;\,\varnothing\big) = \{\iota^n\}$$
$$\mathcal{M}\underline{1}\big((\varnothing,\ldots,\{1\},\ldots,\varnothing)\,;\,\{1\}\big) = \{\pi_j^n\}$$

for $n \geq 0$ and $1 \leq j \leq n$. The input of ι^n above is the constant n-tuple whose entries are all the basepoint. The input of π_j^n above consists of a length-n tuple whose only non-basepoint entry is $\{1\}$ in position j. We will also use

$$\iota^n \in \mathsf{T}\big(\langle * \rangle\,;\,*\big)$$

to denote the unique n-ary operation in the terminal multicategory T, and the unique n-ary basepoint operation in any pointed multicategory.

Lemma 10.2.4. *Suppose* P *is a pointed multicategory with composition* γ*. A pointed multifunctor*

$$F : \mathcal{M}\underline{1} \longrightarrow \mathsf{P}$$

is uniquely determined by

- *an object* $x = F\{1\}$ *and*
- *an operation* $\pi_1^2(x) = F\pi_1^2$

such that

$$\gamma\big(\pi_1^2(x)\,;\,1_x,\iota^0\big) = 1_x.$$

Proof. Since F is a pointed multifunctor, we have

$$F\varnothing = * \quad \text{and} \quad F\iota^n = \iota^n \in \mathsf{P}\big(\langle * \rangle\,;\, *\big)$$

where $*$ denotes the basepoint of P. We let

$$x = F\{1\} \in \mathsf{P}.$$

The nontrivial operations of $\mathcal{M}\underline{1}$ are π_j^n. By equivariance, the value of F on each π_j^n is determined by that of π_1^n. Furthermore, for $n \geq 3$ we have

$$\pi_1^n = \gamma^{\mathcal{M}\underline{1}}\big(\pi_1^2\,;\, \pi_1^2\,,\, \iota^{n-2}\big) \quad \text{in} \quad \mathcal{M}\underline{1}\big(\{1\}, \underbrace{\varnothing, \ldots, \varnothing}_{(n-1)}\,;\, \{1\}\big).$$

Therefore, by multifunctoriality of F, each $F\pi_1^n$ is determined by $F\pi_1^2$. We define

$$\pi_1^2(x) = F\pi_1^2 \in \mathsf{P}\big(x, *\,;\, x\big).$$

Lastly, we also have

$$1_{\{1\}} = \gamma^{\mathcal{M}\underline{1}}\big(\pi_1^2\,;\, 1_{\{1\}}\,,\, \iota^0\big) \quad \text{in} \quad \mathcal{M}\underline{1}\big(\{1\}\,;\, \{1\}\big).$$

Applying F and using multifunctoriality together with preservation of basepoint operations, we have

$$\begin{aligned}
1_x = 1_{F\{1\}} = F(1_{\{1\}}) &= F\big(\gamma^{\mathcal{M}\underline{1}}\big(\pi_1^2\,;\, 1_{\{1\}}\,,\, \iota^0\big)\big) \\
&= \gamma\big(F\pi_1^2\,;\, F(1_{\{1\}})\,,\, F(\iota^0)\big) \\
&= \gamma\big(\pi_1^2(x)\,;\, 1_x\,,\, \iota^0\big).
\end{aligned}$$

□

Lemma 10.2.5. *Suppose* $(\mathsf{C}, \oplus, e)$ *is a small permutative category and suppose* $\mathsf{End}(\mathsf{C})$ *is equipped with its canonical basepoint, e. There is a natural isomorphism of pointed categories*

$$\mathsf{Multicat}_*(\mathcal{M}\underline{1}, \mathsf{End}(\mathsf{C})) \xrightarrow{\cong} (\mathsf{C}, e) \tag{10.2.6}$$

Proof. With the canonical basepoint e, the n-ary basepoint operations ι^n of $\mathsf{End}(\mathsf{C})$ are given by 1_e viewed as morphisms

$$e = \bigoplus_{i=1}^{n} e \xrightarrow{1_e} e.$$

By Lemma 10.2.4, a pointed multifunctor

$$F : \mathcal{M}\underline{1} \longrightarrow \mathsf{End}(\mathsf{C})$$

is determined by

$$x = F\{1\} \quad \text{and} \quad \pi_1^2(x) = F\pi_1^2 \in \mathsf{End}(\mathsf{C})\big(x, e\,;\, x\big) = \mathsf{C}(x \oplus e, x)$$

such that the composite of $\pi_1^2(x)$ with

$$\iota^0 = 1_e \in \mathsf{End}(\mathsf{C})\big(\langle\rangle\,;\, e\big) = \mathsf{C}(e, e) \quad \text{and} \quad 1_x \in \mathsf{End}(\mathsf{C})\big(x\,;\, x\big) = \mathsf{C}(x, x)$$

is the identity on x. This means that the composite

$$x \oplus e \xrightarrow{1_x \oplus 1_e} x \oplus e \xrightarrow{\pi_1^2(x)} x$$

is equal to 1_x.

Therefore, we must also have $\pi_1^2(x) = 1_x$ and hence F is determined uniquely by $x = F\{1\}$. A pointed multinatural transformation

$$\theta : F \longrightarrow G \quad \text{in} \quad \mathsf{Multicat}_*(\mathcal{M}\underline{1}, \mathsf{End}(\mathsf{C}))$$

is likewise completely determined by the component morphism in C

$$\theta_{\{1\}} : F(\{1\}) \longrightarrow G(\{1\}).$$

Since identity morphisms and composition in $\mathsf{Multicat}_*(\mathcal{M}\underline{1}, \mathsf{End}(\mathsf{C}))$ are defined in C, this gives the isomorphism (10.2.6). Naturality in C follows by construction. □

Proposition 10.2.7. *Suppose* P *is a pointed multicategory with basepoint* $*$. *Then a left* $\mathcal{M}\underline{1}$*-module structure on* P

$$\mu : \mathcal{M}\underline{1} \wedge \mathsf{P} \longrightarrow \mathsf{P}.$$

determines and is uniquely determined by operations

$$\pi_1^2(a) \in \mathsf{P}\big(a, * \,;\, a\big) \quad \textit{for} \quad a \in \mathsf{Ob}\,\mathsf{P}$$

such that the following basepoint, unit, and interchange conditions hold for objects a *in* $\mathsf{Ob}\,\mathsf{P}$ *and operations* ϕ *in* $\mathsf{P}\big(\langle b\rangle \,;\, a\big)$ *with* $\langle b\rangle = (b_1, \ldots, b_m)$.

$$\pi_1^2(*) = \iota^2 \quad \textit{in} \quad \mathsf{P}\big(*, * \,;\, *\big) \tag{10.2.8}$$

$$\gamma\big(\pi_1^2(a) \,;\, 1_a, \iota^0\big) = 1_a \quad \textit{in} \quad \mathsf{P}\big(a \,;\, a\big) \tag{10.2.9}$$

$$\gamma\big(\phi \,;\, \langle \pi_1^2(b_j)\rangle_j\big) = \gamma\big(\pi_1^2(a) \,;\, \phi, \iota^m\big) \cdot \xi^{\otimes}_{2,m} \quad \textit{in} \quad \mathsf{P}\big(b_1, *, \ldots, b_m, * \,;\, a\big) \tag{10.2.10}$$

Proof. Suppose P is a left $\mathcal{M}\underline{1}$-module with multiplication μ. Then μ is

- a pointed multifunctor and
- a unital multiplication.

Therefore, μ must give the following assignments for an object a and an m-ary operation ϕ in P:

$$\begin{array}{ccc} (\varnothing, a) \longmapsto * & \iota^n \wedge a \longmapsto \iota^n & \varnothing \wedge \phi \longmapsto \iota^m \\ (\{1\}, a) \longmapsto a & \pi_j^n \wedge a \longmapsto \pi_j^n(a) = \mu(\pi_j^n \wedge a) & \{1\} \wedge \phi \longmapsto \phi \end{array}$$

where we take $\mu(\pi_j^n \wedge a)$ as the definition of $\pi_j^n(a)$.

The basepoint condition (10.2.8) follows because μ is a pointed multifunctor. Now each $\mu(- \wedge a)$ is a pointed multifunctor from $\mathcal{M}\underline{1}$ to P and so Lemma 10.2.4 implies the following.

- The operations $\pi_j^n(a)$ are determined via equivariance by $\pi_1^n(a)$.
- The operations $\pi_1^n(a)$ for $n \geq 3$ are determined by

$$\pi_1^2(a) \in \mathsf{P}\big(a, * \,;\, a\big).$$

- The operation $\pi_1^2(a)$ satisfies the unit condition (10.2.9).

For each m-ary operation

$$\phi \in \mathsf{P}\big(\langle b\rangle \,;\, a\big)$$

the compatibility of μ with composition means that the two sides of (10.2.10) can be rewritten as

$$\gamma\big(\phi \,;\, \langle \pi_1^2(b_j)\rangle_j\big) = \gamma\big(\mu(\{1\} \wedge \phi) \,;\, \mu(\pi_1^2 \wedge \langle b\rangle)\big) = \mu(\pi_1^2 \wedge \phi) \tag{10.2.11}$$

and

$$\gamma\big(\pi_1^2(a)\,;\,\phi,\iota_m\big) = \gamma\big(\mu(\pi_1^2 \wedge a)\,;\,\mu((\{1\},\varnothing)\wedge\phi)\big) = \mu(\pi_1^2 \wedge^{\mathsf{t}} \phi). \tag{10.2.12}$$

Thus, the interchange relation (5.6.15) in $\mathcal{M}\underline{1}\wedge\mathsf{P}$ implies the interchange condition (10.2.10). This finishes the proof that a left $\mathcal{M}\underline{1}$-module structure determines operations satisfying the indicated conditions.

Conversely, suppose given operations $\pi_1^2(a)$ for each a in $\mathsf{Ob}\,\mathsf{P}$, satisfying the three conditions (10.2.8) through (10.2.10). By Lemma 10.2.4, the unit conditions (10.2.9) imply that each $\pi_1^2(a)$ determines a pointed multifunctor

$$\mu_a : \mathcal{M}\underline{1} \longrightarrow \mathsf{P}.$$

We let

- $\mu(\varnothing,-)$ be the constant pointed multifunctor at $* \in \mathsf{P}$,
- $\mu(\{1\},-)$ be the identity pointed multifunctor on P, and
- $\mu(-,a) = \mu_a$ for each object a of P.

By Explanation 5.6.9 this defines a pointed multifunctor

$$\mu : \mathcal{M}\underline{1}\,\#\,\mathsf{P} \longrightarrow \mathsf{P}.$$

Next, because the operations $\mu(\pi_j^n \wedge a)$ in P are determined by multifunctoriality and the operations $\mu(\pi_1^2 \wedge a)$, the interchange conditions (10.2.10) imply, via the equalities (10.2.11) and (10.2.12), that μ satisfies the interchange condition (5.6.16) of Explanation 5.6.14. Thus μ descends to $\mathcal{M}\underline{1}\otimes\mathsf{P}$. Finally, μ descends to $\mathcal{M}\underline{1}\wedge\mathsf{P}$ by the basepoint condition (10.2.8).

We have shown that the three conditions in the statement determine a pointed multifunctor

$$\mu : \mathcal{M}\underline{1}\wedge\mathsf{P} \longrightarrow \mathsf{P}.$$

To see that μ is a left $\mathcal{M}\underline{1}$-module multiplication, we will apply Proposition 10.2.1 together with parts (1) and (3) of Proposition 10.1.28. For this purpose, observe that each of the composites

$$\mathsf{P} \xrightarrow{\lambda^{-1}} \mathsf{S}\wedge\mathsf{P} \xrightarrow{\eta\wedge 1_{\mathsf{P}}} \mathcal{M}\underline{1}\wedge\mathsf{P} \xrightarrow{\mu} \mathsf{P}$$

and

$$\mathcal{M}\underline{1}\wedge\mathsf{P} \xrightarrow{\mu} \mathsf{P} \xrightarrow{\lambda^{-1}} \mathsf{S}\wedge\mathsf{P} \xrightarrow{\eta\wedge 1_{\mathsf{P}}} \mathcal{M}\underline{1}\wedge\mathsf{P}$$

is the identity. The key step, as in the proof of Lemma 10.1.15, is the equality

$$\{1\}\wedge\pi_1^2(a) = \pi_1^2\wedge a \quad \text{in} \quad \mathcal{M}\underline{1}\wedge\mathsf{P}.$$

The above equality is given by the following computation, generalizing that of Lemma 10.1.15 from the case $\mathsf{P} = \mathcal{M}b$:

$$\begin{aligned}
\{1\}\wedge\pi_1^2(a) &= \gamma\big(\gamma\big(\{1\}\wedge\pi_1^2(a)\,;\,\pi_1^2\wedge a\,,\,\pi_1^2\wedge *\big)\,;\,1_{(\{1\},a)}\,,\,\iota^0\wedge a\,,\,\{1\}\wedge\iota^0\,,\,1_{(\varnothing,*)}\big)\\
&\qquad \text{by multifunctoriality of } -\wedge a\\
&= \gamma\big(\gamma\big(\pi_1^2\wedge a\,;\,\{1\}\wedge\pi_1^2(a)\,,\,\varnothing\wedge\pi_1^2(a)\big)\,;\,1_{(\{1\},a)}\,,\{1\}\wedge\iota^0\,,\,\iota^0\wedge a\,,\,1_{(\varnothing,*)}\big)\\
&\qquad \text{by interchange in } \mathcal{M}\underline{1}\wedge\mathsf{P}\\
&= \pi_1^2\wedge a\\
&\qquad \text{by multifunctoriality of } \{1\}\wedge -.
\end{aligned}$$

For $n \geq 3$ we have

$$\pi_1^n = \gamma\big(\pi_1^2\,;\,\pi_1^2\,,\,\iota^{n-2}\big) \quad \text{in} \quad \mathcal{M}\underline{1}\big(\{1\},\underbrace{\varnothing,\ldots,\varnothing}_{n-1};\{1\}\big)$$

and, by definition of μ, we have

$$\pi_1^n(a) = \mu(\pi_1^n \wedge a) = \gamma\big(\pi_1^2(a)\,;\,\pi_1^2(a)\,,\,\iota^{n-2}\big) \quad \text{in} \quad \mathsf{P}\big(a,\underbrace{*,\ldots,*}_{n-1};a\big).$$

Therefore, again following the proof of Lemma 10.1.15, this implies

$$\pi_j^n \wedge a = \{1\} \wedge \pi_j^n(a)$$

for $n \geq 2$ and $1 \leq j \leq n$.

This shows that μ is an isomorphism with inverse $(\eta \wedge 1) \circ \lambda^{-1}$. Therefore, by Proposition 10.2.1 together with parts (1) and (3) of Proposition 10.1.28, μ is the multiplication for the unique left $\mathcal{M}\underline{1}$-module structure on P. $\square$

Definition 10.2.13. Suppose $(\mathsf{C},\oplus,e)$ is a small permutative category and $\mathsf{End}(\mathsf{C})$ is equipped with its canonical basepoint, e. Then the *canonical (left) $\mathcal{M}\underline{1}$-module structure* on $\mathsf{End}(\mathsf{C})$ is

$$\mu : \mathcal{M}\underline{1} \wedge \mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{C})$$

given as follows for objects a in C and morphisms f in $\mathsf{End}(\mathsf{C})$.

$$\begin{array}{ccc} (\varnothing,a) \longmapsto e & \iota^n \wedge a \longmapsto \iota^n = 1_e & \varnothing \wedge f \longmapsto 1_e \\ (\{1\},a) \longmapsto a & \pi_j^n \wedge a \longmapsto 1_a & \{1\} \wedge f \longmapsto f \end{array}$$

Since e is a strict unit for C, the composite

$$a = a \oplus e \xrightarrow{1_a \oplus 1_e} a \oplus e = a \xrightarrow{1_a} a$$

is equal to the identity on a. The interchange relation (10.2.10) holds for each operation f because both sides are equal to f. By Proposition 10.2.7 this uniquely determines a left $\mathcal{M}\underline{1}$-module structure on $\mathsf{End}(\mathsf{C})$. $\diamond$

Lemma 10.2.14. *Taking the canonical module structures of Definition 10.2.13,* End *factors as a* Cat*-enriched multifunctor through* $\mathsf{Mod}^{\mathcal{M}\underline{1}}$*:*

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}} \hookrightarrow \mathsf{Multicat}_*.$$

Proof. The canonical module structure on $\mathsf{End}(\mathsf{C})$ for each small permutative category C is given by Definition 10.2.13. By Proposition 10.1.28 and Definition 10.1.36 the multifunctors and multinatural transformations of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ are those of $\mathsf{Multicat}_*$. Moreover, the monoidal product and braiding of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ are those of $\mathsf{Multicat}_*$. Hence, for $n > 0$, the n-ary operations of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ and their permutation actions are identified with those of $\mathsf{Multicat}_*$ by Proposition 10.1.28 (4) and Definition 10.1.36.

For $n = 0$, the monoidal unit of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is $\mathcal{M}\underline{1}$ and hence a 0-ary operation of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is given by a pointed multifunctor out of $\mathcal{M}\underline{1}$. In contrast, each 0-ary operation of $\mathsf{Multicat}_*$ is given by a pointed multifunctor out of S. On 0-ary operations, the inclusion

$$\mathsf{Mod}^{\mathcal{M}\underline{1}} \hookrightarrow \mathsf{Multicat}_*$$

is given by composition along the unit

$$\eta : \mathsf{S} \longrightarrow \mathcal{M}\underline{1}.$$

For a small permutative category C, this composition induces a natural isomorphism of pointed categories

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}(\mathcal{M}\underline{1}, \mathsf{End}(\mathsf{C})) = \mathsf{Multicat}_*(\mathcal{M}\underline{1}, \mathsf{End}(\mathsf{C})) \cong \mathsf{Multicat}_*(\mathsf{S}, \mathsf{End}(\mathsf{C}))$$

by Lemma 10.2.5.

These observations show that End factors through $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ as an assignment on objects and n-ary operations for each $n \geq 0$. The Cat_*-enriched multifunctor axioms for End hold in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ since the inclusion into $\mathsf{Multicat}_*$ is injective on 2-cells. The Cat-enrichment is obtained by forgetting basepoint objects, that is, multifunctors that factor through T. □

Example 10.2.15 (Non-examples of $\mathcal{M}\underline{1}$-modules)**.** We use Proposition 10.2.7 to give a class of pointed multicategories that are not left $\mathcal{M}\underline{1}$-modules. A similar calculation shows that they also are not right $\mathcal{M}\underline{1}$-modules.

Suppose $(\mathcal{D}, \boxdot, e, \mathrm{T})$ is a small permutative category with null object $\mathrm{T} \neq e$ as basepoint. Recall from Definition 4.3.3 that being null means T is both initial and terminal and, moreover, is fixed by the monoidal product of $\mathcal{D}$. We note two specific cases of interest:

- $(\mathcal{F}, \wedge, \underline{1}, \underline{0})$ (Explanation 8.1.6) and
- $(\mathcal{G}, \oplus, \langle\rangle, *)$ (Proposition 9.1.14).

Let $\mathsf{End}(\mathcal{D})$ be the endomorphism multicategory of $\mathcal{D}$, but equipped with T as its basepoint instead of the monoidal unit e. Thus the nullary basepoint operation

$$\iota^0 \in \mathcal{D}(e, \mathrm{T})$$

is given by the unique morphism to T.

If

$$\mu : \mathcal{M}\underline{1} \wedge \mathsf{End}(\mathcal{D}) \longrightarrow \mathsf{End}(\mathcal{D})$$

is a left $\mathcal{M}\underline{1}$-module structure, then by Proposition 10.2.7 we must have, for each object a of $\mathcal{D}$, an operation

$$\pi_1^2(a) \in \mathsf{End}(\mathcal{D})\big(a, \mathrm{T}\,; a\big) = \mathcal{D}(a \boxdot \mathrm{T}, a)$$

such that the composite with ι^0 and 1_a is the identity on a, as in (10.2.9). The composition in $\mathsf{End}(\mathcal{D})$ is given by that of $\mathcal{D}$, and because T is null any composite

$$a \boxdot e \xrightarrow{1_a \boxdot \iota^0} a \boxdot \mathrm{T} \xrightarrow{\pi_1^2(a)} a$$

is equal to the zero morphism of a. If a is not the basepoint T, this cannot be the identity morphism of a. Therefore, $\mathsf{End}(\mathcal{D})$ equipped with T as its basepoint is not a left $\mathcal{M}\underline{1}$-module. ◇

Example 10.2.16 (Further Non-examples of $\mathcal{M}\underline{1}$-modules)**.** If Q is a nonempty multicategory and $\mathsf{P} = \mathsf{Q}_+$ is the pointed multicategory obtained by adjoining a disjoint basepoint, then P cannot be a left (or right) $\mathcal{M}\underline{1}$-module because, for any object a in Q, the set of operations

$$\mathsf{P}\big(a, *\,; a\big)$$

is empty. Hence there cannot be any operation $\pi_1^2(a)$ as required by Proposition 10.2.7. In particular, the monoidal unit (5.6.22)

$$\mathsf{S} = \mathsf{I} \coprod \mathsf{T}$$

cannot be a left (or right) $\mathcal{M}\underline{1}$-module. ◇

Proposition 5.7.23 shows that the smash product for small pointed multicategories does not restrict, along End, to define a symmetric monoidal structure on the category $\mathsf{PermCat}^{\mathsf{su}}$. The proof there proceeds by showing that there is no permutative category J such that

$$\mathsf{End}(\mathsf{J}) = \mathsf{S},$$

the monoidal unit for $\mathsf{Multicat}_*$. However, Example 10.2.16 shows that S is not a left $\mathcal{M}\underline{1}$-module. Moreover, Lemma 10.2.14 shows that End factors as a Cat-enriched multifunctor

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{End}} \mathsf{Mod}^{\mathcal{M}\underline{1}}.$$

Now we show that this still does not provide a symmetric monoidal structure on $\mathsf{PermCat}^{\mathsf{su}}$.

Proposition 10.2.17. *The symmetric monoidal structure on* $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ *does not restrict to a symmetric monoidal structure on* $\mathsf{PermCat}^{\mathsf{su}}$.

Proof. Suppose to the contrary that the symmetric monoidal structure on $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ restricts to a symmetric monoidal structure on $\mathsf{PermCat}^{\mathsf{su}}$. Then there exists a permutative category C with two objects $\{e, x\}$ such that

$$\mathsf{End}(\mathsf{C}) \cong \mathcal{M}\underline{1},$$

the monoidal unit in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$, as a pointed multicategory, with $\{e, x\}$ corresponding to the objects $\{\varnothing, \{1\}\}$ in $\mathcal{M}\underline{1}$. Since

$$\mathsf{C}(x, x) \neq \varnothing$$

and

$$\begin{aligned}\varnothing &= \mathcal{M}\underline{1}\big(\{1\}, \{1\}\,;\, \{1\}\big)\\ &= \mathsf{End}(\mathsf{C})\big(x, x\,;\, x\big) = \mathsf{C}(x \oplus x, x),\end{aligned}$$

we have that $x \oplus x \neq x$. So $x \oplus x = e$, which implies

$$\begin{aligned}\varnothing \neq \mathsf{C}(x, x) = \mathsf{C}(x \oplus e, x) &= \mathsf{C}(x \oplus x \oplus x, x)\\ &= \mathsf{End}(\mathsf{C})\big(x, x, x\,;\, x\big)\\ &= \mathcal{M}\underline{1}\big(\{1\}, \{1\}, \{1\}\,;\, \{1\}\big).\end{aligned}$$

This contradicts the definition of $\mathcal{M}\underline{1}$. □

The following results describe a closed structure for $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. Recall from Definition 5.7.17 the pointed hom for small pointed multicategories.

Lemma 10.2.18. *For* $\mathsf{M} \in \mathsf{Multicat}_*$ *and* $\mathsf{N} \in \mathsf{Mod}^{\mathcal{M}\underline{1}}$*, the pointed hom multicategory,* $\mathsf{Hom}_*(\mathsf{M}, \mathsf{N})$*, is an* $\mathcal{M}\underline{1}$*-module.*

Proof. The $\mathcal{M}\underline{1}$-module structure morphism for $\mathsf{Hom}_*(\mathsf{M},\mathsf{N})$,

$$\mathcal{M}\underline{1} \wedge \mathsf{Hom}_*(\mathsf{M},\mathsf{N}) \longrightarrow \mathsf{Hom}_*(\mathsf{M},\mathsf{N}),$$

is adjoint to the composite

$$\begin{tikzcd}
(\mathcal{M}\underline{1} \wedge \mathsf{Hom}_*(\mathsf{M},\mathsf{N})) \wedge \mathsf{M} \arrow[d, "\cong"'] & \mathsf{N} \\
\mathcal{M}\underline{1} \wedge (\mathsf{Hom}_*(\mathsf{M},\mathsf{N}) \wedge \mathsf{M}) \arrow[r, "1 \wedge \mathsf{ev}"] & \mathcal{M}\underline{1} \wedge \mathsf{N} \arrow[u]
\end{tikzcd} \tag{10.2.19}$$

where the first isomorphism is the associativity of $\wedge$, the second is given by the evaluation for Hom_*, and the third is the $\mathcal{M}\underline{1}$-module structure of N.

Then the $\mathcal{M}\underline{1}$-module axioms for $\mathsf{Hom}_*(\mathsf{M},\mathsf{N})$ follow from (10.2.19) and the $\mathcal{M}\underline{1}$-module axioms for N. $\square$

Explanation 10.2.20 (Module Structure of $\mathsf{Hom}_*(\mathsf{M},\mathsf{N})$)**.** The characterization of $\mathcal{M}\underline{1}$-modules in Proposition 10.2.7 gives an alternative explanation of the $\mathcal{M}\underline{1}$-module structure on $\mathsf{Hom}_*(\mathsf{M},\mathsf{N})$ in Lemma 10.2.18. For each pointed multifunctor

$$F \in \mathsf{Hom}_*(\mathsf{M},\mathsf{N}),$$

the requisite operation $\pi_1^2(F)$ is a pointed transformation defined by components

$$\big(\pi_1^2(F)\big)_c = \pi_1^2(Fc) \in \mathsf{N}\big(Fc, *\,;\, Fc\big), \tag{10.2.21}$$

where the right hand side of the equality is the operation determined by the $\mathcal{M}\underline{1}$-module structure on N. To show that (10.2.21) defines a pointed transformation satisfying the basepoint, unit, and interchange conditions of Proposition 10.2.7, one uses the fact that F is pointed, together with the corresponding conditions componentwise in N. $\diamond$

Proposition 10.2.22. *The category* $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ *is complete, cocomplete, symmetric monoidal, and closed. The product and internal hom are given by those of* $(\mathsf{Multicat}_*, \wedge, \mathsf{Hom}_*)$.

Proof. Corollary 10.1.14 shows that $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is complete and cocomplete. Definition 10.1.36 describes the symmetric monoidal structure of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. For small pointed multicategories M, N, and P, Lemma 10.2.18 shows that the pointed hom multicategory, $\mathsf{Hom}_*(\mathsf{M},\mathsf{N})$, is an $\mathcal{M}\underline{1}$-module. Recall also from Definition 10.1.36 that $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is full as a subcategory of $\mathsf{Multicat}_*$; this gives the first and final isomorphisms below. Then we have the following, with the second isomorphism from the closed structure of $\mathsf{Multicat}_*$, as in Proposition 5.7.13 and Theorem 5.7.22, and the third isomorphism induced by that of Proposition 10.1.28 (5):

$$\begin{aligned}
\mathsf{Mod}^{\mathcal{M}\underline{1}}\big(\mathsf{M}, \mathsf{Hom}_*(\mathsf{N},\mathsf{P})\big) &\cong \mathsf{Multicat}_*\big(\mathsf{M}, \mathsf{Hom}_*(\mathsf{N},\mathsf{P})\big)\\
&\cong \mathsf{Multicat}_*\big(\mathsf{M}\wedge\mathsf{N}, \mathsf{P}\big)\\
&\cong \mathsf{Multicat}_*\big(\mathsf{M}\wedge_{\mathcal{M}\underline{1}}\mathsf{N}, \mathsf{P}\big)\\
&\cong \mathsf{Mod}^{\mathcal{M}\underline{1}}\big(\mathsf{M}\wedge_{\mathcal{M}\underline{1}}\mathsf{N}, \mathsf{P}\big).
\end{aligned}$$

A similar treatment shows

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}\big(\mathsf{M}\wedge_{\mathcal{M}\underline{1}}\mathsf{N}, \mathsf{P}\big) \cong \mathsf{Mod}^{\mathcal{M}\underline{1}}\big(\mathsf{N}, \mathsf{Hom}_*(\mathsf{M},\mathsf{P})\big).$$

These isomorphisms provide the closed structure for $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. $\square$

Explanation 10.2.23. Recall, from Explanation 5.7.5, that the underlying category of unary operations in each hom object $\mathsf{Hom}(\mathsf{M},\mathsf{N})$ gives the Cat-enrichment of $\mathsf{Multicat}$. The same reasoning shows that the Cat_*-enrichment of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$, in Definition 10.1.36, is given by the underlying categories of the internal hom objects $\mathsf{Hom}_*(\mathsf{M},\mathsf{N})$. ◇

10.3. Elmendorf-Mandell J-Theory and K-Theory

Now we turn to a second K-theory construction, with tuples of finite sets in place of individual finite sets. While moderately more complex, it will yield equivalent symmetric spectra. The additional structure present in $\mathcal{G}_*$-categories will allow us to express the additional multiplicative structure present in ring, bipermutative, braided ring, and E_n-monoidal categories.

Definition 10.3.1. Suppose $\langle a\rangle = (a_1,\ldots,a_q)$ is a tuple of pointed finite sets. Let $\mathcal{T}\langle a\rangle$ denote the smash product of pointed multicategories

$$\mathcal{T}\langle a\rangle = \bigwedge_{k=1}^{q} \mathcal{M}a_k$$

with respect to the basepoints

$$\varnothing \in \mathcal{M}a_k.$$

We let $\varnothing$ also denote the basepoint object of $\mathcal{T}\langle a\rangle$. For the empty tuple we define

$$\mathcal{T}\langle\rangle = \mathcal{M}\underline{1}.$$ ◇

Explanation 10.3.2. Recalling Proposition 10.1.28 (5), $\mathcal{T}\langle a\rangle$ is equivalently defined as the smash product of $\mathcal{M}a_k$ as left $\mathcal{M}\underline{1}$-modules, taking $\mathcal{M}\underline{1}$ for the empty smash product of such. ◇

Recall from Definition 10.1.36 that we let

$$\big(\mathsf{Mod}^{\mathcal{M}\underline{1}},\wedge,\mathcal{M}\underline{1}\big)$$

denote the full subcategory of left $\mathcal{M}\underline{1}$-modules in $\mathsf{Multicat}_*$.

Proposition 10.3.3. *The smash product $\mathcal{T}$ of Definition 10.3.1 defines a pointed functor*

$$\mathcal{T} : (\mathcal{G}^{\mathsf{op}},*) \longrightarrow (\mathsf{Mod}^{\mathcal{M}\underline{1}},\mathsf{T}).$$

Proof. We have $\mathcal{T}$ taking values in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ by definition. To make $\mathcal{T}$ a pointed functor, we define $\mathcal{T}* = \mathsf{T}$, where $*$ is the basepoint of $\mathcal{G}$. Since T is both initial and terminal in $\mathsf{Multicat}_*$, this defines $\mathcal{T}$ on morphisms in $\mathcal{G}$ either from or to the basepoint $*$.

For tuples $\langle\underline{n}\rangle$ and $\langle\underline{m}\rangle$ of length q and p, respectively, the morphisms from $\langle\underline{n}\rangle$ to $\langle\underline{m}\rangle$ in $\mathcal{G}$ are given by pairs $(f,\langle\psi\rangle)$ as described in Definition 9.1.7. We define

$$\mathcal{T}(f,\langle\psi\rangle) : \bigwedge_{\underline{m}_j\in\langle\underline{m}\rangle} \mathcal{M}\underline{m}_j \longrightarrow \bigwedge_{\underline{n}_k\in\langle\underline{n}\rangle} \mathcal{M}\underline{n}_k$$

via the following.

- For each $j \in \overline{p}$, we have

$$\psi_j : \underline{n}_{f^{-1}(j)} \longrightarrow \underline{m}_j$$

and so we have an induced multifunctor

$$\tilde{\psi}_j = \mathcal{M}\psi_j : \mathcal{M}\underline{m}_j \longrightarrow \mathcal{M}(\underline{n}_{f^{-1}(j)})$$

defined by applying ψ_j^{-1} to subsets, as in the proof of Proposition 8.4.7.

- Recall from Lemma 10.1.15 that the partition product (10.1.2) provides

$$\begin{aligned}\prod_{\underline{1},\underline{n}} : \mathcal{M}\underline{1} \wedge \mathcal{M}\underline{n} &\xrightarrow{\cong} \mathcal{M}(\underline{1} \wedge \underline{n}) \cong \mathcal{M}\underline{n}\\ \prod_{\underline{n},\underline{1}} : \mathcal{M}\underline{n} \wedge \mathcal{M}\underline{1} &\xrightarrow{\cong} \mathcal{M}(\underline{n} \wedge \underline{1}) \cong \mathcal{M}\underline{n}\end{aligned} \tag{10.3.4}$$

for each $\underline{n}$ in $\mathcal{F}$.

We define $\mathcal{T}(f, \langle\psi\rangle)$ to be the composite formed by first taking the smash product of $\tilde{\psi}_j$ over $j \in \overline{p}$, then using (10.3.4) for any factors $\mathcal{M}\underline{n}_\varnothing$, and then permuting factors according to f^{-1}:

(10.3.5)

$$\bigwedge_j \mathcal{M}\underline{m}_j \xrightarrow{\wedge_j \tilde{\psi}_j} \bigwedge_j \mathcal{M}(\underline{n}_{f^{-1}(j)}) \xrightarrow{\cong} \bigwedge_{f^{-1}(j) \neq \varnothing} \mathcal{M}(\underline{n}_{f^{-1}(j)}) \xrightarrow[\text{permute}]{\cong} \bigwedge_k \mathcal{M}\underline{n}_k.$$

The associativity (10.1.9) of $\prod$ ensures that iterated applications of (10.3.4) are independent of order.

This definition of $\mathcal{T}$ on morphisms preserves identities because each of the terms in (10.3.5) is an identity when $(f, \langle\psi\rangle) = (1, \langle 1\rangle)$. It preserves composition by naturality (10.1.5) of $\prod$, naturality of the symmetry for $\wedge$ in $\mathsf{Multicat}_*$, and functoriality of the smash product. □

Explanation 10.3.6. We note, in the proof of Proposition 10.3.3 with $\langle \underline{n}\rangle = \langle\rangle$, that

$$\mathcal{T}_{\langle\rangle,\langle \underline{m}\rangle} : \mathcal{G}(\langle\rangle, \langle \underline{m}\rangle) \longrightarrow \mathsf{Multicat}_*(\mathcal{T}\langle \underline{m}\rangle, \mathcal{T}\langle\rangle) = \mathsf{Multicat}_*(\mathcal{T}\langle \underline{m}\rangle, \mathcal{M}\underline{1})$$

is given as follows. By Explanation 9.1.9 (4), a morphism from $\langle\rangle$ to $\langle \underline{m}\rangle$ in $\mathcal{G}$ factors uniquely as

$$\langle\rangle \xrightarrow{(i_p, 1_{\langle \underline{1}\rangle})} \langle \underline{1}\rangle \xrightarrow{(1, \langle\psi\rangle)} \langle \underline{m}\rangle$$

where $\langle \underline{m}\rangle$ has length p, $\langle \underline{1}\rangle$ denotes the constant p-tuple at $\underline{1}$, and i_p is the unique inclusion from $\varnothing$ to $\overline{p}$. The pointed multifunctor induced by $\mathcal{T}$ is

$$\wedge_j \mathcal{M}\underline{m}_j \xrightarrow{\wedge_j \tilde{\psi}_j} \wedge_j \mathcal{M}\underline{1} \xrightarrow{\cong} \mathcal{M}\underline{1}$$

where the unlabeled isomorphism is the iterate of multiplication isomorphisms for $\mathcal{M}\underline{1}$. Each

$$\psi_j : \underline{1} \longrightarrow \underline{m}_j$$

is uniquely determined by an element x of $\underline{m}_j$, namely, the image of $1 \in \underline{1}$. The pointed multifunctor

$$\tilde{\psi}_j = \psi_j^{-1}(-) : \mathcal{M}\underline{m}_j \longrightarrow \mathcal{M}\underline{1}$$

sends a subset $s \subset \underline{m}_j^\flat$ to $\{1\}$ if $x \in s$ and to $\varnothing$ if $x \notin s$. ◇

The main feature of $\mathcal{T}$ over the individual factors $\mathcal{M}$ is stated in the following result. Recall from Proposition 10.1.28 that the smash product of left $\mathcal{M}\underline{1}$-modules is the ordinary smash product of pointed multicategories. The important difference is their units.

Proposition 10.3.7. *The functor*

$$\mathcal{T} : \mathcal{G}^{\mathsf{op}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

is a strictly unital strong symmetric monoidal functor with respect to the concatenation product in $\mathcal{G}$ and the smash product of pointed multicategories.

Proof. The functor $\mathcal{T}$ has been defined in Proposition 10.3.3. Given objects $\langle \underline{n} \rangle$ and $\langle \underline{n}' \rangle$ in $\mathcal{G}$, the monoidal constraint isomorphisms

$$\mathcal{T}^2 : \mathcal{T}\langle \underline{n} \rangle \wedge \mathcal{T}\langle \underline{n}' \rangle \xrightarrow{\cong} \mathcal{T}(\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle) \tag{10.3.8}$$

are defined by the associativity of the smash product or, in the case that $\langle \underline{n} \rangle$ or $\langle \underline{n}' \rangle$ is empty, by the $\mathcal{M}\underline{1}$-module isomorphisms $\prod$ (10.1.16). Naturality of $\mathcal{T}^2$ follows from naturality of the associativity isomorphism and of $\prod$. The unit constraint is the identity because

$$\mathcal{T}\langle\rangle = \mathcal{M}\underline{1}$$

by definition. The associativity axiom (1.1.9) requires an equality of two morphisms

$$\Big(\bigwedge_{\underline{m}_j \in \langle \underline{m} \rangle} \mathcal{M}\underline{m}_j\Big) \wedge \Big(\bigwedge_{\underline{m}'_j \in \langle \underline{m}' \rangle} \mathcal{M}\underline{m}'_j\Big) \wedge \Big(\bigwedge_{\underline{m}''_j \in \langle \underline{m}'' \rangle} \mathcal{M}\underline{m}''_j\Big) \longrightarrow \bigwedge_{\underline{r} \in \big(\langle \underline{m} \rangle \oplus \langle \underline{m}' \rangle \oplus \langle \underline{m}'' \rangle\big)} \mathcal{M}\underline{r}$$

defined by associativity isomorphisms. Therefore, the associativity axiom follows from Mac Lane's Coherence Theorem 1.1.31. The unity axiom (1.1.10) follows from the definition of the unit constraint being the identity and the strict unit isomorphism in $\mathcal{G}$. The symmetry axiom (1.1.18) follows from the Symmetric Coherence Theorem 1.1.41. $\square$

Definition 10.3.9. Let $\mathsf{J}^{\mathcal{T}}$ denote the Cat_*-functor

$$\mathsf{J}^{\mathcal{T}} = \mathsf{Multicat}_*(\mathcal{T}, -) : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}.$$

For each left $\mathcal{M}\underline{1}$-module P, the $\mathcal{G}_*$-category $\mathsf{J}^{\mathcal{T}}\mathsf{P}$ is

$$\mathsf{J}^{\mathcal{T}}\mathsf{P} = \mathsf{Multicat}_*(\mathcal{T}(-), \mathsf{P}) : \mathcal{G} \longrightarrow \mathsf{Cat}.$$

We call this the *$\mathcal{T}$-partition J-theory* of $\mathcal{M}\underline{1}$-modules, or simply the *partition J-theory* when $\mathcal{T}$ is implied by context. $\diamond$

Explanation 10.3.10 (Comparison and Contrast with $\mathsf{J}^{\mathcal{M}}$)**.** Recall the $\mathcal{M}$-partition J-theory, $\mathsf{J}^{\mathcal{M}}$, from Definition 8.4.10. Explanation 8.4.11 describes the components of $\mathsf{J}^{\mathcal{M}}H$ and $\mathsf{J}^{\mathcal{M}}\theta$ for a pointed multifunctor H and a pointed multinatural transformation θ. One has a similar description for $\mathsf{J}^{\mathcal{T}}$, with $\mathcal{M}$ replaced by $\mathcal{T}$ and $\mathcal{F}$ replaced by $\mathcal{G}$.

The key distinction between $\mathsf{J}^{\mathcal{M}}$ and $\mathsf{J}^{\mathcal{T}}$ is Theorem 10.3.17, which shows that $\mathsf{J}^{\mathcal{T}}$ is *symmetric monoidal* as a Cat_*-functor. See Note 10.8.6 for further discussion of this point. The symmetric monoidal structure of $\mathsf{J}^{\mathcal{T}}$ is the reason that our further applications require $\mathcal{G}$, $\mathcal{T}$, and $\mathsf{J}^{\mathcal{T}}$. This is also the reason that the domain of $\mathsf{J}^{\mathcal{T}}$ must be restricted from $\mathsf{Multicat}_*$ to $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. $\diamond$

Now we define the monoidal constraint and unit constraint for $\mathsf{J}^{\mathcal{T}}$.

Definition 10.3.11. For each pair of left $\mathcal{M}\underline{1}$-modules P and Q we define a pointed functor of categories

$$(\mathsf{J}^{\mathcal{T}})^2_{\mathsf{P},\mathsf{Q}} : (\mathsf{J}^{\mathcal{T}}\mathsf{P}) \wedge (\mathsf{J}^{\mathcal{T}}\mathsf{Q}) \longrightarrow \mathsf{J}^{\mathcal{T}}(\mathsf{P} \wedge \mathsf{Q}) \tag{10.3.12}$$

as follows. For each pair of objects $\langle \underline{n} \rangle$ and $\langle \underline{n'} \rangle$ in $\mathcal{G}$, we have

$$\begin{array}{c} \mathsf{Multicat}_*\big(\mathcal{T}\langle \underline{n} \rangle, \mathsf{P}\big) \wedge \mathsf{Multicat}_*\big(\mathcal{T}\langle \underline{n'} \rangle, \mathsf{Q}\big) \\ \Big\downarrow {\scriptstyle \wedge} \\ \mathsf{Multicat}_*\big((\mathcal{T}\langle \underline{n} \rangle) \wedge (\mathcal{T}\langle \underline{n'} \rangle), \mathsf{P} \wedge \mathsf{Q}\big) \\ \Big\downarrow {\scriptstyle (\mathcal{T}^{-2})^*} \\ \mathsf{Multicat}_*\big(\mathcal{T}(\langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle), \mathsf{P} \wedge \mathsf{Q}\big) \end{array} \tag{10.3.13}$$

where $\wedge$ is the smash product of pointed multifunctors and $(\mathcal{T}^{-2})^*$ is precomposition with the inverse of the monoidal constraint for $\mathcal{T}$ (10.3.8).

Then the morphisms (10.3.13) induce a morphism of coends giving the first arrow of the following composite for the Day convolution of $\mathcal{G}_*$-categories, where we write $\mathsf{M}_* = \mathsf{Multicat}_*$ and evaluate at $\langle \underline{m} \rangle$ in $\mathcal{G}$.

$$\begin{array}{c} \big((\mathsf{J}^{\mathcal{T}}\mathsf{P}) \wedge (\mathsf{J}^{\mathcal{T}}\mathsf{Q})\big)_{\langle \underline{m} \rangle} \\ \shortparallel \\ \displaystyle\int^{(\langle \underline{n} \rangle, \langle \underline{n'} \rangle) \in \widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}} \bigvee_{\mathcal{G}^{\flat}(\langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle, \langle \underline{m} \rangle)} \mathsf{M}_*\big(\mathcal{T}\langle \underline{n} \rangle, \mathsf{P}\big) \wedge \mathsf{M}_*\big(\mathcal{T}\langle \underline{n'} \rangle, \mathsf{Q}\big) \\ \Big\downarrow \\ \displaystyle\int^{(\langle \underline{n} \rangle, \langle \underline{n'} \rangle) \in \widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}} \bigvee_{\mathcal{G}^{\flat}(\langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle, \langle \underline{m} \rangle)} \mathsf{M}_*\big(\mathcal{T}(\langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle), \mathsf{P} \wedge \mathsf{Q}\big) \\ \Big\downarrow \\ \mathsf{M}_*\big(\mathcal{T}\langle \underline{m} \rangle, \mathsf{P} \wedge \mathsf{Q}\big) \end{array} \tag{10.3.14}$$

The second arrow in the above diagram is the universal morphism out of the coend that is induced on each summand by

$$\mathsf{M}_*\big(\mathcal{T}(f, \langle \psi \rangle), \mathsf{P} \wedge \mathsf{Q}\big) \quad \text{for} \quad (f, \langle \psi \rangle) \in \mathcal{G}^{\flat}\big(\langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle, \langle \underline{m} \rangle\big).$$

Therefore, (10.3.14) is induced by the pointed functor that, for each summand

$$(f, \langle \psi \rangle) \in \mathcal{G}^{\flat}(\langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle, \langle \underline{m} \rangle),$$

sends pointed multifunctors and pointed multinatural transformations

$$\begin{array}{ccc} \theta : F \longrightarrow F' & \text{in} & \mathsf{M}_*(\mathcal{T}\langle \underline{n} \rangle, \mathsf{P}) \\ & \text{and} & \\ \omega : G \longrightarrow G' & \text{in} & \mathsf{M}_*(\mathcal{T}\langle \underline{n'} \rangle, \mathsf{Q}) \end{array}$$

to the respective composites
(10.3.15)

$$\begin{tikzcd}\mathcal{T}\langle \underline{m}\rangle \arrow[r, "\mathcal{T}(f,\langle\psi\rangle)"] & \mathcal{T}(\langle \underline{n}\rangle \oplus \langle \underline{n}'\rangle) \arrow[r, "\mathcal{T}^{-2}"] & (\mathcal{T}\langle \underline{n}\rangle) \wedge (\mathcal{T}\langle \underline{n}'\rangle) \arrow[r, bend left, "F\wedge G"] \arrow[r, bend right, "F'\wedge G'"'] \arrow[r, phantom, "\Downarrow\theta\wedge\omega"] & \mathsf{P}\wedge\mathsf{Q}.\end{tikzcd}$$

These definitions on summands satisfy the V-cowedge condition (3.5.2) for $\mathsf{V} = \mathsf{Cat}_*$ by the functoriality of $\mathcal{T}$ and naturality of $\mathcal{T}^2$. We define the components

$$((\mathsf{J}^{\mathcal{T}})^2_{\mathsf{P},\mathsf{Q}})_{\langle \underline{m}\rangle}$$

by (10.3.14).

With respect to morphisms in $\mathcal{G}$, each morphism of (10.3.14) is natural. Thus $(\mathsf{J}^{\mathcal{T}^2})_{\mathsf{P},\mathsf{Q}}$ is a morphism of $\mathcal{G}_*$-categories. By construction, these components are 2-natural with respect to pointed multifunctors and pointed multinatural transformations

$$\mathsf{P} \longrightarrow \mathsf{P}' \quad \text{and} \quad \mathsf{Q} \longrightarrow \mathsf{Q}'.$$

This finishes the definition of $(\mathsf{J}^{\mathcal{T}})^2$. ⋄

Now we turn to the unit constraint for $\mathsf{J}^{\mathcal{T}}$. Recall from (9.2.13) that the monoidal unit in $\mathcal{G}_*$-Cat is

$$J = \widehat{\mathcal{G}}(\langle\rangle, -) : \mathcal{G} \longrightarrow \mathsf{Cat}_*.$$

Recall from Explanation 10.3.6 the description of $\mathcal{T}$ on morphisms

$$\langle\rangle \xrightarrow{(i_p,\langle\psi\rangle)} \langle \underline{m}\rangle \quad \text{in} \quad \mathcal{G}.$$

Definition 10.3.16. The unit constraint

$$(\mathsf{J}^{\mathcal{T}})^0 : J \longrightarrow \mathsf{J}^{\mathcal{T}}(\mathcal{M}\underline{1}) = \mathsf{Multicat}_*(\mathcal{T}(-), \mathcal{M}\underline{1})$$

has component at each $\langle \underline{m}\rangle$ given by applying $\mathcal{T}$ to morphisms:

$$\widehat{\mathcal{G}}(\langle\rangle, \langle \underline{m}\rangle) \xrightarrow{(\mathcal{T})_{\langle\rangle,\langle \underline{m}\rangle}} \mathsf{Multicat}_*(\mathcal{T}\langle \underline{m}\rangle, \mathcal{T}\langle\rangle) = \mathsf{Multicat}_*(\mathcal{T}\langle \underline{m}\rangle, \mathcal{M}\underline{1}). \quad ⋄$$

Theorem 10.3.17. *The functor*

$$\mathsf{J}^{\mathcal{T}} = \mathsf{Multicat}_*(\mathcal{T}, -) : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}$$

is a symmetric monoidal Cat_**-functor.*

Proof. The monoidal constraint $(\mathsf{J}^{\mathcal{T}})^2$ is given in Definition 10.3.11. The unit constraint is given in Definition 10.3.16. Each is a Cat_*-natural transformation by construction. By Theorem 2.5.1 it suffices to verify the symmetric monoidal axioms for the underlying functor $\mathsf{J}^{\mathcal{T}}$ and underlying natural transformations $(\mathsf{J}^{\mathcal{T}})^2$ and $(\mathsf{J}^{\mathcal{T}})^0$.

We first check the unity axioms (1.1.10) for $\mathsf{J}^{\mathcal{T}}$. We will show the left unity axiom and the right is similar. The left unity axiom requires that the following diagram commutes for each left $\mathcal{M}\underline{1}$-module P.

$$\begin{tikzcd}
J \wedge (\mathsf{J}^{\mathcal{T}}\mathsf{P}) \arrow[r, "\lambda^J"] \arrow[d, "(\mathsf{J}^{\mathcal{T}})^0 \wedge 1"'] & \mathsf{J}^{\mathcal{T}}\mathsf{P} \\
(\mathsf{J}^{\mathcal{T}}\mathcal{M}\underline{1}) \wedge (\mathsf{J}^{\mathcal{T}}\mathsf{P}) \arrow[r, "(\mathsf{J}^{\mathcal{T}})^2"] & \mathsf{J}^{\mathcal{T}}(\mathcal{M}\underline{1} \wedge \mathsf{P}) \arrow[u, "\cong"']
\end{tikzcd} \tag{10.3.18}$$

In the above diagram, the morphism λ^J denotes the left unit isomorphism for the Day convolution (Definition 3.7.17). The unlabeled isomorphism is induced by the left $\mathcal{M}\underline{1}$-module structure of P.

The morphisms in (10.3.18) are morphisms of $\mathcal{G}_*$-categories. We show they are equal by showing their components at each $\langle \underline{m} \rangle$ in $\mathcal{G}$ are equal. At $\langle \underline{m} \rangle$, the $\mathcal{G}_*$-categories on the left side of (10.3.18) are given by the following coends. Here and below we abbreviate $\mathsf{M}_* = \mathsf{Multicat}_*$.

$$(J \wedge (\mathsf{J}^{\mathcal{T}}\mathsf{P}))\langle \underline{m} \rangle = \int^{(\langle \underline{n} \rangle, \langle \underline{n}' \rangle) \in \widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}} \bigvee_{\mathcal{G}^\flat(\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle, \langle \underline{m} \rangle)} \widehat{\mathcal{G}}(\langle\rangle, \langle \underline{n} \rangle) \wedge \mathsf{M}_*(\mathcal{T}\langle \underline{n}' \rangle, \mathsf{P}) \tag{10.3.19}$$

$$((\mathsf{J}^{\mathcal{T}}\mathcal{M}\underline{1}) \wedge (\mathsf{J}^{\mathcal{T}}\mathsf{P}))\langle \underline{m} \rangle = \int^{(\langle \underline{n} \rangle, \langle \underline{n}' \rangle) \in \widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}} \bigvee_{\mathcal{G}^\flat(\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle, \langle \underline{m} \rangle)} \mathsf{M}_*(\mathcal{T}\langle \underline{n}' \rangle, \mathcal{M}\underline{1}) \wedge \mathsf{M}_*(\mathcal{T}\langle \underline{n}' \rangle, \mathsf{P})$$

To show that the two composites around (10.3.18) are equal, we show that the morphisms out of the coend (10.3.19) are equal by checking that the morphisms out of each summand are equal. Recall from (9.2.7)

$$\widehat{\mathcal{G}}(\langle\rangle, \langle \underline{m} \rangle) = \bigvee_{\mathcal{G}^\flat(\langle\rangle, \langle \underline{m} \rangle)} \mathbf{1}_+,$$

where $\mathbf{1}$ is the terminal category. For each pair $\langle \underline{n} \rangle$ and $\langle \underline{n}' \rangle$ in $\mathcal{G}$, a summand of

$$\bigvee_{\mathcal{G}^\flat(\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle, \langle \underline{m} \rangle)} \widehat{\mathcal{G}}(\langle\rangle, \langle \underline{n} \rangle) \wedge \mathsf{M}_*(\mathcal{T}\langle \underline{n}' \rangle, \mathsf{P})$$

together with an object in that summand are determined by

$$\big((f, \langle \psi \rangle), (i_q, \langle \phi \rangle), F\big) \tag{10.3.20}$$

where q is the length of $\langle \underline{n} \rangle$,

$$\begin{gathered}
(f, \langle \psi \rangle) \in \mathcal{G}^\flat(\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle, \langle \underline{m} \rangle), \\
(i_q, \langle \phi \rangle) \in \mathcal{G}^\flat(\langle\rangle, \langle \underline{n} \rangle), \quad \text{and} \\
F \in \mathsf{M}_*(\mathcal{T}\langle \underline{n}' \rangle, \mathsf{P}).
\end{gathered}$$

The left unit λ^J sends the data (10.3.20) to the object of $\mathsf{M}_*(\mathcal{T}\langle \underline{m} \rangle, \mathsf{P})$ (a pointed multifunctor) given by the composite

$$\mathcal{T}\langle \underline{m} \rangle \xrightarrow{\mathcal{T}(f, \langle \psi \rangle)} \mathcal{T}(\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle) \xrightarrow{\mathcal{T}\left((i_q, \langle \phi \rangle) \oplus 1_{\langle \underline{n}' \rangle}\right)} \mathcal{T}\langle \underline{n}' \rangle \xrightarrow{F} \mathsf{P}. \tag{10.3.21}$$

The composite of other morphisms in (10.3.18) sends the data (10.3.20) first to

$$\big((f,\langle\psi\rangle)\,,\,\mathcal{T}(i_q,\langle\phi\rangle)\,,\,F\big),$$

by $(\mathsf{J}^{\mathcal{T}})^0 \wedge 1$, and then to the following composite, where the final isomorphism is the left $\mathcal{M}\underline{1}$-module structure of P.

(10.3.22)
$$\begin{tikzcd}
\mathcal{T}\langle\underline{m}\rangle \arrow[r, "\mathcal{T}(f{,}\langle\psi\rangle)"] & \mathcal{T}(\langle\underline{n}\rangle\oplus\langle\underline{n'}\rangle) \arrow[d, "\mathcal{T}^{-2}"'] & & \mathsf{P} \\
& (\mathcal{T}\langle\underline{n}\rangle)\wedge(\mathcal{T}\langle\underline{n'}\rangle) \arrow[rr, "\mathcal{T}(i_q{,}\langle\phi\rangle)\wedge F"] & & \mathcal{M}\underline{1}\wedge\mathsf{P} \arrow[u, "\cong"']
\end{tikzcd}$$

The two composites (10.3.21) and (10.3.22) are equal by commutativity of the following diagram in $\mathsf{M}_* = \mathsf{Multicat}_*$, which we explain below.

(10.3.23)
$$\begin{tikzcd}
\mathcal{T}(\langle\underline{n}\rangle\oplus\langle\underline{n'}\rangle) \arrow[rr, "\mathcal{T}\left((i_q{,}\langle\phi\rangle)\oplus 1_{\langle\underline{n'}\rangle}\right)"] \arrow[dd, "\mathcal{T}^{-2}"'] & & \mathcal{T}\langle\underline{n'}\rangle \arrow[rr, "F"] & & \mathsf{P} \\
& & \mathcal{M}\underline{1}\wedge(\mathcal{T}\langle\underline{n'}\rangle) \arrow[u, "\mathcal{T}^2_{\langle\rangle,\langle\underline{n'}\rangle}", "\cong"'] \arrow[drr, "1\wedge F"] & & \\
(\mathcal{T}\langle\underline{n}\rangle)\wedge(\mathcal{T}\langle\underline{n'}\rangle) \arrow[urr, "\mathcal{T}(i_q{,}\langle\phi\rangle)\wedge 1"] \arrow[rrrr, "\mathcal{T}(i_q{,}\langle\phi\rangle)\wedge F"'] & & & & \mathcal{M}\underline{1}\wedge\mathsf{P} \arrow[uu, "\cong"']
\end{tikzcd}$$

In the above diagram, both upward-pointing isomorphisms are given by the left $\mathcal{M}\underline{1}$-module structure. In the case of $\mathcal{T}\langle\underline{n}\rangle$, this module structure is given by the partition product $\prod$ (10.1.2) and is equal to the indicated component of $\mathcal{T}^2$ by definition (10.3.8). Therefore, the trapezoid at left commutes by naturality of $\mathcal{T}^2$ and the trapezoid at right commutes because each pointed multifunctor F is necessarily a left $\mathcal{M}\underline{1}$-module morphism (Proposition 10.1.28 (4)). The lower triangle commutes by functoriality of $\wedge$.

Commutativity of (10.3.23) shows that the data (10.3.20) are sent to the same object of $(\mathsf{J}^{\mathcal{T}}\mathsf{P})\langle\underline{m}\rangle$ by both composites of the unity diagram (10.3.18). A similar computation holds for morphisms, that is, multinatural transformations

$$\theta : F \longrightarrow F' \quad \text{in} \quad \mathsf{M}_*(\mathcal{T}\langle\underline{n'}\rangle, \mathsf{P}),$$

by the 2-cell aspect of Proposition 10.1.28 (4). This completes the proof of the left unity axiom (1.1.10) for $\mathsf{J}^{\mathcal{T}}$.

Now we turn to the associativity and braiding axioms for $\mathsf{J}^{\mathcal{T}}$. The associativity axiom (1.1.9) follows, as in the definition of $(\mathsf{J}^{\mathcal{T}})^2$ (10.3.12), by checking on summands as in (10.3.13). For $\langle \underline{n} \rangle$, $\langle \underline{n}' \rangle$, and $\langle \underline{n}'' \rangle$ in $\mathcal{G}$ we have the following.

$$\begin{tikzcd}
\Big(\mathsf{Multicat}_*\big(\mathcal{T}\langle \underline{n} \rangle, \mathsf{P}\big) \wedge \mathsf{Multicat}_*\big(\mathcal{T}\langle \underline{n}' \rangle, \mathsf{Q}\big)\Big) \wedge \mathsf{Multicat}_*\big(\mathcal{T}\langle \underline{n}'' \rangle, \mathsf{R}\big) \arrow[d, shift right=3em, "\alpha_* \circ \wedge \circ (\wedge, 1)"'] \arrow[d, shift left=3em, "\alpha^* \circ \wedge \circ (1, \wedge) \circ \alpha'"] \\
\mathsf{Multicat}_*\Big(\Big(\big(\mathcal{T}\langle \underline{n} \rangle\big) \wedge \big(\mathcal{T}\langle \underline{n}' \rangle\big)\Big) \wedge \big(\mathcal{T}\langle \underline{n}'' \rangle\big), \mathsf{P} \wedge (\mathsf{Q} \wedge \mathsf{R})\Big) \arrow[d] \\
\mathsf{Multicat}_*\big(\mathcal{T}(\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle \oplus \langle \underline{n}'' \rangle), \mathsf{P} \wedge (\mathsf{Q} \wedge \mathsf{R})\big)
\end{tikzcd}$$

In the above diagram, α' denotes the associator for the smash product of Cat_*-categories (denoted $\alpha^{\otimes}$ in Section 1.4). The morphisms α^* and α_* denote pre- and post-composition with the associativity isomorphism for the smash product in $\mathsf{Multicat}_*$. The upper two morphisms are equal by Cat_*-naturality of

$$\alpha : \wedge \circ (\wedge, 1) \longrightarrow \wedge \circ (1, \wedge) \circ \alpha'.$$

The lower morphism is induced by either of the two equal morphisms

$$\mathcal{T}(\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle \oplus \langle \underline{n}'' \rangle) \longrightarrow ((\mathcal{T}\langle \underline{n} \rangle) \wedge (\mathcal{T}\langle \underline{n}' \rangle)) \wedge (\mathcal{T}\langle \underline{n}'' \rangle)$$

given by the (inverses of) morphisms in the associativity axiom (1.1.9) for $\mathcal{T}$. The remaining morphisms defining $(\mathsf{J}^{\mathcal{T}})^2$ are given by universal morphisms out of the respective coends, and thus the associativity axiom for $\mathsf{J}^{\mathcal{T}}$ follows.

The braiding axiom (1.1.18) follows from naturality of the symmetry isomorphism for $\wedge$ in $\mathsf{Multicat}_*$ together with Proposition 10.1.28 (1) and (2). $\square$

Recall from Definitions 6.3.3 and 6.3.8 the enriched endomorphism multicategory and multifunctor associated to an enriched symmetric monoidal category and functor, respectively. Then Theorem 10.3.17 implies the following.

Corollary 10.3.24. *Taking* Cat_**-enriched endomorphism multicategories,*

$$\mathsf{J}^{\mathcal{T}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}$$

is a Cat_**-enriched multifunctor between* Cat_**-enriched multicategories.*

Recall from Definition 10.2.13 the canonical left $\mathcal{M}\underline{1}$-module structure on $\mathsf{End}(\mathsf{C})$ for a small permutative category C. With this choice of module structure, Lemma 10.2.14 shows that End factors, as a Cat-enriched multifunctor, through the category of left $\mathcal{M}\underline{1}$-modules in $\mathsf{Multicat}_*$. Now we consider the composite with the symmetric monoidal Cat_*-functor $\mathsf{J}^{\mathcal{T}}$ from Definition 10.3.9 and Theorem 10.3.17:

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{End}} \mathsf{Mod}^{\mathcal{M}\underline{1}} \xrightarrow{\mathsf{J}^{\mathcal{T}}} \mathcal{G}_*\text{-}\mathsf{Cat}.$$

By Corollary 10.3.24 and forgetting to Cat, this is a composite of Cat-enriched multifunctors.

Definition 10.3.25. We define the *Elmendorf-Mandell J-theory* as the composite Cat-enriched multifunctor

$$\mathsf{J}^{\mathsf{EM}} = \mathsf{J}^{\mathcal{T}} \circ \mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}. \tag{10.3.26}$$

For a small permutative category C, the associated $\mathcal{G}_*$-category is

$$\mathsf{J}^{\mathsf{EM}}\mathsf{C} = \mathsf{Multicat}_*(\mathcal{T}(-), \mathsf{End}(\mathsf{C})) : \mathcal{G} \longrightarrow \mathsf{Cat}_*. \tag{10.3.27}$$ ⋄

Now we define the Elmendorf-Mandell K-theory multifunctor as a composite of the following. In each case except the first, we have symmetric monoidal sSet-categories and -functors. Recall from Definitions 6.3.3 and 6.3.8 the enriched endomorphism multicategory and multifunctor constructions. Proposition 6.3.10 shows that an enriched symmetric monoidal functor induces an enriched multifunctor on endomorphism multicategories. When applying this result, we will suppress End from the notation.

- By Definition 6.5.1 and Lemma 10.2.14, $\mathsf{PermCat}^{\mathsf{su}}$ is a Cat-enriched multicategory such that

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}} \tag{10.3.28}$$

  is a Cat-enriched multifunctor. Using the same notation, we change enrichment along the nerve functor N as described for enriched multicategories in Section 6.3. Proposition 6.2.9 shows that the change of enrichment makes (10.3.28) a sSet-enriched multifunctor.
- Theorem 10.3.17 shows that

$$\mathsf{J}^{\mathcal{T}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat} \tag{10.3.29}$$

  is a symmetric monoidal Cat_*-functor. Using the same notation, we apply change of enrichment along N, as described for enriched symmetric monoidal categories in Chapter 2. By Theorem 2.4.10 this makes $\mathsf{J}^{\mathcal{T}}$ a symmetric monoidal sSet_*-functor of symmetric monoidal sSet_*-categories. Forgetting basepoints of hom objects (another change of enrichment) and passing to sSet-multicategories via Proposition 6.3.10, we obtain (10.3.29) as a sSet-enriched multifunctor.
- Theorem 9.2.19 shows that taking levelwise nerves induces a symmetric monoidal sSet_*-functor

$$N_* : \mathcal{G}_*\text{-}\mathsf{Cat} \longrightarrow \mathcal{G}_*\text{-}\mathsf{sSet}. \tag{10.3.30}$$

  Forgetting basepoints of hom objects and passing to sSet-multicategories via Proposition 6.3.10, we regard (10.3.30) as a sSet-enriched multifunctor.
- Theorem 9.4.9 shows that

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp} \tag{10.3.31}$$

  is a unital symmetric monoidal sSet_*-functor. Forgetting basepoints of hom objects and passing to sSet-multicategories via Proposition 6.3.10, we obtain (10.3.31) as a sSet-enriched multifunctor.

Definition 10.3.32. The *Elmendorf-Mandell K-theory*

$$\mathsf{K}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp}$$

is the sSet-enriched multifunctor given by the composite

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{End}} \mathsf{Mod}^{\mathcal{M}\underline{1}} \xrightarrow{\mathsf{J}^{\mathcal{T}}} \mathcal{G}_*\text{-}\mathsf{Cat} \xrightarrow{N_*} \mathcal{G}_*\text{-}\mathsf{sSet} \xrightarrow{\mathsf{K}^{\mathcal{G}}} \mathsf{SymSp}$$

of (10.3.28) through (10.3.31). ⋄

We will use the sSet-enriched multifunctoriality of K^{EM} as follows. Recall the following notions from Section 6.1 for a symmetric monoidal category V.

(1) A V-enriched operad is a V-enriched multicategory with one object (Definition 6.1.8).
(2) A V-enriched operad morphism is a V-enriched multifunctor between V-enriched operads (Definition 6.1.10).
(3) Suppose that P is a V-enriched operad and M is a V-enriched multicategory. We say that (X, θ) is an algebra over P if X is an object of M and θ is a multifunctor

$$\theta : \mathsf{P} \longrightarrow \mathsf{M}$$

that sends the single object of P to X. Equivalently,

$$\theta : \mathsf{P} \longrightarrow \mathsf{End}(X)$$

is an operad morphism from P to the endomorphism operad of X (Example 6.1.9 and Definition 6.1.14).

For a Cat-enriched operad P, we let $N\mathsf{P} = \mathsf{P}_N$ denote the sSet-enriched operad given by change of enrichment (Definition 6.2.1) along the nerve functor. The sSet-enriched multifunctoriality of K^{EM} implies the following result.

Theorem 10.3.33. *Suppose* P *is a* Cat*-enriched operad and suppose* (C, θ) *is a* P*-algebra in* $\mathsf{PermCat}^{\mathsf{su}}$. *Then* $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ *is an* $N\mathsf{P}$*-algebra in* SymSp.

Proof. Consider the following composite of multifunctors.

$$\begin{tikzcd} \mathsf{P} \arrow[r, "\theta"] & \mathsf{PermCat}^{\mathsf{su}} \arrow[r, "\mathsf{K}^{\mathsf{EM}}"] & \mathsf{SymSp} \\ * \arrow[r, mapsto] & \mathsf{C} \arrow[r, mapsto] & \mathsf{K}^{\mathsf{EM}}\mathsf{C} \end{tikzcd}$$

Then the composite sSet-enriched operad morphism

$$\begin{tikzcd} N\mathsf{P} \arrow[d, "\theta_N"'] & & \mathsf{End}(\mathsf{K}^{\mathsf{EM}}\mathsf{C}) \\ N\mathsf{End}(\mathsf{C}) \arrow[r, "\mathsf{J}^{\mathsf{EM}}_N"] & N\mathsf{End}(\mathsf{J}^{\mathsf{EM}}\mathsf{C}) \arrow[r, "N_*"] & \mathsf{End}(N_*\mathsf{J}^{\mathsf{EM}}\mathsf{C}) \arrow[u, "\mathsf{K}^{\mathcal{G}}"'] \end{tikzcd}$$

gives $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ the structure of an $N\mathsf{P}$-algebra. □

In Chapters 11, 12, and 13 we show that the higher monoidal structures on small permutative categories described in Part II.2 are encoded via Cat-enriched operad actions. We apply Theorem 10.3.33 in Corollaries 11.3.16, 12.5.3, and 13.5.2 to show that the associated spectra given by K^{EM} have E_n-monoidal structure for $1 \leq n \leq \infty$.

10.4. Elmendorf-Mandell $\mathcal{G}_*$-categories

In this section we define a $\mathcal{G}_*$-category $\mathsf{C}^{\mathcal{G}} = \mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}$ for each small permutative category C, along with two other variants $\mathsf{C}^{\mathcal{G}}_{\cong}$ and $\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}$. These generalize the three constructions of Γ-categories $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$, $\mathsf{C}^{\mathcal{F}}_{\cong}$, and $\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$ of Section 8.3. In Proposition 10.5.1 below we show that $\mathsf{J}^{\mathsf{EM}}\mathsf{C}$ and $\mathsf{C}^{\mathcal{G}} = \mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}$ are naturally isomorphic as $\mathcal{G}_*$-categories.

Recall from Definition 6.5.3 that for $\langle s \rangle \in \mathsf{Prof}(S)$ and $t \in S$, we let $\langle s \circ_k t \rangle$ denote the tuple with t replacing s_k. Likewise for $k \neq \ell$ we let $\langle s \circ_k t \circ_l t' \rangle$ denote the tuple with t replacing s_k and t' replacing s_ℓ.

Definition 10.4.1. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category and $\langle \underline{n} \rangle$ is an object of $\mathcal{G}$ with length $q > 0$. An $\langle \underline{n} \rangle$*-system* in C is a pair

$$(C, \rho) = \{C_{\langle s \rangle}, \rho_{\langle s \rangle;k,t,u}\}$$

consisting of

- a system of objects, with $C_{\langle s \rangle} \in \mathsf{C}$ for each q-tuple of basepoint-free subsets $s_k \subset \underline{n}_k^\flat$, and
- a system of morphisms, with

$$\rho_{\langle s \rangle;k,t,u} : C_{\langle s \circ_k t \rangle} \oplus C_{\langle s \circ_k u \rangle} \longrightarrow C_{\langle s \rangle}$$

  for each $k \in \overline{q}$ and each partition (t, u) of s_k.

We call $\rho_{\langle s \rangle;k,t,u}$ the (t,u)*-gluing morphism of* s_k. These data are subject to the following axioms.

Object Unity: If $s_k = \varnothing$ for any k, then

$$C_{\langle s \rangle} = e. \tag{10.4.2}$$

Gluing Unity: If $s_j = \varnothing$ for any j, or if $t = \varnothing$, or if $u = \varnothing$, then

$$\rho_{\langle s \rangle;k,t,u} = 1. \tag{10.4.3}$$

Gluing Symmetry: For each $k \in \overline{q}$, the following diagram commutes for all partitions (t, u) of s_k.

$$\begin{tikzcd}
C_{\langle s \circ_k t \rangle} \oplus C_{\langle s \circ_k u \rangle} \arrow[r, "\rho_{\langle s \rangle;k,t,u}"] \arrow[d, "\xi"'] & C_{\langle s \rangle} \arrow[d, "1"] \\
C_{\langle s \circ_k u \rangle} \oplus C_{\langle s \circ_k t \rangle} \arrow[r, "\rho_{\langle s \rangle;k,u,t}"] & C_{\langle s \rangle}
\end{tikzcd} \tag{10.4.4}$$

Gluing Associativity: For each $k \in \overline{q}$, the following diagram commutes for all partitions (t, u, v) of s_k.

$$\begin{tikzcd}
C_{\langle s \circ_k t \rangle} \oplus C_{\langle s \circ_k u \rangle} \oplus C_{\langle s \circ_k v \rangle} \arrow[r, "\rho_{\langle s \circ_k t \cup u \rangle;k,t,u} \oplus 1"] \arrow[d, "1 \oplus \rho_{\langle s \circ_k u \cup v \rangle;k,u,v}"'] & C_{\langle s \circ_k t \cup u \rangle} \oplus C_{\langle s \circ_k v \rangle} \arrow[d, "\rho_{\langle s \rangle;k,t \cup u,v}"] \\
C_{\langle s \circ_k t \rangle} \oplus C_{\langle s \circ_k u \cup v \rangle} \arrow[r, "\rho_{\langle s \rangle;k,t,u \cup v}"] & C_{\langle s \rangle}
\end{tikzcd} \tag{10.4.5}$$

Gluing 2-By-2: For each $k \neq \ell$, the following pentagon commutes for all partitions (t, u) of s_k and (v, w) of s_ℓ.

$$\begin{tikzcd}
& C_{\langle s\circ_\ell v\rangle} \oplus C_{\langle s\circ_\ell w\rangle} \arrow[dr, "\rho_{\langle s\rangle;\ell,v,w}"] & \\
C_{\langle s\circ_k t\circ_\ell v\rangle} \oplus C_{\langle s\circ_k u\circ_\ell v\rangle} \oplus C_{\langle s\circ_k t\circ_\ell w\rangle} \oplus C_{\langle s\circ_k u\circ_\ell w\rangle} \arrow[ur, "\rho_{\langle s\circ_\ell v\rangle;k,t,u} \oplus \rho_{\langle s\circ_\ell w\rangle;k,t,u}"] \arrow[d, "1 \oplus \xi \oplus 1"'] & & C_{\langle s\rangle} \\
C_{\langle s\circ_k t\circ_\ell v\rangle} \oplus C_{\langle s\circ_k t\circ_\ell w\rangle} \oplus C_{\langle s\circ_k u\circ_\ell v\rangle} \oplus C_{\langle s\circ_k u\circ_\ell w\rangle} \arrow[dr, "\rho_{\langle s\circ_k t\rangle;\ell,v,w} \oplus \rho_{\langle s\circ_k u\rangle;\ell,v,w}"'] & & \\
& C_{\langle s\circ_k t\rangle} \oplus C_{\langle s\circ_k u\rangle} \arrow[uur, "\rho_{\langle s\rangle;k,t,u}"'] &
\end{tikzcd} \tag{10.4.6}$$

This finishes the definition of an $\langle \underline{n} \rangle$-system in C. We also have the following variant definitions and terms.

- We will also refer to an $\langle \underline{n} \rangle$-system as a *lax* $\langle \underline{n} \rangle$*-system*.
- A *strong* $\langle \underline{n} \rangle$*-system* is an $\langle \underline{n} \rangle$-system for which all of the gluing morphisms $\rho_{\langle s\rangle;k,t,u}$ are isomorphisms.
- A *colax* $\langle \underline{n} \rangle$*-system* is an $\langle \underline{n} \rangle$-system in C^{op}. That is, the direction of each gluing morphism is reversed but the same axioms (with reversed arrows ρ) are satisfied. $\diamond$

Definition 10.4.7. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category and suppose given $\langle \underline{n} \rangle$-systems

$$\{C_{\langle s\rangle}, \rho_{\langle s\rangle;k,t,u}\} \quad \text{and} \quad \{C'_{\langle s\rangle}, \rho'_{\langle s\rangle;k,t,u}\}$$

for $\langle \underline{n} \rangle$ in $\mathcal{G}$ with length $q > 0$. A *morphism of* $\langle \underline{n} \rangle$*-systems*, denoted

$$\{\alpha_{\langle s\rangle}\} : \{C_{\langle s\rangle}, \rho_{\langle s\rangle;k,t,u}\} \longrightarrow \{C'_{\langle s\rangle}, \rho'_{\langle s\rangle;k,t,u}\},$$

consists of component morphisms

$$\alpha_{\langle s\rangle} : C_{\langle s\rangle} \longrightarrow C'_{\langle s\rangle}$$

for each q-tuple of basepoint-free subsets $s_k \subset \underline{n}_k^\flat$. These components are subject to the following axioms.

Unitary: If $s_k = \varnothing$ for any k, then

$$\alpha_{\langle s\rangle} = 1_e. \tag{10.4.8}$$

Gluing Compatibility: The following diagram commutes for all k and for each partition (t,u) of s_k.

$$\begin{tikzcd} C_{\langle s\circ_k t\rangle} \oplus C_{\langle s\circ_k u\rangle} \arrow[r, "\rho_{\langle s\rangle;k,t,u}"] \arrow[d, "\alpha_{\langle s\circ_k t\rangle} \oplus \alpha_{\langle s\circ_k u\rangle}"'] & C_{\langle s\rangle} \arrow[d, "\alpha_{\langle s\rangle}"] \\ C'_{\langle s\circ_k t\rangle} \oplus C'_{\langle s\circ_k u\rangle} \arrow[r, "\rho'_{\langle s\rangle;k,t,u}"] & C'_{\langle s\rangle} \end{tikzcd} \tag{10.4.9}$$

This finishes the definition of a morphism of $\underline{n}$-systems. We also have the following variant definitions.

- A morphism of strong $\langle \underline{n}\rangle$-systems is defined as above, with
$$\{C_{\langle s\rangle}, \rho_{\langle s\rangle;k,t,u}\} \quad \text{and} \quad \{C'_{\langle s\rangle}, \rho'_{\langle s\rangle;k,t,u}\}$$
strong $\langle \underline{n}\rangle$-systems.
- A morphism of colax $\langle \underline{n}\rangle$-systems is defined as above, with
$$\{C_{\langle s\rangle}, \rho_{\langle s\rangle;k,t,u}\} \quad \text{and} \quad \{C'_{\langle s\rangle}, \rho'_{\langle s\rangle;k,t,u}\}$$
colax $\langle \underline{n}\rangle$-systems, but the gluing compatibility axiom (10.4.9) has the variance of ρ reversed as in the following diagram.

$$\begin{tikzcd} C_{\langle s\circ_k t\rangle} \oplus C_{\langle s\circ_k u\rangle} \arrow[d, "\alpha_{\langle s\circ_k t\rangle} \oplus \alpha_{\langle s\circ_k u\rangle}"'] & C_{\langle s\rangle} \arrow[l, "\rho_{\langle s\rangle;k,t,u}"'] \arrow[d, "\alpha_{\langle s\rangle}"] \\ C'_{\langle s\circ_k t\rangle} \oplus C'_{\langle s\circ_k u\rangle} & C'_{\langle s\rangle} \arrow[l, "\rho'_{\langle s\rangle;k,t,u}"'] \end{tikzcd}$$

The identity morphism for a lax, strong, or colax $\langle \underline{n}\rangle$-system $\{C_{\langle s\rangle}, \rho_{\langle s\rangle;k,t,u}\}$ consists of identities
$$\alpha_{\langle s\rangle} = 1_{C_{\langle s\rangle}}$$
for each q-tuple of basepoint-free subsets $s_k \subset \underline{n}_k^\flat$. Composition of morphisms is defined componentwise. This composition is associative and unital because the composition in C is so. $\diamond$

Definition 10.4.10. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category. If $\langle \underline{n}\rangle$ is an object of $\mathcal{G}$ with length $q > 0$, define a pointed category $\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$ as follows.

- The objects of $\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$ are the lax $\langle \underline{n}\rangle$-systems in C.
- The morphisms of $\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$ are the morphisms of lax $\langle \underline{n}\rangle$-systems.
- The basepoint of $\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$ is the constant $\langle \underline{n}\rangle$-system with
$$C_{\langle s\rangle} = e \quad \text{and} \quad \rho_{\langle s\rangle;k,t,u} = 1_e$$
for each $\langle s\rangle$, k, t, and u.

This finishes the definition of $\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$.

For the empty tuple, we define
$$\mathsf{C}^{\mathcal{G}}\langle\rangle = \mathsf{C} \tag{10.4.11}$$
with basepoint given by the unit e. For the basepoint $*$ of $\mathcal{G}$ we define
$$\mathsf{C}^{\mathcal{G}}* = \mathbf{1}, \tag{10.4.12}$$

the terminal category. If $\langle \underline{n} \rangle$ is an object of $\mathcal{G}$ such that some $n_j = 0$, then by the object unity condition (10.4.2) and the gluing unit condition (10.4.3), the only $\langle \underline{n} \rangle$-system is that given by the constant system at $(e, 1_e)$. In this case we define

$$\mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle = \mathsf{C}^{\mathcal{G}} * = \mathbf{1}.$$

We also have the following variant definitions and notation. In each case we take the value at the empty tuple $\langle \rangle$ to be C as in (10.4.11) and the value at $*$ to be $\mathbf{1}$ as in (10.4.12). This also defines the value at tuples $\langle \underline{n} \rangle$ where some $n_j = 0$.

- To emphasize that the objects of $\mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$ are the lax $\langle \underline{n} \rangle$-systems, we also use the notation
$$\mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}\langle \underline{n} \rangle = \mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle.$$
- For $\langle \underline{n} \rangle$ of positive length, and with each $n_j > 0$, we let
$$\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle$$
denote the category of strong $\langle \underline{n} \rangle$-systems and morphisms of such. The basepoint is that of $\mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$.
- For $\langle \underline{n} \rangle$ of positive length, and with each $n_j > 0$, we let
$$\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\langle \underline{n} \rangle$$
denote the category of colax $\langle \underline{n} \rangle$-systems and morphisms of such. The basepoint is that of $\mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$. ◇

Explanation 10.4.13. In contrast with Definition 8.3.9 using $\mathsf{C}^{\mathcal{F}}\underline{n}$ for the category of strong $\underline{n}$-systems, we use the unadorned notation $\mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$ to denote the category of lax $\langle \underline{n} \rangle$-systems because this will be the one we use most frequently in our applications below. See Explanation 10.7.23 for further discussion of this point. ◇

Example 10.4.14. Suppose $p > 0$ and let $\langle \underline{1} \rangle$ denote the constant p-tuple at $\underline{1}$. Then there is a canonical isomorphism of pointed categories

$$\mathsf{C} \cong \mathsf{C}^{\mathcal{G}}\langle \underline{1} \rangle \tag{10.4.15}$$

where an object C of C corresponds to the $\langle \underline{1} \rangle$-system whose only (possibly) nontrivial object is C and whose gluing morphisms are all identities. A morphism h of C corresponds to the morphism of $\langle \underline{1} \rangle$-systems whose only (possibly) nontrivial component is h. ◇

To define $\mathsf{C}^{\mathcal{G}}$ on morphisms of $\mathcal{G}$, we first make the following preliminary definitions. The complete definition of $\mathsf{C}^{\mathcal{G}}$ on morphisms is Definition 10.4.17 below. Recall from Definition 9.1.4 that for an injection

$$f : \overline{q} \hookrightarrow \overline{p}$$

and a q-tuple of pointed finite sets $\langle \underline{n} \rangle$, the p-tuple of pointed finite sets $f_*\langle \underline{n} \rangle$ has jth entry given by $\underline{n}_{f^{-1}(j)}$.

Definition 10.4.16. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category and suppose

$$(f, \langle \psi \rangle) : \langle \underline{n} \rangle \longrightarrow \langle \underline{m} \rangle$$

is a morphism in $\mathcal{G}$ where $\langle \underline{n} \rangle$ has length $q > 0$ and $\langle \underline{m} \rangle$ has length $p > 0$. So

$$f : \overline{q} \hookrightarrow \overline{p}$$

is an injection of sets and

$$\langle \psi \rangle : f_*\langle \underline{n} \rangle \longrightarrow \langle \underline{m} \rangle$$

is a morphism in $\mathcal{F}^p$.

(1) For a p-tuple of basepoint-free subsets $\langle s\rangle$ with

$$s_j \subset (f_*\langle \underline{n}\rangle)_j^\flat = \underline{n}^\flat_{f^{-1}(j)},$$

let $\tilde{f}_*\langle s\rangle$ be the q-tuple obtained by removing entries not indexed by the image of f and then permuting according to (the inverse of) f. Then for each $\langle \underline{n}\rangle$-system

$$(C,\rho) = \{C_{\langle s\rangle}, \rho_{\langle s\rangle;k,t,u}\} \in \mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$$

and each morphism of $\langle \underline{n}\rangle$-systems $\{\alpha_{\langle s\rangle}\}$, we define data for objects and morphisms in $\mathsf{C}^{\mathcal{G}}(f_*\langle \underline{n}\rangle)$ with the following:

$$C^f_{\langle s\rangle} = \begin{cases} C_{\tilde{f}_*\langle s\rangle} & \text{if each } s_j \neq \varnothing, \\ e & \text{otherwise,}\end{cases}$$

$$\rho^f_{\langle s\rangle;k,t,u} = \begin{cases} \rho_{\tilde{f}_*\langle s\rangle;\, f^{-1}(k),\, t,\, u} & \text{if each } s_j \neq \varnothing, \\ 1 & \text{otherwise,}\end{cases}$$

and

$$\alpha^f_{\langle s\rangle} = \begin{cases} \alpha_{\tilde{f}_*\langle s\rangle} & \text{if each } s_j \neq \varnothing, \\ 1 & \text{otherwise.}\end{cases}$$

If, in the definition of $\rho^f_{\langle s\rangle;k,t,u}$, we have $f^{-1}(k) = \varnothing$, then $s_k \subset \overline{1}$ and so at least one of t and u is empty. In this case $\rho^f_{\langle s\rangle;k,t,u}$ is the identity at $C_{\tilde{f}_*\langle s\rangle}$ (if no s_j is empty) or at e (if any s_j is empty). We define a pointed functor

$$\tilde{f} : \mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle \longrightarrow \mathsf{C}^{\mathcal{G}}(f_*\langle \underline{n}\rangle)$$

by $X \longmapsto X^f$ where X is any of

$$C_{\langle s\rangle}, \quad \rho_{\langle s\rangle;k,t,u}, \quad \text{or} \quad \alpha_{\langle s\rangle}.$$

The definition of $\alpha^f_{\langle s\rangle}$ shows that $\tilde{f}$ is functorial and the definition of $C^f_{\langle s\rangle}$ shows that $\tilde{f}$ is pointed.

(2) For each p-tuple of basepoint-free subsets $\langle s\rangle$ with $s_j \subset \underline{m}_j^\flat$, we let $\langle \psi^{-1}(s)\rangle$ be the tuple whose kth component is $\psi_k^{-1}(s_k)$, a subset of the kth entry in $f_*\langle \underline{n}\rangle$. Then for each $f_*\langle \underline{n}\rangle$-system

$$(C,\rho) = \{C_{\langle s\rangle}, \rho_{\langle s\rangle;k,t,u}\} \in \mathsf{C}^{\mathcal{G}}(f_*\langle \underline{n}\rangle)$$

and each morphism of $f_*\langle \underline{n}\rangle$-systems $\{\alpha_{\langle s\rangle}\}$ we define data for objects and morphisms in $\mathsf{C}^{\mathcal{G}}\langle \underline{m}\rangle$ by letting

$$\begin{aligned} C^{\langle\psi\rangle}_{\langle s\rangle} &= C_{\langle \psi^{-1}(s)\rangle}, \\ \rho^{\langle\psi\rangle}_{\langle s\rangle;k,t,u} &= \rho_{\langle \psi^{-1}(s)\rangle;\, k,\, \psi_k^{-1}(t),\, \psi_k^{-1}(u)}, \quad \text{and} \\ \alpha^{\langle\psi\rangle}_{\langle s\rangle} &= \alpha_{\langle \psi^{-1}s\rangle}. \end{aligned}$$

We define a pointed functor

$$\langle \tilde{\psi}\rangle : \mathsf{C}^{\mathcal{G}}(f_*\langle \underline{n}\rangle) \longrightarrow \mathsf{C}^{\mathcal{G}}\langle \underline{m}\rangle$$

by $X \longmapsto X^{\langle\psi\rangle}$ where X is any of

$$\mathsf{C}_{\langle s\rangle}, \quad \rho_{\langle s\rangle;k,t,u}, \quad \text{or} \quad \alpha_{\langle s\rangle}.$$

The definition of $\alpha^{\langle\psi\rangle}_{\langle s\rangle}$ shows that $\langle\tilde{\psi}\rangle$ is functorial and the definition of $\mathsf{C}^{\langle\psi\rangle}_{\langle s\rangle}$ shows that $\langle\tilde{\psi}\rangle$ is pointed.

We use the same definitions and notation above if the systems (C, ρ) are strong or colax. $\diamond$

Now we define $\mathsf{C}^{\mathcal{G}}$, $\mathsf{C}^{\mathcal{G}}_{\cong}$, and $\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}$ on the morphisms of $\mathcal{G}$.

Definition 10.4.17. Suppose $(\mathsf{C}, \oplus, e)$ is a small permutative category and suppose given a morphism

$$\Psi : N \longrightarrow M$$

in $\mathcal{G}$. If either N or M is the basepoint $*$, then define $\mathsf{C}^{\mathcal{G}}\Psi$ as the corresponding zero morphism in Cat_*.

Otherwise, Ψ is of the form

$$\Psi = (f, \langle\psi\rangle) : \langle \underline{n}\rangle \longrightarrow \langle \underline{m}\rangle.$$

If $\langle \underline{m}\rangle$ is the empty tuple, $\langle\rangle$, then so is $\langle \underline{n}\rangle$ and Ψ is either the zero morphism or the identity. In either case $\mathsf{C}^{\mathcal{G}}\Psi$ is uniquely determined.

Now suppose $\langle \underline{m}\rangle$ has length $p > 0$. If $\langle \underline{n}\rangle$ is the empty tuple, then Ψ factors uniquely as a composite (9.1.10)

$$\langle\rangle \longrightarrow \langle \underline{1}\rangle \xrightarrow{\langle\psi\rangle} \langle \underline{m}\rangle$$

for $\langle\psi\rangle$ in $\mathcal{F}^p$ (see Explanation 9.1.9 (4)). Then we define $\mathsf{C}^{\mathcal{G}}\Psi$ to be the composite

$$\mathsf{C}^{\mathcal{G}}\langle\rangle = \mathsf{C} \xrightarrow{\cong} \mathsf{C}^{\mathcal{G}}\langle \underline{1}\rangle \xrightarrow{\langle\tilde{\psi}\rangle} \mathsf{C}^{\mathcal{G}}\langle \underline{m}\rangle$$

of the canonical isomorphism (10.4.15) with $\langle\tilde{\psi}\rangle$ as in Definition 10.4.16 above.

Now suppose, moreover, that $\langle \underline{n}\rangle$ has length $q > 0$. Then we define $\mathsf{C}^{\mathcal{G}}(f, \langle\psi\rangle)$ to be the composite pointed functor

$$\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle \xrightarrow{\tilde{f}} \mathsf{C}^{\mathcal{G}}(f_*\langle \underline{n}\rangle) \xrightarrow{\langle\tilde{\psi}\rangle} \mathsf{C}^{\mathcal{G}}\langle \underline{m}\rangle$$

with $\tilde{f}$ and $\langle\tilde{\psi}\rangle$ as in Definition 10.4.16 above.

This finishes the definition of $\mathsf{C}^{\mathcal{G}}\Psi$, which we sometimes also denote $\mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}\Psi$. We let

$$\mathsf{C}^{\mathcal{G}}_{\cong}\Psi : \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle \longrightarrow \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{m}\rangle$$

and

$$\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\Psi : \mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\langle \underline{n}\rangle \longrightarrow \mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\langle \underline{m}\rangle$$

be the pointed functors given by the same definitions as above for strong, respectively colax, $\langle \underline{n}\rangle$-systems and morphisms. $\diamond$

Taken together, Definitions 10.4.1, 10.4.7, 10.4.10, 10.4.16, and 10.4.17 give assignments on objects and morphisms

$$\begin{aligned} \mathsf{C}^{\mathcal{G}} = \mathsf{C}^{\mathcal{G}}_{\mathrm{lax}} : \mathcal{G} &\longrightarrow \mathsf{Cat}_*, \\ \mathsf{C}^{\mathcal{G}}_{\cong} : \mathcal{G} &\longrightarrow \mathsf{Cat}_*, \quad \text{and} \\ \mathsf{C}^{\mathcal{G}}_{\mathrm{co}} : \mathcal{G} &\longrightarrow \mathsf{Cat}_*. \end{aligned}$$

Now we show these give $\mathcal{G}_*$-categories.

Proposition 10.4.18. *Suppose* C *is a small permutative category. Each of the assignments*

$$\mathsf{C}^{\mathcal{G}} = \mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}, \quad \mathsf{C}^{\mathcal{G}}_{\cong}, \quad \textit{and} \quad \mathsf{C}^{\mathcal{G}}_{\mathrm{co}}$$

defines a $\mathcal{G}_*$*-category.*

Proof. We give the proof for $\mathsf{C}^{\mathcal{G}} = \mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}$; the argument for the other two cases is the same. By definition we have $\mathsf{C}^{\mathcal{G}}* = \mathbf{1}$, so $\mathsf{C}^{\mathcal{G}}$ preserves basepoints. Preservation of identity morphisms $(1, \langle 1 \rangle)$ can be checked directly from Definition 10.4.17.

Now we consider functoriality with respect to a composable pair of morphisms

$$\langle \underline{n} \rangle \xrightarrow{(f, \langle \psi \rangle)} \langle \underline{m} \rangle \xrightarrow{(g, \langle \phi \rangle)} \langle \underline{\ell} \rangle. \tag{10.4.19}$$

where $\langle \underline{n} \rangle$, $\langle \underline{m} \rangle$, and $\langle \underline{\ell} \rangle$ are tuples of length q, p, and r, respectively. If $r = 0$ or $p = 0$, then both p and q must be zero and $(f, \langle \psi \rangle)$ is either a zero or identity morphism so functoriality follows. So now suppose both $p > 0$ and $r > 0$. On tuples of basepoint-free and nonempty subsets $\langle s \rangle$, with s_k a subset of the kth entry in $g_* \langle \underline{m} \rangle$, we have

$$\langle \psi^{-1} \tilde{g}_* \langle s \rangle \rangle = \tilde{g}_* \langle (g_* \langle \psi \rangle)^{-1}(s) \rangle.$$

This shows that the following diagram commutes in Cat_*. In the diagram below, if $q = 0$, we let $f_* \langle \underline{n} \rangle$ and $(gf)_* \langle \underline{n} \rangle$ both be the constant tuple $\langle \underline{1} \rangle$ of length p.

$$\begin{tikzcd}
\mathsf{C}^{\mathcal{G}}(f_*\langle \underline{n} \rangle) \arrow[rr, "\mathsf{C}^{\mathcal{G}}(1, \langle \psi \rangle)"] \arrow[d, "\mathsf{C}^{\mathcal{G}}(g,1)"'] \arrow[drr, "\mathsf{C}^{\mathcal{G}}(g, g_*\langle \psi \rangle)"] & & \mathsf{C}^{\mathcal{G}}\langle \underline{m} \rangle \arrow[d, "\mathsf{C}^{\mathcal{G}}(g,1)"] \\
\mathsf{C}^{\mathcal{G}}((gf)_*\langle \underline{n} \rangle) \arrow[rr, "\mathsf{C}^{\mathcal{G}}(1, g_*\langle \psi \rangle)"] & & \mathsf{C}^{\mathcal{G}}(g_*\langle \underline{m} \rangle)
\end{tikzcd}$$

Then functoriality follows by checking for composites of morphisms $(f, 1)$ with $(g, 1)$ and $(1, \langle \psi' \rangle)$ with $(1, \langle \phi \rangle)$. □

Definition 10.4.20. Suppose C is a small permutative category. We call

- $\mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}$ the *lax Elmendorf-Mandell* $\mathcal{G}_*$*-category* of C,
- $\mathsf{C}^{\mathcal{G}}_{\cong}$ the *strong Elmendorf-Mandell* $\mathcal{G}_*$*-category* of C, and
- $\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}$ the *colax Elmendorf-Mandell* $\mathcal{G}_*$*-category* of C.

When used without an adjective, the phrase *Elmendorf-Mandell* $\mathcal{G}_*$*-category* will refer to the lax case. ◇

In Section 10.7 we give a direct comparison of these three variants of $\mathsf{C}^{\mathcal{G}}$ and show that the respective K-theory symmetric spectra are level equivalent.

10.5. An Equivalent Description of Elmendorf-Mandell J-theory

Now we relate the Elmendorf-Mandell $\mathcal{G}_*$-categories with the J-theory construction J^{EM} via partition multicategories in Definition 10.3.25. This result is a $\mathcal{G}_*$ version of Proposition 8.4.8 for J^{Se} and $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$.

Proposition 10.5.1. *Suppose* C *is a small permutative category. For each* $\langle \underline{n} \rangle \in \mathcal{G}$ *there is an isomorphism of pointed categories*

$$\mathsf{J}^{\mathsf{EM}}\mathsf{C}\langle \underline{n} \rangle = \mathsf{Multicat}_*(\mathcal{T}\langle \underline{n} \rangle, \mathsf{End}(\mathsf{C})) \cong \mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle.$$

These isomorphisms are the components of a pointed natural isomorphism

$$\mathsf{J}^{\mathsf{EM}}\mathsf{C} \cong \mathsf{C}^{\mathcal{G}}.$$

Proof. The first assertion follows, as in the proof of Proposition 8.4.8, from an explicit description of the left hand side. Recall from Definition 10.3.1 and Explanation 10.3.2 that by taking the smash product in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ we have

$$\mathcal{T}\langle \underline{n} \rangle = \wedge_k \mathcal{M}\underline{n}_k \tag{10.5.2}$$

for each $\langle \underline{n} \rangle$ in $\mathcal{G}$ with length $q \geq 0$. For $q = 0$, the empty smash product (10.5.2) is $\mathcal{M}\underline{1}$. In this case we have

$$(\mathsf{J}^{\mathsf{EM}}\mathsf{C})\langle\rangle = \mathsf{Multicat}_*(\mathcal{M}\underline{1}, \mathsf{End}(\mathsf{C})) \cong \mathsf{C} = \mathsf{C}^{\mathcal{G}}\langle\rangle$$

by Lemma 10.2.5. Recalling the proof, each pointed multifunctor on the left is determined by where it sends the object $\{1\}$ in $\mathcal{M}\underline{1}$.

For $\langle \underline{n} \rangle$ of length $q > 0$, we introduce the following notation extending that of Definition 6.5.3. Given a partition $\langle s_k \rangle = (s_{k,1}, \ldots, s_{k,q_k})$ of $s_k \subset \underline{n}_k^\flat$ for some k, and subsets $s_\ell \subset \underline{n}_\ell^\flat$ for $\ell \neq k$, let

$$\langle s \circ_k \langle s_k \rangle \rangle \in \mathsf{Prof}(\mathcal{T}\langle \underline{n} \rangle)$$

be the profile of length q_k whose jth entry is $\langle s \circ_k s_{k,j} \rangle$. Let

$$\langle s \circ_k \iota_{\langle s_k \rangle} \rangle \in \mathcal{T}\langle \underline{n} \rangle\big(\langle s \circ_k \langle s_k \rangle \rangle\,; \langle s \rangle\big)$$

be the tuple of operations whose ℓth entry is the unit 1_{s_ℓ} for $\ell \neq k$, and whose kth entry is $\iota_{\langle s_k \rangle}$. Recalling Explanation 5.6.14, we note that these operations generate the operations of the tensor product $\otimes_k \mathcal{M}\underline{n}_k$ and hence also of the smash product $\mathcal{T}\langle \underline{n} \rangle$.

Moreover, recalling the argument of Proposition 8.4.8 that the operations of each $\mathcal{M}\underline{n}_k$ are generated by 2-ary operations $\iota_{(t,u)}$, we see that the operations of $\mathcal{T}\langle \underline{n} \rangle$ are generated by $\langle s \circ_k \iota_{(t,u)} \rangle$. Therefore, a pointed multifunctor

$$F : \bigwedge_k \mathcal{M}\underline{n}_k \longrightarrow \mathsf{End}(\mathsf{C})$$

determines and is uniquely determined by the following data:

- a family of objects $C_{\langle s \rangle} = F\langle s \rangle \in \mathsf{C}$ for tuples of subsets $s_k \in \underline{n}_k^\flat$, and
- a family of morphisms

$$\rho_{\langle s \rangle;k,t,u} = F\langle s \circ_k \iota_{(t,u)} \rangle : C_{\langle s \circ_k t \rangle} \oplus C_{\langle s \circ_k u \rangle} \longrightarrow C_{\langle s \rangle}.$$

These data are subject to relations for symmetry, compatibility, and interchange detailed in Explanations 5.6.9 and 5.6.14, along with relations for the smash product, Definition 5.6.20. One can verify that these correspond to the axioms of Definition 10.4.1 just as was done in the proof of Proposition 8.4.8.

Similarly, a pointed multinatural transformation $\alpha : F \longrightarrow F'$ determines and is uniquely determined by a family of morphisms in C,

$$\alpha_{\langle s \rangle} : C_{\langle s \rangle} \longrightarrow C'_{\langle s \rangle},$$

satisfying basepoint and naturality axioms corresponding to the axioms for a morphism in $\mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$.

Now we verify naturality of the isomorphisms

$$\mathsf{J}^{\mathsf{EM}}\mathsf{C}\langle \underline{n} \rangle \longrightarrow \mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$$

with respect to morphisms $(f,\langle\psi\rangle)$ in $\mathcal{G}$. For each $p > 0$, naturality for the morphism

$$(i_p, 1) : \langle\rangle \longrightarrow \underline{1}$$

to the constant p-tuple at $\underline{1}$ follows by comparison of Explanation 10.3.6 with Example 10.4.14. Both composites of the naturality square

$$\begin{tikzcd}
\mathsf{Multicat}_*(\mathcal{M}\underline{1}, \mathsf{End}(\mathsf{C})) \arrow[r, "\cong"] \arrow[d, "(\mathcal{T}(i_p,0))^*"', "\cong"] & \mathsf{C}^{\mathcal{G}}\langle\rangle \arrow[d, "\cong"', "\mathsf{C}^{\mathcal{G}}(i_p,0)"] \\
\mathsf{Multicat}_*(\wedge_{k=1}^{p}\mathcal{M}\underline{1}, \mathsf{End}(\mathsf{C})) \arrow[r, "\cong"] & \mathsf{C}^{\mathcal{G}}\langle\underline{1}\rangle
\end{tikzcd}$$

are given as follows.

- A pointed multifunctor F is sent to the $\underline{1}$-system whose only (possibly) nontrivial object is $F\{1\}$.
- A pointed multinatural transformation θ is sent to the morphism of $\underline{1}$-systems whose only (possibly) nontrivial component is $\theta_{\{1\}}$.

Naturality with respect to all other morphisms $(f,\langle\psi\rangle)$ follows by comparing the definition of $\mathcal{T}$ on morphisms, written in (10.3.5), with that of $\mathsf{C}^{\mathcal{G}}$, written in Definitions 10.4.16 and 10.4.17. The morphism denoted $\langle\tilde{\psi}\rangle$ corresponds to $\wedge_j\tilde{\psi}_j$ and the morphism denoted $\tilde{f}$ corresponds to the composite given by permutation and the isomorphisms $\prod_{\underline{1},\underline{m}}$ and $\prod_{\underline{m},\underline{1}}$. □

It follows from Definition 10.3.32 and Proposition 10.5.1 that there is a natural isomorphism of symmetric spectra

$$\mathsf{K}^{\mathsf{EM}}\mathsf{C} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathsf{EM}}\mathsf{C} \cong \mathsf{K}^{\mathcal{G}} N_* \mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}$$

with $\mathsf{C}^{\mathcal{G}}_{\mathrm{lax}} = \mathsf{C}^{\mathcal{G}}$, for small permutative categories C. Using the other two variants of Elmendorf-Mandell $\mathcal{G}_*$-categories in Definition 10.4.20, $\mathsf{C}^{\mathcal{G}}_{\cong}$ and $\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}$, we define the following variants of Elmendorf-Mandell K-theory.

Definition 10.5.3. Suppose C is a small permutative category. We define the symmetric spectra

$$\mathsf{K}^{\mathsf{EM}}_{\cong}\mathsf{C} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{C}^{\mathcal{G}}_{\cong} \quad \text{and}$$
$$\mathsf{K}^{\mathsf{EM}}_{\mathrm{co}}\mathsf{C} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{C}^{\mathcal{G}}_{\mathrm{co}},$$

which are called, respectively, the *strong Elmendorf-Mandell K-theory* and the *colax Elmendorf-Mandell K-theory* of C. ⋄

In Theorems 10.7.16 and 10.7.19, we will show that the three variants of Elmendorf-Mandell K-theory—namely, $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$, $\mathsf{K}^{\mathsf{EM}}_{\cong}\mathsf{C}$, and $\mathsf{K}^{\mathsf{EM}}_{\mathrm{co}}\mathsf{C}$ in Definitions 10.3.32 and 10.5.3—are level equivalent symmetric spectra.

Explanation 10.5.4 (Multifunctoriality of $(-)^{\mathcal{G}}$)**.** By Proposition 10.5.1, the object assignment $\mathsf{C} \longmapsto \mathsf{C}^{\mathcal{G}}$ is part of a Cat-enriched multifunctor

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{(-)^{\mathcal{G}}} \mathcal{G}_*\text{-}\mathsf{Cat}.$$

Here we describe in detail this Cat-enriched multifunctor. With the exception of the case $n = 0$, the description below is essentially the proof of [**EM06**, Theorem 6.1]. First recall from

- Definitions 6.5.1, 6.5.4, and 6.5.11 that $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-enriched multicategory with small permutative categories, multilinear functors, and multilinear transformations, and
- Definition 9.2.8 and Theorem 9.2.15 the symmetric monoidal closed category $\mathcal{G}_*$-Cat with the Day convolution product $\wedge$ (9.2.9) and hom diagram (9.2.10), and with the permutative structure on $\mathcal{G}$ given by concatenation of tuples of pointed finite sets.

For small permutative categories $\mathsf{C}_1, \ldots, \mathsf{C}_n$, and D, the functor

$$\mathsf{PermCat}^{\mathsf{su}}\big(\mathsf{C}_1, \ldots, \mathsf{C}_n\,;\mathsf{D}\big) \xrightarrow{(-)^{\mathcal{G}}} \mathcal{G}_*\text{-}\mathsf{Cat}\big(\mathsf{C}_1^{\mathcal{G}}, \ldots, \mathsf{C}_n^{\mathcal{G}}\,;\mathsf{D}^{\mathcal{G}}\big) \tag{10.5.5}$$

is described as follows.

For $n = 0$, a 0-linear functor

$$\mathbf{1} \xrightarrow{F} \mathsf{D}$$

in $\mathsf{PermCat}^{\mathsf{su}}$ is uniquely determined by a choice of object $x \in \mathsf{D}$. By Lemma 10.2.5, such an object x determines a unique pointed multifunctor

$$U_x : \mathcal{M}\underline{1} \longrightarrow \mathsf{End}(\mathsf{D}) \quad \text{with} \quad U_x\{1\} = x. \tag{10.5.6}$$

With J the monoidal unit of $\mathcal{G}_*$-Cat, the image of F in (10.5.5) is given by the composite

(10.5.7)

$$J \xrightarrow{(\mathsf{J}^{\mathcal{T}})^0} \mathsf{J}^{\mathcal{T}}(\mathcal{M}\underline{1}) = \mathsf{Multicat}_*(\mathcal{T}(-), \mathcal{M}\underline{1}) \xrightarrow{(U_x)_*} \mathsf{Multicat}_*(\mathcal{T}(-), \mathsf{End}(\mathsf{D})) \cong \mathsf{D}^{\mathcal{G}}.$$

More explicitly, recall from (9.2.13) that

$$J\langle \underline{m}\rangle = \widehat{\mathcal{G}}(\langle\rangle, \langle \underline{m}\rangle)$$

for each object $\langle \underline{m}\rangle \in \mathcal{G}$. If $\langle \underline{m}\rangle$ has length 0, then the only nonzero morphism $\langle\rangle \longrightarrow \langle\rangle$ in $\mathcal{G}$ is the identity. The composite (10.5.7) sends $1_{\langle\rangle}$ to the object

$$x = F(*) \in \mathsf{D} = \mathsf{D}^{\mathcal{G}}\langle\rangle.$$

If $\langle \underline{m}\rangle$ has length $p > 0$, then, by Explanation 9.1.9 (4), a morphism

$$(i_p, \langle \psi\rangle) : \langle\rangle \longrightarrow \langle \underline{m}\rangle \in \mathcal{G}$$

is uniquely determined by the pointed functions

$$\psi_i : \underline{1} \longrightarrow \underline{m_i} \quad \text{for} \quad i \in \overline{p}.$$

The composite (10.5.7) sends $(i_p, \langle\psi\rangle)$ to the object $X \in \mathsf{D}^{\mathcal{G}}\langle \underline{m}\rangle$ given as follows. For a p-tuple $\langle s\rangle$ of unpointed finite sets with each $s_i \subset \underline{m_i}^\flat$ for $i \in \overline{p}$, the $\langle s\rangle$-component of X is given by

$$X_{\langle s\rangle} = \begin{cases} x & \text{if } \psi_i(1) \in s_i \text{ for each } i \in \overline{p}, \text{ and} \\ e & \text{otherwise.} \end{cases}$$

For the gluing morphisms, suppose $k \in \overline{p}$ and (t, u) is a partition of s_k. Then the (t, u)-gluing morphism

$$X_{\langle s \circ_k t\rangle} \oplus X_{\langle s \circ_k u\rangle} \xrightarrow{\rho_{\langle s\rangle; k,t,u}} X_{\langle s\rangle}$$

is given by

- 1_x if $\psi_i(1) \in s_i$ for each $i \in \overline{p}$, and

- 1_e otherwise.

For $n > 0$, (10.5.5) sends an n-linear functor

$$\mathsf{C}_1 \times \cdots \times \mathsf{C}_n \xrightarrow{\left(F,\,\{F_j^2\}_{j=1}^n\right)} \mathsf{D} \tag{10.5.8}$$

to a $\mathcal{G}_*$-category morphism

$$\bigwedge_{j=1}^n \mathsf{C}_j^{\mathcal{G}} = \mathsf{C}_1^{\mathcal{G}} \wedge \cdots \wedge \mathsf{C}_n^{\mathcal{G}} \xrightarrow{F^{\mathcal{G}}} \mathsf{D}^{\mathcal{G}}. \tag{10.5.9}$$

The iterated product on the left-hand side and below are assumed to be left normalized (2.5.4). For $p \geq 0$ and each p-tuple of pointed finite sets

$$\langle \underline{m} \rangle = \left(\underline{m_1}, \ldots, \underline{m_p}\right) \in \mathcal{F}^{(p)},$$

the $\mathcal{G}_*$-category morphism $F^{\mathcal{G}}$ has a component functor

$$\Big(\bigwedge_{j=1}^n \mathsf{C}_j^{\mathcal{G}}\Big)\langle \underline{m} \rangle \xrightarrow{F^{\mathcal{G}}\langle \underline{m} \rangle} \mathsf{D}^{\mathcal{G}}\langle \underline{m} \rangle.$$

Since $\bigwedge_{j=1}^n$ is an iterated Day convolution product of $\mathcal{G}_*$-categories, the functor $F^{\mathcal{G}}\langle \underline{m} \rangle$ is uniquely determined by component functors

$$\bigwedge_{j=1}^n \mathsf{C}_j^{\mathcal{G}}\left(\underline{m_{i_{j-1}+1}}, \ldots, \underline{m_{i_j}}\right) \xrightarrow{F^{\mathcal{G}}\langle \underline{m} \rangle} \mathsf{D}^{\mathcal{G}}\langle \underline{m} \rangle \tag{10.5.10}$$

for sequences $(i_0 = 0, i_1, \ldots, i_n)$ with

$$1 \leq i_1 \leq \cdots \leq i_n = p.$$

The smash product of small categories in (10.5.10) is the one in (4.1.7) for $(\mathsf{Cat}, \times, *)$. Now we address two special cases.

- If $p = 0$, then $\langle \underline{m} \rangle$ is the empty tuple. In (10.5.10), each $\mathsf{C}_j^{\mathcal{G}}\langle\rangle = \mathsf{C}_j$ and $\mathsf{D}^{\mathcal{G}}\langle\rangle = \mathsf{D}$ by definition. The component functor $F^{\mathcal{G}}\langle\rangle$ is given by the given functor F in (10.5.8). It is well defined on the smash product in (10.5.10) by the unity axiom of an n-linear functor.
- If $p > 0$ with some $m_i = 0$, then $\langle \underline{m} \rangle = *$ in $\mathcal{G}$, and $\mathsf{D}^{\mathcal{G}}(*) = \mathbf{1}$, the terminal category. The component functor $F^{\mathcal{G}}(*)$ in (10.5.10) is the unique functor to the terminal category.

Now suppose $p > 0$ and each $m_j > 0$. To describe the functor $F^{\mathcal{G}}\langle \underline{m} \rangle$ in (10.5.10), first note that an object in the domain in (10.5.10) is represented by an n-tuple of objects

$$C = \left((C^1, \rho^1), \ldots, (C^n, \rho^n)\right) \in \prod_{j=1}^n \mathsf{C}_j^{\mathcal{G}}\left(\underline{m_{i_{j-1}+1}}, \ldots, \underline{m_{i_j}}\right). \tag{10.5.11}$$

For a p-tuple of pointed finite sets

$$\langle s \rangle = \left(s_1, \ldots, s_p\right) \quad \text{with each} \quad s_i \subset \underline{m_i}^{\flat}, \tag{10.5.12}$$

the object

$$(F^{\mathcal{G}}\langle \underline{m} \rangle)C \in \mathsf{D}^{\mathcal{G}}\langle \underline{m} \rangle$$

has as its $\langle s\rangle$-component the object

$$\big((F^{\mathcal{G}}\langle \underline{m}\rangle)C\big)_{\langle s\rangle} = F\Big\{C^j_{(s_{i_{j-1}+1},\ldots,s_{i_j})}\Big\}_{j=1}^n \in \mathsf{D}. \tag{10.5.13}$$

For a partition

$$t \cup u = s_k \quad \text{for some} \quad i_{r-1}+1 \le k \le i_r \quad \text{and} \quad 1 \le r \le n, \tag{10.5.14}$$

the gluing morphism ρ of $(F^{\mathcal{G}}\langle \underline{m}\rangle)C$ is the following composite in D.

$$\begin{tikzcd}
\big((F^{\mathcal{G}}\langle \underline{m}\rangle)C\big)_{\langle s\circ_k t\rangle} \oplus \big((F^{\mathcal{G}}\langle \underline{m}\rangle)C\big)_{\langle s\circ_k u\rangle} \arrow[r, "\rho_{\langle s\rangle;k,t,u}"] \arrow[d, equal] & \big((F^{\mathcal{G}}\langle \underline{m}\rangle)C\big)_{\langle s\rangle} \arrow[d, equal] \\
F\Big(\ldots,C^r_{(s_{i_{r-1}+1},\ldots,t,\ldots,s_{i_r})},\ldots\Big) \oplus F\Big(\ldots,C^r_{(s_{i_{r-1}+1},\ldots,u,\ldots,s_{i_r})},\ldots\Big) \arrow[d, "F_r^2"'] & F\Big\{C^j_{(s_{i_{j-1}+1},\ldots,s_{i_j})}\Big\}_{j=1}^n \\
F\Big(\ldots,C^r_{(s_{i_{r-1}+1},\ldots,t,\ldots,s_{i_r})} \oplus C^r_{(s_{i_{r-1}+1},\ldots,u,\ldots,s_{i_r})},\ldots\Big) \arrow[ur, "{F(1,\ldots,\rho^r,\ldots,1)}"'] &
\end{tikzcd} \tag{10.5.15}$$

In the above diagram, F_r^2 is the rth linearity constraint of the n-linear functor F, and ρ^r is the gluing morphism in the object (C^r,ρ^r) for the indicated sequences of finite sets and partition (10.5.14).

A morphism $\alpha : C \longrightarrow D$ in the domain in (10.5.10) is represented by an n-tuple of morphisms

$$\Big\{(C^j,\rho_C^j) \xrightarrow{\alpha^j} (D^j,\rho_D^j)\Big\}_{j=1}^n \in \prod_{j=1}^n \mathsf{C}_j^{\mathcal{G}}\big(\underline{m_{i_{j-1}+1}},\ldots,\underline{m_{i_j}}\big).$$

With $\langle s\rangle$ as in (10.5.12), the morphism

$$(F^{\mathcal{G}}\langle \underline{m}\rangle)C \xrightarrow{(F^{\mathcal{G}}\langle \underline{m}\rangle)\alpha} (F^{\mathcal{G}}\langle \underline{m}\rangle)D \in \mathsf{D}^{\mathcal{G}}\langle \underline{m}\rangle$$

has as its $\langle s\rangle$-component the morphism

$$\big((F^{\mathcal{G}}\langle \underline{m}\rangle)\alpha\big)_{\langle s\rangle} = F\Big\{\alpha^j_{(s_{i_{j-1}+1},\ldots,s_{i_j})}\Big\}_{j=1}^n \in \mathsf{D}. \tag{10.5.16}$$

This finishes the description of the $\mathcal{G}_*$-category morphism $F^{\mathcal{G}}$ in (10.5.9).

Suppose θ is the following n-linear transformation between n-linear functors.

$$\begin{tikzcd}
\mathsf{C}_1 \times \cdots \times \mathsf{C}_n \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathsf{D}
\end{tikzcd} \tag{10.5.17}$$

For $n = 0$, θ is uniquely determined by a morphism $f : x \longrightarrow y$, where x and y are the objects determined by F and G. By Lemma 10.2.5, f determines a pointed multinatural transformation

$$U_f : U_x \longrightarrow U_y : \mathcal{M}\underline{1} \longrightarrow \mathsf{End}(\mathsf{D}) \quad \text{with} \quad (U_f)_{\{1\}} = f,$$

where U_x and U_y are as in (10.5.6). This induces a modification

$$(U_f)_* : (U_x)_* \longrightarrow (U_y)_* : \mathsf{J}^{\mathcal{T}}(\mathcal{M}\underline{1}) \longrightarrow \mathsf{Multicat}_*(\mathcal{T}(-),\mathsf{End}(\mathsf{D})) \cong \mathsf{D}^{\mathcal{G}}.$$

Pre-whiskering $(U_f)_*$ with $(\mathsf{J}^{\mathcal{T}})^0$ gives the image of θ in (10.5.5).

Now suppose $n > 0$ in (10.5.17). With C as in (10.5.11) and $\langle s\rangle$ as in (10.5.12), $(\theta^{\mathcal{G}}\langle \underline{m}\rangle)_C$ has as its $\langle s\rangle$-component the morphism

$$\begin{tikzcd}
\big((F^{\mathcal{G}}\langle \underline{m}\rangle)C\big)_{\langle s\rangle} = F\Big\{C^j_{(s_{i_{j-1}+1},\ldots,s_{i_j})}\Big\}_{j=1}^n \arrow[d, "\theta_{\left\{C^j_{(s_{i_{j-1}+1},\ldots,s_{i_j})}\right\}_{j=1}^n}"] \\
\big((G^{\mathcal{G}}\langle \underline{m}\rangle)C\big)_{\langle s\rangle} = G\Big\{C^j_{(s_{i_{j-1}+1},\ldots,s_{i_j})}\Big\}_{j=1}^n
\end{tikzcd}$$

in D. This finishes the description of the functor $(-)^{\mathcal{G}}$ in (10.5.5). $\diamond$

10.6. Equivalence Between Segal K-Theory and Elmendorf-Mandell K-Theory

This section is devoted to proving Theorem 10.6.10, which asserts that the symmetric spectra $\mathsf{K}^{\mathsf{Se}}\mathsf{C}$ and $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ are naturally level equivalent. Our comparison begins with the observation that, for a tuple of pointed finite sets $\langle a\rangle$, we can also form the smash product

$$\wedge_k a_k \cong (\textstyle\prod_k a_k^\flat)_+$$

and apply $\mathcal{M}$. This is the composite of $\mathcal{M}$ with (the opposite of) the functor

$$\wedge : \mathcal{G} \longrightarrow \mathcal{F}$$

discussed in Definition 9.1.15.

The multicategory $\mathcal{T}\langle a\rangle$ relates to $\mathcal{M}(\wedge_k a_k)$ as follows. The following extends the partition product of Definition 10.1.1 to q-fold products for $q > 2$.

Definition 10.6.1. Suppose $\langle a\rangle = (a_1, \ldots, a_q)$ is a q-tuple of pointed finite sets, and $\wedge_k a_k$ is their smash product. For $q > 0$, define

$$\textstyle\prod : \mathcal{T}\langle a\rangle \longrightarrow \mathcal{M}(\wedge_k a_k)$$

by

$$\langle s\rangle \longmapsto \prod_k s_k \subset \prod_k a_k^\flat \subset \wedge_k a_k$$

where $\langle s\rangle$ is a q-tuple of basepoint-free subsets $s_k \subset a_k^\flat$. For $q = 0$, we take $\underline{1}$ as the empty smash product in FinSet_* and define

$$\textstyle\prod = 1_{\mathcal{M}\underline{1}} : \mathcal{M}\underline{1} = \mathcal{T}\langle\rangle \longrightarrow \mathcal{M}\underline{1}.$$

We show that $\prod$ is a well-defined pointed multifunctor in Proposition 10.6.2. $\diamond$

The next result follows by induction on the binary case of Definition 10.1.1, but we also give a detailed direct proof.

Proposition 10.6.2. *In the context of Definition 10.6.1, suppose given a q-tuple of pointed finite sets $\langle a\rangle$ for $q \geq 0$. Then*

$$\textstyle\prod : \mathcal{T}\langle a\rangle \longrightarrow \mathcal{M}(\wedge_k a_k)$$

is a well-defined pointed multifunctor.

Proof. For $q = 0$, $\prod$ is the identity and there is nothing to prove. For $q > 0$, the essential observation that makes $\prod$ well defined is that taking products preserves disjunction. To explain more clearly, recall the following notation from Definition 9.1.4 and the proof of Proposition 10.5.1. Given a partition

$$\langle s_k\rangle = (s_{k,1}, \ldots, s_{k,q_k})$$

of $s_k \subset a_k^\flat$ and subsets $s_\ell \subset a_\ell^\flat$ for $\ell \neq k$,

$$\langle s \circ_k s_{k,j}\rangle \in \mathcal{T}\langle a\rangle$$

denotes the tuple whose ℓth entry is s_ℓ for $\ell \neq k$, and whose kth entry is $s_{k,j}$. Then

$$\langle s \circ_k \langle s_k\rangle\rangle \in \mathsf{Prof}(\mathcal{T}\langle a\rangle)$$

is the profile of length q_k whose jth entry is $\langle s \circ_k s_{k,j}\rangle$. Finally,

$$\langle s \circ_k \iota_{\langle s_k\rangle}\rangle \in \mathcal{T}\big(\langle s \circ_k \langle s_k\rangle\rangle\,;\,\langle s\rangle\big)$$

is the tuple of operations whose ℓth entry is the unit 1_{s_ℓ} for $\ell \neq k$, and whose kth entry is $\iota_{\langle s_k\rangle}$. Recalling Explanation 5.6.14, we note that these operations generate the operations of the smash product $\mathcal{T}\langle a\rangle$.

With this notation, the key observation is the following. Since the $s_{k,j}$ are disjoint for $j \in \{1,\ldots,q_k\}$, the products

$$\prod\langle s \circ_k s_{k,j}\rangle = \prod_{u \in \langle s \circ_k s_{k,j}\rangle} u \quad\text{and}\quad \prod\langle s \circ_k s_{k,i}\rangle = \prod_{u \in \langle s \circ_k s_{k,i}\rangle} u$$

are disjoint for $j \neq i$. Therefore, $\langle \prod\langle s \circ_k s_{k,j}\rangle\rangle_{j=1}^{q_k}$ is a partition of $\prod\langle s\rangle$ and we define

$$\prod\langle s \circ_k \iota_{\langle s_k\rangle}\rangle = \iota_{\langle \prod\langle s\circ_k s_{k,j}\rangle\rangle_{j=1}^{q_k}}.$$

This defines $\prod$ as an assignment on objects and operations. Since there is at most one operation in $\mathcal{M}(\wedge_k a_k)$ with each given input profile and output, commutativity of the multifunctor axioms is automatic. □

Next we have a generalization of the naturality Lemma 10.1.4. Here too the result follows by induction on the binary case, but we also give a detailed direct proof.

Proposition 10.6.3. *The pointed multifunctor $\prod$ is natural with respect to the morphisms of $\mathcal{G}^{\mathsf{op}}$ and defines a natural transformation of functors*

$$\prod : \mathcal{T} \longrightarrow (\mathcal{M} \circ \wedge).$$

Proof. Recalling (9.1.16) and (10.3.5), naturality with respect to a morphism

$$(f, \langle\psi\rangle) : \langle \underline{n}\rangle \longrightarrow \langle \underline{m}\rangle,$$

is verified by commutativity of the outer rectangle in the following diagram. In the upper row we take smash products of left $\mathcal{M}\underline{1}$-modules so that empty smash products are $\mathcal{M}\underline{1}$. In the lower row empty smash products are $\underline{1}$.

$$\begin{tikzcd}
\bigwedge_j \mathcal{M}\underline{m}_j \arrow[r, "\wedge_j\tilde{\psi}_j"] \arrow[d, "\prod"] & \bigwedge_j \mathcal{M}(\underline{n}_{f^{-1}(j)}) \arrow[r, "\cong"] \arrow[d, "\prod"] & \bigwedge_{f^{-1}(j)\neq\varnothing} \mathcal{M}(\underline{n}_{f^{-1}(j)}) \arrow[r, "\cong", "\text{permute}"'] \arrow[d, "\prod"] & \bigwedge_k \mathcal{M}\underline{n}_k \arrow[d, "\prod"] \\
\mathcal{M}[\wedge_j \underline{m}_j] \arrow[r, "\widetilde{\wedge_j\psi_j}"] & \mathcal{M}[\wedge_j (\underline{n}_{f^{-1}(j)})] \arrow[r, "\cong"] & \mathcal{M}\Big[\wedge_{f^{-1}(j)\neq\varnothing} (\underline{n}_{f^{-1}(j)})\Big] \arrow[r, "\cong", "\text{permute}"'] & \mathcal{M}[\wedge_k \underline{n}_k]
\end{tikzcd}$$

In the diagram above, the leftmost square commutes because $\wedge_j\psi_j$ is defined componentwise and hence, as a function on tuples of subsets $\langle t\rangle$ with $t_j \subset \underline{m}_j^\flat$, we have

$$\wedge_j(\psi_j^{-1}(t_j)) = (\wedge_j\psi_j)^{-1}(\wedge_j t_j).$$

The two composites in the middle square are equal because both are given by

$$\langle s\rangle \longmapsto \bigwedge_{f^{-1}(j)\neq\varnothing} s_{f^{-1}(j)}$$

for $s_j \subset \underline{n}_j^\flat$. The two composites in the rightmost square are equal because both are given by permuting factors of a smash product according to the action of f^{-1} on its indexing set. $\square$

There are projections

$$\wedge_k a_k \longrightarrow a_k$$

for each k, but these do *not* give well-defined multifunctors

$$\mathcal{M}(\wedge_k a_k) \longrightarrow \mathcal{M}a_k.$$

Disjoint subsets of a product may project to nondisjoint subsets. However, there is an intermediary given as follows. Recall from Example 5.1.9 we let M^u denote the unary multicategory associated to a category M.

Definition 10.6.4. Suppose $\langle a\rangle = (a_1, \ldots, a_q)$ is a tuple of pointed finite sets for $q > 0$. Let

$$\mathcal{P}\langle a\rangle = \Big(\big(\prod_k a_k^\flat\big)^u\Big)_+$$

denote the pointed multicategory obtained by adjoining a disjoint basepoint T to the discrete unary multicategory whose objects are the elements of the product.

Note that the objects of $\mathcal{P}\langle a\rangle$ correspond to the elements of $\wedge_k a_k$, but $\mathcal{P}\langle a\rangle$ differs from the discrete unary multicategory on $\wedge_k a_k$ because a discrete unary multicateory has no non-identity operations and therefore cannot be a pointed multicategory. $\diamond$

Remark 10.6.5. Note that $\mathcal{P}$ is *not* defined on morphisms of $\mathcal{G}$. For a morphism $\psi : \underline{n} \longrightarrow \underline{m}$ in $\mathcal{F}$, the inverse image of an element $x \in \underline{m}$ is generally not a singleton subset of $\underline{n}$. Therefore, the definitions of $\mathcal{M}$ or $\mathcal{T}$ on morphisms of $\mathcal{F}$ or $\mathcal{G}$, respectively, do not specialize to give well-defined morphisms $\mathcal{P}\langle \underline{m}\rangle \longrightarrow \mathcal{P}\langle \underline{n}\rangle$. $\diamond$

Definition 10.6.6. Suppose $\langle a\rangle$ is a q-tuple of pointed finite sets for $q > 0$. We define two pointed multifunctors

$$\mathcal{T}\langle a\rangle \xleftarrow{\;i\;} \mathcal{P}\langle a\rangle \xrightarrow{\;j\;} \mathcal{M}(\wedge_k a_k)$$

by taking elements of $\prod_k a_k^\flat$ to singletons in two different ways. For each tuple of elements $\langle x\rangle = (x_1, \ldots, x_q) \in \prod_k a_k^\flat$, we define

$$\begin{aligned} i\langle x\rangle &= (\{x_1\}, \ldots, \{x_q\}) \quad \text{and} \\ j\langle x\rangle &= \{(x_1, \ldots, x_q)\}. \end{aligned}$$

So i sends $\langle x\rangle$ to the tuple of the singletons and j sends $\langle x\rangle$ to the singleton of the tuple. $\diamond$

Proposition 10.6.7. *Suppose* $\mathsf{C} = (\mathsf{C}, \oplus, e)$ *is a small permutative category. Then for each tuple of pointed finite sets* $\langle a\rangle = (a_1, \ldots, a_q)$ *with* $q > 0$, *the pointed multifunctors* j *and* i *induce adjunctions of pointed categories*

$$\mathsf{Multicat}_*(\mathcal{T}\langle a\rangle, \mathsf{End}(\mathsf{C})) \xrightarrow{\;i^*\;} \mathsf{Multicat}_*(\mathcal{P}\langle a\rangle, \mathsf{End}(\mathsf{C}))$$

and

$$\mathsf{Multicat}_*(\mathcal{M}(\wedge_k a_k), \mathsf{End}(\mathsf{C})) \xrightarrow{j^*} \mathsf{Multicat}_*(\mathcal{P}\langle a\rangle, \mathsf{End}(\mathsf{C})).$$

Proof. We begin with the assertion about j^*. A pointed multifunctor from $\mathcal{P}\langle a\rangle$ to $\mathsf{End}(\mathsf{C})$ necessarily sends the non-unary operations of the basepoint T to those of $e \in \mathsf{C}$. Therefore,

$$\mathsf{Multicat}_*(\mathcal{P}\langle a\rangle, \mathsf{End}(\mathsf{C})) = \mathsf{Cat}_*(a, \mathsf{C}),$$

where the basepoint of C is e and

$$a = \Big(\prod_k a_k^\flat\Big)_+ \cong \bigwedge_k a_k. \tag{10.6.8}$$

By applying Proposition 8.5.4 to a and using the isomorphism (10.6.8), we see that j^* is a strictly unital symmetric monoidal right adjoint in $\mathsf{PermCat}^{\mathsf{su}}$.

For the assertion about i^*, we apply Proposition 8.5.4 to each individual pointed finite set a_k to obtain strictly unital symmetric monoidal adjunctions

$$\mathsf{Multicat}_*(\mathcal{M}a_k, \mathsf{End}(\mathsf{C})) \longrightarrow \mathsf{Cat}_*(a_k, \mathsf{C}) = \mathsf{Multicat}_*(\mathcal{P}a_k, \mathsf{End}(\mathsf{C})). \tag{10.6.9}$$

By Corollary 5.3.9, End is 2-functorial with respect to strictly unital symmetric monoidal functors. Applying End to both sides of (10.6.9) and using Lemma 5.7.21 we have, therefore, an internal right adjoint in $\mathsf{Multicat}_*$

$$\mathsf{Hom}_*(\mathcal{M}a_k, \mathsf{End}(\mathsf{C})) \longrightarrow \mathsf{Hom}_*(\mathcal{P}a_k, \mathsf{End}(\mathsf{C}))$$

for each $a_k \in \langle a\rangle$.

Therefore, by the $\wedge$-Hom_* adjunction of Theorem 5.7.22 we have, for example, the following sequence of equivalences and right adjoints in Cat_*:

$$\begin{aligned}
\mathsf{Multicat}_*(\mathcal{M}(a_1) \wedge \mathcal{M}(a_2), \mathsf{End}(\mathsf{C})) &\simeq \mathsf{Multicat}_*(\mathcal{M}(a_1), \mathsf{Hom}_*(\mathcal{M}(a_2), \mathsf{End}(\mathsf{C})))\\
&\longrightarrow \mathsf{Multicat}_*(\mathcal{M}(a_1), \mathsf{Hom}_*(\mathcal{P}(a_2), \mathsf{End}(\mathsf{C})))\\
&\simeq \mathsf{Multicat}_*(\mathcal{M}(a_1) \wedge \mathcal{P}(a_2), \mathsf{End}(\mathsf{C}))\\
&\simeq \mathsf{Multicat}_*(\mathcal{P}(a_2), \mathsf{Hom}_*(\mathcal{M}(a_1), \mathsf{End}(\mathsf{C})))\\
&\longrightarrow \mathsf{Multicat}_*(\mathcal{P}(a_2), \mathsf{Hom}_*(\mathcal{P}(a_1), \mathsf{End}(\mathsf{C})))\\
&\simeq \mathsf{Multicat}_*(\mathcal{P}(a_1) \wedge \mathcal{P}(a_2), \mathsf{End}(\mathsf{C})).
\end{aligned}$$

Now the assertion for i^* follows from induction and the identification

$$\wedge_k \mathcal{P}(a_k) \cong \mathcal{P}\langle a\rangle. \qquad \square$$

Recall from Definition 9.1.15 the smash product of pointed finite sets defines a pointed functor

$$\wedge : \mathcal{G} \longrightarrow \mathcal{F}.$$

Composition then provides a change-of-diagram functor

$$\Gamma\text{-}\mathsf{Cat} \xrightarrow{-\circ\wedge} \mathcal{G}_*\text{-}\mathsf{Cat}$$

denoted $\wedge^*$ below.

Theorem 10.6.10. *Suppose C is a small permutative category. The product*

$$\prod : \mathcal{T} \longrightarrow \mathcal{M}(\wedge(-))$$

of Proposition 10.6.2 induces a level equivalence of symmetric spectra

$$\mathsf{K}^{\mathsf{Se}}\mathsf{C} \longrightarrow \mathsf{K}^{\mathsf{EM}}\mathsf{C}$$

that is natural with respect to strictly unital symmetric monoidal functors.

Proof. Recall from Proposition 10.6.3 that $\prod$ is natural with respect to morphisms in $\mathcal{G}$. Therefore, by functoriality of

$$\mathsf{Multicat}_*(-,\mathsf{End}(\mathsf{C})) : (\mathsf{Multicat}_*)^{\mathsf{op}} \longrightarrow \mathsf{Cat}_*$$

and

$$N : \mathsf{Cat}_* \longrightarrow \mathsf{sSet}_*,$$

we have a morphism of $\mathcal{G}_*$-simplicial sets

$$\begin{aligned} N_*\textstyle\prod^* : N_* \wedge^* \mathsf{J}^{\mathsf{Se}}\mathsf{C} &= N_*\mathsf{Multicat}_*\big(\mathcal{M}(\wedge(-)),\mathsf{End}(\mathsf{C})\big) \\ &\longrightarrow N_*\mathsf{Multicat}_*\big(\mathcal{T}(-),\mathsf{End}(\mathsf{C})\big) = N_*\mathsf{J}^{\mathsf{EM}}\mathsf{C} \end{aligned} \tag{10.6.11}$$

where

- N_* denotes the levelwise composition with N and
- $\wedge^*$ is the functor along the top of the following commutative diagram in Cat, induced by the smash product of pointed finite sets.

$$\begin{tikzcd} \Gamma\text{-}\mathsf{Cat} \arrow[r, "\wedge^*"] \arrow[d, "N_*"'] & \mathcal{G}_*\text{-}\mathsf{Cat} \arrow[d, "N_*"] \\ \Gamma\text{-}\mathsf{sSet} \arrow[r, "\wedge^*"] & \mathcal{G}_*\text{-}\mathsf{sSet} \end{tikzcd}$$

Since the above diagram commutes, and since we have

$$\mathsf{K}^{\mathcal{F}} = \mathsf{K}^{\mathcal{G}}\wedge^*$$

by Proposition 9.3.16, then (10.6.11) becomes

$$\begin{aligned} \mathsf{K}^{\mathsf{Se}}\mathsf{C} &= \mathsf{K}^{\mathcal{F}}N_*\mathsf{J}^{\mathsf{Se}}\mathsf{C} = \mathsf{K}^{\mathcal{G}}\wedge^* N_*\mathsf{J}^{\mathsf{Se}}\mathsf{C} \\ &= \mathsf{K}^{\mathcal{G}}N_*\wedge^* \mathsf{J}^{\mathsf{Se}}\mathsf{C} \xrightarrow{\mathsf{K}^{\mathcal{G}}(N_*\prod^*)} \mathsf{K}^{\mathcal{G}}N_*\mathsf{J}^{\mathsf{EM}}\mathsf{C} = \mathsf{K}^{\mathsf{EM}}\mathsf{C}. \end{aligned} \tag{10.6.12}$$

The morphism of symmetric spectra (10.6.12) is natural with respect to morphisms in $\mathsf{PermCat}^{\mathsf{su}}$ by naturality of $N_*\prod^*$ and functoriality of $\mathsf{K}^{\mathcal{G}}$.

We show (10.6.12) is a level equivalence of symmetric spectra by showing that each component of (10.6.11) at $\langle\underline{n}\rangle$ in $\mathcal{G}$,

$$N\mathsf{Multicat}_*\big(\mathcal{M}(\wedge_k\underline{n}_k),\mathsf{End}(\mathsf{C})\big) \longrightarrow N\mathsf{Multicat}_*\big(\mathcal{T}\langle\underline{n}\rangle,\mathsf{End}(\mathsf{C})\big), \tag{10.6.13}$$

is a simplicial homotopy equivalence.

If $\langle\underline{n}\rangle = \langle\rangle$, then $\prod$ is the identity and there is nothing more to prove. Now suppose $\langle\underline{n}\rangle \in \mathcal{G}$ has length $q > 0$. Then we have the following diagram of pointed multifunctors involving i and j of Definition 10.6.6 along with the component of $\prod$ at $\langle\underline{n}\rangle$.

$$\begin{tikzcd} \mathcal{M}(\wedge_k \underline{n}_k) & \\ \mathcal{P}\langle\underline{n}\rangle \arrow[u, "j"] \arrow[r, "i"'] & \mathcal{T}\langle\underline{n}\rangle \arrow[ul, "\prod"'] \end{tikzcd}$$

This diagram in $\mathsf{Multicat}_*$ commutes because, for a tuple $\langle x\rangle$ of elements $x_k \in \underline{n}_k^\flat$, the product of singleton subsets $\{x_k\}$ is the singleton of the tuple $\langle x\rangle$.

Applying the functor

$$\mathsf{Multicat}_*(-, \mathsf{End}(\mathsf{C})) : (\mathsf{Multicat}_*)^{\mathsf{op}} \longrightarrow \mathsf{Cat}_*,$$

we have the following commutative diagram of pointed functors.

$$\begin{tikzcd}
(\mathsf{J}^{\mathsf{Se}}\mathsf{C})(\wedge_k \underline{n}_k) \arrow[d, "j^*"'] \arrow[dr, "\prod^*"] & \\
\mathsf{Multicat}_*(\mathcal{P}\langle \underline{n} \rangle, \mathsf{End}(\mathsf{C})) & (\mathsf{J}^{\mathsf{EM}}\mathsf{C})\langle \underline{n} \rangle \arrow[l, "i^*"]
\end{tikzcd}$$

The vertical and horizontal functors are right adjoints of pointed categories by Proposition 10.6.7 and, therefore, induce simplicial homotopy equivalences on nerves by Proposition 7.2.5. This implies that (10.6.13),

$$(N_* \textstyle\prod^*)_{\langle \underline{n} \rangle} : N(\mathsf{J}^{\mathsf{Se}}\mathsf{C}(\wedge_k \underline{n}_k)) \longrightarrow N(\mathsf{J}^{\mathsf{EM}}\mathsf{C}(\langle \underline{n} \rangle)),$$

is a simplicial homotopy equivalence for each $\langle \underline{n} \rangle$ in $\mathcal{G}$. $\square$

10.7. Comparison of (Co)lax and Strong Elmendorf-Mandell $\mathcal{G}_*$-Categories

Throughout this section, $(\mathsf{C}, \oplus, e, \xi)$ denotes a small permutative category. The main objective of this section is to compare the categories $\mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}\langle \underline{n} \rangle$, $\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle$, and $\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\langle \underline{n} \rangle$ in Definition 10.4.10, which we refer to as, respectively, the *lax*, *strong*, and *colax* versions. Recall that, by definition, if either

- $\langle \underline{n} \rangle = \langle \rangle$ or
- some $n_j = 0$, so $\langle \underline{n} \rangle = *$ in $\mathcal{G}$,

then each of $\mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}\langle \underline{n} \rangle$, $\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle$, and $\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\langle \underline{n} \rangle$ is defined in the same way—either C or $\mathbf{1}$.

After some preliminary definitions, Proposition 10.7.10 shows that $\mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}\langle \underline{n} \rangle$ and $\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle$ are related by an adjunction. Theorem 10.7.16 shows that, as $\langle \underline{n} \rangle$ varies in $\mathcal{G}$, the left adjoints assemble into a morphism of $\mathcal{G}_*$-categories, which induces a level equivalence in K-theory symmetric spectra. Proposition 10.7.22 shows that there is, furthermore, a multifunctorial property with respect to the small permutative category C. Theorem 10.7.19 and Proposition 10.7.27 are the corresponding results for the colax and strong versions. Explanation 10.7.23 discusses some advantages of K^{EM} over $\mathsf{K}^{\mathsf{EM}}_{\cong}$.

Levelwise Adjunctions.

Definition 10.7.1. We define an object (10.7.3) and a morphism (10.7.6) here. In Lemma 10.7.8 below, we show that they are well defined. Suppose

$$\langle \underline{n} \rangle = (\underline{n_1}, \ldots, \underline{n_q}) \in \mathcal{F}^{(q)}$$

with $q > 0$ and each $n_j > 0$. Suppose

$$C = (C, \rho) = \left\{ C_{\langle s \rangle}, \rho_{\langle s \rangle; k,t,u} \right\} \in \mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle \tag{10.7.2}$$

is an $\langle \underline{n} \rangle$-system as in Definition 10.4.1. Define an object

$$\left(\widetilde{C}, \widetilde{\rho} \right) \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle \tag{10.7.3}$$

as follows. For a tuple

$$\langle s\rangle = (s_1, \ldots, s_q) \quad \text{with each} \quad s_j \subset \underline{n_j}^{\flat}, \tag{10.7.4}$$

define the component object

$$\widetilde{C}_{\langle s\rangle} = \bigoplus_{(x_1,\ldots,x_q)\in\prod_{j=1}^q s_j} C_{(\{x_1\},\ldots,\{x_q\})} \in \mathsf{C},$$

with the sequences

$$(x_1, \ldots, x_q) \in \prod_{j=1}^q s_j \subset \prod_{j=1}^q \underline{n}_j^{\flat} \tag{10.7.5}$$

ordered by the lexicographical order of $\wedge_{j=1}^q \underline{n}_j$ in Definition 8.1.5. For a partition (t, u) of s_k with $1 \le k \le q$, the gluing morphism

$$\widetilde{C}_{\langle s\circ_k t\rangle} \oplus \widetilde{C}_{\langle s\circ_k u\rangle} \xrightarrow[\cong]{\widetilde{\rho}_{\langle s\rangle;k,t,u}} \widetilde{C}_{\langle s\rangle} \in \mathsf{C}$$

is

- the identity morphism if t, u, or any s_j is empty, and
- the unique coherence isomorphism in C that permutes factors otherwise.

Moreover, for a morphism

$$\alpha = \left\{\alpha_{\langle s\rangle} : C_{\langle s\rangle} \longrightarrow D_{\langle s\rangle}\right\} : C \longrightarrow D \in \mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle,$$

define a morphism

$$\left(\widetilde{C}, \widetilde{\rho}_C\right) \xrightarrow{\widetilde{\alpha}} \left(\widetilde{D}, \widetilde{\rho}_D\right) \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle \tag{10.7.6}$$

with the components

$$\widetilde{\alpha}_{\langle s\rangle} = \bigoplus_{(x_1,\ldots,x_q)\in\prod_{j=1}^q s_j} \alpha_{(\{x_1\},\ldots,\{x_q\})} : \widetilde{C}_{\langle s\rangle} \longrightarrow \widetilde{D}_{\langle s\rangle} \in \mathsf{C}$$

and $\langle s\rangle$ as in (10.7.4). ⋄

Explanation 10.7.7. In $\langle s\rangle = (s_1, \ldots, s_q)$ in (10.7.4), suppose that either

- each s_j is a one-element set for $1 \le j \le q$, or
- some $s_j = \varnothing$.

Then

$$\widetilde{C}_{\langle s\rangle} = C_{\langle s\rangle} \quad \text{and} \quad \widetilde{\alpha}_{\langle s\rangle} = \alpha_{\langle s\rangle}$$

in Definition 10.7.1. ⋄

Lemma 10.7.8. *The constructions (10.7.3) and (10.7.6) define a functor*

$$\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle \xrightarrow{\widetilde{(-)}} \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle.$$

Proof. The pair $\left(\widetilde{C}, \widetilde{\rho}\right)$ in (10.7.3) satisfies the axioms in Definition 10.4.1 for an object in $\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle$ by the Symmetric Coherence Theorem 1.1.41 in C, since the gluing isomorphisms $\widetilde{\rho}$ are coherence isomorphisms in C. The collection $\widetilde{\alpha}$ in (10.7.6) is compatible with the gluing isomorphisms in $\widetilde{C}$ and $\widetilde{D}$ by the naturality of the structure isomorphisms in C. So $\widetilde{\alpha}$ is a morphism in $\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle$. Identity morphisms and composition in $\mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$ are defined componentwise. So the functoriality of $\oplus$ in C implies that $\widetilde{(-)}$ preserves identity morphisms and composition. □

Definition 10.7.9. For each q-tuple of pointed finite sets $\langle \underline{n} \rangle \in \mathcal{F}^{(q)}$ with $q \geq 0$, define two functors

$$\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle \underset{R^{\mathsf{C}}_{\langle \underline{n} \rangle}}{\overset{L^{\mathsf{C}}_{\langle \underline{n} \rangle}}{\rightleftarrows}} \mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$$

as follows.

- If $\langle \underline{n} \rangle = \langle \rangle$ or if any $n_j = 0$, then $R^{\mathsf{C}}_{\langle \underline{n} \rangle}$ and $L^{\mathsf{C}}_{\langle \underline{n} \rangle}$ are both defined to be identity functors.
- Otherwise:
  - $R^{\mathsf{C}}_{\langle \underline{n} \rangle} = \widetilde{(-)}$ is the functor in Lemma 10.7.8.
  - $L^{\mathsf{C}}_{\langle \underline{n} \rangle}$ is also defined by the assignments (10.7.3) and (10.7.6), applied to the objects and morphisms in $\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle$.

The proof of Lemma 10.7.8 applies to $L^{\mathsf{C}}_{\langle \underline{n} \rangle}$ to show that it is a functor. If C and $\langle \underline{n} \rangle$ are understood, then we abbreviate $L^{\mathsf{C}}_{\langle \underline{n} \rangle}$ and $R^{\mathsf{C}}_{\langle \underline{n} \rangle}$ to, respectively, L and R. ◇

Proposition 10.7.10. *In the context of Definition 10.7.9, the following statements hold.*

(1) $L \dashv R$ is an adjunction with unit a natural isomorphism.
(2) If, in addition, C is a groupoid, then $L \dashv R$ is an adjoint equivalence.

Proof. There is nothing to check if $\langle \underline{n} \rangle = \langle \rangle$ or if any $n_j = 0$. For the remainder of the proof we assume $q > 0$ and each $n_j > 0$. For the first assertion, we

- define the unit and the counit for (L, R) and
- check the triangle identities (I.1.1.11).

The component of the unit

$$1_{\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle} \xrightarrow{\eta} RL$$

at a strong $\langle \underline{n} \rangle$-system

$$C = (C, \rho) = \left\{ C_{\langle s \rangle}, \rho_{\langle s \rangle;k,t,u} \right\} \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle \tag{10.7.11}$$

is the morphism

$$C \xrightarrow{\eta_C} RLC \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle$$

with, for $\langle s \rangle = (s_1, \ldots, s_q)$ as in (10.7.4), components

$$C_{\langle s \rangle} \xrightarrow{(\eta_C)_{\langle s \rangle}} \bigoplus_{(x_1,\ldots,x_q) \in \prod_{j=1}^q s_j} C_{(\{x_1\},\ldots,\{x_q\})} = (RLC)_{\langle s \rangle} \in \mathsf{C}$$

given by

- the identity morphism in each of the two cases in Explanation 10.7.7 and
- composites of inverses of the gluing isomorphisms ρ in C otherwise.

The second case above is well defined by the invertibility of ρ in C and the axioms in Definition 10.4.1. The condition that η_C is a morphism in $\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle$ means that the diagram

$$\begin{tikzcd} C_{\langle s \circ_k t\rangle} \oplus C_{\langle s\circ_k u\rangle} \arrow[r, "\rho_{\langle s\rangle;k,t,u}"] \arrow[d, "(\eta_C)_{\langle s\circ_k t\rangle} \oplus (\eta_C)_{\langle s\circ_k u\rangle}"'] & C_{\langle s\rangle} \arrow[d, "(\eta_C)_{\langle s\rangle}"] \\ (RLC)_{\langle s\circ_k t\rangle} \oplus (RLC)_{\langle s\circ_k u\rangle} \arrow[r, "\widetilde{\rho}_{\langle s\rangle;k,t,u}"] & (RLC)_{\langle s\rangle} \end{tikzcd} \tag{10.7.12}$$

is commutative for each partition (t,u) of s_k with $1 \le k \le q$.

- If either each s_j is a one-element set or some $s_j = \varnothing$, then all four arrows in (10.7.12) are identity morphisms.
- In all other cases, each $(\eta_C)_?$ consists of inverses of ρ in C, and $\widetilde{\rho}$ is a coherence isomorphism in C that permutes factors. So (10.7.12) is commutative by the symmetry and associativity axioms (10.4.4) and (10.4.5) for (C,ρ).

This shows that η_C is a morphism in $\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle$.

To check the naturality of η, suppose

$$\alpha = \left\{\alpha_{\langle s\rangle} : C_{\langle s\rangle} \longrightarrow D_{\langle s\rangle}\right\} : (C,\rho_C) \longrightarrow (D,\rho_C) \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle$$

is a morphism. It suffices to check the naturality of η at a component $\langle s\rangle$ as in (10.7.4), which asserts the commutativity of the following diagram.

$$\begin{tikzcd} C_{\langle s\rangle} \arrow[r, "(\eta_C)_{\langle s\rangle}"] \arrow[d, "\alpha_{\langle s\rangle}"'] & \bigoplus\limits_{(x_1,\ldots,x_q)\in \prod_{j=1}^q s_j} C_{(\{x_1\},\ldots,\{x_q\})} = (RLC)_{\langle s\rangle} \arrow[d, "\oplus\, \alpha_{(\{x_1\},\ldots,\{x_q\})}"] \\ D_{\langle s\rangle} \arrow[r, "(\eta_D)_{\langle s\rangle}"] & \bigoplus\limits_{(x_1,\ldots,x_q)\in \prod_{j=1}^q s_j} D_{(\{x_1\},\ldots,\{x_q\})} = (RLD)_{\langle s\rangle} \end{tikzcd} \tag{10.7.13}$$

In the two cases in Explanation 10.7.7, the horizontal arrows in (10.7.13) are identity morphisms, and the two vertical arrows are equal. In all other cases, (10.7.13) is commutative by

- the fact that each of $(\eta_C)_{\langle s\rangle}$ and $(\eta_D)_{\langle s\rangle}$ is a composite of inverses of gluing morphisms and
- the compatibility of α with the gluing morphisms in C and D.

We have shown that η is a natural transformation. Moreover, it is a natural isomorphism because each component $(\eta_C)_{\langle s\rangle}$ is either an identity morphism or a composite of isomorphisms.

The component of the counit

$$LR \xrightarrow{\ \varepsilon\ } 1_{\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle}$$

at a lax $\langle \underline{n}\rangle$-system $C \in \mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$ as in (10.7.2) is the morphism

$$LRC \xrightarrow{\ \varepsilon_C\ } C \in \mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$$

with components

$$(LRC)_{\langle s\rangle} = \bigoplus_{(x_1,\ldots,x_q)\in\prod_{j=1}^q s_j} C_{(\{x_1\},\ldots,\{x_q\})} \xrightarrow{(\varepsilon_C)_{\langle s\rangle}} C_{\langle s\rangle} \in \mathsf{C}$$

given by

- the identity morphism in each of the two cases in Explanation 10.7.7 and
- composites of the gluing morphisms ρ in C otherwise.

The second case above is well defined by the axioms in Definition 10.4.1. Reusing the diagrams (10.7.12) and (10.7.13) with each η replaced by ε in the opposite direction, we infer that ε is a natural transformation.

Next we check the triangle identities (I.1.1.11). For the triangle identity that asserts the commutativity of the diagram

$$\begin{tikzcd} & LRL \arrow[dr, "\varepsilon L"] & \\ L \arrow[ur, "L\eta"] \arrow[rr, "1_L"] & & L \end{tikzcd}$$

consider an object $C \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle$. Then $\big((\varepsilon L)\circ(L\eta)\big)_C$ has components

$$\begin{tikzcd} (LC)_{\langle s\rangle} = \bigoplus_{(x_1,\ldots,x_q)\in\prod_{j=1}^q s_j} C_{(\{x_1\},\ldots,\{x_q\})} \arrow[d, "\bigoplus(\eta_C)_{(\{x_1\},\ldots,\{x_q\})}", "(L\eta_C)_{\langle s\rangle} ="'] \\ (LRLC)_{\langle s\rangle} = \bigoplus_{(x_1,\ldots,x_q)\in\prod_{j=1}^q s_j} C_{(\{x_1\},\ldots,\{x_q\})} \arrow[d, "\widetilde{\rho}", "(\varepsilon_{LC})_{\langle s\rangle} ="'] \\ (LC)_{\langle s\rangle} = \bigoplus_{(x_1,\ldots,x_q)\in\prod_{j=1}^q s_j} C_{(\{x_1\},\ldots,\{x_q\})} \end{tikzcd}$$

with $\langle s\rangle$ as in (10.7.4).

- In $(L\eta_C)_{\langle s\rangle}$, each factor
$$(\eta_C)_{(\{x_1\},\ldots,\{x_q\})}$$
is an identity morphism by definition, so $(L\eta_C)_{\langle s\rangle}$ is an identity morphism.
- The gluing morphism $\widetilde{\rho}$ in LC is a coherence isomorphism in C that permutes factors. The Symmetric Coherence Theorem 1.1.41 implies that $\widetilde{\rho}$ is the identity morphism in the above diagram.

Therefore, each of $L\eta$ and εL is the identity natural transformation.

The other triangle identity

$$\begin{tikzcd} & RLR \arrow[dr, "R\varepsilon"] & \\ R \arrow[ur, "\eta R"] \arrow[rr, "1_R"] & & R \end{tikzcd}$$

is proved similarly as follows.

- Each component $(\eta_{RC})_{\langle s \rangle}$ is a coherence isomorphism in C that permutes factors. Since its domain and codomain are equal, the uniqueness in Theorem 1.1.41 implies that this is the identity morphism.
- The other natural transformation has components

$$(R\varepsilon_C)_{\langle s \rangle} = \bigoplus_{(x_1,\ldots,x_q) \in \prod_{j=1}^q s_j} (\varepsilon_C)_{(\{x_1\},\ldots,\{x_q\})}.$$

Each displayed component of ε_C is an identity morphism by definition.

Therefore, each of ηR and $R\varepsilon$ is the identity natural transformation. This finishes the proof that (L, R) is an adjunction with unit η and counit ε.

For the second assertion, suppose C is a groupoid. Then each component $(\varepsilon_C)_{\langle s \rangle}$ is either an identity morphism or a composite of isomorphisms. In this case, both the unit η and the counit ε are natural isomorphisms, so (L, R) is an adjoint equivalence. □

Naturality. Next we observe that the left adjoints $L^{\mathsf{C}}_{\langle \underline{n} \rangle}$ are natural with respect to $\langle \underline{n} \rangle \in \mathcal{G}$. To do that, first we provide a simpler description of the left adjoints as follows.

Definition 10.7.14. For each q-tuple of pointed finite sets $\langle \underline{n} \rangle \in \mathcal{F}^{(q)}$, we denote by

$$\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle \xrightarrow{I^{\mathsf{C}}_{\langle \underline{n} \rangle}} \mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$$

the full subcategory inclusion. If C and $\langle \underline{n} \rangle$ are understood, then we abbreviate $I^{\mathsf{C}}_{\langle \underline{n} \rangle}$ to I. ◇

Proposition 10.7.15. *For each $\langle \underline{n} \rangle \in \mathcal{F}^{(q)}$, the two functors*

$$\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle \underset{I}{\overset{L}{\rightrightarrows}} \mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$$

in Definitions 10.7.9 and 10.7.14 are naturally isomorphic. Therefore, Proposition 10.7.10 also holds for the adjunction $I \dashv R$.

Proof. If $q = 0$ or any $n_j = 0$ then L and I are both the identity functor. For the remainder of the proof we assume $q > 0$ and each $n_j > 0$. We define a natural isomorphism

$$L \xrightarrow[\cong]{g} I$$

as follows. For each object $C \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle$ as in (10.7.11) and $\langle s \rangle$ as in (10.7.4), the morphism

$$(LC)_{\langle s \rangle} = \bigoplus_{(x_1,\ldots,x_q) \in \prod_{j=1}^q s_j} C_{(\{x_1\},\ldots,\{x_q\})} \xrightarrow{(g_C)_{\langle s \rangle}} C_{\langle s \rangle} \in \mathsf{C}$$

is given by

- the identity morphism in each of the two cases in Explanation 10.7.7 and
- composites of the gluing morphisms ρ in C otherwise.

The second case above is well defined by the axioms in Definition 10.4.1. Reusing the diagrams (10.7.12) and (10.7.13) with each η replaced by g in the opposite direction, we infer that

- the collection
$$LC \xrightarrow{g_C = \{(g_C)_{\langle s\rangle}\}} C$$
is a morphism in $\mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle$ and
- $g : L \longrightarrow I$ is a natural transformation.

Since the gluing morphisms ρ in C are isomorphisms, each component $(g_C)_{\langle s\rangle}$ is an isomorphism. So g is a natural isomorphism. ☐

Recall that $\mathsf{C}^{\mathcal{G}}$ and, by restriction, $\mathsf{C}^{\mathcal{G}}_{\cong}$ are $\mathcal{G}_*$-categories by Proposition 10.4.18.

Theorem 10.7.16. *For each small permutative category $(\mathsf{C}, \oplus, e, \xi)$, the functors $I^{\mathsf{C}}_{\langle \underline{n}\rangle}$ in Definition 10.7.14 are the components of a $\mathcal{G}_*$-category morphism*

$$\mathsf{C}^{\mathcal{G}}_{\cong} \xrightarrow{I^{\mathsf{C}}} \mathsf{C}^{\mathcal{G}}.$$

Moreover, the induced morphism of symmetric spectra

$$\mathsf{K}^{\mathsf{EM}}_{\cong}\mathsf{C} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{C}^{\mathcal{G}}_{\cong} \xrightarrow{\mathsf{K}^{\mathcal{G}} N_* I^{\mathsf{C}}} \mathsf{K}^{\mathcal{G}} N_* \mathsf{C}^{\mathcal{G}} \cong \mathsf{K}^{\mathsf{EM}}\mathsf{C}$$

is a level equivalence.

Proof. For each morphism

$$\langle \underline{n}\rangle = (\underline{n}_1, \ldots, \underline{n}_q) \xrightarrow{(f, \langle \psi\rangle)} (\underline{m}_1, \ldots, \underline{m}_p) = \langle \underline{m}\rangle \in \mathcal{G}$$

as in Definition 9.1.7, the diagram of functors

(10.7.17)
$$\begin{tikzcd}
\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle \arrow[r, "I^{\mathsf{C}}_{\langle \underline{n}\rangle}"] \arrow[d, "\mathsf{C}^{\mathcal{G}}_{\cong}(f, \langle \psi\rangle)"'] & \mathsf{C}^{\mathcal{G}}\langle \underline{n}\rangle \arrow[d, "\mathsf{C}^{\mathcal{G}}(f, \langle \psi\rangle)"] \\
\mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{m}\rangle \arrow[r, "I^{\mathsf{C}}_{\langle \underline{m}\rangle}"] & \mathsf{C}^{\mathcal{G}}\langle \underline{m}\rangle
\end{tikzcd}$$

is commutative because each $I^{\mathsf{C}}_{?}$ is a full subcategory inclusion. So I^{C} is a morphism of $\mathcal{G}_*$-categories.

For the second assertion, by Proposition 10.7.15, I^{C} is levelwise a left adjoint. After taking the nerve, $N_* I^{\mathsf{C}}$ is a levelwise simplicial homotopy equivalence of $\mathcal{G}_*$-simplicial sets. So $\mathsf{K}^{\mathcal{G}} N_* I^{\mathsf{C}}$ is a level equivalence of symmetric spectra. ☐

Explanation 10.7.18. Using the notation in Definition 10.4.16, we explain the diagram (10.7.17) in more detail. Suppose $q > 0$, $p > 0$, each $n_j > 0$, and each $m_i > 0$. For an object $(C, \rho) \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle$ as in (10.7.11), denote by

$$(C', \rho') \in \mathsf{C}^{\mathcal{G}}\langle \underline{m}\rangle$$

the image of (C, ρ) under either composite in (10.7.17). For a p-tuple of pointed finite sets

$$\langle s\rangle = (s_1, \ldots, s_p) \in \mathcal{F}^{(p)} \quad \text{with each} \quad s_i \subset \underline{m}_i^{\flat},$$

the component $C'_{\langle s\rangle}$ is the object

$$C'_{\langle s\rangle} = \begin{cases} C_{\langle \tilde{f}_* \psi^{-1}(s)\rangle} & \text{if each } \psi_i^{-1}(s_i) \neq \varnothing, \\ e & \text{otherwise} \end{cases}$$

with

$$\langle \tilde{f}_* \psi^{-1}(s)\rangle = \left\{ \psi^{-1}_{f(j)}\left(s_{f(j)}\right)\right\}_{1\le j\le q} \in \mathcal{F}^{(q)}.$$

For a partition

$$t \cup u = s_k \quad \text{with} \quad 1 \le k \le p,$$

the gluing morphism ρ' is

$$\rho'_{\langle s\rangle;k,t,u} = \begin{cases} \rho_{\langle \tilde{f}_*\psi^{-1}(s)\rangle; f^{-1}(k), \psi_k^{-1}(t), \psi_k^{-1}(u)} & \text{if } f^{-1}(k) \neq \varnothing \text{ and each } \psi_i^{-1}(s_i) \neq \varnothing, \\ 1_{C_{\langle \tilde{f}_*\psi^{-1}(s)\rangle}} & \text{if } f^{-1}(k) = \varnothing \text{ and each } \psi_i^{-1}(s_i) \neq \varnothing, \\ 1_e & \text{otherwise.} \end{cases}$$

For a morphism

$$C \xrightarrow{\alpha} D \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle,$$

its image α' under either composite in (10.7.17) has components

$$\alpha'_{\langle s\rangle} = \begin{cases} \alpha_{\langle \tilde{f}_*\psi^{-1}(s)\rangle} & \text{if each } \psi_i^{-1}(s_i) \neq \varnothing, \\ 1_e & \text{otherwise} \end{cases}$$

with $\langle s\rangle \in \mathcal{F}^{(p)}$ as above. ◇

So far we have compared the lax and the strong variants of the Elmendorf-Mandell $\mathcal{G}_*$-categories of C. The next result compares the colax and the strong variants.

Theorem 10.7.19. *The following statements hold for each small permutative category* C *and each* $\langle \underline{n}\rangle \in \mathcal{F}^{(q)}$.

(1) For $q > 0$ and each $n_j > 0$, the constructions (10.7.3) and (10.7.6) define functors

$$\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\langle \underline{n}\rangle \underset{R^{\mathrm{co}}_{\langle \underline{n}\rangle}}{\overset{L^{\mathrm{co}}_{\langle \underline{n}\rangle}}{\rightleftarrows}} \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle,$$

where, in

$$R^{\mathrm{co}}_{\langle \underline{n}\rangle}(C,\rho) = (\widetilde{C},\widetilde{\rho}) \quad \textit{for} \quad (C,\rho) \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n}\rangle,$$

the gluing morphisms go in the colax direction, as in

$$\widetilde{C}_{\langle s\rangle} \xrightarrow[\cong]{\widetilde{\rho}_{\langle s\rangle;k,t,u}} \widetilde{C}_{\langle s\circ_k t\rangle} \oplus \widetilde{C}_{\langle s\circ_k u\rangle} \in \mathsf{C}.$$

(2) The functors $\left(L^{\mathrm{co}}_{\langle \underline{n}\rangle}, R^{\mathrm{co}}_{\langle \underline{n}\rangle}\right)$ form an adjunction with counit a natural isomorphism, where, if $q = 0$ or any $n_j = 0$, we define $L^{\mathrm{co}}_{\langle \underline{n}\rangle}$ and $R^{\mathrm{co}}_{\langle \underline{n}\rangle}$ to both be identity functors. If C is a groupoid, then $\left(L^{\mathrm{co}}_{\langle \underline{n}\rangle}, R^{\mathrm{co}}_{\langle \underline{n}\rangle}\right)$ is, furthermore, an adjoint equivalence.

(3) $R^{\mathrm{co}}_{\langle \underline{n} \rangle}$ *is naturally isomorphic to the full and faithful functor*

$$\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\langle \underline{n} \rangle \xleftarrow{I^{\mathrm{co}}_{\langle \underline{n} \rangle}} \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle$$

that sends each object

$$(C,\rho) \in \mathsf{C}^{\mathcal{G}}_{\cong}\langle \underline{n} \rangle \quad \textit{to} \quad (C,\rho^{\mathrm{co}}) \in \mathsf{C}^{\mathcal{G}}_{\mathrm{co}}\langle \underline{n} \rangle$$

with each component of ρ^{co} *the inverse of the corresponding component of* ρ. *On morphisms,* $I^{\mathrm{co}}_{\langle \underline{n} \rangle}$ *is the identity function.*

(4) *The functors* $I^{\mathrm{co}}_{\langle \underline{n} \rangle}$ *are the components of a* $\mathcal{G}_*$*-category morphism*

$$\mathsf{C}^{\mathcal{G}}_{\mathrm{co}} \xleftarrow{I^{\mathrm{co}}} \mathsf{C}^{\mathcal{G}}_{\cong}.$$

Moreover, the induced morphism of symmetric spectra

$$\mathsf{K}^{\mathsf{EM}}_{\mathrm{co}}\mathsf{C} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{C}^{\mathcal{G}}_{\mathrm{co}} \xleftarrow{\mathsf{K}^{\mathcal{G}} N_* I^{\mathrm{co}}} \mathsf{K}^{\mathcal{G}} N_* \mathsf{C}^{\mathcal{G}}_{\cong} = \mathsf{K}^{\mathsf{EM}}_{\cong}\mathsf{C}$$

is a level equivalence.

Proof. We reuse the proofs of Lemma 10.7.8, Propositions 10.7.10 and 10.7.15, and Theorem 10.7.16 essentially without change. The key point is that, in (10.7.3), $\widetilde{\rho}$ is a coherence isomorphism in C and does not use the gluing morphisms in the given $\langle \underline{n} \rangle$-system (C,ρ). Moreover, for a strong $\langle \underline{n} \rangle$-system (C,ρ), the gluing morphisms ρ are isomorphisms, so they can be inverted to yield colax gluing morphisms. □

Multifunctoriality. Theorem 10.7.16 can be strengthened to include multifunctoriality as follows.

Definition 10.7.20.

- A multilinear functor F (Definition 6.5.4) is *strong* if each linearity constraint F^2_i is a natural isomorphism for $1 \le i \le n$.
- Denote by

$$\mathsf{PermCat}^{\mathsf{sus}}$$

  the Cat-enriched full sub-multicategory of $\mathsf{PermCat}^{\mathsf{su}}$ in Definition 6.5.1 consisting of *strong* multilinear functors.

The definitions (6.6.3) and (6.6.8) show that strong multilinear functors are closed under the symmetric group action and the composition. ◇

Explanation 10.7.21 (Multifunctoriality of $(-)^{\mathcal{G}}_{\cong}$)**.** The Cat-enriched multifunctor

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{(-)^{\mathcal{G}}} \mathcal{G}_*\text{-}\mathsf{Cat}$$

in Explanation 10.5.4 restricts to a Cat-enriched multifunctor

$$\mathsf{PermCat}^{\mathsf{sus}} \xrightarrow{(-)^{\mathcal{G}}_{\cong}} \mathcal{G}_*\text{-}\mathsf{Cat}.$$

For $(-)^{\mathcal{G}}_{\cong}$, the restriction to *strong* multilinear functors is necessary. In more detail, for an n-linear functor F as in (10.5.8), the $\mathcal{G}_*$-category morphism $F^{\mathcal{G}}_{\cong}$ has the same description as $F^{\mathcal{G}}$ in (10.5.9), but with

- each $\mathsf{C}^{\mathcal{G}}_j$ replaced by $(\mathsf{C}_j)^{\mathcal{G}}_{\cong}$ and

- $\mathsf{D}^{\mathcal{G}}$ replaced by $\mathsf{D}^{\mathcal{G}}_{\cong}$.

In this case, in (10.5.11), the components of the gluing morphisms ρ^j are isomorphisms in C_j. This implies that, in the diagram (10.5.15), the morphism

$$F(1,\ldots,\rho^r,\ldots,1)$$

is an isomorphism. Therefore, in order for the gluing morphism $\rho_{\langle s\rangle;k,t,u}$ in (10.5.15) to be an isomorphism, we must assume that the linearity constraints $\{F^2_r\}_{r=1}^n$ are natural isomorphisms. In other words, F is a strong n-linear functor. $\diamond$

Proposition 10.7.22. *The $\mathcal{G}_*$-category morphisms I^{C} in Theorem 10.7.16 are the components of a* Cat*-enriched multinatural transformation*

$$\begin{tikzcd}
\mathsf{PermCat}^{\mathsf{sus}} \arrow[rr, "(-)^{\mathcal{G}}_{\cong}"] \arrow[dr] & \Downarrow I & \mathcal{G}_*\text{-}\mathsf{Cat} \\
& \mathsf{PermCat}^{\mathsf{su}} \arrow[ur, "(-)^{\mathcal{G}}"'] &
\end{tikzcd}$$

with the unlabeled arrow the sub-multicategory inclusion.

Proof. For I, the naturality condition in Definition 5.1.17 states that, for each strong n-linear functor

$$\mathsf{C}_1 \times \cdots \times \mathsf{C}_n \xrightarrow{F} \mathsf{D} \quad \text{with} \quad n \geq 0,$$

the diagram

$$\begin{tikzcd}
(\mathsf{C}_1)^{\mathcal{G}}_{\cong} \otimes \cdots \otimes (\mathsf{C}_n)^{\mathcal{G}}_{\cong} \arrow[r, "F^{\mathcal{G}}_{\cong}"] \arrow[d, "I^{\mathsf{C}_1} \otimes \cdots \otimes I^{\mathsf{C}_n}"'] & \mathsf{D}^{\mathcal{G}}_{\cong} \arrow[d, "I^{\mathsf{D}}"] \\
\mathsf{C}_1^{\mathcal{G}} \otimes \cdots \otimes \mathsf{C}_n^{\mathcal{G}} \arrow[r, "F^{\mathcal{G}}"] & \mathsf{D}^{\mathcal{G}}
\end{tikzcd}$$

in $\mathcal{G}_*$-Cat is commutative. This diagram is commutative by the explicit description of $F^{\mathcal{G}}$ and $F^{\mathcal{G}}_{\cong}$ on

- objects in (10.5.13) and (10.5.15) and
- morphisms in (10.5.16)

and the fact that each $I^{\mathsf{C}}_{\langle \underline{m}\rangle}$, for $\mathsf{C} \in \mathsf{PermCat}^{\mathsf{sus}}$ and $\langle \underline{m}\rangle \in \mathcal{G}$, is an inclusion. $\square$

Explanation 10.7.23 (Lax versus Strong K-Theory)**.** The main reason that we consider the lax version

$$\mathsf{K}^{\mathsf{EM}} \cong \mathsf{K}^{\mathcal{G}} N_*(-)^{\mathcal{G}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp},$$

instead of the strong version

$$\mathsf{K}^{\mathsf{EM}}_{\cong} = \mathsf{K}^{\mathcal{G}} N_*(-)^{\mathcal{G}}_{\cong} : \mathsf{PermCat}^{\mathsf{sus}} \longrightarrow \mathsf{SymSp},$$

as the default Elmendorf-Mandell K-theory is that K^{EM} allows a strictly bigger domain multicategory $\mathsf{PermCat}^{\mathsf{su}}$, as opposed to $\mathsf{PermCat}^{\mathsf{sus}}$. This is a nontrivial difference in practice when further algebraic structures are taken into account. In fact, as we will see in Theorem 11.2.16, monoids in $\mathsf{PermCat}^{\mathsf{su}}$ are small ring categories. On the other hand, the same proof restricted to $\mathsf{PermCat}^{\mathsf{sus}}$ shows that monoids in $\mathsf{PermCat}^{\mathsf{sus}}$ are small ring categories with *invertible* factorization morphisms ∂^l and

∂^r. So general small ring categories, whose factorization morphisms are not necessarily invertible, can only be detected in $\mathsf{PermCat}^{\mathsf{su}}$. Moreover, there are similar remarks for

- Theorem 11.5.5, where a Cat-enriched multifunctor
$$E\mathsf{As} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$$
is a small bipermutative category;
- Theorem 12.4.5, where a Cat-enriched multifunctor
$$\mathsf{Br} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$$
is a small braided ring category; and
- Theorem 13.4.12, where a Cat-enriched multifunctor
$$\mathsf{Mon}^n \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$$
is a small E_n-monoidal category.

In each of the above cases, replacing the codomain with $\mathsf{PermCat}^{\mathsf{sus}}$ yields the corresponding type of categorical structure with invertible factorization morphisms.

To phrase it in another way, while Theorem 10.7.16 shows that, for each small permutative category C, I^{C} induces a level equivalence

$$\mathsf{K}^{\mathsf{EM}}_{\cong}\mathsf{C} \xrightarrow{\mathsf{K}^{\mathcal{G}} N_* I^{\mathsf{C}}} \mathsf{K}^{\mathsf{EM}}\mathsf{C}$$

of K-theory symmetric spectra, many important algebraic structures preserved by K^{EM}—including those in Theorems 11.2.16, 11.5.5, 12.4.5, and 13.4.12—can only be detected in the bigger multicategory $\mathsf{PermCat}^{\mathsf{su}}$, but not in $\mathsf{PermCat}^{\mathsf{sus}}$. Therefore, when algebraic structures on small permutative categories, as parametrized by operads, are taken into account, the lax version K^{EM} has a significant advantage over the strong version $\mathsf{K}^{\mathsf{EM}}_{\cong}$. $\diamond$

Theorem 10.7.19 can also be strengthened to include multifunctoriality using the definitions below.

Definition 10.7.24 (Colax Multilinear Functors)**.** Suppose $\mathsf{C}_1, \ldots, \mathsf{C}_n$, and D are permutative categories.

- A *colax n-linear functor*
$$F : \langle \mathsf{C} \rangle \longrightarrow \mathsf{D}$$
is defined as in Definition 6.5.4 with the direction of each linearity constraint F^2_i reversed, as in
$$F^2_i : F\langle X \circ_i (X_i \oplus X'_i)\rangle \longrightarrow F\langle X \circ_i X_i\rangle \oplus F\langle X \circ_i X'_i\rangle,$$
but the same axioms, with the reversed arrows F^2_i, are satisfied.
- A *colax n-linear transformation* between colax n-linear functors
$$F, F' : \langle \mathsf{C} \rangle \longrightarrow \mathsf{D}$$
is defined as in Definition 6.5.11 with the reserved arrows F^2_i and $(F')^2_i$ in the diagram (6.5.12).
- Denote by
$$\mathsf{PermCat}^{\mathsf{su}}_{\mathsf{co}}$$
the Cat-enriched multicategory consisting of

– small permutative categories,
– colax multilinear functors, and
– colax multilinear transformations.

The proofs in Section 6.6, with the linearity constraints reversed, show that $\mathsf{PermCat}^{\mathsf{su}}_{\mathsf{co}}$ is a well-defined Cat-enriched multicategory. $\diamond$

Explanation 10.7.25. Each strong n-linear functor $\big(F, \{F_i^2\}_{i=1}^n\big)$ as in Definition 10.7.20 yields a colax n-linear functor by taking the inverse of each linearity constraint F_i^2. This assignment gives a Cat-enriched multifunctor

$$\mathsf{PermCat}^{\mathsf{sus}} \xrightarrow{\iota^{\mathsf{co}}} \mathsf{PermCat}^{\mathsf{su}}_{\mathsf{co}} \tag{10.7.26}$$

that is the identity assignment on objects and multilinear transformations. Moreover, the object assignment $\mathsf{C} \longmapsto \mathsf{C}^{\mathcal{G}}_{\mathsf{co}}$ is part of a Cat-enriched multifunctor

$$\mathsf{PermCat}^{\mathsf{su}}_{\mathsf{co}} \xrightarrow{(-)^{\mathcal{G}}_{\mathsf{co}}} \mathcal{G}_*\text{-}\mathsf{Cat}.$$

The assignment on strictly unital colax multilinear functors and colax multilinear transformations are as in Explanation 10.5.4, with

- the linearity constraints F_i^2 reversed and
- $(-)^{\mathcal{G}}_{\mathsf{co}}$ instead of $(-)^{\mathcal{G}}$.

In the current context, each arrow in the diagram (10.5.15) is reversed. $\diamond$

Essentially the same reasoning for Proposition 10.7.22 gives the following result.

Proposition 10.7.27. *The $\mathcal{G}_*$-category morphisms I^{co} in Theorem 10.7.19 (4) are the components of a* Cat*-enriched multinatural transformation*

$$\begin{tikzcd}
\mathsf{PermCat}^{\mathsf{sus}} \arrow[rr, "(-)^{\mathcal{G}}_{\cong}"] \arrow[dr, "\iota^{\mathsf{co}}"'] & \arrow[d, phantom, "\Downarrow I^{\mathsf{co}}"] & \mathcal{G}_*\text{-}\mathsf{Cat} \\
& \mathsf{PermCat}^{\mathsf{su}}_{\mathsf{co}} \arrow[ur, "(-)^{\mathcal{G}}_{\mathsf{co}}"'] &
\end{tikzcd}$$

with ι^{co} as in (10.7.26).

10.8. Notes

10.8.1 (Elmendorf-Mandell K-Theory)**.** Continuing from Note 9.5.1, the constructions $\mathsf{J}^{\mathcal{T}}$, J^{EM}, and K^{EM} were first presented in **[EM06, EM09]**. The first of these uses the $\mathcal{G}_*$-categories $\mathsf{C}^{\mathcal{G}}$ (Section 10.4), and the second uses a description equivalent to that of Section 10.3.

A description of $\mathcal{M}\underline{1}$-modules is also given in **[EM09]** (there called E-modules, where E is the *terminal parameter multicategory for modules*). The important structure result, Proposition 10.1.28, is from **[EM09**, Proposition 5.1**]** (see Note 10.8.4). However, **[EM09]** contains incorrect statements that $\mathsf{J}^{\mathcal{T}}$ (there called J') extends beyond $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ to all of $\mathsf{Multicat}_*$. The error is related to the monoidal unit, and we explain it further in Question A.5.1 below. We identify the necessary corrections in Note 10.8.2 $\diamond$

10.8.2 (Corrections to [**EM09**])**.** As mentioned in Note 10.8.1, and discussed further in Question A.5.1 below, the symmetric monoidal functor $\mathsf{J}^{\mathcal{T}}$ does not extend, as a symmetric monoidal functor, to all of $\mathsf{Multicat}_*$. The restriction of its domain to $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is essential to its satisfying the monoidal unity axioms as in (10.3.18). These, in turn, are essential to the preservation of composition with identity operations in the associated multicategory.

To account for this subtlety, the following statements in [**EM09**] should be corrected:

- In Theorems 1.3 and 5.14, and in Definition 5.15, $\mathbf{Mult}_*$ should be replaced with $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ (there called the category of E-modules).
- On page 2392, lines 1 and 2 (just before the statement of Theorem 1.1), the phrase beginning "with the objects of the larger category being..." should end with "$\mathcal{M}\underline{1}$-modules." Note that $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is bicomplete and symmetric monoidal closed as desired, by Proposition 10.2.22.
- On page 2432, the second to last sentence of section 5, which reads "The lax structure map for the unit is then given by the unique map to the terminal object," should be removed and replaced with the unit constraint $(\mathsf{J}^{\mathcal{T}})^0$ from Definition 10.3.16.

If the unit constraint $(\mathsf{J}^{\mathcal{T}})^0$ is the unique map to the terminal object in $\mathcal{G}_*$-Cat, then the unit axioms of a monoidal functor would imply that every left and right unit isomorphism factors through the zero morphism. But that is not the case.

In the earlier paper [**EM06**], the construction $\mathsf{J}^{\mathcal{T}}$ (there called J) is only discussed as a multifunctor from $\mathsf{PermCat}^{\mathsf{su}}$. Since

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}_*$$

takes values in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ (Lemma 10.2.14), no corrections are necessary. In the proof of [**EM06**, Theorem 6.1], the image of a 0-linear functor, as described in Explanation 10.5.4, is left implicit. ⋄

10.8.3 (Structure of $\mathsf{J}^{\mathcal{T}}$ on $\mathsf{Multicat}_*$)**.** Question A.5.1 and Note 10.8.2 explain that $\mathsf{J}^{\mathcal{T}}$ cannot extend, as a monoidal functor, to all of $\mathsf{Multicat}_*$. However we do have the underlying 2-functor

$$\mathsf{J}^{\mathcal{T}} = \mathsf{Multicat}_*(\mathcal{T}, -)$$

defined on all of $\mathsf{Multicat}_*$. Moreover, the monoidal constraint $(\mathsf{J}^{\mathcal{T}})^2$ and the verifications of the associativity and symmetry axioms in Theorem 10.3.17 are not restricted to left $\mathcal{M}\underline{1}$-modules. It appears that the unit constraint and unity axioms are the essential reasons that $\mathsf{J}^{\mathcal{T}}$ cannot extend to a symmetric monoidal functor on all of $\mathsf{Multicat}_*$. ⋄

10.8.4 (Monoids With Invertible Multiplication)**.** Proposition 10.1.28 shows that the category of $\mathcal{M}\underline{1}$-modules is a full subcategory of $\mathsf{Multicat}_*$. This result holds generally for the subcategory of modules over a monoid M in a monoidal category such that the multiplication map for M is an isomorphism. The result is proved with this generality in [**EM09**, Proposition 5.1]. As noted there, a more general result for algebras over idempotent monads is given in [**Bor94b**, Proposition 4.2.3]. ⋄

10.8.5 (Notation Comparison)**.** For comparison with [**EM06, EM09**], we give the following table of corresponding notation.

[**EM06, EM09**]	Our Notation	Reference
$\mathcal{G}_*$	$\mathcal{G}$	Definition 9.1.7
$q : \underline{r} \longrightarrow \underline{s}$	$f : \overline{p} \hookrightarrow \overline{q}$	Definition 9.1.2
$q_*\langle \mathbf{m} \rangle$	$f_*\langle \underline{n} \rangle$	Definition 9.1.4
$(\alpha, q) : \langle \mathbf{m} \rangle \longrightarrow \langle \mathbf{n} \rangle$	$(f, \langle \psi \rangle) : \langle \underline{n} \rangle \longrightarrow \langle \underline{m} \rangle$	Definition 9.1.7
$K^{\mathrm{Seg}}, K^{\mathrm{New}}$	$\mathsf{K}^{\mathsf{Se}}, \mathsf{K}^{\mathsf{EM}}$	Definitions 8.5.1 and 10.3.32
α_*	$\langle \tilde{\psi} \rangle$	Definition 10.4.16 (2)
E	$\mathcal{M}\underline{1}$	Example 8.4.5
$\overline{\mathsf{C}}\langle \mathbf{m} \rangle$	$\mathsf{C}^{\mathcal{G}}\langle \underline{n} \rangle$	Definition 10.4.10
$p_m, q_{\langle m \rangle}$	i^*, j^*	Proposition 10.6.7
J, J'	$\mathsf{J}^{\mathcal{T}}$	Definition 10.3.9

$\diamond$

10.8.6 (Segal K-Theory and Symmetric Monoidal Structure)**.** In Explanation 9.2.18 we note that Γ-C carries a symmetric monoidal structure defined similarly (via Day convolution) to that of $\mathcal{G}_*$-C. Theorem 9.4.18 shows that the functor

$$\mathsf{K}^{\mathcal{F}} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{SymSp}$$

of Proposition 8.2.6 is symmetric monoidal as a sSet_*-functor. However one encounters difficulties attempting to develop a symmetric monoidal structure for

$$\mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \Gamma\text{-}\mathsf{Cat}.$$

First, as noted in Propositions 5.7.23 and 10.2.17, the symmetric monoidal structure for small pointed multicategories does not restrict to a symmetric monoidal structure for $\mathsf{PermCat}^{\mathsf{su}}$. When one extends to use the symmetric monoidal structure on pointed multicategories, as we do for J^{EM} in Section 10.3, one faces a more fundamental difficulty elaborated in [**EM06**, pp. 181–182]. To produce a pairing

$$\mathsf{C}^{\mathcal{F}}\underline{m} \wedge \mathsf{C}^{\mathcal{F}}\underline{n} \longrightarrow \mathsf{C}^{\mathcal{F}}\underline{mn}$$

on objects, one needs to produce an $(\underline{m} \wedge \underline{n})$-system, in the sense of Definition 8.3.1, from an $\underline{m}$-system and an $\underline{n}$-system. The essential problem is that the (basepoint-free) subsets of $\underline{m} \wedge \underline{n}$ cannot be effectively determined by the subsets of $\underline{m}$ and $\underline{n}$ alone. One can take products of subsets, as in Definition 10.1.1 and Proposition 10.6.2, but these will not produce all the necessary subsets of the product. Efforts to resolve that difficulty were the motivation for the development, in [**EM06, EM09**], of $\mathcal{G}$, $\mathsf{K}^{\mathcal{G}}$, J^{EM}, and K^{EM}.

The preceding discussion can be rephrased in terms of the partition multicategory functors

$$\mathcal{M} : \mathcal{F}^{\mathsf{op}} \longrightarrow \mathsf{Multicat}_* \quad \text{and} \quad \mathcal{T} : \mathcal{G}^{\mathsf{op}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

from Definitions 8.4.1 and 10.3.1: both are symmetric monoidal functors, but only $\mathcal{T}$ is strong (see Proposition 10.1.6, Explanation 10.1.10, and Proposition 10.3.7). The inverse monoidal constraint for $\mathcal{T}$ is used in (10.3.13) and is an essential part of the proof in Theorem 10.3.17 that $\mathsf{J}^{\mathcal{T}}$, and hence also J^{EM}, is a symmetric monoidal functor. ◇

10.8.7 (Multiplicative K-Theory Machines)**.** In addition to the Elmendorf-Mandell K-theory multifunctor, there is also May's construction [**May09a**], which improves the original version in [**May77, May82**]. See also the companion papers [**May09b, May09c**]. A further extension to multiplicative *equivariant* K-theory is given in [**GMMO∞**].

We follow the Elmendorf-Mandell approach in part because it has operadic structure built into it in the sense that K^{EM} is a multifunctor that respects the enrichment. In particular, it allows us to seamlessly include E_n-symmetric spectra for $1 \leq n \leq \infty$ as the target of one construction K^{EM}. ◇

10.8.8 (Lax versus Strong)**.** In Section 10.7 we compare the lax, colax, and strong variant constructions of Elmendorf-Mandell K-theory, showing that they result in level-equivalent symmetric spectra. A similar result for the variant Segal K-theory constructions is recorded in Theorems 8.3.21 and 8.5.2.

The paper [**BO20**] uses the strong Elmendorf-Mandell K-theory $\mathsf{K}^{\mathsf{EM}}_{\cong}$. The conclusions of Theorem 10.7.16 are mentioned in a footnote in [**BO20**, p. 1215]. For the Cat-enriched multifunctor $(-)^{\mathcal{G}}_{\cong}$, the necessity to restrict to *strong* multilinear functors, which we discuss in Explanation 10.7.21, is mentioned in [**BO20**, Remark 3.11]. ◇

10.8.9 (Waldhausen K-Theory)**.** In addition to the Segal K-theory and the equivalent Elmendorf-Mandell K-theory of a small permutative category, there is also Waldhausen K-theory of a small Waldhausen category [**Wal85**]. When the latter satisfies a split cofibration hypothesis, Waldhausen showed that his K-theory functor and Segal's yield equivalent spectra. The main results in [**BO20**] show that this equivalence of K-theory spectra can be promoted to one between multifunctors, using the multifunctorial Waldhausen K-theory in [**BM11, GH06, Zak18**] and the strong Elmendorf-Mandell K-theory multifunctor $\mathsf{K}^{\mathsf{EM}}_{\cong}$. ◇

10.8.10 (Quillen K-Theory)**.** The K-theory constructions of Waldhausen discussed above in Note 10.8.9, along with those of Segal and Elmendorf-Mandell, are generalizations of Quillen's *higher algebraic K-groups* [**Qui73**]. For a commutative ring R, these higher algebraic K groups are given by the homotopy groups of the space $BGL(R)^+$, where $GL(R)$ is the infinite general linear group of R, its classifying space is $BGL(R)$, and $(-)^+$ is Quillen's *plus construction* that abelianizes π_1 while preserving homology groups of a space.

To relate Segal's work to that of Quillen, the permutative category of interest is $\mathsf{S} = \coprod_n GL_n(R)$, the category whose object set is the natural numbers and whose morphisms are the elements of the general linear groups. Quillen gives an explicit construction, related in [**Gra76**] and also [**Wei13**, Section IV.4], that inverts the monoidal sum in S to yield an infinite loop space equivalent to $\mathbb{Z} \times BGL(R)^+$. This space is the zeroth space of the spectrum $\mathsf{K}^{\mathsf{Se}}(\mathsf{S})$. The Segal and Elmendorf-Mandell constructions thus generalize that of Quillen to arbitrary permutative categories. This is why their constructions are regarded as general forms of K-theory. ◇

CHAPTER 11

K-Theory of Ring and Bipermutative Categories

The Elmendorf-Mandell K-theory multifunctor (Definition 10.3.32)

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{K}^{\mathsf{EM}}} \mathsf{SymSp}$$

sends small permutative categories to symmetric spectra. It provides one of the most important connections between category theory, homotopy theory, and algebraic K-theory. Since SymSp is a symmetric monoidal closed category (Theorem 7.6.15), it makes sense to talk about *structured symmetric spectra*, which are symmetric spectra that are algebras of operads. Among the most important structured symmetric spectra are the E_n*-symmetric spectra* for $1 \leq n \leq \infty$.

- An E_1-symmetric spectrum is a strict ring symmetric spectrum, that is, a monoid in SymSp. They form the most basic multiplicative structure in symmetric spectra.
- An E_∞-symmetric spectrum is a strict ring symmetric spectrum whose multiplication is commutative up to all higher coherent homotopies. In a model categorical sense, each E_∞-symmetric spectrum is weakly equivalent to a strictly commutative ring symmetric spectrum. For example, the sphere symmetric spectrum in Definition 7.4.1 is an E_∞-symmetric spectrum.
- An E_n-symmetric spectrum for $1 < n < \infty$ sits somewhere between the E_1 and the E_∞ cases. For example, at each prime p, the Brown-Peterson spectrum is an E_4-symmetric spectrum but not an E_∞-symmetric spectrum. See also Question A.4.2.

In this chapter, we apply the Elmendorf-Mandell K-theory multifunctor K^{EM} to ring categories and bipermutative categories (Chapter II.9) to produce, respectively, strict ring symmetric spectra and E_∞-symmetric spectra. These results are due to Elmendorf-Mandell **[EM06]**. See Corollaries 11.3.16 and 11.6.12 and also Note 11.7.4 for the differences between the presentation in **[EM06]** and this chapter. In Chapters 12 and 13, using our braided ring categories and E_n-monoidal categories (Definitions II.9.5.1 and II.10.7.2), the main results in this chapter will be expanded to the E_n cases for $1 < n < \infty$.

Each of Corollaries 11.3.16 and 11.6.12 is a combination of several key facts. First, the Elmendorf-Mandell K-theory multifunctor respects

- the categorical enrichment in the Cat-enriched multicategory $\mathsf{PermCat}^{\mathsf{su}}$ of small permutative categories and
- the simplicial enrichment in the symmetric monoidal closed category SymSp of symmetric spectra.

The fact that K^{EM} is a multifunctor that respects the enrichment implies that a structure in $\mathsf{PermCat}^{\mathsf{su}}$ that is parametrized by a Cat-enriched operad passes along K^{EM}

to symmetric spectra. Therefore, Corollaries 11.3.16 and 11.6.12 will follow once we know the parameter operads in $\mathsf{PermCat}^{\mathsf{su}}$ for, respectively, ring categories and bipermutative categories in Definitions II.9.1.2 and II.9.3.2.

For the E_1 case, the associative operad As parametrizes

- monoids in a general symmetric monoidal category (Proposition 11.1.15) and
- ring categories in the multicategory $\mathsf{PermCat}^{\mathsf{su}}$ (Theorem 11.2.16).

Both of these statements are consequences of the Coherence Theorem 11.1.7 for As. Combining Theorem 11.2.16 with K^{EM} yields Corollary 11.3.16, which says that, for each small ring category C, $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is a strict ring symmetric spectrum. Along the same lines, the categorical Barratt-Eccles operad $E\mathsf{As}$ is an E_∞-operad that parametrizes

- permutative categories in Cat (Proposition 11.4.26) and
- bipermutative categories in $\mathsf{PermCat}^{\mathsf{su}}$ (Theorem 11.5.5).

Both of these statements are consequences of the Coherence Theorem 11.4.14 for $E\mathsf{As}$. Combining Theorem 11.5.5 with K^{EM} yields Corollary 11.6.12, which says that, for each small bipermutative category C, $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_∞-symmetric spectrum. The following table summaries this discussion.

operad	associative As (11.1.1)	Barratt-Eccles $E\mathsf{As}$ (11.4.10)
$E_?$-operad	E_1 (13.1.23, 13.2.1)	E_∞ (11.6.3)
coherence	11.1.7	11.4.14
in Cat	strict monoidal (11.1.15)	permutative (11.4.26)
in $\mathsf{PermCat}^{\mathsf{su}}$	ring (11.2.16)	bipermutative (11.5.5)
in SymSp	strict ring (11.3.16)	E_∞ (11.6.12)

For open questions related to the Barratt-Eccles operad $E\mathsf{As}$, see Question A.4.3.

Organization. Section 11.1 discusses the associative operad As from scratch. After defining As, Lemma 11.1.4 provides a detailed proof that As is actually an operad. The nth object in As is the nth symmetric group Σ_n. Its operad composition is defined by block sums and block permutations. The Coherence Theorem 11.1.7 describes As in terms of

- two generators, one for the unit and one for the multiplication, and
- two relations, one for unity and one for associativity.

The first application of Theorem 11.1.7 is Proposition 11.1.15, which says that As is the operad for monoids.

Section 11.2 proves Theorem 11.2.16, which says that the associative operad As detects ring category structures on small permutative categories. This is another application of the Coherence Theorem 11.1.7 for As. Section 11.3 proves the main result about strict ring symmetric spectra, Corollary 11.3.16, which says that $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is a strict ring symmetric spectrum for each small ring category C. This result is a consequence of Theorem 11.2.16 and the Elmendorf-Mandell K-theory multifunctor.

Section 11.4 begins by defining the translation category of a set. Applied to the associative operad, this yields the Cat-enriched Barratt-Eccles operad $E\mathsf{As}$. The Coherence Theorem 11.4.14 describes the Barratt-Eccles operad in terms of

- two generating objects for the unit and the multiplication,

- one generating isomorphism for the symmetry isomorphism, and
- generating relations corresponding to those of a permutative category.

The first application of Theorem 11.4.14 is Proposition 11.4.26, which says that EAs is the Cat-enriched operad for permutative categories.

Section 11.5 proves Theorem 11.5.5, which says that the Barratt-Eccles operad EAs detects bipermutative category structures on small permutative categories. This is another application of the Coherence Theorem 11.4.14 for EAs. Section 11.6 proves the main result about E_∞-symmetric spectra, Corollary 11.6.12, which says that $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_∞-symmetric spectrum for each small bipermutative category C. This result is a consequence of Theorem 11.5.5 and the Elmendorf-Mandell K-theory multifunctor.

Reading Guide.

(1) Related to the associative operad, read Definition 11.1.1 and the statements of Theorems 11.1.7 and 11.2.16, Proposition 11.1.15, and Corollary 11.3.16.
(2) Related to the Barratt-Eccles operad, read Definitions 11.4.1, 11.4.10, 11.6.1, and 11.6.5 and the statements of Theorems 11.4.14 and 11.5.5, Propositions 11.4.26 and 11.6.3, and Corollary 11.6.12.
(3) Go back and read the rest of this chapter.

11.1. The Associative Operad

Recall from Definition 5.1.2 that an *operad* is a one-object multicategory. In this section, we describe the associative operad explicitly and also in terms of generators and relations (Theorem 11.1.7). As the first application of Theorem 11.1.7, we observe in Proposition 11.1.15 that the associative operad detects monoid structures. Moreover, in Theorem 11.2.16, we use Theorem 11.1.7 to describe ring category structures on small permutative categories in terms of the associative operad. Recall that Σ_n denotes the nth symmetric group, with identity $\mathrm{id}_n \in \Sigma_n$.

Definition 11.1.1. The *associative operad* As is defined by the following data.

Operations: $\mathsf{As}_n = \Sigma_n$ for $n \geq 0$.

Equivariance: The right symmetric group action

$$\mathsf{As}_n \times \Sigma_n \longrightarrow \mathsf{As}_n$$

is the group multiplication in Σ_n.

Unit: The operad unit is $\mathrm{id}_1 \in \mathsf{As}_1 = \Sigma_1$.

Composition: For $n \geq 1$ and $k_j \geq 0$ for $1 \leq j \leq n$, the operad composition

$$\mathsf{As}_n \times \prod_{j=1}^{n} \mathsf{As}_{k_j} \xrightarrow{\gamma} \mathsf{As}_{k_1+\cdots+k_n}$$

is defined by

$$\gamma\big(\sigma, (\tau_1, \ldots, \tau_n)\big) = \overbrace{\sigma\langle k_1, \ldots, k_n\rangle}^{\text{block permutation}} \cdot (\overbrace{\tau_1 \times \cdots \times \tau_n}^{\text{block sum}}) \tag{11.1.2}$$

for $\sigma \in \Sigma_n$ and $\tau_j \in \Sigma_{k_j}$. On the right-hand side, the block sum and the block permutation are as in, respectively, (II.1.1.8) and (II.1.1.19).

This finishes the definition of As. ◇

Explanation 11.1.3. The permutation in (11.1.2) is given by

$$\begin{aligned}&\big(\sigma\langle k_1,\ldots,k_n\rangle\cdot(\tau_1\times\cdots\times\tau_n)\big)\big(\overbrace{k_1+\cdots+k_{j-1}}^{0 \text{ if } j=1}+i\big)\\&=\underbrace{k_{\sigma^{-1}(1)}+\cdots+k_{\sigma^{-1}(\sigma(j)-1)}}_{0 \text{ if } \sigma(j)=1}+\tau_j(i)\end{aligned}$$

for $1 \le j \le n$ and $1 \le i \le k_j$.

- This permutation first permutes within the jth block via $\tau_j \in \Sigma_{k_j}$ for each $1 \le j \le n$.
- Then it permutes the n resulting consecutive blocks of lengths $k_1,\ldots,k_n$ via $\sigma \in \Sigma_n$.

On the left-hand side, $k_1+\cdots+k_{j-1}+i$ is the ith element in the jth block. On the right-hand side, its image under the permutation is the $\tau_j(i)$th element in the new $\sigma(j)$th block. ◇

Lemma 11.1.4. As *in Definition 11.1.1 is an operad.*

Proof. We check the operad axioms in Definition 5.1.2 for As.

Unity. The right unity axiom (5.1.5) and the left unity axiom (5.1.6) follow from the following permutation equalities.

$$\begin{aligned}\overbrace{\mathrm{id}_1\times\cdots\times\mathrm{id}_1}^{n}&=\mathrm{id}_n\in\Sigma_n\\\sigma\langle 1,\ldots,1\rangle&=\sigma\\\mathrm{id}_1\langle k_1\rangle&=\mathrm{id}_{k_1}\end{aligned}$$

Equivariance. The top equivariance axiom (5.1.7) holds by the following computation for $\sigma,\sigma' \in \Sigma_n$ with $n \ge 1$ and $\tau_j \in \Sigma_{k_j}$ for $1 \le j \le n$.

$$\begin{aligned}&\gamma\big(\sigma,(\tau_1,\ldots,\tau_n)\big)\cdot\sigma'\langle k_{\sigma'(1)},\ldots,k_{\sigma'(n)}\rangle\\&=\sigma\langle k_1,\ldots,k_n\rangle\cdot(\tau_1\times\cdots\times\tau_n)\cdot\sigma'\langle k_{\sigma'(1)},\ldots,k_{\sigma'(n)}\rangle\\&=\sigma\langle k_1,\ldots,k_n\rangle\cdot\sigma'\langle k_{\sigma'(1)},\ldots,k_{\sigma'(n)}\rangle\cdot\big(\tau_{\sigma'(1)}\times\cdots\times\tau_{\sigma'(n)}\big)\\&=(\sigma\sigma')\langle k_{\sigma'(1)},\ldots,k_{\sigma'(n)}\rangle\cdot\big(\tau_{\sigma'(1)}\times\cdots\times\tau_{\sigma'(n)}\big)\\&=\gamma\Big(\sigma\sigma',\big(\tau_{\sigma'(1)},\ldots,\tau_{\sigma'(n)}\big)\Big)\end{aligned}$$

The bottom equivariance axiom (5.1.8) follows from the fact that block sums preserve products in the sense that

$$(\tau_1\times\cdots\times\tau_n)\cdot(\tau_1'\times\cdots\times\tau_n')=\tau_1\tau_1'\times\cdots\times\tau_n\tau_n' \tag{11.1.5}$$

for $\tau_j' \in \Sigma_{k_j}$ and $1 \le j \le n$.

Associativity. For the associativity axiom (5.1.4), suppose further that $\theta_{j,i} \in \Sigma_{l_{j,i}}$ for $1 \le j \le n$ and $1 \le i \le k_j$. For each $1 \le j \le n$, suppose

$$l_j = l_{j,1}+\cdots+l_{j,k_j}.$$

Then there is an equality of permutations

$$(11.1.6)\quad \begin{aligned}&\Big(\sigma\langle k_1,\ldots,k_n\rangle\cdot\prod_{j=1}^n\tau_j\Big)\langle\overbrace{l_{1,1},\ldots,l_{1,k_1},\ldots,l_{n,1},\ldots,l_{n,k_n}}^{((l_{j,i})_{1\le i\le k_j})_{1\le j\le n}}\rangle\\ &=\sigma\langle l_1,\ldots,l_n\rangle\cdot\prod_{j=1}^n\tau_j\langle l_{j,1},\ldots,l_{j,k_j}\rangle\in\Sigma_{l_1+\cdots+l_n}.\end{aligned}$$

Indeed, on each side of the equal sign in (11.1.6), the permutation is given by

$$\Big(\sum_{p=1}^{j-1}l_p\Big)+\Big(\sum_{q=1}^{i-1}l_{j,q}\Big)+h\longmapsto\Big(\sum_{p=1}^{\sigma(j)-1}l_{\sigma^{-1}(p)}\Big)+\Big(\sum_{q=1}^{\tau_j(i)-1}l_{j,\tau_j^{-1}(q)}\Big)+h$$

for $1\le j\le n$, $1\le i\le k_j$, and $1\le h\le l_{j,i}$, with the usual convention that an empty sum means 0.

The associativity axiom (5.1.4) holds in As by the following computation, where the second and the third equalities hold by, respectively, (11.1.6) and (11.1.5).

$$\begin{aligned}&\gamma\Big(\gamma(\sigma,(\tau_1,\ldots,\tau_n)),\big((\theta_{1,i})_{1\le i\le k_1},\ldots,(\theta_{n,i})_{1\le i\le k_n}\big)\Big)\\ &=\Big(\sigma\langle k_1,\ldots,k_n\rangle\cdot\prod_{j=1}^n\tau_j\Big)\langle l_{1,1},\ldots,l_{1,k_1},\ldots,l_{n,1},\ldots,l_{n,k_n}\rangle\cdot\prod_{j=1}^n\prod_{i=1}^{k_j}\theta_{j,i}\\ &=\sigma\langle l_1,\ldots,l_n\rangle\cdot\Big[\prod_{j=1}^n\tau_j\langle l_{j,1},\ldots,l_{j,k_j}\rangle\Big]\cdot\prod_{j=1}^n\prod_{i=1}^{k_j}\theta_{j,i}\\ &=\sigma\langle l_1,\ldots,l_n\rangle\cdot\prod_{j=1}^n\Big[\tau_j\langle l_{j,1},\ldots,l_{j,k_j}\rangle\cdot\prod_{i=1}^{k_j}\theta_{j,i}\Big]\\ &=\gamma\Big(\sigma,\big(\gamma(\tau_j,(\theta_{j,1},\ldots,\theta_{j,k_j}))\big)_{1\le j\le n}\Big)\end{aligned}$$

This proves that As is an operad. □

Coherence of the Associative Operad. An *operad morphism* means a multifunctor (Definition 5.1.12) between two one-object multicategories. So an operad morphism $f:\mathsf{P}\longrightarrow\mathsf{Q}$ between two operads P and Q consists of structure morphisms

$$f_n:\mathsf{P}_n\longrightarrow\mathsf{Q}_n\quad\text{for}\quad n\ge 0$$

that preserve the symmetric group action, operad units, and operad composition in the sense of (5.1.13)–(5.1.15). The subscripts in the structure morphisms will often be omitted. The next coherence result characterizes the associative operad in terms of generators and relations.

Theorem 11.1.7. *The associative operad* As *is operadically generated by the permutations*

$$\mathrm{id}_0\in\Sigma_0\quad\textit{and}\quad\mathrm{id}_2\in\Sigma_2$$

and is subject to the following unity and associativity relations.

$$\begin{aligned} \gamma\big(\mathrm{id}_2, (\mathrm{id}_0, \mathrm{id}_1)\big) &= \mathrm{id}_1 = \gamma\big(\mathrm{id}_2, (\mathrm{id}_1, \mathrm{id}_0)\big) \\ \gamma\big(\mathrm{id}_2, (\mathrm{id}_2, \mathrm{id}_1)\big) &= \mathrm{id}_3 = \gamma\big(\mathrm{id}_2, (\mathrm{id}_1, \mathrm{id}_2)\big) \end{aligned} \tag{11.1.8}$$

Proof. Suppose $\mathsf{F}(\mu, \mathbb{0})$ is the operad freely generated by the elements

$$\mu \in \mathsf{F}_2 \quad \text{and} \quad \mathbb{0} \in \mathsf{F}_0,$$

and suppose $1 \in \mathsf{F}_1$ is the operad unit. Suppose As' is the quotient of $\mathsf{F}(\mu, \mathbb{0})$ by the relations operadically generated by

$$\begin{cases} \mu(\mathbb{0}, 1) = 1 = \mu(1, \mathbb{0}) \\ \mu(\mu, 1) = \mu(1, \mu) \end{cases} \tag{11.1.9}$$

in which we used the juxtaposition notation for operad composition. An element in $\mathsf{F}(\mu, \mathbb{0})$ and its image in As' will be denoted by the same symbol below. We will show that As' is canonically isomorphic to As.

By (11.1.9) and an induction, each operad composite in As'_n involving only

$$\mu_0 = \mathbb{0} \in \mathsf{As}'_0, \quad \mu_1 = 1 \in \mathsf{As}'_1, \quad \text{and} \quad \mu_2 = \mu \in \mathsf{As}'_2$$

yields the same element, which is denoted by $\mu_n \in \mathsf{As}'_n$. This property will be called the *uniqueness of* μ_n below. There is a unique operad morphism

$$\mathsf{F}(\mu, \mathbb{0}) \xrightarrow{\ \phi\ } \mathsf{As} \quad \text{such that} \quad \begin{cases} \phi(\mu) = \mathrm{id}_2 \in \Sigma_2 \\ \phi(\mathbb{0}) = \mathrm{id}_0 \in \Sigma_0. \end{cases}$$

Since the relations (11.1.8) hold in As, ϕ preserves the relations (11.1.9). So it factors through the quotient operad As' to yield an operad morphism

$$\mathsf{As}' \xrightarrow{\ \phi'\ } \mathsf{As} \quad \text{such that} \quad \phi'(\mu_n) = \mathrm{id}_n \tag{11.1.10}$$

for $n \geq 0$.

In the other direction, there is an operad morphism as follows.

$$\begin{array}{ccc} \mathsf{As} & \xrightarrow{\ \varphi\ } & \mathsf{As}' \\ \Sigma_n \ni \sigma & \longmapsto & \mu_n \sigma \in \mathsf{As}'_n \end{array} \tag{11.1.11}$$

Indeed, φ preserves the operad unit and the symmetric group action by definition. The preservation of the operad composition means the commutativity of the diagram

$$\begin{tikzcd} \mathsf{As}_n \times \prod_{j=1}^n \mathsf{As}_{k_j} \arrow[r, "\gamma"] \arrow[d, "\varphi \times \prod_j \varphi"'] & \mathsf{As}_k \arrow[d, "\varphi"] \\ \mathsf{As}'_n \times \prod_{j=1}^n \mathsf{As}'_{k_j} \arrow[r, "\gamma"] & \mathsf{As}'_k \end{tikzcd} \tag{11.1.12}$$

with $k = k_1 + \cdots + k_n$. By

- the equivariance axioms (5.1.7) and (5.1.8) in As and As' and
- the fact that φ preserves the symmetric group action,

it suffices to consider the element

$$\big(\mathrm{id}_n, (\mathrm{id}_{k_j})_{j=1}^n\big) \in \mathsf{As}_n \times \prod_{j=1}^n \mathsf{As}_{k_j}.$$

On this element, the diagram (11.1.12) is commutative by the uniqueness of $\mu_k \in \mathsf{As}'_k$. Finally, observe that the operad morphisms ϕ' in (11.1.10) and φ in (11.1.11) are inverses of each other. □

Monoids as Algebras. An operad, such as As, is regarded as an operad in a permutative category $(\mathsf{C}, \otimes, \mathbb{1})$ with all small coproducts, in which $\otimes$ commutes with small coproducts on each side separately, via the strong symmetric monoidal functor

$$\begin{array}{ccc} \mathsf{Set} & \longrightarrow & \mathsf{C} \\ X & \longmapsto & \coprod_X \mathbb{1}. \end{array} \tag{11.1.13}$$

Recall from Definition I.1.2.28 that a symmetric monoidal category is *closed* if, for each object A, the functor $- \otimes A$ has a right adjoint $[A, -]$, which is called the internal hom. If C is closed, then, by adjunction, $\otimes$ commutes with all small colimits that exist in C on each side separately.

In Proposition 11.1.15 below, we use the C-*enriched endomorphism operad* defined using the internal hom as in Proposition 6.3.6 with $A \in \mathsf{C} = \mathsf{V} = \mathsf{K}$. Its objects are

$$\mathsf{End}(A)_n = [A^{\otimes n}, A] \in \mathsf{C} \quad \text{for} \quad n \geq 0. \tag{11.1.14}$$

Recall from Definition I.1.2.8 that, in a monoidal category $(\mathsf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$, a *monoid* (A, μ, η) consists of

- an object $A \in \mathsf{C}$,
- a multiplication $\mu : A \otimes A \longrightarrow A$, and
- a unit $\eta : \mathbb{1} \longrightarrow A$

that are associative and unital in the sense that the following diagrams are commutative.

$$\begin{tikzcd} (A \otimes A) \otimes A \arrow[r, "\alpha"] \arrow[dd, "\mu\otimes 1_A"'] & A \otimes (A \otimes A) \arrow[d, "1_A\otimes\mu"] \\ & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \otimes A \arrow[r, "\lambda_A"] \arrow[d, "\eta\otimes 1_A"'] & A \arrow[d, equal] \\ A \otimes A \arrow[r, "\mu"] & A \\ A \otimes \mathbb{1} \arrow[r, "\rho_A"] \arrow[u, "1_A\otimes\eta"] & A \arrow[u, equal] \end{tikzcd}$$

The next observation is a consequence of the characterization of the associative operad in Theorem 11.1.7.

Proposition 11.1.15. *Suppose* $(\mathsf{C}, \otimes, \mathbb{1})$ *is a permutative closed category with all small coproducts. Then, for each object* $A \in \mathsf{C}$*, a monoid structure on* A *is uniquely determined by an operad morphism* $\mathsf{As} \longrightarrow \mathsf{End}(A)$*.*

Proof. Given an operad morphism

$$\phi : \mathsf{As} \longrightarrow \mathsf{End}(A),$$

the adjoint of the structure morphism

$$\mathsf{As}_0 = \mathbb{1} \xrightarrow{\phi_0} [\mathbb{1}, A] = \mathsf{End}(A)_0 \tag{11.1.16}$$

is the monoid unit $\eta : \mathbb{1} \longrightarrow A$. The monoid multiplication μ of A is the following composite, with each $\mathbb{1}_?$ a copy of $\mathbb{1}$.

$$\begin{tikzcd} A^{\otimes 2} \arrow[rr, "\mu"] \arrow[d, "\mathrm{id}_2"'] & & A \\ \big(\mathbb{1}_{\mathrm{id}_2} \amalg \mathbb{1}_{(1,2)}\big) \otimes A^{\otimes 2} \arrow[r, equal] & \mathsf{As}_2 \otimes A^{\otimes 2} \arrow[ur, "\phi_2^\#"'] & \end{tikzcd} \tag{11.1.17}$$

The right vertical morphism $\phi_2^\#$ is the adjoint of the structure morphism

$$\mathsf{As}_2 = \mathbb{1}_{\mathrm{id}_2} \amalg \mathbb{1}_{(1,2)} \xrightarrow{\phi_2} \big[A^{\otimes 2}, A\big] = \mathsf{End}(A)_2.$$

The triple (A, μ, η) satisfies the monoid associativity and unity axioms in Definition I.1.2.8 by the corresponding properties (11.1.8) in As.

Conversely, given a monoid (A, μ, η), the assignment in the previous paragraph uniquely defines an operad morphism $\mathsf{As} \longrightarrow \mathsf{End}(A)$ by Theorem 11.1.7. By construction, these two assignments are inverses of each other. □

Remark 11.1.18. For symmetric monoidal categories that are not strict, we will use Proposition 11.1.15 along with Explanation 6.1.19. ◇

11.2. Detecting Ring Categories

In this section, we prove that the associative operad detects ring category structures on small permutative categories (Theorem 11.2.16). This result is due to Elmendorf-Mandell and is an application of the Coherence Theorem 11.1.7 for the associative operad.

Recall from Definition 11.1.1 and Lemma 11.1.4 that the associative operad As has $\mathsf{As}_n = \Sigma_n$, the nth symmetric group, and unit $\mathrm{id}_1 \in \mathsf{As}_1$. Its operad composition γ is given by the product

$$\gamma\big(\sigma, (\tau_1, \ldots, \tau_n)\big) = \overbrace{\sigma\langle k_1, \ldots, k_n\rangle}^{\text{block permutation}} \cdot \big(\overbrace{\tau_1 \times \cdots \times \tau_n}^{\text{block sum}}\big)$$

for $\sigma \in \Sigma_n$ and $\tau_j \in \Sigma_{k_j}$ for $1 \le j \le n$.

- By Theorem 11.1.7, As is operadically generated by $\mathrm{id}_0 \in \mathsf{As}_0$ and $\mathrm{id}_2 \in \mathsf{As}_2$ and is subject to unity and associativity relations.
- By Proposition 11.1.15, for a permutative closed category C with all small coproducts, As detects monoid structure. This means that a monoid structure on an object $A \in \mathsf{C}$ is uniquely determined by an operad morphism $\mathsf{As} \longrightarrow \mathsf{End}(A)$, with $\mathsf{End}(A)$ the C-enriched endomorphism operad in (11.1.14).

The unique object in the one-object multicategory As is denoted by $*$.

For each symmetric monoidal category $(\mathsf{C}, \otimes, \mathbb{1})$ with small coproducts, in which $\otimes$ commutes with small coproducts on each side separately, there is an adjunction

$$\mathsf{Set} \underset{\mathsf{C}(\mathbb{1},?)}{\overset{\coprod_? \mathbb{1}}{\rightleftarrows}} \mathsf{C}. \tag{11.2.1}$$

The right adjoint sends each object $A \in \mathsf{C}$ to the set $\mathsf{C}(\mathbb{1}, A)$ of morphisms $\mathbb{1} \longrightarrow A$ in C. The left adjoint is the functor in (11.1.13) that sends a set X to the coproduct $\coprod_X \mathbb{1}$. It is a strong symmetric monoidal functor. The associative operad As is regarded as a C-enriched operad via the left adjoint.

Example 11.2.2. For the symmetric monoidal category Cat of small categories and functors, the adjunction (11.2.1) becomes

$$\mathsf{Set} \underset{\mathsf{Ob}}{\overset{\mathsf{dis}}{\rightleftarrows}} \mathsf{Cat}. \tag{11.2.3}$$

The left adjoint sends each set X to the discrete category $\mathsf{dis}(X)$ with object set X. The right adjoint Ob is the forgetful functor that sends each small category to its set of objects. The associative operad As is also regarded as a Cat-enriched operad via the strong symmetric monoidal functor dis. So As_n is the discrete category with object set Σ_n. $\diamond$

For the reader's convenience, here we recall from Chapter II.9 the definition of a ring category.

Definition 11.2.4. A *ring category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$

consisting of the following data.

The Additive Structure: $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ is a permutative category.

The Multiplicative Structure: $(\mathsf{C}, \otimes, \mathbb{1})$ is a strict monoidal category.

The Factorization Morphisms: ∂^l and ∂^r are natural transformations

$$\begin{aligned} (A \otimes C) \oplus (B \otimes C) &\xrightarrow{\partial^l_{A,B,C}} (A \oplus B) \otimes C \\ (A \otimes B) \oplus (A \otimes C) &\xrightarrow{\partial^r_{A,B,C}} A \otimes (B \oplus C) \end{aligned} \tag{11.2.5}$$

for objects $A, B, C \in \mathsf{C}$, which are called the *left factorization morphism* and the *right factorization morphism*, respectively.

To simplify the presentation, we often abbreviate $\otimes$ to concatenation, with $\otimes$ always taking precedence over $\oplus$ in the absence of clarifying parentheses. For example, the left factorization morphism is abbreviated to $AC \oplus BC \longrightarrow (A \oplus B)C$. The subscripts in $\xi^{\oplus}$, ∂^l, and ∂^r are sometimes omitted.

The above data are required to satisfy the following seven axioms for all objects A, A', A'', B, B', B'', C, and C' in C.

The Multiplicative Zero Axiom: The diagram of functors

$$\begin{tikzcd} * \times \mathsf{C} \arrow[r, "\cong"] \arrow[d, "\mathbb{0} \times 1_{\mathsf{C}}"'] & \mathsf{C} \arrow[d, "\mathbb{0}"] & \mathsf{C} \times * \arrow[l, "\cong"'] \arrow[d, "1_{\mathsf{C}} \times \mathbb{0}"] \\ \mathsf{C} \times \mathsf{C} \arrow[r, "\otimes"] & \mathsf{C} & \mathsf{C} \times \mathsf{C} \arrow[l, "\otimes"'] \end{tikzcd} \tag{11.2.6}$$

is commutative. In this diagram, the top horizontal isomorphisms drop the $*$ argument. Each $\mathbb{0}$ denotes the constant functor at $\mathbb{0} \in \mathsf{C}$ and $1_{\mathbb{0}}$.

The Zero Factorization Axiom:

$$\begin{aligned} \partial^l_{\mathbb{0},B,C} &= 1_{B\otimes C} & \partial^r_{\mathbb{0},B,C} &= 1_{\mathbb{0}} \\ \partial^l_{A,\mathbb{0},C} &= 1_{A\otimes C} & \partial^r_{A,\mathbb{0},C} &= 1_{A\otimes C} \\ \partial^l_{A,B,\mathbb{0}} &= 1_{\mathbb{0}} & \partial^r_{A,B,\mathbb{0}} &= 1_{A\otimes B} \end{aligned} \tag{11.2.7}$$

The three equalities for ∂^l are called the *left zero factorization axioms*. The three equalities for ∂^r are called the *right zero factorization axioms*.

The Unit Factorization Axiom:

$$\begin{aligned} \partial^l_{A,B,\mathbb{1}} &= 1_{A\oplus B} \\ \partial^r_{\mathbb{1},B,C} &= 1_{B\oplus C} \end{aligned} \tag{11.2.8}$$

These are called, respectively, the *left* and the *right* unit factorization axioms.

The Symmetry Factorization Axiom: The following two diagrams in C are commutative.

$$\begin{tikzcd} AC \oplus BC \arrow[r, "\partial^l"] \arrow[d, "\xi^\oplus"'] & (A\oplus B)C \arrow[d, "\xi^\oplus 1_C"] \\ BC \oplus AC \arrow[r, "\partial^l"] & (B\oplus A)C \end{tikzcd} \qquad \begin{tikzcd} AB \oplus AC \arrow[r, "\partial^r"] \arrow[d, "\xi^\oplus"'] & A(B\oplus C) \arrow[d, "1_A\xi^\oplus"] \\ AC \oplus AB \arrow[r, "\partial^r"] & A(C\oplus B) \end{tikzcd} \tag{11.2.9}$$

These are called, respectively, the *left* and the *right* symmetry factorization axioms.

The Internal Factorization Axiom: The following two diagrams in C are commutative.

(11.2.10)

$$\begin{tikzcd} AB \oplus A'B \oplus A''B \arrow[r, "\partial^l\oplus 1"] \arrow[d, "1\oplus\partial^l"'] & (A\oplus A')B \oplus A''B \arrow[d, "\partial^l"] \\ AB \oplus (A'\oplus A'')B \arrow[r, "\partial^l"] & (A\oplus A'\oplus A'')B \end{tikzcd} \qquad \begin{tikzcd} AB \oplus AB' \oplus AB'' \arrow[r, "\partial^r\oplus 1"] \arrow[d, "1\oplus\partial^r"'] & A(B\oplus B') \oplus AB'' \arrow[d, "\partial^r"] \\ AB \oplus A(B'\oplus B'') \arrow[r, "\partial^r"] & A(B\oplus B'\oplus B'') \end{tikzcd}$$

These are called, respectively, the *left* and the *right* internal factorization axioms.

The External Factorization Axiom: The three diagrams in C below are commutative.

$$\begin{tikzcd} ABC \oplus A'BC \arrow[r, "\partial^l_{A,A',BC}"] \arrow[d, "\partial^l_{AB,A'B,C}"'] & (A\oplus A')BC \arrow[d, equal] \\ (AB\oplus A'B)C \arrow[r, "\partial^l_{A,A',B}1_C"] & (A\oplus A')BC \end{tikzcd} \tag{11.2.11}$$

$$\begin{tikzcd} ABC \oplus AB'C \arrow[r, "\partial^l_{AB,AB',C}"] \arrow[d, "\partial^r_{A,BC,B'C}"'] & (AB\oplus AB')C \arrow[d, "\partial^r 1_C"] \\ A(BC\oplus B'C) \arrow[r, "1_A\partial^l_{B,B',C}"] & A(B\oplus B')C \end{tikzcd} \tag{11.2.12}$$

$$\begin{tikzcd} ABC \oplus ABC' \arrow[r, "\partial^r_{AB,C,C'}"] \arrow[d, "\partial^r_{A,BC,BC'}"'] & AB(C \oplus C') \arrow[d, equal] \\ A(BC \oplus BC') \arrow[r, "1_A \partial^r_{B,C,C'}"] & AB(C \oplus C') \end{tikzcd} \tag{11.2.13}$$

These are called, respectively, the *left*, the *middle*, and the *right* external factorization axioms.

The 2-By-2 Factorization Axiom: The following diagram in C is commutative.

$$\begin{tikzcd} & A(B \oplus B') \oplus A'(B \oplus B') \arrow[dr, "\partial^l"] & \\ AB \oplus AB' \oplus A'B \oplus A'B' \arrow[ur, "\partial^r \oplus \partial^r"] \arrow[d, "1 \oplus \xi^{\oplus} \oplus 1"'] & & (A \oplus A')(B \oplus B') \\ AB \oplus A'B \oplus AB' \oplus A'B' \arrow[r, "\partial^l \oplus \partial^l"'] & (A \oplus A')B \oplus (A \oplus A')B' \arrow[ur, "\partial^r"'] & \end{tikzcd} \tag{11.2.14}$$

This finishes the definition of a ring category.

Moreover, a ring category as above is said to be

- *small* if it has a set of objects and
- *tight* if ∂^l and ∂^r in (11.2.5) are natural isomorphisms. $\diamond$

Definition 11.2.15. For a permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$, a *ring category structure* on C is the additional data $(\otimes, \mathbb{1}, \partial^l, \partial^r)$ such that the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$

is a ring category as in Definition 11.2.4. $\diamond$

Recall from Section 6.6 that $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-enriched multicategory with small permutative categories as objects. The category

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle\, ;\, \mathsf{D}\big) = \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}_1, \ldots, \mathsf{C}_n \rangle\, ;\, \mathsf{D}\big)$$

has

- n-linear functors $\mathsf{C}_1 \times \cdots \times \mathsf{C}_n \longrightarrow \mathsf{D}$ (Definition 6.5.4) as objects and
- multilinear transformations (Definition 6.5.11) as morphisms.

Its operad composition is defined in (6.6.6) for multilinear functors and (6.6.9) for multilinear transformations. Also recall from Definition 5.1.12 the notion of an enriched multifunctor. The next result is [**EM06**, 3.4], which says that the Cat-enriched associative operad detects ring category structures on small permutative categories.

Theorem 11.2.16. *For each small permutative category C, there is a canonical bijective correspondence between*

- *ring category structures on C and*
- Cat*-enriched multifunctors*

$$F : \mathsf{As} \longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \textit{such that} \quad F(*) = \mathsf{C}.$$

Proof. By the adjunction (11.2.3), a Cat-enriched multifunctor $\mathsf{As} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$ is equivalent to a multifunctor $\mathsf{As} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$ with $\mathsf{PermCat}^{\mathsf{su}}$ regarded as a multicategory without its Cat-enrichment, that is, multilinear transformations. Therefore, by Theorem 11.1.7, a Cat-enriched multifunctor

$$F : \mathsf{As} \longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \text{with} \quad F(*) = (\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}) \tag{11.2.17}$$

is equivalent to

- a 0-linear functor, that is, an object
$$F(\mathrm{id}_0) = \mathbb{1} \in \mathsf{C}$$
and
- a 2-linear functor
$$F(\mathrm{id}_2) = (\otimes, \partial^l, \partial^r) : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$$

such that the unity and the associativity relations (11.1.8) are preserved.

- The natural transformations
$$(A \otimes C) \oplus (B \otimes C) \xrightarrow{\partial^l_{A,B,C}} (A \oplus B) \otimes C$$
$$(A \otimes B) \oplus (A \otimes C) \xrightarrow{\partial^r_{A,B,C}} A \otimes (B \oplus C)$$
for $A, B, C \in \mathsf{C}$ are, respectively, the first and the second linearity constraints of the 2-linear functor $\otimes$ as in Definition 6.5.4.
- The preservation of the unity relation in (11.1.8) means the commutativity of the diagram
$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathbb{1} \times 1_{\mathsf{C}}"] \arrow[d, equal] & \mathsf{C} \times \mathsf{C} \arrow[d, "(\otimes, \partial^l, \partial^r)"] & \mathsf{C} \arrow[l, "1_{\mathsf{C}} \times \mathbb{1}"'] \arrow[d, equal] \\ \mathsf{C} \arrow[r, "1_{\mathsf{C}}"] & \mathsf{C} & \mathsf{C} \arrow[l, "1_{\mathsf{C}}"'] \end{tikzcd} \tag{11.2.18}$$
of 1-linear functors. Here $1_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{C}$ is the identity 1-linear functor with the identity linearity constraint (6.6.1).
- The preservation of the associativity relation in (11.1.8) means the commutativity of the diagram
$$\begin{tikzcd} \mathsf{C} \times \mathsf{C} \times \mathsf{C} \arrow[r, "1_{\mathsf{C}} \times (\otimes, \partial^l, \partial^r)"] \arrow[d, "(\otimes, \partial^l, \partial^r) \times 1_{\mathsf{C}}"'] & \mathsf{C} \times \mathsf{C} \arrow[d, "(\otimes, \partial^l, \partial^r)"] \\ \mathsf{C} \times \mathsf{C} \arrow[r, "(\otimes, \partial^l, \partial^r)"] & \mathsf{C} \end{tikzcd} \tag{11.2.19}$$
of 3-linear functors.

We now explain that the data $\{\mathbb{1}, \otimes, \partial^l, \partial^r\}$ and the commutative diagrams (11.2.18) and (11.2.19) are equivalent to a ring category structure on the small permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$.

The commutativity of the diagrams (11.2.18) and (11.2.19) of *functors* means that $\otimes$ is strictly associative with $\mathbb{1}$ as the strict two-sided unit. In other words, $(\mathsf{C}, \otimes, \mathbb{1})$ is a strict monoidal category. At this point, the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$

contains the data part of a ring category in Definition 11.2.4. It remains to check that

- the 2-linear functoriality of $(\otimes, \partial^l, \partial^r)$ and
- the linearity constraints of the commutative diagrams (11.2.18) and (11.2.19)

are equivalent to the ring category axioms (11.2.6)–(11.2.14).

First we consider the 2-linear functor axioms in Definition 6.5.4 for $(\otimes, \partial^l, \partial^r)$.

- Its unity axiom is the multiplicative zero axiom (11.2.6).
- Its constraint unity axiom is the zero factorization axiom (11.2.7).
- Its constraint associativity axiom (6.5.5) is the internal factorization axiom (11.2.10).
- Its constraint symmetry axiom (6.5.6) is the symmetry factorization axiom (11.2.9).
- Its constraint 2-by-2 axiom (6.5.7) with $(i,k) = (2,1)$ is the 2-by-2 factorization axiom (11.2.14). The case $(i,k) = (1,2)$ is equivalent to the case $(i,k) = (2,1)$ by the symmetry axiom (II.1.3.33) for the additive symmetry $\xi^{\oplus}$, which says $\xi^{\oplus}\xi^{\oplus} = 1$.

Next we consider the linearity constraints, in the sense of Definition 6.5.4, in the commutative diagrams (11.2.18) and (11.2.19) of multilinear functors.

- The linearity constraint of the identity 1-linear functor 1_{C} is the identity (6.6.1). Therefore, in terms of their linearity constraints, the commutativity of the diagram (11.2.18) of 1-linear functors is the unit factorization axiom (11.2.8).
- In the commutative diagram (11.2.19) of 3-linear functors, the equality of the first linearity constraints of the two composites is the left external factorization axiom (11.2.11). The equality of the second (respectively, third) linearity constraints is the middle external factorization axiom (11.2.12) (respectively, the right external factorization axiom (11.2.13)).

Therefore, a Cat-enriched multifunctor as in (11.2.17) is equivalent to a ring category structure on the small permutative category C. □

11.3. K-Theory of Ring Categories are Ring Symmetric Spectra

In this section, we prove a theorem of Elmendorf-Mandell that says that the K-theory of each small ring category is a strict ring symmetric spectrum (Corollary 11.3.16). This result is a consequence of Theorem 11.2.16 and the Elmendorf-Mandell K-theory multifunctor. We begin by defining strict ring symmetric spectra and discussing some examples.

Strict Ring Symmetric Spectra. Recall from Definitions 7.6.1 and 7.6.8 and Theorem 7.6.15 the symmetric monoidal closed category

$$\left(\mathsf{SymSp}, \square_S, \mathsf{Hom}_S\right)$$

of symmetric spectra.

Definition 11.3.1. A *strict ring symmetric spectrum* is a monoid in SymSp. ⋄

To explicitly explain the structure of a strict ring symmetric spectrum, first recall from Definitions 7.3.1, 7.3.3, and 7.4.1 the following concepts.

- sSet_*^{Σ} is the symmetric monoidal closed category of symmetric sequences.

- $\square$ is the Day convolution of symmetric sequences.
- $S \in \mathsf{sSet}_*^{\Sigma}$ is the symmetric sphere with monoid multiplication

$$S \square S \xrightarrow{\mu^S} S$$

determined by the canonical isomorphisms

$$S^p \wedge S^q \xrightarrow[\cong]{\mu^S_{p,q}} S^{p+q} \in \mathsf{sSet}_* \quad \text{for} \quad p,q \geq 0.$$

SymSp is the category of left S-modules as in Definition 7.4.2.

In the above context, we may unpack strict ring symmetric spectra as follows.

Proposition 11.3.2. *A strict ring symmetric spectrum is equivalent to a triple*

$$(X, \mu, \eta)$$

consisting of

- *a symmetric spectrum* (X, ρ) *and*
- *morphisms*

$$X \square X \xrightarrow{\mu} X \xleftarrow{\eta} S \in \mathsf{sSet}_*^{\Sigma}$$

such that the following four diagrams in sSet_*^{Σ} *are commutative.*

Compatibility of μ and S-action:

(11.3.3)
$$\begin{tikzcd}
(X \square S) \square X \arrow[r, "\xi\square 1", "\cong"'] \arrow[d, "a"', "\cong"] & (S \square X) \square X \arrow[r, "a", "\cong"'] \arrow[d, "\rho\square 1"'] & S \square (X \square X) \arrow[dd, "1\square\mu"] \\
X \square (S \square X) \arrow[d, "1\square\rho"'] & X \square X \arrow[d, "\mu"'] & \\
X \square X \arrow[r, "\mu"] & X & S \square X \arrow[l, "\rho"']
\end{tikzcd}$$

Compatibility of η and S-action:

(11.3.4)
$$\begin{tikzcd}
S \square S \arrow[r, "\mu^S"] \arrow[d, "1\square\eta"'] & S \arrow[d, "\eta"] \\
S \square X \arrow[r, "\rho"] & X
\end{tikzcd}$$

Associativity:

(11.3.5)
$$\begin{tikzcd}
(X \square X) \square X \arrow[r, "a", "\cong"'] \arrow[dd, "\mu\square 1"'] & X \square (X \square X) \arrow[d, "1\square\mu"] \\
& X \square X \arrow[d, "\mu"] \\
X \square X \arrow[r, "\mu"] & X
\end{tikzcd}$$

Unity:

$$\begin{tikzcd} S \square X \arrow[r, "\rho"] \arrow[d, "\eta\square 1"'] & X \arrow[d, equal] \\ X \square X \arrow[r, "\mu"] & X \\ X \square S \arrow[u, "1\square\eta"] \arrow[r, "\xi"] & S \square X \arrow[u, "\rho"'] \end{tikzcd} \tag{11.3.6}$$

Proof. By Definition I.1.2.8, a monoid in $(\mathsf{SymSp}, \square_S)$ is a triple

$$(X, \overline{\mu}, \overline{\eta})$$

consisting of

- a symmetric spectrum
$$\big(X, \rho : S \square X \longrightarrow X\big)$$
and
- morphisms
$$X \square_S X \xrightarrow{\overline{\mu}} X \xleftarrow{\overline{\eta}} S \in \mathsf{SymSp}$$

such that the associativity and unity diagrams in SymSp below are commutative.

$$\begin{tikzcd} (X \square_S X) \square_S X \arrow[r, "a", "\cong"'] \arrow[dd, "\overline{\mu}\square_S 1"'] & X \square_S (X \square_S X) \arrow[d, "1\square_S\overline{\mu}"] \\ & X \square_S X \arrow[d, "\overline{\mu}"] \\ X \square_S X \arrow[r, "\overline{\mu}"] & X \end{tikzcd} \qquad \begin{tikzcd} S \square_S X \arrow[r, "\rho"] \arrow[d, "\overline{\eta}\square_S 1"'] & X \arrow[d, equal] \\ X \square_S X \arrow[r, "\overline{\mu}"] & X \\ X \square_S S \arrow[u, "1\square_S\overline{\eta}"] \arrow[r, "\xi"] & S \square_S X \arrow[u, "\rho"'] \end{tikzcd} \tag{11.3.7}$$

By Proposition 7.6.2, the monoid multiplication $\overline{\mu}$ is equivalent to a morphism

$$X \square X \xrightarrow{\mu} X \in \mathsf{sSet}_*^{\Sigma} \tag{11.3.8}$$

such that the compatibility diagram (11.3.3) commutes. The monoid unit morphism $\overline{\eta}$ is a morphism

$$S \xrightarrow{\eta} X \in \mathsf{sSet}_*^{\Sigma} \tag{11.3.9}$$

that is compatible with the left S-module action as in Definition 7.4.2. This compatibility condition is the commutative diagram (11.3.4). Since SymSp is the category of left S-modules, the diagrams in (11.3.7) commute in SymSp if and only if they commute in sSet_*^{Σ}. Using

- the structure morphisms μ in (11.3.8) and η in (11.3.9) and
- the universal properties of the coequalizer defining the smash product $\square_S$ (Definition 7.6.1),

the diagrams in (11.3.7) commute in sSet_*^{Σ} if and only if the diagrams (11.3.5) and (11.3.6) commute. $\square$

Explanation 11.3.10 (Strict Ring Symmetric Spectra)**.** As in Explanation 7.6.4, the multiplication morphism μ in (11.3.8) is equivalent to the family of $(\Sigma_p \times \Sigma_q)$-equivariant morphisms

$$X_p \wedge X_q \xrightarrow{\mu_{p,q}} X_{p+q} \in \mathsf{sSet}_* \quad \text{for} \quad p, q \geq 0. \tag{11.3.11}$$

In terms of these morphisms, the compatibility diagram (11.3.3) is equivalent to the diagrams (7.6.5) and (7.6.7) with $f_{p,q} = \mu_{p,q}$. These diagrams are stated below, with $\Sigma_{p,n} = \Sigma_p \times \Sigma_n$ in the second diagram.

$$\begin{tikzcd}
(S^n \wedge X_p) \wedge X_q \arrow[r, "a", "\cong"'] \arrow[d, "\rho_{n,p} \wedge 1"'] & S^n \wedge (X_p \wedge X_q) \arrow[dd, "1 \wedge \mu_{p,q}"] \\
X_{n+p} \wedge X_q \arrow[d, "\mu_{n+p,q}"'] & \\
X_{n+p+q} & S^n \wedge X_{p+q} \arrow[l, "\rho_{n,p+q}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma_{(p+n+q)} \times_{\Sigma_{p+n,q}} \left(\Sigma_{(p+n)} \times_{\Sigma_{p,n}} X_p \wedge S^n\right) \wedge X_q \arrow[dr, "\xi_{p,n} \wedge 1"] \arrow[dd, "a"', "\cong"] & \\
& \Sigma_{(p+n+q)} \times_{\Sigma_{n+p,q}} \left(\Sigma_{(n+p)} \times_{\Sigma_{n,p}} S^n \wedge X_p\right) \wedge X_q \arrow[dd, "\rho_{n,p} \wedge 1"] \\
\Sigma_{(p+n+q)} \times_{\Sigma_{p,n+q}} X_p \wedge \left(\Sigma_{(n+q)} \times_{\Sigma_{n,q}} S^n \wedge X_q\right) \arrow[dd, "1 \wedge \rho_{n,q}"'] & \\
& \Sigma_{(p+n+q)} \times_{\Sigma_{n+p,q}} X_{n+p} \wedge X_q \arrow[dd, "\mu_{n+p,q}"] \\
\Sigma_{(p+n+q)} \times_{\Sigma_{p,n+q}} X_p \wedge X_{n+q} \arrow[dr, "\mu_{p,n+q}"] & \\
& X_{n+p+q}
\end{tikzcd}$$

The unit morphism η in (11.3.9) is equivalent to a sequence of Σ_p-equivariant morphisms

$$S^p \xrightarrow{\eta_p} X_p \in \mathsf{sSet}_* \quad \text{for} \quad p \geq 0. \tag{11.3.12}$$

The compatibility diagram (11.3.4) and the associativity diagram (11.3.5) in sSet_*^{Σ} are equivalent to the diagrams

$$\begin{tikzcd}
S^p \wedge S^q \arrow[r, "\mu^S_{p,q}", "\cong"'] \arrow[d, "1 \wedge \eta_q"'] & S^{p+q} \arrow[d, "\eta_{p+q}"] \\
S^p \wedge X_q \arrow[r, "\rho_{p,q}"] & X_{p+q}
\end{tikzcd}
\qquad
\begin{tikzcd}
(X_n \wedge X_p) \wedge X_q \arrow[r, "a", "\cong"'] \arrow[dd, "\mu_{n,p} \wedge 1"'] & X_n \wedge (X_p \wedge X_q) \arrow[d, "1 \wedge \mu_{p,q}"] \\
& X_n \wedge X_{p+q} \arrow[d, "\mu_{n,p+q}"] \\
X_{n+p} \wedge X_q \arrow[r, "\mu_{n+p,q}"] & X_{n+p+q}
\end{tikzcd}$$

in sSet$_*$ for $n, p, q \geq 0$. The top and bottom halves of the unity diagram (11.3.6) are equivalent to the diagrams

$$\begin{tikzcd} S^p \wedge X_q \arrow[r, "\rho_{p,q}"] \arrow[d, "\eta_p \wedge 1"'] & X_{p+q} \arrow[d, equal] \\ X_p \wedge X_q \arrow[r, "\mu_{p,q}"] & X_{p+q} \end{tikzcd} \qquad \begin{tikzcd} \Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} X_p \wedge S^q \arrow[r, "\xi", "\cong"'] \arrow[d, "1 \wedge \eta_q"'] & \Sigma_{(q+p)} \times_{\Sigma_q \times \Sigma_p} S^q \wedge X_p \arrow[d, "\rho_{q,p}"] \\ \Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} X_p \wedge X_q \arrow[r, "\mu_{p,q}"] & X_{p+q} = X_{q+p} \end{tikzcd}$$

in sSet$_*$ for $p, q \geq 0$. ◇

Using Explanation 11.3.10, we now discuss a few basic examples of strict ring symmetric spectra.

Example 11.3.13 (The Sphere Spectrum)**.** Continuing Example 7.4.9, the symmetric sphere

$$S = \{S^p\}_{p \geq 0}$$

is a strict ring symmetric spectrum. The multiplication morphism $\mu_{p,q}$ in (11.3.11) is given by the canonical isomorphism

$$S^p \wedge S^q \xrightarrow[\cong]{\mu^S_{p,q}} S^{p+q} \quad \text{for} \quad p, q \geq 0.$$

The unit morphism η_p in (11.3.12) is the identity morphism of S^p. The diagrams in Explanation 11.3.10 for the symmetric sphere all follow from the bijection

$$(S^p)_n = \underline{n}^{\wedge p} \cong \underline{n^p} \quad \text{for} \quad n, p \geq 0$$

in Definitions 8.1.5 and 8.2.1 and the Symmetric Coherence Theorem 1.1.41 for sSet$_*$. ◇

Example 11.3.14 (Suspension Spectra of Simplicial Monoids)**.** Suppose (K, μ, η) is a monoid in $(\text{sSet}_*, \wedge, S^0)$. Continuing Example 7.4.10, the suspension spectrum

$$\Sigma^\infty K = \{S^p \wedge K\}_{p \geq 0}$$

is a strict ring symmetric spectrum. The multiplication morphism $\mu_{p,q}$ in (11.3.11) is given by the following composite.

$$\begin{tikzcd} (S^p \wedge K) \wedge (S^q \wedge K) \arrow[rr, "\mu_{p,q}"] \arrow[dr, "\xi_{\mathrm{mid}}"'] & & S^{p+q} \wedge K \\ & (S^p \wedge S^q) \wedge (K \wedge K) \arrow[ur, "\mu^S_{p,q} \wedge \mu"'] & \end{tikzcd}$$

The morphism ξ_{mid} is the unique coherence isomorphism in the symmetric monoidal category sSet$_*$ that swaps the first copy of K and S^q, as in Definition 1.3.2. The unit morphism η_p in (11.3.12) is the composite

$$\begin{tikzcd} S^p \arrow[rr, "\eta_p"] \arrow[dr, "\cong"'] & & S^p \wedge K \\ & S^p \wedge S^0 \arrow[ur, "1 \wedge \eta"'] & \end{tikzcd}$$

with $\cong$ the inverse of the right unit isomorphism in sSet$_*$. The first diagram in Explanation 11.3.10 for $\Sigma^\infty K$ is the outer diagram below in sSet$_*$, where $\wedge$ is omitted to save space.

$$\begin{tikzcd}
(S^n(S^pK))(S^qK) \arrow[rr, "a"] \arrow[d, "a^{-1}1"'] \arrow[drr, phantom, "\text{coherence}"] & & S^n((S^pK)(S^qK)) \arrow[d, "1\xi_{\mathrm{mid}}"] \\
((S^nS^p)K)(S^qK) \arrow[r, "\xi_{\mathrm{mid}}"] \arrow[d, "(\mu^S_{n,p}1)1"'] \arrow[dr, phantom, "\text{nat}"] & ((S^nS^p)S^q)(KK) \arrow[d, "a1"] \arrow[ddl, "(\mu^S_{n,p}1)1"'] & S^n((S^pS^q)(KK)) \arrow[dl, "a^{-1}"'] \arrow[dd, "1(\mu^S_{p,q}\mu)"] \\
(S^{n+p}K)(S^qK) \arrow[d, "\xi_{\mathrm{mid}}"] \arrow[r, phantom, "\text{assoc}"] & (S^n(S^pS^q))(KK) \arrow[d, "(1\mu^S_{p,q})1"] \arrow[r, phantom, "\text{nat}"] & {} \\
(S^{n+p}S^q)(KK) \arrow[d, "\mu^S_{n+p,q}\mu"'] & (S^nS^{p+q})(KK) \arrow[dl, "\mu^S_{n,p+q}\mu"'] \arrow[dr, "1\mu"] \arrow[d, phantom, "\text{functoriality}"] & S^n(S^{p+q}K) \arrow[d, "a^{-1}"] \\
S^{n+p+q}K & {} & (S^nS^{p+q})K \arrow[ll, "\mu^S_{n,p+q}1"]
\end{tikzcd}$$

- The top region commutes by the Symmetric Coherence Theorem 1.1.41 for sSet_*.
- The left triangle commutes by the naturality of ξ_{mid}.
- The right trapezoid commutes by the naturality of the associativity isomorphism a.
- The lower left quadrilateral commutes by the associativity of μ^S and the functoriality of $\wedge$.
- The bottom triangle commutes by the functoriality of $\wedge$.

Other diagrams in Explanation 11.3.10 follow similarly from those for the symmetric sphere, the monoid axioms for (K, μ, η), naturality, functoriality, and Theorem 1.1.41 for sSet_*. $\diamond$

Example 11.3.15 (Eilenberg-Mac Lane Spectra)**.** Suppose R is a ring with unit 1^R. Continuing Example 7.4.11, the Eilenberg-Mac Lane spectrum

$$HR = \left\{R \otimes S^p\right\}_{p \geq 0}$$

is a strict ring symmetric spectrum. The multiplication morphism $\mu_{p,q}$ in (11.3.11) is given on k-simplices by the following assignment on direct sum generators, with $r, r' \in R$ and $i_a, j_b \in \underline{k}^\flat$.

$$\begin{tikzcd}
\Big(\bigoplus\limits_{(\underline{k}^p)^\flat} R\Big) \wedge \Big(\bigoplus\limits_{(\underline{k}^q)^\flat} R\Big) \arrow[r, "(\mu_{p,q})_k"] & \bigoplus\limits_{(\underline{k}^{p+q})^\flat} R \\
\big(r_{i_1,\ldots,i_p}\big) \wedge \big(r'_{j_1,\ldots,j_q}\big) \arrow[r, mapsto] & (rr')_{i_1,\ldots,i_p,j_1,\ldots,j_q}
\end{tikzcd}$$

The unit morphism η_p in (11.3.12) is given on k-simplices by the following morphism.

$$\begin{tikzcd}
(S^p)_k = \underline{k}^{\wedge p} \arrow[r, "(\eta_p)_k"] & \bigoplus\limits_{(\underline{k}^p)^\flat} R \\
(i_1, \ldots, i_p) \arrow[r, mapsto] & 1^R_{i_1,\ldots,i_p} \quad \text{if each } i_j \in \underline{k}^\flat. \\
(i_1, \ldots, i_p) \arrow[r, mapsto] & 0 \quad \text{if some } i_j = 0.
\end{tikzcd}$$

The diagrams in Explanation 11.3.10 are commutative for HR because they are commutative on direct sum generators in $(HR)_p$ for $p \geq 0$. ⋄

K-**Theory Strict Ring Symmetric Spectra.** Recall from Definition 10.3.32 the Elmendorf-Mandell K-theory multifunctor

$$\mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp}.$$

The following result is [**EM06**, 3.5] that describes the K-theory of ring categories.

Corollary 11.3.16. *For each small ring category* C, $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ *is a strict ring symmetric spectrum.*

Proof. Consider the multifunctors

$$\begin{tikzcd}
\mathsf{As} \arrow[r, "F"] & \mathsf{PermCat}^{\mathsf{su}} \arrow[r, "\mathsf{K}^{\mathsf{EM}}"] & \mathsf{SymSp} \\
* \arrow[r, mapsto] & (\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}) \arrow[r, mapsto] & \mathsf{K}^{\mathsf{EM}}\mathsf{C}
\end{tikzcd}$$

with F the multifunctor in Theorem 11.2.16 such that $F(*)$ is the underlying permutative category of C. Since As has a unique object $*$, the composite multifunctor $\mathsf{K}^{\mathsf{EM}} \circ F$ is equivalent to an operad morphism from As to the endomorphism operad of $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$. By Proposition 11.1.15, such an operad morphism is equivalent to a monoid structure on $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$. □

Example 11.3.17 (Endomorphism Ring Categories)**.** Suppose C is a small permutative category. Then

- the endomorphism ring category $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$ in Theorem II.9.2.14 and
- the tight endomorphism ring category $\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$ in Theorem II.9.2.20

are both small. Corollary 11.3.16 applies to each of them to yield a K-theory strict ring symmetric spectrum. ⋄

Example 11.3.18 (Additive Distortion Category)**.** By Example II.9.1.18, the additive distortion category $\mathcal{D}^{\mathrm{ad}}$ in Section I.4.5 is a small ring category, which is also tight. Corollary 11.3.16 applies to $\mathcal{D}^{\mathrm{ad}}$ to yield a K-theory strict ring symmetric spectrum. ⋄

Example 11.3.19 (Ring Categories via Strictification)**.** Suppose C is a small tight bimonoidal category (Definition I.2.1.2). By Corollary II.9.1.19, the equivalent

- right rigid bimonoidal category A in Theorem I.5.5.11 and
- left rigid bimonoidal category A_l in Theorem I.5.5.12

are small tight ring categories. Corollary 11.3.16 applies to each of A and A_l to yield a K-theory strict ring symmetric spectrum. ⋄

11.4. The Barratt-Eccles Operad

In this section, we define the categorical Barratt-Eccles operad $E\mathsf{As}$ and describe it in terms of generators and relations (Theorem 11.4.14). As the first application of Theorem 11.4.14, in Proposition 11.4.26 we observe that the Barratt-Eccles operad detects permutative category structures. Moreover, Theorem 11.5.5 shows that the Barratt-Eccles operad detects bipermutative category structures on small permutative categories.

Translation Categories. In (11.2.3), we observed that the forgetful functor

$$\mathsf{Ob} : \mathsf{Cat} \longrightarrow \mathsf{Set}$$

has a left adjoint dis that sends a set X to the discrete category with object set X. The functor Ob also has a right adjoint given by the translation category functor in the next definition. The translation category functor will be used in the definition of the Barratt-Eccles operad.

Definition 11.4.1. The *translation category functor*

$$E : \mathsf{Set} \longrightarrow \mathsf{Cat}$$

is the functor defined as follows.

- For a set X, the *translation category* EX is the category with
  - object set X and
  - each morphism set a one-element set $*$.
- For a map $f : X \longrightarrow Y$ of sets, the functor

  $$Ef : EX \longrightarrow EY$$

  is
  - the map f on objects and
  - the unique map to a one-element set on morphisms.

This finishes the definition of the translation category functor E ◇

Example 11.4.2 (Principle Bundles)**.** The classifying space (= the geometric realization of the nerve in Example 7.2.8) of the translation category EX is contractible for each set X, since the unique functor $EX \longrightarrow \mathbf{1}$ to the terminal category is an equivalence of categories. For a group G, the regular G-action, which is given by the product in G, induces a free G-action

$$EG \times G \longrightarrow EG. \tag{11.4.3}$$

Here the G-action is *free* in the sense that, for any $x, \sigma \in G$,

$$x\sigma = x \quad \text{implies} \quad \sigma = e$$

with $e \in G$ the identity element. For elements $x, y \in G$, we also denote the unique morphism by

$$yx^{-1} : x \longrightarrow y \in EG. \tag{11.4.4}$$

For each element $\sigma \in G$, the right σ-action in (11.4.3) is the functor

$$EG \xrightarrow[\cong]{\sigma} EG$$

that sends

- an object $x \in EG$ to $x\sigma \in EG$ and
- the unique morphism $yx^{-1} : x \longrightarrow y$ to the unique morphism

$$yx^{-1} = (y\sigma)(x\sigma)^{-1} : x\sigma \longrightarrow y\sigma. \tag{11.4.5}$$

Since the classifying space of EG has a free G-action and is contractible, it is the total space of a universal principal G-bundle. The reader is referred to **[MS74]** for more discussion of principal bundles. ◇

Example 11.4.6 (E as a Right Adjoint)**.** There is an adjunction

$$\mathsf{Cat} \underset{E}{\overset{\mathsf{Ob}}{\rightleftarrows}} \mathsf{Set} \tag{11.4.7}$$

with

- left adjoint Ob the underlying object set functor in (11.2.3),
- E the translation category functor in Definition 11.4.1, and
- $\mathsf{Ob} \circ E = 1_{\mathsf{Set}}$.

Since Ob preserves products, if Q is a Cat-enriched operad, then $\mathsf{Ob}(\mathsf{Q})$ is an operad. By construction, or by the fact that it is a right adjoint, E preserves products. Therefore, if P is an operad in Set, then $E\mathsf{P}$ is a Cat-enriched operad such that

$$\mathsf{Ob}(E\mathsf{P}) = \mathsf{P}. \tag{11.4.8}$$

Moreover, the identity functor on Set induces a Cat-enriched operad morphism

$$\mathsf{P} \xrightarrow{\iota_{\mathsf{P}}} E\mathsf{P}. \tag{11.4.9}$$

On the left-hand side, P is regarded as a Cat-enriched operad via the discrete category functor dis in (11.2.3), which preserves products. ⋄

Definition 11.4.10. The *Barratt-Eccles operad* is the Cat-enriched operad $E\mathsf{As}$, with

- As the associative operad in Definition 11.1.1 and
- E the translation category functor in Definition 11.4.1. ⋄

For $i \neq j \in \{1, \ldots, n\}$, the transposition that swaps i and j is written as (i, j).

Explanation 11.4.11.

(1) For the Cat-enriched operad morphism

$$\mathsf{As} \xrightarrow{\iota_{\mathsf{As}}} E\mathsf{As} \tag{11.4.12}$$

in (11.4.9), each functor

$$\iota_{\mathsf{As}} : \mathsf{As}_n \longrightarrow E\mathsf{As}_n$$

is the identity function on objects. The object set operad of $E\mathsf{As}$ is the associative operad As. On morphisms, the operad structure of $E\mathsf{As}$ is uniquely determined by the fact that each morphism set in each translation category $E\mathsf{As}_n$ has only one element.

(2) Denote by

$$\mathrm{id}_2 \xrightarrow[\cong]{\tau} (1,2) \in E\mathsf{As}_2\big(\mathrm{id}_2; (1,2)\big) \tag{11.4.13}$$

the unique nonidentity isomorphism. Adjacent transpositions $(i, i+1)$ for $1 \leq i < n$ generate the symmetric group Σ_n. Moreover, there is an iterated block sum decomposition

$$(i, i+1) = \overbrace{\mathrm{id}_{i-1} \times}^{\varnothing \text{ if } i=1} (1,2) \overbrace{\times \mathrm{id}_{n-i-1}}^{\varnothing \text{ if } i=n-1} \in \Sigma_n.$$

By the definition (11.1.2) of γ in As, this iterated block sum is an operadic composite of $(1,2)$ and identities. So each morphism in $E\mathsf{As}_n$ decomposes into a categorical composite of isomorphisms of the form ϕv with

- each $v \in \Sigma_n$ and
- each ϕ an operadic composite of one τ and identity morphisms. ⋄

Coherence of the Barratt-Eccles Operad. Theorem 11.1.7 characterizes the associative operad As in terms of generators, namely, $\mathrm{id}_0 \in \Sigma_0$ and $\mathrm{id}_2 \in \Sigma_2$, and the unity and associativity relations. The next result, which is [**Fre17**, 6.3.3], is the extension of that characterization to the Barratt-Eccles operad $E\mathsf{As}$. It is a coherence theorem in the sense that it characterizes the Barratt-Eccles operad in terms of a small number of generators and a few relations. The axioms (11.4.17)–(11.4.20) below are formally identical to those of a permutative category in Definition I.1.2.18. To simplify the notation, we write $y(x_1,\ldots,x_n)$ for the operad composition $\gamma\big(y,(x_1,\ldots,x_n)\big)$. An *enriched operad morphism* is an enriched multifunctor as in Definition 5.1.12 between two enriched multicategories with one object.

Theorem 11.4.14. *Suppose* $(\mathsf{P},\gamma,1)$ *is a* Cat*-enriched operad. Then a* Cat*-enriched operad morphism*

$$f : E\mathsf{As} \longrightarrow \mathsf{P}$$

is uniquely determined by

- *the objects*

$$\mathbb{1} = f(\mathrm{id}_0) \in \mathsf{P}_0 \qquad (11.4.15)$$
$$\mu = f(\mathrm{id}_2) \in \mathsf{P}_2$$

  and
- *the isomorphism*

$$\mu = f(\mathrm{id}_2) \xrightarrow[\cong]{\xi = f(\tau)} f(1,2) = \mu(1,2) = \mu^{\mathsf{op}} \in \mathsf{P}_2(\mu,\mu^{\mathsf{op}}) \qquad (11.4.16)$$

  with

$$\tau : \mathrm{id}_2 \longrightarrow (1,2) \in E\mathsf{As}_2$$

  the isomorphism in (11.4.13).

The above data are subject to the following conditions.

Unity and Associativity on Objects: *The equalities of objects*

$$\mu(\mathbb{1},1) = 1 = \mu(1,\mathbb{1}) \in \mathsf{P}_1 \qquad (11.4.17)$$
$$\mu(\mu,1) = \mu(1,\mu) \in \mathsf{P}_3$$

hold.

The Symmetry Axiom: *The diagram*

$$\begin{tikzcd} \mu \arrow[rr, "1_\mu"] \arrow[dr, "\xi"'] & & \mu \\ & \mu^{\mathsf{op}} \arrow[ur, "\xi^{\mathsf{op}}"'] & \end{tikzcd} \qquad (11.4.18)$$

in P_2 *is commutative, with*

$$\xi^{\mathsf{op}} = \xi(1,2) : \mu^{\mathsf{op}} \longrightarrow \mu$$

the image of ξ *under the right* $(1,2)$*-action.*

The Unit Axiom: *The diagram*

(11.4.19)
$$\begin{tikzcd} 1 = \mu(1,\mathbb{1}) \arrow[rr, "1_1"] \arrow[dr, "\xi(1,\mathbb{1})"'] & & \mu(\mathbb{1},1) = 1 \arrow[dl, equal] \\ & \mu^{\mathsf{op}}(1,\mathbb{1}) & \end{tikzcd}$$

in P_1 *is commutative.*

The Hexagon Axiom: *The diagram*

(11.4.20)
$$\begin{tikzcd} \mu(\mu,1) \arrow[r, "\xi"] \arrow[d, equal] & \mu^{\mathsf{op}}(\mu,1) \arrow[d, equal] \\ \mu(1,\mu) \arrow[d, "\mu(1,\xi)"'] & \mu(\mu^{\mathsf{op}},1)(2,3) \\ \mu(1,\mu^{\mathsf{op}}) \arrow[r, equal] & \mu(\mu,1)(2,3) \arrow[u, "\mu(\xi,1)(2,3)"'] \end{tikzcd}$$

in P_3 *is commutative.*

Proof. We need to prove the necessity and the sufficiency of the data (11.4.15)–(11.4.16) and the axioms (11.4.17)–(11.4.20).

Necessity. Suppose $f : E\mathsf{As} \longrightarrow \mathsf{P}$ is a Cat-enriched operad morphism. We define
$$\begin{cases} \mathbb{1} = f(\mathrm{id}_0) \in \mathsf{P}_0, \\ \mu = f(\mathrm{id}_2) \in \mathsf{P}_2, \quad \text{and} \\ \xi = f(\tau) : \mu \longrightarrow \mu^{\mathsf{op}} \in \mathsf{P}_2 \end{cases}$$
as in (11.4.15)–(11.4.16). Since f preserves the operad units, there is an equality
$$f(\mathrm{id}_1) = 1 \in \mathsf{P}_1.$$

- With ι_{As} as in (11.4.12), the composite
$$\mathsf{As} \xrightarrow{\iota_{\mathsf{As}}} E\mathsf{As} \xrightarrow{f} \mathsf{P}$$
is a Cat-enriched operad morphism. So it preserves the operad relations (11.1.8) in As, which become the equalities (11.4.17) of objects in P.
- Before applying f, the analogues of the axioms (11.4.18)–(11.4.20) hold in $E\mathsf{As}$ because each morphism set $E\mathsf{As}_n(\cdot,\cdot)$ is a one-element set by Definition 11.4.1. Since f is a Cat-enriched operad morphism, after applying f, the axioms hold in P.

Therefore, the data (11.4.15)–(11.4.16) satisfy the axioms (11.4.17)–(11.4.20).

Sufficiency. Suppose given the data

- $\mathbb{1} = f(\mathrm{id}_0) \in \mathsf{P}_0$,
- $\mu = f(\mathrm{id}_2) \in \mathsf{P}_2$ as in (11.4.15), and
- $\xi = f(\tau) : \mu \longrightarrow \mu(1,2) = \mu^{\mathsf{op}} \in \mathsf{P}_2(\mu,\mu^{\mathsf{op}})$ as in (11.4.16)

such that the axioms (11.4.17)–(11.4.20) hold in P. We show that these data extend uniquely to a Cat-enriched operad morphism $f : E\mathsf{As} \longrightarrow \mathsf{P}$ in several stages.

Objects. By Theorem 11.1.7 and the unity and associativity relations (11.4.17), the objects
$$\mathbb{1} = f(\mathrm{id}_0) \quad \text{and} \quad \mu = f(\mathrm{id}_2)$$

uniquely determine a Cat-enriched operad morphism

$$\mathsf{As} \xrightarrow{f} \mathsf{P} \tag{11.4.21}$$

defined by

$$\begin{cases} f(\mathrm{id}_1) = 1 \in \mathsf{P}_1, \\ f(\mathrm{id}_2) = \mu = \mu_2 \in \mathsf{P}_2, \\ f(\mathrm{id}_n) = \mu(\mu_{n-1}, 1) = \mu_n \in \mathsf{P}_n \quad \text{for } n \geq 3, \text{ and} \\ \quad f(\sigma) = \mu_n \sigma \in \mathsf{P}_n \quad \text{for } \sigma \in \Sigma_n. \end{cases}$$

By (11.4.8), this defines the object part of the Cat-enriched operad morphism $f : E\mathsf{As} \longrightarrow \mathsf{P}$.

Morphisms. The identity morphisms $1_{\mathrm{id}_0} \in E\mathsf{As}_0$ and $1_{\mathrm{id}_1} \in E\mathsf{As}_1$ must be sent by f to, respectively, the identity morphisms $1_{\mathbb{1}} \in \mathsf{P}_0$ and $1_1 \in \mathsf{P}_1$. To define f on the other morphisms in $E\mathsf{As}$, note that the coherence theorems for (symmetric) monoidal categories can be applied to

- the objects $\{\mathbb{1}, 1, \mu\}$, which are interpreted as, respectively, the monoidal unit, the identity functor, and the monoidal product, and
- the symmetry isomorphism ξ.

The reason is that the assumed axioms (11.4.17), and also (11.4.18)–(11.4.20) in the symmetric case, are formally identical to those of a (symmetric) strict monoidal category in Definitions I.1.2.1 and I.1.2.18. We apply these coherence theorems as follows.

- By (11.4.17) and Mac Lane's Coherence Theorem I.1.3.3, each iterated operadic composite in P_n involving only the objects

  $$\mu_0 = \mathbb{1} \in \mathsf{P}_0, \quad \mu_1 = 1 \in \mathsf{P}_1, \quad \text{and} \quad \mu_2 = \mu \in \mathsf{P}_2$$

  is equal to μ_n. For example, there are equalities of objects

  $$\begin{aligned} \mu_4 &= \mu\big(\mu(\mu, 1), 1\big) = \mu(\mu, \mu) = \mu\big(\mu(1, \mu), 1\big) \\ &= \mu\big(1, \mu(\mu, 1)\big) = \mu\big(1, \mu(1, \mu)\big) \in \mathsf{P}_4. \end{aligned}$$

- By (11.4.17)–(11.4.20) and the Symmetric Coherence Theorem 1.1.41, for each pair of permutations $\sigma, \theta \in \Sigma_n$, there is a *unique* isomorphism

  $$\mu_n \sigma \xrightarrow[\cong]{\theta\sigma^{-1}} \mu_n \theta \in \mathsf{P}_n \tag{11.4.22}$$

  that is a categorical composite of isomorphisms of the form ϕv, with $v \in \Sigma_n$ and ϕ an operadic composite of
  - one instance of $\xi = f(\tau) : \mu \longrightarrow \mu^{\mathsf{op}}$ in (11.4.16) and
  - identity morphisms.

Regarding $\xi = f(\tau)$ as corresponding to the permutation $(1,2) \in \Sigma_2$, the isomorphism in (11.4.22) corresponds to the permutation $\theta\sigma^{-1} \in \Sigma_n$. If

$$(\sigma, \theta) = (\mathrm{id}_2, (1,2)),$$

then the isomorphism in (11.4.22) is ξ.

By Explanation 11.4.11 (2), for each pair of permutations $\sigma, \theta \in \Sigma_n$, the unique isomorphism

$$\theta\sigma^{-1} : \sigma \longrightarrow \theta \in E\mathsf{As}_n$$

decomposes into a categorical composite of isomorphisms of the form ϕv, with $v \in \Sigma_n$ and ϕ an operadic composite of

- one instance of $\tau : \mathrm{id}_2 \longrightarrow (1,2)$ in (11.4.13) and
- identity morphisms.

The desired $f : E\mathsf{As} \longrightarrow \mathsf{P}$ must be levelwise a functor and preserve the operad structure, namely, the operad units, the equivariant structure, and the operad composition. So we must define the morphism

$$f(\sigma) = \mu_n\sigma \xrightarrow{f(\theta\sigma^{-1})} \mu_n\theta = f(\theta) \in \mathsf{P}_n \tag{11.4.23}$$

as the corresponding isomorphism in (11.4.22). The uniqueness of the isomorphism in (11.4.22) implies the following two statements.

- $f(\theta\sigma^{-1})$ is independent of the choice of a decomposition of $\theta\sigma^{-1} \in E\mathsf{As}_n$ in Explanation 11.4.11 (2).
- $f : E\mathsf{As}_n \longrightarrow \mathsf{P}_n$ is a functor for each n.

The uniqueness of each functor $f : E\mathsf{As}_n \longrightarrow \mathsf{P}_n$ is part of the definitions (11.4.21) and (11.4.23).

It remains to check that the functors $f : E\mathsf{As}_n \longrightarrow \mathsf{P}_n$ preserve the operad structure. Since $f : \mathsf{As} \longrightarrow \mathsf{P}$ in (11.4.21) is a Cat-enriched operad morphism, by (11.4.8) we only need to consider morphisms in $E\mathsf{As}_n$.

Equivariant structure. The assertion that f preserves the symmetric group action means that, for each permutation $\pi \in \Sigma_n$, the diagram

$$\begin{tikzcd} E\mathsf{As}_n \arrow[r, "\pi"] \arrow[d, "f"'] & E\mathsf{As}_n \arrow[d, "f"] \\ \mathsf{P}_n \arrow[r, "\pi"] & \mathsf{P}_n \end{tikzcd} \tag{11.4.24}$$

of functors, with each horizontal arrow the right π-action, is commutative. Since f and π are functors, by the decomposition of morphisms in $E\mathsf{As}_n$ in Explanation 11.4.11 (2), it suffices to consider morphisms of the form $\phi v \in E\mathsf{As}_n$ with $v \in \Sigma_n$ and ϕ an operadic composite of one $\tau : \mathrm{id}_2 \longrightarrow (1,2)$ and identity morphisms. On such a morphism ϕv, the commutativity of (11.4.24) follows from

- the definition (11.4.5) of the equivariant structure on $E\mathsf{As}$ and
- the uniqueness of (11.4.22).

Operad composition. The assertion that f preserves the operad composition means the commutativity of the following diagram of functors.

$$\begin{tikzcd} E\mathsf{As}_n \times \prod_{j=1}^n E\mathsf{As}_{k_j} \arrow[r, "\gamma"] \arrow[d, "f\times\prod_j f"'] & E\mathsf{As}_{k_1+\cdots+k_n} \arrow[d, "f"] \\ \mathsf{P}_n \times \prod_{j=1}^n \mathsf{P}_{k_j} \arrow[r, "\gamma"] & \mathsf{P}_{k_1+\cdots+k_n} \end{tikzcd} \tag{11.4.25}$$

To prove the commutativity of (11.4.25), we use the following facts:

- $E\mathsf{As}$ and P both satisfy the equivariance axioms (6.1.6) and (6.1.7).
- f and γ are functors.
- f preserves the equivariant structure.

- Morphisms in EAs decompose as in Explanation 11.4.11 (2).

Using these facts, it suffices to consider (11.4.25) with

- an identity morphism in n of the $n+1$ factors in $E\mathsf{As}_n \times \prod_{j=1}^{n} E\mathsf{As}_{k_j}$ and
- in the remaining factor, an operadic composite of one $\tau : \mathrm{id}_2 \longrightarrow (1,2)$ and identity morphisms.

In this case, the commutativity of (11.4.25) follows from the uniqueness of (11.4.22). Therefore, $f : E\mathsf{As} \longrightarrow \mathsf{P}$ is a Cat-enriched operad morphism. $\square$

Permutative Categories as Algebras. Proposition 11.1.15 provided a conceptual explanation of the associative operad As as the operad for monoids. The next observation is the analogue of that result for the Barratt-Eccles operad EAs in Definition 11.4.10. We consider the category Cat of small categories and functors as a symmetric monoidal closed category with

- the Cartesian product $\times$ as the monoidal product,
- the terminal category $\mathbf{1}$ as the monoidal unit, and
- internal hom given by diagram categories, which have functors as objects and natural transformations as morphisms.

Similar to Convention 6.4.6, we use the same notation Cat for the associated Cat-enriched multicategory in Proposition 6.3.6. The unique object in EAs, as a one-object Cat-enriched multicategory, is denoted by $*$. For an object A in a permutative closed category C, recall the C-enriched endomorphism operad $\mathsf{End}(A)$ in (11.1.14), with nth object the internal hom object $[A^{\otimes n}, A] \in \mathsf{C}$.

Proposition 11.4.26. *For a small category* C*, a permutative category structure on* C *is uniquely determined by a* Cat*-enriched multifunctor*

$$f : E\mathsf{As} \longrightarrow \mathsf{Cat} \quad \textit{such that} \quad f(*) = \mathsf{C}.$$

Proof. A Cat-enriched multifunctor $f : E\mathsf{As} \longrightarrow \mathsf{Cat}$ such that $f(*) = \mathsf{C}$ is equivalent to a Cat-enriched operad morphism

$$E\mathsf{As} \xrightarrow{\;f\;} \mathsf{End}(\mathsf{C}) = \left\{\mathsf{Cat}(\mathsf{C}^{\times n}, \mathsf{C})\right\}_{n \geq 0}$$

to the Cat-enriched endomorphism operad of C. By Theorem 11.4.14, such a Cat-enriched operad morphism f is uniquely determined by

- the object
$$\mathbb{1} = f(\mathrm{id}_0) \in \mathsf{C},$$
- the functor
$$\otimes = f(\mathrm{id}_2) : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C},$$
and
- the natural isomorphism
$$\otimes \xrightarrow{\;\xi^{\otimes} = f(\tau)\;} \otimes(1,2) = \otimes^{\mathrm{op}}$$

such that the axioms (11.4.17)–(11.4.20) are satisfied.

- The unity and associativity axioms (11.4.17) state that $(\mathsf{C}, \otimes, \mathbb{1})$ is a strict monoidal category as in Definition I.1.2.1.
- The other axioms (11.4.18)–(11.4.20) state that $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a permutative category as in Definition I.1.2.18.

This finishes the proof. $\square$

11.5. Detecting Bipermutative Categories

In this section, we prove that the Barratt-Eccles operad EAs in Definition 11.4.10 detects bipermutative category structures on small permutative categories (Theorem 11.5.5). This result is due to Elmendorf-Mandell and is an application of the Coherence Theorem 11.4.14 for EAs. For the reader's convenience, here we recall from Chapter II.9 the definition of a bipermutative category.

Definition 11.5.1. A *bipermutative category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big)$$

consisting of

- a ring category
$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$
as in Definition 11.2.4 and
- a permutative category structure $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$, with $\xi^{\otimes}$ called the *multiplicative symmetry*.

These data are required to satisfy the following two axioms for objects $A, B, C \in \mathsf{C}$.

The Zero Symmetry Axiom: There is an equality of morphisms

$$\xi^{\otimes}_{A,\mathbb{0}} = 1_{\mathbb{0}} : A \otimes \mathbb{0} = \mathbb{0} \longrightarrow \mathbb{0} = \mathbb{0} \otimes A. \tag{11.5.2}$$

The Multiplicative Symmetry Factorization Axiom: The diagram

$$\begin{tikzcd}
(A \otimes C) \oplus (B \otimes C) \arrow[r, "\partial^l_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}"'] & (A \oplus B) \otimes C \arrow[d, "\xi^{\otimes}_{A \oplus B, C}"] \\
(C \otimes A) \oplus (C \otimes B) \arrow[r, "\partial^r_{C,A,B}"] & C \otimes (A \oplus B)
\end{tikzcd} \tag{11.5.3}$$

is commutative.

This finishes the definition of a bipermutative category. A bipermutative category is *small*, respectively, *tight*, if the underlying ring category is so. ◇

Definition 11.5.4. For a permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$, a *bipermutative category structure* on C is the additional data $(\otimes, \mathbb{1}, \xi^{\otimes}, \partial^l, \partial^r)$ such that the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big)$$

is a bipermutative category as in Definition 11.5.1. ◇

Recall from Section 6.6 that $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-enriched multicategory with small permutative categories as objects. The category

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle\,; \mathsf{D}\big) = \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}_1, \ldots, \mathsf{C}_n \rangle\,; \mathsf{D}\big)$$

has

- n-linear functors
$$\mathsf{C}_1 \times \cdots \times \mathsf{C}_n \longrightarrow \mathsf{D}$$
in Definition 6.5.4 as objects and
- multilinear transformations (Definition 6.5.11) as morphisms.

Also recall from Definition 5.1.12 the notion of an enriched multifunctor. The next result is **[EM06**, 3.8], which says that the Barratt-Eccles operad detects bipermutative category structures on small permutative categories. It extends both

- Theorem 11.2.16 from As to the Barratt-Eccles operad and
- Proposition 11.4.26 from Cat to $\mathsf{PermCat}^{\mathsf{su}}$.

Theorem 11.5.5. *For each small permutative category* C*, there is a canonical bijective correspondence between*

- *bipermutative category structures on* C *and*
- Cat*-enriched multifunctors*

$$F : E\mathsf{As} \longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \textit{such that} \quad F(*) = \mathsf{C}.$$

Proof. A Cat-enriched multifunctor

$$F : E\mathsf{As} \longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \text{such that} \quad F(*) = (\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$$

is equivalent to a Cat-enriched operad morphism

$$F : E\mathsf{As} \longrightarrow \mathsf{End}(\mathsf{C}) = \Big\{ \mathsf{PermCat}^{\mathsf{su}}\big(\langle \overbrace{\mathsf{C}, \ldots, \mathsf{C}}^{n} \rangle ; \mathsf{C}\big)\Big\}_{n \geq 0}$$

to the Cat-enriched endomorphism operad of C. By Theorem 11.4.14, such a Cat-enriched operad morphism is uniquely determined by

- the 0-linear functor, that is, object
$$F(\mathrm{id}_0) = \mathbb{1} \in \mathsf{C},$$
- the 2-linear functor
$$F(\mathrm{id}_2) = (\otimes, \partial^l, \partial^r) : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C},$$
and
- the invertible multilinear transformation
$$(\otimes, \partial^l, \partial^r) \xrightarrow{\xi^{\otimes} = F(\tau)} (\otimes^{\mathsf{op}}, \partial^r, \partial^l)$$

such that the conditions (11.4.17)–(11.4.20) are satisfied, with

$$(\mu, 1, \xi) \quad \text{interpreted as} \quad \big((\otimes, \partial^l, \partial^r), 1_{\mathsf{C}}, \xi^{\otimes}\big).$$

By Theorem 11.2.16 and the first paragraph of its proof, the data $(\otimes, \mathbb{1}, \partial^l, \partial^r)$ and the condition (11.4.17) are equivalent to a ring category structure on the small permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ in the sense of Definition 11.2.15. In particular, $(\mathsf{C}, \otimes, \mathbb{1})$ is a strict monoidal category. Next we consider the remaining data $\xi^{\otimes}$ and the axioms (11.4.18)–(11.4.20).

The invertible multilinear transformation

$$\xi^{\otimes} : \otimes \xrightarrow{\cong} \otimes^{\mathsf{op}}$$

is, by definition, a natural isomorphism that satisfies the two conditions in Definition 6.5.11.

- The three remaining axioms (11.4.18)–(11.4.20) state that $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a permutative category as in Definition I.1.2.18.
- The commutativity of the diagram (6.5.12) for $\xi^{\otimes}$ and $i = 1$ is the multiplicative symmetry factorization axiom (11.5.3) in a bipermutative category. In the presence of the symmetry axiom (11.4.18), the case $i = 2$ of the diagram (6.5.12) is equivalent to the case $i = 1$. So it does not impose any additional restriction.

- The second condition in Definition 6.5.11 states the equalities

$$\zeta^{\otimes}_{-,\mathbb{0}} = 1_{\mathbb{0}} = \zeta^{\otimes}_{\mathbb{0},-}. \tag{11.5.6}$$

  The first equality in (11.5.6) is the zero symmetry axiom (11.5.2) in a bipermutative category. In the presence of the symmetry axiom (11.4.18), the second equality in (11.5.6) is equivalent to the first one. So it does not impose any additional restriction.

Therefore, a Cat-enriched operad morphism

$$F : E\mathsf{As} \longrightarrow \mathsf{End}(\mathsf{C})$$

is equivalent to a bipermutative category structure on C. □

11.6. K-Theory of Bipermutative Categories are E_∞-Symmetric Spectra

In this section, we prove that the K-theory of a small bipermutative category is an E_∞-symmetric spectrum (Corollary 11.6.12). This result is due to Elmendorf-Mandell. It is a consequence of Theorem 11.5.5, the Elmendorf-Mandell K-theory multifunctor, and the fact that $E\mathsf{As}$ is an E_∞-operad (Proposition 11.6.3). Along the way, we record the relationship between strict ring symmetric spectra, E_∞-symmetric spectra, and commutative monoids in SymSp (Proposition 11.6.6).

E_∞-Operads. We first define E_∞-operads and discuss some examples. By Definition 7.2.3, the nerve functor

$$N : \mathsf{Cat} \longrightarrow \mathsf{sSet}$$

is a right adjoint. So it preserves products and takes a Cat-enriched operad to an sSet-enriched operad. A simplicial set is *contractible* if its geometric realization (Definition 7.1.14) is a contractible space.

Definition 11.6.1. An sSet-enriched operad P is an E_∞*-operad* if the following two conditions hold for each $n \geq 0$:

Contractibility: P_n is contractible.

Free Action: The Σ_n-action

$$\mathsf{P}_n \times \Sigma_n \longrightarrow \mathsf{P}_n$$

is free.

A Cat-enriched operad Q is an E_∞*-operad* if the sSet-enriched operad $N\mathsf{Q}$ is an E_∞-operad. ⋄

Example 11.6.2. If P and Q are E_∞-operads, then so is the product operad $\mathsf{P} \times \mathsf{Q}$ because the Cartesian product preserves levelwise contractibility and free symmetric group action. ⋄

Proposition 11.6.3. *The Barratt-Eccles operad* $E\mathsf{As}$ *is an* E_∞*-operad.*

Proof. The free symmetric group action on $E\mathsf{As}$ remains free after applying the nerve functor. It is levelwise contractible because each translation category EX, which has a unique element in each morphism set, is equivalent to the terminal category. □

Explanation 11.6.4 (Simplicial Barratt-Eccles Operad)**.** Using Explanation 7.2.4, we may describe the sSet-enriched Barratt-Eccles operad $N(E\mathsf{As})$ as follows. For $n \geq 0$, the simplicial set $N(E\mathsf{As}_n)$ has the set of k-simplices

$$N(E\mathsf{As}_n)_k = \Sigma_n^{\times(k+1)} \quad \text{for} \quad k \geq 0,$$

with Σ_n the symmetric group on n letters. Each $(k+1)$-tuple of permutations

$$\big(\sigma_0, \ldots, \sigma_k\big) \in \Sigma_n^{\times(k+1)}$$

records $k+1$ objects in the translation category $E\mathsf{As}_n$. Since each morphism set in $E\mathsf{As}_n$ is a one-element set, such a $(k+1)$-tuple determines a unique k-simplex in the nerve $N(E\mathsf{As}_n)$. The face and degeneracy maps are given by

$$\begin{aligned} d_i\big(\sigma_0, \ldots, \sigma_k\big) &= \big(\sigma_0, \ldots, \widehat{\sigma_i}, \ldots, \sigma_k\big) \\ s_i\big(\sigma_0, \ldots, \sigma_k\big) &= \big(\sigma_0, \ldots, \sigma_i, \sigma_i, \ldots, \sigma_k\big) \end{aligned}$$

for $0 \le i \le k$. In other words, the ith face map d_i removes σ_i, and the ith degeneracy map s_i repeats σ_i.

The simplicial operad unit is

$$\mathrm{id}_1 \in \Sigma_1 = N(E\mathsf{As}_1)_0.$$

The right Σ_n-action on $N(E\mathsf{As}_n)$ is given on k-simplices by the diagonal action

$$\big(\sigma_0, \ldots, \sigma_k\big) \cdot \pi = \big(\sigma_0 \pi, \ldots, \sigma_k \pi\big) \quad \text{for} \quad \pi \in \Sigma_n.$$

The operad composition

$$N(E\mathsf{As}_n) \times \prod_{i=1}^{n} N(E\mathsf{As}_{k_i}) \xrightarrow{\ \gamma\ } N(E\mathsf{As}_{k_1+\cdots+k_n})$$

is given on p-simplices by

$$\begin{aligned} &\gamma\Big((\sigma_0, \ldots, \sigma_p); \big\{(\sigma_0^i, \ldots, \sigma_p^i)\big\}_{1 \le i \le n}\Big) \\ &= \Big\{\gamma\big(\sigma_j, (\sigma_j^1, \ldots, \sigma_j^n)\big)\Big\}_{0 \le j \le p} \in \prod_{j=0}^{p} \Sigma_{k_1+\cdots+k_n} \end{aligned}$$

for $p \ge 0$, $\sigma_0, \ldots, \sigma_p \in \Sigma_n$, and $\sigma_0^i, \ldots, \sigma_p^i \in \Sigma_{k_i}$. In the second line above, γ is the operad composition (11.1.2) in the associative operad. $\diamond$

E_∞**-Symmetric Spectra.** Next we define E_∞-symmetric spectra and discuss a few basic examples. Recall from Definition 7.6.22 that each symmetric spectrum X has an *endomorphism simplicial operad* $\mathsf{End}(X)$, which is enriched over sSet_*. For $n \ge 0$, it has

$$\mathsf{End}(X)_n = \mathsf{SymSp}\big(X^{\wedge n} \wedge \Delta^?_+, X\big) \in \mathsf{sSet}_*$$

as its pointed simplicial set of n-ary operations.

Definition 11.6.5.

- An E_∞*-structure* on a symmetric spectrum X is an sSet-enriched operad morphism
$$\mathsf{P} \longrightarrow \mathsf{End}(X)$$
for some E_∞-operad P as in Definition 11.6.1.
- An E_∞*-symmetric spectrum* is a symmetric spectrum equipped with an E_∞-structure. $\diamond$

Recall from Definition I.1.2.23 that a *commutative monoid* is a monoid with a strictly commutative multiplication.

Proposition 11.6.6.

(1) *Each commutative monoid in* SymSp *is an* E_∞*-symmetric spectrum.*

(2) *If a symmetric spectrum has an E_∞-structure via the Barratt-Eccles operad, then it is a strict ring symmetric spectrum.*

Proof. There are morphisms of sSet-enriched operads

$$\mathsf{As} \xrightarrow{\iota} N(E\mathsf{As}) \xrightarrow{\pi} \mathsf{Com},$$

with

$$\mathsf{As}_n = \Sigma_n \quad \text{and} \quad \mathsf{Com}_n = *$$

constant simplicial sets for $n \geq 0$. The morphism ι is the nerve of the Cat-enriched operad morphism ι_{As} in (11.4.12). The morphism π consists of the unique morphisms

$$N(E\mathsf{As}_n) \longrightarrow \mathsf{Com}_n = * \quad \text{for} \quad n \geq 0.$$

A commutative monoid structure on a symmetric spectrum X is equivalent to an sSet-enriched operad morphism

$$\mathsf{Com} \longrightarrow \mathsf{End}(X).$$

Precomposing with π yields an E_∞-structure, since $N(E\mathsf{As})$ is an E_∞-operad by Proposition 11.6.3. The second assertion follows by precomposing a given E_∞-structure

$$N(E\mathsf{As}) \longrightarrow \mathsf{End}(X)$$

with ι and using Proposition 11.1.15. □

Example 11.6.7 (The Sphere Spectrum)**.** The symmetric sphere

$$S = \left\{S^p\right\}_{p \geq 0}$$

is a commutative monoid in SymSp, so it is an E_∞-symmetric spectrum by Proposition 11.6.6. In more detail, by Examples 7.4.9 and 11.3.13, the symmetric sphere is a monoid in SymSp with the canonical isomorphism

$$S^p \wedge S^q \xrightarrow[\cong]{\mu^S_{p,q}} S^{p+q} \quad \text{for} \quad p,q \geq 0.$$

The commutativity of the multiplication

$$S \square_S S \xrightarrow{\mu^S} S \in \mathsf{SymSp}$$

follows from (i) the coequalizer in Definition 7.6.1 that defines $\square_S$ and (ii) the commutative diagram

$$\begin{tikzcd}
\Sigma_{(p+q)} \times_{\Sigma_p \times \Sigma_q} S^p \wedge S^q \arrow[rr, "\xi_{p,q}"] \arrow[dr, "\mu^S_{p,q}"'] & & \Sigma_{(q+p)} \times_{\Sigma_q \times \Sigma_p} S^q \wedge S^p \arrow[dl, "\mu^S_{q,p}"] \\
& S^{p+q} &
\end{tikzcd} \tag{11.6.8}$$

in sSet_* for $p,q \geq 0$. ◇

Example 11.6.9 (Suspension Spectra of Commutative Monoids)**.** For a commutative monoid (K, μ^K, η^K) in $(\mathsf{sSet}_*, \wedge, S^0)$, its suspension spectrum

$$\Sigma^\infty K = \left\{S^p \wedge K\right\}_{p \geq 0}$$

is a commutative monoid in SymSp, so it is an E_∞-symmetric spectrum by Proposition 11.6.6. In more detail, by Examples 7.4.10 and 11.3.14, the suspension spectrum $\Sigma^\infty K$ is a monoid in SymSp with the multiplication morphism

$$\begin{tikzcd}
(S^p \wedge K) \wedge (S^q \wedge K) \arrow[rr, "\mu_{p,q}"] \arrow[dr, "\xi_{\mathrm{mid}}"'] & & S^{p+q} \wedge K \\
& (S^p \wedge S^q) \wedge (K \wedge K) \arrow[ur, "\mu^S_{p,q} \wedge \mu^K"'] &
\end{tikzcd}$$

for $p, q \geq 0$. By the coequalizer in Definition 7.6.1 that defines $\square_S$, the commutativity of the multiplication

$$\Sigma^\infty K \square_S \Sigma^\infty K \xrightarrow{\mu} \Sigma^\infty K \in \mathsf{SymSp}$$

follows from the commutative diagram

$$\begin{tikzcd}
\Sigma_{p+q} \times_{p,q} (S^p \wedge K) \wedge (S^q \wedge K) \arrow[rr, "\xi_{p,q}"] \arrow[d, "\xi_{\mathrm{mid}}"'] & & \Sigma_{q+p} \times_{q,p} (S^q \wedge K) \wedge (S^p \wedge K) \arrow[d, "\xi_{\mathrm{mid}}"] \\
\big(\Sigma_{p+q} \times_{p,q} (S^p \wedge S^q)\big) \wedge (K \wedge K) \arrow[rr, "\xi_{p,q} \wedge \xi"] \arrow[dr, "\mu^S_{p,q} \wedge \mu^K"'] & & \big(\Sigma_{q+p} \times_{q,p} (S^q \wedge S^p)\big) \wedge (K \wedge K) \arrow[dl, "\mu^S_{q,p} \wedge \mu^K"] \\
& S^{p+q} \wedge K &
\end{tikzcd}$$

in sSet_* for $p, q \geq 0$, where $\times_{p,q} = \times_{\Sigma_p \times \Sigma_q}$.

- The top square commutes by the Symmetric Coherence Theorem 1.1.41 for sSet_*.
- The bottom triangle commutes by the commutativity of μ^K and (11.6.8).

$\diamond$

Example 11.6.10 (Eilenberg-Mac Lane Spectra)**.** Suppose R is a commutative ring. Then the Eilenberg-Mac Lane spectrum

$$HR = \left\{R \otimes S^p\right\}_{p \geq 0}$$

is a commutative monoid in SymSp, so it is an E_∞-symmetric spectrum by Proposition 11.6.6. In more detail, by Examples 7.4.11 and 11.3.15, the Eilenberg-Mac Lane spectrum HR is a monoid in SymSp with the multiplication morphism

$$\begin{tikzcd}
\Big(\bigoplus\limits_{(\underline{k}^p)^\flat} R\Big) \wedge \Big(\bigoplus\limits_{(\underline{k}^q)^\flat} R\Big) \arrow[r, "(\mu_{p,q})_k"] & \bigoplus\limits_{(\underline{k}^{p+q})^\flat} R \\
\big(r_{i_1,\ldots,i_p}\big) \wedge \big(r'_{j_1,\ldots,j_q}\big) \arrow[r, mapsto] & (rr')_{i_1,\ldots,i_p,j_1,\ldots,j_q}
\end{tikzcd}$$

for $p,q,k \geq 0$, $r,r' \in R$, and $i_a, j_b \in \underline{k}^\flat$. By the coequalizer in Definition 7.6.1 that defines $\square_S$, the commutativity of the multiplication

$$HR \,\square_S\, HR \xrightarrow{\mu} HR \in \mathsf{SymSp}$$

follows from the commutative diagram

$$\begin{tikzcd}
\Sigma_{p+q} \times_{p,q} \Big(\bigoplus\limits_{(\underline{k^p})^\flat} R \Big) \wedge \Big(\bigoplus\limits_{(\underline{k^q})^\flat} R \Big) \arrow[rr, "(\xi_{p,q})_k"] \arrow[dr, "(\mu_{p,q})_k"'] & & \Sigma_{q+p} \times_{q,p} \Big(\bigoplus\limits_{(\underline{k^q})^\flat} R \Big) \wedge \Big(\bigoplus\limits_{(\underline{k^p})^\flat} R \Big) \arrow[dl, "(\mu_{q,p})_k"] \\
& \bigoplus\limits_{(\underline{k^{p+q}})^\flat} R &
\end{tikzcd}$$

in which $\times_{p,q} = \times_{\Sigma_p \times \Sigma_q}$.

- On direct sum indices, commutativity follows from the commutative diagram (11.6.8).
- On direct sum generators in R, commutativity follows from the assumption that R is a commutative ring. $\diamond$

The E_∞-symmetric spectra in Examples 11.6.7, 11.6.9, and 11.6.10 are commutative monoids in SymSp. On the other hand, the K-theory E_∞-symmetric spectra in Corollary 11.6.12 below are, in general, not commutative monoids.

***K*-Theory E_∞-Symmetric Spectra.** The Elmendorf-Mandell K-theory K^{EM} is the following composite of multifunctors.

(11.6.11)
$$\begin{tikzcd}
\mathsf{PermCat}^{\mathsf{su}} \arrow[rrr, "\mathsf{K}^{\mathsf{EM}}"] \arrow[d, "\mathsf{J}^{\mathsf{EM}}"'] & & & \mathsf{SymSp} \\
\mathcal{G}_*\text{-}\mathsf{Cat} \arrow[rrr, "N_*"] & & & \mathcal{G}_*\text{-}\mathsf{sSet} \arrow[u, "\mathsf{K}^{\mathcal{G}}"']
\end{tikzcd}$$

- J^{EM} is the Elmendorf-Mandell J-theory in Definition 10.3.25, which is a Cat-enriched multifunctor.
- $N : \mathsf{Cat} \longrightarrow \mathsf{sSet}$ is the nerve functor in Definition 7.2.3 from small categories to simplicial sets.
- N_* is the induced change-of-codomain symmetric monoidal sSet_*-functor in Theorem 9.2.19 (2).
- $\mathsf{K}^{\mathcal{G}}$ is the symmetric monoidal sSet_*-functor in Theorem 9.4.9.

The following result is the bipermutative analogue of Corollary 11.3.16 and is essentially [**EM06**, 3.9]. It describes the K-theory of bipermutative categories.

Corollary 11.6.12. *For each small bipermutative category* C, $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ *is an* E_∞*-symmetric spectrum.*

Proof. Consider the multifunctors

$$\begin{array}{ccccc}
E\mathsf{As} & \xrightarrow{\ F\ } & \mathsf{PermCat}^{\mathsf{su}} & \xrightarrow{\ \mathsf{K}^{\mathsf{EM}}\ } & \mathsf{SymSp} \\
* & \longmapsto & (\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}) & \longmapsto & \mathsf{K}^{\mathsf{EM}}\mathsf{C}
\end{array}$$

with

- K^{EM} the Elmendorf-Mandell K-theory multifunctor in (11.6.11) and

- F the Cat-enriched multifunctor in Theorem 11.5.5 such that $F(*)$ is the additive structure of C.

By a change of enrichment (Theorem 2.3.7) via the nerve functor $N : \mathsf{Cat} \longrightarrow \mathsf{sSet}$, $\mathsf{J}^{\mathsf{EM}}_N \circ F_N$ is an sSet-enriched multifunctor. Since $N(E\mathsf{As})$ is an E_∞-operad by Proposition 11.6.3, the composite sSet-enriched operad morphism

$$\begin{tikzcd}
N(E\mathsf{As}) \arrow[d, "F_N"'] & & \mathsf{End}(\mathsf{K}^{\mathsf{EM}}\mathsf{C}) \\
N\mathsf{End}(\mathsf{C}) \arrow[r, "\mathsf{J}^{\mathsf{EM}}_N"] & N\mathsf{End}(\mathsf{J}^{\mathsf{EM}}\mathsf{C}) \arrow[r, "N_*"] & \mathsf{End}(N_*\mathsf{J}^{\mathsf{EM}}\mathsf{C}) \arrow[u, "\mathsf{K}^g"']
\end{tikzcd}$$

gives $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ the structure of an E_∞-symmetric spectrum. □

The braided and E_n analogues of Corollary 11.6.12 are, respectively, Corollaries 12.5.3 and 13.5.2.

Example 11.6.13. By Theorem II.9.3.7, each of the following is a small bipermutative category:

- the finite ordinal category Σ in Proposition I.2.4.8,
- its variant Σ' in Proposition I.2.4.23,
- $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$ of coordinatized finite dimensional complex vector spaces in Example I.2.5.9,
- each small right bipermutative category in Definition I.2.5.2, and
- each small left bipermutative category in Definition I.2.5.11, such as the distortion category $\mathcal{D}$ in Section I.4.2.

Corollary 11.6.12 applies to each of these small bipermutative categories to yield a K-theory E_∞-symmetric spectrum. See also Questions A.5.4 and A.5.6. ◇

Example 11.6.14 (Bipermutative Strictification)**.** Suppose given a small tight symmetric bimonoidal category C (Definition I.2.1.2), such as

- a small distributive symmetric monoidal category in Proposition I.2.3.2,
- the symmetric bimonoidal groupoid Π in Theorem I.2.6.2, and
- the bimonoidal symmetric center of each small tight braided bimonoidal category in Theorem II.4.5.3.

By Corollary II.9.3.12, the equivalent

- right bipermutative category A in Sections I.5.2 through I.5.4 and
- left bipermutative category A_l in Theorem I.5.4.7

are small tight bipermutative categories. Corollary 11.6.12 applies to each of A and A_l to yield a K-theory E_∞-symmetric spectrum. ◇

Example 11.6.15 (Symmetric Center)**.** Suppose C is a small braided ring category in which the left factorization morphism ∂^l is a natural epimorphism. By Theorem II.9.6.4, its symmetric center $\mathsf{C}^{\mathsf{sym}}$ is a small bipermutative category. Corollary 11.6.12 applies to $\mathsf{C}^{\mathsf{sym}}$ to yield a K-theory E_∞-symmetric spectrum. ◇

11.7. Notes

11.7.1 (The Associative Operad)**.** In slightly different form, the material in Section 11.1 for the associative operad As can also be found in [**Yau16**, 14.2]. Further discussion and examples related to As, such as the operads for morphisms and general diagrams of monoids, can be found in [**Yau16**, Ch. 14] and [**Yau20**, Ch. 4.5

and 7]. Moreover, the Cat-enriched associative operad is called the *strict monoidal category operad* and denoted $\mathsf{MCat}_{\mathsf{st}}$ in [**Yau**∞, 18.5]. A more general analogue is the *monoidal category operad*, whose algebras are general, instead of strict, small monoidal categories [**Yau**∞, 18.2–18.3]. $\diamond$

11.7.2 (The Barratt-Eccles Operad)**.** The translation category EG of a group G is also known as the *action groupoid* of G acting on itself. The simplicial Barratt-Eccles operad $N(E\mathsf{As})$ in Explanation 11.6.4 is the simplicial E_∞-operad defined in [**BE74a**]. Due to its importance in infinite loop space theory, the Barratt-Eccles operad is discussed in many other papers, including [**BE74b, BE74c, Ber96, BF04, BF02, Smi89**]. In general symmetric monoidal categories, discussion of E_∞-operads in terms of the Boardman-Vogt W-construction and their algebras can be found in [**Yau20**, Ch. 7 and 11]. See also Question A.4.3.

Moreover, the Barratt-Eccles operad $E\mathsf{As}$ is called the *strict* S-*monoidal category operad* and denoted $\mathsf{MCat}^{\mathsf{S}}_{\mathsf{st}}$ in [**Yau**∞, 21.4]. A more general analogue is the S-*monoidal category operad*, whose algebras are general small symmetric monoidal categories, instead of small permutative categories [**Yau**∞, 21.4.7]. $\diamond$

11.7.3 (Action Operads)**.** The monoidal category operad and the S-monoidal category operad in Notes 11.7.1 and 11.7.2 are examples of the G-*monoidal category operad* $\mathsf{MCat}^{\mathsf{G}}$ for some *action operad* G [**Yau**∞, 19.1]. An $\mathsf{MCat}^{\mathsf{G}}$-algebra is a general small monoidal category with a compatible action by the action operad G on iterated monoidal products [**Yau**∞, 19.2]. Moreover, $\mathsf{MCat}_{\mathsf{st}}$ and $\mathsf{MCat}^{\mathsf{S}}_{\mathsf{st}}$ are examples of the *strict* G-*monoidal category operad* $\mathsf{MCat}^{\mathsf{G}}_{\mathsf{st}}$ in [**Yau**∞, 19.3]. $\diamond$

11.7.4 (Ring and Bipermutative Structures)**.** Theorems 11.2.16 and 11.5.5 are from [**EM06**, 3.4 and 3.8], but the proofs there are organized differently. More specifically, in [**EM06**], the proof of each of these two theorems has essentially three parts:

- Starting from a ring or bipermutative category structure, produce a multifunctor from As or $E\mathsf{As}$ to $\mathsf{PermCat}^{\mathsf{su}}$.
- Starting from a multifunctor from As or $E\mathsf{As}$ to $\mathsf{PermCat}^{\mathsf{su}}$, produce a ring or bipermutative category structure.
- Observe that the previous two constructions are inverses of each other.

On the other hand, our presentation in this chapter emphasizes that, in each case, the correspondence is a consequence of the coherence of the parameter operad that is *not* specific to $\mathsf{PermCat}^{\mathsf{su}}$.

- Both Proposition 11.1.15 and Theorem 11.2.16 use the Coherence Theorem 11.1.7 for the associative operad As.
- Both Proposition 11.4.26 and Theorem 11.5.5 use the Coherence Theorem 11.4.14 for the Barratt-Eccles operad $E\mathsf{As}$.

In particular, our proof of each of Theorems 11.2.16 and 11.5.5 produces the bijective correspondence in one step. The braided and general E_n cases (Theorems 12.4.5 and 13.4.12) will follow a conceptually similar pattern. $\diamond$

11.7.5 (Ring Spectra and Homotopy)**.** For discussion of the relationship between low-dimensional homotopy of ring symmetric spectra and their categorical algebra, see [**JP07, BJP08a, BJP08b, BM11, MT07**]. These papers develop categorical models and strict algebraic invariants for ring spectra whose nonvanishing homotopy groups are concentrated in dimensions 0 and 1. $\diamond$

CHAPTER 12

K-Theory of Braided Ring Categories

In Corollaries 11.3.16 and 11.6.12, we observed that the Elmendorf-Mandell K-theory multifunctor (Definition 10.3.32)

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{K}^{\mathsf{EM}}} \mathsf{SymSp}$$

sends

- small ring categories to strict ring symmetric spectra and
- small bipermutative categories to E_∞-symmetric spectra.

This chapter extends the picture to the E_2 case by showing, in Corollary 12.5.3, that $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_2-symmetric spectrum for each small braided ring category C (Definition II.9.5.1). In Chapter 13, using E_n-monoidal categories and the n-fold monoidal category operad, the results in Chapter 11 and this chapter will be extended to the general E_n cases for $n \geq 1$.

Similar to the strict ring and E_∞ cases in Chapter 11, Corollary 12.5.3 is obtained by combining several key facts. First, the Elmendorf-Mandell K-theory multifunctor respects

- the categorical enrichment in the multicategory $\mathsf{PermCat}^{\mathsf{su}}$ of small permutative categories and
- the simplicial enrichment in the multicategory SymSp of symmetric spectra.

Therefore, a structure in $\mathsf{PermCat}^{\mathsf{su}}$ that is parametrized by a categorical operad passes along K^{EM} to symmetric spectra. For the E_2 case, the braid operad Br in Section 12.1 is an E_2-operad (Theorem 12.2.4) that parametrizes

- braided strict monoidal categories in Cat (Proposition 12.3.22) and
- braided ring categories in $\mathsf{PermCat}^{\mathsf{su}}$ (Theorem 12.4.5).

Both of these statements are consequences of the Coherence Theorem 12.3.10 for Br. Combining Theorem 12.4.5 with K^{EM} yields Corollary 12.5.3 about K-theory E_2-symmetric spectra. The following table summaries the main results in Chapter 11 and this chapter for the associative operad As, the braid operad Br, and the Barratt-Eccles operad $E\mathsf{As}$. The shorthand *smc* stands for *strict monoidal categories*.

operad	associative As (11.1.1)	braid Br (12.1.2)	Barratt-Eccles $E\mathsf{As}$ (11.4.10)
$E_?$-operad	E_1 (13.1.23, 13.2.1)	E_2 (12.2.4)	E_∞ (11.6.3)
coherence	11.1.7	12.3.10	11.4.14
in Cat	smc (11.1.15)	braided smc (12.3.22)	permutative (11.4.26)
in $\mathsf{PermCat}^{\mathsf{su}}$	ring (11.2.16)	braided ring (12.4.5)	bipermutative (11.5.5)
in SymSp	strict ring (11.3.16)	E_2 (12.5.3)	E_∞ (11.6.12)

Organization. Section 12.1 defines the braid operad Br. It generalizes the Barratt-Eccles operad EAs by using braids with specified underlying permutations as morphisms. In Definition 11.4.10, the associative operad As yields the Cat-enriched Barratt-Eccles operad EAs by applying the translation category functor E, which is a right adjoint (Example 11.4.6) and preserves products. On the other hand, the fact that Br is a Cat-enriched operad (Proposition 12.1.10) requires a longer proof. See also Note 12.6.4 for generalizations of the braid operad. Proposition 12.1.11 shows that the braid operad sits between the associative operad and the Barratt-Eccles operad.

Section 12.2 shows that the braid operad Br is an E_2-operad (Theorem 12.2.4). To define E_2-operads and E_n-operads in general, we first recall the little n-cube operad $\mathcal{C}_n$ of Boardman-Vogt and May. An E_n-operad is, by definition, a topological operad that is weakly equivalent to the little n-cube operad. This definition is extended to simplicial operads and categorical operads via, respectively, the geometric realization functor and the classifying space functor. The fact that Br is an E_2-operad is essentially due to Fiedorowicz **[Fie∞]** and is proved in **[Fre17**, 5.2.12**]**. See also Note 12.6.2.

Section 12.3 proves the Coherence Theorem 12.3.10 for the braid operad Br that describes Cat-enriched operad morphisms from Br. Similar to the case of the Barratt-Eccles operad, this coherence result depends on a decomposition of the morphisms in Br (Lemma 12.3.6). This decomposition is a consequence of the fact that braids are generated under braid products and sum braids by the generating braid $s_1 \in B_2$ and its inverse s_1^{-1}. As the first application of the Coherence Theorem 12.3.10, we observe in Proposition 12.3.22 that Br is the categorical operad for small braided strict monoidal categories.

Section 12.4 shows that the braid operad Br detects braided ring category structures on small permutative categories (Theorem 12.4.5). This is another application of the Coherence Theorem 12.3.10 for Br. Section 12.5 proves the main result about E_2-symmetric spectra using the braid operad Br. This result, Corollary 12.5.3, says that $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_2-symmetric spectrum for each small braided ring category C.

Reading Guide.

(1) Read Definition 12.1.2 and the statements of Propositions 12.1.10 and 12.1.11 for the braid operad Br.
(2) For the fact that Br is an E_2-operad, read Definition 12.2.3 and the statement of Theorem 12.2.4.
(3) For the coherence and K-theoretic properties of Br, read Definition 12.5.1 and the statements of Lemma 12.3.6, Theorems 12.3.10 and 12.4.5, Proposition 12.3.22, and Corollary 12.5.3.
(4) Go back and read the rest of this chapter.

12.1. The Braid Operad

In this section, we define the braid operad Br, which is the braided analogue of the Barratt-Eccles operad EAs in Definition 11.4.10. In Proposition 12.1.10, we prove in detail that Br is a Cat-enriched operad. In Proposition 12.1.11, we observe that the Cat-enriched operad morphism As $\longrightarrow$ EAs, from the associative operad to the Barratt-Eccles operad, factors through the braid operad. In subsequent sections, the braid operad will provide braided and E_2 analogues of Propositions 11.4.26 and 11.6.3, Theorem 11.5.5, and Corollary 11.6.12.

Recall the nth braid group B_n in Definition II.1.1.1. It is generated by $s_1, \ldots, s_{n-1}$, and is subject to the braid relations:

$$\begin{aligned} s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2 \text{ and } 1 \leq i, j \leq n-1. \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 1 \leq i \leq n-2. \end{aligned}$$

With Σ_n denoting the nth symmetric group, the canonical group homomorphism

$$\begin{aligned} B_n &\xrightarrow{\pi} \Sigma_n \\ s_i &\longmapsto (i, i+1) \end{aligned}$$

in (II.1.1.12) sends each braid b to its *underlying permutation* $\pi(b) = \overline{b}$. Also recall the notion of a Cat-enriched operad in Definition 6.1.1, with $(\mathsf{Cat}, \times, \mathbf{1})$ the symmetric monoidal category of small categories with the Cartesian product.

Motivation 12.1.1. In the category $E\mathsf{As}_n$, the objects are the permutations in Σ_n, and the same is true for Br_n. For two permutations $\sigma, \theta \in \Sigma_n$, there is a unique morphism

$$\theta\sigma^{-1} : \sigma \longrightarrow \theta \in E\mathsf{As}_n$$

as in (11.4.4). One way to understand this definition is that $\theta\sigma^{-1}$ is the unique permutation that sends σ to θ, in the sense that

$$(\theta\sigma^{-1})\sigma = \theta.$$

On the other hand, morphisms in the braid operad Br are braids. While each braid b has an underlying permutation $\overline{b}$, the latter does not determine the given braid. This fact accounts for the definition of morphisms in (12.1.3) below. ◇

Definition 12.1.2. The *braid operad* Br is defined as follows. Proposition 12.1.10 proves that it is a Cat-enriched operad.

Categories of Operations: For $n \geq 0$, Br_n is the groupoid defined as follows.

- Its object set is Σ_n.
- For permutations $\sigma, \theta \in \Sigma_n$, the morphism set is

$$\mathsf{Br}_n(\sigma; \theta) = \left\{ b \in B_n \;\middle|\; \overline{b} = \theta\sigma^{-1} \in \Sigma_n \right\} \tag{12.1.3}$$

  with $\overline{b}$ the underlying permutation of b.
- For each $\sigma \in \Sigma_n$, the identity morphism

$$1_\sigma \in \mathsf{Br}_n(\sigma; \sigma)$$

  is the identity braid $\mathrm{id}_n \in B_n$.
- The categorical composition in Br_n is given by the multiplication in the braid group B_n, that is,

$$b' \circ b = b'b.$$

Unit: The unit is the identity braid

$$\mathrm{id}_1 \in \mathsf{Br}_1 \cong \mathbf{1}.$$

Symmetric Group Action: For each permutation $v \in \Sigma_n$, the right v-action is the functor

$$\mathsf{Br}_n \xrightarrow[\cong]{v} \mathsf{Br}_n \tag{12.1.4}$$

defined by the assignments

$$\begin{cases} (\sigma \in \Sigma_n) \longmapsto (\sigma v \in \Sigma_n) \\ \big(b \in \mathsf{Br}_n(\sigma;\theta)\big) \longmapsto \big(b \in \mathsf{Br}_n(\sigma v;\theta v)\big). \end{cases}$$

Operad Composition: For $n \geq 1$, $k_1, \ldots, k_n \geq 0$, and $k = k_1 + \cdots + k_n$, the functor

$$\mathsf{Br}_n \times \prod_{j=1}^{n} \mathsf{Br}_{k_j} \xrightarrow{\gamma} \mathsf{Br}_k \tag{12.1.5}$$

is defined on objects by

$$\gamma\big(\sigma, (\sigma_j)_{1\leq j\leq n}\big) = \overbrace{\sigma\langle k_1, \ldots, k_n\rangle}^{\text{block permutation}} \cdot \overbrace{(\sigma_1 \times \cdots \times \sigma_n)}^{\text{block sum}} \in \Sigma_k \tag{12.1.6}$$

for $\sigma \in \Sigma_n$ and $\sigma_j \in \Sigma_{k_j}$. This is the composition in the associative operad As in (11.1.2). For morphisms
- $b \in \mathsf{Br}_n(\sigma;\sigma')$ and
- $b_j \in \mathsf{Br}_{k_j}(\sigma_j;\sigma_j')$ for $1 \leq j \leq n$,

γ is defined by

$$\begin{aligned} \gamma\big(b, (b_j)_{1\leq j\leq n}\big) &= \overbrace{b\langle k_{\sigma^{-1}(1)}, \ldots, k_{\sigma^{-1}(n)}\rangle}^{\text{block braid}} \cdot \overbrace{\Big(\bigoplus_{j=1}^{n} b_{\sigma^{-1}(j)}\Big)}^{\text{sum braid}} \\ &\in \mathsf{Br}_k\Big(\gamma\big(\sigma, (\sigma_j)_{1\leq j\leq n}\big); \gamma\big(\sigma', (\sigma_j')_{1\leq j\leq n}\big)\Big) \end{aligned} \tag{12.1.7}$$

with the sum braid and block braid in Definitions II.1.1.9 and II.1.1.20.

This finishes the definition of the braid operad Br. ◇

Explanation 12.1.8. In (12.1.7), $\gamma\big(b, (b_j)_{1\leq j\leq n}\big)$ is a sum braid followed by a block braid, as discussed in Explanations II.1.1.13 and II.1.1.25.

- The sum braid
$$\bigoplus_{j=1}^{n} b_{\sigma^{-1}(j)} \in B_k$$
is geometrically obtained by placing the given braids $b_{\sigma^{-1}(1)}, \ldots, b_{\sigma^{-1}(n)}$ from left to right in the given order.
- The block braid
$$b\langle k_{\sigma^{-1}(1)}, \ldots, k_{\sigma^{-1}(n)}\rangle \in B_k$$
is obtained from b by replacing its jth string by $k_{\sigma^{-1}(j)}$ parallel strings for $1 \leq j \leq n$.

As we will see shortly in the proof of Proposition 12.1.10, the permutation σ appears in (12.1.7) to account for the fact that b is a morphism $\sigma \longrightarrow \sigma'$ in Br_n. ◇

Example 12.1.9. Consider the groupoids Br_n.

- Br_0 and Br_1 are the terminal category $\mathbf{1}$ because Σ_0, Σ_1, B_0, and B_1 are the trivial groups.
- Br_2 has two objects, id_2 and $(1,2) \in \Sigma_2$. The braid group B_2 is the infinite cyclic group generated by s_1, with underlying permutation $(1,2)$. The morphism sets in Br_2 are as follows.

$$\mathsf{Br}_2\big(\mathrm{id}_2;\mathrm{id}_2\big) = \big\{s_1^{2k}\big\}_{k\in\mathbb{Z}} \qquad \mathsf{Br}_2\big((1,2);(1,2)\big) = \big\{s_1^{2k}\big\}_{k\in\mathbb{Z}}$$
$$\mathsf{Br}_2\big(\mathrm{id}_2;(1,2)\big) = \big\{s_1^{2k+1}\big\}_{k\in\mathbb{Z}} \qquad \mathsf{Br}_2\big((1,2);\mathrm{id}_2\big) = \big\{s_1^{2k+1}\big\}_{k\in\mathbb{Z}}$$

For $n \geq 2$, Br_n has the $n!$ permutations in Σ_n as objects. For any two permutations $\sigma, \theta \in \Sigma_n$, there are infinitely many braids $b \in B_n$ with underlying permutation $\theta\sigma^{-1}$. $\diamond$

Proposition 12.1.10. Br *in Definition 12.1.2 is a* Cat*-enriched operad.*

Proof. We need to check that

- the various parts of Br are well defined, and
- the Cat-enriched operad axioms in Definition 6.1.1 are satisfied.

Br_n *is a category*. For morphisms

$$b_0 : \sigma_0 \longrightarrow \sigma_1 \quad \text{and} \quad b_1 : \sigma_1 \longrightarrow \sigma_2 \in \mathsf{Br}_n,$$

the morphism

$$b_1 b_0 : \sigma_0 \longrightarrow \sigma_2$$

is well defined because its underlying permutation is

$$\overline{(b_1 b_0)} = \overline{b}_1 \overline{b}_0 = (\sigma_2\sigma_1^{-1})(\sigma_1\sigma_0^{-1}) = \sigma_2\sigma_0^{-1} \in \Sigma_n.$$

The unity and associativity of this categorical composition follow from the fact that B_n is a group.

γ *is well defined*. With the notation

$$\underline{k} = (k_1, \ldots, k_n)$$
$$\sigma\underline{k} = (k_{\sigma^{-1}(1)}, \ldots, k_{\sigma^{-1}(n)})$$

for lists of integers, the following equalities in Σ_k prove that the braid in (12.1.7) is a well-defined morphism

$$\gamma\big(\sigma, (\sigma_j)_{1\leq j\leq n}\big) = \sigma\langle \underline{k}\rangle \cdot \prod_{j=1}^{n} \sigma_j \longrightarrow \sigma'\langle \underline{k}\rangle \cdot \prod_{j=1}^{n} \sigma'_j = \gamma\big(\sigma', (\sigma'_j)_{1\leq j\leq n}\big)$$

in Br_k as in (12.1.3).

$$\begin{aligned}
&\Big(\sigma'\langle\underline{k}\rangle \cdot \prod_{j=1}^{n} \sigma'_j\Big) \cdot \Big(\sigma\langle\underline{k}\rangle \cdot \prod_{j=1}^{n} \sigma_j\Big)^{-1} \\
&= \sigma'\langle\underline{k}\rangle \cdot \Big(\prod_{j=1}^{n} \sigma'_j\Big) \cdot \Big(\prod_{j=1}^{n} \sigma_j\Big)^{-1} \cdot \big(\sigma\langle\underline{k}\rangle\big)^{-1} \\
&= \sigma'\langle\underline{k}\rangle \cdot \Big(\prod_{j=1}^{n} \sigma'_j\Big) \cdot \Big(\prod_{j=1}^{n} \sigma_j^{-1}\Big) \cdot \sigma^{-1}\langle\sigma\underline{k}\rangle \\
&= \sigma'\langle\underline{k}\rangle \cdot \Big(\prod_{j=1}^{n} \sigma'_j\Big) \cdot \sigma^{-1}\langle\sigma\underline{k}\rangle \cdot \Big(\prod_{j=1}^{n} \sigma^{-1}_{\sigma^{-1}(j)}\Big) \\
&= \sigma'\langle\underline{k}\rangle \cdot \sigma^{-1}\langle\sigma\underline{k}\rangle \cdot \Big(\prod_{j=1}^{n} \sigma'_{\sigma^{-1}(j)}\Big) \cdot \Big(\prod_{j=1}^{n} \sigma^{-1}_{\sigma^{-1}(j)}\Big) \\
&= (\sigma'\sigma^{-1})\langle\sigma\underline{k}\rangle \cdot \Big(\prod_{j=1}^{n} \sigma'_{\sigma^{-1}(j)}\sigma^{-1}_{\sigma^{-1}(j)}\Big) \\
&= \overline{b}\langle\sigma\underline{k}\rangle \cdot \prod_{j=1}^{n} \overline{b_{\sigma^{-1}(j)}} = \overline{\gamma\big(b, (b_j)_{1\le j\le n}\big)}
\end{aligned}$$

γ is a functor. The fact that γ in (12.1.5) preserves identity morphisms follows from the equalities

$$\mathrm{id}_n\langle\sigma\underline{k}\rangle = \mathrm{id}_k = \bigoplus_{j=1}^{n} \mathrm{id}_{k_{\sigma^{-1}(j)}} \in B_k.$$

For morphisms

$$\begin{gathered}
\sigma \xrightarrow{\;b\;} \theta \xrightarrow{\;b'\;} \varphi \in \mathsf{Br}_n \\
\sigma_j \xrightarrow{\;b_j\;} \theta_j \xrightarrow{\;b'_j\;} \varphi_j \in \mathsf{Br}_{k_j}
\end{gathered}$$

for $1 \le j \le n$, the fact that γ preserves categorical composition is proved by the following equalities in B_k.

$$\begin{aligned}
&\gamma\big(b'b, (b'_jb_j)_{1\le j\le n}\big) \\
&= (b'b)\langle\sigma\underline{k}\rangle \cdot \Big(\bigoplus_{j=1}^{n} b'_{\sigma^{-1}(j)} b_{\sigma^{-1}(j)}\Big) \\
&= b'\langle\theta\underline{k}\rangle \cdot b\langle\sigma\underline{k}\rangle \cdot \Big(\bigoplus_{j=1}^{n} b'_{\sigma^{-1}(j)}\Big) \cdot \Big(\bigoplus_{j=1}^{n} b_{\sigma^{-1}(j)}\Big) \\
&= b'\langle\theta\underline{k}\rangle \cdot \Big(\bigoplus_{j=1}^{n} b'_{\theta^{-1}(j)}\Big) \cdot b\langle\sigma\underline{k}\rangle \cdot \Big(\bigoplus_{j=1}^{n} b_{\sigma^{-1}(j)}\Big) \\
&= \gamma\big(b', (b'_j)_{1\le j\le n}\big) \cdot \gamma\big(b, (b_j)_{1\le j\le n}\big).
\end{aligned}$$

The middle two equalities above use the equality $\overline{b} = \theta\sigma^{-1} \in \Sigma_n$. The first and last equalities are the definition (12.1.7).

For the Cat-enriched operad axioms in Definition 6.1.1 for Br, first observe that, on objects, Br is the associative operad As in Definition 11.1.1. So we only need to check the axioms on morphisms in Br, which are braids with specified underlying permutations.

Symmetric Group Action. To see that the symmetric group action (12.1.4) is an isomorphism of categories, first observe that its assignment on morphisms is well defined because

$$(\theta v)(\sigma v)^{-1} = (\theta v)(v^{-1}\sigma^{-1}) = \theta\sigma^{-1} = \overline{b}.$$

The functoriality of v follows from the definition

$$v(b) = b : \sigma v \longrightarrow \theta v$$

for each morphism $b : \sigma \longrightarrow \theta$ in Br_n. This definition also implies that the symmetric group action on Br_n is compatible with the multiplication and identity in Σ_n.

Unity. The right unity axiom (6.1.4) holds in Br by the equalities

$$\begin{gathered}\overbrace{\mathrm{id}_1 \oplus \cdots \oplus \mathrm{id}_1}^{n} = \mathrm{id}_n\\ b\langle 1, \ldots, 1\rangle = b\end{gathered}$$

in B_n. The left unity axiom (6.1.5) holds in Br because

$$\mathrm{id}_1\langle m\rangle = \mathrm{id}_m \in B_m.$$

Associativity. Suppose given morphisms

- $b \in \mathsf{Br}_n(\sigma;\sigma')$
- $b_j \in \mathsf{Br}_{k_j}(\sigma_j;\sigma'_j)$ for $1 \leq j \leq n$, and
- $b_{j,i} \in \mathsf{Br}_{l_{j,i}}(\sigma_{j,i};\sigma'_{j,i})$ for $1 \leq j \leq n$ and $1 \leq i \leq k_j$,

along with the following notation.

$$\begin{aligned}
\underline{k} &= (k_1, \ldots, k_n) & k &= \sum_{j=1}^{n} k_j\\
\underline{l}_j &= \bigl(l_{j,1}, \ldots, l_{j,k_j}\bigr) & L_j &= \sum_{i=1}^{k_j} l_{j,i}\\
\underline{l} &= \bigl(\underline{l}_1, \ldots, \underline{l}_n\bigr) & \underline{L} &= (L_1, \ldots, L_n)
\end{aligned}$$

$$\theta = \gamma\bigl(\sigma, (\sigma_j)_{1\leq j\leq n}\bigr) = \sigma\langle\underline{k}\rangle \cdot \prod_{j=1}^{n} \sigma_j \in \Sigma_k$$

By definition (12.1.7), we have the following morphisms in Br.

$$\begin{aligned}
\gamma\bigl(b, (b_j)_{1\leq j\leq n}\bigr) &= b\langle\sigma\underline{k}\rangle \cdot \bigoplus_{j=1}^{n} b_{\sigma^{-1}(j)}\\
&\in \mathsf{Br}_k\Bigl(\gamma\bigl(\sigma, (\sigma_j)_{1\leq j\leq n}\bigr); \gamma\bigl(\sigma', (\sigma'_j)_{1\leq j\leq n}\bigr)\Bigr)\\
\gamma\bigl(b_j, (b_{j,i})_{1\leq i\leq k_j}\bigr) &= b_j\langle\sigma_j\underline{l}_j\rangle \cdot \bigoplus_{i=1}^{k_j} b_{j,\sigma_j^{-1}(i)}\\
&\in \mathsf{Br}_{L_j}\Bigl(\gamma\bigl(\sigma_j, (\sigma_{j,i})_{1\leq i\leq k_j}\bigr); \gamma\bigl(\sigma'_j, (\sigma'_{j,i})_{1\leq i\leq k_j}\bigr)\Bigr)
\end{aligned}$$

The following equalities in $B_{L_1+\cdots+L_n}$ prove the associativity axiom (5.1.4) for Br.

$$\begin{aligned}
&\gamma\Big(\gamma\big(b,(b_j)_{1\le j\le n}\big),\big((b_{j,i})_{1\le i\le k_j}\big)_{1\le j\le n}\Big)\\
&= \left[\Big(b\langle\sigma\underline{k}\rangle\cdot\bigoplus_{j=1}^{n} b_{\sigma^{-1}(j)}\Big)\langle\theta\underline{l}\rangle\right]\cdot\left[\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{k_{\sigma^{-1}(j)}} b_{\sigma^{-1}(j),\sigma^{-1}_{\sigma^{-1}(j)}(i)}\right]\\
&= b\langle\sigma\underline{L}\rangle\cdot\left[\bigoplus_{j=1}^{n} b_{\sigma^{-1}(j)}\langle\sigma_{\sigma^{-1}(j)}\underline{l}_{\sigma^{-1}(j)}\rangle\right]\cdot\left[\bigoplus_{j=1}^{n}\bigoplus_{i=1}^{k_{\sigma^{-1}(j)}} b_{\sigma^{-1}(j),\sigma^{-1}_{\sigma^{-1}(j)}(i)}\right]\\
&= b\langle\sigma\underline{L}\rangle\cdot\bigoplus_{j=1}^{n}\left[b_{\sigma^{-1}(j)}\langle\sigma_{\sigma^{-1}(j)}\underline{l}_{\sigma^{-1}(j)}\rangle\cdot\left\{\bigoplus_{i=1}^{k_{\sigma^{-1}(j)}} b_{\sigma^{-1}(j),\sigma^{-1}_{\sigma^{-1}(j)}(i)}\right\}\right]\\
&= b\langle\sigma\underline{L}\rangle\cdot\bigoplus_{j=1}^{n}\gamma\Big(b_{\sigma^{-1}(j)},\big(b_{\sigma^{-1}(j),i}\big)_{1\le i\le k_{\sigma^{-1}(j)}}\Big)\\
&= \gamma\bigg(b,\Big(\gamma\big(b_j,(b_{j,i})_{1\le i\le k_j}\big)\Big)_{1\le j\le n}\bigg)
\end{aligned}$$

This proves that Br is a Cat-enriched operad. □

The braid operad is closely related to the associative operad and the Barratt-Eccles operad. Recall

- the associative operad As with $\mathsf{As}_n = \Sigma_n$ in Definition 11.1.1,
- the Cat-enriched Barratt-Eccles operad $E\mathsf{As}$ with E the translation category functor in Definitions 11.4.1 and 11.4.10, and
- the Cat-enriched operad morphism

$$\mathsf{As} \xrightarrow{\iota_{\mathsf{As}}} E\mathsf{As}$$

  in (11.4.12) that is levelwise the identity function on objects.

We will use Proposition 12.1.11 in Corollary 12.5.2 below.

Proposition 12.1.11. *There is a unique factorization of the* Cat*-enriched operad morphism* ι_{As} *as in the diagram*

$$\begin{tikzcd}
\mathsf{As} \arrow[rr, "\iota_{\mathsf{As}}"] \arrow[dr, "\iota^1"'] & & E\mathsf{As} \\
& \mathsf{Br} \arrow[ur, "\iota^2"'] &
\end{tikzcd}$$

such that both ι^1 *and* ι^2 *are levelwise the identity functions on objects.*

Proof. By Definition 12.1.2, the object set operad of Br is the associative operad As. This defines the operad morphism ι^1. Since the object set operad of the Barratt-Eccles operad $E\mathsf{As}$ is also As, the operad morphism ι^2 is well defined at the object set level. For morphisms, ι^2 is uniquely defined by the fact that, in each translation category $E\mathsf{As}_n$, each morphism set has a single element. This fact also implies that ι^2 is levelwise a functor that preserves the operad unit, symmetric group action, and operad composition. □

Explanation 11.6.4 describes the simplicial Barratt-Eccles operad $N(E\mathsf{As})$, with $N : \mathsf{Cat} \longrightarrow \mathsf{sSet}$ the nerve functor (Definition 7.2.3). The simplicial operad $N\mathsf{Br}$

will play an important role in Theorem 12.2.4 and Corollaries 12.5.2 and 12.5.3. We end this section with an explicit description of it.

Explanation 12.1.12 (Simplicial Braid Operad)**.** For $n, k \geq 0$, a k-simplex in the nerve $N\mathsf{Br}_n$ is a diagram

$$\big((\sigma_i)_{i=0}^k, (b_j)_{j=1}^k\big) = \Big(\sigma_0 \xrightarrow{b_1} \sigma_1 \xrightarrow{b_2} \cdots \xrightarrow{b_k} \sigma_k \Big)$$

with

- each $\sigma_i \in \Sigma_n$ for $0 \leq i \leq k$ and
- each $b_j \in \mathsf{Br}_n(\sigma_{j-1}; \sigma_j)$ for $1 \leq j \leq k$.

In other words, $b_j \in B_n$ is a braid with underlying permutation

$$\overline{b_j} = \sigma_j \sigma_{j-1}^{-1} \in \Sigma_n.$$

In particular, the set of 0-simplices in $N\mathsf{Br}_n$ is

$$(N\mathsf{Br}_n)_0 = \Sigma_n.$$

The face and degeneracy maps are given as in Explanation 7.2.4.

The symmetric group Σ_n acts on $N\mathsf{Br}_n$ diagonally on objects, that is,

$$\big((\sigma_i)_{i=0}^k, (b_j)_{j=1}^k\big) \cdot \pi = \big((\sigma_i \pi)_{i=0}^k, (b_j)_{j=1}^k\big) \quad \text{for} \quad \pi \in \Sigma_n.$$

The unit in the simplicial operad $N\mathsf{Br}$ is

$$\mathrm{id}_1 \in (N\mathsf{Br}_1)_0 = \Sigma_1.$$

The operad composition

$$N\mathsf{Br}_n \times \prod_{\ell=1}^n N\mathsf{Br}_{k_\ell} \xrightarrow{\gamma} N\mathsf{Br}_{k_1+\cdots+k_n}$$

is given on p-simplices by

$$\begin{aligned}
&\gamma\Big(\big((\sigma_i)_{i=0}^p, (b_j)_{j=1}^p\big); \big\{\big((\sigma_i^\ell)_{i=0}^p, (b_j^\ell)_{j=1}^p\big)\big\}_{\ell=1}^n\Big) \\
&= \Big(\big\{\gamma\big(\sigma_i, (\sigma_i^1, \ldots, \sigma_i^n)\big)\big\}_{i=0}^p, \big\{\gamma\big(b_j, (b_j^1, \ldots, b_j^n)\big)\big\}_{j=1}^p\Big).
\end{aligned}$$

The last two γ's are those in, respectively, (12.1.6) and (12.1.7). $\diamond$

12.2. The Braid Operad is an E_2-Operad

In this section, we observe in Theorem 12.2.4 that the braid operad Br in Proposition 12.1.10 is an E_2-operad. This is the E_2 analogue of Proposition 11.6.3, which says that the Barratt-Eccles operad is an E_∞-operad. This assertion will be used in Corollary 12.5.3, which says that the Elmendorf-Mandell K-theory of each small braided ring category is an E_2-symmetric spectrum. To make sense of the concept of an E_2-operad in Theorem 12.2.4, we first define the little n-cube operad $\mathcal{C}_n$ due to Boardman-Vogt [**BV73**, 2.49] and May [**May72**]. It is used to define E_n-operads for $n \geq 1$ in Definition 12.2.3.

Little n-Cubes. The ground symmetric monoidal category is Top, with compactly generated weak Hausdorff spaces as objects and continuous maps as morphisms. See Note 12.6.1 for related references. Suppose $\mathbb{R}$ is the topological space of real numbers. Denote by

- $[0,1] \subseteq \mathbb{R}$ the closed unit interval with interior $(0,1)$ and
- $[0,1]^{\times n} \subseteq \mathbb{R}^n$ the closed unit n-cube with interior $(0,1)^{\times n}$.

A *little n-cube* is a function

$$f = (f^1, \ldots, f^n) : [0,1]^{\times n} \longrightarrow [0,1]^{\times n}$$

such that, for each $1 \leq i \leq n$,

$$f^i : [0,1] \longrightarrow [0,1]$$

is a linear function of the form

$$f^i(t) = a_i + t(b_i - a_i) \quad \text{for} \quad 0 \leq t \leq 1$$

and some $0 \leq a_i < b_i \leq 1$. We sometimes denote f^i by its image $[a_i, b_i]$ and a little n-cube by a product

$$[a_1, b_1] \times \cdots \times [a_n, b_n]. \tag{12.2.1}$$

For $k \geq 0$, a k-tuple $\underline{f} = (f_1, \ldots, f_k)$ of little n-cubes is said to have *pairwise disjoint interiors* if

$$f_i\big((0,1)^{\times n}\big) \cap f_j\big((0,1)^{\times n}\big) = \varnothing \quad \text{for} \quad 1 \leq i < j \leq k. \tag{12.2.2}$$

A k-tuple $\underline{f} = (f_1, \ldots, f_k)$ of little n-cubes with pairwise disjoint interiors is also regarded as a function

$$\coprod_{i=1}^{k} [0,1]^{\times n} \xrightarrow{\ \coprod_i f_i\ } [0,1]^{\times n}.$$

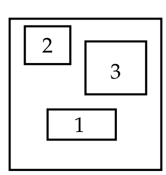
If $k = 0$, then this is regarded as the unique function $\varnothing \longrightarrow [0,1]^{\times n}$. The set of all continuous maps

$$\coprod_{i=1}^{k} [0,1]^{\times n} \longrightarrow [0,1]^{\times n}$$

is a topological space with the compact-open topology. The subset of k-tuples of little n-cubes with pairwise disjoint interiors is given the subspace topology and is denoted by

$$\mathcal{C}_n(k).$$

When $k = 0$, $\mathcal{C}_n(0)$ is the one-point space containing the function $\varnothing \longrightarrow [0,1]^{\times n}$. Below is a picture of an element with three pairwise disjoint little 2-cubes inside the unit square.



$\in \mathcal{C}_2(3)$

The Little n-Cube Operad. For each $n \geq 1$, the sequence of topological spaces

$$\mathcal{C}_n = \{\mathcal{C}_n(k)\}_{k \geq 0}$$

is a Top-enriched operad with the following data.

Unit: The unit is the identity map

$$\big(1 : [0,1]^{\times n} \longrightarrow [0,1]^{\times n}\big) \in \mathcal{C}_n(1).$$

Equivariance: Given $\underline{f} = (f_1, \ldots, f_k) \in \mathcal{C}_n(k)$ and a permutation $\sigma \in \Sigma_k$, define

$$\underline{f}\sigma = \big(f_{\sigma(1)}, \ldots, f_{\sigma(k)}\big) \in \mathcal{C}_n(k).$$

Composition: For $m \geq 1$ and $k_1, \ldots, k_m \geq 0$ with $k = k_1 + \cdots + k_m$, the *composition*

$$\mathcal{C}_n(m) \times \prod_{i=1}^m \mathcal{C}_n(k_i) \xrightarrow{\gamma} \mathcal{C}_n(k)$$

is defined as the composite

$$\begin{tikzcd}
\coprod_{i=1}^m \coprod_{l=1}^{k_i} [0,1]^{\times n} \arrow[rr, "\gamma(\underline{f}, (\underline{g}_i)_{1 \leq i \leq m})"] \arrow[dr, "\amalg_i \underline{g}_i"'] & & {[0,1]^{\times n}} \\
& \coprod_{i=1}^m [0,1]^{\times n} \arrow[ur, "\underline{f}"'] &
\end{tikzcd}$$

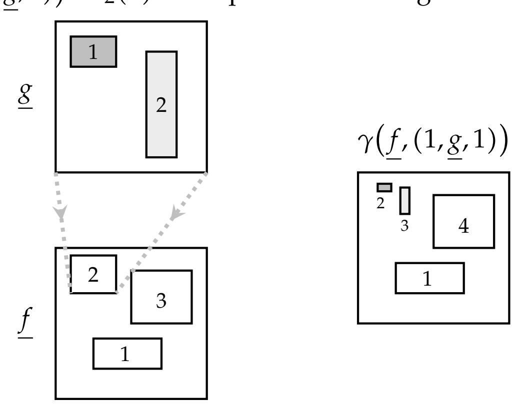
for $\underline{f} \in \mathcal{C}_n(m)$ and $\underline{g}_i \in \mathcal{C}_n(k_i)$ with $1 \leq i \leq m$.

This finishes the definition of $\mathcal{C}_n$.

- The unity axioms (6.1.4) and (6.1.5) hold because the unit in $\mathcal{C}_n(1)$ is the identity map.
- The symmetric group action and the equivariance axioms (6.1.6) and (6.1.7) hold because the symmetric group action on $\mathcal{C}_n(k)$ simply permutes the labels of the k little n-cubes in the k-tuples.
- The associativity axiom (6.1.3) holds because composition of continuous maps is strictly associative.

The Top-enriched operad $\mathcal{C}_n$ is called the *little n-cube operad*.

For example, with $\underline{f} \in \mathcal{C}_2(3)$ and $\underline{g} \in \mathcal{C}_2(2)$ on the left-hand side below, the composite $\gamma\big(\underline{f}, (1, \underline{g}, 1)\big) \in \mathcal{C}_2(4)$ is the picture on the right below.



E_n-Operads. Recall from Example 7.2.8 that the *classifying space* $|N\mathsf{C}|$ of a small category C is the geometric realization of the nerve of that category (Definitions 7.1.14 and 7.2.3). The nerve N preserves limits, in particular, products, because it is a right adjoint. The geometric realization $|-|$ preserves *finite* limits, in particular, products. A detailed proof of this fact is in [**GZ67**, III.3.4]. Therefore, the classifying space functor $|N(-)|$ also preserves products and takes a categorical operad to a topological operad. In the next definition, a *zigzag* means a finite nonempty sequence of arrows, each pointing in either direction.

Definition 12.2.3. Two Top-enriched operads P and Q are *weakly equivalent* if there is a zigzag of Top-enriched operad morphisms

$$\mathsf{P} \xrightarrow{\sim} \mathsf{P}_1 \xleftarrow{\sim} \cdots \xrightarrow{\sim} \mathsf{Q}$$

connecting P and Q in which each operad morphism is levelwise a weak homotopy equivalence. Moreover, for $1 \le n < \infty$, we define the following.

- A Top-enriched operad is an *E_n-operad* if it is weakly equivalent to the little n-cube operad $\mathcal{C}_n$.
- An sSet-enriched operad is an *E_n-operad* if its geometric realization is an E_n-operad.
- A Cat-enriched operad is an *E_n-operad* if its classifying space is an E_n-operad. $\diamond$

The following result is the braid operad analogue of Proposition 11.6.3, which says that the Barratt-Eccles operad is an E_∞-operad. This result is essentially due to Fiedorowicz [**Fie∞**] and is proved in [**Fre17**, 5.2.12]. See also Note 12.6.2.

Theorem 12.2.4. *The braid operad* Br *in Proposition 12.1.10 is an E_2-operad.*

Proof. We refer the reader to [**Fre17**, Ch. 5] for the proof with nice illustrations. Here we provide an overview of the proof. The desired weak equivalence between the classifying space of the braid operad Br and the little 2-cube operad $\mathcal{C}_2$ is given by the following zigzag of Top-enriched operad isomorphisms $\cong$ and weak equivalences $\sim$.

$$\begin{tikzcd}
\mathcal{C}_2 & & |N\mathsf{Br}| \\
\mathsf{Sy}(\widetilde{\mathcal{C}}_2) \arrow[u, "(6)", "\cong"'] & & |N\mathsf{Sy}(E\mathsf{B})| \arrow[u, "\cong", "(1)"'] \\
\mathsf{Sy}(\mathsf{Q}) \arrow[u, "(5)", "\sim"'] \arrow[r, "(4)", "\sim"'] & \mathsf{Sy}(|NE\mathsf{B}|) \arrow[r, "(3)", "\cong"'] & |\mathsf{Sy}(NE\mathsf{B})| \arrow[u, "\cong", "(2)"']
\end{tikzcd} \tag{12.2.5}$$

This diagram involves the following concepts and constructions.

Braided operads. Enriched *braided operads* have the same definition as enriched operads in Definition 6.1.1, except that each symmetric group Σ_n is replaced by the braid group B_n in Definition II.1.1.1. In the equivariance axioms (6.1.6) and (6.1.7), the block permutation and the block sum are replaced by, respectively, the block braid (II.1.1.21) and the sum braid (II.1.1.10).

B_∞-operads. Just as the symmetric groups form the associative operad As in Definition 11.1.1, the braid groups form a braided operad

$$\mathsf{B} = \{B_k\}_{k\ge 0}, \tag{12.2.6}$$

which is called the *braid group operad*.

- Applying the translation category functor E in Definition 11.4.1 yields a Cat-enriched braided operad $E\mathsf{B}$. It inherits from B the free braid group action in each level.
- Applying the classifying space functor, $|NE\mathsf{B}|$ is a Top-enriched braided operad with a levelwise free braid group action. Moreover, each space $|NEB_k|$ is contractible, since each translation category EB_k is equivalent to the terminal category.

A B_∞-*operad* is a Top-enriched braided operad R such that

- each space R_n is contractible, and
- each braid group action

$$\mathsf{R}_n \times B_n \longrightarrow \mathsf{R}_n$$

is free.

So $|NE\mathsf{B}|$ is a B_∞-operad.

Symmetrization. The kernel of the group homomorphism $\pi : B_k \longrightarrow \Sigma_k$ in (II.1.1.12) is called the kth *pure braid group* and is denoted by P_k. Elements in P_k are braids with identity underlying permutation. For an object with a braid group action, the quotient by P_k is called the *symmetrization* and is denoted by Sy. The symmetrization has the following properties.

- It inherits a symmetric group action, since

$$B_k/P_k \cong \Sigma_k.$$

- Sy turns a braided operad into an operad.
- When applied to a small category with a braid group action, Sy commutes with the nerve.
- When applied to a simplicial set with a braid group action, Sy commutes with the geometric realization, which is a coend.
- More conceptually, Sy is the left adjoint of the functor that sends an object with a Σ_k-action to the same object with the pullback B_k-action via the map $\pi : B_k \longrightarrow \Sigma_k$.

The commutativity of the symmetrization with the nerve and the geometric realization yields the isomorphisms of operads (2) and (3) in (12.2.5).

Br *as a symmetrization*. For each $k \geq 0$, there is a functor

$$EB_k \longrightarrow \mathsf{Br}_k$$

defined as follows.

- On objects, it is given by the underlying permutation $\pi : B_k \longrightarrow \Sigma_k$.
- Using the definitions (11.4.4) and (12.1.3), this functor sends a morphism

$$b'b^{-1} : b \longrightarrow b'$$

in the translation category EB_k to the morphism

$$b'b^{-1} : \pi(b) \longrightarrow \pi(b')$$

in the category Br_k.

These functors induce an isomorphism of Cat-enriched operads

$$\mathsf{Sy}(E\mathsf{B}) \xrightarrow{\cong} \mathsf{Br}.$$

Applying the classifying space functor yields the isomorphism of operads (1) in (12.2.5). This is **[Fre17**, 5.2.15].

Universal covering of $\mathcal{C}_2$. The little 2-cube operad $\mathcal{C}_2$ naturally extends to the B_∞-operad $\widetilde{\mathcal{C}}_2$, with each $\widetilde{\mathcal{C}}_2(k)$ the universal covering space of $\mathcal{C}_2(k)$. Passing to the symmetrization of $\widetilde{\mathcal{C}}_2$, the levelwise covering maps

$$\widetilde{\mathcal{C}}_2(k) \longrightarrow \mathcal{C}_2(k) \tag{12.2.7}$$

induce the isomorphism of operads (6) in (12.2.5). This is **[Fre17**, 5.1.6].

Connecting B_∞*-operads.* As noted above, $\widetilde{\mathcal{C}}_2$ and $|NE\mathsf{B}|$ are B_∞-operads. The sequence of spaces

$$\mathsf{Q} = \left\{\mathsf{Q}_k = \widetilde{\mathcal{C}}_2(k) \times |NEB_k|\right\}_{k\geq 0}$$

inherits the structure of a B_∞-operad, with the braid group acting diagonally in each level. Since $\widetilde{\mathcal{C}}_2(k)$ and $|NEB_k|$ are contractible, the projections

$$\widetilde{\mathcal{C}}_2(k) \xleftarrow{\sim} \mathsf{Q}_k \xrightarrow{\sim} |NEB_k|$$

are weak homotopy equivalences and respect the free braid group action. These projections form braided operad morphisms

$$\widetilde{\mathcal{C}}_2 \xleftarrow{\sim} \mathsf{Q} \xrightarrow{\sim} |NE\mathsf{B}|. \tag{12.2.8}$$

Passing to the symmetrization yields the weak equivalences of operads (4) and (5) in (12.2.5). □

12.3. Coherence of the Braid Operad

In this section, we prove a coherence theorem for the braid operad Br; see Theorem 12.3.10. This result is the braid operad extension of Theorem 11.4.14 for the Barratt-Eccles operad. Theorem 12.3.10 uses a decomposition of morphisms in Br in Lemma 12.3.6. Example 12.3.9 is a concrete illustration of that decomposition of morphisms in Br. As the first application, in Proposition 12.3.22, we observe that the braid operad detects braided strict monoidal category structure. It will be used again in Theorem 12.4.5 and Corollary 12.5.3 when we consider the K-theory of braided ring categories.

Decomposition of Morphisms.

Motivation 12.3.1. By Explanation 11.4.11 (2), morphisms in $E\mathsf{As}_n$ for $n \geq 2$ are Cat-enriched operadically generated by the isomorphism

$$\mathrm{id}_2 \xrightarrow[\cong]{\tau} (1,2) \in E\mathsf{As}_2\big(\mathrm{id}_2; (1,2)\big)$$

and identity morphisms. Lemma 12.3.6 below is the analogue for the braid operad Br. By Example 12.1.9, the braid operad Br contains the morphism set

$$\mathsf{Br}_2\big(\mathrm{id}_2; (1,2)\big) = \{s_1^{2k+1}\}_{k\in\mathbb{Z}},$$

where $s_1 \in B_2$ is the generating braid with underlying permutation $(1,2)$. In what follows, the isomorphism

$$\mathrm{id}_2 \xrightarrow[\cong]{s_1} (1,2) \in \mathsf{Br}_2 \tag{12.3.2}$$

will play the role of the isomorphism $\tau \in E\mathsf{As}_2\big(\mathrm{id}_2; (1,2)\big)$.

However, one should be careful that s_1^{-1} can mean either the inverse of the isomorphism in (12.3.2), which is denoted by

$$(s_1)^{-1} : (1,2) \longrightarrow \mathrm{id}_2,$$

or the different isomorphism

$$\mathrm{id}_2 \xrightarrow[\cong]{s_1^{-1}} (1,2) \in \mathsf{Br}_2. \tag{12.3.3}$$

They are related by the equality

$$(s_1)^{-1}(1,2) = s_1^{-1} : \mathrm{id}_2 \longrightarrow (1,2) \in \mathsf{Br}_2 \tag{12.3.4}$$

with $(s_1)^{-1}(1,2)$ the right $(1,2)$-action (12.1.4) on $(s_1)^{-1}$. The equality (12.3.4) holds because, by the symmetric group action (12.1.4) on the morphisms in Br, each side is a morphism $\mathrm{id}_2 \longrightarrow (1,2)$ represented by the braid $s_1^{-1} \in B_2$. There is also an isomorphism

$$(1,2) \xrightarrow[\cong]{s_1} \mathrm{id}_2 \in \mathsf{Br}_2, \tag{12.3.5}$$

which is equal to the right $(1,2)$-action on the s_1 in (12.3.2). ◇

Lemma 12.3.6. *Each morphism in Br_n for $n \geq 2$ decomposes into a categorical composite of isomorphisms of the form ϕv with*

- *each $v \in \Sigma_n$ and*
- *each ϕ an operadic composite of*
  - *one instance of*
  $$s_1^{\pm 1} : \mathrm{id}_2 \longrightarrow (1,2)$$
  in (12.3.2) and (12.3.3) and
  - *identity morphisms.*

Proof. First recall the following three facts:

- By definition (12.1.3), each morphism set $\mathsf{Br}_n(\sigma;\theta)$ consists of braids $b \in B_n$ with underlying permutation $\overline{b} = \theta\sigma^{-1}$.
- By Definition II.1.1.1, $s_i^{(n)} \in B_n$ for $1 \leq i < n$ are the group generators. The underlying permutation of $s_i^{(n)}$ is the adjacent transposition $(i,i+1) \in \Sigma_n$.
- By definition (II.1.1.10), there is a sum braid decomposition
$$\left(s_i^{(n)}\right)^{\pm 1} = \overbrace{\mathrm{id}_{i-1} \oplus}^{\varnothing \text{ if } i=1} s_1^{\pm 1} \overbrace{\oplus\, \mathrm{id}_{n-i-1}}^{\varnothing \text{ if } i=n-1} \in B_n$$
with $s_1 \in B_2$ the generating braid.

These facts imply that each morphism in Br_n decomposes into a categorical composite of isomorphisms of the form

$$\left(s_i^{(n)}\right)^{\pm 1} v : v \longrightarrow (i,i+1)v \in \mathsf{Br}_n \tag{12.3.7}$$

for some $1 \leq i < n$, permutation $v \in \Sigma_n$, and sum braid

$$\left(s_i^{(n)}\right)^{\pm 1} = \overbrace{\mathrm{id}_{i-1} \oplus}^{\varnothing \text{ if } i=1} s_1^{\pm 1} \overbrace{\oplus\, \mathrm{id}_{n-i-1}}^{\varnothing \text{ if } i=n-1} : \mathrm{id}_n \longrightarrow (i,i+1) \in \mathsf{Br}_n. \tag{12.3.8}$$

The block braid induced by an identity braid is an identity braid in the sense that

$$\mathrm{id}_n\langle\sigma\underline{k}\rangle = \mathrm{id}_k \in B_k.$$

By the definition (12.1.7) of γ in Br, the sum braid in (12.3.8) is an operadic composite of one

$$s_1^{\pm 1} : \mathrm{id}_2 \longrightarrow (1,2)$$

in (12.3.2) and (12.3.3) and identity morphisms. $\square$

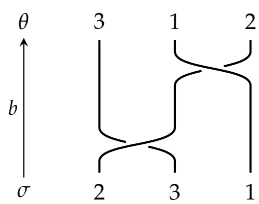
Example 12.3.9. To illustrate the decomposition in Lemma 12.3.6, consider

- the braid $b = s_2 s_1^{-1} \in B_3$ with s_1, s_2 the generating braids in B_3 and
- the cyclic permutations $\sigma = (1,3,2)$ and $\theta = (1,2,3) \in \Sigma_3$.

Since

$$\overline{b} = \sigma = \theta\sigma^{-1} \in \Sigma_3,$$

there is a morphism $b : \sigma \longrightarrow \theta$ in Br_3, as displayed below.



$$\begin{tikzcd} \theta = (2,3)(1,3) \\ (1,2)\sigma = (1,3) \arrow[u, "(\mathrm{id}_1 \oplus s_1)(1,3)"'] \\ \sigma \arrow[u, "(s_1^{-1} \oplus \mathrm{id}_1)\sigma"'] \end{tikzcd}$$

The morphism $b \in \mathsf{Br}_3(\sigma;\theta)$ decomposes as the categorical composite

$$b = \big[(\mathrm{id}_1 \oplus s_1)(1,3)\big]\big[(s_1^{-1} \oplus \mathrm{id}_1)\sigma\big]$$

with $s_1^{\pm 1} : \mathrm{id}_2 \longrightarrow (1,2)$ in (12.3.2) and (12.3.3). Each of the two factors in the above decomposition has the form (12.3.7). This example illustrates that the permutation v and the isomorphism s_1^{-1} cannot be omitted from Lemma 12.3.6. $\diamond$

Coherence. The next result is a coherence theorem for the braid operad in the sense that it describes the braid operad in terms of a small number of generators and a few relations. It is the braid operad analogue of Theorem 11.1.7 for the associative operad As (Definition 11.1.1) and Theorem 11.4.14 for the Barratt-Eccles operad $E\mathsf{As}$ (Definition 11.4.10). The axioms (12.3.13) and (12.3.14) below are formally identical to those of a braided strict monoidal category in Definitions 1.1.1 and 1.1.14. An *enriched operad morphism* is an enriched multifunctor as in Definition 5.1.12 between two enriched multicategories with one object. This result is from [**Fre17**, 6.2.4 and 6.2.6]; see Note 12.6.3.

Theorem 12.3.10. *Suppose* $(\mathsf{P}, \gamma, 1)$ *is a* Cat*-enriched operad. Then a* Cat*-enriched operad morphism*

$$f : \mathsf{Br} \longrightarrow \mathsf{P}$$

is uniquely determined by

- *the objects*

$$\begin{aligned} \mathbb{1} &= f(\mathrm{id}_0) \in \mathsf{P}_0 \\ \mu &= f(\mathrm{id}_2) \in \mathsf{P}_2 \end{aligned} \tag{12.3.11}$$

  and
- *the isomorphism*

$$\mu = f(\mathrm{id}_2) \xrightarrow[\cong]{\xi = f(s_1)} f(1,2) = \mu(1,2) = \mu^{\mathsf{op}} \in \mathsf{P}_2(\mu, \mu^{\mathsf{op}}) \tag{12.3.12}$$

with

$$s_1 : \mathrm{id}_2 \longrightarrow (1,2) \in \mathsf{Br}_2$$

the isomorphism in (12.3.2).

The above data are subject to the following conditions.

Unity and Associativity: *The equalities of objects*

$$\begin{aligned} \mu(\mathbb{1},1) &= 1 = \mu(1,\mathbb{1}) \in \mathsf{P}_1 \\ \mu(\mu,1) &= \mu(1,\mu) \in \mathsf{P}_3 \end{aligned} \tag{12.3.13}$$

hold.

The Hexagon Axiom: *The left and right hexagon diagrams*

$$\begin{tikzcd}[column sep=small] \mu(1,\mu) \arrow[rr, "\xi"] \arrow[d, equal] & & \mu^{\mathsf{op}}(1,\mu) \arrow[d, equal] \\ \mu(\mu,1) \arrow[dr, "\mu(\xi,1)"'] & & \mu(1,\mu^{\mathsf{op}})(1,2) \\ & \mu(\mu^{\mathsf{op}},1) \arrow[r, equal] & \mu(1,\mu)(1,2) \arrow[u, "\mu(1,\xi)(1,2)"'] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \mu(\mu,1) \arrow[rr, "\xi"] \arrow[d, equal] & & \mu^{\mathsf{op}}(\mu,1) \arrow[d, equal] \\ \mu(1,\mu) \arrow[dr, "\mu(1,\xi)"'] & & \mu(\mu^{\mathsf{op}},1)(2,3) \\ & \mu(1,\mu^{\mathsf{op}}) \arrow[r, equal] & \mu(\mu,1)(2,3) \arrow[u, "\mu(\xi,1)(2,3)"'] \end{tikzcd} \tag{12.3.14}$$

in P_3 *are commutative.*

Proof. Similar to the proof of Theorem 11.4.14, there are two directions.

Necessity. Suppose $f : \mathsf{Br} \longrightarrow \mathsf{P}$ is a Cat-enriched operad morphism. We define

$$\begin{cases} \mathbb{1} = f(\mathrm{id}_0) \in \mathsf{P}_0, \\ \mu = f(\mathrm{id}_2) \in \mathsf{P}_2, \quad \text{and} \\ \xi = f(s_1) : \mu \longrightarrow \mu^{\mathsf{op}} \in \mathsf{P}_2 \end{cases}$$

as in (12.3.11)–(12.3.12). Since f preserves the operad units, there is an equality

$$f(\mathrm{id}_1) = 1 \in \mathsf{P}_1.$$

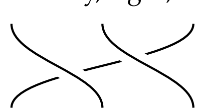
- The object set operad of Br is the associative operad As. So there is a Cat-enriched operad morphism

$$\mathsf{As} \xrightarrow{\iota} \mathsf{Br} \tag{12.3.15}$$

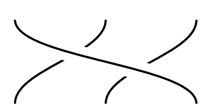
  that is levelwise the identity function on objects, that is, permutations in Σ_n. The composite

$$\mathsf{As} \xrightarrow{\iota} \mathsf{Br} \xrightarrow{f} \mathsf{P}$$

  is a Cat-enriched operad morphism. So it preserves the operad relations (11.1.8) in As, which become the equalities (12.3.13) of objects in P.
- Before applying f, the analogues of the left and right hexagons in (12.3.14) are commutative in Br. In each case, each of the two composites is the left, respectively, right, braid below, read bottom-to-top.



$s_2 s_1 \in B_3$



$s_1 s_2 \in B_3$

  Since f is a Cat-enriched operad morphism, after applying f, the axioms hold in P.

Therefore, the data (12.3.11)–(12.3.12) satisfy the axioms (12.3.13)–(12.3.14).

Sufficiency. Suppose given the data

- $\mathbb{1} = f(\mathrm{id}_0) \in \mathsf{P}_0$,
- $\mu = f(\mathrm{id}_2) \in \mathsf{P}_2$ as in (12.3.11), and
- $\xi = f(s_1) : \mu \longrightarrow \mu(1,2) = \mu^{\mathsf{op}} \in \mathsf{P}_2(\mu, \mu^{\mathsf{op}})$ as in (12.3.12)

such that the axioms (12.3.13)–(12.3.14) hold in P. We show that these data extend uniquely to a Cat-enriched operad morphism $f : \mathsf{Br} \longrightarrow \mathsf{P}$ in several stages.

Objects. By Theorem 11.1.7 and the axiom (12.3.13), the objects

$$\mathbb{1} = f(\mathrm{id}_0) \quad \text{and} \quad \mu = f(\mathrm{id}_2)$$

uniquely determine a Cat-enriched operad morphism

$$\mathsf{As} \xrightarrow{f} \mathsf{P} \tag{12.3.16}$$

defined by

$$\begin{cases} f(\mathrm{id}_1) = 1 \in \mathsf{P}_1, \\ f(\mathrm{id}_2) = \mu = \mu_2 \in \mathsf{P}_2, \\ f(\mathrm{id}_n) = \mu(\mu_{n-1}, 1) = \mu_n \in \mathsf{P}_n \quad \text{for } n \geq 3, \text{ and} \\ \quad f(\sigma) = \mu_n\sigma \in \mathsf{P}_n \quad \text{for } \sigma \in \Sigma_n. \end{cases}$$

By (12.3.15), this defines the object part of the Cat-enriched operad morphism $f : \mathsf{Br} \longrightarrow \mathsf{P}$.

Morphisms. The identity morphisms $1_{\mathrm{id}_0} \in \mathsf{Br}_0$ and $1_{\mathrm{id}_1} \in \mathsf{Br}_1$ must be sent by f to, respectively, the identity morphisms $1_{\mathbb{1}} \in \mathsf{P}_0$ and $1_1 \in \mathsf{P}_1$. To define f on the other morphisms in Br, note that the coherence theorems for (braided) monoidal categories can be applied to

- the objects $\{\mathbb{1}, 1, \mu\}$, which are interpreted as, respectively, the monoidal unit, the identity functor, and the monoidal product, and
- the braiding ξ.

The reason is that the assumed axioms (12.3.13), and also (12.3.14) in the braided case, are formally identical to those of a (braided) strict monoidal category in Definitions 1.1.1 and 1.1.14. We apply these coherence theorems as follows.

- By the axiom (12.3.13) and Mac Lane's Coherence Theorem 1.1.31, each iterated operadic composite in P_n involving only the objects

  $$\mu_0 = \mathbb{1} \in \mathsf{P}_0, \quad \mu_1 = 1 \in \mathsf{P}_1, \quad \text{and} \quad \mu_2 = \mu \in \mathsf{P}_2$$

  is equal to μ_n.
- For each pair of permutations $\sigma, \theta \in \Sigma_n$, consider isomorphisms

  $$\mu_n\sigma \xrightarrow{\cong} \mu_n\theta \in \mathsf{P}_n \tag{12.3.17}$$

  that are categorical composites of isomorphisms of the form ϕv, with $v \in \Sigma_n$ and each ϕ an operadic composite of
  - one

    $$\xi = f(s_1) : \mu \longrightarrow \mu^{\mathsf{op}} \quad \text{or} \quad \xi^{-1}(1,2) : \mu \longrightarrow \mu^{\mathsf{op}}$$

    and
  - identity morphisms.

The hexagon axiom (12.3.14) and the Braided Coherence Theorem 1.1.38 imply that two isomorphisms as in (12.3.17) are equal if their underlying braids are equal. In defining the underlying braids,

- the permutations v are ignored, and
- ξ and $\xi^{-1}(1,2)$ have underlying braids, respectively, s_1 and $s_1^{-1} \in B_2$.

This property will be called the *braided uniqueness* of (12.3.17).

By Lemma 12.3.6, for each pair of permutations $\sigma, \theta \in \Sigma_n$, each isomorphism $b : \sigma \longrightarrow \theta$ in Br_n decomposes into a categorical composite of isomorphisms of the form ϕv with

- each $v \in \Sigma_n$ and
- each ϕ an operadic composite of one $s_1^{\pm 1} : \mathrm{id}_2 \longrightarrow (1,2)$ and identity morphisms.

The desired $f : \mathsf{Br} \longrightarrow \mathsf{P}$ must be levelwise a functor and preserve the operad structure, namely, the operad units, the equivariant structure, and the operad composition. So, by (12.3.4), we must define the isomorphism

$$f(\sigma) = \mu_n \sigma \xrightarrow{f(b)} \mu_n \theta = f(\theta) \in \mathsf{P}_n \tag{12.3.18}$$

as the corresponding isomorphism in (12.3.17) with

$$\begin{aligned} & f\big(s_1^{-1} : \mathrm{id}_2 \longrightarrow (1,2)\big) \\ & = \big(f(s_1) : \mu \longrightarrow \mu^{\mathsf{op}}\big)^{-1}(1,2) \\ & = \big(\xi^{-1}(1,2) : \mu \longrightarrow \mu^{\mathsf{op}}\big). \end{aligned} \tag{12.3.19}$$

The braided uniqueness of (12.3.17) implies the following two statements.

- $f(b)$ is independent of the choice of a decomposition of each morphism $b \in \mathsf{Br}_n$ in Lemma 12.3.6.
- $f : \mathsf{Br}_n \longrightarrow \mathsf{P}_n$ is a functor for each n.

The uniqueness of each functor $f : \mathsf{Br}_n \longrightarrow \mathsf{P}_n$ is part of the definitions (12.3.16) and (12.3.18).

It remains to check that the functors $f : \mathsf{Br}_n \longrightarrow \mathsf{P}_n$ preserve the operad structure. Since $f : \mathsf{As} \longrightarrow \mathsf{P}$ in (12.3.16) is a Cat-enriched operad morphism, by (12.3.15) we only need to consider morphisms in Br_n.

Equivariant structure. The assertion that f preserves the symmetric group action means that, for each permutation $\pi \in \Sigma_n$, the diagram

$$\begin{tikzcd} \mathsf{Br}_n \arrow[r, "\pi"] \arrow[d, "f"'] & \mathsf{Br}_n \arrow[d, "f"] \\ \mathsf{P}_n \arrow[r, "\pi"] & \mathsf{P}_n \end{tikzcd} \tag{12.3.20}$$

of functors, with each horizontal arrow the right π-action, is commutative. Since f and π are functors, by the decomposition of morphisms in Br_n in Lemma 12.3.6, it suffices to consider morphisms of the form $\phi v \in \mathsf{Br}_n$ with $v \in \Sigma_n$ and ϕ an operadic composite of one $s_1^{\pm 1} : \mathrm{id}_2 \longrightarrow (1,2)$ and identity morphisms. On such a morphism ϕv, the commutativity of (12.3.20) follows from

- the definition (12.1.4) of the equivariant structure on Br_n,

- the fact (12.3.4) that
$$(s_1)^{-1}(1,2) = s_1^{-1} : \mathrm{id}_2 \longrightarrow (1,2),$$
- the definition (12.3.18) of f on morphisms, and
- the braided uniqueness of (12.3.17).

Operad composition. The assertion that f preserves the operad composition means the commutativity of the following diagram of functors.

$$\begin{tikzcd} \mathsf{Br}_n \times \prod_{j=1}^{n} \mathsf{Br}_{k_j} \arrow[r, "\gamma"] \arrow[d, "f \times \prod_j f"'] & \mathsf{Br}_{k_1+\cdots+k_n} \arrow[d, "f"] \\ \mathsf{P}_n \times \prod_{j=1}^{n} \mathsf{P}_{k_j} \arrow[r, "\gamma"] & \mathsf{P}_{k_1+\cdots+k_n} \end{tikzcd} \tag{12.3.21}$$

By the equivariance axioms in Br and P and the functoriality of f and γ, it suffices to consider (12.3.21) with

- an identity morphism in n of the $n+1$ factors in $\mathsf{Br}_n \times \prod_{j=1}^{n} \mathsf{Br}_{k_j}$ and
- in the remaining factor, a morphism of the form ϕv with v a permutation and ϕ an operadic composite of one $s_1^{\pm 1} : \mathrm{id}_2 \longrightarrow (1,2)$ and identity morphisms.

In this case, the commutativity of (12.3.21) follows from the braided uniqueness of (12.3.17). Therefore, $f : \mathsf{Br} \longrightarrow \mathsf{P}$ is a Cat-enriched operad morphism. □

Braided Strict Monoidal Categories as Algebras. Proposition 11.1.15 shows that the associative operad As is the operad for monoids. Proposition 11.4.26 shows that the Barratt-Eccles operad $E\mathsf{As}$ is the Cat-enriched operad for permutative categories. The following application of Theorem 12.3.10 is the analogue for the braid operad Br. See also Note 12.6.4.

Proposition 12.3.22. *For a small category C, a braided strict monoidal category structure on C is uniquely determined by a Cat-enriched multifunctor*
$$f : \mathsf{Br} \longrightarrow \mathsf{Cat} \quad \textit{such that} \quad f(*) = \mathsf{C}.$$

Proof. A Cat-enriched multifunctor $f : \mathsf{Br} \longrightarrow \mathsf{Cat}$ such that $f(*) = \mathsf{C}$ is equivalent to a Cat-enriched operad morphism
$$\mathsf{Br} \xrightarrow{\ f\ } \mathrm{End}(\mathsf{C}) = \left\{\mathsf{Cat}(\mathsf{C}^{\times n}, \mathsf{C})\right\}_{n \geq 0}$$
to the Cat-enriched endomorphism operad of C. By Theorem 12.3.10, such a Cat-enriched operad morphism f is uniquely determined by

- the object
$$\mathbb{1} = f(\mathrm{id}_0) \in \mathsf{C},$$
- the functor
$$\otimes = f(\mathrm{id}_2) : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C},$$
and
- the natural isomorphism
$$\otimes \xrightarrow{\ \xi^{\otimes} = f(s_1)\ } \otimes(1,2) = \otimes^{\mathrm{op}}$$

such that the axioms (12.3.13)–(12.3.14) are satisfied.

- The unity and associativity axioms (12.3.13) state that $(\mathsf{C}, \otimes, \mathbb{1})$ is a strict monoidal category as in Definition I.1.2.1.
- The hexagon axiom (12.3.14) state that $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a braided strict monoidal category as in Definition II.1.3.15.

This finishes the proof. □

12.4. Detecting Braided Ring Categories

In this section, we prove that the braid operad detects braided ring category structures on small permutative categories (Theorem 12.4.5). This result is an application of the Coherence Theorem 12.3.10 for the braid operad. In Section 12.5, we use this result to show that the K-theory of a small braided ring category is an E_2-symmetric spectrum (Corollary 12.5.3). For the reader's convenience, here we recall from Chapter II.9 the definition of a braided ring category.

Definition 12.4.1. A *braided ring category* is a tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big)$$

consisting of

- a ring category
$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r)\big)$$
as in Definition 11.2.4 and
- a braided strict monoidal category structure $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$, with $\xi^{\otimes}$ called the *braiding*, as in Definitions 1.1.1 and 1.1.14.

These data are required to satisfy the following axioms for objects $A, B, C \in \mathsf{C}$.

The Zero Braiding Axiom: There are equalities of morphisms as follows.

$$\begin{aligned} \xi^{\otimes}_{A,\mathbb{0}} &= 1_{\mathbb{0}} : A \otimes \mathbb{0} = \mathbb{0} \longrightarrow \mathbb{0} = \mathbb{0} \otimes A \\ \xi^{\otimes}_{\mathbb{0},A} &= 1_{\mathbb{0}} : \mathbb{0} \otimes A = \mathbb{0} \longrightarrow \mathbb{0} = A \otimes \mathbb{0} \end{aligned} \tag{12.4.2}$$

The Braiding Factorization Axiom: The diagram

$$\begin{tikzcd} (A \otimes C) \oplus (B \otimes C) \arrow[r, "\partial^l_{A,B,C}"] \arrow[d, "\xi^{\otimes}_{A,C} \oplus \xi^{\otimes}_{B,C}"'] & (A \oplus B) \otimes C \arrow[d, "\xi^{\otimes}_{A \oplus B, C}"] \\ (C \otimes A) \oplus (C \otimes B) \arrow[r, "\partial^r_{C,A,B}"] \arrow[d, "\xi^{\otimes}_{C,A} \oplus \xi^{\otimes}_{C,B}"'] & C \otimes (A \oplus B) \arrow[d, "\xi^{\otimes}_{C, A \oplus B}"] \\ (A \otimes C) \oplus (B \otimes C) \arrow[r, "\partial^l_{A,B,C}"] & (A \oplus B) \otimes C \end{tikzcd} \tag{12.4.3}$$

is commutative.

This finishes the definition of a braided ring category. A braided ring category is *small*, respectively, *tight*, if the underlying ring category is so. ◇

Definition 12.4.4. For a permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$, a *braided ring category structure* on C is the additional data $(\otimes, \mathbb{1}, \xi^{\otimes}, \partial^l, \partial^r)$ such that the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes, \mathbb{1}, \xi^{\otimes}), (\partial^l, \partial^r)\big)$$

is a braided ring category as in Definition 12.4.1. ◇

Recall from Section 6.6 that $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-enriched multicategory with small permutative categories as objects. The category

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle\,;\mathsf{D}\big) = \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}_1,\ldots,\mathsf{C}_n\rangle\,;\mathsf{D}\big)$$

has

- n-linear functors
$$\mathsf{C}_1\times\cdots\times\mathsf{C}_n \longrightarrow \mathsf{D}$$
as in Definition 6.5.4 as objects and
- multilinear transformations (Definition 6.5.11) as morphisms.

Also recall from Definition 5.1.12 the notion of an enriched multifunctor. The next result is the braid operad analogue of Theorems 11.2.16 and 11.5.5. It says that the braid operad detects braided ring category structures on small permutative categories. It extends Proposition 12.3.22 from Cat to $\mathsf{PermCat}^{\mathsf{su}}$.

Theorem 12.4.5. *For each small permutative category* C*, there is a canonical bijective correspondence between*

- *braided ring category structures on* C *and*
- Cat*-enriched multifunctors*
$$F:\mathsf{Br}\longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \textit{such that} \quad F(*)=\mathsf{C}.$$

Proof. A Cat-enriched multifunctor

$$F:\mathsf{Br}\longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \text{such that} \quad F(*)=(\mathsf{C},\oplus,\mathbb{0},\xi^{\oplus})$$

is equivalent to a Cat-enriched operad morphism

$$F:\mathsf{Br}\longrightarrow \mathsf{End}(\mathsf{C}) = \Big\{\mathsf{PermCat}^{\mathsf{su}}\big(\langle\overbrace{\mathsf{C},\ldots,\mathsf{C}}^{n}\rangle\,;\mathsf{C}\big)\Big\}_{n\geq 0}$$

to the Cat-enriched endomorphism operad of C. By Theorem 12.3.10, such a Cat-enriched operad morphism is uniquely determined by

- the 0-linear functor, that is, object
$$F(\mathrm{id}_0)=\mathbb{1}\in\mathsf{C},$$
- the 2-linear functor
$$F(\mathrm{id}_2)=(\otimes,\partial^l,\partial^r):\mathsf{C}\times\mathsf{C}\longrightarrow\mathsf{C},$$
and
- the invertible multilinear transformation
$$(\otimes,\partial^l,\partial^r)\xrightarrow{\;\xi^{\otimes}=F(s_1)\;}(\otimes^{\mathsf{op}},\partial^r,\partial^l)$$

such that the conditions (12.3.13)–(12.3.14) are satisfied, with

$$(\mu,1,\xi) \quad \text{interpreted as} \quad \big((\otimes,\partial^l,\partial^r),1_{\mathsf{C}},\xi^{\otimes}\big).$$

By Theorem 11.2.16 and the first paragraph of its proof, the data $(\otimes,\mathbb{1},\partial^l,\partial^r)$ and the condition (12.3.13) are equivalent to a ring category structure on the small permutative category $(\mathsf{C},\oplus,\mathbb{0},\xi^{\oplus})$ in the sense of Definition 11.2.15. In particular, $(\mathsf{C},\otimes,\mathbb{1})$ is a strict monoidal category. Next we consider the remaining data $\xi^{\otimes}$ and the hexagon axiom (12.3.14).

The invertible multilinear transformation

$$\xi^{\otimes}:\otimes\xrightarrow{\;\cong\;}\otimes^{\mathsf{op}}$$

is, by definition, a natural isomorphism that satisfies the two conditions in Definition 6.5.11.

- The hexagon axiom (12.3.14) states that $(\mathsf{C}, \otimes, \mathbb{1}, \xi^{\otimes})$ is a braided strict monoidal category as in Definition II.1.3.15.
- Consider the commutativity of the diagram (6.5.12) for $\alpha = \xi^{\otimes}$.
  - The case $i = 1$ is the top half of the braiding factorization axiom (12.4.3) in a braided ring category.
  - The case $i = 2$ is the bottom half of the braiding factorization axiom (12.4.3).
- The second condition in Definition 6.5.11 states the equalities

$$\xi^{\otimes}_{-,\mathbb{0}} = 1_{\mathbb{0}} = \xi^{\otimes}_{\mathbb{0},-}.$$

  These equalities form the zero braiding axiom (12.4.2) in a braided ring category.

Therefore, a Cat-enriched operad morphism

$$F : \mathsf{Br} \longrightarrow \mathsf{End}(\mathsf{C})$$

is equivalent to a braided ring category structure on C. □

12.5. K-Theory of Braided Ring Categories are E_2-Symmetric Spectra

In this section, we prove that the K-theory of a small braided ring category is an E_2-symmetric spectrum (Corollary 12.5.3). This result is the E_2 analogue of Corollaries 11.3.16 and 11.6.12. It is a consequence of Theorem 12.4.5, the Elmendorf-Mandell K-theory multifunctor, and the fact that the braid operad is an E_2-operad (Theorem 12.2.4). Along the way, we record the relationship between strict ring, E_2-, and E_∞-structures on symmetric spectra (Corollary 12.5.2). To make sense of these results, we first define E_n-structure on symmetric spectra.

E_n-Symmetric Spectra. For each symmetric spectrum X, recall from Definition 7.6.22 that it has an *endomorphism simplicial operad* $\mathsf{End}(X)$, which is enriched over sSet_*. For $n \geq 0$, it has

$$\mathsf{End}(X)_n = \mathsf{SymSp}\big(X^{\wedge n} \wedge \Delta^{?}_{+}, X\big) \in \mathsf{sSet}_*$$

as its pointed simplicial set of n-ary operations.

Definition 12.5.1. Suppose $1 \leq n < \infty$.

- An *E_n-structure* on a symmetric spectrum X is an sSet-enriched operad morphism

$$\mathsf{P} \longrightarrow \mathsf{End}(X)$$

  for some sSet-enriched E_n-operad P as in Definition 12.2.3.
- An *E_n-symmetric spectrum* is a symmetric spectrum equipped with an E_n-structure. ◇

Recall

- that a strict ring symmetric spectrum is a monoid in the category SymSp of symmetric spectra (Definition 11.3.1) and
- Definition 11.6.5, which is the E_∞ analogue of Definition 12.5.1.

Applying the factorization (Proposition 12.1.11)

$$\begin{tikzcd} \mathsf{As} \arrow[rr, "\iota_{\mathsf{As}}"] \arrow[dr, "\iota^1"'] & & E\mathsf{As} \\ & \mathsf{Br} \arrow[ur, "\iota^2"'] & \end{tikzcd}$$

and using Propositions 11.1.15 and 11.6.3 and Theorem 12.2.4, we obtain the following relationship between strict ring structure, E_2-structure, and E_∞-structure. This result is a refinement of Proposition 11.6.6 (2).

Corollary 12.5.2. *Suppose X is a symmetric spectrum.*

(1) An E_∞-structure on X via the Barratt-Eccles operad $E\mathsf{As}$ induces an E_2-structure on X by restricting along ι^2.
(2) An E_2-structure on X via the braid operad Br induces a strict ring structure on X by restricting along ι^1.

***K*-Theory E_2-Symmetric Spectra.** Recall the Elmendorf-Mandell K-theory multifunctor in Definition 10.3.32,

$$\mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp}.$$

The following result is the E_2 analogue of Corollaries 11.3.16 and 11.6.12.

Corollary 12.5.3. *For each small braided ring category C, $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_2-symmetric spectrum.*

Proof. Consider the multifunctors

$$\begin{tikzcd} \mathsf{Br} \arrow[r, "F"] & \mathsf{PermCat}^{\mathsf{su}} \arrow[r, "\mathsf{K}^{\mathsf{EM}}"] & \mathsf{SymSp} \\ * \arrow[r, mapsto] & (\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}) \arrow[r, mapsto] & \mathsf{K}^{\mathsf{EM}}\mathsf{C} \end{tikzcd}$$

with F the Cat-enriched multifunctor in Theorem 12.4.5 such that $F(*)$ is the additive structure of C. By Theorem 12.2.4, $N\mathsf{Br}$ is an E_2-operad. As in the proof of Corollary 11.6.12, the composite sSet-enriched operad morphism

$$\begin{tikzcd} N\mathsf{Br} \arrow[d, "F_N"'] & & \mathsf{End}(\mathsf{K}^{\mathsf{EM}}\mathsf{C}) \\ N\mathsf{End}(\mathsf{C}) \arrow[r, "\mathsf{J}^{\mathsf{EM}}_N"] & N\mathsf{End}(\mathsf{J}^{\mathsf{EM}}\mathsf{C}) \arrow[r, "N_*"] & \mathsf{End}(N_*\mathsf{J}^{\mathsf{EM}}\mathsf{C}) \arrow[u, "\mathsf{K}^{\mathcal{G}}"'] \end{tikzcd}$$

gives $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ the structure of an E_2-symmetric spectrum. □

The E_n analogue of Corollary 12.5.3 is Corollary 13.5.2.

Example 12.5.4 (Braided Distortion)**.** The braided distortion category $\mathcal{D}^{\mathsf{br}}$ in Section II.5.2 is a small tight braided ring category by Example II.9.5.9. Corollary 12.5.3 applies to $\mathcal{D}^{\mathsf{br}}$ to yield a K-theory E_2-symmetric spectrum. See also Questions A.5.6 and A.5.7. ◇

Example 12.5.5 (Bimonoidal Drinfeld Center)**.** The bimonoidal Drinfeld center of each small tight ring category is a small tight braided ring category by Corollary II.9.6.1. Corollary 12.5.3 applies to the latter to yield a K-theory E_2-symmetric spectrum. ◇

Example 12.5.6 (Permbraided Strictification)**.** Suppose C is a small tight braided bimonoidal category (Definition II.2.1.29), such as

- a small abelian category with a compatible braided monoidal structure in Theorem II.2.4.22,
- $\mathcal{F}^{\mathsf{any}}$ of Fibonacci anyons in Theorem II.3.4.13,
- $\mathcal{I}^{\mathsf{any}}$ of Ising anyons in Theorem II.3.6.14, and
- the bimonoidal Drinfeld center of a small tight bimonoidal category in Theorem II.4.4.3.

By Corollary II.9.5.10, the equivalent

- right permbraided category A in Theorem II.6.3.6 and
- left permbraided category A_l in Theorem II.6.3.7

are small tight braided ring categories. Corollary 12.5.3 applies to each of A and A_l to yield a K-theory E_2-symmetric spectrum. ⋄

12.6. Notes

12.6.1 (Compactly Generated Spaces)**.** For more point-set level discussion of compactly generated weak Hausdorff spaces, which are used in Section 12.2 to define the little n-cube operads, the reader may consult [**Bro64, GZ67, Kel55, May72, May99, Ste67**]. In the literature, these spaces are sometimes called *k-spaces* and *Kelley spaces*. ⋄

12.6.2 (Br is an E_2-Operad)**.** Theorem 12.2.4 is [**Fre17**, 5.2.12] with one main difference. In [**Fre17**, Ch. 5], Fresse actually uses the *little 2-disc operad* $\mathcal{D}_2$ and its universal covering $\widetilde{\mathcal{D}}_2$, instead of the little 2-cube operad $\mathcal{C}_2$ and its universal covering $\widetilde{\mathcal{C}}_2$ as in Definition 12.2.3 and the diagram (12.2.5). The little 2-disc operad $\mathcal{D}_2$ has a definition similar to that of the little 2-cube operad, but it uses the closed unit 2-disc instead of the closed unit 2-cube. A little 2-disc is defined by radial contraction and translation. There is a weak equivalence, in the sense of Definition 12.2.3, between $\mathcal{C}_2$ and $\mathcal{D}_2$ via the Steiner operad [**Ber96, Ste79**]. Combined with [**Fre17**, 5.2.12], this yields Theorem 12.2.4.

Alternatively, without using the nontrivial weak equivalence between $\mathcal{C}_2$ and $\mathcal{D}_2$, one can directly adapt the argument in [**Fre17**, 5.1.6] for $\mathcal{C}_2$ and $\widetilde{\mathcal{C}}_2$ to obtain (4) and (5) in the diagram (12.2.5). A detailed description of the B_∞-operad $\widetilde{\mathcal{C}}_2$ is in [**Yau∞**, 5.4], which also contains a detailed description of $\widetilde{\mathcal{D}}_2$. Moreover, [**Yau∞**, Ch. 5] contains a general discussion of braided operads. Covering space theory, which is needed to define $\widetilde{\mathcal{C}}_2$ and $\widetilde{\mathcal{D}}_2$, is discussed in many books on elementary topology, such as [**Hat02, Mun00**]. ⋄

12.6.3 (Braid Operad Coherence)**.** Lemma 12.3.6 and Theorem 12.3.10 for Br are from [**Fre17**, 6.2.4 and 6.2.6] with the following differences.

(1) In [**Fre17**], those theorems have an extra unity axiom for the braiding

$$\xi_{\mathbb{1},1} = 1_1 = \xi_{1,\mathbb{1}} \in \mathsf{P}_1,$$

which is stated as

$$c(e,x_1) = id_{x_1} = c(x_1,e)$$

there. By Proposition II.1.3.21, this extra unity axiom is redundant because it is a formal consequence of the other axioms and data.

(2) In [**Fre17**, 6.2.4(a)], the role of the permutation v is not made explicit as in Lemma 12.3.6. The morphism

$$s_i^{(n)} : \mathrm{id}_n \longrightarrow (i,i+1) \in \mathsf{Br}_n$$

in (12.3.8) has domain the identity permutation id_n. In the categorical decomposition of a general morphism in Br_n, the domain of a typical factor is a general permutation and not id_n. Therefore, a general factor in the categorical decomposition has the form $\big(s_i^{(n)}\big)^{\pm 1} v$ in (12.3.7), with a permutation v.

(3) In [**Fre17**, 6.2.4(a)], the role of the isomorphism

$$s_1^{-1} : \mathrm{id}_2 \longrightarrow (1,2)$$

is not made explicit as in Lemma 12.3.6. While $s_1, s_1^{-1} \in B_2$ both have underlying permutation $(1,2)$, they are different as braids and represent two different morphisms $\mathrm{id}_2 \longrightarrow (1,2)$ in Br_2. So s_1^{-1} must be included in that lemma.

The necessity of the permutation v and the isomorphism s_1^{-1} is illustrated in Example 12.3.9. $\diamond$

12.6.4 (Operad for Braided Monoidal Categories)**.** The braid operad Br is called the *strict* B-*monoidal category operad* and denoted $\mathsf{MCat}^{\mathsf{B}}_{\mathsf{st}}$ in [**Yau**∞, 21.1.15]. A more general analogue is the B-*monoidal category operad* $\mathsf{MCat}^{\mathsf{B}}$, whose algebras are general, instead of strict, small braided monoidal categories [**Yau**∞, 21.1.7]. Further coherence properties of braided monoidal categories, in the more general context of action operads, are in [**Yau**∞, 21.2–21.3]. $\diamond$

CHAPTER 13

K-Theory of E_n-Monoidal Categories

In Corollaries 11.3.16, 11.6.12, and 12.5.3, we saw that the Elmendorf-Mandell K-theory multifunctor (Definition 10.3.32)

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\mathsf{K}^{\mathsf{EM}}} \mathsf{SymSp}$$

sends

- small ring categories to strict ring symmetric spectra,
- small bipermutative categories to E_∞-symmetric spectra, and
- small braided ring categories to E_2-symmetric spectra.

This chapter completes the picture with Corollary 13.5.2. It shows that $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_n-symmetric spectrum for each small E_n-monoidal category C (Definition II.10.7.2) for $n \geq 1$.

Similar to the strict ring, E_∞, and E_2 cases in Chapters 11 and 12, Corollary 13.5.2 is obtained by combining several facts. First, the Elmendorf-Mandell K-theory multifunctor respects

- the categorical enrichment in the multicategory $\mathsf{PermCat}^{\mathsf{su}}$ of small permutative categories and
- the simplicial enrichment in the multicategory SymSp of symmetric spectra.

Therefore, a structure in $\mathsf{PermCat}^{\mathsf{su}}$ that is parametrized by a categorical operad passes along K^{EM} to symmetric spectra. For the general E_n case, the n-fold monoidal category operad Mon^n in Section 13.1 is an E_n-operad (Theorem 13.2.1) that parametrizes

- n-fold monoidal categories in Cat (Proposition 13.3.18) and
- E_n-monoidal categories in $\mathsf{PermCat}^{\mathsf{su}}$ (Theorem 13.4.12).

Both of these statements are consequences of the Coherence Theorem 13.3.3 for Mon^n. Combining Theorem 13.4.12 with K^{EM} yields Corollary 13.5.2 about K-theory E_n-symmetric spectra.

Consider the first two cases of Corollary 13.5.2.

- If $n = 1$, then
  - Mon^1 is the associative operad As (Example 13.1.23), and
  - an E_1-monoidal category is a ring category (Example II.10.7.13).

  So Corollary 13.5.2 with $n = 1$ recovers Corollary 11.3.16.
- If $n = 2$, then Corollary 13.5.2 says that $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_2-symmetric spectrum for each small E_2-monoidal category C. So Corollary 13.5.2 with $n = 2$ is an alternative to Corollary 12.5.3 that also yields K-theory E_2-symmetric spectra. The difference is that Corollary 12.5.3 uses the E_2-operad Br and

braided ring categories, while Corollary 13.5.2 with $n = 2$ uses the E_2-operad Mon^2 and E_2-monoidal categories. By Theorem II.10.8.1, braided ring categories are special cases of E_2-monoidal categories.

The following table summaries the main results in Chapters 11 and 12 and this chapter for the associative operad As, the braid operad Br, the n-fold monoidal category operads Mon^n for $n \geq 1$, and the Barratt-Eccles operad $E\mathsf{As}$. The shorthand *smc* stands for *strict monoidal categories*.

operad	As (11.1.1)	Br (12.1.2)	Mon^n (13.1.12)	$E\mathsf{As}$ (11.4.10)
$E_?$-operad	E_1 (13.1.23, 13.2.1)	E_2 (12.2.4)	E_n (13.2.1)	E_∞ (11.6.3)
coherence	11.1.7	12.3.10	13.3.3	11.4.14
in Cat	smc (11.1.15)	braided smc (12.3.22)	n-fold monoidal (13.3.18)	permutative (11.4.26)
in $\mathsf{PermCat}^{\mathsf{su}}$	ring (11.2.16)	braided ring (12.4.5)	E_n-monoidal (13.4.12)	bipermutative (11.5.5)
in SymSp	strict ring (11.3.16)	E_2 (12.5.3)	E_n (13.5.2)	E_∞ (11.6.12)

For open questions related to E_n-operads, see Appendices A.2 and A.4.

Organization. Section 13.1 defines the n-fold monoidal category operad Mon^n and carefully proves in Proposition 13.1.20 that it is a Cat-enriched operad. The constituent categories $\mathsf{Mon}^n(k)$ are those in Definition II.10.5.13. In Theorem II.10.5.18, we saw that the categories $\mathsf{Mon}^n(k)$ provide a decomposition of the free n-fold monoidal category $\mathsf{FMon}^n(\mathsf{C})$ of a small category C. When $n = 1$, Mon^1 is equal to the Cat-enriched associative operad As.

Section 13.2 shows that the n-fold monoidal category operad Mon^n is an E_n-operad. This result is from [**BFSV03**, 3.14]. The assertion (Theorem 13.2.1) that the Cat-enriched operad Mon^n is an E_n-operad means that its classifying space is connected to the little n-cube operad $\mathcal{C}_n$ by a zigzag of topological operad weak equivalences. Explanation 13.2.12 compares (i) the zigzag in Theorem 12.2.4 that connects the classifying space of the braid operad Br with the little 2-cube operad $\mathcal{C}_2$ and (ii) the zigzag in Theorem 13.2.1.

Section 13.3 proves the Coherence Theorem 13.3.3 for the n-fold monoidal category operad Mon^n that describes Cat-enriched operad morphisms from Mon^n. While the details are different, the proof of Theorem 13.3.3 shares the same broad strategy as those of Theorems 11.4.14 and 12.3.10. In particular, the Coherence Theorem II.10.6.8 for $\mathsf{Mon}^n(k)$ is crucial in the sufficiency part of that proof. As the first application of the Coherence Theorem 13.3.3, we observe in Proposition 13.3.18 that Mon^n is the categorical operad for small n-fold monoidal categories.

Section 13.4 shows that Mon^n detects E_n-monoidal category structures on small permutative categories (Theorem 13.4.12). This is another application of the Coherence Theorem 13.3.3 for Mon^n. Section 13.5 proves the main result about E_n-symmetric spectra using the n-fold monoidal category operad Mon^n. This result, Corollary 13.5.2, says that $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_n-symmetric spectrum for each small E_n-monoidal category C. Proposition 13.5.1 shows that an E_∞-structure induces an E_n-structure on a symmetric spectrum.

Reading Guide.

(1) For the n-fold monoidal category operad Mon^n, read Definition 13.1.12 and the statements of Proposition 13.1.20 and Theorem 13.2.1.
(2) For the coherence and K-theoretic properties of Mon^n, read the statements of Theorems 13.3.3 and 13.4.12, Proposition 13.3.18, and Corollary 13.5.2.

(3) Go back and read the rest of this chapter.

13.1. The Iterated Monoidal Category Operad

In this section, we define the n-fold monoidal category operad Mon^n in Definition 13.1.12. Proposition 13.3.18 and Theorem 13.4.12 will show that the Cat-enriched operad Mon^n is to n-fold monoidal categories and E_n-monoidal categories as

- the associative operad (Definition 11.1.1) is to monoids and ring categories (Proposition 11.1.15 and Theorem 11.2.16);
- the Barratt-Eccles operad (Definition 11.4.10) is to permutative categories and bipermutative categories (Proposition 11.4.26 and Theorem 11.5.5); and
- the braid operad (Definition 12.1.2) is to braided strict monoidal categories and braided ring categories (Proposition 12.3.22 and Theorem 12.4.5).

Moreover, Mon^n is an E_n-operad by Theorem 13.2.1. The fact that Mon^n is actually a Cat-enriched operad is proved in Proposition 13.1.20.

***n*-Fold Monoidal Categories.** We first recall some concepts from Chapter II.10 about n-fold monoidal categories.

Definition 13.1.1. For $n \geq 1$, an *n-fold monoidal category* is a tuple

$$\big(\mathsf{C}, \{\otimes_i\}_{1\leq i\leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1\leq i<j\leq n}\big)$$

consisting of the following data.

The Underlying Category: C is a category.

The Unit: $\mathbb{1} \in \mathsf{C}$ is an object, which is called the *unit*.

The Multiplicative Structures: For each $1 \leq i \leq n$,

$$(\mathsf{C}, \otimes_i, \mathbb{1})$$

is a strict monoidal category, which is called the *ith monoidal structure*, with $\otimes_i$ called the *ith product*.

The Exchanges: For each pair (i, j) with $1 \leq i < j \leq n$,

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D) \tag{13.1.2}$$

is a natural transformation for objects $A, B, C, D \in \mathsf{C}$, which is called the *$(i, j)$-exchange*.

These data are required to satisfy the following equalities and commutative diagrams for objects A, A', A'', B, B', B'', C, C', D, and D' in C. The axioms (13.1.3)–(13.1.6) are defined for $1 \leq i < j \leq n$. The axiom (13.1.7) is defined for $1 \leq i < j < k \leq n$.

The Internal Unity Axiom:

$$\eta^{i,j}_{A,B,\mathbb{1},\mathbb{1}} = 1_{A\otimes_j B} = \eta^{i,j}_{\mathbb{1},\mathbb{1},A,B} \tag{13.1.3}$$

The External Unity Axiom:

$$\eta^{i,j}_{A,\mathbb{1},B,\mathbb{1}} = 1_{A\otimes_i B} = \eta^{i,j}_{\mathbb{1},A,\mathbb{1},B} \tag{13.1.4}$$

The Internal Associativity Axiom:

$$\begin{tikzcd}[column sep=-3em]
& (A \otimes_j A') \otimes_i (B \otimes_j B') \otimes_i (C \otimes_j C') \arrow[dl, "1_{A \otimes_j A'} \otimes_i \eta^{i,j}_{B,B',C,C'}"'] \arrow[dr, "\eta^{i,j}_{A,A',B,B'} \otimes_i 1_{C \otimes_j C'}"] & \\
(A \otimes_j A') \otimes_i [(B \otimes_i C) \otimes_j (B' \otimes_i C')] \arrow[dr, "\eta^{i,j}_{A,A',B \otimes_i C, B' \otimes_i C'}"'] & & [(A \otimes_i B) \otimes_j (A' \otimes_i B')] \otimes_i (C \otimes_j C') \arrow[dl, "\eta^{i,j}_{A \otimes_i B, A' \otimes_i B', C, C'}"] \\
& (A \otimes_i B \otimes_i C) \otimes_j (A' \otimes_i B' \otimes_i C') &
\end{tikzcd} \tag{13.1.5}$$

The External Associativity Axiom:

$$\begin{tikzcd}[column sep=-3em]
& (A \otimes_j A' \otimes_j A'') \otimes_i (B \otimes_j B' \otimes_j B'') \arrow[dl, "\eta^{i,j}_{A, A' \otimes_j A'', B, B' \otimes_j B''}"'] \arrow[dr, "\eta^{i,j}_{A \otimes_j A', A'', B \otimes_j B', B''}"] & \\
(A \otimes_i B) \otimes_j [(A' \otimes_j A'') \otimes_i (B' \otimes_j B'')] \arrow[dr, "1_{A \otimes_i B} \otimes_j \eta^{i,j}_{A',A'',B',B''}"'] & & [(A \otimes_j A') \otimes_i (B \otimes_j B')] \otimes_j (A'' \otimes_i B'') \arrow[dl, "\eta^{i,j}_{A,A',B,B'} \otimes_j 1_{A'' \otimes_i B''}"] \\
& (A \otimes_i B) \otimes_j (A' \otimes_i B') \otimes_j (A'' \otimes_i B'') &
\end{tikzcd} \tag{13.1.6}$$

The Triple Exchange Axiom:

$$\begin{tikzcd}[column sep=-3em]
& [(A \otimes_k A') \otimes_j (B \otimes_k B')] \otimes_i [(C \otimes_k C') \otimes_j (D \otimes_k D')] \arrow[dl, "\eta^{j,k}_{A,A',B,B'} \otimes_i \eta^{j,k}_{C,C',D,D'}"'] \arrow[dr, "\eta^{i,j}_{A \otimes_k A', B \otimes_k B', C \otimes_k C', D \otimes_k D'}"] & \\
[(A \otimes_j B) \otimes_k (A' \otimes_j B')] \otimes_i [(C \otimes_j D) \otimes_k (C' \otimes_j D')] \arrow[d, "\eta^{i,k}_{A \otimes_j B, A' \otimes_j B', C \otimes_j D, C' \otimes_j D'}"'] & & [(A \otimes_k A') \otimes_i (C \otimes_k C')] \otimes_j [(B \otimes_k B') \otimes_i (D \otimes_k D')] \arrow[d, "\eta^{i,k}_{A,A',C,C'} \otimes_j \eta^{i,k}_{B,B',D,D'}"] \\
[(A \otimes_j B) \otimes_i (C \otimes_j D)] \otimes_k [(A' \otimes_j B') \otimes_i (C' \otimes_j D')] \arrow[dr, "\eta^{i,j}_{A,B,C,D} \otimes_k \eta^{i,j}_{A',B',C',D'}"'] & & [(A \otimes_i C) \otimes_k (A' \otimes_i C')] \otimes_j [(B \otimes_i D) \otimes_k (B' \otimes_i D')] \arrow[dl, "\eta^{j,k}_{A \otimes_i C, A' \otimes_i C', B \otimes_i D, B' \otimes_i D'}"] \\
& [(A \otimes_i C) \otimes_j (B \otimes_i D)] \otimes_k [(A' \otimes_i C') \otimes_j (B' \otimes_i D')] &
\end{tikzcd} \tag{13.1.7}$$

This finishes the definition of an n-fold monoidal category. It is *small* if it has a set of objects. $\diamond$

By Propositions II.10.1.14 and II.10.1.21, braided strict monoidal categories are special cases of 2-fold monoidal categories, and permutative categories are special cases of n-fold monoidal categories for $n \geq 2$.

For n-fold monoidal categories C and D, an *n-fold monoidal functor* (Definition II.10.3.1)

$$\left(F, \{F_i^2\}_{i=1}^n\right) : \mathsf{C} \longrightarrow \mathsf{D}$$

consists of

- a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ and
- for each $1 \leq i \leq n$, a natural transformation

$$FA \otimes_i FB \xrightarrow{(F_i^2)_{A,B}} F(A \otimes_i B)$$

  for objects $A, B \in \mathsf{C}$, which is called the *ith monoidal constraint*.

These data are required to satisfy the following two conditions.

Monoidality: For each $1 \le i \le n$,

$$(F, F_i^2) : (\mathsf{C}, \otimes_i, \mathbb{1}) \longrightarrow (\mathsf{D}, \otimes_i, \mathbb{1})$$

is a strictly unital monoidal functor.

The Exchange Constraint Axiom: The following diagram in D is commutative for all $A, B, C, D \in \mathsf{C}$ and $1 \le i < j \le n$.

$$\begin{tikzcd}
& (FA \otimes_j FB) \otimes_i (FC \otimes_j FD) \arrow[dl, "(F_j^2)_{A,B} \otimes_i (F_j^2)_{C,D}"'] \arrow[dr, "\eta^{i,j}_{FA,FB,FC,FD}"] & \\
F(A \otimes_j B) \otimes_i F(C \otimes_j D) \arrow[d, "(F_i^2)_{A \otimes_j B, C \otimes_j D}"'] & & (FA \otimes_i FC) \otimes_j (FB \otimes_i FD) \arrow[d, "(F_i^2)_{A,C} \otimes_j (F_i^2)_{B,D}"] \\
F\big((A \otimes_j B) \otimes_i (C \otimes_j D)\big) \arrow[dr, "F\eta^{i,j}_{A,B,C,D}"'] & & F(A \otimes_i C) \otimes_j F(B \otimes_i D) \arrow[dl, "(F_j^2)_{A \otimes_i C, B \otimes_i D}"] \\
& F\big((A \otimes_i C) \otimes_j (B \otimes_i D)\big) &
\end{tikzcd} \tag{13.1.8}$$

There is a monoidal category $(\mathsf{MCat}^n, \times, \mathbf{1})$ of small n-fold monoidal categories and n-fold monoidal functors by Lemma II.10.4.2. Monoids in MCat^n are precisely small $(n+1)$-fold monoidal categories by Theorem II.10.4.5.

An n-fold monoidal functor $(F, \{F_i^2\}_{i=1}^n)$ is *strict* if each F_i^2 is the identity natural transformation. Equivalently, it is a functor that strictly preserves

- the unit $\mathbb{1}$,
- the monoidal products $\{\otimes_i\}_{i=1}^n$, and
- the exchanges $\{\eta^{i,j}\}_{i<j}$.

The wide subcategory of MCat^n with strict n-fold monoidal functors is denoted by $\mathsf{MCat}^n_{\mathsf{st}}$.

Free n-Fold Monoidal Categories. By Proposition II.10.5.9, there is an adjunction

$$\begin{tikzcd} \mathsf{Cat} \arrow[r, shift left, "\mathsf{FMon}^n"] & \mathsf{MCat}^n_{\mathsf{st}} \arrow[l, shift left, "U"] \end{tikzcd} \tag{13.1.9}$$

with

- U the forgetful functor that forgets about the n-fold monoidal structure and
- FMon^n the free n-fold monoidal category functor.

Suppose C is a small category.

- The objects in $\mathsf{FMon}^n(\mathsf{C})$ are generated by the objects in C under the strictly associative products $\{\otimes_i\}_{i=1}^n$, with a strict two-sided unit $\mathbb{1}$.
- The morphisms in $\mathsf{FMon}^n(\mathsf{C})$ are generated by
  - the morphisms in C,
  - the identity morphisms 1_A, and
  - the exchanges

  $$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D)$$

  for objects $A, B, C, D \in \mathsf{FMon}^n(\mathsf{C})$ and $1 \le i < j \le n$, under the strictly associative products $\{\otimes_i\}_{i=1}^n$ and composites.

These data are subject to the relations that make $\mathsf{FMon}^n(\mathsf{C})$ into an n-fold monoidal category.

Decomposition of Free n-Fold Monoidal Categories. For $k \geq 0$ and $n \geq 1$,

$$\mathsf{Mon}^n(k) \subset \mathsf{FMon}^n\{1,\ldots,k\}$$

is the full subcategory in which each object can be written as an iterated $\{\otimes_i\}_{i=1}^n$-product with each object in the discrete category $\{1,\ldots,k\}$ occurring precisely once (Definition II.10.5.13). Since $\{1,\ldots,k\}$ is a discrete category, the morphisms in $\mathsf{Mon}^n(k)$ are iterated composites of finite $\{\otimes_i\}_{i=1}^n$-products of identity morphisms and the exchanges $\{\eta^{i,j}\}_{i<j}$.

- For example, each of

$$\mathsf{Mon}^n(0) = \{\mathbb{1}\} \quad \text{and} \quad \mathsf{Mon}^n(1) = \{1\}$$

is the terminal category.
- The category $\mathsf{Mon}^n(2)$ has objects

$$1 \otimes_i 2 \quad \text{and} \quad 2 \otimes_i 1 \quad \text{for} \quad 1 \leq i \leq n.$$

Its morphisms are generated under composites by the exchanges

$$\begin{aligned} \tau(1) \otimes_i \tau(2) &\xrightarrow{\eta^{i,j}_{\tau(1),\mathbb{1},\mathbb{1},\tau(2)}} \tau(1) \otimes_j \tau(2) \\ \tau(1) \otimes_i \tau(2) &\xrightarrow{\eta^{i,j}_{\mathbb{1},\tau(1),\tau(2),\mathbb{1}}} \tau(2) \otimes_j \tau(1) \end{aligned}$$

for $1 \leq i < j \leq n$ and $\tau \in \Sigma_2$.

A description of $\mathsf{Mon}^n(3)$ is in Example II.10.5.17.

There is a right Σ_k-action on the category $\mathsf{Mon}^n(k)$ that permutes the elements in $\{1,\ldots,k\}$ and changes the subscripts in the generating morphisms accordingly. By Theorem II.10.5.18, for each small category C, there is a natural isomorphism of categories

$$\coprod_{k\geq 0} \mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k} \xrightarrow[\cong]{\phi_{\mathsf{C}}} \mathsf{FMon}^n(\mathsf{C}) \tag{13.1.10}$$

that extends the isomorphism $\mathsf{Mon}^n(1) \times \mathsf{C} \cong \mathsf{C}$. For a small n-fold monoidal category C, the composite of the isomorphism (13.1.10) and the counit of the adjunction (13.1.9) yields the evaluation functors

$$\mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k} \xrightarrow{\theta_k} \mathsf{C} \tag{13.1.11}$$

for $k \geq 0$.

***n*-Fold Monoidal Category Operad.** Recall from Definition 6.1.1 that a Cat-enriched operad is a Cat-enriched multicategory with one object, and $\mathbf{1}$ is the terminal category. We now assemble the categories $\mathsf{Mon}^n(k)$ into an operad.

Definition 13.1.12. For $n \geq 1$, define the data of a Cat-enriched operad

$$\mathsf{Mon}^n = \left(\{\mathsf{Mon}^n(k)\}_{k\geq 0}, \gamma, 1\right)$$

as follows.

Categories and Equivariance: For $k \geq 0$, $\mathsf{Mon}^n(k)$ is the small category with its right Σ_k-action in Definition II.10.5.13.

Unit: The *unit*

$$\mathbf{1} \longrightarrow \mathsf{Mon}^n(1) = \{1\}$$

is the unique functor.

Operad Composition on Objects: For $k \geq 1$, $j_1, \ldots, j_k \geq 0$, and $j = j_1 + \cdots + j_k$, the functor

$$\mathsf{Mon}^n(k) \times \prod_{i=1}^{k} \mathsf{Mon}^n(j_i) \xrightarrow{\gamma} \mathsf{Mon}^n(j) \tag{13.1.13}$$

is defined on objects by

$$\gamma\big(P, (A_i)_{i=1}^k\big) = P(A_1', \ldots, A_k'). \tag{13.1.14}$$

This object is obtained from

$$P = P(1, \ldots, k) \in \mathsf{Mon}^n(k)$$

by replacing each $i \in \{1, \ldots, k\}$ by

$$A_i' = A_i\big(\overline{j_i} + 1, \ldots, \overline{j_i} + j_i\big) \tag{13.1.15}$$

with

$$\overline{j_i} = \begin{cases} 0 & \text{if } i = 1 \\ j_1 + \cdots + j_{i-1} & \text{if } 1 < i \leq k. \end{cases}$$

So A_i' is obtained from

$$A_i = A_i(1, \ldots, j_i) \in \mathsf{Mon}^n(j_i)$$

by replacing each $r \in \{1, \ldots, j_i\}$ by $\overline{j_i} + r$.

Operad Composition on Morphisms: For morphisms

$$\begin{aligned} P &\xrightarrow{f} Q \in \mathsf{Mon}^n(k) \\ A_i &\xrightarrow{g_i} B_i \in \mathsf{Mon}^n(j_i) \end{aligned} \tag{13.1.16}$$

for $1 \leq i \leq k$, the morphism

$$\gamma\big(f, \{g_i\}_{i=1}^k\big) : P(A_1', \ldots, A_k') \longrightarrow Q(B_1', \ldots, B_k')$$

is defined as the composite

$$\begin{tikzcd} P(A_1', \ldots, A_k') \arrow[r, "{f(A_1', \ldots, A_k')}"] & Q(A_1', \ldots, A_k') \arrow[d, "{Q(g_1', \ldots, g_k')}"'] \\ & Q(B_1', \ldots, B_k') \end{tikzcd} \tag{13.1.17}$$

in $\mathsf{Mon}^n(j)$.

- The morphism $f(A_1', \ldots, A_k')$ is obtained from $f = f(1, \ldots, k)$ by replacing each $i \in \{1, \ldots, k\}$ by A_i' in (13.1.15).
- For each $i \in \{1, \ldots, k\}$,

$$g_i' = g_i\big(\overline{j_i} + 1, \ldots, \overline{j_i} + j_i\big) \tag{13.1.18}$$

is obtained from $g_i = g_i(1, \ldots, j_i)$ by replacing each $r \in \{1, \ldots, j_i\}$ by $\overline{j_i} + r$.

- The morphism $Q(g_1',\ldots,g_k')$ is obtained from $Q = Q(1,\ldots,k)$ by replacing each $i \in \{1,\ldots,k\}$ by g_i'.

This finishes the definition of Mon^n. $\diamond$

Example 13.1.19. An example of the composition

$$\mathsf{Mon}^n(2) \times \mathsf{Mon}^n(2) \times \mathsf{Mon}^n(3) \xrightarrow{\gamma} \mathsf{Mon}^n(5)$$

on objects (13.1.14) is

$$\begin{aligned}&\gamma\big(2 \otimes_i 1, \big(1 \otimes_j 2, (2 \otimes_k 3) \otimes_l 1\big)\big)\\ &= \big((4 \otimes_k 5) \otimes_l 3\big) \otimes_i (1 \otimes_j 2) \in \mathsf{Mon}^n(5)\end{aligned}$$

for $i,j,k,l \in \{1,\ldots,n\}$. Consider morphisms

$$\begin{aligned}P = 2 \otimes_i 1 &\xrightarrow{f = \eta^{i,i'}_{2,\mathbb{1},\mathbb{1},1}} 2 \otimes_{i'} 1 = Q \in \mathsf{Mon}^n(2)\\ A_1 = 1 \otimes_j 2 &\xrightarrow{g_1 = \eta^{j,j'}_{\mathbb{1},1,2,\mathbb{1}}} 2 \otimes_{j'} 1 = B_1 \in \mathsf{Mon}^n(2)\\ A_2 = (2 \otimes_k 3) \otimes_l 1 &\xrightarrow{g_2 = \eta^{l,l'}_{(2\otimes_k 3),\mathbb{1},\mathbb{1},1}} (2 \otimes_k 3) \otimes_{l'} 1 = B_2 \in \mathsf{Mon}^n(3)\end{aligned}$$

with $i < i'$, $j < j'$, and $l < l'$ in $\{1,\ldots,n\}$. The morphism $\gamma\big(f,(g_1,g_2)\big)$ in (13.1.17) is the composite

$$\begin{array}{c} P(A_1',A_2') = \big((4 \otimes_k 5) \otimes_l 3\big) \otimes_i (1 \otimes_j 2)\\ {\scriptstyle f(A_1',A_2') =}\Big\downarrow{\scriptstyle \eta^{i,i'}_{((4\otimes_k 5)\otimes_l 3),\mathbb{1},\mathbb{1},(1\otimes_j 2)}}\\ Q(A_1',A_2') = \big((4 \otimes_k 5) \otimes_l 3\big) \otimes_{i'} (1 \otimes_j 2)\\ {\scriptstyle Q(g_1',g_2') =}\Big\downarrow{\scriptstyle \eta^{l,l'}_{(4\otimes_k 5),\mathbb{1},\mathbb{1},3} \otimes_{i'} \eta^{j,j'}_{\mathbb{1},1,2,\mathbb{1}}}\\ Q(B_1',B_2') = \big((4 \otimes_k 5) \otimes_{l'} 3\big) \otimes_{i'} (2 \otimes_{j'} 1)\end{array}$$

in $\mathsf{Mon}^n(5)$. $\diamond$

Proposition 13.1.20. Mon^n *in Definition 13.1.12 is a* Cat*-enriched operad.*

Proof. We must check that

- the composition γ in (13.1.13) is well defined, and
- the operad axioms in Definition 6.1.1 are satisfied.

Functoriality of γ. To see that γ is a functor, first observe that, if f and each g_i are identity morphisms, then $\gamma\big(f,\{g_i\}_{i=1}^k\big)$ in (13.1.17) is the composite of two identity morphisms.

To see that γ preserves composites, consider morphisms

$$\begin{aligned}P &\xrightarrow{f} Q \xrightarrow{\overline{f}} R \in \mathsf{Mon}^n(k)\\ A_i &\xrightarrow{g_i} B_i \xrightarrow{\overline{g}_i} C_i \in \mathsf{Mon}^n(j_i)\end{aligned}$$

for $1 \leq i \leq k$ and the diagram

$$\begin{tikzcd}
P(A_1',\ldots,A_k') \arrow[r, "f(A_1',\ldots,A_k')"] & Q(A_1',\ldots,A_k') \arrow[r, "\overline{f}(A_1',\ldots,A_k')"] \arrow[d, "Q(g_1',\ldots,g_k')"'] & R(A_1',\ldots,A_k') \arrow[d, "R(g_1',\ldots,g_k')"'] \\
& Q(B_1',\ldots,B_k') \arrow[r, "\overline{f}(B_1',\ldots,B_k')"] & R(B_1',\ldots,B_k') \arrow[d, "R(\overline{g}_1',\ldots,\overline{g}_k')"'] \\
& & R(C_1',\ldots,C_k')
\end{tikzcd} \tag{13.1.21}$$

in $\mathsf{Mon}^n(j)$, with $j = j_1 + \cdots + j_k$.

- The square in (13.1.21) is commutative by repeated applications of the axiom (II.10.5.4), which is the naturality of the exchanges $\eta^{i,j}$.
- By the functoriality of each $\otimes_i$ in $\mathsf{Mon}^n(j)$, there is an equality

  $$R(\overline{g}_1'g_1',\ldots,\overline{g}_k'g_k') = R(\overline{g}_1',\ldots,\overline{g}_k') \circ R(g_1',\ldots,g_k').$$

  So the top-right composite in (13.1.21) is the morphism

  $$\gamma\big(\overline{f}f;\{\overline{g}_ig_i\}_{i=1}^k\big).$$

- The other composite in (13.1.21) is the morphism

  $$\gamma\big(\overline{f};\{\overline{g}_i\}_{i=1}^k\big) \circ \gamma\big(f;\{g_i\}_{i=1}^k\big).$$

This shows that γ preserves composites and is a functor.

Equivariance. The equivariance axioms (6.1.6) and (6.1.7) hold because the right Σ_k-action on $\mathsf{Mon}^n(k)$ permutes the elements in $\{1,\ldots,k\}$ from the right.

Unity. The unity axioms (6.1.4) and (6.1.5) follow from the definitions of

- the unit

  $$\mathbf{1} \xrightarrow{\cong} \mathsf{Mon}^n(1) = \{1\}$$

  and
- γ on objects (13.1.14) and morphisms (13.1.17).

Associativity. For the associativity axiom (6.1.3), consider objects

- $P = P(1,\ldots,k) \in \mathsf{Mon}^n(k)$,
- $A_i = A_i(1,\ldots,j_i) \in \mathsf{Mon}^n(j_i)$ for $1 \leq i \leq k$, and
- $C_{i,l} = C_{i,l}(1,\ldots,m_{i,l}) \in \mathsf{Mon}^n(m_{i,l})$ for $1 \leq i \leq k$ and $1 \leq l \leq j_i$,

and the following sums, with the convention that an empty sum is 0.

$$\overline{m}_{i,l} = \sum_{s=1}^{l-1} m_{i,s} \qquad \overline{m}_i = \sum_{r=1}^{i-1}\sum_{l=1}^{j_r} m_{r,l} \qquad M = \sum_{i=1}^{k}\sum_{l=1}^{j_i} m_{i,l}$$

Using the prime notation in (13.1.15) to denote an expression whose labels are shifted up appropriately, the following equalities of objects hold in $\mathsf{Mon}^n(M)$.

$$\begin{aligned}
&\gamma\Big(\gamma\big(P,(A_i)_{i=1}^k\big),\big((C_{i,l})_{l=1}^{j_i}\big)_{i=1}^k\Big)\\
&= \big(P(A_i')_{i=1}^k\big)\big((C_{i,l}')_{l=1}^{j_i}\big)_{i=1}^k\\
&= P\Big(A_i\big(C_{i,l}(\overline{m}_i+\overline{m}_{i,l}+p)_{p=1}^{m_{i,l}}\big)_{l=1}^{j_i}\Big)_{i=1}^k\\
&= P\Big(\big(A_i(C_{i,l}')_{l=1}^{j_i}\big)'\Big)_{i=1}^k\\
&= \gamma\Big(P,\big(\gamma\big(A_i,(C_{i,l})_{l=1}^{j_i}\big)\big)_{i=1}^k\Big)
\end{aligned}$$

This proves the associativity axiom (6.1.3) on objects.

For morphisms, in addition to the morphisms f and $\{g_i\}_{i=1}^k$ in (13.1.16), suppose given morphisms

$$C_{i,l} \xrightarrow{h_{i,l}} D_{i,l} \in \mathsf{Mon}^n(m_{i,l})$$

for $1 \le i \le k$ and $1 \le l \le j_i$. With the prime notation in (13.1.15) and (13.1.18), when applied to the morphisms

$$\Big(f,(g_i)_{i=1}^k,\big((h_{i,l})_{l=1}^{j_i}\big)_{i=1}^k\Big),$$

each composite in the associativity diagram (6.1.3) gives the following composite in $\mathsf{Mon}^n(M)$.

$$\begin{tikzcd}
\big(P(A_i')_{i=1}^k\big)\big((C_{i,l}')_{l=1}^{j_i}\big)_{i=1}^k = P\Big(\big(A_i(C_{i,l}')_{l=1}^{j_i}\big)'\Big)_{i=1}^k \arrow[d, "f\left(\left(A_i(C_{i,l}')_{l=1}^{j_i}\right)'\right)_{i=1}^k"] \\
\big(Q(A_i')_{i=1}^k\big)\big((C_{i,l}')_{l=1}^{j_i}\big)_{i=1}^k = Q\Big(\big(A_i(C_{i,l}')_{l=1}^{j_i}\big)'\Big)_{i=1}^k \arrow[d, "Q\left(\left(g_i(C_{i,l}')_{l=1}^{j_i}\right)'\right)_{i=1}^k"] \\
\big(Q(B_i')_{i=1}^k\big)\big((C_{i,l}')_{l=1}^{j_i}\big)_{i=1}^k = Q\Big(\big(B_i(C_{i,l}')_{l=1}^{j_i}\big)'\Big)_{i=1}^k \arrow[d, "Q\left(\left(B_i(h_{i,l}')_{l=1}^{j_i}\right)'\right)_{i=1}^k"] \\
\big(Q(B_i')_{i=1}^k\big)\big((D_{i,l}')_{l=1}^{j_i}\big)_{i=1}^k = Q\Big(\big(B_i(D_{i,l}')_{l=1}^{j_i}\big)'\Big)_{i=1}^k
\end{tikzcd}$$

This finishes the proof that Mon^n is a Cat-enriched operad. $\square$

Definition 13.1.22. For $n \ge 1$, the Cat-enriched operad Mon^n in Proposition 13.1.20 is called the *n-fold monoidal category operad*. $\diamond$

Example 13.1.23. Mon^1 is canonically isomorphic to the associative operad As in Definition 11.1.1, which is also regarded as a Cat-enriched operad as in Example 11.2.2. Indeed, by definition, each object in $\mathsf{Mon}^1(k)$ can be written as an iterated $\otimes_1$-product with each object in $\{1,\ldots,k\}$ occurring precisely once. Since $\otimes_1$ is strictly associative, the objects in $\mathsf{Mon}^1(k)$ canonically correspond to the permutations in Σ_k, with

$$\Sigma_k \ni \sigma \quad \text{corresponding to} \quad \sigma(1) \otimes_1 \cdots \otimes_1 \sigma(k) \in \mathsf{Mon}^1(k).$$

There are no generating exchanges $\eta^{i,j}$ when $n = 1$, so $\mathsf{Mon}^1(k)$ is the discrete category of Σ_k. Moreover, the right symmetric group action, the unit, and the operad composition (13.1.14) in Mon^1 correspond to those in As. $\diamond$

Explanations 11.6.4 and 12.1.12 describe the simplicial Barratt-Eccles operad $N(E\mathsf{As})$ and the simplicial braid operad $N\mathsf{Br}$, with $N : \mathsf{Cat} \longrightarrow \mathsf{sSet}$ the nerve functor (Definition 7.2.3). The simplicial operad $N\mathsf{Mon}^n$ will play an important role in Theorem 13.2.1, Proposition 13.5.1, and Corollary 13.5.2. We end this section with an explicit description of it.

Explanation 13.1.24 (Simplicial n-Fold Monoidal Category Operad)**.** For $n \geq 1$ and $k, q \geq 0$, a q-simplex in the nerve $N\mathsf{Mon}^n(k)$ is a $(q+1)$-tuple of objects

$$(P_0, \ldots, P_q) \in \big(\mathsf{Mon}^n(k)\big)^{\times(q+1)}$$

such that

$$\mathsf{Mon}^n(k)(P_j, P_{j+1}) \neq \varnothing \quad \text{for} \quad 0 \leq j < q. \tag{13.1.25}$$

By the Coherence Theorem II.10.6.8 for n-fold monoidal categories, each morphism set in $\mathsf{Mon}^n(k)$ is either empty or has only one element. So the condition (13.1.25) specifies a unique q-simplex $(P_j)_{j=0}^q$ in $N\mathsf{Mon}^n(k)$. The face and degeneracy maps are given by

$$\begin{aligned} d_i(P_0, \ldots, P_q) &= (P_0, \ldots, \widehat{P_i}, \ldots, P_q)\\ s_i(P_0, \ldots, P_q) &= (P_0, \ldots, P_i, P_i, \ldots, P_q). \end{aligned}$$

In other words, the ith face map d_i removes P_i, and the ith degeneracy map s_i repeats P_i.

The symmetric group Σ_k acts diagonally on the q-simplices in $N\mathsf{Mon}^n(k)$, that is,

$$(P_0, \ldots, P_q) \cdot \pi = (P_0\pi, \ldots, P_q\pi) \quad \text{for} \quad \pi \in \Sigma_k.$$

Each $P_j\pi$ is obtained from $P_j \in \mathsf{Mon}^n(k)$ by permuting the elements in $\{1, \ldots, k\}$ that appear in P_j via π. The unit in $N\mathsf{Mon}^n$ is

$$1 \in \big(N\mathsf{Mon}^n(1)\big)_0 = \mathsf{Mon}^n(1) = \{1\}.$$

The operad composition

$$N\mathsf{Mon}^n(k) \times \prod_{\ell=1}^{k} N\mathsf{Mon}^n(m_\ell) \xrightarrow{\gamma} N\mathsf{Mon}^n(m_1 + \cdots + m_k)$$

is given on q-simplices by

$$\gamma\Big((P_j)_{j=0}^q; \big\{(P_j^\ell)_{j=0}^q\big\}_{\ell=1}^k\Big) = \Big\{\gamma\big(P_j, (P_j^1, \ldots, P_j^k)\big)\Big\}_{j=0}^q.$$

The γ on the right-hand side is the one in (13.1.14). $\diamond$

13.2. The Iterated Monoidal Category Operad is an E_n-Operad

Recall from Definition 12.2.3 that a Top-enriched operad is an E_n-*operad* if it is weakly equivalent to the little n-cube operad $\mathcal{C}_n$. A Cat-enriched operad is an E_n-operad if its image under the classifying space functor $|N(-)|$ in Example 7.2.8 is an E_n-operad. Theorem 12.2.4 says that the braid operad Br is an E_2-operad. The following analogue for Mon^n in Definition 13.1.22 is the main theorem in [**BFSV03**, 3.14].

Theorem 13.2.1. *For $n \geq 1$, the n-fold monoidal category operad* Mon^n *is an E_n-operad.*

Proof. We refer the reader to [**BFSV03**, 3.14] for the full proof. Here we provide an overview of the proof. The desired weak equivalence between the classifying space of Mon^n and the little n-cube operad $\mathcal{C}_n$ is given by the following zigzag of Top-enriched operad isomorphisms $\cong$ and weak equivalences $\sim$.

(13.2.2)
$$\begin{tikzcd}
\mathcal{C}_n & & |N\mathsf{Mon}^n| \\
\mathsf{D}_n \arrow[u, "(1)", "\sim"'] & & \Big\{\operatorname*{hocolim}_{\mathsf{Mon}^n(k)} *\Big\}_{k\geq 0} \arrow[u, "\sim", "(5)"'] \\
\Big\{\operatorname*{colim}_{\mathsf{Mon}^n(k)} F_k\Big\}_{k\geq 0} \arrow[u, "(2)", "\cong"'] & & \Big\{\operatorname*{hocolim}_{\mathsf{Mon}^n(k)} F_k\Big\}_{k\geq 0} \arrow[ll, "(3)", "\sim"'] \arrow[u, "\sim", "(4)"']
\end{tikzcd}$$

This diagram involves the following concepts and constructions.

The operad D_n. In the little n-cube operad $\mathcal{C}_n$, define *decomposable* elements inductively as follows.

- Each element in $\mathcal{C}_n(0)$ and $\mathcal{C}_n(1)$ is decomposable.
- Inductively, for $k \geq 2$, an element $f = (f_1, \ldots, f_k)$ in $\mathcal{C}_n(k)$ is decomposable if there exists a hyperplane $H \subset \mathbb{R}^n$ that satisfies the following conditions:
  - H is perpendicular to one of the n coordinate axes.
  - H does *not* intersect the interior of the little n-cubes f_i for $1 \leq i \leq k$.
  - H partitions f into two nonempty disjoint subsets
$$S \coprod \big(\{f_1, \ldots, f_k\} \smallsetminus S\big)$$
    such that the following two conditions hold, where $|S|$ is the cardinality of S:
    * $S \in \mathcal{C}_n(|S|)$ lies in the negative side of H and is decomposable.
    * The subset
$$\{f_1, \ldots, f_k\} \smallsetminus S \in \mathcal{C}_n(k - |S|)$$
      lies in the positive side of H and is decomposable.

For example, $\mathcal{C}_n(2)$ and $\mathcal{C}_2(3)$ contain only decomposable elements. For $k \geq 0$, define

(13.2.3)
$$\mathsf{D}_n(k) \subset \mathcal{C}_n(k)$$

as the subspace of decomposable elements. These subspaces form a topological suboperad

$$\mathsf{D}_n = \big\{\mathsf{D}_n(k)\big\}_{k\geq 0} \hookrightarrow \mathcal{C}_n \tag{13.2.4}$$

of the little n-cube operad. A general element $(f_1,\ldots,f_k)$ in $\mathcal{C}_n(k)$ can be shrunk toward the center of each constituent little n-cube f_i to yield a decomposable element. This gives the operad weak equivalence (1) in the diagram (13.2.2).

Separable little cubes. The functor F_k in (13.2.2) requires the following concepts from Definition II.10.6.1. For distinct elements $a \neq b \in \{1,\ldots,k\}$, the restriction functor

$$\mathsf{Mon}^n(k) \longrightarrow \mathsf{Mon}^n\big(\{a,b\}\big) \tag{13.2.5}$$

is the identity on $\{a,b\}$ and sends each $i \in \{1,\ldots,k\} \smallsetminus \{a,b\}$ to $\mathbb{1}$. For an object $A \in \mathsf{Mon}^n(k)$, we write

$$a \otimes_i b \in A$$

if the restriction functor in (13.2.5) sends A to $a \otimes_i b$.

Using the product notation in (12.2.1), suppose

$$f_- = [a_1,b_1]\times\cdots\times[a_n,b_n]$$
$$f_+ = [c_1,d_1]\times\cdots\times[c_n,d_n]$$

are two little n-cubes. For $1 \leq i \leq n$, we write

$$f_- <_i f_+ \quad\text{if}\quad b_i \leq c_i.$$

Geometrically, $f_- <_i f_+$ means that there exists a hyperplane H with the following three properties:

- H is perpendicular to the ith coordinate axis.
- The interior of f_- lies in the negative side of H.
- The interior of f_+ lies in the positive side of H.

For an object $A \in \mathsf{Mon}^n(k)$, an element $(f_1,\ldots,f_k) \in \mathcal{C}_n(k)$ is *A-separable* if

$$a \otimes_i b \in A \quad\text{implies}\quad f_a <_i f_b \quad\text{for}\quad a \neq b \in \{1,\ldots,k\}.$$

Define the contractible subspace

$$G(A) \subset \mathcal{C}_n(k)$$

of A-separable elements. There is an equality

$$\bigcup_{A\in\mathsf{Ob}(\mathsf{Mon}^n(k))} G(A) = \mathsf{D}_n(k) \tag{13.2.6}$$

of subspaces of $\mathcal{C}_n(k)$, where $\mathsf{D}_n(k)$ is the subspace of decomposable elements in (13.2.3).

The functor F_k. For $k \geq 0$, the functor

$$F_k : \mathsf{Mon}^n(k) \longrightarrow \mathsf{Top} \tag{13.2.7}$$

is defined by the subspace

$$F_k(A) = \bigcup_{\mathsf{Mon}^n(k)(Y,A)\neq\varnothing} G(Y) \;\subset\; \mathsf{D}_n(k) \tag{13.2.8}$$

for each object $A \in \mathsf{Mon}^n(k)$. This union is indexed by the set of objects $Y \in \mathsf{Mon}^n(k)$ such that the morphism set $\mathsf{Mon}^n(k)(Y,A)$ is nonempty. By the Coherence Theorem II.10.6.8 for $\mathsf{Mon}^n(k)$, each nonempty morphism set in $\mathsf{Mon}^n(k)$ has a unique element. Therefore, each morphism $B \longrightarrow A$ in $\mathsf{Mon}^n(k)$ has an associated subspace inclusion

$$F_k(B) \hookrightarrow F_k(A).$$

This defines F_k on morphisms in $\mathsf{Mon}^n(k)$.

For each object $A \in \mathsf{Mon}^n(k)$, the identity morphism of A gives the subspace inclusion

$$G(A) \hookrightarrow F_k(A).$$

This is actually a strong deformation retract. Therefore, since $G(A)$ is contractible, so is $F_k(A)$. Moreover, by (13.2.6), the subspace inclusions (13.2.8) induce an isomorphism

$$\underset{\mathsf{Mon}^n(k)}{\mathsf{colim}}\, F_k \xrightarrow{\cong} \mathsf{D}_n(k)$$

for $k \geq 0$. These levelwise isomorphisms form the operad isomorphism (2) in the diagram (13.2.2).

Homotopy colimits. Each functor $F : \mathsf{C} \longrightarrow \mathsf{Top}$ with C a small category has a colimit, since Top has all coproducts and coequalizers. The *homotopy colimit* of F, denoted by $\mathsf{hocolim}_{\mathsf{C}}\, F$, is a modified construction that takes advantage of the notion of homotopy in Top by incorporating cylinders in the identification process. For example, the *homotopy pushout* of a pair of maps

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

is the pushout

$$X \cup_f (A \wedge I_+) \cup_g Y$$

with I the unit interval. This pushout identifies, for each $a \in A$,

- $f(a) \in X$ with $(a,0) \in A \wedge I_+$ and
- $g(a) \in Y$ with $(a,1) \in A \wedge I_+$.

We refer the reader to [**Dug**∞, Part 1] and [**MP12**, 2.1] for an introduction to homotopy colimits. By collapsing the cylinders, there is a canonical map

$$\underset{\mathsf{C}}{\mathsf{hocolim}}\, F \xrightarrow{\alpha_F} \underset{\mathsf{C}}{\mathsf{colim}}\, F \tag{13.2.9}$$

from the homotopy colimit to the colimit. In general, α_F is not a weak homotopy equivalence. However, there are situations, including [**BFSV03**, 6.7–6.9], where α_F is a weak homotopy equivalence. The operad weak equivalence (3) in the diagram (13.2.2) is levelwise the canonical map α_{F_k} in (13.2.9) for the functor F_k in (13.2.7) for $k \geq 0$.

The constant functor. For $k \geq 0$,

$$* : \mathsf{Mon}^n(k) \longrightarrow \mathsf{Top}$$

is the constant functor at the one-point space. Since F_k is objectwise contractible, the natural transformation $F_k \longrightarrow *$ induces a weak homotopy equivalence

$$\underset{\mathsf{Mon}^n(k)}{\mathsf{hocolim}}\, F_k \xrightarrow{\sim} \underset{\mathsf{Mon}^n(k)}{\mathsf{hocolim}}\, * \tag{13.2.10}$$

by the homotopy invariance of homotopy colimits [**Dug**∞, 4.7]. These maps for $k \geq 0$ assemble to form the operad weak equivalence (4) in the diagram (13.2.2).

Homotopy colimits and classifying spaces. The operad weak equivalence (5) in the diagram (13.2.2) follows from the general fact [**Dug**∞, 4.5] that a homotopy colimit has the same weak homotopy type as the geometric realization of its simplicial replacement. For the constant functor $*$, the simplicial replacement is the nerve of the domain category $\mathsf{Mon}^n(k)$. □

Corollary 13.2.11. *The classifying spaces of*

- *the braid operad* Br *and*
- *the 2-fold monoidal category operad* Mon^2

are weakly equivalent Top*-enriched operads.*

Proof. By Theorems 12.2.4 and 13.2.1, both Br and Mon^2 are E_2-operads, so their classifying spaces are weakly equivalent to the little 2-cube operad $\mathcal{C}_2$. In more detail, by the diagrams (12.2.5) and (13.2.2) with $n = 2$, there is a zigzag of Top-enriched operad weak equivalences

$$\begin{tikzcd}
& \mathsf{Sy}(\mathsf{Q}) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \left\{\operatorname*{hocolim}_{\mathsf{Mon}^2(k)} F_k\right\}_{k\geq 0} \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
|N\mathsf{Br}| & & \mathcal{C}_2 & & |N\mathsf{Mon}^2|
\end{tikzcd}$$

that connects the classifying spaces of Br and Mon^2. □

Explanation 13.2.12. Let us compare

- the diagram (12.2.5) that connects $\mathcal{C}_2$ with $|N\mathsf{Br}|$ and
- the diagram (13.2.2) that connects $\mathcal{C}_n$ with $|N\mathsf{Mon}^n|$.

For the braid operad Br, the key concept in the diagram (12.2.5) is that of a B_∞-operad, which is a topological braided operad that is levelwise contractible and has a free braid group action. Both

- $|NE\mathsf{B}|$, which is the levelwise classifying space of the translation category of the braid group operad B in (12.2.6), and
- $\widetilde{\mathcal{C}}_2$, which is the levelwise universal covering of $\mathcal{C}_2$,

are B_∞-operads. Two B_∞-operads can always be connected by a zigzag of braided operad weak equivalences as in (12.2.8). This connection between B_∞-operads yields the key zigzag

$$\mathsf{Sy}(\widetilde{\mathcal{C}}_2) \xleftarrow{\ \sim\ } \mathsf{Sy}(\mathsf{Q}) \xrightarrow{\ \sim\ } \mathsf{Sy}(|NE\mathsf{B}|)$$

of weak equivalences (4) and (5) in the diagram (12.2.5).

For the n-fold monoidal category operad Mon^n, the key concepts in the diagram (13.2.2) are

- the functors
$$F_k : \mathsf{Mon}^n(k) \longrightarrow \mathsf{Top}$$
in (13.2.7) and
- the homotopy colimit of a functor
$$F : \mathsf{C} \longrightarrow \mathsf{Top}$$
with C a small category.

Homotopy colimit maps to colimit via the canonical map (13.2.9) and satisfies homotopy invariance, which yields the weak equivalence (13.2.10). These properties of homotopy colimits yield the key zigzag

$$\Big\{\operatorname*{colim}_{\mathsf{Mon}^n(k)} F_k\Big\}_{k\geq 0} \xleftarrow{\sim} \Big\{\operatorname*{hocolim}_{\mathsf{Mon}^n(k)} F_k\Big\}_{k\geq 0} \xrightarrow{\sim} \Big\{\operatorname*{hocolim}_{\mathsf{Mon}^n(k)} *\Big\}_{k\geq 0}$$

of weak equivalences (3) and (4) in the diagram (13.2.2). ◇

13.3. Coherence of the Iterated Monoidal Category Operad

Recall the coherence results for

- the associative operad As (Definition 11.1.1) in Theorem 11.1.7,
- the Barratt-Eccles operad $E\mathsf{As}$ (Definition 11.4.10) in Theorem 11.4.14, and
- the braid operad Br (Definition 12.1.2) in Theorem 12.3.10.

Each of these coherence results describes a small list of generators and relations for an operad morphism from the operad in question. The main result in this section, Theorem 13.3.3, is the analogue for the n-fold monoidal category operad Mon^n in Definition 13.1.22. It involves

- the objects

$$\begin{gathered} \mathbb{1} \in \mathsf{Mon}^n(0) = \{\mathbb{1}\} \\ 1 \otimes_i 2 \in \mathsf{Mon}^n(2) \end{gathered} \tag{13.3.1}$$

  for $1 \leq i \leq n$ and
- the generating exchange morphisms

$$(1 \otimes_j 2) \otimes_i (3 \otimes_j 4) \xrightarrow{\eta^{i,j}_{1,2,3,4}} (1 \otimes_i 3) \otimes_j (2 \otimes_i 4) \tag{13.3.2}$$

  for $1 \leq i < j \leq n$ in $\mathsf{Mon}^n(4)$.

We will use the juxtaposition notation

$$\gamma\big(y, (x_1, \ldots, x_n)\big) = y(x_1, \ldots, x_n)$$

in Definition 6.1.1 for operad composition. Recall that (i,j) denotes the transposition that swaps i and j. An *enriched operad morphism* is an enriched multifunctor as in Definition 5.1.12 between two enriched multicategories with one object.

Theorem 13.3.3. *Suppose $n \geq 1$ and $(\mathsf{P}, \gamma, 1)$ is a Cat-enriched operad. Then a Cat-enriched operad morphism*

$$f : \mathsf{Mon}^n \longrightarrow \mathsf{P}$$

is uniquely determined by

- *the objects*

$$\begin{gathered} \mathbb{1} = f(\mathbb{1}) \in \mathsf{P}_0 \\ \otimes_i = f(1 \otimes_i 2) \in \mathsf{P}_2 \end{gathered} \tag{13.3.4}$$

  for $1 \leq i \leq n$ and
- *the morphisms*

$$\otimes_i(\otimes_j, \otimes_j) \xrightarrow{\eta^{i,j} = f(\eta^{i,j}_{1,2,3,4})} \otimes_j(\otimes_i, \otimes_i)(2,3) \in \mathsf{P}_4 \tag{13.3.5}$$

  for $1 \leq i < j \leq n$.

The above data are subject to the following conditions.

Unity and Associativity: *The equalities of objects*

$$\begin{aligned} \otimes_i(\mathbb{1},1) = 1 = \otimes_i(1,\mathbb{1}) \in \mathsf{P}_1 \\ \otimes_i(\otimes_i,1) = \otimes_i(1,\otimes_i) \in \mathsf{P}_3 \end{aligned} \tag{13.3.6}$$

hold for $1 \le i \le n$.

n-Fold Monoidal Category Axioms: *The axioms (13.1.3)–(13.1.7) hold in P for the objects and morphisms in (13.3.4)–(13.3.5).*

Proof. As in the proofs of Theorems 11.4.14 and 12.3.10, there are two directions.

Necessity. Suppose $f : \mathsf{Mon}^n \longrightarrow \mathsf{P}$ is a Cat-enriched operad morphism. We define

$$\mathbb{1} \in \mathsf{P}_0, \quad \otimes_i \in \mathsf{P}_2, \quad \text{and} \quad \eta^{i,j} \in \mathsf{P}_4$$

as in (13.3.4)–(13.3.5). By definition, in Mon^n the following relations hold:

- Each $\otimes_i$ is strictly associative with $\mathbb{1}$ as the strict two-sided unit, in the sense of the following equalities.

$$\begin{aligned} \gamma\big(1 \otimes_i 2, (\mathbb{1},1)\big) &= 1 = \gamma\big(1 \otimes_i 2, (1,\mathbb{1})\big) \in \mathsf{Mon}^n(1) \\ \gamma\big(1 \otimes_i 2, (1 \otimes_i 2, 1)\big) &= 1 \otimes_i 2 \otimes_i 3 \\ &= \gamma\big(1 \otimes_i 2, (1, 1 \otimes_i 2)\big) \in \mathsf{Mon}^n(3) \end{aligned}$$

- The n-fold monoidal category axioms (13.1.3)–(13.1.7) hold.

Since f is a Cat-enriched operad morphism, the corresponding axioms, which are (13.3.6) and (13.1.3)–(13.1.7) for the objects and morphisms in (13.3.4)–(13.3.5), hold in P.

Sufficiency. Suppose given the data

- $\mathbb{1} = f(\mathbb{1}) \in \mathsf{P}_0$,
- $\otimes_i = f(1 \otimes_i 2) \in \mathsf{P}_2$ as in (13.3.4), and
- $\eta^{i,j} = f(\eta^{i,j}_{1,2,3,4}) : \otimes_i(\otimes_j,\otimes_j) \longrightarrow \otimes_j(\otimes_i,\otimes_i)(2,3) \in \mathsf{P}_4$ as in (13.3.5)

such that (13.3.6) and the n-fold monoidal category axioms (13.1.3)–(13.1.7) hold in P. We show that these data extend uniquely to a Cat-enriched operad morphism $f : \mathsf{Mon}^n \longrightarrow \mathsf{P}$ in several stages.

Objects. Since f must preserve the operad unit and the symmetric group action, we must define

$$\begin{aligned} f(1) = 1 \in \mathsf{P}_1 \\ f(2 \otimes_i 1) = \otimes_i(1,2) \in \mathsf{P}_2 \end{aligned} \tag{13.3.7}$$

where, on the left-hand side, $1 \in \mathsf{Mon}^n(1) = \{1\}$ is the operad unit in Mon^n. We have already defined f on the objects in $\mathsf{Mon}^n(k)$ for $k \le 2$.

To define f on the objects in $\mathsf{Mon}^n(k)$ for $k > 2$, first we make some preliminary observations.

- In each category $\mathsf{Mon}^n(k)$ with $k \ge 0$, an object X is said to be in *standard form* if the elements in $\{1,\ldots,k\}$ appear in X in increasing order from left to right. For example,

$$1 \otimes_i (2 \otimes_j 3) \in \mathsf{Mon}^n(3)$$

is in standard form, but $(2 \otimes_j 3) \otimes_i 1$ is not. Each object in $\mathsf{Mon}^n(k)$ can be written *uniquely* as

$$X\sigma \tag{13.3.8}$$

with $X \in \mathsf{Mon}^n(k)$ in standard form and $\sigma \in \Sigma_k$. For example,

$$(2 \otimes_j 3) \otimes_i 1 = \big((1 \otimes_j 2) \otimes_i 3\big)(1,2,3) \in \mathsf{Mon}^n(3),$$

with $(1 \otimes_j 2) \otimes_i 3$ in standard form.

- By Definition II.10.5.2 of $\mathsf{FMon}^n(\mathsf{C})$ and the definition (13.1.14) of γ on objects, each object $X \in \mathsf{Mon}^n(k)$ in standard form with $k \geq 3$ can be written as

$$X = \gamma\big(1 \otimes_i 2, (X_1, X_2)\big) \tag{13.3.9}$$

for some objects

$$X_1 \in \mathsf{Mon}^n(h) \quad \text{and} \quad X_2 \in \mathsf{Mon}^n(k-h)$$

in standard form, $1 \leq i \leq n$, and $1 \leq h < k$. For example,

$$(1 \otimes_i 2) \otimes_j (3 \otimes_l 4) = \gamma\big(1 \otimes_j 2, (1 \otimes_i 2, 1 \otimes_l 2)\big) \in \mathsf{Mon}^n(4),$$

with $1 \otimes_i 2$ and $1 \otimes_l 2 \in \mathsf{Mon}^n(2)$ in standard form.

Since f must preserve γ, if fX_1 and fX_2 are already defined, then, with X as in (13.3.9), we must define

$$fX = \gamma\big(\otimes_i, (fX_1, fX_2)\big) \in \mathsf{P}_k. \tag{13.3.10}$$

Well defined on objects. To see that (13.3.10) is well defined, suppose that X in (13.3.9) has another decomposition as

$$X = \gamma\big(1 \otimes_j 2, (X_1', X_2')\big) \tag{13.3.11}$$

for some $1 \leq j \leq n$ and objects X_1', X_2' in standard form. Then $i = j$, and, up to switching the two decompositions (13.3.9) and (13.3.11), X has the form

$$\begin{aligned} X &= (\cdots) \otimes_i (\cdots) \otimes_i (\cdots) \\ &= \gamma\big(1 \otimes_i 2 \otimes_i 3, (X_1, X'', X_2')\big) \end{aligned}$$

with

$$\begin{cases} X_2 = \gamma\big(1 \otimes_i 2, (X'', X_2')\big) \\ X_1' = \gamma\big(1 \otimes_i 2, (X_1, X'')\big). \end{cases} \tag{13.3.12}$$

For example, there are equalities in $\mathsf{Mon}^n(5)$ as follows.

$$\begin{aligned} (1 \otimes_j 2) \otimes_i 3 \otimes_i (4 \otimes_l 5) &= \gamma\Big(1 \otimes_i 2, \big(1 \otimes_j 2, 1 \otimes_i (2 \otimes_l 3)\big)\Big) \\ &= \gamma\Big(1 \otimes_i 2 \otimes_i 3, \big(1 \otimes_j 2, 1, 1 \otimes_l 2\big)\Big) \\ &= \gamma\Big(1 \otimes_i 2, \big((1 \otimes_j 2) \otimes_i 3, 1 \otimes_l 2\big)\Big) \end{aligned}$$

There are equalities in P as follows.

$$\begin{aligned} \gamma\big(\otimes_i, (fX_1', fX_2')\big) &= \gamma\Big(\otimes_i, \big(\gamma(\otimes_i, (fX_1, fX'')), fX_2'\big)\Big) \\ &= \gamma\Big(\otimes_i, \big(fX_1, \gamma(\otimes_i, (fX'', fX_2'))\big)\Big) \\ &= \gamma\big(\otimes_i, (fX_1, fX_2)\big) \end{aligned}$$

The first and the third equalities follow from (13.3.10) and (13.3.12). The second equality holds by the assumed associativity relation in (13.3.6). This shows that the definition (13.3.10) on objects in standard form is well defined.

Since f must preserve the symmetric group action, we must extend (13.3.10) by defining

$$f(X\sigma) = (fX)\sigma \in \mathsf{P}_k \tag{13.3.13}$$

for $X \in \mathsf{Mon}^n(k)$ in standard form and $\sigma \in \Sigma_k$. In particular, f is uniquely defined on the objects in Mon^n.

Preservation of operad axioms on objects. By (13.3.7) and (13.3.13), f preserves the operad unit and the symmetric group action.

To see that f preserves the operad composition on objects, we need to check that the diagram

$$\begin{tikzcd}
\mathsf{Mon}^n(k) \times \prod_{i=1}^{k} \mathsf{Mon}^n(j_i) \arrow[r, "\gamma"] \arrow[d, "f \times \prod_i f"'] & \mathsf{Mon}^n(j) \arrow[d, "f"] \\
\mathsf{P}_k \times \prod_{i=1}^{k} \mathsf{P}_{j_i} \arrow[r, "\gamma"] & \mathsf{P}_j
\end{tikzcd} \tag{13.3.14}$$

is commutative on objects, where $j = j_1 + \cdots + j_k$. By

- the decomposition $X\sigma$ in (13.3.8) for objects in Mon^n,
- the fact that f preserves the symmetric group action on objects, and
- the equivariance axioms (6.1.6) and (6.1.7) for γ,

it suffices to consider objects in $\mathsf{Mon}^n(k) \times \prod_{i=1}^{k} \mathsf{Mon}^n(j_i)$ in standard form. Furthermore, by

- the decomposition (13.3.9) for objects in standard form,
- the associativity and unity axioms (6.1.3)–(6.1.5),
- the assumed unity relation (13.3.6) in P,
- the fact that $\mathbb{1}$ is a strict two-sided unit for $\otimes_i$ in Mon^n, and
- an induction on k,

it suffices to consider an object of the form

$$\big(1 \otimes_i 2, (X_1, X_2)\big) \in \mathsf{Mon}^n(2) \times \mathsf{Mon}^n(j_1) \times \mathsf{Mon}^n(j_2)$$

with $j_1, j_2 \geq 1$ and each of X_1 and X_2 in standard form. On this object, the diagram (13.3.14) is commutative by the definition (13.3.10) of fX.

Morphisms. The identity morphisms

$$1_{\mathbb{1}} \in \mathsf{Mon}^n(0) = \{\mathbb{1}\} \quad \text{and} \quad 1_1 \in \mathsf{Mon}^n(1) = \{1\}$$

must be sent by f to, respectively, the identity morphisms $1_{\mathbb{1}} \in \mathsf{P}_0$ and $1_1 \in \mathsf{P}_1$. By definition, each morphism $\varphi \in \mathsf{Mon}^n(k)$ with $k \geq 2$ decomposes into a categorical composite

$$\varphi = (\phi_r v_r) \circ \cdots \circ (\phi_1 v_1) \tag{13.3.15}$$

with

- each $v_l \in \Sigma_k$ for $1 \leq l \leq r$ and
- each ϕ_l an operadic composite of

- one $\eta^{i,j}_{1,2,3,4}$ with $1 \leq i < j \leq n$ in (13.3.2) and
- identity morphisms of objects in standard form.

Since f is to be a morphism of Cat-enriched operads, we must define

$$f\varphi = \big((f\phi_r)v_r\big) \circ \cdots \circ \big((f\phi_1)v_1\big) \in \mathsf{P}_k \tag{13.3.16}$$

as the corresponding categorical composite with each $f\phi_l$ the corresponding operadic composite of one $\eta^{i,j} = f(\eta^{i,j}_{1,2,3,4})$ in (13.3.5) and identity morphisms. In particular, if f is well defined on morphisms, which we will verify shortly, then it is uniquely defined on morphisms.

Well defined and functorial. To see that f is well defined on morphisms, note that the Coherence Theorem II.10.6.8 for $\mathsf{Mon}^n(k)$ can also be applied to the objects and morphisms

$$\big(\mathbb{1}, \{\otimes_i\}_{i=1}^n, \{\eta^{i,j}\}_{1\leq i<j\leq n}\big)$$

in P. The reason is that the assumed axioms (13.3.6) and (13.1.3)–(13.1.7) are formally identical to those in the categories $\mathsf{Mon}^n(k)$ for $k \geq 0$. We will refer to this property as the *n-fold monoidal coherence in* P. Suppose

$$\varphi = (\phi'_s v'_s) \circ \cdots \circ (\phi'_1 v'_1) \in \mathsf{Mon}^n(k)$$

is another decomposition of φ in (13.3.15) as in the previous paragraph. In the image

$$\big((f\phi'_s)v'_s\big) \circ \cdots \circ \big((f\phi'_1)v'_1\big) \in \mathsf{P}_k, \tag{13.3.17}$$

each $f\phi'_h$ is an operadic composite of one $\eta^{i,j}$ and identity morphisms. By the n-fold monoidal coherence in P, the two morphisms (13.3.16) and (13.3.17), which have the same (co)domain, are equal. This shows that f is well defined on morphisms. The n-fold monoidal coherence in P also implies that

$$f : \mathsf{Mon}^n(k) \longrightarrow \mathsf{P}_k$$

is a functor for each $k \geq 0$.

Preservation of operad units and equivariant structure on morphisms. By definition, f preserves the operad unit on morphisms. To see that f preserves the symmetric group action on morphisms, suppose $\varphi \in \mathsf{Mon}^n(k)$ is as in (13.3.15) and $\pi \in \Sigma_k$. The right π-action in each of Mon^n and P is a functor. Using this fact and the definition (13.3.16) of $f\varphi$, the following computation shows that f preserves the right π-action on morphisms.

$$\begin{aligned}
f(\varphi\pi) &= f\Big(\big[(\phi_r v_r) \circ \cdots \circ (\phi_1 v_1)\big]\pi\Big)\\
&= f\Big((\phi_r v_r)\pi \circ \cdots \circ (\phi_1 v_1)\pi\Big)\\
&= f\Big(\phi_r(v_r\pi) \circ \cdots \circ \phi_1(v_1\pi)\Big)\\
&= (f\phi_r)(v_r\pi) \circ \cdots \circ (f\phi_1)(v_1\pi)\\
&= \big((f\phi_r)v_r\big)\pi \circ \cdots \circ \big((f\phi_1)v_1\big)\pi\\
&= \big[(f\phi_r)v_r \circ \cdots \circ (f\phi_1)v_1\big]\pi\\
&= (f\varphi)\pi
\end{aligned}$$

Preservation of operad composition on morphisms. For the preservation of the operad composition γ by f on morphisms, we use the following facts:

- f is levelwise a functor and preserves the symmetric group action.
- γ in each of Mon^n and P is a functor and satisfies the equivariance axioms (6.1.6) and (6.1.7).
- Morphisms in Mon^n decompose as in (13.3.15).

Using these facts, to prove the commutativity of the diagram (13.3.14) on morphisms, it suffices to consider a morphism

$$\big(\phi, (\phi_i)_{i=1}^k\big) \in \mathsf{Mon}^n(k) \times \prod_{i=1}^k \mathsf{Mon}^n(j_i)$$

such that the following two conditions hold:

- There is only one nonidentity morphism in $\{\phi, \phi_1, \ldots, \phi_k\}$.
- The nonidentity morphism in $\{\phi, \phi_1, \ldots, \phi_k\}$ is an operadic composite of
  - one $\eta^{i,j}_{1,2,3,4}$ with $1 \leq i < j \leq n$ in (13.3.2) and
  - identity morphisms of objects in standard form.

On such a morphism, the diagram (13.3.14) is commutative by the n-fold monoidal coherence in P. □

***n*-Fold Monoidal Categories as Algebras.** Recall the following results:

- The associative operad As is the operad for monoids (Proposition 11.1.15).
- The Barratt-Eccles operad $E\mathsf{As}$ is the Cat-enriched operad for permutative categories (Proposition 11.4.26).
- The braid operad Br is the Cat-enriched operad for braided strict monoidal categories (Proposition 12.3.22).

Proposition 13.3.18 below is the analogue for n-fold monoidal categories and is an application of Theorem 13.3.3.

Proposition 13.3.18. *For $n \geq 1$ and a small category C, an n-fold monoidal category structure on C is uniquely determined by a Cat-enriched multifunctor*

$$f : \mathsf{Mon}^n \longrightarrow \mathsf{Cat} \quad \textit{such that} \quad f(*) = \mathsf{C}.$$

Proof. A Cat-enriched multifunctor $f : \mathsf{Mon}^n \longrightarrow \mathsf{Cat}$ such that $f(*) = \mathsf{C}$ is equivalent to a Cat-enriched operad morphism

$$\mathsf{Mon}^n \xrightarrow{\;f\;} \mathsf{End}(\mathsf{C}) = \big\{\mathsf{Cat}(\mathsf{C}^{\times k}, \mathsf{C})\big\}_{k \geq 0}$$

to the Cat-enriched endomorphism operad of C. By Theorem 13.3.3, such a Cat-enriched operad morphism f is uniquely determined by

- an object
$$\mathbb{1} \in \mathsf{C},$$
- a functor
$$\otimes_i : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$$
  for each $1 \leq i \leq n$, and
- a natural transformation
$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D)$$
  for objects $A, B, C, D \in \mathsf{C}$ and $1 \leq i < j \leq n$

such that the following two statements hold.

- For $1 \le i \le n$, $\otimes_i$ is strictly associative with $\mathbb{1}$ as the strict two-sided unit. In other words, each $(\mathsf{C}, \otimes_i, \mathbb{1})$ is a strict monoidal category.
- The n-fold monoidal category axioms (13.1.3)–(13.1.7) hold in C.

This is the same thing as an n-fold monoidal category structure on C. □

13.4. Detecting E_n-Monoidal Categories

In this section, we prove that the n-fold monoidal category operad detects E_n-monoidal category structures on small permutative categories (Theorem 13.4.12). This result is an application of the Coherence Theorem 13.3.3 for the n-fold monoidal category operad. In Section 13.5, we use this result to show that the K-theory of a small E_n-monoidal category is an E_n-symmetric spectrum (Corollary 13.5.2). For the reader's convenience, here we recall from Chapter II.10 the definition of an E_n-monoidal category.

Definition 13.4.1. For $n \ge 1$, an E_n*-monoidal category* is a tuple

$$\Big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), \{\otimes_i, \partial^{l,i}, \partial^{r,i}\}_{1 \le i \le n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \le i < j \le n}\Big)$$

consisting of the following data.

The Ring Category Structures: For each $1 \le i \le n$, the tuple

$$\big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), (\otimes_i, \mathbb{1}), (\partial^{l,i}, \partial^{r,i})\big) \tag{13.4.2}$$

is a ring category (Definition 11.2.4). The natural transformations

$$\begin{aligned} (A \otimes_i C) \oplus (B \otimes_i C) &\xrightarrow{\partial^{l,i}_{A,B,C}} (A \oplus B) \otimes_i C \\ (A \otimes_i B) \oplus (A \otimes_i C) &\xrightarrow{\partial^{r,i}_{A,B,C}} A \otimes_i (B \oplus C) \end{aligned} \tag{13.4.3}$$

for objects $A, B, C \in \mathsf{C}$, are called, respectively, the *ith left factorization morphism* and the *ith right factorization morphism*.

The n-Fold Monoidal Structure: The tuple

$$\big(\mathsf{C}, \{\otimes_i\}_{1 \le i \le n}, \mathbb{1}, \{\eta^{i,j}\}_{1 \le i < j \le n}\big) \tag{13.4.4}$$

is an n-fold monoidal category (Definition 13.1.1), with (i,j)-*exchange* the natural transformation

$$(A \otimes_j B) \otimes_i (C \otimes_j D) \xrightarrow{\eta^{i,j}_{A,B,C,D}} (A \otimes_i C) \otimes_j (B \otimes_i D) \tag{13.4.5}$$

for objects $A, B, C, D \in \mathsf{C}$ and $1 \le i < j \le n$.

These data are required to satisfy the following axioms for $1 \le i < j \le n$ and objects A, A', B, B', C, C', D, and D' in C.

The Zero Exchange Axiom:

$$\eta^{i,j}_{A,B,C,D} = 1_{\mathbb{0}} \quad \text{if} \quad A, B, C, \text{ or } D \text{ is } \mathbb{0}. \tag{13.4.6}$$

The Exchange Factorization Axiom: The following four diagrams are commutative. They are called, respectively, EF1, EF2, EF3, and EF4.

$$\begin{tikzcd}[column sep=tiny]
& {[(A \otimes_j B) \otimes_i (C \otimes_j D)] \oplus [(A' \otimes_j B) \otimes_i (C \otimes_j D)]} \arrow[dl, "\eta^{i,j}_{A,B,C,D} \oplus \eta^{i,j}_{A',B,C,D}"'] \arrow[dr, "\partial^{l,i}_{A \otimes_j B, A' \otimes_j B, C \otimes_j D}"] & \\
{[(A \otimes_i C) \otimes_j (B \otimes_i D)] \oplus [(A' \otimes_i C) \otimes_j (B \otimes_i D)]} \arrow[d, "\partial^{l,j}_{A \otimes_i C, A' \otimes_i C, B \otimes_i D}"'] & & {[(A \otimes_j B) \oplus (A' \otimes_j B)] \otimes_i (C \otimes_j D)} \arrow[d, "\partial^{l,j}_{A,A',B} \otimes_i 1"] \\
{[(A \otimes_i C) \oplus (A' \otimes_i C)] \otimes_j (B \otimes_i D)} \arrow[dr, "\partial^{l,i}_{A,A',C} \otimes_j 1"'] & & {[(A \oplus A') \otimes_j B] \otimes_i (C \otimes_j D)} \arrow[dl, "\eta^{i,j}_{A \oplus A',B,C,D}"] \\
& {[(A \oplus A') \otimes_i C] \otimes_j (B \otimes_i D)} &
\end{tikzcd} \tag{13.4.7}$$

$$\begin{tikzcd}[column sep=tiny]
& {[(A \otimes_j B) \otimes_i (C \otimes_j D)] \oplus [(A \otimes_j B') \otimes_i (C \otimes_j D)]} \arrow[dl, "\eta^{i,j}_{A,B,C,D} \oplus \eta^{i,j}_{A,B',C,D}"'] \arrow[dr, "\partial^{l,i}_{A \otimes_j B, A \otimes_j B', C \otimes_j D}"] & \\
{[(A \otimes_i C) \otimes_j (B \otimes_i D)] \oplus [(A \otimes_i C) \otimes_j (B' \otimes_i D)]} \arrow[d, "\partial^{r,j}_{A \otimes_i C, B \otimes_i D, B' \otimes_i D}"'] & & {[(A \otimes_j B) \oplus (A \otimes_j B')] \otimes_i (C \otimes_j D)} \arrow[d, "\partial^{r,j}_{A,B,B'} \otimes_i 1"] \\
{(A \otimes_i C) \otimes_j [(B \otimes_i D) \oplus (B' \otimes_i D)]} \arrow[dr, "1 \otimes_j \partial^{l,i}_{B,B',D}"'] & & {[A \otimes_j (B \oplus B')] \otimes_i (C \otimes_j D)} \arrow[dl, "\eta^{i,j}_{A,B \oplus B',C,D}"] \\
& {(A \otimes_i C) \otimes_j [(B \oplus B') \otimes_i D]} &
\end{tikzcd} \tag{13.4.8}$$

$$\begin{tikzcd}[column sep=tiny]
& {[(A \otimes_j B) \otimes_i (C \otimes_j D)] \oplus [(A \otimes_j B) \otimes_i (C' \otimes_j D)]} \arrow[dl, "\eta^{i,j}_{A,B,C,D} \oplus \eta^{i,j}_{A,B,C',D}"'] \arrow[dr, "\partial^{r,i}_{A \otimes_j B, C \otimes_j D, C' \otimes_j D}"] & \\
{[(A \otimes_i C) \otimes_j (B \otimes_i D)] \oplus [(A \otimes_i C') \otimes_j (B \otimes_i D)]} \arrow[d, "\partial^{l,j}_{A \otimes_i C, A \otimes_i C', B \otimes_i D}"'] & & {(A \otimes_j B) \otimes_i [(C \otimes_j D) \oplus (C' \otimes_j D)]} \arrow[d, "1 \otimes_i \partial^{l,j}_{C,C',D}"] \\
{[(A \otimes_i C) \oplus (A \otimes_i C')] \otimes_j (B \otimes_i D)} \arrow[dr, "\partial^{r,i}_{A,C,C'} \otimes_j 1"'] & & {(A \otimes_j B) \otimes_i [(C \oplus C') \otimes_j D]} \arrow[dl, "\eta^{i,j}_{A,B,C \oplus C',D}"] \\
& {[A \otimes_i (C \oplus C')] \otimes_j (B \otimes_i D)} &
\end{tikzcd} \tag{13.4.9}$$

$$\begin{tikzcd}[column sep=tiny]
& {[(A \otimes_j B) \otimes_i (C \otimes_j D)] \oplus [(A \otimes_j B) \otimes_i (C \otimes_j D')]} \arrow[dl, "\eta^{i,j}_{A,B,C,D} \oplus \eta^{i,j}_{A,B,C,D'}"'] \arrow[dr, "\partial^{r,i}_{A \otimes_j B, C \otimes_j D, C \otimes_j D'}"] & \\
{[(A \otimes_i C) \otimes_j (B \otimes_i D)] \oplus [(A \otimes_i C) \otimes_j (B \otimes_i D')]} \arrow[d, "\partial^{r,j}_{A \otimes_i C, B \otimes_i D, B \otimes_i D'}"'] & & {(A \otimes_j B) \otimes_i [(C \otimes_j D) \oplus (C \otimes_j D')]} \arrow[d, "1 \otimes_i \partial^{r,j}_{C,D,D'}"] \\
{(A \otimes_i C) \otimes_j [(B \otimes_i D) \oplus (B \otimes_i D')]} \arrow[dr, "1 \otimes_j \partial^{r,i}_{B,D,D'}"'] & & {(A \otimes_j B) \otimes_i [C \otimes_j (D \oplus D')]} \arrow[dl, "\eta^{i,j}_{A,B,C,D \oplus D'}"] \\
& {(A \otimes_i C) \otimes_j [B \otimes_i (D \oplus D')]} &
\end{tikzcd} \tag{13.4.10}$$

This finishes the definition of an E_n-monoidal category. It is *small* if the category C is small. $\diamond$

Definition 13.4.11. For a permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$, an *$E_n$-monoidal category structure* on C for $n \geq 1$ is the additional data

$$\big(\{\otimes_i, \partial^{l,i}, \partial^{r,i}\}_{1\leq i\leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1\leq i<j\leq n}\big)$$

such that the tuple

$$\Big(\mathsf{C}, (\oplus, \mathbb{0}, \xi^{\oplus}), \{\otimes_i, \partial^{l,i}, \partial^{r,i}\}_{1\leq i\leq n}, \mathbb{1}, \{\eta^{i,j}\}_{1\leq i<j\leq n}\Big)$$

is an E_n-monoidal category as in Definition 13.4.1. ◇

Recall from Section 6.6 that $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-enriched multicategory with small permutative categories as objects. The category

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle\,;\mathsf{D}\big) = \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}_1, \ldots, \mathsf{C}_n\rangle\,;\mathsf{D}\big)$$

has

- n-linear functors
$$\mathsf{C}_1 \times \cdots \times \mathsf{C}_n \longrightarrow \mathsf{D}$$
in Definition 6.5.4 as objects and
- multilinear transformations (Definition 6.5.11) as morphisms.

Also recall from Definition 5.1.12 the notion of an enriched multifunctor. The next result is the Mon^n analogue of Theorems 11.2.16, 11.5.5, and 12.4.5 for, respectively, the associative operad As, the Barratt-Eccles operad $E\mathsf{As}$, and the braid operad Br. It says that the n-fold monoidal category operad detects E_n-monoidal category structures on small permutative categories. It extends Proposition 13.3.18 from Cat to $\mathsf{PermCat}^{\mathsf{su}}$.

Theorem 13.4.12. *For $n \geq 1$ and each small permutative category C, there is a canonical bijective correspondence between*

- *E_n-monoidal category structures on C and*
- Cat*-enriched multifunctors*
$$F : \mathsf{Mon}^n \longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \textit{such that} \quad F(*) = \mathsf{C}.$$

Proof. A Cat-enriched multifunctor

$$F : \mathsf{Mon}^n \longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad \text{such that} \quad F(*) = (\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$$

is equivalent to a Cat-enriched operad morphism

$$F : \mathsf{Mon}^n \longrightarrow \mathsf{End}(\mathsf{C}) = \Big\{\mathsf{PermCat}^{\mathsf{su}}\big(\langle \overbrace{\mathsf{C}, \ldots, \mathsf{C}}^{k}\rangle\,;\mathsf{C}\big)\Big\}_{k\geq 0}$$

to the Cat-enriched endomorphism operad of C. By Theorem 13.3.3, such a Cat-enriched operad morphism is uniquely determined by

- the 0-linear functor, that is, object
$$F(\mathbb{1}) = \mathbb{1} \in \mathsf{C},$$
- the 2-linear functors
$$F(1 \otimes_i 2) = (\otimes_i, \partial^{l,i}, \partial^{r,i}) : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$$
for $1 \leq i \leq n$, and

- the multilinear transformations

$$(A \otimes_l B) \otimes_k (C \otimes_l D) \xrightarrow{\eta^{k,l}_{A,B,C,D} = F(\eta^{k,l}_{1,2,3,4})} (A \otimes_k C) \otimes_l (B \otimes_k D)$$

for $1 \le k < l \le n$ and objects $A, B, C, D \in \mathsf{C}$,

such that

- the unity and associativity conditions (13.3.6) and
- the n-fold monoidal category axioms (13.1.3)–(13.1.7)

hold in C.

The ring category structures. By Theorem 11.2.16 and the first paragraph of its proof, for $i \in \{1, \ldots, n\}$, the data $(\otimes_i, \mathbb{1}, \partial^{l,i}, \partial^{r,i})$ and the conditions (13.3.6) are equivalent to a ring category structure on the permutative category $(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus})$ as in Definition 11.2.15. In particular, each $(\mathsf{C}, \otimes_i, \mathbb{1})$ is a strict monoidal category.

The n-fold monoidal category structure. The n-fold monoidal category axioms (13.1.3)–(13.1.7) guarantee that the tuple

$$\big(\mathsf{C}, \{\otimes_i\}_{1 \le i \le n}, \mathbb{1}, \{\eta^{k,l}\}_{1 \le k < l \le n}\big)$$

is an n-fold monoidal category.

The E_n-monoidal category axioms. For $1 \le k < l \le n$, the multilinear transformation

$$\otimes_k(\otimes_l, \otimes_l) \xrightarrow{\eta^{k,l}} \otimes_l(\otimes_k, \otimes_k)(2,3) \in \mathsf{End}(\mathsf{C})_4$$

is, by definition, a natural transformation that satisfies the two conditions in Definition 6.5.11.

- For $\alpha = \eta^{k,l}$, the commutative diagrams (6.5.12) for $i \in \{1, 2, 3, 4\}$ are, respectively, the axioms (13.4.7)–(13.4.10) in an E_n-monoidal category. Here we use the explicit formulas (6.6.3) and (6.6.8) to unravel the linearity constraints of the domain and codomain 4-linear functors of $\eta^{k,l}$.
- The second condition in Definition 6.5.11 states that

$$\eta^{k,l}_{A,B,C,D} = 1_{\mathbb{0}} \quad \text{if} \quad A, B, C, \text{ or } D \text{ is } \mathbb{0}.$$

This is the zero exchange axiom (13.4.6) in an E_n-monoidal category.

Therefore, a Cat-enriched operad morphism

$$F : \mathsf{Mon}^n \longrightarrow \mathsf{End}(\mathsf{C})$$

is equivalent to an E_n-monoidal category structure on C. □

13.5. K-Theory of E_n-Monoidal Categories are E_n-Symmetric Spectra

In this section, we prove that the K-theory of a small E_n-monoidal category is an E_n-symmetric spectrum (Corollary 13.5.2). This result is the E_n analogue of Corollaries 11.3.16, 11.6.12, and 12.5.3. It is a consequence of Theorem 13.4.12, the Elmendorf-Mandell K-theory multifunctor, and the fact that the n-fold monoidal category operad is an E_n-operad (Theorem 13.2.1). Along the way, we observe that an E_∞-structure induces an E_n-structure (Proposition 13.5.1).

E_n-Symmetric Spectra. An *E_n-symmetric spectrum* (Definition 12.5.1) is a symmetric spectrum X equipped with an sSet-enriched operad morphism

$$\mathsf{P} \longrightarrow \mathsf{End}(X)$$

with

- P an sSet-enriched E_n-operad (Definition 12.2.3) and
- $\mathsf{End}(X)$ the endomorphism simplicial operad (Definition 7.6.22).

The E_∞ analogue is Definition 11.6.5. Recall from Proposition 12.1.11 that the Cat-enriched operad morphism

$$\mathsf{As} \xrightarrow{\iota_{\mathsf{As}}} E\mathsf{As}$$

in (11.4.12), from the associative operad As to the Barratt-Eccles operad $E\mathsf{As}$, factors through the braid operad Br. Recall that $E\mathsf{As}$ is an E_∞-operad (Proposition 11.6.3) and Mon^n is an E_n-operad (Theorem 13.2.1). The following result provides a factorization of ι_{As} through the n-fold monoidal category operad Mon^n for $n \geq 1$. See also Corollary 12.5.2.

Proposition 13.5.1. *There is a commutative diagram of* Cat*-enriched operad morphisms*

$$\begin{tikzcd}
\mathsf{As} \cong \mathsf{Mon}^1 \arrow[r, "i^1"] \arrow[drr, "\iota_{\mathsf{As}} = j^1"'] & \mathsf{Mon}^2 \arrow[r, "i^2"] \arrow[dr, "j^2"] & \mathsf{Mon}^3 \arrow[r, "i^3"] \arrow[d, "j^3"] & \cdots \\
& & E\mathsf{As} & \cdots
\end{tikzcd}$$

with each i^n for $n \geq 1$ the inclusion. As a result,

- *each E_∞-structure on a symmetric spectrum via the Barratt-Eccles operad $E\mathsf{As}$ induces an E_n-structure by restricting along j^n, and*
- *each E_{n+1}-structure on a symmetric spectrum via Mon^{n+1} induces an E_n-structure by restricting along i^n.*

Proof. The isomorphism $\mathsf{As} \cong \mathsf{Mon}^1$ is from Example 13.1.23. Each $\mathsf{Mon}^n(k)$ is the full subcategory of $\mathsf{Mon}^{n+1}(k)$ consisting of objects that can be written without $\otimes_{n+1}$. This inclusion preserves the operad structure in Mon^n and Mon^{n+1} in Definition 13.1.12, so

$$\mathsf{Mon}^n \xrightarrow{i^n} \mathsf{Mon}^{n+1}$$

is a Cat-enriched operad morphism.

To define j^n, recall from (13.3.8) that each object in $\mathsf{Mon}^n(k)$ can be written uniquely in the form $X\sigma$, with

- $X \in \mathsf{Mon}^n(k)$ in standard form, which means that the elements in $\{1, \ldots, k\}$ appear in X in increasing order from left to right, and
- $\sigma \in \Sigma_k$ a permutation.

For $k \geq 0$, the assignment

$$\mathsf{Mon}^n(k) \xrightarrow{j^n_k} E\mathsf{As}_k$$

on objects is defined as

$$j^n_k(X\sigma) = \sigma$$

for $X \in \mathsf{Mon}^n(k)$ in standard form and $\sigma \in \Sigma_k$. These assignments define an operad morphism between the object set operads of Mon^n and $E\mathsf{As}$. On morphisms, j^n_k is uniquely defined by the fact that, in the translation category $E\mathsf{As}_k$, each morphism set has only one element. This property also implies that

- j^n_k is a functor and
- $j^n = \{j^n_k\}_{k \geq 0}$ preserves operad units, symmetric group action, and operad composition.

Therefore,

$$\mathsf{Mon}^n \xrightarrow{j^n} E\mathsf{As}$$

is a Cat-enriched operad morphism.

The equality

$$j^n = j^{n+1} i^n$$

follows from the fact that i^n is the inclusion. This proves the existence of the commutative diagram in the statement. The remaining assertions follow from the existence of the Cat-enriched operad morphisms i^n and j^n. □

***K*-Theory E_n-Symmetric Spectra.** Recall the Elmendorf-Mandell K-theory multifunctor in Definition 10.3.32,

$$\mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp}.$$

The following result is the E_n analogue of Corollaries 11.3.16, 11.6.12, and 12.5.3.

Corollary 13.5.2. *For $n \geq 1$ and each small E_n-monoidal category C, $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_n-symmetric spectrum.*

Proof. Consider the multifunctors

$$\begin{array}{ccccc} \mathsf{Mon}^n & \xrightarrow{F} & \mathsf{PermCat}^{\mathsf{su}} & \xrightarrow{\mathsf{K}^{\mathsf{EM}}} & \mathsf{SymSp} \\ * & \longmapsto & (\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}) & \longmapsto & \mathsf{K}^{\mathsf{EM}}\mathsf{C} \end{array}$$

with F the Cat-enriched multifunctor in Theorem 13.4.12 such that $F(*)$ is the additive structure of C. By Theorem 13.2.1, $N\mathsf{Mon}^n$ is an E_n-operad. As in the proofs of Corollaries 11.6.12 and 12.5.3, the composite sSet-enriched operad morphism

$$\begin{tikzcd} N\mathsf{Mon}^n \arrow[d, "F_N"'] & & & \mathsf{End}(\mathsf{K}^{\mathsf{EM}}\mathsf{C}) \\ N\mathsf{End}(\mathsf{C}) \arrow[r, "\mathsf{J}^{\mathsf{EM}}_N"] & N\mathsf{End}(\mathsf{J}^{\mathsf{EM}}\mathsf{C}) \arrow[r, "N_*"] & \mathsf{End}(N_*\mathsf{J}^{\mathsf{EM}}\mathsf{C}) \arrow[ur, "\mathsf{K}^{\mathcal{G}}"'] \end{tikzcd}$$

gives $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ the structure of an E_n-symmetric spectrum. □

Example 13.5.3. By Proposition II.10.10.2, each small category C freely generates a small E_n-monoidal category $\mathsf{FE}^n(\mathsf{C})$. Corollary 13.5.2 applies to $\mathsf{FE}^n(\mathsf{C})$ to yield a K-theory E_n-symmetric spectrum. See also Questions A.4.2 and A.5.7. ◇

13.6. Notes

13.6.1 (Homotopy Colimits)**.** In addition to [**Dug∞, MP12**], other references for homotopy colimits, which are needed in the proof of Theorem 13.2.1, include [**BK72, CS02, Hir03, Rie14, Shu∞a, Vog73, Vog77**]. $\diamond$

13.6.2 (Iterated Monoidal Category Operads)**.** The nerve of the n-fold monoidal category operad Mon^n in Theorem 13.2.1 is weakly equivalent to

- the nth stage of Berger's filtration [**Ber96**] and
- the nth stage of Smith's filtration [**Smi89**]

of the Barratt-Eccles operad and is closely related to Milgram's model of $\Omega^n\Sigma^n X$ [**Mil66**]. See [**BFSV03**, Section 3]. Moreover, the group completion of the classifying space of an n-fold monoidal category is an n-fold loop space by [**BFSV03**, 2.2 and 3.14]. Proposition 13.3.18, which says that Mon^n-algebras are small n-fold monoidal categories, is a remark in [**BFSV03**, p. 292]. $\diamond$

Bibliography and Indices

APPENDIX A

Open Questions

"I enjoy questions that seem honest, even when they admit or reveal confusion, in preference to questions that appear designed to project sophistication."

– Bill Thurston, MathOverflow user profile

In this chapter, we discuss open questions related to the topics of this work. These open questions provide additional motivation for the main text.

A.1. Bimonoidal Categories

The following questions are about bimonoidal, symmetric bimonoidal, and braided bimonoidal categories in Definitions I.2.1.2 and II.2.1.29.

Question A.1.1 (Functoriality of the Matrix Construction)**.** In Theorem I.8.15.4, we showed that, for each tight symmetric bimonoidal category C, the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory. Denote by $\mathsf{Bi}^{\mathsf{tsy}}$ the full sub-2-category of the 2-category $\mathsf{Bi}^{\mathsf{sy}}$ in Proposition I.7.1.7, with small *tight* symmetric bimonoidal categories as objects. Regard $\mathsf{Bi}^{\mathsf{tsy}}$ as a tricategory with only identity 3-cells. It is claimed in **[SP∞]** that small symmetric monoidal bicategories are the objects of a tricategory, denoted by SMB.

- Extend the assignment

$$\mathsf{C} \longmapsto \mathsf{Mat}^{\mathsf{C}}$$

to a trifunctor

$$\mathsf{Bi}^{\mathsf{tsy}} \longrightarrow \mathsf{SMB}.$$

Among other things, one should carefully verify the tricategory axioms for SMB. For a discussion of tricategories and a detailed verification of the tricategory of small bicategories, the reader is referred to **[JY21**, Ch. 11]. $\diamond$

Question A.1.2 (Bimonoidal Bicategories)**.** Taking the categorification from (commutative) rigs to (symmetric) bimonoidal categories one step further, we could ask about two different monoidal structures, $\boxplus$ and $\boxtimes$, on a bicategory, with $\boxplus$ symmetric.

- Define such a *(braided/sylleptic/symmetric) bimonoidal bicategory*, generalizing the (braided/sylleptic/symmetric) monoidal bicategories in Sections I.6.4 and I.6.5.
- For a tight symmetric bimonoidal category C, prove that the symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$ extends to a symmetric bimonoidal bicategory.
- For a tight braided bimonoidal category C, prove that the monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$ extends to a bimonoidal bicategory.

More precisely, in Theorem I.8.15.4, the monoidal composition $\boxtimes$ in the symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$ involves the matrix tensor product in (I.8.6.3). There should be another symmetric monoidal bicategory structure on the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ in Theorem I.8.4.12, whose monoidal composition uses the matrix direct sum as in Example I.2.5.9. These two symmetric monoidal structures on $\mathsf{Mat}^{\mathsf{C}}$ should make it into a symmetric bimonoidal bicategory. A similar discussion applies in the braided case, with Theorem II.8.4.7 showing that $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory.

- Extend the tricategory SMB (Question A.1.1) to a tricategory SBB with small symmetric bimonoidal bicategories as objects.
- Construct a tricategory BB with small bimonoidal bicategories as objects.
- Extend the assignment $\mathsf{C} \longmapsto \mathsf{Mat}^{\mathsf{C}}$ in
  - Theorem I.8.15.4 in the symmetric case to a trifunctor
$$\mathsf{Bi}^{\mathsf{tsy}} \longrightarrow \mathsf{SBB}$$
    and
  - Theorem II.8.4.7 in the braided case to an analogous trifunctor with codomain BB. $\diamond$

Question A.1.3 (Bimonoidal Bicategorical Centers)**.** Extend the bimonoidal centers in Theorems II.4.4.3 and II.4.5.3 to the bimonoidal bicategorical setting in Question A.1.2. In other words:

- Extend the bimonoidal Drinfeld center in Theorem II.4.4.3 to a bimonoidal bicategory and show that it is a braided bimonoidal bicategory.
- Show that the center of a braided bimonoidal bicategory is a sylleptic bimonoidal bicategory.
- Extend the bimonoidal symmetric center in Theorem II.4.5.3 to a sylleptic bimonoidal bicategory and show that it is a symmetric bimonoidal bicategory.

There are simpler centers of Gray monoids **[BN96]**, braided monoidal 2-categories, and sylleptic monoidal 2-categories **[Cra98]**. As in Theorem II.4.4.3, a tightness assumption is likely necessary for some of these center constructions. $\diamond$

Question A.1.4 (Gray Rings and Bipermutative Gray Monoids)**.** Recall from Sections I.6.6 and I.6.7 that a Gray monoid is a 2-category equipped with a monoid structure under the Gray tensor product. A permutative Gray monoid is a Gray monoid equipped with a compatible Gray symmetry. Symmetric monoidal bicategories can be strictified to permutative Gray monoids; see **[GJO17b]** and the discussion near the end of Section I.6.7.

- Define a *Gray ring* and a *bipermutative Gray monoid* that are analogous to, respectively, a right rigid bimonoidal category (Definition I.5.5.8) and a right bipermutative category (Definition I.2.5.2).
- Along the lines of Theorem I.5.5.11, prove a strictification result from bimonoidal bicategories (Question A.1.2) to Gray rings.
- Along the lines of Theorem I.5.4.6, prove a strictification result from symmetric bimonoidal bicategories to bipermutative Gray monoids.

A bipermutative Gray monoid should be a 2-category equipped with two compatible permutative Gray monoid structures, $\boxplus$ and $\boxtimes$, that interact via distributivity.

In a Gray ring, ⊠ is a Gray monoid structure that is not assumed to be permutative. The following table summaries these (conjectural) concepts.

lax structure	strict structure	strictification
bicategories	2-categories	[**JY21**, 8.4.1]
monoidal bicategories	Gray monoids	[**GPS95, Gur13**]
symmetric monoidal bicategories	permutative Gray monoids	[**GJO17b**]
bimonoidal bicategories	Gray rings	conjecture
symmetric bimonoidal bicategories	bipermutative Gray monoids	conjecture

In each row, the left column can be strictified to the middle column. ◇

Question A.1.5 (Horizontal Bicategories of Double Categories)**.** A number of bicategories, including those of spans and bimodules, are the horizontal bicategories of some double categories.

- For a tight bimonoidal category C, is the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$ in Theorem I.8.4.12 the horizontal bicategory of a double category?
- If so, does the symmetric monoidal bicategory in Theorem I.8.15.4 arise from a symmetric monoidal structure on the double category?

See [**HS**∞] and [**JY21**, 12.3 and 12.4] for a discussion of (monoidal) double categories and their horizontal bicategories. ◇

Question A.1.6 (Braided Sheet Diagrams)**.** String diagrams are graphical reasoning tools in monoidal categories [**JS91a, Sel11**]. Sheet diagrams [**CDH**∞], which we mentioned in Notes I.2.7.5 and I.7.9.2 and Example I.3.10.9, are their analogues for tight bimonoidal categories.

- Develop sheet diagrams for
  - symmetric bimonoidal categories (Definition I.2.1.2) and
  - braided bimonoidal categories (Definition II.2.1.29).

This is, in fact, a coherence question with several parts. More precisely, a *bimonoidal signature* S consists of (i) a set of generating objects and (ii) a set of generating morphisms, each with (co)domain in the free $\{\oplus, \otimes\}$-algebra S^{fr} (Definition I.3.1.2). Given a bimonoidal signature S, one first defines the appropriate braided bimonoidal sheet diagrams and topological deformations corresponding to the axioms of a braided bimonoidal category. Then one constructs a braided bimonoidal category S' with object set S^{fr} and, as morphisms, braided bimonoidal sheet diagrams modulo topological deformations. Finally, one proves that S' is braided bimonoidally equivalent to the free braided bimonoidal category on S. A similar discussion applies in the symmetric case.

In view of the results in Chapter II.3, braided bimonoidal sheet diagrams can be used as graphical reasoning tools in (i) quantum group theory and (ii) the Fibonacci and Ising anyons in topological quantum computation. Sheet diagrams for tight bimonoidal categories in [**CDH**∞] involve the symmetric monoidal string diagrams in [**JS91a**]. Braided bimonoidal sheet diagrams will likely involve both the symmetric monoidal (for the additive structure ⊕) and the braided monoidal (for the multiplicative structure ⊗) string diagrams in [**JS91a**, Ch. 2–3]. The Braided Bimonoidal Coherence Theorem II.5.4.4 will be needed to check the axioms in Definition II.2.1.29 for braided bimonoidal sheet diagrams. Similarly, Laplaza's Coherence Theorems I.3.9.1 and I.4.4.3 will be needed to check the axioms in Definition I.2.1.2 for symmetric bimonoidal sheet diagrams.

The distributivity morphisms δ^l and δ^r in Definitions I.2.1.2 and II.2.1.29 are not invertible in general.

- Is it possible to replace the tightness assumption—that is, the invertibility of δ^l and δ^r—in the sheet diagrams in [**CDH∞**] with flatness in Definitions I.3.9.9 and II.5.4.5?

Related to Section I.7.9, to replace tightness with the much weaker assumption of flatness, one would need to work directly with a flat (symmetric/braided) bimonoidal category and *avoid* using the Strictification Theorems I.5.4.6, I.5.4.7, I.5.5.11, I.5.5.12, II.6.3.6, and II.6.3.7. The reason is that each of these theorems requires the tightness assumption. See Question A.2.8 for further problems about sheet diagrams. ⋄

A.2. E_n-Monoidal Categories

The following questions are about the E_n-monoidal categories in Part II.2.

Question A.2.1 (Coherence of E_n-Monoidal Categories)**.**

- Prove coherence theorems for ring categories (Definition II.9.1.2) along the lines of Theorems I.3.10.7 and I.4.5.8. Each such coherence theorem should say that any reasonable formal diagram in a ring category involving
$$(\oplus, \mathbb{0}, \xi^{\oplus}, \otimes, \mathbb{1}, \partial^l, \partial^r)$$
is commutative, with an assumption on either the common domain or the two paths.
- Prove coherence theorems for bipermutative categories (Definition II.9.3.2) along the lines of Theorems I.3.9.1 and I.4.4.3.
- Prove a coherence theorem for braided ring categories (Definition II.9.5.1) along the lines of Theorem II.5.4.4.
- More generally, prove a coherence theorem for E_n-monoidal categories (Definition II.10.7.2) along the lines of Theorem II.5.4.4. The coherence theorems for n-fold monoidal categories (Theorem II.10.6.8) should be relevant.

As in Theorems I.3.9.1 and I.4.4.3, one may need to assume a monomorphism or an epimorphism condition on the factorization morphisms ∂^l and ∂^r ⋄

Question A.2.2 (n-Monoidal Categories)**.** In an n-fold monoidal category (Definition II.10.1.1), each monoidal structure $\otimes_i$ is strictly associative with a common strict unit $\mathbb{1}$. There is a more general concept called an *n-monoidal category* in [**AM10**, Def. 6.1, 7.1, and 7.24]. It allows each monoidal structure $\otimes_i$ to be non-strict and distinct monoidal units.

- Describe the free n-monoidal category of a small category, along the lines of Proposition II.10.5.9 and Theorem II.10.5.18.
- Generalize the Coherence Theorem II.10.6.8 to n-monoidal categories.

In [**AM10**, Section 6.2], it is stated without detail that, in a 2-monoidal category, each formal diagram is commutative. It is stated there that this coherence result can be deduced from the work in [**Lew72**]. So one possible first step in answering these questions would be to prove in detail this coherence result for 2-monoidal categories. ⋄

Questions A.2.3 through A.2.7 below are all related to each other.

Question A.2.3 (Lax n-Fold Monoidal Categories)**.** Between an n-fold monoidal category and an n-monoidal category (Question A.2.2) is a *lax n-fold monoidal category*. The latter allows each monoidal structure $\otimes_i$ to be nonstrict, and it assumes a common nonstrict monoidal unit $\mathbb{1}$. Analogous to Propositions II.10.1.14 and II.10.1.21, general nonstrict braided monoidal categories should be examples of lax 2-fold monoidal categories, and general nonstrict symmetric monoidal categories should be examples of lax n-fold monoidal categories for $n \geq 2$.

- Describe the free lax n-fold monoidal category of a small category, along the lines of Proposition II.10.5.9 and Theorem II.10.5.18.
- Extend the Coherence Theorem II.10.6.8 to lax n-fold monoidal categories.
- Can n-monoidal categories be strictified to lax n-fold monoidal categories?
- Can lax n-fold monoidal categories be strictified to n-fold monoidal categories?

There are two other possible variants of n-fold monoidal categories. The variant in [**For04**] corresponds to a lax n-fold monoidal category with a common *strict* monoidal unit. The variant in [**FSS07**] corresponds to an n-monoidal category with generally distinct but strict monoidal units. The following table summaries the strictness assumptions of n-fold monoidal categories and its four variants.

monoidal units	strict $\{\otimes_i\}_{1\leq i\leq n}$	nonstrict $\{\otimes_i\}_{1\leq i\leq n}$
common strict	n-fold monoidal	[**For04**]
common nonstrict		lax n-fold monoidal
distinct strict		[**FSS07**]
distinct nonstrict		n-monoidal [**AM10**]

The questions below refer to lax n-fold monoidal categories as defined above, with $\otimes_i$ generally nonstrict and a common nonstrict monoidal unit. ⋄

Question A.2.4 (Unstable Periodic Table of Weak n-Categories)**.** The *periodic table* in [**BD98**] of k-tuply monoidal n-categories is a guiding principle for defining some versions of weak n-categories. In the $n = 1$ column in the periodic table, the values $k = 0, 1, 2$, and ≥ 3 correspond to, respectively, categories, monoidal categories, braided monoidal categories, and symmetric monoidal categories. On the other hand, by Propositions II.10.1.14 and II.10.1.21, braided strict monoidal categories and permutative categories are special examples of, respectively, 2-fold monoidal categories and k-fold monoidal categories for $k \geq 2$. Proposition II.10.2.8 has examples of 2-fold monoidal categories that are not braided strict monoidal categories. The k-fold monoidal category operad Mon^k (Proposition 13.1.20) is an E_k-operad that parametrizes k-fold monoidal categories (Theorem 13.2.1 and Proposition 13.3.18).

- Construct an *unstable* periodic table in which the $n = 1$ column consists of lax k-fold monoidal categories (Question A.2.3) for $k \geq 1$.

The $n = 1$ and $n = 2$ columns of the unstable periodic table should look like this:

	$n = 1$	$n = 2$
$k = 0$	categories	bicategories
$k = 1$	monoidal categories	monoidal bicategories
$k \geq 2$	lax k-fold monoidal categories	k-fold monoidal bicategories

The unstable periodic table does *not* stabilize like the periodic table in [**BD98**], where the $(n = 1, k \geq 3)$ entries are all symmetric monoidal categories. Moreover, the $(n = 2, k \geq 2)$ column in the unstable periodic table contains yet-to-be-defined *k-fold monoidal bicategories*.

- Prove that braided, sylleptic, and symmetric monoidal bicategories in Section I.6.5 are examples of k-fold monoidal bicategories for, respectively, $k = 2, 3$, and ≥ 4. ◇

This question may be regarded as both (i) a litmus test for the correct definition of a k-fold monoidal bicategory and (ii) a conceptual unification of braided, sylleptic, and symmetric monoidal bicategories. Further examples of k-fold monoidal bicategories should arise from the matrix construction in Question A.2.7.

Question A.2.5 (Iterated Gray Monoids)**.** This is a variation of Question A.1.4.

- Analogous to k-fold monoidal categories (Definition II.10.1.1) for $k \geq 1$, define the concept of a *k-fold Gray monoid* that satisfies the following statements:
  - A Gray monoid is precisely a 1-fold Gray monoid.
  - A braided monoidal 2-category [**Cra98**] is an example of a 2-fold Gray monoid, analogous to Proposition II.10.1.14.
  - A sylleptic monoidal 2-category [**Cra98**] is an example of a 3-fold Gray monoid.
  - A permutative Gray monoid is an example of a k-fold Gray monoid for $k \geq 4$, analogous to Proposition II.10.1.21.
- Prove a strictification theorem from k-fold monoidal bicategories (Question A.2.4) to k-fold Gray monoids. This should fit into the following table of strictification theorems.

bicategories	2-categories	[**JY21**, 8.4.1]
monoidal bicategories	Gray monoids	[**GPS95, Gur13**]
braided monoidal bicategories	braided monoidal 2-categories	[**Gur11**]
k-fold monoidal bicategories ($k \geq 1$)	k-fold Gray monoids	conjecture
symmetric monoidal bicategories	permutative Gray monoids	[**GJO17b**]

In each row, the left column can be strictified to the middle column. For the symmetric case, see the discussion near the end of Section I.6.7. ◇

Question A.2.6 (Laplaza E_n-Monoidal Categories)**.** The factorization morphisms $\{\partial^{l,i}, \partial^{r,i}\}_{1 \leq i \leq n}$ in an E_n-monoidal category (Definition II.10.7.2) go in the opposite direction as the distributivity morphisms in a bimonoidal category (Definition I.2.1.2). Moreover, in an E_n-monoidal category, the monoidal structures $\oplus$ and $\{\otimes_i\}_{i=1}^n$ are all strict.

- Define a *Laplaza E_n-monoidal category* with
  - a generally nonstrict additive structure $(\oplus, \mathbb{0})$,
  - a lax n-fold monoidal structure $\left(\{\otimes_i\}_{i=1}^n, \{\eta^{i,j}\}\right)$ (Question A.2.3),

– for each $1 \le i \le n$, a bimonoidal structure $(\oplus, \otimes_i, \lambda_i^\bullet, \rho_i^\bullet, \delta_i^l, \delta_i^r)$ (Definition I.2.1.2), and
– appropriate axioms relating the lax n-fold monoidal structure and the n bimonoidal structures, analogous to those in Definition II.10.7.2.

Laplaza E_n-monoidal categories should contain the following examples:

- An E_n-monoidal category with invertible factorization morphisms should be an example of a Laplaza E_n-monoidal category, analogous to Theorems II.9.1.15, II.9.3.7, and II.9.5.6.
- A Laplaza E_1-monoidal category should be precisely a bimonoidal category (Definition I.2.1.2), analogous to Example II.10.7.13.
- A braided bimonoidal category (Definition II.2.1.29) should be an example of a Laplaza E_2-monoidal category with $\otimes_1 = \otimes_2$, analogous to Theorem II.10.8.1.
- A symmetric bimonoidal category should be an example of a Laplaza E_n-monoidal category for $n \ge 2$ with $\otimes_1 = \cdots = \otimes_n$, analogous to Theorem II.10.9.1.
- Similar to Theorem II.2.4.22, an abelian category with a compatible lax n-fold monoidal structure should be a Laplaza E_n-monoidal category.

Moreover:

- Prove a coherence theorem for Laplaza E_n-monoidal categories, along the lines of Theorem II.5.4.4. This will certainly involve the coherence theorem for lax n-fold monoidal categories in Question A.2.3.
- Prove a strictification theorem for *tight* Laplaza E_n-monoidal categories, along the lines of Theorems II.6.3.6 and II.6.3.7. Here *tight* means that all the distributivity morphisms, δ_i^l and δ_i^r, are natural isomorphisms.
- Is there an analogue of Baez's Conjecture (Theorems I.7.8.1, I.7.8.3, II.7.3.4, and II.7.3.6) for Laplaza E_n-monoidal categories? ◇

Question A.2.7 (Matrix Construction)**.** In Theorem I.8.4.12 we showed that the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory for each tight bimonoidal category C. Moreover, $\mathsf{Mat}^{\mathsf{C}}$ is (i) a monoidal bicategory if C is a tight braided bimonoidal category (Theorem II.8.4.7) and (ii) a symmetric monoidal bicategory if C is a tight symmetric bimonoidal category (Theorem I.8.15.4).

- Show that the matrix construction $\mathsf{Mat}^{\mathsf{C}}$ of a tight Laplaza E_{k+1}-monoidal category (Question A.2.6) is a k-fold monoidal bicategory.

This question asks for a refinement of the table in the introduction of Chapter II.8 as follows.

tight – category C	– bicategory $\mathsf{Mat}^{\mathsf{C}}$	
bimonoidal	plain	I.8.4.12
braided bimonoidal	monoidal	II.8.4.7
Laplaza E_{k+1}-monoidal ($k \ge 1$)	k-fold monoidal	conjecture
symmetric bimonoidal	symmetric monoidal	I.8.15.4

The k-fold monoidal bicategories in the conjectural row refer to the $n = 2$ column in the unstable periodic table in Question A.2.4. Proving that $\mathsf{Mat}^{\mathsf{C}}$ is a k-fold monoidal bicategory will certainly involve the coherence theorem for Laplaza E_{k+1}-monoidal categories in Question A.2.6. The general picture of the table above is that it takes a sum $\oplus$ and a product $\otimes$ to construct the matrix bicategory $\mathsf{Mat}^{\mathsf{C}}$. So any further monoidal structures on the bicategory $\mathsf{Mat}^{\mathsf{C}}$ would have to come from further monoidal structures on C. ⋄

Question A.2.8 (Higher Sheet Diagrams)**.** As in Question A.1.6, develop higher dimensional sheet diagrams for

- n-fold monoidal categories (Definition II.10.1.1),
- n-monoidal categories (Question A.2.2),
- lax n-fold monoidal categories (Question A.2.3),
- E_n-monoidal categories (Definition II.10.7.2), and
- Laplaza E_n-monoidal categories (Question A.2.6).

As discussed in Question A.1.6, each of these items is a coherence question with several parts. ⋄

A.3. Enriched Monoidal Categories

The following questions are about the concepts in Chapters 1, 2, and 3.

Question A.3.1 (Enriched Lax n-Fold Monoidal Categories)**.** In Section 1.4, with V a braided monoidal category, we defined monoidal, braided monoidal, and symmetric monoidal V-categories, with the latter two assuming that V is symmetric. See Lemma 1.3.23 and Explanation 1.3.25.

- For $n \geq 2$ and V a symmetric monoidal category, extend the lax n-fold monoidal categories in Question A.2.3 to the V-enriched setting.
- Extend the results in Chapter 2 to lax n-fold monoidal V-categories.

Analogous to Propositions II.10.1.14 and II.10.1.21, braided monoidal V-categories should be examples of lax 2-fold monoidal V-categories, and symmetric monoidal V-categories should be examples of lax n-fold monoidal V-categories for $n \geq 2$. Theorem II.10.4.5 says that small $(n+1)$-fold monoidal categories are precisely the monoids in the monoidal category MCat^n of small n-fold monoidal categories and n-fold monoidal functors.

- Extend Theorem II.10.4.5 to the V-enriched setting. ⋄

Question A.3.2 (Centers and Enriched Centers)**.** The Drinfeld center of a monoidal category is a braided monoidal category (Theorem II.1.4.27), and the symmetric center of a braided monoidal category is a symmetric monoidal category (Proposition II.1.5.3). Moreover, the Drinfeld center and the symmetric center are generalized to (i) the bimonoidal setting in Theorems II.4.4.3 and II.4.5.3 and (ii) the ring categorical setting in Corollary II.9.6.1 and Theorem II.9.6.4.

- Define a center construction that sends an n-fold monoidal category (Definition II.10.1.1) to an $(n+1)$-fold monoidal category.

Theorem II.10.4.5 should be relevant.

- Repeat the previous question for
  - n-monoidal categories (Question A.2.2),
  - lax n-fold monoidal categories (Question A.2.3),
  - E_n-monoidal categories (Definition II.10.7.2),
  - Laplaza E_n-monoidal categories (Question A.2.6), and
  - lax n-fold monoidal V-categories (Question A.3.1).

As a special case of the last item, the Drinfeld center of a monoidal V-category with V *strict* is studied in [**KYZZ∞, KZ18**]. ⋄

Question A.3.3 (Autonomous Enriched Monoidal Categories)**.** The definition of a monoidal V-category K in [**MP19**, 2.1] assumes that V is braided strict monoidal and K is strict monoidal, so it is the special case of Definition 1.4.2 with both V and K strict. The main theorem in [**MP19**, 1.1] shows that, using their definition, there is a bijective correspondence between (i) some autonomous monoidal V-categories and (ii) some braided oplax monoidal functors from V to the Drinfeld center of an autonomous monoidal category. This bijective correspondence is extended to a 2-equivalence between 2-categories in [**Del**∞, 1.2]. In bicategorical language (Definition I.6.3.9), *autonomy* means that each object $X \in \mathsf{K}$ is equipped with both a left adjoint X_* and a right adjoint X^* that satisfy the triangle identities. A (braided) monoidal functor (F, F^2, F^0) is *oplax* if its monoidal constraint F^2 and unit constraint F^0 go in the opposite directions as those in Definitions 1.1.6 and 1.1.17, with appropriately adjusted axioms. Discussion of autonomous monoidal categories can be found in [**FY92, JS91b, JS93**].

- Extend the 2-equivalence in [**Del**∞, 1.2] to general monoidal V-categories K, with V and K not necessarily strict, as in Definition 1.4.2.

To extend this 2-equivalence to the general nonstrict case, the coherence and strictification results of enriched monoidal categories in Sections 2.5 and 2.6 will likely be necessary. ◇

A.4. Homotopy Theory

Question A.4.1 (Homotopy Theory of Matrix Bicategories)**.** In Example I.8.15.5, we listed some examples of tight symmetric bimonoidal categories C, to which the Bicategorification Theorem I.8.15.4 may be applied to yield a symmetric monoidal bicategory $\mathsf{Mat}^{\mathsf{C}}$.

- What can be said about the homotopy theoretic properties of any of these symmetric monoidal bicategories?
- Consider the previous question for the symmetric bimonoidal bicategories in Question A.1.2.

For instance, for the finite ordinal category Σ in Section I.2.4, Mat^{Σ} may be related to a remark in [**JO12**] about the multiplicative structure on the categorical model for the sphere spectrum. ◇

Question A.4.2 (Categorical Model of BP)**.** The Brown-Peterson spectrum BP has an E_4 structure [**BM13**], but not an E_∞ structure at any prime [**Law18, Sen**∞]. By Corollary 13.5.2, the Elmendorf-Mandell K-theory of each small E_n-monoidal category is an E_n-symmetric spectrum.

- Is there a small E_4-monoidal category (Definition II.10.7.2) whose K-theory is BP?

A positive answer to this question would provide a categorical model of BP. If Question A.5.7 has a positive answer for the 4-fold monoidal category operad Mon^4, then this question also has a positive answer, at least up to weak equivalences. ◇

Question A.4.3 (Boardman-Vogt E_n-Operads)**.** In a symmetric monoidal category C, the commutative operad Com is the operad with each entry the monoidal unit $\mathbb{1}$ and structure morphisms given by the coherence isomorphism $\mathbb{1} \otimes \mathbb{1} \cong \mathbb{1}$. The algebras of Com are precisely commutative monoids. One model of an E_∞-operad

(Definition 11.6.1) is WCom, where W is the Boardman-Vogt W-construction. In the topological setting, it was introduced in [**BV73**]. For a conceptual presentation of the W-construction in terms of coends in a general symmetric monoidal category, see [**Yau20**, Ch. 6–7].

- Describe E_n-operads (Definition 12.2.3) for $n \geq 1$ as a filtration of sub-operads of WCom.
- Compare these models of E_n-operads to
  - the n-fold monoidal category operad Mon^n (Definition 13.1.12),
  - Berger's filtration [**Ber96**] of the Barratt-Eccles operad $N(E\mathsf{As})$ (Definition 11.4.10),
  - Smith's filtration [**Smi89**] of the Barratt-Eccles operad,
  - Batanin's E_n-operads [**Bat07, Bat08**],
  - Fiedorowicz's E_n-operads [**Fie∞b**], and
  - the Fulton-MacPherson E_n-operads [**Fre17, FM94, GJ∞, Sal01**].

In [**Yau20**, 3.2.11 and 6.3.1], the W-construction of an operad in a symmetric monoidal category is defined entrywise as a coend indexed by a substitution category whose objects are trees and whose morphisms correspond to tree substitution. This applies, in particular, to the commutative operad Com. So describing E_n-operads as sub-operads of WCom would provide a combinatorial description of E_n-operads in terms of trees. $\diamond$

A.5. Algebraic K-Theory

Question A.5.1 (Multifunctorial K-Theory of Pointed Multicategories)**.** Contrary to the claim in [**EM09**, Theorem 1.3], Elmendorf-Mandell J-theory J^{EM} (Definition 10.3.25) does *not* extend to a multifunctor on all of $\mathsf{Multicat}_*$, the category of small pointed multicategories, but only to the full subcategory $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ of left $\mathcal{M}\underline{1}$-modules, via the symmetric monoidal Cat_*-functor $\mathsf{J}^{\mathcal{T}}$ (Theorem 10.3.17). Examples 10.2.15 and 10.2.16 present some small pointed multicategories that are not left $\mathcal{M}\underline{1}$-modules.

- Is there a K-theory multifunctor that is objectwise equivalent to Segal K-theory K^{Se} and extends to $\mathsf{Multicat}_*$ via the endomorphism multicategory End in Corollary 5.3.9 and Definition 6.5.1?

The key issue is about the monoidal units. In $\mathsf{Multicat}_*$ the smash unit is $\mathsf{S} = \mathsf{I} \coprod \mathsf{T}$ (Definition 5.6.20), which is different from the monoidal unit $\mathcal{M}\underline{1}$ in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. Unlike Definition 10.3.16 with $\mathsf{J}^{\mathcal{T}}(\mathcal{M}\underline{1})$, the object $\mathsf{J}^{\mathcal{T}}(\mathsf{S})$ is the terminal $\mathcal{G}_*$-category $*$. If the monoidal unit constraint $(\mathsf{J}^{\mathcal{T}})^0$ for $\mathsf{J}^{\mathcal{T}}$ were to be defined as the unique morphism $J \longrightarrow \mathsf{J}^{\mathcal{T}}(\mathsf{S}) = *$ to the terminal object, as stated in the last paragraph in [**EM09**, Section 5], then $\mathsf{J}^{\mathcal{T}}$ cannot satisfy the unity axioms (1.1.10). The reason is that a general left or right unit isomorphism for $\mathsf{J}^{\mathcal{T}}(-)$ does not factor through the zero morphism in $\mathcal{G}_*$-Cat. So with $J \longrightarrow *$ as the unit constraint, $\mathsf{J}^{\mathcal{T}}$ would not be a monoidal functor. $\diamond$

Question A.5.2 (Comparison of K^{Se} and K^{EM} for $\mathcal{M}\underline{1}$-Modules)**.** The Segal and Elmendorf-Mandell K-theory constructions are defined as the following composites, respectively:

$$\mathsf{K}^{\mathsf{Se}} = \mathsf{K}^{\mathcal{F}} N_* \mathsf{J}^{\mathcal{M}} \mathsf{End} \quad \text{and} \quad \mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathcal{T}} \mathsf{End}.$$

The domain of $\mathsf{J}^{\mathcal{M}}$ is the category of small pointed multicategories, $\mathsf{Multicat}_*$, and the domain of $\mathsf{J}^{\mathcal{T}}$ is the category of left $\mathcal{M}\underline{1}$-modules within $\mathsf{Multicat}_*$. Thus both K^{Se} and K^{EM} can be expanded to $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. We will write

$$\mathsf{Mod}^{\mathcal{M}\underline{1}} \xrightarrow{\tilde{\mathsf{K}}^{\mathsf{Se}} = \mathsf{K}^{\mathcal{F}} N_* \mathsf{J}^{\mathcal{M}}} \mathsf{SymSp} \quad \text{and} \quad \mathsf{Mod}^{\mathcal{M}\underline{1}} \xrightarrow{\tilde{\mathsf{K}}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathcal{T}}} \mathsf{SymSp}.$$

- Is there a (natural) level equivalence $\tilde{\mathsf{K}}^{\mathsf{Se}}\mathsf{P} \longrightarrow \tilde{\mathsf{K}}^{\mathsf{EM}}\mathsf{P}$ for each left $\mathcal{M}\underline{1}$-module P?

The level equivalence $\mathsf{K}^{\mathsf{Se}}\mathsf{C} \longrightarrow \mathsf{K}^{\mathsf{EM}}\mathsf{C}$ given in Theorem 10.6.10 for each small permutative category C depends crucially on the adjunctions of Proposition 10.6.7 and these, in turn, depend on Proposition 8.5.4, which gives a strong symmetric monoidal adjunction

$$\mathsf{Cat}_*(a, \mathsf{C}) \underset{R}{\overset{L}{\rightleftarrows}} \mathsf{Multicat}_*(\mathcal{M}a, \mathsf{End}(\mathsf{C}))$$

for each small permutative category C and each pointed finite set a. Therefore the proof of Theorem 10.6.10 does not immediately generalize to $\tilde{\mathsf{K}}^{\mathsf{Se}}$ and $\tilde{\mathsf{K}}^{\mathsf{EM}}$. ◇

Question A.5.3 (Homotopy Theory of $\mathcal{G}_*$-Objects)**.** Notes 8.6.4 and 8.6.6 summarize equivalences of homotopy categories from the work of **[BF78, Tho95, Man10]**:

$$Ho(\mathsf{PermCat}^{\mathsf{su}}) \simeq Ho(\Gamma\text{-}\mathsf{Cat}) \simeq Ho(\Gamma\text{-}\mathsf{sSet}) \simeq Ho(\mathsf{Sp}^{\mathbb{N}}_{\geq 0}),$$

where $\mathsf{Sp}^{\mathbb{N}}_{\geq 0}$ denotes the category of connective sequential spectra. The composite from the homotopy category of small permutative categories to that of connective spectra is induced by Segal K-theory, K^{Se}. Therefore, by Theorem 10.6.10, the composite on homotopy categories is equal to that induced by Elmendorf-Mandell K-theory, K^{EM}. However, K^{EM} factors through $\mathcal{G}_*$-Cat and $\mathcal{G}_*$-sSet instead of Γ-Cat and Γ-sSet.

As diagram categories, $\mathcal{G}_*$-Cat and $\mathcal{G}_*$-sSet have Quillen model structures defined similarly to those of Γ-Cat and Γ-sSet. The functor

$$\wedge : \mathcal{G} \longrightarrow \mathcal{F}$$

is described in Definition 9.1.15 and is used in the comparison of $\mathsf{K}^{\mathcal{F}}$ and $\mathsf{K}^{\mathcal{G}}$ in Proposition 9.3.16. The induced functor from Γ-objects to $\mathcal{G}_*$-objects is symmetric monoidal by Theorem 9.4.18.

- Are the induced functors

$$\Gamma\text{-}\mathsf{Cat} \xrightarrow{\wedge^*} \mathcal{G}_*\text{-}\mathsf{Cat} \quad \text{and} \quad \Gamma\text{-}\mathsf{sSet} \xrightarrow{\wedge^*} \mathcal{G}_*\text{-}\mathsf{sSet}$$

  (monoidal) Quillen equivalences?

Question A.5.7 poses a related question regarding categories of algebras over operads. ◇

Question A.5.4 (K-Theory of Matrix Bicategories)**.** There is a K-theory construction via the following composite on objects only.

$$\mathsf{Bi}^{\mathsf{tsy}} \xrightarrow{\mathsf{Mat}} \mathsf{SMB} \xrightarrow{\text{strictify}} \mathsf{PGray} \xrightarrow{K} \Gamma 2\mathsf{Cat}$$

Mat is the matrix construction in Theorem I.8.15.4 that sends a tight symmetric bimonoidal category to a symmetric monoidal bicategory. The middle arrow is the strictification of symmetric monoidal bicategories to permutative Gray monoids. The right arrow is the K-theory of permutative Gray monoids in **[GJO17b]**.

- How does this compare with the K-theory of tight symmetric bimonoidal categories in [**BDR04**], which is defined using the direct sum instead of the tensor product of matrices?
- What extra structures on Γ-2-categories are there when it is the K-theory of the strictification of $\mathsf{Mat}^{\mathsf{C}}$ for some tight symmetric bimonoidal category C, such as those in Example I.8.15.5 and $\mathsf{Vect}_{\mathsf{c}}^{\mathbb{C}}$ in Example I.2.5.9? $\diamond$

Question A.5.5 (K-Theory of Matrix Symmetric Bimonoidal Bicategories)**.** Repeat Question A.5.4 for

- the matrix symmetric bimonoidal bicategories in Question A.1.2 and
- the bipermutative Gray monoids in Question A.1.4. $\diamond$

Question A.5.6 (K-Theory of Distortion Categories)**.** Recall the finite ordinal category Σ in Definition I.2.4.1. Here we consider Σ as a permutative category with respect to its additive structure $\oplus$. Quillen's $+$-construction $(B\Sigma)^+$ of the classifying space $B\Sigma$ is the sphere spectrum by the Barratt-Priddy-Quillen Theorem [**BP72**]. A different way to say this is that the algebraic K-groups of Σ are the stable homotopy groups of the spheres.

- Can the algebraic K-groups of
  - the distortion category $\mathcal{D}$ in Section I.4.2,
  - the additive distortion category $\mathcal{D}^{\mathsf{ad}}$ in Section I.4.5, and
  - the braided distortion category $\mathcal{D}^{\mathsf{br}}$ in Section II.5.2

  be computed in similar terms?

By Examples 11.3.18, 11.6.13, and 12.5.4, respectively, $\mathsf{K}^{\mathsf{EM}}\mathcal{D}^{\mathsf{ad}}$, $\mathsf{K}^{\mathsf{EM}}\mathcal{D}$, and $\mathsf{K}^{\mathsf{EM}}\mathcal{D}^{\mathsf{br}}$ are strict ring, E_∞-, and E_2-symmetric spectra. Moreover, each of the distortion categories $\mathcal{D}$, $\mathcal{D}^{\mathsf{ad}}$, and $\mathcal{D}^{\mathsf{br}}$ is a Grothendieck construction over Σ by, respectively, Propositions I.4.6.5, I.4.6.7, and II.5.5.3.

- Does that yield a computation of their $(B?)^+$ and algebraic K-groups? $\diamond$

Question A.5.7 (Lifting K-Theory Equivalences to Algebras)**.** The Segal K-theory functor in Definition 8.5.1 induces an equivalence of homotopy categories, from permutative categories to connective symmetric spectra. See Question A.5.3 for further description and a related question about K^{EM}.

- Do the (Quillen) equivalences in Segal's K-theory lift to the categories of algebras over categorical operads, such as the E_2-operad Br and the E_n-operads Mon^n in Theorems 12.2.4 and 13.2.1?

For a categorical operad P and a Cat-enriched multicategory M, such as $\mathsf{PermCat}^{\mathsf{su}}$ in Section 6.6, a P-*algebra in* M is defined as a Cat-enriched multifunctor

$$F : \mathsf{P} \longrightarrow \mathsf{M}.$$

This is equivalent to a Cat-enriched operad morphism

$$\mathsf{P} \longrightarrow \mathsf{End}(A)$$

to the Cat-enriched endomorphism operad of the object $A = F(*)$, with $*$ the unique object in the multicategory P. If the answer to this question is yes for a categorical operad P, then Segal's K-theory induces an equivalence between the homotopy categories of (i) P-algebras in permutative categories and (ii) P-algebras in connective symmetric spectra.

To answer this question, it is tempting to use [**WY19**, Th. 4.4 and 4.6], which give sufficient conditions under which a Quillen equivalence between monoidal model categories lifts to a Quillen equivalence between the categories of algebras over some colored operads. This will *not* work because the domain of Segal's K-theory is the multicategory $\mathsf{PermCat}^{\mathsf{su}}$ (Section 6.6), which is not a symmetric monoidal category, hence also not a monoidal model category. $\diamond$

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List of Main Facts

Part I.1. Symmetric Bimonoidal Categories

Chapter I.1. Basic Category Theory

(1.1.11) An adjunction satisfies the triangle identities.
(p. I.11) A functor is an equivalence if and only if it is fully faithful and essentially surjective.
(1.1.14) Left adjoints preserve colimits. Right adjoints preserve limits.
(1.2.1) A monoidal category satisfies the unity axiom and the pentagon axiom.
(1.2.7) A monoidal category satisfies the left and the right unity properties.
(1.3.3) **Mac Lane's Coherence Theorem**. Any two words of the same length in a monoidal category are connected by a unique canonical map.
(1.3.5) **Mac Lane's Strictification Theorem**. Each monoidal category is adjoint equivalent to a strict monoidal category via strong monoidal functors.
(1.3.8) **Symmetric Coherence Theorem**. Any two permuted words of the same length in a symmetric monoidal category are connected by a unique permuted canonical map.
(1.3.10) **Symmetric Strictification Theorem**. Each symmetric monoidal category is adjoint equivalent to a permutative category via strong symmetric monoidal functors.
(1.3.12) **Epstein's Coherence Theorem**. For each (symmetric) monoidal functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ and F-iterates $G, H : \mathsf{C}^n \longrightarrow \mathsf{D}$, there exists at most one F-coherent map $G \longrightarrow H$.

Chapter I.2. Symmetric Bimonoidal Categories

(2.1.2) A symmetric bimonoidal category has two symmetric monoidal structures, left/right multiplicative zero natural isomorphisms, and left/right distributivity natural monomorphisms, and satisfies 24 axioms.
(2.1.2) A bimonoidal category is defined in the same way as a symmetric bimonoidal category, but without the multiplicative symmetry $\xi^{\otimes}$ and the two axioms that involve $\xi^{\otimes}$. So a bimonoidal category is defined by the other 22 axioms.
(2.1.32) There is a tight symmetric bimonoidal category $\mathsf{Vect}^{\mathbb{C}}$ of finite dimensional complex vector spaces.
(2.2.13) Half of the 24 axioms in a symmetric bimonoidal category are formal consequences of the others.
(2.2.14) One axiom is redundant in a bimonoidal category, which is, therefore, determined by 21 axioms.
(2.3.2) Each distributive symmetric monoidal category yields a tight symmetric bimonoidal category, whose sum is the coproduct.

(2.3.3–2.3.5) Symmetric monoidal closed categories with finite coproducts, the category of modules over a commutative ring, and distributive categories are examples of tight symmetric bimonoidal categories.
(2.4.8) The category Σ of finite ordinals and permutations is a tight symmetric bimonoidal category.
(2.4.23) The variant Σ' of Σ is a tight symmetric bimonoidal category.
(2.5.7) Each right bipermutative category is a tight symmetric bimonoidal category.
(2.5.8–2.5.9) Σ' and $\mathsf{Vect}_c^{\mathbb{C}}$ are right bipermutative categories.
(2.5.16) Each left bipermutative category is a tight symmetric bimonoidal category.
(2.5.17) Σ is a left bipermutative category.
(2.6.2) There is a symmetric bimonoidal groupoid Π with syntax of finite types as objects and Π-terms and Π-combinators as morphisms.

Chapter I.3. Coherence of Symmetric Bimonoidal Categories

(3.1.6) In the elementary graph $\mathsf{Gr}^{\mathsf{el}}(X)$, δ^l and δ^r do not have formal inverses.
(3.1.8) Each prime edge involves at most one nonidentity elementary edge.
(3.1.9) The graph $\mathsf{Gr}(X)$ consists of the vertex set X^{fr} and prime edges.
(3.1.14) Each functor $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$ extends additively and multiplicatively to a graph morphism $\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$.
(3.1.18) The value in C of a path in $\mathsf{Gr}(X)$ is the composite of the images of its constituent prime edges under φ.
(3.1.25) An element in X^{fr} is regular if it has the same support as a formal polynomial whose monomials are distinct in the strict $\{\oplus,\otimes\}$-algebra X^{st}, and whose factors in each monomial are distinct elements in X.
(3.1.29) Any two elements in X^{fr} connected by a path in $\mathsf{Gr}(X)$ have the same support, and one of them is regular if and only if the other one is regular.
(3.2.15) For an element in X^{fr}, the size is equal to the rank if and only if it is a sum, with each summand either in X or a product of two elements in X.
(3.3.6) Each element in X^{fr} has a 0^X-reduction.
(3.3.11–3.3.12) Any two 0^X-reductions of an element in X^{fr} have the same codomain and the same value in a symmetric bimonoidal category.
(3.5.32) Each path in $\mathsf{Gr}(X)$ has a 0^X-reduction.
(3.5.33) Any two parallel paths in $\mathsf{Gr}(X)$ whose domain has the same support as 0^X have the same value in a symmetric bimonoidal category.
(3.6.5) An element in X^{fr} is a polynomial if and only if it is δ-reduced.
(3.6.9) Each element in X^{fr} has a δ-reduction.
(3.7.19) Each path in $\mathsf{Gr}(X)$ that does not contain 0^X has a $(0^X,\delta)$-reduction.
(3.8.5–3.8.7) Each element in X^{fr} has a 1^X-reduction. If the original element is δ-reduced, then all of its 1^X-reductions have the same codomain and the same value in a symmetric bimonoidal category.
(3.8.14) Each $(0^X,\delta)$-free path in $\mathsf{Gr}(X)$ whose (co)domain is δ-reduced has a 1^X-reduction.
(3.9.1) **Laplaza's First Coherence Theorem**. In each symmetric bimonoidal category C satisfying a monomorphism assumption, any two parallel paths in $\mathsf{Gr}(X)$ with a regular domain have the same value in C.
(3.9.9–3.9.10) Theorem I.3.9.1 applies to symmetric bimonoidal categories that are flat, in particular, tight.

(3.10.7) **Bimonoidal Coherence Theorem**. In each bimonoidal category C satisfying a monomorphism assumption, any two parallel paths in $\mathsf{Gr}^{\mathsf{ns}}(X)$ with a non-symmetric regular domain have the same value in C.
(3.10.8) Theorem I.3.10.7 applies to flat, in particular, tight, bimonoidal categories.

Chapter I.4. Coherence of Symmetric Bimonoidal Categories II

(4.2.1) In the distortion category, each object is a finite sequence of nonnegative integers, and each morphism is a finite sequence of permutations.
(4.2.5) The distortion category is a groupoid.
(4.2.12) The additive structure of the distortion category is a permutative category.
(4.2.19) The multiplicative structure of the distortion category is a permutative category.
(4.2.29) The distortion category is a left bipermutative category.
(4.3.1) For a path in $\mathsf{Gr}(X)$, its distortion is defined as its value in the distortion category.
(4.4.3) **Laplaza's Second Coherence Theorem**. In each symmetric bimonoidal category C satisfying a monomorphism assumption, any two parallel paths in $\mathsf{Gr}(X)$ with the same distortion have the same value in C.
(4.4.5) Theorem I.4.4.3 applies to symmetric bimonoidal categories that are flat, in particular, tight.
(4.5.2) In the additive distortion category, each object is a finite sequence of nonnegative integers, and each morphism is a permutation.
(4.5.6) The additive distortion category is a groupoid and a tight bimonoidal category. It faithfully embeds into the distortion category.
(4.5.7) For a path in $\mathsf{Gr}^{\mathsf{ns}}(X)$, its additive distortion is defined as its value in the additive distortion category.
(4.5.8) **Bimonoidal Coherence Theorem II**. In each bimonoidal category C satisfying a monomorphism assumption, any two parallel paths in $\mathsf{Gr}^{\mathsf{ns}}(X)$ with the same additive distortion have the same value in C.
(4.5.9) Theorem I.4.5.8 applies to flat, in particular, tight, bimonoidal categories.
(4.6.5) The distortion category $\mathcal{D}$ is isomorphic to the Grothendieck construction $\int_{\Sigma} F$.
(4.6.7) The additive distortion category $\mathcal{D}^{\mathsf{ad}}$ is isomorphic to the Grothendieck construction $\int_{\Sigma} F^{\mathsf{ad}}$.

Chapter I.5. Strictification of Tight Symmetric Bimonoidal Categories

(5.1.1) A symmetric bimonoidal functor is a functor equipped with two symmetric monoidal functor structures, and satisfies two axioms.
(5.1.10) There is a 1-category $\mathsf{Bi}^{\mathsf{sy}}$ of small symmetric bimonoidal categories and symmetric bimonoidal functors.
(5.1.11) Each symmetric monoidal functor between distributive symmetric monoidal categories induces a symmetric bimonoidal functor.
(5.1.15–5.1.16) Σ and Σ' are isomorphic via symmetric bimonoidal functors.
(5.3.9) Each tight symmetric bimonoidal category C has an associated right bipermutative category A, whose objects are formal polynomials in the objects in C.
(5.4.6–5.4.7) **Bipermutative Strictification Theorems**. Each tight symmetric bimonoidal category is adjoint equivalent to a right/left bipermutative category via symmetric bimonoidal functors.

(5.5.1) A bimonoidal functor is a functor equipped with an additive symmetric monoidal functor structure and a multiplicative monoidal functor structure that satisfies four axioms.
(5.5.4) There is a 1-category Bi of small bimonoidal categories and bimonoidal functors.
(5.5.10) Each tight bimonoidal category C has an associated right rigid bimonoidal category A, whose objects are formal polynomials in the objects in C.
(5.5.11–5.5.12) **Rigid Strictification Theorems**. Each tight bimonoidal category is adjoint equivalent to a right/left rigid bimonoidal category via bimonoidal functors.

Part I.2. Bicategorical Aspects of Symmetric Bimonoidal Categories

Chapter I.6. Definitions from Bicategory Theory

(6.1.2) A bicategory has objects, (identity) 1-cells, (identity) 2-cells, vertical and horizontal compositions, an associator, and two unitors, and satisfies the unity axiom and the pentagon axiom.
(6.1.8) A 2-category is a bicategory whose associator and unitors are identities.
(6.1.10) A 2-category can be described by data and axioms.
(6.1.11) A locally small 2-category is precisely a Cat-category.
(6.1.16) A monoidal category is a one-object bicategory.
(6.2.1) A lax functor has an object assignment, local functors, and two laxity constraints, and satisfies the lax associativity axiom and the lax unity axioms.
(6.2.11) There is a 1-category Bicat with small bicategories as objects and lax functors as morphisms.
(6.2.14) A lax transformation has component 1-cells and natural component 2-cells, and satisfies the lax unity axiom and the lax naturality axiom.
(6.2.26) There is a 2-category $\mathsf{2Cat}$ of small 2-categories, 2-functors, and 2-natural transformations.
(6.3.1) A modification has component 2-cells and satisfies the modification axiom.
(6.3.7) For bicategories B and B' with $\mathsf{Ob}(\mathsf{B})$ a set, there is a bicategory $\mathsf{Bicat}(\mathsf{B},\mathsf{B}')$ with lax functors $\mathsf{B} \longrightarrow \mathsf{B}'$ as objects, lax transformations as 1-cells, and modifications as 2-cells. It is a 2-category if B' is a 2-category. It contains a full subbicategory $\mathsf{Bicat}^{\mathsf{ps}}(\mathsf{B},\mathsf{B}')$ with pseudofunctors as objects and strong transformations as 1-cells.
(6.3.9) An adjunction in a bicategory consists of two 1-cells and two 2-cells, and satisfies two triangle identities.
(6.4.1) A monoidal bicategory has a base bicategory, a monoidal composition, a monoidal identity, a monoidal associator, two monoidal unitors, a pentagonator, and three 2-unitors, and satisfies the non-abelian 4-cocycle condition and two normalization axioms.
(6.5.3) A braided monoidal bicategory is a monoidal bicategory equipped with a braiding and two hexagonators, and satisfies four axioms.
(6.5.7) A sylleptic monoidal bicategory is a braided monoidal bicategory equipped with a syllepsis that satisfies two axioms.
(6.5.9) A symmetric monoidal bicategory is a sylleptic monoidal bicategory that satisfies the triple braid axiom.
(6.6.12) The 1-category $\mathsf{2Cat}$ equipped with the Gray tensor product is a symmetric monoidal closed category Gray.

(6.6.13) A Gray monoid is a monoid in Gray.
(6.7.1) A permutative Gray monoid is a Gray monoid equipped with a Gray symmetry that satisfies three axioms.
(6.7.16) A permutative 2-category is a monoid in $(\mathsf{2Cat}, \times)$ that is equipped with a symmetry 2-natural isomorphism and that satisfies the same three axioms as for permutative Gray monoids.

Chapter I.7. Baez's Conjecture

(7.1.2) A bimonoidal natural transformation is a natural transformation that is a monoidal natural transformation for each of the additive structure and the multiplicative structure.
(7.1.7) There is a 2-category $\mathsf{Bi}^{\mathsf{sy}}$ of small symmetric bimonoidal categories, symmetric bimonoidal functors, and bimonoidal natural transformations.
(7.1.9) Each monoidal natural transformation between symmetric monoidal functors between distributive symmetric monoidal categories is a bimonoidal natural transformation.
(7.2.9) For each symmetric bimonoidal category C, there is a strong symmetric monoidal functor $F_\oplus : \Sigma \longrightarrow \mathsf{C}$ between the additive structures.
(7.3.28) For each flat symmetric bimonoidal category C, there is a symmetric monoidal functor $F_\otimes : \Sigma \longrightarrow \mathsf{C}$ between the multiplicative structures.
(7.4.4) For each flat symmetric bimonoidal category C, $F : \Sigma \longrightarrow \mathsf{C}$ is a robust symmetric bimonoidal functor.
(7.5.8) Epstein's Coherence Theorem I.1.3.12 has a bimonoidal analogue.
(7.6.2–7.6.3) For a symmetric bimonoidal category C and robust symmetric bimonoidal functors $G, H : \Sigma \longrightarrow \mathsf{C}$, there is at most one bimonoidal natural transformation $G \longrightarrow H$, which must be invertible if it exists.
(7.7.9) For each flat symmetric bimonoidal category C and robust symmetric bimonoidal functor $G : \Sigma \longrightarrow \mathsf{C}$, there exists a unique bimonoidal natural transformation $\theta : F \longrightarrow G$, which is, moreover, invertible.
(7.8.1) **Baez's Conjecture**. Σ is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$.
(7.8.3) **Baez's Conjecture, Ver. 2**. Σ' is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fsy}}$.

Chapter I.8. Symmetric Monoidal Bicategorification

(8.1.1) For a category C, $\mathsf{Mat}^{\mathsf{C}}_{m,n}$ has $n \times m$ matrices of objects in C as objects and $n \times m$ matrices of morphisms in C as morphisms.
(8.1.8) The matrix product is a functor.
(8.2.2) There is a natural isomorphism $\ell_A : \mathbb{1}^n A \xrightarrow{\cong} A$ for each flat bimonoidal category C and $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.
(8.2.8) There is a natural isomorphism $r_A : A\mathbb{1}^m \xrightarrow{\cong} A$ for each flat bimonoidal category C and $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.
(8.3.1) There is a natural isomorphism $a : (CB)A \xrightarrow{\cong} C(BA)$ for each tight bimonoidal category C.
(8.4.12) For each tight bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory.
(8.4.14) $\mathsf{Mat}^{\mathsf{C}}_{n,n}$ is a monoidal category.
(8.6.7) The matrix tensor product is a functor.
(8.7.31) The triple $(\boxtimes, \boxtimes^2, \boxtimes^0)$ on $\mathsf{Mat}^{\mathsf{C}}$ is a pseudofunctor.
(8.8.49) The quadruple $(a^{\boxtimes}, a^{\boxtimes^\bullet}, \eta^a, \varepsilon^a)$ is an adjoint equivalence.

(8.9.9) The quadruple $(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$ is an adjoint equivalence.
(8.9.21) The quadruple $(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^{r}, \varepsilon^{r})$ is an adjoint equivalence.
(8.10.4) π is an invertible modification.
(8.11.4) μ is an invertible modification.
(8.11.9) $\lambda^{\boxtimes}$ is an invertible modification.
(8.11.14) $\rho^{\boxtimes}$ is an invertible modification.
(8.12.9) For each tight symmetric bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory.
(8.13.13) For $\sigma \in \Sigma_m$, there is a natural isomorphism $r^{\sigma}_{A} : A\mathbb{1}^{\sigma} \xrightarrow{\cong} A^{\sigma}$ for $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.
(8.13.16) For $\theta \in \Sigma_n$, there is a natural isomorphism $\ell^{\theta}_{A} : \mathbb{1}^{\theta}A \xrightarrow{\cong} {}_{\theta^{-1}}A$ for $A \in \mathsf{Mat}^{\mathsf{C}}_{m,n}$.
(8.13.20) The matrix tensor products $A \boxtimes B$ and $B \boxtimes A$ differ by a column permutation, a row permutation, and a multiplicative symmetry in each entry.
(8.13.44) The quadruple $(\beta, \beta^{\bullet}, \eta^{\beta}, \varepsilon^{\beta})$ is an adjoint equivalence.
(8.14.12) $R_{-|--}$ is an invertible modification.
(8.14.24) $R_{--|-}$ is an invertible modification.
(8.14.26) For each tight symmetric bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ is a braided monoidal bicategory.
(8.15.4) **Bicategorification Theorem**. For each tight symmetric bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ is a symmetric monoidal bicategory.
(8.15.5) Coordinatized 2-vector spaces $\mathsf{2Vect_c} = \mathsf{Mat}^{\mathsf{Vect}^{\mathbb{C}}}$ form a symmetric monoidal bicategory.

Part II.1. Braided Bimonoidal Categories

Chapter II.1. Preliminaries on Braided Structures

(1.1.1) The braid group B_n on n strings is generated by $s_1, \ldots, s_{n-1}$ and subject to two braid relations.
(1.1.9) Sum braids generalize block sums of permutations.
(1.1.12) Each braid has an underlying permutation.
(1.1.20) Block braids generalize block permutations.
(1.2.4) Elementary block braids generalize interval-swapping permutations.
(1.2.14) Elementary block braids are compatible with sum braids.
(1.2.16) Elementary block braids satisfy the hexagon axioms.
(1.3.15) A braided monoidal category is a monoidal category equipped with a braiding that satisfies two hexagon axioms.
(1.3.18) A braided monoidal functor is defined in the same way as a symmetric monoidal functor.
(1.3.21) In each braided monoidal category, the left unit isomorphism uniquely determines the right unit isomorphism, and vice versa, via the braiding.
(1.3.28, 1.3.31) Each braided monoidal category satisfies the third Reidemeister move.
(1.3.36) A symmetric monoidal category is precisely a braided monoidal category whose braiding satisfies the symmetry axiom.
(1.4.27) The Drinfeld center of a monoidal category is a braided monoidal category.
(1.5.3) The symmetric center of a braided monoidal category is a symmetric monoidal category.

(1.6.3) **Braided Coherence Theorem**. Two braided canonical maps with the same (co)domain in a braided monoidal category are equal if their underlying braids are equal.
(1.6.5) **Braided Strictification Theorem**. Each braided monoidal category is adjoint equivalent to a braided strict monoidal category via strong braided monoidal functors.

Chapter II.2. Braided Bimonoidal Categories

(2.1.29) A braided bimonoidal category is a category equipped with an additive symmetric monoidal structure, a multiplicative braided monoidal structure, left/right multiplicative zero natural isomorphisms, and left/right distributivity natural monomorphisms, and satisfies twelve Laplaza's axioms and two additional axioms involving the braiding.
(2.1.37) Tight braided bimonoidal categories are equivalent to BD categories in the sense of Blass and Gurevich.
(2.2.1) Each braided bimonoidal category satisfies all 24 Laplaza axioms.
(2.2.3) A symmetric bimonoidal category is precisely a braided bimonoidal category whose braiding satisfies the symmetry axiom.
(2.3.2) In an Ab-category, composition with a zero morphism yields a zero morphism, and composition commutes with taking the additive inverse of a morphism.
(2.3.7 (1)) For any two objects in an Ab-category, a product, a coproduct, and a direct sum are equivalent.
(2.3.7 (2)) In an Ab-category, the direct sum morphism $f \oplus f'$ can be characterized in terms of the inclusions.
(2.3.7 (3)) In an Ab-category, the sum morphism $f+g$ factors as $\nabla_B(f \oplus g)\Delta_A$.
(2.3.7 (4)) A functor between Ab-categories whose domain has all direct sums is an additive functor if and only if it preserves direct sums.
(2.3.12) For any two objects in a preadditive category, the zero morphism is the unique morphism that factors through the zero object.
(2.3.15) An abelian category is an Ab-category with a zero object, a direct sum for any two objects, and a (co)kernel for each morphism, such that (i) each monomorphism is a kernel and (ii) each epimorphism is a cokernel.
(2.3.17) Each abelian category has all finite (co)limits, with (co)products given by direct sums.
(2.4.22) An abelian category with a compatible braided monoidal structure is a tight braided bimonoidal category.
(2.5.1–2.5.2) An abelian category with a compatible (symmetric) monoidal structure is a tight (symmetric) bimonoidal category.

Chapter II.3. Applications to Quantum Groups and Topological Quantum Computation

(3.1.19) A braided bialgebra is a bialgebra equipped with an R-matrix that satisfies two axioms. A symmetric bialgebra is a braided bialgebra in which the inverse of the R-matrix is its opposite.
(3.1.27–3.1.30) Each cocommutative bialgebra is a symmetric bialgebra with the R-matrix $1 \otimes 1$. Examples include group bialgebras, the universal enveloping bialgebra of a Lie algebra, and Sweedler's 4-dimensional non-(co)commutative bialgebra.

(3.1.33) Each anyonic quantum group is a braided bialgebra.
(3.2.6) The category of left modules over each bialgebra is a monoidal category under the tensor product.
(3.2.12) The category of left modules over each braided bialgebra is a braided monoidal category.
(3.2.13) The category of left modules over each symmetric bialgebra is a symmetric monoidal category.
(3.2.19) The category of left modules over each bialgebra is a tight bimonoidal category. The braided and the symmetric analogues are also true.
(3.3.27) The Fibonacci anyons form a monoidal category.
(3.4.5) The Fibonacci anyons form a braided monoidal category.
(3.4.13) The Fibonacci anyons form a tight braided bimonoidal category.
(3.5.27) The Ising anyons form a monoidal category.
(3.6.7) The Ising anyons form a braided monoidal category.
(3.6.14) The Ising anyons form a tight braided bimonoidal category.

Chapter II.4. Bimonoidal Centers

(4.2.6) The additive structure of the bimonoidal Drinfeld center is a symmetric monoidal category.
(4.3.3) The multiplicative structure of the bimonoidal Drinfeld center is a braided monoidal category.
(4.4.3) For each tight bimonoidal category, the bimonoidal Drinfeld center is a tight braided bimonoidal category.
(4.5.3) For each braided bimonoidal category, the bimonoidal symmetric center is a symmetric bimonoidal category.

Chapter II.5. Coherence of Braided Bimonoidal Categories

(5.1.2) A left permbraided category has an additive permutative structure, a multiplicative braided strict monoidal structure, and identities for $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^{l}, $\xi^{\otimes}_{-,\mathbb{0}}$, and $\xi^{\otimes}_{\mathbb{0},-}$, and satisfies four braided bimonoidal category axioms.
(5.1.8) Each left bipermutative category is a left permbraided category.
(5.1.10) Each left permbraided category is a tight braided bimonoidal category.
(5.1.11) A right permbraided category has an additive permutative structure, a multiplicative braided strict monoidal structure, and identities for $\lambda^{\bullet}$, $\rho^{\bullet}$, δ^{r}, $\xi^{\otimes}_{-,\mathbb{0}}$, and $\xi^{\otimes}_{\mathbb{0},-}$, and satisfies four braided bimonoidal category axioms.
(5.1.17) Each right bipermutative category is a right permbraided category.
(5.1.19) Each right permbraided category is a tight braided bimonoidal category.
(5.2.7) The braided distortion category is a groupoid.
(5.2.13) The additive structure of the braided distortion category is a permutative category.
(5.2.21) The multiplicative structure of the braided distortion category is a braided strict monoidal category.
(5.2.28) In the braided distortion category, the right distributivity morphism δ^{r} has identity braid components.
(5.2.30) The braided distortion category is a left permbraided category.
(5.2.33–5.2.34) The braided distortion category is a tight braided bimonoidal category and satisfies all 24 Laplaza axioms.

(5.3.14) For a path in $\mathsf{Gr}(X)$, its value in a braided bimonoidal category is the composite of the images of its constituent prime edges.
(5.3.15) The braided distortion of a path in $\mathsf{Gr}(X)$ is its value in the braided distortion category.
(5.4.4) **Braided Bimonoidal Coherence Theorem**. In each braided bimonoidal category C satisfying a monomorphism assumption, any two parallel paths in $\mathsf{Gr}(X)$ with the same braided distortion have the same value in C.
(5.4.6) Theorem II.5.4.4 applies to flat, in particular, tight, braided bimonoidal categories.
(5.5.3) The braided distortion category $\mathcal{D}^{\mathsf{br}}$ is isomorphic to the Grothendieck construction $\int_{\Sigma} F^{\mathsf{br}}$.

Chapter II.6. Strictification of Tight Braided Bimonoidal Categories

(6.1.1) A braided bimonoidal functor is a functor equipped with an additive symmetric monoidal structure and a multiplicative braided monoidal structure, and satisfies two axioms.
(6.1.10) There is a category $\mathsf{Bi}^{\mathsf{br}}$ with small braided bimonoidal categories as objects and braided bimonoidal functors as morphisms.
(6.1.12) Each braided monoidal functor that is also an additive functor between abelian categories with a compatible braided monoidal structure canonically extends to a braided bimonoidal functor.
(6.1.15) Each symmetric monoidal functor that is also an additive functor between abelian categories with a compatible symmetric monoidal structure canonically extends to a symmetric bimonoidal functor.
(6.2.39) Each tight braided bimonoidal category has a canonically associated right permbraided category.
(6.3.6–6.3.7) **Permbraided Strictification**. Each tight braided bimonoidal category is adjoint equivalent to a right/left permbraided category via braided bimonoidal functors.

Chapter II.7. The Braided Baez Conjecture

(7.1.4) There is a 2-category $\mathsf{Bi}^{\mathsf{br}}$ of small braided bimonoidal categories, braided bimonoidal functors, and bimonoidal natural transformations.
(7.1.7) Each monoidal natural transformation between braided monoidal functors that are also additive functors between abelian categories with a compatible braided monoidal structure is a bimonoidal natural transformation.
(7.2.4) For each braided bimonoidal category C, there is a strong symmetric monoidal functor $F_{\oplus} : \Sigma \longrightarrow \mathsf{C}$ between the additive structures.
(7.2.9) For each flat braided bimonoidal category C, there is a braided monoidal functor $F_{\otimes} : \Sigma \longrightarrow \mathsf{C}$ between the multiplicative structures.
(7.2.11) For each flat braided bimonoidal category C, $F : \Sigma \longrightarrow \mathsf{C}$ is a robust braided bimonoidal functor.
(7.3.4) **Braided Baez Conjecture**. Σ is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$.
(7.3.6) **Braided Baez Conjecture, Ver. 2**. Σ' is a lax bicolimit of the 2-functor $\varnothing \longrightarrow \mathsf{Bi}_{\mathsf{r}}^{\mathsf{fbr}}$.

Chapter II.8. Monoidal Bicategorification

(8.1.13) For each tight braided bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ is a bicategory.

(8.4.7) For each tight braided bimonoidal category C, $\mathsf{Mat}^{\mathsf{C}}$ is a monoidal bicategory.

Part II.2. E_n-Monoidal Categories

Chapter II.9. Ring, Bipermutative, and Braided Ring Categories

(9.1.15) Tight ring categories form a subclass of tight bimonoidal categories.
(9.1.19) Right and left rigid bimonoidal categories are tight ring categories.
(9.1.20) Each tight ring category is adjoint equivalent to a right, respectively, left, rigid bimonoidal category.
(9.2.14) Each small permutative category has an endomorphism ring category.
(9.2.20) Each small permutative category has an endomorphism tight ring category.
(9.3.7) Each tight bipermutative category yields a tight symmetric bimonoidal category.
(9.3.12) Right and left bipermutative categories are tight bipermutative categories.
(9.3.13) Each tight bipermutative category is adjoint equivalent to a right, respectively, left, bipermutative category.
(9.4.7) In a bipermutative category, about half of the ring category axioms are redundant.
(9.5.4) A bipermutative category is a braided ring category whose braiding satisfies the symmetry axiom.
(9.5.5) In a braided ring category, about half of the ring category axioms are redundant.
(9.5.6) Tight braided ring categories form a subclass of tight braided bimonoidal categories.
(9.5.10) Right and left permbraided categories are tight braided ring categories.
(9.5.11) Each tight braided ring category is adjoint equivalent to a right, respectively, left, permbraided category.
(9.6.1) The bimonoidal Drinfeld center of a tight ring category is a tight braided ring category.
(9.6.4) The symmetric center of a braided ring category with left factorization a natural epimorphism is a bipermutative category.

Chapter II.10. Iterated and E_n-Monoidal Categories

(10.1.9) A 1-fold monoidal category is a strict monoidal category.
(10.1.14) Braided strict monoidal categories form a subclass of 2-fold monoidal categories.
(10.1.21) Permutative categories form a subclass of n-fold monoidal categories for each $n \geq 2$.
(10.2.3) A totally ordered set with a least element forms a permutative category with identity symmetry.
(10.2.8) A totally ordered monoid whose unit is also the least element forms a 2-fold monoidal category.
(10.3.7) A 1-fold monoidal functor is a strictly unital monoidal functor.
(10.3.11) A braided strictly unital monoidal functor is a 2-fold monoidal functor.
(10.3.15) A symmetric strictly unital monoidal functor is an n-fold monoidal functor for each $n \geq 2$.

(10.3.20) The composite of two n-fold monoidal functors is an n-fold monoidal functor.
(10.4.2) MCat^n is a monoidal category.
(10.4.5) Monoids in MCat^n are small $(n+1)$-fold monoidal categories.
(10.4.13) A morphism of monoids in MCat^n is an $(n+1)$-fold monoidal functor with the last monoidal constraint the identity.
(10.5.9) $\mathsf{FMon}^n : \mathsf{Cat} \longrightarrow \mathsf{MCat}^n_{\mathsf{st}}$ is the left adjoint of the forgetful functor.
(10.5.18) $\mathsf{FMon}^n(\mathsf{C})$ decomposes into a coproduct $\coprod_{k\geq 0} \mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k}$.
(10.5.26) $\coprod_{k\geq 0} \mathsf{Mon}^n(k)/\Sigma_k$ is the free n-fold monoidal category on one object.
(10.5.28) There are evaluation functors $\theta_k : \mathsf{Mon}^n(k) \times_{\Sigma_k} \mathsf{C}^{\times k} \longrightarrow \mathsf{C}$ for each small n-fold monoidal category C.
(10.6.8 (1)) Each morphism set in $\mathsf{Mon}^n(k)$ has at most one morphism.
(10.6.8 (2)) There exists a morphism $A \longrightarrow B \in \mathsf{Mon}^n$ if and only if $a \otimes_i b \in A$ implies either $a \otimes_j b \in B$ for some $j \geq i$ or $b \otimes_j a \in B$ for some $j > i$.
(10.6.9) In each n-fold monoidal category, each formal diagram built from identity morphisms, the exchanges $\{\eta^{i,j}\}_{i<j}$, the monoidal products $\{\otimes_i\}_{i=1}^n$, and composites is commutative.
(10.7.13) An E_1-monoidal category is a ring category.
(10.8.1) Braided ring categories form a subclass of E_2-monoidal categories.
(10.9.1) Bipermutative categories form a subclass of E_n-monoidal categories for each $n \geq 2$.
(10.10.2) Each small category has a free E_n-monoidal category.

Part III.1. Enriched Monoidal Categories and Multicategories

Chapter III.1. Enriched Monoidal Categories

(1.1.31) **Mac Lane's Coherence Theorem**. Any two words of the same length in a monoidal category are connected by a unique canonical map.
(1.1.32) **Mac Lane's Strictification Theorem**. Each monoidal category is adjoint equivalent to a strict monoidal category via strong monoidal functors.
(1.1.38) **Braided Coherence Theorem**. Two braided canonical maps with the same (co)domain in a braided monoidal category are equal if their underlying braids are equal.
(1.1.39) **Braided Strictification Theorem**. Each braided monoidal category is adjoint equivalent to a braided strict monoidal category via strong braided monoidal functors.
(1.1.41) **Symmetric Coherence Theorem**. Any two permuted words of the same length in a symmetric monoidal category are connected by a unique permuted canonical map.
(1.1.42) **Symmetric Strictification Theorem**. Each symmetric monoidal category is adjoint equivalent to a permutative category via strong symmetric monoidal functors.
(1.1.44) **Epstein's Coherence Theorem**. For each (symmetric) monoidal functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ and F-iterates $G, H : \mathsf{C}^n \longrightarrow \mathsf{D}$, there exists at most one F-coherent map $G \longrightarrow H$.
(1.2.1) For a monoidal category V, a V-category has hom objects in V satisfying associativity and unity axioms.
(1.2.4) A V-functor satisfies composition and identity axioms.
(1.2.7) A V-natural transformation satisfies a naturality axiom.

(1.2.13) There is a 2-category formed by small V-categories, V-functors, and V-natural transformations.
(1.2.16) The opposite of a V-category is defined if V is braided monoidal.
(1.3.3) The tensor product of V-categories is defined if V is braided monoidal.
(1.3.6) The tensor product of V-categories is 2-functorial.
(1.3.35) The underlying 1-category of V-Cat is monoidal if V is braided, and is symmetric monoidal if V is symmetric.
(1.4.2) A monoidal V-category has associator and unitor V-natural transformations that satisfy unity axioms and a pentagon axiom.
(1.4.7) Composition in a monoidal V-category has an enriched interchange.
(1.4.10) The definition of braided monoidal V-category requires that V be symmetric monoidal.
(1.4.10) A braided monoidal V-category has a V-natural braiding satisfying two hexagon axioms.
(1.4.13) A symmetric monoidal V-category is a braided monoidal V-category satisfying an additional symmetry axiom.
(1.4.17) A monoidal V-functor satisfies associativity and unity axioms.
(1.4.18) A braided monoidal V-functor satisfies a braid axiom. A symmetric monoidal V-functor is a braided monoidal V-functor whose domain and codomain are symmetric monoidal V-categories.
(1.4.22) A monoidal V-natural transformation satisfies monoidal naturality and monoidal unity axioms.
(1.4.25) There are 2-categories formed by each of: monoidal V-categories, braided monoidal V-categories, and symmetric monoidal V-categories with, in each case, the corresponding V-functors and V-natural transformations.
(1.5.1) We use the term Cat-monoidal 2-category to indicate monoidal V-categories when V = Cat.
(1.5.2) The underlying 1-category of a plain/braided/symmetric Cat-monoidal 2-category has the corresponding structure as a 1-category.
(1.5.3) A Cat-monoidal 2-category has a strict form of the data and axioms for a monoidal bicategory. Similar statements hold for the braided and symmetric cases.
(1.5.4) With the Cartesian product, Cat is a symmetric Cat-monoidal 2-category.
(1.5.5) For a braided monoidal category V, V-Cat is a Cat-monoidal 2-category. If V is symmetric, then so is V-Cat.

Chapter III.2. Change of Enrichment

(2.1.2) Change of enrichment along a monoidal functor U is 2-functorial.
(2.1.7) The functor from V-Cat to Cat that takes underlying categories is injective on 2-cells.
(2.2.7) The assignment $E : \mathsf{V} \longmapsto \mathsf{V}\text{-}\mathsf{Cat}$ is 2-functorial with respect to monoidal functors and monoidal natural transformations.
(2.2.11) For small monoidal V, there is a 2-equivalence between V-Cat and $(\mathsf{V}_{\mathsf{st}})$-Cat.
(2.3.7) If U is braided moniodal, then the change of enrichment induced by U is a Cat-monoidal 2-functor. If U is symmetric, then so is the change of enrichment.
(2.3.9) The assignment $E : \mathsf{V} \longmapsto \mathsf{V}\text{-}\mathsf{Cat}$ of braided monoidal categories to Cat-monoidal 2-categories is 2-functorial. A similar result holds for symmetric monoidal categories with E producing symmetric Cat-monoidal categories.

(2.3.16) For small symmetric monoidal V, there is a symmetric Cat-monoidal 2-equivalence between V-Cat and $(\mathsf{V}_{\mathsf{st}})$-Cat.
(2.4.10, 2.4.15) For braided monoidal $U : \mathsf{V} \longrightarrow \mathsf{W}$, change of enrichment along U induces 2-functors between the 2-categories of monoidal V- and W-categories. If U is symmetric, then a similar result holds for braided and symmetric monoidal V- and W-categories.
(2.4.17) The underlying category of a monoidal V-category is monoidal. Similar statements hold for braided and symmetric cases, and for functors and natural transformations.
(2.5.1) Given the data of a monoidal V-category, the enriched monoidal category axioms are satisfied if and only if the underlying data satisfy the ordinary monoidal category axioms. Similar results hold for the braided and symmetric monoidal cases, and also for functors and natural transformations.
(2.5.6) **Enriched (Braided/Symmetric) Monoidal Coherence Theorem**. Any two V-words of the same length in a monoidal V-category are connected by a unique canonical V-map. Similar coherence results hold in the braided and symmetric cases.
(2.5.8) **Enriched Epstein's Coherence Theorem**. For each (symmetric) monoidal V-functor $F : \mathsf{K} \longrightarrow \mathsf{L}$, and F-iterates $G, H : \mathsf{K}^{\otimes n} \longrightarrow \mathsf{L}$, there exists at most one F-coherent map $G \longrightarrow H$.
(2.6.1) **Enriched Monoidal Strictification Theorem**. Each monoidal V-category is adjoint V-equivalent to a strict monoidal V-category via strong monoidal V-functors.
(2.6.3) **Enriched Braided Strictification Theorem**. Each braided monoidal V-category is adjoint V-equivalent to a braided strict monoidal V-category via strong braided monoidal V-functors.
(2.6.4) **Enriched Symmetric Strictification Theorem**. Each symmetric monoidal V-category is adjoint V-equivalent to a strict monoidal V-category via strong symmetric monoidal V-functors.

Chapter III.3. Self-Enrichment and Enriched Yoneda

(3.1.11) Each symmetric monoidal closed V has a canonical enrichment over itself, $\underline{\mathsf{V}}$.
(3.2.1, 3.2.2) A category enriched over symmetric monoidal closed V has co/represented V-functors $\mathcal{Y}^X$ and $\mathcal{Y}_Y$ to $\underline{\mathsf{V}}$.
(3.3.2) The self-enriched category $\underline{\mathsf{V}}$ is symmetric monoidal as a V-category.
(3.3.4) The standard enrichment of a symmetric monoidal functor is symmetric monoidal in the enriched sense.
(3.4.12) **V-Yoneda Bijection Theorem**. For each V-functor $F : \mathsf{C} \longrightarrow \underline{\mathsf{V}}$ and each $X \in \mathsf{C}$, there is a bijection of sets V-nat$(\mathcal{Y}^X, F) \cong \mathsf{V}(\mathbb{1}, FX)$. For each V-functor $G : \mathsf{C}^{\mathsf{op}} \longrightarrow \underline{\mathsf{V}}$ and each $Y \in \mathsf{C}$, there is a bijection of sets V-nat$(\mathcal{Y}_Y, G) \cong \mathsf{V}(\mathbb{1}, GY)$.
(3.5.1) A V-coend is initial among V-cowedges. A V-end is terminal among V-wedges.
(3.5.5) If V is cocomplete, then V-coends are computed by a coequalizer in V. If V is complete, then V-ends are computed by an equalizer in V.
(3.5.12) For V complete symmetric monoidal closed, the mapping object for V-functors to $\underline{\mathsf{V}}$ is given by a V-end.

(3.6.9) V**-Yoneda Lemma**. For a V-functor $F : \mathsf{C} \longrightarrow \mathsf{V}$ with C small, there is a V-natural isomorphism

$$F \xrightarrow{\cong} \mathsf{Map}(\mathcal{Y}^{(-)}, F).$$

(3.7.3) For V complete and cocomplete symmetric monoidal closed, the Day convolution and hom diagram are given by a V-coend and a V-end, respectively.
(3.7.8) V**-Yoneda Density Theorem**. For a V-functor $X : \mathcal{D} \longrightarrow \underline{\mathsf{V}}$ with $\mathcal{D}$ small, there is a V-natural isomorphism

$$\int^x \mathcal{D}(x, -) \otimes X_x \xrightarrow{\cong} X.$$

(3.7.13) There is an isomorphism $\mathsf{Map}(X, Y) \cong \mathsf{Hom}(X, Y)_e$, for V-diagrams X and Y.
(3.7.22) **Day Convolution Theorem**. For a small symmetric monoidal V-category $\mathcal{D}$, the category of $\mathcal{D}$-shaped diagrams in V is symmetric monoidal closed with the Day convolution product, internal hom, and monoidal unit $J = \mathcal{Y}^e$.
(3.7.28) Precomposition with a symmetric monoidal V-functor induces a symmetric monoidal functor between diagram V-categories.
(3.8.4) Change of enrichment along a symmetric monoidal functor $U : \mathsf{V} \longrightarrow \mathsf{W}$ induces a symmetric monoidal functor from $\mathcal{D}$-V to $\mathcal{D}_U$-W.
(3.9.3) If C is tensored and cotensored over V, then $X \otimes -$ and $Y^{(-)}$ extend uniquely to V-functors that are V-adjoint to the respective co/represented V-functors.
(3.9.8) If (F, U) is an adjunction of monoidal functors between symmetric monoidal closed categories and if F^2 is invertible, then F transfers tensor and cotensor structure over its codomain to corresponding structure over its domain.
(3.9.15) The symmetric monoidal closed diagram category $\mathcal{D}$-V is enriched, tensored, and cotensored over V.

Chapter III.4. Pointed Objects, Smash Products, and Pointed Homs

(4.1.6) Smash product with respect to a terminal object T is given by a pushout from a monoidal product.
(4.2.1) Pointed hom with respect to a terminal object T is given by a pullback from an internal hom.
(4.1.5, 4.1.8, 4.2.3) Suppose C is complete and cocomplete symmetric monoidal closed. Then C_* is complete and cocomplete symmetric monoidal closed with respect to the smash product and pointed hom.
(4.3.11) For a small symmetric monoidal category $\mathcal{D}$ with a null object, its pointed unitary enrichment over $(\mathsf{V}_*, \wedge, E)$ is given by taking wedge sums of E over nonzero morphisms in $\mathcal{D}$.
(4.3.19) Assuming the basepoint of V is terminal and the basepoint of $\mathcal{D}$ is null, there is an equivalence of categories between pointed functors from $\mathcal{D}$ to V_* and V_*-enriched functors from the pointed unitary enrichment of $\mathcal{D}$ to the self enrichment of V_*.
(4.3.37) The category of pointed diagrams $\mathcal{D}_*$-V is complete and cocomplete symmetric monoidal closed. Moreover, it is enriched, tensored, and cotensored over V_*.

Chapter III.5. Multicategories

(5.1.2) A multicategory has objects, n-ary operations, symmetric group actions, colored units, and composition that are subject to axioms for symmetry, associativity, unity, and equivariance.
(5.1.2) An operad is a multicategory with one object.
(5.1.11) Each small permutative category C has an endomorphism multicategory with the same objects and with n-ary operations given by morphisms out of n-fold sums in C.
(5.1.12) A multifunctor satisfies axioms for symmetric group action, units, and composition.
(5.1.17) A multinatural transformation satisfies a naturality condition.
(5.1.20) There is a 2-category consisting of small multicategories, multifunctors, and multinatural transformations.
(5.1.21) The initial operad I has a single object and only one operation, which is the unit on its one object.
(5.2.1, 5.2.2) The terminal multicategory T has a single object and a single n-ary operation for each $n \geq 0$. The terminal multicategory is also known as the commutative operad, Com.
(5.3.9) Taking endomorphism operads gives a 2-functor from $\mathsf{PermCat}^{\mathsf{su}}$ to $\mathsf{Multicat}_*$.
(5.4.1) A monad consists of an endofunctor together with multiplication and unit natural transformations such that the associativity and unit diagrams commute.
(5.4.2) A monad algebra consists of an object and a structure morphism such that associativity and unity diagrams commute.
(5.4.13) **Beck's Precise Tripleability Theorem**. An adjunction $L \dashv U$ is strictly monadic if and only if U strictly creates coequalizers for parallel pairs f, g for which (Uf, Ug) has a split coequalizer.
(5.4.18) If T is a monad on a complete and cocomplete category, and if T preserves filtered colimits, then the category of T-algebras is complete and cocomplete.
(5.5.1) A multigraph consists of vertices and multiedges.
(5.5.4, 5.5.9) The forgetful functor from small multicategories to small multigraphs has a left adjoint.
(5.5.11) The category of small multicategories is strictly monadic over the category of small multigraphs.
(5.5.14) The category of small multicategories is complete and cocomplete.
(5.6.9) The sharp product of multicategories is generated by operations $\phi \times d$ and $c \times \psi$ subject to symmetry and compatibility axioms.
(5.6.14) The tensor product of multicategories is generated by those of the sharp product, and subject to an additional interchange relation.
(5.6.14) A multifunctor out of a tensor product of multicategories consists of an assignment on objects that is multifunctorial in each variable separately and that preserves the interchange relation.
(5.7.2, 5.7.4) The internal hom for multicategories has operations given by transformations that satisfy a naturality condition.
(5.7.14) The category of small multicategories is complete and cocomplete symmetric monoidal closed with monoidal product given by the tensor product and closed structure given by the internal hom.
(5.7.22) The category of small pointed multicategories is complete and cocomplete symmetric monoidal closed with monoidal product given by the smash product and closed structure given by the pointed hom.

(5.7.23) The symmetric monoidal structure on $\mathsf{Multicat}_*$ does not restrict along End to a symmetric monoidal structure on $\mathsf{PermCat}^{\mathsf{su}}$.

Chapter III.6. Enriched Multicategories

(6.1.1) A V-enriched multicategory has n-ary operation objects, symmetric group action, colored units, and composition morphisms in V. These satisfy axioms given by commutative diagrams in V for symmetric group action, associativity, unity, and equivariance.
(6.1.8) A V-enriched operad is a V-multicategory with one object.
(6.1.9) An object of an enriched multicategory has a V-enriched endomorphism operad.
(6.1.10) A V-enriched multifunctor satisfies axioms given by commutative diagrams in V for symmetric group action, units, and composition.
(6.1.14) An algebra c over a V-enriched operad P is given by a V-enriched operad morphism $\mathsf{P} \longrightarrow \mathsf{End}(c)$.
(6.1.15) A V-enriched multinatural transformation satisfies a V-naturality diagram in V.
(6.1.18) There is a 2-category consisting of small V-enriched multicategories together with V-enriched multifunctors and multinatural transformations.
(6.2.9) For a symmetric monoidal functor $U : \mathsf{V} \longrightarrow \mathsf{W}$, change of enrichment along U provides a 2-functor from small V-enriched multicategories to small W-enriched multicategories.
(6.3.3, 6.3.6) A symmetric monoidal V-category has a V-enriched endomorphism multicategory with V-objects of n-ary operations given by morphism objects out of n-fold left normalized products in K.
(6.3.10) A symmetric monoidal V-functor induces a V-enriched multifunctor between V-enriched endomorphism multicategories of its domain and codomain.
(6.4.3) The tensor product is a Cat-enriched symmetric monoidal product for the 2-category of small multicategories.
(6.4.4) The smash product is a Cat-enriched symmetric monoidal product for the 2-category of small pointed multicategories.
(6.4.5) Each of $\mathsf{Multicat}$ and $\mathsf{Multicat}_*$ has the structure of a Cat-enriched multicategory induced by the tensor and smash product, respectively.
(6.5.1) The Cat-enriched multicategory structure on $\mathsf{Multicat}_*$ induces a corresponding structure on $\mathsf{PermCat}^{\mathsf{su}}$.
(6.5.4) Multilinear functors of permutative categories consist of functors out of a Cartesian product together with linearity constraints. They are subject to axioms for unity, constraint unity, constraint associativity, constraint symmetry, and constraint 2-by-2.
(6.5.11) Multilinear transformations between multilinear functors satisfy multilinearity conditions with respect to linearity constraints and identities.
(6.5.10, 6.5.13) The categories of n-ary operations in $\mathsf{PermCat}^{\mathsf{su}}$ are canonically isomorphic to the corresponding categories of n-linear functors and transformations.
(6.6.13) The Cat-enriched multicategory structure of $\mathsf{PermCat}^{\mathsf{su}}$ is described explicitly in terms of multilinear functors and transformations.

Part III.2. Algebraic K-Theory

Chapter III.7. Homotopy Theory Background

(7.1.16) The geometric realization of the standard n-simplex is the topological n-simplex.
(7.1.19) The category of simplicial sets is symmetric monoidal closed with the monoidal product given by the levelwise Cartesian product.
(7.1.23) The category of pointed simplicial sets is symmetric monoidal closed with monoidal product given by the levelwise smash product.
(7.2.4) The nerve of a small category is a simplicial set with p-simplices given by strings of p composable arrows.
(7.2.5 (1)) A natural transformation between functors induces a simplicial homotopy on nerves.
(7.2.5 (2)) An adjunction of functors induces a simplicial homotopy equivalence on nerves.
(7.3.7) The category of symmetric sequences is symmetric monoidal closed with monoidal product given by Day convolution.
(7.4.5) The category of symmetric spectra is the category of left modules over the symmetric sphere.
(7.4.6) A symmetric spectrum consists of a symmetric sequence with structure morphisms satisfying unity, associativity, and equivariance axioms.
(7.5.5) The category of symmetric spectra is complete and cocomplete.
(7.6.1) The smash product of symmetric spectra is given by a coequalizer of actions by S.
(7.6.8) The internal hom for symmetric spectra is given by an equalizer of actions by S.
(7.6.15) The category of symmetric spectra is complete and cocomplete symmetric monoidal closed.
(7.8.8) Every level equivalence of symmetric spectra is a stable equivalence.

Chapter III.8. Segal K-Theory of Permutative Categories

(8.1.8) A Γ-object in a pointed category C is a pointed functor from $\mathcal{F}$ to C.
(8.2.6) The construction $\mathsf{K}^{\mathcal{F}}$ is a functor from Γ-simplicial sets to symmetric spectra.
(8.3.13) For a small permutative category C, there are three variant constructions of Γ-categories, $\mathsf{C}^{\mathcal{F}} = \mathsf{C}^{\mathcal{F}}_{\cong}$, $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$, and $\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$.
(8.3.21) For a small permutative category C, each of the Γ-simplicial sets $N_*\mathsf{C}^{\mathcal{F}}_{\cong}$, $N_*\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$, and $N_*\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$ is special, and all three are levelwise weakly-equivalent.
(8.4.5) The partition multicategory $\mathcal{M}\underline{1}$ has two objects, $\varnothing$ and $\{1\}$, with operations given by partitions.
(8.4.7) The partition multicategory $\mathcal{M}$ defines a pointed functor from $\mathcal{F}^{\mathsf{op}}$ to $\mathsf{Multicat}_*$.
(8.4.8, 8.4.10) For a small permutative category C, there is an isomorphism of Γ-categories $\mathsf{J}^{\mathsf{Se}}\mathsf{C} \cong \mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$.

Chapter III.9. Categories of $\mathcal{G}_*$-Objects

(9.1.7) The objects of $\mathcal{G}$ are tuples of objects of $\mathcal{F}$ subject to certain basepoint identifications.
(9.1.15) Smash product of pointed finite sets defines a strict symmetric monoidal functor from $\mathcal{G}$ to $\mathcal{F}$.
(9.2.1) A $\mathcal{G}_*$-object in a pointed category C is a pointed functor from $\mathcal{G}$ to C.

(9.2.15) If C is complete and cocomplete symmetric monoidal closed with terminal basepoint, then the category for $\mathcal{G}_*$-objects in C is complete and cocomplete symmetric monoidal closed with monoidal product given by Day convolution.
(9.2.19) The nerve induces a symmetric monoidal sSet_*-functor from small $\mathcal{G}_*$-categories to $\mathcal{G}_*$-simplicial sets.
(9.3.16) The construction $\mathsf{K}^{\mathcal{G}}$ is a functor from $\mathcal{G}_*$-simplicial sets to symmetric spectra whose restriction along $\wedge^*$ is equal to $\mathsf{K}^{\mathcal{F}}$.
(9.4.9) The functor $\mathsf{K}^{\mathcal{G}}$ is a unital symmetric monoidal sSet-functor.
(9.4.18) The functors $\wedge^*$ and $\mathsf{K}^{\mathcal{F}}$ are symmetric monoidal sSet_*-functors.

Chapter III.10. Elmendorf-Mandell K-Theory of Permutative Categories

(10.1.6) The partition multicategory $\mathcal{M}$ is a symmetric monoidal functor, with monoidal constraint given by the partition product $\prod$.
(10.1.14) The category of left $\mathcal{M}\underline{1}$-modules is complete and cocomplete.
(10.1.15) The partition products $\prod_{\underline{1},b}$ and $\prod_{b,\underline{1}}$ are isomorphisms.
(10.1.28) The 2-category of left $\mathcal{M}\underline{1}$-modules in $\mathsf{Multicat}_*$ is a full sub-2-category and the smash product over $\mathcal{M}\underline{1}$ is isomorphic to that of $\mathsf{Multicat}_*$.
(10.1.36) The category of left $\mathcal{M}\underline{1}$-modules is symmetric monoidal in the Cat_*-enriched sense.
(10.2.14) If C is a small permutative category, $\mathsf{End}(\mathsf{C})$ has a canonical left $\mathcal{M}\underline{1}$-module structure. Taking this structure, End factors through $\mathsf{Mod}^{\mathcal{M}\underline{1}}$.
(10.2.15, 10.2.16) The category of left $\mathcal{M}\underline{1}$-modules is a proper subcategory of $\mathsf{Multicat}_*$. In particular, the monoidal unit S is not an $\mathcal{M}\underline{1}$-module.
(10.2.17) The symmetric monoidal structure on $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ does not restrict to $\mathsf{PermCat}^{\mathsf{su}}$.
(10.2.22) The symmetric monoidal structure on $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is closed.
(10.3.3) The smash product of partition multicategories, $\mathcal{T}$, defines a pointed functor from $\mathcal{G}^{\mathsf{op}}$ to $\mathsf{Mod}^{\mathcal{M}\underline{1}}$.
(10.3.7) The functor $\mathcal{T}$ is strictly unital strong symmetric monoidal.
(10.3.13) The monoidal constraint for the partition J-theory $\mathsf{J}^{\mathcal{T}}$ uses the inverse monoidal constraint for $\mathcal{T}$.
(10.3.17) The partition J-theory $\mathsf{J}^{\mathcal{T}}$ is a symmetric monoidal Cat_*-functor.
(10.3.25) Elmendorf-Mandell J-theory $\mathsf{J}^{\mathsf{EM}} = \mathsf{J}^{\mathcal{T}} \circ \mathsf{End}$ is a Cat-enriched multifunctor.
(10.3.32) Elmendorf-Mandell K-theory $\mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} N_* \mathsf{J}^{\mathsf{EM}}$ is a sSet-enriched multifunctor.
(10.3.33) The multifunctor K^{EM} preserves enriched operad actions.
(10.4.18) For a small permutative category C there are three variant constructions of $\mathcal{G}_*$-categories, $\mathsf{C}^{\mathcal{G}} = \mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}$, $\mathsf{C}^{\mathcal{G}}_{\cong}$, and $\mathsf{C}^{\mathcal{G}}_{\mathrm{co}}$.
(10.5.1) For a small permutative category C, there is an isomorphism of $\mathcal{G}_*$-categories $\mathsf{J}^{\mathsf{EM}}\mathsf{C} \cong \mathsf{C}^{\mathcal{G}}_{\mathrm{lax}}$.
(10.6.10) There is a level equivalence of symmetric spectra $\mathsf{K}^{\mathsf{Se}}\mathsf{C} \longrightarrow \mathsf{K}^{\mathsf{EM}}\mathsf{C}$ for each small permutative category C. It is natural with respect to strictly unital symmetric monoidal functors.
(10.7.16) There is a level equivalence $\mathsf{K}^{\mathsf{EM}}_{\cong}\mathsf{C} \longrightarrow \mathsf{K}^{\mathsf{EM}}_{\mathsf{lax}}$ for each small permutative category C.
(10.7.19 (4)) There is a level equivalence $\mathsf{K}^{\mathsf{EM}}_{\cong} \longrightarrow \mathsf{K}^{\mathsf{EM}}_{\mathsf{co}}\mathsf{C}$ for each small permutative category C.

(10.7.22, 10.7.27) The $\mathcal{G}_*$-category morphisms $\mathsf{C}^{\mathcal{G}}_{\cong} \longrightarrow \mathsf{C}^{\mathcal{G}}$ and $\mathsf{C}^{\mathcal{G}}_{\cong} \longrightarrow \mathsf{C}^{\mathcal{G}}_{\mathsf{co}}$ are components of Cat-enriched multinatural transformations.

Chapter III.11. *K*-Theory of Ring and Bipermutative Categories

(11.1.4) As is an operad.
(11.1.7) As is generated by id_0 and id_2, which are subject to unity and associativity relations.
(11.1.15) As is the operad for monoids.
(11.2.16) As detects ring category structures on small permutative categories.
(11.3.2) A strict ring symmetric spectrum is a symmetric spectrum equipped with multiplication and unit morphisms in symmetric sequences that satisfy compatibility, associativity, and unity axioms.
(11.3.13–11.3.15) The sphere spectrum, the suspension spectrum of a monoid in pointed simplicial sets, and the Eilenberg-Mac Lane spectrum of a ring are strict ring symmetric spectra.
(11.3.16) $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is a strict ring symmetric spectrum for each small ring category C.
(11.3.17) For each small permutative category, the *K*-theory of its (tight) endomorphism ring category is a strict ring symmetric spectrum.
(11.3.18) The *K*-theory of the additive distortion category is a strict ring symmetric spectrum.
(11.3.19) For each small tight bimonoidal category, the *K*-theory of its right/left rigid bimonoidal strictification is a strict ring symmetric spectrum.
(11.4.7) The translation category functor E is a right adjoint.
(11.4.11) Each morphism in the Barratt-Eccles operad decomposes into a categorical composite of ϕv with v a permutation and ϕ an operadic composite of one $\tau : \mathrm{id}_2 \longrightarrow (1,2)$ and identity morphisms.
(11.4.14) The Barratt-Eccles operad is generated by two objects and one isomorphism, which are subject to relations that are formally identical to those of a permutative category.
(11.4.26) The Barratt-Eccles operad is the Cat-enriched operad for permutative categories.
(11.5.5) The Barratt-Eccles operad detects bipermutative category structures on small permutative categories.
(11.6.3) The Barratt-Eccles operad is an E_∞-operad.
(11.6.6 (1)) Each commutative monoid in SymSp is an E_∞-symmetric spectrum.
(11.6.6 (2)) Each E_∞-symmetric spectrum via the Barratt-Eccles operad has a strict ring structure.
(11.6.7) The symmetric sphere is a commutative monoid in SymSp.
(11.6.9) The suspension spectrum of a commutative monoid in sSet_* is a commutative monoid in SymSp.
(11.6.10) The Eilenberg-Mac Lane spectrum of a commutative ring is a commutative monoid in SymSp.
(11.6.12) $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_∞-symmetric spectrum for each small bipermutative category C.
(11.6.13) The *K*-theory of each small right/left bipermutative category is an E_∞-symmetric spectrum. For example, this applies to the finite ordinal category Σ, its variant Σ', $\mathsf{Vect}^{\mathbb{C}}_{\mathsf{c}}$, and the distortion category.

(11.6.14) For each small tight symmetric bimonoidal category, the K-theory of its right/left bipermutative strictification is an E_∞-symmetric spectrum. For example, this applies to small distributive symmetric monoidal categories, the symmetric bimonoidal groupoid Π, and the bimonoidal symmetric center of a small tight braided bimonoidal category.
(11.6.15) For each small braided ring category whose left factorization morphism is a natural epimorphism, the K-theory of its symmetric center is an E_∞-symmetric spectrum.

Chapter III.12. K-Theory of Braided Ring Categories

(12.1.10) The braid operad is a Cat-enriched operad.
(12.2.4) The braid operad is an E_2-operad.
(12.3.6) Each morphism in the braid operad admits a categorical decomposition into isomorphisms of the form ϕv with v a permutation and ϕ an operadic composite of one $s_1^{\pm 1} : \mathrm{id}_2 \longrightarrow (1,2)$ and identity morphisms.
(12.3.10) The braid operad is generated by two objects and one isomorphism, which are subject to relations that are formally identical to those of a braided strict monoidal category.
(12.3.22) The braid operad is the Cat-enriched operad for braided strict monoidal categories.
(12.4.5) The braid operad detects braided ring category structures on small permutative categories.
(12.5.2 (1)) An E_∞-symmetric spectrum via the Barratt-Eccles operad has an E_2-structure.
(12.5.2 (2)) An E_2-symmetric spectrum via the braid operad has a strict ring structure.
(12.5.3) $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_2-symmetric spectrum for each small braided ring category C.
(12.5.4) The K-theory of the braided distortion category is an E_2-symmetric spectrum.
(12.5.5) For each small tight ring category, the K-theory of its bimonoidal Drinfeld center is an E_2-symmetric spectrum.
(12.5.6) For each small tight braided bimonoidal category, the K-theory of its right/left permbraided strictification is an E_2-symmetric spectrum. For example, this applies to a small abelian category with a compatible braided monoidal structure, Fibonacci anyons, Ising anyons, and the bimonoidal Drinfeld center of a small tight bimonoidal category.

Chapter III.13. K-Theory of E_n-Monoidal Categories

(13.1.20) Mon^n is a Cat-enriched operad.
(13.1.23) Mon^1 is the associative operad.
(13.2.1) Mon^n is an E_n-operad.
(13.3.3) Mon^n is generated by the objects $\mathbb{1}$ and $\{1 \otimes_i 2\}_{i=1}^n$ and the exchange morphisms $\{\eta_{1,2,3,4}^{i,j}\}_{1 \le i < j \le n}$, which are subject to relations that are formally identical to those of an n-fold monoidal category.
(13.3.18) Mon^n is the Cat-enriched operad for n-fold monoidal categories.
(13.4.12) Mon^n detects E_n-monoidal category structures on small permutative categories.

(13.5.1) The canonical Cat-enriched operad morphism $\mathsf{As} \longrightarrow E\mathsf{As}$ factors through Mon^n. As a result, an E_∞-symmetric spectrum via the Barratt-Eccles operad induces an E_n-structure. An E_{n+1}-structure via Mon^{n+1} induces an E_n-structure.
(13.5.2) $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$ is an E_n-symmetric spectrum for each small E_n-monoidal category C.
(13.5.3) For each small category, the K-theory of its free E_n-monoidal category is an E_n-symmetric spectrum.

List of Notations

Part I.1

Chapter I.1	Page	Description
$\mathsf{Ob}(\mathsf{C})$, $\mathsf{Ob}\,\mathsf{C}$	I.7	objects in a category C
$\mathsf{C}(X,Y)$, $\mathsf{C}(X;Y)$	I.8	set of morphisms $X \longrightarrow Y$
1_X	I.8	identity morphism
$g \circ f$, gf	I.8	composition of morphisms
$\cong$, $\xrightarrow{\cong}$	I.8	an isomorphism
$F : \mathsf{C} \longrightarrow \mathsf{D}$	I.8	a functor
Id_{C}, 1_{C}	I.9	identity functor
$\mathbf{1}$	I.9	terminal category
θ_X	I.9	a component of a natural transformation θ
1_F	I.9	identity natural transformation
$\phi\theta$	I.9	vertical composition
$\theta' * \theta$	I.9	horizontal composition
(L, R, ϕ), $L \dashv R$	I.10	an adjunction
η, ε	I.10	unit and counit of an adjunction
$\mathsf{colim}\,F$	I.11	colimit
$\mathsf{lim}\,F$	I.11	limit
$\varnothing$, $\varnothing^{\mathsf{C}}$	I.12	an initial object
$\coprod$, $\amalg$	I.12	a coproduct
$\otimes$	I.14	monoidal product
$\mathbb{1}$	I.14	monoidal unit
α	I.14	associativity isomorphism
λ, ρ	I.15	unit isomorphisms
(X, μ, η)	I.15	a monoid
(Y, Δ, ε)	I.16	a comonoid
(F, F^2, F^0)	I.16	a monoidal functor
ξ	I.18	symmetry isomorphism
$(\mathsf{Set}, \times, *)$	I.19	category of sets
$(\mathsf{Cat}, \times, \mathbf{1})$	I.19	category of small categories
$(\mathsf{Vect}^{\Bbbk}, \otimes, \Bbbk)$	I.19	category of $\Bbbk$-vector spaces
$[-,-]$	I.19	internal hom
e, $-$, $u \,\square\, v$	I.19	words
$\sigma(-)$	I.20	a left permutation
$w\sigma$	I.20	a permuted word

Chapter I.2		
$(\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$	I.25	additive structure
$(\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$	I.25	multiplicative structure
$\lambda^{\bullet}$, $\rho^{\bullet}$	I.25	multiplicative zeros

δ^l, δ^r	I.25	distributivity morphisms
$\mathsf{Vect}^{\mathbb{C}}$	I.30	finite dimensional complex vector spaces
$\alpha^{-\oplus}$	I.30	inverse of $\alpha^{\oplus}$
$\mathsf{Mod}(R)$	I.37	category of R-modules
Σ	I.38	category of finite ordinals and permutations
Σ_n	I.38	symmetric group on n letters
$\sigma \oplus \tau$	I.38	block sum of permutations
M^T	I.42	transpose of M
Σ'	I.43	a variant of Σ
$\mathsf{Vect}_{\mathsf{c}}^{\mathbb{C}}$	I.46	coordinatized version of $\mathsf{Vect}^{\mathbb{C}}$
$\mathbb{C}^m$	I.48	$\mathbb{C} \oplus \cdots \oplus \mathbb{C}$ with m copies of $\mathbb{C}$
Π	I.51	symmetric bimonoidal groupoid of syntax of finite types

Chapter I.3

S^{fr}	I.58	free $\{\oplus, \otimes\}$-algebra of S
$G = (V, E)$	I.58	a graph with vertex set V and edge set E
$u \longrightarrow v$	I.58	an edge with domain u and codomain v
$(e_n, \ldots, e_1)$	I.58	a path consisting of the edges $e_1, \ldots, e_n$
$v_0 \twoheadrightarrow v_n$	I.58	a path with domain v_0 and codomain v_n
$0^X, 1^X$	I.59	additive zero and multiplicative unit in X
$\mathsf{Gr}^{\mathsf{el}}(X)$	I.59	elementary graph
$\mathsf{E}_{\mathsf{el}}(X)$	I.59	set of elementary edges
$\mathsf{E}_{\mathsf{el}}^{\mathsf{fr}}(X)$	I.60	free $\{\oplus, \otimes\}$-algebra of $\mathsf{E}_{\mathsf{el}}(X)$
$\mathsf{E}^{\mathsf{pr}}(X)$	I.60	set of prime edges
$\mathsf{Gr}(X)$	I.60	graph of X
$\varphi : \mathsf{Gr}(X) \longrightarrow \mathsf{C}$	I.61	graph morphism extending $\varphi : X \longrightarrow \mathsf{Ob}(\mathsf{C})$
φP	I.62	value in C of a path P
X^{st}	I.63	strict $\{\oplus, \otimes\}$-algebra of X
supp	I.63	support $X^{\mathsf{fr}} \longrightarrow X^{\mathsf{st}}$
norm	I.65	norm $X^{\mathsf{fr}} \longrightarrow \mathbb{Z}_+$
rank	I.66	rank $X^{\mathsf{fr}} \longrightarrow \mathbb{Z}_+$
size	I.66	size $X^{\mathsf{fr}} \longrightarrow \mathbb{Z}_+$
$P \oplus 1_c, 1_c \oplus P$	I.73	sum of the paths P and 1_c
$P \otimes 1_c, 1_c \otimes P$	I.73	product of the paths P and 1_c
(IH)	I.117	induction hypothesis
$\mathsf{Gr}^{\mathsf{ns}}(X)$	I.132	nonsymmetric graph of X
X^{ns}	I.133	nonsymmetric strict $\{\oplus, \otimes\}$-algebra of X
nsupp	I.133	nonsymmetric support

Chapter I.4

id_m	I.143	identity permutation in Σ_m
$\mathcal{D}$	I.143	distortion category
$\underline{r}$	I.143	a finite sequence $(r_1, \ldots, r_m)$
$\lvert\underline{r}\rvert$	I.143	length of $\underline{r}$
$\varnothing$	I.143	empty sequence
$\underline{\sigma}$	I.143	a morphism $(\sigma; \sigma_1, \ldots, \sigma_m)$ in $\mathcal{D}$
ϑ	I.156	$X \longrightarrow \mathsf{Ob}(\mathcal{D})$ and $\mathsf{Gr}(X) \longrightarrow \mathcal{D}$
$\mathcal{D}^{\mathsf{ad}}$	I.165	additive distortion category
ϑ	I.167	$X \longrightarrow \mathsf{Ob}(\mathcal{D}^{\mathsf{ad}})$ and $\mathsf{Gr}^{\mathsf{ns}}(X) \longrightarrow \mathcal{D}^{\mathsf{ad}}$
$\int_{\mathsf{C}} F$	I.168	Grothendieck construction of $F : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$
$\Sigma^{\times n}$	I.169	n-fold Cartesian product of Σ
F^{ad}	I.170	functor $\Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$

Chapter I.7

Chapter I.8

$(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^a, \varepsilon^a)$	I.383	monoidal associator
$(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^\ell, \varepsilon^\ell)$	I.387	left monoidal unitor
$(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^r, \varepsilon^r)$	I.390	right monoidal unitor
π	I.392	pentagonator
μ	I.400	middle 2-unitor
$\lambda^{\boxtimes}$	I.402	left 2-unitor
$\rho^{\boxtimes}$	I.404	right 2-unitor
${}_\theta A$	I.409	row permutation of A by θ
A^σ	I.409	column permutation of A by σ
$\mathbb{1}^\sigma$	I.409	permutation matrix of σ
r^σ_A	I.410	natural isomorphism $A\mathbb{1}^\sigma \xrightarrow{\cong} A^\sigma$
ℓ^θ_A	I.411	natural isomorphism $\mathbb{1}^\theta A \xrightarrow{\cong} {}_{\theta^{-1}}A$
$\tau_{m,n}$	I.412	permutation in Σ_{mn} that transposes an $n \times m$ matrix
$(\beta, \beta^\bullet, \eta^\beta, \varepsilon^\beta)$	I.418	braiding
$h_{m\vert n,p}$	I.420	comparison 2-cell for $R_{-\vert--}$
$R_{-\vert--}$	I.421	left hexagonator
$h_{m,n\vert p}$	I.424	comparison 2-cell for $R_{--\vert-}$
$R_{--\vert-}$	I.425	right hexagonator
ν	I.428	syllepsis

Part II.1
Chapter II.1

B_n	II.8	braid group on n strings
$s_1, \ldots, s_{n-1}$	II.8	generating braids in B_n
$s_i^{(n)}$	II.8	s_i in B_n
$\mathrm{id}, \mathrm{id}_n$	II.8	identity braid in B_n
$\mathcal{I}$	II.8	unit interval $[0,1]$
$\oplus$	II.9	sum braid
$\pi(b), \overline{b}$	II.10	underlying permutation of b
$\sigma\langle \underline{k} \rangle$	II.10	block permutation induced by $\sigma \in \Sigma_n$
$b\langle \underline{k} \rangle$	II.11	block braid induced by $b \in B_n$
$\tau\langle m, n \rangle$	II.13	interval-swapping permutation in Σ_{m+n}
$b^{\oplus}_{m,n}$	II.14	elementary block braid induced by $s_1 \in B_2$
$\left(\overline{\mathsf{C}}, \overline{\otimes}, \overline{\mathbb{1}}, \overline{\alpha}, \overline{\lambda}, \overline{\rho}, \overline{\xi}\right)$	II.26	Drinfeld center of C
$(A; \beta^A)$	II.26	an object in $\overline{\mathsf{C}}$
$\mathsf{C}^{\mathsf{sym}}$	II.35	symmetric center of C
$\mathsf{br}(\phi)$	II.37	underlying braid of ϕ

Chapter II.2

$(\oplus, \mathbb{0}, \alpha^{\oplus}, \lambda^{\oplus}, \rho^{\oplus}, \xi^{\oplus})$	II.45	additive structure
$(\otimes, \mathbb{1}, \alpha^{\otimes}, \lambda^{\otimes}, \rho^{\otimes}, \xi^{\otimes})$	II.45	multiplicative structure
$\lambda^\bullet, \rho^\bullet$	II.46	multiplicative zeros
δ^l, δ^r	II.46	distributivity morphisms
Ab	II.50	category of abelian groups
$\mathbb{0} : A \longrightarrow B$	II.51	zero morphism
i_1, i_2	II.51	inclusions
p_1, p_2	II.51	projections
$A \oplus B$	II.51	direct sum of objects A and B
$f \oplus f'$	II.52	direct sum morphism

Chapter II.3

$\mathsf{Vect}^{\Bbbk}$	II.71	category of $\Bbbk$-vector spaces

$A^{\otimes n}$	II.71	$A \otimes \cdots \otimes A$ with n copies of A and $A^{\otimes 0} = \Bbbk$
$A^{\oplus n}$	II.71	$A \oplus \cdots \oplus A$ with n copies of A and $A^{\oplus 0} = \mathbb{0}$
$\mathsf{Vect}^{\Bbbk}_{\otimes}$	II.71	$\mathsf{Vect}^{\Bbbk}$ with the tensor product
$\mathsf{Vect}^{\Bbbk}_{\oplus}$	II.71	$\mathsf{Vect}^{\Bbbk}$ with the direct sum
$(A, \mu, \eta, \Delta, \varepsilon)$	II.72	a bialgebra
$\sum_i s'_i \otimes s''_i$	II.73	an element in $A^{\otimes 2}$
S^{op}	II.73	$\xi^{\otimes} S$
S_{12}, S_{13}, S_{23}	II.73	elements obtained from $S \in A^{\otimes 2}$ by inserting 1
$\sum_{(x)} x^{(1)} \otimes x^{(2)}$	II.74	Sweedler's notation for comultiplication
Δ^{op}	II.74	opposite comultiplication $\xi^{\otimes}\Delta$
$\Bbbk G$	II.75	group bialgebra of G
$U\mathfrak{g}$	II.76	universal enveloping bialgebra of $\mathfrak{g}$
H_4	II.77	Sweedler's 4-dimensional bialgebra
$\mathbb{C}\mathbb{Z}_n$	II.78	anyonic quantum groups
$\mathsf{Mod}(A)_{\otimes}$	II.80	$\mathsf{Mod}(A)$ with the tensor product
$\mathsf{Mod}(A)_{\oplus}$	II.83	$\mathsf{Mod}(A)$ with the direct sum
$\mathsf{Vect}^{\mathbb{C}}_{\mathsf{sk}}$	II.85	abelian category with linear maps $\mathbb{C}^m \longrightarrow \mathbb{C}^n$ as morphisms
$\mathcal{F}^{\mathsf{any}}$	II.86	Fibonacci anyons
$\mathbb{0} = (0;0)$	II.86	additive zero in $\mathcal{F}^{\mathsf{any}}$
$\mathbb{1} = (1;0)$	II.86	vacuum in $\mathcal{F}^{\mathsf{any}}$
$\tau = (0;1)$	II.86	non-abelian anyon in $\mathcal{F}^{\mathsf{any}}$
q	II.90	reciprocal of the golden ratio
z	II.92	$e^{3\pi i/5}$
$\mathcal{I}^{\mathsf{any}}$	II.95	Ising anyons
$\mathbb{0} = (0;0;0)$	II.95	additive zero in $\mathcal{I}^{\mathsf{any}}$
$\mathbb{1} = (1;0;0)$	II.95	vacuum in $\mathcal{I}^{\mathsf{any}}$
$\sigma = (0;1;0)$	II.95	non-abelian anyon in $\mathcal{I}^{\mathsf{any}}$
$\psi = (0;0;1)$	II.95	fermion in $\mathcal{I}^{\mathsf{any}}$
w	II.105	$e^{\pi i/8}$

Chapter II.4

$\overline{\mathsf{C}}^{\mathsf{bi}}$	II.113	bimonoidal Drinfeld center
$(\overline{\oplus}, \overline{\mathbb{0}}, \alpha^{\overline{\oplus}}, \lambda^{\overline{\oplus}}, \rho^{\overline{\oplus}}, \xi^{\overline{\oplus}})$	II.113	additive structure of $\overline{\mathsf{C}}^{\mathsf{bi}}$
$(\overline{\otimes}, \overline{\mathbb{1}}, \alpha^{\overline{\otimes}}, \lambda^{\overline{\otimes}}, \rho^{\overline{\otimes}}, \xi^{\overline{\otimes}})$	II.113	multiplicative structure of $\overline{\mathsf{C}}^{\mathsf{bi}}$
$\overline{\lambda}^{\bullet}, \overline{\rho}^{\bullet}$	II.113	multiplicative zeros of $\overline{\mathsf{C}}^{\mathsf{bi}}$
$\overline{\delta}^{l}, \overline{\delta}^{r}$	II.113	distributivity morphisms of $\overline{\mathsf{C}}^{\mathsf{bi}}$
$(A; \beta^{A})$	II.114	an object in $\overline{\mathsf{C}}^{\mathsf{bi}}$
$\mathsf{C}^{\mathsf{sym}}$	II.127	bimonoidal symmetric center

Chapter II.5

$\mathcal{D}^{\mathsf{br}}$	II.137	braided distortion category
$\underline{r}$	II.137	an object in $\mathcal{D}^{\mathsf{br}}$
$\varnothing$	II.137	empty sequence in $\mathcal{D}^{\mathsf{br}}$
$\underline{\sigma}$	II.137	a morphism in $\mathcal{D}^{\mathsf{br}}$
$\int_{\mathsf{C}} F$	II.156	Grothendieck construction of $F : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{Cat}$
$\mathcal{B}$	II.157	braid category
F^{br}	II.157	functor $\Sigma^{\mathsf{op}} \longrightarrow \mathsf{Cat}$

Chapter II.6

$(F, F^2_{\oplus}, F^0_{\oplus}, F^2_{\otimes}, F^0_{\otimes})$	II.164	a braided bimonoidal functor
$F_{\oplus}$	II.164	additive structure $(F, F^2_{\oplus}, F^0_{\oplus})$
$F_{\otimes}$	II.164	multiplicative structure $(F, F^2_{\otimes}, F^0_{\otimes})$

$\mathsf{Bi}^{\mathsf{br}}$	II.166	category of small braided bimonoidal categories
$(-)_{\mathsf{rt}}$	II.169	right normalized bracketing
$\underline{a}$	II.169	an object in A
π	II.169	realization function $\mathsf{Ob}(\mathsf{A}) \longrightarrow \mathsf{Ob}(\mathsf{C})$
$\cong^{\oplus}_{\mathsf{ML}}$	II.170	a Mac Lane coherence isomorphism
$\cong_{\mathsf{Lap}}$	II.172	a Laplaza coherence isomorphism
A	II.180	associated right permbraided category
π	II.181	functor $\mathsf{A} \longrightarrow \mathsf{C}$
ι	II.181	functor $\mathsf{C} \longrightarrow \mathsf{A}$

Chapter II.7

$\mathsf{Bi}^{\mathsf{br}}$	II.190	2-category of small braided bimonoidal categories
$\mathsf{Bi}^{\mathsf{fbr}}_{\mathsf{r}}$	II.191	full sub-2-category of $\mathsf{Bi}^{\mathsf{br}}$ with flat objects and robust 1-cells
Σ	II.193	category of $n \geq 0$ and permutations
$(-)_{\mathsf{lt}}$	II.194	left normalized bracketing
$\overline{n}$	II.194	left normalized sum of n copies of $\mathbb{1}$
θ^{G}	II.199	unique bimonoidal natural transformation $F \longrightarrow G$

Chapter II.8

$(\mathsf{B}, 1, c, a, \ell, r)$	II.205	a bicategory
$\mathsf{Mat}^{\mathsf{C}}_{m,n}$	II.206	category of $n \times m$ matrices in C
$A = (A_{ji})$	II.206	a matrix with (j,i)-entry A_{ji}
$\mathbb{0}_{m,n}$	II.207	0 matrix in $\mathsf{Mat}^{\mathsf{C}}_{m,n}$
BA	II.207	matrix product
$g \star f$	II.207	matrix product of morphisms
$\mathbb{1}^{n}$	II.207	$n \times n$ identity matrix
$\zeta^{\ell}, \zeta^{r}, \ell, r, a$	II.208	natural isomorphisms in $\mathsf{Mat}^{\mathsf{C}}$
$(1_{\boxtimes}, 1^{2}_{\boxtimes}, 1^{0}_{\boxtimes})$	II.214	monoidal identity
$A \boxtimes B$	II.215	matrix tensor product
$(\boxtimes, \boxtimes^{2}, \boxtimes^{0})$	II.218	monoidal composition
$(a^{\boxtimes}, a^{\boxtimes\bullet}, \eta^{a}, \varepsilon^{a})$	II.223	monoidal associator
$(\ell^{\boxtimes}, \ell^{\boxtimes\bullet}, \eta^{\ell}, \varepsilon^{\ell})$	II.225	left monoidal unitor
$(r^{\boxtimes}, r^{\boxtimes\bullet}, \eta^{r}, \varepsilon^{r})$	II.227	right monoidal unitor
π	II.227	pentagonator
μ	II.228	middle 2-unitor
$\lambda^{\boxtimes}$	II.228	left 2-unitor
$\rho^{\boxtimes}$	II.228	right 2-unitor

Part II.2
Chapter II.9

$(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}, \otimes, \mathbb{1}, \partial^{l}, \partial^{r})$	II.238	a ring category
$\partial^{l}, \partial^{r}$	II.239	left and right factorization morphisms
$\mathsf{End}(A)$	II.243	endomorphism rig of a monoid
$\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{D})$	II.244	category of strictly unital symmetric monoidal functors
$\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$	II.245	endomorphism ring category
$(\boxplus, \mathbb{0}, \xi^{\boxplus})$	II.245	additive structure of $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$
$\mathsf{Perm}^{\mathsf{sug}}(\mathsf{C};\mathsf{C})$	II.249	a full subcategory of $\mathsf{Perm}^{\mathsf{su}}(\mathsf{C};\mathsf{C})$
$(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}, \otimes, \mathbb{1}, \partial^{l}, \partial^{r})$	II.251	a bipermutative category
$(\mathsf{C}, \oplus, \mathbb{0}, \xi^{\oplus}, \otimes, \mathbb{1}, \partial^{l}, \partial^{r})$	II.259	a braided ring category
$\overline{\mathsf{C}}^{\mathsf{bi}}$	II.262	bimonoidal Drinfeld center
$\mathsf{C}^{\mathsf{sym}}$	II.263	symmetric center

Chapter II.10

Part III.1

Chapter III.1

Chapter III.2

Chapter III.3

Chapter III.4

Chapter III.5

$\langle c\rangle\sigma$	III.186	right permutation of $\langle c\rangle$ by σ
1_c	III.186	c-colored unit
$\mathsf{End}(X)$	III.189	endomorphism operad of X
$\mathsf{End}(\mathsf{C})$	III.189	endomorphism multicategory of C
$F : \mathsf{M} \longrightarrow \mathsf{N}$	III.189	multifunctor from M to N
$\alpha : F \longrightarrow G$	III.191	multinatural transformation from F to G
$\beta\alpha$	III.191	vertical composition
$\alpha' * \alpha$	III.191	horizontal composition
$\mathsf{Multicat}$	III.191	2-category of small multicategories
I	III.191	initial operad
$\mathsf{T} = \mathsf{Com}$	III.191	terminal multicategory; commutative operad
$(\mathsf{M}, *^{\mathsf{M}}, \iota^{\mathsf{M}})$	III.194	pointed multicategory
$\mathsf{Multicat}_*$	III.195	2-category of small pointed multicategories
$\mathsf{PermCat}^{\mathsf{su}}$	III.195	2-category with strictly unital symmetric monoidal functors
(T, μ, η)	III.198	monad T with multiplication μ and unit η
$\mathsf{Alg}(T)$	III.198	category of T-algebras
Δ_W	III.199	constant functor at W
$\mathsf{Vt}X$	III.204	vertices of a multigraph X
$X(\langle c\rangle ; c')$	III.204	multiedges with source $\langle c\rangle$ and target c'
MGraph, MGraph^C	III.204	category of small multigraphs; subcategory with vertex set C
$\mathsf{Multicat}^C$	III.205	subcategory of multicategories with object set C
(L^C, U^C)	III.205	left adjoint to forgetful $U^C : \mathsf{Multicat}^C \longrightarrow \mathsf{MGraph}^C$
f_0^*	III.205	change of object/vertex set by function $f_0 : C \longrightarrow D$
(L, U)	III.206	left adjoint to forgetful $U : \mathsf{Multicat} \longrightarrow \mathsf{MGraph}$
$\langle c\rangle \otimes \langle d\rangle$, $\langle c\rangle \otimes^{\mathsf{t}} \langle d\rangle$	III.210	product and transposed product of tuples
$\xi^{\otimes}$	III.210	transpose permutation
$X \,\&\, Y$	III.210	product of multigraphs
$\mathsf{M} \,\#\, \mathsf{N}$	III.211	sharp product of multicategories
$\phi \otimes \psi$, $\phi \otimes^{\mathsf{t}} \psi$	III.212	product and transposed product of operations
$\xi^{\otimes}$	III.212	transposition bijection for operations
$\mathsf{M} \otimes \mathsf{N}$	III.213	Boardman-Vogt tensor product of multicategories
$\wedge$, $\mathsf{S} = \mathsf{I}_+$	III.215	smash product and smash unit for pointed multicategories
$\mathsf{M} \vee \mathsf{N}$	III.215	wedge product of pointed multicategories
$\langle Fc\rangle$, $\langle Fc\rangle^{\mathsf{t}}$, $\langle F\rangle\phi$	III.215	applying a tuple of operations $\langle F\rangle$ to a profile $\langle c\rangle$ and operation ϕ
$\mathsf{Hom}(\mathsf{M}, \mathsf{N})$	III.216	internal hom for multicategories
$\alpha : \langle F\rangle \longrightarrow G$	III.216	transformation in $\mathsf{Hom}(\mathsf{M}, \mathsf{N})$
$\mathsf{Hom}_*(\mathsf{M}, \mathsf{N})$	III.225	pointed hom multicategory

Chapter III.6

$(\mathsf{V}, \otimes, \xi)$	III.230	permutative category, monoidal product, and symmetry
$(\mathsf{M}, \gamma, 1)$	III.230	V-enriched multicategory, composition, and unit
$\mathsf{M}(\langle c\rangle ; c')$	III.230	n-ary operation object with input profile $\langle c\rangle$ and output c'
$\langle c\rangle\sigma$	III.230	right permutation of $\langle c\rangle$ by σ
1_c	III.230	c-colored unit
$\mathsf{End}(c)$	III.233	endomorphism V-enriched operad of c
$F : \mathsf{M} \longrightarrow \mathsf{N}$	III.233	V-enriched multifunctor from M to N
P	III.234	V-enriched operad
(c, θ)	III.234	P-algebra
$\alpha : F \longrightarrow G$	III.234	V-enriched multinatural transformation
$\beta\alpha$	III.235	vertical composition
$\alpha' * \alpha$	III.235	horizontal composition
$\mathsf{V}\text{-}\mathsf{Multicat}$	III.235	2-category of small V-enriched multicategories

Part III.2
Chapter III.7

$X \wedge A$, X^A	III.274	simplicial tensor and cotensor for symmetric sequence X
$\mathsf{Map}_\Sigma(X,Y)$, $\mathsf{Hom}_\Sigma(X,Y)$	III.274	symmetric mapping object and symmetric hom object
$i_n \dashv \mathsf{ev}_n$	III.274	evaluation at $\underline{n}$ for symmetric sequences, and left adjoint i_n
$X^{\wedge k}$	III.275	right normalized iterated smash product
$\mathsf{Sym}(X)$	III.275	symmetric sequence given by smash powers of $X \in \mathsf{sSet}_*$
$S = \mathsf{Sym}(S^1)$	III.276	symmetric sphere
(A,μ,η), (M,θ)	III.276	commutative monoid and left module
SymSp	III.276	category of symmetric spectra
ρ, $\rho^{\Sigma}_{p,q}$, $\rho_{p,q}$	III.276	structure morphisms for symmetric spectrum
$\Sigma^\infty K$	III.277	suspension spectrum of $K \in \mathsf{sSet}_*$
HR	III.278	Eilenberg-Mac Lane spectrum of commutative ring R
$(A \otimes -, \pi, \nu)$	III.279	monad associated to monoid A
$X \wedge Y = X \square_S Y$	III.282	smash product of symmetric spectra
$\mathsf{Hom}_S(X,Y)$	III.284	internal hom for symmetric spectra
$F_n \dashv \mathsf{ev}_n$	III.285	evaluation at $\underline{n}$ for symmetric spectra, and left adjoint F_n
$\mathsf{Map}_S(X,Y)$	III.287	mapping simplicial set for symmetric spectra X and Y
$\mathsf{End}(X)$	III.287	endomorphism simplicial operad for symmetric spectrum X
$(\mathcal{L},\mathcal{R})$	III.287	weak factorization system
$\mathsf{M}, \mathcal{W}, \mathcal{C}, \mathcal{F}$	III.288	model category, weak equivalences, cofibrations, and fibrations
A^c, A^f	III.288	cofibrant replacement and fibrant replacement
$\mathsf{Cyl}(A)$, $\mathsf{Path}(A)$	III.288	cylinder object and path object
$\gamma : \mathsf{M} \longrightarrow \mathit{Ho}\mathsf{M}$	III.290	homotopy category and canonical functor of M
$\mathsf{M} \simeq_Q \mathsf{N}$	III.292	Quillen equivalence between model categories
$\mathsf{Cell}(\mathcal{I})$	III.293	relative $\mathcal{I}$-cell complexes
$f \square g$	III.293	pushout product of f and g

Chapter III.8

FinSet	III.300	category of pointed finite sets and pointed morphisms
$\mathcal{F}$	III.300	small skeleton of FinSet with objects $\underline{n} = \{0,1,\ldots,n\}$
$a^\flat$	III.301	punctured finite set
$\lvert a \rvert$	III.301	cardinality of a
$L : \underline{m} \wedge \underline{n} \cong \underline{mn}$	III.301	lexicographic order bijection
$X : (\mathcal{F}, \underline{0}) \longrightarrow (\mathsf{C}, *)$	III.302	Γ-object in C
Γ-C	III.302	category of Γ-objects in C
$p_n : X\underline{n} \longrightarrow \prod X\underline{1}$	III.303	nth Segal map
Γ-sSet	III.303	category of Γ-simplicial sets
Γ-Cat	III.303	2-category of Γ-categories
$N_* : \Gamma\text{-}\mathsf{Cat} \longrightarrow \Gamma\text{-}\mathsf{sSet}$	III.304	functor induced by composition with N
$\overline{S}$	III.304	the $\mathcal{F}$-sphere
h_i, $\eta_{m,n,i}$, $\eta_{m,n}$	III.304	structure morphisms for $\underline{m} \wedge X\underline{n} \longrightarrow X\underline{mn}$
$\mathsf{K}^{\mathcal{F}} X$	III.305	K-theory symmetric spectrum of Γ-simplicial set X
$(\mathsf{C},\rho) = \{\mathsf{C}_s, \rho_{s,t}\}$	III.306	$\underline{n}$-system in C with (s,t)-gluing morphsims $\rho_{s,t}$
$\{\alpha_s\}$	III.307	morphism of $\underline{n}$-systems
$\mathsf{C}^{\mathcal{F}}\underline{n} = \mathsf{C}^{\mathcal{F}}_{\cong}\underline{n}$, $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}\underline{n}$, $\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}\underline{n}$	III.308	pointed categories of strong, lax, and colax $\underline{n}$-systems in C
$\mathsf{C}^{\mathcal{F}}\psi$	III.309	pointed functor associated to $\psi : \underline{n} \longrightarrow \underline{m}$
$\mathsf{C}^{\mathcal{F}} = \mathsf{C}^{\mathcal{F}}_{\cong}$, $\mathsf{C}^{\mathcal{F}}_{\mathrm{lax}}$, $\mathsf{C}^{\mathcal{F}}_{\mathrm{co}}$	III.311	strong, lax, and colax Segal Γ-categories of C
P_v, Q_v, $v \in \{\cong, \mathrm{lax}, \mathrm{co}\}$	III.311	adjoint functors comparing Segal Γ-categories
$\mathcal{M}a$	III.314	partition multicategory of a
$2^{a^\flat}$	III.314	set of basepoint-free subsets of a
$\iota_{\langle s \rangle}$	III.314	unique operation in $(\mathcal{M}a)\big(\langle s \rangle\,;\, t\big)$ if $\langle s \rangle$ is a partition of t
$\mathcal{M}\underline{1}$	III.315	partition multicategory of $\underline{1}$
$\mathsf{J}^{\mathcal{M}} = \mathsf{Multicat}_*(\mathcal{M}, -)$	III.319	$\mathcal{M}$-partition J-theory

Chapter III.12

Chapter III.13

Appendix III.A

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